Boundary Value Problems for Evolutions of Willmore Type

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Abstract

Geometric gradient flows of energy functionals involving the curvature of a given object have become an indispensable tool both to understand problems in pure mathematics and to model a wide range of phenomena in the natural sciences. A prominent example of such a functional is the Willmore energy which is given by the integrated squared mean curvature of the given surface. Its L^2–gradient flow, the Willmore flow, is a parabolic quasilinear evolution law of fourth order.
While there is an abundance of results on the behaviour of curves and closed surfaces, there remains a huge number of open questions regarding boundary value problems for the Willmore flow.

The first part of the thesis is devoted to the Willmore flow of two- or higher-dimensional immersed compact open hypersurfaces in Euclidean space with Navier boundary conditions which demand the boundary to remain fixed during the evolution and the mean curvature to vanish on the boundary.
We initiate the research on this flow showing existence of strong solutions in anisotropic Sobolev spaces given a sufficiently smooth initial surface that has zero mean curvature on the boundary and is close to an appropriate reference manifold. The regularity of the initial immersion corresponds to the trace space of the solution space. Considering motions that are given as graphs over the fixed reference geometry we may write the evolution in terms of a scalar function describing the position of the evolving surface with respect to the reference manifold. The required analysis is technically elaborate as the evolution needs to be translated to local charts.

In the second part of the thesis we study planar networks composed of three immersed curves that meet in one or two triple junctions and may or may not have endpoints fixed in the plane. The elastic energy of such a configuration is given by the sum of the Willmore energies of the single curves each containing a positively weighted length penalisation term. Its L^2–gradient flow leads to a system of Willmore type evolution laws with natural nonlinear coupled boundary conditions. Hereby, the curves need to stay attached but the junctions are allowed to move. The major difficulties lie in the tangential degrees of freedom which are due to geometric nature of the problem.
We show existence of solutions in the strong and classical sense, namely in anisotropic Sobolev spaces and parabolic Hölder spaces. In both cases compatibility and regularity assumptions on the initial network are required. We further establish uniqueness of solutions in both function space settings in a purely geometric sense showing that any two observable motions solving the flow are reparametrisations of each other. The parabolic nature of the problem allows us to show in addition that solutions are smooth for positive times.
As a main result we show that the flow exists globally in time if the length of each curve remains uniformly bounded away from zero and if at least one angle at the triple junction stays uniformly bounded away from zero, π and 2π. The proof relies on energy estimates and the existence of solutions in the Sobolev setting on time intervals of uniform length quantifiable in terms of the initial network.

Zusammenfassung

Geometrische Gradientenflüsse von Krümmungsenergien sind ein unentbehrliches Hilfsmittel, so–wohl zum Verständnis von Fragestellungen in der reinen Mathematik als auch zur Modellierung eines breiten Spektrums von Phänomenen in den Naturwissenschaften. Ein bekanntes Beispiel solcher Energiefunktionale ist die Willmore-Energie, die durch die integrierte quadrierte mittlere

Krümmung der betrachteten Fläche gegeben ist. Der zugehörige L^2–Gradientenfluss, Willmore-Fluss genannt, ist eine parabolische quasilineare Evolutionsgleichung vierter Ordnung.
Gegenüber einer Fülle an Resultaten über das Verhalten von Kurven und geschlossenen Flächen gibt es eine Vielzahl offener Probleme über Randwertprobleme für den Willmore-Fluss.

Der erste Teil der Arbeit befasst sich mit dem Willmore-Fluss von zwei- oder höherdimensionalen immersierten kompakten offenen Hyperflächen im Euklidischen Raum mit Navier-Randbedingungen, die der Forderung entsprechen, dass der Rand während der Evolution fixiert bleibt und die mittlere Krümmung am Rand verschwindet.
Wir beginnen die Forschung zu diesem Fluss, indem wir Existenz starker Lösungen in anisotropen Sobolevräumen zeigen für hinreichend glatte Anfangsflächen, deren mittlere Krümmung am Rand verschwindet und die nahe an einer geeigneten Referenzmannigfaltigkeit liegen. Die Regularität der Immersion zum Anfangszeitpunkt entspricht dem Spurraum des Lösungsraums. Indem wir Bewegungen betrachten, die als Graph über einer fixierten Referenzgeometrie gegeben sind, können wir die Evolution mittels einer skalarwertigen Funktion ausdrücken, die die Position der evolvierenden Fläche zur Referenzmannigfaltigkeit beschreibt. Die benötigte Analysis ist technisch aufwendig, da die Evolution in lokalen Koordinaten betrachtet werden muss.

Im zweiten Teil der Arbeit studieren wir planare Netzwerke zusammengesetzt aus drei immersierten Kurven, die sich in einem oder zwei Tripelpunkten treffen, wobei im ersten Fall die verbleibenden Endpunkte fixiert sind. Die elastische Energie einer solchen Konfiguration ist durch die Summe der Willmore-Energien der einzelnen Kurven gegeben, die jede einen positiv gewichteten Term beinhalten, der die Länge der jeweiligen Kurve bestraft. Der zugehörige L^2–Gradientenfluss führt zu einem System von Willmore-ähnlichen Evolutionsgleichungen mit natürlichen nichtlinearen gekoppelten Randbedingungen. Die Tripelpunkte müssen dabei erhalten bleiben, dürfen sich jedoch bewegen. Die hauptsächlichen Schwierigkeiten liegen in den aufgrund der geometrischen Natur des Problems auftretenden tangentialen Freiheitsgraden.

Wir zeigen Existenz von Lösungen im starken und klassischen Sinne, nämlich in anisotropen Sobolevräumen und in parabolischen Hölderräumen. In beiden Fällen werden Kompatibilitäts- und Regularitätsbedingungen an das Anfangsnetzwerk benötigt. Außerdem weisen wir geometrische Eindeutigkeit von Lösungen sowohl in Hölder- als auch in Sobolevräumen nach, indem wir zeigen, dass zwei beobachtbare Lösungen des Flusses durch Umparametrisierung ineinander überführt werden können. Die parabolische Struktur des Problems erlaubt es uns außerdem zu zeigen, dass Lösungen für positive Zeiten glatt sind.
Als Hauptresultat zeigen wir, dass der Fluss für alle Zeiten existiert, sofern die Länge jeder Kurve nach unten durch eine positive Zahl beschränkt bleibt und solange sich an jedem Tripelpunkt wenigstens ein Winkel in einem zeitunabhängigen, abgeschlossenen Teilintervall von $(0, \pi)$ befindet. Der Beweis basiert auf Energieabschätzungen und der Existenz von Lösungen in Sobolevräumen in gleichmäßigen, mittels einer Norm des Anfangswertes quantifizierbaren Zeitintervallen.

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Contents

Introduction

The significance of geometric flows and state of the art

In the recent decades the field of geometric evolution equations has evolved into a flourishing research area at the border between analysis and geometry with stupendous significance both in pure and applied mathematics. One distinguishes between *extrinsic* and *intrinsic* flows. Extrinsic geometric flows are evolutions of manifolds that are embedded or immersed in an ambient space. The evolution law is usually given in terms of geometric quantities of the manifold, such as for instance the mean curvature, and typically affects both the immersion and the metric of the manifold. In contrast, intrinsic flows are formulated on manifolds independent of any surrounding space merely in terms of intrinsic quantities with the *Ricci flow* being their probably most prominent example. The Ricci flow was first discovered by Hamilton who initiated a research program [73] revealing its deep connection to topology and paving the way for the proof, by Perelman [119, 120, 121], of the Poincaré conjecture, which was at the time one of the most famous open problems in mathematics.

Even earlier, Hamilton's techniques proved to be a powerful tool in the study of the *mean curvature flow*, the gradient flow of the area functional and probably the most famous representative of an extrinsic geometric flow. A surface embedded in some ambient space moves with normal velocity equal to its mean curvature which decreases the area of the surface most efficiently. This flow has originally been suggested to model the evolution of interfaces in soap, see [143], and the motion of grain boundaries in crystalline materials, see [109]. We refer to [60] for an overview of the mean curvature flow and its various applications in the natural sciences. Also from the mathematical point of view astonishing properties of the flow have been established in the recent decades initiated by the results [59, 69, 77] by Gage, Hamilton, Grayson and Huisken. We emphasise that this is a very non-complete list and refer the reader to [30, 50, 99, 127] for results and references regarding the mean curvature flow.

In this thesis we study boundary value problems for extrinsic flows of *Willmore* type arising as gradient flows of energy functionals involving the *mean curvature* of the surface, an extrinsic quantity measuring how the surface is curved with respect to the ambient space. Given an immersion $f : M \to \mathbb{R}^{n+1}$ of an n–dimensional smooth compact manifold M we may define a Riemannian metric g on M by considering the pull-back of the Euclidean metric under f. The volume measure on M with respect to g is then denoted by $\mathrm{d}V_g$. The Willmore functional, originally proposed by Thomsen in [139] and later extensively studied by Willmore in [148, 149], associates to each sufficiently smooth immersion $f : M \to \mathbb{R}^{n+1}$ an energy $W(f)$ given by

$$W(f) = \frac{1}{2}\int_M H^2 \,\mathrm{d}V_g$$

describing the total bending energy of the object $f(M)$ in terms of its mean curvature H given by the sum of the principal curvatures of the surface.
We remark that there are different conventions in the literature regarding the normalisation factor in front of the integral.
The *Willmore flow* is then given by the L^2–gradient flow of the Willmore energy. A sufficiently smooth one-parameter family of immersions $f : [0,T) \times M \to \mathbb{R}^n$ with $T > 0$ or $T = \infty$ is a solution to the Willmore flow if its normal velocity satisfies the fourth order evolution law

$$\langle \partial_t f, \nu\rangle = -\langle \Delta_g \boldsymbol{H}, \nu\rangle - Q(A)H \tag{WF}$$

where ν is the normal along f, $\boldsymbol{H} = H\nu$ the mean curvature vector, Δ_g the Laplace Beltrami operator and $Q(A)$ is quadratic in the second fundamental form A. The existence time T is called the life span of the solution f.

Energies involving the principal curvatures of two-dimensional surfaces embedded in $\mathbb{R}^3$ naturally appear in various physical contexts and are thus not of purely mathematical interest. Applications include for example the description of elastic shells going back to the work of Poisson [123] and Germain [67], as well as models for biological membranes due to Canham [27] and Helfrich [76]. Generalised Willmore energies of Canham-Helfrich type typically involve also the Gaussian curvature of the surface and occur for example when two different phases are separated by a curved membrane in three dimensional Euclidean space, see for instance [55] and the references therein. An example of a model involving the bending energy of lipid bilayers is given in [54]. For further information on modelling aspects and mathematical results related to Canham-Helfrich models we refer to [18, 23, 39, 54, 55, 80, 93, 105, 110, 112] and the references therein.

In the case $n = 1$ an immersion $\gamma : M \to \mathbb{R}^2$ describes a planar curve defined on an interval $M = I$ and the Willmore energy of γ coincides (up to multiplication by a constant) with the *elastic energy* $E(\gamma)$ given by

$$E(\gamma) = \int_I \kappa^2 \,\mathrm{d}s\,.$$ [1]

Here we denote by $\mathrm{d}s = |\gamma_x|\,\mathrm{d}x$ and $\partial_s = |\gamma_x|^{-1}\,\partial_x$ the arclength measure and arclength derivative, respectively. The unit normal ν is the counter-clockwise rotation of $\gamma_x\,|\gamma_x|^{-1}$ by the angle $\frac{\pi}{2}$ and the curvature of γ is given by $\kappa = \langle \partial_s^2\gamma, \nu\rangle$. The study of the elastic energy and its stationary points, the *elastica*, goes back to D. Bernoulli and Euler who used the curvature integral to model the bending of thin inextensible elastic rods [56]. A discussion of classical results and applications can be found in [142]. We refer to [87] for a classification of elastica in $\mathbb{R}^2$. A common variant of the elastic energy is given by

$$E_\mu(\gamma) = \int_I \kappa^2 + \mu \,\mathrm{d}s\,, \qquad \mu \geq 0\,,$$

with the additional term acting as a length penalisation. The L^2–gradient flow of an energy E_μ, $\mu > 0$, is also referred to as *elastic flow*. Its evolution equation is given by

$$\langle \partial_t\gamma, \nu\rangle = -2\partial_s^2\kappa - \kappa^3 + \mu\kappa\,. \tag{EF}$$

In the following we give an overview of results for the minimisation and the L^2–gradient flow of E_μ and W that are most relevant regarding the approach and techniques considered in this thesis. However, we emphasise that this discussion does not claim to be a complete presentation of all related contributions available in the literature.

The study of the evolutionary problem of E_μ was initiated by Polden in [124, 125] in the case $\mu > 0$ where the author shows long time existence of the elastic flow of closed planar curves and sub-convergence to an elastica. In [49] the result was then extended to closed curves in higher codimension where the authors prove global existence of the flow in the case $\mu \geq 0$ and in addition sub-convergence up to translation to an elastica in the case $\mu > 0$. For related results on the elastic motion of closed curves we refer to [81, 87, 117, 118, 144, 146].

Also in the case of closed surfaces, namely compact n–dimensional manifolds without boundary, both the stationary and the evolutionary problem associated to the Willmore energy are rather

[1] In contrast to the higher dimensional case we consider the elastic energy without the normalisation factor $\frac{1}{2}$.

well understood, especially in the case $n = 2$ in which the energy is invariant under conformal transformations. It is well-known that for any immersion $f : M \to \mathbb{R}^3$ of a closed 2–dimensional surface M there holds $\frac{1}{2}W(f) \geq 4\pi$ with equality only for round spheres, see [28, 148]. Willmore conjectured in [148] that for any immersion f of the 2–dimensional torus in $\mathbb{R}^3$ there holds $\frac{1}{2}W(f) \geq 2\pi^2$, and that the minimum of W among all immersed tori in $\mathbb{R}^3$ is attained by the Clifford torus (and its conformal images). This conjecture was proven by Marques and Neves in [103] building on the partial result by Li and Yau [90]. For other results related to the existence and characterisation of closed embedded surfaces minimizing the functional W we refer to [20, 26, 128, 134, 148, 149] and the references therein.

The study of the L^2-gradient flow of W for surfaces was initiated by Kuwert and Schätzle in [84] where the authors give a lower bound on the life span of any smooth solution to the Willmore flow of a closed 2–dimensional surface M in $\mathbb{R}^3$ (and also in higher co-dimension) depending only on how much the curvature of the initial surface is concentrated in space. This work was extended in [83, 85] by the same authors where global existence of the flow and convergence to a sphere, the global minimizer of the energy, is proven for immersions of the sphere $M = \mathbb{S}^2$ with initial surface $f_0 : \mathbb{S}^2 \to \mathbb{R}^3$ satisfying $\frac{1}{2}W(f_0) \leq 8\pi$. This is a generalisation of the earlier work by Simonett [135] who proved long time existence and convergence to a sphere under the assumption that the initial surface is close to a sphere in the $C^{2+\alpha}$–topology. We refer to [86] for stability estimates related to the results in [83, 84, 85] on the Willmore flow with small initial energy.
Numerical simulations by Mayer and Simonett in [104] gave rise to the conjecture that the flow can develop singularities in finite time if the smallness assumption is violated. The onset of singularities and the optimality of the constant 8π was analytically confirmed by Blatt in [21] where the author proves that there are immersions $f_0 : \mathbb{S}^2 \to \mathbb{R}^3$ with $\frac{1}{2}W(f_0)$ arbitrarily close to 8π which become singular under the Willmore flow in the sense that either the diameter does not stay bounded or that locally the curvature blows up. In [29] the authors exclude that such blow-ups are compact, and further establish another stability result for the Willmore flow with initial surface near a local minimiser. However, it is in general still an open question whether singularities develop in finite or infinite time and how the flow behaves near the maximal time of existence. In this context, it is worth mentioning the contribution [106] where the authors show the existence of finite time singularities to a locally constrained Willmore flow. An improvement of the just mentioned result is given in [22].

While the evolutionary and stationary problem associated to E_μ and W are well understood in the case of closed manifolds, much less is known regarding the behaviour of manifolds with boundary (in particular in higher dimension). Regarding the applications it is in particular important to understand the evolution of more complicated configurations composed of several individual open surfaces or curves meeting with a common boundary and forming a connected cluster or network, respectively.
This thesis gives a contribution to both the Willmore flow of open surfaces and the Willmore flow of networks. Before presenting the results of this thesis we briefly summarise what is known about the statics and dynamics of the energies E_μ and W in the case of open manifolds.

In comparison to the closed case the problems become more involved as boundary conditions need to be imposed. We refer to [18, 112] for a discussion of possible choices. The major part of the present literature considers either *clamped* or *Navier* boundary conditions. The Navier conditions demand the position of the boundary to be fixed and the mean curvature of the manifold to vanish on the boundary. These are natural conditions arising from the first variation of the functional. As suggested by its name, clamped boundary conditions correspond to fixing the position of the boundary and the angle with which the manifold meets its boundary.

For results on the existence of open elastica, namely the minimisation problem of the elastic energy in one dimension subject to clamped or Navier conditions, we refer to [41, 42, 92, 130] and the references therein. Numerical simulations on the elastic flow of open curves subject to Navier or clamped boundary conditions, can be found in [17]. The following results on the long time behaviour of the elastic flow of open curves in $\mathbb{R}^n$ are valid in the case of a positive length penalisation parameter $\mu > 0$. In [91] long time existence of global smooth solutions and sub-convergence to a stationary point is shown in the clamped case which means that the boundary points and the tangents at the boundary are fixed during the evolution. The details for short time existence of the flow are carried out in [137]. In [40] the authors improved the result in [91] showing convergence of the flow to an elastica. In [34] and [39] long time existence of the flow with Navier conditions as well as sub-convergence to stationary points are shown, in the former case with fixed length.
The elastic flow of open curves with infinite length has been considered in [113] where the authors show global existence of the flow and sub-convergence to a stationary point of finite energy in the case $\mu > 0$. By considering a general framework for gradient flows of geometric energy functionals defined on planar curves the contribution [114] improves the results in [91, 113] turning sub-convergence into convergence. Recently, a global existence result for the vanilla elastic flow corresponding to the case $\mu = 0$ has been published in [147] for open curves with boundary points moving freely on parallel lines.

While the stationary and evolutionary problem associated to the elastic energy of open curves is rather well understood, the higher dimensional analogues are still subjects of ongoing research. However, there are a number of results concerning the minimisation of the energy W among certain classes of open surfaces. In [31] existence of smooth immersed Willmore surfaces of revolution in $\mathbb{R}^3$ with prescribed boundary and fixed conormal is obtained by rotating the graph of a positive smooth even function. The results are generalised in the contribution [33]. Analogous results to [31] in the case of Navier conditions can be found in [43]. In [32] the authors extend the contributions in [43] showing the existence of unstable smooth immersed Willmore surfaces of revolution with Navier boundary conditions. Further results on the existence of open Willmore surfaces and their properties can for instance be found in [44, 52, 115, 145].

There are a huge number of open questions on the Willmore flow of open surfaces. As in the associated stationary problem appropriate conditions at the boundary need to be considered. A variety of possible sets of boundary constraints is proposed in [18] where the authors provide a numerical analysis of gradient flows for energy functionals quadratic in the principal curvatures of a given surface in $\mathbb{R}^3$. Here also free and semi-free boundary conditions are treated. A short time existence result for the Willmore flow of an embedded open surface in $\mathbb{R}^3$ with boundary curve freely moving on the boundary of a fixed domain is given in [1, 108]. Here also the line tension of the boundary curve is taken into account.

New results contributed in this thesis

In the first part of the thesis we give a local existence result for the Willmore flow of compact immersed hypersurfaces with boundary. We hereby consider a time-dependent family of immersions $f : [0, T] \times M \to \mathbb{R}^{n+1}$ defined on a smooth compact manifold M with boundary of arbitrary dimension $n \geq 2$. Imposing the constraint that the boundary stays fixed during the evolution the first variation of the Willmore energy gives rise to the condition that the mean curvature vanishes on the boundary throughout the flow. These conditions are referred to as the Navier conditions. Together with the evolution law (WF) and an appropriate initial condition we obtain an initial-boundary value problem for f on the manifold M. The motion equation is parabolic quasilinear of fourth order in f and degenerate, as only the normal movement is specified. To find a more convenient formulation of the problem we introduce a reference geometry $\varphi : M \to \mathbb{R}^{n+1}$ and consider initial immersions that are "close" to φ in a C^1–sense and coincide with φ on the boundary of M. We further do not allow for tangential movement of the boundary. The Navier conditions for the evolution f then yield

$$\begin{aligned} f &= \varphi \qquad \text{on } [0,T] \times \partial M\,, \\ H &= 0 \qquad \text{on } [0,T] \times \partial M\,. \end{aligned}$$

For small time the evolution can be written in terms of a height function indicating the position of the surface with respect to the reference geometry which results in an initial boundary value problem for a scalar function defined on M. The concept of solving higher dimensional flows by considering solutions that are given as graphs over an initial or reference surface has for instance been used in [51, 78, 138] in the case of mean curvature flow.
As we are confronted with a higher order evolution law on a manifold with boundary, technically elaborate methods have to be accessed. While other contributions on existence of geometric flows for surfaces with boundary use existence of weak solutions for the linearised problem and regularity estimates, see for instance [47, 62], we prove existence of strong solutions in anisotropic Sobolev spaces

$$W_p^1\left((0,T); L_p(M)\right) \cap L_p\left((0,T); W_p^4(M)\right)\,.$$

The evolution law should then be understood to hold in an almost everywhere sense. The integration parameter $p \in (4+n, \infty)$ is chosen in such a way that the boundary conditions hold pointwise in time and space. In particular, the initial datum needs to be compatible in the sense that it satisfies the Navier conditions. Furthermore, one needs to impose a regularity requirement on the initial value corresponding to the solution space which is however in comparison rather mild. This might prove helpful in further research concerning global existence of the flow using similar methods as in the second part of this thesis. To our best knowledge we hereby provide the first result on the higher-dimensional Willmore flow with Navier conditions.
Generally speaking, our existence proof follows a classical procedure. We show well-posedness of an associated linear evolution problem on the manifold and deduce existence of the original system by an application of the contraction mapping principle. However, to conduct these steps the problem has to be translated to local charts. In doing so, it is crucial to consider a coordinate system that is well-adapted to the geometry taking into account that the boundary of M is curved. Taking advantage of the compactness of the manifold the normal collar coordinates introduced in [12, 72, 131] are a suitable candidate allowing for uniform bounds on the metric and a localised characterisation of the solution space in terms of the spaces on the chart domains. To prove well-posedness of the linearised system on the local level the classical theory [136] by Solonnikov provides a suitable framework in the Sobolev setting. To obtain the necessary contraction property rigorous estimates have to be derived.
The strategy pursued in the first part of this thesis may be applied to an entire class of parabolic initial-boundary value problems formulated on compact manifolds, namely such that fit into the setting in [136].
It is worth mentioning that in [71] the authors develop a framework treating a class of second order boundary value problems on manifolds with boundary and of bounded geometry that fit into the setting discussed in [3], in some sense the time-independent analogue to the problems studied in [136].

The second part of the thesis is based on the publications [64, 65] of the present author in collaboration with Prof. Dr. Harald Garcke and Alessandra Pluda, PhD. We contribute to the question how singular structures behave under flows of Willmore type. Networks are an example of such generalised submanifolds in one dimension. We consider the situation of three immersed planar curves meeting in one or two triple junctions leading to two prototypes which we refer to as *Triod* and *Theta network*. The elastic energy of a network $\gamma = \left(\gamma^1, \gamma^2, \gamma^3\right)$ composed of curves $\gamma^i : [0,1] \to \mathbb{R}^2$ is then given by

$$E_\mu(\gamma) = \sum_{i=1}^{3} E_{\mu^i}(\gamma^i) = \sum_{i=1}^{3} \int_{[0,1]} \left(\kappa^i\right)^2 + \mu^i \,\mathrm{d}s^i\,,$$

for given parameters $\mu^i \geq 0$ where the geometric quantities with superscript i refer to the curve γ^i. Elastic energies of this type appear in beam and rod models of Euler-Bernoulli type in situations where beams and rods form networks. Such structures consist of interconnected elastic elements and involve coupling conditions which take into account possible constraints on the geometry of the admissible configurations. Physically, this problem would occur if the network relaxes in a very viscous medium. More realistic evolutions would also include inertia effects which however falls out of the scope of this work.

We consider the L^2–gradient flow of the elastic energy of Triods and Theta networks allowing the triple junctions to move but fixing the other endpoints in the case of Triods. This problem was proposed by Barrett, Garcke and Nürnberg in [16] where the conditions at the junctions and the endpoints are derived also in the case that constraints on the angles are imposed at the junctions. The authors further propose weak formulations to study finite element methods for the flows and provide several numerical simulations exhibiting interesting features of the flow.

In this thesis we consider the case without angle conditions. Each curve of the network satisfies the evolution law (EF). At the triple junctions the first variation gives rise to the following conditions:

$$\begin{aligned}\gamma^1 &= \gamma^2 = \gamma^3\,,\\ \kappa^1 &= \kappa^2 = \kappa^3 = 0\,,\\ \sum_{i=1}^{3} \kappa^i_s \nu^i &- \mu^i \tau^i = 0\,.\end{aligned}$$

The condition $\kappa^i = 0$ also appears in the elastic flow of open curves. The first condition ensures that the curves stay attached during the evolution while the last condition reflects the balance of forces. In the case of Triods the Navier conditions are imposed at every endpoint.

The major challenges one is facing in the study of the resulting evolution problems, which we refer to as the *geometric problems*, originate from their geometric nature. Indeed, the flow is invariant with respect to reparametrisation and uniqueness can only be expected up to reparametrisation of the curves. To prove existence of the flow with PDE methods one carefully needs to introduce an auxiliary problem with a particular choice of tangential velocity. Furthermore, additional boundary conditions need to be imposed due to the tangential degrees of freedom at the junctions and endpoints. In doing so, two objectives need to be fulfilled simultaneously. On the one hand, the extra conditions need to be of "tangential nature", namely such that they can always be obtained by reparametrising the curves appropriately. On the other hand, the boundary conditions in the resulting auxiliary system, referred to as the *analytic problem*, need to be compatible and independent enough. This property is reflected in the so-called Lopatinskii-Shapiro condition, the crucial requirement in the linear theory by Solonnikov [136], that is used to show well-posedness of a linearised system associated to the analytic problem, as it allows for coupled boundary conditions mixing the unknowns in a non-trivial way. Existence and uniqueness of the analytic problem is then obtained via contraction estimates in analogy to the strategy applied in the higher dimensional problem studied in the first part of the thesis. Hereby, the theory in [136] allows us to show both existence of strong solutions in the anisotropic Sobolev spaces

$$W^1_p\left((0,T); L_p\left((0,1); (\mathbb{R}^2)^3\right)\right) \cap L_p\left((0,T); W^4_p\left((0,1); (\mathbb{R}^2)^3\right)\right)$$

with $p \in (5, \infty)$, and of classical solutions using the so-called parabolic Hölder spaces

$$C^{\frac{4+\alpha}{4}, 4+\alpha}\left([0,T] \times [0,1]; (\mathbb{R}^2)^3\right)$$

with $\alpha \in (0,1)$. The value of p guarantees that all boundary conditions can be evaluated in time and space. In both settings, compatibility conditions need to be imposed on the initial network. In contrast to the Sobolev setting the high regularity of classical solutions leads to additional conditions involving fourth order derivatives of the curves. The proof of the Lopatinskii-Shapiro condition gives rise to a geometric requirement on the initial network which we refer to as the *non-degeneracy condition*: at each triple junction at least one angle needs to be different from zero, π and 2π. In other words, the tangents are not allowed to be linearly dependent. As it turns out this condition is not only crucial for the existence of the flow but further plays a fundamental role in the characterisation of the long time behaviour of the evolution.

While the existence of strong and classical solutions to the geometric problem follows quite easily from the respective results for the analytic problem, it is non-trivial to show that solutions to the geometric problem are indeed unique in a purely geometric sense. We hereby follow the methods in [66]. Exploiting the parabolic nature of the equation we further show that solutions are smooth for positive times. The results in the work [64] cover also the existence and uniqueness of classical solutions to the flow with prescribed angles at the junctions.

The analysis of the long time behaviour relies on a priori estimates on the curvature that are obtained with energy methods similar to the techniques in [49]. As a result we obtain that for any smooth solution of the flow the L^2-norm of the second arclength derivative of the curvature is bounded uniformly in time. Difficulties in deriving this estimate arise due to the nonlinear boundary conditions and the fact that the tangential movement can not be neglected. However, imposing a uniform version of the non-degeneracy condition, the tangential velocity at the junction points can be expressed in terms of the normal velocity which is composed of purely geometric terms. In contrast to [49] we do not need uniform bounds on higher order derivatives of the curvature. This is due to our delicate existence result which allows for rather low initial regularity, namely networks of class $W_p^{4-4/p}\left((0,1); (\mathbb{R}^2)^3\right)$. The existence time is proportional to the norm of the initial network in this space and in inverse proportion to the lengths of the single curves composing the network. Using the a priori bound we then show that the flow can be extended as long as the network is non-degenerate and as long as the curves have positive length. Here it is crucial to have a bound on the length of each curve which forces us to restrict to parameters $\mu^i > 0$ in the energy. In this way we establish the following characterisation of possible singular behaviours of the flow. If the flow does not exist globally, then, for a subsequence of times approaching the maximal time of existence, one curve disappears or the angles become degenerate at one triple junction. Both scenarios can happen simultaneously, both in finite or in infinite time.

Numerical simulations by R. Nürnberg based on the methods developed in [16] give rise to the conjecture that singularities as characterised in our result may indeed occur in finite time. We refer to [19] for a detailed description of the numerical methods used by the authors Barrett, Garcke and Nürnberg to approximate curvature driven interface evolutions.

It is worth mentioning that the presented results on Triods and Theta networks may be generalised to planar configurations composed of more than three immersed curves that meet in triple junctions and may have endpoints fixed in the plane. We refer to [65] for more details on this aspect.

For results regarding the long time behaviour of the (mean) curvature flow of networks we refer to the contributions [98, 100, 101, 102, 122] by Mantegazza, Novaga and their coauthors. Short time existence results for geometric flows of triple junction clusters have been obtained by several authors, see for instance [25, 47, 58, 61, 62, 63, 66, 68, 132].

Recently, also the stationary problem corresponding to the elastic energy E_μ in the class of Theta networks has been considered [37, 38] prescribing the angles at the junctions to be of 120 degrees in order to avoid minimising sequences to shrink to a point. In the recent contribution [116] the authors study a second order gradient flow of the p–elastic energy of networks which is obtained by replacing the integration coefficient $p = 2$ by general values of $p \in (1, \infty)$. For this weaker notion of solutions the authors establish global existence and convergence to a stationary point of the energy.

After the completion of [65] we got aware of the work "Flow of elastic networks: long–time existence result" by Dall'Acqua, Lin, Pozzi, see [35], which had appeared as a preprint shortly before. Here the authors give a long time existence result under the hypothesis that smooth solutions exist for a uniform short time interval using techniques that are considerably different than the ones used in our work [65]. The required short time existence result has been delivered in addition by the same authors in their recent contribution [36].

Structure of the thesis

The body of the thesis is divided into two parts which comprise the novel results while most of the supplementing material is collected in the appendices.

Part I is devoted to the Willmore flow of compact open surfaces with Navier conditions and consists of two chapters. In Chapter 1 we derive the evolution problem for a one-parameter family of immersed compact hypersurfaces with boundary. Considering evolutions that arise as graphs with respect to a fixed reference geometry we obtain an initial-boundary value problem for a scalar function on the reference manifold. We further introduce the notion of strong solutions to this graph formulation and the implied requirements on the initial surface.
In Chapter 2 the existence of such strong graph solutions is deduced from well-posedness of an associated linearised problem with the help of a fixed point argument both fundamentally relying on localisation methods and a suitable characterisation of the solution space.

The elastic flow of networks is studied in Part II which is partitioned in three chapters. In the preliminary Chapter 3 we provide a derivation of the L^2–gradient flow of the elastic energy in the classes of Triods and Theta networks and discuss the difficulties that arise due to the tangential degrees of freedom. Taking into account that the evolution problem is invariant with respect to reparametrisation we introduce a geometric notion of solution both in the classical and strong setting and explain the implications on the initial networks. We conclude the chapter with the fundamental a priori estimates.
In Chapter 4 we show existence and uniqueness of strong and classical solutions to the geometric evolution problems for Theta networks and Triods in a purely geometric sense. The main part of the chapter is however concerned with the existence and uniqueness of classical and strong solutions to an auxiliary problem, the analytic problem, which can be treated with classical PDE methods. To obtain a purely geometric result suitable reparametrisations have to be constructed.
Chapter 5 is dedicated to give a characterisation of the long time behaviour of the flow. In consideration of the geometric nature of the problem we introduce a suitable notion of maximal solutions that allows for different parametrisations of the flow in adjacent time intervals. We conclude the main part of the thesis with some simulations which have been kindly provided by Prof. Dr. R. Nürnberg to illustrate the scenarios proposed by our result.

The appendices provide all needed background information that is not "new" mathematical content and can be considered as a reference work to the results established in this thesis. Concepts from both differential geometry and the theory of function spaces are needed. A reader who is rather familiar with these areas may directly start reading the new contributions in this thesis listed in Part I and Part II. We remark that some auxiliary results that are (more or less) known in the

literature have been moved to the appendix. This concerns in particular results related to function spaces that are used both in Part I and Part II of the thesis. Clear references to these results are given whenever they are used.
Chapter A is devoted to the collection of definitions and results around the topic of (Riemannian) manifolds. Hereby, we mainly follow the concepts given in [89] carefully referring to the respective sections in the book. Chapter B and Chapter C introduce the classes of function spaces that appear in the existence results including some of their properties. While Chapter B is devoted to function spaces on domains, Chapter C is concerned with their counterparts on compact manifolds with particular emphasis on their characterisations by local coordinates.

Notation

Let us comment on some notational conventions adopted throughout the thesis.
We let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of natural numbers while $\mathbb{N}_0$ denotes the natural numbers including 0. The Euclidean norm of a vector $x \in \mathbb{R}^n$ with $n \in \mathbb{N}$ is denoted by $|x|$ or $\|x\|$. The Euclidean scalar product of two vectors $x, y \in \mathbb{R}^n$ is be denoted by $\langle x, y\rangle$. If U is a subset of $\mathbb{R}^n$ we denote by $\overline{U}$ and ∂U the closure and the boundary, respectively, of the set with respect to the Euclidean topology on $\mathbb{R}^n$. The set $\mathbb{H}^n := \{x \in \mathbb{R}^n : x_n \geq 0\}$ refers to the closed upper half-space and $\mathrm{int}\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$ to the open upper half-space. Given a set $U \subset \mathbb{R}^n$ that is open in $\mathbb{H}^n$ we call $U \cap \mathrm{int}\mathbb{H}^n$ the *interior* of U and denote it by $\mathrm{int}\, U$. We sometimes use the convention $\mathrm{int}\, U := U$ in the case that $U \subset \mathbb{R}^n$ is open in $\mathbb{R}^n$. We use the following convention concerning Sobolev (Slobodeckij) spaces on open subsets of the closed upper half-space. Given $p \in (1, \infty)$, $s \geq 0$, and a set $U \subset \mathbb{H}^n$ that is open in $\mathbb{H}^n$, the space $W^s_p(U)$ is defined by

$$W^s_p(U) := W^s_p(\mathrm{int} U) \,.$$

In the case that $s > 0$ is non-integer, these spaces are referred to by both the terms *Sobolev-Slobodeckij space* and *Slobodeckij space*.
In Chapters 1 and 2 a smooth manifold M of dimension $n \in \mathbb{N}$ is a topological manifold M of dimension $n \in \mathbb{N}$ endowed with a fixed smooth structure $\mathcal{A}$, and hence actually a pairing $(M, \mathcal{A})$. The smooth structure $\mathcal{A}$ is usually suppressed in the notation. Any charts we consider are elements of the smooth structure. Whenever we mention the term "manifold" without further specification, its boundary may be empty or non-empty. We further use the *Einstein summation convention* which yields that if one index appears exactly twice in a product, once as an upper index and once as a lower index, the considered term should be summed over all values the respective index can take in the context.
In Part II we denote differentiation with respect to time or space with subscript: if $\eta : [0, T] \times [0, 1] \to \mathbb{R}^2$ is a curve we write $\eta_x := \partial_x \eta$ and $\eta_t := \partial_t \eta$. We remark that there are some notational differences in Part II in comparison to [64, 65]. These are mainly due to the ambition to keep the notation consistent with the preceding chapters on the higher dimensional problem. In particular, it is worth mentioning that there are different conventions regarding the expression "curve" in the literature. In contrast to the convention adopted in [64, 65] the term *curve* shall in this thesis refer to the function and not the image of the function in the plane. This is in consistency to [49] and the considerations in Chapter 1. Introducing an equivalence relation on the space of open curves by identifying curves that are equal up to reparametrisation one obtains a one-to-one correspondence between the notion of "curve" introduced here and the one in [64, 65]. In particular, the meaning and statement of the mathematical results are not affected.

Part I

Willmore Flow with Navier conditions

Main result

In the first part of the thesis we study the Willmore flow of compact open surfaces with Navier boundary conditions. Given a smooth compact oriented manifold M with boundary of dimension $n \in \mathbb{N}$, $n \geq 2$, we associate to each sufficiently smooth immersion $f : M \to \mathbb{R}^{n+1}$ the Willmore energy

$$W(f) = \frac{1}{2} \int_M H_f^2 \,\mathrm{d}V_g$$

where H_f is the mean curvature of (M, f) and $\mathrm{d}V_g$ denotes the volume measure of M with respect to the induced metric g given by the pull-back of the Euclidean metric under f. A thorough introduction to immersed hypersurfaces (M, f) in Euclidean space is given in Section 1.1. The Willmore flow arises as the L^2–gradient flow of the Willmore energy. In the case that one fixes the boundary of the immersed surfaces and also prohibits tangential movement of the boundary, the computations in Section 1.2 yield that a smooth one-parameter family f of smooth immersions $f(t) : M \to \mathbb{R}^{n+1}$, $t \in [0, T]$, is a solution to the gradient flow with initial value f_0 if f satisfies

$$\begin{cases} \langle \partial_t f, \nu \rangle &= -\langle \Delta_g \boldsymbol{H}_f, \nu \rangle - Q(A_f) H_f & \text{in } [0,T] \times M\,, \\ H_f &= 0 & \text{on } [0,T] \times \partial M\,, \\ f &= f_0 & \text{on } [0,T] \times \partial M\,, \\ f(0) &= f_0 & \text{on } \{0\} \times M\,. \end{cases} \tag{W}$$

Here $\Delta_{g(t)}$ denotes the Laplace Beltrami operator with respect to the metric $g(t)$, $\nu(t)$ is the outer unit normal to $(M, f(t))$, $\boldsymbol{H}_f(t,p) = H_f(t,p)\nu(t,p)$ the mean curvature vector of $(M, f(t))$ at $p \in M$ and $Q\left(A_{f(t)}\right)$ is a term that is quadratic in the second fundamental form $A_{f(t)}$ of $(M, f(t))$. The boundary conditions in (W) are referred to as *Navier boundary conditions*. The requirements on the initial immersion $f_0 : M \to \mathbb{R}^{n+1}$ are specified below.

We show the existence of a strong solution f to (W) in the space

$$\mathbb{E}_T := W_p^1\left((0,T); L_p(M)\right) \cap L_p\left((0,T); W_p^4(M)\right)$$

that is naturally reflecting the structure of the evolution law which is a (degenerate) parabolic quasilinear equation of fourth order in f as $\boldsymbol{H}_f(t,p) = \Delta_{g(t)} f(t,p)$. In particular, all appearing derivatives in the motion equation exist in an almost everywhere sense. Considering values $p \in (4+n, \infty)$ the third order derivatives of f are continuous with respect to space and time which yields in particular that the boundary conditions are valid pointwise. Hence, the regularity of the solution implies that the boundary conditions are also fulfilled by the initial value which yields that only immersions satisfying these *compatibility conditions* can be admissible initial values. Also, the regularity of the initial immersion is determined by the solution space: given a function f in the space $\mathbb{E}_T$ its initial value $f(0)$ lies in the so-called *trace space* given by $W_p^{4-4/p}(M)$.
To prove existence of strong solutions to (W) we let $\varphi : M \to \mathbb{R}^{n+1}$ be a given smooth immersion and consider evolutions $f(t) : M \to \mathbb{R}^{n+1}$ that are given as graphs over the so called *reference surface* (M, φ), namely immersions of the form

$$f(t) = \varphi + \varrho(t)\nu_\varphi$$

with ν_φ the unit normal field to (M, φ) and $\varrho(t)$ a so called *height function*. To guarantee that $f(t)$ is an immersion the height function $\varrho(t)$ needs to satisfy a smallness condition depending on the geometry of (M, φ) given in Corollary 1.42. To simplify the analysis we demand the boundary of the initial value to be equal to the boundary of the reference surface which translates to the condition $\varrho = 0$ on $[0, T] \times \partial M$. In this way one obtains the evolution problem (1.6) for the height function ϱ which we refer to as the *graph formulation* of (W). Details on the notion of graph solutions and the conditions on the initial value are given in Section 1.3.

The main result of Part I is the following existence theorem.

Main Theorem 1 (Existence of the Willmore flow with Navier conditions)**.** *Let M be a smooth compact oriented manifold with boundary of dimension $n \in \mathbb{N}$, $n \geq 2$, $\varphi : M \to \mathbb{R}^{n+1}$ a smooth immersion and $p \in (4 + n, \infty)$. There exists a constant $\varepsilon_0 > 0$ and a time $T > 0$ such that for all $\varrho_0 \in W_p^{4-4/p}(M)$ with $\|\varrho_0\|_{W_p^{4-4/p}(M)} < \varepsilon_0$, $\varrho_0 = 0$ on ∂M, and $H_{\varphi+\varrho_0\nu_\varphi} = 0$ on ∂M, there exists $\varrho \in \mathbb{E}_T$ such that the functions $f(t) := \varphi + \varrho(t)\nu_\varphi$, $t \in [0, T]$, form a family of immersions solving the equations* (W) *with initial datum $f_0 = \varphi + \varrho_0\nu_\varphi$.*

One crucial point lies in verifying that the boundary conditions in (W) are compatible and independent enough. This is guaranteed by the so called Lopatinskii-Shapiro condition, see Proposition 2.47, of the associated linear parabolic system of fourth order that arises by considering the full linearisation of the graph formulation of (W) around the reference geometry, see Section 2.2. To show existence of the linear system on the entire manifold we analyse the localised problems on the respective chart domains. It is then a non-trivial task to compose these localised solutions to a function on the manifold solving the entire problem. This is done in Section 2.3. The existence of the graph formulation of (W) shown in Section 2.4 is then deduced from the existence of the linear system by a contraction argument using the Fréchet differentiability of the nonlinear quantities in (W) in the corresponding spaces, see Section 2.1.

Chapter 1

Preliminaries on the Willmore flow with Navier conditions

This chapter is devoted to formulate the Willmore flow of compact immersed surfaces with Navier conditions and to provide an overview of the framework and techniques that are used to prove the existence result for the flow in Chapter 2.
In Section 1.1 we study properties of immersed hypersurfaces and compute the change of geometric quantities for a given one-parameter family of immersions that evolves only in normal direction which is used when deriving the Euler–Lagrange equations for the Willmore functional in Section 1.2. To this end we show in Subsection 1.1.3 that each smooth one-parameter family of immersions of a given smooth manifold with boundary that keeps the boundary fixed can be smoothly reparametrised in such a way that the reparametrised family only evolves in normal direction. The Willmore flow with Navier conditions is then derived in Section 1.2 as the L^2–gradient flow of the Willmore energy in the class of immersions with a given boundary. In Section 1.3 we present the notion of strong graph solutions and derive the conditions on the initial value that are implied by this approach. Section 1.4 gives an overview of the localisation techniques that are used in Chapter 2 to show existence of strong graph solutions to the Willmore flow with Navier conditions in the formulation (1.6).

1.1 A preface on flows of immersed hypersurfaces

This section provides the required background information on immersed hypersurfaces. Details on smooth manifolds can be found in Chapter A which mainly follows [89, Chapters 1, 3, 5, 14, 15, 16]. We remark that the notion of immersed submanifolds introduced in [89, Chapter 5] is different from what is discussed in this section. The immersed submanifolds in $\mathbb{R}^{n+1}$ defined in [89] are precisely the images of *injective* immersions $f : M \to \mathbb{R}^{n+1}$ of smooth manifolds M which is shown in [89, Proposition 5.18].

1.1.1 Basic properties of immersed hypersurfaces

In this subsection we study immersed submanifolds in Euclidean space of co-dimension one. Given a smooth manifold M with or without boundary of dimension $n \in \mathbb{N}$, a map $f \in C^1\left(M; \mathbb{R}^{n+1}\right)$ is called *immersion* if it has injective differential $\mathrm{d}f_p : T_pM \to \mathbb{R}^{n+1}$ at every point $p \in M$. The definition of $\mathrm{d}f_p$ is given in Remark A.6. A C^k*–immersion* or *immersion of regularity* C^k, $k \in \mathbb{N}$, is an immersion f that satisfies $f \in C^k\left(M; \mathbb{R}^{n+1}\right)$. Although the image $f(M)$ may not be a manifold in the subspace topology of $\mathbb{R}^{n+1}$, many geometric quantities such as the mean curvature may be defined due to injectivity of the differential inducing a Riemannian metric on M. For details on C^k–Riemannian metrics on smooth manifolds, $k \in \mathbb{N}_0$, we refer to Section A.4.

Proposition 1.1 (Riemannian metric induced by an immersion)**.** *Suppose that M is a smooth manifold of dimension $n \in \mathbb{N}$ and let $f : M \to \mathbb{R}^{n+1}$ be a C^k–immersion, $k \in \mathbb{N}$. Given $p \in M$ the pullback metric $g_p := (\mathrm{d}f_p)^* \left(\langle \cdot\,, \cdot \rangle_{\mathbb{R}^{n+1}}\right)$ defined by $g_p(v,w) := \langle \mathrm{d}f_p(v), \mathrm{d}f_p(w)\rangle_{\mathbb{R}^{n+1}}$ for $v, w \in T_pM$ is a positive definite symmetric bilinear form on T_pM. The family $g := (g_p)_{p\in M}$ is a C^{k-1}–Riemannian metric on M. It is also referred to as the first fundamental form.*

Proof. As the linear map $\mathrm{d}f_p : T_pM \to \mathbb{R}^{n+1}$ is injective for every $p \in M$, it is straightforward to see that the pullback g_p of the Euclidean scalar product is itself a positive definite symmetric bilinear form. For a chart (U, ϕ) the function $g_{ij} \circ \phi^{-1}$ is given by

$$\begin{aligned}\left(g_{ij} \circ \phi^{-1}\right)(x) &= g_{\phi^{-1}(x)} \left(\frac{\partial}{\partial x^i}_{|\phi^{-1}(x)}, \frac{\partial}{\partial x^j}_{|\phi^{-1}(x)} \right) \\ &= \left\langle \mathrm{d}f_{|\phi^{-1}(x)} \left(\frac{\partial}{\partial x^i}_{|\phi^{-1}(x)} \right), \mathrm{d}f_{|\phi^{-1}(x)} \left(\frac{\partial}{\partial x^j}_{|\phi^{-1}(x)} \right) \right\rangle_{\mathbb{R}^{n+1}} \\ &= \left\langle \frac{\partial \left(f \circ \phi^{-1}\right)}{\partial x_i}(x), \frac{\partial \left(f \circ \phi^{-1}\right)}{\partial x_j}(x) \right\rangle_{\mathbb{R}^{n+1}}.\end{aligned}$$

Thus $x \mapsto \left(g_{ij} \circ \phi^{-1}\right)(x)$ has regularity C^{k-1} on $\phi(U)$ which shows that g defines a C^{k-1}–Riemannian metric on M. □

Given a smooth manifold M and an immersion $f : M \to \mathbb{R}^{n+1}$ the metric and its inverse are often endowed with a subscript f to indicate the immersion they refer to, that is $(g_{ij,f})$ and (g^{ij}_f).

Proposition 1.2 (Christoffel symbols induced by an immersion)**.** *Let M be a smooth manifold of dimension $n \in \mathbb{N}$ and suppose that $f : M \to \mathbb{R}^{n+1}$ is a C^k–immersion, $k \geq 2$, with induced metric g and associated Levi-Civita connection. The Christoffel symbols at $p \in M$ are given by*

$$\Gamma^l_{ij}(p) = g^{lm}(p) \left\langle \frac{\partial^2 \left(f \circ \phi^{-1}\right)}{\partial x_i \partial x_j}(\phi(p)), \frac{\partial \left(f \circ \phi^{-1}\right)}{\partial x_m}(\phi(p)) \right\rangle_{\mathbb{R}^{n+1}}.$$

Proof. This follows from

$$\begin{aligned}\frac{\partial g_{im}}{\partial x_j}_{|p} &= \frac{\partial}{\partial x_j} \left(g_{im} \circ \phi^{-1}\right)(\phi(p)) = \frac{\partial}{\partial x_j} \left\langle \frac{\partial \left(f \circ \phi^{-1}\right)}{\partial x_i}, \frac{\partial \left(f \circ \phi^{-1}\right)}{\partial x_m} \right\rangle_{\mathbb{R}^{n+1}} (\phi(p)) \\ &= \left\langle \frac{\partial^2 \left(f \circ \phi^{-1}\right)}{\partial x_i \partial x_j}(\phi(p)), \frac{\partial \left(f \circ \phi^{-1}\right)}{\partial x_m}(\phi(p)) \right\rangle + \left\langle \frac{\partial^2 \left(f \circ \phi^{-1}\right)}{\partial x_m \partial x_j}(\phi(p)), \frac{\partial \left(f \circ \phi^{-1}\right)}{\partial x_i}(\phi(p)) \right\rangle\end{aligned}$$

combined with Proposition A.35. □

To study how an immersed surface behaves with respect to its ambient space we introduce the notion of a unit normal field. Its construction relies on the so-called cross-product.

Definition 1.3. Given $n \in \mathbb{N}$ we define $\psi : \left(\mathbb{R}^{n+1}\right)^n \to \mathbb{R}^{n+1}$ via

$$\left(v^1, \ldots, v^n\right) \mapsto \psi\left(v^1, \ldots, v^n\right) := v^1 \times \cdots \times v^n := \left(\det\left(v^1, \ldots, v^n, e_j\right)\right)_{j=1}^{n+1}$$

where $e_j \in \mathbb{R}^{n+1}$ is the vector given by $(e_j)_i = \delta_{ij}$.

Given vectors $v^1, \ldots, v^n \in \mathbb{R}^{n+1}$ the expression $\psi\left(v^1, \ldots, v^n\right)$ is called the *cross-product* of $v^1, \ldots, v^n$. It is straightforward to check that $\left\langle v^1 \times \cdots \times v^n, v^j \right\rangle_{\mathbb{R}^{n+1}} = 0$ for all $j \in \{1, \ldots, n\}$. Moreover, if $v^1, \ldots, v^n$ are linearly independent, the cross product $v^1 \times \cdots \times v^n$ is non zero and $\left(v^1, \ldots, v^n, v^1 \times \cdots \times v^n\right)$ form a positively oriented basis of $\mathbb{R}^{n+1}$.

Proposition 1.4 (Unit normal field to immersed hypersurfaces)**.** *Suppose that M is a smooth oriented manifold of dimension $n \in \mathbb{N}$ and let $f : M \to \mathbb{R}^{n+1}$ be a C^k–immersion, $k \in \mathbb{N}$. Then there exists a function $\nu : M \to \mathbb{R}^{n+1}$ of regularity C^{k-1} such that for all $p \in M$ the vector $\nu(p)$ has unit norm and satisfies for all $v \in T_pM$ the identity*

$$\langle \nu(p), (\mathrm{d}f_p)(v) \rangle_{\mathbb{R}^{n+1}} = 0\,.$$

Moreover, $\nu(p)$ is the unique unit vector normal to $\mathrm{d}f_p\,(T_pM)$ such that for every positively oriented basis $v_1, \ldots, v_n$ of T_pM, the vectors $\mathrm{d}f_p(v_1), \ldots, \mathrm{d}f_p(v_n), \nu(p)$ form a positively oriented basis of $\mathbb{R}^{n+1}$. The function ν is called the unit normal field to (M, f).

Proof. Let p in M be fixed. As $\mathrm{d}f_p$ has full rank, the image $(\mathrm{d}f_p)(T_pM)$ is a n–dimensional subspace of $\mathbb{R}^{n+1}$. Let N_pM denote the one-dimensional subspace of $\mathbb{R}^{n+1}$ that is the orthogonal complement to $(\mathrm{d}f_p)(T_pM)$ with respect to the Euclidean inner product on $\mathbb{R}^{n+1}$. Let $(v_1, \ldots, v_n)$ be any positively oriented basis of T_pM and let $\nu(p) \in N_pM$ be the unit vector such that $(\mathrm{d}f_p(v_1), \ldots, \mathrm{d}f_p(v_n), \nu(p))$ is positively oriented in $\mathbb{R}^{n+1}$. This is well-defined as $\mathrm{d}f_p(v_1), \ldots, \mathrm{d}f_p(v_n), \eta$ are linearly independent for every $\eta \in N_pM \setminus \{0\}$ and as there are only two vectors in N_pM with unit norm which differ by a sign. The vector $\nu(p)$ is independent of the choice of the positively oriented basis of T_pM. Indeed, let $(w_1, \ldots, w_n)$ be another positively oriented basis of T_pM and let B, $B^{-1} \in Gl(n; \mathbb{R})$ be the transformation matrices. By linearity of $\mathrm{d}f_p$ the matrices that change between the bases $(\mathrm{d}f_p(v_1), \ldots, \mathrm{d}f_p(v_n), \nu(p))$ and $(\mathrm{d}f_p(w_1), \ldots, \mathrm{d}f_p(w_n), \nu(p))$ are given by the block matrices $\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} B^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, respectively. Hence, the basis $(\mathrm{d}f_p(v_1), \ldots, \mathrm{d}f_p(v_n), \nu(p))$ is positively oriented in $\mathbb{R}^{n+1}$ if and only if $(\mathrm{d}f_p(w_1), \ldots, \mathrm{d}f_p(w_n), \nu(p))$ is. Thus $\nu(p)$ is well-defined for every $p \in M$. It remains to show that $\nu : M \to \mathbb{R}^{n+1}$ is of regularity C^{k-1}. Let $q \in M$ and (U, ϕ) be a positively oriented chart around q. The vector

$$\widetilde{\nu}(p) := \frac{\mathrm{d}f_p\big(\partial x^1_{|p}\big) \times \cdots \times \mathrm{d}f_p\big(\partial x^n_{|p}\big)}{\Big\| \mathrm{d}f_p\big(\partial x^1_{|p}\big) \times \cdots \times \mathrm{d}f_p\big(\partial x^n_{|p}\big) \Big\|}$$

is a unit vector in N_pM and $\mathrm{d}f_p\big(\partial x^1_{|p}\big), \ldots, \mathrm{d}f_p\big(\partial x^n_{|p}\big), \widetilde{\nu}(p)$ form a positively oriented basis of $\mathbb{R}^{n+1}$. By uniqueness of a vector with these properties we conclude that $\nu(p) = \widetilde{\nu}(p)$ for all $p \in U$. As f is of regularity C^k, the map $p \mapsto \widetilde{\nu}(p) = \nu(p)$ is of regularity C^{k-1} on U. This shows the claim. □

Given a smooth oriented n–manifold M that is immersed in $\mathbb{R}^{n+1}$ via f, there are two choices to define the unit normal field. In contrast to the preceding proof we could have chosen $\nu(p) \in N_pM$ such that the vectors $\mathrm{d}f_p(v_1), \ldots, \mathrm{d}f_p(v_n), -\nu(p)$ are positively oriented in $\mathbb{R}^{n+1}$. This corresponds to the distinction between outward- and inward-pointing normal. The following definitions of the Weingarten map, the second fundamental form and the mean curvature depend on the precise choice of ν up to a sign. The definition of the mean curvature vector is invariant under the choice of ν.

Proposition 1.5. *Suppose that M is a smooth oriented manifold of dimension $n \in \mathbb{N}$ and $f : M \to \mathbb{R}^{n+1}$ a C^k–immersion, $k \geq 2$. Then for every $p \in M$ the differential $\mathrm{d}\nu_p$ maps T_pM into $\mathrm{d}f_p\,(T_pM)$.*

Proof. Let p be in M and (U, ϕ) be a chart around p. Then for all $x \in \phi(U)$,

$$1 = \big\langle \big(\nu \circ \phi^{-1}\big)(x), \big(\nu \circ \phi^{-1}\big)(x) \big\rangle_{\mathbb{R}^{n+1}}\,.$$

Differentiating this identity with respect to x_i, $i \in \{1, \dots, n\}$, we obtain

$$0 = \left\langle \left(\nu \circ \phi^{-1}\right)(x), \frac{\partial \left(\nu \circ \phi^{-1}\right)}{\partial x_i}(x) \right\rangle_{\mathbb{R}^{n+1}} = \left\langle \nu\left(\phi^{-1}(x)\right), \mathrm{d}\nu_{\phi^{-1}(x)}\left(\partial x^i_{|p}\right) \right\rangle_{\mathbb{R}^{n+1}}$$

due to the definition of $\mathrm{d}\nu_p$ given in Remark A.6. This shows the claim. □

We now introduce a $(2,0)$-tensor field, the second fundamental form, that indicates how the considered manifold is curved with respect to its ambient space $\mathbb{R}^{n+1}$. Contracting with the metric one obtains the associated $(1,1)$-tensor, the Weingarten map. The *mean curvature* is then given by the trace of the Weingarten map.

Definition 1.6 (Weingarten map and second fundamental form)**.** Suppose that M is a smooth oriented manifold of dimension $n \in \mathbb{N}$, $f : M \to \mathbb{R}^{n+1}$ a C^k–immersion, $k \geq 2$, and ν the unit normal field to (M, f). The linear map $W_p := -\left(\mathrm{d}f_p\right)^{-1} \circ \mathrm{d}\nu_p : T_pM \to T_pM$ is called the *Weingarten map* of (M, f) at $p \in M$. The bilinear form $A_p : T_pM \times T_pM \to \mathbb{R}$ defined by $A_p(v, w) := g_p\left(W_p(v), w\right)$ for $v, w \in T_pM$ is called the *second fundamental form* of (M, f) at p. We refer to the vector $A_p\nu(p)$ as the *vector-valued second fundamental form* of (M, f) at p.

Proposition 1.7 (Second fundamental form in local coordinates)**.** *Suppose that M is a smooth oriented manifold of dimension $n \in \mathbb{N}$ and $f : M \to \mathbb{R}^{n+1}$ a C^k–immersion, $k \geq 2$. Then for every chart (U, ϕ) and every $p \in U$ the second fundamental form has the representation*

$$a_{ij}(p) := A_p\left(\frac{\partial}{\partial x^i}_{|p}, \frac{\partial}{\partial x^j}_{|p}\right) = \left\langle \frac{\partial^2 \left(f \circ \phi^{-1}\right)}{\partial x_i \partial x_j}(\phi(p)), \nu(p) \right\rangle_{\mathbb{R}^{n+1}}.$$

In particular, A_p is symmetric. Moreover, the local representation of the vector-valued second fundamental form is given by

$$a_{ij}(p)\nu(p) = \frac{\partial^2 \left(f \circ \phi^{-1}\right)}{\partial x_i \partial x_j}(\phi(p)) - \Gamma^l_{ij}(p)\frac{\partial \left(f \circ \phi^{-1}\right)}{\partial x_l}(\phi(p)).$$

Proof. Let p be in M and (U, ϕ) be a chart around p. Then we obtain

$$\begin{aligned} A_p\left(\partial x^i_{|p}, \partial x^j_{|p}\right) &= g_p\left(W_p\left(\partial x^i_{|p}\right), \partial x^j_{|p}\right) = \left\langle \mathrm{d}f_p\left(W_p\left(\partial x^i_{|p}\right)\right), \mathrm{d}f_p\left(\partial x^j_{|p}\right) \right\rangle_{\mathbb{R}^{n+1}} \\ &= -\left\langle \mathrm{d}\nu_p\left(\partial x^i_{|p}\right), \mathrm{d}f_p\left(\partial x^j_{|p}\right) \right\rangle_{\mathbb{R}^{n+1}} \\ &= -\left\langle \frac{\partial \left(\nu \circ \phi^{-1}\right)}{\partial x_i}(\phi(p)), \frac{\partial \left(f \circ \phi^{-1}\right)}{\partial x_j}(\phi(p)) \right\rangle_{\mathbb{R}^{n+1}}. \end{aligned}$$

Observe that for every $x \in \phi(U)$ and every $j \in \{1, \dots, n\}$,

$$0 = \left\langle \nu(\phi^{-1}(x)), \mathrm{d}f_{\phi^{-1}(x)}\left(\partial x^j_{|\phi^{-1}(x)}\right) \right\rangle_{\mathbb{R}^{n+1}}.$$

Differentiating this identity with respect to x_i yields the first formula. The symmetry follows by Schwarz' Theorem. Let $b^m_{ij}(p) \in \mathbb{R}$ be such that

$$\frac{\partial^2 \left(f \circ \phi^{-1}\right)}{\partial x_i \partial x_j}(\phi(p)) = b^m_{ij}(p)\frac{\partial \left(f \circ \phi^{-1}\right)}{\partial x_m}(\phi(p)) + a_{ij}(p)\nu(p).$$

Contracting the expression

$$\left\langle \frac{\partial^2 \left(f \circ \phi^{-1}\right)}{\partial x_i \partial x_j}(\phi(p)), \frac{\partial \left(f \circ \phi^{-1}\right)}{\partial x_r}(\phi(p)) \right\rangle = b^m_{ij}(p)g_{mr}(p)$$

with the inverse metric $g^{rl}(p)$ we obtain $b^l_{ij}(p) = \Gamma^l_{ij}(p)$ and the desired formula. □

Thus, for all $p \in M$ the Weingarten map $W_p : T_pM \to T_pM$ is self-adjoint. In particular, all its eigenvalues $\kappa_1(p), \dots, \kappa_n(p)$, the *principal curvatures*, are real.

Definition 1.8 (Mean curvature (vector)). Let M be a smooth oriented manifold of dimension $n \in \mathbb{N}$, $f : M \to \mathbb{R}^{n+1}$ a C^k–immersion, $k \geq 2$, and ν the unit normal field to (M, f). The *mean curvature $H(p)$ of (M, f) at $p \in M$* is defined to be the trace of the linear map W_p. We call $\boldsymbol{H}(p) := H(p)\nu(p)$ the *mean curvature vector*.

The principal curvatures are independent of the choice of basis of T_pM but their signs do depend on the choice of normal. Thus, the sign of the mean curvature depends on the choice of ν. We observe though, that the mean curvature vector is independent of the choice of ν.

Proposition 1.9 (Mean curvature (vector) in local coordinates). *Let M be a smooth oriented manifold of dimension $n \in \mathbb{N}$ and $f : M \to \mathbb{R}^{n+1}$ a C^k–immersion, $k \geq 2$. Then for every chart (U, ϕ) and every $p \in U$ the mean curvature has the representation*

$$H(p) = a_{ij}(p) g^{ij}(p) .$$

The mean curvature vector in local coordinates is given by

$$\boldsymbol{H}(p) = g^{ij}(p) \left(\frac{\partial^2 \left(f \circ \phi^{-1}\right)}{\partial x_i \partial x_j} (\phi(p)) - \Gamma^l_{ij}(p) \frac{\partial \left(f \circ \phi^{-1}\right)}{\partial x_l} (\phi(p)) \right) = (\Delta_g f)(p) .$$

Proof. Let $p \in M$. The trace of the linear map $W_p : T_pM \to T_pM$ is the trace of its representation matrix with respect to any fixed basis of T_pM. Given a chart (U, ϕ) there exist coefficients $w_i^j(p) \in \mathbb{R}$ such that

$$W_p \left(\frac{\partial}{\partial x^i}_{|p} \right) = w_i^j(p) \frac{\partial}{\partial x^j}_{|p} .$$

Observe that

$$a_{il}(p) = g_p \left(W_p \left(\frac{\partial}{\partial x^i}_{|p} \right), \frac{\partial}{\partial x^l}_{|p} \right) = w_i^m(p) g_p \left(\frac{\partial}{\partial x^m}_{|p}, \frac{\partial}{\partial x^l}_{|p} \right) = w_i^m(p) g_{ml}(p) .$$

Contracting both sides with $g^{lj}(p)$ implies $w_i^j(p) = a_{il}(p) g^{lj}(p)$ which yields the desired formula for $H(p)$. The expression for the mean curvature vector then follows from Proposition 1.7 and the identity (A.7). □

The following proposition shows that immersions are "stable" with respect to small deviations in the C^1–norm. This is needed in the derivation of the Willmore flow in Section 1.2 and its graph formulation in Section 1.3.

Proposition 1.10. *Let M be a smooth compact manifold of dimension $n \in \mathbb{N}$ and $\varphi : M \to \mathbb{R}^{n+1}$ a smooth immersion. Let (U_α, ϕ_α), $\alpha \in \{1, \dots, N\}$, be charts covering M. Then there exists $\delta > 0$ such that every function $f \in C^1\big(M; \mathbb{R}^{n+1}\big)$, that satisfies for all $\alpha \in \{1, \dots, N\}$*

$$\left\| \nabla \left(f \circ \phi_\alpha^{-1}\right) - \nabla \left(\varphi \circ \phi_\alpha^{-1}\right) \right\|_{C\left(\overline{\phi_\alpha(U_\alpha)}; \mathbb{R}^{(n+1)n}\right)} < \delta ,$$

is an immersion.

Proof. Let $f \in C^1\big(M; \mathbb{R}^{n+1}\big)$ and $\alpha \in \{1, \dots, N\}$ be given. For all $q \in U_\alpha$ the rank of the linear map $\mathrm{d}f_q : T_qM \to \mathbb{R}^{n+1}$ is the rank of its representation matrix with respect to the basis

$\left((\mathrm{d}\phi_\alpha)_q\right)^{-1}(e_j)$, $j \in \{1, \ldots, n\}$, and the standard basis of $\mathbb{R}^{n+1}$. This matrix is given by

$$J^f(q) := \left(\frac{\partial\left(f^i \circ \phi_\alpha^{-1}\right)}{\partial x_j}(\phi_\alpha(q))\right)_{i\in\{1,\ldots,n+1\}, j\in\{1,\ldots,n\}}.$$

Let $q \in M$ and $\alpha = \alpha(q) \in \{1, \ldots, n\}$ with $q \in U_\alpha$ be given. As φ is an immersion, the matrix

$$J^\varphi(q) := \left(\frac{\partial\left(\varphi^i \circ \phi_\alpha^{-1}\right)}{\partial x_j}(\phi_\alpha(q))\right)_{i\in\{1,\ldots,n+1\}, j\in\{1,\ldots,n\}}$$

has rank n, and thus there exists an index $l_q \in \{1, \ldots, n+1\}$ such that

$$J^\varphi_{l_q}(q) := \left(\frac{\partial\left(\varphi^i \circ \phi_\alpha^{-1}\right)}{\partial x_j}(\phi_\alpha(q))\right)_{i\in\{1,\ldots,n+1\}\setminus\{l_q\}, j\in\{1,\ldots,n\}}$$

has non-zero determinant. Let $\mu_q > 0$ be such that $\left|\det\left(J^\varphi_{l_q}(q)\right)\right| > \mu_q$. As φ is smooth and the determinant is continuous in its entries, there exists a constant $\delta_q > 0$ such that for all $x \in B_{\delta_q}\left(\phi_\alpha(q)\right) \cap \phi_\alpha(U_\alpha)$ there holds

$$\left|\det\left(J^\varphi_{l_q}\left(\phi_\alpha^{-1}(x)\right)\right)\right| > \frac{\mu_q}{2}.$$

As M is compact, there exist finitely many points $q_1, \ldots, q_K$, $K \in \mathbb{N}$, such that M is covered by the sets

$$V_{q_k} := \phi^{-1}_{\alpha(q_k)}\left(B_{\delta_{q_k}}(\phi_{\alpha(q_k)}(q_k)) \cap \phi_{\alpha(q_k)}\left(U_{\alpha(q_k)}\right)\right).$$

Let $R > 1$ be such that for all $i \in \{1, \ldots, n+1\}$, $j \in \{1, \ldots, n\}$, $\alpha \in \{1, \ldots, N\}$ and $x \in \phi_\alpha(U_\alpha)$, $\left|\frac{\partial\left(\varphi^i \circ \phi_\alpha^{-1}\right)}{\partial x_j}(x)\right| < R$. Let further $\varepsilon := \min\left\{\frac{\mu_{q_k}}{4} : k \in \{1, \ldots, K\}\right\}$. As $\det : [-2R, 2R]^{n^2} \to \mathbb{R}$ is uniformly continuous, there exists $\delta \in (0,1)$ such that for all matrices (a_{ij}), $(b_{ij}) \in [-2R, 2R]^{n^2}$ with entries satisfying $|a_{ij} - b_{ij}| < \delta$, there holds

$$|\det(a_{ij}) - \det(b_{ij})| < \varepsilon.$$

Suppose that $f \in C^1\left(M; \mathbb{R}^{n+1}\right)$ satisfies for all $\alpha \in \{1, \ldots, N\}$,

$$\left\|\nabla\left(f \circ \phi_\alpha^{-1}\right) - \nabla\left(\varphi \circ \phi_\alpha^{-1}\right)\right\|_{C\left(\overline{\phi_\alpha(U_\alpha)}; \mathbb{R}^{(n+1)n}\right)} < \delta.$$

Let $p \in M$ be given and $k \in \{1, \ldots, K\}$ such that $p \in V_{q_k}$ which implies in particular

$$\left|\det J^\varphi_{l_{q_k}}(p)\right| > \frac{\mu_{q_k}}{2} \geq 2\varepsilon.$$

As for all $i \in \{1, \ldots, n+1\} \setminus \{l_{q_k}\}$, $j \in \{1, \ldots, n\}$,

$$\left|\frac{\partial\left(f^i \circ \phi_{\alpha(q_k)}\right)}{\partial x_j}\left(\phi_{\alpha(q_k)}(p)\right) - \frac{\partial\left(\varphi^i \circ \phi_{\alpha(q_k)}\right)}{\partial x_j}\left(\phi_{\alpha(q_k)}(p)\right)\right| < \delta,$$

we conclude that

$$\left|\det J^f_{l_{q_k}}(p) - \det J^\varphi_{l_{q_k}}(p)\right| < \varepsilon$$

which yields $\left|\det J^f_{l_{q_k}}(p)\right| > \varepsilon$. This shows that $\mathrm{d}f_p$ has full rank. □

1.1.2 Evolution of geometric quantities under normal flows

To compute the first variation of the Willmore energy we have to study how geometric quantities, as for instance the mean curvature, of a one-parameter family of immersions evolve in dependence on the *velocity* of the evolution. As it turns out, the computations simplify strongly if one assumes that the immersions only evolve in the direction that is normal to the respective surface, see Definition 1.11 for the precise terms. In Subsection 1.1.3 we show that this assumption is not restrictive.

Definition 1.11 (Normal velocity)**.** Let M be a smooth oriented manifold of dimension $n \in \mathbb{N}$ and $f : J \times M \to \mathbb{R}^{n+1}$, $J \subset \mathbb{R}$ an open interval, a smooth one-parameter family of smooth immersions $f(t) : M \to \mathbb{R}^{n+1}$, $t \in J$. The function $\partial_t f$ is called *velocity vector* of f. Denoting by $\nu(t)$ the unit normal field to $(M, f(t))$ we call $V(t) := \langle \partial_t f, \nu \rangle$ the *normal velocity of* f and $\boldsymbol{V}(t) := V(t)\nu(t)$ the *normal velocity vector of* f. If $\partial_t f(t) = \boldsymbol{V}(t)$ we say that the velocity vector $\partial_t f$ is *normal along* f and call f a *normal flow*.

In the following we let M be a smooth oriented manifold and $f : J \times M \to \mathbb{R}^{n+1}$ a smooth one-parameter family of smooth immersions such that the velocity vector is normal along f. We compute the evolution of the important geometric quantities in local coordinates. Hereby, the dependence on t indicates that the quantity refers to the geometry on M induced by $f(t)$. Let (U, ϕ) be a chart around $p \in M$ and let $t \in J$.

Lemma 1.12 (Evolution of the metric)**.** *The metric evolves as*

$$\partial_t g_{ij}(t,p) = -2V(t,p)a_{ij}(t,p)\,.$$

Proof.

$$\begin{aligned}\partial_t g_{ij}(t,p) &= \partial_t \left\langle \frac{\partial\left(f(t)\circ\phi^{-1}\right)}{\partial x_i}(\phi(p)), \frac{\partial\left(f(t)\circ\phi^{-1}\right)}{\partial x_j}(\phi(p))\right\rangle \\ &= \left\langle \frac{\partial\left(\boldsymbol{V}(t)\circ\phi^{-1}\right)}{\partial x_i}, \frac{\partial\left(f(t)\circ\phi^{-1}\right)}{\partial x_j}\right\rangle(\phi(p)) + \left\langle \frac{\partial\left(f(t)\circ\phi^{-1}\right)}{\partial x_i}, \frac{\partial\left(\boldsymbol{V}(t)\circ\phi^{-1}\right)}{\partial x_j}\right\rangle(\phi(p))\,.\end{aligned}$$

As $\boldsymbol{V}$ is normal along f it holds

$$\begin{aligned}\left\langle \frac{\partial\left(\boldsymbol{V}(t)\circ\phi^{-1}\right)}{\partial x_i}, \frac{\partial\left(f(t)\circ\phi^{-1}\right)}{\partial x_j}\right\rangle(\phi(p)) &= -\left\langle \boldsymbol{V}(t,p), \frac{\partial^2\left(f(t)\circ\phi^{-1}\right)}{\partial x_i \partial x_j}(\phi(p))\right\rangle \\ &= -V(t,p)a_{ij}(t,p)\,.\end{aligned}$$

□

Lemma 1.13 (Evolution of the inverse metric)**.** *The evolution of the inverse metric is given by*

$$\partial_t g^{ij}(t,p) = 2g^{im}(t,p)g^{kj}(t,p)a_{mk}(t,p)V(t,p)\,.$$

Proof. Differentiating $g^{im}(t,p)g_{mk}(t,p) = \delta^i_k$ with respect to t yields

$$\left(\partial_t g^{im}(t,p)\right)g_{mk}(t,p) = -g^{im}(t,p)\partial_t g_{mk}(t,p)\,.$$

Multiplying by $g^{kj}(t,p)$ and summing gives

$$\partial_t g^{ij}(t,p) = 2g^{im}(t,p)g^{kj}(t,p)a_{mk}(t,p)V(t,p)\,.$$

□

Lemma 1.14 (Evolution of the component of the Riemannian volume form)**.** *The component* $g(t,p) := \det\left(g_{ij}(t,p)\right)$ *of the Riemannian volume form evolves as*

$$\partial_t g(t,p) = -2g(t,p)V(t,p)H(t,p)\,.$$

Proof. Using the following derivation rule for the determinant of a time dependent matrix

$$\frac{\mathrm{d}}{\mathrm{d}t}\det A(t) = \det A(t)\operatorname{tr}\left(A(t)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}A(t)\right)$$

we obtain

$$\begin{aligned}\partial_t g(t,p) &= -2g(t,p)V(t,p)\operatorname{tr}\left(\left(g^{ij}(t,p)\right)_{ij}\left(a_{lm}(t,p)\right)_{lm}\right) = -2g(t,p)V(t,p)g^{ij}(t,p)a_{ij}(t,p)\\ &= -2g(t,p)V(t,p)H(t,p)\,.\end{aligned}$$

□

Lemma 1.15 (Evolution of the Riemannian volume form)**.** *Let* $\mathrm{d}V_{g_t}$ *denote integration with respect to the Riemannian volume form on* (M,g_t) *where* g_t *is the Riemannian metric on* M *induced by* $f(t)$*. Then formally we have*

$$\partial_t \mathrm{d}V_{g_t} = -V(t)H(t)\mathrm{d}V_{g_t}\,.$$

Proof. In local coordinates we formally have $\mathrm{d}V_{g_t} = \sqrt{g(t)}\mathrm{d}\lambda^n$ with $\mathrm{d}\lambda^n$ denoting integration with respect to the n–dimensional Lebesgue measure. Thus the measure $\mathrm{d}V_{g_t}$ evolves as

$$\partial_t\left(\mathrm{d}V_{g_t}\right) = \partial_t\sqrt{g(t)}\mathrm{d}\lambda^n = \frac{1}{2\sqrt{g(t)}}\partial_t g(t)\,\mathrm{d}\lambda^n = -\sqrt{g(t)}V(t)H(t)\,\mathrm{d}\lambda^n = -V(t)H(t)\,\mathrm{d}V_{g_t}\,.$$

□

Lemma 1.16 (Evolution of the second fundamental form)**.** *The evolution law of the second fundamental form is given by*

$$\left\langle \partial_t\left(a_{ij}(t,p)\nu(t,p)\right), \nu(t,p)\right\rangle = \left\langle \frac{\partial^2\left(\boldsymbol{V}(t)\circ\phi^{-1}\right)}{\partial x_i\partial x_j}(\phi(p)) - \Gamma^l_{ij}(t)\frac{\partial\left(\boldsymbol{V}(t)\circ\phi^{-1}\right)}{\partial x_l}(\phi(p)), \nu(t,p)\right\rangle\,.$$

Proof. By Proposition 1.7 we have

$$a_{ij}(t,p)\nu(t,p) = \frac{\partial^2\left(f(t)\circ\phi^{-1}\right)}{\partial x_i\partial x_j}(\phi(p)) - \Gamma^l_{ij}(t)\frac{\partial\left(f(t)\circ\phi^{-1}\right)}{\partial x_l}(\phi(p))\,.$$

Differentiating with respect to time gives

$$\begin{aligned}\partial_t\left(a_{ij}(t,p)\nu(t,p)\right) =& \frac{\partial^2\left(\boldsymbol{V}(t)\circ\phi^{-1}\right)}{\partial x_i\partial x_j}(\phi(p)) - \Gamma^l_{ij}(t)\frac{\partial\left(\boldsymbol{V}(t)\circ\phi^{-1}\right)}{\partial x_l}(\phi(p))\\ &- \partial_t\left(\Gamma^l_{ij}(t)\right)\frac{\partial\left(f(t)\circ\phi^{-1}\right)}{\partial x_l}(\phi(p))\,.\end{aligned}$$

The normal component of this expression is given by

$$\left\langle \partial_t\left(a_{ij}(t,p)\nu(t,p)\right), \nu(t,p)\right\rangle = \left\langle \frac{\partial^2\left(\boldsymbol{V}(t)\circ\phi^{-1}\right)}{\partial x_i\partial x_j}(\phi(p)) - \Gamma^l_{ij}(t)\frac{\partial\left(\boldsymbol{V}(t)\circ\phi^{-1}\right)}{\partial x_l}(\phi(p)), \nu(t,p)\right\rangle\,.$$

□

Lemma 1.17 (Evolution of the squared mean curvature)**.** *The squared mean curvature evolves as*

$$\begin{aligned}\partial_t \langle \boldsymbol{H}(t,p), \boldsymbol{H}(t,p)\rangle =& 4 g^{im}(t,p) g^{kj}(t,p) a_{mk}(t,p) a_{ij}(t,p) H(t,p) V(t,p) \\ &+ 2 \langle \Delta_{M,g_t} \boldsymbol{V}(t,p), \boldsymbol{H}(t,p)\rangle \,.\end{aligned}$$

Proof.

$$\begin{aligned}&\partial_t \langle \boldsymbol{H}(t,p), \boldsymbol{H}(t,p)\rangle = 2 \langle \partial_t \boldsymbol{H}(t,p), \boldsymbol{H}(t,p)\rangle = 2H(t,p) \langle \partial_t \boldsymbol{H}(t,p), \nu(t,p)\rangle \\ &= 2H(t,p) \left(\partial_t \left(g^{ij}(t,p)\right) a_{ij}(t,p) + g^{ij}(t,p) \langle \partial_t \left(a_{ij}(t,p)\nu(t,p)\right), \nu(t,p)\rangle\right) \\ &= 4\, g^{im}(t,p) g^{kj}(t,p) a_{mk}(t,p) a_{ij}(t,p) V(t,p) H(t,p) + 2 \langle \Delta_{M,g_t} \boldsymbol{V}(t,p), \boldsymbol{H}(t,p)\rangle \,.\end{aligned}$$

□

We accentuate one term appearing in Lemma 1.17 that is quadratic in the second fundamental form and therefore denoted by $Q(A)$.

Definition 1.18 (The term $Q(A)$)**.** Let M be a smooth oriented manifold of dimension $n \in \mathbb{N}$ and $f : M \to \mathbb{R}^{n+1}$ be a smooth immersion with induced metric $g = (\mathrm{d}f)^* (\langle \cdot, \cdot \rangle)$ on M, second fundamental form $(a_{ij})_{ij}$ and mean curvature H. We denote by $Q(A, f)$ the term

$$Q(A, f) := Q(A_f) := 2 g^{im} g^{kj} a_{mk} a_{ij} - \frac{1}{2} H^2 = 2 g^{im} g^{kj} a_{mk} a_{ij} - \frac{1}{2} g^{ij} g^{lk} a_{ij} a_{lk} \,.$$

Proposition 1.19 (Representation of $Q(A)$ in terms of the principal curvatures)**.** *Let M be a smooth oriented manifold of dimension $n \in \mathbb{N}$ and $f : M \to \mathbb{R}^{n+1}$ be a smooth immersion with induced metric $g = (\mathrm{d}f)^* (\langle \cdot, \cdot \rangle)$ on M and second fundamental form $(a_{ij})_{ij}$ and mean curvature H. The term $Q(A, f)$ can be written in terms of the principal curvatures $\kappa_1, \ldots, \kappa_n$ as*

$$Q(A, f) = 2 \sum_{i=1}^{n} \kappa_i^2 - \frac{1}{2} H^2 = 2 \sum_{i=1}^{n} \kappa_i^2 - \frac{1}{2} \left(\sum_{i=1}^{n} \kappa_i\right)^2 \,.$$

Proof. Given $p \in M$ Proposition 1.9 yields that with respect to the canonical basis on the tangent space T_pM the Weingarten map W_p is represented by the real and symmetric matrix $\mathcal{W}(p) = \left(g^{ij}(p) a_{jk}(p)\right)_{ik}$. In particular, there exists an orthogonal matrix $U(p) \in \mathbb{R}^{n \times n}$ such that $\mathcal{W}(p) = U(p) D(p) U(p)^T$ where $D(p) = \operatorname{diag}(\kappa_1(p), \ldots, \kappa_n(p))$. As the trace is invariant with respect to coordinate transformations, we find for any $p \in M$,

$$g^{im}(p) g^{kj}(p) a_{mk}(p) a_{ij}(p) = \operatorname{tr}\left(\mathcal{W}(p)^2\right) = \operatorname{tr}\left(U(p) D(p)^2 U(p)^T\right) = \operatorname{tr}\left(D(p)^2\right) = \sum_{i=1}^{n} \kappa_i(p)^2 \,.$$

This proves the claim. □

1.1.3 Reparametrisation to normal flows

This subsection shows that every smooth one-parameter family of immersions of a smooth compact oriented manifold with boundary that keeps the boundary fixed can be reparametrised by time-dependent diffeomorphisms such that the reparametrised evolution is normal. To this end one needs to solve an equation on the manifold M of the type

$$\frac{\partial \Psi}{\partial t}(t, q) = Y(t, \Psi(t, q))$$

with Y a smooth time-dependent vector field on M. The theory presented in [89, Chapter 9] on "Integral Curves and Flows" provides a suitable framework.

Definition 1.20 (Integral curves and flows)**.** Let M be a smooth manifold of dimension $n \in \mathbb{N}$ and V be a smooth vector field on M. An *integral curve of* V is a smooth curve $\gamma : J \to M$ defined on an open interval $J \subset \mathbb{R}$ such that for all $t \in J$,

$$\gamma'(t) = V_{\gamma(t)} .$$

The integral curve is called *maximal* if it cannot be extended to an integral curve defined on a larger open interval. A *flow domain* for M is an open subset $\mathcal{D} \subset \mathbb{R} \times M$ such that for all $p \in M$ the set $\mathcal{D}^{(p)} = \{t \in \mathbb{R} : (t,p) \in \mathcal{D}\}$ is an open interval containing 0. A *flow on* M *with infinitesimal generator* V is a smooth map $\theta : \mathcal{D} \to M$ defined on a flow domain $\mathcal{D}$ for M such that for all $p \in M$, $\theta(0,p) = p$ and

$$\frac{\partial}{\partial t}\theta\left(\cdot, p\right)_{|t=0} = V(p) ,$$

and for all $s \in \mathcal{D}^{(p)}$ and $t \in \mathcal{D}^{\theta(s,p)}$ with $s + t \in \mathcal{D}^{(p)}$,

$$\theta(t, \theta(s,p)) = \theta(t+s,p) .$$

The flow is *maximal* if it cannot be extended to a flow defined on a larger flow domain.

Theorem 1.21 (Fundamental Theorem on Flows)**.** *[89, Theorem 9.12, Theorem 9.34]*
Let M be a smooth manifold of dimension $n \in \mathbb{N}$. Suppose that V is a smooth vector field on M that is tangential to the boundary ∂M in the case that ∂M is nonempty, that is, $V(p) \in T_p\partial M$ for all $p \in \partial M$. There is a unique smooth maximal flow $\theta : \mathcal{D} \to M$ with infinitesimal generator V. It satisfies the following properties.

(a) For each $p \in M$ the curve $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ is the unique maximal integral curve of V starting at p where $\mathcal{D}^{(p)} := \{t \in \mathbb{R} : (t,p) \in \mathcal{D}\}$ and $\theta^{(p)}(t) := \theta(p,t)$.

(b) If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{\theta(s,p)}$ is the interval $\mathcal{D}^{(p)} - s := \{t - s : t \in \mathcal{D}^{(p)}\}$.

(c) For each $t \in \mathbb{R}$ the set $M_t := \{p \in M : (t,p) \in \mathcal{D}\}$ is open in M and $\theta\left(t, \cdot\right) : M_t \to M_{-t}$ is a smooth diffeomorphism with inverse $\theta(-t, \cdot)$.

Proof. The case of manifolds without boundary is treated in [89, Theorem 9.12]. If the manifold has nonempty boundary and the vector field is tangential to the boundary, this is precisely the statement of [89, Theorem 9.34]. □

Definition 1.22 (Time-dependent vector fields)**.** Let M be a smooth manifold of dimension $n \in \mathbb{N}$. A smooth *time-dependent vector field* on M is a smooth map $V : J \times M \to TM$ with $J \subset \mathbb{R}$ an interval and $V(t,p) \in T_pM$ for each $(t,p) \in J \times M$. If the boundary of M is nonempty, we say that V is tangential to ∂M if $V(t,p) \in T_p\partial M$ for all $(t,p) \in J \times \partial M$.

Definition 1.23 (Integral curves of time-dependent vector fields)**.** Let M be a smooth manifold of dimension $n \in \mathbb{N}$, $J \subset \mathbb{R}$ an interval and $V : J \times M \to TM$ a smooth time-dependent vector field on M. An *integral curve* of V is a differentiable curve $\gamma : J_0 \to M$ where $J_0 \subset J$ is an interval contained in J such that

$$\gamma'(t) = V(t, \gamma(t)) \qquad \text{for all } t \in J_0 .$$

The following theorem is a consequence of [89, Theorem 9.34] and [89, Theorem 9.48].

Theorem 1.24 (Theorem on time-dependent flows for manifolds with boundary)**.** *Let M be a smooth manifold with boundary of dimension $n \in \mathbb{N}$, let $J \subset \mathbb{R}$ be an open interval and let*

$V : J \times M \to TM$ be a smooth time-dependent vector field on M tangential to ∂M. There exists an open subset $\mathcal{E} \subset J \times J \times M$ and a smooth map $\psi : \mathcal{E} \to M$ called the time-dependent flow of V, with the following properties.

(a) For each $t_0 \in J$ and $p \in M$, the set $\mathcal{E}^{(t_0,p)} = \{t \in J : (t, t_0, p) \in \mathcal{E}\}$ is an open interval containing t_0, and the smooth curve $\psi^{(t_0,p)} : \mathcal{E}^{(t_0,p)} \to M$ defined by $\psi^{(t_0,p)}(t) = \psi(t, t_0, p)$ is the unique maximal integral curve of V with initial condition $\psi^{(t_0,p)}(t_0) = p$.

(b) If $t_1 \in \mathcal{E}^{(t_0,p)}$ and $q = \psi^{(t_0,p)}(t_1)$, then $\mathcal{E}^{(t_1,q)} = \mathcal{E}^{(t_0,p)}$ and $\psi^{(t_1,q)} = \psi^{(t_0,p)}$.

(c) For each $(t_1, t_0) \in J \times J$, the set $M_{t_1,t_0} = \{p \in M : (t_1, t_0, p) \in \mathcal{E}\}$ is open in M, and the map $\psi_{t_1,t_0} : M_{t_1,t_0} \to M$ defined by $\psi_{t_1,t_0}(p) = \psi(t_1, t_0, p)$ is a diffeomorphism from M_{t_1,t_0} onto M_{t_0,t_1} with inverse ψ_{t_0,t_1}.

(d) If $p \in M_{t_1,t_0}$ and $\psi_{t_1,t_0}(p) \in M_{t_2,t_1}$, then $p \in M_{t_2,t_0}$ and $\psi_{t_2,t_1} \circ \psi_{t_1,t_0}(p) = \psi_{t_2,t_0}(p)$.

Proof. This result follows exactly by the same arguments as in the case of manifolds without boundary shown in [89, Theorem 9.48]. One has to apply [89, Theorem 9.34] instead of [89, Theorem 9.12]. To this end we observe that the vector field V induces a smooth vector field $\widetilde{V}$ on the smooth manifold $J \times M$ defined by

$$\widetilde{V}(s, p) = \left(\frac{\partial}{\partial s}_{|s}, V(s, p) \right) \in T_{(s,p)}\,(J \times M) = T_s J \times T_p M\,.$$

As the vector field $\widetilde{V}$ is tangential to the boundary $\partial\,(J \times M) = J \times \partial M$, we may apply [89, Theorem 9.34] to obtain the existence of a unique smooth maximal flow $\widetilde{\theta}$ with infinitesimal generator $\widetilde{V}$. The remaining proof is exactly analogous to the proof of [89, Theorem 9.48]. □

Proposition 1.25. *Let M be a smooth compact manifold with boundary of dimension $n \in \mathbb{N}$. Let J be an open interval containing 0 and $Y : J \times M \to TM$ a smooth time-dependent vector field on M tangential to ∂M. Then there exists a smooth function $\Psi : J \times M \to M$ that is a family of smooth diffeomorphisms $\Psi(t, \cdot)$ of M, $t \in J$, with $\Psi(0, \cdot) = \mathrm{id}_M$ and such that for all $t \in J$, $q \in M$ it holds*

$$\frac{\partial \Psi}{\partial t}(t, q) = Y(t, \Psi(t, q))\,.$$

If M is oriented, the diffeomorphisms $\Psi(t, \cdot)$ are orientation preserving.

Proof. We show that for every $p \in M$ the set $\mathcal{E}^{(0,p)}$ is equal to the whole interval J. To this end let $I \subset J$ be a bounded open interval with $0 \in I$ and $\overline{I} \subset J$. Given $(t, p) \in \overline{I} \times M$ there exists $\varepsilon_{t,p} > 0$ such that

$$(t - \varepsilon_{p,t}, t + \varepsilon_{t,p}) \subset \mathcal{E}^{(t,p)}\,.$$

In particular, $(t + \varepsilon_{t,p}, t, p)$ lies in the open set $\mathcal{E}$ and thus there exists $\widetilde{\varepsilon}_{t,p} > 0$ and an open neighbourhood U_p of p in M such that

$$(t + \varepsilon_{t,p} - \widetilde{\varepsilon}_{t,p}, t + \varepsilon_{t,p} + \widetilde{\varepsilon}_{t,p}) \times (t - \widetilde{\varepsilon}_{t,p}, t + \widetilde{\varepsilon}_{t,p}) \times U_p \subset \mathcal{E}\,.$$

As $\overline{I} \times M$ is compact, there exist finitely many (t_i, p_i), $i \in \{1, \ldots, N\}$, such that

$$\overline{I} \times M \subset \bigcup_{i=1}^{N} (t_i - \widetilde{\varepsilon}_{t_i,p_i}, t_i + \widetilde{\varepsilon}_{t_i,p_i}) \times U_p\,.$$

Let $\varepsilon := \min\{\widetilde{\varepsilon}_{t_i,p_i}, \varepsilon_{t_i,p_i} : i \in \{1,\dots,N\}\}$. Assume that there exists $t \in \overline{I}$ and $p \in M$ such that

$$b := \sup \mathcal{E}^{(t,p)} < \sup \overline{I}^{1}\,.$$

Let $t_0 \in \mathcal{E}^{(t,p)}$ be such that $b - \frac{\varepsilon}{2} < t_0 < b$ and define $q := \psi^{(t,p)}(t_0)$. Then assertion (a) in Theorem 1.24 implies that $(t_0, t_0, q) \in \mathcal{E}$. Thus there exists an index $i \in \{1,\dots,N\}$ such that

$$(t_0, t_0, q) \in (t_i + \varepsilon_{t_i,p_i} - \widetilde{\varepsilon}_{t_i,p_i}, t_i + \varepsilon_{t_i,p_i} + \widetilde{\varepsilon}_{t_i,p_i}) \times (t_i - \widetilde{\varepsilon}_{t_i,p_i}, t_i + \widetilde{\varepsilon}_{t_i,p_i}) \times U_{p_i}$$

which implies $\left(t_0 + \frac{\varepsilon}{2}, t_0, q\right) \in \mathcal{E}$ and thus, using the statement (b) in Theorem 1.24,

$$t_0 + \frac{\varepsilon}{2} \in \mathcal{E}^{(t_0,q)} = \mathcal{E}^{(t,p)}$$

contradicting the assumption $b = \sup \mathcal{E}^{(t,p)}$. This shows $\overline{I} \subset \mathcal{E}^{(t,p)}$ for every $t \in \overline{I}$, $p \in M$ and every bounded interval $I \subset J$. Hence we obtain $\mathcal{E}^{(t,p)} = J$ for all $p \in M$ and all $t \in J$. As $\psi^{(0,p)} : J \to M$ is the unique maximal integral curve of Y with initial condition $\psi^{(0,p)}(0) = p$, it holds for all $p \in M$ and $t \in J$ that

$$\frac{\partial \psi^{(0,p)}}{\partial t}(t) = Y(t, \psi^{(0,p)}(t))\,.$$

Define $\Psi : J \times M \to M$ by $\Psi(t,p) := \psi^{(0,p)}(t) = \psi(t,0,p)$. Then Ψ is smooth and satisfies $\Psi(0,\cdot) = \mathrm{id}_M$ and

$$\frac{\partial \Psi}{\partial t}(t,p) = Y(t, \Psi(t,p))\,.$$

Moreover, $\Psi(t,\cdot)$ is a diffeomorphism of M for every $t \in J$. Indeed, assertion (c) of Theorem 1.24 implies that $p \mapsto \Psi(t,p) = \psi(t,0,p) = \psi_{t,0}(p)$ is a diffeomorphism from $M_{t,0}$ to $M_{0,t}$ where $M_{t,0} = M_{0,t} = M$ as $\mathcal{E}^{(t,p)} = J$ for all $t \in J$ and all $p \in M$. To show uniqueness assume that $\widetilde{\Psi} : J \times M \to M$ satisfies the same properties as the mapping Ψ. In particular, for every $p \in M$ the map $t \mapsto \widetilde{\Psi}(t,p)$ is the unique maximal integral curve of Y starting in p. This immediately implies $\widetilde{\Psi} = \Psi$ on $J \times M$.

Assume further that M is an oriented manifold. Let

$$\widetilde{J} := \{t \in J \,:\, \Psi(t,\cdot) \text{ is orientation preserving}\}\,.$$

Then $\widetilde{J}$ is non-empty as $\Psi(0,\cdot) = \mathrm{id}_M$ is orientation preserving. We show that $\widetilde{J}$ is open and closed in J which by connectedness of the interval J then yields $J = \widetilde{J}$. Let $t \in \widetilde{J}$ be fixed and $(V_1,\psi_1),\dots,(V_N,\psi_N)$ be positively oriented charts with $M = \bigcup_{i=1}^N V_i$. Let $i,j \in \{1,\dots,N\}$ be such that $\Psi(t)(V_i) \cap V_j \neq \emptyset$. As $\Psi(t)$ is orientation preserving, it holds

$$\det D\left(\psi_j \circ \Psi(t) \circ \psi_i^{-1}\right) > 0 \qquad \text{on} \qquad \psi_i(V_i \cap \Psi(t)^{-1}(V_j))\,.$$

Smoothness of Ψ implies that there is $\varepsilon_{ij} > 0$ such that for all $\tau \in (t - \varepsilon_{ij}, t + \varepsilon_{ij}) \cap J$ the set $\Psi(\tau, V_i) \cap V_j$ is non-empty and

$$\det D\left(\psi_j \circ \Psi(\tau) \circ \psi_i^{-1}\right) > 0 \qquad \text{on} \qquad \psi_i(V_i \cap \left(\Psi(\tau)^{-1}\right)(V_j))\,.$$

Let $\varepsilon := \min\{\varepsilon_{ij} \,:\, i,j \in \{1,\dots,N\}\}$ and $\tau \in (t - \varepsilon, t + \varepsilon) \cap J$ be fixed. Given $p \in M$ we consider positively oriented charts (U,ϕ), (V,ψ) around p and $\Psi(\tau,p)$, respectively. There exist $i,j \in \{1,\dots,N\}$ such that $p \in V_i$, $\Psi(\tau,p) \in V_j$. In particular $\Psi(\tau,V_i) \cap V_j$ is non-empty and the determinant of $D\left(\psi_j \circ \Psi(\tau) \circ \psi_i^{-1}\right)(\psi_i(p))$ is positive. Observe that

$$D\left(\psi_j \circ \Psi(\tau) \circ \psi_i^{-1}\right)(\psi_i(p))$$

[1] The case that $a := \inf \mathcal{E}^{(t,p)} > \inf \overline{I}$ can be treated with similar arguments. In the above reasoning one has to replace $t + \varepsilon_{t,p}$ by $t - \varepsilon_{t,p}$.

$$= D\left(\psi_j \circ \psi^{-1}\right)\left(\psi\left(\Psi(\tau, p)\right)\right) D\left(\psi \circ \Psi(\tau) \circ \phi^{-1}\right)(\phi(p))\, D\left(\phi \circ \psi_i^{-1}\right)(\psi_i(p))$$

where both $D\left(\psi_j \circ \psi^{-1}\right)\left(\psi\left(\Psi(\tau, p)\right)\right)$ and $D\left(\phi \circ \psi_i^{-1}\right)(\psi_i(p))$ have positive determinant as all involved charts are positively oriented. This shows that the determinant of $D\left(\psi \circ \Psi(\tau) \circ \phi^{-1}\right)(\phi(p))$ is positive and hence $(t-\varepsilon, t+\varepsilon) \cap J \subset \widetilde{J}$. To show that $\widetilde{J}$ is closed in J we let $t_n \in \widetilde{J}$, $n \in \mathbb{N}$, be such that $|t_n - t| \to 0$ for $n \to \infty$ and some $t \in J$. Let $p \in M$ and (U, ϕ), (V, ψ) be positively oriented charts around p and $\Psi(t,p)$, respectively. As V is open in M and $\tau \mapsto \Psi(\tau, p)$ is continuous, there exists $N \in \mathbb{N}$ such that $\Psi(t_n, p) \in V$ for all $n \geq N$. Thus for all $n \geq N$ the determinant of $D\left(\psi \circ \Psi(t_n) \circ \phi^{-1}\right)(\phi(p))$ is positive and by continuity

$$0 \leq \lim_{n\to\infty} \det D\left(\psi \circ \Psi(t_n) \circ \phi^{-1}\right)(\phi(p)) = \det D\left(\psi \circ \Psi(t) \circ \phi^{-1}\right)(\phi(p))\,.$$

On the other hand we have $\det D\left(\psi \circ \Psi(t) \circ \phi^{-1}\right)(\phi(p)) \neq 0$ as $\Psi(t)$ is a smooth diffeomorphism. This shows $t \in J$. □

Proposition 1.26 (Reparametrisation to normal flows)**.** *Let M be a smooth compact oriented manifold with boundary of dimension $n \in \mathbb{N}$ and $J \subset \mathbb{R}$ an open or half-open interval containing 0. Suppose that f is a smooth one-parameter family of smooth immersions $f(t) : M \to \mathbb{R}^{n+1}$, $t \in J$, such that*

$$(t,p) \mapsto f(t,p) := f(t)(p) \in C^\infty\left(J \times M; \mathbb{R}^{n+1}\right)$$

and suppose that $\partial_t f(t,p) = 0$ for all $(t,p) \in J \times \partial M$. Then there exists a smooth one-parameter family Φ of smooth orientation preserving diffeomorphisms $\Phi(t) : M \to M$ such that $\Phi_0 = \mathrm{id}_M$,

$$(t,p) \mapsto \Phi(t,p) := \Phi(t)(p) \in C^\infty\left(J \times M; M\right)$$

and

$$\partial_t\left(f(t) \circ \Phi(t)\right)(p) = \left\langle \left(\partial_t f\right)(t, \Phi(t,p)), \nu(t, \Phi(t,p))\right\rangle \nu(t, \Phi(t,p))\,.$$

Proof. We adapt the proof of [99, Proposition 1.3.4] to our situation. There exists a function $X \in C^\infty\left(J \times M; \mathbb{R}^{n+1}\right)$ such that for all $t \geq 0$ and $p \in M$, $X(t,p) \in \left(\mathrm{d}f(t)\right)_p(T_pM)$ and

$$\partial_t f(t,p) = \langle \partial_t f(t,p), \nu(t,p)\rangle \nu(t,p) + X(t,p)\,.$$

We define a smooth time-dependent vector field Y on M by

$$Y(t,p) := -\left(\left(\mathrm{d}f(t)\right)_p\right)^{-1}(X(t,p)) \in T_pM$$

and set $Y(t,p) := Y(0,p)$ for all $t \in (-\delta, 0)$, $p \in M$, in the case that $J = [0,a)$ with $a \in (0, \infty]$. Here $\delta > 0$ is some constant. The hypothesis $\partial_t f = 0$ on $J \times \partial M$ implies $Y = 0$ on $J \cup (-\delta, 0) \times \partial M$. By Proposition 1.25 there exists a smoothly time-dependent family of smooth orientation preserving diffeomorphisms $\Phi(t)$ of M such that $\Phi_0 = \mathrm{id}_M$ and $\partial_t \Phi(t,p) = Y(t, \Phi(t,p))$. This implies

$$\begin{aligned}\partial_t\left(f(t, \Phi(t,p))\right) &= \left(\partial_t f\right)(t, \Phi(t,p)) + \left(\mathrm{d}f(t)\right)_{\Phi(t,p)}\left(\partial_t \Phi(t,p)\right)\\ &= \left\langle \left(\partial_t f\right)(t, \Phi(t,p)), \nu(t, \Phi(t,p))\right\rangle \nu(t, \Phi(t,p)) + X(t, \Phi(t,p)) - X(t, \Phi(t,p))\,.\end{aligned}$$

□

1.2 The Willmore energy and its gradient flow

This section is devoted to the first variation of the Willmore energy in the class of immersed hypersurfaces with given fixed boundary and the formal derivation of the L^2–gradient flow of the functional. We remark that from the geometrical viewpoint it would be meaningful to allow for variations that reparametrise the boundary tangentially. This corresponds to tangential movement of the boundary in the evolutionary problem which is not visible for the observer of the geometric evolution. Being interested in the question of *existence* of the flow we exclude tangential movements of the boundary for simplification.

Proposition 1.26 shows that every one-parameter family of immersions with zero velocity on the boundary can be reparametrised by a family of smooth orientation preserving diffeomorphisms in such a way, that the resulting flow is normal. As the Willmore energy is invariant under smooth orientation preserving diffeomorphisms, we may use the evolution laws derived in Section 1.1.2 to compute the first variation, which is given in Proposition 1.29. In Proposition 1.32 we give a characterisation of the set of immersions at which the Willmore energy has an L^2–gradient leading to the flow given in (1.3).

Definition 1.27 (Willmore energy)**.** Let M be a smooth compact oriented manifold of dimension $n \in \mathbb{N}$ and $f : M \to \mathbb{R}^{n+1}$ a C^k–immersion, $k \geq 2$, inducing the C^{k-1}–Riemannian metric $g = (\mathrm{d}f)^* \left(\langle \cdot, \cdot \rangle_{\mathbb{R}^{n+1}}\right)$ on M. The *Willmore energy* of (M, f) is defined as

$$W(f) := \frac{1}{2} \int_M H^2 \omega_g = \frac{1}{2} \int_M H^2 \, \mathrm{d}V_g \tag{1.1}$$

where ω_g, $\mathrm{d}V_g$ and H are the Riemannian volume form, the induced volume measure and the mean curvature of (M, g), respectively.

We remark that the squared mean curvature is continuous on the compact manifold M and thus integrable with respect to the volume measure.

Proposition 1.28 (Invariance of the Willmore energy under diffeomorphisms)**.** *Let M be a smooth compact oriented manifold of dimension $n \in \mathbb{N}$. Suppose that $f : M \to \mathbb{R}^{n+1}$ is a C^k–immersion, $k \geq 2$, and $\Phi : M \to M$ a smooth orientation preserving diffeomorphism. Then the Willmore energy of (M, f) is equal to the Willmore energy of $(M, f \circ \Phi)$.*

Proof. The metrics induced by the immersions f and $\tilde{f} := f \circ \Phi$ are denoted by g and $\tilde{g}$, respectively. Let (U_l, ϕ_l), $l \in \{1, \ldots, N\}$, be positively oriented charts covering M and let (ψ_l) be a smooth partition of unity subordinate to this covering. As Φ is a smooth orientation preserving diffeomorphism, the charts $\left(\Phi(U_l), \phi_l \circ \Phi^{-1}\right)$, $l \in \{1, \ldots, N\}$, are positively oriented and cover M. The functions $\left(\psi_l \circ \Phi^{-1}\right)$ form a smooth partition of unity subordinate to this covering. Thus the Willmore energy of (M, f) is given by

$$\begin{aligned}
W(f) &= \sum_{l=1}^{N} \int_{\phi_l(U_l)} \left(\psi_l H_g^2 (\omega_g)_{1,\ldots,n}\right) \left(\phi_l^{-1}(x)\right) \, \mathrm{d}x \\
&= \sum_{l=1}^{N} \int_{(\phi_l \circ \Phi^{-1})(\Phi(U_l))} \left((\psi_l \circ \Phi^{-1}) H_g^2 (\omega_g)_{1,\ldots,n}\right) \left(\Phi\left(\phi_l^{-1}(x)\right)\right) \, \mathrm{d}x \\
&= \sum_{l=1}^{N} \int_{\phi_l(U_l)} \psi_l \left(\phi_l^{-1}(x)\right) \left(H_g^2 (\omega_g)_{1,\ldots,n}\right) \left(\Phi\left(\phi_l^{-1}(x)\right)\right) \, \mathrm{d}x \,.
\end{aligned}$$

Thus it is enough to show for all $x \in \phi_l(U_l)$,

$$\left(H_g^2 (\omega_g)_{1,\ldots,n}\right) \left(\Phi\left(\phi_l^{-1}(x)\right)\right) = \left(H_{\tilde{g}}^2 (\omega_{\tilde{g}})_{1,\ldots,n}\right) \left(\phi_l^{-1}(x)\right) .$$

With respect to the chart (U_l, ϕ_l) the function $\widetilde{g}_{ij}$ is given by

$$\begin{aligned}\widetilde{g}_{ij}\left(\phi_l^{-1}(x)\right) &= \left\langle \frac{\partial\left(\tilde{f}\circ\phi_l^{-1}\right)}{\partial x_i}(x), \frac{\partial\left(\tilde{f}\circ\phi_l^{-1}\right)}{\partial x_j}(x)\right\rangle \\ &= \left\langle \frac{\partial\left(f\circ\left(\phi_l\circ\Phi^{-1}\right)^{-1}\right)}{\partial x_i}(x), \frac{\partial\left(f\circ\left(\phi_l\circ\Phi^{-1}\right)^{-1}\right)}{\partial x_j}(x)\right\rangle\end{aligned}$$

and thus equal to $g_{ij}\left(\Phi\left(\phi_l^{-1}(x)\right)\right)$ expressed with respect to $\left(\Phi(U_l), \phi_l\circ\Phi^{-1}\right)$. This implies in particular

$$(\omega_g)_{1,\dots,n}\left(\Phi\left(\phi_l^{-1}(x)\right)\right) = (\omega_{\tilde{g}})_{1,\dots,n}\left(\phi_l^{-1}(x)\right).$$

For any $p \in M$ the unit normal $\widetilde{\nu}(p)$ to $(M, \tilde{f})$ at $p \in M$ is equal to the unit normal to (M, f) at $\Phi(p)$. This implies that the second fundamental form $\widetilde{a}_{ij}\left(\phi_l^{-1}(x)\right)$ of $(M, \tilde{f})$ expressed in the chart (U_l, ϕ_l) is equal to $a_{ij}\left(\Phi\left(\phi_l^{-1}(x)\right)\right)$, as

$$\begin{aligned}\widetilde{a}_{ij}\left(\phi_l^{-1}(x)\right) &= \left\langle \frac{\partial^2\left(\tilde{f}\circ\phi_l^{-1}\right)}{\partial x_i\partial x_j}(x), \widetilde{\nu}\left(\phi_l^{-1}(x)\right)\right\rangle \\ &= \left\langle \frac{\partial^2\left(f\circ\left(\phi_l\circ\Phi^{-1}\right)^{-1}\right)}{\partial x_i\partial x_j}(x), \nu\left(\Phi\left(\phi_l^{-1}(x)\right)\right)\right\rangle = a_{ij}\left(\Phi\left(\phi_l^{-1}(x)\right)\right).\end{aligned}$$

The formula for the mean curvature in local coordinates given in Proposition 1.9 implies

$$H_g\left(\Phi\left(\phi_l^{-1}(x)\right)\right) = H_{\tilde{g}}\left(\phi_l^{-1}(x)\right).$$

□

Proposition 1.29 (First variation of the Willmore energy)**.** *Let M be a smooth compact oriented manifold with boundary of dimension $n \in \mathbb{N}$ and $J \subset \mathbb{R}$ an open interval. Suppose that f is a one-parameter family of smooth immersions $f(t): M \to \mathbb{R}^{n+1}$ such that*

$$(t,p) \mapsto f(t,p) := f(t)(p) \in C^\infty\left(J \times M; \mathbb{R}^{n+1}\right)$$

and $\partial_t f(t,p) = 0$ for all $(t,p) \in J \times \partial M$. The Willmore energy of $(M, f(t))$ evolves as

$$\frac{\mathrm{d}}{\mathrm{d}t}W(f(t)) = \int_M \langle \Delta_{M,g_t}\boldsymbol{H}(t) + Q(A)(t)\boldsymbol{H}(t), \boldsymbol{V}(t)\rangle\,\mathrm{d}V_{g_t} + \int_{\partial M} H(t) g_t\left(\nabla_{M,g_t}V(t), N(t)\right)\mathrm{d}V_{\tilde{g}_t}$$

where $N(t)$ is the outer unit conormal along ∂M with respect to g_t, $\tilde{g}_t$ is the metric on ∂M induced by g_t, and $\boldsymbol{V}(t) = V(t)\nu(t) = \langle\partial_t f(t), \nu(t)\rangle\,\nu(t)$ is the normal velocity vector of f.

Proof. By Proposition 1.26 there exists a smooth one-parameter family Φ of smooth orientation preserving diffeomorphisms $\Phi(t): M \to M$ such that $\Phi_0 = \mathrm{id}_M$ and with $\tilde{f} := f \circ \Phi$,

$$\partial_t \tilde{f}(t) = \langle(\partial_t f)(t, \Phi(t,p)), \nu(t, \Phi(t,p))\rangle\,\nu(t, \Phi(t,p)).$$

We observe that the normal $\widetilde{\nu}(t,p)$ to $(M, \tilde{f}(t))$ is given by $\widetilde{\nu}(t,p) = \nu(t, \Phi(t,p))$ and in particular the normal velocity of $\tilde{f}$ equals

$$\widetilde{V}(t,p) = \left\langle\partial_t\tilde{f}(t,p), \widetilde{\nu}(t,p)\right\rangle = \langle(\partial_t f)(t, \Phi(t,p)), \nu(t, \Phi(t,p))\rangle$$

which shows that $\tilde{f}$ is a normal flow. By Proposition 1.28 we obtain for all $t \in J$, $W(f(t)) = W(f(t)\circ\Phi(t)) = W\left(\tilde{f}(t)\right)$ and hence $\frac{\mathrm{d}}{\mathrm{d}t}W(f(t)) = \frac{\mathrm{d}}{\mathrm{d}t}W\left(\tilde{f}(t)\right)$. The integral $W(\tilde{f}(t))$ can be expressed by a finite sum of Lebesgue integrals on parametrisation domains of M in $\mathbb{R}^n$ as described in A.20. The chosen covering and partition of unity as well as the domains of integration do not depend on time. All geometric quantities that appear in the following calculations refer to $(M, \tilde{f}(t))$, $t \in J$. Using a theorem on differentiation of parameter dependent Lebesgue integrals we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} W\left(\tilde{f}(t)\right) = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_M H(t)^2 \,\mathrm{d}V_{g_t} = \frac{1}{2} \int_M \partial_t \left(H(t)^2\right) \,\mathrm{d}V_{g_t} + \frac{1}{2} \int_M H(t)^2 \,\partial_t \left(\mathrm{d}V_{g_t}\right).$$

Using $\partial_t H(t)^2 = \partial_t \langle \boldsymbol{H}(t), \boldsymbol{H}(t) \rangle$ and the computations in Section 1.1.2 we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} W\left(\tilde{f}(t)\right) = \int_M \langle \boldsymbol{H}(t), \Delta_{M,g_t} \boldsymbol{V}(t) \rangle + \langle Q(A)(t) \boldsymbol{H}(t), \boldsymbol{V}(t) \rangle \,\mathrm{d}V_{g_t}.$$

Applying the product rule for the divergence shown in Proposition A.33 and the Divergence Theorem A.34 to each component yields

$$\begin{aligned}
\int_M \langle \boldsymbol{H}, \Delta_{M,g_t} \boldsymbol{V} \rangle \,\mathrm{d}V_{g_t} &= \sum_{i=1}^{n+1} \int_M \boldsymbol{H}^i \left(\Delta_{M,g_t} \boldsymbol{V}^i\right) \mathrm{d}V_{g_t} \\
&= \sum_{i=1}^{n+1} \int_{\partial M} \boldsymbol{H}^i g_t \left(\nabla_{M,g_t} \boldsymbol{V}^i, N\right) \mathrm{d}V_{\tilde{g}_t} - \int_M g_t \left(\nabla_{M,g_t} \boldsymbol{H}^i, \nabla_{M,g_t} \boldsymbol{V}^i\right) \mathrm{d}V_{g_t} \\
&= \sum_{i=1}^{n+1} \int_M \left(\Delta_{M,g_t} \boldsymbol{H}^i\right) \boldsymbol{V}^i \mathrm{d}V_{g_t} + \int_{\partial M} \boldsymbol{H}^i g_t \left(\nabla_{M,g_t} \boldsymbol{V}^i, N\right) \mathrm{d}V_{\tilde{g}_t} \\
&\quad - \int_{\partial M} \boldsymbol{V}^i g_t \left(\nabla_{M,g_t} \boldsymbol{H}^i, N\right) \mathrm{d}V_{\tilde{g}_t}.
\end{aligned}$$

The second boundary integral vanishes as the velocity is zero at the boundary. To simplify the first boundary term we observe

$$\nabla_{M,g_t} \boldsymbol{V}^i(t,p) = \nu^i(t,p) g^{lm}(t,p) \frac{\partial V(t)}{\partial x^l}(p) \frac{\partial}{\partial x^m}_{|p} + V(t,p) g^{lm}(t,p) \frac{\partial \nu^i(t)}{\partial x^l}(p) \frac{\partial}{\partial x^m}_{|p}$$

where the second term vanishes as $V = 0$ on $[0,\infty) \times \partial M$. Hence we obtain

$$\sum_{i=1}^{n+1} \boldsymbol{H}(t)^i g_t \left(\nabla_{M,g_t} \boldsymbol{V}(t)^i, N(t)\right) = H(t) g_t \left(\nabla_{M,g_t} V(t), N(t)\right).$$

This shows the desired identity for $\frac{\mathrm{d}}{\mathrm{d}t} W(\tilde{f}(t))$. The integrands $\mathcal{G}_{\tilde{f}}(t)$ and $\mathcal{B}_{\tilde{f}}(t)$ in the first and second integral, respectively, depend only on geometric quantities for $(M, \tilde{f}(t))$ and satisfy the identity $\mathcal{G}_{\tilde{f}}(t,p) = \mathcal{G}_f(t, \Phi(t,p))$. As the metrics induced by $f(t)$ and $\tilde{f}(t)$, respectively, satisfy $g_{\tilde{f}(t)}(p) = g_{f(t)}(\Phi(t)(p))$ and as integration is independent of the chosen charts, we obtain the desired formula for $\frac{\mathrm{d}}{\mathrm{d}t} W(f(t))$. □

The Willmore flow arises as the L^2–gradient flow of the Willmore energy and is derived in the following. We remark that the approach presented here should be considered as formal and refer to [68, 126] for details on rigorous gradient flow structures for geometric flows.
Let $\varphi : M \to \mathbb{R}^{n+1}$ be a given smooth immersion. As we want to keep the boundary fixed during the flow prohibiting also any tangential movement, the *set of admissible immersions* is given by

$$\mathcal{A}_\varphi := \left\{ f : M \to \mathbb{R}^{n+1} : f \text{ is a smooth immersion and } f_{|\partial M} = \varphi_{|\partial M} \right\}.$$

Definition 1.30 (L^2–gradient of W in $\mathcal{A}_\varphi$)**.** Let M be a smooth compact oriented manifold with boundary of dimension $n \in \mathbb{N}$ and let $f_0 \in \mathcal{A}_\varphi$ be given. We say that the Willmore energy $W : \mathcal{A}_\varphi \to \mathbb{R}$ has an *L^2–gradient at* f_0, denoted by $\operatorname{grad}_{\mathcal{A}_\varphi} W(f_0)$, if there exists a smooth function $\operatorname{grad}_{\mathcal{A}_\varphi} W(f_0) : M \to \mathbb{R}$ such that for all smooth one-parameter families of immersions $f : J \times M \to \mathbb{R}^{n+1}$ with $J \subset \mathbb{R}$ an open interval such that $0 \in J$, $f(0) = f_0$ and $f(t) \in \mathcal{A}_\varphi$ for all $t \in J$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} W(f(t))_{|t=0} = \int_M \operatorname{grad}_{\mathcal{A}_\varphi} W(f_0) \left\langle \partial_t f(t)_{|t=0}, \nu_0 \right\rangle \mathrm{d}V_{g_0}$$

where ν_0 is the unit normal field to (M, f_0) and g_0 the metric on M induced by f_0. We define the *domain* of $\operatorname{grad}_{\mathcal{A}_\varphi} W$ by

$$\mathcal{D}(\operatorname{grad}_{\mathcal{A}_\varphi} W) := \left\{ f \in \mathcal{A}_\varphi : W \text{ has an } L^2\text{–gradient at } f \right\} .$$

To study the L^2–gradient of W we make use of Lemma A.25 which can be seen as a version of the Fundamental Lemma of Calculus of Variations on manifolds.

Lemma 1.31 (Uniqueness of the L^2–gradient.)**.** *Let M be a smooth compact oriented manifold with boundary of dimension $n \in \mathbb{N}$ and let $f_0 \in \mathcal{A}_\varphi$ be given. Then the L^2–gradient of the Willmore energy $W : \mathcal{A}_\varphi \to \mathbb{R}$ at f_0 is unique if it exists.*

Proof. Assume that there exist smooth functions $\mathcal{G}_i : M \to \mathbb{R}$, $i \in \{1, 2\}$, both satisfying the defining property of the L^2–gradient. Let $\psi \in C^\infty(M)$ with $\psi \equiv 0$ on ∂M be given. By Proposition 1.10 there exists $\varepsilon > 0$ such that

$$f(t, p) := f_0(p) + t\psi(p)\nu_0(p)$$

defines a smooth one-parameter family of smooth immersions $f : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}^{n+1}$ with $f(t) \in \mathcal{A}_\varphi$ and $f(0) = f_0$. The defining property of the L^2–gradient yields

$$0 = \int_M (\mathcal{G}_1 - \mathcal{G}_2)(p) \left\langle \psi(p)\nu_0(p), \nu_0(p) \right\rangle \mathrm{d}V_{g_0} = \int_M (\mathcal{G}_1 - \mathcal{G}_2)(p)\psi(p) \, \mathrm{d}V_{g_0} .$$

As this identity is valid for all $\psi \in C^\infty(M)$ with $\psi \equiv 0$ on ∂M, Lemma A.25 yields that $\mathcal{G}_1(p) = \mathcal{G}_2(p)$ for all $p \in M$. □

Proposition 1.32 (Characterisation of the L^2–gradient of W)**.** *Let M be a smooth compact oriented manifold with boundary of dimension $n \in \mathbb{N}$ and let $f_0 \in \mathcal{A}_\varphi$ be given. The Willmore energy W has an L^2–gradient at f_0 if and only if f_0 satisfies $H_{f_0}(p) = 0$ for all $p \in \partial M$. In this case, the L^2–gradient is given by*

$$\operatorname{grad}_{\mathcal{A}_\varphi} W(f_0) = \langle \Delta_{M,g_0} \boldsymbol{H}_0 + Q(A)_0 \boldsymbol{H}_0, \nu_0 \rangle$$

where $\boldsymbol{H}_0$ and ν_0 denote the mean curvature vector and unit normal field of (M, f_0), respectively, and $Q(A)_0 = Q(A, f_0)$ is the expression defined in Definition 1.18. In other words,

$$\mathcal{D}(\operatorname{grad}_{\mathcal{A}_\varphi} W) = \{ f \in \mathcal{A}_\varphi : H_f = 0 \text{ on } \partial M \} .$$

Proof. Suppose that f_0 satisfies the boundary condition $H_{f_0} = 0$ on ∂M. Let $J \subset \mathbb{R}$ be an open interval with $0 \in J$ and $f : J \times M \to \mathbb{R}^{n+1}$ be a smooth one-parameter family of immersions such that $f(0) = f_0$ and $f(t) \in \mathcal{A}_\varphi$. In particular, $\partial_t f(t) = 0$ on ∂M and Proposition 1.29 immediately implies that W has an L^2–gradient at f_0 which, by uniqueness, is given by the desired identity. Suppose on the other hand that W has an L^2–gradient at f_0. We show that

$$\operatorname{grad}_{\mathcal{A}_\varphi} W(f_0) = \langle \Delta_{M,g_0} \boldsymbol{H}_0 + Q(A)_0 \boldsymbol{H}_0, \nu_0 \rangle \tag{1.2}$$

on M. As all involved quantities are smooth, it is enough to show the identity for every interior point. Assume by contradiction that there exists $p \in M \setminus \partial M$ such that

$$\mathcal{G}(p) := \operatorname{grad}_{\mathcal{A}_\varphi} W(f_0)(p) - \langle \Delta_{M,g_0} \boldsymbol{H}_0(p) + Q(A)_0(p)\boldsymbol{H}_0(p), \nu_0(p)\rangle > 0\,.^{1}$$

There exist a constant $\delta > 0$ and open neighbourhoods U and V around p with $p \in U \subset \overline{U}^M \subset V \subset \overline{V}^M \subset M \setminus \partial M$ such that $\mathcal{G} \geq \delta$ on V. Here, $\overline{U}^M$ and $\overline{V}^M$ denote the closure of U and V in M, respectively. Let $\psi \in C^\infty(M)$ be a function satisfying $0 \leq \psi \leq 1$ on M, $\operatorname{supp}\psi \subset V$ and $\psi \equiv 1$ on $\overline{U}^M$. The existence of such a bump function follows from [89, Proposition 2.25]. In particular, $\psi \equiv 0$ on ∂M and $f(t,p) := f_0(p) + t\psi(p)\nu_0(p)$ is a smooth one-parameter family of smooth immersions for $|t| < \varepsilon$, $\varepsilon > 0$ sufficiently small, with $f(t) \in \mathcal{A}_\varphi$ and $\partial_t f(t) = 0$ on ∂M. Observe that $V(0) = \big\langle (\partial_t f(t))_{|t=0}, \nu_0\big\rangle = \psi$ on M. The defining property of the L^2–gradient yields

$$\frac{\mathrm{d}}{\mathrm{d}t} W(f(t))_{|t=0} = \int_M \operatorname{grad}_{\mathcal{A}_\varphi} W(f_0)(q)\psi(q)\,\mathrm{d}V_{g_0}\,.$$

For any boundary point $q \in \partial M$ there exists a boundary chart (U_q, ϕ_q) around q such that $\psi \equiv 0$ on U_q which implies $\nabla_{M,g_0} V(0,q) = \nabla_{M,g_0}\psi(q) = 0$. Proposition 1.29 thus implies

$$\frac{\mathrm{d}}{\mathrm{d}t} W(f(t))_{|t=0} = \int_M \langle \Delta_{M,g_0} \boldsymbol{H}_0(q) + Q(A)_0(q)\boldsymbol{H}_0(q), \nu_0(q)\rangle\, \psi(q)\mathrm{d}V_{g_0}$$

and hence

$$0 = \int_M \mathcal{G}(q)\psi(q)\,\mathrm{d}V_{g_0} \geq \int_{\overline{U}^M} \mathcal{G}(q)\psi(q)\,\mathrm{d}V_{g_0} \geq \delta \int_{\overline{U}^M} 1\,\mathrm{d}V_{g_0} > 0\,,$$

a contradiction. This shows the identity (1.2). It remains to show that f_0 satisfies the boundary condition $H_{f_0} = 0$ on ∂M. Observe that the identity (1.2) implies that for all smooth one-parameter families of immersions $f : J \times M \to \mathbb{R}^{n+1}$, $J \subset \mathbb{R}$ an open interval with $0 \in J$, $f(0) = f_0$ and $\partial_t f(t) = 0$ on ∂M,

$$0 = \int_{\partial M} H_0 g_0\left(\nabla_{M,g_0} V(0), N(0)\right)\,\mathrm{d}V_{\tilde{g}_0}$$

where $N(0)$ is the smooth outer unit conormal of (M, g_0). Assume that there exists a point $q \in \partial M$ such that $H_0(q) > 0$. Let (U, ϕ) be a boundary chart around q such that for all $p \in U$, $H_0(p) \geq \delta > 0$. The function $x^n := \phi^n : U \to \mathbb{R}$ is smooth and satisfies $x^n = 0$ on $U \cap \partial M$. By [89, Proposition 15.33] for every $p \in U \cap \partial M$ the outer unit conormal $N(0,p)$ of (M, f_0) equals

$$N(0,p) = \frac{-\nabla_{M,g_0} x^n(p)}{|\nabla_{M,g_0} x^n(p)|_{g_0}}\,.$$

Let $\mathcal{U}$ be an open neighbourhood of q in M such that the closure of $\mathcal{U}$ in M is contained in U and let $\psi \in C^\infty(M)$ be a function with $\psi \equiv 1$ on $\mathcal{U}$, $0 \leq \psi \leq 1$ on M and $\operatorname{supp}\psi \subset U$. Then for $|t| < \varepsilon$, $\varepsilon > 0$ sufficiently small, the function

$$f(t,p) := f_0(p) + t\psi(p)x^n(p)\nu_0(p)$$

defines a smooth one-parameter family of immersions $f : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}^{n+1}$ with $f(t) = f_0$ on ∂M and $V(0,p) = \psi(p)x^n(p)$ for any $p \in M$. In the case that p is a boundary point, we observe that

$$\nabla_{M,g_0} V(0,p) = \psi(p)\nabla_{M,g_0} x^n(p) + x^n(p)\nabla_{M,g_0}\psi(p) = \psi(p)\nabla_{M,g_0} x^n(p)\,.$$

[1]The case $\mathcal{G}(p) < 0$ for some $p \in M \setminus \partial M$ can be treated analogously.

This yields

$$
\begin{aligned}
0 &= -\int_{\partial M} H_0(p) g_0 \left(\nabla_{M,g_0} V(0,p), N(0,p)\right) \mathrm{d}V_{\tilde{g}_0} = \int_{\partial M} H_0(p)\psi(p) \left|\nabla_{M,g_0} x^n(p)\right|_{g_0} \mathrm{d}V_{\tilde{g}_0} \\
&\geq \delta \int_{\mathcal{U}} \left|\nabla_{M,g_0} x^n(p)\right|_{g_0} \mathrm{d}V_{\tilde{g}_0} > 0\,,
\end{aligned}
$$

a contradiction. This completes the proof. □

The characterisation of the L^2–gradient of the Willmore energy yields the following gradient flow.

Definition 1.33 (Smooth solution to the Willmore flow with Navier conditions)**.** Let M be a smooth compact oriented manifold with boundary of dimension $n \in \mathbb{N}$ and $\varphi : M \to \mathbb{R}^{n+1}$ be a given smooth immersion. A *smooth solution to the L^2–gradient flow of W in $\mathcal{A}_\varphi$* is a smooth one-parameter family of smooth immersions $f : J \times M \to \mathbb{R}^{n+1}$ with $J \subset \mathbb{R}$ an interval and $f \in C^\infty\left(J \times M; \mathbb{R}^{n+1}\right)$, such that for all $t \in J$, $f(t) \in \mathcal{D}(\operatorname{grad}_{\mathcal{A}_\varphi} W)$ and for all $p \in M$ and $t \in J$,

$$
V(t,p) = \langle \partial_t f(t,p), \nu(t,p) \rangle = -\operatorname{grad}_{\mathcal{A}_\varphi} W(f(t))\,.
$$

The L^2–gradient flow of W is referred to as *Willmore flow with Navier conditions.*

The characterisation of $\mathcal{D}(\operatorname{grad}_{\mathcal{A}_\varphi} W)$ given in Proposition 1.32 immediately implies the following observation.

Corollary 1.34. *Let M be a smooth compact oriented manifold with boundary of dimension $n \in \mathbb{N}$ and $\varphi : M \to \mathbb{R}^{n+1}$ be a given smooth immersion. A smooth one-parameter family of smooth immersions $f : J \times M \to \mathbb{R}^{n+1}$, with $J \subset \mathbb{R}$ an interval and $f \in C^\infty\left(J \times M; \mathbb{R}^{n+1}\right)$, is a smooth solution to the L^2–gradient flow of W in $\mathcal{A}_\varphi$ if and only if f satisfies*

$$
\begin{cases}
\langle \partial_t f, \nu \rangle &= -\langle \Delta_g \boldsymbol{H}, \nu \rangle - Q(A)\, H \quad \text{in } J \times M\,, \\
H &= 0 \qquad\qquad\qquad\qquad\quad \text{on } J \times \partial M\,, \\
f &= \varphi \qquad\qquad\qquad\qquad\quad \text{on } J \times \partial M
\end{cases}
\tag{1.3}
$$

where $\nu(t,p)$ and $H(t,p)$ are the normal field and mean curvature of $(M, f(t))$, $t \in J$, at the point $p \in M$, respectively, $\boldsymbol{H}(t,p) = H(t,p)\nu(t,p)$, g_t is the metric on M induced by $f(t)$, $Q(A)$ is the expression defined in Definition 1.18 and further

$$
\left(\Delta_g \boldsymbol{H}\right)(t,p) := \left(\Delta_{g_t} \boldsymbol{H}(t)\right)(p) = \Delta_{g_t}\left(\Delta_{g_t} f(t)\right)(p) =: \Delta_g^2 f(t,p)\,.
$$

Expanding the motion equation in local coordinates, one obtains a quasilinear equation of fourth order of parabolic type, that is *degenerate* in the sense that only the *normal component of the time derivative* is prescribed. The parameter t is interpreted as time and the interval J is of the form $J = [0,T]$, $J = [0,T)$ or $J = (0,T)$ for $T > 0$. The following corollary shows that the Willmore energy is indeed decreasing along the flow.

Corollary 1.35. *Let M be a smooth compact oriented manifold with boundary of dimension $n \in \mathbb{N}$ and $\varphi : M \to \mathbb{R}^{n+1}$ be a given smooth immersion. Given an interval $J \subset \mathbb{R}$ and a smooth solution $f \in C^\infty\left(J \times M; \mathbb{R}^{n+1}\right)$ to the L^2–gradient flow of W in $\mathcal{A}_\varphi$ it holds for all $t \in J$,*

$$
\frac{\mathrm{d}}{\mathrm{d}t} W(f(t)) \leq 0\,.
$$

Proof. Proposition 1.29 and Corollary 1.34 yield for every $t \in J$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} W(f(t)) &= \int_M \langle \Delta_{M,g_t} \boldsymbol{H}(t) + Q(A)(t)\boldsymbol{H}(t), \boldsymbol{V}(t)\rangle \,\mathrm{d}V_{g_t} + \int_{\partial M} H(t) g_t \left(\nabla_{M,g_t} V(t), N(t)\right) \mathrm{d}V_{\tilde{g}_t} \\ &= \int_M \left(\langle \Delta_{M,g_t} \boldsymbol{H}(t), \nu(t)\rangle + Q(A)(t)H(t)\right) \langle \partial_t f(t), \nu(t)\rangle \,\mathrm{d}V_{g_t} \\ &= -\int_M \left(\langle \Delta_{M,g_t} \boldsymbol{H}(t), \nu(t)\rangle + Q(A)(t)H(t)\right)^2 \mathrm{d}V_{g_t} \leq 0\,.\end{aligned}$$

□

The following proposition yields a more explicit representation of the motion equation in terms of the principal curvatures. This equivalent formulation is used in many application oriented contributions in the literature, as for instance in [18, 19].

Proposition 1.36 (An equivalent formulation of the L^2–gradient of W)**.** *Let M be a smooth compact oriented manifold of dimension $n \in \mathbb{N}$ and $f : M \to \mathbb{R}^{n+1}$ be a smooth immersion inducing the metric $g = (\mathrm{d}f)^* (\langle \cdot, \cdot \rangle)$ on M. Let further $\nu(p)$, $(a_{ij}(p))_{ij}$, $H(p)$ and $\boldsymbol{H}(p)$ be the normal field, second fundamental form, mean curvature and mean curvature vector of (M, f) at $p \in M$ as defined in Proposition 1.4, Definition 1.6 and Definition 1.8, respectively, and let*

$$Q(A) = 2 g^{im} g^{kj} a_{mk} a_{ij} - \frac{1}{2} H^2$$

be the expression defined in Definition 1.18. Denoting by $\kappa_1, \ldots, \kappa_n$ the principal curvatures of (M, f) it holds

$$-\langle \Delta_g \boldsymbol{H}, \nu \rangle - Q(A)H = -\Delta_g H - H \sum_{i=1}^n \kappa_i^2 + \frac{1}{2} H^3\,.$$

Proof. Proposition 1.19 yields

$$g^{im} g^{kj} a_{mk} a_{ij} = \sum_{i=1}^n \kappa_i^2 \tag{1.4}$$

and in particular

$$-Q(A)H = -2H \sum_{i=1}^n \kappa_i^2 + \frac{1}{2} H^3\,.$$

Using the product rule for the divergence given in Proposition A.33 we find

$$\Delta_g \boldsymbol{H} = \Delta_g (H\nu) = \nu \Delta_g H + 2g \left(\nabla_g H, \nabla_g \nu^j\right)_{j=1}^{n+1} + H \Delta_g \nu$$

with $\nu = \left(\nu^1, \ldots, \nu^{n+1}\right)$, where for all $j \in \{1, \ldots, n+1\}$,

$$g\left(\nabla_g H, \nabla_g \nu^j\right) = \mathrm{d}\nu^j \left(\nabla_g H\right)\,.$$

Proposition 1.5 yields for every $p \in M$ that $\mathrm{d}\nu_p \left(\nabla_g H(p)\right)$ lies in $\mathrm{d}f_p(T_p M)$. Hence we obtain

$$-\langle \Delta_g \boldsymbol{H}, \nu \rangle = -\Delta_g H - H \langle \Delta_g \nu, \nu \rangle - 2 \langle \mathrm{d}\nu \left(\nabla_g H\right), \nu \rangle = -\Delta_g H - H \langle \Delta_g \nu, \nu \rangle\,.$$

Given $p \in M$ and a chart (U, ϕ) around p we have

$$\Delta_g \nu(p) = g^{ij}(p) \left(\frac{\partial^2 (\nu \circ \phi^{-1})}{\partial x_i \partial x_j} (\phi(p)) - \Gamma_{ij}^l (p) \frac{\partial (\nu \circ \phi^{-1})}{\partial x_l} (\phi(p)) \right)\,.$$

As $\frac{\partial(\nu\circ\phi^{-1})}{\partial x_l}(\phi(p)) = \mathrm{d}\nu_p\left(\frac{\partial}{\partial x^l}\right)$ lies in $\mathrm{d}f_p(T_pM)$, we obtain

$$\left\langle \frac{\partial(\nu\circ\phi^{-1})}{\partial x_l}(\phi(p)), \nu(p)\right\rangle = 0\,.$$

Differentiating this identity yields further

$$\left\langle \frac{\partial^2(\nu\circ\phi^{-1})}{\partial x_i \partial x_j}\left(\phi(p)\right), \nu(p)\right\rangle = -\left\langle \frac{\partial(\nu\circ\phi^{-1})}{\partial x_i}(\phi(p)), \frac{\partial(\nu\circ\phi^{-1})}{\partial x_j}(\phi(p))\right\rangle$$

which allows us to conclude that

$$\begin{aligned}\langle \Delta_g \nu(p), \nu(p)\rangle &= -g^{ij}(p)\left\langle \frac{\partial(\nu\circ\phi^{-1})}{\partial x_i}(\phi(p)), \frac{\partial(\nu\circ\phi^{-1})}{\partial x_j}(\phi(p))\right\rangle \\ &= -g^{ij}(p)\left\langle \mathrm{d}\nu_p\left(\frac{\partial}{\partial x^i}\right), \mathrm{d}\nu_p\left(\frac{\partial}{\partial x^j}\right)\right\rangle = -g^{ij}(p)g_p\left(W_p\left(\frac{\partial}{\partial x^i}\right), W_p\left(\frac{\partial}{\partial x^j}\right)\right)\end{aligned}$$

where $W_p = -\left(\mathrm{d}f_p\right)^{-1}\circ \mathrm{d}\nu_p$ denotes the Weingarten map. Proposition 1.9 yields the local representation

$$W_p\left(\frac{\partial}{\partial x^j}\right) = g^{lm}(p)a_{jm}(p)\frac{\partial}{\partial x^l}$$

which implies together with the identity (1.4)

$$\begin{aligned}\langle \Delta_g \nu(p), \nu(p)\rangle &= -g^{ij}(p)g^{lm}(p)a_{jm}(p)g_p\left(W_p\left(\frac{\partial}{\partial x^i}\right), \frac{\partial}{\partial x^l}\right) \\ &= -g^{ij}(p)g^{lm}(p)a_{jm}(p)a_{il}(p) = -\sum_{i=1}^n \kappa_i^2\,.\end{aligned}$$

We finally conclude

$$\begin{aligned}-\left\langle \Delta_g \boldsymbol{H}, \nu\right\rangle - Q(A)H &= -\Delta_g H - H\left\langle \Delta_g \nu, \nu\right\rangle - Q(A)H \\ &= -\Delta_g H + H\sum_{i=1}^n \kappa_i^2 - 2H\sum_{i=1}^n \kappa_i^2 + \frac{1}{2}H^3 = -\Delta_g H - H\sum_{i=1}^n \kappa_i^2 + \frac{1}{2}H^3\,.\end{aligned}$$

□

1.3 Strong graph solutions and conditions on the initial value

In this section we introduce the concept of strong graph solutions used in Chapter 2 to prove an existence result for the Willmore flow with Navier conditions given in system (1.3), and derive necessary conditions on the initial value.

One major difficulty in the formulation (1.3) lies in the fact that the geometry on M induced by the immersion $f(t)$ changes with time. This can be overcome by considering a fixed smooth metric on M which leads to the so-called *reference geometry*. The concept of graph solutions then suggests to study evolutions of initial surfaces that can be described as a graph over the reference geometry. The continuity of the motion indicates that such evolutions should be expressible as time-dependent graphs over the reference surface, at least in a short time interval.

Definition 1.37 (Smooth reference geometry)**.** Given $n \in \mathbb{N}$ a *smooth reference geometry of dimension* n is a tuple (M, φ) with M a smooth compact oriented manifold with boundary of dimension n and $\varphi : M \to \mathbb{R}^{n+1}$ a smooth immersion.

Let (M,φ) be a given smooth reference geometry. We consider graph solutions of the form

$$f(t)=\varphi+\varrho(t)\nu_\varphi$$

with $\varrho(t):M\to\mathbb{R}$ a suitable *height function* and ν_φ the unit normal to (M,φ). Proposition 1.10 guarantees that $f(t)$ is an immersion provided that for a finite covering of M the local representations of the differential of $\varrho(t)\nu_\varphi$ are uniformly small. A sufficient condition in terms of ϱ is given in Corollary 1.42. It is worth mentioning that this criterion depends in particular on the differential of the normal ν_φ and hence on the curvature of the reference surface.

We intend to prove existence of such graph solutions in the solution space

$$W_p^{1,4}\left((0,T)\times M;\mathbb{R}^{n+1}\right):=W_p^1\left((0,T);L_p(M;\mathbb{R}^{n+1})\right)\cap L_p\left((0,T);W_p^4(M;\mathbb{R}^{n+1})\right) \tag{1.5}$$

on the smooth Riemannian manifold (M,g_φ) with $p\in(4+n,\infty)$. Details on function spaces on smooth compact Riemannian manifolds can be found in Chapter C. We refer in particular to Section C.2 for a discussion of anisotropic Sobolev spaces on compact manifolds and to Section 1.4 for further explanations on the handling of these spaces in the context of this thesis. Roughly speaking, a function f lies in the space (1.5) if there exists a suitable covering of M by charts (U,ϕ) such that $f\circ\phi^{-1}$ lies in the corresponding space on $\phi(U)$. Anisotropic Sobolev spaces on domains are discussed in Section B.3.
In particular, the value of p guarantees for a smooth domain Ω the continuous embedding

$$W_p^1\left((0,T);L_p(\Omega)\right)\cap L_p\left((0,T);W_p^4(\Omega)\right)\hookrightarrow C\left([0,T];C^3\left(\overline{\Omega}\right)\right)$$

as shown in Proposition B.35. This ensures that the boundary conditions appearing in system (1.3) can be formulated pointwise in time and space. In particular, a solution f of regularity (1.5) to the flow satisfies $f(0)=\varphi$ and $H_{f(0)}=0$ on ∂M. Proposition B.35 further yields that $f(0)\circ\phi^{-1}$ lies in the Slobodeckij space $W_p^{4-4/p}(\phi(U))$ for every chart (U,ϕ) which corresponds to the property $f(0)\in W_p^{4-4/p}(M)$. Details on Slobodeckij spaces on domains and compact manifolds are discussed in Section B.2 and Subsection C.1.2, respectively.
We thus need to impose *compatibility conditions* on the initial value. To this end we observe that for immersions f of the form $f=\varphi+\varrho\nu_\varphi$ with height function $\varrho:M\to\mathbb{R}$ the condition $f=\varphi$ on ∂M is equivalent to $\varrho=0$ on ∂M. This leads to the definition of admissible initial height functions.

Definition 1.38 (Admissible initial height function)**.** Let (M,φ) be a smooth reference geometry of dimension $n\in\mathbb{N}$ and $p\in(4+n,\infty)$. A *p–admissible initial height function* is a function $\varrho_0\in X_0:=W_p^{4-4/p}(M)$ with $\varrho_0=0$ on ∂M such that $f_0:=\varphi+\varrho_0\nu_\varphi$ is an immersion that satisfies $H_{f_0}=0$ on ∂M.

Definition 1.39 (Strong graph solution)**.** Let (M,φ) be a smooth reference geometry of dimension $n\in\mathbb{N}$. Given $p\in(4+n,\infty)$ and a p–admissible initial height function ϱ_0, a *strong graph solution to the Willmore flow with Navier conditions* in $[0,T]$ for $T>0$ with initial value ϱ_0 is a function

$$\varrho\in\mathbb{E}_T:=W_p^1\left((0,T);L_p(M)\right)\cap L_p\left((0,T);W_p^4(M)\right)$$

such that $f^\varrho(t):=\varphi+\varrho(t)\nu_\varphi\in C^3\big(M;\mathbb{R}^{n+1}\big)$ is an immersion for every $t\in[0,T]$ and

$$\begin{cases}\langle\partial_t f^\varrho,\nu^\varrho\rangle &=-\langle\Delta_{g^\varrho}\boldsymbol{H}^\varrho,\nu^\varrho\rangle-Q\left(A\right)^\varrho H^\varrho \quad \text{on } (0,T)\times M\,,\\ H^\varrho &=0 \quad \text{on } [0,T]\times\partial M\,,\\ \varrho &=0 \quad \text{on } [0,T]\times\partial M\,,\\ \varrho(0) &=\varrho_0 \quad \text{on } \{0\}\times M\end{cases} \tag{1.6}$$

where g_t^ϱ is the metric on M induced by $f^\varrho(t)$, $\nu^\varrho(t,p)$ and $H^\varrho(t,p)$ are the normal field and mean curvature of $(M, f^\varrho(t))$ at the point p, respectively, $\boldsymbol{H}^\varrho = H^\varrho \nu^\varrho$, $Q(A)^\varrho(t,p)$ the quantity $Q(A)_{f^\varrho(t)}(p)$ defined in 1.18 and

$$\left(\Delta_g \boldsymbol{H}^\varrho\right)(t,p) := \big(\Delta_{g_t^\varrho} \boldsymbol{H}^\varrho(t)\big)(p) = \big(\Delta_{g_t^\varrho}\big(\Delta_{g_t^\varrho} f^\varrho(t)\big)\big)(p) =: \Delta^2_{g^\varrho} f^\varrho(t,p)\,.$$

We remark that a strong graph solution satisfies the motion equation in an almost everywhere sense while the boundary conditions are valid pointwise. This is made precise in Section 2.1.
In Corollary 1.42 we show a criterion depending only on the norm of ϱ in the solution space $\mathbb{E}_T$ to guarantee that the functions $f^\varrho(t)$ are immersions.

1.4 Notation and results related to the localisation procedure

In this section we collect notational conventions and results related to the localisation procedure that is used in Chapter 2 to show existence of graph solutions.

We note that a smooth reference geometry (M,φ) as defined in Definition 1.37 is in particular a smooth Riemannian manifold (M, g_φ) with induced metric g_φ as defined in Section 1.1. Geometric quantities related to the geometry on M induced by g_φ are denoted with subscript φ. We emphasise that all function spaces on M appearing in Chapter 2 are taken with respect to the reference geometry induced by g_φ.
A lot of estimates rely on certain "nice" properties of geometric terms related to the reference geometry. These are based on the concepts of the *normal covering* and the *uniform localisation system* of (M,φ).

Proposition 1.40 (Normal covering and localisation system of the reference surface)**.** *Given a smooth reference geometry (M,φ) of dimension $n \in \mathbb{N}$, $n \geq 2$, there exists a finite covering $\mathcal{T}$ of M of positively oriented charts (U_α, ϕ_α), $\alpha \in \{1,\dots,N\}$, such that for all $\alpha \in \{1,\dots,N\}$,*

(i) all derivatives of $g_{ij,\varphi} \circ \phi_\alpha^{-1}$ and $g_\varphi^{ij} \circ \phi_\alpha^{-1}$ are uniformly bounded on $\overline{\phi_\alpha(U_\alpha)}$;

(ii) there exists $Q > 1$ such that for all $v \in \mathbb{R}^n$, $q \in U_\alpha$,

$$Q^{-1}|v|^2 \leq \sum_{i,j=1}^n g_{ij,\varphi}(q) v^i v^j \leq Q|v|^2\,,$$
$$Q^{-1}|v|^2 \leq \sum_{i,j=1}^n g_\varphi^{ij}(q) v^i v^j \leq Q|v|^2\,;$$

(iii) for all continuous functions $f : M \to \mathbb{R}$, $f \circ \phi_\alpha^{-1}$ is uniformly bounded on $\overline{\phi_\alpha(U_\alpha)}$;

(iv) if (U_α, ϕ_α) is an interior chart, then $\phi_\alpha(U_\alpha)$ is a smooth domain in $\mathbb{R}^n$;

(v) if (U_α, ϕ_α) is a boundary chart, then $\phi_\alpha(U_\alpha) \cap \operatorname{int}\mathbb{H}^n$ is a smooth domain.

The covering $\mathcal{T}$ is referred to as a smooth normal covering. If $(\psi_\alpha)_{\alpha\in\{1,\dots,N\}}$ is a smooth partition of unity subordinate to the covering $\mathcal{T}$, we call

$$\mathcal{C} := \{(U_\alpha, \phi_\alpha, \psi_\alpha) \,:\, \alpha \in \{1,\dots,N\}\}$$

a uniform localisation system.

Proof. The existence of $\mathcal{T}$ is shown in Proposition A.44 using normal coordinates on the smooth Riemannian manifold (M, g_φ) as introduced in Section A.4. We refer to [89, Theorem 2.23] for the existence of a smooth partition of unity. The notion of uniform localisation systems is defined in Definition A.45. □

Given $\alpha \in \{1, \dots, N\}$ we refer to *the interior* $\mathrm{int}\phi_\alpha(U_\alpha)$ of $\phi_\alpha(U_\alpha)$ as the set $\phi_\alpha(U_\alpha)$ in the case of an interior chart, and as $\phi_\alpha(U_\alpha) \cap \mathrm{int}\mathbb{H}^n$ in the case of a boundary chart, respectively, using the convention $W_p^s(\phi_\alpha(U_\alpha)) := W_p^s(\mathrm{int}(\phi_\alpha(U_\alpha))$. The items (iv) and (v) yield that the interior of $\phi_\alpha(U_\alpha)$ is a smooth domain which allows to make use of Theorem C.29 and the existence result given in [136, Theorem 5.4].

In the following we let (M, φ) be a smooth reference geometry of dimension $n \in \mathbb{N}$, $n \geq 2$, with normal covering $\mathcal{T}$. We hereby restrict to the case of dimensions $n \geq 2$ for several reasons. The existence of the Willmore flow of open curves in $\mathbb{R}^n$ has already been studied in [137]. Moreover, the existence of a uniform localisation system requires the hypothesis $n \geq 2$. However, in the case of curves, a localisation procedure as performed in the case of higher-dimensional objects is not necessary, since one may simply parametrise the curves on intervals as done in Part II.

We endow the boundary ∂M with the smooth structure and Riemannian metric induced by the smooth structure and Riemannian metric g_φ on M, respectively. Details are given in Proposition A.15 and Proposition A.26, respectively. Furthermore, we let $\mathcal{J} := \{\alpha \in \{1, \dots, N\} : (U_\alpha, \phi_\alpha)$ is a boundary chart$\}$, $\iota : \partial M \to M$ be the inclusion mapping and $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection onto the first $n-1$ components. For $\alpha \in \mathcal{J}$ we introduce the notations $V_\alpha := U_\alpha \cap \partial M$ and $\sigma_\alpha := \pi \circ \phi_\alpha \circ \iota$. Proposition A.46 shows that the charts $(V_\alpha, \sigma_\alpha)$, $\alpha \in \mathcal{J}$, form a normal covering of ∂M. One readily checks the identities

$$\phi_\alpha(U_\alpha \cap \partial M) = \sigma_\alpha(V_\alpha) \times \{0\} = \partial\phi_\alpha(U_\alpha) \cap \partial\mathbb{H}^n \subset \partial\phi_\alpha(U_\alpha)\,.$$

In particular, for every $x' \in \sigma_\alpha(V_\alpha)$ it holds

$$\sigma_\alpha^{-1}(x') = \phi_\alpha^{-1}((x', 0))\,.$$

Furthermore, one easily verifies that given a smooth partition of unity (ψ_α) on M subordinate to the covering (U_α, ϕ_α), the functions $\psi_\alpha \circ \iota$ are a partition of unity on ∂M subordinate to the covering $(V_\alpha, \sigma_\alpha)$.

To obtain estimates with constants *independent* of the considered time interval we make use of the embeddings in Corollary B.38 and Corollary B.39. To this end we endow

$$\mathbb{E}_{T,\alpha} := W_p^1\left((0,T); L_p\left(\phi_\alpha(U_\alpha)\right)\right) \cap L_p\left((0,T); W_p^4\left(\phi_\alpha(U_\alpha)\right)\right)$$

with the norm $|||\cdot|||_{\mathbb{E}_{T,\alpha}}$ which, given $\varrho_\alpha \in \mathbb{E}_{T,\alpha}$, is defined as

$$|||\varrho_\alpha|||_{\mathbb{E}_{T,\alpha}} = \|\varrho_\alpha\|_{\mathbb{E}_{T,\alpha}} + \|\varrho_\alpha(0)\|_{W_p^{4-4/p}(\phi_\alpha(U_\alpha))}$$

with $\|\varrho_\alpha\|_{\mathbb{E}_{T,\alpha}}$ as in Definition B.26 and $\|\varrho_\alpha(0)\|_{W_p^{4-4/p}(\phi_\alpha(U_\alpha))}$ as in Proposition B.19, respectively. Proposition B.36 shows that $|||\cdot|||_{\mathbb{E}_{T,\alpha}}$ and $\|\cdot\|_{\mathbb{E}_{T,\alpha}}$ are equivalent norms on $\mathbb{E}_{T,\alpha}$. More details on the required function spaces and embeddings are given in Chapter B.
The essential ingredient in the localisation procedure is the characterisation of the solution space

$$\mathbb{E}_T := W_p^1\left((0,T); L_p(M)\right) \cap L_p\left((0,T); W_p^4(M)\right)$$

as introduced in Subsection C.2 with the help of the corresponding spaces on the chart domains of the smooth normal covering $\mathcal{T}$. It is shown in Proposition C.16 that equivalently to the geometric norm we may consider the norm $|||\cdot|||_{\mathbb{E}_T}$ on $\mathbb{E}_T$ which, given $\varrho \in \mathbb{E}_T$, is defined by

$$|||\varrho|||_{\mathbb{E}_T} := \|\varrho\|_{\mathbb{E}_T} + \|\varrho(0)\|_{W_p^{4-4/p}(M)}$$

where

$$\|\varrho\|_{\mathbb{E}_T} := \|\varrho\|^{\mathcal{T}}_{W_p^{1,4}((0,T)\times M)} := \sum_{\alpha=1}^{N} \left\|(\psi_\alpha \varrho) \circ \phi_\alpha^{-1}\right\|_{\mathbb{E}_{T,\alpha}} , \tag{1.7}$$

and

$$\|\varrho(0)\|_{W_p^{4-4/p}(M)} := \sum_{\alpha=1}^{N} \left\|(\psi_\alpha \varrho(0)) \circ \phi_\alpha^{-1}\right\|_{W_p^{4-4/p}(\phi_\alpha(U_\alpha))} \tag{1.8}$$

is the norm in Proposition C.11. In this way, we have

$$|||\varrho|||_{\mathbb{E}_T} = \sum_{\alpha=1}^{N} |||(\psi_\alpha \varrho) \circ \phi_\alpha^{-1}|||_{\mathbb{E}_{T,\alpha}} .$$

The characterisation of the solution space $\mathbb{E}_T$ in Proposition C.16 allows us to work with the "localised" spaces $\mathbb{E}_{T,\alpha}$. Indeed, to show that a function $\eta : (0,T) \times M \to \mathbb{R}$ lies in $\mathbb{E}_T$, it is enough to show that for all $\alpha \in \{1,\dots,N\}$ the function $(t,x) \mapsto \eta_\alpha(t,x) := \eta\left(t, \phi_\alpha^{-1}(x)\right)$ is an element of $\mathbb{E}_{T,\alpha}$. Conversely, given $\eta \in \mathbb{E}_T$, one has $\eta_\alpha \in \mathbb{E}_{T,\alpha}$ for all $\alpha \in \{1,\dots,N\}$ with

$$\|\eta_\alpha\|_{\mathbb{E}_{T,\alpha}} \leq C(Q)\, \|\eta\|_{\mathbb{E}_T} .$$

Analogous properties hold for the spaces

$$\begin{aligned} X_0 &:= W_p^{4-4/p}(M) , \\ X_T &:= L_p\left((0,T); L_p(M)\right) , \\ Y_{1,T} &:= W_p^{1/2-1/4p,\,2-1/p}\left((0,T) \times \partial M\right) , \\ Y_{2,T} &:= W_p^{1-1/4p,\,4-1/p}\left((0,T) \times \partial M\right) \end{aligned}$$

and their "localised" versions

$$\begin{aligned} X_{0,\alpha} &:= W_p^{4-4/p}\left(\phi_\alpha(U_\alpha)\right) , \\ X_{T,\alpha} &:= L_p\left((0,T); L_p\left(\phi_\alpha(U_\alpha)\right)\right) , \\ Y_{1,T,\alpha} &:= W_p^{1/2-1/4p,\,2-1/p}\left((0,T) \times \partial\phi_\alpha(U_\alpha)\right) , \\ Y_{2,T,\alpha} &:= W_p^{1-1/4p,\,4-1/p}\left((0,T) \times \partial\phi_\alpha(U_\alpha)\right) . \end{aligned}$$

The constant Q is introduced in Proposition 1.40 and refers to the normal covering $\mathcal{T}$. We use the notation $C(Q)$ in the estimates given in Chapter 2 to indicate that the respective constants depend on the choice of the normal covering of the reference geometry.
In Section 2.3 we are concerned with the extension of functions defined on subsets of the manifold to functions on the entire manifold while maintaining their regularity. To this end we make use of so-called cut-off functions $\xi_\alpha \in C^\infty(M)$ with the properties $\operatorname{supp}\xi_\alpha \subset U_\alpha$, $0 \leq \xi_\alpha \leq 1$ on M and $\xi \equiv 1$ on $\operatorname{supp}\psi_\alpha$. The existence of such functions follows from [89, Proposition 2.2.5]. The property

$$\operatorname{dist}\left(\operatorname{supp}\left(\xi_\alpha \circ \phi_\alpha^{-1}\right) \cap \overline{\phi_\alpha(U_\alpha)}, \partial\phi_\alpha\left(U_\alpha\right) \cap \operatorname{int}\mathbb{H}^n\right) > 0$$

as illustrated in Figure 1.1 then allows us to use the extension results in Subsection C.1.3 and Subsection C.2.2. We refer to Chapter C for more details on properties of function spaces on compact manifolds.

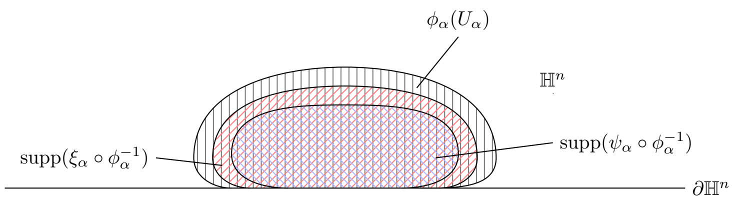

Figure 1.1: The cut-off function ξ_α.

One crucial aspect in the graph formulation is to ensure that the height functions $\varrho(t)$ are "small enough" to guarantee that the functions $f^\varrho(t) := \varphi + \varrho(t)\nu_\varphi$ are immersions. A sufficient criterion on the height function is given in Corollary 1.42 using the following refinement of Proposition 1.10.

Proposition 1.41. *Let (M, φ) be a smooth reference geometry of dimension $n \in \mathbb{N}$, $n \geq 2$, with normal covering $\mathcal{T}$ as in Proposition 1.40. Then there exists $\delta > 0$ such that every function $f \in C^1\left(M; \mathbb{R}^{n+1}\right)$ that satisfies for all $\alpha \in \{1, \dots, N\}$,*

$$\left\|\nabla\left(f \circ \phi_\alpha^{-1}\right) - \nabla\left(\varphi \circ \phi_\alpha^{-1}\right)\right\|_{C\left(\overline{\phi_\alpha(U_\alpha)}; \mathbb{R}^{(n+1)n}\right)} < \delta\,,$$

is an immersion with $\det\left((g_{ij,f}(p))_{ij}\right) \in \left[\frac{1}{2}Q^{-n}, 2Q^n\right]$ for all $p \in M$, and there exists a constant $\sigma = \sigma(\varphi) > 0$ such that for all $x \in \phi_\alpha\left(U_\alpha\right)$ and all $\alpha \in \{1, \dots, N\}$,

$$\left\|\psi\left(\nabla\left(f \circ \phi_\alpha^{-1}\right)(x)\right)\right\| \geq \sigma\,,$$

where ψ is defined in Definition 1.3.

Proof. By Proposition 1.10 there exists a constant $\widetilde{\delta} > 0$ such that every $f \in C^1\left(M; \mathbb{R}^{n+1}\right)$ that satisfies for all $\alpha \in \{1, \dots, N\}$,

$$\left\|\nabla\left(f \circ \phi_\alpha^{-1}\right) - \nabla\left(\varphi \circ \phi_\alpha^{-1}\right)\right\|_{C\left(\overline{\phi_\alpha(U_\alpha)}; \mathbb{R}^{(n+1)n}\right)} < \widetilde{\delta}\,,$$

is an immersion. Proposition 1.40 implies that for all $\alpha \in \{1, \dots, N\}$ and all $x \in \phi_\alpha\left(U_\alpha\right)$, there holds

$$\det\left(g_{ij,\varphi}\left(\phi_\alpha^{-1}(x)\right)\right) \in \left[Q^{-n}, Q^n\right]\,.$$

Let $R > 1$ be such that for all $i \in \{1, \dots, n+1\}$, $j \in \{1, \dots, n\}$, $\alpha \in \{1, \dots, N\}$ and $x \in \phi_\alpha(U_\alpha)$, $\left|\frac{\partial\left(\varphi^i \circ \phi_\alpha^{-1}\right)}{\partial x_j}(x)\right| < R$. As $\left|g_{ij,\varphi}\left(\phi_\alpha^{-1}(x)\right)\right| \leq R^2$ and $\det : \left[-2R^2, 2R^2\right]^{n^2} \to \mathbb{R}$ is continuous, there exists $\widehat{\delta} \in (0,1)$ such that for all (a_{ij}), $(b_{ij}) \in \left[-2R^2, 2R^2\right]^{n^2}$ with entries satisfying $|a_{ij} - b_{ij}| < \widehat{\delta}$, there holds

$$\left|\det\left((a_{ij})_{ij}\right) - \det\left((b_{ij})_{ij}\right)\right| < \frac{1}{2}\min\left\{Q^n, Q^{-n}\right\}\,.$$

Suppose that $f : M \to \mathbb{R}^{n+1}$ is such that for all $\alpha \in \{1, \dots, N\}$,

$$\left\|\nabla\left(f \circ \phi_\alpha^{-1}\right) - \nabla\left(\varphi \circ \phi_\alpha^{-1}\right)\right\|_{C\left(\overline{\phi_\alpha(U_\alpha)}; \mathbb{R}^{(n+1)n}\right)} < \min\left\{\widehat{\delta}(4R)^{-1}, \widetilde{\delta}\right\}\,.$$

Then f is an immersion and the metric $(g_{ij,f})$ is well-defined. Furthermore, there holds

$$\begin{aligned}&\left|g_{ij,\varphi}\left(\phi_\alpha^{-1}(x)\right) - g_{ij,f}\left(\phi_\alpha^{-1}(x)\right)\right| \\ &\leq \left|\left\langle \frac{\partial\left(\varphi \circ \phi_\alpha^{-1}\right)}{\partial x_i}(x) - \frac{\partial\left(f \circ \phi_\alpha^{-1}\right)}{\partial x_i}(x), \frac{\partial\left(\varphi \circ \phi_\alpha^{-1}\right)}{\partial x_j}(x)\right\rangle\right|\end{aligned}$$

$$+ \left| \left\langle \frac{\partial \left(f \circ \phi_\alpha^{-1}\right)}{\partial x_i}(x), \frac{\partial \left(\varphi \circ \phi_\alpha^{-1}\right)}{\partial x_j}(x) - \frac{\partial \left(f \circ \phi_\alpha^{-1}\right)}{\partial x_j}(x) \right\rangle \right| < \widehat{\delta}\,.$$

In particular, we obtain

$$\left|\det\left((g_{ij,\varphi})_{ij}\right) - \det\left((g_{ij,f})_{ij}\right)\right| < \frac{1}{2}\min\left\{Q^n, Q^{-n}\right\}$$

and thus $\det\left((g_{ij,f})_{ij}\right) \in \left[\frac{1}{2}Q^{-n}, 2Q^n\right]$. It remains to show the last assertion. As φ is an immersion, the columns in $\nabla\left(\varphi \circ \phi_\alpha^{-1}\right)(x)$ are linearly independent for all $\alpha \in \{1, \ldots, N\}$ and $x \in \overline{\phi_\alpha(U_\alpha)}$, and thus

$$\left\|\psi\left(\nabla\left(\varphi \circ \phi_\alpha^{-1}\right)(x)\right)\right\| = \varepsilon_{\alpha,x} > 0\,.$$

By compactness of $\overline{\phi_\alpha(U_\alpha)}$ and continuity of the function $x \mapsto \left\|\psi\left(\nabla\left(\varphi \circ \phi_\alpha^{-1}\right)(x)\right)\right\|$ on $\overline{\phi_\alpha(U_\alpha)}$, we obtain

$$\boldsymbol{\varepsilon} := \min_{\alpha \in \{1,\ldots,N\}} \min_{x \in \overline{\phi_\alpha(U_\alpha)}} \varepsilon_{\alpha,x} > 0\,.$$

Uniform continuity of $\|\psi(\cdot)\|$ on $[-2R, 2R]^{(n+1)\times n}$ yields that there exists $\boldsymbol{\delta} \in (0, \widetilde{\delta})$ such that for all $v, w \in [-2R, 2R]^{(n+1)\times n}$ with $|v_i^j - w_i^j| < \boldsymbol{\delta}$ for all $j \in \{1, \ldots, n\}$, $i \in \{1, \ldots, n+1\}$, there holds

$$\big|\|\psi(v)\| - \|\psi(w)\|\big| < \frac{1}{2}\boldsymbol{\varepsilon}\,.$$

Suppose that $f : M \to \mathbb{R}^{n+1}$ is such that for all $\alpha \in \{1, \ldots, N\}$,

$$\left\|\nabla\left(f \circ \phi_\alpha^{-1}\right) - \nabla\left(\varphi \circ \phi_\alpha^{-1}\right)\right\|_{C\left(\overline{\phi_\alpha(U_\alpha)};\mathbb{R}^{(n+1)n}\right)} < \boldsymbol{\delta}\,.$$

Let $\alpha \in \{1, \ldots, N\}$ and $x \in \overline{\phi_\alpha(U_\alpha)}$ be given. As for all $i \in \{1, \ldots, n+1\}$, $j \in \{1, \ldots, n\}$,

$$\left|\frac{\partial\left(f^i \circ \phi_\alpha^{-1}\right)}{\partial x_j}(x) - \frac{\partial\left(\varphi^i \circ \phi_\alpha^{-1}\right)}{\partial x_j}(x)\right| \le \left\|\nabla\left(f \circ \phi_\alpha^{-1}\right) - \nabla\left(\varphi \circ \phi_\alpha^{-1}\right)\right\|_{C\left(\overline{\phi_\alpha(U_\alpha)};\mathbb{R}^{(n+1)n}\right)} < \boldsymbol{\delta}\,,$$

we conclude

$$\begin{aligned}
&\left\|\psi\left(\nabla\left(f \circ \phi_\alpha^{-1}\right)(x)\right)\right\| \\
&\ge \left\|\psi\left(\nabla\left(\varphi \circ \phi_\alpha^{-1}\right)(x)\right)\right\| - \left|\left\|\psi\left(\nabla\left(f \circ \phi_\alpha^{-1}\right)(x)\right)\right\| - \left\|\psi\left(\nabla\left(\varphi \circ \phi_\alpha^{-1}\right)(x)\right)\right\|\right| \\
&\ge \left\|\psi\left(\nabla\left(\varphi \circ \phi_\alpha^{-1}\right)(x)\right)\right\| - \frac{1}{2}\boldsymbol{\varepsilon} = \varepsilon_{\alpha,x} - \frac{1}{2}\boldsymbol{\varepsilon} \ge \frac{1}{2}\boldsymbol{\varepsilon}\,.
\end{aligned}$$

□

Corollary 1.42 (The neighbourhood for the height function)**.** *Let (M, φ) be a smooth reference geometry of dimension $n \in \mathbb{N}$, $n \ge 2$, with normal covering $\mathcal{T}$ as in Proposition 1.40, and $p \in (4+n, \infty)$. Given $T_0 > 0$ there exists a constant $\varepsilon = \varepsilon(T_0, Q) > 0$ such that for all $T \in (0, T_0]$, all $\varrho \in \mathbb{E}_T$ with $|||\varrho|||_{\mathbb{E}_T} < \varepsilon$ and all $t \in [0, T]$ the function $f^\varrho(t) := \varphi + \varrho(t)\nu_\varphi$ is an immersion such that for every $p \in M$ the metric satisfies*

$$\det\left((g_{ij,f}(p))_{ij}\right) \in \left[\frac{1}{2}Q^{-n}, 2Q^n\right]\,.$$

Furthermore, there exists a constant $\sigma > 0$ such that for all $\varrho \in \mathbb{E}_T$ with $|||\varrho|||_{\mathbb{E}_T} < \varepsilon$, and all $\alpha \in \{1, \ldots, N\}$, $x \in \overline{\phi_\alpha(U_\alpha)}$, $t \in [0, T]$,

$$\left\|\psi\left(\nabla\left(f^\varrho(t) \circ \phi_\alpha^{-1}\right)(x)\right)\right\| \ge \sigma$$

where ψ is the function defined in Definition 1.3. For $T \in (0, T_0]$ we define

$$\boldsymbol{U}_T := \left\{\varrho \in \mathbb{E}_T \,:\, |||\varrho|||_{\mathbb{E}_T} < \varepsilon\right\}\,.$$

Proof. By Proposition 1.10 there exists $\delta > 0$ such that every function $f \in C^1\big(M;\mathbb{R}^{n+1}\big)$ that satisfies for all $\alpha \in \{1,\dots,N\}$,

$$\left\|\nabla\left(f\circ\phi_\alpha^{-1}\right)-\nabla\left(\varphi\circ\phi_\alpha^{-1}\right)\right\|_{C\left(\overline{\phi_\alpha(U_\alpha)};\mathbb{R}^{n+1}\right)}<\delta$$

is an immersion with $\det\big((g_{ij,f}(p))_{ij}\big) \in \big[\frac{1}{2}Q^{-n}, 2Q^n\big]$. Moreover, we may choose δ such that $\left\|\psi\left(\nabla\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)(x)\right)\right\| \geq \sigma$ for a constant $\sigma > 0$. Given any $T \in (0,T_0]$ and $\alpha \in \{1,\dots,N\}$, Corollary B.38 yields the continuous embedding

$$\mathbb{E}_{T,\alpha} \hookrightarrow C\big([0,T];C^3\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$$

and there exists a constant $C_\alpha(T_0) > 0$ such that for all $\eta_\alpha \in \mathbb{E}_{T,\alpha}$,

$$\|\eta_\alpha\|_{C\left([0,T];C^3\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \leq C_\alpha(T_0)\, |||\eta_\alpha|||_{\mathbb{E}_{T,\alpha}}.$$

Proposition C.11 and C.16 imply that there exists a constant $C(Q) > 0$ such that for all $\varrho \in \mathbb{E}_T$, the functions $\varrho_\alpha(t,x) := \varrho\left(t,\phi_\alpha^{-1}(x)\right)$ satisfy

$$|||\varrho_\alpha|||_{\mathbb{E}_{T,\alpha}} \leq C(Q)\, |||\varrho|||_{\mathbb{E}_T}.$$

Let

$$C(T_0) := \max\left\{C_\alpha(T_0)\left\|\nu_\varphi\circ\phi_\alpha^{-1}\right\|_{C^1\left(\overline{\phi_\alpha(U_\alpha)}\right)} : \alpha \in \{1,\dots,N\}\right\}$$

and let $\varepsilon > 0$ be so small that $2C(Q)C(T_0)\varepsilon < \delta$ where $C(Q)$ is the constant appearing in the estimate above. We notice that $\varepsilon > 0$ is independent of T. Let $T \in (0,T_0]$, $\varrho \in \mathbb{E}_T$ with $|||\varrho|||_{\mathbb{E}_T} < \varepsilon$ and $t \in [0,T]$ be arbitrary. Then for any $\alpha \in \{1,\dots,N\}$ there holds

$$\begin{aligned}
&\left\|\nabla\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)-\nabla\left(\varphi\circ\phi_\alpha^{-1}\right)\right\|_{C\left(\overline{\phi_\alpha(U_\alpha)};\mathbb{R}^{n+1}\right)}\\
&=\left\|\nabla\left(\varrho(t)\circ\phi_\alpha^{-1}\right)\left(\nu_\varphi\circ\phi_\alpha^{-1}\right)+\left(\varrho(t)\circ\phi_\alpha^{-1}\right)\nabla\left(\nu_\varphi\circ\phi_\alpha^{-1}\right)\right\|_{C\left(\overline{\phi_\alpha(U_\alpha);\mathbb{R}^{n+1}}\right)}\\
&\leq 2\,\|\varrho_\alpha\|_{C\left([0,T];C^3\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)}\left\|\nu_\varphi\circ\phi_\alpha^{-1}\right\|_{C^1\left(\overline{\phi_\alpha(U_\alpha)}\right)} \leq 2C_\alpha(T_0)\left\|\nu_\varphi\circ\phi_\alpha^{-1}\right\|_{C^1\left(\overline{\phi_\alpha(U_\alpha)}\right)} |||\varrho_\alpha|||_{\mathbb{E}_{T,\alpha}}\\
&\leq 2C(T_0)C(Q)\,|||\varrho|||_{\mathbb{E}_T} < \delta\,.
\end{aligned}$$

This proves the claim. □

We note that the size of ε is in inverse proportion to the C^1–norm of the normal ν_φ which is itself a measure of the curvature of the reference geometry.

Corollary 1.43. *Let (M,φ) be a smooth reference geometry of dimension $n \in \mathbb{N}$, $n \geq 2$, with normal covering $\mathcal{T}$ as in Proposition 1.40, and $p \in (4+n,\infty)$. Given $T_0 > 0$, $T \in (0,T_0]$ and $\alpha \in \{1,\dots,N\}$ we let*

$$\boldsymbol{U}_{T,\alpha} := \left\{\varrho_\alpha \in \mathbb{E}_{T,\alpha} : |||\varrho_\alpha|||_{\mathbb{E}_{T,\alpha}} \leq C(Q)\varepsilon\right\}$$

with $C(Q)$ and ε as in the proof of Corollary 1.42. There exists $R = R(Q,T_0) > 0$ such that for all $\alpha \in \{1,\dots,N\}$,

$$\boldsymbol{U}_{T,\alpha} \subset \left\{f \in C\big([0,T];C^3\big(\overline{\phi_\alpha(U_\alpha)}\big)\big) : \|f\|_{C\left([0,T];C^3\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \leq R\right\}.$$

Proof. Given $\alpha \in \{1,\dots,N\}$ we let $C_\alpha(T_0) > 0$ be as in the proof of Corollary 1.42 such that for all $\eta_\alpha \in \mathbb{E}_{T,\alpha}$,

$$\|\eta_\alpha\|_{C\left([0,T];C^3\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \leq C_\alpha(T_0)\, |||\eta_\alpha|||_{\mathbb{E}_{T,\alpha}},$$

and set $\boldsymbol{C}(T_0) := \max\left\{C_\alpha(T_0) : \alpha \in \{1,\dots,N\}\right\}$. Then the claim follows with $R := \boldsymbol{C}(T_0)C(Q)\varepsilon$. □

Chapter 2

Graph solutions to the Willmore flow with Navier conditions

This chapter is devoted to prove Main Theorem 1 on the existence of the Willmore flow (W) with Navier conditions. As described in Section 1.3 we show this result by passing to the graph formulation of (W) which is given in (1.6). The proof of Main Theorem 1 is then a consequence of the existence of strong solutions to (1.6).
The existence of solutions to (1.6) is deduced from the well-posedness of the associated linear problem shown in Section 2.3 and the mapping properties of the nonlinear operator derived in Section 2.1 and Section 2.2 by means of the Banach Fixed-Point Theorem. To this end, one needs to construct a mapping whose fixed points are precisely the solutions of (1.6) which is done in Section 2.4. The proof of Main Theorem 1 is then given at the end of Section 2.4.
The well-posedness of the linearised problem associated to (1.6) relies on a localisation technique which translates the boundary value problem on the manifold to "localised problems" on the chart domains. We refer the reader to Section 1.4 for notation and results required for the localisation procedure.

We consider a smooth reference geometry (M, φ) of dimension $n \in \mathbb{N}$, $n \geq 2$, with normal covering $\mathcal{T} = \{(U_\alpha, \phi_\alpha) : \alpha \in \{1, \dots, N\}\}$ and localisation system $\mathcal{C} = \{(U_\alpha, \phi_\alpha, \psi_\alpha) : \alpha \in \{1, \dots, N\}\}$ as introduced in Section 1.4 with associated constant Q. The corresponding covering of the boundary ∂M is denoted by $(V_\alpha, \sigma_\alpha)$, $\alpha \in \mathcal{J}$, with $V_\alpha = U_\alpha \cap \partial M$, $\sigma_\alpha = \pi \circ \phi_\alpha \circ \iota$, $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection onto the first $n-1$ components and $\iota : \partial M \to M$ the embedding. Furthermore, given $T_0 > 0$ and $T \in (0, T_0]$ we denote by

$$\boldsymbol{U}_T := \left\{\varrho \in \mathbb{E}_T \,:\, |||\varrho|||_{\mathbb{E}_T} < \varepsilon\right\}$$

the neighbourhood constructed in Corollary 1.42. Given a function $\varrho \in \boldsymbol{U}_T$ we mark the geometric quantities of $(M, f^\varrho(t))$, with $f^\varrho(t) : M \to \mathbb{R}^{n+1}$ defined by $f^\varrho(t) := \varphi + \varrho(t)\nu_\varphi$, with superscript ϱ.

2.1 Regularity properties of the graph formulation

In this section we study the regularity properties of the quantities in system (1.6) seen as functions in the height function $\varrho \in \boldsymbol{U}_T$. The main insight is that the nonlinearities describing the motion equation lie in the space X_T while the terms describing the boundary condition $H = 0$ are elements of $Y_{1,T}$ which are introduced in Section 1.4. The acquired estimates hold with constants independent of the considered time interval.

Lemma 2.1 (Space regularity of first order terms)**.** *Let $\alpha \in \{1, \dots, N\}$, T_0 be positive, $T \in (0, T_0]$ and $\varrho \in \boldsymbol{U}_T$. Every component of the function $(t, x) \mapsto \nabla\left(f^\varrho(t) \circ \phi_\alpha^{-1}\right)(x)$ as well as the function*

$(t,x) \mapsto g^{\varrho}_{ij}\left(t, \phi_\alpha^{-1}(x)\right)$ lie in the space $C\big([0,T]; C^2\left(\overline{\phi_\alpha(U_\alpha)}\right)\big)$ with norms controlled by

$$C(T_0)\, |||(t,x) \mapsto \varrho\left(t, \phi_\alpha^{-1}(x)\right)|||_{\mathbb{E}_{T,\alpha}}.$$

Proof. We observe that for $x \in \phi_\alpha(U_\alpha)$,

$$\begin{aligned}&\frac{\partial\left(f^{\varrho}(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i}(x)\\&= \frac{\partial\left(\varphi\circ\phi_\alpha^{-1}\right)}{\partial x_i}(x) + \frac{\partial\left(\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i}(x)\nu_\varphi\left(\phi_\alpha^{-1}(x)\right) + \varrho\left(t,\phi_\alpha^{-1}(x)\right)\frac{\partial\left(\nu_\varphi\circ\phi_\alpha^{-1}\right)}{\partial x_i}(x)\,.\end{aligned}$$

Proposition B.35 implies

$$(t,x) \mapsto \left(\varrho(t)\circ\phi_\alpha^{-1}\right)(x) \in C\big([0,T]; C^3\left(\overline{\phi_\alpha(U_\alpha)}\right)\big)$$

and thus smoothness of the reference geometry yields

$$(t,x) \mapsto \frac{\partial\left(f^{\varrho}(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i}(x) \in C\big([0,T]; C^2\left(\overline{\phi_\alpha(U_\alpha)};\mathbb{R}^{n+1}\right)\big).$$

As $C\big([0,T]; C^2\left(\overline{\phi_\alpha\left(U_\alpha\right)}\right)\big)$ is a Banach algebra, we obtain

$$(t,x) \mapsto g^{\varrho}_{ij}\left(t,\phi_\alpha^{-1}(x)\right) = \left\langle \frac{\partial\left(f^{\varrho}(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i}(x), \frac{\partial\left(f^{\varrho}(t)\circ\phi_\alpha^{-1}\right)}{\partial x_j}(x)\right\rangle \in C\big([0,T]; C^2\left(\overline{\phi_\alpha(U_\alpha)}\right)\big).$$

□

Lemma 2.2 (Time regularity of first order terms)**.** *Let T_0 be positive and $T \in (0,T_0]$. There exists a constant $\tau > \frac{1}{2} - \frac{1}{4p}$ such that for every $\varrho \in \mathbb{E}_T$, $\alpha \in \{1,\dots,N\}$ and $i \in \{1,\dots,n\}$ there holds*

$$(t,x) \mapsto \varrho\left(t,\phi_\alpha^{-1}(x)\right),\ (t,x) \mapsto \frac{\partial\left(\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i}(x) \in C^\tau\left([0,T]; C\big(\overline{\phi_\alpha(U_\alpha)}\big)\right).$$

The norms of the above functions in the space $C^\tau\left([0,T]; C\big(\overline{\phi_\alpha(U_\alpha)}\big)\right)$ are controlled by

$$C(T_0)\, |||(t,x) \mapsto \varrho\left(t, \phi_\alpha^{-1}(x)\right)|||_{\mathbb{E}_{T,\alpha}}.$$

Proof. Let $T \in (0,T_0)$, $\varrho \in \mathbb{E}_T$, $\alpha \in \{1,\dots,N\}$ and $i \in \{1,\dots,n\}$ be given. Then $(t,x) \mapsto \varrho_\alpha(t,x) := \varrho\left(t,\phi_\alpha^{-1}(x)\right)$ lies in $\mathbb{E}_{T,\alpha} := W^{1,4}_p\left((0,T)\times\phi_\alpha\left(U_\alpha\right)\right)$ and by Corollary B.39 there holds for all $\theta \in \left(\frac{1+n/p}{4-4/p}, 1\right)$ and all $\delta \in (0, 1-1/p)$,

$$\mathbb{E}_{T,\alpha} \hookrightarrow C^{(1-\theta)(1-1/p-\delta)}\big([0,T]; C^1\left(\overline{\phi_\alpha\left(U_\alpha\right)}\right)\big)$$

with embedding constant independent of T. As $\theta \searrow \frac{1+n/p}{4-4/p}$ and $\delta \searrow 0$ there holds

$$(1-\theta)\left(1-1/p-\delta\right) \nearrow \frac{3}{4} - \frac{4+n}{4p} > \frac{1}{2}$$

due to $p \in (4+n,\infty)$. This yields the claim. □

Lemma 2.3 (Space regularity of the inverse metric)**.** *Let T_0 be positive, $T \in (0,T_0]$, $\alpha \in \{1,\dots,N\}$ and $i, j \in \{1,\dots,n\}$. The function*

$$g^{ij} : \boldsymbol{U}_T \to C\big([0,T]; C^2\left(\overline{\phi_\alpha\left(U_\alpha\right)}\right)\big), \quad \varrho \mapsto \left((t,x) \mapsto g^{ij,\varrho}\left(t,\phi_\alpha^{-1}(x)\right)\right)$$

is well-defined and Lipschitz continuous with constant $C(Q,T_0,\varepsilon)$.

Proof. Given $t \in [0,T]$ and $x \in \overline{\phi_\alpha(U_\alpha)}$, there holds

$$g^{ij}(\varrho)(t,x) = g^{ij,\varrho}\left(t,\phi_\alpha^{-1}(x)\right) = T\left(\left(g^\varrho_{kl}\left(t,\phi_\alpha^{-1}(x)\right)\right)_{kl}\right)_{ij}$$

with $T : Gl(n;\mathbb{R}) \to Gl(n;\mathbb{R}) \subset \mathbb{R}^{n\times n}$, $A \mapsto T(A) := A^{-1}$. We observe that T is infinitely often Fréchet differentiable with

$$T^{(k)}(A)\left(B_1,\dots,B_k\right) = (-1)^k \sum_{\sigma\in S_k} A^{-1}\prod_{j=1}^k B_{\sigma(j)}A^{-1}\,,$$

for $B_1,\dots,B_k \in \mathbb{R}^{n\times n}$ and $A \in Gl(n;\mathbb{R})$ where S_k denotes the set of permutations of the set $\{1,\dots,k\}$. Given $k,l \in \{1,\dots,n\}$ we observe that

$$\begin{aligned} g^\varrho_{kl}\left(t,\phi_\alpha^{-1}(x)\right) =& g_{kl,\varphi}\left(t,\phi_\alpha^{-1}(x)\right) + \frac{\partial\left(\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_k}(x)\frac{\partial\left(\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_l}(x) \\ &- 2a_{kl,\varphi}\left(t,\phi_\alpha^{-1}(x)\right)\varrho\left(t,\phi_\alpha^{-1}(x)\right). \end{aligned}$$

For $\varrho,\eta \in \boldsymbol{U}_T$ and $\tau \in [0,1]$ there holds $\tau\varrho + (1-\tau)\eta \in \boldsymbol{U}_T$ and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} g^{\tau\varrho+(1-\tau)\eta}_{kl}\left(t,\phi_\alpha^{-1}(x)\right) =& -2a_{kl,\varphi}\left(t,\phi_\alpha^{-1}(x)\right)\left(\varrho\left(t,\phi_\alpha^{-1}(x)\right) - \eta\left(t,\phi_\alpha^{-1}(x)\right)\right) \\ &+ \left(\frac{\partial\left(\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_k}(x) - \frac{\partial\left(\eta(t)\circ\phi_\alpha^{-1}\right)}{\partial x_k}(x)\right)p_l(\tau,\varrho,\eta)(t,x) \\ &+ \left(\frac{\partial\left(\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_l}(x) - \frac{\partial\left(\eta(t)\circ\phi_\alpha^{-1}\right)}{\partial x_l}(x)\right)p_k(\tau,\varrho,\eta)(t,x)\,, \end{aligned}$$

where for $k \in \{1,\dots,n\}$,

$$p_k(\tau,\varrho,\eta)(t,x) = \tau\frac{\partial\left(\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_k}(x) + (1-\tau)\frac{\partial\left(\eta(t)\circ\phi_\alpha^{-1}\right)}{\partial x_k}(x)\,.$$

Corollary 1.43 implies for all $t \in [0,T]$, $x \in \overline{\phi_\alpha(U_\alpha)}$ and ϱ, $\eta \in \boldsymbol{U}_T$,

$$\sup_{\tau\in[0,1]}\left|p_k\left(\tau,\varrho,\eta\right)(t,x)\right| \le \tau\left\|\varrho_\alpha\right\|_{C\left([0,T];C^3\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} + (1-\tau)\left\|\eta_\alpha\right\|_{C\left([0,T];C^3\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \le R$$

which allows us to conclude

$$\left|\frac{\mathrm{d}}{\mathrm{d}\tau}g^{\tau\varrho+(1-\tau)\eta}_{kl}\left(t,\phi_\alpha^{-1}(x)\right)\right| \le C\left(\varphi,R\right)\left\|\varrho_\alpha-\eta_\alpha\right\|_{C\left([0,T];C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \le C\left(R,T_0,Q\right)\left|\!\left|\!\left|\varrho-\eta\right|\!\right|\!\right|_{\mathbb{E}_T}.$$

With $A(\varrho)(t,x) := \left(g^\varrho_{kl}\left(t,\phi_\alpha^{-1}(x)\right)\right)_{kl}$ we observe further that

$$\begin{aligned} \left|g^{ij}(\varrho)(t,x) - g^{ij}(\eta)(t,x)\right| &= \left|T\left(A(\varrho)(t,x)\right)_{ij} - T\left(A(\eta)(t,x)\right)_{ij}\right| \\ &= \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}T\left(A(\tau\varrho+(1-\tau\eta))(t,x)\right)_{ij}\mathrm{d}\tau\right| \\ &\le \int_0^1\left\|\frac{\mathrm{d}}{\mathrm{d}\tau}T\left(A(\tau\varrho+(1-\tau\eta))(t,x)\right)\right\|_2\mathrm{d}\tau\,. \end{aligned}$$

The formula for derivatives of T yields that $\frac{\mathrm{d}}{\mathrm{d}\tau}T\left(A(\tau\varrho+(1-\tau\eta))(t,x)\right)$ equals

$$-\left(A(\tau\varrho+(1-\tau\eta))(t,x)\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(A(\tau\varrho+(1-\tau\eta))(t,x)\right)\left(A(\tau\varrho+(1-\tau\eta))(t,x)\right)^{-1}.$$

Given any $v \in \boldsymbol{U}_T$, Corollary 1.42 and Corollary 1.43 yield

$$\left\|A(v)(t,x)^{-1}\right\|_2 = \det\left((g^v_{kl}(t,x))_{kl}\right)^{-1} \left\|\mathrm{cof}\left((g^v_{kl}(t,x))_{kl}\right)\right\| \leq 2Q^n R^n = C(Q,R)\,, \tag{2.1}$$

where cof denotes the cofactor-matrix. This allows us to conclude

$$\left\|g^{ij}(\varrho) - g^{ij}(\eta)\right\|_{C\left([0,T];C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \leq C(Q,R,T_0)\, |||\varrho-\eta|||_{\mathbb{E}_T}\,.$$

To estimate $g^{ij}(\varrho) - g^{ij}(\eta)$ in $C\big([0,T];C^1\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$ we observe that for $\varrho \in \boldsymbol{U}_T$,

$$\begin{aligned}\frac{\partial}{\partial x_m} T\left(A(\varrho)(t,x)\right) &= \left(T^{(1)}(A(\varrho)(t,x))\right)\left(\frac{\partial A(\varrho)}{\partial x_m}(t,x)\right)\\ &= A(\varrho)(t,x)^{-1}\,\frac{\partial A(\varrho)}{\partial x_m}(t,x)\, A(\varrho)(t,x)^{-1}\,,\end{aligned}$$

where for $k,l \in \{1,\dots,n\}$ there exist smooth functions $c_i : \overline{\phi_\alpha(U_\alpha)} \to \mathbb{R}$, $i \in \{1,2,3\}$, such that

$$\begin{aligned}\frac{\partial}{\partial x_m} A(\varrho)(t,x)_{kl} =& c_1(x) + c_2(x)\varrho_\alpha(t,x) + c_3(x)\frac{\partial \varrho_\alpha(t)}{\partial x_m}(x) + \frac{\partial^2 \varrho_\alpha(t)}{\partial x_k \partial x_m}(x)\frac{\partial \varrho_\alpha(t)}{\partial x_l}(x)\\ &+ \frac{\partial^2 \varrho_\alpha(t)}{\partial x_l \partial x_m}(x)\frac{\partial \varrho_\alpha(t)}{\partial x_k}(x)\,.\end{aligned}$$

In particular, we observe for $\varrho,\eta \in \boldsymbol{U}_T$, $t \in [0,T]$ and $x \in \overline{\phi_\alpha(U_\alpha)}$ using Corollary 1.43, Corollary B.38 and Proposition C.16,

$$\left\|\frac{\partial A(\varrho)}{\partial x_m}(t,x) - \frac{\partial A(\eta)}{\partial x_m}(t,x)\right\|_2 \leq C(R)\left\|\varrho_\alpha - \eta_\alpha\right\|_{C\left([0,T];C^3\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \leq C(R,Q,T_0)\, |||\varrho-\eta|||_{\mathbb{E}_T}\,.$$

Using (2.1) we conclude for all $t \in [0,T]$, $x \in \overline{\phi_\alpha(U_\alpha)}$, $\varrho,\eta \in \boldsymbol{U}_T$,

$$\left\|T^{(1)}\left(A(\varrho)(t,x)\right)\left(\frac{\partial A(\varrho)}{\partial x_m}(t,x) - \frac{\partial A(\eta)}{\partial x_m}(t,x)\right)\right\|_2 \leq C(Q,R,T_0)\, |||\varrho-\eta|||_{\mathbb{E}_T}\,.$$

Moreover, we observe that $\left(T^{(1)}\left(A(\varrho)(t,x)\right) - T^{(1)}\left(A(\eta)(t,x)\right)\right)\left(\frac{\partial A(\varrho)}{\partial x_m}(t,x)\right)$ equals

$$\begin{aligned}&\left(T\left(A(\varrho)(t,x)\right) - T\left(A(\eta)(t,x)\right)\right)\left(\frac{\partial A(\eta)}{\partial x_m}(t,x)\right)A(\varrho)(t,x)^{-1}\\ &+ A(\eta)(t,x)^{-1}\left(\frac{\partial A(\eta)}{\partial x_m}(t,x)\right)\left(T\left(A(\varrho)(t,x)\right) - T\left(A(\eta)(t,x)\right)\right).\end{aligned}$$

Lipschitz continuity of $g^{ij} : \boldsymbol{U}_T \to C\big([0,T];C\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$ implies

$$\left\|T\left(A(\varrho)(t,x)\right) - T\left(A(\eta)(t,x)\right)\right\|_2 \leq C(Q,R,T_0)\, |||\varrho-\eta|||_{\mathbb{E}_T}$$

which allows us to conclude that for all $t \in [0,T]$, $x \in \overline{\phi_\alpha(U_\alpha)}$, $\varrho,\eta \in \boldsymbol{U}_T$, $m \in \{1,\dots,n\}$,

$$\begin{aligned}\left|\frac{\partial}{\partial x_m} g^{ij}(\varrho)(t,x) - \frac{\partial}{\partial x_m} g^{ij}(\eta)(t,x)\right| \leq& \left\|T^{(1)}\left(A(\varrho)(t,x)\right)\left(\frac{\partial A(\varrho)}{\partial x_m}(t,x) - \frac{\partial A(\eta)}{\partial x_m}(t,x)\right)\right\|_2\\ &+ \left\|\left(T^{(1)}(A(\varrho)(t,x)) - T^{(1)}(A(\eta)(t,x))\right)\left(\frac{\partial A(\varrho)}{\partial x_m}(t,x)\right)\right\|_2\\ \leq& C(Q,R,T_0)\, |||\varrho-\eta|||_{\mathbb{E}_T}\,.\end{aligned}$$

Given $r,m \in \{1,\dots,n\}$, the product and chain rule for Fréchet derivatives imply

$$\frac{\partial}{\partial x_r}\left(T^{(1)}\left(A(\varrho)(t,x)\right)\left(\frac{\partial A(\varrho)}{\partial x_m}(t,x)\right)\right)$$

$$= \left(T^{(2)}(A(\varrho)(t,x))\right)\left(\frac{\partial A(\varrho)}{\partial x_r}(t,x), \frac{\partial A(\varrho)}{\partial x_m}(t,x)\right) + T^{(1)}\left(A(\varrho)(t,x)\right)\left(\frac{\partial^2 A(\varrho)}{\partial x_r \partial x_m}(t,x)\right).$$

Similar arguments as before yield for $t \in [0,T]$, $x \in \overline{\phi_\alpha(U_\alpha)}$, ϱ, $\eta \in \boldsymbol{U}_T$, $r, l \in \{1, \ldots, n\}$,

$$\left|\frac{\partial^2}{\partial x_r \partial x_m} g^{ij}(\varrho)(t,x) - \frac{\partial^2}{\partial x_r \partial x_m} g^{ij}(\eta)(t,x)\right| \leq \left\| \frac{\partial}{\partial x_r}\left(T^{(1)}\left(A(\varrho)(t,x)\right)\left(\frac{\partial A(\varrho)}{\partial x_m}(t,x)\right)\right)\right\|_2$$
$$\leq C(Q,R,T_0) \, |||\varrho - \eta|||_{\mathbb{E}_T} .$$

□

Lemma 2.4 (Time regularity of the inverse metric)**.** *Let T_0 be positive, $T \in (0,T_0]$, $\alpha \in \{1,\ldots,N\}$ and $i, j \in \{1,\ldots,n\}$. There exists a constant $\tau > \frac{1}{2} - \frac{1}{4p}$ such that*

$$g^{ij} : \boldsymbol{U}_T \to C^\tau\left([0,T]; C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right), \varrho \mapsto \left((t,x) \mapsto g^{ij,\varrho}\left(t, \phi_\alpha^{-1}(x)\right)\right)$$

is well-defined and Lipschitz continuous with constant $C(Q,T_0,\varepsilon)$.

Proof. Lemma 2.2 implies that there exists $\tau > \frac{1}{2} - \frac{1}{4p}$ such that for all $\varrho \in \boldsymbol{U}_T$, the functions $(t,x) \mapsto \varrho\left(t, \phi_\alpha^{-1}(x)\right)$, $(t,x) \mapsto \nabla\left(\varrho(t) \circ \phi_\alpha^{-1}\right)(x)$ lie in $C^\tau\left([0,T]; C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)$. Using the formula for the inverse A^{-1} of a matrix A in terms of the cofactor-matrix of A, one obtains for $t \in [0,T]$ and $x \in \overline{\phi_\alpha(U_\alpha)}$,

$$g^{ij}(\varrho)(t,x) = \det\left(\left(g_{kl}^\varrho\left(t, \phi_\alpha^{-1}(x)\right)\right)_{kl}\right)^{-1} p\left(\varrho\left(t, \phi_\alpha^{-1}(x)\right), \nabla\left(\varrho(t) \circ \phi_\alpha^{-1}\right)(x), x\right)$$

with $p \in C^2\left(\mathbb{R}^n; C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)$ a polynomial in the components $\varrho\left(t, \phi_\alpha^{-1}(x)\right)$, $\partial_i\left(\varrho(t) \circ \phi_\alpha^{-1}\right)(x)$, $i \in \{1,\ldots,n\}$, with coefficients that are time-independent and smooth on $\overline{\phi_\alpha(U_\alpha)}$. Applying Proposition B.3 to p and $K := [-R,R]^{n+1}$, we obtain

$$(t,x) \mapsto p\left(\varrho\left(t, \phi_\alpha^{-1}(x)\right), \nabla\left(\varrho(t) \circ \phi_\alpha^{-1}\right)(x), x\right) \in C^\tau\left([0,T]; C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)$$

with norm bounded by $C(Q,T_0) \, |||\varrho|||_{\mathbb{E}_T}$. Similarly, one sees that there exists a function $q \in C^2\left(\mathbb{R}^n; C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)$, polynomial in $\varrho\left(t, \phi_\alpha^{-1}(x)\right), \nabla\left(\varrho(t) \circ \phi_\alpha^{-1}\right)(x)$ with smooth coefficients independent of time such that

$$\det\left(\left(g_{kl}^\varrho\left(t, \phi_\alpha^{-1}(x)\right)\right)_{kl}\right) = q\left(\varrho\left(t, \phi_\alpha^{-1}(x)\right), \nabla\left(\varrho(t) \circ \phi_\alpha^{-1}\right)(x), x\right)$$

which allows us to conclude that

$$(t,x) \mapsto \det\left(\left(g_{kl}^\varrho\left(t, \phi_\alpha^{-1}(x)\right)\right)_{kl}\right) \in C^\tau\left([0,T]; C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)$$

with norm bounded by $C(Q,T_0) \, |||\varrho|||_{\mathbb{E}_T}$. Corollary 1.42 implies that for all $\varrho \in \boldsymbol{U}_T$, all $t \in [0,T]$ and all $x \in \overline{\phi_\alpha(U_\alpha)}$, there holds

$$\det\left(\left(g_{kl}^\varrho\left(t, \phi_\alpha^{-1}(x)\right)\right)_{kl}\right) \geq \frac{1}{2} Q^{-n} .$$

Proposition B.4 then yields

$$(t,x) \mapsto \det\left(\left(g_{kl}^\varrho\left(t, \phi_\alpha^{-1}(x)\right)\right)_{kl}\right)^{-1} \in C^\tau\left([0,T]; C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)$$

with norm bounded by $C(Q,T_0) \, |||\varrho|||_{\mathbb{E}_T}$ which allows us to conclude that g^{ij} is well-defined. Given $\varrho, \eta \in \boldsymbol{U}_T$, the properties shown in Proposition B.3 and B.4 imply that

$$\left\|g^{ij}(\varrho) - g^{ij}(\eta)\right\|_{C^\tau\left([0,T]; C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)}$$

$$\begin{aligned}
&\leq C(Q,T_0,\varepsilon)\left\|(t,x)\mapsto\left(\varrho\left(t,\phi_\alpha^{-1}(x)\right)-\eta\left(t,\phi_\alpha^{-1}(x)\right)\right)\right\|_{C^\tau\left([0,T];C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)}\\
&\quad+C\left(T_0,Q,\varepsilon\right)\left\|(t,x)\mapsto\nabla\left((\varrho(t)-\eta(t))\circ\phi_\alpha^{-1}\right)(x)\right\|_{C^\tau\left([0,T];C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)}\\
&\leq C(Q,T_0,\varepsilon)\left|\!\left|\!\left|\varrho-\eta\right|\!\right|\!\right|_{\mathbb{E}_T}.
\end{aligned}$$

□

Lemma 2.5 (Space regularity of the normal)**.** *Let T_0 be positive, $T\in(0,T_0]$ and $\alpha\in\{1,\dots,N\}$. The function*

$$\nu:\boldsymbol{U}_T\to C\left([0,T];C^2\left(\overline{\phi_\alpha(U_\alpha)};\mathbb{R}^{n+1}\right)\right),\quad \varrho\mapsto\left((t,x)\mapsto\nu^\varrho\left(t,\phi_\alpha^{-1}(x)\right)\right)$$

is well-defined and Lipschitz continuous with constant $C(Q,T_0,\varepsilon,\sigma)$.

Proof. As in Definition 1.3 we consider the n-linear mapping $\psi:\left(\mathbb{R}^{n+1}\right)^n\to\mathbb{R}^{n+1}$ given by

$$\psi\left(v^1,\dots,v^n\right)_i:=\det\left(v^1,\dots,v^n,e_i\right)$$

for $i\in\{1,\dots,n+1\}$ and $v^j\in\mathbb{R}^{n+1}$, $j\in\{1,\dots,n\}$. Then ψ is differentiable with

$$(D\psi)(v)\left(y^1,\dots,y^n\right)=\sum_{j=1}^n\psi\left(v^1,\dots,v^{j-1},y^j,v^{j+1},\dots,v^n\right),$$

or, equivalently, for $j\in\{1,\dots,n\}$, $i\in\{1,\dots,n+1\}$,

$$\partial_{v_i^j}\psi(v)=\psi\left(v^1,\dots,v^{j-1},e_i,v^{j+1},\dots,v_n\right). \tag{2.2}$$

On $U:=\left\{\left(v^1,\dots,v^n\right)\in\left(\mathbb{R}^{n+1}\right)^n:\psi\left(v^1,\dots,v^n\right)\neq 0\right\}$ we define

$$F:U\to\mathbb{R}^{n+1},\quad F\left(v^1,\dots,v^n\right):=\frac{\psi\left(v^1,\dots,v^n\right)}{\left\|\psi\left(v^1,\dots,v^n\right)\right\|}$$

and observe that $\nu(\varrho)(t,x)=F\left(\nabla\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)(x)\right)$. Then F is differentiable on U with, using the abbreviation $v=\left(v^1,\dots,v^n\right)$,

$$\partial_{v_i^j}F(v)=\frac{1}{\|\psi(v)\|}\left(\partial_{v^j}\psi(v)-\left\langle\partial_{v_i^j}\psi(v),\psi(v)\right\rangle\frac{\psi(v)}{\|\psi(v)\|^2}\right).$$

Let ϱ, $\eta\in\boldsymbol{U}_T$, $\tau\in[0,1]$, $x\in\overline{\phi_\alpha(U_\alpha)}$ and $t\in[0,T]$ be given. Then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}F\left(\nabla\left(f^{\tau\varrho+(1-\tau)\eta}(t)\circ\phi_\alpha^{-1}\right)(x)\right)\\
&=(DF)\left(\nabla\left(f^{\tau\varrho+(1-\tau)\eta}(t)\circ\phi_\alpha^{-1}\right)(x)\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\nabla\left(f^{\tau\varrho+(1-\tau)\eta}(t)\circ\phi_\alpha^{-1}\right)(x)\right).
\end{aligned}$$

Given $v\in\boldsymbol{U}_T$, $x\in\overline{\phi_\alpha(U_\alpha)}$ and $t\in[0,T]$, Corollary 1.42 implies that there exists a constant $\sigma>0$ such that

$$\left\|\psi\left(\nabla\left(f^v(t)\circ\phi_\alpha^{-1}\right)(x)\right)\right\|^{-1}\leq\sigma^{-1}.$$

Using Corollary 1.43 we obtain that every component of $DF\left(\nabla\left(f^v(t)\circ\phi_\alpha^{-1}\right)(x)\right)$ is bounded by a constant $C(R,\sigma)=C(Q,T_0,\varepsilon,\sigma)$. As further

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\nabla\left(f^{\tau\varrho+(1-\tau)\eta}(t)\circ\phi_\alpha^{-1}\right)(x)\right)=&\nu_\varphi\left(\phi_\alpha^{-1}(x)\right)\left(\nabla\left(\varrho(t)\circ\phi_\alpha^{-1}\right)(x)-\nabla\left(\eta(t)\circ\phi_\alpha^{-1}\right)(x)\right)\\
&+\left(\varrho\left(t,\phi_\alpha^{-1}(x)\right)-\eta\left(t,\phi_\alpha^{-1}(x)\right)\right)\nabla\left(\nu_\varphi\circ\phi_\alpha^{-1}\right)(x)
\end{aligned} \tag{2.3}$$

we conclude for all $t \in [0,T]$, $x \in \overline{\phi_\alpha(U_\alpha)}$ and $\varrho, \eta \in \boldsymbol{U}_T$,

$$\begin{aligned}\|\nu(\varrho)(t,x) - \nu(\eta)(t,x)\| &= \left\| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau} F\big(\nabla\big(f^{\tau\varrho+(1-\tau)\eta}(t)\circ\phi_\alpha^{-1}\big)(x)\big)\mathrm{d}\tau \right\| \\ &\leq C(Q,T_0,\varepsilon,\sigma)\, \|\varrho_\alpha - \eta_\alpha\|_{C\left([0,T];C^3\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \\ &\leq C(Q,T_0,\varepsilon,\sigma)\, \|\!|\varrho - \eta|\!\|_{\mathbb{E}_T}\,.\end{aligned}$$

Furthermore, we observe that

$$\left| \frac{\partial}{\partial x_l}\nu(\varrho)(t,x) - \frac{\partial}{\partial x_l}\nu(\eta)(t,x) \right| \tag{2.4}$$

$$\leq \left| (DF)\left(\nabla\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)(x)\right)\left(\frac{\partial}{\partial x_l}\nabla\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)(x) - \frac{\partial}{\partial x_l}\nabla\left(f^\eta(t)\circ\phi_\alpha^{-1}\right)(x)\right)\right| \tag{2.5}$$

$$+ \left| \left((DF)\left(\nabla\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)(x)\right) - (DF)\left(\nabla\left(f^\eta(t)\circ\phi_\alpha^{-1}\right)(x)\right)\right)\frac{\partial}{\partial x_l}\nabla\left(f^\eta(t)\circ\phi_\alpha^{-1}\right)(x)\right|\,. \tag{2.6}$$

As the absolute value of every component of $(DF)\left(\nabla\left(f^v(t)\circ\phi_\alpha^{-1}\right)(x)\right)$ is bounded by a constant $C(Q,T_0,\varepsilon,\sigma)$, we obtain for every $t \in [0,T]$, $x \in \overline{\phi_\alpha\left(U_\alpha\right)}$ that (2.5) can be estimated by

$$C(Q,T_0,\varepsilon,\sigma)\, \|\varrho_\alpha - \eta_\alpha\|_{C\left([0,T];C^2\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \leq C(Q,T_0,\varepsilon,\sigma)\, \|\!|\varrho-\eta|\!\|_{\mathbb{E}_T}\,.$$

The difference $(DF)\left(\nabla\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)(x)\right) - (DF)\left(\nabla\left(f^\eta(t)\circ\phi_\alpha^{-1}\right)(x)\right)$ in the term (2.6) can be written as

$$\begin{aligned}&\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}(DF)\left(\nabla\big(f^{\tau\varrho+(1-\tau)\eta}(t)\circ\phi_\alpha^{-1}\big)(x)\right)\mathrm{d}\tau \\ &= \int_0^1 \left(D^2F\right)\left(\nabla\big(f^{\tau\varrho+(1-\tau)\eta}(t)\circ\phi_\alpha^{-1}\big)(x)\right)\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\nabla\big(f^{\tau\varrho+(1-\tau)\eta}(t)\circ\phi_\alpha^{-1}\big)(x)\right)\right)\mathrm{d}\tau\,.\end{aligned}$$

Second order derivatives of F are of the form

$$\partial_{v_l^k}\partial_{v_i^j}F(v) = \mathfrak{p}\left(\psi(v), \partial_{v_i^j}\psi(v), \partial_{v_l^k}\psi(v), \partial_{v_l^k}\partial_{v_i^j}\psi(v), \|\psi(v)\|^{-1}\right),$$

where $j,k \in \{1,\dots,n\}$, $i,l \in \{1,\dots,n+1\}$ and $\mathfrak{p}$ is a polynomial. It is easily deduced from (2.2) that $\partial_{v_l^k}\partial_{v_i^j}\psi(v)$ is a bilinear expression in e_l and e_i with coefficients that are polynomial in the components of $v^1,\dots,v^n$. Thus, for every $\tau \in [0,1]$, $\varrho,\eta \in \boldsymbol{U}_T$, $t \in [0,T]$ and $x \in \overline{\phi_\alpha\left(U_\alpha\right)}$, the absolute value of every component of $\left(D^2F\right)\left(\nabla\left(f^{\tau\varrho+(1-\tau)\eta}(t)\circ\phi_\alpha^{-1}\right)(x)\right)$ can be bounded by a constant $C(Q,T_0,\varepsilon,\sigma)$. As further for every $\eta \in \boldsymbol{U}_T$, $t \in [0,T]$, $x \in \overline{\phi_\alpha\left(U_\alpha\right)}$ there holds

$$\left\| \frac{\partial}{\partial x_l}\nabla\left(f^\eta(t)\circ\phi_\alpha^{-1}\right)(x)\right\| \leq C(Q,T_0,\varepsilon)\,,$$

the identity (2.3) implies that the term (2.6) is estimated by

$$C(Q,T_0,\varepsilon,\sigma)\, \|\varrho_\alpha - \eta_\alpha\|_{C\left([0,T];C^1\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \leq C(Q,T_0,\varepsilon,\sigma)\, \|\!|\varrho-\eta|\!\|_{\mathbb{E}_T}\,.$$

Finally, the second order spacial derivatives of $\nu(\varrho)$ are given by

$$\begin{aligned}&\frac{\partial^2}{\partial x_k \partial x_l}\nu(\varrho)(t,x) \\ &= \left(D^2F\right)\left(\nabla\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)(x)\right)\left(\frac{\partial}{\partial x_k}\nabla\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)(x), \frac{\partial}{\partial x_l}\nabla\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)(x)\right)\end{aligned}$$

$$+ (DF)\left(\nabla\left(f^{\varrho}(t)\circ\phi_\alpha^{-1}\right)(x)\right)\frac{\partial^2}{\partial x_l \partial x_k}\nabla\left(f^{\varrho}(t)\circ\phi_\alpha^{-1}\right)(x)\,.$$

We observe that

$$\begin{aligned}&\left(D^2F\right)\left(\nabla\left(f^{\varrho}(t)\circ\phi_\alpha^{-1}\right)(x)\right)-\left(D^2F\right)\left(\nabla\left(f^{\eta}(t)\circ\phi_\alpha^{-1}\right)(x)\right)\\ &=\int_0^1\left(D^3F\right)\left(\nabla\left(f^{\tau\varrho+(1-\tau)\eta}(t)\right)(x)\right)\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\nabla\left(f^{\tau\varrho+(1-\tau)\eta}(t)\right)(x)\right)\mathrm{d}\tau\,,\end{aligned}$$

where third order derivatives $D^3F(v)$, $v\in U$, are polynomial expressions in derivatives of $\psi(v)$ up to order 3 and $\|\psi(v)\|^{-1}$ which are themselves polynomial in $v^1,\dots,v^n$. In particular, every component of $\left(D^3F\right)\left(\nabla\left(f^{\varrho}(t)\circ\phi_\alpha^{-1}\right)(x)\right)$ is bounded by a constant $C(Q,T_0,\varepsilon,\sigma)$. Moreover, there holds for $\varrho,\eta\in\boldsymbol{U}_T$, $t\in[0,T]$, $x\in\overline{\phi_\alpha(U_\alpha)}$,

$$\begin{aligned}\left|\frac{\partial^2}{\partial x_l\partial x_k}\nabla\left(f^{\varrho}(t)\circ\phi_\alpha^{-1}\right)(x)-\frac{\partial^2}{\partial x_l\partial x_k}\nabla\left(f^{\varrho}(t)\circ\phi_\alpha^{-1}\right)(x)\right| &\le C\left\|\varrho_\alpha-\eta_\alpha\right\|_{C\left([0,T];C^3\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)}\\ &\le C(Q,T_0)\,|||\varrho-\eta|||_{\mathbb{E}_T}\,.\end{aligned}$$

Similar arguments as before then show that second order derivatives of $\nu(\varrho)-\nu(\eta)$ are bounded in absolute value by $C(Q,T_0,\varepsilon,\sigma)\,|||\varrho-\eta|||_{\mathbb{E}_T}$. This concludes the proof. □

Corollary 2.6 (Space regularity of second order terms). *Let T_0 be positive, $T\in(0,T_0]$, $\alpha\in\{1,\dots,N\}$ and $\varrho\in\boldsymbol{U}_T$. The Christoffel symbols and the components of the second fundamental form $(t,x)\mapsto a_{ij}^{\varrho}\left(t,\phi_\alpha^{-1}(x)\right)$ are elements of $C\left([0,T];C^1\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)$ with norms controlled by*

$$C(T_0)\,|||(t,x)\mapsto\varrho\left(t,\phi_\alpha^{-1}(x)\right)|||_{\mathbb{E}_{T,\alpha}}\,.$$

Proof. Both statements follow from Lemma 2.1, Lemma 2.3 and Lemma 2.5 using

$$\Gamma_{ij}^{l,\varrho}\left(t,\phi_\alpha^{-1}(x)\right)=\frac{1}{2}g^{ml,\varrho}\left(t,\phi_\alpha^{-1}(x)\right)\left(\frac{\partial\left(g_{im,t}^{\varrho}\circ\phi_\alpha^{-1}\right)}{\partial x_j}+\frac{\partial\left(g_{jm,t}^{\varrho}\circ\phi_\alpha^{-1}\right)}{\partial x_i}-\frac{\partial\left(g_{ij,t}^{\varrho}\circ\phi_\alpha^{-1}\right)}{\partial x_l}\right)(x)$$

and

$$a_{ij}^{\varrho}\left(t,\phi_\alpha^{-1}(x)\right)=\left\langle\frac{\partial^2\left(f^{\varrho}(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i\partial x_j}(x),\nu^{\varrho}\left(t,\phi_\alpha^{-1}(x)\right)\right\rangle\,.$$

□

Lemma 2.7 (Time regularity of the normal). *Let T_0 be positive, $T\in(0,T_0]$, $\alpha\in\{1,\dots,N\}$ and $i,j\in\{1,\dots,n\}$. There exists a constant $\tau>\frac{1}{2}-\frac{1}{4p}$ such that*

$$\nu:\boldsymbol{U}_T\to C^{\tau}\left([0,T];C\left(\overline{\phi_\alpha(U_\alpha)};\mathbb{R}^{n+1}\right)\right),\varrho\mapsto\left((t,x)\mapsto\varrho\left(t,\phi_\alpha^{-1}(x)\right)\right)$$

is well-defined and Lipschitz continuous with constant $C(Q,T_0,\varepsilon,\sigma)$.

Proof. By Lemma 2.2 there exists $\tau>\frac{1}{2}-\frac{1}{4p}$ such that for all $\varrho\in\boldsymbol{U}_T$, $(t,x)\mapsto\varrho\left(t,\phi_\alpha^{-1}(x)\right)$, $(t,x)\mapsto\nabla\left(\varrho(t)\circ\phi_\alpha^{-1}\right)(x)$ lie in $C^{\tau}\left([0,T];C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)$. Recalling the function ψ defined in Definition 1.3 there holds for $t\in[0,T]$, $x\in\overline{\phi_\alpha(U_\alpha)}$,

$$\nu(\varrho)(t,x)=\frac{G\left(\varrho\left(t,\phi_\alpha^{-1}(x)\right),\nabla\left(\varrho(t)\circ\phi_\alpha^{-1}\right)(x),x\right)}{\left\|G\left(\varrho\left(t,\phi_\alpha^{-1}(x)\right),\nabla\left(\varrho(t)\circ\phi_\alpha^{-1}\right)(x),x\right)\right\|}$$

with $G\left(\varrho\left(t,\phi_\alpha^{-1}(x)\right),\nabla\left(\varrho(t)\circ\phi_\alpha^{-1}\right)(x),x\right)=\psi\left(\nabla\left(f^{\varrho}(t)\circ\phi_\alpha^{-1}\right)(x)\right)$. In particular, G lies in the space $C^2\left(\mathbb{R}^{n+1};C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)$ being a polynomial in $\varrho\left(t,\phi_\alpha^{-1}(x)\right)$, $\partial_{x_i}\left(\varrho(t)\circ\phi_\alpha^{-1}\right)(x)$, $i\in$

$\{1, \ldots, n\}$ with time-independent coefficients depending on the reference geometry, smooth on $\overline{\phi_\alpha(U_\alpha)}$. Proposition B.3 applied to G and $K := [-R, R]^{n+1}$ yields

$$(t,x) \mapsto G\left(\varrho\left(t, \phi_\alpha^{-1}(x)\right), \nabla\left(\varrho(t) \circ \phi_\alpha^{-1}\right)(x), x\right) \in C^\tau\left([0,T]; C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)$$

with norm bounded by $C(Q, T_0) \, |||\varrho|||_{\mathbb{E}_T}$. Corollary 1.42 implies that there exists a constant $\sigma > 0$ such that for all $\varrho \in \boldsymbol{U}_T$, $t \in [0,T]$ and $x \in \overline{\phi_\alpha(U_\alpha)}$,

$$\left\|G\left(\varrho\left(t, \phi_\alpha^{-1}(x)\right), \nabla\left(\varrho(t) \circ \phi_\alpha^{-1}\right)(x), x\right)\right\| \geq \sigma \, .$$

Thus, by Proposition B.4 we obtain also

$$(t,x) \mapsto \left\|G\left(\varrho\left(t, \phi_\alpha^{-1}(x)\right), \nabla\left(\varrho(t) \circ \phi_\alpha^{-1}\right)(x), x\right)\right\|^{-1} \in C^\tau\left([0,T]; C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)$$

with norm bounded by $C(Q, T_0, \sigma) \, |||\varrho|||_{\mathbb{E}_T}$. The statements in Proposition B.3 and B.4 yield further for ϱ, $\eta \in C^\tau\left([0,T]; C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)$,

$$\|\nu(\varrho) - \nu(\eta)\|_{C^\tau\left([0,T]; C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \leq C(Q, T_0, \varepsilon, \sigma) \, |||\varrho - \eta|||_{\mathbb{E}_T} \, .$$

□

Proposition 2.8 (The first component of the nonlinear operator)**.** *Given* $T_0 > 0$ *and* $T \in (0, T_0]$, *the function*

$$N_{T,1} : \boldsymbol{U}_T \to X_T \; , \qquad \varrho \mapsto \langle \partial_t f^\varrho, \nu^\varrho \rangle + \left\langle \Delta^2_{g^\varrho} f^\varrho, \nu^\varrho \right\rangle + Q(A)^\varrho \langle \Delta_{g^\varrho} f^\varrho, \nu^\varrho \rangle$$

is well-defined. The norm of $N_{T,1}(\varrho)$ *in* X_T *is bounded by* $C(T_0) \, |||\varrho|||_{\mathbb{E}_T}$.

Proof. Let $\varrho \in \boldsymbol{U}_T$ be given. Corollary 1.42 shows that for all $t \in [0,T]$, $f^\varrho(t) : M \to \mathbb{R}^{n+1}$ is an immersion and in particular, the expressions Δ_{g^ϱ} and ν^ϱ are well-defined. By Proposition C.16 it is enough to show that for all $\alpha \in \{1, \ldots, N\}$, the function $(t,x) \mapsto N_{T,1}(\varrho)\left(t, \phi_\alpha^{-1}(x)\right)$ lies in $X_{T,\alpha}$. Observe that

$$(t,x) \mapsto \langle \partial_t f^\varrho, \nu^\varrho \rangle \left(t, \phi_\alpha^{-1}(x)\right) = \partial_t \varrho\left(t, \phi_\alpha^{-1}(x)\right) \left\langle \nu_\varphi\left(\phi_\alpha^{-1}(x)\right), \nu^\varrho\left(t, \phi_\alpha^{-1}(x)\right)\right\rangle$$

lies in $X_{T,\alpha}$, as $(t,x) \mapsto \partial_t \varrho\left(t, \phi_\alpha^{-1}(x)\right)$ and $(t,x) \mapsto \nu^\varrho\left(t, \phi_\alpha^{-1}(x)\right)$ are elements of $X_{T,\alpha}$ and $C\left([0,T]; C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)$, respectively, by Lemma 2.1 and Lemma 2.5, respectively. Furthermore, we have

$$\begin{aligned}&\left(\Delta_{g^\varrho} f^\varrho\right)\left(t, \phi_\alpha^{-1}(x)\right)\\ &= g^{ij,\varrho}\left(t, \phi_\alpha^{-1}(x)\right)\left(\frac{\partial^2\left(f^\varrho(t) \circ \phi_\alpha^{-1}\right)}{\partial x_i \partial x_j}(x) - \Gamma_{ij}^{l,\varrho}\left(t, \phi_\alpha^{-1}(x)\right) \frac{\partial\left(f^\varrho(t) \circ \phi_\alpha^{-1}\right)}{\partial x_l}(x)\right),\end{aligned}$$

and thus $(t,x) \mapsto \left(\Delta_{g^\varrho} f^\varrho\right)\left(t, \phi_\alpha^{-1}(x)\right) \in C\left([0,T]; C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)$ by Lemma 2.1, Lemma 2.3 and Corollary 2.6. Similarly, Lemma 2.1, Lemma 2.3 and Corollary 2.6 imply that

$$(t,x) \mapsto Q(A)^\varrho\left(t, \phi_\alpha^{-1}(x)\right) = \left(2 g^{im,\varrho} g^{kj,\varrho} a^\varrho_{mk} a^\varrho_{ij} - a^\varrho_{ij} g^{ij,\varrho} a^\varrho_{kl} g^{kl,\varrho}\right)\left(t, \phi_\alpha^{-1}(x)\right)$$

lies in $C\left([0,T]; C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)$. Furthermore, we observe that

$$\left(\Delta_g^2 f^\varrho\right)\left(t, \phi_\alpha^{-1}(x)\right) = g^{ij,\varrho}\left(t, \phi_\alpha^{-1}(x)\right) \frac{\partial^2\left(\Delta_{g^\varrho} f^\varrho(t) \circ \phi_\alpha^{-1}\right)}{\partial x_i \partial x_j}(x)$$

$$- g^{ij,\varrho}\left(t,\phi_\alpha^{-1}(x)\right)\Gamma_{ij}^{l,\varrho}\left(t,\phi_\alpha^{-1}(x)\right)\frac{\partial\left(\Delta_{g^\varrho} f^\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_l}(x)\,.$$

Lemma 2.1 and Corollary 2.6 yield that $(t,x)\mapsto g^{ij,\varrho}\left(t,\phi_\alpha^{-1}(x)\right)$, $(t,x)\mapsto\Gamma_{ij}^{l,\varrho}\left(t,\phi_\alpha^{-1}(x)\right)$ and every component of $(t,x)\mapsto\nabla\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)(x)$ lie in $C\big([0,T];C^0\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$. Using the formula for $\left(\Delta_{g^\varrho} f^\varrho(t)\circ\phi_\alpha^{-1}\right)$ and the fact that for all $\alpha\in\mathbb{N}_0^n$ with $|\alpha|=4$ there holds

$$(t,x)\mapsto\frac{\partial^\alpha\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}(x)\in L_p\left((0,T);L_p\left(\phi_\alpha(U_\alpha)\right)\right),$$

we conclude $(t,x)\mapsto\left(\Delta_g^2 f^\varrho\right)\left(t,\phi_\alpha^{-1}(x)\right)\in X_{T,\alpha}$. □

Proposition 2.9 (The second component of the nonlinear operator)**.** *Given $T_0>0$ and $T\in(0,T_0]$, the function*

$$N_{T,2}:\boldsymbol{U}_T\to Y_{1,T},\quad \varrho\mapsto\langle\Delta_{g^\varrho} f^\varrho,\nu^\varrho\rangle$$

is well-defined. The norm of $N_{T,2}(\varrho)$ in $Y_{1,T}$ is bounded by $C(T_0)\,|\!|\!|\varrho|\!|\!|_{\mathbb{E}_T}$.

Proof. Given $\varrho\in\boldsymbol{U}_T$, Corollary 1.42 implies that the expressions $\Delta_{g_t^\varrho}$ and $\nu^\varrho(t)$ are well-defined for all $t\in[0,T]$. Proposition A.15, A.46 and C.16 imply that it is enough to check

$$(t,x')\mapsto\langle\Delta_{g^\varrho} f^\varrho,\nu^\varrho\rangle\left(t,\sigma_\alpha^{-1}(x')\right)\in W_p^{1/2-1/4p,2-1/p}\left((0,T)\times\sigma_\alpha(V_\alpha)\right).$$

Lemma 2.5, Lemma 2.3 and Lemma 2.1 imply that the functions $(t,x)\mapsto\nu^\varrho\left(t,\phi_\alpha^{-1}(x)\right)$, $(t,x)\mapsto g^{ij,\varrho}\left(t,\phi_\alpha^{-1}(x)\right)$ and $(t,x)\mapsto\frac{\partial\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_l}(x)$ lie in $C^0\big([0,T];C^2\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$. Theorem C.29 implies that

$$(t,x)\mapsto\frac{\partial^2\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i\partial x_j}(x)\in Y_{1,T,\alpha}\,. \tag{2.7}$$

This allows us to conclude that

$$(t,x)\mapsto\Gamma_{ij}^{l,\varrho}\left(t,\phi_\alpha^{-1}(x)\right)=g^{lm,\varrho}\left(t,\phi_\alpha^{-1}(x)\right)\left\langle\frac{\partial^2\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i\partial x_j}(x),\frac{\partial\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_m}(x)\right\rangle$$

lies in $L_p\big((0,T);W_p^{2-1/p}\big(\partial\phi_\alpha(U_\alpha)\big)\big)$ and further

$$(t,x)\mapsto\langle\Delta_{g^\varrho} f^\varrho,\nu^\varrho\rangle\left(t,\phi_\alpha^{-1}(x)\right)\in L_p\big((0,T);W_p^{2-1/p}\big(\partial\phi_\alpha(U_\alpha)\big)\big)\,.$$

As $\sigma_\alpha(V_\alpha)\times\{0\}=\partial\phi_\alpha(U_\alpha)\cap\partial\mathbb{H}^n$ and $\sigma_\alpha^{-1}(x')=\phi_\alpha^{-1}(x',0)$ for all $x'\in\sigma_\alpha(V_\alpha)$, this yields, invoking Proposition C.23,

$$(t,x')\mapsto\langle\Delta_{g^\varrho} f^\varrho,\nu^\varrho\rangle\left(t,\sigma_\alpha^{-1}(x')\right)\in L_p\big((0,T);W_p^{2-1/p}\big(\sigma_\alpha(V_\alpha)\big)\big)\,.$$

Lemma 2.2 implies that there exists $\tau>\frac{1}{2}-\frac{1}{4p}$ such that

$$(t,x)\mapsto\nabla\left(f^\varrho(t)\circ\phi_\alpha^{-1}\right)(x)\in C^\tau\left([0,T];C\big(\overline{\phi_\alpha\left(U_\alpha\right)};\mathbb{R}^{(n+1)\times n}\big)\right).$$

Combining this with (2.7), Proposition B.3 and Proposition C.31, we obtain

$$(t,x)\mapsto\Gamma_{ij}^{l,\varrho}\left(t,\phi_\alpha^{-1}(x)\right)\in W_p^{1/2-1/4p}\left((0,T);L_p\left(\partial\phi_\alpha(U_\alpha)\right)\right)$$

and, using further the formula for the Laplace Beltrami operator,

$$(t,x)\mapsto\langle\Delta_{g^\varrho} f^\varrho,\nu^\varrho\rangle\left(t,\phi_\alpha^{-1}(x)\right)\in W_p^{1/2-1/4p}\left((0,T);L_p\left(\partial\phi_\alpha(U_\alpha)\right)\right).$$

This yields in particular

$$(t,x)\mapsto\langle\Delta_{g^\varrho} f^\varrho,\nu^\varrho\rangle\left(t,\sigma_\alpha^{-1}(x)\right)\in W_p^{1/2-1/4p}\left((0,T);L_p\left(\sigma_\alpha(V_\alpha)\right)\right).$$

□

2.2 Linearisation of the graph formulation

This section is devoted to prove the following result.

Proposition 2.10 (Differentiability of the nonlinear operator)**.** *Let T_0 be positive and $T \in (0, T_0]$. The function*

$$N_T := (N_{T,1}, N_{T,2}) : \boldsymbol{U}_T \to \boldsymbol{F}_T := X_T \times Y_{1,T}$$

with $N_{T,i}$, $i \in \{1, 2\}$, defined in Proposition 2.8 and Proposition 2.9, respectively, is Fréchet differentiable with Lipschitz continuous derivative, that is, $N_T \in C^1(\boldsymbol{U}_T, \boldsymbol{F}_T)$ with Fréchet derivative $DN_T \in C^{0,1}(\boldsymbol{U}_T, \mathcal{L}(\mathbb{E}_T, \boldsymbol{F}_T))$.

To this end, we show existence of the directional derivative $\partial_u N_T(v) \in \boldsymbol{F}_T$ for all $v \in \boldsymbol{U}_T$ and all $u \in \mathbb{E}_T$, linearity and continuity of $u \mapsto DN_T(v)u := \partial_u N_T(v) \in \boldsymbol{F}_T$ on $\mathbb{E}_T$ and, finally, Lipschitz continuity of $v \mapsto DN_T(v) \in \mathcal{L}(\mathbb{E}_T, \boldsymbol{F}_T)$ on $\boldsymbol{U}_T$. In the following, we formally derive the candidate for the directional derivative $\partial_u N_T(v)$, namely

$$\partial_u N_T(v)(t,p) := \frac{\mathrm{d}}{\mathrm{d}\delta} N_T(v + \delta u)(t,p)\,\Big|_{\delta=0}.$$

The expression will be calculated pointwise for every $t \in [0, T]$ and $p \in M$. Regularity issues will be addressed afterwards. Given $v \in \boldsymbol{U}_T$ and $u \in \mathbb{E}_T$ we let $\delta_0 > 0$ be so small that $v + \delta u \in \boldsymbol{U}_T$ for all $\delta \in (-\delta_0, \delta_0)$. In particular,

$$f^{v+\delta u}(t) := \varphi + (v(t) + \delta u(t))\,\nu_\varphi$$

is an immersion for all $t \in [0, T]$ and all $\delta \in (-\delta_0, \delta_0)$. All geometric quantities of $\left(M, f^{v+\delta u}(t)\right)$ are marked with superscript $v + \delta u$. The geometric expressions induced by the reference geometry are indicated by subscript φ. As most of the precise expressions appearing in the linearised terms are not relevant for the further considerations, we introduce the following notation to combine terms of the same structure.

Definition 2.11 (Lipschitz condition of order k)**.** Let T_0 be positive, $T \in (0, T_0]$, $m \in \mathbb{N}$, $k \in \{0, 1, 2, 3\}$ and $\alpha \in \{1, \dots, N\}$. A function

$$c : \boldsymbol{U}_T \to C\big([0,T]; C\big(\overline{\phi_\alpha(U_\alpha)}; \mathbb{R}^m\big)\big)$$

is said to satisfy the *Lipschitz condition of order k* if it holds for all $\varrho, \eta \in \boldsymbol{U}_T$,

$$\|c(\varrho) - c(\eta)\|_{C\left([0,T]; C\left(\overline{\phi_\alpha(U_\alpha)}; \mathbb{R}^m\right)\right)} \le C(T_0)\, |||\varrho - \eta|||_{\mathbb{E}_T}$$

and if there exists a Lipschitz continuous function

$$\mathfrak{c} : [0,T] \times \overline{\phi_\alpha(U_\alpha)} \times \mathbb{R} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^{n^k} \to \mathbb{R}^m$$

such that for all $\varrho \in \boldsymbol{U}_T$,

$$c(\varrho)(t,x) = \mathfrak{c}\left(t, x, \varrho_\alpha(t,x), \nabla\varrho_\alpha(t,x), \cdots, \nabla^k \varrho_\alpha(t,x)\right)$$

where $\varrho_\alpha(t) := \varrho(t) \circ \phi_\alpha^{-1}$ and $\nabla^k \varrho_\alpha(t,x)$ is the vertical vector that consists of $\partial^\beta \varrho_\alpha(t,x) := \frac{\partial^\beta \varrho_\alpha}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}(t,x)$ for any possible choices of $\beta \in \mathbb{N}_0^n$, $|\beta| = k$. If c satisfies the Lipschitz condition of order k, we often denote the order in superscript by writing c^k instead of c.

Lemma 2.12. *Let T_0 be positive, $T \in (0, T_0]$, $m = 1$, $\alpha \in \{1, \dots, N\}$, and $k, l \in \{0, 1, 2, 3\}$. If c_1 and c_2 satisfy the Lipschitz condition of order l and k, respectively, then $c_1 + c_2$ and $c_1 c_2$ satisfy the Lipschitz condition of order* $\max\{l, k\}$.

Proof. The functions $c_1 + c_2,\ c_1 c_2 : \boldsymbol{U}_T \to C\big([0,T]; C\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$ depend on derivatives of the argument of order at most $\max\{l, k\}$. The Lipschitz continuity of $c_1 + c_2$ is straightforward. The estimate for the product $c_1 c_2$ is easily obtained using that for all $\varrho \in \boldsymbol{U}_T$, $i \in \{1, 2\}$, there holds

$$\|c_i(\varrho)\|_{C\left([0,T];C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \leq C(T_0)\, |\!|\!|\varrho|\!|\!|_{\mathbb{E}_T} \leq C(T_0)\varepsilon \,.$$

□

Lemma 2.13. *Let T_0 be positive, $T \in (0, T_0]$, $m \in \mathbb{N}$, $k \in \{0, 1, 2, 3\}$, $\alpha \in \{1, \dots, N\}$, $M \in \mathbb{N}$, $\beta^j \in \mathbb{N}_0^n$, $j \in \{1, \dots, M\}$ with $|\beta^j| \leq k$ and $\mathfrak{a} \in C\big(\overline{\phi_\alpha(U_\alpha)}; \mathbb{R}^m\big)$. Then the function*

$$c : \boldsymbol{U}_T \to \Big\{ f : [0,T] \times \overline{\phi_\alpha(U_\alpha)} \to \mathbb{R}^m \Big\}$$

defined by

$$c(\varrho)(t,x) = \mathfrak{a}(x) \prod_{j=1}^{M} \frac{\partial^{\beta^j} \left(\varrho(t) \circ \phi_\alpha^{-1}\right)}{\partial x_1^{\beta_1^j} \cdots \partial x_n^{\beta_n^j}}(x)$$

satisfies the Lipschitz condition of order k.

Proof. Proposition B.35 implies for all $\varrho \in \boldsymbol{U}_T$ that $c(\varrho) \in C\big([0,T]; C\big(\overline{\phi_\alpha(U_\alpha)}; \mathbb{R}^m\big)\big)$. Let $\varrho, \eta \in \boldsymbol{U}_T$ be given. In the case $M = 1$, Corollary B.38 implies

$$\begin{aligned}
\|c(\varrho) - c(\eta)\|_{C\left([0,T];C\left(\overline{\phi_\alpha(U_\alpha)};\mathbb{R}^m\right)\right)} &\leq \|\mathfrak{a}\|_{C\left(\overline{\phi_\alpha(U_\alpha)};\mathbb{R}^m\right)} \left\| \frac{\partial^{\beta^1} \left((\varrho - \eta) \circ \phi_\alpha^{-1}\right)}{\partial x_1^{\beta_1^1} \cdots \partial x_n^{\beta_n^1}} \right\|_{C\left([0,T];C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \\
&\leq \|\mathfrak{a}\|_{C\left(\overline{\phi_\alpha(U_\alpha)};\mathbb{R}^m\right)} \|\varrho - \eta\|_{C\left([0,T];C^3\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \\
&\leq C(T_0)\, |\!|\!|\varrho - \eta|\!|\!|_{\mathbb{E}_T} \,.
\end{aligned}$$

Suppose that the claim is shown for some $M - 1 \in \mathbb{N}_0$ and let c be of the given form. Then

$$c(\varrho)(t,x) = c_1(\varrho)(t,x) \frac{\partial^{\beta^M} \left((\varrho - \eta) \circ \phi_\alpha^{-1}\right)}{\partial x_1^{\beta_1^M} \cdots \partial x_n^{\beta_n^M}}$$

with c_1 satisfying the Lipschitz condition of order k. The case $M = 1$ implies that $c = c_1 c_2$ with both c_i satisfying the Lipschitz condition of order k. The claim now follows from Lemma 2.12. □

The following corollaries are immediate consequences of Lemma 2.3 and Lemma 2.5, respectively.

Corollary 2.14 (Lipschitz conditions of the inverse metric)**.** *Let T_0 be positive, $T \in (0, T_0]$, $\alpha \in \{1, \dots, N\}$, and $i, j, k \in \{1, \dots, n\}$. The functions*

$$\begin{aligned}
g^{ij} &: \boldsymbol{U}_T \to C\big([0,T]; C\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)\,, \quad \varrho \mapsto \big((t,x) \mapsto g^{ij,\varrho}\left(t, \phi_\alpha^{-1}(x)\right)\big) \\
\frac{\partial g^{ij}}{\partial x_k} &: \boldsymbol{U}_T \to C\big([0,T]; C\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)\,, \quad \varrho \mapsto \Big((t,x) \mapsto \frac{\partial}{\partial x_k}\left(g^{ij,\varrho}(t) \circ \phi_\alpha^{-1}\right)(x)\Big)
\end{aligned}$$

satisfy Lipschitz conditions of order 1 *and* 2*, respectively*

Proof. This follows from Lemma 2.3. □

Corollary 2.15 (Lipschitz conditions of the normal)**.** *Let T_0 be positive, $T \in (0,T_0]$, $\alpha \in \{1,\dots,N\}$ and $i \in \{1,\dots,n\}$. The functions*

$$\nu : \boldsymbol{U}_T \to C\big([0,T]; C\big(\overline{\phi_\alpha(U_\alpha)}; \mathbb{R}^{n+1}\big)\big), \quad \varrho \mapsto \big((t,x) \mapsto \nu^\varrho\left(t, \phi_\alpha^{-1}(x)\right)\big)$$
$$\frac{\partial \nu}{\partial x_i} : \boldsymbol{U}_T \to C\big([0,T]; C\big(\overline{\phi_\alpha(U_\alpha)}; \mathbb{R}^{n+1}\big)\big), \quad \varrho \mapsto \left((t,x) \mapsto \frac{\partial\left(\nu^\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i}(x)\right)$$

satisfy Lipschitz conditions of order 1 and 2, respectively.

Proof. This follows from Lemma 2.5. □

The following calculations are performed in a chart domain U_α, with $(U_\alpha,\phi_\alpha) \in \mathcal{T}$, for given $v \in \boldsymbol{U}_T$, $u \in \mathbb{E}_T$ and $\delta_0 = \delta_0(u,v) > 0$ as above. For the sake of readability the composition of the involved functions with the parametrisation ϕ_α^{-1} is not explicitly written. In the following we denote by c^k, $k \in \{1,2,3\}$, a function defined on $\boldsymbol{U}_T$ that satisfies the Lipschitz condition of order k as introduced in Definition 2.11. These functions appear in the linearisations and may change from line to line. Their precise form is not important. Hereby, we make use of Lemma 2.12, 2.13 and Corollaries 2.14 and 2.15 without explicitly mentioning it. Furthermore, the coefficients may contain terms depending on the reference geometry (M,φ).

Lemma 2.16 (Linearisation of the tangential vectors)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\frac{\partial f^{v+\delta u}(t)}{\partial x^i}(p)_{|\delta=0} = \frac{\partial u(t)}{\partial x^i}(p)\nu_\varphi(p) + u(t,p)\frac{\partial \nu_\varphi}{\partial x^i}(p)\,.$$

Proof. This is straightforward. □

Lemma 2.17 (Linearisation of the metric)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\delta} g_{ij}^{v+\delta u}(t,p)_{|\delta=0} = c^1(v)(t,\phi_\alpha(p))\frac{\partial u(t)}{\partial x^i}(p) + c^1(v)(t,\phi_\alpha(p))\frac{\partial u(t)}{\partial x^j}(p) + c^1(v)(t,\phi_\alpha(p))u(t,p)\,.$$

Proof. The formula

$$g_{ij}^{v+\delta u}(t,p) = \left\langle \frac{\partial f^{v+\delta u}(t)}{\partial x^i}(p), \frac{\partial f^{v+\delta u}(t)}{\partial x^j}(p)\right\rangle$$

and the previous lemma imply

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta} g_{ij}^{v+\delta u}(t,p)_{|\delta=0} =& \frac{\partial u(t)}{\partial x^i}\left\langle \nu_\varphi, \frac{\partial f^v(t)}{\partial x^j}\right\rangle + \frac{\partial u(t)}{\partial x^j}\left\langle \nu_\varphi, \frac{\partial f^v(t)}{\partial x^i}\right\rangle \\ &+ u(t)\left(\left\langle \frac{\partial \nu_\varphi}{\partial x^i}, \frac{\partial f^v(t)}{\partial x^j}\right\rangle + \left\langle \frac{\partial \nu_\varphi}{\partial x^j}, \frac{\partial f^v(t)}{\partial x^i}\right\rangle\right).\end{aligned}$$

□

Lemma 2.18 (Linearisation of the inverse metric)**.**

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta} g^{ij,v+\delta u}(t,p)_{|\delta=0} &= -g^{mj,v}(t,p)g^{ki,v}(t,p)\frac{\mathrm{d}}{\mathrm{d}\delta} g_{km}^{v+\delta u}(t,p)_{|\delta=0} \\ &= \sum_{i=1}^n c_i^1(v)(t,\phi_\alpha(p))\frac{\partial u(t)}{\partial x^i}(p) + c^1(v)(t,\phi_\alpha(p))u(t,p)\,.\end{aligned}$$

Proof. This follows from differentiating the identity

$$g_{km}^{v+\delta u}(t,p)g^{mj,v+\delta u}(t,p) = \delta_k^j$$

with respect to δ, evaluating in $\delta = 0$ and contracting with $g^{ki,v}(t,p)$. □

Lemma 2.19 (Linearisation of second derivatives of the immersion)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\frac{\partial^2 f^{v+\delta u}(t)}{\partial x^i \partial x^j}(p)_{\big|\delta=0} = \frac{\partial^2 u(t)}{\partial x^i \partial x^j}(p)\nu_\varphi(p) + \frac{\partial u(t)}{\partial x^i}(p)\frac{\partial \nu_\varphi}{\partial x^j}(p) + \frac{\partial u(t)}{\partial x^j}(p)\frac{\partial \nu_\varphi}{\partial x^i}(p) + u(t,p)\frac{\partial^2 \nu_\varphi}{\partial x^i \partial x^j}(p)\,.$$

Proof. This follows directly by differentiating the identity in Lemma 2.16. □

Lemma 2.20 (Linearisation of the Christoffel symbols)**.**

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta}\Gamma^{l,v+\delta u}_{ij}(t,p)_{\big|\delta=0} =& c^1(v)(t,\phi_\alpha(p))\frac{\partial^2 u(t)}{\partial x^i \partial x^j}(p) + \sum_{l=1}^{n} c^2_l(v)(t,\phi_\alpha(p))\frac{\partial u(t)}{\partial x^l}(p) \\ &+ c^2(v)(t,\phi_\alpha(p))u(t,p)\,.\end{aligned}$$

Proof. This follows from the previous Lemmas and the formula

$$\Gamma^{l,v+\delta u}_{ij}(t,p) = g^{lm,v+\delta u}(t,p)\left\langle \frac{\partial^2 f^{v+\delta u}(t)}{\partial x^i \partial x^j}(p), \frac{\partial f^{v+\delta u}(t)}{\partial x^m}(p)\right\rangle\,.$$

□

Lemma 2.21 (Linearisation of the second fundamental form)**.**

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta}\left(a^{v+\delta u}_{ij}(t,p)\nu^{v+\delta u}(t,p)\right)_{\big|\delta=0} =& \frac{\mathrm{d}}{\mathrm{d}\delta}\left(\frac{\partial^2 f^{v+\delta u}(t)}{\partial x^i \partial x^j}(p)\right)_{\big|\delta=0} - \Gamma^{l,v}_{ij}(p)\frac{\mathrm{d}}{\mathrm{d}\delta}\left(\frac{\partial f^{v+\delta u}(t)}{\partial x^l}(p)\right)_{\big|\delta=0} \\ &- \frac{\mathrm{d}}{\mathrm{d}\delta}\Gamma^{l,v+\delta u}_{ij}(t,p)_{\big|\delta=0}\frac{\partial f^v(t)}{\partial x^l}(p)\,.\end{aligned}$$

Proof. This follows from the previous Lemmas and the representation of the second fundamental form in local coordinates given in Proposition 1.7. □

Lemma 2.22 (Linearisation of the normal)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\nu^{v+\delta u}(t,p)_{\big|\delta=0} = \sum_{i=1}^{n} c^1_i(v)\left(t,\phi_\alpha(p)\right)\frac{\partial u(t)}{\partial x^i}(p) + c^1(v)\left(t,\phi_\alpha(p)\right)u(t,p)\,, \tag{2.8}$$

where $c^1_i, c^1 : \boldsymbol{U}_T \to C\big([0,T];C^2\big(\overline{\phi_\alpha(U_\alpha)};\mathbb{R}^{n+1}\big)\big)$ *are Lipschitz continuous with Lipschitz constant* $C(Q,T_0,\varepsilon,\sigma)$*.*

Proof. As in the proof of Lemma 2.5 we study the n-linear mapping $\psi : \left(\mathbb{R}^{n+1}\right)^n \to \mathbb{R}^{n+1}$ given by $\psi\left(v^1,\dots,v^n\right)_i := \det\left(v^1,\cdots,v^n,e_i\right)$, $i \in \{1,\dots,n+1\}$. The Leibniz formula implies that ψ is smooth and that for all $m \in \mathbb{N}$, $i_1,\dots,i_m \in \{1,\dots,n+1\}$, $j_1,\dots,j_m \in \{1,\dots,n\}$, $\partial_{v^{j_1}_{i_1}}\cdots\partial_{v^{j_m}_{i_m}}\psi(v)$ is m-linear in the unit vectors $e_{i_1},\dots,e_{i_m}$ with coefficients that are polynomial in the components of $v^1,\dots,v^n$. In particular,

$$\left(D\psi\right)(v)\left(y^1,\dots,y^n\right) = \sum_{j=1}^{n}\psi\left(v^1,\dots,v^{j-1},y^j,v^{j+1},\dots,v^n\right).$$

The proof of Proposition 1.4 implies that $\nu^{v+\delta u}(t,p) = \frac{\psi(h(\delta))}{\|\psi(h(\delta))\|}$ where $h(\delta) := \left(\frac{\partial f^{v+\delta u}(t)}{\partial x^j}(p)\right)_{j=1}^{n}$. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\nu^{v+\delta u}(t,p)_{\big|\delta=0} = \frac{1}{\|\psi(h(0))\|}\left(\frac{\mathrm{d}}{\mathrm{d}\delta}\psi(h(\delta))_{|\delta=0} - \left\langle\frac{\mathrm{d}}{\mathrm{d}\delta}\psi(h(\delta))_{|\delta=0},\nu^v(t,p)\right\rangle\nu^v(t,p)\right)$$

as $\frac{\psi(h(0))}{\|\psi(h(0))\|} = \nu^v(t,p)$. Using the linearisation of the tangent vectors calculated in Lemma 2.16,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\delta}\psi(h(\delta))_{|\delta=0} = (D\psi)\,(h(0))\,h'(0) \\
&= \sum_{j=1}^{n}\psi\left(\frac{\partial f^v(t)}{\partial x^1}(p),\dots,\frac{\partial f^v(t)}{\partial x^{j-1}}(p),\frac{\mathrm{d}}{\mathrm{d}\delta}\frac{\partial f^{v+\delta u}(t)}{\partial x^j}(p)_{\big|\delta=0},\frac{\partial f^v(t)}{\partial x^{j+1}}(p),\dots,\frac{\partial f^v(t)}{\partial x^n}(p)\right) \\
&= \sum_{j=1}^{n}\frac{\partial u(t)}{\partial x^j}(p)\psi\left(\frac{\partial f^v(t)}{\partial x^1}(p),\dots,\frac{\partial f^v(t)}{\partial x^{j-1}}(p),\nu_\varphi(p),\frac{\partial f^v(t)}{\partial x^{j+1}}(p),\dots,\frac{\partial f^v(t)}{\partial x^n}(p)\right) \\
&\quad + u(t,p)\sum_{j=1}^{n}\psi\left(\frac{\partial f^v(t)}{\partial x^1}(p),\dots,\frac{\partial f^v(t)}{\partial x^{j-1}}(p),\frac{\partial \nu_\varphi}{\partial x^j}(p),\frac{\partial f^v(t)}{\partial x^{j+1}}(p),\dots,\frac{\partial f^v(t)}{\partial x^n}(p)\right).
\end{aligned}$$

Exemplary, we show that

$$c : \boldsymbol{U}_T \to C\big([0,T];C^2\big(\overline{\phi_\alpha\,(U_\alpha)};\mathbb{R}^{n+1}\big)\big)\,,\quad v\mapsto \big((t,x)\mapsto \psi\left(\nabla\left(f^v(t)\circ\phi_\alpha^{-1}\right)(x)\right)\big)$$

is Lipschitz continuous with constant $C(Q,T_0,\varepsilon)$. We note that c is well-defined due to the continuous embedding $\mathbb{E}_{T,\alpha}\hookrightarrow C\big([0,T];C^2\big(\overline{\phi_\alpha\,(U_\alpha)}\big)\big)$. The Leibniz formula implies that each component of ψ is a polynomial in its arguments. As further

$$\left\|(t,x)\mapsto v\left(t,\phi_\alpha^{-1}(x)\right)\right\|_{C\left([0,T];C^3\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \le C(Q,T_0,\varepsilon)$$

by Corollary 1.43, it is straightforward to verify that c is Lipschitz continuous on $\boldsymbol{U}_T$ with values in $C\big([0,T];C^2\big(\overline{\phi_\alpha\,(U_\alpha)}\big)\big)$. Corollary 1.42 and Corollary 1.43 imply that there exists a constant $C(R)>0$ such that for all $v\in\boldsymbol{U}_T$, $t\in[0,T]$ and $x\in\overline{\phi_\alpha\,(U_\alpha)}$,

$$\sigma \le \|c(v)(t,x)\| \le C(R)\,.$$

On the compact set $\overline{B_{C(R)}(0)}\setminus\overline{B_{\sigma/2}(0)}$, the function $p : y\mapsto \|y\|^{-1}$ is smooth and, in particular, all its derivatives are Lipschitz continuous. Thus, the map $(t,x)\mapsto p\,(c(v)(t,x))$ lies in $C\big([0,T];C^2\big(\overline{\phi_\alpha\,(U_\alpha)}\big)\big)$. Furthermore, for all $v,w\in\boldsymbol{U}_T$, $t\in[0,T]$, $x\in\overline{\phi_\alpha\,(U_\alpha)}$, there holds

$$\begin{aligned}
|p\,(c(v)(t,x)) - p\,(c(w)(t,x))| &\le L(\varepsilon,\sigma)\,\|c(v)(t,x)-c(w)(t,x)\| \\
&\le L(\varepsilon,\sigma)\,\|c(v)-c(w)\|_{C\left([0,T];C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \\
&\le C(Q,T_0,\varepsilon,\sigma)\,|||v-w|||_{\mathbb{E}_T}\,.
\end{aligned}$$

The first and second order derivatives of $p\,(c(v)(t,x)) - p\,(c(w)(t,x))$ can be estimated with the fundamental theorem of calculus using that for all $\varrho\in\boldsymbol{U}_T$, $t\in[0,T]$, $x\in\overline{\phi_\alpha\,(U_\alpha)}$ and $k\in\mathbb{N}$, all components of $\left(D^k p\right)(c(\varrho)(t,x))$ are bounded in absolute value by a constant $C(\varepsilon,\sigma)$. □

Lemma 2.23 (Linearisation of the velocity)**.**

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\delta}\left\langle \partial_t f^{v+\delta u},\nu^{v+\delta u}\right\rangle(t,p)_{\big|\delta=0} &= \partial_t u(t,p)\,\langle\nu_\varphi(p),\nu^v(t,p)\rangle \\
&\quad + \partial_t v(t,p)\sum_{i=1}^{n}\left(c_i^1(v)\,(t,\phi_\alpha(p))\,\frac{\partial u(t)}{\partial x^i}(p) + c^1(v)\,(t,\phi_\alpha(p))\,u(t,p)\right).
\end{aligned}$$

Proof. This follows from the previous lemma and the identity

$$\partial_t f^{v+\delta u}(t,p) = (\partial_t v(t,p) + \delta\partial_t u(t,p))\,\nu_\varphi(p)\,.$$

□

Lemma 2.24 (Linearisation of the mean curvature)**.**

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta}\left\langle \Delta_{g^{v+\delta u}} f^{v+\delta u}, \nu^{v+\delta u}\right\rangle(t,p)_{|\delta=0} =& g^{ij,v}(t,p)\left\langle \nu_\varphi(p), \nu^v(t,p)\right\rangle\left(\frac{\partial^2 u(t)}{\partial x^i \partial x^j}(p) - \Gamma^{l,v}_{ij}(t,p)\frac{\partial u(t)}{\partial x^l}(p)\right)\\ &+\sum_{i=1}^n c_i^2(v)\,(t,\phi_\alpha(p))\,\frac{\partial u(t)}{\partial x^i}(p) + c^2(v)\,(t,\phi_\alpha(p))\,u(t,p)\,,\end{aligned}$$

where second order derivatives of v appear only linearly in the coefficient functions $c^2(v)$, $c_i^2(v)$.

Proof. Observe that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta}\left\langle \Delta_{g^{v+\delta u}} f^{v+\delta u}, \nu^{v+\delta u}\right\rangle(t,p)_{|\delta=0} =& \left\langle \frac{\mathrm{d}}{\mathrm{d}\delta}\Delta_{g^{v+\delta u}} f^{v+\delta u}(t,p)_{|\delta=0}, \nu^v(t,p)\right\rangle\\ &+\left\langle \Delta_{g_t^v} f^v(t,p), \frac{\mathrm{d}}{\mathrm{d}\delta}\nu^{v+\delta u}(t,p)_{|\delta=0}\right\rangle .\end{aligned}$$

As $\Delta_{g^{v+\delta u}} f^{v+\delta u} = g^{ij,v+\delta u} a_{ij}^{v+\delta u}\nu^{v+\delta u}$, the linearisations in 2.16, 2.18, 2.19 and 2.21 imply

$$\begin{aligned}&\left\langle \frac{\mathrm{d}}{\mathrm{d}\delta}\Delta_{g^{v+\delta u}} f^{v+\delta u}(t,p)_{|\delta=0}, \nu^v(t,p)\right\rangle\\ &= \frac{\mathrm{d}}{\mathrm{d}\delta}\, g^{ij,v+\delta u}(t,p)_{|\delta=0}\, a_{ij}^v(t,p) + g^{ij,v}(t,p)\left\langle \frac{\mathrm{d}}{\mathrm{d}\delta}\left(\frac{\partial^2 f^{v+\delta u}(t)}{\partial x^i \partial x^j}(p)\right)_{|\delta=0}, \nu^v(t,p)\right\rangle\\ &\quad - g^{ij,v}(t,p)\Gamma^{l,v}_{ij}(t,p)\left\langle \frac{\mathrm{d}}{\mathrm{d}\delta}\left(\frac{\partial f^{v+\delta u}(t)}{\partial x^l}(p)\right)_{|\delta=0}, \nu^v(t,p)\right\rangle\\ &= g^{ij,v}(t,p)\left\langle \nu_\varphi(p), \nu^v(t,p)\right\rangle\left(\frac{\partial^2 u(t)}{\partial x^i \partial x^j}(p) - \Gamma^{l,v}_{ij}(t,p)\frac{\partial u(t)}{\partial x^l}(p)\right) + \sum_{i=1}^n c_i^2(v)\,(t,\phi_\alpha(p))\,\frac{\partial u(t)}{\partial x^i}(p)\\ &\quad + c^2(v)\,(t,\phi_\alpha(p))\,u(t,p)\,.\end{aligned}$$

We observe that second order derivatives of v_α appear only linearly in $a_{ij}^v(t,p)$, $\Gamma^{l,v}_{ij}(t,p)$ and $\Delta_{g_t^v} f^v(t,p)$. ☐

Lemma 2.25 (Linearisation of $Q(A)$)**.**

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta}Q(A)^{v+\delta u}(t,p)_{|\delta=0} =& \sum_{i,j=1}^n c_{ij}^1(v)\,(t,\phi_\alpha(p))\,\frac{\partial^2 u(t)}{\partial x^i \partial x^j}(p) + \sum_{i=1}^n c_i^2(v)\,(t,\phi_\alpha(p))\,\frac{\partial u(t)}{\partial x^i}(p)\\ &+ c^2(v)\,(t,\phi_\alpha(p))\,u(t,p)\,.\end{aligned}$$

Proof. The expression $Q(A)^{v+\delta u}$ is given by the formula

$$Q(A)^{v+\delta u} = 2g^{im,v+\delta u} g^{kj,v+\delta u} a_{mk}^{v+\delta u} a_{ij}^{v+\delta u} - \frac{1}{2}\left(H^{v+\delta u}\right)^2 .$$

The claim follows from the linearisations in the Lemmas 2.18, 2.21, 2.22 and 2.24 using $H^{v+\delta u} = \left\langle \Delta_{g^{v+\delta u}} f^{v+\delta u}, \nu^{v+\delta u}\right\rangle$ and $a_{ij}^{v+\delta u} = \left\langle a_{ij}^{v+\delta u}\nu^{v+\delta u}, \nu^{v+\delta u}\right\rangle$. ☐

Lemma 2.26 (Linearisation of the leading order term)**.**

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\delta}\left(\left\langle \Delta^2_{g^{v+\delta u}} f^{v+\delta u}, \nu^{v+\delta u}\right\rangle(t,p)\right)_{|\delta=0}\\ &= \left\langle \nu_\varphi(p), \nu^v(t,p)\right\rangle \Delta^2_{g_t^v} u(t,p) + \sum_{1\le|\beta|\le 3} c_\beta^{5-|\beta|}(v)\,(t,\phi_\alpha(p))\,D^\beta u(t,p) + c^4(v)\,(t,\phi_\alpha(p))\,u(t,p)\end{aligned}$$

where the expression $c^4(v)$ comprises terms of the form

$$\sum_{|\beta|=4} c^1_\beta(v)\partial^\beta_x v_\alpha + c^3(v)\,. \tag{2.9}$$

Proof. Using the linearisation of the unit normal we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta}\left(\left\langle \Delta^2_{g^{v+\delta u}} f^{v+\delta u}, \nu^{v+\delta u}\right\rangle(t,p)\right)_{|\delta=0} =& \left\langle \frac{\mathrm{d}}{\mathrm{d}\delta}\left(\Delta^2_{g^{v+\delta u}} f^{v+\delta u}(t,p)\right)_{|\delta=0}, \nu^v(t,p)\right\rangle \\ &+\sum_{i=1}^n c^4_i(v)\,(t,\phi_\alpha(p))\,\frac{\partial u(t)}{\partial x^i}(p) + c^4(v)\,(t,\phi_\alpha(p))\,u(t,p)\end{aligned}$$

with terms $c^4(v)$, $c^4_i(v)$ of the form (2.9). The identity

$$\Delta^2_{g^{v+\delta u}} f^{v+\delta u}(t,p) = g^{ij,v+\delta u}(t,p)\left(\frac{\partial^2 \boldsymbol{H}^{v+\delta u}(t)}{\partial x^i\partial x^j}(p) - \Gamma^{l,v+\delta u}_{ij}(t,p)\frac{\partial \boldsymbol{H}^{v+\delta u}(t)}{\partial x^l}(p)\right)$$

implies using the linearisation in Lemma 2.18,

$$\begin{aligned}&\left\langle \frac{\mathrm{d}}{\mathrm{d}\delta}\left(\Delta^2_{g^{v+\delta u}} f^{v+\delta u}(t,p)\right)_{|\delta=0}, \nu^v(t,p)\right\rangle \\ =& \left\langle \Delta_{g^v_t}\left(\frac{\mathrm{d}}{\mathrm{d}\delta}\boldsymbol{H}^{v+\delta u}(t,p)_{|\delta=0}\right), \nu^v(t,p)\right\rangle + \frac{\mathrm{d}}{\mathrm{d}\delta} g^{ij,v+\delta u}(t,p)_{|\delta=0}\left\langle \frac{\partial^2 \boldsymbol{H}(t)}{\partial x^i\partial x^j}(p), \nu^v(t,p)\right\rangle \\ =& \left\langle \Delta_{g^v_t}\left(\frac{\mathrm{d}}{\mathrm{d}\delta}\boldsymbol{H}^{v+\delta u}(t,p)_{|\delta=0}\right), \nu^v(t,p)\right\rangle + \sum_{i=1}^n c^4_i(v)\,(t,\phi_\alpha(p))\,\frac{\partial u(t)}{\partial x^i}(p) + c^4(v)\,(t,\phi_\alpha(p))\,u(t,p)\end{aligned}$$

with coefficients $c^4(v)$, $c^4_i(v)$ of the form (2.9). Observe that

$$\begin{aligned}\left\langle \frac{\partial}{\partial x^l}\frac{\mathrm{d}}{\mathrm{d}\delta}\boldsymbol{H}^{v+\delta u}(t,p)_{|\delta=0}, \nu^v(t,p)\right\rangle =& \frac{\partial}{\partial x^l}\left\langle \frac{\mathrm{d}}{\mathrm{d}\delta}\boldsymbol{H}^{v+\delta u}(t,p)_{|\delta=0}, \nu^v(t,p)\right\rangle \\ &-\left\langle \frac{\mathrm{d}}{\mathrm{d}\delta}\boldsymbol{H}^{v+\delta u}(t,p)_{|\delta=0}, \frac{\partial \nu^v(t)}{\partial x^l}(p)\right\rangle\end{aligned}$$

where

$$\begin{aligned}\left\langle \frac{\mathrm{d}}{\mathrm{d}\delta}\boldsymbol{H}^{v+\delta u}(t,p)_{|\delta=0}, \frac{\partial \nu^v(t)}{\partial x^l}(p)\right\rangle =& \sum_{i,j=1}^n c^2_{ij}(v)\,(t,\phi_\alpha(p))\,\frac{\partial^2 u(t)}{\partial x^i\partial x^j}(p) + \sum_{i=1}^n c^2_i(v)\,(t,\phi_\alpha(p))\,\frac{\partial u(t)}{\partial x^i}(p) \\ &+ c^2(v)\,(t,\phi_\alpha(p))\,u(t,p)\,.\end{aligned}$$

The calculations in Lemma 2.22 and 2.24 imply

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\delta}\boldsymbol{H}^{v+\delta u}(t,p)_{|\delta=0} =& \left\langle \nu_\varphi(p), \nu^v(t,p)\right\rangle \nu^v(t,p)\Delta_{g^v_t} u(t,p) + \sum_{i=1}^n c^2_i(v)\,(t,\phi_\alpha(p))\,\frac{\partial u(t)}{\partial x^i}(p) \\ &+ c^2(v)\,(t,\phi_\alpha(p))\,u(t,p)\,,\end{aligned}$$

with second order derivatives of v_α appearing linearly in the terms $c^2(v)$ and $c^2_i(v)$. Hence, we obtain

$$\begin{aligned}&g^{ij,v}(t,p)\Gamma^{l,v}_{ij}(t,p)\left\langle \frac{\partial}{\partial x^l}\frac{\mathrm{d}}{\mathrm{d}\delta}\boldsymbol{H}^{v+\delta u}(t,p)_{|\delta=0}, \nu^v(t,p)\right\rangle \\ =& \left\langle \nu_\varphi(p), \nu^v(t,p)\right\rangle g^{ij,v}(t,p)\Gamma^{l,v}_{ij}(t,p)\frac{\partial}{\partial x^l}\left(\Delta_{g^v_t} u(t,p)\right)\end{aligned}$$

$$+\sum_{i,j} c_{ij}^2(v)\,(t,\phi_\alpha(p))\,\frac{\partial^2 u(t)}{\partial x^i \partial x^j}(p)+\sum_i c_i^3(v)\,(t,\phi_\alpha(p))\,\frac{\partial u(t)}{\partial x^i}(p)+c^3(v)\,(t,\phi_\alpha(p))\,u(t,p)\,.$$

Finally, we observe

$$\begin{aligned}
&\left\langle g^{ij,v}(t,p)\frac{\partial^2}{\partial x^i\partial x^j}\left(\frac{\mathrm{d}}{\mathrm{d}\delta}\boldsymbol{H}^{v+\delta u}(t,p)_{|\delta=0}\right),\nu^v(t,p)\right\rangle\\
=&g^{ij,v}(t,p)\frac{\partial}{\partial x^i}\left\langle\frac{\partial}{\partial x^j}\left(\frac{\mathrm{d}}{\mathrm{d}\delta}\boldsymbol{H}^{v+\delta u}(t,p)_{|\delta=0}\right),\nu^v(t,p)\right\rangle\\
&-g^{ij,v}(t,p)\left\langle\frac{\partial}{\partial x^j}\left(\frac{\mathrm{d}}{\mathrm{d}\delta}\boldsymbol{H}^{v+\delta u}(t,p)_{|\delta=0}\right),\frac{\partial \nu^v(t)}{\partial x^i}(p)\right\rangle\\
=&\langle\nu_\varphi(p),\nu^v(t,p)\rangle\, g^{ij,v}(t,p)\frac{\partial^2}{\partial x^i\partial x^j}\left(\Delta_{g_t^v}u(t)\right)(p)+\sum_{i,j,l=1}^n c_{ijl}^2(v)(t,\phi_\alpha(p))\frac{\partial^3 u(t)}{\partial x^i\partial^j\partial x^l}(p)\\
&+\sum_{i,j=1}^n c_{ij}^3(v)\,(t,\phi_\alpha(p))\,\frac{\partial^2 u(t)}{\partial x^i\partial x^j}(p)+\sum_{i=1}^n c_i^4(v)\,(t,\phi_\alpha(p))\,\frac{\partial u(t)}{\partial x^i}(p)+c^4(v)\,(t,\phi_\alpha(p))\,u(t,p)\,.
\end{aligned}$$

The fourth order coefficient functions $c^4(v)$, $c_i^4(v)$ are of the form (2.9) as second derivatives of v_α appear linearly in the linearisation of the mean curvature vector. □

We now investigate the regularity of the formal expression $\frac{\mathrm{d}}{\mathrm{d}\delta}N_T\,(v+\delta u)\,(t,p)_{|\delta=0}$.

Lemma 2.27 (Gâteaux differentiability of $N_{T,1}$)**.** *Let T_0 be positive, $T\in(0,T_0]$, and $N_{T,1}$ be the operator defined in Proposition 2.8. For all $v\in\boldsymbol{U}_T$ and all $u\in\mathbb{E}_T$ there holds $\partial_u N_{T,1}(v)\in X_T$ where*

$$\partial_u N_{T,1}(v)(t,p):=\frac{\mathrm{d}}{\mathrm{d}\delta}N_{T,1}\,(v+\delta u)\,(t,p)_{\big|\delta=0}$$

for $t\in[0,T]$ and $p\in M$. Here $\frac{\mathrm{d}}{\mathrm{d}\delta}N_{T,1}\,(v+\delta u)\,(t,p)_{\big|\delta=0}$ refers to the formal expression obtained by composing the terms computed in the preceding lemmas. Moreover, for all $v\in\boldsymbol{U}_T$, the map $\mathbb{E}_T\ni u\mapsto\partial_u N_{T,1}(v)\in X_T$ is linear and continuous.

Proof. Let $v\in\boldsymbol{U}_T$ and $u\in\mathbb{E}_T$ be given. To show $\partial_u N_{T,1}(v)\in X_T$, Proposition C.16 implies that it is enough to prove for all $\alpha\in\{1,\dots,N\}$,

$$(t,x)\mapsto\partial_u N_{T,1}(v)\left(t,\phi_\alpha^{-1}(x)\right)=\frac{\mathrm{d}}{\mathrm{d}\delta}N_{T,1}\,(v+\delta u)\left(t,\phi_\alpha^{-1}(x)\right)_{\big|\delta=0}\in X_{T,\alpha}\,.$$

As $v_\alpha(t,x):=v\left(t,\phi_\alpha^{-1}(x)\right)$ lies in $\boldsymbol{U}_{T,\alpha}$, Definition 2.11 implies that all coefficient functions of the form $c^k(v)$ with $k\in\{0,1,2,3\}$ satisfy

$$(t,x)\mapsto c^k(v)\,(t,x)\in C\big([0,T];C\big(\overline{\phi_\alpha(U_\alpha)}\big)\big).$$

Moreover, Lemma 2.5 implies $(t,x)\mapsto\nu^v\left(t,\phi_\alpha^{-1}(x)\right)\in C\big([0,T];C\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$. As $\mathbb{E}_{T,\alpha}$ embeds continuously into $C\big([0,T];C^3\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$, Lemma 2.23 implies

$$(t,x)\mapsto\frac{\mathrm{d}}{\mathrm{d}\delta}\left\langle\partial_t f^{v+\delta u},\nu^{v+\delta u}\right\rangle\left(t,\phi_\alpha^{-1}(x)\right)_{\big|\delta=0}\in X_{T,\alpha}\,.$$

Similarly, Lemma 2.25 implies

$$(t,x)\mapsto\frac{\mathrm{d}}{\mathrm{d}\delta}Q(A)^{v+\delta u}\left(t,\phi_\alpha^{-1}(x)\right)_{\big|\delta=0}\in C\big([0,T];C\big(\overline{\phi_\alpha(U_\alpha)}\big)\big).$$

Using Lemma 2.1, Lemma 2.3, Lemma 2.5, Corollary 2.6 and the expression derived in Lemma 2.24 one observes

$$(t,x) \mapsto \frac{\mathrm{d}}{\mathrm{d}\delta} \left\langle \Delta_{g^{v+\delta u}} f^{v+\delta u}, \nu^{v+\delta u} \right\rangle \left(t, \phi_\alpha^{-1}(x)\right)_{|\delta=0} \in C\big([0,T]; C\big(\overline{\phi_\alpha(U_\alpha)}\big)\big).$$

In the proof of Proposition 2.8 it is shown that

$$(t,x) \mapsto Q(A)^v \left(t, \phi_\alpha^{-1}(x)\right), \qquad (t,x) \mapsto \langle \Delta_{g^v} f^v, \nu^v \rangle \left(t, \phi_\alpha^{-1}(x)\right) \in C\big([0,T]; C\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$$

which allows us to conclude that

$$(t,x) \mapsto \frac{\mathrm{d}}{\mathrm{d}\delta} Q(A)^{v+\delta u} \left(t, \phi_\alpha^{-1}(x)\right) \left\langle \Delta_{g^v} f^{v+\delta u}, \nu^{v+\delta u} \right\rangle \left(t, \phi_\alpha^{-1}(x)\right)_{|\delta=0} \in X_{T,\alpha}\,.$$

As fourth order derivatives of v_α and $u(t) \circ \phi_\alpha^{-1}$ appear only linearly in the formula derived in Lemma 2.26 with coefficients in $C\big([0,T]; C\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$, we obtain

$$(t,x) \mapsto \frac{\mathrm{d}}{\mathrm{d}\delta} \left(\left\langle \Delta^2_{g^{v+\delta u}} f^{v+\delta u}, \nu^{v+\delta u} \right\rangle \left(t, \phi_\alpha^{-1}(x)\right) \right)_{|\delta=0} \in X_{T,\alpha}\,.$$

This shows that $\partial_u N_{T,1}(v)$ lies in X_T. The formulas in local coordinates show that $\partial_u N_{T,1}(v)$ is linear in u. Continuity of $u \mapsto \partial_u N_{T,1}(v) \in X_T$ on $\mathbb{E}_T$ follows from the above reasoning using once more Proposition C.16 and the following estimate for $f \in X_{T,\alpha}$ and $g \in C\big([0,T]; C\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$,

$$\|fg\|_{X_{T,\alpha}} \leq \|f\|_{X_{T,\alpha}} \|g\|_{C\left([0,T]; C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)}\,.$$

□

Lemma 2.28 (Gâteaux differentiability of $N_{T,2}$)**.** *Let T_0 be positive, $T \in (0,T_0]$, and $N_{T,2}$ be the operator defined in Proposition 2.9. For all $v \in \boldsymbol{U}_T$ and all $u \in \mathbb{E}_T$ there holds $\partial_u N_{T,2}(v) \in Y_{1,T}$ where*

$$\partial_u N_{T,2}(v)(t,p) := \frac{\mathrm{d}}{\mathrm{d}\delta} N_{T,2}\left(v + \delta u\right)(t,p)_{|\delta=0}$$

for $t \in [0,T]$ and $p \in \partial M$. Here $\frac{\mathrm{d}}{\mathrm{d}\delta} N_{T,2}\left(v+\delta u\right)(t,p)_{|\delta=0}$ is the formal expression obtained by composing the terms in the preceding lemmas. Moreover, for all $v \in \boldsymbol{U}_T$, the map $\mathbb{E}_T \ni u \mapsto \partial_u N_{T,2}(v) \in Y_{1,T}$ is linear and continuous.

Proof. Let $v \in \boldsymbol{U}_T$ and $u \in \mathbb{E}_T$ be given. To show $\partial_u N_{T,2}(v) \in Y_{1,T}$, Proposition A.15, A.46 and C.16 yield that it is enough to prove

$$(t,x') \mapsto \partial_u N_{T,2}(v)\left(t, \sigma_\alpha^{-1}(x')\right) \in W_p^{1/2-1/4p,\,2-1/p}\left((0,T) \times \sigma_\alpha(V_\alpha)\right).$$

Lemma 2.1 and Lemma 2.5 imply that $(t,x) \mapsto g_{ij}^v\left(t, \phi_\alpha^{-1}(x)\right)$ and $(t,x) \mapsto \nu^v\left(t,\phi_\alpha^{-1}(x)\right)$ lie in $C\big([0,T]; C^2\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$. Theorem C.29 yields

$$(t,x) \mapsto \frac{\partial^2 \left(u(t) \circ \phi_\alpha^{-1}\right)}{\partial x_i \partial x_j}(x)\,, (t,x) \mapsto \frac{\partial^2 \left(v(t) \circ \phi_\alpha^{-1}\right)}{\partial x_i \partial x_j}(x) \in W_p^{1/2-1/4p,\,2-1/p}\left((0,T) \times \partial\phi_\alpha(U_\alpha)\right). \tag{2.10}$$

Furthermore, Proposition B.35 yields that $(t,x) \mapsto u\left(t,\phi_\alpha^{-1}(x)\right)$, $(t,x) \mapsto v\left(t,\phi_\alpha^{-1}(x)\right)$, $(t,x) \mapsto \nabla\left(u(t) \circ \phi_\alpha^{-1}\right)(x)$ and $(t,x) \mapsto \nabla\left(v(t)\circ\phi_\alpha^{-1}\right)(x)$ lie in $C\big([0,T]; C^2\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$.
As the product of functions $f \in C\big([0,T]; C^2\big(\overline{\phi_\alpha\left(U_\alpha\right)}\big)\big)$ and $g \in L_p\big((0,T); W_p^{2-1/p}\big(\partial\phi_\alpha(U_\alpha)\big)\big)$ satisfies $fg \in L_p\big((0,T); W_p^{2-1/p}\big(\partial\phi_\alpha(U_\alpha)\big)\big)$, Lemma 2.24 allows us to conclude that

$$(t,x) \mapsto \partial_u N_{T,2}(v)\left(t,\phi_\alpha^{-1}(x)\right) = \frac{\mathrm{d}}{\mathrm{d}\delta} \left\langle \Delta_{g_t^{v+\delta u}} f^{v+\delta u}(t)\left(\phi_\alpha^{-1}(x)\right), \nu^{v+\delta u}\left(t,\phi_\alpha^{-1}(x)\right) \right\rangle_{|\delta=0}$$

lies in $L_p\big((0,T);W_p^{2-1/p}\big(\partial\phi_\alpha(U_\alpha)\big)\big)$. As $\sigma_\alpha(V_\alpha)\times\{0\}=\partial\phi_\alpha(U_\alpha)\cap\partial\mathbb{H}^n$ and $\sigma_\alpha^{-1}(x')=\phi_\alpha^{-1}(x',0)$ for all $x'\in\sigma_\alpha(V_\alpha)$, Proposition C.23 yields in particular

$$(t,x')\mapsto\partial_u N_{T,2}(v)\left(t,\sigma_\alpha^{-1}(x')\right)\in L_p\big((0,T);W_p^{2-1/p}\big(\sigma_\alpha(V_\alpha)\big)\big).$$

To show that $(t,x')\mapsto\partial_u N_{T,2}(v)\left(t,\sigma_\alpha^{-1}(x')\right)$ lies in $W_p^{1/2-1/4p}\left((0,T);L_p\left(\sigma_\alpha(V_\alpha)\right)\right)$ it is enough to prove

$$(t,x)\mapsto\partial_u N_{T,2}(v)\left(t,\phi_\alpha^{-1}(x)\right)\in W_p^{1/2-1/4p}\left((0,T);L_p\left(\partial\phi_\alpha(U_\alpha)\right)\right)$$

since $\sigma_\alpha\left(V_\alpha\right)\times\{0\}\subset\partial\phi_\alpha\left(U_\alpha\right)$ and $\sigma_\alpha^{-1}(x')=\phi_\alpha^{-1}\left(x',0\right)$ for all $x'\in\sigma_\alpha\left(V_\alpha\right)$. Lemma 2.2 implies that there exists $\tau>\frac{1}{2}-\frac{1}{4p}$ such that for all $i\in\{1,\dots,n\}$, the functions $(t,x)\mapsto u\left(t,\phi_\alpha^{-1}(x)\right)$, $(t,x)\mapsto v\left(t,\phi_\alpha^{-1}(x)\right)$, $(t,x)\mapsto\frac{\partial\left(u(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i}(x)$ and $(t,x)\mapsto\frac{\partial\left(v(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i}(x)$ are elements of the space $C^\tau\big([0,T];C\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$. The precise arguments can be found in the proof of Lemma 2.30 below. According to Proposition B.3 all terms in $\partial_u N_{T,2}(v)$, that contain derivatives of u and v of order at most one, lie in $C^\tau\big([0,T];C\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$. As second order derivatives appear only linearly in $\partial_u N_{T,2}(v)$, the property (2.10) and Proposition C.31 yield

$$(t,x)\mapsto\partial_u N_{T,2}(v)\left(t,\phi_\alpha^{-1}(x)\right)\in W_p^{1/2-1/4p}\left((0,T);L_p\left(\partial\phi_\alpha(U_\alpha)\right)\right).$$

This shows that $\partial_u N_{T,2}(v)$ lies in $Y_{1,T}$. The representation of $\partial_u N_{T,2}(v)$ in a boundary chart (U_α,ϕ_α) is linear in u which implies that the map $\mathbb{E}_T\ni u\mapsto\partial_u N_{T,2}(v)\in Y_{1,T}$ is linear. The continuity of this map follows from the arguments above and Proposition C.16 using the respective estimates in Theorem C.29, Proposition B.35, Lemma 2.2, Proposition C.31 and the Banach algebra property of $C^\tau\big([0,T];C\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$. $\square$

To prove Lipschitz continuity of $v\mapsto DN_T(v)$ with a constant independent of $T\in(0,T_0]$ we consider the norm $|||\cdot|||_{\mathbb{E}_T}$ on $\mathbb{E}_T$. Given a Banach space X and an operator $T\in\mathcal{L}\left(\mathbb{E}_T,X\right)$ we define

$$|||T|||_{\mathcal{L}(\mathbb{E}_T,X)}:=\sup_{u\in\mathbb{E}_T,\,|||u|||_{\mathbb{E}_T}\le1}\|Tx\|_X\ .$$

Lemma 2.29 (Lipschitz continuity of $DN_{T,1}$)**.** *Given $T_0>0$ and $T\in(0,T_0]$, the map*

$$\begin{aligned}DN_{T,1}:\big(\boldsymbol{U}_T,\,|||\cdot|||_{\mathbb{E}_T}\big)&\to\Big(\mathcal{L}\left(\mathbb{E}_T,X_T\right),\,|||\cdot|||_{\mathcal{L}(\mathbb{E}_T,X_T)}\Big)\\ v&\mapsto DN_{T,1}(v)=\left(u\mapsto\partial_u N_{T,1}(v)\right)\end{aligned}$$

is Lipschitz continuous with Lipschitz constant $C(Q,T_0,\varepsilon)$.

Proof. Let $v,w\in\boldsymbol{U}_T$, $u\in\mathbb{E}_T$ and $\alpha\in\{1,\dots,N\}$ be given. Proposition C.16 implies that it is enough to show

$$\left\|(t,x)\mapsto\left(DN_{T,1}(v)(u)-DN_{T,1}(w)(u)\right)\left(t,\phi_\alpha^{-1}(x)\right)\right\|_{X_{T,\alpha}}\le C(T_0)\,|||v_\alpha-w_\alpha|||_{\mathbb{E}_{T,\alpha}}\,|||u_\alpha|||_{\mathbb{E}_{T,\alpha}}\,,$$

where $v_\alpha(t,x):=v\left(t,\phi_\alpha^{-1}(x)\right)$ and analogously for u_α and w_α. All terms that need to be estimated are given in Lemma 2.23– 2.26. To estimate these expressions we observe that for $f\in X_{T,\alpha}$ and $g\in C\big([0,T];C\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$ there holds

$$\|fg\|_{X_{T_\alpha}}\le\|f\|_{X_{T_\alpha}}\|g\|_{C\left([0,T];C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)}\ .$$

Furthermore, Corollary B.38 implies for all functions $f_\alpha\in\mathbb{E}_{T,\alpha}$,

$$\|f_\alpha\|_{C\left([0,T];C^3\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)}\le C(T_0)\,|||f_\alpha|||_{\mathbb{E}_{T,\alpha}}\,.$$

The terms coming from the linearisation of the velocity in Lemma 2.23 are of the shape

$$\begin{aligned}&\partial_t u_\alpha(t,x)\left(c^1(v)-c^1(w)\right)(t,x)\\&+\partial_t\left(v_\alpha-w_\alpha\right)(t,x)\left(\sum_{i=1}^n c_i^1(v)(t,x)\frac{\partial u_\alpha}{\partial x_i}(t,x)+c^1(v)(t,x)u_\alpha(t,x)\right)\\&+\partial_t w_\alpha(t,x)\left(\sum_{i=1}^n\left(c_i^1(v)-c_i^1(w)\right)(t,x)\frac{\partial u_\alpha}{\partial x_i}(t,x)+\left(c^1(v)-c^1(w)\right)(t,x)u_\alpha(t,x)\right).\end{aligned}$$

Using Definition 2.11 these terms can be estimated in $X_{T,\alpha}$ by

$$\begin{aligned}&C(T_0)\left\|\partial_t u_\alpha\right\|_{X_{T_\alpha}}\left|\!\left|\!\left|v_\alpha-w_\alpha\right|\!\right|\!\right|_{\mathbb{E}_{T,\alpha}}+C(T_0)\left\|\partial_t\left(v_\alpha-w_\alpha\right)\right\|_{X_{T,\alpha}}\left|\!\left|\!\left|v_\alpha\right|\!\right|\!\right|_{\mathbb{E}_{T,\alpha}}\left|\!\left|\!\left|u_\alpha\right|\!\right|\!\right|_{\mathbb{E}_{T,\alpha}}\\&+C(T_0)\left\|\partial_t w_\alpha\right\|_{X_{T,\alpha}}\left|\!\left|\!\left|v_\alpha-w_\alpha\right|\!\right|\!\right|_{\mathbb{E}_{T,\alpha}}\left|\!\left|\!\left|u_\alpha\right|\!\right|\!\right|_{\mathbb{E}_{T,\alpha}}\leq C(T_0,Q,\varepsilon)\left|\!\left|\!\left|v_\alpha-w_\alpha\right|\!\right|\!\right|\left|\!\left|\!\left|u_\alpha\right|\!\right|\!\right|_{\mathbb{E}_{T,\alpha}},\end{aligned}$$

where we used that the norms $|\!|\!|v_\alpha|\!|\!|_{\mathbb{E}_{T,\alpha}}$, $|\!|\!|w_\alpha|\!|\!|_{\mathbb{E}_{T,\alpha}}$ are bounded by a constant $C(Q,\varepsilon)$ for all $v,w\in\boldsymbol{U}_T$. Lemma 2.24, 2.25 and 2.26 show that the remaining terms are of the shape

$$\begin{aligned}&\sum_{0\leq|\beta|\leq 4}\left(c_\beta^3(v)-c_\beta^3(w)\right)(t,x)D^\beta u_\alpha(t,x)\\&+\sum_{|\beta|=4}\left(c_\beta^3(v)-c_\beta^3(w)\right)(t,x)D^\beta v_\alpha(t,x)u_\alpha(t,x)\\&+\sum_{|\beta|=4}\sum_{i=1}^n\left(c_{\beta,i}^3(v)-c_{\beta,i}^3(w)\right)(t,x)D^\beta v_\alpha(t,x)\frac{\partial u_\alpha}{\partial x_i}(t,x)\\&+\sum_{|\beta|=4}c_\beta^3(w)(t,x)\left(D^\beta v_\alpha-D^\beta w_\alpha\right)(t,x)u_\alpha(t,x)\\&+\sum_{|\beta|=4}\sum_{i=1}^n c_{\beta,i}^3(w)(t,x)\left(D^\beta v_\alpha-D^\beta w_\alpha\right)(t,x)\frac{\partial u_\alpha}{\partial x_i}(t,x)\,.\end{aligned}$$

Using the arguments stated above and again Definition 2.11 all of these terms can be estimated by $C(Q,T_0,\varepsilon)\,|\!|\!|v_\alpha-w_\alpha|\!|\!|_{\mathbb{E}_{T,\alpha}}\,|\!|\!|u_\alpha|\!|\!|_{\mathbb{E}_{T,\alpha}}$. □

Lemma 2.30 (Lipschitz continuity of $DN_{T,2}$)**.** *Let T_0 be positive and $T\in(0,T_0]$. The map $v\mapsto DN_{T,2}(v)=(u\mapsto\partial_u N_{T,2}(v))\in\mathcal{L}\left(\mathbb{E}_T,Y_{1,T}\right)$ is Lipschitz continuous on $\boldsymbol{U}_T$ with Lipschitz constant $C(Q,T_0,\varepsilon,\sigma)$.*

Proof. We refine the estimates and arguments performed in the proof of Lemma 2.28. We first show for $v,w\in\boldsymbol{U}_T$ the estimate

$$|\!|\!|DN_{T,2}(v)-DN_{T,2}(w)|\!|\!|_{\mathcal{L}\left(\mathbb{E}_T,L_p\left((0,T);W_p^{2-1/p}(\partial M)\right)\right)}\leq C(T_0,Q,\varepsilon,\sigma)\,|\!|\!|v-w|\!|\!|_{\mathbb{E}_T}\,. \tag{2.11}$$

Proposition C.16 and the arguments in the proof of Lemma 2.28 imply that it is enough to show for every boundary chart $(U_\alpha,\phi_\alpha)\in\mathcal{T}$,

$$\begin{aligned}&\left\|(t,x)\mapsto\left(\partial_u N_{T,2}(v)-\partial_u N_{T,2}(w)\right)\left(t,\phi_\alpha^{-1}(x)\right)\right\|_{L_p\left((0,T);W_p^{2-1/p}(\partial\phi_\alpha(U_\alpha))\right)}\\\leq&C(Q,T_0)\,|\!|\!|u|\!|\!|_{\mathbb{E}_T}\,|\!|\!|v-w|\!|\!|_{\mathbb{E}_T}\,.\end{aligned}$$

The proof of Lemma 2.24 implies for $v\in\boldsymbol{U}_T$, $t\in[0,T]$, $x\in\overline{\phi_\alpha\left(U_\alpha\right)}$,

$$\partial_u N_{T,2}(v)\left(t,\phi_\alpha^{-1}(x)\right)$$

$$
\begin{aligned}
=&g^{ij}(v)\left(t,\phi_\alpha^{-1}(x)\right)\left\langle \nu_\varphi\left(\phi_\alpha^{-1}(x)\right),\nu(v)\left(t,\phi_\alpha^{-1}(x)\right)\right\rangle \frac{\partial^2\left(u(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i\partial x_j}(x) &(2.12)\\
&-g^{ij}(v)\left(t,\phi_\alpha^{-1}(x)\right)\left\langle \nu_\varphi\left(\phi_\alpha^{-1}(x)\right),\nu(v)\left(t,\phi_\alpha^{-1}(x)\right)\right\rangle \Gamma_{ij}^{l,v}\left(t,\phi_\alpha^{-1}(x)\right)\frac{\partial\left(u(t)\circ\phi_\alpha^{-1}\right)}{\partial x_l}(x) &(2.13)\\
&+\frac{\mathrm{d}}{\mathrm{d}\delta}g^{ij,v+\delta u}\left(t,\phi_\alpha^{-1}(x)\right)_{|\delta=0}\,a_{ij}^v\left(t,\phi_\alpha^{-1}(x)\right) &(2.14)\\
&+\left\langle \Delta_{g_t^v}f^v(t)\left(\phi_\alpha^{-1}(x)\right),\frac{\mathrm{d}}{\mathrm{d}\delta}\nu^{v+\delta u}\left(t,\phi_\alpha^{-1}(x)\right)_{|\delta=0}\right\rangle . &(2.15)
\end{aligned}
$$

Moreover, we recall that given $f\in C\big([0,T];C^2\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$, $g\in L_p\big((0,T);W_p^{2-1/p}\big(\partial\phi_\alpha(U_\alpha)\big)\big)$, there holds $fg\in L_p\big((0,T);W_p^{2-1/p}\big(\partial\phi_\alpha(U_\alpha)\big)\big)$ with

$$
\|fg\|_{L_p\left((0,T);W_p^{2-1/p}(\partial\phi_\alpha(U_\alpha))\right)}\le \|f\|_{C\left([0,T];C^2\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)}\|g\|_{L_p\left((0,T);W_p^{2-1/p}(\partial\phi_\alpha(U_\alpha))\right)} .
$$

Furthermore, Theorem C.30 yields for all $\varrho\in\mathbb{E}_T$ the estimate

$$
\left\|(t,x)\mapsto\frac{\partial^2\left(\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i\partial x_j}(x)\right\|_{L_p\left((0,T);W_p^{2-1/p}(\partial\phi_\alpha(U_\alpha))\right)}\le C(Q,T_0)\,|||\varrho|||_{\mathbb{E}_T} .
$$

Given v, $w\in\boldsymbol{U}_T$ we estimate the terms in the difference $\partial_u N_{T,2}(v)-\partial_u N_{T,2}(w)$ in the space $Z_T^\alpha:=L_p\big((0,T);W_p^{2-1/p}\big(\partial\phi_\alpha(U_\alpha)\big)\big)$. Lemma 2.3 and Lemma 2.5 imply that the expression corresponding to (2.12) can be estimated by

$$
\begin{aligned}
&\Big(\left\|\left(g^{ij}(v)-g^{ij}(w)\right)\left\langle\left(\nu_\varphi\circ\phi_\alpha^{-1}\right),\nu(v)\right\rangle\right\|_{C\left([0,T];C^2\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)}\\
&\quad+\left\|g^{ij}(w)\left\langle\left(\nu_\varphi\circ\phi_\alpha^{-1}\right),\nu(v)-\nu(w)\right\rangle\right\|_{C\left([0,T];C^2\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)}\Big)\left\|\frac{\partial^2\left(u\circ\phi_\alpha^{-1}\right)}{\partial x_i\partial x_j}\right\|_{Z_T^\alpha}\\
&\le C(Q,R,T_0)\,|||v-w|||_{\mathbb{E}_T}\left(|||v|||_{\mathbb{E}_T}+|||w|||_{\mathbb{E}_T}\right)|||u|||_{\mathbb{E}_T}\le C(Q,T_0,\varepsilon)\,|||v-w|||_{\mathbb{E}_T}\,|||u|||_{\mathbb{E}_T} .
\end{aligned}
$$

As for $v\in\boldsymbol{U}_T$,

$$
\Gamma_{ij}^{l,v}\left(t,\phi_\alpha^{-1}(x)\right)=g^{lm,v}\left(t,\phi_\alpha^{-1}(x)\right)\left\langle\frac{\partial^2\left(f^v(t)\circ\phi_\alpha^{-1}\right)(x)}{\partial x_i\partial x_j}(x),\frac{\partial\left(f^v(t)\circ\phi_\alpha^{-1}\right)}{\partial x_m}(x)\right\rangle ,
$$

and further,

$$
a_{ij}^v\left(t,\phi_\alpha^{-1}(x)\right)=\left\langle\frac{\partial^2\left(f^v(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i\partial x_j}(x),\nu^v\left(t,\phi_\alpha^{-1}(x)\right)\right\rangle ,
$$

Lemma 2.3 and 2.5 imply that also the terms corresponding to (2.13) and (2.14) are estimated by

$$
C(Q,T_0,\varepsilon)\,|||v-w|||_{\mathbb{E}_T}\,|||u|||_{\mathbb{E}_T} .
$$

Finally, Lemma 2.22 implies that the term corresponding to (2.15) is of the form

$$
\begin{aligned}
&\sum_{i,j,k=1}^n\left(c_{ijk}^1(v)(t,x)\frac{\partial^2\left(v(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i\partial x_j}(x)-c_{ijk}^1(w)(t,x)\frac{\partial^2\left(w(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i\partial x_j}(x)\right)\frac{\partial\left(u(t)\circ\phi_\alpha^{-1}\right)}{\partial x_k}(x)\\
&+\sum_{i,j=1}^n\left(c_{ij}^1(v)(t,x)\frac{\partial^2\left(v(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i\partial x_j}(x)-c_{ij}^1(w)(t,x)\frac{\partial^2\left(w(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i\partial x_j}(x)\right)u\left(t,\phi_\alpha^{-1}(x)\right) ,
\end{aligned}
\tag{2.16}
$$

where the functions $c_{ijk}^1, c_{ij}^1:\boldsymbol{U}_T\to C\big([0,T];C^2\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$ are Lipschitz continuous with constant $C(Q,T_0,\varepsilon,\sigma)$. The arguments above yield that the expression (2.16) can be estimated in

$L_p\big((0,T);W_p^{2-1/p}\big(\partial\phi_\alpha\left(U_\alpha\right)\big)\big)$ by $C(Q,T_0,\varepsilon,\sigma)\,\|\!|v-w|\!\|_{\mathbb{E}_T}\,\|\!|u|\!\|_{\mathbb{E}_T}$. This implies the validity of estimate (2.11). To show for $v,w\in \boldsymbol{U}_T$ the estimate

$$\|\!|DN_{T,2}(v)-DN_{T,2}(w)|\!\|_{\mathcal{L}\left(\mathbb{E}_T,W_p^{1/2-1/4p}((0,T);L_p(\partial M))\right)}\le C(Q,T_0,\varepsilon,\sigma)\,\|\!|v-w|\!\|_{\mathbb{E}_T}\,,$$

the arguments performed in the proof of Lemma 2.28 imply that it is enough to show for all $u\in\mathbb{E}_T$,

$$\begin{aligned}&\left\|(t,x)\mapsto\left(\partial_u N_{T,2}(v)-\partial_u N_{T,2}(w)\right)\left(t,\phi_\alpha^{-1}(x)\right)\right\|_{W_p^{1/2-1/4p}((0,T);L_p(\partial\phi_\alpha(U_\alpha)))}\\ &\le C(Q,T_0,\varepsilon,\sigma)\,\|\!|u|\!\|_{\mathbb{E}_T}\,\|\!|v-w|\!\|_{\mathbb{E}_T}\,.\end{aligned}$$

To prove this we observe that Theorem C.30 implies for all $\varrho\in\mathbb{E}_T$,

$$(t,x)\mapsto\frac{\partial^2\left(\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i\partial x_j}(x)\in W_p^{1/2-1/4p}\left((0,T);L_p\left(\partial\phi_\alpha\left(U_\alpha\right)\right)\right)$$

with norm bounded by $C(Q,T_0)\,\|\!|\varrho|\!\|_{\mathbb{E}_T}$. Moreover, Lemma 2.2 implies that there exists a constant $\tau>\frac{1}{2}-\frac{1}{4p}$ such that for all $i\in\{1,\dots,n\}$, $\varrho\in\mathbb{E}_T$, the functions $(t,x)\mapsto\varrho\left(t,\phi_\alpha^{-1}(x)\right)$, $(t,x)\mapsto\frac{\partial\left(\varrho(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i}(x)$ are elements of the space $C^\tau\big([0,T];C\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$ with norm bounded by $C(Q,T_0)\,\|\!|\varrho|\!\|_{\mathbb{E}_T}$. As second order derivatives of u and v appear only linearly in $\partial_u N_{T,2}(v)$, Proposition B.3 and C.31 imply that it is enough to show that for $u\in\mathbb{E}_T$, and $v\in\boldsymbol{U}_T$, the expression $\partial_u N_{T,2}(v)\left(t,\phi_\alpha^{-1}(x)\right)$ is of the form

$$\begin{aligned}\partial_u N_{T,2}(v)\left(t,\phi_\alpha^{-1}(x)\right)=&f^{ij}(v)\left(t,\phi_\alpha^{-1}(x)\right)\frac{\partial^2\left(u(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i\partial x_j}(x)\\ &-h^{k,ij}(v)\left(t,\phi_\alpha^{-1}(x)\right)\frac{\partial^2\left(v(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i\partial x_j}(x)\frac{\partial\left(u(t)\circ\phi_\alpha^{-1}\right)}{\partial x_k}(x)\\ &-h^{ij}(v)\left(t,\phi_\alpha^{-1}(x)\right)\frac{\partial^2\left(v(t)\circ\phi_\alpha^{-1}\right)}{\partial x_i\partial x_j}(x)\,u\left(t,\phi_\alpha^{-1}(x)\right)\end{aligned}$$

with $f^{ij},h^{k,ij},h^{ij}:\left(\boldsymbol{U}_T,\|\!|\cdot|\!\|_{\mathbb{E}_T}\right)\to C^\tau\big([0,T];C\big(\overline{\phi_\alpha\left(U_\alpha\right)}\big)\big)$ Lipschitz continuous. This follows from the formulas (2.12)– (2.15), Proposition B.3 and B.4 and Lemma 2.4 and 2.7. ☐

Proof of Proposition 2.10. The statement in Proposition 2.10 follows combining Lemma 2.27, Lemma 2.28, Lemma 2.29 and Lemma 2.30. ☐

2.3 Well-posedness of the linearised graph formulation

In Section 2.4 we show existence of solutions to the graph formulation with the help of a contraction argument and well-posedness of an associated linearised problem. To this end we consider the full linearisation of the operator $N_T=(N_{T,1},N_{T,2})$ defined in Proposition 2.10 around the reference geometry $\varphi:M\to\mathbb{R}^{n+1}$ which corresponds to

$$DN_T(0)=\left(DN_{T,1}(0),DN_{T,2}(0)\right)\in\mathcal{L}\left(\mathbb{E}_T,X_T\times Y_{1,T}\right).$$

In the following we show that this leads to a system of the following form

$$\begin{cases}\left(\partial_t+A\right)u=\partial_t u+\Delta_{g_\varphi}^2 u+\text{ lower order terms in }u&=f\qquad\text{in }[0,T]\times M\,,\\ \qquad B_1 u=\Delta_{g_\varphi}u+\text{ lower order terms in }u&=h_1\qquad\text{on }[0,T]\times\partial M\,,\\ \qquad B_2 u=u&=h_2\qquad\text{on }[0,T]\times\partial M\,,\\ \qquad u&=u_0\qquad\text{on }\{0\}\times M\,.\end{cases}$$

2.3.1 Preliminaries on the linearised operators

In this subsection we collect some important properties of the operators appearing in the linearised graph formulation.

Definition 2.31 (Linearised operators)**.** Let T_0 be positive and $T \in (0, T_0]$. We define $A \in \mathcal{L}(\mathbb{E}_T, X_T)$ and $B_1 \in \mathcal{L}(\mathbb{E}_T, Y_{1,T})$ by

$$\begin{aligned} A &:= DN_{T,1}(0) - \partial_t\,, \\ B_1 &:= DN_{T,2}(0)\,. \end{aligned}$$

Corollary 2.32 (Linearised interior operator in local coordinates)**.** *Given $T_0 > 0$, $T \in (0, T_0]$ and $\alpha \in \{1, \ldots, N\}$ there exist smooth functions*

$$c_{\beta,\alpha} \circ \phi_\alpha^{-1}\,, d_{\beta,\alpha} \circ \phi_\alpha^{-1} \in C^\infty\big(\overline{\phi_\alpha(U_\alpha)}\big)$$

such that for all $u \in \mathbb{E}_T$ and almost every $t \in (0, T)$, $x \in \phi_\alpha(U_\alpha)$,

$$\begin{aligned} (Au)\left(t, \phi_\alpha^{-1}(x)\right) &= \sum_{|\beta| \leq 4} c_{\beta,\alpha}\left(\phi_\alpha^{-1}(x)\right) D^\beta \left(u(t) \circ \phi_\alpha^{-1}\right)(x) \\ &= \Delta_{g_\varphi}^2 u\left(t, \phi_\alpha^{-1}(x)\right) + \sum_{|\beta| \leq 3} d_{\beta,\alpha}\left(\phi_\alpha^{-1}(x)\right) D^\beta \left(u(t) \circ \phi_\alpha^{-1}\right)(x) \end{aligned}$$

where $D := i\left(\partial_{x_1}, \ldots, \partial_{x_n}\right)^T$.

Proof. This follows from Lemma 2.24– 2.26. □

Corollary 2.33 (Linearised boundary operator in local coordinates)**.** *Let T_0 be positive, $T \in (0, T_0]$, $\alpha \in \{1, \ldots, N\}$ and $c_{i,\alpha}^2 := c_i^2(0)$, $c_\alpha^2 := c^2(0)$, $i \in \{1, \ldots, n\}$, be the functions appearing in Lemma 2.24. Then for all $u \in \mathbb{E}_T$, $x \in \phi_\alpha(U_\alpha) \cap \partial\mathbb{H}^n$ and $t \in [0, T]$ it holds*

$$(B_1 u)\left(t, \phi_\alpha^{-1}(x)\right) = \Delta_{g_\varphi} u(t)\left(\phi_\alpha^{-1}(x)\right) + \sum_{i=1}^n c_{i,\alpha}^2(x) \frac{\partial u(t)}{\partial x^i}\left(\phi_\alpha^{-1}(x)\right) + c_\alpha^2(x) u\left(t, \phi_\alpha^{-1}(x)\right).$$

In particular, there exist coefficient functions

$$b_{\gamma,\alpha} \circ \phi_\alpha^{-1} \in C^\infty\big(\overline{\phi_\alpha(U_\alpha)}\big)$$

such that for $x \in \partial\phi_\alpha(U_\alpha) \cap \partial\mathbb{H}^n$ and $t \in [0, T]$,

$$(B_1 u)\left(t, \phi_\alpha^{-1}(x)\right) = \sum_{|\gamma| \leq 2} b_{\gamma,\alpha}\left(\phi_\alpha^{-1}(x)\right) D^\gamma \big(u(t) \circ \phi_\alpha^{-1}\big)(x)$$

where $D := i\left(\partial_{x_1}, \ldots, \partial_{x_n}\right)^T$. Given $u \in \mathbb{E}_T$, $\alpha, \beta \in \{1, \ldots, N\}$, $q \in U_\alpha \cap U_\beta \cap \partial M$ and $t \in [0, T]$, then it holds

$$(B_1 u)(t, q) = \sum_{|\gamma| \leq 2} b_{\gamma,\alpha}(q) D^\gamma \big(u(t) \circ \phi_\alpha^{-1}\big)(\phi_\alpha(q)) = \sum_{|\gamma| \leq 2} b_{\gamma,\beta}(q) D^\gamma \big(u(t) \circ \phi_\beta^{-1}\big)(\phi_\beta(q))\,.$$

Proof. This is a direct consequence of Proposition 2.9 and Lemma 2.24 using that the mean curvature is independent of the choice of coordinates. □

Definition 2.34. Let T_0 be positive and $T \in (0, T_0]$. For $\alpha \in \{1, \dots, N\}$ let $c_{\beta,\alpha}$ be the coefficients in Corollary 2.32. We define a differential operator on the interior of $\phi_\alpha(U_\alpha)$ via

$$\mathcal{A}_\alpha(x, D) := \sum_{|\beta| \le 4} c_{\beta,\alpha}\left(\phi_\alpha^{-1}(x)\right) D^\beta$$

where $D := i\left(\partial_{x_1}, \dots, \partial_{x_n}\right)^T$.

Proposition 2.35. *Given $\alpha \in \{1, \dots, N\}$ and*

$$\eta_\alpha \in \mathbb{E}_{T,\alpha} := W_p^1\left((0,T); L_p\left(\phi_\alpha(U_\alpha)\right)\right) \cap L_p\left((0,T); W_p^4\left(\phi_\alpha(U_\alpha)\right)\right),$$

the function

$$\left(A_\alpha \eta_\alpha\right)(t, x) := \mathcal{A}_\alpha(x, D)\eta_\alpha(t, x)$$

lies in $X_{T,\alpha}$ and the hereby induced operator $A_\alpha : \mathbb{E}_{T,\alpha} \to X_{T,\alpha}$ is linear and continuous.

Proof. For almost every $t \in (0, T)$ it holds $\eta_\alpha(t) \in W_p^4\left(\phi_\alpha(U_\alpha)\right)$. As the coefficients g_α^{ij} are uniformly bounded on $\phi_\alpha(U_\alpha)$, we obtain for almost every $t \in (0, T)$,

$$x \mapsto \mathcal{A}_\alpha(x, D)\eta_\alpha(t, x) = (A_\alpha \eta_\alpha)(t, x) \in L_p\left(\phi_\alpha(U_\alpha)\right).$$

To show that $A_\alpha \eta_\alpha : (0, T) \to L_p\left(\phi_\alpha(U_\alpha)\right)$ is strongly measurable, we consider a sequence of simple functions $f_n : (0, T) \to W_p^4\left(\phi_\alpha(U_\alpha)\right)$ such that for almost every $t \in (0, T)$ it holds $\|f_n(t) - \eta_\alpha(t)\|_{W_p^4(\phi_\alpha(U_\alpha))} \to 0$ as $n \to \infty$. Then we observe that

$$\|(A_\alpha f_n)(t) - (A_\alpha \eta_\alpha)(t)\|^p_{L_p(\phi_\alpha(U_\alpha))} \le C(Q)\, \|f_n(t) - \eta_\alpha(t)\|^p_{W_p^4(\phi_\alpha(U_\alpha))} \to 0$$

as $n \to \infty$. As further

$$\int_0^T \|(A_\alpha \eta_\alpha)(t)\|^p_{L_p(\phi_\alpha(U_\alpha))}\, \mathrm{d}t \le C(Q) \int_0^T \|\eta_\alpha(t)\|^p_{W_p^4(\phi_\alpha(U_\alpha))}\, \mathrm{d}t \le C(Q)\, \|\eta_\alpha\|_{\mathbb{E}_{T,\alpha}},$$

we obtain $A_\alpha \eta_\alpha \in X_{T,\alpha}$. Linearity of $A_\alpha : \mathbb{E}_{T,\alpha} \to X_{T,\alpha}$ follows immediately from linearity of $\mathcal{A}_\alpha$ and the above estimate implies that A_α is continuous. □

Definition 2.36. Let T_0 be positive, $T \in (0, T_0]$, $\alpha \in \{1, \dots, N\}$ and $b_{\gamma,\alpha}$ be the coefficients in Corollary 2.33. We define a differential operator on $\phi_\alpha(U_\alpha)$ via

$$\mathcal{B}_{1,\alpha}(x, D) := \sum_{|\gamma| \le 2} b_{\gamma,\alpha}\left(\phi_\alpha^{-1}(x)\right) D^\gamma$$

where $D := i\left(\partial_{x_1}, \dots, \partial_{x_n}\right)^T$.

Corollary 2.37. *Given $T_0 > 0$, $T \in (0, T_0]$, $u \in \mathbb{E}_T$, $\alpha, \beta \in \{1, \dots, N\}$, $q \in U_\alpha \cap U_\beta \cap \partial M$ and $t \in [0, T]$, there holds the identity*

$$\left(B_1 u\right)(t, q) = \mathcal{B}_{1,\alpha}\left(\phi_\alpha(q), D\right)\left(u(t) \circ \phi_\alpha^{-1}\right)(\phi_\alpha(q)) = \mathcal{B}_{1,\beta}\left(\phi_\beta(q), D\right)\left(u(t) \circ \phi_\beta^{-1}\right)(\phi_\beta(q))$$

and in particular it holds for any $q \in \partial M$, $t \in [0, T]$,

$$\left(B_1 u\right)(t, q) = \sum_{\alpha=1}^N \psi_\alpha(q)\mathcal{B}_{1,\alpha}\left(\phi_\alpha(q), D\right)\left(u(t) \circ \phi_\alpha^{-1}\right)\left(\phi_\alpha(q)\right).$$

Proof. This is a direct consequence of Corollary 2.33. □

Proposition 2.38. *Given $\alpha \in \{1, \dots, N\}$, $T_0 > 0$, $T \in (0, T_0]$, and $\eta_\alpha \in \mathbb{E}_{T,\alpha}$ the function*

$$(B_{1,\alpha}\eta_\alpha)(t,x) := \mathcal{B}_{1,\alpha}(x,D)\eta_\alpha(t,x)\,, \qquad t \in (0,T)\,,\ x \in \partial\phi_\alpha(U_\alpha)\,,$$

lies in $Y_{1,T,\alpha} := W_p^{1/2-1/4p, 2-1/p}((0,T) \times \partial\phi_\alpha(U_\alpha))$ and the hereby induced operator $B_{1,\alpha} : \mathbb{E}_{T,\alpha} \to Y_{1,T,\alpha}$ is linear and continuous.

Proof. Let $\eta_\alpha \in \mathbb{E}_{T,\alpha}$ be given. As $\mathrm{int}\phi_\alpha(U_\alpha)$ is a smooth domain in $\mathbb{R}^n$ with boundary given by $\partial(\mathrm{int}\phi_\alpha(U_\alpha)) = \partial\phi_\alpha(U_\alpha)$, Theorem C.30 implies for all $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \le 2$,

$$\partial^\gamma \eta_\alpha \in Y_{1,T,\alpha}$$

with $\|\partial^\gamma\eta_\alpha\|_{Y_{1,T,\alpha}} \le C(T_0)\,|||\eta_\alpha|||_{\mathbb{E}_{T,\alpha}}$. As the coefficient functions are time-independent and smooth, one readily checks that $B_{1,\alpha}\eta_\alpha$ lies in $Y_{1,T,\alpha}$ and $\|B_{1,\alpha}\eta_\alpha\|_{Y_{1,T,\alpha}} \le C(T_0)\,|||\eta_\alpha|||_{\mathbb{E}_{T,\alpha}}$ with constant $C(T_0)$ depending on the coefficients that are expressions involving the geometry of (M,φ). In particular, $B_{1,\alpha} : \mathbb{E}_{T,\alpha} \to Y_{1,T,\alpha}$ is a continuous linear operator. □

Proposition 2.39. *Given $T_0 > 0$, $T \in (0, T_0]$, $\alpha \in \{1, \dots, N\}$ and $\eta_\alpha \in \mathbb{E}_{T,\alpha}$ the function*

$$(B_{2,\alpha}\eta_\alpha)(t,x) := \eta_\alpha(t,x)\,, \qquad t \in (0,T)\,,\ x \in \partial\phi_\alpha(U_\alpha)\,,$$

lies in $Y_{2,T,\alpha} := W_p^{1-1/4p, 4-1/p}((0,T) \times \partial\phi_\alpha(U_\alpha))$ and the hereby induced operator $B_{2,\alpha} : \mathbb{E}_{T,\alpha} \to Y_{2,T,\alpha}$ is linear and continuous.

Proof. Let $\eta_\alpha \in \mathbb{E}_{T,\alpha}$ be given. Theorem C.30 implies

$$B_{2,\alpha}\eta_\alpha = (\eta_\alpha)_{|\partial\Omega} \in Y_{2,T,\alpha}$$

with $\|\eta_\alpha\|_{Y_{2,T,\alpha}} \le C(T_0)\,|||\eta_\alpha|||_{\mathbb{E}_{T,\alpha}}$. In particular, $B_{2,\alpha} : \mathbb{E}_{T,\alpha} \to Y_{2,T,\alpha}$ is a continuous linear operator. □

Proposition 2.40. *Given $T_0 > 0$, $T \in (0, T_0]$ and $u \in \mathbb{E}_T$, the expression $B_2 u$ defined by $B_2 u(t,q) := u(t,q)$ for $t \in (0,T)$, $q \in \partial M$, lies in $Y_{2,T} := W_p^{1-1/4p, 4-1/p}((0,T) \times \partial M)$ and the hereby induced operator $B_2 : \mathbb{E}_T \to Y_{2,T}$ is linear and continuous.*

Proof. Let $u \in \mathbb{E}_T$ be given. To show $B_2 u \in Y_{2,T}$ Proposition C.16 implies that it is enough to check for all $\alpha \in \mathcal{J}$,

$$(t,x') \mapsto (B_2 u)\left(t, \sigma_\alpha^{-1}(x')\right) \in W_p^{1-1/4p, 4-1/p}((0,T) \times \sigma_\alpha(V_\alpha))\,. \tag{2.18}$$

The function $(t,x) \mapsto u_\alpha(t,x) := u\left(t, \phi_\alpha^{-1}(x)\right)$ lies in $\mathbb{E}_{T,\alpha}$ with $\|u_\alpha\|_{\mathbb{E}_{T,\alpha}} \le C(Q)\,\|u\|_{\mathbb{E}_T}$. Using the results in Proposition 2.39, we obtain $B_{2,\alpha}u_\alpha \in Y_{2,T,\alpha}$ with the estimate $\|B_{2,\alpha}u_\alpha\|_{Y_{2,T,\alpha}} \le C(T_0)\,|||u_\alpha|||_{\mathbb{E}_{T,\alpha}}$. Observing that $\partial\phi_\alpha(U_\alpha) \cap \partial\mathbb{H}^n = \sigma_\alpha(V_\alpha) \times \{0\}$ and $\sigma_\alpha^{-1}(x') = \phi_\alpha^{-1}(x', 0)$ for all $x' \in \sigma_\alpha(V_\alpha)$, Proposition C.23 implies (2.18) as for $t \in [0,T]$ and $x' \in \sigma_\alpha(V_\alpha)$,

$$(B_2 u)\left(t, \sigma_\alpha^{-1}(x')\right) = B_{2,\alpha}u_\alpha(t, (x',0))\,.$$

As $(V_\alpha, \sigma_\alpha)$, $\alpha \in \mathcal{J}$, is a normal covering of ∂M, we obtain

$$\begin{aligned}
\|B_2 u\|_{Y_{2,T}} &\le C(Q) \sum_{\alpha \in \mathcal{J}} \left\|(t,x') \mapsto (B_2 u)\left(t, \sigma_\alpha^{-1}(x')\right)\right\|_{p, 1-1/4p, 4-1/p} \\
&\le C(Q) \sum_{\alpha \in \mathcal{J}} \|B_{2,\alpha}u_\alpha\|_{Y_{2,T,\alpha}} \le C(Q,T_0) \sum_{\alpha \in \mathcal{J}} |||u_\alpha|||_{\mathbb{E}_{T,\alpha}} \le C(Q,T_0)\,|||u|||_{\mathbb{E}_T}\,.
\end{aligned}$$

The mapping B_2 is obviously linear on $\mathbb{E}_T$. This completes the proof. □

Proposition 2.41 (The linear compatibility operator)**.** *Let T_0 be positive and $T \in (0, T_0]$. The operator*

$$\begin{aligned}
&\mathcal{G}_T : X_T \times Y_{1,T} \times Y_{2,T} \times X_0 \to W_p^{2-5/p}(\partial M) \times W_p^{4-5/p}(\partial M)\,,\\
&\qquad \mathcal{G}_T^1\left(f, h_1, h_2, u_0\right)(q) := \sum_{\alpha=1}^{N} \psi_\alpha(q)\mathcal{B}_{1,\alpha}\left(\phi_\alpha(q), D\right)\left(u_0 \circ \phi_\alpha^{-1}\right)\left(\phi_\alpha(q)\right) - h_1(0,q)\,,\\
&\qquad \mathcal{G}_T^2\left(f, h_1, h_2, u_0\right)(q) := u_0(q) - h_2(0,q)
\end{aligned}$$

is linear and continuous. In particular, $\mathbb{F}_T := \ker \mathcal{G}_T$ is a closed linear subspace of the space $X_T \times Y_{1,T} \times Y_{2,T} \times X_0$.

Proof. Let $(h_1, h_2, u_0) \in Y_{1,T} \times Y_{2,T} \times X_0$ be given. Proposition C.16 implies

$$(t, x') \mapsto h_1\left(t, \sigma_\alpha^{-1}(x')\right) \in W_p^{1/2-1/4p, 2-1/p}\left((0,T) \times \sigma_\alpha(V_\alpha)\right)$$

and Proposition B.35 yields $h_1(0) \circ \sigma_\alpha^{-1} \in W_p^{2-5/p}\left(\sigma_\alpha(V_\alpha)\right)$. This yields $h_1(0) \in W_p^{2-5/p}(\partial M)$. Analogously, one obtains $h_2(0) \in W_p^{4-5/p}(\partial M)$. Theorem C.27 implies for all $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq 2$,

$$\left(\partial_x^\beta\left(u_0 \circ \phi_\alpha^{-1}\right)\right)_{|\partial\phi_\alpha(U_\alpha)} \in W_p^{4-|\beta|-5/p}\left(\partial\phi_\alpha(U_\alpha)\right). \tag{2.19}$$

From this we immediately obtain $u_0 \circ \sigma_\alpha^{-1} \in W_p^{4-5/p}\left(\sigma_\alpha(V_\alpha)\right)$ which yields that $\mathcal{G}_T^2$ is well-defined. As the coefficients $b_{\gamma,\alpha} \circ \phi_\alpha^{-1}$ are smooth on $\overline{\phi_\alpha\left(U_\alpha\right)}$, we conclude that

$$x' \mapsto \psi_\alpha\left(\phi_\alpha^{-1}(x', 0)\right)\mathcal{B}_{1,\alpha}((x', 0), D)\left(u_0 \circ \phi_\alpha^{-1}\right)((x', 0))$$

lies in $W_p^{2-5/p}\left(\sigma_\alpha\left(V_\alpha\right)\right)$. Lemma C.14 applied to the manifold ∂M with localisation system $(V_\alpha, \sigma_\alpha, \psi_\alpha \circ \iota)$ yields

$$q \mapsto \psi_\alpha(q)\mathcal{B}_{1,\alpha}\left(\phi_\alpha(q), D\right)\left(u_0 \circ \phi_\alpha^{-1}\right)\left(\phi_\alpha(q)\right) \in W_p^{2-5/p}\left(\partial M\right).$$

This shows that $\mathcal{G}_T$ is well-defined and linear. The continuity follows from the estimates shown in Proposition C.16, Proposition B.35, Theorem C.27, the arguments in Lemma C.14, Proposition C.1 and Lemma C.13. □

Definition 2.42 (Linear compatibility conditions)**.** Let T_0 be positive and $T \in (0, T_0]$. Given an initial datum $u_0 \in X_0$ we say that a tuple $(h_1, h_2) \in Y_{1,T} \times Y_{2,T}$ satisfies the *linear compatibility conditions with respect to* u_0 if for all $f \in X_T$ it holds $(f, h_1, h_2, u_0) \in \mathbb{F}_T$, that is, for all $q \in \partial M$,

$$\sum_{\alpha=1}^{N} \psi_\alpha(q)\mathcal{B}_{1,\alpha}\left(\phi_\alpha(q), D\right)\left(u_0 \circ \phi_\alpha^{-1}\right)\left(\phi_\alpha(q)\right) = h_1(0,q)\,, \tag{2.20}$$

$$u_0(q) = h_2(0,q)\,. \tag{2.21}$$

In particular, we have

$$\mathbb{F}_T = \{\ (f, h_1, h_2, u_0) \in X_T \times Y_{1,T} \times Y_{2,T} \times X_0 : (h_1, h_2) \text{ satisfies the linear compatibility conditions with respect to } u_0\}\ .$$

Proposition 2.43 (The linear operator)**.** *Let T_0 be positive and $T \in (0, T_0]$. The operator*

$$L_T : \mathbb{E}_T \to \mathbb{F}_T, \qquad L_T u := \begin{pmatrix} \partial_t + A \\ B_1 \\ B_2 \\ |_{t=0} \end{pmatrix} u\,.$$

is well-defined, linear and continuous.

Proof. Proposition C.11 implies that to show $u_{|t=0} \in X_0$ it is enough to prove

$$x \mapsto u(0, \phi_\alpha^{-1}(x)) \in W_p^{4-4/p}(\phi_\alpha(U_\alpha)) = X_{0,\alpha}\,.$$

By Proposition C.16, $(t,x) \mapsto u_\alpha(t,x) := u\left(t, \phi_\alpha^{-1}(x)\right)$ lies in $\mathbb{E}_{T,\alpha} = W_p^{1,4}((0,T) \times \phi_\alpha(U_\alpha))$ and Corollary B.38 implies $x \mapsto u_\alpha(0,x) \in X_{0,\alpha}$ with

$$\|x \mapsto u_\alpha(0,x)\|_{X_{0,\alpha}} \leq C(T_0)\, \|\!|u|\!\|_{\mathbb{E}_{T,\alpha}}\,.$$

In particular, $u_{|t=0}$ lies in X_0 and the norm estimates in Proposition C.16 and C.11 imply

$$\left\|u_{|t=0}\right\|_{X_0} \leq C(Q) \sum_{\alpha=1}^{N} \|(u_\alpha)(0)\|_{X_{0,\alpha}} \leq C(Q,T_0) \sum_{\alpha=1}^{N} \|\!|u_\alpha|\!\|_{\mathbb{E}_{T,\alpha}} \leq C(Q,T_0)\, \|\!|u|\!\|_{\mathbb{E}_T}\,.$$

Combining these considerations with the results in Lemma 2.27, Lemma 2.28 and Proposition 2.40, we conclude that $L_T : \mathbb{E}_T \to X_T \times Y_{1,T} \times Y_{2,T} \times X_0$ is well-defined, linear and continuous. It remains to show that $(B_1 u, B_2 u)$ satisfies the linear compatibility conditions with respect to $u_{|t=0}$. As $\mathbb{E}_{T,\alpha}$ embeds continuously into $C\big([0,T]; C^3\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$ by Proposition B.35, spacial differentiation and temporal evaluation of u_α can be interchanged. Corollary 2.37 then yields for any $q \in \partial M$,

$$\begin{aligned}
(B_1 u)(0,q) &= \sum_{\alpha=1}^{N} \psi_\alpha(q) \mathcal{B}_{1,\alpha}(\phi_\alpha(q), D)\, u_\alpha(0)(\phi_\alpha(q)) \\
&= \sum_{\alpha=1}^{N} \psi_\alpha(q) \mathcal{B}_{1,\alpha}(\phi_\alpha(q), D) \left(u_{|t=0} \circ \phi_\alpha^{-1}\right)(\phi_\alpha(q))
\end{aligned}$$

which shows $\mathcal{G}_T^1(L_T(u)) = 0$. The identity (2.21) is straightforward. □

2.3.2 Existence and uniqueness in the chart domains

To prove that the operator $L_T : \mathbb{E}_T \to \mathbb{F}_T$ is invertible, we consider the associated boundary value problem in the chart domains. To show that this localised problem admits a unique solution in the space $\mathbb{E}_{T,\alpha}$, we use [136, Theorem 5.4]. To this end we need to verify that the localised operators satisfy the conditions listed in [136, §1]. We note that Theorem B.21 and Proposition B.31 imply that the spaces used in [136] are equal to the ones used in this thesis with equivalent norms. In the following we let $(U_\alpha, \phi_\alpha) \in \mathcal{T}$ be an interior or boundary chart. The theory in [136] is formulated for systems of parabolic equations. This becomes crucial when considering the elastic flow of networks in Part II. As we are concerned with a scalar equation, the numbers s_k and t_j introduced in [136, §1] are not essential. However, it is readily checked that with the choice $s_1 = 4$, $t_1 = 0$ and $b = 2$ our problem fits the situation considered in [136, §1]. One crucial aspect is to verify the *uniform parabolicity condition* imposed on the differential operator $\partial_t + \mathcal{A}_\alpha(x,D)$. To this end, we observe that the calculations in Section 2.2 imply the following result.

Corollary 2.44. *The principal part of $\mathcal{A}_\alpha(x,D)$ is given by*

$$(\mathcal{A}_\alpha)_\#(x,D) = \sum_{i,j,k,l=1}^{n} g_\alpha^{ij}(x) g_\alpha^{kl}(x) \partial_{x^i} \partial_{x^j} \partial_{x^k} \partial_{x^l}$$

where

$$g_\alpha^{ij}(x) := g_\varphi^{ij}\left(\phi_\alpha^{-1}(x)\right) = \left\langle \frac{\partial\left(\varphi \circ \phi_\alpha^{-1}\right)}{\partial x_i}(x), \frac{\partial\left(\varphi \circ \phi_\alpha^{-1}\right)}{\partial x_j}(x) \right\rangle.$$

Proof. This follows from Lemma 2.26. □

Proposition 2.45 (Uniform parabolicity)**.** *For every $\alpha \in \{1, \dots, N\}$, $x \in \phi_\alpha(U_\alpha)$ and all $\xi \in \mathbb{R}^n$ the estimate*

$$Q^{-2}|\xi|^4 \le (\mathcal{A}_\alpha)_\# (x, i\xi) \le Q^2|\xi|^4$$

is valid. In particular, the differential operator $\partial_t + \mathcal{A}_\alpha(x, D)$ satisfies the uniform parabolicity condition given in [136, §1].

Proof. Let $\alpha \in \{1, \dots, N\}$, $x \in \phi_\alpha(U_\alpha)$ and $\xi \in \mathbb{R}^n$ be given. The estimate (A.12) implies

$$Q^{-1}|\xi|^2 \le \sum_{i,j=1}^{n} g_\alpha^{ij}(x)\xi_i\xi_j \le Q|\xi|^2$$

with $Q > 1$ and hence

$$Q^{-2}|\xi|^4 \le \left(\sum_{i,j=1}^{n} g_\alpha^{ij}(x)\xi_i\xi_j\right)^2 = \sum_{i,j,k,l=1}^{n} g_\alpha^{ij}(x)g_\alpha^{kl}(x)\xi_i\xi_j\xi_k\xi_l = (\mathcal{A}_\alpha)_\# (x, i\xi) \le Q^2|\xi|^4 .$$

In particular, the roots p_s of the polynomial $(\mathcal{A}_\alpha)_\# (x, i\xi) + p$ with respect to the variable p are real and satisfy the inequality

$$\Re p_s = p_s \le -Q^{-2}|\xi|^4$$

which shows that $\partial_t + \mathcal{A}_\alpha(x, D)$ satisfies the uniform parabolicity condition in [136, §1] with constant $\delta = Q^{-2}$. □

The framework developed in [136] allows for more complicated initial value conditions that are merged in the matrix $\mathcal{C}$ introduced in [136, §1]. In our case $\mathcal{C}$ is equal to the constant 1 and all required properties are trivially fulfilled with $\varrho = 0$. We further need to verify the so called *complementary condition* on the boundary operators, given in [136, §1, p. 11], which follows from the *Lopatinskii–Shapiro condition*, see [53, pages 11-15]. Furthermore, compatibility conditions on the data need to be imposed. The calculations in Section 2.2 imply the following result for the boundary operator.

Corollary 2.46. *The principal part of $\mathcal{B}_{1,\alpha}(x, D)$ is given by*

$$(\mathcal{B}_{1,\alpha})_\# (x, D) = \sum_{i,j=1}^{n} g_\alpha^{ij}(x)\partial_{x^i}\partial_{x^j}$$

where $g_\alpha^{ij}(x) := g_\varphi^{ij}\left(\phi_\alpha^{-1}(x)\right)$.

Proof. This follows from Lemma 2.24. □

To ease notation we omit the subscript α in the result below.

Proposition 2.47 (Lopatinskii–Shapiro condition)**.** *For any fixed $t \in [0, T]$, $x_0 \in \partial\phi(U)$, $h \in \mathbb{R}^2$, $\eta \in \mathbb{R}^n$ with $\langle \eta, \nu_{\partial\phi(U)}(x_0)\rangle = 0$ and $\lambda \in \overline{\mathbb{C}}_+ := \{z \in \mathbb{C} : \Re(z) \ge 0\}$ with $|\eta| + |\lambda| \ne 0$ the ordinary differential equation*

$$\begin{cases} \lambda v(s) + \mathcal{A}_\#(x_0, \eta + i\nu_{\partial\phi(U)}(x_0)\partial_s)v(s) &= 0 , \qquad s > 0 , \\ \mathcal{B}_{1,\#}(x_0, \eta + i\nu_{\partial\phi(U)}(x_0)\partial_s)v(0) &= h_1 , \\ v(0) &= h_2 \end{cases} \tag{2.22}$$

admits in the space $C_0(\mathbb{R}_+, \mathbb{C}) := \{f : \mathbb{R}_+ \to \mathbb{C} : \lim_{y\to\infty} f(y) = 0\}$ *a unique solution* v *where* $\nu_{\partial\phi(U)}(x_0)$ *denotes the outer unit normal to* $\partial\phi(U)$ *at* x_0*.*

Proof. Let $t \in [0,T]$ and $x_0 \in \partial\phi(U)$ be fixed. In all our previous considerations we have written the differential operator

$$\mathcal{A}(x_0, D) = \sum_{|\beta|\le 4} c_\beta(x_0) i^{|\beta|} \partial_{e_1}^{\beta_1} \cdots \partial_{e_n}^{\beta_n} = \sum_{|\beta|\le 4} c_\beta(x_0) i^{|\beta|} \partial_e^\beta$$

with respect to the standard basis $\{e_1, \dots, e_n\}$ of the vector-space $\mathbb{R}^n$ where $D := i(\partial_{e_1}, \dots, \partial_{e_n})^T$ and $c_\beta(x_0) := c_{\beta,\alpha}\left(\phi_\alpha^{-1}(x_0)\right)$, see Definition 2.34. Note that the coefficients are now constant as x_0 is fixed. We now perform a coordinate transformation in $\mathbb{R}^n$ which does not change the operator itself, but only its representation. To this end, let $\tau_n := \tau_n(x_0) := \nu_{\partial\phi(U)}(x_0)$ and $\tau_1, \dots, \tau_{n-1} \in T_{x_0}\partial\phi(U)$ be such that $\{\tau_1, \dots, \tau_n\}$ forms an orthonormal basis of $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ and let $M(x_0) \in \mathbb{R}^{n\times n}$ be the matrix that satisfies $M(x_0)(\tau_1, \dots, \tau_n)^T = (e_1, \dots, e_n)^T$. Defining $\partial_\tau := (\partial_{\tau_1}, \dots, \partial_{\tau_n})^T$ we have $M(x_0)\partial_\tau = \partial_e$ as the directional derivative is linear in the direction. This enables us to rewrite the differential operator $\mathcal{A}(x_0, D)$ as

$$\mathcal{A}(x_0, i\partial_e) = \tilde{\mathcal{A}}(x_0, i\partial_\tau) = \sum_{|\beta|\le 4} c_\beta(x_0) i^{|\beta|} \left(M(x_0)\partial_\tau\right)^\beta$$

with principal part

$$\begin{aligned}\tilde{\mathcal{A}}_\#(x_0, i\partial_\tau) &= g^{ij}(x_0) g^{kl}(x_0) \left(M(x_0)\partial_\tau\right)_i \left(M(x_0)\partial_\tau\right)_j \left(M(x_0)\partial_\tau\right)_k \left(M(x_0)\partial_\tau\right)_l \\ &= \left[(M(x_0)\partial_\tau)^T G^{-1}(x_0) \left(M(x_0)\partial_\tau\right)\right]^2 = \left(\partial_\tau^T M(x_0)^T G^{-1}(x_0) M(x_0)\partial_\tau\right)^2 .\end{aligned}$$

Note that $A(x_0) := (a_{ij})_{i,j=1}^n (x_0) := M^T(x_0) G^{-1}(x_0) M(x_0)$ is symmetric and positive definite. Let $A'(x_0) \in \mathbb{R}^{(n-1)\times(n-1)}$, $b \in \mathbb{R}^{n-1}$ and $a_{nn} \in \mathbb{R}$ be such that

$$A(x_0) = \begin{pmatrix} A'(x_0) & b \\ b^T & a_{nn} \,. \end{pmatrix}$$

The entry a_{nn} is positive as $A(x_0)$ is positive definite. Now let $h \in \mathbb{R}^2$, $\eta \in \mathbb{R}^n$ and $\lambda \in \overline{\mathbb{C}}_+$ be such that $\langle \eta, \nu_{\partial\phi(U)}(x_0) \rangle = 0$ and $|\eta| + |\lambda| \ne 0$. In the new coordinates we have $\nu_{\partial\phi(U)}(x_0) = (0, \dots, 0, 1)^T$ and hence $\eta = (\eta', 0)^T$ for some $\eta' \in \mathbb{R}^{n-1}$. The ordinary differential equation thus becomes

$$\lambda v(s) + \left((\eta', i\partial_s) A(x_0) (\eta', i\partial_s)^T\right)^2 v(s) = 0\,, \qquad s > 0\,, \tag{2.23}$$

$$\left((\eta', i\partial_s) A(x_0) (\eta', i\partial_s)^T v(s)\right)_{|s=0} = h_1\,, \tag{2.24}$$

$$v(0) = h_2\,. \tag{2.25}$$

Equation (2.23) is a homogeneous linear fourth order ODE for $v(s)$ with constant coefficients in $\mathbb{C}$. To find a fundamental system of the solution space we need to exhibit the roots of the characteristic polynomial in $z \in \mathbb{C}$

$$P(z) := \lambda + \left((\eta', iz) A(x_0) (\eta', iz)^T\right)^2 = \lambda + \left(-a_{nn} z^2 + 2ib^T\eta' z + (\eta')^T A'(x_0) \eta'\right)^2 .$$

The four roots z_k, $k \in \{1,2,3,4\}$, of $P(z)$ are given by

$$z_{1/2/3/4} = \frac{1}{a_{nn}} \left(ib^T\eta' \pm \sqrt{a_{nn}(\eta')^T A'(x_0)\eta' - (b^T\eta')^2 \pm a_{nn}\sqrt{-\lambda}} \right).$$

Lemma 2.48 and Lemma 2.49 imply that

$$\Re\left(\sqrt{a_{nn}(\eta')^T A'(x_0)\eta' - (b^T\eta')^2 \pm a_{nn}\sqrt{-\lambda}}\right)$$

is non zero.

Case 1: $\lambda \neq 0$.

In this case the four roots z_k are distinct. Thus, $\{e^{z_k \cdot s} : k \in \{1,2,3,4\}\}$ forms a fundamental system of the solution space to equation (2.23). Since $\Re\left(\sqrt{a_{nn}(\eta')^T A'(x_0)\eta' - (b^T\eta')^2 \pm a_{nn}\sqrt{-\lambda}}\right)$ is non-zero, there are exactly two z_k, $k \in \{1,2,3,4\}$ such that $\Re(z_k) < 0$. We may assume $\Re(z_1) < 0$ and $\Re(z_3) < 0$. As we demand that $\lim_{s\to\infty} v(s) = 0$, the solution to our system is of the form

$$v(s) = ce^{z_1 \cdot s} + de^{z_3 \cdot s} \qquad c, d \in \mathbb{C}\,,$$

where $c, d \in \mathbb{C}$ need to be chosen such that the boundary conditions (2.24) and (2.25) are satisfied. The second boundary condition implies $c + d = h_2$. Equation (2.24) is equivalent to

$$(\eta')^T A'(x_0)\eta' v(0) + 2ib^T\eta' v'(0) - a_{nn}v''(0) = h_1\,.$$

Differentiating $v(s) = ce^{z_1 \cdot s} + de^{z_3 \cdot s}$ and using $d = h_2 - c$ gives

$$\begin{aligned} h_1 &= (\eta')^T A'(x_0)\eta' h_2 + 2ib^T\eta' h_2 z_3 - a_{nn}h_2 z_3^2 + c\left(2ib^T\eta' z_1 - 2ib^T\eta' z_3 - a_{nn}z_1^2 + a_{nn}z_3^2\right) \\ &= (\eta')^T A'(x_0)\eta' h_2 + 2ib^T\eta' h_2 z_3 - a_{nn}h_2 z_3^2 - 2c\sqrt{-\lambda}\,, \end{aligned}$$

where we used the equations for z_1 and z_3 combined with the fact that both $\Re(z_1) < 0$ and $\Re(z_3) < 0$. As $\lambda \neq 0$ we get a unique solution $c \in \mathbb{C}$ and thus a unique solution $v \in C_0(\mathbb{R}_+, \mathbb{C})$ of the ODE (2.23) - (2.25).

Case 2: $\lambda = 0$.

In this case there holds $\eta' \neq 0$. As $a_{nn}A'(x_0) - bb^T$ is positive definite, we have

$$a_{nn}(\eta')^T A'(x_0)\eta' - \left(b^T\eta'\right)^2 > 0$$

and hence the roots

$$z_{1/2} = \frac{1}{a_{nn}}\left(ib^T\eta' \pm \sqrt{a_{nn}(\eta')^T A'(x_0)\eta' - (b^T\eta')^2}\right)$$

are of multiplicity 2 and $\{e^{z_1 \cdot s}, se^{z_1 \cdot s}, e^{z_2 \cdot s}, se^{z_2 \cdot s}\}$ forms a fundamental system of (2.23). Furthermore, we have $\Re z_1 > 0$ and $\Re z_2 < 0$. Using equation (2.25) we know that $v \in C_0(\mathbb{R}_+, \mathbb{C})$ is of the form

$$v(s) = h_2 e^{z_2 \cdot s} + dse^{z_2 \cdot s}, \qquad d \in \mathbb{C}\,.$$

Since $v'(0) = h_2 z_2 + d$ and $v''(0) = h_2 z_2^2 + 2dz_2$, equation (2.24) is equivalent to

$$\begin{aligned} h_1 &= (\eta')^T A'(x_0)\eta' h_2 + 2ib^T\eta' h_2 z_2 - a_{nn}h_2 z_2^2 + 2d\left(ib^T\eta' - a_{nn}z_2\right) \\ &= 2d\left(ib^T\eta' - a_{nn}z_2\right) = 2d\sqrt{a_{nn}(\eta')^T A'(x_0)\eta' - (b^T\eta')^2}\,, \end{aligned}$$

where we used the identity $P(z_2) = 0$ in the case $\lambda = 0$. By Lemma 2.49,

$$\Re\left(\sqrt{a_{nn}(\eta')^T A'(x_0)\eta' - (b^T\eta')^2 \pm a_{nn}\sqrt{-\lambda}}\right) = \sqrt{a_{nn}(\eta')^T A'(x_0)\eta' - (b^T\eta')^2}$$

is non-zero. This shows the existence of a unique solution v to (2.23) - (2.25) in the case $\lambda = 0$. □

Lemma 2.48. *Let $A \in \mathbb{R}^{n\times n}$ be positive definite and $A' \in \mathbb{R}^{(n-1)\times(n-1)}$, $b \in \mathbb{R}^{n-1}$, $a_{nn} \in \mathbb{R}\setminus\{0\}$ be such that*

$$A = \begin{pmatrix} A' & b \\ b^T & a_{nn} \end{pmatrix} .$$

Then the matrix $a_{nn}A' - bb^T$ is positive definite.

Proof. A matrix is positive semi-definite if and only if all determinants of $k \times k$ sub-matrices that are obtained by erasing lines and columns are non negative. The determinant of a block matrix $D \in \mathbb{R}^{k\times k}$ with $k \geq 1$ of the form

$$D = \begin{pmatrix} D' & e \\ f^T & d_{kk} \end{pmatrix}$$

with $D' \in \mathbb{R}^{(k-1)\times(k-1)}$, $e, f \in \mathbb{R}^{k-1}$, $d_{kk} \in \mathbb{R}\setminus\{0\}$, is given by

$$\det D = \det\left(d_{kk}D' - ef^T\right) .$$

This shows that the determinant of the $(n-1-l)\times(n-1-l)$–sub-matrix of $a_{nn}A' - bb^T$ that is obtained by erasing the lines $i_1,\dots,i_l \in \{1,\dots,n-1\}$ and the columns $j_1,\dots,j_l \in \{1,\dots,n-1\}$ with $l \in \{0,\dots,n-2\}$, is equal to the determinant of the $(n-l)\times(n-l)$–sub-matrix of A obtained by erasing the respective l lines and l columns of A. As A is positive semi-definite, these determinants are all non negative. Thus $a_{nn}A' - bb^T$ is positive definite being a positive semi-definite matrix with positive determinant. □

Lemma 2.49. *Let $C \in \mathbb{R}^{(n-1)\times(n-1)}$ be positive definite, $c \in \mathbb{R}\setminus\{0\}$ and $\eta' \in \mathbb{R}^{n-1}$ and $\lambda \in \overline{\mathbb{C}}_+$ such that $|\eta| + |\lambda| \neq 0$. Then*

$$\Re\left(\sqrt{(\eta')^T C\eta' \pm c\sqrt{-\lambda}}\right) \neq 0 .$$

Proof. Suppose that there exists $s \in \mathbb{R}$ such that

$$\sqrt{(\eta')^T C\eta' + c\sqrt{-\lambda}} = is .$$

In the case that η' is non-zero, this implies $c\sqrt{-\lambda} = -\varepsilon^2$ for some $\varepsilon > 0$ and hence $\lambda = -\frac{\varepsilon^4}{c^2}$ which contradicts the assumption that λ has non negative real part. If $\eta' = 0$ the identity $\lambda = -\frac{s^4}{c^2}$ enforces $s = 0$ and thus $\lambda = 0$ contradicting $|\eta'| + |\lambda| \neq 0$. □

The existence of the localised problem in the chart domains is now a direct consequence of [136, Theorem 5.4].

Theorem 2.50 (Existence and uniqueness in chart domains)**.** *Let $\alpha \in \{1,\dots,N\}$, $p \in (4+n,\infty)$, $T_1 > 0$ and $T \in (0,T_1]$. Then for every*

$$(f_\alpha, h_{1,\alpha}, h_{2,\alpha}, \eta_{0,\alpha}) \in X_{T,\alpha} \times Y_{1,T,\alpha} \times Y_{2,T,\alpha} \times X_{0,\alpha}$$

satisfying for $i \in \{1,2\}$, $x \in \partial\phi_\alpha(U_\alpha)$,

$$\mathcal{B}_{i,\alpha}(x,D)\eta_{0,\alpha}(x) = h_{i,\alpha}(0,x) ,$$

there is a unique $\eta_\alpha \in \mathbb{E}_{T,\alpha}$ such that

$$(\partial_t + A_\alpha)\,\eta_\alpha = f_\alpha , \quad B_{i,\alpha}\eta_\alpha = h_{i,\alpha} , \quad (\eta_\alpha)_{|t=0} = \eta_{0,\alpha} , \tag{2.26}$$

with the estimate

$$\|\eta_\alpha\|_{\mathbb{E}_{T,\alpha}} \le C(T)\left(\|f_\alpha\|_{X_{T,\alpha}} + \|h_{1,\alpha}\|_{Y_{1,T,\alpha}} + \|h_{2,\alpha}\|_{Y_{2,T,\alpha}} + \|\eta_{0,\alpha}\|_{X_{0,\alpha}}\right).$$

Moreover, there holds the uniform estimate

$$\|\eta_\alpha\|_{\mathbb{E}_{T,\alpha}} \le C(T_1)\left(\|f_\alpha\|_{X_{T,\alpha}} + |||h_{1,\alpha}|||_{Y_{1,T,\alpha}} + |||h_{2,\alpha}|||_{Y_{2,T,\alpha}} + \|\eta_{0,\alpha}\|_{X_{0,\alpha}}\right). \tag{2.27}$$

Proof. We verify the conditions required to apply [136, Theorem 5.4]. The coefficients of the differential operators $\mathcal{A}_\alpha(x, D)$ and $\mathcal{B}_{i,\alpha}(x, D)$, $i \in \{1, 2\}$, are smooth on $\overline{\phi_\alpha(U_\alpha)}$ and $\mathrm{int}\phi_\alpha(U_\alpha)$ is a smooth domain. With the choices $l = 4$, $b = 2$, $\sigma_1 = 2$, $\sigma_2 = 0$, $s_1 = 4$, $t_1 = 0$ and $\varrho = 0$ the boundary value problem (2.26) fits into the setting discussed in [136] once we have checked that compatibility conditions of order 3 are fulfilled. These are specified in [136, §14] and precisely mean that the initial value $\eta_{0,\alpha}$ satisfies for $i \in \{1, 2\}$,

$$(\mathcal{B}_{i,\alpha}(x, D)\eta_{0,\alpha})_{|\partial\phi_\alpha(U_\alpha)} = (h_{i,\alpha})_{|t=0}\,.$$

Due to the embeddings $Y_{i,T,\alpha} \hookrightarrow C\big([0, T] \times \overline{\phi_\alpha(U_\alpha)}\big)$ and $X_{0,\alpha} \hookrightarrow C^3\big(\overline{\phi_\alpha(U_\alpha)}\big)$, the evaluations are well-defined, which is also discussed in [136, p.134 f.]. By [136, Theorem 5.4] there exists a unique $\eta_\alpha \in \mathbb{E}_{T,\alpha}$ satisfying the identities (2.26) and the estimate

$$\|\eta_\alpha\|_{\mathbb{E}_{T,\alpha}} \le C(T)\left(\|f_\alpha\|_{X_{T,\alpha}} + \|h_{1,\alpha}\|_{Y_{1,T,\alpha}} + \|h_{2,\alpha}\|_{Y_{2,T,\alpha}} + \|\eta_{0,\alpha}\|_{X_{0,\alpha}}\right).$$

To show the uniform estimate (2.27) let $\eta_\alpha \in \mathbb{E}_{T,\alpha}$ with $(\partial_t + A_\alpha)\,\eta_\alpha = f_\alpha$, $B_{i,\alpha}\eta_\alpha = h_{i,\alpha}$ and $(\eta_\alpha)_{|t=0} = \eta_{0,\alpha}$ be given. Proposition B.37 implies the existence of $\boldsymbol{E}h_{i,\alpha} \in Y_{i,T_1,\alpha}$ with $(\boldsymbol{E}h_{i,\alpha})_{|(0,T)} = h_{i,\alpha}$ and

$$\|\boldsymbol{E}h_{i,\alpha}\|_{Y_{i,T_1,\alpha}} \le C(T_1)\, |||h_{i,\alpha}|||_{Y_{i,T,\alpha}}\,.$$

Let $\widehat{f}_\alpha \in X_{T_1,\alpha}$ be the trivial extension of f_α. Then $\big(\widehat{f}_\alpha, \boldsymbol{E}h_{1,\alpha}, \boldsymbol{E}h_{2,\alpha}, \eta_{0,\alpha}\big)$ lies in $\mathbb{F}_{T_1,\alpha}$ and there exists $\widehat{\eta}_\alpha \in \mathbb{E}_{T_1,\alpha}$ with $(\partial_t + A_\alpha)\,\widehat{\eta}_\alpha = \widehat{f}_\alpha$, $B_{i,\alpha}\widehat{\eta}_\alpha = \boldsymbol{E}h_{i,\alpha}$, $(\widehat{\eta}_\alpha)_{|t=0} = \eta_{0,\alpha}$ and

$$\begin{aligned}\|\widehat{\eta}_\alpha\|_{\mathbb{E}_{T_1,\alpha}} &\le C(T_1)\left(\left\|\widehat{f}_\alpha\right\|_{X_{T_1,\alpha}} + \|\boldsymbol{E}h_{1,\alpha}\|_{Y_{1,T_1,\alpha}} + \|\boldsymbol{E}h_{2,\alpha}\|_{Y_{2,T_1,\alpha}} + \|\eta_{0,\alpha}\|_{X_0}\right)\\ &\le C(T_1)\left(\|f_\alpha\|_{X_{T,\alpha}} + |||h_{1,\alpha}|||_{Y_{1,T,\alpha}} + |||h_{2,\alpha}|||_{Y_{2,T,\alpha}} + \|\eta_{0,\alpha}\|_{X_0}\right).\end{aligned}$$

By uniqueness, $(\widehat{\eta}_\alpha)_{|(0,T)} = \eta_\alpha$ and we obtain

$$\|\eta_\alpha\|_{\mathbb{E}_{T,\alpha}} \le \|\widehat{\eta}_\alpha\|_{\mathbb{E}_{T_1,\alpha}} \le C(T_1)\left(\|f_\alpha\|_{X_{T,\alpha}} + |||h_{1,\alpha}|||_{Y_{1,T,\alpha}} + |||h_{2,\alpha}|||_{Y_{2,T,\alpha}} + \|\eta_{0,\alpha}\|_{X_0}\right).$$

□

2.3.3 Existence and uniqueness on the manifold

This section shows that the operator L_T defined in Proposition 2.43 is an isomorphism provided that T is sufficiently small. The crucial ingredients are embeddings of anisotropic spaces as shown in Proposition B.35 with constants *independent* of the considered time interval. As discussed in Corollary B.38 the norm $|||\cdot|||_{\mathbb{E}_T}$ defined in (1.7) and (1.8) allows for uniform in time estimates. We observe that the estimate (2.27) still holds when replacing $\|\cdot\|_{\mathbb{E}_{T,\alpha}}$ by $|||\cdot|||_{\mathbb{E}_{T,\alpha}}$.

Lemma 2.51. *Let (M, φ) be a smooth reference geometry of dimension $n \ge 2$, T be positive and $p \in (4 + n, \infty)$. Given $h_2 \in Y_{2,T} = W_p^{1-1/4p,4-1/p}((0, T) \times \partial M)$ and $\eta_0 \in X_0 = W_p^{4-4/p}(M)$ with $(\eta_0)_{|\partial M} = (h_2)_{|t=0}$, there exists $w \in \mathbb{E}_T$ such that $B_2 w = h_2$ and $w_{|t=0} = \eta_0$.*

Proof. Let $\xi_\alpha \in C^\infty(M)$ satisfy $\operatorname{supp}\xi_\alpha \subset U_\alpha$ and $\xi_\alpha \equiv 1$ on $\operatorname{supp}\psi_\alpha$. Definition C.10 implies for all $\alpha \in \{1,\dots,N\}$ that $g_\alpha := (\psi_\alpha\eta_0)\circ\phi_\alpha^{-1}$ lies in $W_p^{4-4/p}(\operatorname{int}\phi_\alpha(U_\alpha))$. In the case that (U_α,ϕ_α) is a boundary chart, we observe that

$$\operatorname{dist}\left(\operatorname{supp} g_\alpha \cap \overline{\phi_\alpha(U_\alpha)}, \partial\phi_\alpha(U_\alpha)\cap\operatorname{int}\mathbb{H}^n\right) > 0\,.$$

Thus, Lemma C.13 implies that the trivial extension Eg_α lies in $W_p^{4-4/p}(\operatorname{int}\mathbb{H}^n)$. We write g_α instead of Eg_α for simplicity. Proposition C.16 applied to the manifold ∂M with normal covering $(U_\alpha\cap\partial M, \pi\circ\phi_\alpha\circ\iota)$, implies that

$$(t,x') \mapsto \widehat{f}^0_\alpha(t,x') := \psi_\alpha\left(\phi_\alpha^{-1}(x',0)\right) h_2(t,\phi_\alpha^{-1}(x',0))$$

lies in $W_p^{1-1/4p,4-1/p}((0,T)\times\pi(\phi_\alpha(U_\alpha\cap\partial M)))$. The set $\pi(\phi_\alpha(U_\alpha\cap\partial M))$ is open in $\mathbb{R}^{n-1}$ and, denoting by $i:\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}\times\{0\}\subset\mathbb{R}^n$ the inclusion, we obtain

$$\operatorname{dist}\left(\operatorname{supp}\left(\psi_\alpha\circ\phi_\alpha^{-1}\circ i\right)\cap\overline{\pi\left(\phi_\alpha(U_\alpha\cap\partial M)\right)}, \partial\pi\left(\phi_\alpha(U_\alpha\cap\partial M)\right)\cap\mathbb{R}^{n-1}\right) > 0\,.$$

Lemma C.21 implies for the trivial extension

$$(t,x') \mapsto \widehat{f}^0_\alpha(t,x') \in W_p^{1-1/4p,4-1/p}\left((0,T)\times\mathbb{R}^{n-1}\right).$$

In particular, we deduce

$$(t,(x',0)) \mapsto f^0_\alpha(t,(x',0)) := \widehat{f}^0_\alpha(t,x') \in W_p^{1-1/4p,4-1/p}\left((0,T)\times\left(\mathbb{R}^{n-1}\times\{0\}\right)\right).$$

Let $\nu := -e_n$ be the outer unit normal to $\partial\mathbb{H}^n = \mathbb{R}^{n-1}\times\{0\}$. Corollary C.28 implies for $j\in\{1,2,3\}$,

$$x \mapsto \widetilde{f}^j_\alpha(x) := \operatorname{tr}\frac{\partial^j g_\alpha}{\partial\nu^j} \in W_p^{4-j-5/p}\left(\mathbb{R}^{n-1}\times\{0\}\right)$$

where tr denotes restriction to the boundary $\partial\mathbb{H}^n$. By [45, Theorem 4.5] there exist functions

$$f^j_\alpha \in W_p^{\frac{4-j-1/p}{4},4-j-1/p}\left((0,T)\times(\mathbb{R}^{n-1}\times\{0\})\right)$$

with $\left(f^j_\alpha\right)_{|t=0} = \widetilde{f}^j_\alpha$. Using [70, Théorème 4.2] we deduce existence of $w_\alpha \in W_p^{1,4}\left((0,T)\times\mathbb{R}^n_+\right)$ with

$$\left((w_\alpha)_{|t=0}, \operatorname{tr}w_\alpha, \operatorname{tr}\frac{\partial w_\alpha}{\partial\nu}, \operatorname{tr}\frac{\partial^2 w_\alpha}{\partial\nu^2}, \operatorname{tr}\frac{\partial^3 w_\alpha}{\partial\nu^3}\right) = \left(g_\alpha, f^0_\alpha, f^1_\alpha, f^2_\alpha, f^3_\alpha\right).$$

In the case that (U_α,ϕ_α) is an interior chart we have

$$\operatorname{dist}\left(\operatorname{supp} g_\alpha\cap\overline{\phi_\alpha(U_\alpha)}, \partial\phi_\alpha(U_\alpha)\right) > 0$$

and Lemma C.13 implies that the trivial extension lies in $W_p^{4-4/p}(\mathbb{R}^n)$. Using [45, Theorem 4.5] there exists $w_\alpha \in W_p^{1,4}((0,T)\times\mathbb{R}^n)$ with $(w_\alpha)_{|t=0} = g_\alpha = (\psi_\alpha\eta_0)\circ\phi_\alpha^{-1}$. For $t\in(0,T)$ and $q\in M$ we define

$$w(t,q) := \sum_{\alpha=1}^N \xi_\alpha(q)w_\alpha(t,\phi_\alpha(q)) = \sum_{\alpha=1}^N \left(\left(\xi_\alpha\circ\phi_\alpha^{-1}\right)w_\alpha(t)\right)\circ\phi_\alpha\,.$$

As $\left(\xi_\alpha\circ\phi_\alpha^{-1}\right)w_\alpha$ lies in $W_p^{1,4}((0,T)\times\operatorname{int}\phi_\alpha(U_\alpha))$ and

$$\operatorname{dist}\left(\operatorname{supp}\left(\xi_\alpha\circ\phi_\alpha^{-1}\right)\cap\overline{\phi_\alpha(U_\alpha)}, \partial\phi_\alpha(U_\alpha)\cap\operatorname{int}\mathbb{H}^n\right) \geq \delta > 0$$

for every $t\in(0,T)$ and a constant δ depending only on ξ_α, Lemma C.22 implies that the trivial extension of $\xi_\alpha(w_\alpha\circ\phi_\alpha)$ to the entire manifold lies in $\mathbb{E}_T$. We conclude that w lies in $\mathbb{E}_T \hookrightarrow C([0,T]\times M)$ and for all $q\in M$,

$$w(0,q) = \sum_{\alpha=1}^N \xi_\alpha(q)w_\alpha(0,\phi_\alpha(q)) = \sum_{\alpha=1}^N \xi_\alpha(q)\psi_\alpha(q)\eta_0(q) = \eta_0(q)\,.$$

We obtain for all $t \in (0,T)$ and $q \in \partial M$ with $J_q := \{\alpha \in \{1,\dots,N\} : q \in (U_\alpha, \phi_\alpha)\}$,

$$w(t,q) = \sum_{\alpha \in J_q} \xi_\alpha(q)(\mathrm{tr} w_\alpha)(t,\phi_\alpha(q)) = \sum_{\alpha \in J_q} \xi_\alpha(q) f^0_\alpha(t,\phi_\alpha(q)) = \sum_{\alpha \in J_q} \xi_\alpha(q)\psi_\alpha(q) h_2(t,q) = h_2(t,q)\,.$$

□

Lemma 2.52. *Let (M,φ) be a smooth reference geometry of dimension $n \geq 2$, T be positive and $p \in (4+n,\infty)$. Given $h \in Y_{1,T} = W_p^{1/2-1/4p,2-1/p}((0,T)\times\partial M)$ with $h(0)=0$, there exists $u \in \mathbb{E}_T$ such that*

$$B_1 u = h\,, \qquad B_2 u = 0\,, \quad u_{|t=0} = 0\,.$$

Proof. Let $\xi_\alpha \in C^\infty(M)$ satisfy $\operatorname{supp}\xi_\alpha \subset U_\alpha$ and $\xi_\alpha \equiv 1$ on $\operatorname{supp}\psi_\alpha$. Furthermore, let $\nu := -e_n$ be the outer unit normal to $\partial\mathbb{H}^n$. If (U_α,ϕ_α) is a boundary chart, we define

$$(t,x) \mapsto h_\alpha(t,x) := \psi_\alpha\left(\phi_\alpha^{-1}(x)\right) h\left(t,\phi_\alpha^{-1}(x)\right) b_\alpha(x)$$

where for $x \in \phi_\alpha(U_\alpha)$

$$b_\alpha(x) := \left(g_\alpha^{nn}(x)\right)^{-1}\,.$$

We remark that $b_\alpha \in C^\infty\big(\overline{\phi_\alpha(U_\alpha)}\big)$ due to Proposition A.44. As in the proof of Lemma 2.51, the results in Proposition C.16 and Lemma C.21 imply that the trivial extension of h_α, which is denoted by h_α, lies in $W_p^{1/2-1/4p,2-1/p}\left((0,T)\times\left(\mathbb{R}^{n-1}\times\{0\}\right)\right)$. By [70, Théorème 4.2] there exists a function $u_\alpha \in W_p^{1,4}\left((0,T)\times\mathbb{R}^n_+\right)$ such that

$$\left((u_\alpha)_{|t=0}, \mathrm{tr} u_\alpha, \mathrm{tr}\frac{\partial u_\alpha}{\partial\nu}, \mathrm{tr}\frac{\partial^2 u_\alpha}{\partial\nu^2}, \mathrm{tr}\frac{\partial^3 u_\alpha}{\partial\nu^3}\right) = (0,0,0,h_\alpha,0)$$

where tr denotes the restriction to $\partial\mathbb{H}^n$. In the case that (U_α,ϕ_α) is an interior chart, we set $u_\alpha \equiv 0$. Lemma C.22 implies that the trivial extension of $(t,q) \mapsto \xi_\alpha(q) u_\alpha(t,\phi_\alpha(q))$ to the entire manifold M, which is denoted by $\widetilde{u}_\alpha$, lies in $\mathbb{E}_T$, in particular

$$(t,q) \mapsto u(t,q) := \sum_{\alpha=1}^N \widetilde{u}_\alpha(t,q) = \sum_{\alpha=1}^N \xi_\alpha(q) u_\alpha(t,\phi_\alpha(q)) \in \mathbb{E}_T\,.$$

It is readily checked that $u_{|t=0} = 0$ and $B_2 u = 0$. By Lemma 2.28 the operator $B_1 : \mathbb{E}_T \to Y_{1,T}$ is linear and hence $B_1 u = \sum_{\alpha=1}^N B_1(\widetilde{u}_\alpha) \in Y_{1,T}$. Let $t \in (0,T)$, $q \in \partial M$ and $\alpha \in \{1,\dots,N\}$ be given. If q does not lie in U_α, there exists a boundary chart $(U_\beta,\phi_\beta) \in \mathcal{T}$ and a subset $V_\beta \subset U_\beta$ open in M such that $q \in V_\beta$ and $V_\beta \cap \operatorname{supp}\xi_\alpha = \emptyset$. In particular, $D^\gamma\big(\xi_\alpha \circ \phi_\beta^{-1}\big)(\phi_\beta(q)) = 0$ for all $\gamma \in \mathbb{N}_0^n$ and Corollary 2.33 implies

$$(B_1\widetilde{u}_\alpha)(t,q) = \sum_{|\gamma|\leq 2} b_{\gamma,\beta}(q) D^\gamma\big(\widetilde{u}_\alpha(t)\circ\phi_\beta^{-1}\big)(\phi_\beta(q)) = 0\,.$$

If q lies in U_α, then Corollary 2.33 implies

$$(B_1\widetilde{u}_\alpha)(t,q) = \sum_{|\gamma|\leq 2} b_{\gamma,\alpha}(q) D^\gamma\left(\widetilde{u}_\alpha(t)\circ\phi_\alpha^{-1}\right)(\phi_\alpha(q))\,.$$

As $u_\alpha \in C\big([0,T];C^3\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$ satisfies $u_\alpha(t,x) = 0$ for all $t \in [0,T]$, $x \in \mathbb{R}^{n-1}\times\{0\}$, we have for all $\gamma' \in \mathbb{N}_0^{n-1}$, $|\gamma'| \leq 2$,

$$D^{(\gamma',0)}\left(u_\alpha(t)\right)(\phi_\alpha(q)) = 0\,.$$

Furthermore, for all $t \in [0,T]$ and $x \in \mathbb{R}^{n-1} \times \{0\}$,

$$\frac{\partial u_\alpha}{\partial x_n}(t,x) = -\frac{\partial u_\alpha}{\partial \nu}(t,x) = 0\,,$$

which implies for all $t \in [0,T]$, $x \in \mathbb{R}^{n-1} \times \{0\}$, $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \le 2$ and $\gamma_n \in \{0,1\}$,

$$D^\gamma \left(u_\alpha(t)\right)(x) = 0\,.$$

We conclude that

$$\begin{aligned}(B_1 \widetilde{u}_\alpha)(t,q) &= b_{2e_n,\alpha}(q) D^{2e_n}\left(\widetilde{u}_\alpha(t) \circ \phi_\alpha^{-1}\right)(\phi_\alpha(q)) = g_\alpha^{nn}\left(\phi_\alpha(q)\right) D^{2e_n}\left(u_\alpha(t)\left(\xi_\alpha \circ \phi_\alpha^{-1}\right)\right)(\phi_\alpha(q)) \\ &= g_\alpha^{nn}(\phi_\alpha(q))\xi_\alpha(q)\frac{\partial^2 u_\alpha(t)}{\partial x_n^2}(\phi_\alpha(q)) = g_\alpha^{nn}(\phi_\alpha(q))\xi_\alpha(q)\frac{\partial^2 u_\alpha}{\partial \nu^2}(t,\phi_\alpha(q)) \\ &= g_\alpha^{nn}(\phi_\alpha(q))\xi_\alpha(q) h_\alpha(t,q) = g_\alpha^{nn}(\phi_\alpha(q))\xi_\alpha(q)\psi_\alpha(q)h(t,q)b_\alpha(\phi_\alpha(q)) = \psi_\alpha(q)h(t,q)\,.\end{aligned}$$

This shows for all $t \in [0,T]$, $q \in \partial M$,

$$(B_1 u)(t,q) = \sum_{\alpha=1}^N \psi_\alpha(q) h(t,q) = h(t,q)\,.$$

□

Proposition 2.53 (Extension of initial and boundary data). *Let (M,φ) be a smooth reference geometry of dimension $n \ge 2$, T_1 be positive, $T \in (0,T_1]$ and $p \in (4+n,\infty)$. For any $(f,h_1,h_2,u_0) \in \mathbb{F}_T$ there exists $v \in \mathbb{E}_T$ such that*

$$B_1 v = h_1\,, \qquad B_2 v = h_2\,, \qquad v_{|t=0} = u_0\,.$$

Proof. The identity (2.21) and Lemma 2.51 imply that there exists $w \in \mathbb{E}_T$ such that $B_2 w = h_2$ and $w_{|t=0} = u_0$. Lemma 2.28 yields that $h_1 - B_1 w$ lies in $Y_{1,T}$. To show that $(h_1 - B_1 w)_{|t=0} = 0$ on ∂M we recall that for any $\alpha \in \{1,\dots,N\}$ there holds

$$(t,x) \mapsto w\left(t, \phi_\alpha^{-1}(x)\right) \in C\big([0,T]; C^3\big(\overline{\phi_\alpha\left(U_\alpha\right)}\big)\big)$$

which allows us to interchange spacial differentiation and temporal evaluation. Given $q \in \partial M$, Corollary 2.37 and the identity (2.20) yield

$$\begin{aligned}(B_1 w)(0,q) &= \sum_{\alpha=1}^N \psi_\alpha(q)\mathcal{B}_{1,\alpha}\left(\phi_\alpha(q), D\right)\left(w_{|t=0} \circ \phi_\alpha^{-1}\right)(\phi_\alpha(q)) \\ &= \sum_{\alpha=1}^N \psi_\alpha(q)\mathcal{B}_{1,\alpha}\left(\phi_\alpha(q), D\right)\left(u_0 \circ \phi_\alpha^{-1}\right)(\phi_\alpha(q)) = h_1(0,q)\,.\end{aligned}$$

Lemma 2.52 implies the existence of $u \in \mathbb{E}_T$ with $B_1 u = h_1 - B_1 w$, $B_2 u = 0$ and $u_{|t=0} = 0$. The function $v := u + w \in \mathbb{E}_T$ fulfils the desired properties. □

To prove that the operator $L_T : \mathbb{E}_T \to \mathbb{F}_T$ is surjective we make use of the following Lemma.

Lemma 2.54. *Let (M,φ) be a smooth reference geometry of dimension $n \ge 2$, T_1 be positive and $p \in (4+n,\infty)$. There exists a time $\widetilde{T}_0 \in (0,T_1]$ such that for all $T \in (0,\widetilde{T}_0]$ and all $f \in X_T$ there exists $\eta \in \mathbb{E}_T$ with $L_T \eta = (f,0,0,0)$.*

Proof. Given $\alpha \in \{1, \dots, N\}$ we let $\xi_\alpha \in C^\infty(M)$ satisfy $0 \le \xi_\alpha \le 1$, $\operatorname{supp} \xi_\alpha \subset U_\alpha$ and $\xi_\alpha \equiv 1$ on $\operatorname{supp} \psi_\alpha$. Let $T \in (0, T_1]$ and $(F, G) \in Z_T$ be given where

$$Z_T := \{(F, G) \in X_T \times Y_{1,T} : G(0) = 0\} \, .$$

Lemma C.17 and Lemma C.18 imply

$$\begin{aligned}(t,x) \mapsto F_\alpha(t,x) &:= \psi_\alpha\left(\phi_\alpha^{-1}(x)\right) F\left(t, \phi_\alpha^{-1}(x)\right) \in X_{T,\alpha}\, ,\\ (t,x) \mapsto G_\alpha(t,x) &:= \psi_\alpha\left(\phi_\alpha^{-1}(x)\right) G\left(t, \phi_\alpha^{-1}(x)\right) \in Y_{1,T,\alpha}\end{aligned}$$

with $\|F_\alpha\|_{X_{T,\alpha}} \le C(Q)\, \|F\|_{X_T}$ and $\|G_\alpha\|_{Y_{1,T,\alpha}} \le C(Q)\, \|G\|_{Y_{1,T}}$. Theorem 2.50 implies that there is a unique $\eta_\alpha \in \mathbb{E}_{T,\alpha}$ such that

$$(\partial_t + A_\alpha)\, \eta_\alpha = F_\alpha\, , \quad B_{1,\alpha}\eta_\alpha = G_\alpha\, , \quad B_{2,\alpha}\eta_\alpha = 0\, , \quad (\eta_\alpha)_{|t=0} = 0$$

and

$$|||\eta_\alpha|||_{\mathbb{E}_{T,\alpha}} \le C(T_1)\big(\, \|F_\alpha\|_{X_{T,\alpha}} + \|G_\alpha\|_{Y_{1,T,\alpha}}\,\big)\, .$$

As $\operatorname{dist}\left(\operatorname{supp}\left(\xi_\alpha \circ \phi_\alpha^{-1}\right) \cap \overline{\phi_\alpha(U_\alpha)}, \partial\phi_\alpha(U_\alpha) \cap \operatorname{int}\mathbb{H}^n\right) > 0$, Lemma C.22 implies that the trivial extension of

$$(t, q) \mapsto \widetilde{\eta}_\alpha(t, q) := \xi_\alpha(q)\eta_\alpha\left(t, \phi_\alpha(q)\right)$$

to the entire manifold lies in $\mathbb{E}_T$ and in particular,

$$(t, q) \mapsto \eta(t, q) := \sum_{\alpha=1}^N \widetilde{\eta}_\alpha(t, q) = \sum_{\alpha=1}^N \xi_\alpha(q)\eta_\alpha\left(t, \phi_\alpha(q)\right) \in \mathbb{E}_T\, .$$

The function η satisfies $\eta_{|t=0} = 0$. As further $\phi_\alpha\left(U_\alpha \cap \partial M\right) \subset \partial\phi_\alpha\left(U_\alpha\right)$, we conclude for all $t \in [0, T]$, $q \in \partial M$,

$$\left(B_2\eta\right)(t, q) = \eta(t, q) = \sum_{\alpha=1}^N \xi_\alpha(q)\eta_\alpha\left(t, \phi_\alpha(q)\right) = 0\, .$$

For almost every $t \in (0, T)$ and $x \in \phi_\alpha(U_\alpha)$, Corollary 2.32 implies

$$\begin{aligned}&\partial_t\widetilde{\eta}_\alpha\left(t, \phi_\alpha^{-1}(x)\right) + A\left(\widetilde{\eta}_\alpha\right)\left(t, \phi_\alpha^{-1}(x)\right) = \partial_t\widetilde{\eta}_\alpha\left(t, \phi_\alpha^{-1}(x)\right) + \sum_{|\beta|\le 4} c_{\beta,\alpha}\left(\phi_\alpha^{-1}(x)\right) D^\beta\left(\widetilde{\eta}_\alpha(t) \circ \phi_\alpha^{-1}\right)(x)\\ &= \xi_\alpha\left(\phi_\alpha^{-1}(x)\right)\partial_t\eta_\alpha(t, x) + \sum_{|\beta|\le 4} c_{\beta,\alpha}\left(\phi_\alpha^{-1}(x)\right) D^\beta\left(\left(\xi_\alpha \circ \phi_\alpha^{-1}\right)\eta_\alpha(t)\right)(x)\\ &= \xi_\alpha\left(\phi_\alpha^{-1}(x)\right)\left(\partial_t\eta_\alpha + A_\alpha\eta_\alpha\right)(t, x) + R^1_{T,\alpha}(F)(t, x)\end{aligned}$$

where

$$R^1_{T,\alpha}(F)(t, x) := \sum_{|\beta|\le 4}\ \sum_{\substack{\gamma+\delta=\beta\,,\\ \gamma\ne 0}} \binom{\beta}{\gamma} c_{\beta,\alpha}\left(\phi_\alpha^{-1}(x)\right) D^\gamma\left(\xi_\alpha \circ \phi_\alpha^{-1}\right)(x) D^\delta\left(\eta_\alpha(t)\right)(x)\, .$$

The functions $c_{\beta,\alpha} \circ \phi_\alpha^{-1}$, $\xi_\alpha \circ \phi_\alpha^{-1}$ are smooth on $\overline{\phi_\alpha(U_\alpha)}$. Furthermore, Corollary B.38 yields the embedding

$$\left(\mathbb{E}_{T,\alpha},\, |||\cdot|||_{\mathbb{E}_{T,\alpha}}\right) \hookrightarrow C\big([0, T]; C^3\big(\overline{\phi_\alpha(U_\alpha)}\big)\big) \tag{2.28}$$

with uniform constant $C(T_1)$. We thus obtain $R^1_{T,\alpha}(F) \in X_{T,\alpha}$ with

$$\left\|R^1_{T,\alpha}(F)\right\|^p_{X_{T,\alpha}} = \int_0^T \left\|R^1_{T,\alpha}(F)(t)\right\|^p_{L_p(\phi_\alpha(U_\alpha))} \mathrm{d}t \le C(c_{\beta,\alpha}, \xi_\alpha) \sum_{|\gamma|\le 3} \int_0^T \|D^\gamma\eta_\alpha(t)\|^p_{L_p(\phi_\alpha(U_\alpha))}\, \mathrm{d}t$$

$$\le C \sum_{|\gamma|\le 3} \int_0^T \|D^\gamma \eta_\alpha(t)\|^p_{C(\overline{\phi_\alpha(U_\alpha)})}\,\mathrm{d}t \le C\int_0^T \|\eta_\alpha\|^p_{C\left([0,T];C^3(\overline{\phi_\alpha(U_\alpha)})\right)}\,\mathrm{d}t \le C(T_1)T\,|||\eta_\alpha|||^p_{\mathbb{E}_{T,\alpha}}$$
$$\le C(T_1)T\big(\|F_\alpha\|_{X_{T,\alpha}} + \|G_\alpha\|_{Y_{1,T,\alpha}}\big)^p .$$

Using Lemma C.19 it is readily checked that

$$(t,q)\mapsto R^1_T(F)(t,q) := \sum_{\alpha=1}^N R^1_{T,\alpha}(F)(t,\phi_\alpha(q))$$

lies in X_T with the estimate

$$\big\|R^1_T(F)\big\|_{X_T} \le C(Q)\sum_{\alpha=1}^N \big\|R^1_{T,\alpha}(F)\big\|_{X_{T,\alpha}} \le C(Q,T_1)T^{1/p}\sum_{\alpha=1}^N \|F_\alpha\|_{X_{T,\alpha}} + \|G_\alpha\|_{Y_{1,T,\alpha}}$$
$$\le C(Q,T_1)T^{1/p}\big(\|F\|_{X_T} + \|G\|_{Y_{1,T}}\big).$$

For every $t\in[0,T]$ and $x\in\partial\phi_\alpha(U_\alpha)\cap\partial\mathbb{H}^n$, Corollary 2.33 implies

$$(B_1\widetilde{\eta}_\alpha)\big(t,\phi_\alpha^{-1}(x)\big) = \sum_{|\gamma|\le 2} b_{\gamma,\alpha}\big(\phi_\alpha^{-1}(x)\big)\,D^\gamma\big(\widetilde{\eta}_\alpha(t)\circ\phi_\alpha^{-1}\big)(x)$$
$$= \sum_{|\gamma|\le 2} b_{\gamma,\alpha}\big(\phi_\alpha^{-1}(x)\big)\,D^\gamma\big(\big(\xi_\alpha\circ\phi_\alpha^{-1}\big)\eta_\alpha(t)\big)(x) = \xi_\alpha\big(\phi_\alpha^{-1}(x)\big)(B_{1,\alpha}\eta_\alpha)(t,x) + R^2_{T,\alpha}(G)(t,x)$$

where for $t\in[0,T]$, $x\in\partial\phi_\alpha(U_\alpha)$

$$R^2_{T,\alpha}(G)(t,x) := \sum_{|\gamma|\le 2} b_{\gamma,\alpha}\big(\phi_\alpha^{-1}(x)\big)\sum_{\substack{0\le\delta\le\gamma,\\ \delta\ne 0}}\binom{\gamma}{\delta}D^\delta\big(\xi_\alpha\circ\phi_\alpha^{-1}\big)(x)D^{\gamma-\delta}\eta_\alpha(t,x)\,.$$

Moreover, $(B_1\widetilde{\eta}_\alpha)(t,q) = 0$ for any $t\in[0,T]$ and $q\in\partial M$ with $q\notin U_\alpha$. The embedding (2.28) yields for every $t\in[0,T]$, $|\gamma|\le 1$, $D^\gamma\eta_\alpha(t)\in C^2\big(\overline{\phi_\alpha(U_\alpha)}\big)\subset W_p^{2-1/p}(\partial\phi_\alpha(U_\alpha))$ and further $D^\gamma\eta_\alpha\in C\big([0,T];C^2\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$. Hence, we obtain $R^2_{T,\alpha}(G)\in L_p\big((0,T);W_p^{2-1/p}\big(\partial\phi_\alpha(U_\alpha)\big)\big)$ with

$$\big\|R^2_{T,\alpha}(G)\big\|_{L_p((0,T);W_p^{2-1/p}(\partial\phi_\alpha(U_\alpha)))} \le C\sum_{|\gamma|\le 1}\int_0^T \|D^\gamma\eta_\alpha(t)\|^p_{W_p^{2-1/p}(\partial\phi_\alpha(U_\alpha))}\,\mathrm{d}t$$
$$\le C\sum_{|\gamma|\le 1}\int_0^T \|D^\gamma\eta_\alpha(t)\|^p_{C^2\left(\overline{\phi_\alpha(U_\alpha)}\right)}\,\mathrm{d}t \le CT\sum_{|\gamma|\le 1}\|D^\gamma\eta_\alpha\|^p_{C\left([0,T];C^2\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)}$$
$$\le CT\,\|\eta_\alpha\|^p_{C([0,T];C^3\left(\overline{\phi_\alpha(U_\alpha)}\right))} \le C(T_1)T\,|||\eta|||^p_{\mathbb{E}_{T,\alpha}} \le C(T_1)T\big(\|F_\alpha\|_{X_{T,\alpha}} + \|G_\alpha\|_{Y_{1,T,\alpha}}\big)^p .$$

Furthermore, Corollary B.39 yields that there exists $\sigma\in(1/2,1)$ such that

$$\big(\mathbb{E}_{T,\alpha},\,|||\cdot|||_{\mathbb{E}_{T,\alpha}}\big)\hookrightarrow C^\sigma\big([0,T];C^1\big(\overline{\phi_\alpha(U_\alpha)}\big)\big)$$

with uniform constant $C(T_1)$. Using further Proposition C.32 we obtain that $R^2_{T,\alpha}(G)$ lies in $W_p^{1/2-1/4p}((0,T);L_p(\partial\phi_\alpha(U_\alpha)))$ and that there exists $\varepsilon\in(0,1)$ such that

$$\big\|R^2_{T,\alpha}(G)\big\|_{W_p^{1/2-1/4p}((0,T);L_p(\partial\phi_\alpha(U_\alpha)))} \le CT^\varepsilon\sum_{|\gamma|\le 1}\|D^\gamma\eta_\alpha\|_{C^\sigma([0,T];L_p(\partial\phi_\alpha(U_\alpha)))}$$
$$\le CT^\varepsilon\sum_{|\gamma|\le 1}\|D^\gamma\eta_\alpha\|_{C^\sigma\left([0,T];C\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \le CT^\varepsilon\,\|\eta_\alpha\|_{C^\sigma\left([0,T];C^1\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \le C(T_1)T^\varepsilon\,|||\eta_\alpha|||_{\mathbb{E}_{T,\alpha}}$$

$$\leq C(T_1)T^{\varepsilon}\big(\, \|F_\alpha\|_{X_{T,\alpha}} + \|G_\alpha\|_{Y_{1,T,\alpha}} \big) .$$

We conclude that there exists $\varepsilon \in (0,1)$ such that

$$\left\| R^2_{T,\alpha}(G) \right\|_{Y_{1,T,\alpha}} \leq C(T_1)T^{\varepsilon}\big(\, \|F_\alpha\|_{X_{T,\alpha}} + \|G_\alpha\|_{Y_{1,T,\alpha}} \big) \leq C(Q,T_1)T^{\varepsilon}\big(\, \|F\|_{X_T} + \|G\|_{Y_{1,T}} \big) .$$

In particular, we obtain

$$(t,x') \mapsto R^2_{T,\alpha}(G)\,(t,(x',0)) \in W_p^{1/2-1/4p,2-1/p}\left((0,T)\times \sigma_\alpha\left(V_\alpha\right)\right) .$$

As for all $\delta \in \mathbb{N}_0^n$

$$\operatorname{dist}\left(\operatorname{supp}\left(D^\delta\left(\xi_\alpha \circ \phi_\alpha^{-1}\right)\right) \cap \left(\overline{\sigma_\alpha\left(V_\alpha\right)} \times \{0\}\right), \partial\sigma_\alpha\left(V_\alpha\right)\times\{0\}\right) > 0 \,,$$

Lemma C.22 yields that the trivial in space extension of

$$(t,q) \to R^2_{T,\alpha}(G)\,(t,\phi_\alpha(q))$$

to the entire manifold ∂M lies in $Y_{1,T}$ with norm bounded by $C(Q)\left\| R^2_{T,\alpha}(G)\right\|_{Y_{1,T,\alpha}}$. In particular,

$$(t,q) \mapsto R^2_T(G)(t,q) := \sum_{\alpha=1}^{N} R^2_{T,\alpha}(G)\,(t,\phi_\alpha(q))$$

lies in $Y_{1,T}$ with

$$\left\| R^2_T(G)\right\|_{Y_{1,T}} \leq C(Q)\sum_{\alpha=1}^{N}\left\| R^2_{T,\alpha}(G)\right\|_{Y_{1,T,\alpha}} \leq C(Q,T_1)T^{\varepsilon}\big(\, \|F\|_{X_T} + \|G\|_{Y_{1,T}} \big) .$$

We define $R_T : Z_T \to X_T \times Y_{1,T}$ by

$$R_T((F,G)) := \left(R^1_T(F), R^2_T(G)\right) .$$

As $R^2_{T,\alpha}(G)(0) = 0$ for all $\alpha \in \{1,\dots,N\}$, we obtain $R^2_T(G)(0) = 0$ which yields $R_T\left(Z_T\right) \subset Z_T$. The above estimates show that R_T is continuous with

$$\|R_T(F,G)\|_{X_T\times Y_{1,T}} \leq C(Q,T_1)T^{\varepsilon}\,\|(F,G)\|_{X_T\times Y_{1,T}}$$

and furthermore, the linearity of the local solution operators from Theorem 2.50 yield that R_T is linear. Let $\widetilde{T}_0 \in (0,T_1]$ be such that $C(Q,T_1)\widetilde{T}_0^{\varepsilon} < 1$. Then for all $T \in (0,\widetilde{T}_0]$ the operator $R_T \in \mathcal{L}\left(Z_T\right)$ satisfies $\|R_T\|_{\mathcal{L}(Z_T)} < 1$, and thus $\mathrm{Id}_{Z_T} + R_T \in \mathcal{L}\left(Z_T\right)$ is invertible for all $T \in (0,\widetilde{T}_0]$. The above calculations imply for almost every $t \in (0,T)$, $q \in M$,

$$\begin{aligned}
&\partial_t\eta(t,q) + A(\eta)(t,q) = \sum_{\alpha=1}^{N} \partial_t\widetilde{\eta}_\alpha(t,q) + A(\widetilde{\eta}_\alpha)(t,q) \\
&= \sum_{\alpha=1}^{N} \xi_\alpha(q)\left(\partial_t\eta_\alpha + A_\alpha\eta_\alpha\right)(t,\phi_\alpha(q)) + R^1_{T,\alpha}(F)(t,\phi_\alpha(q)) = F(t,q) + R^1_T(F)(t,q) \,,
\end{aligned}$$

and for $t \in [0,T]$ and $q \in \partial M$,

$$\begin{aligned}
&\left(B_1\eta\right)(t,q) = \sum_{\alpha=1}^{N}\left(B_1\widetilde{\eta}_\alpha\right)(t,q) = \sum_{\alpha=1}^{N}\xi_\alpha(q)\left(B_{1,\alpha}\eta_\alpha\right)(t,\phi_\alpha(q)) + R^2_{T,\alpha}(G)\,(t,\phi_\alpha(q)) \\
&= \sum_{\alpha=1}^{N}\xi_\alpha(q)\psi_\alpha(q)G(t,q) + R^2_T(G)(t,q) = G(t,q) + R^2_T(G)(t,q) \,,
\end{aligned}$$

which yields

$$(\partial_t + A, B_1)\,\eta = (\mathrm{Id}_{Z_T} + R_T)\,(F, G)\,.$$

Given $T \in (0, \widetilde{T}_0]$ and $(f, 0) \in Z_T$ there exists $(F, G) \in Z_T$ with

$$(\mathrm{Id}_{Z_T} + R_T)\,(F, G) = (f, 0)\,.$$

The above argumentation yields that there exists $\eta \in \mathbb{E}_T$ with

$$L_T\eta = ((\mathrm{Id}_{Z_T} + R_T)\,(F, G), 0, 0) = (f, 0, 0, 0)\,.$$

□

Lemma 2.55 (Surjectivity of the linear operator)**.** *Let (M, φ) be a smooth reference geometry of dimension $n \geq 2$, T_1 be positive and $p \in (4+n, \infty)$. There exists a time $\widetilde{T}_0 \in (0, T_1]$ such that for all $T \in (0, \widetilde{T}_0]$ the map $L_T \in \mathcal{L}(\mathbb{E}_T, \mathbb{F}_T)$ is surjective.*

Proof. Let $\widetilde{T}_0 \in (0, T_1]$ be as in Lemma 2.54, $T \in (0, \widetilde{T}_0]$ and $(f, h_1, h_2, u_0) \in \mathbb{F}_T$ be given. Proposition 2.53 implies the existence of $v \in \mathbb{E}_T$ with $B_i v = h_i$, $i \in \{1, 2\}$, and $v_{|t=0} = u_0$. By Lemma 2.27, the function $f - (\partial_t + A)v$ lies in X_T and Lemma 2.54 yields that there exists $\eta \in \mathbb{E}_T$ with $L_T\eta = (f - (\partial_t + A)v, 0, 0, 0)$. Then $u := v + \eta$ lies in $\mathbb{E}_T$ and by linearity of L_T we obtain $L_T u = (f, h_1, h_2, u_0)$. □

Lemma 2.56 (Injectivity of the linear operator)**.** *Let (M, φ) be a smooth reference geometry of dimension $n \geq 2$, T_1 be positive and $p \in (4+n, \infty)$. There exists a time $\widehat{T}_0 \in (0, T_1]$ such that for all $T \in (0, \widehat{T}_0]$, the operator $L_T \in \mathcal{L}(\mathbb{E}_T, \mathbb{F}_T)$ is injective.*

Proof. Let $T \in (0, T_1]$ and $u \in \mathbb{E}_T$ with $L_T u = (0, 0, 0, 0) \in \mathbb{F}_T$. Proposition C.16 implies for every chart $(U_\alpha, \phi_\alpha) \in \mathcal{T}$ that the function

$$(t, x) \mapsto u_\alpha(t, x) := \psi_\alpha\left(\phi_\alpha^{-1}(x)\right) u\left(t, \phi_\alpha^{-1}(x)\right)$$

lies in $\mathbb{E}_{T,\alpha}$. Furthermore, using Corollary 2.32 we obtain for almost every $t \in (0, T)$, $x \in \phi_\alpha(U_\alpha)$,

$$\begin{aligned}(\partial_t + A_\alpha)\,(u_\alpha)\,(t, x) &= \psi_\alpha\left(\phi_\alpha^{-1}(x)\right)\left(\partial_t u\left(t, \phi_\alpha^{-1}(x)\right) + \mathcal{A}_\alpha(x, D)u\left(t, \phi_\alpha^{-1}(x)\right)\right) + R^1_{T,\alpha}(u)(t, x)\\ &= \psi_\alpha\left(\phi_\alpha^{-1}(x)\right)\left(\partial_t u\left(t, \phi_\alpha^{-1}(x)\right) + (Au)\left(t, \phi_\alpha^{-1}(x)\right)\right) + R^1_{T,\alpha}(u)(t, x)\\ &= R^1_{T,\alpha}(u)(t, x)\end{aligned}$$

where for $t \in (0, T)$, $x \in \phi_\alpha(U_\alpha)$

$$R^1_{T,\alpha}(u)(t, x) := \sum_{|\beta| \leq 4} c_{\beta,\alpha}\left(\phi_\alpha^{-1}(x)\right) \sum_{\substack{\gamma+\delta=\beta\,,\\ \gamma \neq 0}} \binom{\beta}{\gamma} D^\gamma\left(\psi_\alpha \circ \phi_\alpha^{-1}\right)(x) D^\delta\left(u(t) \circ \phi_\alpha^{-1}\right)(x)\,.$$

The functions $c_{\beta,\alpha} \circ \phi_\alpha^{-1}$ and $\psi_\alpha \circ \phi_\alpha^{-1}$ are smooth on $\overline{\phi_\alpha(U_\alpha)}$ and Corollary B.38 implies the embedding

$$\left(\mathbb{E}_{T,\alpha},\, \|\!| \cdot |\!\|_{\mathbb{E}_{T,\alpha}}\right) \hookrightarrow C\left([0, T]; C^3\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)$$

with embedding constant $C(T_1)$ independent of $T \in (0, T_1]$. Using

$$\|\!|(t, x) \mapsto u(t, \phi_\alpha^{-1}(x)|\!\|_{\mathbb{E}_{T,\alpha}} \leq C(Q)\, \|\!|u|\!\|_{\mathbb{E}_T}$$

we obtain $R^1_{T,\alpha}(u) \in X_{T,\alpha}$ with $\left\|R^1_{T,\alpha}(u)\right\|^p_{X_{T,\alpha}} \leq C(Q, T_1)T\, \|\!|u|\!\|^p_{\mathbb{E}_T}$. We further have for $t \in (0, T)$, $x \in \partial\phi_\alpha(U_\alpha)$,

$$(B_{1,\alpha}u_\alpha)\,(t, x) = \mathcal{B}_{1,\alpha}(x, D)\left(\psi_\alpha\left(\phi_\alpha^{-1}(x)\right) u\left(t, \phi_\alpha^{-1}(x)\right)\right)$$

$$= \psi_\alpha\left(\phi_\alpha^{-1}(x)\right)\mathcal{B}_{1,\alpha}(x,D)u\left(t,\phi_\alpha^{-1}(x)\right) + R_{T,\alpha}^2(u)(t,x)$$

where for $t \in (0,T)$, $x \in \partial\phi_\alpha(U_\alpha)$

$$R_{T,\alpha}^2(u)(t,x) = \sum_{|\gamma|\le 2} b_{\gamma,\alpha}\left(\phi_\alpha^{-1}(x)\right) \sum_{\substack{0\le\delta\le\gamma\\ \delta\neq 0}} \binom{\gamma}{\delta} D^\delta\left(\psi_\alpha\circ\phi_\alpha^{-1}\right)(x) D^{\gamma-\delta}u\left(t,\phi_\alpha^{-1}(x)\right).$$

The next step is to show that there exists $\varepsilon > 0$ such that

$$\left\|R_{T,\alpha}^2(u)\right\|_{Y_{1,T,\alpha}} \le C(Q,T_1)T^\varepsilon \left\|\left|u\right|\right\|_{\mathbb{E}_T}.$$

Proposition C.16 implies that $(t,x) \mapsto \mathfrak{u}_\alpha(t,x) := u\left(t,\phi_\alpha^{-1}(x)\right)$ lies in $\mathbb{E}_{T,\alpha}$ and Corollary B.38 implies $\mathfrak{u}_\alpha \in C\big([0,T];W_p^{4-4/p}\big(\phi_\alpha(U_\alpha)\big)\big)$ with

$$\left\|\mathfrak{u}_\alpha\right\|_{C\left([0,T];W_p^{4-4/p}(\phi_\alpha(U_\alpha))\right)} \le C(T_1)\left\|\left|\mathfrak{u}_\alpha\right|\right\|_{\mathbb{E}_{T,\alpha}}.$$

To estimate the norm of $R_{T,\alpha}^2(u)$ in $L_p\big((0,T);W_p^{2-1/p}\big(\partial\phi_\alpha(U_\alpha)\big)\big)$, we observe that Corollary B.38 yields for every $t \in [0,T]$ and $\gamma \in \mathbb{N}_0^n$, $|\gamma| \le 1$,

$$\left\|D^\gamma\mathfrak{u}_\alpha(t)\right\|_{W_p^{2-1/p}(\partial\phi_\alpha(U_\alpha))} \le C_\alpha\left\|\mathfrak{u}_\alpha(t)\right\|_{C^3\left(\overline{\phi_\alpha(U_\alpha)}\right)} \le C(T_1)\left\|\left|\mathfrak{u}_\alpha\right|\right\|_{\mathbb{E}_{T,\alpha}}.$$

As $b_{\gamma,\alpha}\circ\phi_\alpha^{-1}$ and $\psi_\alpha\circ\phi_\alpha^{-1}$ are smooth on $\overline{\phi_\alpha(U_\alpha)}$, we obtain

$$\begin{aligned}
&\left\|R_{T,\alpha}^2(u)\right\|_{L_p\left((0,T);W_p^{2-1/p}(\partial\phi_\alpha(U_\alpha))\right)}^p \le C\sum_{|\gamma|\le 1}\int_0^T \left\|D^\gamma\left(u(t)\circ\phi_\alpha^{-1}\right)\right\|_{W_p^{2-1/p}(\partial\phi_\alpha(U_\alpha))}^p \,\mathrm{d}t\\
&\le C\sum_{|\gamma|\le 1}\int_0^T \left\|D^\gamma\left(u(t)\circ\phi_\alpha^{-1}\right)\right\|_{C^2\left(\overline{\phi_\alpha(U_\alpha)}\right)}^p \,\mathrm{d}t \le CT\left\|\mathfrak{u}_\alpha\right\|_{C\left([0,T];C^3\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)}\\
&\le C(T_1)T\left\|\left|\mathfrak{u}_\alpha\right|\right\|_{\mathbb{E}_{T,\alpha}}^p \le C(Q,T_1)T\left\|\left|u\right|\right\|_{\mathbb{E}_T}^p.
\end{aligned}$$

Corollary B.39 yields that there exists $\sigma \in (1/2, 1)$ such that

$$\left(\mathbb{E}_{T,\alpha}, \left\|\left|\cdot\right|\right\|_{\mathbb{E}_{T,\alpha}}\right) \hookrightarrow C^\sigma\left([0,T];C^1\left(\overline{\phi_\alpha\left(U_\alpha\right)}\right)\right)$$

with uniform constant $C(T_1)$. Proposition C.32 then implies

$$R_{T,\alpha}^2(u) \in W_p^{1/2-1/4p}\left((0,T);L_p\left(\partial\phi_\alpha\left(U_\alpha\right)\right)\right)$$

and

$$\begin{aligned}
&\left\|R_{T,\alpha}^2(u)\right\|_{W_p^{1/2-1/4p}((0,T);L_p(\partial\phi_\alpha(U_\alpha)))}^p \le CT^\varepsilon\sum_{|\gamma|\le 1}\left\|D^\gamma\mathfrak{u}_\alpha\right\|_{C^\sigma([0,T];L_p(\partial\phi_\alpha(U_\alpha)))}\\
&\le CT^\varepsilon\left\|\mathfrak{u}_\alpha\right\|_{C^\sigma\left([0,T];C^1\left(\overline{\phi_\alpha(U_\alpha)}\right)\right)} \le C(T_1)T^\varepsilon\left\|\left|\mathfrak{u}_\alpha\right|\right\|_{\mathbb{E}_{T,\alpha}} \le C(Q,T_1)T^\varepsilon\left\|\left|u\right|\right\|_{\mathbb{E}_T}
\end{aligned}$$

for some $\varepsilon \in (0,1)$. This shows the desired estimate $\left\|R_{T,\alpha}^2(u)\right\|_{Y_{1,T,\alpha}} \le C(Q,T_1)T^\varepsilon\left\|\left|u\right|\right\|_{\mathbb{E}_T}$. For every $x \in \phi_\alpha\left(U_\alpha\right)\cap\partial\mathbb{H}^n$ it holds for $i \in \{1,2\}$,

$$\mathcal{B}_{i,\alpha}(x,D)u\left(t,\phi_\alpha^{-1}(x)\right) = \left(B_i u\right)\left(t,\phi_\alpha^{-1}(x)\right) = 0.$$

As $\psi_\alpha\circ\phi_\alpha^{-1}$ vanishes on $\partial\phi_\alpha\left(U_\alpha\right)\cap\mathrm{int}\mathbb{H}^n$, we obtain for all $t \in (0,T)$ and $x \in \partial\phi_\alpha(U_\alpha)$,

$$\left(B_{1,\alpha}u_\alpha\right)(t,x) = \psi_\alpha\left(\phi_\alpha^{-1}(x)\right)\mathcal{B}_{1,\alpha}(x,D)u\left(t,\phi_\alpha^{-1}(x)\right) + R_{T,\alpha}^2(u)(t,x) = R_{T,\alpha}^2(u)(t,x)$$

and

$$(B_{2,\alpha}u_\alpha)(t,x) = u_\alpha(t,x) = \psi_\alpha\left(\phi_\alpha^{-1}(x)\right) u\left(t,\phi_\alpha^{-1}(x)\right) = 0\,.$$

Proposition C.16 and the uniform estimate in Theorem 2.50 imply

$$\begin{aligned}\|\!|u|\!\|_{\mathbb{E}_T} &\leq C(Q)\sum_{\alpha=1}^{N} \|\!|u_\alpha|\!\|_{\mathbb{E}_{T,\alpha}} \\ &\leq C(Q,T_1)\sum_{\alpha=1}^{N} \|(\partial_t + A_\alpha)\,u_\alpha\|_{X_{T,\alpha}} + \left\|(u_\alpha)_{|t=0}\right\|_{X_{0,\alpha}} + \|B_{1,\alpha}u_\alpha\|_{Y_{1,T,\alpha}} + \|B_{2,\alpha}u_\alpha\|_{Y_{2,T,\alpha}} \\ &\leq C(Q,T_1)\sum_{\alpha=1}^{N} \left\|R^1_{T,\alpha}(u)\right\|_{X_{T,\alpha}} + \left\|R^2_{T,\alpha}(u)\right\|_{Y_{1,T,\alpha}} \leq C(Q,T_1)T^\varepsilon\,\|\!|u|\!\|_{\mathbb{E}_T}\,.\end{aligned}$$

Choosing $\widehat{T}_0 \in (0,T_1]$ so small that $C(Q,T_1)\widehat{T}_0^\varepsilon < 1$ we obtain $u \equiv 0$. □

To obtain that the inverse of the linear operator is continuous with operator norm independent of the considered time interval, we have to continuously extend in time a given right hand side. To this end we make use of the extension operator constructed in Proposition B.37 and Proposition C.25 that is continuous with a constant independent of $T \in (0,T_0)$ once we endow $Y_{i,T}$, $i \in \{1,2\}$, with the norms

$$\|\!|h_i|\!\|_{Y_{i,T}} := \|h_i\|_{Y_{i,T}} + \|(h_i)\,(0)\|_{W_p^{2i-5/p}(\partial M)}\,.$$

Theorem 2.57 (Uniform well-posedness of the linear problem)**.** *Let (M,φ) be a smooth reference geometry of dimension $n \geq 2$, $p \in (4+n,\infty)$, and T_1 be positive. There exists a time $T_0 \in (0,T_1]$ such that for all $T \in (0,T_0]$ the map $L_T \in \mathcal{L}(\mathbb{E}_T,\mathbb{F}_T)$ defined in Proposition 2.43 is an isomorphism. Furthermore, there exists a constant $C(T_0)$ independent of $T \in (0,T_0]$ such that for all $u \in \mathbb{E}_T$ with $L_Tu = (f,h_1,h_2,u_0) \in \mathbb{F}_T$ there holds*

$$\|\!|u|\!\|_{\mathbb{E}_T} \leq C(T_0)\left(\|f\|_{X_T} + \|\!|h_1|\!\|_{Y_{1,T}} + \|\!|h_2|\!\|_{Y_{2,T}} + \|u_0\|_{X_0}\right). \tag{2.29}$$

Proof. Let T_1 be positive and $T_0 := \min\{\widetilde{T}_0,\widehat{T}_0\}$ with $\widetilde{T}_0$ and $\widehat{T}_0$ as in Lemma 2.55 and 2.56, respectively. Given $T \in (0,T_0]$, Proposition 2.43, Lemma 2.55 and 2.56 imply that the map $L_T : \mathbb{E}_T \to \mathbb{F}_T$ is a bijective continuous linear operator between Banach spaces. The Open Mapping Theorem [24, Corollary 2.7] yields that $L_T^{-1} : \mathbb{F}_T \to \mathbb{E}_T$ is continuous. In particular, given $u \in \mathbb{E}_T$ with $L_Tu = (f,h_1,h_2,u_0) \in \mathbb{F}_T$ there holds the estimate

$$\|\!|u|\!\|_{\mathbb{E}_T} \leq C(T)\left(\|f\|_{X_T} + \|\!|h_1|\!\|_{Y_{1,T}} + \|\!|h_2|\!\|_{Y_{2,T}} + \|u_0\|_{X_0}\right). \tag{2.30}$$

It remains to show that the constant in (2.30) is independent of $T \in (0,T_0]$. Let $T \in (0,T_0]$, $(f,h_1,h_2,u_0) \in \mathbb{F}_T$ be given and $u \in \mathbb{E}_T$ with $L_Tu = (f,h_1,h_2,u_0)$. Proposition C.25 implies the existence of $\boldsymbol{E}h_i \in Y_{i,T_0}$, $i \in \{1,2\}$, with $(\boldsymbol{E}h_i)_{|(0,T)} = h_i$ and

$$\|\!|\boldsymbol{E}h_i|\!\|_{Y_{i,T_0}} \leq C(T_0)\,\|\!|h_i|\!\|_{Y_{i,T}}\,. \tag{2.31}$$

In particular, $(f,\boldsymbol{E}h_1,\boldsymbol{E}h_2,u_0)$ lies in $\mathbb{F}_{T_0}$ and there exists $\widetilde{u} \in \mathbb{E}_{T_0}$ with $L_{T_0}\widetilde{u} = (f,\boldsymbol{E}h_1,\boldsymbol{E}h_2,u_0)$. The estimates (2.30) and (2.31) imply

$$\begin{aligned}\|\!|\widetilde{u}|\!\|_{\mathbb{E}_{T_0}} &\leq C(T_0)\left(\|f\|_{X_{T_0}} + \|\!|\boldsymbol{E}h_1|\!\|_{Y_{1,T_0}} + \|\!|\boldsymbol{E}h_2|\!\|_{Y_{2,T_0}} + \|u_0\|_{X_0}\right) \\ &\leq C(T_0)\left(\|f\|_{X_T} + \|\!|h_1|\!\|_{Y_{1,T}} + \|\!|h_2|\!\|_{Y_{2,T}} + \|u_0\|_{X_0}\right).\end{aligned}$$

By definition of the operators L_T and L_{T_0} we have

$$L_T\left(\widetilde{u}_{|(0,T)}\right) = (L_{T_0}\widetilde{u})_{|(0,T)} = (f, \boldsymbol{E}h_1, \boldsymbol{E}h_2, u_0)_{|(0,T)} = (f, h_1, h_2, u_0)\,.$$

As L_T is an isomorphism, this yields $u = \widetilde{u}_{|(0,T)}$ in $\mathbb{E}_T$ and in particular

$$|||u|||_{\mathbb{E}_T} \leq |||\widetilde{u}|||_{\mathbb{E}_{T_0}} \leq C(T_0)\left(\|f\|_{X_T} + |||h_1|||_{Y_{1,T}} + |||h_2|||_{Y_{2,T}} + \|u_0\|_{X_0}\right).$$

□

2.4 Existence of graph solutions

This section is devoted to deduce the existence of strong solutions to the graph formulation (1.6) from the well-posedness of the linearised system associated to (1.6) with the help of a contraction argument. To this end, we observe that a function $\varrho \in \boldsymbol{U}_T$ with $\varrho_{|t=0} = \varrho_0$ is a solution the graph formulation (1.6) of (W) with initial datum ϱ_0 if and only if

$$L_T(\varrho) = (\partial_t\varrho + A(\varrho) - N_{T,1}(\varrho), B_1(\varrho) - N_{T,2}(\varrho), 0, \varrho_0)\ .$$

Once it is confirmed that the right hand side satisfies the linear compatibility conditions (2.20) and (2.21), we may invert the operator L_T and end up with a fixed point equation

$$\varrho = K_T(\varrho) = L_T^{-1}\left((\partial_t\varrho + A(\varrho) - N_{T,1}(\varrho), B_1(\varrho) - N_{T,2}(\varrho), 0, \varrho_0)\right).$$

It then remains to verify that K_T is a contractive self-mapping on an appropriate closed subset of the space $\mathbb{E}_T$ for T sufficiently small. In the following we show that the subset we consider in the proof of Main Theorem 1 is non-empty. To this end we need the following Lemma.

Lemma 2.58. *Let $T_1 > 0$, $T \in (0, T_1]$, $p \in (4+n, \infty)$, $\alpha \in \{1, \dots, N\}$ and A_α be the operator defined in Proposition 2.35. Then for every $f_\alpha \in L_p\left((0,T); L_p\left(\phi_\alpha(U_\alpha)\right)\right)$, $\eta_{0,\alpha} \in W_p^{4-4/p}\left(\phi_\alpha(U_\alpha)\right)$ there exists a function $\eta_\alpha \in \mathbb{E}_{T,\alpha}$ such that*

$$(\partial_t + A_\alpha)\,\eta_\alpha = f_\alpha, \qquad (\eta_\alpha)_{|t=0} = \eta_{0,\alpha}\,,$$

satisfying the estimate

$$\|\eta_\alpha\|_{\mathbb{E}_{T,\alpha}} \leq C(T_1)\left(\|f_\alpha\|_{L_p((0,T);L_p(\phi_\alpha(U_\alpha)))} + \|\eta_{0,\alpha}\|_{W_p^{4-4/p}(\phi_\alpha(U_\alpha))}\right) \tag{2.32}$$

with constant $C(T_1)$ independent of T.

Proof. To make use of [136, Theorem 5.5] we need to consider an associated Cauchy problem in the full space $\mathbb{R}^n$. To this end we extend the coefficients $\widetilde{c}_{\beta,\alpha} := c_{\beta,\alpha} \circ \phi_\alpha^{-1} \in C^\infty\left(\overline{\phi_\alpha(U_\alpha)}\right)$ of $\mathcal{A}_\alpha(x, D)$ continuously to $\widetilde{c}_\beta \in C(\mathbb{R}^n)$. As $W_\alpha := \phi_\alpha(U_\alpha)$ is a bounded smooth domain, there exists $\delta > 0$ such that the nearest point projection

$$P_{\partial W_\alpha} : (W_\alpha)_\delta \setminus W_\alpha \to \partial W_\alpha$$

is well-defined and continuous where

$$(W_\alpha)_\delta := \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial W_\alpha) < \delta\}\,.$$

Let $\psi \in C_0^\infty(\mathbb{R};\mathbb{R})$ satisfy $\psi = 1$ on $[0, \frac{\delta}{2}]$ and $\psi = 0$ on $[\delta, \infty)$ and let $x_0 \in W_\alpha$ be fixed. For $|\beta| = 4$ we define

$$\widetilde{c}_\beta(x) = \chi_{W_\alpha}(x)\widetilde{c}_{\beta,\alpha}(x)$$

$$+ (1-\chi_{W_\alpha}(x)) \left(\psi\left(\operatorname{dist}(x,\partial W_\alpha)\right)\widetilde{c}_{\beta,\alpha}\left(P_{\partial W_\alpha}(x)\right) + \left(1-\psi\left(\operatorname{dist}(x,\partial W_\alpha)\right)\right)\widetilde{c}_{\beta,\alpha}(x_0)\right)$$

where $\chi_{W_\alpha}(x) = 1$ if $x \in W_\alpha$ and zero otherwise. In the case $|\beta| = k < 4$ we set

$$\widetilde{c}_\beta(x) = \chi_{W_\alpha}(x)\widetilde{c}_{\beta,\alpha}(x) + (1-\chi_{W_\alpha}(x))\,\psi\left(\operatorname{dist}(x,\partial W_\alpha)\right)\widetilde{c}_{\beta,\alpha}\left(P_{\partial W_\alpha}(x)\right).$$

Then the operator

$$\widetilde{A}_\alpha : W_p^{1,4}\left((0,T)\times\mathbb{R}^n\right) \to L_p\left((0,T);L_p(\mathbb{R}^n)\right),\ \eta \mapsto (\widetilde{A}_\alpha\eta)(t,x) := \sum_{|\beta|\le 4}\widetilde{c}_\beta(x)D^\beta\eta(t,x)$$

has continuous coefficients and Proposition 2.45 yields for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$ the estimate

$$Q^{-2}|\xi|^4 \le \sum_{|\beta|=4}\widetilde{c}_\beta(x)\xi^\beta \le Q^2|\xi|^4\,.$$

Let $\widetilde{f}_\alpha \in L_p\left((0,T_1);L_p(\mathbb{R}^n)\right)$ be the trivial extension of f_α. By Definition B.14 there exists a function $\widetilde{\eta}_{0,\alpha} \in W_p^{4-4/p}(\mathbb{R}^n)$ such that $R\widetilde{\eta}_{0,\alpha} = \eta_{0,\alpha}$ and $\|\widetilde{\eta}_{0,\alpha}\|_{W_p^{4-4/p}(\mathbb{R}^n)} \le 2\,\|\eta_{0,\alpha}\|_{W_p^{4-4/p}(\phi_\alpha(U_\alpha))}$ where R denotes the restriction to the domain $\phi_\alpha(U_\alpha)$. By [136, Theorem 5.5] there exists a unique solution $\widetilde{\eta}_\alpha \in W_p^{1,4}\left((0,T_1)\times\mathbb{R}^n\right)$ to the initial value problem

$$(\partial_t + \widetilde{A}_\alpha)\widetilde{\eta}_\alpha = \widetilde{f}_\alpha\,, \qquad (\widetilde{\eta}_\alpha)_{|t=0} = \widetilde{\eta}_{0,\alpha}\,,$$

satisfying the estimate

$$\|\widetilde{\eta}_\alpha\|_{W_p^{1,4}((0,T_1)\times\mathbb{R}^n)} \le C(T_1)\left(\|\widetilde{f}_\alpha\|_{L_p((0,T_1);L_p(\mathbb{R}^n))} + \|\widetilde{\eta}_{0,\alpha}\|_{W_p^{4-4/p}(\mathbb{R}^n)}\right).$$

Then the restricted function $\eta_\alpha := R\widetilde{\eta}_\alpha \in \mathbb{E}_{T,\alpha}$ is a desired solution to our problem satisfying the uniform estimate (2.32). □

Proposition 2.59. *Let (M,φ) be a smooth reference geometry of dimension $n \in \mathbb{N}$, $n \ge 2$, $p \in (4+n,\infty)$, T_0 be positive, $T \in (0,T_0)$ and $r > 0$. There exists $\varepsilon_0(r) > 0$ such that for all $\varrho_0 \in W_p^{4-4/p}(M)$ with $\|\varrho_0\|_{W_p^{4-4/p}(M)} < \varepsilon_0(r)$, the set*

$$\mathbb{X}_{T,r} := \left\{\varrho \in \mathbb{E}_T : \ |||\varrho|||_{\mathbb{E}_T} \le r \text{ and } \varrho_{|t=0} = \varrho_0\right\} \tag{2.33}$$

is non-empty. For $0 < r_1 \le r_2$ one may choose $\varepsilon_0(r_1) \le \varepsilon_0(r_2)$.

Proof. Let $\varrho_0 \in W_p^{4-4/p}(M)$ be given. For all $\alpha \in \{1,\dots,N\}$ the function $(\psi_\alpha\varrho_0)\circ\phi_\alpha^{-1}$ lies in $W_p^{4-4/p}(\phi_\alpha(U_\alpha))$. By Lemma 2.58 there exists $\widetilde{\varrho}_\alpha \in \mathbb{E}_{T_0,\alpha} = W_p^{1,4}\left((0,T_0)\times\phi_\alpha(U_\alpha)\right)$ such that

$$(\widetilde{\varrho}_\alpha)_{|t=0} = (\psi_\alpha\varrho_0)\circ\phi_\alpha^{-1}\,, \qquad \|\widetilde{\varrho}_\alpha\|_{\mathbb{E}_{T_0,\alpha}} \le C(T_0)\left\|(\psi_\alpha\varrho_0)\circ\phi_\alpha^{-1}\right\|_{W_p^{4-4/p}(\phi_\alpha(U_\alpha))}\,.$$

Let $\xi_\alpha \in C^\infty(M)$ satisfy $\operatorname{supp}\xi_\alpha \subset U_\alpha$ and $\xi_\alpha \equiv 1$ on $\operatorname{supp}\psi_\alpha$. As $(t,x)\mapsto \xi_\alpha\left(\phi_\alpha^{-1}(x)\right)\widetilde{\varrho}_\alpha(t,x)$ lies in $\mathbb{E}_{T_0,\alpha}$ and

$$\operatorname{dist}\left(\operatorname{supp}\left(\xi_\alpha\circ\phi_\alpha^{-1}\right)\widetilde{\varrho}_\alpha(t)\cap\overline{\phi_\alpha(U_\alpha)},\partial\phi_\alpha(U_\alpha)\cap\operatorname{int}\mathbb{H}^n\right) \ge \delta$$

for every $t \in (0,T_0)$ and a constant δ independent of t, Lemma C.22 implies that the trivial extension $E\left(\xi_\alpha\left(\widetilde{\varrho}_\alpha\circ\phi_\alpha\right)\right)$ of $\xi_\alpha\left(\widetilde{\varrho}_\alpha\circ\phi_\alpha\right)$ to the entire manifold M lies in $\mathbb{E}_{T_0} = W_p^{1,4}\left((0,T_0)\times M\right)$ with

$$\left\|E\left(\xi_\alpha\left(\widetilde{\varrho}_\alpha\circ\phi_\alpha\right)\right)\right\|_{\mathbb{E}_{T_0}} \le C(Q)\left\|\left(\xi_\alpha\circ\phi_\alpha^{-1}\right)\widetilde{\varrho}_\alpha\right\|_{\mathbb{E}_{T_0,\alpha}} \le C(Q)\,\|\widetilde{\varrho}_\alpha\|_{\mathbb{E}_{T_0,\alpha}}\,.$$

This allows us to conclude that the function ϱ defined by

$$\varrho(t,p) := \sum_{\alpha=1}^{N} \xi_\alpha(p)\widetilde{\varrho}_\alpha\left(t,\phi_\alpha(p)\right)$$

lies in $\mathbb{E}_{T_0}$ satisfying the estimate

$$\begin{aligned}\|\varrho\|_{\mathbb{E}_{T_0}} &\leq \sum_{\alpha=1}^{N} \left\|E\left(\xi_\alpha\left(\widetilde{\varrho}_\alpha\circ\phi_\alpha\right)\right)\right\|_{\mathbb{E}_{T_0}} \leq C(Q)\sum_{\alpha=1}^{N}\|\widetilde{\varrho}_\alpha\|_{\mathbb{E}_{T_0,\alpha}} \\ &\leq C(Q,T_0)\sum_{\alpha=1}^{N}\left\|(\psi_\alpha\varrho_0)\circ\phi_\alpha^{-1}\right\|_{W_p^{4-4/p}(\phi_\alpha(U_\alpha))} = C(Q,T_0)\,\|\varrho_0\|_{W_p^{4-4/p}(M)}\,.\end{aligned}$$

In particular, ϱ lies in $\mathbb{E}_T$ and is continuous on $[0,T]\times M$ with initial value

$$\varrho(0,p) = \sum_{\alpha=1}^{N}\xi_\alpha(p)\widetilde{\varrho}_\alpha\left(0,\phi_\alpha(p)\right) = \sum_{\alpha=1}^{N}\xi_\alpha(p)\left(\psi_\alpha\varrho_0\right)(p) = \varrho_0(p)$$

and

$$|||\varrho|||_{\mathbb{E}_T} = \|\varrho\|_{\mathbb{E}_T} + \|\varrho_0\|_{W_p^{4-4/p}(M)} \leq C(Q,T_0)\,\|\varrho_0\|_{W_p^{4-4/p}(M)} + \|\varrho_0\|_{W_p^{4-4/p}(M)}\,. \tag{2.34}$$

Suppose that $\|\varrho_0\|_{W_p^{4-4/p}(M)} < \varepsilon_0(r)$ with $0 < \varepsilon_0(r) < r\left(C(Q,T_0)+1\right)^{-1}$ and $C(Q,T_0)$ as in (2.34). The above arguments then yield the existence of $\varrho\in\mathbb{X}_{T,r}$. □

The uniform estimate (2.29) in Theorem 2.57 gives a bound in terms of the norm $|||\cdot|||_{Y_{1,T}}$. As the estimates on the Lipschitz constant of $N_{T,2} : \boldsymbol{U}_T \to Y_{1,T}$ are valid with respect to the norm $\|\cdot\|_{Y_{1,T}}$, an additional argument on the norm of $B_1(\varrho)(0)$ in $W_p^{2-5/p}(M)$ is needed.

Lemma 2.60. *Let (M,φ) be a smooth reference geometry of dimension $n\in\mathbb{N}$, $n\geq 2$, $p\in(4+n,\infty)$, T_0 be positive, $T\in(0,T_0)$ and $\varrho\in\mathbb{E}_T$ with $\varrho_{|t=0}=\varrho_0$ be given. There exists a constant $C=C(Q)>0$ such that*

$$\left\|\left(B_1\varrho\right)(0)\right\|_{W_p^{2-5/p}(\partial M)} \leq C\,\|\varrho_0\|_{W_p^{4-4/p}(M)}\,.$$

Proof. Let $(U_\alpha,\phi_\alpha)\in\mathcal{T}$ be a boundary chart. Proposition B.35 implies that

$$x\mapsto\varrho\left(0,\phi_\alpha^{-1}(x)\right) = \varrho_0\left(\phi_\alpha^{-1}(x)\right)\in W_p^{4-4/p}\left(\phi_\alpha\left(U_\alpha\right)\right)$$

and Theorem C.27 yields

$$x\mapsto\mathcal{B}_{1,\alpha}(x,D)\varrho_0\left(\phi_\alpha^{-1}(x)\right) = B_1(\varrho)(0)\left(\phi_\alpha^{-1}(x)\right)\in W_p^{2-5/p}\left(\partial\phi_\alpha\left(U_\alpha\right)\right) \tag{2.35}$$

with norm bounded by $C(Q,\varphi)\,\|\varrho_0\|_{W_p^{4-4/p}(M)}$. As the charts (V_α,σ_α), $\alpha\in\mathcal{J}$, form a normal covering of ∂M with $\sigma_\alpha\left(V_\alpha\right)\times\{0\} = \partial\phi_\alpha\left(U_\alpha\right)\cap\partial\mathbb{H}^n$ and $\sigma_\alpha^{-1}(x') = \phi_\alpha^{-1}\left((x',0)\right)$ for all $x'\in\sigma_\alpha\left(V_\alpha\right)$, Proposition C.12 yields that (2.35) implies in particular

$$x'\mapsto B_1(\varrho)(0)\left(\sigma_\alpha^{-1}(x')\right)\in W_p^{2-5/p}\left(\sigma_\alpha\left(V_\alpha\right)\right)$$

and

$$\left\|\left(B_1\varrho\right)(0)\right\|_{W_p^{2-5/p}(\partial M)} \leq C(Q)\sum_{(V_\alpha,\sigma_\alpha)}\left\|B_1(\varrho)(0)\circ\sigma_\alpha^{-1}\right\|_{W_p^{2-5/p}(\sigma_\alpha(V_\alpha))} \leq C(Q)\,\|\varrho_0\|_{W_p^{4-4/p}(M)}\,.$$

□

Proof of Main Theorem 1. Let $T_0 > 0$ be given. We intend to deduce existence of graph solutions from the well-posedness of the linearised graph formulation via a contraction argument. As the nonlinear operator N_T is only defined for functions in $\mathbb{E}_T$ that are small with respect to the norm $|||\cdot|||_{\mathbb{E}_T}$, we have to include this in the choice of the complete metric space $\mathbb{X}_{T,r}$ on which the Banach Fixed-Point Theorem should be applied meaning that the parameter r has to be chosen small enough. This is also mirrored in the necessity of the initial value to have small norm ε_0. Furthermore, smallness of the considered time interval $[0,T]$ is needed to obtain a contractive function. To make the choice of these parameters more precise, let $\varepsilon = \varepsilon(T_0, Q) > 0$ be as in Corollary 1.42, $C(T_0)$ be the constant from Theorem 2.57, $C > 0$ as in Lemma 2.60 and $C(Q, T_0, \varepsilon, \sigma)$ be a constant such that

$$\|DN_{T,1}\|_{C^{0,1}(\boldsymbol{U}_T;\mathcal{L}(\mathbb{E}_T,X_T))} + \|DN_{T,2}\|_{C^{0,1}(\boldsymbol{U}_T;\mathcal{L}(\mathbb{E}_T,Y_{1,T}))} \leq C(Q, T_0, \varepsilon, \sigma)\,,$$

where the operator norm in $\mathcal{L}(\mathbb{E}_T, Z)$ with $Z \in \{X_T, Y_{1,T}\}$ is considered with respect to $|||\cdot|||_{\mathbb{E}_T}$, $\|\cdot\|_{X_T}$ and $\|\cdot\|_{Y_{1,T}}$. The existence of $C(Q, T_0, \varepsilon, \sigma)$ follows from Lemma 2.29 and 2.30. Let further

$$r \in \left(0, \min\left\{\varepsilon, 1, \frac{1}{2}\left(C(T_0)C(Q, T_0, \varepsilon, \sigma)\right)^{-1}\right\}\right)$$

be fixed and ϱ_0 a given admissible initial height function with $\|\varrho_0\|_{W_p^{4-4/p}(M)} < \varepsilon_0$ where

$$\varepsilon_0 \in \left(0, \min\left\{\varepsilon_0(r)\,(C+1)^{-1}, r\,(4C(T_0)\,(C+1))^{-1}\right\}\right),$$

with $\varepsilon_0(r)$ as in Proposition 2.59. As $N_T(0)$ is a smooth time-independent function on (M, φ), there exists a constant $C(\varphi) > 0$, such that

$$\|N_{T,1}(0)\|_{X_T} + \|N_{T,2}(0)\|_{Y_{1,T}} \leq C(\varphi)T^{1/p}\,.$$

We consider the time interval $[0,T]$ with

$$T \in \left(0, \min\left\{T_0, \left(4r^{-1}C(T_0)C(\varphi)\right)^{-p}\right\}\right).$$

Proposition 2.59 implies that with this choice of r, ϱ_0 and T, the set $\mathbb{X}_{T,r}$ defined in (2.33) is a non-empty metric space, that is complete being a closed subset of the Banach space $\mathbb{E}_T$. Moreover, $\mathbb{X}_{T,r}$ forms a subset of $\boldsymbol{U}_T$ as defined in Corollary 1.42 which yields that

$$N_T = (N_{T,1}, N_{T,2}) : \mathbb{X}_{T,r} \to X_T \times Y_{1,T}$$

is well-defined and Fréchet differentiable with Lipschitz continuous derivative DN_T as shown in Section 2.1. In particular, every function $\varrho \in \mathbb{X}_{T,r}$ induces a family of immersions $f^\varrho(t) = \varphi + \varrho(t)\nu_\varphi$, $t \in [0,T]$, of M into $\mathbb{R}^{n+1}$. The graph formulation is equivalent to $\varrho \in \mathbb{X}_{T,r}$ solving

$$(-N_{T,1}(\varrho), -N_{T,2}(\varrho), B_2(\varrho)) = (0,0,0)$$

which is equivalent to

$$L_T(\varrho) = (\partial_t\varrho + A(\varrho) - N_{T,1}(\varrho), B_1(\varrho) - N_{T,2}(\varrho), 0, \varrho_0)\,. \tag{2.36}$$

In order to apply Theorem 2.57 we need to verify that for all $\varrho \in \mathbb{X}_{T,r}$,

$$(\partial_t\varrho + A(\varrho) - N_{T,1}(\varrho), B_1(\varrho) - N_{T,2}(\varrho), 0, \varrho_0) \in \mathbb{F}_T$$

which is equivalent to saying that $(B_1(\varrho) - N_{T,2}(\varrho), 0)$ satisfies the linear compatibility conditions with respect to ϱ_0, which are given in the identities (2.20) and (2.21). As ϱ_0 is an admissible initial height function, there holds $H_{f^{\varrho_0}} = N_{T,2}(\varrho)(0) = 0$ and thus

$$B_1(\varrho)(0) - N_{T,2}(\varrho)(0) = B_1(\varrho)(0)$$

which is the identity (2.20). The condition (2.21) is satisfied as $B_2(\varrho)(0) = (\varrho_0)_{|\partial M} = 0$. This allows us to conclude that the mapping

$$K_T : \mathbb{X}_{r,T} \to \mathbb{E}_T\,, \varrho \mapsto L_T^{-1}\left(\partial_t\varrho + A(\varrho) - N_{T,1}(\varrho), B_1(\varrho) - N_{T,2}(\varrho), 0, \varrho_0\right)$$

is well-defined. Given $\varrho \in \mathbb{X}_{r,T}$, Lipschitz continuity of $DN_T : \boldsymbol{U}_T \to \mathcal{L}(\mathbb{E}_T, \boldsymbol{F}_T)$ with $\boldsymbol{F}_T = X_T \times Y_{1,T}$ implies the estimate

$$\begin{aligned}
&\|\partial_t\varrho + A(\varrho) - N_{T,1}(\varrho)\|_{X_T} + \|B_1(\varrho) - N_{T,2}(\varrho)\|_{Y_{1,T}}\\
&= \|DN_{T,1}(0)\varrho - N_{T,1}(\varrho)\|_{X_T} + \|DN_{T,2}(0)\varrho - N_{T,2}(\varrho)\|_{Y_{1,T}} = \|DN_T(0)\varrho - N_T\varrho\|_{\boldsymbol{F}_T}\\
&\le \left\|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}N_T(t\varrho) - DN_T(0)\varrho\,\mathrm{d}t\right\|_{\boldsymbol{F}_T} + \|N_T(0)\|_{\boldsymbol{F}_T}\\
&\le \int_0^1 \|\!|DN_T(t\varrho) - DN_T(0)|\!\|_{\mathcal{L}(\mathbb{E}_T,\boldsymbol{F}_T)}\,\mathrm{d}t\,\|\!|\varrho|\!\|_{\mathbb{E}_T} + C(\varphi)T^{1/p}\\
&\le \|DN_T\|_{C^{0,1}(\boldsymbol{U}_T;\mathcal{L}(\mathbb{E}_T,\boldsymbol{F}_T))}\int_0^1 t\,\mathrm{d}t\,\|\!|\varrho|\!\|_{\mathbb{E}_T}^2 + C(\varphi)T^{1/p} \le C(Q,T_0,\varepsilon,\sigma)\frac{1}{2}r^2 + C(\varphi)T^{1/p}\,.
\end{aligned}$$

Lemma 2.60 and the uniform estimate (2.29) in Theorem 2.57 then imply

$$\begin{aligned}
&\|\!|K_T\varrho|\!\|_{\mathbb{E}_T}\\
&\le C(T_0)\left(\|\partial_t\varrho + A(\varrho) - N_{T,1}(\varrho)\|_{X_T} + \|B_1(\varrho) - N_{T,2}(\varrho)\|_{Y_{1,T}} + (C+1)\,\|\varrho_0\|_{W_p^{4-4/p}(M)}\right)\\
&\le C(T_0)\left(C(Q,T_0,\varepsilon,\sigma)\frac{1}{2}r^2 + C(\varphi)T^{1/p} + (C+1)\,\varepsilon_0\right) \le \frac{r}{4} + \frac{r}{4} + \frac{r}{4} < r\,.
\end{aligned}$$

As for all $\varrho \in \mathbb{X}_{r,T}$, there holds $(K_T\varrho)(0) = \varrho_0$, we obtain in particular $K_T\varrho \in \mathbb{X}_{r,T}$. Furthermore, the map $K_T : \mathbb{X}_{r,T} \to \mathbb{X}_{r,T}$ is contractive as for ϱ, $\xi \in \mathbb{X}_{r,T}$, there holds

$$\begin{aligned}
&\|N_{T,1}\varrho - \partial_t\varrho - A\varrho - N_{T,1}\xi + \partial_t\xi + A\xi\|_{X_T}\\
&= \left\|\int_0^1 DN_{T,1}(t\varrho + (1-t)\xi)(\varrho - \xi) - DN_{T,1}(0)(\varrho - \xi)\,\mathrm{d}t\right\|_{X_T}\\
&\le \int_0^1 \|\!|DN_{T,1}(t\varrho + (1-t)\xi) - DN_{T,1}(0)|\!\|_{\mathcal{L}(\mathbb{E}_T,X_T)}\,\mathrm{d}t\,\|\!|\varrho - \xi|\!\|_{\mathbb{E}_T}\\
&\le \|DN_{T,1}\|_{C^{0,1}(\boldsymbol{U}_T;\mathcal{L}(\mathbb{E}_T,X_T))}\int_0^1 \|\!|t\varrho + (1-t)\xi|\!\|_{\mathbb{E}_T}\,\mathrm{d}t\,\|\!|\varrho - \xi|\!\|_{\mathbb{E}_T}\\
&\le \|DN_{T,1}\|_{C^{0,1}(\boldsymbol{U}_T;\mathcal{L}(\mathbb{E}_T,X_T))}\,r\,\|\!|\varrho - \xi|\!\|_{\mathbb{E}_T}\,,
\end{aligned}$$

and analogously

$$\|N_{T,2}\varrho - B_1\varrho - N_{T,2}\xi + B_1\xi\|_{Y_{1,T}} \le \|DN_{T,2}\|_{C^{0,1}(\boldsymbol{U}_T;\mathcal{L}(\mathbb{E}_T,Y_{1,T}))}\,r\,\|\!|\varrho - \xi|\!\|_{\mathbb{E}_T}\,,$$

which yields, as $(N_{T,2}(\varrho) - N_1(\varrho) - N_{T,2}(\xi) + B_1\xi)_{|t=0} = 0$,

$$\|\!|K_T\varrho - K_T\xi|\!\|_{\mathbb{E}_T} \le C(T_0)C(Q,T_0,\varepsilon,\sigma)r\,\|\!|\varrho - \xi|\!\|_{\mathbb{E}_T} < \frac{1}{2}\,\|\!|\varrho - \xi|\!\|_{\mathbb{E}_T}\,.$$

The Banach Fixed-Point Theorem [151, Theorem 1.A] implies the existence of a unique fixed point $\boldsymbol{\varrho}$ of K_T. As $K_T\boldsymbol{\varrho} = \boldsymbol{\varrho}$ is equivalent to $\boldsymbol{\varrho}$ satisfying the identity (2.36), which is itself equivalent to the graph formulation, and as further the functions $f^{\boldsymbol{\varrho}}(t) = \varphi + \boldsymbol{\varrho}(t)\nu_\varphi$ form a family of immersions of M into $\mathbb{R}^{n+1}$, we conclude that $\boldsymbol{\varrho}$ is a strong graph solution to the Willmore flow with Navier conditions in $[0,T]$ with initial value ϱ_0. The family of immersion $f^{\boldsymbol{\varrho}}(t)$, $t \in [0,T]$, form a solution to (W) with initial datum $f^{\varrho_0} = \varphi + \varrho_0\nu_\varphi$. □

Part II

Elastic Flow of Networks

Main results

This part is devoted to the study of the elastic flow of planar networks subject to natural boundary conditions. We hereby consider two types of networks $\gamma = \left(\gamma^1, \gamma^2, \gamma^3\right)$ composed of three regular curves $\gamma^i : [0,1] \to \mathbb{R}^2$. The curves of a *Theta network* intersect each other at their endpoints in two triple junctions $\gamma^1(0) = \gamma^2(0) = \gamma^3(0)$ and $\gamma^1(1) = \gamma^2(1) = \gamma^3(1)$. In the case of the *Triod* the curves meet in one triple junction $\gamma^1(0) = \gamma^2(0) = \gamma^3(0)$ and have arbitrary endpoints $P^i = \gamma^i(1)$. Given $\mu = \left(\mu^1, \mu^2, \mu^3\right) \in \mathbb{R}^3$ we consider the elastic energy

$$E_\mu(\gamma) := \sum_{i=1}^{3} \int_{[0,1]} \left(\left(\kappa^i\right)^2 + \mu^i\right) \mathrm{d}s^i$$

where s^i denotes the arclength parameter and κ^i the curvature of the curve γ^i. Details on the geometry of curves are given in Section 3.1.

We are interested in the L^2–gradient flow of E_μ: an initial network of Theta or Triod type evolves with a normal velocity that induces the steepest descent of the energy E_μ with respect to the L^2–inner product while maintaining the structure (Theta or Triod) of the network. Hereby, we allow for movement of the triple junctions. In the case of Triods the endpoints P^1, P^2, P^3 are supposed to stay fixed during the evolution. In Section 3.2 we derive the first variation of the energy E_μ in the classes of Theta networks and Triods, respectively, and give details on the formal gradient flow structure. Each curve γ^i of the network evolves with normal velocity

$$\left\langle \gamma^i_t, \nu^i \right\rangle = -2\kappa^i_{ss} - \left(\kappa^i\right)^3 + \mu^i \kappa^i =: \overline{V}\left(\gamma^i\right).$$

Here ν^i denotes the unit normal vector of the curve γ^i given by the counter-clockwise rotation by $\frac{\pi}{2}$ of the unit tangent τ^i. Differentiation with respect to time and the arclength parameter is indicated with subscripts t and s, respectively. Moreover, boundary conditions of different orders naturally arise from the first variation. On a time interval $[0,T]$, $T > 0$, the evolution problem for Theta networks is given by

$$\begin{cases} \left\langle \gamma^i_t, \nu^i \right\rangle = -\left(2\kappa^i_{ss} + \left(\kappa^i\right)^3 - \mu^i\kappa^i\right) & \text{in } [0,T]\times[0,1] \quad \text{(motion)}\,, \\ \gamma^1 = \gamma^2 = \gamma^3 & \text{on } [0,T]\times\{0,1\} \quad \text{(concurrency condition)}\,, \\ \kappa^i = 0 & \text{on } [0,T]\times\{0,1\} \quad \text{(curvature condition)}\,, \\ \sum_{i=1}^3 2\kappa^i_s\nu^i - \mu^i\tau^i = 0 & \text{on } [0,T]\times\{0,1\} \quad \text{(third order condition)}\,, \\ \gamma(0,[0,1]) = \sigma([0,1]) & \text{(initial value)}\,, \end{cases} \tag{Θ}$$

while the flow for Triods reads as

$$\begin{cases} \left\langle \gamma^i_t, \nu^i \right\rangle = -\left(2\kappa^i_{ss} + \left(\kappa^i\right)^3 - \mu^i\kappa^i\right) & \text{in } [0,T]\times[0,1] \quad \text{(motion)}\,, \\ \gamma^1 = \gamma^2 = \gamma^3 & \text{on } [0,T]\times\{0\} \quad \text{(concurrency condition)}\,, \\ \gamma^i = P^i & \text{on } [0,T]\times\{1\} \quad \text{(endpoints)}\,, \\ \kappa^i = 0 & \text{on } [0,T]\times\{0,1\} \quad \text{(curvature condition)}\,, \\ \sum_{i=1}^3 2\kappa^i_s\nu^i - \mu^i\tau^i = 0 & \text{on } [0,T]\times\{0\} \quad \text{(third order condition)}\,, \\ \gamma(0,[0,1]) = \sigma([0,1]) & \text{(initial value)}\,. \end{cases} \tag{$\mathbb{T}$}$$

The requirements on the initial network σ depend on the desired regularity of the solution γ and are specified below.

Let us draw the attention to the geometric nature of the evolution problems (Θ) and $(\mathbb{T})$, which we refer to as *geometric problems*. Indeed, as shown in Proposition 3.31, any reparametrisation (with appropriate properties on the boundary) of a given solution to system (Θ) or $(\mathbb{T})$ again solves the respective system - the problems are invariant with respect to reparametrisation. As a consequence, uniqueness of solutions can only be expected in terms of the *images* parametrised by the curves which is referred to as *geometric uniqueness*.
The results in Part II rely on the publications [64, 65]. The first main result is the existence and geometric uniqueness of strong and classical solutions to the evolution problems (Θ) and $(\mathbb{T})$ given suitable initial networks.
Regarding strong solutions we consider a certain range of integration parameters to guarantee that spacial derivatives up to order three can be evaluated pointwise. In particular, all boundary conditions need to hold at initial time. In the case of classical solutions more compatibility conditions need to be imposed at the boundary points of the initial network which is due to the high regularity of the solution. Details on the notions of solution and the requirements at initial time are discussed in Section 3.4.
We hereby exclude one degenerate geometric situation. At each triple junction we require the so-called *non-degeneracy condition* which demands that the angles α^1, α^2, and α^3 between the tangents τ^2 and τ^3, τ^3 and τ^1, and τ^1 and τ^2, respectively, at the triple junction satisfy

$$\max\left\{\left|\sin\alpha^1\right|, \left|\sin\alpha^2\right|, \left|\sin\alpha^3\right|\right\} > 0\,. \tag{ND}$$

In other words, at least one of the angles should be different from 0, π and 2π.
The following theorem summarises the results in Theorem 4.30, Theorem 4.32, Theorem 4.37, Theorem 4.38, Theorem 4.58, Theorem 4.62, and Theorem 4.63.

Main Theorem 2 (Existence, uniqueness and regularisation of strong solutions to the elastic flow of networks)**.** *Let* $p \in (5,\infty)$*.*

- *Given a Theta network* $\sigma \in W_p^{4-4/p}\left((0,1);(\mathbb{R}^2)^3\right)$ *satisfying all boundary conditions appearing in system* (Θ) *and further the non-degeneracy condition* (ND) *in both triple junctions, there exist* $T>0$ *and a function*

$$\gamma \in W_p^{1,4}\left((0,T)\times(0,1);(\mathbb{R}^2)^3\right)$$

solving (Θ) *with initial datum* σ *such that for all* $t\in[0,T]$*,* $\gamma(t)$ *is a Theta network composed of regular curves fulfilling the non-degeneracy condition* (ND) *in both triple junctions, and such that for all* $\varepsilon\in(0,T)$*,*

$$\gamma \in C^\infty\left([\varepsilon,T]\times[0,1];(\mathbb{R}^2)^3\right).$$

- *Given a Triod* $\sigma \in W_p^{4-4/p}\left((0,1);(\mathbb{R}^2)^3\right)$ *with endpoints* P^1*,* P^2*,* P^3 *satisfying all boundary conditions appearing in system* $(\mathbb{T})$ *and further the non-degeneracy condition* (ND) *in the triple junction, there exist* $T>0$ *and a function*

$$\gamma \in W_p^{1,4}\left((0,T)\times(0,1);(\mathbb{R}^2)^3\right)$$

solving $(\mathbb{T})$ *with endpoints* P^1*,* P^2*,* P^3 *and initial datum* σ *such that for all* $t\in[0,T]$*,* $\gamma(t)$ *is a Triod composed of regular curves fulfilling the non-degeneracy condition* (ND) *in the triple junction, and such that for all* $\varepsilon\in(0,T)$*,*

$$\gamma \in C^\infty\left([\varepsilon,T]\times[0,1];(\mathbb{R}^2)^3\right).$$

- *Solutions to system* (Θ) *and* $(\mathbb{T})$ *are unique in the following sense. Given* $T > 0$ *and two solutions* $\gamma, \eta \in W_p^{1,4}\left((0,T)\times(0,1);(\mathbb{R}^2)^3\right)$ *both solving system* (Θ) *or system* $(\mathbb{T})$*, such that for* $i \in \{1,2,3\}$ *there exist* C^1*–diffeomorphisms* $\zeta_0^i : [0,1] \to [0,1]$ *with* $\zeta_0^i \in W_p^{4-4/p}((0,1);\mathbb{R})$ *and* $\gamma^i(0) = \eta^i(0) \circ \zeta_0^i$*, then there exists* $\zeta \in W_p^{1,4}\left((0,T)\times(0,1);\mathbb{R}^3\right)$ *with* $\zeta(0) = \zeta_0$ *such that for all* $t \in [0,T]$*,* $i \in \{1,2,3\}$*,* $\zeta^i(t) : [0,1] \to [0,1]$ *is a* C^1*–diffeomorphism with* $\gamma^i(t) = \eta^i(t) \circ \zeta^i(t)$*.*

The following result is the combination of Theorem 4.53, Theorem 4.54, Theorem 4.66, Theorem 4.68, and Theorem 4.69.

Main Theorem 3 (Existence and uniqueness of classical solutions to the elastic flow of networks)**.** *Let* $\alpha \in (0,1)$*.*

- *Let* $\sigma \in C^{4+\alpha}\left([0,1];(\mathbb{R}^2)^3\right)$ *be a Theta network satisfying all boundary conditions appearing in system* (Θ)*, the non-degeneracy condition* (ND) *in both triple junctions, and further*

$$\sin\left(\alpha^1\right)\overline{V}\left(\sigma^1\right) + \sin\left(\alpha^2\right)\overline{V}\left(\sigma^2\right) + \sin\left(\alpha^3\right)\overline{V}\left(\sigma^3\right) = 0$$

  at both triple junctions. Then there exist $T > 0$ *and a function*

$$\gamma \in C^{\frac{4+\alpha}{4},4+\alpha}\left([0,T]\times[0,1];(\mathbb{R}^2)^3\right)$$

  solving (Θ) *with initial datum* σ *such that for all* $t \in [0,T]$*,* $\gamma(t)$ *is a Theta network composed of regular curves fulfilling the non-degeneracy condition* (ND) *in both triple junctions.*

- *Let* $\sigma \in C^{4+\alpha}\left([0,1];(\mathbb{R}^2)^3\right)$ *be a Triod with endpoints* P^1*,* P^2*,* P^3 *satisfying all boundary conditions appearing in system* $(\mathbb{T})$*, the non-degeneracy condition* (ND) *in the triple junction, and further*

$$\sin\left(\alpha^1\right)\overline{V}\left(\sigma^1\right)(0) + \sin\left(\alpha^2\right)\overline{V}\left(\sigma^2\right)(0) + \sin\left(\alpha^3\right)\overline{V}\left(\sigma^3\right)(0) = 0$$

  and

$$\overline{V}\left(\sigma^1\right)(1) = \overline{V}\left(\sigma^2\right)(1) = \overline{V}\left(\sigma^3\right)(1) = 0$$

  where α^i *are the angles at the triple junction* $\sigma^1(0)$*. Then there exist* $T > 0$ *and a function*

$$\gamma \in C^{\frac{4+\alpha}{4},4+\alpha}\left([0,T]\times[0,1];(\mathbb{R}^2)^3\right)$$

  solving $(\mathbb{T})$ *with endpoints* P^1*,* P^2*,* P^3 *such that for all* $t \in [0,T]$*,* $\gamma(t)$ *is a Triod composed of regular curves fulfilling the non-degeneracy condition* (ND) *in the triple junction.*

- *Given* $T > 0$ *and two classical solutions* $\gamma, \eta \in C^{\frac{4+\alpha}{4},4+\alpha}\left([0,T]\times[0,1];(\mathbb{R}^2)^3\right)$ *to system* (Θ) *or* $(\mathbb{T})$*, respectively, such that for* $i \in \{1,2,3\}$ *there exist* C^1*–diffeomorphisms* $\zeta_0^i : [0,1] \to [0,1]$ *with* $\zeta_0^i \in C^{4+\alpha}([0,1];\mathbb{R})$ *and* $\gamma^i(0) = \eta^i(0) \circ \zeta_0^i$*, then there exists* $\zeta \in C^{\frac{4+\alpha}{4},4+\alpha}\left([0,T]\times[0,1];\mathbb{R}^3\right)$ *with* $\zeta(0) = \zeta_0$ *such that for all* $t \in [0,T]$*,* $i \in \{1,2,3\}$*,* $\zeta^i(t) : [0,1] \to [0,1]$ *is a* C^1*–diffeomorphism with* $\gamma^i(t) = \eta^i(t) \circ \zeta^i(t)$*.*

We notice that the existence results are valid for any choice of $\mu \in \mathbb{R}^3$ covering in particular the "pure" Willmore flow of networks which corresponds to $\mu = 0$. In both results the time of existence is proportional to the norm of the initial network in the corresponding trace space and in inverse proportion to the lengths of the single curves composing the network.
To investigate the long time behaviour of the flow we need to introduce a suitable notion of maximal solutions. As we are interested in the geometric motion of the networks as *sets* in the plane, we

allow for different parametrisations γ_j in adjacent time intervals $[t_j, t_{j+1}]$ that are then composing the maximal solution $(\gamma_n)_{n\in\mathbb{N}_0}$ with partition $(t_n)_{n\in\mathbb{N}_0}$. While the initial parametrisation γ_0 needs to fulfil merely

$$\gamma_0 \in W_p^{1,4}\left((0,t_0)\times(0,1);(\mathbb{R}^2)^3\right),$$

all subsequent parametrisations γ_j for $j \in \mathbb{N}$ are required to be smooth which is justified by the regularisation property in Main Theorem 2 shown in Subsection 4.1.2. We refer to Subsection 5.1 for the concept of the so-called jointed smooth solutions on which our notion of maximal solutions is based. Existence of maximal solutions and uniqueness up to reparametrisations in the class $W_p^{1,4}\left((0,T)\times(0,1);\mathbb{R}^3\right)$ is established in Theorem 5.9.

Main Theorem 4 (Long time behaviour of the elastic flow of networks)**.** *Let $p \in (5, 10]$ and $\mu^i > 0$ for all $i \in \{1, 2, 3\}$. Let further σ be a Theta network or Triod satisfying the respective conditions in Main Theorem 2 and $(\gamma_n)_{n\in\mathbb{N}_0}$ be a maximal solution to the considered system with initial datum σ in the maximal time interval $[0, T_{max})$ with $T_{max} \in (0,\infty]$. Then*

$$T_{max} = \infty$$

or at least one of the following happens:

(i) the inferior limit of the length of at least one curve is zero as the time approaches T_{max}. That is, there exists $i \in \{1, 2, 3\}$ such that the lengths $\ell^i(t)$ of the ith curve at time t satisfy

$$\liminf_{t\nearrow T_{max}} \ell^i(t) = 0\,.$$

(ii) the angles $\alpha^i(t)$, $i \in \{1, 2, 3\}$, $t \in [0, T_{max})$, between the tangent vectors at the triple junction (Triod) or at one of the triple junctions (Theta) satisfy

$$\liminf_{t\nearrow T_{max}} \max\left\{\left|\sin\alpha^1(t)\right|, \left|\sin\alpha^2(t)\right|, \left|\sin\alpha^3(t)\right|\right\} = 0\,.$$

Hereby, it is not excluded that several scenarios happen simultaneously.

Let us give some comments on the structure of Part II and the methods used to prove the above results.
The major challenges one is facing in the study of the evolution problems (Θ) and $(\mathbb{T})$ arise from the tangential degrees of freedom. Appropriate "tangential conditions" need to be imposed on the curves which on the one hand do not affect the geometric nature of the problem and on the other hand allow for the application of classical PDE methods. To turn the degenerate motion law into a parabolic quasilinear equation of fourth order, a suitable tangential velocity needs to be specified. The resulting evolution equations are of the form

$$\gamma^i_t = -2\frac{\gamma^i_{xxxx}}{|\gamma^i_x|^4} + f\left(\gamma^i_{xxx}, \gamma^i_{xx}, \gamma^i_x, \left|\gamma^i_x\right|^{-1}\right)$$

for a nonlinear function f. Moreover, the boundary value problem is under-determined in the sense that at each triple junction and each fixed endpoint one additional boundary condition needs to be chosen. As it turns out it is convenient to demand

$$\left\langle \gamma^i_{xx}, \tau^i \right\rangle = 0$$

which is a purely tangential condition as it can always be obtained by parametrising the curves appropriately. Further details related to the tangential degrees of freedom are discussed in Section 3.3.

In this way one obtains an auxiliary boundary value problem referred to as the *analytic problem*. Using the same strategy as for the Willmore flow of open surfaces studied in Chapter 2, existence and uniqueness of strong and classical solutions to the analytic problem is deduced from well-posedness of an associated linearised system with the help of contraction estimates. Details are given in Section 4.1. The well-posedness of the linearised system relies on the theory in [136] which allows for coupled boundary conditions of different order and can be applied in both function space settings. In doing so, one crucial aspect is the verification of the Lopatinskii–Shapiro condition which uses the non-degeneracy condition (ND) as an essential ingredient. Besides, one important feature of the parabolic structure of the analytic problems is that strong solutions are *smooth* for positive times which is shown in Subsection 4.1.2.
It is worth mentioning that the requirements on the initial value to the auxiliary problem are slightly different than the ones stated in Main Theorem 2 and Main Theorem 3, respectively. To deduce existence of classical and strong solutions to problems (Θ) and ($\mathbb{T}$) one needs to show that every initial datum that is admissible to one of the geometric problems can be reparametrised in such a way that it satisfies the requirements needed to solve the respective analytic problem. This is studied in Section 4.2, where also uniqueness of the geometric problems up to reparametrisation is shown in both function space settings following the approach in [66, Chapter 5].

Our notion of maximal solutions focuses on the observable evolution of the network which may a priori be composed of several parametrisations defined on adjacent time intervals. However, parametrising the curves with constant speed equal to the length we obtain one parametrisation for the entire time interval, which is in particular a smooth solution to the considered system for positive times. This is shown in Proposition 5.8.
The long time existence result relies on the energy estimates shown in Section 3.5 and the low initial regularity required for the existence of strong solutions which yield that as long as the lengths of the single curves of the evolution are uniformly bounded from below and as long as at each junction at least one angle stays uniformly bounded away from zero, π and 2π, the flow can be extended. We shortly sketch the contradiction argument the proof is based on. Given a solution γ to (Θ) or ($\mathbb{T}$) existing until a maximal time $T_{max} < \infty$ and satisfying the negations of item (i) and item (ii) in Main Theorem 4, Theorem 3.55 yields the a priori estimate

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\|\kappa^i_{ss}(t)\right\|^2_{L_2(\gamma^i(t))} \leq C\,.$$

Hereby, the negation of (ii), namely the uniform version of the non-degeneracy condition (ND), is needed in order to express the tangential velocity at the junction points in terms of the normal velocity with constants uniformly bounded in time.
Corollary 3.28 yields that thanks to the assumption $\mu^i > 0$, $i \in \{1, 2, 3\}$, the gradient flow structure provides an upper bound on the global length translating to a uniform in time and space upper bound on the first derivative of the parametrisation. Using interpolation inequalities we thereby obtain that the solution γ is bounded in $W^4_2\left((0,1);(\mathbb{R}^2)^3\right)$ uniformly on the maximal time interval of existence. Restricting to values $p \in (5, 10]$ the latter space embeds into the trace space $W^{4-4/p}_p\left((0,1);(\mathbb{R}^2)^3\right)$. Theorem 4.30 shows that the time of existence depends only on a lower bound on the length of the curves of the initial network and its norm in the trace space which then allows us to extend the flow beyond the maximal time of existence to obtain a contradiction. Details are given in Sections 5.13 and 5.15.
We remark that the possible singular behaviours described in Main Theorem 4 are not merely technical assumptions but in fact quite realistic scenarios for the nature of potential singularities. This is illustrated by the simulations in Section 5.5 which have been kindly provided by Prof. Dr. Robert Nürnberg from Università di Trento and are based on the work [16].

Chapter 3

Preliminaries on the elastic flow of networks

3.1 A preface on regular open curves

In the following we always consider the Euclidean norm $|\cdot| = \sqrt{\langle \cdot , \cdot \rangle}$ on $\mathbb{R}$ and $\mathbb{R}^2$, respectively, with its induced topology. Furthermore, we endow all embedded submanifolds of $\mathbb{R}^k$, $k \in \{1,2\}$, with the smooth atlas that is induced by the maximal smooth atlas containing the chart $\left(\mathbb{R}^k, \mathrm{id}\right)$, $k \in \{1,2\}$. Given a differentiable function $f : [0,T] \times [a,b] \to \mathbb{R}^2$, with $a, b \in \mathbb{R}$, $a < b$, differentiation with respect to $t \in [0,T]$ or $y \in [a,b]$ is mostly denoted by a subscript, that is, $f_t := \partial_t f$, $f_y := \partial_y f$.

Definition 3.1 (Regular open curves)**.** A *regular open curve* in the plane $\mathbb{R}^2$ is a C^1–immersion $\gamma : [0,1] \to \mathbb{R}^2$.

We notice that a C^1–function $\gamma : [0,1] \to \mathbb{R}^2$ is an immersion if and only if $|\gamma_x(x)| \neq 0$ for all $x \in [0,1]$. The interval $[0,1]$ serves as a "prototype" of a smooth compact one-dimensional manifold with boundary. Given $a, b \in \mathbb{R}$ with $a < b$ and a C^1–immersion $\eta : [a,b] \to \mathbb{R}^2$, the function $\gamma : [0,1] \to \mathbb{R}^2$ defined by $\gamma(x) := \eta\left(a + x(b-a)\right)$ is a C^1–immersion with $\gamma\left([0,1]\right) = \eta\left([a,b]\right)$.

Definition 3.2 (Reparametrisation of open curves)**.** Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, $\gamma : [a,b] \to \mathbb{R}^2$ and $\eta : [c,d] \to \mathbb{R}^2$ be regular open curves. The curve η is called a *reparametrisation of the curve* γ, if there exists a C^1–diffeomorphism $\theta : [a,b] \to [c,d]$ with $\eta\left(\theta(x)\right) = \gamma(x)$.

Two regular open curves that are reparametrisations of each other have the same image. One easily checks that *being a reparametrisation* defines an equivalence relation $\sim$ on the set of regular open curves. This precisely corresponds to identifying curves that have the same image. One also says that such curves are "equal up to reparametrisation". The equivalence class of a regular open curve γ with respect to $\sim$ is denoted by $[\gamma]$.

Definition 3.3 (Regularity of open curves)**.** Let $k \in \mathbb{N}$, $\alpha \in [0,1)$ and $p \in [1,\infty]$ with $p > \frac{1}{k-1}$. A regular open curve $\gamma : [0,1] \to \mathbb{R}^2$ is of class $C^{k+\alpha}$ (or W^k_p, respectively), if $\gamma \in C^{k+\alpha}\left([0,1];\mathbb{R}^2\right)$ (or $\gamma \in W^k_p\left((0,1);\mathbb{R}^2\right)$, respectively).

A function F that maps a regular open curve $\gamma : [a,b] \to \mathbb{R}^2$ (of regularity X with X one of the spaces in Definition 3.3) to a function $F(\gamma) : [a,b] \to \mathbb{R}^d$, $d \in \mathbb{N}$, is said to be *geometric* or *invariant under reparametrisation* if for any reparametrisation $\eta : [c,d] \to \mathbb{R}^2$ of γ with $\eta \circ \theta = \gamma$, $\theta : [a,b] \to [c,d]$, such that η is of regularity X, there holds $F(\eta)(\theta(x)) = F(\gamma)(x)$. Examples for such geometric quantities are the tangent, the unit normal, the curvature, the length functional and the elastic energy.
In the following we study how the general geometric notions regarding immersed submanifolds (M,f) defined in Section 1.1 simplify in the case $M = [0,1]$. Given a regular open curve γ :

$[0,1] \to \mathbb{R}^2$ we consider the interval $[0,1]$ as a Riemannian manifold with boundary and induced metric $g := (\mathrm{d}\gamma_x)^* (\langle \cdot , \cdot \rangle)$ and denote by ∂_x the canonical tangent vector attached to each point of the interval $[0,1]$. The component of the Riemannian volume form ω_g of $([0,1], g)$ is given by $(\omega_g)_x(\partial_x) = \sqrt{g(x)} = |\gamma_x(x)|$.

Definition 3.4 (Tangent and normal vector)**.** The *unit tangent vector* of a regular open curve $\gamma : [0,1] \to \mathbb{R}^2$ at the point x is given by $\tau(x) := \frac{\gamma_x(x)}{|\gamma_x(x)|}$. The *unit normal vector* ν at the point x is given by the counter-clock wise rotation of $\tau(x)$ by $\pi/2$.

Definition 3.5 (Length functional)**.** The *length* of a regular open curve $\gamma : [0,1] \to \mathbb{R}^2$ is given by

$$L(\gamma) = \int_{[0,1]} \omega_g = \int_0^1 (\omega_g)_x(\partial_x)\,\mathrm{d}x = \int_0^1 |\gamma_x(x)|\,\mathrm{d}x\,.$$

It is straight forward to check that the length of the curve γ is invariant under reparametrisation. In fact, if $f = F(\gamma)$ is a function invariant under reparametrisation, the (constant) quantity $\int_{[0,1]} F(\gamma)\,\omega_g$ is invariant under reparametrisation. Indeed if $\theta : [0,1] \to [c,d]$ is a C^1–diffeomorphism and $\eta := \gamma \circ \theta^{-1}$, one observes

$$\int_0^1 F(\gamma)(x)|\gamma_x(x)|\,\mathrm{d}x = \int_0^1 F(\eta)(\theta(x))|\eta_y(\theta(x))||\theta_x(x)|\,\mathrm{d}x = \int_c^d F(\eta)(y)|\eta_y(y)|\,\mathrm{d}y\,. \tag{3.1}$$

An important tool to describe the geometry of curves is the *arclength parameter.* If $\gamma : [0,1] \to \mathbb{R}^2$ is a regular open curve, we let

$$s(x) := \int_0^x |\gamma_x(y)|\,\mathrm{d}y$$

which gives a C^1–diffeomorphism $s : [0,1] \to [0, L(\gamma)]$ with $\frac{\mathrm{d}s}{\mathrm{d}x} = |\gamma_x(x)|$ and inverse $x : [0, L(\gamma)] \to [0,1]$. The curve $\widetilde{\gamma} := \gamma \circ x : [0, L(\gamma)] \to \mathbb{R}^2$ satisfies $\gamma([0,1]) = \widetilde{\gamma}([0, L(\gamma)])$ and

$$\partial_s \widetilde{\gamma}(s) = \gamma_x(x(s)) \frac{\mathrm{d}x}{\mathrm{d}s} = \gamma_x(x(s))|\gamma_x(x(s))|^{-1} = \tau(x(s))\,. \tag{3.2}$$

The curve $\widetilde{\gamma}$ is said to be *parametrised by arclength.* The identity (3.2) yields the formal rules $\partial_s = \frac{\partial_x}{|\gamma_x|}$ and $\mathrm{d}s = |\gamma_x|\,\mathrm{d}x$. With an abuse of notation we consider functions f on the interval $[0,1]$ as functions in the arclength parameter $s = s(x)$ without explicitly distinguishing between f and $f \circ x$. Rigorous arguments can be obtained by composing the considered functions with the diffeomorphism x or its inverse s, respectively. The expression $\partial_s f$ is referred to as the *arclength derivative* of f and may take s or x as an argument. Rigorously, one has for $k \in \mathbb{N}$, and $f \in C^k([0,1]; \mathbb{R})$, $\partial_s^k f(s) := \partial_s^k (f \circ x)(s)$ and $\partial_s^k f(x) := \partial_s^k (f \circ x)(s(x))$.

It is also common to write the so-called *arclength measure* $\mathrm{d}s$ when integrating with respect to the volume element ω_g on $[0,1]$ induced by γ. Given a function f on $[0,1]$ that is integrable with respect to ω_g, c.f Definition A.24, it holds

$$\int_{[0,1]} f\omega_g = \int_0^1 f(x)|\gamma_x(x)|\,\mathrm{d}x = \int_0^{L(\gamma)} f(x(s))\,\mathrm{d}s =: \int_{[0,1]} f\,\mathrm{d}s\,. \tag{3.3}$$

Given $f \in C^1([0,1]; \mathbb{R})$ we have

$$\begin{aligned}\int_{[0,1]} \partial_s f(s)\,\mathrm{d}s &= \int_0^1 \partial_s (f \circ x)(s(x))\,|\gamma_x(x)|\,\mathrm{d}x = \int_0^1 \partial_x f(x)\,(\partial_s x)(s(x))\,|\gamma_x(x)|\,\mathrm{d}x \\ &= \int_0^1 \partial_x f(x)\,\mathrm{d}x = f(1) - f(0)\,. \end{aligned} \tag{3.4}$$

Definition 3.6 (L_p–spaces)**.** Let $\gamma : [0,1] \to \mathbb{R}^2$ be a regular open curve with arclength parameter s and let $f : (0,1) \to \mathbb{R}$ be Lebesgue measurable. We define for $p \in [1,\infty)$,

$$\|f\|^p_{L_p(\gamma,\mathrm{d}s)} := \int_{[0,1]} |f|^p \,\mathrm{d}s = \int_0^1 |f(x)|^p |\gamma_x(x)| \,\mathrm{d}x$$

and

$$L_p(\gamma, \mathrm{d}s) := \left\{ f : (0,1) \to \mathbb{R} \text{ Lebesgue measurable with } \|f\|_{L_p(\gamma,\mathrm{d}s)} < \infty \right\}.$$

For $p = \infty$ we let $L_\infty(\gamma, \mathrm{d}s) := L_\infty((0,1), \lambda)$ with λ the Lebesgue measure on $\mathbb{R}$.

Lemma 3.7. *Let $\gamma : [0,1] \to \mathbb{R}^2$ and $\eta : [0,1] \to \mathbb{R}^2$ be regular open curves of regularity $k \in \mathbb{N}$ with arclength parameters s, $\tilde{s}$, respectively, and let $\theta : [0,1] \to [0,1]$ be a C^k–diffeomorphism with $\gamma = \eta \circ \theta$ and $\theta(y) = y$ for $y \in \{0,1\}$. Given $f \in C^k([0,1];\mathbb{R})$ there holds for all $0 \le n \le k$, $p \in [1,\infty)$,*

$$\int_{[0,1]} |\partial_s^n f|^p \,\mathrm{d}s = \int_{[0,1]} \left|\partial_{\tilde{s}}^n \left(f \circ \theta^{-1}\right)\right|^p \mathrm{d}\tilde{s}\,.$$

Proof. We inductively show for $0 \le n \le k$ the identity

$$\partial_s^n (f \circ x) \circ s = \partial_{\tilde{s}}^n \left(f \circ \theta^{-1} \circ y\right) \circ \tilde{s} \circ \theta\,. \tag{3.5}$$

This is trivial in the case $n = 0$. If $n = 1$ we have for $\hat{x} \in [0,1]$,

$$\partial_s (f \circ x)(s(\hat{x})) = (\partial_x f)(\hat{x}) \left|\gamma_x(\hat{x})\right|^{-1},$$
$$\partial_{\tilde{s}} \left(f \circ \theta^{-1} \circ y\right)(\tilde{s}(\theta(\hat{x}))) = \partial_y \left(f \circ \theta^{-1}\right)(\theta(\hat{x})) \left|\eta_y(\theta(\hat{x}))\right|^{-1} = (\partial_x f)(\widehat{x}) \left|\theta_x(\widehat{x})\right|^{-1} \left|\eta_y(\theta(\hat{x}))\right|^{-1}.$$

Suppose that (3.5) is shown for all $0 \le j \le n-1$ for some $n \le k-1$. Then we conclude

$$\begin{aligned}
\partial_s^n (f \circ x) \circ s &= \partial_s \left(\partial_s^{n-1} (f \circ x)\right) \circ s = \partial_s \left(\partial_s^{n-1} (f \circ x) \circ s \circ x\right) \circ s \\
&= \partial_{\tilde{s}} \left(\partial_s^{n-1} (f \circ x) \circ s \circ \theta^{-1} \circ y\right) \circ \tilde{s} \circ \theta \\
&= \partial_{\tilde{s}} \left(\partial_{\tilde{s}}^{n-1} \left(f \circ \theta^{-1} \circ y\right) \circ \tilde{s} \circ \theta \circ \theta^{-1} \circ y\right) \circ \tilde{s} \circ \theta = \partial_{\tilde{s}}^n \left(f \circ \theta^{-1} \circ y\right) \circ \tilde{s} \circ \theta\,.
\end{aligned}$$

Per definition we have $\partial_s^n f(x) = \partial_s^n (f \circ x)(s(x))$ and $\partial_{\tilde{s}}^n \left(f \circ \theta^{-1}\right)(y) = \partial_{\tilde{s}}^n \left(f \circ \theta^{-1} \circ y\right)(\tilde{s}(y))$ and thus (3.3) yields for any $0 \le n \le k$,

$$\begin{aligned}
\int_{[0,1]} |\partial_s^n f|^p \,\mathrm{d}s &= \int_0^1 \left|\partial_s^n (f \circ x)(s(x))\right|^p |\gamma_x(x)| \,\mathrm{d}x \\
&= \int_0^1 \left|\partial_{\tilde{s}}^n \left(f \circ \theta^{-1} \circ y\right)(\tilde{s}(\theta(x)))\right|^p |\eta_y(\theta(x))| \, |\theta_x(x)| \,\mathrm{d}x \\
&= \int_0^1 \left|\partial_{\tilde{s}}^n \left(f \circ \theta^{-1} \circ y\right)(\tilde{s}(y))\right|^p |\eta_y(y)| \,\mathrm{d}y = \int_{[0,1]} \left|\partial_{\tilde{s}}^n \left(f \circ \theta^{-1}\right)\right|^p \mathrm{d}\tilde{s}\,.
\end{aligned}$$

□

Definition 3.8. A regular open curve $\gamma : [0,1] \to \mathbb{R}^2$ is said to be *parametrised with constant speed equal to the length* if for all $x \in [0,1]$ there holds $|\gamma_x(x)| = L(\gamma)$.

Given a regular open curve $\gamma : [0,1] \to \mathbb{R}^2$ with length $L(\gamma)$, the curve $\eta := \gamma \circ \phi^{-1}$ with $\phi : [0,1] \to [0,1]$ denoting the C^1–diffeomorphism

$$\phi(x) := L(\gamma)^{-1} \int_0^x |\gamma_x(y)| \,\mathrm{d}y$$

is parametrised with constant speed equal to the length.

Definition 3.9 (Curvature (vector))**.** Given a regular open curve $\gamma \in C^2\left([0,1];\mathbb{R}^2\right)$ the *curvature vector* $\boldsymbol{\kappa}(x)$ at a point x is given by

$$\boldsymbol{\kappa}(x) = \left\langle \frac{\gamma_{xx}(x)}{|\gamma_x(x)|^2}, \nu(x) \right\rangle \nu(x) = \partial_s^2(\gamma(x(s)))\,.$$

The *curvature* $\kappa(x)$ at x is given by

$$\kappa(x) = \langle \boldsymbol{\kappa}(x), \nu(x) \rangle = \left\langle \frac{\gamma_{xx}(x)}{|\gamma_x(x)|^2}, \nu(x) \right\rangle\,.$$

One easily checks that the curvature is invariant under reparametrisation. That is, given a regular open curve $\eta \in C^2\left([0,1];\mathbb{R}^2\right)$, such that $\gamma = \eta \circ \theta$ for a C^2–diffeomorphism $\theta : [0,1] \to [0,1]$, the curvatures κ_γ and κ_η of γ and η, respectively, relate via

$$\kappa_\gamma = \kappa_\eta \circ \theta\,. \tag{3.6}$$

We record the so-called *Frenet formulas* for the arclength derivative of the tangent and the normal of a regular open curve given by

$$\tau_s = \kappa\nu\,, \qquad\qquad \nu_s = -\kappa\tau\,. \tag{3.7}$$

Corollary 3.10. *Let $\gamma : [0,1] \to \mathbb{R}$ and $\eta : [0,1] \to \mathbb{R}$ be smooth regular open curves with arclength parameters s, $\tilde{s}$ and curvature κ_γ and κ_η, respectively, such $\gamma = \eta\circ\theta$ for a smooth diffeomorphism $\theta : [0,1] \to [0,1]$ with $\theta(y) = y$ for $y \in \{0,1\}$. Then for all $m \in \mathbb{N}$, $\alpha_l\,, k_l \in \mathbb{N}_0$, $l \in \{1,\dots,m\}$ and $p \in [1,\infty]$, there holds*

$$\left\| \prod_{l=1}^{m} \left(\partial_s^{k_l}\kappa_\gamma\right)^{\alpha_l} \right\|_{L_p(\gamma,\mathrm{d}s)} = \left\| \prod_{l=1}^{m} \left(\partial_{\tilde{s}}^{k_l}\kappa_\eta\right)^{\alpha_l} \right\|_{L_p(\eta,\mathrm{d}\tilde{s})}\,.$$

Proof. Let $\alpha\,, n \in \mathbb{N}_0$ be arbitrary. Lemma 3.7 and identity (3.6) yield for all $p \in [1,\infty)$,

$$\begin{aligned}\|(\partial_s^n\kappa_\gamma)^\alpha\|_{L_p(\gamma,\mathrm{d}s)}^p &= \int_{[0,1]} |\partial_s^n\kappa_\gamma|^{\alpha p}\,\mathrm{d}s = \int_{[0,1]} \left|\partial_{\tilde{s}}^n\left(\kappa_\gamma\circ\theta^{-1}\right)\right|^{\alpha p}\mathrm{d}\tilde{s} = \int_{[0,1]} |\partial_{\tilde{s}}\kappa_\eta|^{\alpha p}\,\mathrm{d}\tilde{s}\\ &= \|(\partial_{\tilde{s}}^n\kappa_\eta)^\alpha\|_{L_p(\eta,\mathrm{d}\tilde{s})}^p\,.\end{aligned}$$

Moreover, the identity (3.5) yields

$$\begin{aligned}\|(\partial_s^n\kappa_\gamma)^\alpha\|_{L_\infty(\gamma,\mathrm{d}s)} &= \sup_{x\in[0,1]} |\partial_s^n\left(\kappa_\gamma\circ x\right)(s(x))|^\alpha = \sup_{x\in[0,1]} |\partial_{\tilde{s}}^n\left(\kappa_\eta\circ y\right)\left(\tilde{s}\left(\theta(x)\right)\right)|^\alpha\\ &= \sup_{y\in[0,1]} |\partial_{\tilde{s}}^n\left(\kappa_\eta\circ y\right)(\tilde{s}(y))|^\alpha = \|(\partial_{\tilde{s}}^n\kappa_\eta)^\alpha\|_{L_p(\eta,\mathrm{d}\tilde{s})}\,.\end{aligned}$$

This shows the claim in the case $m = 1$. The general case follows from the identity

$$\prod_{l=1}^{m}\left(\partial_s^{k_l}\left(\kappa_\gamma\circ x\right)\right)^{\alpha_l}\circ s = \prod_{l=1}^{m}\left(\partial_{\tilde{s}}^{k_l}\left(\kappa_\eta\circ y\right)\right)^{\alpha_l}\circ\tilde{s}\circ\theta$$

which is a direct consequence of (3.5). □

Corollary 3.11 (Hölder's inequality on curves)**.** *Let $\gamma : [0,1] \to \mathbb{R}^2$ be a regular open curve with arclength parameter s and curvature κ. Given $m \in \mathbb{N}$ and $\alpha_l\,, k_l \in \mathbb{N}_0$, $p_l \in [1,\infty]$, $l \in \{1,\dots,m\}$, such that $\sum_{l=1}^m \frac{1}{p_l} = 1$, there holds*

$$\left\| \prod_{l=1}^{m} \left(\partial_s^{k_l}\kappa\right)^{\alpha_l} \right\|_{L_1(\gamma,\mathrm{d}s)} \leq \prod_{l=1}^{m} \left\| \left(\partial_s^{k_l}\kappa\right)^{\alpha_l} \right\|_{L_{p_l}(\gamma,\mathrm{d}s)}\,.$$

Proof. Due to Corollary 3.10 we may assume without loss of generality that $\gamma : [0,1] \to \mathbb{R}^2$ satisfies $|\gamma_x(x)| = L$ for all $x \in [0,1]$ where $L = L(\gamma)$ denotes the length of the curve γ. Then the usual Hölder inequality [4, Lemma 3.18] yields

$$\begin{aligned}
\left\| \prod_{l=1}^{m} \left(\partial_s^{k_l} \kappa\right)^{\alpha_l} \right\|_{L_1(\gamma, \mathrm{d}s)} &= \int_0^1 \left(\prod_{l=1}^{m} \left|\partial_s^{k_l} (\kappa \circ x)(s(x))\right|^{\alpha_l} \right) |\gamma_x(x)| \,\mathrm{d}x \\
= L \left\| \prod_{l=1}^{m} \left(\partial_s^{k_l} (\kappa \circ x) \circ s\right)^{\alpha_l} \right\|_{L_1((0,1))} &\le L \prod_{l=1}^{m} \left\| \left(\partial_s^{k_l} (\kappa \circ x) \circ s\right)^{\alpha_l} \right\|_{L_{p_l}((0,1))} \\
= \prod_{l=1}^{m} L^{1/p_l} \left(\int_0^1 \left|\partial_s^{k_l} (\kappa \circ x) \circ s\right|^{\alpha_l p_l} \mathrm{d}x \right)^{1/p_l} &= \prod_{l=1}^{m} \left(\int_0^1 \left|\partial_s^{k_l} (\kappa \circ x)(s(x))\right|^{\alpha_l p_l} |\gamma_x(x)| \,\mathrm{d}x \right)^{1/p_l} \\
= \prod_{l=1}^{m} \left\| \left(\partial_s^{k_l} \kappa\right)^{\alpha_l} \right\|_{L_{p_l}(\gamma, \mathrm{d}s)} .
\end{aligned}$$

□

Definition 3.12 (Elastic energy)**.** Let $\gamma \in C^2\left([0,1];\mathbb{R}^2\right)$ be a regular open curve. Given $\mu \in \mathbb{R}$ the *elastic energy* E_μ *with parameter* μ of γ is defined by

$$E_\mu(\gamma) := \int_{[0,1]} \left(|\boldsymbol{\kappa}|^2 + \mu\right) \mathrm{d}s = \int_0^1 \left(|\boldsymbol{\kappa}(x)|^2 + \mu\right) |\gamma_x(x)| \,\mathrm{d}x \,.$$

To compute the first variation of E_μ in the class

$$\mathcal{A} := \left\{\gamma : [0,1] \to \mathbb{R}^2 \text{ is a smooth immersion}\right\}$$

one needs to understand how geometric quantities of a one-parameter family of regular open curves evolve in dependence of the *velocity* of the evolution. Such considerations have already been undertaken in Section 1.1.2 in the case of higher dimensional flows restricting to immersions that evolve only in *normal* direction. While this approach is justified in the case that the boundary stays fixed during the evolution, c.f Proposition 1.26, the *tangential* velocity has to be taken into account when allowing for movement of the boundary.

Given an interval $J \subset \mathbb{R}$ and a smooth one-parameter family $\gamma : J \times [0,1] \to \mathbb{R}^2$ of regular open curves $\gamma(t) \in \mathcal{A}$, $t \in J$, we let $\tau(t)$, $\nu(t)$ and $\kappa(t)$ denote the unit tangent, unit normal and curvature of $\gamma(t)$, respectively. Furthermore, we let $\mathrm{d}s_t$ and ∂_{s_t} be the arclength measure and derivative associated to the curve $\gamma(t)$. We remark that the dependence of the arclength measure and derivative on the parameter $t \in J$ is usually omitted.

Definition 3.13 (Normal and tangential velocity)**.** Let $J \subset \mathbb{R}$ be an interval and $\gamma : J \times [0,1] \to \mathbb{R}^2$ be a smooth one-parameter family of regular open curves $\gamma(t) \in \mathcal{A}$, $t \in J$. The function $\partial_t \gamma$ is called *velocity vector* of γ. The *normal velocity* and *tangential velocity* of γ at $(t,x) \in J \times [0,1]$ are given by $V(t,x) := V(\gamma)(t,x) := \langle \partial_t \gamma(t,x), \nu(t,x)\rangle$ and $T(t,x) := T(\gamma)(t,x) := \langle \partial_t \gamma(t,x), \tau(t,x)\rangle$, respectively.

The following evolution formulas are well-known in the literature, see for example [49, Lemma 2.1]. All appearing quantities refer to γ and are defined on $J \times [0,1]$. The reader shall be reminded that composition of the respective functions with the diffeomorphism x is necessary to make sense of the arclength derivatives.

Lemma 3.14 (Evolution formulas)**.** *Let $J \subset \mathbb{R}$ be an interval and $\gamma : J \times [0,1] \to \mathbb{R}^2$ be a smooth one-parameter family of regular open curves $\gamma(t) \in \mathcal{A}$, $t \in J$, satisfying the evolution law*

$$\partial_t \gamma = V\nu + T\tau \qquad \text{on } J \times [0,1]\,. \tag{3.8}$$

Then the following formulas hold:

$$\partial_t \partial_s = \partial_s \partial_t + (\kappa V - T_s)\,\partial_s\,, \tag{3.9}$$
$$\partial_t\,(\mathrm{d}s) = (T_s - \kappa V)\,\mathrm{d}s\,, \tag{3.10}$$
$$\partial_t \tau = (V_s + T\kappa)\,\nu\,, \tag{3.11}$$
$$\partial_t \nu = -\,(V_s + T\kappa)\,\tau\,, \tag{3.12}$$
$$\partial_t \kappa = \langle \partial_t \boldsymbol{\kappa}, \nu\rangle = V_{ss} + T\kappa_s + \kappa^2 V\,. \tag{3.13}$$

Proof. The identities follow from straightforward computations using the Frenet formulas (3.7) and the evolution law (3.8). ☐

Lemma 3.15. *Let $J \subset \mathbb{R}$ be an interval and $\gamma : J \times [0,1] \to \mathbb{R}^2$ be a smooth one-parameter family of regular open curves $\gamma(t) \in \mathcal{A}$, $t \in J$, satisfying the evolution law* (3.8)*. Then it holds for all $t \in J$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\mu(\gamma(t)) = \int_{[0,1]} V(t)\left(2\kappa_{ss}(t) + \kappa^3(t) - \mu\kappa(t)\right)\,\mathrm{d}s \\ + \left[2\kappa(t)V_s(t) - 2\kappa_s(t)V(t) + \kappa^2(t)T(t) + \mu T(t)\right]_0^1\,.$$

Proof. This is a special case of [16, Lemma 2.1]. The proof can be found in [17, Lemma 2.2]. It follows from the formulas (3.10) and (3.13) using integration by parts with respect to the arclength measure. ☐

3.2 The elastic energy and its gradient flow

In this section we compute the first variation of the elastic energy of Triods and Theta networks which gives rise to the evolution problems (3.20) and (3.21) that can be understood as L^2–gradient flows of the elastic energy.

Definition 3.16 (Theta network)**.** A *Theta network* $\gamma = \left(\gamma^1, \gamma^2, \gamma^3\right)$ is composed of three regular open curves $\gamma^i : [0,1] \to \mathbb{R}^2$ whose images intersect each other at their endpoints in the triple junctions $O^1 = \gamma^1(0) = \gamma^2(0) = \gamma^3(0)$ and $O^2 = \gamma^1(1) = \gamma^2(1) = \gamma^3(1)$.

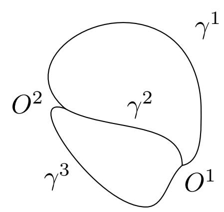

Figure 3.1: A Theta network.

Definition 3.17. A *Triod* $\gamma = \left(\gamma^1, \gamma^2, \gamma^3\right)$ is composed of three regular open curves $\gamma^i : [0,1] \to \mathbb{R}^2$ whose images meet in one triple junction $O = \gamma^1(0) = \gamma^2(0) = \gamma^3(0)$. The other endpoints $P^1 = \gamma^1(1)$, $P^2 = \gamma^2(1)$ and $P^3 = \gamma^3(1)$ are arbitrary.

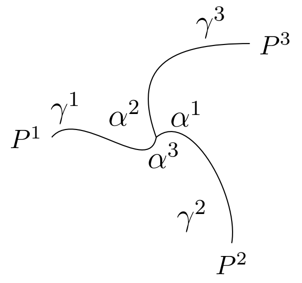

Figure 3.2: A Triod.

We remark that the images of the curves composing the network may intersect also at other points than the triple junctions. Also each curve may have self-intersections as the curves are only required to be *immersed* but not necessarily embedded.

Definition 3.18 (Regularity of networks)**.** Let $k \in \mathbb{N}$, $\alpha \in [0,1)$ and $p \in [1,\infty]$ with $p > \frac{1}{k-1}$. A Triod or Theta network $\gamma = \left(\gamma^1, \gamma^2, \gamma^3\right)$ is of class $C^{k+\alpha}$ (or W^k_p, respectively), if each of its curves γ^i has the respective regularity.

Definition 3.19 (Concurrency condition and angles)**.** Three regular open curves $\gamma^i : [0,1] \to \mathbb{R}^2$ are said to satisfy the *concurrency condition* in $y \in \{0,1\}$ if $O = \gamma^1(y) = \gamma^2(y) = \gamma^3(y)$. In this case, the point O is called *triple junction* and we denote by $\alpha^1(y)$, $\alpha^2(y)$ and $\alpha^3(y)$ the angles between the tangent vectors $\tau^2(y)$ and $\tau^3(y)$, $\tau^3(y)$ and $\tau^1(y)$, and $\tau^1(y)$ and $\tau^2(y)$, respectively, where τ^i is the tangent to the curve γ^i for $i \in \{1,2,3\}$.

The convention regarding the angles is illustrated in Figure 3.2.

Definition 3.20 (Elastic energy of networks)**.** Let $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ be a Triod or a Theta network of class C^2. Given $\mu = \left(\mu^1, \mu^2, \mu^3\right) \in \mathbb{R}^3$ the *elastic energy E_μ with parameters μ^i, $i \in \{1,2,3\}$*, is defined by

$$E_\mu(\gamma) := \sum_{i=1}^{3} E_{\mu^i}\left(\gamma^i\right) = \sum_{i=1}^{3} \int_{[0,1]} \left(|\boldsymbol{\kappa}^i|^2 + \mu^i\right) \mathrm{d}s^i$$

where $\boldsymbol{\kappa}^i$ is the curvature vector of the curve γ^i. Here the notation $\mathrm{d}s^i$ should be understood as integration with respect to the volume form ω_{g^i} of $\left([0,1], \gamma^i\right)$, see (3.3).

We adopt the following notation. Given an interval $J \subset \mathbb{R}$ and a smooth one-parameter family $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$ of Triods or Theta networks, we let $\tau^i(t)$, $\nu^i(t)$ and $\kappa^i(t)$ be the unit tangent, unit normal and curvature of the curve $\gamma^i(t)$, respectively. We remark that the space variable is usually not written explicitly. Also the dependence on $t \in J$ and $i \in \{1,2,3\}$ of the arclength measure $\mathrm{d}s^i_t$ and the arclength derivative $\partial_{s^i_t}$ associated to the curve $\gamma^i(t)$ is usually omitted. If a quantity $f^i(t)$ related to $\gamma^i(t)$ is differentiated with respect to the arclength parameter related to $\gamma^i(t)$ we simply write $\partial_s f^i(t)$ or $f^i_s(t)$. Furthermore, we let $V^i, T^i : J \times [0,1] \to \mathbb{R}$, $i \in \{1,2,3\}$, be the normal and tangential velocity of γ^i, respectively, which yields the evolution law

$$\partial_t \gamma^i = V^i \nu^i + T^i \tau^i \qquad \text{on } J \times [0,1]\,,\ i \in \{1,2,3\}\,. \tag{3.14}$$

The first variation in the class of *Theta networks*

$$\mathcal{A}_\Theta := \left\{ \left(\gamma^1, \gamma^2, \gamma^3\right) \,:\, \gamma^i : [0,1] \to \mathbb{R}^2 \text{ smooth immersion}\,, \gamma^1(y) = \gamma^2(y) = \gamma^3(y)\,, y \in \{0,1\} \right\}$$

gives the following result which is a special case of [16, Lemma 2.2].

Proposition 3.21 (First variation of the elastic energy of Theta networks)**.** *Let $J \subset \mathbb{R}$ be an interval and $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$ be a smooth one-parameter family of Theta networks $\gamma(t) \in \mathcal{A}_\Theta$, $t \in J$, satisfying the evolution law* (3.14)*. Then it holds for all $t \in J$,*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E_\mu(\gamma(t)) = & \sum_{i=1}^{3} \int_{[0,1]} V^i(t) \big(2\kappa^i_{ss}(t) + \left(\kappa^i\right)^3(t) - \mu^i \kappa^i(t)\big)\, \mathrm{d}s^i \\
& + \left\langle \gamma^1_t(t), \sum_{i=1}^{3} (\mu^i - (\kappa^i(t)^2) \gamma^i_s(t) \right\rangle_0^1 - 2 \left\langle \gamma^1_t(t), \sum_{i=1}^{3} \kappa^i_s(t) \nu^i(t) \right\rangle_0^1 \\
& + 2 \sum_{i=1}^{3} \left\langle \boldsymbol{\kappa}^i(t), (\gamma^i_s)_t(t) \right\rangle_0^1 .
\end{aligned}$$

Proof. Lemma 3.15 implies for $t \in J$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E_\mu(\gamma(t)) = & \sum_{i=1}^{3} \int_{[0,1]} V^i(t) \big(2\kappa^i_{ss}(t) + \left(\kappa^i\right)^3(t) - \mu^i \kappa^i(t)\big)\, \mathrm{d}s^i \\
& + \sum_{i=1}^{3} \left[2\kappa^i(t) V^i_s(t) - 2\kappa^i_s(t) V^i(t) + (\kappa^i)^2(t) T^i(t) + \mu^i T^i(t) \right]_0^1 .
\end{aligned}$$

Differentiating the concurrency condition in time we obtain for $y \in \{0,1\}$ the identities

$$\gamma^1_t(y) = \gamma^2_t(y) = \gamma^3_t(y)$$

which imply due to $T^i(t) = \langle \gamma^i_t(t), \tau^i(t) \rangle = \langle \gamma^i_t(t), \gamma^i_s(t) \rangle$ the identity

$$\sum_{i=1}^{3} \left[\left(\mu^i + (\kappa^i(t))^2\right) T^i(t) \right]_0^1 = \left\langle \gamma^1_t(t), \sum_{i=1}^{3} (\mu^i + (\kappa^i(t)^2) \gamma^i_s(t) \right\rangle_0^1 .$$

Similarly, we observe that $V^i(t) = \langle \gamma^i_t(t), \nu^i(t) \rangle$ implies

$$\sum_{i=1}^{3} \kappa^i_s(t) V^i(t) = \left\langle \gamma^1_t(t), \sum_{i=1}^{3} \kappa^i_s(t) \nu^i(t) \right\rangle_0^1 .$$

Finally, the identity (3.11) yields $\left(\gamma^i_s\right)_t = V^i_s(t) \nu^i(t) + T^i(t) \boldsymbol{\kappa}^i(t)$ which implies

$$\sum_{i=1}^{3} \left[2\kappa^i(t) V^i_s(t) \right]_0^1 = \sum_{i=1}^{3} 2 \left\langle \boldsymbol{\kappa}^i(t), (\gamma^i_s)_t(t) \right\rangle_0^1 - \sum_{i=1}^{3} \left[2 \left(\kappa^i(t)\right)^2 T^i(t) \right]_0^1 .$$

$\square$

We obtain different boundary terms when considering the first variation of the elastic energy in the class of *Triods* with given endpoints P^1, P^2, P^3, represented by the admissible set

$$\mathcal{A}_{\mathbb{T}} := \left\{ \left(\gamma^1, \gamma^2, \gamma^3\right) \,:\, \gamma^i : [0,1] \to \mathbb{R}^2 \text{ smooth immersion}\,, \gamma^1(0) = \gamma^2(0) = \gamma^3(0)\,, \gamma^i(1) = P^i \right\}\,.$$

Proposition 3.22 (First variation of the elastic energy of Triods)**.** *Let $J \subset \mathbb{R}$ be an interval and $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$ be a smooth one-parameter family of Triods $\gamma(t) \in \mathcal{A}_{\mathbb{T}}$, $t \in J$, satisfying the evolution law* (3.14)*. Then it holds for all $t \in J$,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_\mu(\gamma(t)) = &\sum_{i=1}^{3} \int_{[0,1]} V^i(t)\big(2\kappa^i_{ss}(t) + \left(\kappa^i\right)^3(t) - \mu^i\kappa^i(t)\big)\,\mathrm{d}s^i \\ &+ \left\langle \gamma^1_t(t,0), \sum_{i=1}^{3} \left(\mu^i - \kappa^i(t,0)^2\right)\gamma^i_s(t,0) \right\rangle \\ &- 2\left\langle \gamma^1_t(t,0), \sum_{i=1}^{3} \kappa^i_s(t,0)\nu^i(t,0) \right\rangle \\ &+ 2\sum_{i=1}^{3} \left\langle \boldsymbol{\kappa}^i(t,0), (\gamma^i_s)_t(t,0)\right\rangle + 2\sum_{i=1}^{3} \left\langle \boldsymbol{\kappa}^i(t,1), (\gamma^i_s)_t(t,1)\right\rangle\,.\end{aligned}$$

Proof. As in the previous proof we have for $t \in J$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_\mu(\gamma(t)) = &\sum_{i=1}^{3} \int_{[0,1]} V^i(t)\big(2\kappa^i_{ss}(t) + \left(\kappa^i\right)^3(t) - \mu^i\kappa^i(t)\big)\,\mathrm{d}s^i \\ &+ \sum_{i=1}^{3} \left[2\left\langle \boldsymbol{\kappa}^i(t), (\gamma^i_s)_t(t)\right\rangle - 2\kappa^i_s(t)V^i(t) - (\kappa^i)^2(t)T^i(t) + \mu^i T^i(t)\right]_0^1\,.\end{aligned}$$

At the triple junction $\gamma^1(t,0)$ we may rewrite the boundary term as in the previous proof to obtain the desired terms. As $\gamma^i(t,1) = P^i$ for all $t \in J$ and $i \in \{1,2,3\}$, differentiating with respect to time yields

$$0 = T^i(t,1) = V^i(t,1)$$

for all $t \in J$ and $i \in \{1,2,3\}$ which implies that the boundary term at $x = 1$ reduces to

$$2\sum_{i=1}^{3} \left\langle \boldsymbol{\kappa}^i(t,1), (\gamma^i_s)_t(t,1)\right\rangle\,.$$

□

The elastic flow of networks can be understood as the L^2-gradient flow of the elastic energy E_μ. Depending on the class of networks one is considering, the gradient flow structure gives rise to certain conditions at the boundary that need to be fulfilled by the flow. Suitable conditions at the junctions have been derived in [16, Theorem 2.1]. Similarly as in Section 1.2 we formally derive the gradient flow and give details on the resulting boundary conditions for networks being of Theta or Triod type.

Definition 3.23 (L^2-gradient of E_μ)**.** Let $\mathcal{A} \in \{\mathcal{A}_\Theta, \mathcal{A}_{\mathbb{T}}\}$ be an admissible set. We say that the elastic energy $E_\mu : \mathcal{A} \to \mathbb{R}$ has an L^2*–gradient* at $\gamma_0 \in \mathcal{A}$, denoted by $\mathrm{grad}_{\mathcal{A}}E_\mu(\gamma_0)$, if there exist smooth functions $\mathrm{grad}_{\mathcal{A}}E_\mu(\gamma_0)^i : [0,1] \to \mathbb{R}$ for $i \in \{1,2,3\}$ such that for all smooth one-parameter families $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$ with $J \subset \mathbb{R}$ an open interval with $0 \in J$, $\gamma(0) = \gamma_0$ and $\gamma(t) \in \mathcal{A}$

for all $t \in J$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\mu(\gamma(t))_{|t=0} = \sum_{i=1}^{3} \int_{[0,1]} \mathrm{grad}_{\mathcal{A}} E_\mu(\gamma_0)^i \left\langle (\gamma_t^i)_{|t=0}, \nu_0^i \right\rangle \mathrm{d}s_0^i \,.$$

Here ν_0^i and s_0^i denote the unit normal and arclength parameter of the curve γ_0^i, respectively. We define the *domain* of $\mathrm{grad}_{\mathcal{A}} E_\mu$ by

$$\mathcal{D}\left(\mathrm{grad}_{\mathcal{A}} E_\mu\right) := \left\{ \gamma \in \mathcal{A} : \, E_\mu \text{ has an } L^2\text{–gradient at } \gamma \right\} \,.$$

Lemma 3.24. *Let $\mathcal{A} \in \{\mathcal{A}_\Theta, \mathcal{A}_\mathbb{T}\}$ be an admissible set and $\gamma \in \mathcal{A}$ be given. The L^2–gradient $\mathrm{grad}_{\mathcal{A}} E_\mu(\gamma_0)$ is unique if it exists.*

Proof. Suppose that E_μ has an L^2–gradient at $\gamma_0 \in \mathcal{A}$ and that there exist smooth functions $F^i : [0,1] \to \mathbb{R}$, $i \in \{1,2,3\}$, satisfying the characterising property of $\mathrm{grad}_{\mathcal{A}} E_\mu(\gamma_0)$. We observe that for all $\varphi = (\varphi^1, \varphi^2, \varphi^3) \in C_0^\infty\left((0,1);\mathbb{R}^3\right)$ and a suitably small $\varepsilon(\varphi) > 0$,

$$\gamma^i(t) := \gamma_0^i + t\varphi^i \nu_0^i \,, \qquad t \in (-\varepsilon(\varphi), \varepsilon(\varphi)) \,, \; i \in \{1,2,3\} \,,$$

defines an admissible variation of γ_0. Indeed, the map $\gamma^i(t) : [0,1] \to \mathbb{R}^2$ is an immersion for t sufficiently small since $\min_{x\in[0,1]} |\gamma_{0,x}^i(x)| > 0$ and $\max_{x\in[0,1]} \left|\partial_x \left(\varphi^i(x)\nu_0^i(x)\right)\right| \leq C$. Furthermore, the type of network is preserved as φ is compactly supported in $(0,1)$. Thus the functions $G^i := \mathrm{grad}_{\mathcal{A}} E(\gamma_0)^i - F^i$ satisfy for all $\varphi \in C_0^\infty\left((0,1);\mathbb{R}^3\right)$ the identity

$$0 = \sum_{i=1}^{3} \int_{[0,1]} G^i \varphi^i \,\mathrm{d}s^i = \sum_{i=1}^{3} \int_0^1 G^i(x)\varphi^i(x) |(\gamma_0^i)_x(x)| \,\mathrm{d}x$$

which implies $G^i = 0$ for all $i \in \{1,2,3\}$. □

We now use the results obtained in Proposition 3.21 and 3.22 to derive conditions at the triple junction and the given endpoints that characterise the Theta networks and Triods at which E_μ has an L^2–gradient.
The following result has already been shown in [16, Theorem 2.1]. We prove it here for completeness.

Proposition 3.25. *The elastic energy $E_\mu : \mathcal{A}_\Theta \to \mathbb{R}$ has an L^2–gradient at $\gamma \in \mathcal{A}_\Theta$ if and only if γ satisfies at both triple junctions $O^1 = \gamma^1(0)$ and $O^2 = \gamma^1(1)$,*

$$\kappa^i(y) = 0 \,, \qquad i \in \{1,2,3\} \,, y \in \{0,1\} \,, \tag{3.15a}$$

$$\sum_{i=1}^{3} 2\kappa_s^i(y)\nu^i(y) - \mu^i \tau^i(y) = 0 \,, \qquad y \in \{0,1\} \,. \tag{3.15b}$$

The L^2–gradient is given by

$$\mathrm{grad}_{\mathcal{A}_\Theta} E_\mu(\gamma)^i = 2\kappa_{ss}^i + (\kappa^i)^3 - \mu^i \kappa^i \,.$$

In other words

$$\mathcal{D}\left(\mathrm{grad}_{\mathcal{A}_\Theta} E_\mu\right) = \{\gamma \in \mathcal{A}_\Theta : \, \gamma \text{ satisfies the identities (3.15)}\} \,.$$

The identity (3.15a) is referred to as *curvature condition*, while (3.15b) is called *third order condition.*

Proof. Suppose that $\gamma_0 \in \mathcal{A}_\Theta$ satisfies (3.15a) and (3.15b). Then Proposition 3.21 implies for all admissible variations $(\gamma(t))$ with $\gamma(0) = \gamma_0$ the identity

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\mu(\gamma(t))_{|t=0} = \sum_{i=1}^{3} \int_{[0,1]} \left(2\kappa^i_{0,ss} + \left(\kappa^i_0\right)^3 - \mu^i {\kappa_0}^i\right) \left\langle \left(\gamma^i_t\right)_{|t=0}, \nu^i_0 \right\rangle \mathrm{d}s^i_0$$

which shows that E_μ has an L^2-gradient at γ_0 given by

$$\mathrm{grad}_{\mathcal{A}_\Theta} E_\mu(\gamma_0)^i = 2\kappa^i_{0,ss} + (\kappa^i_0)^3 - \mu^i \kappa^i_0 \,.$$

Conversely, suppose that E_μ has an L^2–gradient at $\gamma_0 \in \mathcal{A}_\Theta$. We observe that for all $\psi^i \in C^\infty\big([0,1];\mathbb{R}^2\big)$, $i \in \{1,2,3\}$, with $\psi^1(y) = \psi^2(y) = \psi^3(y)$ and $|t| > 0$ sufficiently small, the curves $t \mapsto \gamma^i(t) := \gamma^i_0 + t\psi^i$ define networks $\gamma(t) = \big(\gamma^1(t), \gamma^2(t), \gamma^3(t)\big)$ in $\mathcal{A}_\Theta$. With the choice $\psi^i = \varphi^i \nu^i_0$, $\varphi^i \in C^\infty_0((0,1);\mathbb{R})$, we infer from Proposition 3.21,

$$0 = \sum_{i=1}^{3} \int_{[0,1]} \left(\mathrm{grad}_{\mathcal{A}_\Theta} E_\mu(\gamma_0)^i - \left(2\kappa^i_{0,ss} + \left(\kappa^i_0\right)^3 - \mu^i \kappa^i_0\right)\right) \varphi^i \,\mathrm{d}s^i_0$$

for all $\varphi^i \in C^\infty_0((0,1);\mathbb{R})$ which implies for $i \in \{1,2,3\}$,

$$\mathrm{grad}_{\mathcal{A}_\Theta} E_\mu(\gamma_0)^i = 2\kappa^i_{0,ss} + \left(\kappa^i_0\right)^3 - \mu^i \kappa^i_0 \,.$$

Furthermore, this yields that all boundary terms in the corresponding expression in Proposition 3.21 need to vanish for all admissible variations. In particular, for all $\psi^i \in C^\infty\big([0,1];\mathbb{R}^2\big)$,

$$0 = \left\langle \psi^1, \sum_{i=1}^{3} \left(- 2\kappa^i_{0,s}\nu^i_0 - (\kappa^i_0)^2 \tau^i_0 + \mu^i \tau^i_0 \right) \right\rangle_0^1 + 2\sum_{i=1}^{3} \left\langle \boldsymbol{\kappa}^i_0, \psi^i_s \right\rangle_0^1 . \tag{3.16}$$

We let $\psi^2 = \psi^3 = 0$ and $\psi^1 \in C^\infty\big([0,1];\mathbb{R}^2\big)$ be such that close to $y \in \{0,1\}$, $\psi^1(x) = \int_y^x \nu^1_0(\tilde{x})|(\gamma^1_0)_x(\tilde{x})| \,\mathrm{d}\tilde{x}$. Then $\psi^1(0) = \psi^1(1) = 0$ and $\psi^1_s(0) = \nu^1_0(0)$, $\psi^1_s(1) = \nu^1_0(1)$ and thus (3.16) yields $\kappa^1_0(y) = 0$ for $y \in \{0,1\}$. An analogous argument for $i \in \{2,3\}$ yields that γ_0 fulfils (3.15a). Thus the property (3.16) reduces to

$$0 = \left\langle \psi^1, \sum_{i=1}^{3} \left(- 2\kappa^i_{0,s}\nu^i_0 + \mu^i \tau^i_0 \right) \right\rangle_0^1$$

for all choices of $\psi^1(0)\,, \psi^1(1) \in \mathbb{R}^2$ which implies that γ_0 fulfils (3.15b). □

Proposition 3.26. *The elastic energy $E_\mu : \mathcal{A}_\mathbb{T} \to \mathbb{R}$ has an L^2–gradient at $\gamma \in \mathcal{A}_\mathbb{T}$ if and only if at the triple junction $O = \gamma^1(0)$ and at the endpoints $P^i = \gamma^i(1)$, respectively, it holds*

$$\kappa^i(0) = 0\,, \qquad i \in \{1,2,3\}\,, \tag{3.17a}$$

$$\sum_{i=1}^{3} 2\kappa^i_s(0)\nu^i(0) - \mu^i \tau^i(0) = 0\,, \tag{3.17b}$$

$$\kappa^i(1) = 0\,, \qquad i \in \{1,2,3\}\,. \tag{3.17c}$$

The L^2–gradient is given by

$$\mathrm{grad}_{\mathcal{A}_\mathbb{T}} E_\mu(\gamma)^i = \left(2\kappa^i_{ss} + (\kappa^i)^3 - \mu^i \kappa^i\right).$$

In other words

$$\mathcal{D}\left(\operatorname{grad}_{\mathcal{A}_{\mathbb{T}}} E_\mu\right) = \{\gamma \in \mathcal{A}_{\mathbb{T}} : \gamma \text{ satisfies the identities (3.17)}\} .$$

The identities (3.17a) and (3.17c) are referred to as *curvature condition*, while (3.17b) is called *third order condition*.

Proof. As in the previous proof it is straightforward to verify that if $\gamma_0 \in \mathcal{A}_{\mathbb{T}}$ satisfies the identities (3.17), then E_μ has an L^2-gradient at γ_0 which is given by the desired identity. Conversely, suppose that E_μ has an L^2-gradient at $\gamma_0 \in \mathcal{A}_{\mathbb{T}}$. Then for all $\psi^i \in C^\infty\left([0,1];\mathbb{R}^2\right)$, $i \in \{1,2,3\}$, with $\psi^1(0) = \psi^2(0) = \psi^3(0)$ and $\psi^i(1) = 0$, the curves $t \mapsto \gamma^i(t) := \gamma_0^i + t\psi^i$ define Triods $\gamma(t) = \left(\gamma^1(t), \gamma^2(t), \gamma^3(t)\right)$ for $|t|$ sufficiently small. With the special choice $\psi^i = \varphi^i \nu_0^i$ with an arbitrary $\varphi^i \in C_0^\infty((0,1);\mathbb{R})$, Proposition 3.22 implies that

$$\operatorname{grad}_{\mathcal{A}_{\mathbb{T}}} E_\mu(\gamma_0)^i = 2\kappa_{0,ss}^i + (\kappa_0^i)^3 - \mu^i \kappa_0^i .$$

We thus have for all $\psi^i \in C^\infty\left([0,1];\mathbb{R}^2\right)$, $i \in \{1,2,3\}$, with $\psi^1(0) = \psi^2(0) = \psi^3(0)$ and $\psi^1(1) = \psi^2(1) = \psi^3(1) = 0$,

$$\begin{aligned} 0 = &\left\langle \psi^1(0), \sum_{i=1}^3 (-2\kappa_{s,0}^i(0)\nu_0^i(0) - \kappa_0^i(0)^2 \tau_0^i(O) + \mu^i \tau_0^i(0) \right\rangle + 2\sum_{i=1}^3 \left\langle \boldsymbol{\kappa}_0^i(0), \psi_s^i(0)\right\rangle \\ &+ 2\sum_{i=1}^3 \left\langle \boldsymbol{\kappa}_0^i(1), \psi_s^i(1)\right\rangle . \end{aligned}$$

Arguing precisely as in the proof of Proposition 3.25 we conclude that γ_0 satisfies the identities (3.17). □

We are interested in the evolution problems that arise as L^2–gradient flows of the elastic energy. The characterisation of the L^2–gradient of the elastic energy E_μ yields the following flow.

Definition 3.27 (Smooth solution to the gradient flow)**.** Let $\mathcal{A} \in \{\mathcal{A}_\Theta, \mathcal{A}_{\mathbb{T}}\}$ be a set of admissible networks. A *smooth solution to the gradient flow of* E_μ *in* $\mathcal{A}$ is a smooth function $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$ with $J \subset \mathbb{R}$ an interval such that for all $t \in J$, $\gamma(t) \in \mathcal{D}(\operatorname{grad}_{\mathcal{A}} E_\mu)$ and for all $t \in J$ and $i \in \{1,2,3\}$,

$$V^i(t) = \left\langle \gamma_t^i(t), \nu^i(t)\right\rangle = -\operatorname{grad}_{\mathcal{A}} E_\mu(\gamma(t))^i = -\left(2\kappa_{ss}^i(t) + \left(\kappa^i(t)\right)^3 - \mu^i \kappa^i(t)\right). \tag{3.18}$$

Corollary 3.28. *Let $\mu \in \mathbb{R}^3$, $\mathcal{A} \in \{\mathcal{A}_\Theta, \mathcal{A}_{\mathbb{T}}\}$ be a set of admissible networks and $\gamma : J \times [0,1] \to \left(\mathbb{R}^2\right)^3$ be a smooth solution to the gradient flow of E_μ in $\mathcal{A}$. Then for all $t \in J$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\mu(\gamma(t)) = -\sum_{i=1}^3 \int_{[0,1]} \left(2\kappa_{ss}^i(t) + \left(\kappa^i\right)^3(t) - \mu^i \kappa^i(t)\right)^2 \mathrm{d}s^i . \tag{3.19}$$

In particular, if $\mu^i \geq 0$ for all $i \in \{1,2,3\}$, then for all $j \in \{1,2,3\}$ and $\tau, t \in J$ with $\tau \leq t$ there holds

$$\left\|\kappa^j(t)\right\|^2_{L_2\left(\gamma^j(t), \mathrm{d}s_t^j\right)} \leq \sum_{i=1}^3 \left\|\kappa^i(t)\right\|^2_{L_2\left(\gamma^i(t), \mathrm{d}s_t^i\right)} \leq E_\mu(\gamma(\tau)) .$$

In the case $\mu^i > 0$ for all $i \in \{1,2,3\}$ we further have for all $i \in \{1,2,3\}$ and $\tau, t \in J$ with $\tau \leq t$,

$$L\left(\gamma^i(t)\right) \leq \frac{1}{\mu^i} E_\mu(\gamma(\tau)) .$$

Proof. Let $t \in J$ be given. As $\gamma(t)$ lies in $\mathcal{D}\left(\mathrm{grad}_{\mathcal{A}} E_\mu\right)$, Definition 3.23 yields

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\mu(\gamma(t)) = \sum_{i=1}^{3} \int_{[0,1]} \mathrm{grad}_{\mathcal{A}} E_\mu(\gamma(t))^i V^i(t)\,\mathrm{d}s^i\,.$$

The identity (3.19) is now a consequence of (3.18). In particular, $t \mapsto E_\mu(t)$ is decreasing. Let τ, $t \in J$ with $\tau \le t$ be given. If $\mu^i \ge 0$ for all $i \in \{1,2,3\}$, we obtain for all $j \in \{1,2,3\}$,

$$\left\|\kappa^j(t)\right\|^2_{L_2\left(\gamma^j(t),\mathrm{d}s^j_t\right)} \le \sum_{i=1}^{3} \left\|\kappa^i(t)\right\|^2_{L_2\left(\gamma^i(t),\mathrm{d}s^i_t\right)} \le \sum_{i=1}^{3} E_{\mu^i}\left(\gamma^i(t)\right) = E_\mu(\gamma(t)) \le E_\mu(\gamma(\tau))\,.$$

In the case $\mu^i > 0$ for all $i \in \{1,2,3\}$ we further have for all $i \in \{1,2,3\}$,

$$L\left(\gamma^i(t)\right) \le \frac{1}{\mu^i} E_{\mu^i}\left(\gamma^i(t)\right) \le \frac{1}{\mu^i} E_\mu(\gamma(t)) \le \frac{1}{\mu^i} E_\mu(\gamma(\tau))\,.$$

This completes the proof. □

The parameter t is often interpreted as time and the interval J is of the form $J = [0,T]$, $J = [0,T)$ or $J = (0,T)$ for $T > 0$. Thus one can view a smooth solution γ to the gradient flow of E_μ in $\mathcal{A}$ as the evolution of the initial network $\gamma(0) \in \mathcal{A}$ in the considered class of admissible networks that decreases the elastic energy most efficiently with respect to the L^2–inner product.

This gives rise to two different problems depending on the set of networks one is interested in. The corresponding boundary conditions follow from the characterisation of $\mathcal{D}(\mathrm{grad}_{\mathcal{A}} E_\mu)$ in the different cases that have been established in Section 3.2.

Definition 3.29 (Elastic flow for Theta networks)**.** A *smooth solution to the elastic flow for Theta networks* is a smooth function $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$, with $J \subset \mathbb{R}$ an interval, such that for all $t \in J$ and $i \in \{1,2,3\}$, $\gamma^i(t)$ is a regular open curve and for $i \in \{1,2,3\}$,

$$\begin{cases} \left\langle \gamma^i_t, \nu^i \right\rangle = -\left(2\kappa^i_{ss} + \left(\kappa^i\right)^3 - \mu^i \kappa^i\right) & \text{in } J \times [0,1] \quad \text{(motion)}\,, \\ \gamma^1 = \gamma^2 = \gamma^3 & \text{on } J \times \{0,1\} \quad \text{(concurrency condition)}\,, \\ \kappa^i = 0 & \text{on } J \times \{0,1\} \quad \text{(curvature condition)}\,, \\ \sum_{i=1}^{3} 2\kappa^i_s \nu^i - \mu^i \tau^i = 0 & \text{on } J \times \{0,1\} \quad \text{(third order condition)}\,. \end{cases} \tag{3.20}$$

Definition 3.30 (Elastic flow for Triods)**.** A *smooth solution to the elastic flow for Triods with given endpoints* P^1*,* P^2*,* P^3 is a smooth function $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$, with $J \subset \mathbb{R}$ an interval, such that for all $t \in J$ and $i \in \{1,2,3\}$, $\gamma^i(t)$ is a regular open curve and for $i \in \{1,2,3\}$,

$$\begin{cases} \left\langle \gamma^i_t, \nu^i \right\rangle = -\left(2\kappa^i_{ss} + \left(\kappa^i\right)^3 - \mu^i \kappa^i\right) & \text{in } J \times [0,1] \quad \text{(motion)}\,, \\ \gamma^1 = \gamma^2 = \gamma^3 & \text{on } J \times \{0\} \quad \text{(concurrency condition)}\,, \\ \gamma^i = P^i & \text{on } J \times \{1\} \quad \text{(endpoints)}\,, \\ \kappa^i = 0 & \text{on } J \times \{0,1\} \quad \text{(curvature condition)}\,, \\ \sum_{i=1}^{3} 2\kappa^i_s \nu^i - \mu^i \tau^i = 0 & \text{on } J \times \{0\} \quad \text{(third order condition)}\,. \end{cases} \tag{3.21}$$

In the systems (3.20) and (3.21) there are no additional constraints imposed on the curves besides the concurrency and the fixed endpoint constraint, that are needed to maintain the shape of the network. As the remaining boundary conditions "naturally" arise from the first variation of the energy, the problems (3.20) and (3.21) are also referred to as the flows with *natural boundary conditions.*

The boundary problems (3.20) and (3.21) are *geometric* meaning that they are invariant under reparametrisation.

Proposition 3.31 (Invariance of the elastic flow under reparametrisation)**.** *Suppose that* $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$*,* $J \subset \mathbb{R}$ *an interval, is a smooth solution to the gradient flow of* E_μ *in* $\mathcal{A}$ *with* $\mathcal{A} \in \{\mathcal{A}_\Theta, \mathcal{A}_\mathbb{T}\}$ *and* $\theta^i : J \times [0,1] \to [0,1]$*,* $i \in \{1,2,3\}$*, is a smooth one-parameter family of smooth diffeomorphisms* $\theta^i(t) : [0,1] \to [0,1]$ *with* $\theta^i(t,0) = 0$ *and* $\theta^i(t,1) = 1$ *for* $t \in J$*. Then* $\widetilde{\gamma} : J \times [0,1] \to (\mathbb{R}^2)^3$ *defined by* $\widetilde{\gamma}^i(t,x) := \gamma^i\left(t, \theta^i(t,x)\right)$*,* $i \in \{1,2,3\}$*, is a smooth solution to the gradient flow of* E_μ *in* $\mathcal{A}$*.*

Proof. The function $\widetilde{\gamma}$ is smooth being a composition of smooth functions and for every $i \in \{1,2,3\}$ and $t \in J$, $\widetilde{\gamma}^i(t)$ is a regular open curve being a reparametrisation of $\gamma^i(t)$. As $\theta^i(t,0) = 0$ and $\theta^i(t,1) = 1$, the topology of the network, that is the Triod or Theta shape, respectively, is preserved. Given $t \in J$ we let $s^i(t)$ and $\widetilde{s}^i(t)$ be the arclength parameters of the curves $\gamma^i(t)$ and $\widetilde{\gamma}^i(t)$, respectively, with inverse functions $x^i(t)$ and $\widetilde{x}^i(t)$. Then one observes for $x \in [0,1]$,

$$\begin{aligned}
&\partial_{\widetilde{s}^i(t)}\left(\widetilde{\gamma}^i(t) \circ \widetilde{x}^i(t)\right)\left(\widetilde{s}^i(t,x)\right) \\
&= \left|(\widetilde{\gamma}^i_x)(t,x)\right|^{-1} \widetilde{\gamma}^i_x(t,x) = \left|\gamma^i_x(t,\theta^i(t,x))\theta^i_x(t,x)\right|^{-1} \gamma^i_x\left(t,\theta^i(t,x)\right)\theta^i_x(t,x) \\
&= \left|\gamma^i_x(t,\theta^i(t,x))\right|^{-1} \gamma^i_x\left(t,\theta^i(t,x)\right) = \partial_{s^i(t)}\left(\gamma^i(t) \circ x^i(t)\right)\left(s^i(t,\theta^i(t,x))\right).
\end{aligned}$$

Inductively, one sees that this identity holds for arbitrary order of arclength derivative. This yields that the quantities appearing in the motion equation and the boundary conditions are all geometric, that is, preserved under reparametrisation. Denoting by $\nu^i(t)$ and $\widetilde{\nu}^i(t)$ the unit normals to $\gamma^i(t)$ and $\widetilde{\gamma}^i(t)$, $t \in J$, respectively, we observe for $t \in J$ and $x \in [0,1]$,

$$\begin{aligned}
\left\langle \widetilde{\gamma}^i_t(t,x), \widetilde{\nu}^i(t,x)\right\rangle &= \left\langle \gamma^i_t\left(t,\theta^i(t,x)\right) + \gamma^i_x\left(t,\theta^i(t,x)\right)\partial_t\theta^i(t,x), \nu^i\left(t,\theta^i(t,x)\right)\right\rangle \\
&= \left\langle \gamma^i_t\left(t,\theta^i(t,x)\right), \nu^i\left(t,\theta^i(t,x)\right)\right\rangle .
\end{aligned}$$

Thus we may conclude that if γ is a smooth solution to the gradient flow of E_μ in $\mathcal{A}$ then so is its reparametrisation $\widetilde{\gamma}$. □

In particular, *uniqueness* of solutions to the gradient flow of E_μ in an admissible set $\mathcal{A}$ can only be expected up to (time dependent) reparametrisations which precisely corresponds to the evolution of the *images* of the network being unique. This notion is referred to as *geometric uniqueness*.

3.3 Tangential degrees of freedom

In this section we study the tangential degrees of freedom in the evolution problems (3.20) and (3.21), which we also refer to as the *geometric problems*. The evolution law is *degenerate* in the sense that only the normal component is specified. Indeed, the observable motion of the image of the network is determined by the normal velocity of each curve and the requirement that the shape of the network, Triod or Theta, is maintained during the evolution. In doing so, we allow for motion of the triple junctions which gives rise to tangential movement. While the normal velocity is invariant under reparametrisation as shown in Proposition 3.31, different solutions γ and $\widetilde{\gamma}$ to the elastic flow in a class $\mathcal{A}$ with $[\gamma(t)] = [\widetilde{\gamma}(t)]$ in general have different tangential velocities in the interior of the curves as this does not affect the geometric evolution. At the triple junctions though, the tangential movement is uniquely determined by the normal velocities of the curves and the concurrency constraint with the exception of one degenerate geometric situation.

Definition 3.32 (Non-degeneracy condition)**.** Three regular open curves $\gamma^i : [0,1] \to \mathbb{R}^2$, $i \in \{1,2,3\}$, that satisfy the concurrency condition in $y \in \{0,1\}$ are said to fulfil the *non-degeneracy* *condition* in y if the angles $\alpha^1(y)$, $\alpha^2(y)$, and $\alpha^3(y)$ between $\tau^2(y)$ and $\tau^3(y)$, $\tau^3(y)$ and $\tau^1(y)$, and $\tau^1(y)$ and $\tau^2(y)$, respectively, satisfy

$$\max\left\{\left|\sin\alpha^1(y)\right|, \left|\sin\alpha^2(y)\right|, \left|\sin\alpha^3(y)\right|\right\} > 0\,. \tag{3.22}$$

The above condition appeared first in [64, Definition 3.2].

Definition 3.33 (Non-degenerate smooth solutions)**.** Let $J \subset \mathbb{R}$ be an interval. A smooth solution $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$ to system (3.20) or (3.21) is called *non-degenerate* if at every $y \in \{0,1\}$ in which $\gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y)$ holds for all $t \in J$, the curves $\gamma^1(t)$, $\gamma^2(t)$ and $\gamma^3(t)$ satisfy the non-degeneracy condition in y for all $t \in J$.

An equivalent formulation of (3.22) is

$$\operatorname{span}\left\{\nu^1(y), \nu^2(y), \nu^3(y)\right\} = \mathbb{R}^2$$

which is satisfied if and only if at least one angle at the triple junction is different from 0, π and 2π. As the three angles at the junction sum up to 2π, this is equivalent to the requirement that at most one angle is equal to 0 or π. The non-degeneracy condition appears in the definition of geometrically admissible initial networks as it is needed to prove the validity of the *Lopatinskii–Shapiro condition* which is itself the crucial ingredient to the existence of the evolution, see Proposition 4.6. Its uniform version plays an important role regarding the long time behaviour of the evolution, see Chapter 5. We formulate it as in [65, (6.9)].

Definition 3.34 (Uniform non-degeneracy condition)**.** Let $J \subset \mathbb{R}$ be an interval and $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$ be a one-parameter family of triples $\left(\gamma^1(t), \gamma^2(t), \gamma^3(t)\right)$ of regular open curves $\gamma^i(t) : [0,1] \to \mathbb{R}^2$, $i \in \{1,2,3\}$. Let $y \in \{0,1\}$ be such that for every $t \in J$, the curves $\gamma^1(t)$, $\gamma^2(t)$, $\gamma^3(t)$ satisfy the concurrency condition in y. Then γ is said to fulfil the *uniform non-degeneracy condition* in y if the angles $\alpha^1(t,y)$, $\alpha^2(t,y)$, and $\alpha^3(t,y)$ between the tangent vectors $\tau^2(t,y)$ and $\tau^3(t,y)$, $\tau^3(t,y)$ and $\tau^1(t,y)$, and $\tau^1(t,y)$ and $\tau^2(t,y)$, respectively, satisfy

$$\rho := \inf_{t\in J} \max\left\{\left|\sin\alpha^1(t,y)\right|, \left|\sin\alpha^2(t,y)\right|, \left|\sin\alpha^3(t,y)\right|\right\} > 0\,. \tag{3.23}$$

Proposition 3.35. *[65, Lemma 6.11]. Let $J \subset \mathbb{R}$ be an interval and $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$ be a smooth solution to one of the geometric problems* (3.20) *or* (3.21) *such that γ satisfies the uniform non-degeneracy condition in every $y \in \{0,1\}$ where $\gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y)$ holds for every $t \in J$. Then for every $t \in J$ the tangential velocities $T^i(t,y)$, $i \in \{1,2,3\}$, are linear combinations of the normal velocities $V^i(t,y)$, $i \in \{1,2,3\}$, with coefficients uniformly bounded with respect to $t \in J$.*

Proof. All quantities in the following are evaluated in y. To improve readability we omit this in the notation. Differentiating the concurrency condition with respect to t yields for every $t \in J$,

$$V^1(t)\nu^1(t) + T^1(t)\tau^1(t) = V^2(t)\nu^2(t) + T^2(t)\tau^2(t) = V^3(t)\nu^3(t) + T^3(t)\tau^3(t)\,.$$

Testing these identities with $\tau^1(t)$, $\tau^2(t)$, $\tau^3(t)$ implies

$$\begin{pmatrix} -\left\langle\tau^1(t),\tau^2(t)\right\rangle & 1 & 0 \\ 0 & -\left\langle\tau^2(t),\tau^3(t)\right\rangle & 1 \\ 1 & 0 & -\left\langle\tau^3(t),\tau^1(t)\right\rangle \end{pmatrix} \begin{pmatrix} T_1(t) \\ T_2(t) \\ T_3(t) \end{pmatrix} = \begin{pmatrix} \left\langle\nu^1(t),\tau^2(t)\right\rangle V_1(t) \\ \left\langle\nu^2(t),\tau^3(t)\right\rangle V_2(t) \\ \left\langle\nu^3(t),\tau^1(t)\right\rangle V_3(t) \end{pmatrix}.$$

The 3×3–matrix on the left hand side is denoted by $M(t)$ in the following. Its determinant is given by

$$\det M(t) = 1 - \left\langle\tau^2(t),\tau^1(t)\right\rangle\left\langle\tau^3(t),\tau^2(t)\right\rangle\left\langle\tau^1(t),\tau^3(t)\right\rangle = 1 - \cos\alpha^1(t)\cos\alpha^2(t)\cos\alpha^3(t)\,.$$

Using (3.23) we obtain

$$\inf_{t\in J}\det M(t) = 1-\sup_{t\in J}\left(\cos\alpha^1(t)\cos\alpha^2(t)\cos\alpha^3(t)\right)$$
$$\geq 1-\sup_{t\in J}\min\left\{|\cos\alpha^1(t)|,|\cos\alpha^2(t)|,|\cos\alpha^3(t)|\right\}\geq 1-\sqrt{1-\rho^2}>0\,.$$

In particular, for every $t\in J$ there exists a unique solution $T(t)=\left(T^1(t),T^2(t),T^3(t)\right)$ to the above system. By Cramer's rule each component $T^i(t)$ of the unique solution $T(t)$ can be expressed as a linear combination of $V^1(t)$, $V^2(t)$ and $V^3(t)$ with coefficients that are polynomials in the entries of $M(t)$, $\langle\nu^1(t),\tau^2(t)\rangle$, $\langle\nu^2(t),\tau^3(t)\rangle$, $\langle\nu^3(t),\tau^1(t)\rangle$ and $(\det M(t))^{-1}$. We refer to Lemma 4.64 for explicit the formulas. The condition (3.23) ensures that these coefficients are uniformly bounded with respect to $t\in J$. □

To construct an explicit solution to the elastic flow in a given admissible class, one has to remove the tangential degrees of freedom that are present in the problem which corresponds to fixing a parametrisation of the flow.

In the following we specify a tangential velocity for each curve and impose additional boundary conditions that turn the degenerate problems into systems of quasilinear parabolic boundary value problems of fourth order, also called the *analytic problems*. In doing so one needs to keep in mind that the tangential component is already determined at the junction points as shown in Proposition 3.35. Furthermore, the boundary constraints need to be *tangential* in the sense that they do not change the geometry of the problem but can always be obtained by suitable reparametrisation of the curves. These issues are studied in Theorem 4.59.

Given a smooth regular open curve $\gamma:[0,1]\to\mathbb{R}^2$ and $\mu\in\mathbb{R}$ we explicitly compute the expression

$$\overline{V}(\gamma)\nu := -\left(2\kappa_{ss}+\kappa^3-\mu\kappa\right)\nu \tag{3.24}$$

in terms of γ which gives

$$\begin{aligned}
\overline{V}(\gamma)\nu &= -2\kappa_{ss}\nu-\kappa^3\nu+\mu\kappa\nu\\
&= -2\frac{\gamma_{xxxx}}{|\gamma_x|^4}+12\frac{\gamma_{xxx}\langle\gamma_{xx},\gamma_x\rangle}{|\gamma_x|^6}+5\frac{\gamma_{xx}|\gamma_{xx}|^2}{|\gamma_x|^6}+8\frac{\gamma_{xx}\langle\gamma_{xxx},\gamma_x\rangle}{|\gamma_x|^6}-35\frac{\gamma_{xx}\langle\gamma_{xx},\gamma_x\rangle^2}{|\gamma_x|^8}\\
&\quad-\left\langle -2\frac{\gamma_{xxxx}}{|\gamma_x|^4}+12\frac{\gamma_{xxx}\langle\gamma_{xx},\gamma_x\rangle}{|\gamma_x|^6}+5\frac{\gamma_{xx}|\gamma_{xx}|^2}{|\gamma_x|^6}+8\frac{\gamma_{xx}\langle\gamma_{xxx},\gamma_x\rangle}{|\gamma_x|^6}-35\frac{\gamma_{xx}\langle\gamma_{xx},\gamma_x\rangle^2}{|\gamma_x|^8},\tau\right\rangle\tau\\
&\quad+\mu\frac{\gamma_{xx}}{|\gamma_x|^2}-\left\langle\mu\frac{\gamma_{xx}}{|\gamma_x|^2},\tau\right\rangle\tau\,.
\end{aligned}\tag{3.25}$$

Definition 3.36. [64, Definition 3.5]**.** Given a smooth regular open curve $\gamma:[0,1]\to\mathbb{R}^2$ and $\mu\in\mathbb{R}$ we define

$$\begin{aligned}
\overline{T}(\gamma) :=& \left\langle -2\frac{\gamma_{xxxx}}{|\gamma_x|^4}+12\frac{\gamma_{xxx}\langle\gamma_{xx},\gamma_x\rangle}{|\gamma_x|^6}+5\frac{\gamma_{xx}|\gamma_{xx}|^2}{|\gamma_x|^6}+8\frac{\gamma_{xx}\langle\gamma_{xxx},\gamma_x\rangle}{|\gamma_x|^6}-35\frac{\gamma_{xx}\langle\gamma_{xx},\gamma_x\rangle^2}{|\gamma_x|^8},\tau\right\rangle\\
&+\mu\left\langle\frac{\gamma_{xx}}{|\gamma_x|^2},\tau\right\rangle\,.
\end{aligned}\tag{3.26}$$

Theorem 4.59 shows that given parameters $\mu^i\in\mathbb{R}$, $i\in\{1,2,3\}$, and a one-parameter family of networks $\eta:J\times[0,1]\to(\mathbb{R}^2)^3$ solving the elastic flow in an admissible set $\mathcal{A}$ (in the strong or classical sense) there exists a reparametrisation γ of η satisfying the evolution law

$$\gamma^i_t=\overline{V}\left(\gamma^i\right)\nu^i+\overline{T}\left(\gamma^i\right)\tau^i \qquad \text{in } J\times[0,1]\,,\ i\in\{1,2,3\}\,. \tag{3.27}$$

This suggests to look for a one-parameter family of networks $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$ satisfying (3.27) together with the boundary conditions in the respective geometric problem. The evolution laws (3.27) yield a system of six scalar parabolic equations of fourth order. To obtain a unique strong solution one would expect to have two independent conditions for each equation at each boundary point $y \in \{0,1\}$. The boundary conditions in both of the geometric problems (3.20) and (3.21) give nine scalar conditions for each boundary point. Thus, the resulting non-degenerate system is under-determined as for each curve one condition at each boundary point is missing. To remove these tangential degrees of freedom one has to carefully choose conditions on the curves that on the one hand yield a well-posed PDE and on the other hand do not affect the geometric problem. As in [64, (3.4)] it turns out that the so-called *tangential second order condition* given by

$$\left\langle \gamma^i_{xx}, \tau^i \right\rangle = 0 \qquad \text{on } J \times \{0,1\}\,,\ i \in \{1,2,3\} \tag{3.28}$$

satisfies these requirements, see Proposition 4.18, Proposition 4.19 and Theorem 4.59.

The resulting auxiliary boundary value problems are analysed in detail in Section 4.1. Existence and uniqueness of solutions with appropriate initial data is shown in different function space settings. To answer the question of existence of solutions to the geometric problem given an initial network, one has to find a reparametrisation of the initial network satisfying the additional tangential conditions at the boundary that are necessary to solve the analytic problem. This is done in Proposition 4.56. This then yields a solution to the geometric problem as shown in Theorem 4.58. By constructing suitable reparametrisations we show that the solutions to problems (3.20) and (3.21) are unique up to reparametrisation, see Theorem 4.61 and Theorem 4.62.

3.4 Notions of solutions and conditions on the initial network

In this section we introduce the concepts of classical and strong solutions to the evolution problems (3.20) and (3.21) carefully explaining the resulting requirements on the initial networks.
To derive the geometric problems from the variational point of view we have restricted ourselves to parametrisations that are smooth in time and space mainly to simplify technical aspects arising from regularity issues. Indeed, the aim was to derive the right conditions at the junctions and the endpoints. Turning now to the question of *existence of solutions* we address the issue of regularity more carefully.
A priori, there are several notions of solutions that could be considered from the point of view of analysis, roughly divided into the categories of strong and weak solutions. To obtain a notion of weak solutions one would need to integrate the gradient flow equation (3.18) by parts using the boundary conditions that are valid in the respective class of admissible networks. We do not want to go into detail here as we intend to show existence of strong solutions, that are, roughly speaking, time-dependent parametrisations solving the resulting boundary value problem at least almost everywhere in time and space. More precisely, all appearing derivatives of the parametrisations in the motion equation should exist at least in the sense of distributional derivatives and the derivatives of lower order appearing in the boundary conditions should exist pointwise in space such that one can actually evaluate at the end-points.
The L^2–gradient flow of E_μ in the different classes of networks gives rise to a quasilinear parabolic-type equation of fourth order as discussed in Section 3.3. We shall prove existence in two natural settings for this type of equation, namely in Hölder Spaces and Sobolev Spaces. In the latter case the derivatives exist in a distributional sense while in the former they exist as classical derivatives which is why the corresponding solutions are referred to as *classical solutions*. The solution spaces

are given by

$$C^{\frac{4+\alpha}{4},4+\alpha}\left([0,T]\times[0,1];(\mathbb{R}^2)^3\right)$$

for $T>0$ and $\alpha\in(0,1)$ and

$$W_p^1\left((0,T);L_p((0,1);(\mathbb{R}^2)^3)\right)\cap L_p\left((0,T);W_p^4((0,1);(\mathbb{R}^2)^3)\right)$$

respectively, where we restrict to $p\in(5,\infty)$. The definition of these spaces with corresponding norms are given in Definitions B.7 and B.26, respectively.
As explained before, both types of solutions are rather strong bringing the disadvantage that several conditions need to be imposed on the initial object. These are the so called *compatibility conditions* and are derived in the following. Initial networks satisfying these conditions are called *geometrically admissible* to distinguish them from the *analytically admissible* initial values that are admissible for the auxiliary boundary value problem derived in Section 3.3. The advantage in the Sobolev setting is that the regularity and compatibility conditions required on the initial network are weaker with respect to the ones in the Hölder setting. Of course, the downside is that also the solution one obtains is less regular than the one arising from Hölder theory. Fortunately, one can use the parabolic nature of the problem to show that solutions to the L^2–gradient flows are smooth for positive times, also when starting from a Sobolev initial network. This is shown in Subsection 4.1.2.

Let $p\in(5,\infty)$ and $\gamma\in W_p^1\left((0,T);L_p((0,1);(\mathbb{R}^2)^3)\right)\cap L_p\left((0,T);W_p^4((0,1);(\mathbb{R}^2)^3)\right)$ be such that $\gamma^i(t)$, $i\in\{1,2,3\}$, are regular open curves solving one of the systems (3.20) and (3.21). Proposition B.35 implies that the solution space is continuously embedded in $C\left([0,T];C^3\left([0,1];(\mathbb{R}^2)^3\right)\right)$ and thus all boundary conditions are valid pointwise in time and space. Evaluation in $t=0$ yields that the initial network $\gamma(0)$ satisfies all appearing boundary conditions. Its regularity is determined by the trace space of the solution space which is given by the Slobodeckij space $W_p^{4-4/p}\left((0,1);(\mathbb{R}^2)^3\right)$ defined in Definition B.14.

Definition 3.37 (Geometrically admissible initial network (Sobolev setting))**.** Let $p\in(5,\infty)$. A *geometrically p–admissible initial network* to one of the systems (3.20) and (3.21) is a function $\sigma\in W_p^{4-4/p}\left((0,1);(\mathbb{R}^2)^3\right)$ such that $\sigma^i:[0,1]\to\mathbb{R}^2$, $i\in\{1,2,3\}$, are regular open curves satisfying all boundary conditions appearing in the respective system and further the non-degeneracy condition in all $y\in\{0,1\}$ with $\sigma^1(y)=\sigma^2(y)=\sigma^3(y)$.

If σ is a geometrically p–admissible initial network to system (3.20) or (3.21) and $\theta^i:[0,1]\to[0,1]$, $i\in\{1,2,3\}$, are diffeomorphisms with $\theta^i\in W_p^{4-4/p}((0,1))$, $\theta^i(y)=y$ for $y\in\{0,1\}$, then $\varrho^i:=\sigma^i\circ\theta^i$ defines a geometrically p–admissible initial network $\varrho=\left(\varrho^1,\varrho^2,\varrho^3\right)$ to the same system which is a consequence of the following Lemma.

Lemma 3.38. *Let $p\in(5,\infty)$, $f,g\in W_p^{4-4/p}((0,1))$ be such that $g:[0,1]\to[0,1]$ is a C^1–diffeomorphism. Then $f\circ g$ and g^{-1} lie in $W_p^{4-4/p}((0,1))$.*

Proof. We note that [151, Corollary 4.37] yields for all $y\in[0,1]$ the differentiation rule $g_y^{-1}(y)=\left(g_x\left(g^{-1}(y)\right)\right)^{-1}$. As $W_p^{4-4/p}((0,1))\hookrightarrow C^3([0,1])$ by Proposition B.35, we obtain in particular $g^{-1}\in C^3([0,1])\hookrightarrow W_p^3((0,1))$ and

$$g_{yyy}^{-1}(y)=3\left(g_x\left(g^{-1}(y)\right)\right)^{-5}\left(g_{xx}\left(g^{-1}(y)\right)\right)^2-\left(g_x\left(g^{-1}(y)\right)\right)^{-4}g_{xxx}\left(g^{-1}(y)\right).$$

As g is a C^1–diffeomorphism, it is straightforward to show that $g_{xxx}\circ g^{-1}$ lies in $W_p^{1-4/p}((0,1))$. Proposition C.32 and the Banach algebra property of $W_p^{1-4/p}((0,1))$ shown in Proposition B.24

yield $g^{-1}_{yyy} \in W_p^{1-4/p}((0,1))$ and hence $g^{-1} \in W_p^{4-4/p}((0,1))$. Similar arguments may be used for $f \circ g \in C^3([0,1]) \hookrightarrow W^3_p((0,1))$ as

$$\partial_x^3 \left(f \circ g\right)(x) = f_{xxx}(g(x)) \left(g_x(x)\right)^3 + 3 f_{xx}(g(x)) g_x(x) g_{xx}(x) + f_x(g(x)) g_{xxx}(x) \,.$$

□

In the Hölder setting additional conditions appear at the boundary points. Let $\alpha \in (0,1)$ and $\gamma \in C^{\frac{4+\alpha}{4}, 4+\alpha}\left([0,T] \times [0,1]; (\mathbb{R}^2)^3\right)$ be such that $\gamma^i(t)$, $i \in \{1,2,3\}$, are regular open curves solving system (3.20) or (3.21). Suppose that the curves $\gamma^i(t)$ satisfy the concurrency condition in $y \in \{0,1\}$ and let $\tau^i_0 := \tau^i(0,y)$ and $\nu^i_0 := \nu^i_0(0,y)$ for $i \in \{1,2,3\}$. Due to the regularity of γ there holds for all $t \in [0,T]$,

$$\gamma^1_t(t,y) = \gamma^2_t(t,y) = \gamma^3_t(t,y) \,,$$

and thus in particular

$$\gamma^1_t(0,y) = \gamma^2_t(0,y) = \gamma^3_t(0,y) \,. \tag{3.29}$$

Let α^1_0, α^2_0 and α^3_0 denote the angles between the tangents τ^2_0 and τ^3_0, τ^3_0 and τ^1_0, and τ^1_0 and τ^2_0, respectively. Basic trigonometric relations imply the identities

$$\begin{aligned} \sin\left(\alpha^1_0\right)\tau^1_0 + \sin\left(\alpha^2_0\right)\tau^2_0 + \sin\left(\alpha^3_0\right)\tau^3_0 &= 0 \,, \\ \sin\left(\alpha^1_0\right)\nu^1_0 + \sin\left(\alpha^2_0\right)\nu^2_0 + \sin\left(\alpha^3_0\right)\nu^3_0 &= 0 \,. \end{aligned}$$

Given $i \in \{1,2,3\}$ we let

$$V^i_0 := V^i(0,y) = -\left(2\kappa^i_{ss}(0,y) + \left(\kappa^i\right)^3(0,y) - \mu^i \kappa^i(0,y)\right)$$

be the normal velocity of γ in y at time 0. Testing (3.29) with ν^1_0 and multiplying by $\sin\left(\alpha^1_0\right)$ gives

$$\begin{aligned} \sin\left(\alpha^1_0\right) V^1_0 &= \left\langle \gamma^2_t(0,y), \sin\left(\alpha^1_0\right)\nu^1_0 \right\rangle = -\sin\left(\alpha^2_0\right) V^2_0 - \sin\left(\alpha^3_0\right)\left\langle \gamma^2_t(0,y), \nu^3_0 \right\rangle \\ &= -\sin\left(\alpha^2_0\right) V^2_0 - \sin\left(\alpha^3_0\right) V^3_0 \,. \end{aligned}$$

In the case that γ is a classical solution to (3.21) in $[0,T]$ with given endpoints P^1, P^2, P^3, there holds for all $t \in [0,T]$, $i \in \{1,2,3\}$,

$$\gamma^i_t(t,1) = 0$$

and in particular, $\gamma^i_t(0,1) = 0$. Testing with the normal $\nu^i(0,1)$ to the curve $\gamma^i(0)$ at the point $y = 1$ yields for all $i \in \{1,2,3\}$,

$$0 = V^i(0,1) = -\left(2\kappa^i_{ss}(0,1) + \left(\kappa^i\right)^3(0,1) - \mu^i\kappa^i(0,1)\right).$$

Definition 3.39 (Geometrical fourth order condition (junction))**.** Let $\mu^i \in \mathbb{R}$, $\gamma^i \in C^4\left([0,1];\mathbb{R}^2\right)$, $i \in \{1,2,3\}$, be regular open curves and $y \in \{0,1\}$ such that $\gamma^1(y) = \gamma^2(y) = \gamma^3(y)$. Let further $\alpha^i(y)$ denote the angle between $\tau^{i+1}(y)$ and $\tau^{i-1}(y)$ (mod3). We say that γ satisfies the *geometrical fourth order condition* in $y \in \{0,1\}$ if

$$\sin\left(\alpha^1(y)\right)\overline{V}\left(\gamma^1\right)(y) + \sin\left(\alpha^2(y)\right)\overline{V}\left(\gamma^2\right)(y) + \sin\left(\alpha^3(y)\right)\overline{V}\left(\gamma^3\right)(y) = 0$$

with $\overline{V}\left(\gamma^i\right) = -\left(2\kappa^i_{ss} + \left(\kappa^i\right)^3 - \mu^i\kappa^i\right)$ as in (3.24).

Definition 3.40 (Geometrical fourth order condition (endpoints))**.** Let $\mu^i \in \mathbb{R}$ be given and $\gamma^i \in C^4\left([0,1];\mathbb{R}^2\right)$, $i \in \{1,2,3\}$, be regular open curves with $\gamma^i(1) = P^i$ for given points $P^1, P^2, P^3 \in \mathbb{R}^2$. We say that γ satisfies the *geometrical fourth order condition* in $y = 1$ if for all $i \in \{1,2,3\}$, the curvature κ^i of γ^i satisfies

$$0 = \overline{V}\left(\gamma^i\right)(1) = -\left(2\kappa^i_{ss}(1) + \left(\kappa^i\right)^3(1) - \mu^i\kappa^i(1)\right).$$

Definition 3.41 (Geometrically admissible initial network (Hölder setting))**.** Let $\alpha \in (0,1)$. A *geometrically α–admissible initial network* to systems (3.20) or (3.21), respectively, is a function $\sigma \in C^{4+\alpha}\left([0,1];(\mathbb{R}^2)^3\right)$ such that $\sigma^i : [0,1] \to \mathbb{R}^2$, $i \in \{1,2,3\}$, are regular open curves satisfying all boundary conditions appearing in the respective system, the applicable fourth order condition in $y = 0$ and $y = 1$ and further the non-degeneracy condition in all $y \in \{0,1\}$ with $\sigma^1(y) = \sigma^2(y) = \sigma^3(y)$.

If σ is a geometrically α–admissible initial network to system (3.20) or (3.21) and $\theta^i : [0,1] \to [0,1]$, $i \in \{1,2,3\}$, are diffeomorphisms with $\theta^i \in C^{4+\alpha}([0,1])$, $\theta^i(y) = y$, $y \in \{0,1\}$, then $\varrho^i := \sigma^i \circ \theta^i$ defines a geometrically α–admissible initial network $\varrho = \left(\varrho^1, \varrho^2, \varrho^3\right)$ to the same system. This follows from the following lemma and the fact that the conditions listed in Definition 3.41 are invariant under reparametrisation.

Lemma 3.42. *Let $\alpha \in (0,1)$ and $f, g \in C^{4+\alpha}([0,1])$ be such that $g : [0,1] \to [0,1]$ is a C^1–diffeomorphism. Then $f \circ g$ and g^{-1} lie in $C^{4+\alpha}([0,1])$.*

Proof. We observe that given $\alpha \in (0,1)$, $h \in C^{\alpha}([0,1])$ and $g : [0,1] \to [0,1]$ a C^1–diffeomorphism, the function $h \circ g$ lies in $C^{\alpha}([0,1])$ which follows from

$$\begin{aligned}\sup_{x,y\in[0,1],x\neq y} \frac{|h(g(x)) - h(g(y))|}{|x-y|^{\alpha}} &= \sup_{x,y\in[0,1],x\neq y} \frac{|h(g(x)) - h(g(y))|}{|g(x)-g(y)|^{\alpha}} \frac{|g(x)-g(y)|^{\alpha}}{|x-y|^{\alpha}} \\ &\leq \sup_{x,y\in[0,1],x\neq y} \frac{|h(x)-h(y)|}{|x-y|^{\alpha}} \|g\|^{\alpha}_{C^1([0,1])} \leq \|h\|_{C^{\alpha}([0,1])} \|g\|^{\alpha}_{C^1([0,1])} .\end{aligned}$$

Given $f, g \in C^{4+\alpha}([0,1])$, $g : [0,1] \to [0,1]$ a C^1–diffeomorphism, the chain rule yields $f \circ g \in C^4([0,1])$. Computing $\partial_x^4 (f \circ g)$ explicitly and using the above observation combined with the Banach algebra property of $C^{\alpha}([0,1])$, one verifies $f \circ g \in C^{4+\alpha}([0,1])$. Similarly, one obtains $g^{-1} \in C^{4+\alpha}([0,1])$ by using the differentiation rules for g^{-1} shown in [151, Corollary 4.37]. □

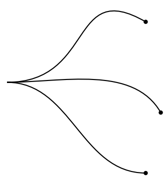

Figure 3.3: A non-admissible initial network.

Definition 3.43 (Strong solution to the geometric problem)**.** Let $p \in (5,\infty)$ and σ a geometrically p–admissible initial network to systems (3.20) or (3.21). Given $T > 0$ a *strong solution* to the considered system in $[0,T]$ with initial datum σ is a function

$$\gamma \in W^1_p\left((0,T); L_p\left((0,1);(\mathbb{R}^2)^3\right)\right) \cap L_p\left((0,T); W^4_p\left((0,1);(\mathbb{R}^2)^3\right)\right)$$

such that there exist diffeomorphisms $\theta^i : [0,1] \to [0,1]$ with $\theta^i \in W_p^{4-4/p}((0,1))$, $\theta^i(y) = y$ for $y \in \{0,1\}$ and $\gamma^i(0) = \sigma^i \circ \theta^i$, and such that for all $t \in [0,T]$, $\gamma^i(t) : [0,1] \to \mathbb{R}^2$, $i \in \{1,2,3\}$, are regular open curves that fulfil the non-degeneracy condition in each $y \in \{0,1\}$ in which $\gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y)$ holds for all $t \in [0,T]$, solve the motion equation (3.18) almost everywhere in $(0,T)\times(0,1)$ and satisfy the boundary conditions of the considered system pointwise.

Definition 3.44 (Classical solution to the geometric problem)**.** Let $\alpha \in (0,1)$ and σ a geometrically α–admissible initial network to systems (3.20) or (3.21). Given $T > 0$ a *classical solution* to the considered system in $[0,T]$ with initial datum σ is a function

$$\gamma \in C^{\frac{4+\alpha}{4},4+\alpha}\left([0,T]\times[0,1];(\mathbb{R}^2)^3\right)$$

such that there exist diffeomorphisms $\theta^i : [0,1] \to [0,1]$ with $\theta^i \in C^{4+\alpha}([0,1])$, $\theta^i(y) = y$ for $y \in \{0,1\}$ and $\gamma^i(0) = \sigma^i \circ \theta^i$, and such that for all $t \in [0,T]$, $\gamma^i(t) : [0,1] \to \mathbb{R}^2$, $i \in \{1,2,3\}$, are regular open curves that fulfil the non-degeneracy condition in each $y \in \{0,1\}$ in which $\gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y)$ holds for all $t \in [0,T]$ and satisfy the motion equation (3.18) and the boundary conditions of the considered system pointwise.

Lemma 3.45. *Let T be positive and γ be a strong or classical or non-degenerate smooth solution to systems* (3.20) *or* (3.21) *in* $[0,T]$. *Then there holds*

$$\boldsymbol{c}(\gamma) := \min_{i\in\{1,2,3\}} \min_{t\in[0,T],x\in[0,1]} \left|\gamma^i_x(t,x)\right| > 0\,. \tag{3.30}$$

Moreover, γ satisfies the uniform non-degeneracy condition in each $y \in \{0,1\}$ in which $\gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y)$ holds for all $t \in [0,T]$.

Proof. Proposition B.35 yields that γ lies in $C\left([0,T];C^1\left([0,1];(\mathbb{R}^2)^3\right)\right)$. For every $t \in [0,T]$ and $i \in \{1,2,3\}$, the curve $\gamma^i(t)$ is regular and thus $\min_{x\in[0,1]}\left|\gamma^i_x(t,x)\right| > 0$. Hence, for every $i \in \{1,2,3\}$, $(t,x) \mapsto \left|\gamma^i_x(t,x)\right|$ is a positive and continuous function on the compact set $[0,T]\times[0,1]$ which attains its positive minimum. This yields the identity (3.30). Let $y \in \{0,1\}$ be such that for all $t \in [0,T]$, $\gamma(t)$ satisfies the concurrency condition in y and let $\alpha^1(t,y)$, $\alpha^2(t,y)$, and $\alpha^3(t,y)$ denote the angles between the tangent vectors $\tau^2(t,y)$ and $\tau^3(t,y)$, $\tau^3(t,y)$ and $\tau^1(t,y)$, and $\tau^1(t,y)$ and $\tau^2(t,y)$, respectively. By definition we have for every $t \in [0,T]$,

$$\max\left\{\left|\sin\alpha^1(t,y)\right|, \left|\sin\alpha^2(t,y)\right|, \left|\sin\alpha^3(t,y)\right|\right\} > 0\,.$$

Modulo three there holds the identity

$$\alpha^i(t,y) = \arccos\left(\left\langle \frac{\gamma^{i+1}_x(t,y)}{\left|\gamma^{i+1}_x(t,y)\right|}, \frac{\gamma^{i-1}_x(t,y)}{\left|\gamma^{i-1}_x(t,y)\right|}\right\rangle\right), \quad t \in [0,T]\,,$$

which yields in particular that

$$t \mapsto \max\left\{\left|\sin\alpha^1(t,y)\right|, \left|\sin\alpha^2(t,y)\right|, \left|\sin\alpha^3(t,y)\right|\right\}$$

is a positive and continuous function on the compact set $[0,T]$ whose minimum is attained. □

3.5 A priori estimates for the geometric problem

In this section we derive an a priori estimate for smooth solutions of (3.20) and (3.21) which is a key ingredient in the study of the long time behaviour of the flow in Chapter 5. The computations follow [65, Section 6]. We use the following version of the *Gagliardo–Nirenberg inequality.*

Theorem 3.46 (Gagliardo–Nirenberg inequality)**.** *Let $m \in \mathbb{N}$ be given. Suppose that u belongs to $L_2((0,1))$ and $\partial^m_x u$ belongs to $L_2((0,1))$. Then for all $0 \le j < m$ there exist positive constants C_1, C_2 depending only on m and j such that for all $p \in [2,\infty]$ there holds*

$$\left\|\partial^j_x u\right\|_{L_p((0,1))} \le C_1 \left\|\partial^m_x u\right\|^{\sigma}_{L_2((0,1))} \left\|u\right\|^{1-\sigma}_{L_2((0,1))} + C_2 \left\|u\right\|_{L_2((0,1))}$$

where

$$\sigma = \frac{j + 1/2 - 1/p}{m}\,,$$

and $\frac{1}{\infty} := 0$.

Proof. This is a special case of [111, Theorem 1]. □

Corollary 3.47 (Gagliardo–Nirenberg inequality on curves)**.** *Let $\gamma : [0,1] \to \mathbb{R}^2$ be a smooth regular open curve with arclength parameter s, curvature κ and length $L := L(\gamma)$. Given $m \in \mathbb{N}$ and $0 \le j < m$ there exist positive constants C_1, C_2 depending only on m and j (and not on the curve γ) such that for all $p \in [2,\infty]$ there holds*

$$\left\|\partial_s^j \kappa\right\|_{L_p(\gamma,\mathrm{d}s)} \le C_1 \left\|\partial_s^m \kappa\right\|_{L_2(\gamma,\mathrm{d}s)}^{\sigma} \left\|\kappa\right\|_{L_2(\gamma,\mathrm{d}s)}^{1-\sigma} + \frac{C_2}{L^{m\sigma}} \left\|\kappa\right\|_{L_2(\gamma,\mathrm{d}s)} \tag{3.31}$$

where

$$\sigma = \frac{j + 1/2 - 1/p}{m}\,,$$

and $\frac{1}{\infty} := 0$.

Proof. Corollary 3.10 yields that we may assume without loss of generality that γ is parametrised with constant speed equal to the length, that is, $|\gamma_x(x)| = L$ for all $x \in [0,1]$. Moreover, κ lies in $C^\infty([0,1];\mathbb{R})$ and one easily shows by induction that for all $k \in \mathbb{N}$,

$$\partial_s^k (\kappa \circ x)(s(x)) = L^{-k} \partial_x^k \kappa(x)\,.$$

Thus, for all $p \in [1,\infty)$ and $k \in \mathbb{N}$, we obtain

$$\begin{aligned}\left\|\partial_s^k \kappa\right\|_{L_p(\gamma,\mathrm{d}s)}^p &= \int_0^1 \left|\partial_s^k (\kappa \circ x)(s(x))\right|^p |\gamma_x(x)|\,\mathrm{d}x = L^{-kp+1} \int_0^1 \left|\partial_x^k \kappa(x)\right|^p \mathrm{d}x \\ &= L^{-kp+1} \left\|\partial_x^k \kappa\right\|_{L_p((0,1))}^p\,,\end{aligned}$$

and thus

$$\left\|\partial_s^k \kappa\right\|_{L_p(\gamma,\mathrm{d}s)} = L^{-k+\frac{1}{p}} \left\|\partial_x^k \kappa\right\|_{L_p((0,1))}\,.$$

In the case $p = \infty$ we have

$$\left\|\partial_s^k \kappa\right\|_{L_\infty(\gamma,\mathrm{d}s)} = \sup_{x\in[0,1]} \left|\partial_s^k (\kappa \circ x)(s(x))\right| = L^{-k} \sup_{x\in[0,1]} \left|\partial_x^k \kappa(x)\right| = L^{-k+\frac{1}{p}} \left\|\partial_x^k \kappa\right\|_{L_\infty((0,1))}\,.$$

Hence Corollary 3.47 yields for all $m \in \mathbb{N}$, $0 \le j < m$ and $p \in [2,\infty]$,

$$\begin{aligned}\left\|\partial_s^j \kappa\right\|_{L_p(\gamma,\mathrm{d}s)} &\le L^{-j+\frac{1}{p}} \left(C_1 \left\|\partial_x^m \kappa\right\|_{L_2((0,1))}^{\sigma} \left\|\kappa\right\|_{L_2((0,1))}^{1-\sigma} + C_2 \left\|\kappa\right\|_{L_2((0,1))}\right) \\ &= L^{-j+\frac{1}{p}} \left(C_1 L^{m\sigma-\frac{\sigma}{2}} \left\|\partial_s^m \kappa\right\|_{L_2(\gamma,\mathrm{d}s)}^{\sigma} L^{\frac{\sigma-1}{2}} \left\|\kappa\right\|_{L_2(\gamma,\mathrm{d}s)}^{1-\sigma} + C_2 L^{-\frac{1}{2}} \left\|\kappa\right\|_{L_2(\gamma,\mathrm{d}s)}\right) \\ &= C_1 L^{-j+\frac{1}{p}+m\sigma-\frac{1}{2}} \left\|\partial_s^m \kappa\right\|_{L_2(\gamma,\mathrm{d}s)}^{\sigma} \left\|\kappa\right\|_{L_2(\gamma,\mathrm{d}s)}^{1-\sigma} + C_2 L^{-j+\frac{1}{p}-\frac{1}{2}} \left\|\kappa\right\|_{L_2(\gamma,\mathrm{d}s)} \\ &= C_1 \left\|\partial_s^m \kappa\right\|_{L_2(\gamma,\mathrm{d}s)}^{\sigma} \left\|\kappa\right\|_{L_2(\gamma,\mathrm{d}s)}^{1-\sigma} + \frac{C_2}{L^{\sigma m}} \left\|\kappa\right\|_{L_2(\gamma,\mathrm{d}s)}\,.\end{aligned}$$

□

Furthermore, we use Young's inequality in the following version.

Lemma 3.48. *Let $a, b, \varepsilon > 0$ and $p, p' \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Then it holds*

$$ab \le \varepsilon a^p + C(\varepsilon, p, p')\, b^{p'}\,.$$

Proof. The identity (3-11) in [4, Lemma 3.18] yields

$$ab = (p\varepsilon a^p)^{1/p} \left((p\varepsilon)^{-p'/p} b^{p'}\right)^{1/p'} \leq \frac{1}{p}(p\varepsilon a^p) + \frac{1}{p'}\left((p\varepsilon)^{-p'/p} b^{p'}\right) = \varepsilon a^p + \frac{1}{p'}(p\varepsilon)^{-p'/p} b^{p'}.$$

□

We use the following notation throughout this section. Given an interval $J \subset \mathbb{R}$ and a smooth solution $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$ to (3.20) or (3.21), we let $\tau^i(t)$, $\nu^i(t)$ and $\kappa^i(t)$ be the unit tangent, unit normal and curvature of the curve $\gamma^i(t)$. The dependence of the arclength measure $\mathrm{d}s_t^i$ and the arclength derivative $\partial_{s_t^i}$ on $t \in J$ and $i \in \{1,2,3\}$ is usually omitted. If a function $f^i(t) : [0,1] \to \mathbb{R}$ related to $\gamma^i(t)$ is derived with respect to the arclength parameter s_t^i we simply write $\partial_s f^i(t)$ or $f_s^i(t)$. In particular, we omit the composition with the diffeomorphism $x^i(t) : \left[0, L\left(\gamma^i(t)\right)\right] \to [0,1]$. Furthermore, we let V^i, $T^i : J \times [0,1] \to \mathbb{R}$, $i \in \{1,2,3\}$, be the normal and tangential velocity of γ^i, respectively. Given $p \in [1,\infty]$ we use the abbreviation $L_p\left(\gamma^i(t)\right) := L_p\left(\gamma^i(t), \mathrm{d}s_t^i\right)$ in the following. Furthermore, we define for $k \in \mathbb{N}_0$, $p \in [1,\infty]$,

$$\left\|\partial_s^k \kappa(t)\right\|_{L_p(\gamma(t))}^p := \sum_{i=1}^3 \left\|\partial_s^k \kappa^i(t)\right\|_{L_p(\gamma^i(t))}^p,$$
$$\left\|\partial_s^k \kappa(t)\right\|_{L_\infty(\gamma(t))} := \sum_{i=1}^3 \left\|\partial_s^k \kappa^i(t)\right\|_{L_\infty(\gamma^i(t))}.$$

As in [102] we introduce some notation that will be helpful in the following arguments.

Definition 3.49. Given $h \in \mathbb{N}_0$ and $\lambda \in \mathbb{N}$ we denote by $\mathfrak{p}_\lambda\left(\partial_s^h \kappa\right)$ a polynomial in $\kappa, \dots, \partial_s^h \kappa$ such that every monomial it contains is of the form

$$C \prod_{l=0}^h \left(\partial_s^l \kappa\right)^{\alpha_l}, \quad \sum_{l=0}^h (l+1)\,\alpha_l = \lambda,$$

with $C > 0$, $\alpha_l \in \mathbb{N}_0$ for $l \in \{0, \dots, h\}$ and $\alpha_h \in \mathbb{N}$ for at least one monomial. The variable λ is referred to as the *order* of the polynomial.

Lemma 3.50. *Given $h, h_1, h_2 \in \mathbb{N}_0$ and $\lambda, \lambda_1, \lambda_2 \in \mathbb{N}$ there holds*

$$\partial_s\left(\mathfrak{p}_\lambda\left(\partial_s^h \kappa\right)\right) = \mathfrak{p}_{\lambda+1}\left(\partial_s^{h+1}\kappa\right), \tag{3.32}$$
$$\mathfrak{p}_{\lambda_1}\left(\partial_s^{h_1}\kappa\right)\mathfrak{p}_{\lambda_2}\left(\partial_s^{h_2}\kappa\right) = \mathfrak{p}_{\lambda_1+\lambda_2}\left(\partial_s^{\max\{h_1,h_2\}}\kappa\right). \tag{3.33}$$

Proof. This follows easily from the definition. □

Using the general formulas stated in Lemma 3.14 we derive explicit evolution equations for the curvature and its first and second arclength derivative.

Lemma 3.51. *Let $J \subset \mathbb{R}$ be an interval and $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$ be a smooth solution to* (3.20) *or* (3.21)*. Omitting the dependence on $i \in \{1,2,3\}$, $t \in J$ and $x \in [0,1]$ there holds*

$$\partial_t \kappa = -2\partial_s^4\kappa - 5\kappa^2\partial_s^2\kappa - 6\kappa\left(\partial_s\kappa\right)^2 - \kappa^5 + T\partial_s\kappa + \mu\left(\partial_s^2\kappa + \kappa^3\right), \tag{3.34}$$
$$\partial_t\partial_s\kappa = -2\partial_s^5\kappa - 5\kappa^2\partial_s^3\kappa + T\partial_s^2\kappa + \mu\partial_s^3\kappa + \mathfrak{p}_6\left(\partial_s^2\kappa\right) + \mathfrak{p}_4\left(\partial_s\kappa\right), \tag{3.35}$$
$$\partial_t\partial_s^2\kappa = -2\partial_s^6\kappa - 5\kappa^2\partial_s^4\kappa + T\partial_s^3\kappa + \mu\partial_s^4\kappa + \mathfrak{p}_7\left(\partial_s^3\kappa\right) + \mathfrak{p}_5\left(\partial_s^2\kappa\right) \tag{3.36}$$

Proof. The identity (3.34) directly follows from (3.13). Using (3.9), (3.13) and (3.34) one obtains

$$\begin{aligned}\partial_t\partial_s\kappa &= \partial_s\partial_t\kappa + (\kappa V - \partial_s T)\,\partial_s\kappa = \partial_s^3 V + T\partial_s^2\kappa + 3\kappa\partial_s\kappa V + \kappa^2\partial_s V \\ &= -2\partial_s^5\kappa - 5\kappa^2\partial_s^3\kappa - 24\kappa\partial_s\kappa\partial_s^2\kappa + T\partial_s^2\kappa - 6\left(\partial_s\kappa\right)^3 - 6\kappa^4\partial_s\kappa + \mu\left(4\kappa^2\partial_s\kappa + \partial_s^3\kappa\right),\end{aligned}$$

and

$$\begin{aligned}\partial_t\partial_s^2\kappa =& \partial_s\partial_t\partial_s\kappa + (\kappa V - \partial_s T)\,\partial_s^2\kappa \\ =& \partial_s\left(-2\partial_s^5\kappa - 5\kappa^2\partial_s^3\kappa + T\partial_s^2\kappa + \mu\partial_s^3\kappa + \mathfrak{p}_6\left(\partial_s^2\kappa\right) + \mathfrak{p}_4\left(\partial_s\kappa\right)\right) + (\kappa V - \partial_2 T)\,\partial_s^2\kappa \\ =& -2\partial_s^6\kappa - 5\kappa^2\partial_s^4\kappa + 10\kappa\partial_s\kappa\partial_s^3\kappa + T\partial_s^3\kappa + \mu\partial_s^4\kappa + \mathfrak{p}_7\left(\partial_s^3\kappa\right) + \mathfrak{p}_5\left(\partial_s^2\kappa\right) - 2\kappa\left(\partial_s^2\kappa\right)^2 \\ & - \kappa^4\partial_s^2\kappa + \mu\kappa^2\partial_s^2\kappa = -2\partial_s^6\kappa - 5\kappa^2\partial_s^4\kappa + T\partial_s^3\kappa + \mu\partial_s^4\kappa + \mathfrak{p}_7\left(\partial_s^3\kappa\right) + \mathfrak{p}_5\left(\partial_s^2\kappa\right).\end{aligned}$$

□

Proposition 3.52. *Let $J \subset \mathbb{R}$ be an interval. If $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$ is a smooth solution to* (3.20)*, then for every $t \in J$ it holds*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^3\int_{[0,1]}|\partial_s^2\kappa^i(t)|^2\,\mathrm{d}s^i \\ =&\sum_{i=1}^3\int_{[0,1]} -|2\partial_s^4\kappa^i(t)|^2 - 2\mu^i|\partial_s^3\kappa^i(t)|^2 + \mathfrak{p}_{10}\left(\partial_s^3\kappa^i(t)\right) + \mathfrak{p}_8\left(\partial_s^2\kappa^i(t)\right)\,\mathrm{d}s^i \\ &+ \sum_{y\in\{0,1\}}\sum_{i=1}^3 T^i(t,y)\left(\mathfrak{p}_6\left(\partial_s^3\kappa^i(t,y)\right) + \mathfrak{p}_4\left(\partial_s\kappa^i(t,y)\right)\right) + \mathfrak{p}_5\left(\partial_s^2\kappa^i(t,y)\right).\end{aligned}$$

If $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$ is a smooth solution to (3.21)*, then for every $t \in J$ it holds*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^3\int_{[0,1]}|\partial_s^2\kappa^i(t)|^2\,\mathrm{d}s^i \\ =&\sum_{i=1}^3\int_{[0,1]} -|2\partial_s^4\kappa^i(t)|^2 - 2\mu^i|\partial_s^3\kappa^i(t)|^2 + \mathfrak{p}_{10}\left(\partial_s^3\kappa^i(t)\right) + \mathfrak{p}_8\left(\partial_s^2\kappa^i(t)\right)\,\mathrm{d}s^i \\ &+ \sum_{i=1}^3 T^i(t,0)\left(\mathfrak{p}_6\left(\partial_s^3\kappa^i(t,0)\right) + \mathfrak{p}_4\left(\partial_s\kappa^i(t,0)\right)\right) + \mathfrak{p}_5\left(\partial_s^2\kappa^i(t,0)\right).\end{aligned}$$

Proof. Let $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$ be a smooth solution to (3.20) or (3.21). As $\gamma^i : J \times [0,1] \to \mathbb{R}^2$, $i \in \{1,2,3\}$, is a smooth one-parameter family of regular open curves, all integrals appearing in this proof are well-defined and integration and differentiation may be interchanged. Using (3.10) and (3.36) we obtain for every $i \in \{1,2,3\}$, $t \in J$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{[0,1]}|\partial_s^2\kappa^i(t)|^2\,\mathrm{d}s^i \\ =&\int_{[0,1]}\partial_s^2\kappa^i(t)\{2\partial_t\partial_s^2\kappa^i(t) + \partial_s^2\kappa^i(t)\left(\partial_s T^i(t) - \kappa^i(t)V^i(t)\right)\}\,\mathrm{d}s^i \\ =&\int_{[0,1]}\partial_s^2\kappa^i(t)\Big\{-4\partial_s^6\kappa^i(t) - 10\,\kappa^i(t)^2\partial_s^4\kappa^i(t) + 2\mu^i\,\partial_s^4\kappa^i(t) - 36\left(\partial_s\kappa^i(t)\right)^2\partial_s^2\kappa^i(t) \\ &+ \mathfrak{p}_7\left((\partial_s^3\kappa^i(t)\right) + \mathfrak{p}_5\left(\partial_s^2\kappa^i(t)\right) + 2T^i(t)\partial_s^3\kappa^i(t) + \partial_s T\partial_s^2\kappa^i(t)\Big\}\,\mathrm{d}s^i.\end{aligned}$$

Using the integration by parts formula (3.4) we obtain

$$\begin{aligned}
&\int_{[0,1]} \partial_s^2\kappa^i(t)\left(-10\kappa^i(t)^2\partial_s^4\kappa^i(t) - 36\left(\partial_s\kappa^i(t)\right)^2\partial_s^2\kappa^i(t) + 2\mu^i\,\partial_s^4\kappa^i(t)\right)\mathrm{d}s^i \\
&= \int_{[0,1]} 10\left(\partial_s^3\kappa^i(t)\right)^2\kappa^i(t)^2 + \mathfrak{p}_{10}\left(\partial_s^3\kappa^i(t)\right) - 2\mu^i\left(\partial_s^3\kappa^i(t)\right)^2\mathrm{d}s^i \\
&\quad + \left[-10\,\partial_s^3\kappa^i(t)\partial_s^2\kappa^i(t)\kappa^i(t)^2 - 12\left(\partial_s\kappa^i(t)\right)^3\partial_s^2\kappa^i(t) + 2\mu^i\partial_s^3\kappa^i(t)\partial_s^2\kappa^i(t)\right]_0^1
\end{aligned}$$

and

$$\int_{[0,1]} -4\partial_s^2\kappa^i(t)\partial_s^6\kappa^i(t)\,\mathrm{d}s^i = \int_{[0,1]} -4\left(\partial_s^4\kappa^i(t)\right)^2\mathrm{d}s^i + \left[4\partial_s^4\kappa^i(t)\partial_s^3\kappa^i(t) - 4\partial_s^2\kappa^i(t)\partial_s^5\kappa^i(t)\right]_0^1 .$$

Using further

$$\partial_s\left(T^i(t)\left(\partial_s^2\kappa^i(t)\right)^2\right) = 2\partial_s^2\kappa^i(t)\partial_s^3\kappa^i(t)T^i(t) + \left(\partial_s^2\kappa^i(t)\right)^2\partial_s T^i(t)$$

we find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^3\int_{[0,1]}|\partial_s^2\kappa^i(t)|^2\,\mathrm{d}s^i \\
=&\sum_{i=1}^3\int_{[0,1]} -|2\partial_s^4\kappa^i(t)|^2 - 2\mu^i|\partial_s^3\kappa^i(t)|^2 + \mathfrak{p}_{10}\left(\partial_s^3\kappa^i(t)\right) + \mathfrak{p}_8\left(\partial_s^2\kappa^i(t)\right)\mathrm{d}s^i \\
&+\sum_{i=1}^3\left[4\partial_s^4\kappa^i(t)\partial_s^3\kappa^i(t) - 4\partial_s^2\kappa^i(t)\partial_s^5\kappa^i(t) - 12\left(\partial_s\kappa^i(t)\right)^3\partial_s^2\kappa^i(t) - 10\,\partial_s^3\kappa^i(t)\partial_s^2\kappa^i(t)\kappa^i(t)^2\right]_0^1 \\
&+\sum_{i=1}^3\left[2\mu^i\partial_s^3\kappa^i(t)\partial_s^2\kappa^i(t) + T^i(t)\left(\partial_s^2\kappa^i(t)\right)^2\right]_0^1 .
\end{aligned}$$

We focus now on the boundary terms. Let $y \in \{0,1\}$ be such that γ satisfies the concurrency condition in y. Using the fact that $\kappa^i(t,y) = 0$, it remains to deal with

$$\sum_{i=1}^3\left(4\partial_s^3\kappa^i\partial_s^4\kappa^i - 4\partial_s^2\kappa^i\partial_s^5\kappa^i - 12\left(\partial_s\kappa^i\right)^3\partial_s^2\kappa^i + 2\mu^i\,\partial_s^2\kappa^i\partial_s^3\kappa^i + T^i\left(\partial_s^2\kappa^i\right)^2\right)_{|(t,y)} . \tag{3.37}$$

All the following calculations are performed in (t,y) for arbitrary $t \in J$. For the sake of notation we often omit the dependence on (t,y) in the following. Differentiating in time the curvature condition $\kappa^i(t,y) = 0$ yields together with (3.34),

$$\begin{aligned}
0 = \partial_t\kappa^i &= -2\partial_s^4\kappa^i - 5\left(\kappa^i\right)^2\partial_s^2\kappa^i - 6\kappa^i\left(\partial_s\kappa^i\right)^2 + T^i\partial_s\kappa^i - \left(\kappa^i\right)^5 + \mu^i\left(\partial_s^2\kappa^i + \left(\kappa^i\right)^3\right) \\
&= -2\partial_s^4\kappa^i + T^i\partial_s\kappa^i + \mu^i\partial_s^2\kappa^i .
\end{aligned}$$

Thus we obtain

$$-\sum_{i=1}^3 2\partial_t\kappa^i\partial_s^3\kappa^i = \sum_{i=1}^3\left(4\partial_s^3\kappa^i\partial_s^4\kappa^i - 2T^i\partial_s\kappa^i\partial_s^3\kappa^i - 2\mu^i\,\partial_s^2\kappa^i\partial_s^3\kappa^i\right) = 0 . \tag{3.38}$$

Moreover, differentiating in time both the concurrency condition and the third order condition implies for $i,j \in \{1,2,3\}$,

$$\gamma_t^i(t,y) = V^i(t,y)\nu^i(t,y) + T^i(t,y)\tau^i(t,y) = V^j(t,y)\nu^j(t,y) + T^j(t,y)\tau^j(t,y) = \gamma_t^j(t,y)$$

and

$$\partial_t \sum 2\partial_s\kappa^i(t,y)\nu^i(t,y) - \mu^i\tau^i(t,y) = 0$$

which yields using (3.11), (3.12) and (3.35),

$$\begin{aligned}
0 &= \left\langle V^1\nu^1 + T^1\tau^1 \,,\, \partial_t \sum_{i=1}^3 2\partial_s\kappa^i\nu^i - \mu^i\tau^i \right\rangle \\
&= \sum_{i=1}^3 \left\langle V^i\nu^i + T^i\tau^i \,,\, \partial_t(2\partial_s\kappa^i\nu^i - \mu^i\tau^i)\right\rangle \\
&= \sum_{i=1}^3 \left\langle V^i\nu^i + T^i\tau^i, \left(2\partial_t\partial_s\kappa^i - \mu^i\left(\partial_s V^i + T^i\kappa^i\right)\right)\nu^i - 2\partial_s\kappa^i(\partial_s V^i + T^i\kappa^i)\tau^i\right\rangle \\
&= \sum_{i=1}^3 \left(\left(2\partial_t\partial_s\kappa^i - \mu^i\partial_s V^i\right)V^i - 2\partial_s\kappa^i\partial_s V^i T^i\right) \\
&= \sum_{i=1}^3 2\left(4\partial_s^5\kappa^i + 12\left(\partial_s\kappa^i\right)^3 - 2T^i\partial_s^2\kappa^i - 4\mu^i\,\partial_s^3\kappa^i + \left(\mu^i\right)^2\partial_s\kappa^i\right)\partial_s^2\kappa^i \\
&\qquad + 4T^i\partial_s\kappa^i\partial_s^3\kappa^i - 2\mu^i\,T^i(\partial_s\kappa^i)^2\,,
\end{aligned}$$

that is

$$\begin{aligned}
\sum_{i=1}^3 \Big(&-4\partial_s^2\kappa^i\partial_s^5\kappa^i - 12\left(\partial_s\kappa^i\right)^3\partial_s^2\kappa^i + 4\mu^i\,\partial_s^2\kappa^i\partial_s^3\kappa^i + 2T^i\left(\partial_s^2\kappa^i\right)^2 \\
&- \left(\mu^i\right)^2\partial_s\kappa^i\partial_s^2\kappa^i - 2T^i\partial_s\kappa^i\partial_s^3\kappa^i + \mu^i T^i(\partial_s\kappa^i)^2\Big) = 0\,.
\end{aligned}$$

Combined with (3.38) we obtain

$$\begin{aligned}
\sum_{i=1}^3 \Big(&4\partial_s^3\kappa^i\partial_s^4\kappa^i - 4\partial_s^2\kappa^i\partial_s^5\kappa^i - 12\left(\partial_s\kappa^i\right)^3\partial_s^2\kappa^i + 2\mu^i\,\partial_s^3\kappa^i\partial_s^2\kappa^i + 2T^i(\partial_s^2\kappa^i)^2 \\
&- \left(\mu^i\right)^2\partial_s\kappa^i\partial_s^2\kappa^i - 4T^i\partial_s\kappa^i\partial_s^3\kappa^i + \mu^i T^i(\partial_s\kappa^i)^2\Big) = 0\,.
\end{aligned}$$

Hence we can express the sum (3.37) as

$$\sum_{i=1}^3 \left(-T^i(\partial_s^2\kappa^i)^2 + \left(\mu^i\right)^2\partial_s\kappa^i\partial_s^2\kappa^i + 4T^i\partial_s\kappa^i\partial_s^3\kappa^i - \mu^i T^i\left(\partial_s\kappa^i\right)^2\right).$$

Combined with the previous computations this gives the desired result.
If γ is a smooth solution to (3.21), it is easy to see that at $y=1$ the contribution is zero. Indeed, we have $\kappa^i(t,1)=0$ and $\gamma^i_t(t,1)=0$ for all $t\in J$ and $i\in\{1,2,3\}$, and thus

$$V^i(t,1) = T^i(t,1) = \partial_t\kappa^i(t,1) = 0$$

which yields $\partial_s^2\kappa^i(t,1)=0$ and using (3.34) also $\partial_s^4\kappa^i(t,1)=0$. □

Lemma 3.53. *Let $J\subset\mathbb{R}$ be an interval and $\gamma : J\times[0,1]\to(\mathbb{R}^2)^3$ be a smooth solution to* (3.20) *or* (3.21) *with parameters $\mu^i>0$, $i\in\{1,2,3\}$, such that*

$$\ell := \inf_{t\in J}\min_{i\in\{1,2,3\}} L\left(\gamma^i(t)\right) > 0\,.$$

Then for all $t \in J$ and $i \in \{1,2,3\}$ there holds

$$\int_{[0,1]} \mathfrak{p}_{10}\left(\partial_s^3 \kappa^i(t)\right) \mathrm{d}s^i \leq \|\partial_s^4 \kappa^i(t)\|^2_{L_2(\gamma^i(t),\mathrm{d}s^i)} + C\|\kappa^i(t)\|^2_{L_2(\gamma^i(t),\mathrm{d}s^i)} + C\|\kappa^i(t)\|^{18}_{L_2(\gamma^i(t),\mathrm{d}s^i)}\,,$$
$$\int_{[0,1]} \mathfrak{p}_{8}\left(\partial_s^2 \kappa^i(t)\right) \mathrm{d}s^i \leq \frac{\mu_i}{2}\|\partial_s^3 \kappa^i(t)\|^2_{L_2(\gamma^i(t),\mathrm{d}s^i)} + C\|\kappa^i(t)\|^2_{L_2(\gamma^i(t),\mathrm{d}s^i)} + C\|\kappa^i(t)\|^{14}_{L_2(\gamma^i(t),\mathrm{d}s^i)}$$

for some positive constants $C > 0$ independent of $i \in \{1,2,3\}$ and $t \in J$.

Proof. As $\gamma^i : J \times [0,1] \to \mathbb{R}^2$, $i \in \{1,2,3\}$, is a smooth one-parameter family of regular open curves, all integrals in the following are well-defined. To obtain the desired estimates we adapt [102, pages 260-261] to our situation. Let $t \in J$, $i \in \{1,2,3\}$ and $m \in \{1,2\}$ be given. We remark that in this proof the constants C and C_m may vary from line to line. Furthermore, we use the abbreviation $L_2\left(\gamma^i(t)\right) := L_2\left(\gamma^i(t), \mathrm{d}s^i_t\right)$. Every monomial of $\mathfrak{p}_{2m+6}\left(\partial_s^{m+1}\kappa^i(t)\right)$ is of the form

$$\prod_{l=0}^{m+1} \left(\partial_s^l \kappa^i(t)\right)^{\alpha_l}$$

with $\alpha_l \in \mathbb{N}_0$ and $\sum_{l=0}^{m+1} (l+1)\,\alpha_l = 2m+6$. We define $I := \{l \in \{0,\dots,m+1\} : \alpha_l \neq 0\}$ and for every $l \in I$ we set

$$\beta_l := \frac{2m+6}{(l+1)\alpha_l}\,.$$

We observe that $\beta_l \geq 1$, $\sum_{l \in I} \frac{1}{\beta_l} = 1$ and $\alpha_l \beta_l > 2$ for every $l \in I$. Thus Hölder's inequality on curves shown in Corollary 3.11 implies

$$\left\| \prod_{l \in I} \left(\partial_s^l \kappa^i(t)\right)^{\alpha_l} \right\|_{L_1(\gamma^i(t))} \leq \prod_{l \in I} \left\| \left(\partial_s^l \kappa^i(t)\right)^{\alpha_l} \right\|_{L_{\beta_l}(\gamma^i(t))} = \prod_{l \in I} \|\partial_s^l \kappa^i(t)\|^{\alpha_l}_{L_{\alpha_l\beta_l}(\gamma^i(t))}\,.$$

Applying the version of the Gagliardo–Nirenberg inequality shown in Corollary 3.31 we obtain for every $l \in I$,

$$\begin{aligned}\left\|\partial_s^l \kappa^i(t)\right\|_{L_{\alpha_l\beta_l}(\gamma^i(t))} \leq & C_1(l,m)\left\|\partial_s^{m+2}\kappa^i(t)\right\|^{\sigma_l}_{L_2(\gamma^i(t))} \left\|\kappa^i(t)\right\|^{1-\sigma_l}_{L_2(\gamma^i(t))} \\ & + C_2(l,m) L\left(\gamma^i(t)\right)^{-(m+2)\sigma_l} \left\|\kappa^i(t)\right\|_{L_2(\gamma^i(t))}\end{aligned}$$

with coefficient σ_l given by

$$\sigma_l = \frac{l + \frac{1}{2} - \frac{1}{\alpha_l\beta_l}}{m+2}\,.$$

As all single lengths $L\left(\gamma^i(t)\right)$ of the network are uniformly bounded from below in $t \in J$ and $i \in \{1,2,3\}$, we may define

$$C(m) := \max_{l \in I, i \in \{1,2,3\}, t \in J} \left\{ C_1(l,m)^{\alpha_l}, C_2(l,m)^{\alpha_l} L\left(\gamma^i(t)\right)^{-(m+2)\sigma_l\alpha_l} \right\} < \infty\,.$$

Elementary computations then yield

$$\begin{aligned}&\left\| \prod_{l \in I} \left(\partial_s^l \kappa^i(t)\right)^{\alpha_l} \right\|_{L_1(\gamma^i(t))} \leq \prod_{l \in I} \|\partial_s^l \kappa^i(t)\|^{\alpha_l}_{L_{\alpha_l\beta_l}(\gamma^i(t))} \\ &\leq C(m) \prod_{l \in I} \|\kappa^i(t)\|^{(1-\sigma_l)\alpha_l}_{L_2(\gamma^i(t))} \left(\|\partial_s^{m+2}\kappa^i(t)\|^{\sigma_l}_{L_2(\gamma^i(t))} + \|\kappa^i(t)\|^{\sigma_l}_{L_2(\gamma^i(t))} \right)^{\alpha_l} \\ &\leq C_m \prod_{l \in I} \|\kappa^i(t)\|^{(1-\sigma_l)\alpha_l}_{L_2(\gamma^i(t))} \left(\left\|\partial_s^{m+2}\kappa^i(t)\right\|_{L_2(\gamma^i(t))} + \left\|\kappa^i(t)\right\|_{L_2(\gamma^i(t))} \right)^{\sigma_l\alpha_l}\end{aligned}$$

$$= C_m \left\|\kappa^i(t)\right\|_{L_2(\gamma^i(t))}^{\sum_{l\in I}(1-\sigma_l)\alpha_l} \left(\left\|\partial_s^{m+2}\kappa^i(t)\right\|_{L_2(\gamma^i(t))} + \left\|\kappa^i(t)\right\|_{L_2(\gamma^i(t))}\right)^{\sum_{l\in I}\sigma_l\alpha_l} .$$

Observe that the exponent $\sum_{l\in I}\sigma_l\alpha_l$ can be strictly estimated from above by 2 which follows from

$$\begin{aligned}
\sum_{l\in I}\sigma_l\alpha_l &= \frac{1}{m+2}\sum_{l\in I}\alpha_l\left(l+\frac{1}{2}\right) - \frac{1}{\beta_l} = \frac{1}{m+2}\left(\sum_{l\in I}\alpha_l(l+1) - \sum_{l\in I}\frac{1}{2}\alpha_l - 1\right)\\
&= \frac{1}{m+2}\left(2m+5-\sum_{l\in I}\frac{1}{2}\alpha_l\right) \leq \frac{2m+5}{m+2} - \frac{1}{2(m+2)}\sum_{l\in I}\alpha_l\frac{l+1}{m+2}\\
&= \frac{2m+5}{m+2} - \frac{1}{2(m+2)^2}\sum_{l\in I}\alpha_l(l+1) = \frac{2m+5}{m+2} - \frac{2m+6}{2(m+2)^2} = \frac{2m+5}{m+2} - \frac{m+3}{(m+2)^2}\\
&= \frac{2m+4}{m+2} - \frac{1}{(m+2)^2} = 2 - \frac{1}{(m+2)^2} < 2\,.
\end{aligned}$$

Applying Young's inequality with $p := \frac{2}{\sum_{l\in I}\sigma_l\alpha_l} > 1$ and $q := \frac{2}{2-\sum_{l\in I}\sigma_l\alpha_l}$ we obtain

$$\begin{aligned}
\left\|\prod_{l\in I}\left(\partial_s^l\kappa^i(t)\right)^{\alpha_l}\right\|_{L_1(\gamma^i(t))} &\leq C_m\varepsilon\left(\left\|\partial_s^{m+2}\kappa^i(t)\right\|_{L_2(\gamma^i(t))} + \left\|\kappa^i(t)\right\|_{L_2(\gamma^i(t))}\right)^2 + C_{m,\varepsilon}\left\|\kappa^i(t)\right\|_{L_2(\gamma^i(t))}^{2(2m+5)}\\
&\leq C_m\varepsilon\left(\left\|\partial_s^{m+2}\kappa^i(t)\right\|_{L_2(\gamma^i(t))}^2 + \left\|\kappa^i(t)\right\|_{L_2(\gamma^i(t))}^2\right) + C_{m,\varepsilon}\left\|\kappa^i(t)\right\|_{L_2(\gamma^i(t))}^{2(2m+5)},
\end{aligned}$$

where we used that the exponent $q\sum_{l\in I}(1-\sigma_l)\alpha_l$ simplifies to $2(2m+5)$. Indeed, we compute

$$\begin{aligned}
2\frac{\sum_{l\in I}(1-\sigma_l)\alpha_l}{2-\sum_{l\in I}\sigma_l\alpha_l} &= 2\frac{\sum_{l\in I}\alpha_l - \frac{1}{m+2}\sum_{l\in I}\left(\alpha_l(l+1) - \frac{1}{2}\alpha_l - \frac{1}{\beta_l}\right)}{2 - \frac{1}{m+2}\sum_{l\in I}\left(\alpha_l(l+1) - \frac{1}{2}\alpha_l - \frac{1}{\beta_l}\right)}\\
&= 2\frac{(m+2)\sum_{l\in I}\alpha_l - (2m+5) + \frac{1}{2}\sum_{l\in I}\alpha_l}{2(m+2) - 2m - 5 + \frac{1}{2}\sum_{l\in I}\alpha_l}\\
&= \frac{(2m+5)\left(\frac{1}{2}\sum_{l\in I}\alpha_l - 1\right)}{\frac{1}{2}\sum_{l\in I}\alpha_l - 1} = 2(2m+5)\,.
\end{aligned}$$

In the case $m=2$ we choose $\varepsilon > 0$ such that the coefficient in front of $\left\|\partial_s^4\kappa^i(t)\right\|_{L_2(\gamma^i(t))}^2$ is bounded by 1. To obtain the desired estimate in the case $m=1$, we consider the above reasoning with $\varepsilon > 0$ small enough such that the coefficient of the highest order term is bounded by $\frac{1}{2}\min_{i\in\{1,2,3\}}\mu_i$. □

Lemma 3.54. *Let $J \subset \mathbb{R}$ be an interval and $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$ be a smooth solution to* (3.20) *or* (3.21) *with parameters $\mu^i > 0$, $i \in \{1,2,3\}$, such that*

(i) the lengths of the three curves are uniformly bounded away from zero, that is,

$$\ell := \inf_{t\in J}\min_{i\in\{1,2,3\}} L\left(\gamma^i(t)\right) > 0\,,$$

(ii) the uniform non-degeneracy condition (3.23) *is satisfied in every $y \in \{0,1\}$ where $\gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y)$ holds for all $t \in J$, that is,*

$$\rho := \inf_{t\in J}\max\left\{\left|\sin\alpha^1(t,y)\right|, \left|\sin\alpha^2(t,y)\right|, \left|\sin\alpha^3(t,y)\right|\right\} > 0$$

where $\alpha^1(t,y)$, $\alpha^2(t,y)$, and $\alpha^3(t,y)$ denote the angles between the tangent vectors $\tau^2(t,y)$ and $\tau^3(t,y)$, $\tau^3(t,y)$ and $\tau^1(t,y)$, and $\tau^1(t,y)$ and $\tau^2(t,y)$, respectively.

If $y \in \{0,1\}$ is such that $\gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y)$ for all $t \in J$, then the following estimates hold for all $t \in J$:

$$\sum_{i=1}^{3} T^i(t,y)\mathfrak{p}_6\left(\partial_s^3 \kappa^i(t,y)\right) \leq \left\|\partial_s^4 \kappa(t)\right\|^2_{L_2(\gamma(t))} + C\left\|\kappa(t)\right\|^2_{L_2(\gamma(t))} + C\left\|\kappa(t)\right\|^{18}_{L_2(\gamma(t))},$$

$$\sum_{i=1}^{3} T^i(t,y)\mathfrak{p}_4\left(\partial_s \kappa^i(t,y)\right) \leq \frac{\boldsymbol{\mu}}{4}\left\|\partial_s^3 \kappa(t)\right\|^2_{L_2(\gamma(t))} + C\left\|\kappa(t)\right\|^2_{L_2(\gamma(t))} + C\left\|\kappa(t)\right\|^{14}_{L_2(\gamma(t))},$$

$$\sum_{i=1}^{3} \mathfrak{p}_5\left(\partial_s^2 \kappa^i(t,y)\right) \leq \frac{\boldsymbol{\mu}}{4}\left\|\partial_s^3 \kappa(t)\right\|^2_{L_2(\gamma(t))} + C\left\|\kappa(t)\right\|^2_{L_2(\gamma(t))} + C\left\|\kappa(t)\right\|^{2\delta}_{L_2(\gamma(t))}$$

where $\boldsymbol{\mu} := \min_{i\in\{1,2,3\}} \mu_i$ and $\delta > 0$ is a given value. The constants C depend on ℓ and ρ.

Proof. Let $y \in \{0,1\}$ be such that $\gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y)$ for all $t \in J$. As γ satisfies the uniform non-degeneracy condition in y, Proposition 3.35 yields that for every $t \in J$, $i \in \{1,2,3\}$, the tangential velocities $T^i(t,y)$ are linear combinations of the normal velocities $V^j(t,y)$, $j \in \{1,2,3\}$, with coefficients that are uniformly bounded with respect to $t \in J$. Let $t \in J$ and $i \in \{1,2,3\}$ be given. As $\kappa^i(t,y) = 0$ we obtain

$$\begin{aligned}\left|T^i(t,y)\right| &\leq C\sum_{i=1}^{3}\left|V^i(t,y)\right| \leq C\sum_{i=1}^{3}\left|\partial_s^2\kappa^i(t,y)\right| \leq C\sum_{i=1}^{3}\left\|\partial_s^2\kappa^i(t)\right\|_{C([0,1])}\\ &= C\sum_{i=1}^{3}\left\|\partial_s^2\kappa^i(t)\right\|_{L_\infty\left(\gamma^i(t),\mathrm{d}s_t^i\right)}\end{aligned}$$

with constant $C = C(\rho)$ independent of $t \in J$. Given $t \in J$ we conclude that the term $\sum_{i=1}^{3} T^i(t,0)\mathfrak{p}_6\left(\partial_s^3\kappa^i(t,0)\right)$ is controlled by a finite sum of terms of the form

$$C(\rho)\prod_{l=0}^{3}\left\|\partial_s^l\kappa(t)\right\|^{\alpha_l}_{L_\infty(\gamma(t))}, \quad \sum_{l=0}^{3}(l+1)\alpha_l = 9\,.$$

Similarly, for any $t \in J$ the expression $\sum_{i=1}^{3} T^i(t,0)\mathfrak{p}_4\left(\partial_s\kappa^i(t,0)\right)$ is bounded by a finite sum of terms of the type

$$C(\rho)\prod_{l=0}^{2}\left\|\partial_s^l\kappa(t)\right\|^{\alpha_l}_{L_\infty(\gamma^i(t))}, \quad \sum_{l=0}^{2}(l+1)\alpha_l = 7\,,$$

and every monomial of $\mathfrak{p}_5\left(\partial_s^2\kappa^i(t,0)\right)$, $t \in J$, is bounded by an expression of the form

$$\prod_{l=0}^{2}\left\|\partial_s^l\kappa^i(t)\right\|^{\alpha_l}_{L_\infty(\gamma^i(t))}, \quad \sum_{l=0}^{2}(l+1)\alpha_l = 5\,.$$

The following arguments are adaptations of [102, pages 261-262]. The version of the Gagliardo–Nirenberg inequality shown in Corollary 3.31 yields for $m \in \{2,3\}$, $l \in \{0,\dots,m\}$, $i \in \{1,2,3\}$, $t \in J$,

$$\left\|\partial_s^l\kappa^i(t)\right\|_{L_\infty(\gamma^i(t))} \leq C\left(l,m,\ell\right)\left\|\kappa^i(t)\right\|^{1-\sigma_l}_{L_2(\gamma^i(t))}\left(\left\|\partial_s^{m+1}\kappa^i(t)\right\|^{\sigma_l}_{L_2(\gamma^i(t))} + \left\|\kappa^i(t)\right\|^{\sigma_l}_{L_2(\gamma^i(t))}\right)$$

with $\sigma_l := \frac{l+\frac{1}{2}}{m+1}$ and hence we obtain for any $\alpha_l \in \mathbb{N}_0$,

$$\left\|\partial_s^l\kappa(t)\right\|^{\alpha_l}_{L_\infty(\gamma(t))} \leq C\sum_{i=1}^{3}\left\|\partial_s^l\kappa^i(t)\right\|^{\alpha_l}_{L_\infty(\gamma^i(t))}$$

$$\le C\,(m,\ell) \sum_{i=1}^{3} \left\|\kappa^i(t)\right\|_{L_2(\gamma^i(t))}^{(1-\sigma_l)\alpha_l} \left(\left\|\partial_s^{m+1}\kappa^i(t)\right\|_{L_2(\gamma^i(t))}^{\sigma_l} + \left\|\kappa^i(t)\right\|_{L_2(\gamma^i(t))}^{\sigma_l}\right)^{\alpha_l}$$

$$\le C\,(m,\ell) \sum_{i=1}^{3} \left\|\kappa^i(t)\right\|_{L_2(\gamma^i(t))}^{(1-\sigma_l)\alpha_l} \left(\left\|\partial_s^{m+1}\kappa^i(t)\right\|_{L_2(\gamma^i(t))} + \left\|\kappa^i(t)\right\|_{L_2(\gamma^i(t))}\right)^{\sigma_l\alpha_l}$$

$$\le C\,(m,\ell)\, \|\kappa(t)\|_{L_2(\gamma(t))}^{(1-\sigma_l)\alpha_l} \left(\left\|\partial_s^{m+1}\kappa(t)\right\|_{L_2(\gamma(t))} + \|\kappa(t)\|_{L_2(\gamma(t))}\right)^{\sigma_l\alpha_l}.$$

We conclude that an expression of the form $\prod_{l=0}^{m} \left\|\partial_s^l\kappa(t)\right\|_{L_\infty(\gamma(t))}^{\alpha_l}$ with $m \in \mathbb{N}$ and $\alpha_l \in \mathbb{N}_0$, $l \in \{0,\dots,m\}$, is bounded by

$$C\,(m,\ell)\, \|\kappa(t)\|_{L_2(\gamma(t))}^{\sum(1-\sigma_l)\alpha_l} \left(\left\|\partial_s^{m+1}\kappa(t)\right\|_{L_2(\gamma(t))} + \|\kappa(t)\|_{L_2(\gamma(t))}\right)^{\sum\sigma_l\alpha_l},$$

where in all cases of $m \in \mathbb{N}_0$ and $\lambda(m) = \sum_{l=0}^{m}(l+1)\alpha_l$ relevant for this proof there holds

$$\sum_{l=0}^{m} \sigma_l\alpha_l < 2\,.$$

Indeed, suppose that $m \in \{2,3\}$ and $\alpha_l \in \mathbb{N}_0$, $l \in \{0,\dots,m\}$, satisfy $\sum_{l=0}^{m}(l+1)\alpha_l(l+1) = \lambda(m)$ with $\lambda(3) = 9$ and $\lambda(2) \in \{5,7\}$. Indeed, a similar computation as in the proof of Lemma 3.53 yields

$$\sum_{l=0}^{m} \sigma_l\alpha_l \le \lambda(m) \left(\frac{1}{m+1} - \frac{1}{2(m+1)^2}\right),$$

where the right hand side takes the values $\frac{63}{32}$, $\frac{35}{18}$ and $\frac{25}{18}$ in the respective cases. Young's inequality with $p := \frac{2}{\sum\sigma_l\alpha_l} > 1$ and $q := \frac{2}{2-\sum\sigma_l\alpha_l}$ implies

$$\prod_{l=0}^{m} \left\|\partial_s^l\kappa(t)\right\|_{L_\infty(\gamma(t))}^{\alpha_l} \le C\,(m,\ell,\varepsilon)\, \|\kappa(t)\|_{L_2(\gamma(t))}^{2\delta} + C\,(m,\ell)\,\varepsilon \left(\left\|\partial_s^{m+1}\kappa(t)\right\|_{L_2(\gamma(t))} + \|\kappa(t)\|_{L_2(\gamma(t))}\right)^2$$

$$\le C\,(m,\ell,\varepsilon)\, \|\kappa(t)\|_{L_2(\gamma(t))}^{2\delta} + C\,(m,\ell)\,\varepsilon \left(\left\|\partial_s^{m+1}\kappa(t)\right\|_{L_2(\gamma(t))}^2 + \|\kappa(t)\|_{L_2(\gamma(t))}^2\right)$$

with $\delta := \frac{q}{2}\sum_{l=0}^{m}(1-\sigma_l)\,\alpha_l > 0$. A computation shows that

$$\delta = \frac{\sum_{l=0}^{m}\alpha_l\,(m+1+1/2) - \lambda(m)}{2(m+1) - \lambda(m) + \frac{1}{2}\sum_{l=0}^{m}\alpha_l},$$

thus $\delta = 7$ if $m = 2$ and $\lambda = 7$, and $\delta = 9$ if $m = 3$ and $\lambda = 9$. The claim follows upon choosing $\varepsilon > 0$ appropriately in the respective cases. □

Theorem 3.55 (A priori estimate for smooth solutions)**.** *Let $J \subset \mathbb{R}$ be an interval and $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$ be a smooth solution to* (3.20) *or* (3.21) *with parameters $\mu^i > 0$, $i \in \{1,2,3\}$, such that*

(i) the lengths of the three curves are uniformly bounded away from zero, that is,

$$\ell := \inf_{t\in J} \min_{i\in\{1,2,3\}} L\left(\gamma^i(t)\right) > 0\,,$$

(ii) the uniform non-degeneracy condition (3.23) *is satisfied in every $y \in \{0,1\}$ where $\gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y)$ holds for all $t \in J$, that is,*

$$\rho := \inf_{t\in J} \max\left\{\left|\sin\alpha^1(t,y)\right|, \left|\sin\alpha^2(t,y)\right|, \left|\sin\alpha^3(t,y)\right|\right\} > 0$$

where $\alpha^1(t,y)$, $\alpha^2(t,y)$, and $\alpha^3(t,y)$ denote the angles between the tangent vectors $\tau^2(t,y)$ and $\tau^3(t,y)$, $\tau^3(t,y)$ and $\tau^1(t,y)$, and $\tau^1(t,y)$ and $\tau^2(t,y)$, respectively.

Then for all $\tau, t \in J$ with $\tau < t$ it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{3} \left\| \partial_s^2 \kappa^i(t) \right\|_{L_2(\gamma^i(t))}^2 \leq C \left(E_\mu(\gamma(\tau)) \right).$$

Proof. Proposition 3.52, Lemma 3.53 and Lemma 3.54 yield with $\boldsymbol{\mu} = \min_{i \in \{1,2,3\}} \mu^i$ for any $t \in J$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{3} \left\| \partial_s^2 \kappa^i(t) \right\|_{L_2(\gamma^i(t))}^2 \leq -2 \left\| \partial_s^4 \kappa(t) \right\|_{L_2(\gamma(t))}^2 - \boldsymbol{\mu} \left\| \partial_s^3 \kappa(t) \right\|_{L_2(\gamma(t))}^2 + C \left(\|\kappa(t)\|_{L_2(\gamma(t))}^2 \right),$$

where by Corollary 3.28 there holds for any $\tau, t \in J$ with $\tau < t$,

$$\|\kappa(t)\|_{L_2(\gamma(t))}^2 \leq E_\mu(\gamma(\tau)).$$

This completes the proof. □

Chapter 4

Existence and uniqueness of the elastic flow of networks

4.1 Existence and uniqueness of the analytic problem

In this section we study the auxiliary evolution problems associated to (3.20) and (3.21). As discussed in Section 3.3 these are obtained by imposing the tangential velocity defined in (3.26) for every curve of the network and further the tangential second order condition (3.28) at the boundary which, together with the curvature condition, is equivalent to the so-called *second order condition*

$$\gamma^i_{xx} = 0 \qquad \text{on } J \times \{0,1\}\,,\ i \in \{1,2,3\}\,. \tag{4.1}$$

Definition 4.1 (Smooth solution to the analytic problem for Theta networks)**.** A smooth solution to the analytic problem associated to (3.20) is a smooth function $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$, with $J \subset \mathbb{R}$ an interval, such that for all $t \in J$ and $i \in \{1,2,3\}$, $\gamma^i(t)$ is a regular open curve and for $i \in \{1,2,3\}$,

$$\begin{cases} \gamma^i_t = \overline{V}\left(\gamma^i\right)\nu^i + \overline{T}\left(\gamma^i\right)\tau^i & \text{in } J \times [0,1] \quad (\text{motion})\,, \\ \gamma^1 = \gamma^2 = \gamma^3 & \text{on } J \times \{0,1\} \quad (\text{concurrency condition})\,, \\ \gamma^i_{xx} = 0 & \text{on } J \times \{0,1\} \quad (\text{second order condition})\,, \\ \sum_{i=1}^3 2\kappa^i_s \nu^i - \mu^i \tau^i = 0 & \text{on } J \times \{0,1\} \quad (\text{third order condition})\,. \end{cases} \tag{4.2}$$

Definition 4.2 (Smooth solution to the analytic problem for Triods)**.** A smooth solution to the analytic problem associated to (3.21) with given endpoints P^1, P^2, P^3 is a smooth function $\gamma : J \times [0,1] \to (\mathbb{R}^2)^3$, with $J \subset \mathbb{R}$ an interval, such that for all $t \in J$ and $i \in \{1,2,3\}$, $\gamma^i(t)$ is a regular open curve and for $i \in \{1,2,3\}$,

$$\begin{cases} \gamma^i_t = \overline{V}\left(\gamma^i\right)\nu^i + \overline{T}\left(\gamma^i\right)\tau^i & \text{in } J \times [0,1] \quad (\text{motion})\,, \\ \gamma^1 = \gamma^2 = \gamma^3 & \text{on } J \times \{0\} \quad (\text{concurrency condition})\,, \\ \gamma^i = P^i & \text{on } J \times \{1\} \quad (\text{endpoints})\,, \\ \gamma^i_{xx} = 0 & \text{on } J \times \{0,1\} \quad (\text{second order condition})\,, \\ \sum_{i=1}^3 2\kappa^i_s \nu^i - \mu^i \tau^i = 0 & \text{on } J \times \{0\} \quad (\text{third order condition})\,. \end{cases} \tag{4.3}$$

Depending on whether one intends to prove the existence of solutions to (4.2) and (4.3) in Hölder spaces or in Sobolev spaces, different conditions on the initial network have to be imposed. As discussed in Section 3.4 the initial value of a solution in $W^1_p\left((0,T); L_p\left((0,1);(\mathbb{R}^2)^3\right)\right) \cap L_p\left((0,T); W^4_p\left((0,1);(\mathbb{R}^2)^3\right)\right)$, $p \in (5,\infty)$, to (4.2) or (4.3) is of regularity $W^{4-4/p}_p\left((0,1);(\mathbb{R}^2)^3\right)$ and satisfies all boundary conditions appearing in (4.2) or (4.3), respectively.

Definition 4.3 (Analytically admissible initial network (Sobolev))**.** Let $p \in (5,\infty)$. An *analytically p–admissible initial network* to system (4.2) or (4.3) is a function $\varphi \in W_p^{4-4/p}\left((0,1);(\mathbb{R}^2)^3\right)$ such that $\varphi^i : [0,1] \to \mathbb{R}^2$, $i \in \{1,2,3\}$, are regular open curves satisfying all boundary conditions appearing in the respective system and further the non-degeneracy condition in all $y \in \{0,1\}$ with $\varphi^1(y) = \varphi^2(y) = \varphi^3(y)$.

Let $\alpha \in (0,1)$, $T > 0$ be given and $\gamma \in C^{\frac{4+\alpha}{4},4+\alpha}\left([0,T] \times [0,1];(\mathbb{R}^2)^3\right)$ be such that $\gamma^i(t)$, $i \in \{1,2,3\}$, are regular open curves solving system (4.2) or (4.3). Suppose that the curves $\gamma^i(t)$ satisfy the concurrency condition in $y \in \{0,1\}$. Deriving in time yields that the normal and tangential velocities V and T satisfy in $(0,y)$,

$$V^1\nu^1 + T^1\tau^1 = V^2\nu^2 + T^2\tau^2 = V^3\nu^3 + T^3\tau^3 \,.$$

In the case that γ is a classical solution to (4.3), differentiating the condition $\gamma^i(t,1) = P^i$ with respect to time yields for all $i \in \{1,2,3\}$,

$$0 = \gamma_t^i(0,1) = V^i(0,1)\nu^i(0,1) + T^i(0,1)\tau^i(0,1) \,.$$

As γ is a solution to (3.27) we have $V^i = \overline{V}\left(\gamma^i\right)$ and $T^i = \overline{T}\left(\gamma^i\right)$ with $\overline{V}$ and $\overline{T}$ as in (3.24) and (3.26), respectively.

Definition 4.4 (Analytical fourth order condition (junction))**.** Let $\mu^i \in \mathbb{R}$, $\gamma^i \in C^4\left([0,1];\mathbb{R}^2\right)$, $i \in \{1,2,3\}$, be regular open curves and $y \in \{0,1\}$ such that $\gamma^1(y) = \gamma^2(y) = \gamma^3(y)$ and $\overline{V}^i := \overline{V}\left(\gamma^i\right)$, $\overline{T}^i := \overline{T}\left(\gamma^i\right)$ the expressions defined in (3.24) and (3.26), respectively, for the curve γ^i and the parameter μ^i. We say that γ satisfies the *analytical fourth order condition* in $y \in \{0,1\}$ if

$$\overline{V}^1(y)\nu^1(y) + \overline{T}^1(y)\tau^1(y) = \overline{V}^2(y)\nu^2(y) + \overline{T}^2(y)\tau^2(y) = \overline{V}^3(y)\nu^3(y) + \overline{T}^3(y)\tau^3(y) \,.$$

Definition 4.5 (Analytical fourth order condition (endpoints))**.** Let $\mu^i \in \mathbb{R}$, $\gamma^i \in C^4\left([0,1];\mathbb{R}^2\right)$, $i \in \{1,2,3\}$, be regular open curves with $\gamma^i(1) = P^i$ for given endpoints $P^1, P^2, P^3 \in \mathbb{R}^2$. We say that γ satisfies the *analytical fourth order condition* in $y = 1$ if for all $i \in \{1,2,3\}$,

$$0 = \overline{V}\left(\gamma^i\right)(1)\nu^i(1) + \overline{T}\left(\gamma^i\right)(1)\tau^i(1)$$

where $\overline{V}$ and $\overline{T}$ are the expressions defined in (3.24) and (3.26), respectively.

Definition 4.6 (Analytically admissible initial network (Hölder))**.** Let $\alpha \in (0,1)$. An *analytically α–admissible initial network* to system (4.2) or (4.3), respectively, is a function $\varphi \in C^{4+\alpha}\left([0,1];(\mathbb{R}^2)^3\right)$ such that $\varphi^i : [0,1] \to \mathbb{R}^2$, $i \in \{1,2,3\}$, are regular open curves satisfying all boundary conditions appearing in the respective system, the applicable analytical fourth order condition in $y = 0$ and $y = 1$ and further the non-degeneracy condition in all $y \in \{0,1\}$ with $\varphi^1(y) = \varphi^2(y) = \varphi^3(y)$.

Definition 4.7 (Strong solution to the analytic problem)**.** Let $p \in (5,\infty)$ and φ be an analytically p–admissible initial network to system (4.2) or (4.3). Given $T > 0$ a *strong solution* to the considered system in $[0,T]$ with initial datum φ is a function

$$\gamma \in W_p^1\left((0,T);L_p\left((0,1);(\mathbb{R}^2)^3\right)\right) \cap L_p\left((0,T);W_p^4\left((0,1);(\mathbb{R}^2)^3\right)\right)$$

with $\gamma(0) = \varphi$ such that for all $t \in [0,T]$, $\gamma^i(t) : [0,1] \to \mathbb{R}^2$, $i \in \{1,2,3\}$, are regular open curves that solve the motion equation (3.27) almost everywhere in $(0,T) \times (0,1)$, satisfy the boundary conditions in the respective system pointwise and fulfil the non-degeneracy condition in each $y \in \{0,1\}$ in which $\gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y)$ holds for all $t \in [0,T]$.

Definition 4.8 (Classical solution to the analytic problem)**.** Let $\alpha \in (0,1)$ and φ be an analytically α–admissible initial network to system (4.2) or (4.3), respectively. Given $T > 0$ a *classical solution* to the considered system in $[0,T]$ with initial datum φ is a function

$$\gamma \in C^{\frac{4+\alpha}{4},4+\alpha}\left([0,T]\times[0,1];(\mathbb{R}^2)^3\right)$$

with $\gamma(0) = \varphi$ such that for all $t \in [0,T]$, $\gamma^i(t) : [0,1] \to \mathbb{R}^2$, $i \in \{1,2,3\}$, are regular open curves that satisfy the motion equation and the boundary conditions in the respective system pointwise and fulfil the non-degeneracy condition in each $y \in \{0,1\}$ in which $\gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y)$ holds for all $t \in [0,T]$.

Remark 4.9. In the following sections we show existence and uniqueness of strong and classical solutions to the analytic problems. For the sake of readability we concentrate on the system (4.3). As all terms appearing in system (4.2) are also present in the problem (4.3), it is straightforward to adapt the following arguments to the case of Theta networks. Indeed, this is just a matter of suitably redefining the operators and spaces appearing in the respective proofs for system (4.3). No additional estimates are needed. The main results are always stated for (4.2) and (4.3) while the proofs are given exemplarily for the system (4.3).

4.1.1 Existence and uniqueness of strong solutions

In this subsection we show existence and uniqueness of strong solutions to systems (4.2) and (4.3) following [65, Section 3]. All arguments are performed for system (4.3) and can be adapted to system (4.2) in a straight forward manner.
In the following given $p \in (1,\infty)$, $r,s \in (0,\infty)$, $T > 0$ and $y \in \mathbb{R}$, we identify the Sobolev (Slobodeckij) space $W_p^s(\{y\})$ on the zero-dimensional manifold $\{y\}$ with $\mathbb{R}$ via the isometric isomorphism $f \mapsto f(y)$, see (C.17), which induces an isometric isomorphism

$$W_p^r\left((0,T);L_p(\{y\})\right) \cap L_p\left((0,T);W_p^s(\{y\})\right) \to W_p^r\left((0,T)\right), \quad f \mapsto \left(t \mapsto f(t,y)\right).$$

Given $p \in (5,\infty)$ and $T > 0$ we consider the space

$$\boldsymbol{E}_T := W_p^1\left((0,T);L_p\left((0,1);(\mathbb{R}^2)^3\right)\right) \cap L_p\left((0,T);W_p^4\left((0,1);(\mathbb{R}^2)^3\right)\right).$$

Whenever we are referring to an *analytically p–admissible initial network*, we hereby imply that we consider an analytically p–admissible initial network to the system (4.3) with given endpoints $P^i = \varphi^i(1)$, $i \in \{1,2,3\}$.

Well-posedness of the linearised problem

This paragraph is devoted to show well-posedness of a linearised system associated to (4.3) which is obtained by linearising the motion equation and the third order condition around a fixed analytically p–admissible initial network φ and considering the principal part of the respective linearisation. This results in the following system:

$$\begin{cases} u_t^i(t,x) + \frac{2}{|\varphi_x^i(x)|^4} u_{xxxx}^i(t,x) = f^i(t,x) & \text{for } (t,x) \in J\times[0,1]\,, i \in \{1,2,3\}\,,\\ u^1(t,0) = u^2(t,0) = u^3(t,0) & \text{for } t \in J\,,\\ u(t,1) = h_2(t) & \text{for } t \in J\,,\\ u_{xx}(t,y) = 0 & \text{for } (t,y) \in J\times\{0,1\}\,,\\ \sum_{i=1}^3 \frac{2}{|\varphi_x^i(0)|^3}\left\langle u_{xxx}^i(t,0), \nu_0^i(0)\right\rangle \nu_0^i(0) = h_1(t) & \text{for } t \in J\,, \end{cases} \tag{4.4}$$

for a suitable right hand side (f, h_1, h_2) and a suitably regular function u which are discussed in the following.

The boundary conditions in (4.3) that are already linear form part of the solution space.

Proposition 4.10. *Given $p \in (5, \infty)$ and $T > 0$ the operator*

$$\begin{aligned}\mathcal{B}_T : \boldsymbol{E}_T \to W_p^{1-1/4p}\left((0,T);(\mathbb{R}^2)^2\right) \times W_p^{1/2-1/4p}\left((0,T);(\mathbb{R}^2)^6\right),\\ u \mapsto \left(u^1(\cdot,0) - u^2(\cdot,0), u^2(\cdot,0) - u^3(\cdot,0), u_{xx}(\cdot,0), u_{xx}(\cdot,1)\right)\end{aligned}$$

is well-defined, linear and continuous. Its kernel

$$\mathbb{E}_T := \ker\left(\mathcal{B}_T\right)$$

is a closed linear subspace of $\boldsymbol{E}_T$.

Proof. This is a direct consequence of Theorem C.29. □

The linearisation of the third order condition makes use of the following simplified formula.

Lemma 4.11 (Third order condition)**.** *Let $\gamma : [0,1] \to \mathbb{R}^2$ be a regular open curve with $\gamma_{xx}(0) = 0$ and $\mu \in \mathbb{R}$. Then there holds*

$$2\kappa_s(0)\nu(0) - \mu\tau(0) = \frac{2}{|\gamma_x(0)|^3}\left\langle \gamma_{xxx}(0), \nu(0)\right\rangle \nu(0) - \mu\tau(0)\,.$$

Proof. Omitting the evaluation in $x = 0$ there holds

$$\kappa_s\nu = \partial_s\boldsymbol{\kappa} - \kappa\nu_s = \partial_s\boldsymbol{\kappa} = \partial_s\left(\frac{1}{|\gamma_x|^2}\left\langle\gamma_{xx},\nu\right\rangle\nu\right) = \frac{1}{|\gamma_x|^3}\left\langle\gamma_{xxx},\nu\right\rangle\nu\,.$$

□

Lemma 4.12. *Let $p \in (5,\infty)$ and φ be an analytically p–admissible initial network. Then for all $i \in \{1,2,3\}$,*

$$x \mapsto \left|\varphi_x^i(x)\right|^{-1} \in C\left([0,1];\mathbb{R}\right)$$

and

$$\boldsymbol{c} := \boldsymbol{c}(\varphi) := \min_{i\in\{1,2,3\}}\min_{x\in[0,1]}\left|\varphi_x^i(x)\right| > 0\,. \tag{4.5}$$

Proof. Definition 4.3 implies for $i \in \{1,2,3\}$, $\varphi^i \in C^1\left([0,1];\mathbb{R}^2\right)$ and $\varphi_x^i \neq 0$ on $[0,1]$ which yields $\boldsymbol{c}^i := \min_{x\in[0,1]}\left|\varphi_x^i(x)\right| > 0$ and in particular

$$x \mapsto \left|\varphi_x^i(x)\right|^{-1} \in C\left([0,1];\mathbb{R}\right)$$

and $\boldsymbol{c} = \min\{\boldsymbol{c}^1, \boldsymbol{c}^2, \boldsymbol{c}^3\} > 0$. □

Proposition 4.13 (The linearised operator)**.** *Let $p \in (5,\infty)$ and φ be an analytically p–admissible initial network with unit normal denoted by ν_0^i. Given $T > 0$ the operators*

$$L_T^1 : \mathbb{E}_T \to X_T := L_p\left((0,T);L_p\left((0,1);(\mathbb{R}^2)^3\right)\right), L_T^1(u) := \left(u_t^i + \frac{2}{|\varphi_x^i|^4}u_{xxxx}^i\right)_{i\in\{1,2,3\}},$$

$$L_T^2 : \mathbb{E}_T \to Y_{1,T} := W_p^{1/4-1/4p}\left((0,T);\mathbb{R}^2\right), \qquad L_T^2(u) := \sum_{i=1}^{3}\frac{2}{|\varphi_x^i(0)|^3}\left\langle u_{xxx}^i(\cdot,0), \nu_0^i(0)\right\rangle\nu_0^i(0)\,,$$

$$L_T^3 : \mathbb{E}_T \to Y_{2,T} := W_p^{1-1/4p}\left((0,T);(\mathbb{R}^2)^3\right), \qquad L_T^3(u) := u\,(\cdot,1)\,,$$
$$L_T^4 : \mathbb{E}_T \to X_0 := W_p^{4-4/p}\left((0,1);(\mathbb{R}^2)^3\right), \qquad L_T^4(u) := u\,(0,\cdot)$$

are well-defined, linear and continuous. We let $L_T := \left(L_T^1, L_T^2, L_T^3, L_T^4\right)$.

Proof. The statements about the operators L_T^3 and L_T^4 follow directly from Theorem C.29 and Proposition B.35, respectively. That L_T^1 is well-defined, linear and continuous is a consequence of Lemma 4.12. The statement about L_T^2 follows from $|\varphi_x^i(0)|^{-1} \in \mathbb{R}$ and Theorem C.29. □

Proposition 4.14 (The linear compatibility operator)**.** *Let* $p \in (5,\infty)$ *and* φ *be an analytically* p*–admissible initial network with unit normal denoted by* ν_0^i*. Given* $T > 0$ *the operator*

$$\mathcal{G}_T := \left(\mathcal{G}_T^1, \mathcal{G}_T^2, \mathcal{G}_T^3, \mathcal{G}_T^4\right) : X_T \times Y_{1,T} \times Y_{2,T} \times X_0 \to (\mathbb{R}^2)^2 \times (\mathbb{R}^2)^6 \times \mathbb{R}^2 \times (\mathbb{R}^2)^3$$

defined by

$$\begin{aligned}
\mathcal{G}_T^1\left((f,h_1,h_2,\psi)\right) &:= \left(\psi^1(0) - \psi^2(0), \psi^2(0) - \psi^3(0)\right),\\
\mathcal{G}_T^2\left((f,h_1,h_2,\psi)\right) &:= \left(\psi_{xx}(0), \psi_{xx}(1)\right),\\
\mathcal{G}_T^3\left((f,h_1,h_2,\psi)\right) &:= \sum_{i=1}^{3} \frac{2}{|\varphi_x^i(0)|^3} \left\langle \psi_{xxx}^i(0), \nu_0^i(0)\right\rangle \nu_0^i(0) - h_1(0)\,,\\
\mathcal{G}_T^4\left((f,h_1,h_2,\psi)\right) &:= \psi(1) - h_2(0)
\end{aligned}$$

is well-defined, linear and continuous. In particular,

$$\mathbb{F}_T := \ker\left(\mathcal{G}_T\right)$$

is a closed linear subspace of $X_T \times Y_{1,T} \times Y_{2,T} \times X_0$.

Proof. This follows from the continuous embeddings $X_0 \hookrightarrow C^3\left([0,1];(\mathbb{R}^2)^3\right)$, $Y_{1,T} \hookrightarrow C\left([0,T];\mathbb{R}^2\right)$ and $Y_{2,T} \hookrightarrow C\left([0,T];(\mathbb{R}^2)^3\right)$ which are a consequence of Theorem B.20 due to $p \in (5,\infty)$. □

Definition 4.15 (Linear compatibility conditions)**.** Let $p \in (5,\infty)$, φ be an analytically p–admissible initial network and $T > 0$. Given $\psi \in X_0$ we say that a tuple $(h_1,h_2) \in Y_{1,T} \times Y_{2,T}$ satisfies the *linear compatibility conditions* with respect to ψ if for all $f \in X_T$ there holds $(f,h_1,h_2,\psi) \in \mathbb{F}_T$.

As a consequence of Proposition 4.13 we obtain

Corollary 4.16. *Let* $p \in (5,\infty)$*,* φ *be an analytically* p*–admissible initial network and* $T > 0$*. The operator* $L_T : \mathbb{E}_T \to \mathbb{F}_T$ *is well-defined, linear and continuous.*

Proof. Due to Proposition 4.13 it is enough to show for $u \in \mathbb{E}_T$ that $\mathcal{G}_T\left(L_T(u)\right) = 0$. As shown in Proposition B.35, $\mathbb{E}_T$ embeds continuously into $C\left([0,T];C^3\left([0,1];(\mathbb{R}^2)^3\right)\right)$ which yields that all spacial derivatives of u up to order three can be evaluated pointwise in time and space. In particular, $u \in \mathbb{E}_T = \ker\left(\mathcal{B}_T\right)$ yields

$$\left(\mathcal{G}_T^1\left(L_T(u)\right), \mathcal{G}_T^2\left(L_T(u)\right)\right) = \mathcal{B}_T(u)_{|t=0} = 0\,.$$

Furthermore, we have

$$\mathcal{G}_T^3\left(L_T(u)\right) = \sum_{i=1}^{3} \frac{2}{|\varphi_x^i(0)|^3} \left\langle \left(L_T^4(u)\right)_{xxx}^i(0), \nu_0^i(0)\right\rangle \nu_0^i(0) - L_T^2(u)(0) = 0$$

and

$$\mathcal{G}^4_T\left(L_T(u)\right) = L^4_T(u)(1) - L^3_T(u)(0) = u(0,1) - u(0,1) = 0\,.$$

□

To prove that the linear operator $L_T : \mathbb{E}_T \to \mathbb{F}_T$ is invertible we use [136, Theorem 5.4] on well-posedness of boundary value problems for parabolic systems. To this end we need to verify that the boundary value problem associated to $L_T \in \mathcal{L}\left(\mathbb{E}_T, \mathbb{F}_T\right)$ satisfies the conditions listed in [136, § 1]. We observe that we have a system for $m = 6$ scalar unknown functions $\left(u^1_1, u^1_2, u^2_1, u^2_2, u^3_1, u^3_2\right)$. Adopting the notation used in [136, § 1] the linear differential operators $l_{kj}(t,x,\partial_t,\partial_x)$, $k,j \in \{1,\dots,6\}$, $(t,x) \in Q := [0,T]\times(0,1)$ are given by $l_{kj} = 0$ for $k \neq j$ and

$$\begin{aligned} l_{11}(t,x,\partial_t,\partial_x) = l_{22}(t,x,\partial_t,\partial_x) &= \partial_t + \frac{2}{|\varphi^1_x(x)|^4}\partial^4_x\,,\\ l_{33}(t,x,\partial_t,\partial_x) = l_{44}(t,x,\partial_t,\partial_x) &= \partial_t + \frac{2}{|\varphi^2_x(x)|^4}\partial^4_x\,,\\ l_{55}(t,x,\partial_t,\partial_x) = l_{66}(t,x,\partial_t,\partial_x) &= \partial_t + \frac{2}{|\varphi^3_x(x)|^4}\partial^4_x\,. \end{aligned}$$

As the operators are of fourth order in space, we have $b = 2$ and $r = 6$. Moreover, we choose $s_k = 4$ and $t_j = 0$ for every $k,j \in \{1,\dots,6\}$ which are divisible by $2b = 4$ and satisfy $\sum_{j=1}^6 s_j + t_j = 24 = 2br$. Given $(t,x) \in Q$ and $\xi \in \mathbb{R}$ there holds

$$l_{11}(t,x,p\lambda^4, i\xi\lambda) = p\lambda^4 + \frac{2}{\left|\varphi^1_x(x)\right|^4}\xi^4\lambda^4$$

and the analogous identities for the other diagonal elements of (l_{kj}). In particular, the degree of the polynomial $l_{kj}(t,x,p\lambda^4,i\xi\lambda)$ with respect to λ does not exceed $s_k + t_j$ and every l_{kj} is equal to its principal part.

Proposition 4.17 (Uniform parabolicity)**.** *Let $p \in (5,\infty)$, $T > 0$ and φ be an analytically p-admissible initial network. Then there exists a constant $\delta > 0$ such that for every $(t,x) \in [0,T]\times(0,1)$ and $\xi \in \mathbb{R}$ the roots p_s of the polynomial*

$$L(t,x,p,i\xi) := \det\left(\left(l_{kj}(t,x,p,i\xi)\right)_{k,j\in\{1,\dots,6\}}\right)$$

with respect to p satisfy $\Re p_s \leq -\delta\xi^4$.

Proof. Given $(t,x) \in [0,T]\times(0,1)$ and $\xi \in \mathbb{R}$ the expression

$$L(t,x,p,i\xi) = \prod_{i=1}^{3}\left(p + \frac{2}{\left|\varphi^i_x(x)\right|^4}\xi^4\right)^2$$

is a polynomial of degree six in p with roots of multiplicity two given by $p_i = -\frac{2}{|\varphi^i_x(x)|^4}\xi^4$ which satisfy for all $i \in \{1,2,3\}$,

$$\Re p_i = p_i = -\frac{2}{\left|\varphi^i_x(x)\right|^4}\xi^4 \leq -\frac{2}{C^4}\xi^4$$

with $C := \|\varphi\|_{C^1([0,1];(\mathbb{R}^2)^3)}$. □

We sort the scalar boundary operators $B_{q,j}$, $q \in \{1,\dots,12\}$, $j \in \{1,\dots,6\}$, in $x = 0$, in the notation of [136, § 1] by decreasing order. Given $q \in \{1,2\}$ and $j^i_k \in \{1,\dots,6\}$ depending on $i \in \{1,2,3\}$ and $k \in \{1,2\}$, the highest order operators B_{q,j^i_k} are of the form

$$B_{q,j^i_k}(t,0,\partial_t,\partial_x) = \frac{2}{\left|\varphi^i_x(0)\right|^3}\left(\nu^i_0\right)_k(0)\left(\nu^i_0\right)_q(0)\left(\partial^3_x\right)_{|x=0}$$

with degree $\beta_{q,j_k^i} = 3 = \sigma_q$, and are thus equal to their principal parts. Here $j_1^1 = 1$, $j_2^1 = 2$, $j_1^2 = 3$, $j_2^2 = 4$, $j_1^3 = 5$, $j_2^3 = 6$. Corresponding to the second order condition we define for $q \in \{3, \dots, 8\}$ the operator $B_{q,q-2}$ by

$$B_{q,q-2}(t, 0, \partial_t, \partial_x) = \left(\partial_x^2\right)_{|x=0}$$

and $B_{q,j} = 0$ else and thus $\sigma_q = \max_j \beta_{q,j} = 2$ and the operators are equal to their principal parts. The concurrency condition is represented by the operators $B_{q,j}$, $q \in \{9, 10, 11, 12\}$ and $j \in \{1, \dots, 6\}$, which are defined by

$$B_{q,q-8}(t, 0, \partial_t, \partial_x) = \left(\partial_x^0\right)_{|x=0}\,, \quad B_{q,q-6}(t, 0, \partial_t, \partial_x) = -\left(\partial_x^0\right)_{|x=0}$$

and $B_{q,j} = 0$ else, being equal to their principal parts with $\sigma_q = 0$. The boundary operators at $x = 0$ satisfy the *complementary condition* in [136, page 18] which follows from the *Lopatinskii–Shapiro condition*, see [53, pages 11-15], shown in the following Proposition. The proof fundamentally relies on the fact that the analytically p–admissible initial network φ satisfies the *non-degeneracy condition* (3.22).

Proposition 4.18 (Lopatinskii–Shapiro condition at $x = 0$)**.** *[64, Lemma 3.14]. Let $p \in (5, \infty)$ and φ be an analytically p–admissible initial network with unit normal ν_0^i. For any $\lambda \in \overline{\mathbb{C}_+} := \{z \in \mathbb{C} : \Re(z) \geq 0\}$ with $|\lambda| \neq 0$ every solution $v \in C^4\left([0, \infty); (\mathbb{C}^2)^3\right)$ to the ordinary differential equation*

$$\lambda v^i(s) + \frac{1}{|\varphi_x^i(0)|^4} v_{ssss}^i(s) = 0\,, \quad s > 0\,, \quad i \in \{1, 2, 3\}\,, \tag{4.6a}$$

$$\sum_{i=1}^3 \frac{1}{|\varphi_x^i(0)|^3} \left\langle v_{sss}^i(0), \nu_0^i(0)\right\rangle \nu_0^i(0) = 0\,, \tag{4.6b}$$

$$v_{ss}^i(0) = 0\,, \qquad i \in \{1, 2, 3\}\,, \tag{4.6c}$$

$$v^1(0) - v^2(0) = 0\,, \tag{4.6d}$$

$$v^2(0) - v^3(0) = 0 \tag{4.6e}$$

which satisfies $\lim_{s\to\infty} \left|v^i(s)\right| = 0$ for $i \in \{1, 2, 3\}$ has to be the trivial solution.

Proof. In this proof we use the notation $\nu_0^i := \nu_0^i(0)$, $\tau_0^i := \tau_0^i(0)$ and $\varphi_x^i := \varphi_x^i(0)$. Let $\lambda \in \overline{\mathbb{C}_+} \setminus \{0\}$ be given and $\left(v^1, v^2, v^3\right)$ be a solution to (4.6) such that $\lim_{s\to\infty} \left|v^i(s)\right| = 0$ for $i \in \{1, 2, 3\}$. Due to the specific exponential form of the solutions to (4.6a), all derivatives of v^i up to order four decay to zero for s tending to infinity. We test the equation (4.6a) by $\left|\varphi_x^i\right| \left\langle \overline{v}^i(s), \nu_0^i\right\rangle \nu_0^i$, integrate with respect to $s \in [0, \infty)$ and sum with respect to $i \in \{1, 2, 3\}$ which yields

$$0 = \sum_{i=1}^3 \lambda \left|\varphi_x^i\right| \int_0^\infty \left|\left\langle v^i(s), \nu_0^i\right\rangle\right|^2 \mathrm{d}s + \sum_{i=1}^3 \frac{1}{|\varphi_x^i|^3} \int_0^\infty \left\langle v_{ssss}^i(s), \nu_0^i\right\rangle \left\langle \overline{v}^i(s), \nu_0^i\right\rangle \mathrm{d}s\,. \tag{4.7}$$

We integrate by parts twice the second term in (4.7) using the decay of v^i and its derivatives at infinity to obtain

$$\sum_{i=1}^3 \frac{1}{|\varphi_x^i|^3} \int_0^\infty \left\langle v_{ssss}^i(s), \nu_0^i\right\rangle \left\langle \overline{v}^i(s), \nu_0^i\right\rangle \mathrm{d}s = \sum_{i=1}^3 \frac{1}{|\varphi_x^i|^3} \int_0^\infty \left|\left\langle v_{ss}^i(s), \nu_0^i\right\rangle\right|^2 \mathrm{d}s$$
$$+ \sum_{i=1}^3 \frac{1}{|\varphi_x^i|^3} \left\langle \overline{v}^i(0), \nu_0^i\right\rangle \left\langle v_{sss}^i(0), \nu_0^i\right\rangle - \sum_{i=1}^3 \frac{1}{|\varphi_x^i|^3} \left\langle \overline{v}_s^i(0), \nu_0^i\right\rangle \left\langle v_{ss}^i(0), \nu_0^i\right\rangle\,.$$

The second boundary term vanishes due to (4.6c). Using (4.6b), (4.6d) and (4.6e) the first boundary term becomes

$$\sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^3} \langle \overline{v}^i(0), \nu_0^i \rangle \langle v_{sss}^i(0), \nu_0^i \rangle = \left\langle \overline{v}^1(0), \sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^3} \langle v_{sss}^i(0), \nu_0^i \rangle \nu_0^i \right\rangle = 0$$

which allows us to conclude

$$0 = \sum_{i=1}^{3} \lambda \left|\varphi_x^i\right| \int_0^\infty \left|\langle v^i(s), \nu_0^i \rangle\right|^2 \mathrm{d}s + \sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^3} \int_0^\infty \left|\langle v_{ss}^i(s), \nu_0^i \rangle\right|^2 \mathrm{d}s \,. \tag{4.8}$$

If $\Re(\lambda) > 0$, taking the real part of (4.8) yields for all $i \in \{1, 2, 3\}$ and $s \in [0, \infty)$, $\langle v^i(s), \nu_0^i \rangle = 0$. In the case that $\Re(\lambda) = 0$, we have $|\Im(\lambda)| > 0$ and the same result is obtained by taking the imaginary part of (4.8). Using (4.6d) and (4.6e) we find for all $a^i \in \mathbb{C}$ that

$$\left\langle v^1(0), \sum_{i=1}^{3} a^i \nu_0^i \right\rangle = 0 \,.$$

As φ satisfies the non-degeneracy condition at $x = 0$ the vectors $\nu_0^i \in \mathbb{R}^2$, $i \in \{1, 2, 3\}$, are linearly independent over $\mathbb{R}$ and thus also linearly independent over $\mathbb{C}$. In particular, $\{\nu_0^1, \nu_0^2, \nu_0^3\}$ form a basis of $\mathbb{C}^2$ which enforces $v^1(0) = 0$.
In the same way as before, testing equation (4.6a) by $\left|\varphi_x^i\right| \langle \overline{v}^i(s), \tau_0^i \rangle \tau_0^i$ we get

$$0 = \sum_{i=1}^{3} \lambda \left|\varphi_x^i\right| \int_0^\infty \left|\langle v^i(s), \tau_0^i \rangle\right|^2 \mathrm{d}s + \sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^3} \int_0^\infty \left|\langle v_{ss}^i(s), \tau_0^i \rangle\right|^2 \mathrm{d}s \tag{4.9}$$

$$+ \sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^3} \langle \overline{v}^i(0), \tau_0^i \rangle \langle v_{sss}^i(0), \tau_0^i \rangle - \sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^3} \langle \overline{v}_s^i(0), \tau_0^i \rangle \langle v_{ss}^i(0), \tau_0^i \rangle \,. \tag{4.10}$$

Again the second boundary term vanishes due to (4.6c) and the first one due to $v^i(0) = 0$. In the case that $\Re(\lambda) > 0$ considering the real part of (4.9) yields for all $s \in [0, \infty)$ and $i \in \{1, 2, 3\}$, $\langle v^i(s), \tau_0^i \rangle = 0$. If $\Re(\lambda) = 0$ the same result follows by taking the imaginary part of (4.9) using $\Im(\lambda) > 0$. As τ_0^i and ν_0^i form a basis of $\mathbb{C}^2$, we conclude for $s \in [0, \infty)$ and $i \in \{1, 2, 3\}$ that $v^i(s) = 0$. □

The boundary operators at $x = 1$ correspond to the second order and the endpoint condition. For $q \in \{1, \ldots, 6\}$ we have

$$B_{q,q}(t, 1, \partial_t, \partial_x) = \left(\partial_x^2\right)_{x=1}$$

and $B_{q,j} = 0$ for $j \in \{1, \ldots, 6\} \setminus \{q\}$ which yields that all $B_{q,j}$ are equal to their principal parts and $\sigma_q = 2$. For $q \in \{7, \ldots, 12\}$ we have

$$B_{q,q-6}(t, 1, \partial_t, \partial_x) = \left(\partial_x^0\right)_{|x=1}$$

and $B_{q,j} = 0$ else which yields $\sigma_q = 0$ and all $B_{q,j}$ are equal to their principal parts. The complementary condition for the boundary operators at $x = 1$ again follows from the Lopatinskii–Shapiro condition.

Proposition 4.19 (Lopatinskii–Shapiro condition at $x = 1$)**.** *[64, Lemma 4.8]. Let $p \in (5, \infty)$ and φ be an analytically p–admissible initial network. For any value of $\lambda \in \overline{\mathbb{C}_+} := \{z \in \mathbb{C} : \Re(z) \geq 0\}$ with $|\lambda| \neq 0$ every solution $v \in C^4\big([0, \infty); (\mathbb{C}^2)^3\big)$ to the ordinary differential equation*

$$\lambda v^i(s) + \frac{1}{|\varphi_x^i(1)|^4} v_{ssss}^i(s) = 0 \,, \quad s > 0 \,, \quad i \in \{1, 2, 3\} \,, \tag{4.11a}$$

$$v_{ss}^i(0) = 0\,, \qquad i \in \{1,2,3\}\,, \tag{4.11b}$$

$$v^i(0) = 0\,, \qquad i \in \{1,2,3\} \tag{4.11c}$$

which satisfies $\lim_{s\to\infty} \left|v^i(s)\right| = 0$ *for* $i \in \{1,2,3\}$ *has to be the trivial solution.*

Proof. Let $\lambda \in \overline{\mathbb{C}_+} \setminus \{0\}$ be given and $\left(v^1, v^2, v^3\right)$ be a solution to (4.11) with $\lim\limits_{s\to\infty} \left|v^i(s)\right| = 0$ for $i \in \{1,2,3\}$. As in the proof of Proposition 4.18 we conclude that all derivatives of v^i up to order four decay to zero for s tending to infinity. Testing equation (4.11a) with $\overline{v}^i$ and integrating with respect to $s \in [0,\infty)$ gives for every $i \in \{1,2,3\}$,

$$0 = \lambda \int_0^\infty \left\langle v^i(s), \overline{v}^i(s)\right\rangle \mathrm{d}s + \frac{1}{|\varphi_x^i(1)|^4} \int_0^\infty \left\langle v_{ssss}^i(s), \overline{v}^i(s)\right\rangle \mathrm{d}s\,. \tag{4.12}$$

Applying integration by parts twice to the second integral in (4.12) yields

$$0 = \lambda \int_0^\infty \left|v^i(s)\right|^2 \mathrm{d}s + \frac{1}{|\varphi_x^i(1)|^4} \int_0^\infty \left|v_{ss}^i(s)\right|^2 \mathrm{d}s\,. \tag{4.13}$$

If $\Re\lambda > 0$, taking the real part of (4.13) yields $v^i(s) = 0$ for all $s \in [0,\infty)$. In the case that $\Re\lambda = 0$, we have $\Im\lambda \neq 0$ and $v^i(s) = 0$ for all $s \in [0,\infty)$ follows from taking the imaginary part of (4.13). □

It remains to verify the complementary condition in [136, page 12] for the initial matrix $\mathcal{C}$ representing the initial conditions for the system associated to L_T. In our case there holds for $\alpha, j \in \{1,\ldots,6\}$,

$$C_{\alpha,j}\left(x, \partial_t, \partial_x\right) = \delta_{\alpha,j}$$

and thus $\mathcal{C} = (C_{\alpha,j}) = \mathrm{Id} \in \mathbb{R}^{6\times 6}$, $\rho_\alpha = 0$ and $\mathcal{C}$ is equal to its principal part. It remains to verify that the rows of

$$\widehat{\mathcal{L}}\left(0,x,p,0\right) := L\left(0,x,p,0\right)\left(l_{kj}(0,x,p,0)\right)^{-1} = \left(\prod_{i=1}^3 p^2\right) p^{-1}\mathrm{Id}_{6\times 6} = p^5 \mathrm{Id}_{6\times 6}$$

are linearly independent modulo p^6 which is trivially fulfilled.
The interval $(0,1)$ is a bounded smooth domain and trivially satisfies the conditions listed in [136, § 13]. Due to Lemma 4.12 the coefficients in l_{kj} are continuous on $[0,T] \times [0,1]$ for any $T > 0$. Furthermore, the compatibility conditions of order 3 given in (4.19) in [136, § 14] precisely correspond to the linear compatibility conditions in Definition 4.15. As $p \in (5,\infty)$ there holds for all $q \in \{1,\ldots,12\}$ that

$$\frac{4-\sigma_q}{4} - \frac{5}{4p} \in (0,1)\,.$$

Finally, it is shown in Theorem B.21 and Proposition B.31 that the spaces used in [136] coincide with the ones used in this thesis with equivalent norms (with equivalence constants that may depend on the time interval).

Theorem 4.20 (Well-posedness of the linearised problem in Sobolev setting)**.** *Let* $p \in (5,\infty)$, $T > 0$ *and* φ *be an analytically* p*–admissible initial network. Then the operator* $L_T \in \mathcal{L}\left(\mathbb{E}_T, \mathbb{F}_T\right)$ *is bijective with continuous inverse* L_T^{-1}*.*

Proof. This follows from [136, Theorem 5.4] with $l = 4 > 3 = \max_q \sigma_q$. □

The continuity of the inverse follows from the Inverse Function Theorem [24, Corollary 2.7]. As in Chapter 2 we need to establish estimates with constants that are independent of the considered time interval. Therefore, we consider on $\mathbb{E}_T$ the norm

$$|||u|||_{\mathbb{E}_T} := \|u\|_{\mathbb{E}_T} + \|u(0)\|_{X_0} = \|u\|_{\boldsymbol{E}_T} + \|u(0)\|_{W_p^{4-4/p}((0,1);(\mathbb{R}^2)^3)} = |||u|||_{\boldsymbol{E}_T}$$

that is equivalent to the norm

$$\|u\|_{\boldsymbol{E}_T} = \|u\|_{W_p^1((0,T);L_p((0,1);(\mathbb{R}^2)^3))\cap L_p\left((0,T);W_p^4((0,1);(\mathbb{R}^2)^3)\right)}$$

due to Proposition B.36. Furthermore, the spaces $Y_{i,T}$, $i \in \{1,2\}$, are endowed with the norms

$$|||h_i|||_{Y_{i,T}} := \|h_i\|_{Y_{i,T}} + |h_i(0)| \,.$$

Corollary B.22 implies that these are equivalent to the usual Slobodeckij norms $\|\cdot\|_{Y_{i,T}}$ as defined in Proposition B.19 and thus we have an equivalent norm on $\mathbb{F}_T$ given by

$$|||\,(f, h_1, h_2, \psi)\,|||_{\mathbb{F}_T} := \|f\|_{X_T} + |||h_1|||_{Y_{1,T}} + |||h_2|||_{Y_{2,T}} + \|\psi\|_{X_0} \,.$$

Theorem 4.21 (Uniform well-posedness of the linear problem)**.** *Let $p \in (5,\infty)$, φ be an analytically p–admissible initial network and $T_0 > 0$. For all $T \in (0, T_0]$ the map $L_T \in \mathcal{L}(\mathbb{E}_T, \mathbb{F}_T)$ is a linear isomorphism. Furthermore, there exists a constant $C(T_0)$ independent of $T \in (0, T_0]$ such that for all $u \in \mathbb{E}_T$ with $L_T(u) = (f, h_1, h_2, \psi) \in \mathbb{F}_T$ there holds*

$$|||u|||_{\mathbb{E}_T} \leq C(T_0)\left(\|f\|_{X_T} + |||h_1|||_{Y_{1,T}} + |||h_2|||_{Y_{2,T}} + \|\psi\|_{X_0}\right). \tag{4.14}$$

Proof. Given $T \in (0, T_0]$ Theorem 4.20 yields that $L_T \in \mathcal{L}(\mathbb{E}_T, \mathbb{F}_T)$ is a bijective linear operator with continuous inverse L_T^{-1}. It remains to show that the operator norm of L_T^{-1} is uniformly bounded in $T \in (0, T_0]$. Let $T \in (0, T_0]$ and $(f, h_1, h_2, \psi) \in \mathbb{F}_T$ be given and $u \in \mathbb{E}_T$ with $L_T u = (f, h_1, h_2, \psi)$. Due to Proposition B.23 there exists an extension $\boldsymbol{E}h_i \in Y_{i,T_0}$ for $i \in \{1,2\}$ with $(\boldsymbol{E}h_i)_{|(0,T)} = h_i$ and

$$|||\boldsymbol{E}h_i|||_{Y_{i,T_0}} \leq c(T_0)\, |||h_i|||_{Y_{i,T}} \,. \tag{4.15}$$

Extending f by 0 to $\boldsymbol{E}f \in X_{T_0}$ we observe that $(\boldsymbol{E}f, \boldsymbol{E}h_1, \boldsymbol{E}h_2, \psi)$ lies in $\mathbb{F}_{T_0}$. As L_{T_0} is a linear isomorphism, there exists a unique $\widetilde{u} \in \mathbb{E}_{T_0}$ with $L_{T_0}\widetilde{u} = (\boldsymbol{E}f, \boldsymbol{E}h_1, \boldsymbol{E}h_2, \psi)$. We notice that

$$L_T(u) = (f, h_1, h_2, \psi) = (\boldsymbol{E}f, \boldsymbol{E}h_1, \boldsymbol{E}h_2, \psi)_{|(0,T)} = (L_{T_0}\widetilde{u})_{|(0,T)} = L_T\left(\widetilde{u}_{|(0,T)}\right)$$

which yields $u = \widetilde{u}_{|(0,T)}$ as L_T is an isomorphism. Using the equivalence of the norms on $\mathbb{E}_{T_0}$ and $\mathbb{F}_{T_0}$, continuity of $L_{T_0}^{-1}$ and (4.15) we obtain

$$\begin{aligned} |||u|||_{\mathbb{E}_T} &= |||\widetilde{u}_{|(0,T)}|||_{\mathbb{E}_T} \leq |||L_{T_0}^{-1}\left((\boldsymbol{E}f, \boldsymbol{E}h_1, \boldsymbol{E}h_2, \psi)\right)|||_{\mathbb{E}_{T_0}} \\ &\leq C(T_0)\,|||\,(\boldsymbol{E}f, \boldsymbol{E}h_1, \boldsymbol{E}h_2, \psi)\,|||_{\mathbb{F}_{T_0}} \leq C(T_0)\left(\|f\|_{X_T} + |||h_1|||_{Y_{1,T}} + |||h_2|||_{Y_{2,T}} + \|\psi\|_{X_0}\right). \end{aligned}$$

□

The contraction argument

Proposition 4.22. *Let $p \in (5,\infty)$, $T_0 > 0$ and φ be an analytically p–admissible initial network with $\boldsymbol{c}$ as in* (4.5)*. Given $T \in (0, T_0]$ the set*

$$\mathbb{E}_T^{\varphi} := \left\{\gamma \in \mathbb{E}_T : \gamma_{|t=0} = \varphi\right\}$$

is a non-empty complete metric space in the norm $|||\cdot|||_{\mathbb{E}_T}$ and there exists $\mathcal{E}\varphi \in \mathbb{E}_T^{\varphi}$ such that $|||\mathcal{E}\varphi|||_{\mathbb{E}_T} \leq C(T_0, \boldsymbol{c})\, \|\varphi\|_{X_0}$.

Proof. Let $T \in (0, T_0]$ be given. Proposition B.35 yields the continuous embedding

$$\mathbb{E}_T \hookrightarrow C\big([0,T]; W_p^{4-4/p}\big((0,1);(\mathbb{R}^2)^3\big)\big)$$

which implies that $\mathbb{E}_T^\varphi$ is a well-defined closed subset of the Banach space $\big(\mathbb{E}_T, \|\!|\cdot|\!\|_{\mathbb{E}_T}\big)$ and thus a complete metric space in $\|\!|\cdot|\!\|_{\mathbb{E}_T}$. Let $h_1 := \sum_{i=1}^3 \mu^i \tau_0^i(0)$ with $\tau_0^i(0)$ the unit tangent to φ^i in $x=0$. As φ is an analytically p–admissible initial network we conclude upon using Lemma 4.11 that $\mathcal{G}_T\left((0,h_1,\varphi(1),\varphi)\right) = 0$ and thus $(0,h_1,\varphi(1),\varphi) \in \mathbb{F}_T$. Theorem 4.21 yields

$$\mathcal{E}\varphi := L_T^{-1}\left((0,h_1,\varphi(1),\varphi)\right) \in \mathbb{E}_T$$

with $(\mathcal{E}\varphi)_{|t=0} = L_T^4\left(L_T^{-1}\left((0,h_1,\varphi(1),\varphi)\right)\right) = \varphi$ and

$$\|\!|\mathcal{E}\varphi|\!\|_{\mathbb{E}_T} \leq C\left(T_0\right)\Big(\|\!|h_1|\!\|_{Y_{1,T}} + \|\!|\varphi(1)|\!\|_{Y_{2,T}} + \|\varphi\|_{X_0}\Big).$$

The claim now follows from

$$\|\!|h_1|\!\|_{Y_{1,T}} \leq \big(T_0^{1/p}+1\big)\left|h_1\right| \leq C\left(T_0,\boldsymbol{c}\right)\|\varphi_x\|_{C([0,1];(\mathbb{R}^2)^3)} \leq C\left(T_0,\boldsymbol{c}\right)\|\varphi\|_{X_0}$$

and

$$\|\!|\varphi(1)|\!\|_{Y_{2,T}} \leq \big(T_0^{1/p}+1\big)\left|\varphi(1)\right| \leq \big(T_0^{1/p}+1\big)\|\varphi\|_{C([0,1];(\mathbb{R}^2)^3)} \leq C\left(T_0\right)\|\varphi\|_{X_0}.$$

□

Given $T>0$ and $M>0$ we let

$$\overline{B_{M,T}} := \left\{\gamma \in \boldsymbol{E}_T : \|\!|\gamma|\!\|_{\boldsymbol{E}_T} \leq M\right\}.$$

Lemma 4.23. *Let $p \in (5,\infty)$, $T_0 > 0$, $M > 0$ and φ be an analytically p–admissible initial network with $\boldsymbol{c}$ as in* (4.5). *There exists a time $\widetilde{T}(\boldsymbol{c},M) \in (0,T_0]$ such that for all $T \in \big(0,\widetilde{T}(\boldsymbol{c},M)\big]$, $\gamma \in \mathbb{E}_T^\varphi \cap \overline{B_{M,T}}$ and all $i \in \{1,2,3\}$ the curve $\gamma^i(t)$ is regular with*

$$\inf_{t\in[0,T],x\in[0,1]}\left|\gamma_x^i(t,x)\right| \geq \frac{\boldsymbol{c}}{2} \tag{4.16}$$

and satisfies the non-degeneracy condition in $y=0$. Moreover, given a polynomial $\mathfrak{p}$ there exists a constant $C>0$ depending on $\boldsymbol{c}$ such that

$$\sup_{t\in[0,T],x\in[0,1]} \mathfrak{p}\left(|\gamma_x^i(t,x)|^{-1}\right) \leq C(\boldsymbol{c}).$$

Proof. Let $\theta \in \left(\frac{1/p+1}{4-4/p},1\right)$ and $\delta \in (0,1-1/p)$ be fixed and $\alpha := (1-\theta)(1-1/p-\delta)$. Corollary B.39 yields for all $T\in(0,T_0]$ the continuous embedding

$$i_T : \mathbb{E}_T \hookrightarrow C^\alpha\left([0,T]; C^1\left([0,1];(\mathbb{R}^2)^3\right)\right)$$

with $\|i_T(\gamma)\|_{C^\alpha([0,T];C^1([0,1];(\mathbb{R}^2)^3))} \leq C\left(T_0\right)\|\!|\gamma|\!\|_{\mathbb{E}_T}$. Given $T\in(0,T_0]$ and $\gamma \in \mathbb{E}_T^\varphi \cap \overline{B_{M,T}}$ we have for $i\in\{1,2,3\}$, $t\in[0,T]$ and $x\in[0,1]$,

$$\left|\gamma_x^i(t,x)\right| \geq \left|\varphi_x^i(x)\right| - \left|\gamma_x^i(t,x)-\gamma_x^i(0,x)\right| \geq \boldsymbol{c} - \left\|\gamma^i(t)-\gamma^i(0)\right\|_{C^1([0,1];\mathbb{R}^2)},$$

where

$$\left\|\gamma^i(t)-\gamma^i(0)\right\|_{C^1([0,1];\mathbb{R}^2)} \leq t^\alpha\left\|\gamma^i\right\|_{C^\alpha([0,T];C^1([0,1];\mathbb{R}^2))} \leq T^\alpha C\left(T_0\right)\|\!|\gamma|\!\|_{\mathbb{E}_T} \leq T^\alpha M C\left(T_0\right).$$

Choosing $\widehat{T} = \widehat{T}(\boldsymbol{c}, M) \in (0, T_0]$ so small that $\widehat{T}^{\alpha} M C(T_0) \leq \frac{\boldsymbol{c}}{2}$, we obtain for all $T \in \big(0, \widehat{T}(\boldsymbol{c}, M)\big]$, $\gamma \in \mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$, $t \in [0,T]$ and $i \in \{1,2,3\}$, that $\gamma^i(t)$ is a regular curve satisfying the estimate (4.16).

Let $T \in (0, \widehat{T}(\boldsymbol{c}, M)]$ and $\gamma \in \mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$ be given. For $t \in [0,T]$ and $i \in \{1,2,3\}$ we let $\alpha^i(t) \in [0, 2\pi]$ denote the angle between $\frac{\gamma_x^{i+1}(t,0)}{|\gamma_x^{i+1}(t,0)|}$ and $\frac{\gamma_x^{i-1}(t,0)}{|\gamma_x^{i-1}(t,0)|}$ where the index i should be read mod 3. As φ satisfies the non-degeneracy condition in $y = 0$, there exists $i \in \{1,2,3\}$ such that $\alpha^i(0) \in (0, \pi)$. Without loss of generality we may assume $\alpha^3(0) \in (0, \pi)$. In particular, there exists $\varepsilon \in (0, 1/2)$ such that

$$\left\langle \frac{\gamma_x^1(0,0)}{|\gamma_x^1(0,0)|}, \frac{\gamma_x^2(0,0)}{|\gamma_x^2(0,0)|} \right\rangle \in [-1+\varepsilon, 1-\varepsilon] .$$

To improve readability we omit the dependence on $y = 0$ in the following. We observe for $t \in [0,T]$ that

$$\begin{aligned} &\left| \left\langle \frac{\gamma_x^1(0)}{|\gamma_x^1(0)|}, \frac{\gamma_x^2(0)}{|\gamma_x^2(0)|} \right\rangle - \left\langle \frac{\gamma_x^1(t)}{|\gamma_x^1(t)|}, \frac{\gamma_x^2(t)}{|\gamma_x^2(t)|} \right\rangle \right| \\ &\leq \left| \left\langle \frac{\gamma_x^1(0)}{|\gamma_x^1(0)|} - \frac{\gamma_x^1(t)}{|\gamma_x^1(t)|}, \frac{\gamma_x^2(0)}{|\gamma_x^2(0)|} \right\rangle \right| + \left| \left\langle \frac{\gamma_x^1(t)}{|\gamma_x^1(t)|}, \frac{\gamma_x^2(t)}{|\gamma_x^2(t)|} - \frac{\gamma_x^2(0)}{|\gamma_x^2(0)|} \right\rangle \right| , \end{aligned}$$

where the first term is bounded by

$$\begin{aligned} &\left| \frac{\gamma_x^1(0)}{|\gamma_x^1(0)|} - \frac{\gamma_x^1(t)}{|\gamma_x^1(t)|} \right| \left| \frac{\gamma_x^2(0)}{|\gamma_x^2(0)|} \right| \leq C(T_0, \boldsymbol{c}, M) \left| \frac{\gamma_x^1(0)}{|\gamma_x^1(0)|} - \frac{\gamma_x^1(t)}{|\gamma_x^1(t)|} \right| \\ &\leq C(T_0, \boldsymbol{c}, M) \Big(|\gamma_x^1(0)| \left| |\gamma_x^1(0)|^{-1} - |\gamma_x^1(t)|^{-1} \right| + |\gamma_x^1(0) - \gamma_x^1(t)| \, |\gamma_x^1(t)|^{-1} \Big) \\ &\leq C(T_0, \boldsymbol{c}, M) \, |\gamma_x^1(0) - \gamma_x^1(t)| \leq C(T_0, \boldsymbol{c}, M) \, T^{\alpha} \, \|\gamma\|_{C^{\alpha}([0,T]; C^1([0,1];\mathbb{R}^2))} \leq C(T_0, \boldsymbol{c}, M) \, T^{\alpha} . \end{aligned}$$

As the second term can be estimated similarly, we obtain the existence of a constant $C(T_0, \boldsymbol{c}, M)$ such that for all $\gamma \in \mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$, $T \in (0, \widehat{T}(\boldsymbol{c}, M)]$ and $t \in [0,T]$,

$$\left| \left\langle \frac{\gamma_x^1(0)}{|\gamma_x^1(0)|}, \frac{\gamma_x^2(0)}{|\gamma_x^2(0)|} \right\rangle - \left\langle \frac{\gamma_x^1(t)}{|\gamma_x^1(t)|}, \frac{\gamma_x^2(t)}{|\gamma_x^2(t)|} \right\rangle \right| \leq C(T_0, \boldsymbol{c}, M) \, T^{\alpha} . \tag{4.17}$$

We let $\widetilde{T}(\boldsymbol{c}, M) \in (0, \widehat{T}(\boldsymbol{c}, M)]$ be so small that with the constant in (4.17) it holds

$$C(T_0, \boldsymbol{c}, M) \left(\widetilde{T}(\boldsymbol{c}, M) \right)^{\alpha} \leq \frac{\varepsilon}{2} .$$

Then for all $T \in \big(0, \widetilde{T}(\boldsymbol{c}, M)\big]$, $\gamma \in \mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$ and $t \in [0,T]$,

$$\left\langle \frac{\gamma_x^1(t)}{|\gamma_x^1(t)|}, \frac{\gamma_x^2(t)}{|\gamma_x^2(t)|} \right\rangle \in [-1 + \varepsilon/2, 1 - \varepsilon/2]$$

which shows in particular that γ satisfies the non-degeneracy condition in $y = 0$. □

Proposition 4.24 (The nonlinear operator)**.** *Let $p \in (5, \infty)$, φ be an analytically p–admissible initial value with $\boldsymbol{c}$ as in* (4.5)*. Let $T_0 := 1$, $M > 0$, $\widetilde{T}(\boldsymbol{c}, M)$ be as in Lemma 4.23 and $\overline{V}$, $\overline{T}$ be the expressions defined in* (3.24) *and* (3.26)*. For all $T \in \big(0, \widetilde{T}(\boldsymbol{c}, M)\big]$ the operators*

$$N_T^1 : \mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}} \to X_T \, , \gamma \mapsto \left(\gamma_t^i - \overline{V}\left(\gamma^i\right) \nu^i - \overline{T}\left(\gamma^i\right) \tau^i \right)_{i \in \{1,2,3\}}$$

and

$$N_T^2 : \mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}} \to Y_{1,T} \, , \gamma \mapsto \sum_{i=1}^{3} \frac{2}{|\gamma_x^i(\cdot,0)|^3} \left\langle \gamma_{xxx}^i(\cdot,0), \nu^i(\cdot,0) \right\rangle \nu^i(\cdot,0) - \mu^i \tau^i(\cdot,0)$$

are well-defined. We let $N_T := \left(N_T^1, N_T^2\right)$.

Proof. Let $T \in \big(0, \widetilde{T}(\boldsymbol{c}, M)\big]$ be given. The identities (3.25) and (3.26) yield for $i \in \{1,2,3\}$,

$$\big(N_T^1(\gamma)\big)^i = \gamma_t^i + 2\frac{\gamma^i_{xxxx}}{|\gamma^i_x|^4} - 12\frac{\gamma^i_{xxx}\left\langle \gamma^i_{xx}, \gamma^i_x \right\rangle}{|\gamma^i_x|^6} - 5\frac{\gamma^i_{xx}\left|\gamma^i_{xx}\right|^2}{|\gamma^i_x|^6} - 8\frac{\gamma^i_{xx}\left\langle \gamma^i_{xxx}, \gamma^i_x \right\rangle}{|\gamma^i_x|^6} + 35\frac{\gamma^i_{xx}\left\langle \gamma^i_{xx}, \gamma^i_x \right\rangle^2}{|\gamma^i_x|^8}$$
$$- \mu\frac{\gamma^i_{xx}}{|\gamma^i_x|^2}$$

which lies in $L_p\left((0,T); L_p\left((0,1);\mathbb{R}^2\right)\right)$ for all $\gamma \in \mathbb{E}_T^\varphi \cap \overline{B_{M,T}}$ due to (4.16), $\gamma_t, \gamma_{xxxx} \in X_T$ and the embedding $\mathbb{E}_\mathbb{T} \hookrightarrow C\left([0,T]; C^3\left([0,1]; (\mathbb{R}^2)^3\right)\right)$. To prove that N_T^2 is well-defined we let $T \in \big(0, \widetilde{T}(\boldsymbol{c}, M)\big]$ and $\gamma \in \mathbb{E}_T^\varphi \cap \overline{B_{M,T}}$ be given. Theorem C.29 yields

$$t \mapsto \gamma^i_{xxx}(t,0) \in W_p^{1/4 - 1/4p}\left((0,T);\mathbb{R}^2\right).$$

Due to the high value of p there exist $\theta \in \left(\frac{1/p+1}{4-4/p}, 1\right)$ and $\delta \in (0, 1 - 1/p)$ such that

$$\alpha := (1-\theta)\left(1 - 1/p - \delta\right) > 1/4 - 1/4p\,.$$

Corollary B.39 yields for $i \in \{1,2,3\}$, $\gamma^i_x \in C^\alpha\left([0,T]; C\left([0,1];\mathbb{R}^2\right)\right)$ and due to (4.16) we may apply Proposition B.4 to obtain

$$(t,x) \mapsto \left|\gamma^i_x(t,x)\right|^{-1} \in C^\alpha\left([0,T]; C([0,1])\right)$$

and in particular $t \mapsto \left|\gamma^i_x(t,0)\right|^{-1} \in C^\alpha([0,T])$ and $t \mapsto \gamma^i_x(t,0) \in C^\alpha\left([0,T];\mathbb{R}^2\right)$. The claim now follows from the Banach algebra property of $C^\alpha([0,T])$ and Proposition C.32. ☐

Given $p \in (5,\infty)$ and an analytically p–admissible initial network φ, we now deduce existence of strong solutions to the analytic problem (4.3) from the uniform well-posedness of the linearised system represented by the isomorphism L_T. To this end we notice that given $M > 0$ and $T \in \big(0, \widetilde{T}(\boldsymbol{c}, M)\big]$ with $\widetilde{T}(\boldsymbol{c}, M)$ as in Lemma 4.23, a function $\gamma \in \mathbb{E}_T^\varphi \cap \overline{B_{M,T}}$ is a solution to (4.3) if and only if

$$L_T(\gamma) = \left(L_T^1(\gamma) - N_T^1(\gamma), L_T^2(\gamma) - N_T^2(\gamma), \varphi(1), \varphi\right)$$

which is equivalent to the fixed point equation

$$\gamma = K_T(\gamma) := L_T^{-1}\left(\left(L_T^1(\gamma) - N_T^1(\gamma), L_T^2(\gamma) - N_T^2(\gamma), \varphi(1), \varphi\right)\right)$$

once it is shown that

$$\mathcal{G}_T\left(\left(L_T^1(\gamma) - N_T^1(\gamma), L_T^2(\gamma) - N_T^2(\gamma), \varphi(1), \varphi\right)\right) = 0\,.$$

To obtain the existence of a unique fixed point γ of K_T, the constant M has to be chosen appropriately to ensure that K_T is a self-mapping, and T is required to be sufficiently small such that K_T is contractive. These arguments are made precise in the following.

Proposition 4.25 (The contraction operator)**.** *Let $p \in (5,\infty)$, φ be an analytically p–admissible initial value with $\boldsymbol{c}$ as in* (4.5)*, $T_0 := 1$, $M > 0$ and $\widetilde{T}(\boldsymbol{c}, M)$ be as in Lemma 4.23. Then for all $T \in \big(0, \widetilde{T}(\boldsymbol{c}, M)\big]$ and $\gamma \in \mathbb{E}_T^\varphi \cap \overline{B_{M,T}}$ it holds*

$$\left(L_T^1(\gamma) - N_T^1(\gamma), L_T^2(\gamma) - N_T^2(\gamma), \varphi(1), \varphi\right) \in \mathbb{F}_T\,.$$

In particular, the operator

$$K_T : \mathbb{E}_T^\varphi \cap \overline{B_{M,T}} \to \mathbb{E}_T^\varphi, \gamma \mapsto L_T^{-1}\left(\left(L_T^1(\gamma) - N_T^1(\gamma), L_T^2(\gamma) - N_T^2(\gamma), \varphi(1), \varphi\right)\right)$$

is well-defined.

Proof. Proposition 4.13 and 4.24 imply

$$\left(L_T^1(\gamma) - N_T^1(\gamma), L_T^2(\gamma) - N_T^2(\gamma), \varphi(1), \varphi\right) \in X_T \times Y_{1,T} \times Y_{2,T} \times X_0 \,.$$

As $\mathbb{F}_T = \ker(\mathcal{G}_T)$ and $L_T \in \mathcal{L}(\mathbb{E}_T, \mathbb{F}_T)$ is an isomorphism, the claim follows from

$$\mathcal{G}_T\left(\left(L_T^1(\gamma) - N_T^1(\gamma), L_T^2(\gamma) - N_T^2(\gamma), \varphi(1), \varphi\right)\right) = 0$$

using further that $L_T^4\left(K_T(\gamma)\right) = \varphi$ for $\gamma \in \mathbb{E}_T^\varphi$. By Definition 4.3 of analytically p–admissible initial networks we immediately obtain for $i \in \{1,2,4\}$,

$$\mathcal{G}_T^i\left(\left(L_T^1(\gamma) - N_T^1(\gamma), L_T^2(\gamma) - N_T^2(\gamma), \varphi(1), \varphi\right)\right) = 0$$

noticing that time and space evaluation are well-defined for derivatives of γ up to order three. As φ satisfies the third order condition, we find

$$\begin{aligned}
&\mathcal{G}_T^3\left(\left(L_T^1(\gamma) - N_T^1(\gamma), L_T^2(\gamma) - N_T^2(\gamma), \varphi(1), \varphi\right)\right)\\
&= \sum_{i=1}^{3} \frac{2}{\left|\varphi_x^i(0)\right|^3} \left\langle \varphi_{xxx}^i(0), \nu_0^i(0)\right\rangle \nu_0^i(0) - L_T^2(\gamma)_{|t=0} + N_T^2(\gamma)_{|t=0} = N_T^2(\gamma)_{|t=0}\\
&= \sum_{i=1}^{3} \frac{2}{\left|\varphi_x^i(0)\right|^3} \left\langle \varphi_{xxx}^i(0), \nu_0^i(0)\right\rangle \nu_0^i(0) - \mu^i \tau_0^i(0) = 0
\end{aligned}$$

where ν_0^i and τ_0^i are the unit normal and tangent to φ^i, respectively. □

Proposition 4.26 (Contraction Estimates I)**.** *Let $p \in (5,\infty)$, φ be an analytically p–admissible initial value with $\boldsymbol{c}$ as in* (4.5)*, $T_0 := 1$, $M > 0$ and $\widetilde{T}(\boldsymbol{c}, M)$ be as in Lemma 4.23. There exists a constant $\sigma \in (0,1)$ and a constant $C > 0$ depending on $\boldsymbol{c}$ and M such that for all $T \in \left(0, \widetilde{T}(\boldsymbol{c}, M)\right]$, $\gamma, \widetilde{\gamma} \in \mathbb{E}_T^\varphi \cap \overline{B_{M,T}}$ it holds*

$$\left\| L_T^1(\gamma) - N_T^1(\gamma) - \left(L_T^1(\widetilde{\gamma}) - N_T^1(\widetilde{\gamma})\right)\right\|_{X_T} \leq C(\boldsymbol{c}, M) T^\sigma \,|||\gamma - \widetilde{\gamma}|||_{\mathbb{E}_T} \,.$$

Proof. Let $T \in \left(0, \widetilde{T}(\boldsymbol{c}, M)\right]$ and $\gamma, \widetilde{\gamma} \in \mathbb{E}_T^\varphi \cap \overline{B_{M,T}}$ be fixed. Furthermore, we consider fixed parameters $\theta \in \left(\frac{1/p+1}{4-4/p}, 1\right)$ and $\delta \in (0, 1 - 1/p)$ and let $\alpha := (1-\theta)(1-1/p)$. Corollary B.39 then yields for all $u \in \boldsymbol{E}_T$ that u lies in $C^\alpha\left([0,T]; C^1\left([0,1]; (\mathbb{R}^2)^3\right)\right)$ with

$$\|u\|_{C^\alpha([0,T];C^1([0,1];(\mathbb{R}^2)^3))} \leq C\, |||u|||_{\boldsymbol{E}_T} \tag{4.18}$$

with a constant $C = C(\theta, \delta)$. We denote by $\mathfrak{p}$ a polynomial of some degree in one or several variables that are specified in the brackets. The precise shape of $\mathfrak{p}$ is not important, it may change from line to line while maintaining the same notation. The highest order term is given by

$$\left(\frac{2}{|\varphi_x^i|^4} - \frac{2}{|\gamma_x^i|^4}\right)\left(\gamma_{xxxx}^i - \widetilde{\gamma}_{xxxx}^i\right) + \left(\frac{2}{|\widetilde{\gamma}_x^i|^4} - \frac{2}{|\gamma_x^i|^4}\right)\widetilde{\gamma}_{xxxx}^i. \tag{4.19}$$

Using for $a, b \in \mathbb{R}$ the identity

$$\frac{1}{|a|^4} - \frac{1}{|b|^4} = (|b| - |a|)\left(\frac{1}{|a|^2|b|} + \frac{1}{|a||b|^2}\right)\left(\frac{1}{|a|^2} + \frac{1}{|b|^2}\right)$$

the term (4.19) can be written as

$$\left(|\varphi_x^i| - |\gamma_x^i|\right)\mathfrak{p}\left(\left|\varphi_x^i\right|^{-1}, \left|\gamma_x^i\right|^{-1}\right)\left(\gamma_{xxxx}^i - \widetilde{\gamma}_{xxxx}^i\right) + \left(|\widetilde{\gamma}_x^i| - |\gamma_x^i|\right)\mathfrak{p}\left(\left|\widetilde{\gamma}_x^i\right|^{-1}, \left|\gamma_x^i\right|^{-1}\right)\widetilde{\gamma}_{xxxx}^i \,. \tag{4.20}$$

Using (4.5), (4.16) and (4.18) the first term in (4.20) can be estimated as follows:

$$\begin{aligned}
&\left\|\left(|\varphi_x^i|-|\gamma_x^i|\right)\mathfrak{p}\left(|\varphi_x^i|^{-1},|\gamma_x^i|^{-1}\right)\left(\gamma_{xxxx}^i-\widetilde{\gamma}_{xxxx}^i\right)\right\|_{X_T}\\
&\leq C(\boldsymbol{c})\sup_{t\in[0,T],x\in[0,1]}\left|\varphi_x^i(x)-\gamma_x^i(t,x)\right|\left\|\gamma_{xxxx}^i-\widetilde{\gamma}_{xxxx}^i\right\|_{X_T}\\
&\leq C(\boldsymbol{c})T^{\alpha}\sup_{t\in[0,T],x\in[0,1]}t^{-\alpha}\left|\gamma_x^i(0,x)-\gamma_x^i(t,x)\right|\,|||\gamma-\widetilde{\gamma}|||_{\mathbb{E}_T}\\
&\leq C(\boldsymbol{c})T^{\alpha}\sup_{t\in[0,T]}t^{-\alpha}\left\|\gamma^i(t)-\gamma^i(0)\right\|_{C^1([0,1];\mathbb{R}^2)}\,|||\gamma-\widetilde{\gamma}|||_{\mathbb{E}_T}\\
&\leq C(\boldsymbol{c})T^{\alpha}\left\|\gamma^i\right\|_{C^{\alpha}([0,T];C^1([0,1];\mathbb{R}^2))}\,|||\gamma-\widetilde{\gamma}|||_{\mathbb{E}_T}\leq C(\boldsymbol{c})T^{\alpha}\,|||\gamma|||_{\mathbb{E}_T}\,|||\gamma-\widetilde{\gamma}|||_{\mathbb{E}_T}\\
&\leq C(\boldsymbol{c},M)T^{\alpha}\,|||\gamma-\widetilde{\gamma}|||_{\mathbb{E}_T}\,.
\end{aligned}$$

Similarly, we obtain for the second term in (4.20) the estimate

$$\begin{aligned}
&\left\|\left(|\widetilde{\gamma}_x^i|-|\gamma_x^i|\right)\mathfrak{p}\left(|\widetilde{\gamma}_x^i|^{-1},|\gamma_x^i|^{-1}\right)\widetilde{\gamma}_{xxxx}^i\right\|_{X_T}\\
&\leq C(\boldsymbol{c})\sup_{t\in[0,T],x\in[0,1]}\left|\widetilde{\gamma}_x^i(t,x)-\gamma_x^i(t,x)\right|\left\|\widetilde{\gamma}_{xxxx}^i\right\|_{X_T}\\
&\leq C(\boldsymbol{c},M)\sup_{t\in[0,T],x\in[0,1]}\left|\left(\widetilde{\gamma}_x^i(t,x)-\gamma_x^i(t,x)\right)-\left(\widetilde{\gamma}_x^i(0,x)-\gamma_x^i(0,x)\right)\right|\\
&\leq C(\boldsymbol{c},M)T^{\alpha}\sup_{t\in[0,T]}t^{-\alpha}\left\|\left(\widetilde{\gamma}^i(t)-\gamma^i(t)\right)-\left(\widetilde{\gamma}^i(0)-\gamma^i(0)\right)\right\|_{C^1([0,1];\mathbb{R}^2)}\\
&\leq C(\boldsymbol{c},M)T^{\alpha}\left\|\widetilde{\gamma}^i-\gamma^i\right\|_{C^{\alpha}([0,T];C^1([0,1];\mathbb{R}^2))}\leq C(\boldsymbol{c},M)T^{\alpha}\,|||\gamma-\widetilde{\gamma}|||_{\mathbb{E}_T}\,.
\end{aligned}$$

Up to multiplication by constant coefficients, all the other terms of the expression $L_T^1(\gamma)-N_T^1(\gamma)-\left(L_T^1(\widetilde{\gamma})-N_T^1(\widetilde{\gamma})\right)$ are of the form

$$\frac{a^i\left\langle b^i,c^i\right\rangle}{|\gamma_x^i|^j}-\frac{\widetilde{a}^i\left\langle\widetilde{b}^i,\widetilde{c}^i\right\rangle}{|\widetilde{\gamma}_x^i|^j}\tag{4.21}$$

with $j\in\{2,6,8\}$ and with a^i,b^i,c^i and $\widetilde{a}^i,\widetilde{b}^i,\widetilde{c}^i$ spacial derivatives up to order three of γ^i and $\widetilde{\gamma}^i$, respectively. Adding and subtracting the expression

$$\frac{\widetilde{a}^i\left\langle b^i,c^i\right\rangle}{|\gamma_x^i|^j}+\frac{\widetilde{a}^i\left\langle\widetilde{b}^i,c^i\right\rangle}{|\gamma_x^i|^j}+\frac{\widetilde{a}^i\left\langle\widetilde{b}^i,\widetilde{c}^i\right\rangle}{|\gamma_x^i|^j}$$

to (4.21), we obtain

$$\frac{\left(a^i-\widetilde{a}^i\right)\left\langle b^i,c^i\right\rangle}{|\gamma_x^i|^j}+\frac{\widetilde{a}^i\left\langle b^i-\widetilde{b}^i,c^i\right\rangle}{|\gamma_x^i|^j}+\frac{\widetilde{a}^i\left\langle\widetilde{b}^i,c^i-\widetilde{c}^i\right\rangle}{|\gamma_x^i|^j}+\widetilde{a}^i\left\langle\widetilde{b}^i,\widetilde{c}^i\right\rangle\left(|\gamma_x^i|^{-j}-|\widetilde{\gamma}_x^i|^{-j}\right).\tag{4.22}$$

Using (4.16) and the uniform embedding

$$\left(\boldsymbol{E}_T,\,|||\cdot|||_{\boldsymbol{E}_T}\right)\hookrightarrow\left(C\left([0,T];C^3\left([0,1];(\mathbb{R}^2)^3\right)\right),\|\cdot\|_{C([0,T];C^3([0,1];(\mathbb{R}^2)^3))}\right)$$

with embedding constant independent of T as shown in Corollary B.38, the first three terms in (4.22) can be estimated in X_T by

$$C(\boldsymbol{c},M)T^{1/p}\,|||\gamma-\widetilde{\gamma}|||_{\mathbb{E}_T}\,.$$

Given $p,q\in\mathbb{R}^2$ the identities

$$\frac{1}{|p|^2}-\frac{1}{|q|^2}=(|p|-|q|)\left(\frac{1}{|p|^2|q|}+\frac{1}{|p||q|^2}\right),$$

$$
\begin{aligned}
\frac{1}{|p|^6} - \frac{1}{|q|^6} &= \left(\frac{1}{|p|^2} - \frac{1}{|q|^2}\right)\left(\frac{1}{|p|^4} + \frac{1}{|p|^2|q|^2} + \frac{1}{|q|^4}\right),\\
\frac{1}{|p|^8} - \frac{1}{|q|^8} &= \left(\frac{1}{|p|^2} - \frac{1}{|q|^2}\right)\left(\frac{1}{|p|^2} + \frac{1}{|q|^2}\right)\left(\frac{1}{|p|^4} + \frac{1}{|q|^4}\right)
\end{aligned}
$$

together with Corollary B.38 and (4.16) yield that the norm $\|\cdot\|_{X_T}$ of the last term of (4.22) is bounded by

$$
\begin{aligned}
&C\,(M)\,T^{1/p} \sup_{t\in[0,T],x\in[0,1]} \left|\gamma^i_x(t,x) - \widetilde{\gamma}^i_x(t,x)\right| \mathfrak{p}\left(\left|\gamma^i_x(t,x)\right|^{-1}, \left|\widetilde{\gamma}^i_x(t,x)\right|^{-1}\right)\\
&\leq C(\boldsymbol{c},M)T^{1/p}\,\|\gamma - \widetilde{\gamma}\|_{C([0,T];C^3([0,1];(\mathbb{R}^2)^3))} \leq C(\boldsymbol{c},M)T^{1/p}\,|||\gamma - \widetilde{\gamma}|||_{\mathbb{E}_T}\,.
\end{aligned}
$$

The claim now follows with $\sigma = \min\{\alpha, 1/p\}$ and an appropriate constant $C(\boldsymbol{c},M)$. □

Proposition 4.27 (Contraction Estimates II)**.** *Let $p \in (5,\infty)$, φ be an analytically p–admissible initial value with $\boldsymbol{c}$ as in* (4.5)*, $T_0 := 1$, $M > 0$ and $\widetilde{T}(\boldsymbol{c},M)$ be as in Lemma 4.23. There exists a constant $\sigma \in (0,1)$ and a constant $C > 0$ depending on $\boldsymbol{c}$, $\|\varphi\|_{X_0}$ and M such that for all $T \in \left(0,\widetilde{T}(\boldsymbol{c},M)\right]$, $\gamma, \widetilde{\gamma} \in \mathbb{E}^{\varphi}_T \cap \overline{B_{M,T}}$ it holds*

$$
|||L^2_T(\gamma) - N^2_T(\gamma) - \left(L^2_T(\widetilde{\gamma}) - N^2_T(\widetilde{\gamma})\right)|||_{Y_{1,T}} \leq C\left(\boldsymbol{c}, \|\varphi\|_{X_0}, M\right) T^{\sigma}\,|||\gamma - \widetilde{\gamma}|||_{\mathbb{E}_T}\,.
$$

Proof. Let $T \in \left(0,\widetilde{T}(\boldsymbol{c},M)\right]$, and $\gamma, \widetilde{\gamma} \in \mathbb{E}^{\varphi}_T \cap \overline{B_{M,T}}$ be fixed. We note that $\gamma_{|t=0} = \widetilde{\gamma}_{|t=0} = \varphi$ yields

$$
L^2_T(\gamma)_{|t=0} - N^2_T(\gamma)_{|t=0} - \left(L^2_T(\widetilde{\gamma})_{|t=0} - N^2_T(\widetilde{\gamma})_{|t=0}\right) = N^2_T(\widetilde{\gamma})_{|t=0} - N^2_T(\gamma)_{|t=0} = 0\,.
$$

Thus it is enough to estimate the term in the usual sub-multiplicative norm $\|\cdot\|_{Y_{1,T}}$ on $Y_{1,T}$. To improve readability we omit in the following the evaluation of the terms in $x = 0$. Furthermore, we let $b(\gamma) := \frac{1}{2}L^2_T(\gamma) - \frac{1}{2}N^2_T(\gamma)$, $\beta := 1/4 - 1/4p$ and ν^i, τ^i, and $\tilde{\nu}^i, \tilde{\tau}^i$ be the unit normal and tangent of γ^i and $\widetilde{\gamma}^i$, respectively. The unit normal and tangent of φ are denoted by ν^i_0 and τ^i_0, respectively. Adding and subtracting the terms

$$
\begin{aligned}
&\sum_{i=1}^{3}\frac{1}{|\varphi^i_x|^3}\langle\widetilde{\gamma}^i_{xxx} - \gamma^i_{xxx}, \nu^i\rangle\nu^i_0 + \sum_{i=1}^{3}\frac{1}{|\varphi^i_x|^3}\langle\widetilde{\gamma}^i_{xxx} - \gamma^i_{xxx}, \nu^i\rangle\nu^i + \sum_{i=1}^{3}\frac{1}{|\gamma^i_x|^3}\left\langle\widetilde{\gamma}^i_{xxx}, \nu^i\right\rangle\nu^i\\
&+\sum_{i=1}^{3}\frac{1}{|\gamma^i_x|^3}\left\langle\widetilde{\gamma}^i_{xxx}, \tilde{\nu}^i\right\rangle\nu^i + \sum_{i=1}^{3}\frac{1}{|\gamma^i_x|^3}\left\langle\widetilde{\gamma}^i_{xxx}, \tilde{\nu}^i\right\rangle\tilde{\nu}^i
\end{aligned}
$$

to the expression

$$
\begin{aligned}
b(\widetilde{\gamma}) - b(\gamma) = &\sum_{i=1}^{3}\frac{1}{|\varphi^i_x|^3}\left\langle(\widetilde{\gamma}^i_{xxx} - \gamma^i_{xxx}), \nu^i_0\right\rangle\nu^i_0 + \sum_{i=1}^{3}\frac{1}{|\gamma^i_x|^3}\left\langle\gamma^i_{xxx}, \nu^i\right\rangle\nu^i\\
&-\sum_{i=1}^{3}\frac{1}{|\widetilde{\gamma}^i_x|^3}\left\langle\widetilde{\gamma}^i_{xxx}, \tilde{\nu}^i\right\rangle\tilde{\nu}^i - \frac{\mu}{2}\sum_{i=1}^{3}\left(\frac{\gamma^i_x}{|\gamma^i_x|} - \frac{\widetilde{\gamma}^i_x}{|\widetilde{\gamma}^i_x|}\right)
\end{aligned}
\tag{4.23}
$$

we obtain

$$
\sum_{i=1}^{3}\frac{1}{|\gamma^i_x|^3}\left\langle\widetilde{\gamma}^i_{xxx}, \nu^i - \tilde{\nu}^i\right\rangle\nu^i + \sum_{i=1}^{3}\frac{1}{|\gamma^i_x|^3}\left\langle\widetilde{\gamma}^i_{xxx}, \tilde{\nu}^i\right\rangle\left(\nu^i - \tilde{\nu}^i\right) \tag{4.24}
$$

$$
+\sum_{i=1}^{3}\frac{1}{|\varphi^i_x|^3}\left\langle(\widetilde{\gamma}^i_{xxx} - \gamma^i_{xxx}), \nu^i_0 - \nu^i\right\rangle\nu^i_0 + \sum_{i=1}^{3}\frac{1}{|\varphi^i_x|^3}\left\langle(\widetilde{\gamma}^i_{xxx} - \gamma^i_{xxx}), \nu^i\right\rangle\left(\nu^i_0 - \nu^i\right) \tag{4.25}
$$

$$+\sum_{i=1}^{3}\left(\frac{1}{|\gamma_x^i|^3}-\frac{1}{|\widetilde{\gamma}_x^i|^3}\right)\langle\widetilde{\gamma}_{xxx}^i,\widetilde{\nu}^i\rangle\,\widetilde{\nu}^i+\sum_{i=1}^{3}\left(\frac{1}{|\varphi_x^i|^3}-\frac{1}{|\gamma_x^i|^3}\right)\langle\widetilde{\gamma}_{xxx}^i-\gamma_{xxx}^i,\nu^i\rangle\,\nu^i \tag{4.26}$$

$$-\frac{\mu}{2}\sum_{i=1}^{3}\left(\frac{\gamma_x^i}{|\gamma_x^i|}-\frac{\widetilde{\gamma}_x^i}{|\widetilde{\gamma}_x^i|}\right). \tag{4.27}$$

Due to $p\in(5,\infty)$ Proposition B.24 implies that the space $W_p^{\beta}((0,T))$ is a Banach algebra which allows us to consider the terms in (4.24) to (4.27) individually. Theorem C.30 implies that there exists a constant $C>0$ independent of T such that

$$\left\|\gamma_{xxx}^i(0)-\widetilde{\gamma}_{xxx}^i(0)\right\|_{W_p^{\beta}((0,T);\mathbb{R}^2)}\le C\,\|\!|\gamma^i-\widetilde{\gamma}^i|\!\|_{W_p^{1,4}((0,T)\times(0,1);\mathbb{R}^2)}\le C\,\|\!|\gamma-\widetilde{\gamma}|\!\|_{\mathbb{E}_T}$$

and further

$$\left\|\gamma_{xxx}^i(0)\right\|_{W_p^{\beta}((0,T);\mathbb{R}^2)}\le C\,\|\!|\gamma|\!\|_{\mathbb{E}_T}\le C(M)$$

and analogously for $\widetilde{\gamma}$. As in the proof of Proposition 4.24 we choose $\theta\in\left(\frac{1/p+1}{4-4/p},1\right)$ and $\delta\in(0,1-1/p)$ such that $\alpha:=(1-\theta)\,(1-1/p-\delta)>\beta=1/4-1/4p$. Then Corollary B.39 yields that $\gamma_x(0)$, $\widetilde{\gamma}_x(0)$ lie in $C^{\alpha}\left([0,T];(\mathbb{R}^2)^3\right)$ and

$$\|\gamma_x(0)\|_{C^{\alpha}([0,T];(\mathbb{R}^2)^3)}\le C\,\|\!|\gamma|\!\|_{\mathbb{E}_T}$$

and analogously for $\widetilde{\gamma}_x(0)$ and $\gamma_x(0)-\widetilde{\gamma}_x(0)$. The estimate (4.16) allows us to apply Proposition B.4 to obtain for $i\in\{1,2,3\}$, that $\left|\gamma_x^i(0)\right|^{-1}$ and $\left|\widetilde{\gamma}_x^i(0)\right|^{-1}$ lie in $C^{\alpha}([0,T])$ with norms bounded by a constant $C(\boldsymbol{c},M)$ and further

$$\left\|\left|\gamma_x^i(0)\right|^{-1}-\left|\widetilde{\gamma}_x^i(0)\right|^{-1}\right\|_{C^{\alpha}([0,T])}\le C(\boldsymbol{c},M)\left\|\gamma_x^i(0)-\widetilde{\gamma}_x^i(0)\right\|_{C^{\alpha}([0,T])}\le C(\boldsymbol{c},M)\,\|\!|\gamma-\widetilde{\gamma}|\!\|_{\mathbb{E}_T}.$$

Using for $a,b\in\mathbb{R}^2$ the identity

$$\left(\frac{1}{|a|^3}-\frac{1}{|b|^3}\right)=\left(\frac{1}{|a|}-\frac{1}{|b|}\right)\left(\frac{1}{|a|^2}+\frac{1}{|a||b|}+\frac{1}{|b|^2}\right)$$

and the Banach algebra property of $C^{\alpha}([0,T])$ we obtain

$$\left\|\left|\gamma_x^i(0)\right|^{-3}-\left|\widetilde{\gamma}_x^i(0)\right|^{-3}\right\|_{C^{\alpha}([0,T])}\le C(\boldsymbol{c},M)\,\|\!|\gamma-\widetilde{\gamma}|\!\|_{\mathbb{E}_T}.$$

Denoting by $R\in\mathbb{R}^{2\times2}$ the counter-clockwise rotation by the angle $\frac{\pi}{2}$, we find $\nu^i=R\tau^i=R\left(\frac{\gamma_x^i}{|\gamma_x^i|}\right)$. As Hölder spaces are stable under products, we find that $\nu^i(0)$ and $\widetilde{\nu}^i(0)$ lie in $C^{\alpha}\left([0,T];\mathbb{R}^2\right)$ with norms bounded by a constant $C(\boldsymbol{c},M)$ and further

$$\begin{aligned}\left\|\nu^i(0)-\widetilde{\nu}^i(0)\right\|_{C^{\alpha}([0,T];\mathbb{R}^2)}&\le\left\|\gamma_x^i(0)\right\|_{C^{\alpha}([0,T];\mathbb{R}^2)}\left\|\left|\gamma_x^i(0)\right|^{-1}-\left|\widetilde{\gamma}_x^i(0)\right|^{-1}\right\|_{C^{\alpha}([0,T])}\\&\quad+\left\|\left|\widetilde{\gamma}_x^i(0)\right|^{-1}\right\|_{C^{\alpha}([0,T])}\left\|\gamma_x^i(0)-\widetilde{\gamma}_x^i(0)\right\|_{C^{\alpha}([0,T];\mathbb{R}^2)}\\&\le C(\boldsymbol{c},M)\,\|\!|\gamma-\widetilde{\gamma}|\!\|_{\mathbb{E}_T}.\end{aligned}$$

Combining these estimates with Proposition C.32 and the Banach algebra properties of $C^{\alpha}([0,T])$ and $W_p^{\beta}((0,T))$ we obtain that there exist constants $\sigma\in(0,1)$ and $C\left(\boldsymbol{c},\|\varphi\|_{X_0},M\right)>0$ such that all terms in (4.24) to (4.27) can be estimated by

$$C\left(\boldsymbol{c},\|\varphi\|_{X_0},M\right)T^{\sigma}\,\|\!|\gamma-\widetilde{\gamma}|\!\|_{\mathbb{E}_T}.$$

□

Corollary 4.28. *Let $p \in (5,\infty)$, φ be an analytically p–admissible initial value with $\boldsymbol{c}$ as in* (4.5)*, $T_0 := 1$, $M > 0$ and $\widetilde{T}(\boldsymbol{c}, M)$ be as in Lemma 4.23. There exists a time $T_*\left(\boldsymbol{c}, \|\varphi\|_{X_0}, M\right) \in \left(0, \widetilde{T}(\boldsymbol{c}, M)\right]$ such that for every $T \in \left(0, T_*\left(\boldsymbol{c}, \|\varphi\|_{X_0}, M\right)\right]$ the operator*

$$K_T : \left(\mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}, \, |||\cdot|||_{\mathbb{E}_T}\right) \to \left(\mathbb{E}_T^{\varphi}, \, |||\cdot|||_{\mathbb{E}_T}\right)$$

is a contraction.

Proof. Proposition 4.25 yields that K_T is well-defined. Moreover, Theorem 4.21 with $T_0 = 1$ and Proposition 4.26 and 4.27 imply that there exist constants $C\left(\boldsymbol{c}, \|\varphi\|_{X_0}, M\right) > 0$ and $\sigma > 0$ such that for all $T \in \left(0, \widetilde{T}(\boldsymbol{c}, M)\right]$, $\gamma, \widetilde{\gamma} \in \mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$,

$$\begin{aligned}
&|||K_T(\gamma) - K_T(\widetilde{\gamma})|||_{\mathbb{E}_T} \\
&\leq C \left(\left\| L_T^1(\gamma) - N_T^1(\gamma) - \left(L_T^1(\widetilde{\gamma}) - N_T^1(\widetilde{\gamma})\right) \right\|_{X_T} + |||L_T^2(\gamma) - N_T^2(\gamma) - \left(L_T^2(\widetilde{\gamma}) - N_T^2(\widetilde{\gamma})\right)|||_{Y_{1,T}} \right) \\
&\leq C\left(\boldsymbol{c}, \|\varphi\|_{X_0}, M\right) T^{\sigma} |||\gamma - \widetilde{\gamma}|||_{\mathbb{E}_T}.
\end{aligned}$$

Now choose $T_* = T_*\left(\boldsymbol{c}, \|\varphi\|_{X_0}, M\right) \in \left(0, \widetilde{T}(\boldsymbol{c}, M)\right]$ such that $C\left(\boldsymbol{c}, \|\varphi\|_{X_0}, M\right) T_*^{\sigma} < 1$. □

Proposition 4.29. *Let $p \in (5,\infty)$ and φ be an analytically p–admissible initial value with $\boldsymbol{c}$ as in* (4.5)*. There exists a positive radius $M(\varphi)$ and a positive time $\boldsymbol{T}(\varphi)$ both depending on $\boldsymbol{c}$ and $\|\varphi\|_{X_0}$ such that for all $T \in (0, \boldsymbol{T}(\varphi)]$ the set $\mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$ is non-empty and the operator*

$$K_T : \left(\mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}, \, |||\cdot|||_{\mathbb{E}_T}\right) \to \left(\mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}, \, |||\cdot|||_{\mathbb{E}_T}\right)$$

is a contractive self-mapping.

Proof. Let $T_0 = 1$. Proposition 4.22 implies that there exists $\mathcal{E}\varphi \in \mathbb{E}_1^{\varphi}$ with $|||\mathcal{E}\varphi|||_{\mathbb{E}_1} \leq C_1 \|\varphi\|_{X_0}$ for a constant $C_1 > 0$ and Corollary B.39 yields for fixed $\theta \in \left(\frac{1+1/p}{4-4/p}, 1\right)$, $\delta \in (0, 1 - 1/p)$ with $\alpha := (1-\theta)(1 - 1/p - \delta)$ that $\mathcal{E}\varphi$ lies in $C^{\alpha}\left([0,1]; C^1\left([0,1]; (\mathbb{R}^2)^3\right)\right)$ with

$$\|\mathcal{E}\varphi\|_{C^{\alpha}([0,1];C^1([0,1];(\mathbb{R}^2)^3))} \leq C_2 \, |||\mathcal{E}\varphi|||_{\mathbb{E}_1} \leq C_3 \|\varphi\|_{X_0}$$

for a constant $C_3 > 0$. We let $\widehat{T} = \widehat{T}\left(\boldsymbol{c}, \|\varphi\|_{X_0}\right) \in (0,1]$ be so small that $\widehat{T}^{\alpha} C_3 \|\varphi\|_{X_0} \leq c/2$. Then for all $t \in [0, \widehat{T}]$, $i \in \{1,2,3\}$, the curve $(\mathcal{E}\varphi)^i(t)$ is regular and $\mathcal{E}\varphi$ lies in $\mathbb{E}_{\hat{T}}^{\varphi}$ for all $T \in (0, \widehat{T}]$, where we do not distinguish between $\mathcal{E}\varphi$ and $\mathcal{E}\varphi_{|[0,T]}$. In particular, for all $T \in (0, \widehat{T}]$, $N_T(\mathcal{E}\varphi) \in X_T \times Y_{1,T}$ is well-defined and $N_{\widehat{T}}(\mathcal{E}\varphi)_{|[0,T]} = N_T(\mathcal{E}\varphi)$. This yields for all $T \in (0, \widehat{T}]$ that

$$K_T(\mathcal{E}\varphi) = L_T^{-1}\left(\left(L_T^1(\mathcal{E}\varphi) - N_T^1(\mathcal{E}\varphi), L_T^2(\mathcal{E}\varphi) - N_T^2(\mathcal{E}\varphi), \varphi(1), \varphi\right)\right)$$

lies in $\mathbb{E}_T$ and satisfies $K_T(\mathcal{E}\varphi) = K_{\hat{T}}(\mathcal{E}\varphi)_{|[0,T]}$. Let $M := 2\max\left\{ |||\mathcal{E}\varphi|||_{\mathbb{E}_{\hat{T}}}, |||K_{\hat{T}}(\mathcal{E}\varphi)|||_{\mathbb{E}_{\hat{T}}} \right\}$. Then for all $T \in (0, \widehat{T}]$, $\mathcal{E}\varphi$ lies in $\mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$ and $|||K_T(\mathcal{E}\varphi)|||_{\mathbb{E}_T} \leq M/2$. Let $T_*\left(\boldsymbol{c}, \|\varphi\|_{X_0}, M\right)$ be the corresponding time in Corollary 4.28. Given $T \in \left(0, \min\left\{\widehat{T}, T_*\left(\boldsymbol{c}, \|\varphi\|_{X_0}, M\right)\right\}\right]$ and $\gamma \in \mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$ we have for some $\sigma \in (0,1)$,

$$|||K_T(\gamma) - K_T(\mathcal{E}\varphi)|||_{\mathbb{E}_T} \leq C\left(\boldsymbol{c}, \|\varphi\|_{X_0}, M\right) T^{\sigma} |||\gamma - \mathcal{E}\varphi|||_{\mathbb{E}_T} \leq C\left(\boldsymbol{c}, \|\varphi\|_{X_0}, M\right) T^{\sigma} 2M. \quad (4.28)$$

We now choose $\boldsymbol{T}\left(\boldsymbol{c}, \|\varphi\|_{X_0}\right) \in \left(0, \min\left\{\widehat{T}, T_*\left(\boldsymbol{c}, \|\varphi\|_{X_0}, M\right)\right\}\right]$ so small that

$$C\left(\boldsymbol{c}, \|\varphi\|_{X_0}, M\right) \boldsymbol{T}\left(\boldsymbol{c}, \|\varphi\|_{X_0}\right)^{\sigma} 2M \leq M/2.$$

As $\boldsymbol{T}\left(\boldsymbol{c}, \|\varphi\|_{X_0}\right) \leq T_*\left(\boldsymbol{c}, \|\varphi\|_{X_0}, M\right)$, Corollary 4.28 yields that for all $T \in \left(0, \boldsymbol{T}\left(\boldsymbol{c}, \|\varphi\|_{X_0}\right)\right]$,

$$K_T : \left(\mathbb{E}_T^\varphi \cap \overline{B_{M,T}}, \, \|\!|\cdot|\!\|_{\mathbb{E}_T}\right) \to \left(\mathbb{E}_T^\varphi, \, \|\!|\cdot|\!\|_{\mathbb{E}_T}\right)$$

is contractive and given $\gamma \in \mathbb{E}_T^\varphi \cap \overline{B_{M,T}}$ the estimate (4.28) yields

$$\|\!|K_T(\gamma)|\!\|_{\mathbb{E}_T} \leq \|\!|K_T\left(\mathcal{E}\varphi\right)|\!\|_{\mathbb{E}_T} + \|\!|K_T(\gamma) - K_T\left(\mathcal{E}\varphi\right)|\!\|_{\mathbb{E}_T} \leq {}^{M}\!/2 + {}^{M}\!/2 \leq M\,.$$

□

Theorem 4.30 (Existence and uniqueness of strong solutions to the analytic problem (Triods))**.** *Let $p \in (5, \infty)$ and φ be an analytically p–admissible initial value to system* (4.3) *with given endpoints P^1, P^2, P^3. There exists a positive time $\boldsymbol{T}$ depending on $\|\varphi\|_{X_0}$ and $\boldsymbol{c}(\varphi)$ such that for all $T \in (0, \boldsymbol{T}]$ there exists a strong solution to system* (4.3) *in* $[0, T]$ *with initial datum φ which is unique in $\boldsymbol{E}_T \cap \overline{B_{M,T}}$ with M as in the proof of Proposition 4.29.*

Proof. Let $M(\varphi)$ and $\boldsymbol{T}(\varphi)$ be the radius and time as in Proposition 4.29 and let $T \in (0, \boldsymbol{T}]$ be given. By the Banach Fixed-Point Theorem [151, Theorem 1.A], the operator K_T possesses a unique fixed point $\gamma \in \mathbb{E}_T^\varphi \cap \overline{B_{M,T}}$ which is by construction a strong solution to the system (4.3) in $[0, T]$ with endpoints P^1, P^2, P^3 and initial datum φ. Indeed, we emphasise that Lemma 4.16 yields that for all $t \in [0, \boldsymbol{T}]$, $i \in \{1, 2, 3\}$, the curve $\gamma^i(t)$ is regular and satisfies the non-degeneracy condition in $y = 0$. If $\widetilde{\gamma}$ is another strong solution to (4.3) in $[0, T]$ with endpoints P^1, P^2, P^3 and initial datum φ satisfying $\|\!|\widetilde{\gamma}|\!\|_{\boldsymbol{E}_T} \leq M$ then $\widetilde{\gamma}$ is a fixed point of K_T in $\mathbb{E}_T^\varphi \cap \overline{B_{M,T}}$ and thus equal to γ. □

Remark 4.31. The dependence of the existence time $\boldsymbol{T} = \boldsymbol{T}(\varphi)$ in Theorem 4.30 on the initial datum φ via $\|\varphi\|_{X_0}$ and $\boldsymbol{c}(\varphi)$ is expressed in the following relation. As $\|\varphi\|_{X_0} \to \infty$ and/or $\boldsymbol{c}(\varphi) \to 0$ there holds $\boldsymbol{T}(\varphi) \to 0$.

We obtain the corresponding result in the case of Theta networks.

Theorem 4.32 (Existence and uniqueness of strong solutions to the analytic problem (Theta))**.** *Let $p \in (5, \infty)$ and φ be an analytically p–admissible initial value to system* (4.2)*. There exists a positive time $\boldsymbol{T}$ depending on $\|\varphi\|_{X_0}$ and $\boldsymbol{c}(\varphi)$ such that for all $T \in (0, \boldsymbol{T}]$ there exists a strong solution to system* (4.2) *in* $[0, T]$ *with initial datum φ which is unique in $\boldsymbol{E}_T \cap \overline{B_{M,T}}$ with M defined accordingly as in the proof of Proposition 4.29.*

Proof. This follows from a suitable adaptation of the arguments used for the Triods as explained in Remark 4.9. □

To prove a refinement of the above theorem we introduce the following notation.

Definition 4.33. Given $p \in (5, \infty)$ and $\eta \in X_0 = W_p^{4-4/p}\left((0,1); (\mathbb{R}^2)^3\right)$ we let

$$|\eta|_{X_0} := \sum_{j=1}^{3} \left\|\partial_x^j \eta\right\|_{L_p((0,1);(\mathbb{R}^2)^3)} + \left[\partial_x^3 \eta\right]_{W_p^{1-4/p}((0,1);(\mathbb{R}^2)^3)}\,.$$

In the following we show that the existence time in Theorem 4.32 depends on $\|\varphi\|_{X_0}$ only via $|\varphi|_{X_0}$. This is due to the translational invariance of the problem (4.2) for the Theta network.

Theorem 4.34 (A refinement of Theorem 4.32)**.** *Let $p \in (5, \infty)$ and φ be an analytically p–admissible initial value to system* (4.2)*. There exists a positive time $\boldsymbol{T}$ depending on $|\varphi|_{X_0}$ and*

$\boldsymbol{c}(\varphi)$ such that for all $T \in (0, \boldsymbol{T}]$ there exists a strong solution to system (4.2) in $[0,T]$ with initial datum φ which is unique in $\boldsymbol{E}_T \cap \overline{B_{M,T}}$ where M is defined accordingly as in the proof of Proposition 4.29.

Proof. Let $p \in (5, \infty)$, φ be an analytically p–admissible initial value to system (4.2) and $v := -\varphi^1(0)$. We let $(\varphi^v)^i : [0,1] \to \mathbb{R}^2$ be defined by $(\varphi^v)^i(x) := \varphi^i(x) + v$ and observe that $\varphi^v := \big((\varphi^v)^1, (\varphi^v)^2, (\varphi^v)^3\big)$ defines an analytically p–admissible initial value to system (4.2). By Theorem 4.32 there exists a time $\boldsymbol{T}(\varphi^v)$ depending on $\boldsymbol{c}(\varphi^v)$, $|\varphi^v|_{X_0}$ and $\|\varphi^v\|_{L_p((0,1);(\mathbb{R}^2)^3)}$ such that for all $T \in (0, \boldsymbol{T}(\varphi^v)]$ there exists a strong solution to (4.2) in $[0,T]$ with initial datum φ^v. As φ^v satisfies $\varphi^v(0) = (0,0,0)$, we have for all $x \in [0,1]$,

$$|\varphi^v(x)| \leq \int_0^x |\varphi^v_x(y)| \,\mathrm{d}y \leq \int_0^1 |\varphi^v_x(y)| \,\mathrm{d}y \leq \|\varphi^v_x\|_{L_p((0,1);(\mathbb{R}^2)^3)}$$

which yields in particular

$$\|\varphi^v\|_{L_p((0,1);(\mathbb{R}^2)^3)} \leq \|\varphi^v_x\|_{L_p((0,1);(\mathbb{R}^2)^3)} \,.$$

This shows that the existence time $\boldsymbol{T}(\varphi^v)$ only depends on $|\varphi^v|_{X_0} = |\varphi|_{X_0}$ and $\boldsymbol{c}(\varphi^v) = \boldsymbol{c}(\varphi)$. Given $T \in (0, \boldsymbol{T}(\varphi^v)]$ and a strong solution η to (4.2) in $[0,T]$ with initial datum φ^v, the function $\eta^{-v} := \big((\eta^{-v})^1, (\eta^{-v})^2, (\eta^{-v})^3\big)$ with $(\eta^{-v})^i(t,x) := \eta^i(t,x) - v$ for $t \in [0,T]$, $x \in [0,1]$, is a strong solution to (4.2) in $[0,T]$ with initial datum φ. Combined with the above arguments this yields the claim. □

4.1.2 Parabolic regularisation

In this subsection, following [65, Section 4], we show that every strong solution to (4.2) or (4.3), respectively, is smooth for positive times. To this end we use the classical theory in [136] on solutions to linear parabolic systems in *parabolic Hölder spaces*, see Definition B.7. Again, all arguments are performed exemplarily for system (4.3).

Lemma 4.35. *Let $p \in (5, \infty)$ and T be positive. There exists $\alpha \in (0,1)$ such that for all $u \in \boldsymbol{E}_T$,*

$$u\,, u_x\,, u_{xx} \in C^{\frac{\alpha}{4},\alpha}\left([0,T] \times [0,1]; (\mathbb{R}^2)^3\right)$$

and

$$t \mapsto u_x(t,0) \in C^{\frac{1+\alpha}{4}}\left([0,T]; (\mathbb{R}^2)^3\right).$$

Proof. Proposition B.35 yields

$$\boldsymbol{E}_T \hookrightarrow C\big([0,T]; W_p^{4-4/p}\left((0,1); (\mathbb{R}^2)^3\right)\big)$$

and by Theorem B.25 we have for all $\delta \in (0, 1 - 1/p)$,

$$\boldsymbol{E}_T \hookrightarrow C^{1-1/p-\delta}\left([0,T]; L_p\left((0,1); (\mathbb{R}^2)^3\right)\right).$$

Thus Proposition B.1 gives for all $\delta \in (0, 1 - 1/p)$ and all $\theta \in (0,1)$ the continuous embedding

$$\boldsymbol{E}_T \hookrightarrow C^{(1-1/p-\delta)(1-\theta)}\big([0,T]; W_p^{\theta(4-4/p)}\left((0,1); (\mathbb{R}^2)^3\right)\big). \tag{4.29}$$

In particular, Theorem B.20 yields for all $\delta \in (0, 1 - 1/p)$ and all $\theta \in \left(\frac{1+1/p}{4-4/p}, 1\right)$ the continuous embedding

$$\boldsymbol{E}_T \hookrightarrow C^{(1-1/p-\delta)(1-\theta)}\left([0,T]; C^1\left([0,1]; (\mathbb{R}^2)^3\right)\right).$$

As $p \in (5,\infty)$ there exists $\delta^* \in (0, 1 - 1/p)$ such that the interval $\left(\frac{1+1/p}{4-4/p}, \frac{1-2/p-2\delta^*}{2-2/p-2\delta^*}\right)$ is non-empty and all $\theta \in \left(\frac{1+1/p}{4-4/p}, \frac{1-2/p-2\delta^*}{2-2/p-2\delta^*}\right)$ satisfy $(1 - 1/p - \delta^*)(1-\theta) > \frac{1}{2}$. In particular, we obtain for all $u \in \boldsymbol{E}_T$ and all $\alpha \in (0,1)$,

$$u_x \in C^{\frac{1+\alpha}{4}}\left([0,T]; C\left([0,1]; (\mathbb{R}^2)^3\right)\right).$$

The embedding (4.29) and Theorem B.20 imply for all $\delta \in (0, 1 - 1/p)$ and all $\theta \in \left(\frac{2+1/p}{4-4/p}, 1\right)$ the continuous embedding

$$\boldsymbol{E}_T \hookrightarrow C^{(1-1/p-\delta)(1-\theta)}\left([0,T]; C^2\left([0,1]; (\mathbb{R}^2)^3\right)\right).$$

Due to $p \in (5,\infty)$ we may choose $\delta_* \in (0, 1 - 1/p)$ such that $\left(\frac{2+1/p}{4-4/p}, \frac{3-4/p-4\delta_*}{4-4/p-4\delta_*}\right)$ is non-empty. Since all $\theta \in \left(\frac{2+1/p}{4-4/p}, \frac{3-4/p-4\delta_*}{4-4/p-4\delta_*}\right)$ satisfy $(1 - 1/p - \delta_*)(1-\theta) > \frac{1}{4}$, we obtain for all $\alpha \in (0,1)$,

$$u\,, u_x\,, u_{xx} \in C^{\frac{\alpha}{4}}\left([0,T]; C\left([0,1]; (\mathbb{R}^2)^3\right)\right).$$

Finally, Proposition B.35 yields for all $\alpha \in (0,1)$,

$$u\,, u_x\,, u_{xx} \in C\left([0,T]; C^1\left([0,1]; (\mathbb{R}^2)^3\right)\right) \hookrightarrow C\left([0,T]; C^{\alpha}\left([0,1]; (\mathbb{R}^2)^3\right)\right).$$

□

Proposition 4.36 (Hölder regularity of strong solutions)**.** *Let $p \in (5,\infty)$, T be positive and φ an analytically p–admissible initial network to* (4.3) *with endpoints P^1, P^2, P^3. Suppose that $\gamma \in \boldsymbol{E}_T$ is a strong solution to the analytic problem* (4.3) *in $[0,T]$ with endpoints P^1, P^2, P^3 and initial datum φ. Then there exists $\alpha \in (0,1)$ such that for all $\varepsilon \in (0,T)$,*

$$\gamma \in C^{\frac{4+\alpha}{4}, 4+\alpha}\left([\varepsilon, T] \times [0,1]; (\mathbb{R}^2)^3\right).$$

Proof. Let $\varepsilon \in (0,T)$ be fixed and $\eta \in C_0^\infty((0,\infty);\mathbb{R})$ with $\eta \equiv 1$ on $[\varepsilon, T]$. Corollary B.32 yields that the function $g = \left(g^1, g^2, g^3\right)$ defined by $g^i = \eta\gamma^i$, $i \in \{1,2,3\}$, lies in $\boldsymbol{E}_T$ and satisfies a parabolic boundary value problem of the following form: for $t \in (0,T)$, $x \in (0,1)$, $y \in \{0,1\}$ and $i \in \{1,2,3\}$,

$$\begin{cases} g_t^i(t,x) + \frac{2}{|\gamma_x^i(t,x)|^4} g_{xxxx}^i(t,x) + f\left(\gamma_x^i, \gamma_{xx}^i, g_x^i, g_{xx}^i, g_{xxx}^i\right)(t,x) &= \eta'(t)\gamma^i(t,x)\,, \\ g^1(t,0) - g^2(t,0) &= 0\,, \\ g^1(t,0) - g^3(t,0) &= 0\,, \\ g^i(t,1) &= \eta(t)\varphi^i(1)\,, \\ g_{xx}^i(t,y) &= 0\,, \\ \sum_{i=1}^3 \frac{2}{|\gamma_x^i(t,0)|^3}\left\langle g_{xxx}^i(t,0), \nu^i(t,0)\right\rangle \nu^i(t,0) - \frac{\mu^i}{|\gamma_x^i(t,0)|} g_x^i(t,0) &= 0\,, \\ g^i(0,x) &= 0 \end{cases} \tag{4.30}$$

where the evolution equation is valid almost everywhere in $(0,T) \times (0,1)$ with lower order terms given by

$$\begin{aligned} f\left(\gamma_x^i, \gamma_{xx}^i, g_x^i, g_{xx}^i, g_{xxx}^i\right)(t,x) = &- 12\frac{\left\langle \gamma_{xx}^i, \gamma_x^i\right\rangle}{|\gamma_x^i|^6} g_{xxx}^i - 8\frac{\gamma_{xx}^i}{|\gamma_x^i|^6}\left\langle g_{xxx}^i, \gamma_x^i\right\rangle \\ &- \left(5\frac{|\gamma_{xx}^i|^2}{|\gamma_x^i|^6} - 35\frac{\left\langle \gamma_{xx}^i, \gamma_x^i\right\rangle^2}{|\gamma_x^i|^8} + \mu\frac{1}{|\gamma_x^i|^2}\right) g_{xx}^i\,. \end{aligned}$$

Due to the embedding $\boldsymbol{E}_T \hookrightarrow C\left([0,T]; C^3\left([0,1];(\mathbb{R}^2)^3\right)\right)$ the boundary conditions are well-defined and hold for every $t\in[0,T]$. With $R:\mathbb{R}^2\to\mathbb{R}^2$ denoting the rotation by the angle $\frac{\pi}{2}$ we have $\nu^i(t,x)=\left|\gamma^i_x(t,x)\right|^{-1}R\left(\gamma^i_x(t,x)\right)$.
The boundary value problem (4.30) is linear in the components of g and the *principal parts* of the differential operators both in the interior and at the boundary are of precisely the same structure as the ones of the linearised problem associated to $L_T\in\mathcal{L}\left(\mathbb{E}_T,\mathbb{F}_T\right)$ studied in Subsection 4.1.1. The only difference is that the coefficients of the principal parts of the evolution equation and the third order condition in (4.30) are now time-dependent, namely given in terms of $\gamma(t,x)$ instead of $\varphi(x)=\gamma(0,x)$ as in the case of L_T.
We replace φ and ν_0 by $\gamma(t)$ and $\nu(t)$ in the definitions of $l_{jj}(t,x,\partial_t,\partial_x)$ and $B_{q,j}\left(t,0,\partial_t,\partial_x\right)$ for $q\in\{1,2\}$ and $j\in\{1,\dots,6\}$, respectively, and add the lower order terms appearing in (4.30). As the structure of the principal parts is unchanged, we may consider precisely the same values of s_k, t_j, σ_q and ρ_α as in Subsection 4.1.1 to apply the theory in [136]. Given $\xi\in\mathbb{R}$, the roots of the polynomial $\det\left(\left(l_{kj}(t,x,p,i\xi)\right)\right)$ with respect to p given by $p_i=-\frac{2}{|\gamma^i_x(t,x)|^4}\xi^4$, $i\in\{1,2,3\}$, satisfy

$$\max_{t\in[0,T],x\in[0,1]} -\frac{2}{\left|\gamma^i_x(t,x)\right|^4}\xi^4\leq -\frac{2}{C^4}\xi^4$$

with $C=\|\gamma\|_{C([0,T];C^1([0,1];(\mathbb{R}^2)^3))}$. This yields that the system is uniformly parabolic. The complementary conditions for the boundary operators at $(t,y)\in[0,T]\times\{0,1\}$ follow from the validity of the Lopatinskii–Shapiro condition in $y\in\{0,1\}$ shown in Propositions 4.18 and 4.19 by replacing $\varphi^i_x(y)$ and $\nu^i_0(y)$ by $\gamma^i_x(t,y)$ and $\nu^i(t,y)$ respectively. In particular, the coefficients of the considered ordinary differential equation remain constant and all arguments can be performed in the same way. In the case $y=0$ one uses that $\gamma(t)$ satisfies the non-degeneracy condition in $y=0$ for every $t\in[0,T]$.
The complementary condition for the initial value is trivially fulfilled.

Let $\alpha\in(0,1)$ be given. In order to apply the existence result [136, Theorem 4.9] in Hölder spaces with $l=4+\alpha$ it remains to verify the conditions on the coefficients and the right hand side. Lemma 3.45 yields for all $i\in\{1,2,3\}$,

$$\min_{t\in[0,T],x\in[0,1]}\left|\gamma^i_x(t,x)\right|\geq \boldsymbol{c}(\gamma)>0\,. \tag{4.31}$$

The Banach algebra structure of Hölder spaces, Lemma 4.35, Proposition B.4, Proposition B.5 and (4.31) then imply that the coefficients in the motion equation and the third order condition are of regularity $C^{\frac{\alpha}{4},\alpha}([0,T]\times[0,1])$ and $C^{\frac{1+\alpha}{4}}([0,T])$, respectively.
The regularity requirements on the right hand side follow from Lemma 4.35.
As $\eta(0)=\eta'(0)=0$ the compatibility conditions of order four as stated in [136, (4.19)] are fulfilled. By [136, Theorem 4.9] there exists a unique solution $\widetilde{g}\in C^{\frac{4+\alpha}{4},4+\alpha}\left([0,T]\times[0,1];(\mathbb{R}^2)^3\right)$ to system (4.30). As the function $\widetilde{g}$ solves (4.30) in the space $\boldsymbol{E}_T$, the uniqueness assertion in [136, Theorem 5.4] yields $g=\widetilde{g}$. This shows the claim as g is equal to γ on $[\varepsilon,T]\times[0,1]$. □

Theorem 4.37 (Smoothness of strong solutions for positive times (Triod))**.** *Let $p\in(5,\infty)$, T be positive and φ be an analytically p–admissible initial network to* (4.3)*. Suppose that $\gamma\in\boldsymbol{E}_T$ is a strong solution to the system* (4.3) *in $[0,T]$ with initial datum φ. Then for all $\varepsilon\in(0,T)$ it holds*

$$\gamma\in C^\infty\left([\varepsilon,T]\times[0,1];(\mathbb{R}^2)^3\right).$$

Proof. We show inductively that there exists $\alpha\in(0,1)$ such that for all $k\in\mathbb{N}$ and $\varepsilon\in(0,T)$,

$$\gamma\in C^{\frac{2k+2+\alpha}{4},2k+2+\alpha}\left([\varepsilon,T]\times[0,1];(\mathbb{R}^2)^3\right). \tag{4.32}$$

The case $k = 1$ is precisely the statement of Proposition 4.36. Suppose that there exists $\alpha \in (0,1)$ and $k \in \mathbb{N}$ such that (4.32) is valid for all $\varepsilon \in (0,T)$. Let $\varepsilon \in (0,T)$ be arbitrary and $\eta \in C_0^\infty((\varepsilon/2,\infty);\mathbb{R})$ be a cut–off function with $\eta \equiv 1$ on $[\varepsilon, T]$. By induction hypothesis we have

$$(t,x) \mapsto \gamma(t+\varepsilon/2, x) \in C^{\frac{2k+2+\alpha}{4}, 2k+2+\alpha}\left([0,T-\varepsilon/2]\times[0,1];(\mathbb{R}^2)^3\right) \tag{4.33}$$

and hence we obtain

$$(t,x) \mapsto g(t,x) := \eta(t+\varepsilon/2)\,\gamma(t+\varepsilon/2,x) \in C^{\frac{2k+2+\alpha}{4}, 2k+2+\alpha}\left([0,T-\varepsilon/2]\times[0,1];(\mathbb{R}^2)^3\right).$$

Furthermore, g is a classical solution to system (4.30) in $[0,T-\varepsilon/2]\times[0,1]$. Proposition B.9 and the properties (3.30) and (4.33) imply that the coefficients in the motion equation and its right hand side are of regularity $C^{\frac{2k+\alpha}{4},2k+\alpha}([0,T-\varepsilon/2]\times[0,1])$. As $t \mapsto \gamma_x(t,0)$ lies in $C^{\frac{2k+1+\alpha}{4}}\left([0,T-\varepsilon/2];(\mathbb{R}^2)^3\right)$, Proposition B.3, Proposition B.4 and (3.30) imply that the coefficients in the third order condition are of class $C^{\frac{2k+1+\alpha}{4}}([0,T-\varepsilon/2])$. Hence the coefficients and the right hand side fulfil the regularity requirements of [136, Theorem 4.9] in the case $l = 2(k+1)+2+\alpha$. As $\eta^{(j)}(0) = 0$ for all $j \in \mathbb{N}$, compatibility conditions of order $2(k+1)+2$ as stated in [136, (4.19)] are satisfied. Thus [136, Theorem 4.9] yields that there exists a solution

$$\widetilde{g} \in C^{\frac{2(k+1)+2+\alpha}{4}, 2(k+1)+2+\alpha}\left([0,T-\varepsilon/2]\times[0,1];(\mathbb{R}^2)^3\right)$$

to system (4.30) which by uniqueness needs to be equal to g. Hence we obtain

$$(t,x) \mapsto \eta(t)\gamma(t,x) \in C^{\frac{2(k+1)+2+\alpha}{4}, 2(k+1)+2+\alpha}\left([\varepsilon/2,T]\times[0,1];(\mathbb{R}^2)^3\right)$$

which completes the induction as $\eta \equiv 1$ on $[\varepsilon, T]$. This shows the claim as by Definition B.7 of the parabolic Hölder spaces it holds

$$\bigcap_{k\in\mathbb{N}} C^{\frac{2k+2+\alpha}{4}, 2k+2+\alpha}\left([\varepsilon,T]\times[0,1];(\mathbb{R}^2)^3\right) = C^\infty\left([\varepsilon,T]\times[0,1];(\mathbb{R}^2)^3\right).$$

□

In the case of Theta networks we obtain the corresponding result.

Theorem 4.38 (Smoothness of strong solutions for positive times (Theta))**.** *Let $p \in (5,\infty)$, T be positive and φ an analytically p–admissible initial network to* (4.2)*. Suppose that $\gamma \in \boldsymbol{E}_T$ is a strong solution to system* (4.2) *in $[0,T]$ with initial datum φ. Then for all $\varepsilon \in (0,T)$ it holds*

$$\gamma \in C^\infty\left([\varepsilon,T]\times[0,1];(\mathbb{R}^2)^3\right).$$

Proof. This follows adapting the arguments used for Triods as explained in Remark 4.9. □

4.1.3 Existence and uniqueness of classical solutions

In this subsection we adapt the arguments in Subsection 4.1.1 to prove that given $\alpha \in (0,1)$ and an analytically α–admissible initial network φ to system (4.2) or (4.3), respectively, there exists a time $T > 0$ such that there exists a classical solution γ to the considered system in $[0,T]$ with initial datum φ. This result is given in [64, Section 3.3, Section 3.4]. The solution space $\boldsymbol{E}_T$, $T > 0$ then becomes the parabolic Hölder space

$$\boldsymbol{E}_T := C^{\frac{4+\alpha}{4}, 4+\alpha}\left([0,T]\times[0,1];(\mathbb{R}^2)^3\right)$$

defined in Definition B.7. We remark that regardless of the regularisation result shown in Subsection 4.1.2, the existence of classical solutions to (4.3) does complement the results in Subsection 4.1.1 and Subsection 4.1.2. Indeed, given an analytically p–admissible initial value to (4.3) smoothness of the corresponding solution is only warranted *away* from the initial datum. In contrast, the time regularity of a classical solution γ to (4.3) is characterised by the space $C^{\frac{4+\alpha}{4}}\left([0,T];C\left([0,1]\right)\right)$ and is thus valid up to time zero.

In analogy to the case of strong solutions the existence of classical solutions relies on the uniform well-posedness of the associated linearised problem (4.4) and a fixed point argument. We adapt the arguments in Subsection 4.1.1 using the same symbols for the operators and spaces to highlight the similarities. Again, all proofs are done in an exemplary manner for system (4.3).

Well-posedness of the linearised problem

We prove well-posedness of the linear system (4.4) associated to the analytic problem (4.3) in parabolic Hölder spaces proceeding as in Subsection 4.1.1.

Proposition 4.39. *Given $\alpha \in (0,1)$ and $T>0$ the operator*

$$\begin{aligned}\mathcal{B}_T : \boldsymbol{E}_T &\to C^{\frac{4+\alpha}{4}}\left([0,T];(\mathbb{R}^2)^2\right) \times C^{\frac{2+\alpha}{4}}\left([0,T];(\mathbb{R}^2)^6\right),\\ u &\mapsto \left(u^1(\cdot,0)-u^2(\cdot,0), u^2(\cdot,0)-u^3(\cdot,0), u_{xx}(\cdot,0), u_{xx}(\cdot,1)\right)\end{aligned}$$

is well-defined, linear and continuous. Its kernel $\mathbb{E}_T := \ker\left(\mathcal{B}_T\right)$ is a closed linear subspace of $\boldsymbol{E}_T$.

Proof. This is a direct consequence of Proposition B.8. □

Lemma 4.40. *Let $\alpha \in (0,1)$ and φ be an analytically α–admissible initial network to (4.3) with $R := \|\varphi\|_{C^{4+\alpha}([0,1];(\mathbb{R}^2)^3)}$. Then*

$$\boldsymbol{c} := \boldsymbol{c}(\varphi) := \min_{i\in\{1,2,3\}} \min_{x\in[0,1]} \left|\varphi^i_x(x)\right| > 0$$

and for all $j\in\mathbb{N}$ it holds $x \mapsto \left|\varphi^i_x\right|^{-j}(x) := \left|\varphi^i_x(x)\right|^{-j} \in C^{1+\alpha}\left([0,1]\right)$ with

$$\left\|\left|\varphi^i_x\right|^{-j}\right\|_{C^{1+\alpha}([0,1])} \leq C\left(\boldsymbol{c},R\right).$$

Proof. Definition 4.6 implies for $i\in\{1,2,3\}$, $\varphi^i \in C^{4+\alpha}\left([0,1];\mathbb{R}^2\right)$ and $\varphi^i_x \neq 0$ on $[0,1]$ which yields

$$\boldsymbol{c}^i := \min_{x\in[0,1]} \left|\varphi^i_x(x)\right| > 0$$

and in particular $\boldsymbol{c} = \min\left\{\boldsymbol{c}^1,\boldsymbol{c}^2,\boldsymbol{c}^3\right\} > 0$. As the norm of $\varphi^i_x \in C^3\left([0,1];\mathbb{R}^2\right)$ is uniformly bounded away from zero, we conclude that $x\mapsto \left|\varphi^i_x(x)\right|^{-1}$ lies in $C^1\left([0,1];\mathbb{R}\right)$ with

$$\partial_x\left(\left|\varphi^i_x(x)\right|^{-1}\right) = -\frac{\left\langle \varphi^i_{xx}(x),\varphi^i_x(x)\right\rangle}{\left|\varphi^i_x(x)\right|^3}.$$

As we have in particular $\varphi^i_x \in C^\alpha\left([0,1];\mathbb{R}^2\right)$ with $\left|\varphi^i_x(x)\right| \geq \boldsymbol{c}$ for all $x\in[0,1]$, Proposition B.4 yields $\left|\varphi^i_x\right|^{-1} \in C^\alpha\left([0,1]\right)$ with

$$\left\|\left|\varphi^i_x\right|^{-1}\right\|_{C^\alpha([0,1])} \leq C\left(\boldsymbol{c}\right)\left\|\varphi^i_x\right\|_{C^\alpha([0,1];\mathbb{R}^2)} \leq C\left(\boldsymbol{c},R\right).$$

As also φ^i_{xx} lies in $C^\alpha\left([0,1];\mathbb{R}^2\right)$, the Banach algebra property of $C^\alpha\left([0,1]\right)$ yields that $x \mapsto \partial_x\left(\left|\varphi^i_x(x)\right|^{-1}\right)$ lies in $C^\alpha\left([0,1]\right)$ with

$$\left\|x \mapsto \partial_x\left(\left|\varphi^i_x(x)\right|^{-1}\right)\right\|_{C^\alpha([0,1])} \leq C\left(\boldsymbol{c}, R\right).$$

Hence we conclude that $\left|\varphi^i_x\right|^{-1}$ lies in $C^{1+\alpha}\left([0,1]\right)$ with

$$\left\|\left|\varphi^i_x\right|^{-1}\right\|_{C^{1+\alpha}([0,1])} \leq C\left(\boldsymbol{c}, R\right).$$

This shows the claim in the case $j=1$. The general case follows from the Banach algebra property of $C^{1+\alpha}\left([0,1]\right)$. □

Proposition 4.41 (The linearised operator)**.** *Let $\alpha \in (0,1)$ and φ be an analytically α–admissible initial network with unit normal denoted by ν^i_0. Given $T>0$ the operators*

$$L^1_T : \mathbb{E}_T \to X_T := C^{\frac{\alpha}{4},\alpha}\left([0,T]\times[0,1];(\mathbb{R}^2)^3\right), L^1_T(u) := \left(u^i_t + \frac{2}{|\varphi^i_x|^4}u^i_{xxxx}\right)_{i\in\{1,2,3\}},$$

$$L^2_T : \mathbb{E}_T \to Y_{1,T} := C^{\frac{1+\alpha}{4}}\left([0,T];\mathbb{R}^2\right), \qquad L^2_T(u) := \sum_{i=1}^{3} \frac{2}{|\varphi^i_x(0)|^3}\left\langle u^i_{xxx}(\cdot,0), \nu^i_0(0)\right\rangle \nu^i_0(0),$$

$$L^3_T : \mathbb{E}_T \to Y_{2,T} := C^{\frac{4+\alpha}{4}}\left([0,T];(\mathbb{R}^2)^3\right), \qquad L^3_T(u) := u\left(\cdot,1\right),$$

$$L^4_T : \mathbb{E}_T \to X_0 := C^{4+\alpha}\left([0,1];(\mathbb{R}^2)^3\right), \qquad L^4_T(u) := u\left(0,\cdot\right)$$

are well-defined, linear and continuous. We let $L_T := \left(L^1_T, L^2_T, L^3_T, L^4_T\right)$.

Proof. The statements about the operators L^3_T and L^4_T follow directly from Proposition B.8. That L^1_T is well-defined, linear and continuous is a consequence of Proposition B.8, the regularity of $x \mapsto \left|\varphi^i_x(x)\right|^{-1}$ shown in Lemma 4.40 and the Banach algebra property of parabolic Hölder spaces shown in Proposition B.9. The statement about L^2_T follows from Proposition B.8 and the Banach algebra property of Hölder spaces. □

To guarantee well-posedness of the system (4.4) in parabolic Hölder spaces given an initial datum ψ, the right hand side (f, h_1, h_2) needs to fulfil certain requirements in order to be *compatible* with the initial datum ψ. In comparison to the conditions discussed in Proposition 4.14, we need to ask additional conditions involving fourth order derivatives of the initial datum. Indeed suppose that $u \in C^{\frac{4+\alpha}{4},4+\alpha}\left([0,T]\times[0,1];(\mathbb{R}^2)^3\right)$ is a classical solution to (4.4). Deriving the concurrency condition in time yields for all $t \in [0,T]$, $i,l \in \{1,2,3\}$,

$$u^i_t(t,0) = f^i(t,0) - \frac{2}{\left|\varphi^i_x(0)\right|^4}u^i_{xxxx}(t,0) = f^l(t,0) - \frac{2}{\left|\varphi^l_x(0)\right|^4}u^l_{xxxx}(t,0) = u^l_t(t,0).$$

Evaluating this identity in $t=0$ then gives additional compatibility conditions

$$f^i(0,0) - \frac{2}{\left|\varphi^i_x(0)\right|^4}\psi^i_{xxxx}(0) = f^l(0,0) - \frac{2}{\left|\varphi^l_x(0)\right|^4}\psi^l_{xxxx}(0).$$

Differentiating the condition $u^i(t,1) = h^i_2(t)$ with respect to time yields

$$\partial_t h^i_2(t) = u^i_t(t,1) = f^i(t,1) - \frac{2}{\left|\varphi^i_x(1)\right|^4}u^i_{xxxx}(t,1)$$

and thus

$$\partial_t h^i_2(0) = f^i(0,1) - \frac{2}{\left|\varphi^i_x(1)\right|^4}\psi^i_{xxxx}(1).$$

As all involved functions can be derived in the classical sense and evaluated pointwise, it is straight forward to check that given $\alpha \in (0,1)$, $T > 0$ and φ an analytically α–admissible initial datum with unit normal denoted by ν_0^i, the linear compatibility operator

$$\mathcal{G}_T : X_T \times Y_{1,T} \times Y_{2,T} \times X_0 \to (\mathbb{R}^2)^2 \times (\mathbb{R}^2)^6 \times \mathbb{R}^2 \times (\mathbb{R}^2)^3 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$$

defined by

$$\begin{aligned}
\mathcal{G}_T^1\left((f,h_1,h_2,\psi)\right) &:= \left(\psi^1(0)-\psi^2(0), \psi^2(0)-\psi^3(0)\right),\\
\mathcal{G}_T^2\left((f,h_1,h_2,\psi)\right) &:= \left(\psi_{xx}(0), \psi_{xx}(1)\right),\\
\mathcal{G}_T^3\left((f,h_1,h_2,\psi)\right) &:= \sum_{i=1}^3 \frac{2}{|\varphi_x^i(0)|^3}\left\langle \psi_{xxx}^i(0), \nu_0^i(0)\right\rangle \nu_0^i(0) - h_1(0),\\
\mathcal{G}_T^4\left((f,h_1,h_2,\psi)\right) &:= \psi(1) - h_2(0),\\
\mathcal{G}_T^5\left((f,h_1,h_2,\psi)\right) &:= f^1(0,0) - \frac{2}{|\varphi_x^1(0)|^4}\psi_{xxxx}^1(0) - f^2(0,0) + \frac{2}{|\varphi_x^2(0)|^4}\psi_{xxxx}^2(0),\\
\mathcal{G}_T^6\left((f,h_1,h_2,\psi)\right) &:= f^2(0,0) - \frac{2}{|\varphi_x^2(0)|^4}\psi_{xxxx}^2(0) - f^3(0,0) + \frac{2}{|\varphi_x^3(0)|^4}\psi_{xxxx}^3(0),\\
\mathcal{G}_T^7\left((f,h_1,h_2,\psi)\right) &:= f^1(0,1) - \frac{2}{|\varphi_x^1(1)|^4}\psi_{xxxx}^1(1) - \partial_t h_2^1(0),\\
\mathcal{G}_T^8\left((f,h_1,h_2,\psi)\right) &:= f^2(0,1) - \frac{2}{|\varphi_x^2(1)|^4}\psi_{xxxx}^2(1) - \partial_t h_2^2(0),\\
\mathcal{G}_T^9\left((f,h_1,h_2,\psi)\right) &:= f^3(0,1) - \frac{2}{|\varphi_x^3(1)|^4}\psi_{xxxx}^3(1) - \partial_t h_2^3(0)
\end{aligned}$$

is well-defined, linear and continuous. In particular, $\mathbb{F}_T := \ker(\mathcal{G}_T)$ is a closed linear subspace of $X_T \times Y_{1,T} \times Y_{2,T} \times X_0$. Given $\psi \in X_0$ we say that $(f,h_1,h_2) \in X_T \times Y_{1,T} \times Y_{2,T}$ satisfies the *linear compatibility conditions with respect to* ψ if $(f,h_1,h_2,\psi) \in \mathbb{F}_T$. As in Corollary 4.16 one checks that for all $u \in \mathbb{E}_T$ and $i \in \{1,2,3,4\}$, $\mathcal{G}_T^i\left(L_T(u)\right) = 0$. Given $u \in \mathbb{E}_T$ one has for all $t \in [0,T]$, $i \in \{1,2\}$,

$$0 = u_t^i(t,0) - u_t^{i+1}(t,0) = L_T^{1,i}(u)(t,0) - \frac{2}{|\varphi_x^i(0)|^4}u_{xxxx}^i(t,0) - L_T^{1,i+1}(u)(t,0) + \frac{2}{|\varphi_x^{i+1}(0)|^4}u_{xxxx}^{i+1}(t,0)$$

which yields in particular $\mathcal{G}_T^j\left(L_T(u)\right) = 0$ for $j \in \{5,6\}$. Furthermore, for all $u \in \mathbb{E}_T$, $i \in \{1,2,3\}$, $t \in [0,T]$, there holds

$$u_t^i(t,1) = L_T^{1,i}(u)(t,1) - \frac{2}{|\varphi_x^i(1)|^4}u_{xxxx}^i(t,1)$$

and thus

$$\begin{aligned}
0 &= L_T^{1,i}(u)(0,1) - \frac{2}{|\varphi_x^i(1)|^4}u_{xxxx}^i(0,1) - u_t^i(0,1)\\
&= L_T^{1,i}(u)(0,1) - \frac{2}{|\varphi_x^i(1)|^4}\left(L_T^{4,i}\right)_{xxxx}(1) - \partial_t L_T^{3,i}(u)(0)
\end{aligned}$$

which yields $\mathcal{G}_T^{6+i}\left(L_T(u)\right) = 0$.

To prove that the operator $L_T : \mathbb{E}_T \to \mathbb{F}_T$ is invertible also with this choice of solution space, we apply [136, Theorem 4.9] on well-posedness of linear parabolic systems in parabolic Hölder spaces. We hereby endow $\mathbb{E}_T$ and $\mathbb{F}_T$ with the norms

$$\|u\|_{\mathbb{E}_T} := \|u\|_{\frac{4+\alpha}{4},4+\alpha} := \|u\|_{C^{\frac{4+\alpha}{4},4+\alpha}([0,T]\times[0,1];(\mathbb{R}^2)^3)}$$

and

$$\begin{aligned}\|(f,h_1,h_2,\psi)\|_{\mathbb{F}_T} &:= \|f\|_{\frac{\alpha}{4},\alpha} + \|h_1\|_{C^{\frac{1+\alpha}{4}}} + \|h_2\|_{C^{\frac{4+\alpha}{4}}} + \|\psi\|_{C^{4+\alpha}} \\ &:= \|f\|_{C^{\frac{\alpha}{4},\alpha}([0,T]\times[0,1];(\mathbb{R}^2)^3)} + \|h_1\|_{C^{\frac{1+\alpha}{4}}([0,T];\mathbb{R}^2)} + \|h_2\|_{C^{\frac{4+\alpha}{4}}([0,T];(\mathbb{R}^2)^3)} + \|\psi\|_{C^{4+\alpha}([0,1];(\mathbb{R}^2)^3)}\,.\end{aligned}$$

Theorem 4.42 (Well-posedness of the linearised system in parabolic Hölder spaces)**.** *Let $\alpha \in (0,1)$, T be positive and φ be an analytically α–admissible initial value to* (4.3)*. Then the operator $L_T \in \mathcal{L}(\mathbb{E}_T, \mathbb{F}_T)$ defined in Proposition 4.41 is bijective with continuous inverse L_T^{-1}.*

Proof. We intend to apply [136, Theorem 4.9] with $l := 4+\alpha$. The results in Subsection 4.1.1 show that the evolution equation is uniformly parabolic and that the boundary operators satisfy the complementary condition. We notice that these requirements are independent of the choice of solution space. Lemma 4.40 shows that the coefficients in the evolution equation belong to the class $C^{\frac{\alpha}{4},\alpha}([0,T]\times[0,1])$. The coefficients of the boundary operators are constant and thus trivially fulfil the regularity requirements stated in [136, Theorem 4.9]. The compatibility conditions of order four in [136, §14] precisely correspond to the linear compatibility conditions encoded in the space $\mathbb{F}_T$. Thus [136, Theorem 4.9] allows us to conclude that for all $(f,h_1,h_2,\psi) \in \mathbb{F}_T$ there exists a unique $u \in \mathbb{E}_T$ with $L_T(u) = (f,h_1,h_2,\psi)$ satisfying the estimate

$$\|u\|_{\frac{4+\alpha}{4},4+\alpha} \leq C\Big(\|f\|_{\frac{\alpha}{4},\alpha} + \|h_1\|_{C^{\frac{1+\alpha}{4}}([0,T];\mathbb{R}^2)} + \|h_2\|_{C^{\frac{4+\alpha}{4}}([0,T];(\mathbb{R}^2)^3)} + \|\psi\|_{C^{4+\alpha}([0,1];(\mathbb{R}^2)^3)} \Big).$$

□

The constant C in Theorem 4.42 a priori depends on T. In the fixed point argument it needs to be guaranteed that the operator norm of $L_T^{-1} \in \mathcal{L}(\mathbb{F}_T, \mathbb{E}_T)$ is bounded by a constant $C(T_0)$ independent of $T \in (0,T_0]$ where $T_0 > 0$ is some fixed time slot. As in Subsection 4.1.1 this can be achieved with the help of extension operators. In contrast to the Sobolev setting a change of norms on $\mathbb{E}_T$ and $\mathbb{F}_T$ is not necessary in the case of parabolic Hölder spaces.

Theorem 4.43 (Uniform well-posedness of the linearised system in parabolic Hölder spaces)**.** *Let $\alpha \in (0,1)$ and φ be an analytically α–admissible initial network to* (4.3)*. Given $T_0 > 0$ there exists a constant $C(T_0) > 0$ such that*

$$\sup_{t\in(0,T_0]} \left\|L_T^{-1}\right\|_{\mathcal{L}(\mathbb{F}_T,\mathbb{E}_T)} \leq C(T_0)\,.$$

Proof. Let $T_0 > 0$ and $T \in (0,T_0]$ be fixed. Given $(f,h_1,h_2,\psi) \in \mathbb{F}_T$ we let $u \in \mathbb{E}_T$ be the unique element with $L_T u = (f,h_1,h_2,\psi)$. Moreover, with E_{2T_0} and $\boldsymbol{E}_{2T_0}$ denoting the temporal extension operators on (parabolic) Hölder spaces defined in Lemma B.6 and Lemma B.10, respectively, we obtain

$$(\boldsymbol{E}_{2T_0} f, E_{2T_0} h_1, E_{2T_0} h_2, \psi) \in X_{2T_0} \times Y_{1,2T_0} \times Y_{2,2T_0} \times X_0$$

with

$$\|(\boldsymbol{E}_{2T_0} f, E_{2T_0} h_1, E_{2T_0} h_2, \psi)\|_{X_{2T_0}\times Y_{1,2T_0}\times Y_{2,2T_0}\times X_0} \leq C(T_0)\,\|(f,h_1,h_2,\psi)\|_{\mathbb{F}_T}\,.$$

As

$$\mathcal{G}_{2T_0}\left((\boldsymbol{E}_{2T_0} f, E_{2T_0} h_1, E_{2T_0} h_2, \psi)\right) = \mathcal{G}_T\left((f,h_1,h_2,\psi)\right) = 0$$

we obtain $(\boldsymbol{E}_{2T_0} f, E_{2T_0} h_1, E_{2T_0} h_2, \psi) \in \mathbb{F}_{2T_0}$. As $L_{2T_0} \in \mathcal{L}(\mathbb{E}_{2T_0}, \mathbb{F}_{2T_0})$ is a linear isomorphism due to Theorem 4.42, there exists a unique $\widetilde{u} \in \mathbb{E}_{2T_0}$ with

$$L_{2T_0}(\widetilde{u}) = (\boldsymbol{E}_{2T_0} f, E_{2T_0} h_1, E_{2T_0} h_2, \psi)\,.$$

We notice that

$$L_T(u) = (f, h_1, h_2, \psi) = (\boldsymbol{E}_{2T_0} f, E_{2T_0} h_1, E_{2T_0} h_2, \psi)_{|[0,T]} = L_{2T_0}\left(\widetilde{u}\right)_{|[0,T]} = L_T\left(\widetilde{u}_{|[0,T]}\right)$$

and thus $u = \widetilde{u}_{|[0,T]}$ which allows us to conclude

$$\begin{aligned}\|u\|_{\mathbb{E}_T} \le \|\widetilde{u}\|_{\mathbb{E}_{2T_0}} &\le \left\|L_{2T_0}^{-1}\right\|_{\mathcal{L}\left(\mathbb{F}_{2T_0},\mathbb{E}_{2T_0}\right)} \|(\boldsymbol{E}_{2T_0} f, E_{2T_0} h_1, E_{2T_0} h_2, \psi)\|_{\mathbb{F}_{2T_0}} \\ &\le \left\|L_{2T_0}^{-1}\right\|_{\mathcal{L}\left(\mathbb{F}_{2T_0},\mathbb{E}_{2T_0}\right)} C\left(T_0\right) \|(f, h_1, h_2, \psi)\|_{\mathbb{F}_T} \,.\end{aligned}$$

□

The contraction argument

Proposition 4.44. *Let $\alpha \in (0,1)$ and φ be an analytically α–admissible initial network to (3.21). Then for any $T > 0$ the set*

$$\mathbb{E}_T^{\varphi} := \left\{\gamma \in \mathbb{E}_T : \gamma_{|t=0} = \varphi\right\}$$

is a non-empty complete metric space in the norm $\|\cdot\|_{\mathbb{E}_T}$ and there exists $\mathcal{E}\varphi \in \mathbb{E}_T^{\varphi}$ such that $\|\mathcal{E}\varphi\|_{\mathbb{E}_T} \le \|\varphi\|_{X_0}$.

Proof. Let $T > 0$ be given. As for any $\gamma, \eta \in \mathbb{E}_T$ there holds

$$\|\gamma(0) - \eta(0)\|_{C^{4+\alpha}([0,1];(\mathbb{R}^2)^3)} \le \|\gamma - \eta\|_{\mathbb{E}_T} \,,$$

we conclude that $\mathbb{E}_T^{\varphi}$ is a closed subset of the Banach space $\left(\mathbb{E}_T, \|\cdot\|_{\mathbb{E}_T}\right)$ and hence a complete metric space in the norm $\|\cdot\|_{\mathbb{E}_T}$. Moreover, we observe that $\mathcal{E}\varphi(t) := \varphi(t)$, $t \in [0,T]$, defines an element $\mathcal{E}\varphi \in \mathbb{E}_T^{\varphi}$ with $\|\mathcal{E}\varphi\|_{\mathbb{E}_T} = \|\varphi\|_{X_0}$. □

Given $T > 0$ and $M > 0$ we let

$$\overline{B_{M,T}} := \left\{\gamma \in \mathbb{E}_T := \|\gamma\|_{\mathbb{E}_T} \le M\right\} \,.$$

The following result is completely analogous to Lemma 4.16.

Lemma 4.45. *Let $\alpha \in (0,1)$, $T_0 > 0$, $M > 0$ and φ be an analytically α–admissible initial datum to (4.3) with $\boldsymbol{c}$ as in Lemma 4.40. There exists a time $\widetilde{T}(\boldsymbol{c}, M) \in (0, T_0]$ such that for all $T \in \left(0, \widetilde{T}(\boldsymbol{c}, M)\right]$, $\gamma \in \mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$ and all $i \in \{1,2,3\}$ the curve $\gamma^i(t)$ is regular with*

$$\inf_{t\in[0,T]\,,x\in[0,1]} \left|\gamma_x^i(t,x)\right| \ge \frac{\boldsymbol{c}}{2} \tag{4.34}$$

and satisfies the non-degeneracy condition in $y = 0$.

Proof. Let $T \in (0, T_0]$ and $\gamma \in \mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$ be given. Then for any $t \in [0,T]$, $x \in [0,1]$, $i \in \{1,2,3\}$, there holds

$$\left|\gamma_x^i(t,x)\right| \ge \left|\varphi_x^i(x)\right| - \left|\gamma_x^i(t,x) - \gamma_x^i(0,x)\right| \,.$$

As γ_x^i lies in $C^{\frac{3+\alpha}{4}}\left([0,T]; C\left([0,1];\mathbb{R}^2\right)\right)$ by Proposition B.8, we obtain for any $t \in [0,T]$, $x \in [0,1]$, $i \in \{1,2,3\}$,

$$\left|\gamma_x^i(t,x)\right| \ge \boldsymbol{c} - t^{\frac{3+\alpha}{4}} \left\|\gamma_x^i\right\|_{C^{\frac{3+\alpha}{4}}([0,T];C([0,1];\mathbb{R}^2))} \ge \boldsymbol{c} - T^{\frac{3+\alpha}{4}} M \,.$$

The claim then follows arguing as in the proof of Lemma 4.16. □

In the following we set $T_0 := 1$.

Corollary 4.46. *Let $\alpha \in (0,1)$, $M > 0$, φ be an analytically α–admissible initial datum to* (4.3) *with $\boldsymbol{c}$ as in Lemma 4.40 and $\widetilde{T}(\boldsymbol{c}, M)$ be as in Lemma 4.45. Then for all $T \in \left(0, \widetilde{T}(\boldsymbol{c}, M)\right]$, $\gamma \in \mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$, $i \in \{1,2,3\}$ and $j \in \mathbb{N}$, there holds*

$$(t,x) \mapsto \left|\gamma_x^i\right|^{-j}(t,x) := \left|\gamma_x^i(t,x)\right|^{-j} \in C^{\frac{\alpha}{4},\alpha}\left([0,T]\times[0,1]\right)$$

with

$$\left\|\left|\gamma_x^i\right|^{-j}\right\|_{\frac{\alpha}{4},\alpha} \le C(\boldsymbol{c}, M)$$

and further

$$t \mapsto \left|\gamma_x^i(0)\right|^{-j}(t) := \left|\gamma_x^i(t,0)\right|^{-j} \in C^{\frac{1+\alpha}{4}}\left([0,T]\right)$$

with

$$\left\|\left|\gamma_x^i(0)\right|^{-j}\right\|_{C^{\frac{1+\alpha}{4}}([0,T])} \le C(\boldsymbol{c}, M).$$

Proof. Let $T \in \left(0, \widetilde{T}(\boldsymbol{c}, M)\right]$, $\gamma \in \mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$ and $i \in \{1,2,3\}$ be given. Proposition B.8 yields

$$\gamma_x^i \in C^{\frac{3+\alpha}{4},3+\alpha}\left([0,T]\times[0,1];\mathbb{R}^2\right) \hookrightarrow C^{\frac{\alpha}{4},\alpha}\left([0,T]\times[0,1];\mathbb{R}^2\right).$$

The estimate (4.34), Proposition B.3 and Proposition B.5 yield

$$(t,x) \mapsto \left|\gamma_x^i\right|^{-1}(t,x) \in C^{\frac{\alpha}{4},\alpha}\left([0,T]\times[0,1]\right)$$

with

$$\left\|\left|\gamma_x^i\right|^{-1}\right\|_{\frac{\alpha}{4},\alpha} \le C(\boldsymbol{c})\left\|\gamma_x^i\right\|_{\frac{\alpha}{4},\alpha} \le C(\boldsymbol{c}, M).$$

The corresponding statement for general $j \in \mathbb{N}$ then follows from the Banach algebra property of parabolic Hölder spaces shown in Proposition B.9. As in particular

$$\gamma_x^i \in C^{\frac{1+\alpha}{4}}\left([0,T]; C\left([0,1];\mathbb{R}^2\right)\right),$$

we obtain using (4.34) and Proposition B.3 that

$$(t,x) \mapsto \left|\gamma_x^i\right|^{-1}(t,x) \in C^{\frac{1+\alpha}{4}}\left([0,T]; C\left([0,1];\mathbb{R}^2\right)\right)$$

with

$$\left\|\left|\gamma_x^i\right|^{-1}\right\|_{C^{\frac{1+\alpha}{4}}([0,T];C([0,1]))} \le C(\boldsymbol{c}, M)$$

and thus in particular

$$t \mapsto \left|\gamma_x^i(t,0)\right|^{-1} \in C^{\frac{1+\alpha}{4}}\left([0,T]\right)$$

with norm bounded by $C(\boldsymbol{c}, M)$. The Banach algebra property of Hölder spaces then yields the corresponding statement for general $j \in \mathbb{N}$. □

To infer the existence of (4.3) in parabolic Hölder spaces from the well-posedness of the associated linear problem (4.4) in parabolic Hölder spaces, we introduce the operator N_T representing the nonlinearities appearing in (4.3) in analogy to Proposition 4.24. As in 4.25 we may then define the contraction operator K_T.

Proposition 4.47. *Let $\alpha \in (0,1)$, φ be an analytically α–admissible initial value to* (4.3) *with $\boldsymbol{c}$ as in Lemma 4.40, $T_0 := 1$, $M > 0$, $\widetilde{T}(\boldsymbol{c}, M)$ be as in Lemma 4.45 and $\overline{V}$, $\overline{T}$ be the expressions defined in* (3.24) *and* (3.26)*. For all $T \in \left(0, \widetilde{T}(\boldsymbol{c}, M)\right]$ the operator*

$$\left(N_T^1, N_T^2\right) : \mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}} \to X_T \times Y_{1,T} = C^{\frac{\alpha}{4},\alpha}\left([0,T]\times[0,1];(\mathbb{R}^2)^3\right) \times C^{\frac{1+\alpha}{4}}\left([0,T];\mathbb{R}^2\right),$$

$$\left(N_T^1(\gamma)\right)^i := \gamma_t^i - \overline{V}\left(\gamma^i\right)\nu^i - \overline{T}\left(\gamma^i\right)\tau^i\,,$$
$$N_T^2(\gamma) := \sum_{i=1}^{3}\frac{2}{\left|\gamma_x^i(\cdot,0)\right|^3}\left\langle\gamma_{xxx}^i\left(\cdot,0\right),\nu^i\left(\cdot,0\right)\right\rangle\nu^i\left(\cdot,0\right)$$

is well-defined. Moreover, given $T\in\left(0,\widetilde{T}(\boldsymbol{c},M)\right]$, $\gamma\in\mathbb{E}_T^{\varphi}\cap\overline{B_{M,T}}$,

$$\mathcal{G}_T\left(\left(L_T^1(\gamma)-N_T^1(\gamma),L_T^2(\gamma)-N_T^2(\gamma),\varphi(1),\varphi\right)\right)=0$$

and the operator

$$K_T:\mathbb{E}_T^{\varphi}\cap\overline{B_{M,T}}\to\mathbb{E}_T^{\varphi}\,,\gamma\mapsto L_T^{-1}\left(\left(L_T^1(\gamma)-N_T^1(\gamma),L_T^2(\gamma)-N_T^2(\gamma),\varphi(1),\varphi\right)\right)$$

is well-defined.

Proof. Let $T\in\left(0,\widetilde{T}(\boldsymbol{c},M)\right]$ and $\gamma\in\mathbb{E}_T^{\varphi}\cap\overline{B_{M,T}}$ be given. As in Proposition 4.24 we have for $i\in\{1,2,3\}$

$$\begin{aligned}\left(N_T^1(\gamma)\right)^i =&\gamma_t^i+2\frac{\gamma_{xxxx}^i}{|\gamma_x^i|^4}-12\frac{\gamma_{xxx}^i\left\langle\gamma_{xx}^i,\gamma_x^i\right\rangle}{|\gamma_x^i|^6}-5\frac{\gamma_{xx}^i\left|\gamma_{xx}^i\right|^2}{|\gamma_x^i|^6}-8\frac{\gamma_{xx}^i\left\langle\gamma_{xxx}^i,\gamma_x^i\right\rangle}{|\gamma_x^i|^6}+35\frac{\gamma_{xx}^i\left\langle\gamma_{xx}^i,\gamma_x^i\right\rangle^2}{|\gamma_x^i|^8}\\&-\mu\frac{\gamma_{xx}^i}{|\gamma_x^i|^2}\end{aligned}$$

which lies in $C^{\frac{\alpha}{4},\alpha}\left([0,T]\times[0,1];\mathbb{R}^2\right)$ due to Corollary 4.46, Proposition B.8 and the Banach algebra property of $C^{\frac{\alpha}{4},\alpha}\left([0,T]\times[0,1]\right)$. Corollary 4.46 further yields $t\mapsto\gamma_x^i(t,0)\in C^{\frac{1+\alpha}{4}}\left([0,T];\mathbb{R}^2\right)$, and $t\mapsto\left|\gamma_x^i(t,0)\right|^{-j}\in C^{\frac{1+\alpha}{4}}\left([0,T]\right)$ for $j\in\{1,3\}$. As further $t\mapsto\gamma_{xxx}^i(t,0)\in C^{\frac{1+\alpha}{4}}\left([0,T];\mathbb{R}^2\right)$ due to Proposition B.8, the Banach algebra property of $C^{\frac{1+\alpha}{4}}\left([0,T]\right)$ allows us to conclude that $N_{T,2}$ is well-defined. In particular, on using Proposition 4.41 we conclude that

$$\left(L_T^1(\gamma)-N_T^1(\gamma),L_T^2(\gamma)-N_T^2(\gamma),\varphi(1),\varphi\right)\in X_T\times Y_{1,T}\times Y_{2,T}\times X_0\,.$$

As φ is an analytically α–admissible initial value to (4.3), the arguments used in Proposition 4.25 show for $j\in\{1,2,3,4\}$ that

$$\mathcal{G}_T^j\left(\left(L_T^1(\gamma)-N_T^1(\gamma),L_T^2(\gamma)-N_T^2(\gamma),\varphi(1),\varphi\right)\right)=0\,.$$

As φ further satisfies the analytical fourth order condition in $y=0$, namely for $i,l\in\{1,2,3\}$,

$$\overline{V}\left(\varphi^i\right)(0)\nu_0^i(0)+\overline{T}\left(\varphi^i\right)(0)\tau_0^i(0)=\overline{V}\left(\varphi^l\right)(0)\nu_0^l(0)+\overline{T}\left(\varphi^l\right)(0)\tau_0^l(0)\,,$$

we obtain for any $i,l\in\{1,2,3\}$ due to $\gamma_{|t=0}=\varphi$,

$$\begin{aligned}&\left(L_T^1(\gamma)-N_T^1(\gamma)\right)^i(0,0)-\frac{2}{\left|\varphi_x^i(0)\right|^4}\varphi_{xxxx}^i(0)=\overline{V}\left(\gamma^i\right)(0,0)\nu^i(0,0)+\overline{T}\left(\gamma^i\right)(0,0)\tau^i(0,0)\\&=\overline{V}\left(\varphi^i\right)(0)\nu_0^i(0)+\overline{T}\left(\varphi^i\right)(0)\tau_0^i(0)=\overline{V}\left(\varphi^l\right)(0)\nu_0^l(0)+\overline{T}\left(\varphi^l\right)(0)\tau_0^l(0)\\&=\overline{V}\left(\gamma^l\right)(0,0)\nu^l(0,0)+\overline{T}\left(\gamma^l\right)(0,0)\tau^l(0,0)=\left(L_T^1(\gamma)-N_T^1(\gamma)\right)^l(0,0)-\frac{2}{\left|\varphi_x^l(0)\right|^4}\varphi_{xxxx}^l(0)\end{aligned}$$

which yields for $j\in\{5,6\}$,

$$\mathcal{G}_T^j\left(\left(L_T^1(\gamma)-N_T^1(\gamma),L_T^2(\gamma)-N_T^2(\gamma),\varphi(1),\varphi\right)\right)=0\,.$$

As φ satisfies the analytical fourth order condition in $y=1$, namely for $i\in\{1,2,3\}$,

$$\overline{V}\left(\varphi^i\right)(1)\nu_0^i(1)+\overline{T}\left(\varphi^i\right)(1)\tau_0^i(1)=0\,,$$

we obtain for $\gamma \in \mathbb{E}_T^\varphi$, $i \in \{1,2,3\}$,

$$\begin{aligned}&\left(L_T^1(\gamma) - N_T^1(\gamma)\right)^i(0,1) - \frac{2}{|\varphi_x^i(1)|^4}\varphi_{xxxx}^i(1) - \partial_t\varphi^i(1)\\ &= \overline{V}\left(\gamma^i\right)(0,1)\nu^i(0,1) + \overline{T}\left(\gamma^i\right)(0,1)\tau^i(0,1) = \overline{V}\left(\varphi^i\right)(1)\nu_0^i(1) + \overline{T}\left(\varphi^i\right)(1)\tau_0^i(1) = 0\end{aligned}$$

and thus for $i \in \{1,2,3\}$,

$$\mathcal{G}_T^{6+i}\left(\left(L_T^1(\gamma) - N_T^1(\gamma), L_T^2(\gamma) - N_T^2(\gamma), \varphi(1), \varphi\right)\right) = 0\,.$$

This shows that

$$\left(L_T^1(\gamma) - N_T^1(\gamma), L_T^2(\gamma) - N_T^2(\gamma), \varphi(1), \varphi\right) \in \mathbb{F}_T = \ker\left(\mathcal{G}_T\right)$$

and thus $K_T : \mathbb{E}_T^\varphi \cap \overline{B_{M,T}} \to \mathbb{E}_T$ is well-defined due to Theorem 4.42. Given $\gamma \in \mathbb{E}_T^\varphi \cap \overline{B_{M,T}}$ we have $L_T^4\left(K_T(\gamma)\right) = \varphi$ and thus $K_T(\gamma) \in \mathbb{E}_T^\varphi$. □

Lemma 4.48. *Let $\alpha \in (0,1)$, $T \in [0,1]$ and φ be an analytically α–admissible initial datum to* (4.3). *There exist constants $\varepsilon \in (0,1)$ and $C > 0$ independent of T such that for all $\xi\,,\eta \in \mathbb{E}_T^\varphi$, $k \in \{1,2,3\}$*

$$\left\|\partial_x^{4-k}\xi - \partial_x^{4-k}\eta\right\|_{\frac{\alpha}{4},\alpha} \leq CT^\varepsilon\left\|\partial_x^{4-k}\xi - \partial_x^{4-k}\eta\right\|_{\frac{k+\alpha}{4},k+\alpha} \leq CT^\varepsilon\left\|\xi - \eta\right\|_{\mathbb{E}_T}\,.$$

Proof. We let $k \in \{1,2,3\}$ and $T \in [0,1]$ be given. Furthermore, we fix a coefficient $\theta_k \in \left[\frac{\alpha}{k+\alpha}, \frac{k}{k+\alpha}\right)$ such that $\theta_k\left(k+\alpha\right)$ is not an integer. As $0 < \theta_k\left(k+\alpha\right) < k+\alpha$ we may apply [94, Proposition 1.1.3 iii)] to conclude that $C^{\theta_k(k+\alpha)}\left([0,1]\right)$ belongs to the class J_{θ_k} between $C\left([0,1]\right)$ and $C^{k+\alpha}\left([0,1]\right)$, which by [94, Definition 1.1.1] means that

$$C^{k+\alpha}\left([0,1]\right) \subset C^{\theta_k(k+\alpha)}\left([0,1]\right) \subset C\left([0,1]\right)$$

and that there exists a constant $C > 0$ such that for all $f \in C^{k+\alpha}\left([0,1]\right)$,

$$\|f\|_{C^{\theta_k(k+\alpha)}([0,1])} \leq C\,\|f\|_{C^{k+\alpha}([0,1])}^{\theta_k}\,\|f\|_{C([0,1])}^{1-\theta_k}\,.$$

Proposition B.1 with $X_0 := C\left([0,1]\right)$, $X_1 := C^{k+\alpha}\left([0,1]\right)$, $Y := C^{\theta_k(k+\alpha)}\left([0,1]\right)$ and $\sigma := \theta_k$ yields for all $\beta \in (0,1)$ the continuous embedding

$$C\left([0,T]; C^{k+\alpha}\left([0,1]\right)\right) \cap C^\beta\left([0,T]; C\left([0,1]\right)\right) \hookrightarrow C^{(1-\theta_k)\beta}\left([0,T]; C^{\theta_k(k+\alpha)}\left([0,1]\right)\right) \tag{4.35}$$

with embedding constant independent of T. Given $\delta \in [0,1)$ and $\mu > 0$ we let

$$C_0^\delta\left([0,T]; C^\mu\left([0,1]\right)\right) := \left\{g \in C^\delta\left([0,T]; C^\mu\left([0,1]\right)\right) : g(0,x) = 0 \text{ for all } x \in [0,1]\right\}\,.$$

It is then straightforward to see that (4.35) restricts to a continuous embedding

$$C_0\left([0,T]; C^{k+\alpha}\left([0,1]\right)\right) \cap C_0^\beta\left([0,T]; C\left([0,1]\right)\right) \hookrightarrow C_0^{(1-\theta_k)\beta}\left([0,T]; C^{\theta_k(k+\alpha)}\left([0,1]\right)\right) \tag{4.36}$$

with embedding constant independent of T. We define

$$X_T^k := \left\{g \in C^{\frac{k+\alpha}{4},k+\alpha}\left([0,T]\times[0,1]; (\mathbb{R}^2)^3\right) : g(0,x) = 0 \text{ for all } x \in [0,1]\right\}$$

and let $\beta_k := \frac{k+\alpha}{4} \in (0,1)$. As

$$X_T^k \hookrightarrow C_0\left([0,T]; C^{k+\alpha}\left([0,1]; (\mathbb{R}^2)^3\right)\right) \cap C_0^{\beta_k}\left([0,T]; C\left([0,1]; (\mathbb{R}^2)^3\right)\right),$$

we conclude that

$$X_T^k \hookrightarrow C_0^{(1-\theta_k)\beta_k}\big([0,T];C^{\theta_k(k+\alpha)}\big([0,1];(\mathbb{R}^2)^3\big)\big) \hookrightarrow C_0^{(1-\theta_k)\beta_k}\left([0,T];C^{\alpha}\left([0,1];(\mathbb{R}^2)^3\right)\right)$$

with embedding constant C independent of T, where we used that $\theta_k\,(k+\alpha)\geq\alpha$. Given $\xi\,,\eta\in\mathbb{E}_T^{\varphi}$ there holds $\partial_x^{4-k}\xi-\partial_x^{4-k}\eta\in X_T^k$ and thus Proposition B.2 yields with $\varepsilon_k:=(1-\theta_k)\,\beta_k-\frac{\alpha}{4}>0$,

$$\begin{aligned}
&\left\|\partial_x^{4-k}\xi-\partial_x^{4-k}\eta\right\|_{C^{\frac{\alpha}{4}}([0,T];C([0,1];(\mathbb{R}^2)^3))}\leq\left\|\partial_x^{4-k}\xi-\partial_x^{4-k}\eta\right\|_{C^{\frac{\alpha}{4}}([0,T];C^{\alpha}([0,1];(\mathbb{R}^2)^3))}\\
&\leq 2T^{\varepsilon_k}\left\|\partial_x^{4-k}\xi-\partial_x^{4-k}\eta\right\|_{C^{(1-\theta_k)\beta_k}([0,T];C^{\alpha}([0,1];(\mathbb{R}^2)^3))}\leq 2CT^{\varepsilon_k}\left\|\partial_x^{4-k}\xi-\partial_x^{4-k}\eta\right\|_{\frac{k+\alpha}{4},k+\alpha}\\
&\leq 2CT^{\varepsilon_k}\left\|\xi-\eta\right\|_{\mathbb{E}_T}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\left\|\partial_x^{4-k}\xi-\partial_x^{4-k}\eta\right\|_{C([0,T];C^{\alpha}([0,1];(\mathbb{R}^2)^3))}\leq\left\|\partial_x^{4-k}\xi-\partial_x^{4-k}\eta\right\|_{C^{\frac{\alpha}{4}}([0,T];C^{\alpha}([0,1];(\mathbb{R}^2)^3))}\\
&\leq 2CT^{\varepsilon_k}\left\|\partial_x^{4-k}\xi-\partial_x^{4-k}\eta\right\|_{\frac{k+\alpha}{4},k+\alpha}\leq 2CT^{\varepsilon_k}\left\|\xi-\eta\right\|_{\mathbb{E}_T}\,.
\end{aligned}$$

□

Proposition 4.49 (Contraction estimates I)**.** *Let $\alpha\in(0,1)$, φ be an analytically α–admissible initial value to* (4.3) *with $\boldsymbol{c}$ as in Lemma 4.40, $T_0:=1$, $M\geq\|\varphi\|_{4+\alpha}$ and $\widetilde{T}(\boldsymbol{c},M)$ be as in Lemma 4.45. There exist $\varepsilon\in(0,1)$ and a constant $C>0$ depending on $\boldsymbol{c}$ and M such that for all $T\in\big(0,\widetilde{T}(\boldsymbol{c},M)\big]$, γ, $\widetilde{\gamma}\in\mathbb{E}_T^{\varphi}\cap\overline{B_{M,T}}$ it holds*

$$\left\|L_T^1(\gamma)-N_T^1(\gamma)-\left(L_T^1\left(\widetilde{\gamma}\right)-N_T^1\left(\widetilde{\gamma}\right)\right)\right\|_{\frac{\alpha}{4},\alpha}\leq C(\boldsymbol{c},M)T^{\varepsilon}\left\|\gamma-\widetilde{\gamma}\right\|_{\mathbb{E}_T}\,.$$

Proof. Let $T\in\big(0,\widetilde{T}(\boldsymbol{c},M)\big]$ and γ, $\widetilde{\gamma}\in\mathbb{E}_T^{\varphi}\cap\overline{B_{M,T}}$ be fixed. As in Proposition 4.26 we denote by $\mathfrak{p}$ a polynomial of some degree depending on one or several variables that are specified in the brackets. We allow $\mathfrak{p}$ to change from line to line while maintaining the same notation. As in Proposition 4.26 the highest order term is of the form

$$\left(\left|\varphi_x^i\right|-\left|\gamma_x^i\right|\right)\mathfrak{p}\left(\left|\varphi_x^i\right|^{-1},\left|\gamma_x^i\right|^{-1}\right)\left(\gamma_{xxxx}^i-\widetilde{\gamma}_{xxxx}^i\right)+\left(\left|\widetilde{\gamma}_x^i\right|-\left|\gamma_x^i\right|\right)\mathfrak{p}\left(\left|\widetilde{\gamma}_x^i\right|^{-1},\left|\gamma_x^i\right|^{-1}\right)\widetilde{\gamma}_{xxxx}^i\,. \tag{4.37}$$

The Banach algebra property of parabolic Hölder spaces shown in Proposition B.9 yields that the first term in (4.37) can be estimated by

$$\left\|\left|\varphi_x^i\right|-\left|\gamma_x^i\right|\right\|_{\frac{\alpha}{4},\alpha}\left\|\mathfrak{p}\left(\left|\varphi_x^i\right|^{-1},\left|\gamma_x^i\right|^{-1}\right)\right\|_{\frac{\alpha}{4},\alpha}\left\|\gamma-\widetilde{\gamma}\right\|_{\mathbb{E}_T}\,.$$

Using again the Banach algebra property of parabolic Hölder spaces combined with Lemma 4.40 and Corollary 4.46 we obtain

$$\left\|\mathfrak{p}\left(\left|\varphi_x^i\right|^{-1},\left|\gamma_x^i\right|^{-1}\right)\right\|_{\frac{\alpha}{4},\alpha}\leq C(\boldsymbol{c},M).$$

Identifying φ with its constant in time extension $\mathcal{E}\varphi\in\mathbb{E}_T^{\varphi}$, Lemma 4.48 applied to $\varphi\,,\gamma\in\mathbb{E}_T^{\varphi}\cap\overline{B_{M,T}}$ yields that there exists $\varepsilon\in(0,1)$ such that with suitable constants $C(\boldsymbol{c},M)>0$ independent of T there holds

$$\left\|\left|\varphi_x^i\right|-\left|\gamma_x^i\right|\right\|_{\frac{\alpha}{4},\alpha}\leq C(\boldsymbol{c},M)\left\|\varphi_x^i-\gamma_x^i\right\|_{\frac{\alpha}{4},\alpha}\leq C(\boldsymbol{c},M)T^{\varepsilon}\left\|\gamma-\varphi\right\|_{\mathbb{E}_T}\,.$$

Similarly, we may estimate the second term in (4.37) by

$$\left\|\left|\widetilde{\gamma}_x^i\right|-\left|\gamma_x^i\right|\right\|_{\frac{\alpha}{4},\alpha}\left\|\mathfrak{p}\left(\left|\widetilde{\gamma}_x^i\right|^{-1},\left|\gamma_x^i\right|^{-1}\right)\right\|_{\frac{\alpha}{4},\alpha}\left\|\widetilde{\gamma}_{xxxx}^i\right\|_{\frac{\alpha}{4},\alpha}\leq C(\boldsymbol{c},M)\left\|\widetilde{\gamma}_x^i-\gamma_x^i\right\|_{\frac{\alpha}{4},\alpha}\left\|\widetilde{\gamma}\right\|_{\mathbb{E}_T}$$

$$\leq C(\boldsymbol{c},M)T^{\varepsilon}\left\|\gamma-\widetilde{\gamma}\right\|_{\mathbb{E}_T}\,,$$

which allows us to conclude that (4.37) is bounded in $\|\cdot\|_{\frac{\alpha}{4},\alpha}$ by an expression of the form $C(\boldsymbol{c},M)T^{\varepsilon}\left\|\gamma-\widetilde{\gamma}\right\|_{\mathbb{E}_T}$. As shown in Proposition 4.26 the remainder of the expression $L^1_T(\gamma)-N^1_T(\gamma)-\left(L^1_T(\widetilde{\gamma})-N^1_T(\widetilde{\gamma})\right)$ is composed of terms of the form

$$\frac{\left(a^i-\widetilde{a}^i\right)\left\langle b^i,c^i\right\rangle}{|\gamma^i_x|^j}+\frac{\widetilde{a}^i\left\langle b^i-\widetilde{b}^i,c^i\right\rangle}{|\gamma^i_x|^j}+\frac{\widetilde{a}^i\left\langle \widetilde{b}^i,c^i-\widetilde{c}^i\right\rangle}{|\gamma^i_x|^j}+\widetilde{a}^i\left\langle \widetilde{b}^i,\widetilde{c}^i\right\rangle\left(\left|\gamma^i_x\right|^{-j}-\left|\widetilde{\gamma}^i_x\right|^{-j}\right) \tag{4.38}$$

with $j\in\{2,6,8\}$, and a^i, b^i, c^i and $\widetilde{a}^i$, $\widetilde{b}^i$, $\widetilde{c}^i$ spacial derivatives of order at least one and at most three of γ^i and $\widetilde{\gamma}^i$, respectively. Corollary 4.46, Lemma 4.48 and the Banach algebra property of parabolic Hölder spaces yield that all expressions comprised by the first three terms in (4.38) can be estimated by $C(\boldsymbol{c},M)T^{\varepsilon}\left\|\gamma-\widetilde{\gamma}\right\|_{\mathbb{E}_T}$. The arguments in Proposition 4.26 imply that the last term in (4.38) is of the form

$$\widetilde{a}^i\left\langle \widetilde{b}^i,\widetilde{c}^i\right\rangle\left|\left|\gamma^i_x\right|-\left|\widetilde{\gamma}^i_x\right|\right|\mathfrak{p}\left(\left|\gamma^i_x\right|^{-1},\left|\widetilde{\gamma}^i_x\right|^{-1}\right)$$

which can be estimated by $C(\boldsymbol{c},M)T^{\varepsilon}\left\|\gamma-\widetilde{\gamma}\right\|_{\mathbb{E}_T}$ due to the arguments above recalling that

$$\left\|\mathfrak{p}\left(\left|\gamma^i_x\right|^{-1},\left|\widetilde{\gamma}^i_x\right|^{-1}\right)\right\|_{\frac{\alpha}{4},\alpha}\leq C(\boldsymbol{c},M)$$

and

$$\left\|\left|\left|\gamma^i_x\right|-\left|\widetilde{\gamma}^i_x\right|\right|\right\|_{\frac{\alpha}{4},\alpha}\leq C(\boldsymbol{c},M)\left\|\gamma^i_x-\widetilde{\gamma}^i_x\right\|_{\frac{\alpha}{4},\alpha}\leq C(\boldsymbol{c},M)T^{\varepsilon}\left\|\gamma-\widetilde{\gamma}\right\|_{\mathbb{E}_T}\,.$$

□

Proposition 4.50 (Contraction estimates II)**.** *Let $\alpha\in(0,1)$, φ be an analytically α–admissible initial value to* (4.3) *with $\boldsymbol{c}$ as in Lemma 4.40, $T_0:=1$, $M\geq\|\varphi\|_{4+\alpha}$ and $\widetilde{T}(\boldsymbol{c},M)$ be as in Lemma 4.45. There exist $\varepsilon\in(0,1)$ and a constant $C>0$ depending on $\boldsymbol{c}$ and M such that for all $T\in\left(0,\widetilde{T}(\boldsymbol{c},M)\right]$, γ, $\widetilde{\gamma}\in\mathbb{E}^{\varphi}_T\cap\overline{B_{M,T}}$ it holds*

$$\left\|L^2_T(\gamma)-N^2_T(\gamma)-\left(L^2_T(\widetilde{\gamma})-N^2_T(\widetilde{\gamma})\right)\right\|_{C^{\frac{1+\alpha}{4}}([0,T];\mathbb{R}^2)}\leq C(\boldsymbol{c},M)T^{\varepsilon}\left\|\gamma-\widetilde{\gamma}\right\|_{\mathbb{E}_T}\,.$$

Proof. Let $M\geq\|\varphi\|_{4+\alpha}$ be given and $T\in\left(0,\widetilde{T}(\boldsymbol{c},M)\right]$, $\gamma\,,\widetilde{\gamma}\in\mathbb{E}^{\varphi}_T\cap\overline{B_{M,T}}$ be fixed. To improve readability we omit in the following the evaluation of the terms in $x=0$. We let ν^i_0, τ^i_0, ν^i, τ^i and $\widetilde{\nu}^i$, $\widetilde{\tau}^i$, $i\in\{1,2,3\}$, be the unit normal and tangent of φ^i, γ^i and $\widetilde{\gamma}^i$, respectively. The arguments in Proposition 4.27 yield that the expression $L^2_T(\gamma)-N^2_T(\gamma)-\left(L^2_T(\widetilde{\gamma})-N^2_T(\widetilde{\gamma})\right)$ can be written as

$$\sum_{i=1}^{3}\frac{1}{|\gamma^i_x|^3}\left\langle\widetilde{\gamma}^i_{xxx},\nu^i-\widetilde{\nu}^i\right\rangle\nu^i+\sum_{i=1}^{3}\frac{1}{|\gamma^i_x|^3}\left\langle\widetilde{\gamma}^i_{xxx},\widetilde{\nu}^i\right\rangle\left(\nu^i-\widetilde{\nu}^i\right) \tag{4.39}$$

$$+\sum_{i=1}^{3}\frac{1}{|\varphi^i_x|^3}\left\langle(\widetilde{\gamma}^i_{xxx}-\gamma^i_{xxx}),\nu^i_0-\nu^i\right\rangle\nu^i_0+\sum_{i=1}^{3}\frac{1}{|\varphi^i_x|^3}\left\langle(\widetilde{\gamma}^i_{xxx}-\gamma^i_{xxx}),\nu^i\right\rangle\left(\nu^i_0-\nu^i\right) \tag{4.40}$$

$$+\sum_{i=1}^{3}\left(\frac{1}{|\gamma^i_x|^3}-\frac{1}{|\widetilde{\gamma}^i_x|^3}\right)\left\langle\widetilde{\gamma}^i_{xxx},\widetilde{\nu}^i\right\rangle\widetilde{\nu}^i+\sum_{i=1}^{3}\left(\frac{1}{|\varphi^i_x|^3}-\frac{1}{|\gamma^i_x|^3}\right)\left\langle\widetilde{\gamma}^i_{xxx}-\gamma^i_{xxx},\nu^i\right\rangle\nu^i \tag{4.41}$$

$$-\frac{\mu}{2}\sum_{i=1}^{3}\left(\frac{\gamma^i_x}{|\gamma^i_x|}-\frac{\widetilde{\gamma}^i_x}{|\widetilde{\gamma}^i_x|}\right). \tag{4.42}$$

Due to the Banach algebra property of $C^{\frac{1+\alpha}{4}}([0,T])$ we may consider the terms in (4.39) to (4.42) individually. Proposition B.8 implies

$$\left\|\gamma^i_{xxx}(0)-\widetilde{\gamma}^i_{xxx}(0)\right\|_{C^{\frac{1+\alpha}{4}}([0,T];\mathbb{R}^2)} \le \left\|\gamma^i-\widetilde{\gamma}^i\right\|_{\frac{4+\alpha}{4},4+\alpha} \le \|\gamma-\widetilde{\gamma}\|_{\mathbb{E}_T}\ ,$$

and further

$$\left\|\gamma^i_{xxx}(0)\right\|_{C^{\frac{1+\alpha}{4}}([0,T];\mathbb{R}^2)} \le \left\|\gamma^i\right\|_{\frac{4+\alpha}{4},4+\alpha} \le \|\gamma\|_{\mathbb{E}_T} \le M$$

and analogously for $\widetilde{\gamma}$. Furthermore, Lemma 4.40 and Lemma 4.45 yield for all $j\in\mathbb{N}$,

$$\left\|\left|\varphi^i_x(0)\right|^{-j}\right\|_{C^{\frac{1+\alpha}{4}}([0,T])} \le C(\boldsymbol{c},M)$$

and

$$\left\|\left|\gamma^i_x(0)\right|^{-j}\right\|_{C^{\frac{1+\alpha}{4}}([0,T])} \le C(\boldsymbol{c},M)$$

and analogously for $\widetilde{\gamma}$. Given $\xi\,,\eta\in\mathbb{E}^\varphi_T\cap\overline{B_{M,T}}$ Proposition B.8 implies

$$\xi^i_x(0)-\eta^i_x(0)\in C_0^{\frac{3+\alpha}{4}}\left([0,T];\mathbb{R}^2\right)$$

and thus using Proposition B.2 we conclude

$$\left\|\xi^i_x(0)-\eta^i_x(0)\right\|_{C^{\frac{1+\alpha}{4}}([0,T];\mathbb{R}^2)} \le 2T^{\frac{2+\alpha}{4}}\left\|\xi^i_x(0)-\eta^i_x(0)\right\|_{C^{\frac{3+\alpha}{4}}([0,T];\mathbb{R}^2)} \le 2T^{\frac{2+\alpha}{4}}\|\xi-\eta\|_{\mathbb{E}_T}\ .$$

Using Lemma 4.45 and the arguments in the proof of Proposition B.4 we find

$$\begin{aligned}&\left\|\left|\xi^i_x(0)\right|^{-1}-\left|\eta^i_x(0)\right|^{-1}\right\|_{C^{\frac{1+\alpha}{4}}([0,T])}\\ &\le C\Big(\boldsymbol{c},\left\|\xi^i_x(0)\right\|_{C^{\frac{1+\alpha}{4}}([0,T];\mathbb{R}^2)},\left\|\eta^i_x(0)\right\|_{C^{\frac{1+\alpha}{4}}([0,T];\mathbb{R}^2)}\Big)\left\|\xi^i_x(0)-\eta^i_x(0)\right\|_{C^{\frac{1+\alpha}{4}}([0,T];\mathbb{R}^2)}\\ &\le C(\boldsymbol{c},M)T^{\frac{2+\alpha}{4}}\|\xi-\eta\|_{\mathbb{E}_T}\ .\end{aligned}$$

Using further the identity

$$\left|\xi^i_x(0)\right|^{-3}-\left|\eta^i_x(0)\right|^{-3}=\left(\left|\xi^i_x(0)\right|^{-1}-\left|\eta^i_x(0)\right|^{-1}\right)\mathfrak{p}\left(\left|\xi^i_x(0)\right|^{-1},\left|\eta^i_x(0)\right|^{-1}\right)$$

shown in Proposition 4.27, we find

$$\left\|\left|\xi^i_x(0)\right|^{-3}-\left|\eta^i_x(0)\right|^{-3}\right\|_{C^{\frac{1+\alpha}{4}}([0,T])} \le C(\boldsymbol{c},M)T^{\frac{2+\alpha}{4}}\|\xi-\eta\|_{\mathbb{E}_T}\ .$$

Finally, we observe

$$\begin{aligned}&\left\|\xi^i_x(0)\left|\xi^i_x(0)\right|^{-1}-\eta^i_x(0)\left|\eta^i_x(0)\right|^{-1}\right\|_{C^{\frac{1+\alpha}{4}}([0,T];\mathbb{R}^2)}\\ &\le\left\|\xi^i_x(0)\right\|_{\frac{1+\alpha}{4}}\left\|\left|\xi^i_x(0)\right|^{-1}-\left|\eta^i_x(0)\right|^{-1}\right\|_{\frac{1+\alpha}{4}}+\left\|\xi^i_x(0)-\eta^i_x(0)\right\|_{\frac{1+\alpha}{4}}\left\|\left|\eta^i_x(0)\right|^{-1}\right\|_{\frac{1+\alpha}{4}}\\ &\le C(\boldsymbol{c},M)T^{\frac{2+\alpha}{4}}\|\xi-\eta\|_{\mathbb{E}_T}\ .\end{aligned}$$

As the unit normal is obtained by applying the counter-clockwise rotation $R\in\mathbb{R}^{2\times 2}$ by the angle $\frac{\pi}{2}$ to the unit tangent, the claim follows combining the above observations with the respective choices of $\xi\,,\eta\in\{\varphi,\gamma,\widetilde{\gamma}\}$. □

Corollary 4.51. *Let $\alpha\in(0,1)$, φ be an analytically α–admissible initial value to* (4.3) *with $\boldsymbol{c}$ as in Lemma 4.40, $T_0:=1$, $M\ge\|\varphi\|_{4+\alpha}$ and $\widetilde{T}(\boldsymbol{c},M)$ be as in Lemma 4.45. There exists a time $T_*(\boldsymbol{c},M)\in\big(0,\widetilde{T}(\boldsymbol{c},M)\big]$ such that for every $T\in\big(0,T_*(\boldsymbol{c},M)\big]$ the operator*

$$K_T:\left(\mathbb{E}^\varphi_T\cap\overline{B_{M,T}},\|\cdot\|_{\mathbb{E}_T}\right)\to\left(\mathbb{E}^\varphi_T,\|\cdot\|_{\mathbb{E}_T}\right)$$

is a contraction.

Proof. Let $M \geq \|\varphi\|_{4+\alpha}$, $T \in \big(0, \widetilde{T}(\boldsymbol{c}, M)\big]$ and $\gamma\,, \widetilde{\gamma} \in \mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$ be given. Then K_T is well-defined due to Proposition 4.47. Moreover, Theorem 4.43 and the contraction estimates in Proposition 4.49 and Proposition 4.50 yield that there exist constants $C(\boldsymbol{c}, M) > 0$ and $\varepsilon \in (0,1)$ such that

$$\|K_T(\gamma) - K_T(\widetilde{\gamma})\|_{\mathbb{E}_T} \leq C(\boldsymbol{c}, M) T^{\varepsilon} \|\gamma - \widetilde{\gamma}\|_{\mathbb{E}_T} \,.$$

Now choose $T_* = T_*(\boldsymbol{c}, M) \in \big(0, \widetilde{T}(\boldsymbol{c}, M)\big]$ such that $C(\boldsymbol{c}, M) T_*^{\varepsilon} < 1$. □

Proposition 4.52. *Let $\alpha \in (0,1)$ and φ be an analytically α–admissible initial value to* (4.3) *with $\boldsymbol{c}$ as in Lemma 4.40. There exists a radius $M(\varphi) \geq \|\varphi\|_{4+\alpha} > 0$ and a positive time $\boldsymbol{T}(\varphi)$ depending on $\boldsymbol{c}$ and $\|\varphi\|_{4+\alpha}$ such that for all $T \in (0, \boldsymbol{T}(\varphi))$ the set $\mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$ is non-empty and the operator*

$$K_T : \left(\mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}, \|\cdot\|_{\mathbb{E}_T}\right) \to \left(\mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}, \|\cdot\|_{\mathbb{E}_T}\right)$$

is a contractive self-mapping.

Proof. We identify $\varphi \in C^{4+\alpha}\left([0,1]; (\mathbb{R}^2)^3\right)$ with its constant in time extension $\mathcal{E}\varphi \in \mathbb{E}_1$ and observe that for all $t \in [0,1]$, $i \in \{1,2,3\}$, $(\mathcal{E}\varphi)^i\,(t)$ is a regular open curve. In particular, for all $T \in [0,1]$, the expression $N_T\,(\mathcal{E}\varphi)$ is well-defined and so is $K_T\,(\mathcal{E}\varphi) \in \mathbb{E}_T$. Furthermore, for all $T \in [0,1]$ there holds $K_T\,(\mathcal{E}\varphi) = K_1\,(\mathcal{E}\varphi)_{|[0,T]}$. As the operator norm of L_T^{-1} is uniformly bounded with respect to $T \in [0,1]$ and as the norm of $N_1\,(\mathcal{E}\varphi)$ depends only on $\|\varphi\|_{4+\alpha}$ and $\boldsymbol{c}$, the radius

$$M := 2 \max\left\{\|\varphi\|_{4+\alpha}\,, \|K_1\,(\mathcal{E}\varphi)\|_{\mathbb{E}_1}\right\}$$

depends only on $\|\varphi\|_{4+\alpha}$ and $\boldsymbol{c}$. Moreover, for all $T \in (0,1]$, $\mathcal{E}\varphi$ lies in $\mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$ and satisfies $\|K_T\,(\mathcal{E}\varphi)\|_{\mathbb{E}_T} \leq M/2$. Let $T_*(\boldsymbol{c}, M)$ be the corresponding time in Corollary 4.51. Then for all $T \in \big(0, T_*(\boldsymbol{c}, M)\big]$, $\gamma \in \mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$ it holds

$$\|K_T(\gamma) - K_T\,(\mathcal{E}\varphi)\|_{\mathbb{E}_T} \leq C(\boldsymbol{c}, M) T^{\varepsilon} \|\gamma - \mathcal{E}\varphi\|_{\mathbb{E}_T} \leq C(\boldsymbol{c}, M) T^{\varepsilon} 2M \,. \tag{4.43}$$

We let $\boldsymbol{T}(\varphi) \in \big(0, T_*(\boldsymbol{c}, M)\big]$ be so small that for all $T \in (0, \boldsymbol{T}(\varphi)]$, the constants in (4.43) satisfy $C(\boldsymbol{c}, M) T^{\varepsilon} 2M \leq M/2$. Let $T \in (0, \boldsymbol{T}(\varphi)]$ be given. Due to Corollary 4.51 the map $K_T : \left(\mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}, \|\cdot\|_{\mathbb{E}_T}\right) \to \left(\mathbb{E}_T^{\varphi}, \|\cdot\|_{\mathbb{E}_T}\right)$ is a contraction and using (4.43) we find for all $\gamma \in \mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$,

$$\|K_T(\gamma)\|_{\mathbb{E}_T} \leq \|K_T(\gamma) - K_T\,(\mathcal{E}\varphi)\|_{\mathbb{E}_T} + \|K_T\,(\mathcal{E}\varphi)\|_{\mathbb{E}_T} \leq M/2 + M/2 \leq M \,.$$

□

Theorem 4.53 (Existence and uniqueness of classical solutions to the analytic problem (Triod))**.** *Let $\alpha \in (0,1)$ and φ be an analytically α–admissible initial datum to system* (4.3)*. There exists a positive time $\boldsymbol{T}$ depending only on $\|\varphi\|_{4+\alpha}$ and $\boldsymbol{c}(\varphi)$ such that for all $T \in (0, \boldsymbol{T}]$ there exists a classical solution to system* (4.3) *in $[0,T]$ with initial datum φ which is unique in $\mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$ with M as in Proposition 4.52.*

Proof. Let $M = M(\varphi)$ and $\boldsymbol{T}(\varphi)$ as in Proposition 4.52. The Banach Fixed-Point Theorem [151, Theorem 1.A] yields that for all $T \in (0, \boldsymbol{T}(\varphi)]$ the map K_T possesses a unique fixed point in $\mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$ which is by construction a classical solution to system (4.3). The uniqueness assertion follows from the uniqueness of the fixed point. □

We state the corresponding result in the case of Theta networks.

Theorem 4.54 (Existence and uniqueness of classical solutions to the analytic problem (Theta))**.** *Let $\alpha \in (0,1)$ and φ be an analytically α–admissible initial datum to system* (4.2)*. There exists* *a positive time $\boldsymbol{T}$ depending only on $\|\varphi\|_{4+\alpha}$ and $\boldsymbol{c}(\varphi)$ such that for all $T \in (0, \boldsymbol{T}]$ there exists a classical solution to the system (4.2) in $[0,T]$ with initial datum φ which is unique in $\mathbb{E}_T^{\varphi} \cap \overline{B_{M,T}}$ with M as in Proposition 4.52.*

Proof. As explained in Remark 4.9 the proof for system (4.3) can be adapted to prove the analogous result for system (4.2). □

4.2 Existence and uniqueness of the geometric problem

In this section we turn to the question of existence and uniqueness of the geometric problems (3.20) and (3.21) given appropriate initial networks. The arguments given in [64, Section 3.5] and [65, Section 5] are worked out in detail. For the sake of readability and clarity, we again show all results in an exemplary manner for system (3.21). As all conditions appearing in system (3.20) also appear in (3.21), the arguments can easily be adapted to prove the corresponding conclusions for system (3.20) which are stated at the end of each subsection.

4.2.1 Existence and uniqueness of strong solutions

In this subsection we show that given $p \in (5, \infty)$ and any geometrically p–admissible initial network σ to the elastic flow for Triods (3.21) as defined in Definition 3.37, there exists a strong solution γ solving (3.21) as defined in Definition 3.43 in a (possibly small) time interval $[0,T]$ with initial datum σ. Furthermore, strong solutions to (3.21) are geometrically unique in the sense that any two solutions γ and $\widetilde{\gamma}$ to (3.21) with the same initial datum σ coincide as sets in their common time interval of existence. We hereby follow the arguments in [66, Chapter 5].

The existence of strong solutions to the geometric problem is a consequence of the existence of the auxiliary analytic problem (4.3) shown in Theorem 4.30 once it is shown that every geometrically p–admissible initial network to (3.21) can be reparametrised to an analytically p–admissible initial network to system (4.3). To this end, we need the following lemma.

Lemma 4.55. *Given a, $b \in \mathbb{R}$ there exists a smooth diffeomorphism $\theta : [0,1] \to [0,1]$ such that $\theta(0) = 0$, $\theta(1) = 1$, $\theta_x(0) = \theta_x(1) = 1$, $\theta_{xx}(0) = a$, $\theta_{xx}(1) = b$ and $\theta_x(x) \geq \frac{1}{8}$ for all $x \in [0,1]$.*

Proof. We let p and q be the second order polynomials on $\mathbb{R}$ determined by the constraints $p(0) = 0$, $p_x(0) = 1$, $p_{xx}(0) = a$ and $q(1) = 1$, $q_x(1) = 1$, $q_{xx}(1) = b$, respectively. Let $\delta \in (0, 1/6)$ be such that for all $x \in [0, 3\delta]$ it holds $p(x) \leq \frac{1}{4}$ and $p_x(x) \in \left[\frac{3}{4}, \frac{5}{4}\right]$, and for all $x \in [1-3\delta, 1]$, $q(x) \geq \frac{3}{4}$, and $q_x(x) \in \left[\frac{3}{4}, \frac{5}{4}\right]$. Let g be the linear function on $\mathbb{R}$ with $g(3\delta) = p(3\delta)$ and $g(1-3\delta) = q(1-3\delta)$. Then the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} p(x) & x \in [0, 3\delta]\,, \\ g(x) & x \in [3\delta, 1-3\delta]\,, \\ q(x) & x \in [1-3\delta, 1]\,, \end{cases}$$

satisfies $f\left([0,1]\right) = [0,1]$ and $f \in C^{0,1}([0,1];\mathbb{R})$. Furthermore, f is differentiable almost everywhere with $f_x(x) \geq \frac{1}{2}$ for almost every $x \in [0,1]$. Let $(\psi^{\varepsilon})_{\varepsilon>0}$ be the Standard–Dirac sequence on $\mathbb{R}$. Then for every $\varepsilon > 0$ the convolution $f * \psi^{\varepsilon}$ lies in $C^{\infty}([0,1];\mathbb{R})$ and the sequence $(f * \psi^{\varepsilon})$ converges to f in $C([0,1];\mathbb{R})$ as $\varepsilon \searrow 0$. Moreover, for almost every $x \in (\varepsilon, 1-\varepsilon)$ we have

$$\left(f * \psi^{\varepsilon}\right)_x(x) = \int_{-1}^{1} \psi^1(y) f_x(x - \varepsilon y)\,\mathrm{d}y \geq \frac{1}{2}\,.$$

Let $\eta \in C_0^\infty((0,1))$ be a cut-off function satisfying $0 \le \eta \le 1$ on $[0,1]$, $\eta \equiv 1$ on $[2\delta, 1-2\delta]$, $\operatorname{supp}\eta \subset (\delta, 1-\delta)$ and $|\partial_x^k \eta| \le C_k \delta^{-k}$ for all $k \in \mathbb{N}$, see for example [4, 4.19]. Furthermore, we let $\varepsilon \in (0, \delta/2)$ be so small that $\|f * \psi^\varepsilon - f\|_{C([0,1];\mathbb{R})} < \frac{\delta}{C_1 10}$. We define $\theta : [0,1] \to \mathbb{R}$ by

$$\theta(x) := (1-\eta(x))\, f(x) + \eta(x)\,(f * \psi^\varepsilon)\,(x)\,.$$

It follows from the construction that θ is smooth on $[0,1]$ and satisfies the constraints at the boundary points. For $x \in [0,\delta] \cup [1-\delta, 1]$ it holds that $\theta_x(x) = f_x(x) \ge \frac{3}{4}$. By the choice of ε we have for every $x \in (\delta, 2\delta) \cup (1-2\delta, 1-\delta)$,

$$\begin{aligned} \left|(f * \psi^\varepsilon)_x\,(x) - f(x)\right| &\le \int_{-1}^{1} \psi^1(y)\,|f_x(x-\varepsilon y) - f_x(x)|\,\mathrm{d}y \\ &\le \sup_{x,y \in (0,3\delta)\cup(1-3\delta,1)} |f_x(x) - f_x(y)| \le \frac{1}{2}\,. \end{aligned}$$

Thus for almost every $x \in (\delta, 2\delta) \cup (1-2\delta, 1-\delta)$ we obtain

$$\theta_x(x) = f_x(x) + \eta(x)\left((f * \psi^\varepsilon)_x\,(x) - f_x(x)\right) + \eta_x(x)\left((f * \psi^\varepsilon)\,(x) - f(x)\right) \ge \frac{3}{4} - \frac{1}{2} - \frac{1}{10} \ge \frac{1}{8}\,.$$

Moreover, we observe for almost every $x \in [2\delta, 1-2\delta] \subset (\varepsilon, 1-\varepsilon)$,

$$\theta_x(x) = (f * \psi^\varepsilon)_x\,(x) \ge \frac{1}{2}\,.$$

By continuity of θ_x the estimates hold pointwise in the respective sets. As $\theta(0) = 0$, $\theta(1) = 1$ and θ is increasing, there holds $\theta([0,1]) = [0,1]$. The statement now follows from [151, Corollary 4.37, Theorem 4.G]. ☐

Proposition 4.56. *Let $p \in (5,\infty)$ and σ be a geometrically p–admissible initial network to system* (3.21) *with given endpoints P^1, P^2, P^3. There exist smooth diffeomorphisms $\theta^i : [0,1] \to [0,1]$, $i \in \{1,2,3\}$, such that $\varphi = \left(\varphi^1, \varphi^2, \varphi^3\right)$ defined by $\varphi^i := \sigma^i \circ \theta^i$ is an analytically p–admissible initial network to* (4.3) *with endpoints P^1, P^2, P^3.*

Proof. Suppose that σ is a geometrically p–admissible initial network to system (3.21) with given endpoints P^1, P^2, P^3. Lemma 4.55 implies that there exist smooth diffeomorphisms $\theta^i : [0,1] \to [0,1]$, $i \in \{1,2,3\}$, such that $\theta^i(0) = 0$, $\theta^i(1) = 1$, $\theta^i_x(0) = \theta^i_x(1) = 1$ and $\theta^i_{xx}(y) = -\left|\sigma^i_x(y)\right|^{-2}\left\langle \sigma^i_{xx}(y), \sigma^i_x(y)\right\rangle$ for $y \in \{0,1\}$. Lemma 3.38 yields that $\varphi^i := \sigma^i \circ \theta^i$ lies in $W_p^{4-4/p}\left((0,1);\mathbb{R}^2\right)$. Moreover, $\varphi^1(0) = \varphi^2(0) = \varphi^3(0)$, $\varphi^i(1) = \sigma^i(1) = P^i$, $i \in \{1,2,3\}$, and φ^i is a regular open curve being a reparametrisation of σ^i. As the normal and the tangent as well as the curvature and its arclength derivative are invariant under reparametrisation, the non-degeneracy condition (3.22) at $y = 0$ as well as the third order condition $\sum_{i=1}^3 2\kappa^i_s(0)\nu^i(0) - \mu^i\tau^i(0) = 0$ and the curvature condition $\kappa^i(y) = 0$, $y \in \{0,1\}$, remain satisfied. The special choices of $\theta^i_x(y)$ and $\theta^i_{xx}(y)$, $y \in \{0,1\}$, guarantee that the reparametrised curves φ^i satisfy the second order condition (4.1). Indeed, for $y \in \{0,1\}$, we have

$$\begin{aligned} \varphi^i_{xx}(y) &= \sigma^i_{xx}(y)\theta^i_x(y)^2 + \sigma^i_x(y)\theta^i_{xx}(y) = \sigma^i_{xx}(y) - \sigma^i_x(y)\left|\sigma^i_x(y)\right|^{-2}\left\langle \sigma^i_{xx}(y), \sigma^i_x(y)\right\rangle \\ &= \left|\sigma^i_x(y)\right|^2 \boldsymbol{\kappa}^i(y) = 0\,. \end{aligned}$$

This shows that φ is an analytically p–admissible initial network to (4.3) with endpoints P^1, P^2, P^3. ☐

Remark 4.57. Let $p \in (5,\infty)$ and σ be a geometrically p–admissible initial network to system (3.20). Defining smooth diffeomorphisms $\theta^i : [0,1] \to [0,1]$, $i \in \{1,2,3\}$, in precise analogy to Proposition 4.56 we obtain that $\varphi^i := \sigma^i \circ \theta^i$, $i \in \{1,2,3\}$, is an analytically p–admissible initial value to (4.2).

The following result is in analogy to [66, Theorem 5.2].

Theorem 4.58 (Existence of strong solutions to the geometric problem)**.** *Let $p \in (5,\infty)$ and σ be a geometrically p–admissible initial network to system* (3.21) *with given endpoints P^1, P^2, P^3. Then there exists a positive time $\boldsymbol{T}$ such that for all $T \in (0,\boldsymbol{T}]$ there exists a strong solution to system* (3.21) *in $[0,T]$ with endpoints P^1, P^2, P^3 and initial datum σ.*

Proof. Given endpoints P^1, P^2, P^3 and a geometrically p–admissible initial network σ to the corresponding system (3.21), Proposition 4.56 implies that there exist reparametrisations φ^i of σ^i, $i \in \{1,2,3\}$, such that $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ is an analytically p–admissible initial value to system (4.3) with endpoints P^1, P^2, P^3. Theorem 4.30 yields the existence of a positive time $\boldsymbol{T}$ such that for all $T \in (0,\boldsymbol{T}]$ there exists a strong solution γ to (4.3) in $[0,T]$ with endpoints P^1, P^2, P^3 and initial value φ. Comparing Definitions 3.43 and 4.7 we observe that being a strong solution to the analytic problem (4.3) in $[0,T]$ with endpoints P^1, P^2, P^3 and initial datum φ, the function γ is in particular a strong solution to the geometric problem (3.21) in $[0,T]$ with endpoints P^1, P^2, P^3 and initial datum σ. □

In the following result we show that any strong solution to the geometric problem (3.21) can be reparametrised in such a way that it evolves with the tangential velocity chosen in (3.26) and satisfies the additional boundary condition (3.28). We hereby follow the arguments in [66, Theorem 5.3].

Theorem 4.59. *Let $p \in (5,\infty)$, T be positive and η be a strong solution to* (3.21) *in the time interval $[0,T]$ with endpoints P^1, P^2, P^3. There exists a time $T_R \in (0,T]$ and a function $\psi \in W^{1,4}_p\left((0,T_R)\times(0,1);\mathbb{R}^3\right)$ such that for all $t \in [0,T_R]$, $i \in \{1,2,3\}$, $\psi^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism such that $\gamma = (\gamma^1,\gamma^2,\gamma^3)$ with $\gamma^i(t,x) := \eta^i\left(t,\psi^i(t,x)\right)$ is a strong solution to* (4.3) *in $[0,T_R]$ with endpoints P^1, P^2, P^3 and initial datum $\eta(0)\circ\psi(0)$. The existence time T_R depends on $\|\|\eta\|\|_{\boldsymbol{E}_T}$ and $\boldsymbol{c}(\eta) = \min_{i\in\{1,2,3\}} \min_{x\in[0,1],t\in[0,T]} \left|\eta^i_x(t,x)\right|$.*

Proof. As $\eta(0)$ is a geometrically p–admissible initial value to (3.21), Proposition 4.56 yields that there exist smooth diffeomorphisms $\theta^i : [0,1] \to [0,1]$, $i \in \{1,2,3\}$, such that $\varphi = (\varphi^1,\varphi^2,\varphi^3)$ defined by $\varphi^i := \eta^i(0)\circ\theta^i$ is an analytically p–admissible initial value to (4.3) with endpoints P^1, P^2, P^3. We thus take θ as an initial datum for the unknown ψ. Suppose that $\psi \in W^{1,4}_p\left((0,T)\times(0,1);\mathbb{R}^3\right)$ is such that for all $t \in [0,T]$, $i \in \{1,2,3\}$, $\psi^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism. Then Lemma B.45 implies that the composition γ with $\gamma^i(t,x) := \eta^i\left(t,\psi^i(t,x)\right)$ lies in $\boldsymbol{E}_T$ and the derivatives can be calculated by the chain rule. This allows us to derive necessary and sufficient conditions for the reparametrisation ψ to guarantee that γ is a strong solution to (4.3). Indeed, for almost every $t \in (0,T)$, $x \in (0,1)$,

$$\gamma^i_t(t,x) = \eta^i_t\left(t,\psi^i(t,x)\right) + \eta^i_x\left(t,\psi^i(t,x)\right)\psi^i_t(t,x)\,. \tag{4.44}$$

Testing (4.44) with the normal $\nu^i_\eta\left(t,\psi^i(t,x)\right) = \nu^i_\gamma(t,x)$ yields the requirement that the normal velocity of γ^i is equal to the normal velocity of η^i which is fulfilled regardless of the choice of reparametrisation ψ. On the contrary, ψ does play a role in determining the tangential velocity of the evolution. Testing (4.44) with $\eta^i_x\left(t,\psi^i(t,x)\right)$ yields

$$\psi^i_t(t,x) = \left|\eta^i_x\left(t,\psi^i(t,x)\right)\right|^{-2}\left(\left\langle\gamma^i_t(t,x) - \eta^i_t\left(t,\psi^i(t,x)\right), \eta^i_x\left(t,\psi^i(t,x)\right)\right\rangle\right).$$

To guarantee that γ^i solves (3.27) we need to require

$$\left|\eta^i_x\left(t,\psi^i(t,x)\right)\right|^{-1}\left\langle\gamma^i_t(t,x),\eta^i_x\left(t,\psi^i(t,x)\right)\right\rangle=\overline{T}\left(\gamma^i\right)(t,x)\,.$$

The expression $\overline{T}$ is defined in formula (3.26) and can be expressed in terms of η^i and ψ^i. Indeed,

$$\overline{T}\left(\gamma^i\right)(t,x)=\frac{-2\psi^i_{xxxx}(t,x)}{\left|\psi^i_x(t,x)\right|^4\left|\eta^i_x\left(t,\psi^i(t,x)\right)\right|^3}+g\left(\psi^i,\eta^i\right)(t,x)$$

where $g\left(\psi^i,\eta^i\right)(t,x)$ comprises terms depending on $\left(\partial^k_x\eta^i\right)\left(t,\psi^i(t,x)\right)$, $k\in\{1,2,3,4\}$, and further $\partial^m_x\psi^i(t,x)$, $m\in\{1,2,3\}$, with $\left(\partial^4_x\eta^i\right)\left(t,\psi^i(t,x)\right)$ appearing linearly. Hence, we obtain the evolution equation

$$\psi^i_t(t,x)=\frac{-2\psi^i_{xxxx}(t,x)}{\left|\psi^i_x(t,x)\right|^4\left|\eta^i_x\left(t,\psi^i(t,x)\right)\right|^4}-F\left(\psi^i,\eta^i\right)(t,x)$$

with

$$F\left(\eta^i,\psi^i\right)(t,x):=-\left|\eta^i_x\left(t,\psi^i(t,x)\right)\right|^{-1}\left(g\left(\psi^i,\eta^i\right)(t,x)-\left\langle\eta^i_t\left(t,\psi^i(t,x)\right),\tau^i\left(t,\psi^i(t,x)\right)\right\rangle\right).$$

Comparing systems (3.21) and (4.3) then yields that the functions $(t,x)\mapsto\eta^i(t,\psi^i(t,x))$ with $i\in\{1,2,3\}$ form a strong solution to (4.3) with initial datum $\eta(0)\circ\theta$ if and only if for all $t\in[0,T]$, $x\in[0,1]$ and $y\in\{0,1\}$,

$$\begin{cases}\psi^i_t(t,x)+\frac{2\psi^i_{xxxx}(t,x)}{|(\eta^i_x)(t,\psi^i(t,x))|^4|\psi^i_x(t,x)|^4}+F\left(\psi^i,\eta^i\right)(t,x) &=0\,,\\ \psi^i(t,y)&=y\,,\\ \psi^i_{xx}(t,y)+\left|\eta^i_x(t,y)\right|^{-2}\left\langle\eta^i_{xx}(t,y),\eta^i_x(t,y)\right\rangle\psi^i_x(t,y)^2&=0\,,\\ \psi^i(0,x)&=\theta^i(x)\,.\end{cases}\tag{4.45}$$

If there exists $T_R\in(0,T)$ and a solution $\psi\in W^{1,4}_p\left((0,T_R)\times(0,1);\mathbb{R}\right)$ such that for all $t\in[0,T_R]$, $i\in\{1,2,3\}$, $\psi^i(t):[0,1]\to[0,1]$ is a C^1–diffeomorphism, then every curve $\gamma^i(t)=\eta^i(t)\circ\psi^i(t)$ is regular being a reparametrisation of $\eta^i(t)$ and inherits all *geometric* boundary conditions in (4.3) from η. Also the non-degeneracy condition in $y=0$ remains satisfied. It remains to show that in a short time interval, system (4.45) admits a strong solution ψ with the desired properties. This is shown in Lemma 4.60.

Observe that due to Proposition 4.56 the initial datum θ to (4.45) is smooth and satisfies the boundary conditions of system (4.45). Furthermore, its norm in $W^{4-4/p}_p\left((0,1);\mathbb{R}^3\right)$ depends on $\boldsymbol{c}(\eta)$ and $|||\eta|||_{\mathbb{E}_T}$. To obtain existence of a strong solution to (4.45), it is enough to require that the initial datum θ lies in $W^{4-4/p}_p\left((0,1);\mathbb{R}^3\right)$, satisfies the boundary conditions appearing in system (4.45) and that for all $i\in\{1,2,3\}$, $\theta^i:[0,1]\to[0,1]$ is a C^1–diffeomorphism, see Lemma 4.60. □

Lemma 4.60. *Let $p\in(5,\infty)$, T be positive and η be a strong solution to* (3.21) *in $[0,T]$ with endpoints P^1, P^2, P^3. Given $i\in\{1,2,3\}$ we let $\theta^i:[0,1]\to[0,1]$ be a C^1–diffeomorphism such that $\theta^i\in W^{4-4/p}_p((0,1))$ for $y\in\{0,1\}$, $\theta^i(y)=y$ and*

$$\theta^i_{xx}(y)=-\left\langle\eta^i_{xx}(0,y),\eta^i_x(0,y)\right\rangle\theta^i_x(y)^2\left|\eta^i_x(0,y)\right|^{-2}\,.$$

Then there exists a time $T_R\in(0,T]$ and C^1–diffeomorphisms $\psi^i(t):[0,1]\to[0,1]$, $i\in\{1,2,3\}$, $t\in[0,T_R]$, such that $\psi=\left(\psi^1,\psi^2,\psi^3\right)\in W^{1,4}_p\left((0,T_R)\times(0,1);\mathbb{R}^3\right)=:E_{T_R}$ is a strong solution *to* (4.45)*. The existence time* T_R *depends on* $|||\eta|||_{\boldsymbol{E}_T}$*,* $\boldsymbol{c}(\eta)$*, and further on* $\|\theta\|_{W_p^{4-4/p}((0,1);\mathbb{R}^3)}$ *and* $\boldsymbol{c}(\theta) := \min_{i\in\{1,2,3\},x\in[0,1]} \left|\theta_x^i(x)\right|$ *and* ψ *satisfies*

$$\boldsymbol{c}(\psi) := \min_{i\in\{1,2,3\}} \min_{t\in[0,T],x\in[0,1]} \left|\psi^i(t,x)\right| \geq \frac{1}{2}\boldsymbol{c}(\theta)\,, \tag{4.46}$$

$$|||\psi|||_{E_{T_R}} \leq C\big(c(\theta)\,, \|\theta\|_{W_p^{4-4/p}((0,1);\mathbb{R}^3)}\,, \boldsymbol{c}(\eta),\, |||\eta|||_{\boldsymbol{E}_T}\big). \tag{4.47}$$

Proof. We observe that the expression $F\left(\psi^i,\eta^i\right)(t,x)$ in system (4.45) contains terms of the form $f^i\left(t,\psi^i(t,x)\right)$ with $f^i \in L_p\left((0,T);L_p\left((0,1)\right)\right)$. To remove the dependence of f^i on the solution $\psi^i(t,x)$, it is convenient to consider the associated problem for the inverse $\xi = \left(\xi^1,\xi^2,\xi^3\right)$ defined by $\xi^i(t) := \psi^i(t)^{-1}$. Indeed, suppose that $\psi \in W_p^{1,4}\left((0,T_R)\times(0,1);\mathbb{R}^3\right)$ is a solution to (4.45) with $\psi^i(t):[0,1]\to[0,1]$ a C^1–diffeomorphism. Then Lemma B.46 yields $\xi \in W_p^{1,4}\left((0,T_R)\times(0,1);\mathbb{R}^3\right)$ and for $i\in\{1,2,3\}$, $t\in[0,T_R]$, $y\in[0,1]$,

$$\xi_t^i(t,y) = -\psi_t^i\left(t,\xi^i(t,y)\right)\xi_y^i(t,y) = \frac{2\psi_{xxxx}^i\left(t,\xi^i(t,y)\right)}{\left|\eta_x^i(t,y)\right|^4}\left(\xi_y^i(t,y)\right)^5 + F\left(\psi^i,\eta^i\right)\left(t,\xi^i(t,y)\right)\xi_y^i(t,y)\,.$$

The arguments in Lemma B.46 yield

$$\begin{aligned}\xi_{yyyy}^i(t,y) =& -\xi_y^i(t,y)^5\psi_{xxxx}^i\left(t,\xi^i(t,y)\right) + 10\xi_y^i(t,y)^6\psi_{xx}^i\left(t,\xi^i(t,y)\right)\psi_{xxx}^i\left(t,\xi^i(t,y)\right)\\ &- 15\xi_y^i(t,y)^7\psi_{xx}^i\left(t,\xi^i(t,y)\right)^3\end{aligned}$$

and hence

$$\xi_t^i(t,y) = \frac{-2\xi_{yyyy}^i(t,y)}{\left|\eta_x^i(t,y)\right|^4} + F\left(\psi^i,\eta^i\right)\left(t,\xi^i(t,y)\right)\xi_y^i(t,y) + p\left(\xi^i,\psi^i\right)(t,y)\left|\eta_x^i(t,y)\right|^{-4}$$

where $p\left(\xi^i,\psi^i\right)(t,y)$ is a polynomial in $\xi_y^i(t,y)$, $\psi_{xx}^i\left(t,\xi^i(t,y)\right)$ and $\psi_{xxx}^i\left(t,\xi^i(t,y)\right)$. Using the formula for $\xi_{yyy}^i(t,y)$ shown in Lemma B.46, we replace $\psi_{xxx}^i\left(t,\xi^i(t,y)\right)$ by

$$-\xi_{yyy}^i(t,y)\xi_y^i(t,y)^{-4} + 3\xi_y^i(t,y)\psi_{xx}^i\left(t,\xi^i(t,y)\right)^2\,.$$

Then $\psi_{xx}^i\left(t,\xi^i(t,y)\right)$ is replaced by $-\xi_y^i(t,y)^{-3}\xi_{yy}^i(t,y)$ and finally, we insert $\psi_x^i\left(t,\xi^i(t,y)\right) = \xi_y^i(t,y)^{-1}$. Transforming also the boundary conditions we obtain that for all $t\in[0,T_R]$, $y\in[0,1]$, $z\in\{0,1\}$,

$$\begin{cases} \xi_t^i(t,y) + \frac{2\xi_{yyyy}^i(t,y)}{|(\eta_x^i)(t,y)|^4} + G\left(\xi^i,\eta^i\right)(t,y) &= 0\,,\\ \xi^i(t,z) &= z\,,\\ \xi_{yy}^i(t,z) - \left|\eta_x^i(t,z)\right|^{-2}\left\langle\eta_{xx}^i(t,z),\eta_x^i(t,z)\right\rangle\xi_y^i(t,z) &= 0\,,\\ \xi^i(0,y) &= \left(\theta^i\right)^{-1}(y) \end{cases} \tag{4.48}$$

where $G\left(\xi^i,\eta^i\right)(t,y)$ is a polynomial in the components of $\partial_y^m\xi_y^i(t,y)$, $m\in\{1,2,3\}$, $\xi_y^i(t,y)^{-1}$, $\left(\partial_x^k\eta^i\right)(t,y)$, $k\in\{1,2,3,4\}$, $\left|\left(\eta_x^i\right)(t,y)\right|^{-1}$ and $\left(\eta_t^i\right)(t,y)$ with $\eta_{xxxx}^i(t,y)$ and $\eta_t^i(t,y)$ appearing linearly. If $\xi\in W_p^{1,4}\left((0,T_R)\times(0,1);\mathbb{R}^3\right)$ is a strong solution to (4.48) with $\xi^i(t):[0,1]\to[0,1]$ a C^1–diffeomorphism for every $t\in[0,T_R]$, $i\in\{1,2,3\}$, then we can reverse the above argumentation to conclude that ξ^{-1} is a strong solution to (4.45) in $[0,T_R]$ with the desired properties.

We give some remarks on how to prove existence of strong solutions to (4.48). First, we observe that Lemma 3.38 yields that the C^1–diffeomorphism $\left(\theta^i\right)^{-1}:[0,1]\to[0,1]$ is an element of

$W_p^{4-4/p}\left((0,T);\mathbb{R}^3\right)$ and satisfies the boundary conditions in (4.48) evaluated in $t=0$. Similar arguments as in Subsection 4.1.1 yield that the associated linearised problem

$$\begin{cases} \xi_t^i(t,y)+\frac{2\xi^i_{yyyy}(t,y)}{|(\eta_x^i)(t,y)|^4} &= f^i(t,y)\,,\\ \xi^i(t,z) &= z\,,\\ \xi^i_{yy}(t,z) &= b^i(t,z)\,,\\ \xi^i(0,y) &= \phi^i(y)\,, \end{cases} \tag{4.49}$$

$t\in[0,T]$, $y\in[0,1]$, $z\in\{0,1\}$, admits a unique solution $\xi\in W_p^{1,4}\left((0,T)\times(0,1);\mathbb{R}^3\right)$ for every given right hand side $f\in L_p\left((0,T);L_p\left((0,1);\mathbb{R}^3\right)\right)$, $b\in W_p^{1/2-1/4p}\left((0,T);L_p\left(\{0,1\};\mathbb{R}^3\right)\right)\simeq W_p^{1/2-1/4p}\left((0,T);\mathbb{R}^6\right)$ and $\phi\in W_p^{4-4/p}\left((0,1);\mathbb{R}^3\right)$ such that the linear compatibility conditions of order three in [136, § 14] are satisfied. Indeed, Proposition B.35 yields that the motion equation is uniformly parabolic. The boundary operators in (4.49) coincide with the ones considered in Subsection 4.1.1 for the boundary point $x=1$. In particular, Proposition 4.19 implies that the complementary condition is satisfied. The existence of (4.49) is then a direct consequence of [136, Theorem 5.4]. Moreover, the norm of the solution is uniformly bounded with respect to the right hand side if one considers the modified norms as in Subsection 4.1.1, see (4.14).

Existence of strong solutions to (4.48) is obtained via contraction estimates in the set

$$X_{\boldsymbol{T},M}:=\left\{\xi\in E_{\boldsymbol{T}}:\xi^i(t,z)=z \text{ for } z\in\{0,1\}\,,\xi(0)=\theta^{-1}\,,\,|||\xi|||_{E_{\boldsymbol{T}}}\leq M\right\}$$

where $E_{\boldsymbol{T}}:=W_p^{1,4}\left((0,\boldsymbol{T})\times(0,1);\mathbb{R}^3\right)$ and $\boldsymbol{T}\in(0,T)$ and $M>0$ are defined appropriately in analogy to the strategy in 4.1.1. In particular, the time $\boldsymbol{T}$ needs to be chosen sufficiently small in dependence on $\boldsymbol{c}\left(\theta^{-1}\right)$ to guarantee that for all $\xi\in X_{\boldsymbol{T},M}$, $i\in\{1,2,3\}$, $t\in[0,\boldsymbol{T}]$, $\xi^i(t):[0,1]\to[0,1]$ is a C^1–diffeomorphism. This can be achieved using Corollary B.38 and the fact that $\theta^i:[0,1]\to[0,1]$ is a diffeomorphism for $i\in\{1,2,3\}$. In analogy to Lemma 4.23 one has for all $\xi\in X_{\boldsymbol{T},M}$,

$$\min_{i\in\{1,2,3\}}\min_{t\in[0,T],x\in[0,1]}\left|\xi_x^i(t,x)\right|\geq\frac{1}{2}\boldsymbol{c}\left(\theta^{-1}\right).$$

To apply the Banach Fixed-Point Theorem [151, Theorem 1.A] analogously as in Subsection 4.1.1, it is enough to show that for ξ, $\zeta\in X_{M,\boldsymbol{T}}$,

$$\begin{aligned} \left\|G\left(\eta^i,\xi^i\right)-G\left(\eta^i,\zeta^i\right)\right\|_{L_p((0,T);L_p((0,1)))}&\leq C(M)T^\alpha\,|||\xi-\zeta|||_{E_{\boldsymbol{T}}}\,,\\ |||\,\left|\eta^i\left(\cdot,z\right)\right|^{-2}\left\langle\eta^i_{xx}(\cdot,z),\eta^i_x(\cdot,z)\right\rangle\left(\xi^i_y(\cdot,z)-\zeta^i_y\left(\cdot,z\right)\right)|||_{W_p^{1/2-1/4p}((0,T))}&\leq C(M)T^\alpha\,|||\xi-\zeta|||_{E_{\boldsymbol{T}}} \end{aligned}$$

for some $\alpha\in(0,1)$. The argument for the boundary term follows from the Banach algebra property of $W_p^{1/2-1/4p}((0,T))$ shown in Proposition B.24, the embedding in Proposition C.32 using that ξ_y and ζ_y lie in $C^{\frac{1}{2}}\left([0,T];\mathbb{R}^3\right)$ due to the arguments in Lemma 4.35. The estimate for G follows from the fact that $G\left(\eta^i,\xi^i\right)$ is a polynomial in $\xi^i_y\,,\xi^i_{yy}\,,\xi^i_{yyy}\,,\left|\xi^i_y\right|^{-1}\in C\left([0,T];C([0,1])\right)$ with coefficients depending on products of $\partial_x^k\eta^i(t,y)$, $k\in\{1,2,3,4\}$, $\left|\eta^i_x(t,y)\right|^{-1}$ and $\eta^i_t(t,y)$, with $\partial_x^4\eta^i$ and η^i_t appearing only linearly. Denoting by $D_\xi G\left(\eta^i,\xi^i\right)$ the Fréchet derivative of G with respect to $\xi^i_y\,,\xi^i_{yy}\,,\xi^i_{yyy}\,,\left|\xi^i_y\right|^{-1}$, one obtains an estimate of the form

$$\begin{aligned} &\left\|G\left(\eta^i,\xi^i\right)-G\left(\eta^i,\zeta^i\right)\right\|_{L_p((0,T);L_p((0,1)))}\\ &\leq\left\|\int_0^1\left(D_\xi G\right)\left(\eta^i,\tau\xi^i+(1-\tau)\zeta^i\right)\mathrm{d}\tau\left(\xi^i_y-\zeta^i_y,\xi^i_{yy}-\zeta^i_{yy},\xi^i_{yyy}-\zeta^i_{yyy},\left(\xi^i_y\right)^{-1}-\left(\zeta^i_y\right)^{-1}\right)\right\|_{L_p}\\ &\leq T^{1/p}C\big(\,|||\eta|||_{W_p^{1,4}((0,T)\times(0,1);(\mathbb{R}^2)^3)},M\big)\,|||\xi-\zeta|||_{E_{\boldsymbol{T}}}\,. \end{aligned}$$

In analogy to the arguments in Proposition 4.26, Proposition 4.27, Proposition 4.29 and Theorem 4.30 one shows that there exists a time $T_R \in (0, \boldsymbol{T})$ depending on $\boldsymbol{c}(\eta)$, $\|\eta\|_{\boldsymbol{E}_T}$, $\boldsymbol{c}(\theta)$, and $\|\theta\|_{W_p^{4-4/p}((0,1);\mathbb{R}^3)}$, such that there exists a strong solution $\xi \in E_{T_R}$ to (4.48) in $[0, T_R]$ satisfying

$$\|\xi\|_{E_{T_R}} \leq C\big(c(\theta), \|\theta\|_{W_p^{4-4/p}((0,1);\mathbb{R}^3)}, \boldsymbol{c}(\eta), \|\eta\|_{\boldsymbol{E}_T}\big).$$

The differentiation rules shown in Lemma B.46 show that $\psi \in E_{T_R}$ satisfies the desired estimates. □

Geometric uniqueness of strong solutions to (3.21) now follows from uniqueness of the auxiliary problem (4.3) and the existence of a suitable family of reparametrisations as constructed in Theorem 4.59. Hereby, the term *geometric uniqueness* means that given two solutions to the geometric problem (3.21) with initial values describing the same set in the plane, the solutions parametrise the same image at every time in their common interval of existence.

Theorem 4.61 (Local geometric uniqueness of strong solutions)**.** *Let $p \in (5, \infty)$, T be positive and suppose that both η and $\widetilde{\eta}$ are strong solutions to* (3.21) *in* $[0, T]$ *with endpoints P^1, P^2, P^3 such that for $i \in \{1,2,3\}$ there exist C^1–diffeomorphisms $\zeta_0^i : [0,1] \to [0,1]$ with $\zeta_0^i \in W_p^{4-4/p}((0,1);\mathbb{R})$, $\zeta_0^i(y) = y$ for $y \in \{0,1\}$, and $\eta^i(0) = \widetilde{\eta}^i(0) \circ \zeta_0^i$. Then there exists a time $T_g \in (0,T)$ and a function $\zeta \in W_p^{1,4}\big((0,T_g) \times (0,1);\mathbb{R}^3\big)$ with $\zeta(0) = \zeta_0$ such that for all $t \in [0, T_g]$, $i \in \{1,2,3\}$, $\zeta^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism with $\eta^i(t) = \widetilde{\eta}^i(t) \circ \zeta^i(t)$. The existence time T_g depends on $\boldsymbol{c}(\eta)$, $\boldsymbol{c}(\widetilde{\eta})$, $\|\eta\|_{\boldsymbol{E}_T}$, $\|\widetilde{\eta}\|_{\boldsymbol{E}_T}$, $\boldsymbol{c}(\zeta_0)$ and $\|\zeta_0\|_{W_p^{4-4/p}((0,1);\mathbb{R}^3)}$.*

Proof. Theorem 4.59 yields that there exists $T_\eta \in (0,T]$ and $\psi \in W_p^{1,4}\big((0,T_\eta) \times (0,1);\mathbb{R}^3\big)$ such that for all $t \in [0, T_\eta]$, $i \in \{1,2,3\}$, $\psi^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism such that $\gamma = \big(\gamma^1, \gamma^2, \gamma^3\big)$ with $\gamma^i(t,x) := \eta^i\big(t, \psi^i(t,x)\big)$ is a strong solution to (4.3) in $(0, T_\eta)$ with initial datum $\varphi := \eta(0) \circ \theta$ where θ is the function constructed in Proposition 4.56. Then for $i \in \{1,2,3\}$, $\widetilde{\theta}^i := \zeta_0^i \circ \theta^i : [0,1] \to [0,1]$ is a C^1–diffeomorphism with $\widetilde{\theta}^i(y) = y$ for $y \in \{0,1\}$ and Lemma 3.38 yields $\widetilde{\theta}^i \in W_p^{4-4/p}((0,1))$. Furthermore, for $y \in \{0,1\}$ it holds

$$\begin{aligned}
&\widetilde{\theta}^i_{xx}(y) + \left|\widetilde{\eta}^i_x(0,y)\right|^{-2} \left\langle \widetilde{\eta}^i_{xx}(0,y), \widetilde{\eta}^i_x(0,y)\right\rangle \widetilde{\theta}^i(y)^2 \\
&= \left(\zeta_0^i\right)_{xx}(y) + \left(\zeta_0^i\right)_x(y)\theta^i_{xx}(y) + \left|\eta^i_x(0,y)\right|^{-2} \left|\left(\zeta_0^i\right)_x(y)\right|^4 \left\langle \eta^i_{xx}(0,y), \eta^i_x(0,y)\right\rangle \left(\left(\zeta_0^i\right)_x(y)\right)^{-3} \\
&\quad + \left|\eta^i_x(0,y)\right|^{-2} \left|\left(\zeta_0^i\right)_x(y)\right|^4 \left\langle \eta^i_x(0,y), \eta^i_x(0,y)\right\rangle \left(\left(\zeta_0^i\right)_x(y)\right)^{-4} \left(\zeta_0^i\right)_{xx}(y) \\
&= \left(\zeta_0^i\right)_{xx}(y) - \left(\zeta_0^i\right)_{xx}(y) + \left(\zeta_0^i\right)_x(y)\big(\theta^i_{xx}(y) + \left|\eta^i_x(0,y)\right|^{-2} \left\langle \eta^i_{xx}(0,y), \eta^i_x(0,y)\right\rangle\big) = 0\,.
\end{aligned}$$

Hence, $\widetilde{\theta}$ satisfies the boundary conditions of system (4.45) with $\widetilde{\eta}$ instead of η. The arguments in Theorem 4.59 and Lemma 4.60 yield that there exists $T_{\widetilde{\eta}} \in (0,T]$ and a function $\vartheta \in W_p^{1,4}\big((0,T_{\widetilde{\eta}}) \times (0,1);\mathbb{R}^3\big)$ such that for all $t \in [0, T_{\widetilde{\eta}}]$, $i \in \{1,2,3\}$, $\vartheta^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism and $\widetilde{\gamma}^i(t,x) := \widetilde{\eta}^i\big(t, \vartheta^i(t,x)\big)$ defines a strong solution $\widetilde{\gamma}$ to (4.3) with initial datum $\widetilde{\eta}(0) \circ \vartheta(0) = \varphi$. For $\tau \in (0, \min\{T_\eta, T_{\widetilde{\eta}}\})$ with τ tending to zero, the norms $\|\widetilde{\gamma}\|_{\mathbb{E}_\tau}$ and $\|\gamma\|_{\mathbb{E}_\tau}$ tend to $\|\varphi\|_{X_0} \leq M/2$ with $M = M(\varphi)$ defined in the proof of Proposition 4.29. In particular, there exists a time $T_g \in (0, \min\{T_\eta, T_{\widetilde{\eta}}, \boldsymbol{T}\}]$ such that γ and $\widetilde{\gamma}$ lie in $\boldsymbol{E}_{T_g} \cap \overline{B_{M,T_g}}$ with M as in the proof of Proposition 4.29 and $\boldsymbol{T} = \boldsymbol{T}(\varphi)$ as in Theorem 4.30. In fact, this even holds for $T_g = \min\{T_\eta, T_{\widetilde{\eta}}, \boldsymbol{T}\}$. As solutions to (4.3) with initial datum φ are unique in $\boldsymbol{E}_{T_g} \cap \overline{B_{M,T_g}}$ by Theorem 4.30, we obtain $\widetilde{\gamma} = \gamma$ in $\boldsymbol{E}_{T_g}$. Finally, for $t \in [0, T_g]$ and $i \in \{1,2,3\}$, the maps $\zeta^i(t) := \vartheta^i(t) \circ \psi^i(t)^{-1} : [0,1] \to [0,1]$ are C^1–diffeomorphisms with $\zeta^i(0) = \zeta_0^i$, $\eta^i(t) = \widetilde{\eta}^i(t) \circ \zeta^i(t)$ and Lemma B.45 yields that $\zeta = \big(\zeta^1, \zeta^2, \zeta^3\big)$ lies in $W_p^{1,4}\big((0,T_g) \times (0,1);\mathbb{R}^3\big)$. □

Theorem 4.62 (Geometric uniqueness of strong solutions)**.** *Let $p \in (5,\infty)$, T be positive and suppose that both η and $\widetilde{\eta}$ are strong solutions to (3.21) in $[0,T]$ with endpoints P^1, P^2, P^3 such that for $i \in \{1,2,3\}$ there exist C^1–diffeomorphisms $\zeta_0^i : [0,1] \to [0,1]$ with $\zeta_0^i \in W_p^{4-4/p}((0,1);\mathbb{R})$, $\zeta_0^i(y) = y$ for $y \in \{0,1\}$, and $\eta^i(0) = \widetilde{\eta}^i(0) \circ \zeta_0^i$. Then there exists $\zeta \in W_p^{1,4}\left((0,T)\times(0,1);\mathbb{R}^3\right)$ with $\zeta(0) = \zeta_0$ such that for all $t \in [0,T]$, $i \in \{1,2,3\}$, $\zeta^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism with $\eta^i(t) = \widetilde{\eta}^i(t) \circ \zeta^i(t)$.*

Proof. The proof relies on the a priori estimates (4.50) and (4.51) that are derived in the following.

Let $\tau \in (0,T]$ be given and $\zeta \in W_p^{1,4}\left((0,\tau)\times(0,1);\mathbb{R}^3\right)$ with $\zeta(0) = \zeta_0$ be such that for all $t \in [0,\tau]$, $i \in \{1,2,3\}$, $\zeta^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism with $\eta^i(t) = \widetilde{\eta}^i(t) \circ \zeta^i(t)$. Lemma B.45 yields for every $t \in [0,\tau]$, $x \in [0,1]$,

$$\partial_x \eta^i(t,x) = \partial_y \widetilde{\eta}^i\left(t,\zeta^i(t,x)\right)\partial_x \zeta^i(t,x)$$

and thus in particular

$$\left|\partial_x \zeta^i(t,x)\right| = \left|\partial_x \eta^i(t,x)\right|\left|\partial_y \widetilde{\eta}^i\left(t,\zeta^i(t,x)\right)\right|^{-1} \geq C\boldsymbol{c}(\eta)\,|||\widetilde{\eta}|||_{\boldsymbol{E}_T}^{-1}. \tag{4.50}$$

Lemma B.45 yields that we may apply the chain rule to differentiate $\eta^i(t) = \widetilde{\eta}^i(t)\circ\zeta^i(t)$ once with respect to t and up to four times with respect to x. The resulting formulas imply

$$|||\zeta|||_{W_p^{1,4}((0,\tau)\times(0,1))} \leq C\left(|||\eta|||_{\boldsymbol{E}_T}, |||\widetilde{\eta}|||_{\boldsymbol{E}_T}, \boldsymbol{c}(\widetilde{\eta})\right) + \|\zeta_0\|_{W_p^{4-4/p}((0,1);\mathbb{R}^3)}. \tag{4.51}$$

In particular, the estimates (4.50) and (4.51) are independent of τ and the choice of ζ. We now construct the function ζ with the help of Theorem 4.61 and estimates (4.50) and (4.51).

Throughout this proof, an *admissible triple with existence time* τ is a triple $(\varrho,\widetilde{\varrho},\sigma_0)$ such that ϱ and $\widetilde{\varrho}$ are strong solutions to (3.21) in $[0,\tau]$ for some $\tau \in (0,T]$ with $|||\varrho|||_{\boldsymbol{E}_\tau} \leq |||\eta|||_{\boldsymbol{E}_T}$, $|||\widetilde{\varrho}|||_{\boldsymbol{E}_\tau} \leq |||\widetilde{\eta}|||_{\boldsymbol{E}_T}$, $\boldsymbol{c}(\varrho) \geq \boldsymbol{c}(\eta)$, $\boldsymbol{c}(\widetilde{\varrho}) \geq \boldsymbol{c}(\widetilde{\eta})$, and $\sigma_0^i : [0,1] \to [0,1]$ is a C^1–diffeomorphism, $i \in \{1,2,3\}$, with $\sigma \in W_p^{4-4/p}\left((0,1);\mathbb{R}^3\right)$, $\sigma^i(y) = y$, $y \in \{0,1\}$, $\varrho^i(0) = \widetilde{\varrho}^i(0)\circ\sigma_0^i$ and $c(\sigma_0) \geq C\boldsymbol{c}(\eta)\,|||\widetilde{\eta}|||_{\boldsymbol{E}_T}^{-1}$, and

$$\|\sigma\|_{W_p^{4-4/p}((0,1);\mathbb{R}^3)} \leq C\left(|||\eta|||_{\boldsymbol{E}_T}, |||\widetilde{\eta}|||_{\boldsymbol{E}_T}, \boldsymbol{c}(\widetilde{\eta})\right) + \|\zeta_0\|_{W_p^{4-4/p}((0,1);\mathbb{R}^3)}$$

with C and $C\left(|||\eta|||_{\boldsymbol{E}_T}, |||\widetilde{\eta}|||_{\boldsymbol{E}_T}, \boldsymbol{c}(\widetilde{\eta})\right)$ being the constants in (4.50) and (4.51), respectively.

The arguments in Theorem 4.61 imply that there exists a time $\boldsymbol{T} \in (0,T]$ such that for all admissible triples $(\varrho,\widetilde{\varrho},\sigma_0)$ with existence time $\tau \in (0,T]$ there exists $\sigma \in W_p^{1,4}\left((0,\tau\wedge\boldsymbol{T})\times(0,1);\mathbb{R}^3\right)$ with $\sigma(0) = \sigma_0$ such that for all $i \in \{1,2,3\}$, $t \in [0,\tau\wedge\boldsymbol{T}]$, $\sigma^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism with $\varrho^i(t) = \widetilde{\varrho}^i(t)\circ\sigma^i(t)$.
By assumption, $(\eta,\widetilde{\eta},\zeta_0)$ is an admissible triple with existence time T and thus there exists $\zeta^{(0)} \in W_p^{1,4}\left((0,T\wedge\boldsymbol{T})\times(0,1);\mathbb{R}^3\right)$ with $\zeta^{(0)}(0) = \zeta_0$ such that for all $i \in \{1,2,3\}$, $t \in [0,T\wedge\boldsymbol{T}]$, $\zeta^{(0),i}(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism with $\eta^i(t) = \widetilde{\eta}^i(t)\circ\zeta^{(0),i}(t)$. If $\boldsymbol{T} = T$ there is nothing left to prove. Suppose that $\boldsymbol{T} < T$ and define for $t \in [0,T-\boldsymbol{T}]$, $\eta^{(1)}(t) := \eta(t+\boldsymbol{T})$, $\widetilde{\eta}^1(t) := \widetilde{\eta}(t+\boldsymbol{T})$. The estimates (4.50) and (4.51) yield that $\left(\eta^{(1)},\widetilde{\eta}^{(1)},\zeta^{(0)}(\boldsymbol{T})\right)$ is an admissible triple with existence time $T-\boldsymbol{T}$. In particular, there exists $\zeta^{(1)} \in W_p^{1,4}\left((0,\boldsymbol{T}\wedge(T-\boldsymbol{T}))\times(0,1);\mathbb{R}^3\right)$ with $\zeta^{(1)}(0) = \zeta^{(0)}(\boldsymbol{T})$ such that for all $t \in [0,\boldsymbol{T}\wedge(T-\boldsymbol{T})]$, $i \in \{1,2,3\}$, $\zeta^{(1),i}(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism with $\eta^{(1),i}(t) = \widetilde{\eta}^{(1),i}(t)\circ\zeta^{(1),i}(t)$. Then Lemma B.33 yields that

$$\zeta(t) := \zeta^{(0)}(t)\chi_{[0,\boldsymbol{T}]}(t) + \zeta^{(1)}(t-\boldsymbol{T})\,\chi_{(\boldsymbol{T},(2\boldsymbol{T})\wedge T]}(t)$$

defines a function $\zeta \in W_p^{1,4}\left((0,(2\boldsymbol{T})\wedge T)\times(0,1);\mathbb{R}^3\right)$. Furthermore, $\zeta(0) = \zeta_0$ and for all $t \in [0,(2\boldsymbol{T})\wedge T]$, $i \in \{1,2,3\}$, $\zeta^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism with $\eta^i(t) = \widetilde{\eta}^i(t)\circ\zeta^i(t)$. The claim follows upon repeating this argument $k \in \mathbb{N}$ times until $k\boldsymbol{T} \geq T$. □

We state the according result in the case of Theta networks.

Theorem 4.63 (Geometric existence and uniqueness of strong solutions (Theta))**.** *Let $p \in (5,\infty)$. Given a geometrically p–admissible initial network σ to system* (3.20) *there exists a positive time $\boldsymbol{T}$ such that for all $T \in (0,\boldsymbol{T}]$ there exists a strong solution to system* (3.20) *in* $[0,T]$ *with initial datum σ. If both η and $\widetilde{\eta}$ are strong solutions to* (3.20) *in* $[0,T]$, $T>0$, *such that for $i \in \{1,2,3\}$ there exist C^1–diffeomorphisms $\zeta_0^i : [0,1] \to [0,1]$ with $\zeta_0^i \in W_p^{4-4/p}((0,1);\mathbb{R})$, $\zeta_0^i(y) = y$ for $y \in \{0,1\}$, and $\eta^i(0) = \widetilde{\eta}^i(0) \circ \zeta_0^i$, then there exists $\zeta \in W_p^{1,4}\left((0,T)\times(0,1);\mathbb{R}^3\right)$ with $\zeta(0) = \zeta_0$ such that for all $t \in [0,T]$, $i \in \{1,2,3\}$, $\zeta^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism with $\eta^i(t) = \widetilde{\eta}^i(t) \circ \zeta^i(t)$.*

Proof. This result is obtained by adapting the arguments in this subsection to the case of Theta networks. □

4.2.2 Existence and uniqueness of classical solutions

This subsection is devoted to prove existence of classical solutions to (3.21) given a geometrically α–admissible initial network. As in Subsection 4.2.1 we show that any geometrically α–admissible initial network to (3.21) with endpoints P^1, P^2, P^3 possesses a reparametrisation that is an analytically α–admissible initial value to (4.3). In contrast to the Sobolev setting we have to take into account the geometrical and analytical fourth order conditions introduced in Definitions 3.39 and 3.40 and Definitions 4.4 and 4.5, respectively, that arise due to the increased regularity of the solutions. This issue is addressed in the following lemma that is in a way analogous to [46, Lemma 2.3].

Lemma 4.64. *Let $\eta^i : [0,1] \to \mathbb{R}^2$, $i \in \{1,2,3\}$, be regular open curves with $\eta^1(0) = \eta^2(0) = \eta^3(0)$ satisfying the non-degeneracy condition in $x=0$. We denote by τ^i and ν^i, $i \in \{1,2,3\}$, the unit tangent and unit normal to η^i in the point* 0*, respectively, and let α^1, α^2, and α^3 be the angles between τ^2 and τ^3, τ^3 and τ^1, and τ^1 and τ^2, respectively. Then for any real numbers V^i, T^i, $i \in \{1,2,3\}$, the equations*

$$T^1\tau^1 + V^1\nu^1 = T^2\tau^2 + V^2\nu^2 = T^3\tau^3 + V^3\nu^3 \tag{4.52}$$

are equivalent to

$$V^i - \sin\left(\alpha^{i-1}\right)T^{i+1} - \cos\left(\alpha^{i-1}\right)V^{i+1} = 0\,, \qquad i \in \{1,2,3\}\,, \tag{4.53a}$$

$$V^i + \sin\left(\alpha^{i+1}\right)T^{i-1} - \cos\left(\alpha^{i+1}\right)V^{i-1} = 0\,, \qquad i \in \{1,2,3\}\,, \tag{4.53b}$$

$$\sin\alpha^1 V^1 + \sin\alpha^2 V^2 + \sin\alpha^3 V^3 = 0\,. \tag{4.53c}$$

In the case that $\sin\left(\alpha^i\right) \neq 0$ for all $i \in \{1,2,3\}$, the equations (4.52) *and the conditions* (4.53) *for real numbers V^i, T^i, $i \in \{1,2,3\}$, are equivalent to*

$$\sin\left(\alpha^i\right)T^i - \cos\left(\alpha^{i+1}\right)V^{i+1} + \cos\left(\alpha^{i-1}\right)V^{i-1} = 0\,, \qquad i \in \{1,2,3\}\,, \tag{4.54a}$$

$$\sin\alpha^1 V^1 + \sin\alpha^2 V^2 + \sin\alpha^3 V^3 = 0\,. \tag{4.54b}$$

The indices in (4.53) *and* (4.54) *should be read* $\mod 3$, *that is* $1-1 \equiv 3$ *and* $3+1 \equiv 1$.

Proof. Let $V := \left(V^1, V^2, V^3\right)$ and $T := \left(T^1, T^2, T^3\right)$ be given. The non-degeneracy condition yields $\operatorname{span}\left\{\tau^1, \tau^2, \tau^3\right\} = \mathbb{R}^2$ and we may assume without loss of generality that τ^1 and τ^2 form a basis of $\mathbb{R}^2$. Testing the identity (4.52) with τ^1 and τ^2 yields that (4.52) is equivalent to

$(V,T) \in \ker A$ where A is the matrix defined by

$$A := \begin{pmatrix} 0 & -\langle \nu^2, \tau^1 \rangle & 0 & 1 & -\langle \tau^2, \tau^1 \rangle & 0 \\ \langle \nu^1, \tau^2 \rangle & 0 & 0 & \langle \tau^1, \tau^2 \rangle & -1 & 0 \\ 0 & 0 & -\langle \nu^3, \tau^1 \rangle & 1 & 0 & -\langle \tau^3, \tau^1 \rangle \\ \langle \nu^1, \tau^2 \rangle & 0 & -\langle \nu^3, \tau^2 \rangle & \langle \tau^1, \tau^2 \rangle & 0 & -\langle \tau^3, \tau^2 \rangle \end{pmatrix}.$$

The dimension of the image $\mathrm{rg}\,(A)$ is equal to four. Indeed, a vector $\left(x^1, x^2, y^1, y^2\right)^T \in \mathbb{R}^4$ lies in the image of A if and only if there exists $(V,T) \in \mathbb{R}^6$ such that

$$T^1\tau^1 + V^1\nu^1 - T^2\tau^2 - V^2\nu^2 = \sum_{i=1}^{2} x^i\tau^i =: x\,,$$

$$T^1\tau^1 + V^1\nu^1 - T^3\tau^3 - V^3\nu^3 = \sum_{i=1}^{2} y^i\tau^i =: y\,.$$

As τ^i and ν^i are linearly independent for all $i \in \{1,2,3\}$, any vectors $x, y \in \mathbb{R}^2$ can be generated and thus we conclude $\mathrm{rg}(A) = \mathbb{R}^4$ which yields $\dim\ker(A) = 2$.

In the next step we show that any $(V,T) \in \ker(A)$ satisfies (4.53). Let $i \in \{1,2,3\}$ be given. Testing the identity

$$T^i\tau^i + V^i\nu^i = T^{i+1}\tau^{i+1} + V^{i+1}\nu^{i+1}$$

with $\sin\left(\alpha_{i+1}\right)\tau^{i-1}$ yields

$$\sin\left(\alpha^{i+1}\right)\cos\left(\alpha^{i+1}\right)T^i - \sin\left(\alpha^{i+1}\right)^2 V^i = \sin\left(\alpha^{i+1}\right)\cos\left(\alpha^i\right)T^{i+1} + \sin\left(\alpha^{i+1}\right)\sin\left(\alpha^i\right)V^{i+1}\,,$$

and similarly, testing with $\cos\left(\alpha^{i+1}\right)\nu^{i-1}$ gives

$$\cos\left(\alpha^{i+1}\right)\sin\left(\alpha^{i+1}\right)T^i + \cos\left(\alpha^{i+1}\right)^2 V^i = -\cos\left(\alpha^{i+1}\right)\sin\left(\alpha^i\right)T^{i+1} + \cos\left(\alpha^{i+1}\right)\cos\left(\alpha^i\right)V^{i+1}\,.$$

Subtracting the first equation from the second we obtain (4.53a) using appropriate properties of sin and cos and the fact that $\alpha^1 + \alpha^2 + \alpha^3 = 2\pi$. Similarly, testing

$$T^i\tau^i + V^i\nu^i = T^{i-1}\tau^{i-1} + V^{i-1}\nu^{i-1}$$

with $\sin\left(\alpha^{i-1}\right)\tau^{i+1}$ and $\cos\left(\alpha^{i-1}\right)\nu^{i+1}$, respectively, gives

$$\sin\left(\alpha^{i-1}\right)\cos\left(\alpha^{i-1}\right)T^i + \sin\left(\alpha^{i-1}\right)^2 V^i = \sin\left(\alpha^{i-1}\right)\cos\left(\alpha^i\right)T^{i-1} - \sin\left(\alpha^{i-1}\right)\sin\left(\alpha^i\right)V^{i-1}\,,$$

$$-\cos\left(\alpha^{i-1}\right)\sin\left(\alpha^{i-1}\right)T^i + \cos\left(\alpha^{i-1}\right)^2 V^i = \sin\left(\alpha^i\right)\cos\left(\alpha^{i-1}\right)T^{i-1} + \cos\left(\alpha^{i-1}\right)\cos\left(\alpha^i\right)V^{i-1}$$

which implies (4.53b). As $\sum_{i=1}^{3}\sin\left(\alpha^i\right)\nu^i = 0$ and (V,T) satisfies (4.52), we find further that

$$\begin{aligned} \sin\alpha^1 V^1 &= \left\langle T^1\tau^1 + V^1\nu^1, \sin\alpha^1\nu^1 \right\rangle = \left\langle T^2\tau^2 + V^2\nu^2, -\sin\alpha^2\nu^2 - \sin\alpha^3\nu^3 \right\rangle \\ &= -\sin\alpha^2 V^2 - \sin\alpha^3\left\langle T^3\tau^3 + V^3\nu^3, \nu^3 \right\rangle = -\sin\alpha^2 V^2 - \sin\alpha^3 V^3 \end{aligned}$$

which allows us to conclude that (4.52) implies (4.53). On the other hand we observe that $(V,T) \in \mathbb{R}^6$ satisfies (4.53) if and only if $(V,T) \in \ker(B)$ where B is the matrix given by

$$B := \begin{pmatrix} \sin\alpha^1 & \sin\alpha^2 & \sin\alpha^3 & 0 & 0 & 0 \\ 1 & -\cos\alpha^3 & 0 & 0 & -\sin\alpha^3 & 0 \\ 1 & 0 & -\cos\alpha^2 & 0 & 0 & \sin\alpha^2 \\ 0 & 1 & -\cos\alpha^1 & 0 & 0 & -\sin\alpha^1 \\ -\cos\alpha^3 & 1 & 0 & \sin\alpha^3 & 0 & 0 \\ -\cos\alpha^2 & 0 & 1 & -\sin\alpha^2 & 0 & 0 \\ 0 & -\cos\alpha^1 & 1 & 0 & \sin\alpha^1 & 0 \end{pmatrix}.$$

As $\sin\alpha^i = 0$ for at most one $i \in \{1,2,3\}$, we may assume without loss of generality that $\sin\alpha^1 \neq 0$ and $\sin\alpha^3 \neq 0$. It is then straightforward to check that the first, fourth, fifth and sixth column of B are linearly independent. In particular, $\dim \operatorname{rg}(B) \geq 4$ and thus $\dim\ker(B) \leq 2$. As $\ker(A) \subset \ker(B)$ and $\dim\ker(A) = 2$, we conclude that $\dim\ker(B) = 2$ and $\ker(A) = \ker(B)$. This shows that the equations (4.52) are equivalent to the conditions (4.53).

We now show that any $(V,T) \in \mathbb{R}^6$ satisfying (4.52) also fulfils (4.54). To this end, we let $i \in \{1,2,3\}$ be given and observe that testing the identity

$$T^i\tau^i + V^i\nu^i = T^{i+1}\tau^{i+1} + V^{i+1}\nu^{i+1}$$

with $\cos\left(\alpha^{i+1}\right)\tau^{i-1}$ and $\sin\left(\alpha^{i+1}\right)\nu^{i-1}$ yields

$$\begin{aligned}\cos\left(\alpha^{i+1}\right)^2 T^i &= \cos\left(\alpha^{i+1}\right)\sin\left(\alpha^{i+1}\right)V^i + \cos\left(\alpha^{i+1}\right)\cos\left(\alpha^{i}\right)T^{i+1} + \cos\left(\alpha^{i+1}\right)\sin\left(\alpha^{i}\right)V^{i+1}\,,\\ \sin\left(\alpha^{i+1}\right)^2 T^i &= -\sin\left(\alpha^{i+1}\right)\cos\left(\alpha^{i+1}\right)V^i - \sin\left(\alpha^{i+1}\right)\sin\left(\alpha^{i}\right)T^{i+1} + \sin\left(\alpha^{i+1}\right)\cos\left(\alpha^{i}\right)V^{i+1}\end{aligned}$$

and hence

$$T^i = \cos\left(\alpha^{i-1}\right)T^{i+1} - \sin\left(\alpha^{i-1}\right)V^{i+1}\,. \tag{4.55}$$

Similarly, testing

$$T^i\tau^i + V^i\nu^i = T^{i-1}\tau^{i-1} + V^{i-1}\nu^{i-1}$$

with $\cos\left(\alpha^{i-1}\right)\tau^{i+1}$ and $\sin\left(\alpha^{i-1}\right)\nu^{i+1}$ yields

$$T^i = \cos\left(\alpha^{i+1}\right)T^{i-1} + \sin\left(\alpha^{i+1}\right)V^{i-1}\,. \tag{4.56}$$

Thus, any $(V,T) \in \ker(A)$ satisfies (4.53a), (4.53b), (4.55) and (4.56). In particular, inserting (4.53a) in

$$\sin\left(\alpha^{i-1}\right)T^i = \cos\left(\alpha^{i-1}\right)\sin\left(\alpha^{i-1}\right)T^{i+1} - \sin\left(\alpha^{i-1}\right)^2 V^{i+1}$$

yields

$$\sin\left(\alpha^{i-1}\right)T^i = \cos\left(\alpha^{i-1}\right)V^i - V^{i+1}\,.$$

Using (4.53b) we obtain

$$\sin\left(\alpha^{i-1}\right)T^i = -\cos\left(\alpha^{i-1}\right)\sin\left(\alpha^{i+1}\right)T^{i-1} + \cos\left(\alpha^{i-1}\right)\cos\left(\alpha^{i+1}\right)V^{i-1} - V^{i+1}\,.$$

Multiplying by $\cos\left(\alpha^{i+1}\right)$ and using (4.56) gives

$$\sin\left(\alpha^{i}\right)T^i = \cos\left(\alpha^{i+1}\right)V^{i+1} - \cos\left(\alpha^{i-1}\right)V^{i-1}\,.$$

This shows $\ker(A) \subset \ker(C)$ where C is given by

$$C := \begin{pmatrix} \sin\alpha^1 & \sin\alpha^2 & \sin\alpha^3 & 0 & 0 & 0\\ 0 & -\cos\alpha^2 & \cos\alpha^3 & \sin\alpha^1 & 0 & 0\\ \cos\alpha^1 & 0 & -\cos\alpha^3 & 0 & \sin\alpha^2 & 0\\ -\cos\alpha^1 & \cos\alpha^2 & 0 & 0 & 0 & \sin\alpha^3 \end{pmatrix}.$$

If $\sin\left(\alpha^i\right) \neq 0$ for all $i \in \{1,2,3\}$, one easily shows that the first, fourth, fifth and sixth column of C are linearly independent. In particular, $\dim\ker(C) = 2$ and as $\ker(A) \subset \ker(C)$ and $\dim\ker(A) = 2$, we conclude $\ker(A) = \ker(C)$. This shows the claim. □

Lemma 4.65. *Let $\alpha \in (0,1)$ and σ be a geometrically α–admissible initial network to system* (3.21) *with given endpoints P^1, P^2, P^3. There exist smooth diffeomorphisms $\theta^i : [0,1] \to [0,1]$, $i \in \{1,2,3\}$, such that $\varphi = \left(\varphi^1,\varphi^2,\varphi^3\right)$ defined by $\varphi^i := \sigma^i \circ \theta^i$ is an analytically α–admissible initial network to* (4.3) *with endpoints P^1, P^2, P^3.*

Proof. Let $\alpha \in (0,1)$ and σ be a geometrically α–admissible initial network to system (3.21) with given endpoints P^1, P^2, P^3. We proceed as in Lemma 4.55 and Proposition 4.56 taking into account the fourth order conditions arising in the Hölder setting. As in Lemma 4.55 the idea is to consider suitable polynomials p^i and q^i, $i \in \{1,2,3\}$, near the boundary points $y=0$ and $y=1$, respectively, which are then smoothly connected by a function g^i to a function θ^i on $[0,1]$ in such a way that the function θ^i satisfies $\theta^i_x > 0$ in $[0,1]$ and that $\sigma^i \circ \theta^i$ has the desired properties. In analogy to Lemma 4.55 and Proposition 4.56 we demand that p^i and q^i, $i \in \{1,2,3\}$, are polynomials on $\mathbb{R}$ with $p^i(0)=0$, $q^i(1)=1$, $p^i_x(0)=1=q^i_x(1)$, $p^i_{xx}(0) = -\left|\sigma^i_x(0)\right|^{-2}\left\langle\sigma^i_{xx}(0),\sigma^i_x(0)\right\rangle$, $q^i_{xx}(1) = -\left|\sigma^i_x(1)\right|^{-2}\left\langle\sigma^i_{xx}(1),\sigma^i_x(1)\right\rangle$. These conditions are sufficient to guarantee that $\varphi^i := \sigma^i \circ \theta^i$, $i \in \{1,2,3\}$, are regular open curves with $\varphi^i \in C^{4+\alpha}\left([0,1];\mathbb{R}^2\right)$ satisfying all boundary conditions appearing in (4.3) and the non-degeneracy condition in $y=0$. To ensure that φ further satisfies the respective analytical fourth order conditions in $y=0$ and $y=1$, we need to impose conditions on $p^i_{xxx}(0)$, $p^i_{xxxx}(0)$, $q^i_{xxx}(0)$ and $q^i_{xxxx}(0)$ that are derived in the following. We let α^1, α^2, and α^3 denote the angles between the tangents $\tau^2(0)$ and $\tau^3(0)$, $\tau^3(0)$ and $\tau^1(0)$, and $\tau^1(0)$ and $\tau^2(0)$, respectively. Moreover, we define for $i \in \{1,2,3\}$,

$$\overline{V}^i(0) := \overline{V}\left(\sigma^i\right)(0) = -\left(2\kappa^i_{ss}(0) + \left(\kappa^i(0)\right)^3 - \mu^i\kappa^i(0)\right).$$

As σ satisfies the geometrical fourth order condition in $y=0$, there holds

$$\sin\alpha^1\overline{V}^1(0) + \sin\alpha^2\overline{V}^2(0) + \sin\alpha^3\overline{V}^3(0) = 0. \tag{4.57}$$

Given smooth diffeomorphisms $\theta^i : [0,1] \to [0,1]$, $i \in \{1,2,3\}$, we let $\overline{T}^i(\theta) := \overline{T}\left(\sigma^i \circ \theta^i\right)(0)$ where $\overline{T}$ is the expression defined in (3.26). As the curvature is invariant under reparametrisation, the curves $\sigma^i \circ \theta^i$, $i \in \{1,2,3\}$, satisfy the analytical fourth order condition in $y=0$ (cf. Definition 4.4) if and only if

$$\overline{V}^1(0)\nu^1(0) + \overline{T}^1(\theta)\tau^1(0) = \overline{V}^2(0)\nu^2(0) + \overline{T}^2(\theta)\tau^2(0) = \overline{V}^3(0)\nu^3(0) + \overline{T}^3(\theta)\tau^3(0).$$

Suppose that $\sin\left(\alpha^i\right) \neq 0$ for all $i \in \{1,2,3\}$. Then Lemma 4.64 yields that the curves $\sigma^i \circ \theta^i$, $i \in \{1,2,3\}$, satisfy the analytical fourth order condition in $y=0$ if and only if

$$\overline{T}^1(\theta) = \left(\sin\alpha^1\right)^{-1}\left(\cos\alpha^2\overline{V}^2(0) - \cos\alpha^3\overline{V}^3(0)\right), \tag{4.58a}$$

$$\overline{T}^2(\theta) = \left(\sin\alpha^2\right)^{-1}\left(\cos\alpha^3\overline{V}^3(0) - \cos\alpha^1\overline{V}^1(0)\right), \tag{4.58b}$$

$$\overline{T}^3(\theta) = \left(\sin\alpha^3\right)^{-1}\left(\cos\alpha^1\overline{V}^1(0) - \cos\alpha^2\overline{V}^2(0)\right). \tag{4.58c}$$

As the expression $\overline{T}^i(\theta)$ is of the form

$$\overline{T}^i(\theta) = \frac{2}{\left|\sigma^i_x(0)\right|^3}\theta^i_{xxxx}(0) + f\left(\sigma^i, \theta^i_{xxx}(0), \theta^i_{xx}(0), \theta^i_x(0), \theta^i(0)\right),$$

where $\partial^l_x\theta^i(0) = \partial^l_x p^i(0)$ for $l \in \{0,1,2,3,4\}$, and as the right hand side of (4.58) is independent of θ, we impose $p^i_{xxx}(0) = 1$ for $i \in \{1,2,3\}$ and determine $p^i_{xxxx}(0)$, $i \in \{1,2,3\}$, such that (4.58) is fulfilled.

It remains to consider the case that there exists $i \in \{1,2,3\}$ such that $\sin\left(\alpha^i\right) = 0$. As σ satisfies the non-degeneracy condition in $y=0$, at most one of the angles α^1, α^2, α^3 lies in $\{0,\pi,2\pi\}$ and we may assume without loss of generality that $\sin\alpha^1 = 0$. Lemma 4.64 yields that $\sigma^i \circ \theta^i$, $i \in \{1,2,3\}$, then satisfy the analytical fourth order condition in $y=0$ if and only if for all $i \in \{1,2,3\}$ the identities (4.53a) and (4.53b) are fulfilled by $\left(\overline{V}(0), \overline{T}(\theta)\right)$. Using (4.57) and $\sin\alpha^2 \neq 0$, $\sin\alpha^3 \neq 0$, one easily shows that $\left(\overline{V}(0), \overline{T}(\theta)\right)$ satisfy (4.53a) and (4.53b) for all $i \in \{1,2,3\}$ if and only if

$$\overline{T}^1(\theta) = \left(\sin\alpha^2\right)^{-1}\left(\overline{V}^3(0) - \cos\alpha^2\overline{V}^1(0)\right),$$

$$\overline{T}^2(\theta) = \left(\sin\alpha^3\right)^{-1}\left(\overline{V}^1(0) - \cos\alpha^3\overline{V}^2(0)\right),$$
$$\overline{T}^3(\theta) = \left(\sin\alpha^2\right)^{-1}\left(\cos\alpha^2\overline{V}^3(0) - \overline{V}^1(0)\right).$$

Arguing analogously as in the previous case we determine p^i in such a way that $\overline{T}(\theta)$ fulfils the above identities.
Let us turn to the fourth order conditions in the point $y = 1$. Given smooth diffeomorphisms $\theta^i : [0,1] \to [0,1]$, $i \in \{1,2,3\}$, the curves $\sigma^i \circ \theta^i$, $i \in \{1,2,3\}$, satisfy the analytical fourth order condition in $y = 1$ if and only if for all $i \in \{1,2,3\}$,

$$\overline{V}\left(\sigma^i\right)(1)\nu^i(1) + \overline{T}\left(\sigma^i \circ \theta^i\right)(1)\tau^i(1) = 0\,.$$

As σ satisfies the geometrical fourth order condition in $y = 1$, namely, for all $i \in \{1,2,3\}$,

$$\overline{V}\left(\sigma^i\right)(1) = 0\,,$$

it remains to show that we may choose $q^i_{xxx}(1)$ and $q^i_{xxxx}(1)$ in such a way that any diffeomorphism θ^i with $\partial_x^l\theta^i(1) = \partial_x^l q^i(1)$, $l \in \{0,1,2,3,4\}$, satisfies

$$\overline{T}\left(\sigma^i \circ \theta^i\right)(1) = 0\,.$$

The choice of $q^i(1)$, $q^i_x(1)$ and $q^i_{xx}(1)$ yields that $\partial_x^2\left(\sigma^i \circ \theta^i\right)(1) = 0$ for any diffeomorphism θ^i coinciding with q^i near $y = 1$. In particular, formula (3.26) for $\overline{T}\left(\sigma^i \circ \theta^i\right)(1)$ reduces to

$$\overline{T}\left(\sigma^i \circ \theta^i\right)(1) = -2\left|\sigma^i_x(1)\right|^{-5}\left\langle\left(\sigma^i \circ \theta^i\right)_{xxxx}(1), \sigma^i_x(1)\right\rangle\,.$$

Setting $q^i_{xxx}(1) = 0$ we obtain

$$\begin{aligned}&\overline{T}\left(\sigma^i \circ \theta^i\right)(1)\\ &= -2\left|\sigma^i_x(1)\right|^{-5}\left\langle\sigma^i_{xxxx}(1) + 6\sigma^i_{xxx}(1)q^i_{xx}(1) + 3\sigma^i_{xx}(1)(q^i_{xx}(1))^2 + \sigma^i_x(1)q^i_{xxxx}(1), \sigma^i_x(1)\right\rangle\,.\end{aligned}$$

Hence, the choice

$$q^i_{xxxx}(1) = -\left|\sigma^i_x(1)\right|^{-2}\left\langle\sigma^i_{xxxx}(1) + 6\sigma^i_{xxx}(1)q^i_{xx}(1) + 3\sigma^i_{xx}(1)(q^i_{xx}(1))^2, \sigma^i_x(1)\right\rangle$$

guarantees the desired property.
Given p^i and q^i the desired diffeomorphisms $\theta^i : [0,1] \to [0,1]$, $i \in \{1,2,3\}$, are then constructed as in Lemma 4.55. □

In analogy to Theorem 4.58 we obtain

Theorem 4.66 (Existence of classical solutions to the geometric problem)**.** *Let* $\alpha \in (0,1)$ *and* σ *be a geometrically* α*–admissible initial value to system* (3.21) *with given endpoints* P^1, P^2, P^3. *Then there exists a positive time* $\boldsymbol{T}$ *such that for all* $T \in (0,\boldsymbol{T}]$ *there exists a classical solution to system* (3.21) *in* $[0,T]$ *with endpoints* P^1, P^2, P^3 *and initial datum* σ.

Proof. Given $\alpha \in (0,1)$ and a geometrically α–admissible initial value σ to system (3.21) with endpoints P^1, P^2, P^3, Lemma 4.65 yields that there exist smooth diffeomorphisms $\theta^i : [0,1] \to [0,1]$, $i \in \{1,2,3\}$, such that $\varphi = \left(\varphi^1,\varphi^2,\varphi^3\right)$ defined by $\varphi^i := \sigma^i \circ \theta^i$ is an analytically α–admissible initial datum to (4.3) with endpoints P^1, P^2, P^3. Theorem 4.53 yields that there exists a positive time $\boldsymbol{T}$ such that for all $T \in [0,\boldsymbol{T}]$ there exists a classical solution to (4.3) in $[0,T]$ with endpoints P^1, P^2, P^3 and initial datum φ, which is in particular a classical solution to (3.21) in $[0,T]$ with endpoints P^1, P^2, P^3 and initial datum σ. □

We prove the analogon to Theorem 4.59 in the Hölder setting. We hereby make use of the fact that parabolic Hölder spaces are stable under composition with diffeomorphisms. More precisely, given $T > 0$, $\alpha \in (0,1)$, $f, g \in C^{\frac{4+\alpha}{4}, 4+\alpha}([0,T]\times[0,1];\mathbb{R})$ such that for all $t \in [0,T]$, $g(t) : [0,1] \to [0,1]$ is a diffeomorphism, the function $(t,x) \mapsto f(t, g(t,x))$ lies in $C^{\frac{4+\alpha}{4}, 4+\alpha}([0,T]\times[0,1];\mathbb{R})$.

Theorem 4.67. *Let $\alpha \in (0,1)$, T be positive and η be a classical solution to (3.21) in the time interval $[0,T]$ with endpoints P^1, P^2, P^3. There exists a time $T_R \in (0,T]$ and a function $\psi \in C^{\frac{4+\alpha}{4}, 4+\alpha}\left([0,T_R]\times[0,1];\mathbb{R}^3\right)$ such that for all $t \in [0,T_R]$, $i \in \{1,2,3\}$, $\psi^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism such that $\gamma = \left(\gamma^1, \gamma^2, \gamma^3\right)$ with $\gamma^i(t,x) := \eta^i\left(t, \psi^i(t,x)\right)$ is a classical solution to (4.3) in $[0,T_R]$ with endpoints P^1, P^2, P^3 and initial datum $\eta(0) \circ \psi(0)$. The existence time T_R depends on $\|\eta\|_{\frac{4+\alpha}{4}, 4+\alpha}$ and $\boldsymbol{c}(\eta) = \min_{i\in\{1,2,3\}, x\in[0,1], t\in[0,T]} \left|\eta^i_x(t,x)\right|$.*

Proof. Let α^1, α^2 and α^3 denote the angles between the unit tangents $\tau^2(0,0)$ and $\tau^3(0,0)$, $\tau^3(0,0)$ and $\tau^1(0,0)$, and $\tau^1(0,0)$ and $\tau^2(0,0)$ to $\eta(0)$ in $y = 0$, respectively. As η is a classical solution to (3.21), deriving the concurrency condition in time yields

$$\eta^1_t(t,0) = \eta^2_t(t,0) = \eta^3_t(t,0)\,.$$

Testing this identity with $\sin\left(\alpha^1\right)\nu^1(0,0)$ and using the relation $\sum_{i=1}^3 \sin\left(\alpha^i\right)\nu^i(0,0) = 0$ yields that $\eta(0)$ satisfies the geometrical fourth order condition in $y = 0$. Similarly, differentiating the condition $\eta^i(t,1) = P^i$ with respect to time gives after evaluation in $t = 0$ and testing with $\nu^i(0,1)$ that the normal velocity $V\left(\eta^i\right)$ of η^i satisfies

$$V\left(\eta^i\right)(0,1) = \overline{V}\left(\eta^i(0)\right)(1) = 0\,.$$

Hence, $\eta(0)$ satisfies also the geometrical fourth order condition in $y = 1$, and is in fact a geometrically α–admissible initial network to system (3.21) with endpoints P^1, P^2, P^3. Thus, Lemma 4.65 implies that there exist smooth diffeomorphisms $\theta^i : [0,1] \to [0,1]$, $i \in \{1,2,3\}$, such that $\varphi = \left(\varphi^1, \varphi^2, \varphi^3\right)$ defined by $\varphi^i := \eta^i(0) \circ \theta^i$ is an analytically α–admissible initial network to (4.3) with endpoints P^1, P^2, P^3. As in Theorem 4.59 we derive necessary and sufficient conditions for the unknown ψ. If $\psi \in C^{\frac{4+\alpha}{4}, 4+\alpha}\left([0,T]\times[0,1];\mathbb{R}^3\right)$ is such that for all $t \in [0,T]$, $i \in \{1,2,3\}$, $\psi^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism, then $\gamma^i(t,x) := \eta^i\left(t, \psi^i(t,x)\right)$ defines a function $\gamma \in C^{\frac{4+\alpha}{4}, 4+\alpha}\left([0,T]\times[0,1];(\mathbb{R}^2)^3\right)$. Testing the pointwise identity

$$\gamma^i_t(t,x) = \eta^i_t\left(t, \psi^i(t,x)\right) + \eta^i_x\left(t, \psi^i(t,x)\right)\psi^i_t(t,x) \tag{4.59}$$

with $\nu^i_\eta\left(t, \psi^i(t,x)\right) = \nu^i_\gamma(t,x)$ yields that the normal velocities $V\left(\gamma^i\right)$ and $V\left(\eta^i\right)$ satisfy the identity

$$V\left(\gamma^i\right)(t,x) = V\left(\eta^i\right)\left(t, \psi^i(t,x)\right)$$

regardless of the precise choice of the reparametrisations ψ. Testing (4.59) with $\eta^i_x\left(t, \psi^i(t,x)\right)$ yields

$$\psi^i_t(t,x) = \left|\eta^i_x\left(t, \psi^i(t,x)\right)\right|^{-1}\left(T\left(\gamma^i\right)(t,x) - T\left(\eta^i\right)\left(t, \psi^i(t,x)\right)\right)$$

where $T\left(\gamma^i\right)$ and $T\left(\eta^i\right)$ are the tangential velocities of γ^i and η^i, respectively. As in the proof of Theorem 4.59 we find that the curves $(t,x) \mapsto \eta^i\left(t, \psi^i(t,x)\right)$, $i \in \{1,2,3\}$, form a classical solution to (4.3) in $[0,T]$ with endpoints P^1, P^2, P^3 and initial datum $\eta(0) \circ \theta$ if and only if the diffeomorphisms $\psi^i(t) : [0,1] \to [0,1]$ form a classical solution ψ to system (4.45) which is equivalent to

$$\begin{cases} \psi^i_t(t,x) - \left|\eta^i_x\left(t, \psi^i(t,x)\right)\right|^{-1}\left(\overline{T}\left(\gamma^i(t)\right)(x) - T\left(\eta^i\right)\left(t, \psi^i(t,x)\right)\right) &= 0\,, \\ \psi^i(t,y) &= y\,, \\ \psi^i_{xx}(t,y) + \left|\eta^i_x(t,y)\right|^{-2}\left\langle \eta^i_{xx}(t,y), \eta^i_x(t,y)\right\rangle \psi^i_x(t,y)^2 &= 0\,, \\ \psi^i(0,x) &= \theta^i(x)\,, \end{cases} \tag{4.60}$$

for $t \in [0,T]$, $x \in [0,1]$, $y \in \{0,1\}$, $i \in \{1,2,3\}$, where $\gamma^i(t,x) := \eta^i\left(t, \psi^i(t,x)\right)$ and $\overline{T}\left(\gamma^i(t)\right)(x)$ is the expression defined in formula (3.26) which depends on $\partial_x^k \eta^i\left(t, \psi^i(t,x)\right)$ and $\partial_x^k \psi^i(t,x)$, $k \in \{1,2,3,4\}$. Turning to the existence of classical solutions to (4.60) we verify that the smooth diffeomorphisms $\theta^i : [0,1] \to [0,1]$, $i \in \{1,2,3\}$, constructed in Lemma 4.65 satisfy the compatibility conditions for (4.60) that are required in the Hölder setting. Indeed, deriving the condition $\psi^i(t,y) = y$ with respect to $t \in [0,T]$ yields that in addition to the boundary conditions in (4.60) the diffeomorphisms θ^i need to be such that

$$\overline{T}\left(\eta^i(0) \circ \theta^i\right)(y) = T\left(\eta^i\right)(0,y) \tag{4.61}$$

holds for $i \in \{1,2,3\}$ and $y \in \{0,1\}$. As the boundary conditions $\theta^i(y) = y$ and

$$\theta^i_{xx}(y) + \left|\eta^i_x(0,y)\right|^{-2} \left\langle \eta^i_{xx}(0,y), \eta^i_x(0,y)\right\rangle \theta^i_x(y)^2 = 0$$

for $i \in \{1,2,3\}$ and $y \in \{0,1\}$ follow directly from the construction of θ, it remains to verify (4.61). As η is a classical solution to (3.21), we have for all $t \in [0,T]$,

$$\eta^1_t(t,0) = \eta^2_t(t,0) = \eta^3_t(t,0)$$

and thus in $(t,y) = (0,0)$ it holds

$$V\left(\eta^1\right)\nu^1 + T\left(\eta^1\right)\tau^1 = V\left(\eta^2\right)\nu^2 + T\left(\eta^2\right)\tau^2 = V\left(\eta^3\right)\nu^3 + T\left(\eta^3\right)\tau^3 .$$

Arguing as in the proof of Lemma 4.65 to distinguish the different scenarios for the values of $\sin\left(\alpha^i\right)$, Lemma 4.64 yields that for all $i \in \{1,2,3\}$, the term $T\left(\eta^i\right)(0,0)$ is a linear combination of $V\left(\eta^1\right)(0,0)$, $V\left(\eta^2\right)(0,0)$ and $V\left(\eta^3\right)(0,0)$ with coefficients only depending on the angles α^1, α^2, α^3. The same identities are fulfilled by $\overline{T}\left(\eta^i(0) \circ \theta^i\right)(0)$, $i \in \{1,2,3\}$, due to the proof of Lemma 4.65 and the identity $V\left(\eta^i\right)(0,0) = \overline{V}\left(\eta^i(0)\right)(0)$. This shows that (4.61) is valid for all $i \in \{1,2,3\}$ in the case $y = 0$. The construction of θ^i in the proof of Lemma 4.65 further yields

$$\overline{T}\left(\eta^i(0) \circ \theta^i\right)(1) = 0 ,$$

and testing the identity $\eta^i_t(0,1) = 0$ with $\tau^i(0,1)$ gives

$$T\left(\eta^i\right)(0,1) = 0$$

which allows us to conclude that θ satisfies the compatibility conditions for system (4.60) in the Hölder setting.

To show that there exist a time $T_R \in (0,T]$ and a family of diffeomorphisms ψ solving (4.60) in the classical sense we proceed as in the case of strong solutions and consider the problem for the inverse $\xi = \left(\xi^1, \xi^2, \xi^3\right)$ defined by $\xi^i(t) := \left(\psi^i(t)\right)^{-1}$ which is given in (4.48). In the following we briefly sketch how to prove existence of classical solutions to (4.48) with initial datum θ^{-1}. As in the proof of Lemma 4.60 one shows that the curves $\left(\theta^i\right)^{-1}$, $i \in \{1,2,3\}$, satisfy the boundary conditions appearing in (4.48). In the Hölder setting we have to verify in addition the fourth order compatibility condition for the initial value coming from $\xi^i_t(0,z) = 0$ for $z \in \{0,1\}$. As

$$\xi^i_t(t,y) = -\psi^i_t\left(t, \xi^i(t,y)\right)\xi^i_y(t,y) ,$$

the additional fourth order condition for θ^{-1} is again given by (4.61) for $i \in \{1,2,3\}$, $y \in \{0,1\}$. The validity of this condition is shown above.
To obtain existence of a classical solution to (4.48) we consider the associated linearised system given in (4.49) for a suitable right hand side $f \in C^{\frac{\alpha}{4},\alpha}\left([0,T] \times [0,1]; \mathbb{R}^3\right)$, $b \in C^{\frac{2+\alpha}{4}}\left([0,T]; \mathbb{R}^6\right)$

and $\phi \in C^{4+\alpha}\left([0,1];\mathbb{R}^3\right)$ satisfying the linear compatibility conditions of order four in [136, § 14]. Then Lemma 3.45, Proposition 4.19 and [136, Theorem 4.9] yield the existence of a classical solution to (4.49) with norm uniformly bounded with respect to the right hand side in analogy to Theorem 4.43. Existence of classical solutions is obtained via contraction estimates in the set

$$X_{\boldsymbol{T},M} := \left\{\xi \in E_{\boldsymbol{T}} : \xi^i(t,z) = z \text{ for } z \in \{0,1\}, \xi(0) = \theta^{-1}, \|\xi\|_{E_{\boldsymbol{T}}} \leq M\right\}$$

with $E_{\boldsymbol{T}} := C^{\frac{4+\alpha}{4},4+\alpha}\left([0,T]\times[0,1];\mathbb{R}^3\right)$ and $\boldsymbol{T} \in (0,T]$, $M > 0$ defined appropriately in dependence on $\left\|\theta^{-1}\right\|_{4+\alpha}$ and $\boldsymbol{c}\left(\theta^{-1}\right)$ to guarantee that for all $\xi \in X_{\boldsymbol{T},M}$, $i \in \{1,2,3\}$, $t \in [0,\boldsymbol{T}]$, $\xi^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism. We refer to the arguments in Lemma 4.60 and Section 4.1.3. The existence time $T_R \in (0,\boldsymbol{T})$ of the classical solution ξ to (4.48) depends on $\boldsymbol{c}(\eta)$, $\|\eta\|_{E_{T_R}}$, $\boldsymbol{c}(\theta)$ and $\|\theta\|_{4+\alpha}$. □

We remark that as in the case of strong solutions, smoothness of θ is not necessary to solve (4.60) in the classical sense. In fact, it is enough ot require that $\theta \in C^{4+\alpha}\left([0,1];\mathbb{R}^3\right)$ satisfies the boundary conditions appearing in (4.60) and in addition the conditions on $\overline{T}\left(\eta^i(0)\circ\theta^i\right)(y)$, $y \in \{0,1\}$, derived in Lemma 4.65 with $\sigma^i := \eta^i(0)$.
In analogy to Theorem 4.61 and Theorem 4.62 we obtain

Theorem 4.68 (Geometric uniqueness of classical solutions)**.** *Let $\alpha \in (0,1)$, T be positive and suppose that both η and $\widetilde{\eta}$ are classical solutions to* (3.21) *in $[0,T]$ with endpoints P^1, P^2, P^3 such that for $i \in \{1,2,3\}$ there exist C^1–diffeomorphisms $\zeta_0^i : [0,1] \to [0,1]$ with $\zeta_0^i \in C^{4+\alpha}([0,1])$, $\zeta_0^i(y) = y$ for $y \in \{0,1\}$, and $\eta^i(0) = \widetilde{\eta}^i(0)\circ\zeta_0^i$. Then there exists $\zeta \in C^{\frac{4+\alpha}{4},4+\alpha}\left([0,T]\times[0,1];\mathbb{R}^3\right)$ with $\zeta(0) = \zeta_0$ such that for all $t \in [0,T]$, $i \in \{1,2,3\}$, $\zeta^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism with $\eta^i(t) = \widetilde{\eta}^i(t)\circ\zeta^i(t)$.*

Proof. By Theorem 4.67 there exists a time $T_\eta \in [0,T]$ and $\psi \in C^{\frac{4+\alpha}{4},4+\alpha}\left([0,T]\times[0,1];\mathbb{R}^3\right)$ with $\psi(0) = \theta$ such that for all $t \in [0,T_\eta]$, $i \in \{1,2,3\}$, $\psi^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism such that $\gamma = \left(\gamma^1,\gamma^2,\gamma^3\right)$ with $\gamma^i(t,x) := \eta^i\left(t,\psi^i(t,x)\right)$ is a classical solution to (4.3) in $[0,T_\eta]$ with initial datum $\varphi = \eta(0)\circ\theta$ where θ is the function constructed in Lemma 4.65. Then for $i \in \{1,2,3\}$, $\widetilde{\theta}^i := \zeta_0^i\circ\theta^i : [0,1] \to [0,1]$ is a C^1–diffeomorphism with $\widetilde{\theta}^i(y) = y$, $y \in \{0,1\}$, and $\widetilde{\theta}^i \in C^{4+\alpha}([0,1])$. It is shown in the proof of Theorem 4.61 that $\widetilde{\theta}$ satisfies the boundary conditions of system (4.60) with $\widetilde{\eta}$ instead of η. As angles and curvature are invariant under reparametrisation and as both $T\left(\eta^i\right)(0,0)$ and $T\left(\widetilde{\eta}^i\right)(0,0)$ are given by a linear combination of $\overline{V}\left(\eta^i(0)\right)(0) = \overline{V}\left(\widetilde{\eta}^i(0)\right)(0)$ with coefficients only depending on the angles between the tangents at the junction of $\eta(0)$, we conclude that

$$T\left(\eta^i\right)(0,0) = T\left(\widetilde{\eta}^i\right)(0,0).$$

As further

$$\overline{T}\big(\widetilde{\eta}^i(0)\circ\widetilde{\theta}^i\big)(0) = \overline{T}\big(\eta^i(0)\circ(\zeta_0^i)^{-1}\circ\zeta_0^i\circ\theta^i\big)(0) = \overline{T}\left(\eta^i(0)\circ\theta^i\right)(0),$$

we conclude as in the proof of Theorem 4.67 that $\widetilde{\theta}$ satisfies the fourth order compatibility condition

$$\overline{T}\big(\widetilde{\eta}^i(0)\circ\widetilde{\theta}^i\big)(0) = T\left(\widetilde{\eta}^i\right)(0,0).$$

Furthermore, the fixed endpoint condition implies

$$T\left(\widetilde{\eta}^i\right)(0,1) = 0$$

and the construction of θ in the proof of Lemma 4.65 yields

$$0 = \overline{T}\left(\eta^i(0)\circ\theta^i\right)(1) = \overline{T}\big(\widetilde{\eta}^i(0)\circ\widetilde{\theta}^i\big)(1).$$

Hence, $\widetilde{\theta}$ satisfies the compatibility conditions of order four to system (4.60) with $\widetilde{\eta}$ instead of η and Theorem 4.67 yields that there exists $T_{\widetilde{\eta}} \in (0, T]$, $\vartheta \in C^{\frac{4+\alpha}{4}, 4+\alpha}\left([0,T] \times [0,1]; \mathbb{R}^3\right)$ with $\vartheta(0) = \widetilde{\theta}$ such that for all $t \in [0,T]$, $i \in \{1,2,3\}$, $\vartheta^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism with $\widetilde{\gamma}^i(t,x) := \widetilde{\eta}^i\left(t, \vartheta^i(t,x)\right)$ defining a classical solution $\widetilde{\gamma}$ to (4.3) with initial datum $\widetilde{\eta}(0)\circ\vartheta(0) = \varphi$. By definition of the parabolic Hölder norms there exists a time $T_g \in (0, \min\{T_\eta, T_{\widetilde{\eta}}, \boldsymbol{T}\}]$ such that γ and $\widetilde{\gamma}$ lie in $\boldsymbol{E}_{T_g} \cap \overline{B_{M,T_g}}$ with $M = M(\varphi)$ as in the proof of Proposition 4.52 and $\boldsymbol{T} = \boldsymbol{T}(\varphi)$ as in Theorem 4.53. In fact, one may choose $T_g = \min\{T_\eta, T_{\widetilde{\eta}}, \boldsymbol{T}\}$. The uniqueness assertion in Theorem 4.53 yields for all $t \in [0, T_g]$ and $i \in \{1,2,3\}$ the identity $\eta^i(t) = \widetilde{\eta}^i(t) \circ \zeta^{(0),i}(t)$ with $\zeta^{(0),i}(t) := \vartheta^i(t) \circ \left(\psi^i(t)\right)^{-1}$. This shows "local geometric uniqueness". To show "global geometric uniqueness" we need to construct reparametrisations ζ^i, $i \in \{1,2,3\}$, on the entire time interval $[0,T]$. To this end we may use exactly the same arguments as in the proof of Theorem 4.62. The idea is to concatenate the diffeomorphisms from the local geometric uniqueness using that the existence time T_g does not depend on the time slot one is considering which is due to the characterisations of the existence times in Theorem 4.67, Theorem 4.53 and the a priori estimates (4.50) and (4.51). The only difference to the proof of Theorem 4.62 is that we need to verify that the concatenated diffeomorphism lies in the space $C^{\frac{4+\alpha}{4}, 4+\alpha}\left([0,T] \times [0,1]; \mathbb{R}^3\right)$. We briefly sketch the required argument. Suppose that $2T_g \leq T$, $\zeta^{(0),i}, \zeta^{(1),i} \in C^{\frac{4+\alpha}{4}, 4+\alpha}([0,T_g] \times [0,1]; \mathbb{R})$ are such that for all $t \in [0, T_g]$, $\zeta^{(0),i}(t)$ and $\zeta^{(1),i}(t)$ are C^1–diffeomorphisms of the interval $[0,1]$ satisfying $\zeta^{(0),i}(T_g) = \zeta^{(1),i}(0)$ and the identities

$$
\begin{aligned}
\eta^i(t) &= \widetilde{\eta}^i(t) \circ \zeta^{(0),i}(t)\,, && t \in [0, T_g]\,,\\
\eta^i(t) &= \widetilde{\eta}^i(t) \circ \zeta^{(1),i}(t - T_g)\,, && t \in [T_g, 2T_g]\,.
\end{aligned}
$$

Deriving these identities once with respect to time and four times with respect to space one verifies that

$$
\zeta^i(t) := \zeta^{(0),i}(t)\chi_{[0,T_g]}(t) + \zeta^{(1),i}(t - T_g)\chi_{(T_g, 2T_g]}(t)\,, \qquad t \in [0, 2T_g]
$$

defines a function in $C^{\frac{4+\alpha}{4}, 4+\alpha}([0, 2T_g] \times [0,1]; \mathbb{R})$ using the respective regularities of η^i and $\widetilde{\eta}^i$. □

In the case of Theta networks we obtain the analogous result.

Theorem 4.69 (Geometric existence and uniqueness of classical solutions (Theta))**.** *Let $\alpha \in (0,1)$. Given a geometrically α–admissible initial network σ to system* (3.20) *there exists a positive time $\boldsymbol{T}$ such that for all $T \in (0, \boldsymbol{T}]$ there exists a classical solution to system* (3.20) *in $[0,T]$ with initial datum σ. If both η and $\widetilde{\eta}$ are classical solutions to* (3.20) *in $[0,T]$, $T > 0$, such that for $i \in \{1,2,3\}$ there exist C^1–diffeomorphisms $\zeta_0^i : [0,1] \to [0,1]$ with $\zeta_0^i \in C^{4+\alpha}([0,1])$, $\zeta_0^i(y) = y$ for $y \in \{0,1\}$, and $\eta^i(0) = \widetilde{\eta}^i(0) \circ \zeta_0^i$, then there exists $\zeta \in C^{\frac{4+\alpha}{4}, 4+\alpha}\left([0,T] \times [0,1]; \mathbb{R}^3\right)$ with $\zeta(0) = \zeta_0$ such that for all $t \in [0,T]$, $i \in \{1,2,3\}$, $\zeta^i(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism with $\eta^i(t) = \widetilde{\eta}^i(t) \circ \zeta^i(t)$.*

Proof. As explained in Remark 4.9 we may again adapt the arguments in this subsection to obtain the result also for system (3.20). □

Chapter 5

Long time behaviour of the elastic flow of networks

This chapter is devoted to prove the long time existence result Main Theorem 4 given in the introductory section of Part II following [65, Section 7]. In Section 5.1 we introduce our notion of solutions. We refer to Section 5.4 for further comments on our approach of defining maximal solutions. Using the a priori estimates derived in Section 3.5, the particular long time existence theorem for Triods and Theta networks is then shown in Sections 5.2 and 5.3, respectively. In Section 5.5 we conclude the chapter with some conjectures on possible scenarios for different initial values.

5.1 Existence and uniqueness of maximal solutions

This section is dedicated to the study of maximal solutions. Taking into account the geometric nature of the evolution problems (3.20) and (3.21) we hereby allow a priori that the geometric evolution is composed of different parametrisations defined on adjacent time intervals. The regularisation result in Subsection 4.1.2 then guarantees the existence of one parametrisation on the entire time interval as shown in Corollary 5.8. In Proposition 5.9 we show that given a geometrically admissible initial network there exists a maximal solution which is unique up to reparametrisations.

Definition 5.1 (Jointed strong solution)**.** Let $p \in (5, \infty)$, T be positive and σ be a geometrically p–admissible initial datum to (3.20) or (3.21), respectively. A *jointed strong solution to the considered system in* $[0, T]$ *with initial datum* σ is a collection $(\gamma_0, \ldots, \gamma_N)$, $N \in \mathbb{N}_0$, with $\gamma_j : [t_{j-1}, t_j] \times [0, 1] \to (\mathbb{R}^2)^3$, $j \in \{0, \ldots, N\}$, $0 =: t_{-1} < t_0 < \cdots < t_N = T$, such that γ_0 is a strong solution to the considered system with initial datum σ, and for all $j \in \{1, \ldots, N\}$, $t \mapsto \gamma_j (t + t_{j-1})$ is a strong solution to the considered system in $[0, t_j - t_{j-1}]$ with initial datum $\gamma_{j-1} (t_{j-1})$. We call $(t_0, \ldots, t_N)$ the *partition* of the solution $(\gamma_0, \ldots, \gamma_N)$.

Definition 5.2. Let $p \in (5, \infty)$ and T be positive. Given two jointed strong solutions $(\gamma_0, \ldots, \gamma_N)$, $(\eta_0, \ldots, \eta_M)$, $N, M \in \mathbb{N}_0$, to one of the systems (3.20) and (3.21) in $[0, T]$, we say that $(\gamma_0, \ldots, \gamma_N)$ *and* $(\eta_0, \ldots, \eta_M)$ *describe the same evolution* if for all $j \in \{0, \ldots, N\}$, $k \in \{0, \ldots, M\}$, $t \in [t_{j-1}, t_j] \cap [t_{k-1}, t_k]$, it holds $\left[\gamma_j^i(t)\right] = \left[\eta_k^i(t)\right]$. In this case we write $(\gamma_0, \ldots, \gamma_N) \sim (\eta_0, \ldots, \eta_M)$.

Proposition 5.3 (Existence and Uniqueness of jointed strong solutions)**.** *Let* $p \in (5, \infty)$ *and* σ *be a geometrically* p*–admissible initial datum to system* (3.20) *or system* (3.21)*. There exists* $T > 0$ *such that there exists a jointed strong solution to the considered system in* $[0, T]$ *with initial datum* σ*. It is unique in the following sense: If* $(\gamma_0, \ldots, \gamma_N)$ *and* $(\eta_0, \ldots, \eta_N)$*,* $N \in \mathbb{N}_0$*, are jointed strong solutions to the considered system in* $[0, T]$*,* $T > 0$*, with initial datum* σ *and partition*

$(t_0,\dots,t_N)$, then for all $j\in\{0,\dots,N\}$ there exists $\zeta_j\in W^{1,4}_p\left((t_{j-1},t_j)\times(0,1);\mathbb{R}^3\right)$ such that for all $t\in[t_{j-1},t_j]$, $i\in\{1,2,3\}$, $\zeta^i_j(t):[0,1]\to[0,1]$ is a C^1–diffeomorphism with $\gamma^i_j(t)=\eta^i_j(t)\circ\zeta^i_j(t)$. In particular, $(\gamma_0,\dots,\gamma_N)\sim(\eta_0,\dots,\eta_N)$.

Proof. We write the proof in an exemplary manner for system (3.21). The result for (3.20) is obtained by using Theorem 4.63 instead of Theorem 4.58 and 4.62.

The existence of a jointed strong solution to (3.21) with endpoints P^1, P^2, P^3 and initial datum σ follows from Theorem 4.58 as every strong solution to (3.21) with endpoints P^1, P^2, P^3 and initial datum σ is in particular a jointed strong solution. Let $(\gamma_0,\dots,\gamma_N)$ and $(\eta_0,\dots,\eta_N)$, $N\in\mathbb{N}_0$, be jointed strong solutions to (3.21) with endpoints P^1, P^2, P^3, initial datum σ and partition $(t_0,\dots,t_N)$. We show the uniqueness statement inductively with respect to $j\in\{0,\dots,N\}$. As γ_0 and η_0 are strong solutions to (3.21) in $[0,t_0]$ with endpoints P^1, P^2, P^3 and initial datum σ, Lemma 3.38 and Definition 3.43 yield that there exist C^1–diffeomorphisms $\xi^i_0:[0,1]\to[0,1]$, $\xi^i_0\in W^{4-4/p}_p((0,1);\mathbb{R})$, $\xi^i_0(y)=y$, $y\in\{0,1\}$, such that $\gamma^i_0=\eta^i_0\circ\xi^i_0$. Theorem 4.62 implies that there exists $\zeta_0\in W^{1,4}_p\left((0,t_0)\times(0,1);\mathbb{R}^3\right)$ with $\zeta^i_0(0)=\xi^i_0$ such that for all $t\in[0,t_0]$, $i\in\{1,2,3\}$, $\zeta^i_0(t):[0,1]\to[0,1]$ is a C^1–diffeomorphism with $\gamma^i_0(t)=\eta^i_0(t)\circ\zeta^i_0(t)$. Let $j\in\{0,\dots,N-1\}$ be given and $\zeta_j\in W^{1,4}_p\left((t_{j-1},t_j)\times(0,1);\mathbb{R}^3\right)$ be such that for all $t\in[t_{j-1},t_j]$, $i\in\{1,2,3\}$, $\zeta^i_j(t):[0,1]\to[0,1]$ is a C^1–diffeomorphism with $\gamma^i_j(t)=\eta^i_j(t)\circ\zeta^i_j(t)$. In particular, there holds for all $i\in\{1,2,3\}$ that $\gamma^i_j(t_j)=\eta^i_j(t_j)\circ\zeta^i_j(t_j)$. As $t\mapsto\gamma_{j+1}(t+t_j)$ and $t\mapsto\eta_{j+1}(t+t_j)$ are strong solutions to (3.21) in $[0,t_{j+1}-t_j]$ with endpoints P^1, P^2, P^3, and initial values $\gamma_j(t_j)$ and $\eta_j(t_j)$, respectively, Lemma 3.38 and Definition 3.43 again yield the existence of C^1–diffeomorphisms $\xi^i_{j+1}:[0,1]\to[0,1]$, $i\in\{1,2,3\}$, with $\xi^i_{j+1}\in W^{4-4/p}_p((0,1);\mathbb{R})$, $\xi^i_{j+1}(y)=y$, $y\in\{0,1\}$, such that $\gamma^i_{j+1}(t_j)=\eta^i_{j+1}(t_j)\circ\xi^i_{j+1}$. Theorem 4.62 then implies the existence of $\tilde\zeta_{j+1}\in W^{1,4}_p\left((0,t_{j+1}-t_j)\times(0,1);\mathbb{R}^3\right)$, $\tilde\zeta^i_{j+1}(0)=\xi^i_{j+1}$, such that for all $t\in[0,t_{j+1}-t_j]$, $i\in\{1,2,3\}$, $\tilde\zeta^i_{j+1}(t):[0,1]\to[0,1]$ is a C^1–diffeomorphism with $\gamma^i_{j+1}(t+t_j)=\eta^i_{j+1}(t+t_j)\circ\tilde\zeta^i_{j+1}(t)$. This completes the induction as $\zeta^i_{j+1}(t):=\tilde\zeta^i_{j+1}(t-t_j)$, $i\in\{1,2,3\}$, $t\in[t_j,t_{j+1}]$, has the desired properties. □

Definition 5.4 (Jointed smooth solution)**.** Let $p\in(5,\infty)$, T be positive and σ be a geometrically p–admissible initial datum to (3.20) or (3.21), respectively. A *jointed smooth solution to the considered system in* $[0,T]$ *with initial datum* σ is a jointed strong solution $(\gamma_0,\dots,\gamma_N)$ to the considered system with initial datum σ such that for all $0<\varepsilon<t_0$, $\gamma_0\in C^\infty\left([\varepsilon,t_0]\times[0,1];(\mathbb{R}^2)^3\right)$, and for all $j\in\{1,\dots,N\}$, $\gamma_j\in C^\infty\left([t_{j-1},t_j]\times[0,1];(\mathbb{R}^2)^3\right)$. We call $(t_0,\dots,t_N)$ the *partition* of the solution $(\gamma_0,\dots,\gamma_N)$.

Proposition 5.5 (Existence of jointed smooth solutions)**.** *Let $p\in(5,\infty)$ and σ be a geometrically p–admissible initial datum to* (3.20) *or* (3.21)*, respectively. Then there exists a time $T>0$ such that there exists a jointed smooth solution to the considered system in $[0,T]$ with endpoints P^1, P^2, P^3 and initial datum σ.*

Proof. We prove the statement exemplarily for system (3.21). The statement for (3.20) is obtained using Remark 4.57, Theorem 4.32 and Theorem 4.38.

Let σ be a geometrically p–admissible initial datum to (3.21) with endpoints P^1, P^2, P^3. Proposition 4.56 yields the existence of smooth diffeomorphisms $\theta^i:[0,1]\to[0,1]$, $i\in\{1,2,3\}$, such that $\varphi=\left(\varphi^1,\varphi^2,\varphi^3\right)$ defined by $\varphi^i:=\sigma^i\circ\theta^i$ is an analytically p–admissible initial network to (4.3) with endpoints P^1, P^2, P^3. Theorem 4.30 implies that there exists $T>0$ such that there exists a strong solution γ to (4.3) in $[0,T]$ with initial datum φ which satisfies by Theorem 4.37 for all $\varepsilon\in(0,T)$, $\gamma\in C^\infty\left([\varepsilon,T]\times[0,1];(\mathbb{R}^2)^3\right)$. In particular, γ is a jointed smooth solution to (3.21) in $[0,T]$ with endpoints P^1, P^2, P^3 and initial datum σ. □

Proposition 5.6. *Let $p \in (5,\infty)$, T be positive and σ be a geometrically p–admissible initial datum to* (3.20) *or* (3.21)*, respectively, and suppose that $(\gamma_0,\dots,\gamma_N)$, $N\in\mathbb{N}_0$, is a jointed smooth solution to the considered system in $[0,T]$ with initial datum σ. Then there exists $\eta:[0,T]\times[0,1]\to(\mathbb{R}^2)^3$ such that for all $t\in[0,T]$, $i\in\{1,2,3\}$, $\eta^i(t)$ is a regular open curve parametrised with constant speed equal to its length and such that for all $\varepsilon\in(0,T)$, $\eta_{[\varepsilon,T]}$ is a smooth solution to the considered system with $\eta_{[\varepsilon,T]}\sim(\gamma_0,\dots,\gamma_N)_{|[\varepsilon,T]}$.*

Proof. Suppose that $(\gamma_0,\dots,\gamma_N)$, $N\in\mathbb{N}_0$, is a jointed smooth solution to (3.21). The proof for system (3.20) is completely analogous. As γ_j is a strong solution to (3.21) in $[t_{j-1},t_j]$, Lemma 3.45 yields that

$$\boldsymbol{c}(\gamma_j) := \min_{i\in\{1,2,3\}}\min_{t\in[0,T],x\in[0,1]}\left|\left(\gamma_j^i\right)_x(t,x)\right| > 0\,.$$

Given $j\in\{0,\dots,N\}$, $t\in[t_{j-1},t_j]$ where $t_{-1}:=0$, $i\in\{1,2,3\}$, we reparametrise the regular open curve $\gamma_j^i(t)$ with constant speed equal to its length

$$\ell_j^i(t) := L\left(\gamma_j^i(t)\right) = \int_0^1\left|\left(\gamma_j^i\right)_x(t,x)\right|\,\mathrm{d}x \geq \boldsymbol{c}(\gamma_j)\,,$$

that is, we consider $\eta_j^i(t) := \gamma_j^i(t)\circ\left(\phi_j^i(t)\right)^{-1}$ with $\phi_j^i(t):[0,1]\to[0,1]$ denoting the C^1–diffeomorphism

$$\phi_j^i(t,x) := \ell_j^i(t)^{-1}\int_0^x\left|\left(\gamma_j^i\right)_x(t,y)\right|\,\mathrm{d}y\,.$$

As for $j\in\{1,\dots,N\}$, $i\in\{1,2,3\}$, $(t,x)\mapsto\gamma_j^i(t,x)$ is smooth in $[t_{j-1},t_j]\times[0,1]$, we conclude that $t\mapsto\ell_j^i(t)$ is smooth on $[t_{j-1},t_j]$ and that $(t,x)\mapsto\phi_j^i(t,x)$ and $(t,x)\mapsto\phi_j^i(t)^{-1}(x)$ are smooth on $[t_{j-1},t_j]\times[0,1]$. Hence for all $j\in\{1,\dots,N\}$, $i\in\{1,2,3\}$, $(t,x)\mapsto\eta_j^i(t,x)$ is smooth on $[t_{j-1},t_j]\times[0,1]$, and analogously, for all $\varepsilon\in(0,t_0)$, $(t,x)\mapsto\eta_0^i(t,x)$ is smooth on $[\varepsilon,t_0]\times[0,1]$. To show that the function $\eta:[0,T]\times[0,1]\to(\mathbb{R}^2)^3$ defined by

$$\eta(t) := \begin{cases}\sum_{j=0}^N\eta_j(t)\chi_{[t_{j-1},t_j)}(t) & t\in[0,T)\,,\\ \eta_N(T) & t=T\,,\end{cases}$$

is smooth on $[\varepsilon,T]$ for all $\varepsilon\in(0,T)$, it is enough to show that for all $j\in\{0,\dots,N-1\}$, $\eta_j(t_j)=\eta_{j+1}(t_j)$. Indeed, smoothness of η then follows from the existence of smooth solutions to the corresponding analytic problem with uniform existence time and the results on geometric uniqueness. Let $j\in\{0,\dots,N-1\}$ be given. As γ_{j+1} is a strong solution to (3.21) with initial datum $\gamma_j(t_j)$, there exist C^1–diffeomorphisms $\xi^i:[0,1]\to[0,1]$, $\xi^i(y)=y$, $y\in\{0,1\}$, $i\in\{1,2,3\}$, such that $\gamma_j^i(t_j)\circ\xi^i=\gamma_{j+1}^i(t_j)$ which yields in particular $\left[\eta_j^i(t_j)\right]=\left[\eta_{j+1}^i(t_j)\right]$. As both $\eta_j^i(t_j)$ and $\eta_{j+1}^i(t_j)$ are parametrisations with constant speed equal to

$$\ell_j^i(t_j) = L\left(\gamma_j^i(t_j)\right) = L\left(\gamma_{j+1}^i(t_j)\right) = \ell_{j+1}^i(t_j)\,,$$

we conclude $\eta_j^i(t_j)=\eta_{j+1}^i(t_j)$ using [15, Lemma 2.1.14]. Similar arguments as in the proof of Proposition 3.31 yield that for all $\varepsilon\in(0,T)$, $\eta_{[\varepsilon,T]}$ is a strong solution (and hence a smooth solution) to (3.21) with endpoints P^1, P^2, P^3. The property $\eta_{|[\varepsilon,T]}\sim(\gamma_0,\dots,\gamma_N)$ directly follows from the construction of η. □

Definition 5.7 (Maximal solution)**.** Let $p\in(5,\infty)$ and σ be a geometrically p–admissible initial value to (3.20) or (3.21), respectively. A *maximal solution* to the respective system in $[0,T)$, $T\in(0,\infty]$, with initial datum σ is a collection $(\gamma_n)_{n\in\mathbb{N}_0}$ with $\gamma_n:[t_{n-1},t_n]\times[0,1]\to(\mathbb{R}^2)^3$, $t_{-1}:=0$, $t_{n-1}<t_n$ for all $n\in\mathbb{N}_0$ and $\lim\limits_{n\to\infty}t_n=T$, such that for all $N\in\mathbb{N}_0$, $(\gamma_0,\dots,\gamma_N)$ is a

jointed smooth solution to the considered system with initial datum σ, and such that in the case $T \in (0,\infty)$, there does not exist a jointed smooth solution to to the considered system in $[0,T]$ with endpoints P^1, P^2, P^3 and initial datum σ. We call $(t_n)_{n\in\mathbb{N}_0}$ the *partition* of the solution and $T_{max} := T$ the *maximal time of existence.*

Corollary 5.8. *Let $p \in (5,\infty)$ and σ be a geometrically p–admissible initial value to* (3.20) *or* (3.21)*, respectively, and suppose that $(\gamma_n)_{n\in\mathbb{N}_0}$ is a maximal solution to the considered system in the maximal time interval $[0,T_{max})$, $T_{max} \in (0,\infty]$, with initial datum σ and partition $(t_n)_{n\in\mathbb{N}_0}$. Then there exists $\eta : [0,T_{max}) \times [0,1] \to (\mathbb{R}^2)^3$ such that for all $t \in [0,T_{max})$, $i \in \{1,2,3\}$, $\eta^i(t)$ is a regular open curve parametrised with constant speed equal to its length $\ell^i(t)$ and such that for all $0 < \varepsilon < T < T_{max}$, $\eta_{|[\varepsilon,T]}$ is a smooth solution to the considered system with $\eta_{|[\varepsilon,T]} \sim (\gamma_0,\dots,\gamma_N)_{|[\varepsilon,T]}$ with $N = \min\{n \in \mathbb{N}_0 : t_n \geq T\}$.*

Proof. For all $N \in \mathbb{N}_0$, $(\gamma_0,\dots,\gamma_N)$ is a jointed smooth solution to the considered system in $[0,t_N]$ with initial datum σ. Proposition 5.6 yields the existence of $\eta^N : [0,t_N] \times [0,1] \to (\mathbb{R}^2)^3$ such that for all $t \in [0,t_N]$, $i \in \{1,2,3\}$, $\eta^{N,i}(t)$ is a regular open curve parametrised with constant speed equal to its length and such that for all $\varepsilon \in (0,t_N)$, $\eta^N_{|[\varepsilon,t_N]}$ is a smooth solution to the considered system in $[\varepsilon,t_N]$ with $\eta^N_{|[\varepsilon,t_N]} \sim (\gamma_0,\dots,\gamma_N)_{|[\varepsilon,T]}$. Given $N, M \in \mathbb{N}_0$ with $N \geq M$, the maps η^N and η^M coincide on $[0,t_M]$ due to [15, Lemma 2.1.14]. Indeed, given $t \in [0,t_M]$, $i \in \{1,2,3\}$, both $\eta^{N,i}(t)$ and $\eta^{M,i}(t)$ parametrise the same image with constant speed equal to the length. This allows us to define $\eta : [0,T_{max}) \times [0,1] \to (\mathbb{R}^2)^3$ by

$$\eta(t) := \sum_{N\in\mathbb{N}_0} \chi_{[t_{N-1},t_N)}(t)\eta^N(t)\,,$$

where $t_{-1} := 0$. It is then a direct consequence of Proposition 5.6 that η has the desired properties. □

Proposition 5.9 (Existence and uniqueness of maximal solutions)**.** *Let $p \in (5,\infty)$ and σ be a geometrically p–admissible initial value to* (3.20) *or* (3.21)*, respectively. There exists a maximal solution $(\gamma_n)_{n\in\mathbb{N}_0}$ to the considered system in the maximal time interval $[0,T_{max})$, $T_{max} \in (0,\infty]$, with initial datum σ and partition $(t_n)_{n\in\mathbb{N}}$. It is unique in the following sense: If $(\eta_0,\dots,\eta_N)$ is a jointed smooth solution to the considered system in $[0,T]$, $T < T_{max}$, with initial datum σ and partition $(t_0,\dots,t_N)$, then $(\eta_0,\dots,\eta_N) \sim (\gamma_0,\dots,\gamma_N)$.*

Proof. We carry out the details for system (3.21). The same arguments apply to problem (3.20). Given a geometrically p–admissible initial value σ to (3.21) with given endpoints P^1, P^2, P^3, we define

$$A_\sigma := \{T > 0 : \text{ there exists a jointed smooth solution to (3.21) in } [0,T] \text{ with endpoints } P^1, P^2, P^3 \text{ and initial datum } \sigma\}\,.$$

Proposition 5.5 yields that A_σ is non-empty and hence $T_{max} := \sup A_\sigma \in (0,\infty]$. Let $(T_n)_{n\in\mathbb{N}_0}$ be a sequence with $T_n \in A_\sigma$ and $T_n < T_{n+1}$ for $n \in \mathbb{N}_0$ such that $\lim\limits_{n\to\infty} T_n = T_{max}$. For every $n \in \mathbb{N}_0$ we have a jointed smooth solution $\left(\gamma^n_0,\dots,\gamma^n_{N_n}\right)$, $N_n \in \mathbb{N}_0$, to (3.21) in $[0,T_n]$ with endpoints P^1, P^2, P^3, initial datum σ and partition $\left(t^n_0,\dots,t^n_{N_n}\right)$, $t_{N_n} = T_n$. Without loss of generality (by replacing given partitions by finer ones and increasing the collection of networks accordingly) we may assume that there exists a sequence $(t_n)_{n\in\mathbb{N}_0}$, $t_{-1} := 0$, $t_{n-1} < t_n$, $n \in \mathbb{N}_0$, such that for all $n \in \mathbb{N}_0$, $\left(\gamma^n_0,\dots,\gamma^n_{N_n}\right)$ has partition $(t_0,\dots,t_{N_n})$. By increasing the amount of time steps $T_n \in A_\sigma$ we may further assume that $t_n = T_n$ and hence $N_n = n$. Given $n \in \mathbb{N}_0$ we define $\gamma_n := \gamma^n_n$.

We show inductively for all $n \in \mathbb{N}_0$ that $(\gamma_0, \dots, \gamma_n)$ is a jointed smooth solution to (3.21) in $[0, t_n]$ with endpoints P^1, P^2, P^3 and initial datum σ. The case $n = 0$ follows by construction. Suppose that $n \in \mathbb{N}_0$ is such that $(\gamma_0, \dots, \gamma_n)$ is a jointed smooth solution to (3.21) in $[0, t_n]$ with endpoints P^1, P^2, P^3 and initial datum σ. As $\left(\gamma_0^{n+1}, \dots, \gamma_{n+1}^{n+1}\right)$ is a jointed smooth solution to (3.21) in $[0, t_{n+1}]$ with endpoints P^1, P^2, P^3 and initial datum σ, we have $\gamma_{n+1} = \gamma_{n+1}^{n+1} \in C^\infty\left([t_n, t_{n+1}] \times [0,1]; (\mathbb{R}^2)^3\right)$ and $t \mapsto \gamma_{n+1}(t + t_n)$ is a strong solution to (3.21) in $[0, t_{n+1} - t_n]$ with endpoints P^1, P^2, P^3 and initial datum $\gamma_n^{n+1}(t_n)$. As both $\left(\gamma_0^{n+1}, \dots, \gamma_n^{n+1}\right)$ and $(\gamma_0, \dots, \gamma_n)$ are jointed strong solutions to (3.21) in $[0, t_n]$ with endpoints P^1, P^2, P^3 and initial datum σ, Proposition 5.3 yields that there exist C^1–diffeomorphisms $\xi^i : [0,1] \to [0,1]$, $\xi^i \in W_p^{4-4/p}((0,1); \mathbb{R})$, $i \in \{1,2,3\}$, such that $\gamma_n^{n+1,i}(t_n) = \gamma_n^i(t_n) \circ \xi^i$. Hence $t \mapsto \gamma_{n+1}(t + t_n)$ is a strong solution to (3.21) in $[0, t_{n+1} - t_n]$ with endpoints P^1, P^2, P^3 and initial datum $\gamma_n(t_n)$. This completes the induction.

To prove maximality of $(\gamma_n)_{n \in \mathbb{N}_0}$ we assume that T_{max} is finite and that $(\eta_0, \dots, \eta_N)$, $N \in \mathbb{N}_0$, is a jointed smooth solution to (3.21) in $[0, T_{max}]$ with endpoints P^1, P^2, P^3, initial datum σ and partition $(\tau_0, \dots, \tau_N)$. By Proposition 5.6 there exists $\eta \in C^\infty\left([\varepsilon, T_{max}] \times [0,1]; (\mathbb{R}^2)^3\right)$, $\varepsilon := \frac{\tau_0}{2}$, such that for all $t \in [\varepsilon, T_{max}]$, $x \in [0,1]$, $\left|\eta_x^i(t,x)\right| = \ell^i(t) := L\left(\eta^i(t)\right)$ and such that η is a smooth solution to (3.21) with endpoints P^1, P^2, P^3. Lemma 3.45 yields that

$$\boldsymbol{c} := \inf_{t \in [\varepsilon, T_{max}], i \in \{1,2,3\}} \ell^i(t) > 0$$

and that η satisfies the uniform non-degeneracy condition in $[\varepsilon, T_{max}]$. Furthermore, Proposition B.31 implies $\eta \in W_p^{1,4}\left((\varepsilon, T_{max}) \times (0,1); (\mathbb{R}^2)^3\right)$. In particular, η is a strong solution to (3.21) in $[\varepsilon, T_{max}]$ with initial datum $\eta_0(\varepsilon)$ and Corollary B.38 yields for every $t \in [\varepsilon, T_{max}]$,

$$\|\eta(t)\|_{W_p^{4-4/p}((0,1);(\mathbb{R}^2)^3)} \le C \, \|\eta\|_{W_p^{1,4}((\varepsilon,T_{max}) \times (0,1);(\mathbb{R}^2)^3)} \le \boldsymbol{C} < \infty \, .$$

Given $t \in [\varepsilon, T_{max}]$, $i \in \{1,2,3\}$ and $y \in \{0,1\}$ we further observe

$$\eta_{xx}^i(t,y) = \left(\ell^i(t)\right)^2 \eta_{ss}^i(t,y) = \left(\ell^i(t)\right)^2 \kappa^i(t,y) = 0$$

as $\eta^i(t)$ is parametrised with constant speed equal to $\ell^i(t)$ and as η is a smooth solution to (3.21). As for any $t \in [\varepsilon, T_{max}]$, $\eta(t)$ further satisfies the non-degeneracy condition in $y = 0$, we conclude that $\eta(t)$ is an analytically p–admissible initial network to (4.3) for every $t \in [\varepsilon, T_{max}]$. Theorem 4.30 yields that there exists a time $T = T(\boldsymbol{c}, \boldsymbol{C})$ such that for all $t \in [\varepsilon, T_{max}]$ there exists a strong solution to (4.3) in $[0, T]$ with initial datum $\eta(t)$. Let $\tau := T_{max} - T/2$ and $\tilde{\varrho}$ be a strong solution to (4.3) in $[0, T]$ with initial datum $\eta(\tau)$. Theorem 4.37 yields for all $\delta \in (0, T)$ that $\tilde{\varrho} \in C^\infty\left([\delta, T] \times [0,1]; (\mathbb{R}^2)^3\right)$. As both $\tilde{\varrho}$ and $t \mapsto \eta(t + \tau)$ are strong solutions to (3.21) with endpoints P^1, P^2, P^3 and initial datum $\eta(\tau)$ in $[0, T_{max} - \tau]$, Theorem 4.62 yields in particular that there exist diffeomorphisms $\zeta^i : [0,1] \to [0,1]$, $\zeta^i \in W_p^{4-4/p}((0,1); \mathbb{R})$, $i \in \{1,2,3\}$, such that $\tilde{\varrho}^i(T_{max} - \tau) = \eta^i(T_{max}) \circ \zeta^i$.
With $\varrho(t) := \tilde{\varrho}(t - \tau)$ for all $t \in [\tau, T + \tau]$ we thus find that $(\eta_0)_{|[0,\varepsilon]}, \eta_{|[\varepsilon, T_{max}]}, \varrho_{|[T_{max}, T_{max} + T/2]}$ form a jointed smooth solution to (3.21) with endpoints P^1, P^2, P^3 and initial datum σ which yields $T_{max} + T/2 \in A_\sigma$, a contradiction to $T_{max} = \sup A_\sigma$.

This allows us to conclude that $(\gamma_n)_{n \in \mathbb{N}_0}$ is a maximal solution to (3.21) in the maximal time interval $[0, T_{max})$ with endpoints P^1, P^2, P^3 and initial datum σ.

The uniqueness result follows from Proposition 5.3. □

Remark 5.10. Using contraction estimates and the theory [136] in parabolic Hölder spaces $C^{\frac{k+\alpha}{4}, k+\alpha}([0,T] \times [0,1])$, $T > 0$, of arbitrary order $k \in \mathbb{N}$, $k \ge 4$, one shows that if γ, η

are strong solutions to (3.20) or (3.21), respectively, in $(0,T)$ with $\gamma^i(0) = \eta^i(0) \circ \zeta_0^i$, $i \in \{1,2,3\}$, with C^1–diffeomorphisms $\zeta_0^i : [0,1] \to [0,1]$, $\zeta^i \in W_p^{4-4/p}((0,1);\mathbb{R})$, such that $\gamma, \eta \in C^\infty\left([\varepsilon,T]\times[0,1];(\mathbb{R}^2)^3\right)$ for all $\varepsilon \in (0,T)$, then the one-parameter family of diffeomorphisms $\zeta \in W_p^{1,4}\left((0,T)\times(0,1);\mathbb{R}^3\right)$ constructed in Theorem 4.62 satisfies $\zeta \in C^\infty\left([\varepsilon,T]\times[0,1];\mathbb{R}^3\right)$ for all $\varepsilon \in (0,T)$. We omit the details here as this property is not relevant for our further considerations.

5.2 Long time behaviour of the elastic flow of Triods

This section is devoted to prove the long time existence result in the case of Triods. The proof relies on a contradiction argument which essentially says that as long as the lengths of the curves are uniformly bounded from below and as long as at least one angle at the triple junction is uniformly bounded away from zero, π and 2π, the flow can be extended. Indeed, under these assumptions, the a priori estimate shown in Section 3.5 applies. Restricting to positive parameters μ^i, the gradient flow structure further yields an upper bound on the length which ultimately results in a uniform in time bound on the solution in the space $W_2^4\left((0,1);(\mathbb{R}^2)^3\right)$. This space embeds into the initial data space $X_0 = W_p^{4-4/p}\left((0,1);(\mathbb{R}^2)^3\right)$ for values of $p \in (5,10]$ as shown in Lemma 5.11. As the time interval of existence depends on the initial value only via its norm in X_0 and the lengths of its curves, the energy estimates allow us to overcome the maximal time of existence by restarting the flow shortly before.

Lemma 5.11. *Given $p \in (5,10]$ there holds the continuous embedding*

$$W_2^4((0,1)) \hookrightarrow W_p^{4-4/p}((0,1)).$$

Proof. Let $p \in (5,10]$ be given. Then [140, Theorem 4.6.1.c)] yields the following continuous embedding of Besov spaces:

$$B_{22}^4((0,1)) \hookrightarrow B_{pp}^{4-4/p}((0,1)).$$

As $4 - 4/p$ is not integer-valued for all $p \in (5,10]$, Theorem B.19 yields that the Besov space $B_{pp}^{4-4/p}((0,1))$ coincides with the Sobolev-Slobodeckij space $W_p^{4-4/p}((0,1))$ with equivalent norms. Moreover, [140, Theorem 4.6.1b)] yields the identity $B_{22}^4((0,1)) = H_2^4((0,1))$ with equivalent norms where $H_2^4((0,1))$ denotes the respective Bessel potential space. Due to [140, Definition 4.2.1, Remark 4.2.2/3] the space $H_2^4((0,1))$ coincides with the Sobolev space $W_2^4((0,1))$ with equivalent norms. □

The following definition introduces some notation appearing in the long time existence result.

Definition 5.12. Let $p \in (5,\infty)$ and σ be a geometrically p–admissible initial datum to (3.21) with endpoints P^1, P^2, P^3. Suppose that $(\gamma_n)_{n\in\mathbb{N}_0}$ is a maximal solution to (3.21) with endpoints P^1, P^2, P^3, initial datum σ and partition $(t_n)_{n\in\mathbb{N}_0}$ in the maximal time interval $[0,T_{max})$ with $T_{max} \in (0,\infty]$. Given $n \in \mathbb{N}_0$, $t \in [t_{n-1},t_n]$ where $t_{-1} := 0$, $i \in \{1,2,3\}$, we let $\ell^i(t) := L\left(\gamma_n^i(t)\right)$ be the length of the i-th curve of the evolving Triod at time t, and $\alpha^1(t,0)$, $\alpha^2(t,0)$, $\alpha^3(t,0)$ be the angles between the tangent vectors $\tau_n^2(t,0)$ and $\tau_n^3(t,0)$, $\tau_n^3(t,0)$ and $\tau_n^1(t,0)$, and $\tau_n^1(t,0)$ and $\tau_n^2(t,0)$, respectively, to the curves $\gamma_n^i(t)$, $i \in \{1,2,3\}$, at the triple junction $\gamma_n^1(t,0)$.

Theorem 5.13 (Long time behaviour of the elastic flow of Triods)**.** *Let $p \in (5,10]$ and σ be a geometrically p–admissible initial datum to* (3.21) *with given endpoints P^1, P^2, P^3 and parameters $\mu^i > 0$ for all $i \in \{1,2,3\}$. Suppose that $(\gamma_n)_{n\in\mathbb{N}_0}$ is a maximal solution to* (3.21) *with endpoints*

P^1, P^2, P^3, parameters $\mu^i > 0$, $i \in \{1,2,3\}$, and initial datum σ in the maximal time interval $[0, T_{max})$ with $T_{max} \in (0, \infty]$. Then

$$T_{max} = \infty$$

or at least one of the following happens:

(i) the inferior limit of the length of at least one curve is zero as the time approaches T_{max}, that is, there exists $i \in \{1,2,3\}$ such that

$$\liminf_{t \nearrow T_{max}} \ell^i(t) = 0\,.$$

(ii) the angles $\alpha^i(t,0)$, $i \in \{1,2,3\}$, $t \in [0, T_{max})$, between the tangent vectors at the triple junction satisfy

$$\liminf_{t \nearrow T_{max}} \max\left\{\left|\sin\alpha^1(t,0)\right|, \left|\sin\alpha^2(t,0)\right|, \left|\sin\alpha^3(t,0)\right|\right\} = 0\,.$$

Proof. Let $(\gamma_n)_{n\in\mathbb{N}_0}$ be a maximal solution to (3.21) with endpoints P^1, P^2, P^3 and initial datum σ in the maximal time interval $[0, T_{max})$. Suppose that $T_{max} < \infty$ and that none of the two scenarios happens, that is, there exists $\delta \in (0, T_{max})$ such that

$$\ell_\delta := \inf_{t\in(T_{max}-\delta, T_{max})} \min_{i\in\{1,2,3\}} \ell^i(t) > 0\,,$$

and

$$\rho_\delta := \inf_{t\in(T_{max}-\delta, T_{max})} \max\left\{\left|\sin\alpha^1(t,0)\right|, \left|\sin\alpha^2(t,0)\right|, \left|\sin\alpha^3(t,0)\right|\right\} > 0\,.$$

Let $N \in \mathbb{N}_0$ be such that $t_N \geq T_{max} - \delta$. As for all $j \in \{0, \ldots, N\}$, $t \mapsto \gamma_j(t + t_{j-1})$ is a strong solution to (3.21) with endpoints P^1, P^2, P^3 in $[0, t_j - t_{j-1}]$, Lemma 3.45 yields in particular for all $j \in \{0, \ldots, N\}$,

$$\ell_j := \inf_{t\in[t_{j-1}, t_j]} \min_{i\in\{1,2,3\}} \ell^i(t) \geq \boldsymbol{c}(\gamma_j) > 0\,,$$

and

$$\rho_j := \inf_{t\in[t_{j-1}, t_j]} \max\left\{\left|\sin\alpha^1(t,0)\right|, \left|\sin\alpha^2(t,0)\right|, \left|\sin\alpha^3(t,0)\right|\right\} > 0\,.$$

This allows us to conclude that

$$\ell := \inf_{t\in[0, T_{max})} \min_{i\in\{1,2,3\}} \ell^i(t) = \min\{\ell_\delta, \ell_0, \ldots, \ell_N\} > 0\,, \tag{5.1}$$

and

$$\rho := \inf_{t\in[0, T_{max})} \max\left\{\left|\sin\alpha^1(t,0)\right|, \left|\sin\alpha^2(t,0)\right|, \left|\sin\alpha^3(t,0)\right|\right\} = \min\{\rho_\delta, \rho_0, \ldots, \rho_N\} > 0\,. \tag{5.2}$$

Let $\eta : [0, T_{max}) \times [0,1] \to (\mathbb{R}^2)^3$ be the constant speed parametrisation constructed in Corollary 5.8 and let $\varepsilon := \frac{t_0}{2}$. Then $\eta \in C^\infty\left([\varepsilon, T_{max}) \times [0,1]; (\mathbb{R}^2)^3\right)$ is a smooth solution to (3.21) with endpoints P^1, P^2, P^3 in $[\varepsilon, T_{max})$. As η and $(\gamma_n)_{n\in\mathbb{N}}$ describe the same evolution of Triods, we conclude that η satisfies the uniform non-degeneracy condition in $[\varepsilon, T_{max})$ and that the lengths of the curves are uniformly bounded from below on $[\varepsilon, T_{max})$. Thus the a priori estimate shown in Theorem 3.55 yields for all $t \in [\varepsilon, T_{max})$, $i \in \{1,2,3\}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_s^2\kappa^i(t)\right\|^2_{L_2(\eta^i(t))} \leq C\left(E_\mu\left(\eta(\varepsilon)\right)\right)$$

with $\kappa^i(t)$ denoting the curvature of $\eta^i(t)$. Hence we obtain for all $\tau \in (\varepsilon, T_{max})$, $i \in \{1,2,3\}$,

$$\begin{aligned}\left\|\partial_s^2\kappa^i(\tau)\right\|^2_{L_2(\eta^i(\tau))} &= \int_\varepsilon^\tau \frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_s^2\kappa^i(t)\right\|^2_{L_2(\eta^i(t))}\,\mathrm{d}t + \left\|\partial_s^2\kappa^i(\varepsilon)\right\|^2_{L_2(\eta^i(\varepsilon))}\\ &\le (\tau-\varepsilon)\,C\left(E_\mu\left(\eta(\varepsilon)\right)\right) + C\left(\eta(\varepsilon)\right) \le (T_{max}-\varepsilon)\,C\left(E_\mu\left(\eta(\varepsilon)\right)\right) + C\left(\eta(\varepsilon)\right) < \infty \end{aligned} \tag{5.3}$$

as T_{max} is assumed to be finite. Furthermore, the decrease of the elastic energy due to the gradient flow structure (see Corollary 3.28) yields

$$\sup_{t\in[\varepsilon,T_{max})}\max_{i\in\{1,2,3\}}\left\|\kappa^i\left(t\right)\right\|^2_{L_2(\eta^i(t))} \le E_\mu\left(\eta(\varepsilon)\right) \tag{5.4}$$

and

$$\sup_{t\in[\varepsilon,T_{max})}\max_{i\in\{1,2,3\}}\ell^i\left(t\right) = \sup_{t\in[\varepsilon,T_{max})}\max_{i\in\{1,2,3\}}\left|\eta^i(t,x)\right| \le \frac{1}{\mu^i}E_\mu\left(\eta\left(\varepsilon\right)\right). \tag{5.5}$$

We let $\tau \in (\varepsilon, T_{max})$ and $i \in \{1,2,3\}$ be fixed and denote by s^i_τ and x^i_τ the arclength parameter of $\eta^i(\tau)$ and its inverse. As $\eta^i(\tau)$ is parametrised with constant speed equal to its length $\ell^i\left(\tau\right)$, we obtain the following differentiation rule for any $k \in \mathbb{N}_0$ and $x \in [0,1]$:

$$\partial_s^k\left(\eta^i(\tau)\circ x^i_\tau\right)\left(s^i_\tau(x)\right) = \ell^i(\tau)^{-k}\partial_x^k\eta^i(\tau,x)\,. \tag{5.6}$$

In particular, we have for any $x \in [0,1]$,

$$\begin{aligned}\kappa^i\left(\tau,x\right) &= \partial_s^2\left(\eta^i\left(\tau\right)\circ x^i_\tau\right)\left(s^i_\tau(x)\right) = \ell^i(\tau)^{-2}\partial_x^2\eta^i\left(\tau,x\right),\\ \partial_s^2\kappa^i(\tau,x) &= \partial_s^4\left(\eta^i\left(\tau\right)\circ x^i_\tau\right)\left(s^i_\tau(x)\right) = \ell^i(\tau)^{-4}\partial_x^4\eta^i\left(\tau,x\right),\end{aligned}$$

which yields together with (5.3), (5.4), and (5.5)

$$\begin{aligned}\left\|\partial_x^2\eta^i\left(\tau\right)\right\|^2_{L_2((0,1);\mathbb{R}^2)} &= \ell^i(\tau)^4\int_0^1\left|\kappa^i(\tau,x)\right|^2\,\mathrm{d}x = \ell^i(\tau)^3\left\|\kappa^i(\tau)\right\|^2_{L_2(\eta^i(\tau))} \le C\left(E_\mu\left(\eta\left(\varepsilon\right)\right)\right),\\ \left\|\partial_x^4\eta^i(\tau)\right\|^2_{L_2((0,1);\mathbb{R}^2)} &= \ell^i(\tau)^8\int_0^1\left|\partial_s^2\kappa^i(\tau,x)\right|^2\,\mathrm{d}x = \ell^i(\tau)^7\left\|\partial_s^2\kappa^i(\tau)\right\|^2_{L_2(\eta^i(\tau))} \le C\left(\eta(\varepsilon)\right).\end{aligned}$$

The classical Gagliardo-Nirenberg inequality stated in Theorem 3.46 yields with constants C_1, C_2 independent of τ,

$$\begin{aligned}\left\|\partial_x^3\eta^i(\tau)\right\|_{L_2((0,1);\mathbb{R}^2)} &\le C_1\left\|\partial_x^4\eta^i(\tau)\right\|^\sigma_{L_2((0,1);\mathbb{R}^2)}\left\|\partial_x^2\eta^i(\tau)\right\|^{1-\sigma}_{L_2((0,1);\mathbb{R}^2)} + C_2\left\|\partial_x^2\eta^i(\tau)\right\|_{L_2((0,1);\mathbb{R}^2)}\\ &\le C\left(\eta(\varepsilon)\right).\end{aligned}$$

As the points P^1, P^2, P^3 are fixed in the plane and as the single lengths $\ell^i(\tau)$, $i \in \{1,2,3\}$, are bounded from above uniformly in $\tau \in [\varepsilon, T_{max})$ due to (5.5), there exists a constant $R > 0$ independent of $\tau \in [\varepsilon, T_{max})$, $x \in [0,1]$, such that

$$\sup_{\tau\in[\varepsilon,T_{max}),x\in[0,1]}\left|\eta^i\left(\tau,x\right)\right| \le R\,.$$

Hence we obtain $\eta(\tau) \in W_2^4\left((0,1);(\mathbb{R}^2)^3\right)$ for all $\tau \in [\varepsilon, T_{max})$ with

$$\sup_{\tau\in[\varepsilon,T_{max})}\left\|\eta(\tau)\right\|_{W_2^4((0,1);(\mathbb{R}^2)^3)} \le C\left(\eta(\varepsilon)\right).$$

Lemma 5.11 yields for all $p \in (5,10]$,

$$W_2^4\left((0,1);(\mathbb{R}^2)^3\right) \hookrightarrow W_p^{4-4/p}\left((0,1);(\mathbb{R}^2)^3\right)$$

with

$$\sup_{\tau\in[\varepsilon,T_{max})} \|\eta(\tau)\|_{W_p^{4-4/p}((0,1);(\mathbb{R}^2)^3)} \leq C(\eta(\varepsilon)) \leq \boldsymbol{C} \tag{5.7}$$

for a constant $\boldsymbol{C} > 0$. As η is a solution to (3.21) in $[\varepsilon, T_{max})$, the Triod $\eta(\tau)$ is a geometrically p–admissible initial network to (3.21) with endpoints P^1, P^2, P^3 for all $\tau \in [\varepsilon, T_{max})$. In fact, we further have for all $y \in \{0,1\}$, $\tau \in [\varepsilon, T_{max})$,

$$\partial_x^2 \eta^i(\tau, y) = \ell^i(\tau)^2 \partial_s^2 \eta^i(\tau, y) = \ell^i(\tau)^2 \kappa^i(\tau, y) = 0$$

which yields that for all $\tau \in [\varepsilon, T_{max})$, $\eta(\tau)$ is an analytically p–admissible initial value to (4.3) with endpoints P^1, P^2, P^3. Due to (5.1) and (5.7) Theorem 4.30 yields that there exists one uniform existence time $T = T(\ell, \boldsymbol{C})$ such that for all $\tau \in [\varepsilon, T_{max})$ there exists a solution $\tilde{\varrho}_\tau$ to (4.3) in $[0,T]$ with endpoints P^1, P^2, P^3 and initial datum $\eta(\tau)$, which satisfies for all $\delta \in (0,T)$, $\tilde{\varrho}_\tau \in C^\infty\left([\delta,T]\times[0,1];(\mathbb{R}^2)^3\right)$ due to Theorem 4.37. Let $\tau := T_{max} - T/2$ and $\tilde{\varrho}$ be the solution to (4.3) in $[0,T]$ with endpoints P^1, P^2, P^3 and initial datum $\eta(\tau)$. Given $t \in [\tau, T_{max} + T/2]$ we let $\varrho(t) := \tilde{\varrho}(t-\tau)$ and further $\tilde{\tau} := T_{max} - T/4$. As both $\tilde{\varrho}$ and $t \mapsto \eta(t+\tau)$ are strong solutions to (3.21) in $[0, T/4]$ with endpoints P^1, P^2, P^3 and initial datum $\eta(\tau)$, Theorem 4.62 yields that there exist diffeomorphisms $\zeta^i : [0,1] \to [0,1]$, $\zeta^i \in W_p^{4-4/p}((0,1);\mathbb{R})$, $i \in \{1,2,3\}$, such that $\tilde{\varrho}^i(T/4) = \varrho^i(\tilde{\tau}) = \eta^i(\tilde{\tau}) \circ \zeta^i$. We thus conclude that

$$\left((\gamma_0)_{|[0,\varepsilon]}, \eta_{|[\varepsilon,\tilde{\tau}]}, \varrho_{[\tilde{\tau}, T_{max}+T/2]}\right)$$

is a jointed smooth solution to (3.21) in the time interval $[0, T_{max} + T/2]$ with endpoints P^1, P^2, P^3 and initial datum σ. This contradicts the maximality of T_{max}. □

5.3 Long time behaviour of the elastic flow of Theta networks

In this section we prove the analogon to Theorem 5.13 for the elastic flow of Theta networks given by system (3.20). While the strategy of the proof remains the same, there is one remarkable difficulty compared to the elastic flow of Triods (3.21). Indeed, the presence of fixed endpoints in the flow (3.21) serves as a key ingredient to find a uniform in time and space L_∞-bound on the parametrisations which is needed to guarantee that the flow can be restarted closed to T_{max} with a uniform time of existence. These arguments are no longer possible in the case of Theta networks. However, we can make use of the translational invariance of (3.20) which implicates that the existence time of the analytic problem (4.2) is independent of the position of the Theta network. This is shown in Theorem 4.34.

Definition 5.14. Let $p \in (5,\infty)$ and σ be a geometrically p–admissible initial datum to (3.20). Suppose that $(\gamma_n)_{n\in\mathbb{N}_0}$ is a maximal solution to (3.20) with initial datum σ and partition $(t_n)_{n\in\mathbb{N}_0}$ in the maximal time interval $[0, T_{max})$ with $T_{max} \in (0,\infty]$. Given $n \in \mathbb{N}_0$, $t \in [t_{n-1}, t_n]$ where $t_{-1} := 0$, $i \in \{1,2,3\}$, we let $\ell^i(t) := L\left(\gamma_n^i(t)\right)$ be the length of the i-th curve of the evolving Theta network at time t, and for $y \in \{0,1\}$, we let $\alpha^1(t,y)$, $\alpha^2(t,y)$, $\alpha^3(t,y)$ denote the angles between the tangent vectors $\tau_n^2(t,y)$ and $\tau_n^3(t,y)$, $\tau_n^3(t,y)$ and $\tau_n^1(t,y)$, and $\tau_n^1(t,y)$ and $\tau_n^2(t,y)$, respectively, to the curves $\gamma_n^i(t)$, $i \in \{1,2,3\}$, at the triple junction $\gamma_n^1(t,y)$.

Theorem 5.15 (Long time behaviour of the elastic flow of Theta networks)**.** *Let $p \in (5,10]$ and σ be a geometrically p–admissible initial datum to* (3.20) *with parameters $\mu^i > 0$ for all $i \in \{1,2,3\}$.*

Suppose that $(\gamma_n)_{n\in\mathbb{N}_0}$ *is a maximal solution to* (3.20) *with parameters* $\mu^i > 0$, $i \in \{1,2,3\}$, *and initial datum* σ *in the maximal time interval* $[0, T_{max})$ *with* $T_{max} \in (0,\infty]$. *Then*

$$T_{max} = \infty$$

or at least one of the following happens:

(i) the inferior limit of the length of at least one curve is zero as the time approaches T_{max}, *that is, there exists* $i \in \{1,2,3\}$ *such that*

$$\liminf_{t\nearrow T_{max}} \ell^i(t) = 0\,.$$

(ii) the angles $\alpha^i(t,y)$, $i \in \{1,2,3\}$, $t \in [0,T_{max})$, *between the tangent vectors at one of the triple junctions* $\gamma^i(t,y)$, $y \in \{0,1\}$, *satisfy*

$$\liminf_{t\nearrow T_{max}} \max\left\{\left|\sin\alpha^1(t,y)\right|, \left|\sin\alpha^2(t,y)\right|, \left|\sin\alpha^3(t,y)\right|\right\} = 0\,.$$

Proof. We sketch how to adapt the proof of Theorem 5.13. Given a maximal solution to (3.20) with initial datum σ in the maximal time interval $[0,T_{max})$ we suppose that none of the scenarios happen. Then the constant speed parametrisation $\eta : [0,T_{max}) \times [0,1] \to (\mathbb{R}^2)^3$ as constructed in Corollary 5.8 is a smooth solution to (3.20) in $[\varepsilon, T_{max})$ for $\varepsilon := \frac{t_0}{2}$ satisfying the uniform non-degeneracy condition on $[\varepsilon, T_{max})$ in both triple junctions and further

$$\ell := \inf_{t\in[0,T_{max})} \min_{i\in\{1,2,3\}} \ell^i(t) > 0\,. \tag{5.8}$$

Using the a priori estimate in Theorem 3.55 and the bounds due to the gradient flow structure shown in Corollary 3.28 we obtain $\partial_x\eta(\tau) \in W_2^3\left((0,1);(\mathbb{R}^2)^3\right)$ for all $\tau \in [\varepsilon, T_{max})$ with

$$\sup_{\tau\in[\varepsilon,T_{max})} \|\partial_x\eta(\tau)\|_{W_2^3((0,1);(\mathbb{R}^2)^3)} \le C\left(\eta(\varepsilon)\right).$$

On the lines of Lemma 5.11 one has for all $p \in (5,10]$ the continuous embedding

$$W_2^3\left((0,1);(\mathbb{R}^2)^3\right) \hookrightarrow W_p^{3-4/p}\left((0,1);(\mathbb{R}^2)^3\right)$$

which yields for a constant $\boldsymbol{C} > 0$,

$$\sup_{\tau\in[\varepsilon,T_{max})} |\eta(\tau)|_{W_p^{4-4/p}((0,1);(\mathbb{R}^2)^3)} = \sup_{\tau\in[\varepsilon,T_{max})} \|\partial_x\eta(\tau)\|_{W_p^{3-4/p}((0,1);(\mathbb{R}^2)^3)} \le \boldsymbol{C}\,. \tag{5.9}$$

Due to (5.8) and (5.9) the refined short time existence result for the analytic problem for Theta networks given in Theorem 4.34 yields that there exists a uniform existence time $T = T(\ell, \boldsymbol{C}) > 0$ such that for all $\tau \in [\varepsilon, T_{max})$ there exists a solution to the analytic problem for Theta networks (4.2) in $[0,T]$ with initial datum $\eta(\tau)$ which is smooth away from zero. As in the proof ot Theorem 5.13 one shows that this allows us to extend the given maximal solution beyond T_{max} contradicting the maximality of T_{max}. □

5.4 A remark on the definition of maximal solutions

Let us give some remarks on the definition of maximal solutions to the elastic flow of Triods and Theta networks represented by the systems (3.21) and (3.20), respectively. Our notion of maximal solution requires that away from the initial value the evolution is piecewise parametrised by *smooth* parametrisations. This might seem to be a restriction on the evolutions one is considering. Indeed, in analogy to Definition 5.7 of maximal (smooth) solutions one may introduce the notion of a *maximal strong solution* without requiring the piecewise parametrisations to be smooth. The existence of maximal strong solutions given an admissible initial datum can be shown analogously to Proposition 5.9. A priori, it is possible that given a p–admissible initial network to (3.20) or to (3.21), respectively, the maximal strong solution exists longer than the smooth one. As these two maximal solutions are equal up to reparametrisation on finite time intervals due to Proposition 5.3, this would mean that up to some certain point in time, the evolution can be described by a (jointed) *smooth* solution which then suddenly loses its regularity ending up in being merely a (jointed) *strong* solution. Using again the a priori estimates for smooth solutions the following result shows that this can *not* be the case. As a result we conclude that the long time existence results stated in Theorems 5.13 and 5.15 describing the possibilities of geometric singularities of a given evolution are independent of the notion of maximal solution.

Proposition 5.16. *Let $p \in (5,\infty)$, σ be a geometrically p–admissible initial network to* (3.20) *or* (3.21)*, respectively. Let T be positive and $(\gamma_0,\dots,\gamma_N)$, $N \in \mathbb{N}_0$, be a jointed strong solution to the considered system in $[0,T]$ with parameters $\mu^i > 0$, $i \in \{1,2,3\}$, and initial datum σ. Then there exists a jointed smooth solution $(\eta_0,\dots,\eta_M)$, $M \in \mathbb{N}_0$, to the considered system in $[0,T]$ with initial datum σ, such that $(\gamma_0,\dots,\gamma_N) \sim (\eta_0,\dots,\eta_M)$.*

Proof. We give a detailed proof for system (3.21) which works analogously for (3.20). Given $p \in (5,\infty)$ and a geometrically p–admissible initial network σ to (3.21) with endpoints P^1, P^2, P^3 and parameters $\mu^i > 0$, $i \in \{1,2,3\}$, we let

$$A_\sigma := \{\tau > 0 : \text{ there exists a jointed smooth solution to (3.21) in } [0,\tau] \text{ with endpoints } P^1, P^2, P^3 \text{ and initial datum } \sigma\} .$$

By Proposition 5.5 there holds $T_{max} = \sup A_\sigma \in (0,\infty]$. We aim to show that $T_{max} \geq T$. Assume by contradiction that $T_{max} < T$. Proposition 5.9 yields the existence of a maximal (smooth) solution $(\eta_n)_{n\in\mathbb{N}_0}$ to (3.21) in the maximal time interval $[0,T_{max})$ with endpoints P^1, P^2, P^3, initial datum σ and partition $(\tau_n)_{n\in\mathbb{N}_0}$. Without loss of generality we may assume that for all $j \in \{0,\dots,N\}$, there exists $k(j) \in \mathbb{N}_0$ such that $\tau_{k(j)} = t_j$. Given $n \in \mathbb{N}_0$, $t \in [\tau_{n-1},\tau_n]$, there exists $j \in \{0,\dots,N\}$ such that $t_{j-1} \leq \tau_{n-1} \leq \tau_n \leq t_j$, which allows us to define $\gamma_n(t) := \gamma_j(t)$ on the interval $[\tau_{n-1},\tau_n]$. In particular, for all $n \in \mathbb{N}_0$ we have $\tau_n < T_{max} < T$ and thus both $(\gamma_0,\dots,\gamma_n)$ and $(\eta_0,\dots,\eta_n)$ are jointed strong solutions to (3.21) in $[0,\tau_n]$ with endpoints P^1, P^2, P^3 and initial datum σ. Proposition 5.3 yields for all $n \in \mathbb{N}_0$, $k \in \{0,\dots,n\}$, $t \in [\tau_{k-1},\tau_k]$ the existence of C^1–diffeomorphisms $\xi_k^i(t) : [0,1] \to [0,1]$, $i \in \{1,2,3\}$, $\xi_k^i(t) \in W_p^{4-4/p}((0,1))$, such that $\eta_k^i(t) = \gamma_k^i(t) \circ \xi_k^i(t)$. By Corollary 5.8 there exists $\eta : [0,T_{max}) \times [0,1] \to (\mathbb{R}^2)^3$ such that for all $n \in \mathbb{N}_0$, $\eta_{|[0,\tau_n]} \sim (\eta_0,\dots,\eta_n)$, and for all $t \in [\tau_{n-1},\tau_n]$, $i \in \{1,2,3\}$, $\eta^i(t)$ is parametrised with constant speed equal to its length

$$\ell^i(t) = L\left(\eta^i(t)\right) = L\left(\eta_n^i(t)\right) = L\left(\gamma_n^i(t)\right)$$

and such that for $\varepsilon := \frac{\tau_0}{2}$, $\eta_{|[\varepsilon,T_{max})}$ is a smooth solution to (3.21) with endpoints P^1, P^2, P^3 and initial datum σ. As $t \mapsto \gamma_j(t + t_{j-1})$, $t \in [0, t_j - t_{j-1}]$, is a strong solution to (3.21) with endpoints P^1, P^2, P^3 for all $j \in \{0,\dots,N\}$, Lemma 3.45 yields

$$\boldsymbol{c}(\gamma_j) := \min_{i\in\{1,2,3\}} \min_{t\in[t_{j-1},t_j],x\in[0,1]} \left|\left(\gamma_j^i\right)_x(t,x)\right| > 0$$

and

$$\rho(\gamma_j) := \min_{t\in[t_{j-1},t_j]} \max\left\{\left|\sin\alpha^1(t,0)\right|, \left|\sin\alpha^2(t,0)\right|, \sin\left|\alpha^3(t,0)\right|\right\} > 0$$

where $\alpha^1(t,0)$, $\alpha^2(t,0)$, $\alpha^3(t,0)$ denote the angles between the tangent vectors $\tau^2(t,0)$ and $\tau^3(t,0)$, $\tau^3(t,0)$ and $\tau^1(t,0)$, and $\tau^1(t,0)$ and $\tau^2(t,0)$, respectively, and in particular

$$\boldsymbol{c} := \min\left\{\boldsymbol{c}\left(\gamma_j\right) : j \in \{0,\dots,N\}\right\} > 0$$

and

$$\rho := \min\left\{\rho\left(\gamma_j\right) : j \in \{0,\dots,N\}\right\} > 0\,.$$

Given any $t \in [\varepsilon, T_{max})$ there exists $n \in \mathbb{N}_0$ and $j \in \{0,\dots,N\}$ such that $t \in [\tau_{n-1},\tau_n] \subset [t_{j-1},t_j]$. As

$$\left[\eta^i(t)\right] = \left[\eta^i_n(t)\right] = \left[\gamma^i_n(t)\right]$$

and as the length $\ell^i(t)$ and the angles $\alpha^i(t,0)$ are invariant with respect to reparametrisation, we obtain that

$$\inf_{t\in[\varepsilon,T_{max})}\min_{i\in\{1,2,3\}} L\left(\eta^i(t)\right) \geq \min_{j\in\{0,\dots,N\}} \boldsymbol{c}\left(\gamma_j\right) = \boldsymbol{c} > 0$$

and that $\eta_{|[\varepsilon,T_{max})}$ satisfies the uniform non-degeneracy condition

$$\inf_{t\in[\varepsilon,T_{max})} \max\left\{\left|\sin\alpha^1(t,0)\right|, \left|\sin\alpha^2(t,0)\right|, \left|\sin\alpha^3(t,0)\right|\right\} \geq \rho > 0\,.$$

The a priori estimate shown in Theorem 3.55 yields for the curvature $\kappa^i(t)$ of $\eta^i(t)$, $t \in [\varepsilon, T_{max})$, $i \in \{1,2,3\}$, the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_s^2\kappa^i(t)\right\|^2_{L_2(\eta^i(t))} \leq C\left(E_\mu\left(\eta(\varepsilon)\right)\right).$$

Using further the bound on the length and the curvature shown in Corollary 3.28, the same arguments as in the proof of Theorem 5.13 yield

$$\sup_{t\in[\varepsilon,T_{max})} \|\eta(t)\|_{W_p^{4-4/p}((0,1);(\mathbb{R}^2)^3)} \leq C\left(\eta(\varepsilon)\right) \leq \boldsymbol{C}$$

and thus a uniform existence time $\boldsymbol{T} = \boldsymbol{T}\left(\boldsymbol{c},\boldsymbol{C}\right)$ for (4.3) for all initial networks $\eta(t)$ with $t \in [\varepsilon, T_{max})$ which allows us to overcome the maximal existence time T_{max}. Indeed, as in the proof of Theorem 5.13 one shows that there exists a jointed smooth solution to (3.21) in $[0, T_{max} + \boldsymbol{T}/2]$ with endpoints P^1, P^2, P^3 and initial datum σ which yields $T_{max} + \boldsymbol{T}/2 \in A_\sigma$ and thus contradicts the definition of T_{max}.

We hence conclude that there exists a jointed smooth solution to (3.21) in $[0,T]$ with endpoints P^1, P^2, P^3 and initial datum σ. The uniqueness assertion follows from Proposition 5.3. □

5.5 Conjectures and simulations

We conclude this section with some unproven remarks and conjectures related to the long time behaviour of the elastic flow of networks with given parameters $\mu^1 = \mu^2 = \mu^3 > 0$. These presumptions are supported by simulations which have been kindly provided by Prof. Dr. Robert Nürnberg from Università di Trento based on the methods developed in [16]. The simulations shown in Figure 5.1, 5.2 and 5.3 are done with the parameters $\mu^i = 2$ for $i \in \{1,2,3\}$. The underlying parameters for Figure 5.4 are given by $\mu^i = 0.2$, $i \in \{1,2,3\}$.

First of all, we note that the scenarios described in items (*i*) and (*ii*) in Theorems 5.13 and 5.15 can be considered as *singularities* of the flow. Indeed, in both situations the network stops being a non-degenerate element of the considered class and thus no longer satisfies the requirements included in the notion of solution we consider in this thesis. However, it might be conceivable to continue the flow in a “weaker notion” to allow for degenerate networks or a change of topology from three to two curves. Furthermore, it is worth pointing out that none of the potential scenarios listed in Theorem 5.13 excludes the others. Indeed it is possible that both singular behaviours occur as the time approaches T_{max}, regardless of $T_{\max}$ being finite or infinite. This is suggested by the simulations in Figures 5.2 and 5.4.

Let us give some remarks about Theorem 5.13 on the elastic flow of Triods. In the case that the three endpoints P^1, P^2, P^3 are pairwise distinct, it is a priori evident that the length of *at most* one curve can go to zero. This corresponds to the situation that the junction collapses to one of the fixed endpoints in finite or infinite time. In this case we can not exclude that the limiting network is degenerate. However, there are scenarios of evolving Triods with fixed *distinct* endpoints which are conjectured to exist globally in time without developing a singularity. It is suspected that this occurs when the initial Triod is "close" to a *Steiner configuration*. Given pairwise distinct endpoints P^1, P^2, P^3, the Steiner configuration is the unique Triod with endpoints P^1, P^2, P^3 consisting of three straight lines that meet in a triple junction forming angles of 120 degrees. The Steiner configuration has minimal length among all Triods with endpoints P^1, P^2, P^3. As all its curves are straight, the Steiner configuration is in particular a minimiser of E_μ in the class of Triods with endpoints P^1, P^2, P^3. It further satisfies the boundary conditions in (3.21) as the curvature is zero and the third order condition reduces to Young's law. The conjectured stability of the Steiner configuration is supported by the simulation in Figure 5.1: starting with a perturbed Steiner configuration the evolving Triod exists globally in time and converges to the Steiner configuration.

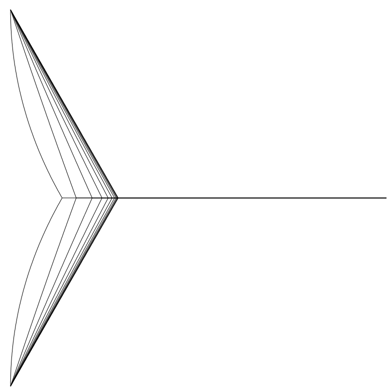

Figure 5.1: An evolving Triod existing globally and converging to the Steiner configuration.

We remark that our analysis allows for the situation that the three endpoints P^1, P^2, P^3 coincide. In this case the Triod actually has the shape of a Theta network. However, only one of the two triple junctions is allowed to move while the "artificial" junction $P^1 = P^2 = P^3$ stays fixed during the evolution. Figure 5.2 shows a simulation that starts with a symmetric double bubble. The left triple junction is allowed to move while the right one stays fixed. The simulation suggests that the double bubble shrinks to a point in *finite* time and thus provides an example where the two scenarios (i) and (ii) in Theorem 5.13 happen simultaneously.

In the case of Theta networks it is suspected that all evolutions develop a singularity. This conjecture is supported by recent results on the minimisation problem naturally associated to the flow. In [38, Chapter 3] it is shown that

$$\inf \{E_\mu(\gamma) : \gamma \text{ is a Theta network}\} = 0 \,,$$

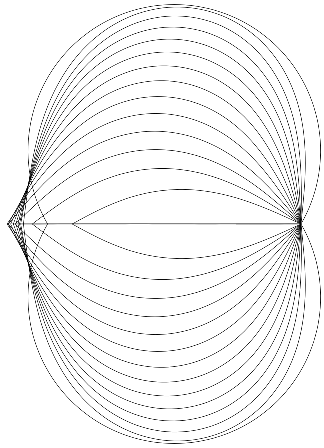

Figure 5.2: An evolving double bubble with one junction fixed converging to a point in finite time.

where the infimum is not attained. This result is illustrated in Figure 5.4 which is discussed below. However, if one enlarges the admissible class allowing for modified Theta networks consisting of *two curves* that meet in one quadruple junction, the "figure eight", the unique closed planar elastica with one self-intersection as shown in [87, Theorem 0.1 (a)], is a stationary point of the energy E_μ. Figure 5.3 shows a simulation of an evolving Theta network that converges to the figure eight. It is remarkable that according to the simulation the singularity occurs in finite time. The starting network is a drawn-out rotated double bubble with a relatively short straight curve in the middle. As the time approaches the maximal time of existence, the short curve in the middle eventually shrinks to a point which corresponds to the singularity described in case (i) of Theorem 5.15. We note that the angles at the junction remain bounded away from 0, π and 2π.

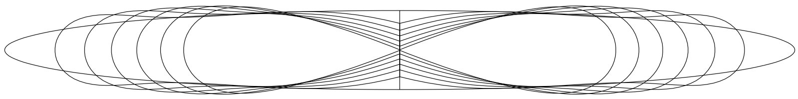

Figure 5.3: An evolving Theta network converging to the figure eight in finite time.

Figure 5.4 shows an example of an evolving Theta network where both singular behaviours occur simultaneously. According to the simulation we presume that the singularities develop in finite time. Starting with a symmetric double bubble the network first flattens and expands. At some point it stops expanding and starts shrinking faster and faster in a possibly self-similar way. The simulation suggests that the network eventually shrinks to a point with parallel tangent vectors.

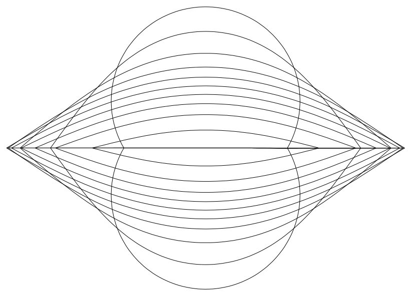

Figure 5.4: An evolving double bubble shrinking to a point in finite time.

Finally, we give some comments on the long time behaviour in the case $\mu^i \geq 0$, $i \in \{1,2,3\}$, which includes in particular the plain Willmore flow of networks corresponding to $\mu^i = 0$ for $i \in \{1,2,3\}$. While these cases are covered in the existence results in Chapter 4, our method of proof regarding the long time behaviour does not apply directly. Indeed, the upper bounds on the single lengths resulting from Corollary 3.28 in the case $\mu^i > 0$ for all $i \in \{1,2,3\}$ are an essential requirement in our proof. However, using that the lengths can grow at most linearly, the arguments can be adapted to cover also the case $\mu^i \geq 0$ for $i \in \{1,2,3\}$. We refer to [35, Remark 6.3] for more details on this aspect.
Based on the methods developed in [16], Prof. Dr. Robert Nürnberg has provided a simulation in the case $\mu^i = 0$, $i \in \{1,2,3\}$, which is shown in Figure 5.5. Starting with a symmetric double bubble one observes that the network flattens and expands. The simulation suggests that the network exists globally expanding further and further and becoming degenerate in infinite time.

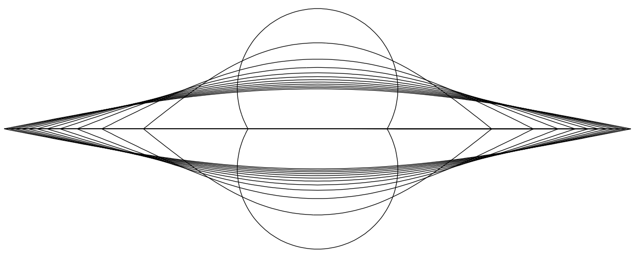

Figure 5.5: An evolving double bubble in the case $\mu^i = 0$, $i \in \{1,2,3\}$, existing globally and becoming degenerate.

Lastly, it is worth mentioning that Theorem 5.15 on the long time behaviour of the elastic flow of Theta networks does not answer the question whether the network "escapes to infinity" as the time of existence approaches T_{max}. However, we conjecture that such a scenario does not happen.

Appendices

Appendix A

Smooth manifolds

This chapter provides some background information on smooth manifolds. We begin by recalling the notions of topological manifolds with and without boundary, smooth structures and differentiability of functions between manifolds following [89, Chapter 1, 5].
Hereby, we use the following convention concerning the dimension. Whenever the considered manifold has a boundary, the dimension shall be a natural number. If we are concerned with a manifold without boundary, the dimension is allowed to be a natural number or zero.
Let us recall the definition of topological manifolds with and without boundary and the notion of *smooth structures* that allow to consider not merely continuous but also differentiable functions on manifolds.
A *topological n–manifold* is a second countable Hausdorff space M that is locally Euclidean of dimension n: for each point $p \in M$ there exists an open subset $U \subset M$ and a homeomorphism $\phi : U \to V$ onto an open subset V of $\mathbb{R}^n$. We call (U, ϕ) *chart* (around p), the components $\phi = (x^1, \dots, x^n)$ *local coordinates* on U and $\phi^{-1} : V \to U$ *local parametrisation* around p. A *compact n–manifold* is a topological n–manifold that is compact. Given two charts (U, ϕ), (W, ψ) with $U \cap W \neq \emptyset$, the homeomorphism $\psi \circ \phi^{-1} : \phi(U \cap W) \to \psi(U \cap W)$ is called the *transition map* from ϕ to ψ.
Two charts (U, ϕ) and (W, ψ) are called *smoothly compatible* if the transition map from ϕ to ψ is a smooth diffeomorphism. Hereby, a bijective map $F : U \to V$ between open subsets U and V of $\mathbb{R}^n$ is called *smooth diffeomorphism* if $F \in C^\infty(U; \mathbb{R}^n)$ and $F^{-1} \in C^\infty(V; \mathbb{R}^n)$. An *atlas* $\mathcal{A}$ for M is a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a *smooth atlas* if any two charts in $\mathcal{A}$ are smoothly compatible. A smooth atlas is *maximal* if it is not properly contained in any larger smooth atlas. A maximal smooth atlas is called *smooth structure* on M. A *smooth n–manifold* is a pair $(M, \mathcal{A})$ with M a topological n–manifold and $\mathcal{A}$ a smooth structure on M. Given a topological manifold M, it is shown in [89, Proposition 1.17] that any smooth atlas $\mathcal{A}$ is contained in a unique maximal smooth atlas, called the *smooth structure determined* by $\mathcal{A}$.
The closed n–dimensional upper half-space $\mathbb{H}^n \subset \mathbb{R}^n$ is defined as

$$\mathbb{H}^n := \left\{(x^1, \dots, x^n) \in \mathbb{R}^n \,:\, x^n \geq 0\right\} .$$

As a subset of $\mathbb{R}^n$ the topological interior and boundary are given by

$$\begin{aligned} \mathrm{int}\mathbb{H}^n &= \left\{(x^1, \dots, x^n) \in \mathbb{R}^n \,:\, x^n > 0\right\} , \\ \partial\mathbb{H}^n &= \left\{(x^1, \dots, x^n) \in \mathbb{R}^n \,:\, x^n = 0\right\} . \end{aligned}$$

Notice that $\mathbb{H}^n$ is a topological space endowed with the subspace topology inherited by $\mathbb{R}^n$. A set $U \subset \mathbb{H}^n$ is (relatively) open in $\mathbb{H}^n$ if there exists an open subset V of $\mathbb{R}^n$ such that $V \cap \mathbb{H}^n = U$. In particular every open subset of $\mathbb{H}^n$ is Lebesgue measurable. To define smooth manifolds with boundary one needs to specify differentiability of functions defined on open subsets of $\mathbb{H}^n$. A

function $f : U \to \mathbb{R}$ defined on an open subset U of $\mathbb{H}^n$ has *regularity* C^k on U, $k \in \mathbb{N}$, if for each point $x \in U$ there exists an open subset $\widetilde{U}$ of $\mathbb{R}^n$ containing x and a function $\widetilde{f} \in C^k(\widetilde{U};\mathbb{R})$ that agrees with f on $\widetilde{U} \cap \mathbb{H}^n \subset U$. In this case we write $f \in C^k(U;\mathbb{R})$. In case $f \in C^k(U;\mathbb{R})$ for all $k \in \mathbb{N}$ we say that f is *smooth* and write $f \in C^\infty(U;\mathbb{R})$. A function $f : U \to V$ between open subsets U and V of $\mathbb{H}^n$ is called *smooth diffeomorphism* if it is bijective and both f and f^{-1} are smooth.
A *topological n–manifold with boundary* is then a second countable Hausdorff space M in which every point has a neighbourhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to an open subset of $\mathbb{H}^n$. An *interior chart* (U, ϕ) for M is an open subset $U \subset M$ together with a homeomorphism $\phi : U \to \phi(U)$ onto an open subset of $\mathbb{R}^n$. We call (U, ϕ) *boundary chart* if $U \subset M$ is open and $\phi : U \to \phi(U)$ is a homeomorphism onto an open subset $\phi(U)$ of $\mathbb{H}^n$ such that $\phi(U) \cap \partial\mathbb{H}^n \neq \emptyset$. If (U, ϕ) is a chart around p we call $\phi = (x^1, \ldots, x^n)$ *local coordinates* on U and $\phi^{-1} : V \to U$ *local parametrisation* around p. A point $p \in M$ is called *interior point* of M if it is in the domain of some interior chart. It is a *boundary point* of M if it is in the domain of a boundary chart that maps p to $\partial\mathbb{H}^n$. The set of all boundary points of M is denoted by ∂M, the *boundary of M*. The *interior* int$M$ of M is the set of all interior points. Theorem [89, Theorem 1.37] shows that ∂M and int$M$ are disjoint sets whose union is M.
As in the case of manifolds without boundary we call two charts (U, ϕ) and (W, ψ) *smoothly compatible* if the transition map $\psi \circ \phi^{-1} : \phi(U \cap W) \to \psi(U \cap W)$ is a smooth diffeomorphism. An *atlas* $\mathcal{A}$ for M is a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a *smooth atlas* if any two charts in $\mathcal{A}$ are smoothly compatible. A smooth atlas is *maximal* if it is not properly contained in any larger smooth atlas. Such an atlas is called *smooth structure* on M. A *smooth n–manifold with boundary* is a pair $(M, \mathcal{A})$ with M a topological n–manifold with boundary and $\mathcal{A}$ a smooth structure on M. Every smooth atlas $\mathcal{A}$ on a topological n–manifold with boundary is contained in a unique maximal smooth atlas, called the *smooth structure determined* by $\mathcal{A}$.

Given two smooth manifolds $(M, \mathcal{A})$ and $(N, \mathcal{B})$ with or without boundary of dimension m and n, respectively, a function $f : M \to N$ is *of regularity* C^k, $k \in \mathbb{N}$, if for every $p \in M$ there exist charts $(U, \phi) \in \mathcal{A}$ around p and $(V, \psi) \in \mathcal{B}$ around $f(p)$ such that $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ has regularity C^k. In this case we write $f \in C^k(M;N)$ and it is readily checked that f is continuous and that $\psi \circ f \circ \phi^{-1}$ has regularity C^k on $\phi(U \cap f^{-1}(V))$ for all charts $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$. In the case that $f \in C^k(M;N)$ for all $k \in \mathbb{N}$ we say that f is *smooth* and write $f \in C^\infty(M;N)$. A *C^k–diffeomorphism from M to N* is a bijective function $f \in C^k(M;N)$ with $f^{-1} \in C^k(N;M)$. We say that M and N are *C^k–diffeomorphic* if there exists a C^k–diffeomorphism between them. A *local C^k–diffeomorphism* from M to N is a function $f \in C^k(M;N)$ such that every point $p \in M$ has a neighbourhood U with $f(U)$ open in N and such that $f_{|U} : U \to f(U)$ is a C^k–diffeomorphism.
To characterise functions on manifolds by their local behaviour on the manifold we need partitions of unity, see [89, page 43].

Definition A.1 (Partition of unity)**.** Let M be a topological space and let $\mathcal{U} := (U_\alpha)_{\alpha \in A}$ be an arbitrary open cover of M, indexed by a set A. A *partition of unity subordinate to $\mathcal{U}$* is a family $(\psi_\alpha)_{\alpha \in A}$ of continuous functions $\psi_\alpha : M \to \mathbb{R}$ such that

(i) $0 \leq \psi_\alpha(p) \leq 1$ for all $p \in M$ and all $\alpha \in A$,

(ii) $\operatorname{supp} \psi_\alpha := \overline{\{p \in M : \psi_\alpha(p) \neq 0\}} \subset U_\alpha$ for each $\alpha \in A$,

(iii) every point in M has a neighbourhood that intersects $\operatorname{supp} \psi_\alpha$ only for finitely many α,

(iv) $\sum_{\alpha \in A} \psi_\alpha(p) = 1$ for all $p \in M$.

If M is a smooth manifold with or without boundary, a *smooth partition of unity* is one which satisfies $\psi_\alpha \in C^\infty(M;\mathbb{R})$.

Theorem A.2 (Existence of partitions of unity)**.** *Suppose M is a smooth manifold with or without boundary and $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ is an open cover indexed by a set A. Then there exists a smooth partition of unity on M subordinate to the covering $\mathcal{U}$.*

Proof. This is shown in [89, Theorem 2.23]. □

A.1 Linear concepts associated to manifolds

In the following we recall the concept of *tangent spaces* to a smooth manifold that allow to introduce the *differential* of functions between manifolds, a coordinate independent derivative which can be generalised to the *covariant derivative of tensor fields.* We hereby use the definition of tangent spaces given in [89, Chapter 3, pages 50-65]. An overview of common alternative definitions can be found in [89, Chapter 3, pages 71-75].

Let $(M, \mathcal{A})$ be a smooth n–manifold. Given $p \in M$ a $\mathbb{R}$–linear map $v : C^\infty(M;\mathbb{R}) \to \mathbb{R}$ is called a *derivation at p* if for all $f, g \in C^\infty(M;\mathbb{R})$ the product rule

$$v(fg) = f(p)v(g) + v(f)g(p)$$

is satisfied. The set of all derivations at p is called the *tangent space to M at p.* It is denoted by T_pM. An element of T_pM is called a *tangent vector* at p. The tangent space T_pM is a vector space over $\mathbb{R}$ with respect to pointwise addition and scalar multiplication. A derivation $v \in T_pM$ vanishes on the constant functions and for $f, g \in C^\infty(M;\mathbb{R})$ with $f(p) = g(p) = 0$ there holds $v(fg) = 0$.

Given two smooth manifolds M and N, a function $F \in C^\infty(M, N)$ and a point p in M, the *differential of F at p* is the linear map $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ defined as $\mathrm{d}F_p(v)f := v(f \circ F)$ for $v \in T_pM$ and $f \in C^\infty(N;\mathbb{R})$.

If F is a smooth diffeomorphism, the differential $\mathrm{d}F_p$ is an isomorphism of vector spaces.

Further properties of the differential are discussed in [89, Chapter 3]. The differential of functions $F \in C^1(M;\mathbb{R}^m)$, $m \in \mathbb{N}$, is defined in Remark A.6.

It is worth mentioning that given an open subset U of a smooth manifold M, the differential of the inclusion $i : U \to M$ induces an isomorphism between the tangent spaces at every point in U. This corresponds to the property that derivations act *locally* in the sense that they only depend on the values of the function in a small neighbourhood of the considered point, see [89, Proposition 3.8, Proposition 3.9].

This yields in particular that the tangent space to a smooth manifold without boundary of dimension n at any point is an n–dimensional vector space, see [89, Proposition 3.10].

The same holds true in the case of smooth manifolds with boundary as the differential of the inclusion $i : \mathbb{H}^n \to \mathbb{R}^n$ is an isomorphism of the vector spaces $T_p\mathbb{H}^n$ and $T_p\mathbb{R}^n$. We refer to [89, Lemma 3.11, Proposition 3.12] for the precise arguments.

Every chart of a smooth manifold induces a basis of the tangent space of each point in the domain of the chart. To understand how these basis vectors act as derivations one has to think about the manifold structure of a vector space. The following result is taken from [89, Example 1.24]. We give some details here as we need it several times.

Proposition A.3. *Let V be a vector space of dimension n. Then there exists a topology and a smooth structure on V independent of any choice of basis with respect to which V is a smooth manifold without boundary of dimension n.*

Proof. Let $\mathcal{E} := (E_1, \ldots, E_n)$ be a basis of V and let $w \in V$ be a vector. Then there exist coefficients $w^i \in \mathbb{R}$ such that $w = w^i E_i$ and we can assert a norm to $w \in V$ by setting $\|w\|^2_{\mathcal{E},V} := \sum_{i=1}^n (w^i)^2$. With respect to this topology the mapping E that sends $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ to the vector $x^i E_i$ is a linear isomorphism and in particular continuous. Thus the topology induced by $\|\cdot\|_{\mathcal{E},V}$ is Hausdorff and second countable and V is locally Euclidean of dimension n. If $\widetilde{\mathcal{E}} = \big(\widetilde{E}_1, \ldots, \widetilde{E}_n\big)$ is another basis of V, we have two norms on $\mathbb{R}^n$ via $\|x\|_{\mathcal{E},\mathbb{R}^n} := \big\|x^i E_i\big\|_{\mathcal{E},V}$ and $\|x\|_{\widetilde{\mathcal{E}},\mathbb{R}^n} := \big\|x^i \widetilde{E}_i\big\|_{\widetilde{\mathcal{E}},V}$. As all norms on $\mathbb{R}^n$ are equivalent, the two bases $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ induce the same topology on V and thus the topology is independent of the chosen basis. Moreover, the transition map $\widetilde{E}^{-1} \circ E : \mathbb{R}^n \to \mathbb{R}^n$ is smooth being linear and invertible. Thus any two charts $\big(V, E^{-1}\big)$ and $\big(V, \widetilde{E}^{-1}\big)$ are smoothly compatible and their collection defines a smooth structure on V. □

Proposition A.4. *Let V be an n–dimensional vector space with its standard smooth manifold structure introduced in A.3. For each $a \in V$ the map $v \mapsto \big(Dv_{|a} : C^\infty(V;\mathbb{R}) \to \mathbb{R}\big)$ acting on $f \in C^\infty(V;\mathbb{R})$ via*

$$Dv_{|a} f := \frac{\mathrm{d}}{\mathrm{d}t} f(a + tv)_{|t=0} \,,$$

is a canonical isomorphism from V to $T_a V$. In particular, with $e_1, \ldots, e_n$ denoting the standard basis on $\mathbb{R}^n$, the n derivations $\frac{\partial}{\partial x^i}_{|a} := (De_i)_{|a}$ form a basis of the tangent space $T_a \mathbb{R}^n$.

Proof. It is straightforward to verify that $Dv_{|a}$ is a derivation at a for every $v \in V$ and that $v \mapsto Dv_{|a}$ is linear. The arguments in [89, Proposition 3.2] imply that the map is an isomorphism which does not depend on any choice of basis for V. □

Proposition A.5 (Basis on the tangent space induced by local coordinates)**.** *Let $(M, \mathcal{A})$ be a smooth n–manifold, $p \in M$ and $(U, \phi) \in \mathcal{A}$ a coordinate chart around p. Then the n derivations*

$$\partial x^i_{|p} := \frac{\partial}{\partial x^i}_{|p} := (\mathrm{d}\phi_p)^{-1} \left(\frac{\partial}{\partial x^i}_{|\phi(p)} \right)$$

form a basis of the tangent space $T_p M$ and for any $f \in C^\infty(M;\mathbb{R})$ it holds that

$$\frac{\partial f}{\partial x^i}_{|p} := \frac{\partial}{\partial x^i}_{|p} f = \frac{\partial \left(f \circ \phi^{-1} \right)}{\partial x_i} (\phi(p)) \,.$$

In the case $p \in \partial M$ the derivation of $f \circ \phi^{-1}$ with respect to the n-th component should be understood as a one-sided derivative. Given $v \in T_p M$, the coefficients v^i in the representation $v = v^i \frac{\partial}{\partial x^i}_{|p}$ are called the components of v with respect to the chart (U, ϕ).

Proof. Suppose that $p \in M$ is an interior point. As the coordinate map $\phi : U \to \phi(U)$ is a smooth diffeomorphism, the differential $\mathrm{d}\phi_p : T_p U \to T_{\phi(p)} \phi(U)$ is a linear isomorphism. Using the identifications $T_p U \cong T_p M$ and $T_{\phi(p)} \phi(U) \cong T_{\phi(p)} \mathbb{R}^n$ we conclude that the preimages $(\mathrm{d}\phi_p)^{-1} \left(\frac{\partial}{\partial x^i}_{|\phi(p)} \right)$, $i \in \{1, \ldots, n\}$, of the basis $\frac{\partial}{\partial x^i}_{|\phi(p)}$ of $T_{\phi(p)} \mathbb{R}^n$ form a basis of $T_p M$. Moreover, for every $f \in C^\infty(M;\mathbb{R})$ we have

$$\begin{aligned}
(\mathrm{d}\phi_p)^{-1} \left(\frac{\partial}{\partial x^i}_{|\phi(p)} \right) f &= \left(\mathrm{d}\phi^{-1}_{\phi(p)} \right) \left(\frac{\partial}{\partial x^i}_{|\phi(p)} \right) f = \frac{\partial}{\partial x^i}_{|\phi(p)} \left(f \circ \phi^{-1} \right) \\
&= (De_i)_{|\phi(p)} \left(f \circ \phi^{-1} \right) = \frac{\partial \left(f \circ \phi^{-1} \right)}{\partial x_i} (\phi(p)) \,.
\end{aligned}$$

In the case that p is a boundary point, one has the identifications $T_p U \cong T_p M$ and $T_{\phi(p)} \phi(U) \cong T_{\phi(p)} \mathbb{H}^n \cong T_{\phi(p)} \mathbb{R}^n$. Following the proof of [89, Lemma 3.11] one sees that the preimage of $\frac{\partial}{\partial x^n}_{|\phi(p)}$

under $\mathrm{d}i_{\phi(p)} : T_{\phi(p)}\mathbb{H}^n \to T_{\phi(p)}\mathbb{R}^n$ acts on $\widetilde{f} \in C^\infty(\mathbb{H}^n;\mathbb{R})$ by

$$\mathrm{d}i^{-1}_{\phi(p)}\left(\frac{\partial}{\partial x^n}_{|\phi(p)}\right)\widetilde{f} = \lim_{h\searrow 0}\frac{1}{h}\left(\widetilde{f}(\phi(p)+he_n) - \widetilde{f}(\phi(p))\right).$$

□

Remark A.6 (Differential of non-smooth real-valued functions)**.** Let M be a smooth n–manifold, $p \in M$ and $(U,\phi) \in \mathcal{A}$ a coordinate chart around p. Given $f \in C^\infty(M;\mathbb{R})$ and $g \in C^\infty(\mathbb{R};\mathbb{R})$ Proposition A.5 yields

$$\mathrm{d}f_p\left(\frac{\partial}{\partial x^i}_{|p}\right)g = \frac{\partial}{\partial x^i}_{|p}(g\circ f) = \frac{\partial}{\partial x_i}\left(g\circ f\circ\phi^{-1}\right)(\phi(p)) = g'(f(p))\frac{\partial}{\partial x_i}\left(f\circ\phi^{-1}\right)(\phi(p))$$

where in the case $p \in \partial M$ derivation with respect to the n-th component should be understood as a one-sided derivative. Using the identification $T_{f(p)}\mathbb{R} \cong \mathbb{R}$ due to Proposition A.4 we obtain $\mathrm{d}f_p\left(\frac{\partial}{\partial x^i}_{|p}\right) = \frac{\partial}{\partial x_i}\left(f\circ\phi^{-1}\right)(\phi(p))$. Based on this observation we may now define the differential of a function $f \in C^1(M;\mathbb{R}^m)$, $m \in \mathbb{N}$, as follows: given $v = a^i\frac{\partial}{\partial x^i}_{|p} \in T_pM$ we set

$$\mathrm{d}f_p(v) := a^i\mathrm{d}f_p\left(\frac{\partial}{\partial x^i}_{|p}\right) := a^i\frac{\partial}{\partial x^i}_{|p}f := a^i\frac{\partial f}{\partial x^i}_{|p} := a^i\frac{\partial f}{\partial x^i}(p) := a^i\frac{\partial}{\partial x_i}\left(f\circ\phi^{-1}\right)(\phi(p))$$

where in the case $p \in \partial M$ the expression $\frac{\partial}{\partial x_n}\left(f\circ\phi^{-1}\right)(\phi(p))$ is a one-sided derivative. One readily checks that the term $\mathrm{d}f_p(v)$ is independent of the chosen chart (U,ϕ) around p and that $\mathrm{d}f_p : T_pM \to \mathbb{R}^m$ is linear.

Proposition A.7. *Let $(M,\mathcal{A})$ be a smooth n-manifold with boundary, let $p \in \partial M$ be a boundary point and $(U,\phi) \in \mathcal{A}$ a chart around p. The expression*

$$T_pM^+ := (\mathrm{d}\phi_p)^{-1}\left(\mathrm{d}i^{-1}_{\phi(p)}(\mathrm{int}\mathbb{H}^n)\right)$$

is independent of the chosen chart (U,ϕ). Hereby, we use the identifications $T_{\phi(p)}\phi(U) \cong T_{\phi(p)}\mathbb{H}^n$ and $\mathrm{int}\mathbb{H}^n \subset \mathbb{R}^n \cong T_{\phi(p)}\mathbb{R}^n$ and denote by $i : \mathbb{H}^n \to \mathbb{R}^n$ the inclusion map. Using the identification $T_pM \cong T_pU$ we view T_pM^+ as a subset of T_pM.

Proof. Let $p \in \partial M$ and (U,ϕ), (V,ψ) be two charts around p. Suppose that

$$v \in (\mathrm{d}\phi_p)^{-1}\left(\mathrm{d}i^{-1}_{\phi(p)}(\mathrm{int}\mathbb{H}^n)\right).$$

Then there exist $a^1,\dots,a^n \in \mathbb{R}$ with $a^n > 0$ such that $\mathrm{d}(i\circ\phi)_p(v) = (a^1,\dots,a^n)$ where we identified $\mathbb{R}^n$ and $T_{\phi(p)}\mathbb{R}^n$. This implies

$$\mathrm{d}\psi_p(v) = \mathrm{d}\left(\psi\circ\phi^{-1}\right)_{\phi(p)}\left(\mathrm{d}i^{-1}_{\phi(p)}(a^1,\dots,a^n)\right) = \mathrm{d}\left(\psi\circ\phi^{-1}\right)_{\phi(p)}\left(a^i\frac{\partial}{\partial x^i}_{|\phi(p)}\right)$$

where $\frac{\partial}{\partial x^n}_{|\phi(p)}$ should be understood as a one-sided derivative. Given $f \in C^\infty(\psi(V);\mathbb{R})$ the derivation $\mathrm{d}\psi_p(v) \in T_{\psi(p)}\psi(V) \cong T_{\psi(p)}\mathbb{H}^n$ acts as

$$\begin{aligned}\mathrm{d}\psi_p(v)f &= a^i\mathrm{d}\left(\psi\circ\phi^{-1}\right)_{\phi(p)}\frac{\partial}{\partial x^i}_{|\phi(p)}f = a^i\frac{\partial}{\partial x^i}_{|\phi(p)}\left(f\circ\psi\circ\phi^{-1}\right)\\ &= a^i\frac{\partial(\psi\circ\phi^{-1})^j}{\partial x_i}(\phi(p))\frac{\partial}{\partial x^j}_{|\psi(p)}f.\end{aligned}$$

Using again the identification $T_{\psi(p)}\mathbb{R}^n \cong \mathbb{R}^n$ we obtain

$$\mathrm{d}i_{\psi(p)}\left(\mathrm{d}\psi_p(v)\right) = \left(a^i \frac{\partial(\psi\circ\phi^{-1})^1}{\partial x_i}(\phi(p)), \dots, a^i \frac{\partial(\psi\circ\phi^{-1})^n}{\partial x_i}(\phi(p)) \right).$$

As $\left(\psi\circ\phi^{-1}\right)(\phi(U\cap V)\cap(\mathbb{R}^{n-1}\times\{0\}))\subset\mathbb{R}^{n-1}\times\{0\}$ we conclude that for all $i\in\{1,\dots,n-1\}$,

$$\frac{\partial(\psi\circ\phi^{-1})^n}{\partial x_i}(\phi(p)) = 0\,.$$

Furthermore, the property $\left(\psi\circ\phi^{-1}\right)(\phi(U\cap V))\subset\mathbb{H}^n$ and $\psi(p)\in\partial\mathbb{H}^n$ implies

$$\frac{\partial(\psi\circ\phi^{-1})^n}{\partial x_n}(\phi(p)) \geq 0\,.$$

As the matrix $D\left(\psi\circ\phi^{-1}\right)(\phi(p))$ is invertible, we conclude

$$\frac{\partial(\psi\circ\phi^{-1})^n}{\partial x_n}(\phi(p)) > 0$$

which shows $\mathrm{d}i_{\psi(p)}(\mathrm{d}\psi_p(v))\in\mathrm{int}\mathbb{H}^n$. □

In the following we introduce the most important *bundles* on a smooth manifold $(M,\mathcal{A})$ of dimension n.

The *tangent bundle* of M is defined as the disjoint union

$$TM := \bigsqcup_{p\in M} T_pM = \bigcup_{p\in M}\{(p,v) : v\in T_pM\}\,.$$

We denote by $\pi : TM\to M$ the projection defined by $\pi(p,v)=p$. Given $p\in M$ we often use the identification $\{(p,v) : v\in T_pM\}\cong T_pM$.
It is shown in [89, Proposition 3.18] that there is a natural topology and smooth structure on the tangent bundle turning it into a $2n$–dimensional smooth manifold. Given $(U,\varphi)\in\mathcal{A}$ we define a chart $\left(\pi^{-1}(U),\widetilde{\varphi}\right)$ for TM by $\widetilde{\varphi} : \pi^{-1}(U)\to\mathbb{R}^{2n}$,

$$\widetilde{\varphi}\left(p, v^i \frac{\partial}{\partial x^i}_{|p}\right) = \left(\varphi^1(p),\dots,\varphi^n(p),v^1,\dots,v^n\right).$$

If M is a manifold with boundary and (U,φ) is a boundary chart for M, then rearranging the coordinates in $\left(\pi^{-1}(U),\widetilde{\varphi}\right)$ yields a boundary chart for TM.

A *vector field* on M is then a *section in the bundle* TM, that is, a mapping $X : M\to TM$ such that $X(p)\in T_pM$ for all $p\in M$. The *components* of X with respect to a chart $(U,\phi)\in\mathcal{A}$ are the functions $X^i : M\to\mathbb{R}$ such that $X(p) = X^i(p)\frac{\partial}{\partial x^i}_{|p}$ for all $p\in U$. Here we used the identification $\{(p,v) : v\in T_pM\}\cong T_pM$. We define $\mathcal{X}^k(M)$ to be the set of C^k–vector fields on M.
One easily observes that a vector field $X : M\to TM$ is of regularity C^k, $k\in\mathbb{N}_0$, if there exists an atlas $\widetilde{\mathcal{A}}\subset\mathcal{A}$ such that the components of X with respect to every chart in the atlas are of regularity C^k.

Similarly to the tangent bundle one defines the *cotangent bundle* of M as the disjoint union

$$T^*M := \bigsqcup_{p\in M}(T_pM)^* = \bigcup_{p\in M}\{(p,\eta) : \eta\in(T_pM)^*\}\,,$$

where $(T_pM)^*$ denotes the dual space of T_pM. We denote again by $\pi : T^*M\to M$ the projection defined by $\pi(p,\eta)=p$.

One easily verifies that given a chart $(U,\phi)\in\mathcal{A}$ and $p\in U$, the linear functions $\mathrm{d}x^i_p : T_pM\to\mathbb{R}$, $i\in\{1,\dots,n\}$, form a basis of $(T_pM)^*$ where $x^i : U\to\mathbb{R}$ is the i-th component of $\phi : U\to\mathbb{R}$. This is the dual basis to the basis $\frac{\partial}{\partial x^i}_{|p}$, $i\in\{1,\dots,n\}$, of T_pM associated to the chart (U,ϕ). In particular the components of a cotangent vector $\eta\in(T_pM)^*$ with respect to the basis $\mathrm{d}x^i_p$, $i\in\{1,\dots,n\}$, are given by $\eta^i=\eta\left(\frac{\partial}{\partial x^i}_{|p}\right)$.
In analogy to the tangent bundle there is a natural topology and smooth structure on the cotangent bundle turning it into a $2n$–dimensional smooth manifold. Given $(U,\varphi)\in\mathcal{A}$ we define a chart $\left(\pi^{-1}(U),\widetilde{\varphi}\right)$ for T^*M by $\widetilde{\varphi}:\pi^{-1}(U)\to\mathbb{R}^{2n}$,

$$\widetilde{\varphi}\,(p,\eta)=\left(\varphi^1(p),\dots,\varphi^n(p),\eta\left(\frac{\partial}{\partial x^1}_{|p}\right),\dots,\eta\left(\frac{\partial}{\partial x^n}_{|p}\right)\right).$$

If M is a n–manifold with boundary and (U,ϕ) is a boundary chart for M, then rearranging the coordinates in $\left(\pi^{-1}(U),\widetilde{\varphi}\right)$ yields a boundary chart for T^*M.

Given $r,s\in\mathbb{N}_0$ such that $r+s\in\mathbb{N}$ and $p\in M$, a (r,s)-*tensor* on T_pM is a $(r+s)$-linear mapping

$$\eta:\underbrace{T_pM\times\cdots\times T_pM}_{r}\times\underbrace{(T_pM)^*\times\cdots\times(T_pM)^*}_{s}\to\mathbb{R}\,.$$

We denote by $T^s_r\,(T_pM)$ the vector space of (r,s)-tensors on T_pM and call its elements r-*times covariant* and s-*times contravariant* adopting the convention $T^0_0(T_pM)=\mathbb{R}$.
In the case $r+s\in\mathbb{N}$ one easily checks that given a chart (U,ϕ) around $p\in M$ the family

$$\left\{\mathrm{d}x^{i_1}_p\otimes\cdots\otimes\mathrm{d}x^{i_r}_p\otimes\frac{\partial}{\partial x^{j_1}}_{|p}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_s}}_{|p}: i_1,\dots,i_r,j_1,\dots,j_s\in\{1,\dots,n\}\right\}$$

forms a basis of the n^{r+s}-dimensional vector space $T^s_r(T_pM)$ where $\frac{\partial}{\partial x^i}_{|p}$ should be understood as an element of $((T_pM)^*)^*$ via the evaluation mapping.

Again one shows that the (r,s)-*tensor bundle* $T^s_r(M)$ on M defined as the disjoint union

$$T^s_r(M):=\bigsqcup_{p\in M}T^s_r\,(T_pM)=\bigcup_{p\in M}\{(p,\eta):\eta\in T^s_r\,(T_pM)\}$$

possesses a natural smooth structure of a $(n+n^{r+s})$-manifold. A map $T:M\to T^s_rM$ is said to be a (r,s)-*tensor field* if $T(p)\in T^s_r(T_pM)$ for all $p\in M$ identifying as usual $\{(p,\eta):\eta\in T^s_r\,(T_pM)\}$ and $T^s_r(T_pM)$. Given a chart $(U,\phi)\in\mathcal{A}$ there exist functions $T^{j_1\dots j_s}_{i_1\dots i_r}:U\to\mathbb{R}$, called the *components* in the chart (U,ϕ), such that for all $q\in U$,

$$T(q)=T^{j_1\cdots j_s}_{i_1\dots i_r}(q)\,\mathrm{d}x^{i_1}_q\otimes\cdots\otimes\mathrm{d}x^{i_r}_q\otimes\frac{\partial}{\partial x^{j_1}}_{|q}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_s}}_{|q}\,.$$

A $(0,0)$-tensor field is a function $f:M\to\mathbb{R}$ with component in the chart (U,ϕ) given by $f_{|U}$.
As in the case of vector fields one easily observes that a (r,s)-tensor field $T:M\to T^s_rM$ is of class C^k, $k\in\mathbb{N}_0$, if around every point $p\in M$ there exists a chart $(U,\phi)\in\mathcal{A}$ such that the components $T^{j_1,\dots,j_s}_{i_1,\dots,i_r}$ in this chart are of regularity C^k.
We define the notion of *linear connections* as in [75, Chapter 1.1]. Given $X\in T_pM$ and $f\in C^1(M;\mathbb{R})$ we hereby set $X(f):=\mathrm{d}f_p(X)$ as introduced in Remark A.6.

Definition A.8 (Linear connection)**.** Let $(M,\mathcal{A})$ be a smooth n–manifold. A *linear connection* D on M is a map $D:TM\times\mathcal{X}^1(M)\to TM$ such that

1. for all $p\in M$, $X\in T_pM$, $Y\in\mathcal{X}^1(M)$, $D(X,Y)\in T_pM$,

2. for all $p \in M$, $D : T_pM \times \mathcal{X}^1(M) \to T_pM$ is bilinear,
3. for all $p \in M$, $X \in T_pM$, $Y \in \mathcal{X}^1(M)$ and $f \in C^1(M;\mathbb{R})$,
$$D(X, fY) = X(f)Y(p) + f(p)D(X,Y)\,,$$
4. for all $X \in \mathcal{X}^k(M)$, $Y \in \mathcal{X}^{k+1}(M)$, $k \in \mathbb{N}_0$, the vector field $D(X,Y)$ defined by $p \mapsto D(X(p),Y)$ is of regularity C^k.

We write $D_X(Y) := D(X,Y)$ and call $D_X(Y)$ the *covariant derivative* of Y with respect to X. Given a chart $(U,\phi) \in \mathcal{A}$ we set
$$\nabla_i := D_{\frac{\partial}{\partial x^i}}\,.$$
The functions $\Gamma^l_{ij} : U \to \mathbb{R}$ that satisfy for all $i,j \in \{1,\dots,n\}$, $p \in U$,
$$\nabla_i\left(\frac{\partial}{\partial x^j}\right)(p) = \Gamma^l_{ij}(p)\frac{\partial}{\partial x^l}_{|p}$$
are called the *Christoffel symbols* of D in (U,ϕ).

It is easy to verify that $\Gamma^l_{ij} \in C^\infty(U;\mathbb{R})$ and that for all $p \in U$, $X = X^i \frac{\partial}{\partial x^i}_{|p} \in T_pM$ and $Y \in \mathcal{X}^1(M)$ with $Y = Y^j \frac{\partial}{\partial x^j}$ on U,
$$D_X(Y)(p) = X^i(p)(\nabla_i Y)(p) = X^i(p)\left(\frac{\partial Y^j}{\partial x^i}(p) + \Gamma^j_{i\alpha}(p)Y^\alpha(p)\right)\frac{\partial}{\partial x^j}_{|p}\,.$$

Definition A.9 (Covariant derivative of tensor fields)**.** Let $(M,\mathcal{A})$ be a smooth n–manifold, D a linear connection on M, $r,s \in \mathbb{N}_0$ with $r+s \in \mathbb{N}$ and T a (r,s)-tensor field on M of class C^1. Given $p \in M$, $X \in T_pM$ and $(U,\phi) \in \mathcal{A}$ a chart around p, we define $D_X(T)$ to be the (r,s)-tensor on T_pM defined by $(D_X(T))(p) = X^i(p)(\nabla_i T)(p)$ where
$$(\nabla_i T)^{j_1\dots j_s}_{i_1\dots i_r}(p) := \frac{\partial T^{j_1\dots j_s}_{i_1\dots i_r}}{\partial x^i}(p) - \sum_{k=1}^r \Gamma^\alpha_{ii_k}(p)T^{j_1\dots j_s}_{i_1\dots i_{k-1}\alpha i_{k+1} i_r}(p) + \sum_{k=1}^s \Gamma^{j_k}_{i\alpha}(p)T^{j_1\dots j_{k-1}\alpha j_{k+1}\dots j_s}_{i_1\dots i_r}(p)\,.$$
Given a (r,s)-tensor field T of class C^k, $k \in \mathbb{N}$, we let ∇T be the $(r+1,s)$-tensor field of class C^{k-1} whose components in a chart $(U,\phi) \in \mathcal{A}$ are given by
$$(\nabla T)^{j_1\dots j_s}_{i_1\dots i_{r+1}} = (\nabla_{i_1} T)^{j_1\dots j_s}_{i_2\dots i_{r+1}}\,.$$
The expression ∇T is called the *covariant derivative* of the tensor field T.

In the case that T is a $(0,0)$-tensor of regularity C^1, (that is, a function in $C^1(M;\mathbb{R})$) the expression $D_X(T)$ with $X \in T_pM$, $p \in M$ in a chart $(U,\phi) \in \mathcal{A}$ reduces to
$$(D_X T)(p) = X^i(p)(\nabla_i T)(p) = X^i(p)\frac{\partial T}{\partial x^i}(p) = (\mathrm{d}T_p)(X)\,, \tag{A.1}$$
the directional derivative of T with respect to X as defined in Remark A.6. The covariant derivative ∇T is the $(1,0)$-tensor field with representation in (U,ϕ) given by
$$\nabla T(p) = \frac{\partial T}{\partial x^i}(p)\mathrm{d}x^i_p = \mathrm{d}T_p\,,$$
the differential of T. If T lies in $C^2(M;\mathbb{R})$, the components of the $(2,0)$-tensor field $\nabla^2 T := \nabla(\nabla T)$ in a chart (U,ϕ) are given by
$$\left(\nabla^2 T\right)_{ij}(p) = (\nabla(\nabla T))_{ij}(p) = (\nabla_i(\nabla T))_j(p) = \frac{\partial^2\left(T\circ\phi^{-1}\right)}{\partial x_i \partial x_j}(\phi(p)) - \Gamma^k_{ij}(p)\frac{\partial\left(T\circ\phi^{-1}\right)}{\partial x_k}(\phi(p))\,.$$

Definition A.10 (Higher covariant derivatives of functions)**.** Let $(M, \mathcal{A})$ be a smooth n–manifold and let D be a linear connection on M. Given $f \in C^m(M;\mathbb{R})$, $m \in \mathbb{N}_0$, we define the k-th covariant derivative $\nabla^k f$ of f with respect to the connection D by

$$\begin{aligned} \nabla^0 f &:= f\,, \\ \nabla^k f &:= \nabla\left(\nabla^{k-1} f\right), \qquad\qquad 1 \le k \le m\,. \end{aligned}$$

The following proposition follows from [97, Lemma 1.3].

Proposition A.11 (Higher covariant derivatives in local coordinates)**.** *Let $(M, \mathcal{A})$ be a smooth n-manifold, D a linear connection on M and $f \in C^m(M;\mathbb{R})$ with $m \in \mathbb{N}$. Given $1 \le k \le m$ the components of the $(k,0)$-tensor $\nabla^k f$ in the chart $(U,\phi) \in \mathcal{A}$ are of the form*

$$\begin{aligned} \left(\nabla^k f\right)_{i_1\dots i_k}(p) &= \frac{\partial^k (f\circ\phi^{-1})}{\partial x_{i_1}\cdots\partial x_{i_k}}(\phi(p)) \\ &+ \sum_{l=1}^{k-1}\sum_{j_1,\dots,j_l=1}^{n} S^{j_1\dots j_l}_{i_1\dots i_k}\left(\Gamma(p), D\Gamma(p),\dots, D^{k-l-1}\Gamma(p)\right)\frac{\partial^l (f\circ\phi^{-1})}{\partial x_{j_1}\cdots\partial x_{j_l}}(\phi(p)) \end{aligned} \tag{A.2}$$

where every expression $S^{j_1\dots j_l}_{i_1\dots i_k}\left(\Gamma(p), D\Gamma(p),\dots, D^{k-l-1}\Gamma(p)\right)$ in the sum represents a polynomial in the Christoffel symbols $\left(\Gamma^{\alpha}_{\beta\gamma}(p)\right)$ and their derivatives in the chart (U,ϕ) up to order $k-l-1$.

Proof. As in [97, Lemma 1.3] the proof follows by induction with respect to $1 \le k \le m$. Let $(U,\phi) \in \mathcal{A}$ be given. The case $k = 1$ follows directly from (A.1). Suppose that $1 \le k \le m-1$ and that the components of $\nabla^k f$ in the chart (U,ϕ) are given by the formula (A.2). By definition of the covariant derivative we obtain

$$\begin{aligned} \left(\nabla^{k+1} f\right)_{i_1\dots i_{k+1}}(p) &= \left(\nabla\left(\nabla^k f\right)\right)_{i_1\dots i_{k+1}}(p) = \left(\nabla_{i_1}\left(\nabla^k f\right)\right)_{i_2\dots i_{k+1}}(p) \\ &= \frac{\partial(\nabla^k f)_{i_2\dots i_{k+1}}}{\partial x^{i_1}}(p) - \sum_{l=2}^{k+1}\Gamma^{\alpha}_{i_1 i_l}(p)(\nabla^k f)_{i_2\dots i_{l-1}\alpha i_{l+1} i_{k+1}}(p)\,. \end{aligned}$$

Inserting the formula (A.2) for the components of $\nabla^k f$ yields

$$\left(\nabla^{k+1} f\right)_{i_1\dots i_{k+1}}(p) = \frac{\partial^{k+1}(f\circ\phi^{-1})}{\partial x_{i_1}\cdots\partial x_{i_{k+1}}}(\phi(p)) - \sum_{l=2}^{k+1}\Gamma^{\alpha}_{i_1 i_l}(p)(\nabla^k f)_{i_2\dots i_{l-1}\alpha i_{l+1} i_{k+1}}(p) \tag{A.3a}$$

$$+\sum_{l=1}^{k-1}\sum_{j_1,\dots,j_l=1}^{n} S^{j_1\dots j_l}_{i_1\dots i_k}\left(\Gamma(p), D\Gamma(p),\dots, D^{k-l-1}\Gamma(p)\right)\frac{\partial^{l+1}(f\circ\phi^{-1})}{\partial x_{i_1}\partial x_{j_1}\cdots\partial x_{j_l}}(\phi(p)) \tag{A.3b}$$

$$+\sum_{l=1}^{k-1}\sum_{j_1,\dots,j_l=1}^{n}\frac{\partial}{\partial x_{i_1}}\left(S^{j_1\dots j_l}_{i_1\dots i_k}\left(\Gamma, D\Gamma,\dots, D^{k-l-1}\Gamma\right)\circ\phi^{-1}\right)(\phi(p))\frac{\partial^l(f\circ\phi^{-1})}{\partial x_{j_1}\cdots\partial x_{j_l}}(\phi(p))\,. \tag{A.3c}$$

The second term on the right hand side of (A.3a) is already in the desired form. In the expression (A.3b) derivatives of $f\circ\phi^{-1}$ of order $k' \in \{2,\dots,k\}$ appear with coefficients that are polynomial in the Christoffel symbols and their derivatives up to order at most $(k+1)-k'-1$. Finally, we notice that also the terms in (A.3c) fit into the desired formula as

$$\frac{\partial}{\partial x^{i_1}}\left(S^{j_1\dots j_l}_{i_1\dots i_k}\left(\Gamma, D\Gamma,\dots, D^{k-l-1}\Gamma\right)\circ\phi^{-1}\right)(\phi(p))$$

is a polynomial expression in the components of Γ and their derivatives up to order at most $k-l-1+1 = (k+1)-l-1$. □

A.2 Embedded submanifolds and the boundary of a manifold

In this section we discuss properties of embedded submanifolds with particular emphasis on the features of the boundary of a smooth manifold. Unless stated otherwise the considered manifolds may be with or without boundary in the following.
Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds of dimension n and m, respectively. A map $f : N \to M$ is called *smooth immersion* if it is smooth and has injective differential $\mathrm{d}f_p : T_pN \to T_{f(p)}M$ at every point $p \in N$. A *smooth embedding* of N into M is a smooth immersion $f : N \to M$ that is a homeomorphism onto its image $f(N) \subset M$ with respect to the subspace topology on $f(N)$. A *smooth embedded submanifold (with boundary)* of M of dimension $0 \leq m \leq n$ $(1 \leq m \leq n)$ is a subset $S \subset M$ that is an m–dimensional topological manifold (with boundary) in the subspace topology endowed with a smooth structure with respect to which the inclusion map $i : S \hookrightarrow M$ is a smooth embedding.
The following result shows that smooth embedded submanifolds of M are precisely the images of smooth embeddings.

Proposition A.12. *Let $(M, \mathcal{A})$ be a smooth n–manifold. Let $(N, \mathcal{B})$ be a smooth manifold (with boundary) of dimension $0 \leq m \leq n$ $(1 \leq m \leq n)$. Suppose that $f : N \to M$ is a smooth embedding. Then $f(N)$ is a smooth embedded submanifold (with boundary) of M of dimension m.*

Proof. With respect to the subspace topology inherited from M the set $f(N)$ is Hausdorff and second countable. As $f : N \to f(N)$ is a homeomorphism with respect to the subspace topology on $f(N)$, the set $\mathcal{B}_f := \{(f(U), \phi \circ f^{-1}) : (U, \phi) \in \mathcal{B}\}$ is an atlas for $f(N)$. All charts within $\mathcal{B}_f$ are smoothly compatible, $f : (N, \mathcal{B}) \to (f(N), \mathcal{B}_f)$ is a smooth diffeomorphism and possible boundary points of N are mapped to boundary points of $f(N)$. Thus the inclusion $i : (f(N), \mathcal{B}_f) \to (M, \mathcal{A})$ is a smooth embedding being the composition of the smooth diffeomorphism $f^{-1} : (f(N), \mathcal{B}_f) \to (N, \mathcal{B})$ and the smooth embedding $f : (N, \mathcal{B}) \to (M, \mathcal{A})$. □

Embedded submanifolds without boundary can be characterised by the existence of certain charts.

Definition A.13. Let $(M, \mathcal{A})$ be a smooth n–manifold. Given a subset $N \subset M$ and $0 \leq m \leq n$ we say that N satisfies the *local m-slice condition* if each point of N is contained in the domain of a chart $(U, \phi) \in \mathcal{A}$ such that

$$\phi(N \cap U) = \{(x^1, \dots, x^m, x^{m+1}, \dots, x^n) \in \phi(U) : x^{m+1} = \dots = x^n = 0\}.$$

Any such chart is called *slice chart* for N in M and the corresponding coordinates are called *slice coordinates.*

Theorem A.14 (Local slice criterion for embedded submanifolds)**.** *Let $(M, \mathcal{A})$ be a smooth n–manifold and let $0 \leq m \leq n$. Every smooth embedded m–dimensional submanifold of M without boundary satisfies the local m-slice condition. Conversely, if N is a subset of M that satisfies the local m-slice condition, then with the subspace topology N is a topological manifold of dimension m and it has a smooth structure turning it into an m-dimensional smooth embedded submanifold of M. This structure is given by the maximal smooth atlas containing the charts*

$$\{(N \cap U, \pi \circ \phi \circ i) : (U, \phi) \in \mathcal{A} \text{ is a slice chart for } N\}$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^m$ denotes the projection on the first m components if $m \geq 1$, or $\pi : \mathbb{R}^n \to \{0\}$, if $m = 0$, respectively, and $i : N \to M$ is the inclusion mapping.

Proof. This is shown in [89, Theorem 5.8]. □

There is an analogue to the local m-slice condition for embedded submanifolds with boundary, see [89, Theorem 5.51].
Given a smooth manifold with boundary, the boundary is itself a smooth embedded submanifold without boundary.

Proposition A.15 (Induced structure on the boundary)**.** *Let $(M, \mathcal{A})$ be a smooth manifold with boundary of dimension $n \in \mathbb{N}$. Then ∂M is a smooth embedded submanifold of M without boundary of dimension $n-1$ with smooth structure given by*

$$\mathcal{A}_{\partial M} := \{(\partial M \cap U, \pi \circ \phi \circ \iota) : (U, \phi) \in \mathcal{A} \ \textit{ is a boundary chart}\}$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denotes the projection on the first $n-1$ components if $n \geq 2$, or $\pi : \mathbb{R} \to \{0\}$, if $n = 1$, and $\iota : \partial M \to M$ is the inclusion mapping.

Proof. The set $\partial M \subset M$ satisfies the local $(n-1)$-slice condition. For any $p \in \partial M$ there exists a chart $(U, \phi) \in \mathcal{A}$ around p such that $\phi(U)$ is open in $\mathbb{H}^n$ and $\phi(p) \in \partial \mathbb{H}^n$. By [89, Theorem 1.37] every boundary chart (U, ϕ) satisfies $\phi(U \cap \partial M) = \phi(U) \cap \partial \mathbb{H}^n$. □

Proposition A.16. *Let $(M, \mathcal{A})$ be a smooth manifold with boundary of dimension $n \in \mathbb{N}$. Then for every $p \in \partial M$ the space $T_p \partial M$ can be identified with a subspace of $T_p M$ of dimension $n-1$.*

Proof. The boundary ∂M is a smooth embedded submanifold of M of dimension $n-1$ and the inclusion map $\iota : \partial M \to M$ is a smooth embedding. In particular for every $p \in \partial M$ the differential $\mathrm{d}\iota_p : T_p \partial M \to T_p M$ is injective which allows us to identify $T_p \partial M$ with the $n-1$ dimensional subspace $\mathrm{d}\iota_p\,(T_p \partial M)$ of $T_p M$. □

Given a boundary point p and a chart $(U, \phi) \in \mathcal{A}$ around p, it follows from [79, Chapter 6.3, Lemma 2] that if the n–th component of $v \in T_p M$ with respect to the chart (U, ϕ) is positive (negative, zero), then it is positive (negative, zero) with respect to *every* chart around p. Moreover, the vector v lies in $T_p \partial M$ if and only if its n–th component with respect to one (then every) chart around p is equal to zero. The vector v is called *inward-pointing* (*outward-pointing*) if its n–th component is positive (negative).

Definition A.17 (Outward-pointing vector fields)**.** Let $(M, \mathcal{A})$ be a smooth manifold with boundary of dimension $n \in \mathbb{N}$. An *outward-pointing vector field along* ∂M is a map $N : \partial M \to TM$ such that for all $p \in \partial M$ the vector $N(p)$ lies in $T_p M$ and is outward-pointing. The vector field N is of regularity C^k, $k \in \mathbb{N}_0$, if around every boundary point p there is a chart $(U, \phi) \in \mathcal{A}$ such that the components N^i in the representation $N(q) = N^i(q) \frac{\partial}{\partial x^i}_{|q}$, $q \in U \cap \partial M$, are of regularity C^k on $U \cap \partial M$.

A.3 Orientability of manifolds and integration of differential forms

In the following we introduce the concept of orientability of a manifold and the integration of differential forms on compact manifolds.

Given a real vector space V of dimension $n \in \mathbb{N}$, two bases $(E_1, \ldots, E_n)$ and $(\widetilde{E}_1, \ldots, \widetilde{E}_n)$ for V are *consistently oriented* if the transition matrix (B_i^j) determined by the relations $E_i = B_i^j \widetilde{E}_j$ has

positive determinant. This is an equivalence relation on the set of all ordered bases and there are exactly two equivalence classes. An *orientation* for a real vector space of positive dimension is an equivalence class of ordered bases. A vector space together with an orientation is called *oriented vector space.* Any basis that is in the given orientation is called *(positively) oriented.* Any basis that is not in the given orientation is said to be negatively oriented.
An orientation for a zero dimensional vector space is simply a choice of one of the numbers ± 1.

Given a smooth manifold $(M, \mathcal{A})$ of dimension n, a *pointwise orientation* on M is a choice of orientation on each tangent space. The manifold M is called *orientable* if there exists a pointwise orientation on M such that for every $p \in M$ there is a chart $(U, \phi) \in \mathcal{A}$ such that $\left(\frac{\partial}{\partial x^1}_{|q}, \dots, \frac{\partial}{\partial x^n}_{|q}\right)$ is positively oriented for all $q \in U$. The manifold M together with its orientation is called *oriented manifold.* A chart (U, ϕ) is called positively (negatively) oriented if $\left(\frac{\partial}{\partial x^1}_{|q}, \dots, \frac{\partial}{\partial x^n}_{|q}\right)$ is positively (negatively) oriented for every $q \in U$.

We notice that every 0–dimensional manifold is orientable by just assigning (arbitrarily) 1 or -1 to each point.

Given an oriented smooth manifold $(M, \mathcal{A})$ and two charts $(U, \phi), (V, \psi) \in \mathcal{A}$ there holds the relation

$$\frac{\partial}{\partial y^i}_{|p} = \frac{\partial \left(\phi^j \circ \psi^{-1}\right)}{\partial y_i} (\psi(p)) \frac{\partial}{\partial x^j}_{|p} .$$

Thus a chart $(V, \psi) \in \mathcal{A}$ is positively oriented if and only if for all positively oriented charts $(U, \phi) \in \mathcal{A}$ with $p \in U$, $\det D \left(\phi \circ \psi^{-1}\right)(\psi(p)) > 0$. As the expression $\det D \left(\phi \circ \psi^{-1}\right)(x) \in \mathbb{R} \backslash \{0\}$ depends continuously on $x \in \psi(U \cap V)$, a chart $(U, \phi) \in \mathcal{A}$ with connected domain U is either positively or negatively oriented.
The set $\mathcal{A}_O := \{(U, \phi) \in \mathcal{A} : (U, \phi) \text{ is positively oriented}\}$ is a smooth atlas for M and it is the maximal smooth atlas containing all positively oriented charts.

A smooth local diffeomorphism $\Phi : M \to M$ of an oriented smooth manifold $(M, \mathcal{A})$ is called *orientation preserving* if for each $p \in M$ and each positively oriented basis $(v_1, \dots, v_n)$ of T_pM, the elements $(\mathrm{d}\Phi_p(v_1), \dots, \mathrm{d}\Phi_p(v_n))$ form a positively oriented basis of $T_{\Phi(p)}M$.
One easily observes that a smooth local diffeomorphism $\Phi : M \to M$ is orientation preserving if and only if for all $p \in M$ and all positively oriented charts (U, ϕ) and $(V, \psi) \in \mathcal{A}$ around p and $\Phi(p)$, respectively, the determinant of $D\left(\psi \circ \Phi \circ \phi^{-1}\right)(\phi(p))$ is positive.
Furthermore, given an orientation preserving diffeomorphism $\Phi : M \to M$, the inverse Φ^{-1} is orientation preserving and for every positively oriented chart $(U, \phi) \in \mathcal{A}$, the chart $\left(\Phi(U), \phi \circ \Phi^{-1}\right)$ is positively oriented.

We remark that the notion of orientability of manifolds presented in [89, Chapter 15] is slightly more general than the definition presented here. In the case that ∂M is empty or the dimension of M is greater than 1, these two definitions are equivalent which follows from [89, Proposition 15.6].

Proposition A.18 (Induced orientation on the boundary)**.** *[89, Proposition 15.24]*
Let $(M, \mathcal{A})$ be an oriented smooth manifold with boundary of dimension $n \in \mathbb{N}$. Suppose that N is any smooth outward-pointing vector field along ∂M and let $\iota : \partial M \to M$ denote the inclusion mapping. Then ∂M is orientable and ∂M has a unique orientation such that for all $p \in \partial M$ the vectors $(v_1, \dots, v_{n-1})$ form a positively oriented basis for $T_p\partial M$ if and only if $(N(p), \mathrm{d}\iota_p(v_1), \dots, \mathrm{d}\iota_p(v_{n-1}))$ is a positively oriented basis for T_pM. Moreover the orientation is independent of the choice of the outward-pointing vector field N.

Proof. This result is shown in [89, Proposition 15.24]. In the case that $n = 1$ one assigns the number $+1$ (or -1) to $p \in \partial M$ if $N(p)$ is positively (or negatively) oriented, respectively, as explained in [89, Remark 15.21]. □

The orientability of a manifold is an important tool needed to define the integration of differential forms on manifolds which are introduced in the following.

Let V be a real vector space of dimension $n \in \mathbb{N}$ with dual space V^*. Given $m \in \mathbb{N}$ an *alternating m–form* on V is a multi-linear map $\omega : V^m \to \mathbb{R}$ such that $\omega(v_1, \dots, v_m) = 0$ for all $v_1, \dots, v_m \in V$ with $v_i = v_j$ for some $i \neq j$. The set of all alternating m–forms on V is denoted by $\Lambda^m(V^*)$.
As V is finite dimensional, the space of all m–multi-linear maps can be identified with the space $T^m(V^*)$ of m–covariant tensors on V of which the alternating m–covariant tensors $\Lambda^m(V^*)$ form a subspace. By convention, $T^0(V) = \Lambda^0(V) = \mathbb{R}$.
In [89, Proposition 14.9] it is shown that $\Lambda^n(V^*)$ is one dimensional and given $\omega \in \Lambda^n(V^*)$, a basis $e_1, \dots, e_n$ of V and $v_i = a_i^j e_j$ with $a_i^j \in \mathbb{R}$ it holds that $\omega(v_1, \dots, v_n) = \det((a_i^j))\omega(e_1, \dots, e_n)$.
Given $\varphi_1, \dots, \varphi_k \in V^*$, $k \in \mathbb{N}$, the *exterior product* $\varphi_1 \wedge \cdots \wedge \varphi_k \in \Lambda^k(V^*)$ is defined as

$$(\varphi_1 \wedge \cdots \wedge \varphi_k)(v_1, \dots, v_k) = \det\left(\left(\varphi_i(v_j)\right)_{i,j=1}^k\right) \tag{A.4}$$

for all $v_1, \dots, v_k \in V$. It is shown in [89, Proposition 14.8, 14.11, 14.12] that given $m, j \in \mathbb{N}$, there is a unique bilinear mapping, called *wedge product* or *exterior product*,

$$\wedge : \Lambda^m(V^*) \times \Lambda^j(V^*) \to \Lambda^{m+j}(V^*), \qquad (\omega, \sigma) \mapsto \omega \wedge \sigma,$$

such that for all $\psi_1, \dots, \psi_m, \eta_1, \dots, \eta_j \in V^*$,

$$(\psi_1 \wedge \cdots \wedge \psi_m) \wedge (\eta_1 \wedge \cdots \wedge \eta_j) = \psi_1 \wedge \cdots \wedge \psi_m \wedge \eta_1 \wedge \cdots \wedge \eta_j$$

in the sense of (A.4). Moreover, if $v_1, \dots, v_n$ is a basis of V with dual basis $v_1^*, \dots, v_n^*$, then

$$\left\{v_{i_1}^* \wedge \cdots \wedge v_{i_m}^* \ : \ 1 \leq i_1 < \cdots < i_m \leq n\right\}$$

forms a basis of $\Lambda^m(V^*)$.
In analogy to the concept of tensor fields, we now introduce functions that assign to every $p \in M$ an alternating m–form on T_pM.

Definition A.19 (Differential forms)**.** Let $(M, \mathcal{A})$ be a smooth n–manifold with $n \in \mathbb{N}$ and let $0 \leq m \leq n$. A *differential form* of degree m on M is a section in the bundle

$$\Lambda^m((TM)^*) = \bigsqcup_{p \in M} \Lambda^m\left((T_pM)^*\right).$$

A differential form ω has regularity C^k, $k \in \mathbb{N}_0$, with respect to a chart $(U, \phi) \in \mathcal{A}$ if for all $1 \leq \mu_1, \dots, \mu_m \leq n$ the *components*

$$\omega_{\mu_1,\dots,\mu_m} : U \to \mathbb{R}, \qquad p \mapsto \omega_p\left(\frac{\partial}{\partial x^{\mu_1}}_{|p}, \dots, \frac{\partial}{\partial x^{\mu_m}}_{|p}\right)$$

are of regularity C^k. The differential form ω has regularity C^k on M if it has regularity C^k with respect to the charts of an atlas $\widetilde{\mathcal{A}} \subset \mathcal{A}$. In this case we write $\omega \in \Omega_{(k)}^m M$. The *exterior product* of $\omega \in \Omega_{(k)}^m(M)$ and $\sigma \in \Omega_{(k)}^j(M)$ with $0 \leq j + m \leq n$ is the form $\omega \wedge \sigma \in \Omega_{(k)}^{m+j}(M)$ defined by $(\omega \wedge \sigma)_p := \omega_p \wedge \sigma_p$ for all $p \in M$. The *interior product* $X \lrcorner \omega \in \Omega_{(k)}^{m-1}(M)$ of $X \in \mathcal{X}^k(M)$ and $\omega \in \Omega_{(k)}^m(M)$ is the differential form defined by

$$(X \lrcorner \omega)_p (v_1, \dots, v_{m-1}) := \omega_p (X(p), v_1, \dots, v_{m-1})$$

for $p \in M$ and $v_1, \dots, v_{m-1} \in T_pM$.

We observe that the tangential bundle of degree 0 can be identified with the set of real-valued functions on the manifold.

Definition A.20 (Integrability of n–forms)**.** Let $(M, \mathcal{A})$ be a smooth compact oriented n–manifold with $n \in \mathbb{N}$, let $(U_i, \phi_i) \in \mathcal{A}$, $i \in \{1, \dots, N\}$, be positively oriented charts that cover M and (ψ_i) a smooth partition of unity subordinate to this covering. A differential form ω of degree n is said to be *integrable* if

$$g_i := (\psi_i \omega_{1,\dots,n}) \circ \phi_i^{-1} : \phi_i(U_i) \to \mathbb{R}$$

is Lebesgue integrable for all $i \in \{1, \dots, N\}$. In this case we define

$$\int_M \omega := \sum_{i=1}^N \int_{\phi_i(U_i)} g_i(x) \, \mathrm{d}x \, .$$

It is well-known in the literature that this definition is independent of the chosen covering of positively oriented charts and the partition of unity, see for example [89, Proposition 16.5]. According to [89, page 406], in the case that $M = \{y_1, \dots, y_N\}$ is a compact oriented 0–dimensional manifold, the integral of a 0–form on M, that is a real-valued function $f : M \to \mathbb{R}$, is defined by

$$\int_M f = \sum_{i=1}^N \pm f(y_i)$$

where points with positive orientation get positive signs, and points with negative orientation negative ones.

Proposition A.21 (Integrability of continuous n-forms)**.** *Let $(M, \mathcal{A})$ be a smooth compact oriented n–manifold, $n \in \mathbb{N}$. Every continuous n–form on M is integrable.*

Proof. Let $(U_i, \phi_i) \in \mathcal{A}$, $i \in \{1, \dots, N\}$ be any covering of M by positively oriented charts and let (ψ_i) be a smooth partition of unity subordinate to this covering. Then it is enough to check that

$$g_i(x) = \psi_i \left(\phi_i^{-1}(x)\right) \omega_{\phi_i^{-1}(x)} \left(\frac{\partial}{\partial x^1}_{|\phi_i^{-1}(x)}, \dots, \frac{\partial}{\partial x^n}_{|\phi_i^{-1}(x)} \right)$$

is Lebesgue integrable on $\phi_i(U_i)$ for every i. The functions $\psi_i \circ \phi_i^{-1}$ and $\omega_{1,\dots,n} \circ \phi_i^{-1}$ have regularity C^∞ and C^0 on $\phi_i(U_i)$, respectively. The support of g_i is contained in $\phi_i(\operatorname{supp} \psi_i)$. The set $\operatorname{supp} \psi_i$ is compact in M being a closed subset of the compact set M. Hence $\phi_i(\operatorname{supp} \psi_i)$ is compact in $\mathbb{R}^n$ and g_i is bounded being a continuous function with compact support which yields

$$\int_{\phi_i(U_i)} |g_i(x)| \, \mathrm{d}x = \int_{\phi_i(\operatorname{supp} \psi_i)} |g_i(x)| \, \mathrm{d}x \le \|g_i\|_\infty \, \lambda^n \left(\phi_i \left(\operatorname{supp} \psi_i\right)\right) < \infty \, ,$$

where λ^n denotes the Lebesgue measure on $\mathbb{R}^n$. □

A.4 Riemannian manifolds and normal coordinates

This section is devoted to Riemannian manifolds and the construction of normal coordinates.
Let $(M, \mathcal{A})$ be a smooth n–manifold. A *C^k–Riemannian metric g* on M, $k \in \mathbb{N}_0$, is a family of positive definite symmetric bilinear forms $g_p : T_pM \times T_pM \to \mathbb{R}$ such that for all charts $(U, \phi) \in \mathcal{A}$ and all $i, j \in \{1, \dots, n\}$ the map

$$g_{ij} : U \to \mathbb{R} \, , \qquad p \mapsto g_p \left(\frac{\partial}{\partial x^i}_{|p}, \frac{\partial}{\partial x^j}_{|p} \right)$$

is of regularity C^k. In the case that all g_{ij} are smooth we say that g is a *smooth Riemannian metric*. A *smooth Riemannian manifold* is a smooth manifold with a smooth Riemannian metric. A *Riemannian manifold of regularity* C^k, $k \in \mathbb{N}_0$, is a smooth manifold endowed with a C^k–Riemannian metric g which is then a $(2,0)$-tensor field $g : M \to T^0_2(M)$ of class C^k. Moreover, for every chart $(U,\phi) \in \mathcal{A}$ and every $p \in U$ the matrix $G(p) := (g_{ij}(p))$ is positive definite with inverse $G^{-1}(p) =: \left(g^{ij}(p)\right)$. On every tangent space T_pM the metric induces a norm by $|v|_g := \sqrt{g_p(v,v)}$ for $v \in T_pM$.
The additional structure provided by a Riemannian metric allows to generalise concepts that are known from Euclidean geometry. Given a Riemannian manifold $(M,\mathcal{A},g)$ of regularity C^k, $k \in \mathbb{N}_0$, the *gradient* of a function $f \in C^l(M;\mathbb{R})$, $l \in \mathbb{N}$, is the unique vector field $\mathrm{grad} f$ on M such that

$$g_p\left(\mathrm{grad}_p f, w\right) = \mathrm{d}f_p(w)$$

for all $p \in M$ and all $w \in T_pM$, where $\mathrm{d}f_p$ for non-smooth f is introduced in Remark A.6. Instead of $\mathrm{grad} f$ we also write $\nabla_{M,g} f$ to indicate the dependence on the metric. One easily checks that the representation of $\mathrm{grad} f$ with respect to a chart $(U,\phi) \in \mathcal{A}$ is given by

$$\mathrm{grad}_p f = g^{ij}(p) \frac{\partial f}{\partial x^i}_{|p} \frac{\partial}{\partial x^j}_{|p} . \tag{A.5}$$

In particular, $\mathrm{grad} f$ lies in $\mathcal{X}^{\min\{l-1,k\}}(M)$.
The Riemannian metric further allows to define an integral on the manifold which applies to any continuous function on the manifold. This is due to the existence of the *Riemannian volume form* that yields a one to one correspondence between n–forms and scalar functions of a given regularity.

Proposition A.22 (Existence of Riemannian volume form)**.** *Let $(M,\mathcal{A},g)$ be an oriented Riemannian manifold of regularity C^k with $k \in \mathbb{N}_0$. Then there exists a unique differential form $\omega_g \in \Omega^n_{(k)}M$, called the Riemannian volume form, such that $(\omega_g)_p(v_1,\ldots,v_n) = 1$ for every positively oriented orthonormal basis of T_pM and every $p \in M$. The component of ω in any positively oriented chart $(U,\phi) \in \mathcal{A}$ is given by*

$$(\omega_g)_p \left(\frac{\partial}{\partial x^1}_{|p}, \ldots, \frac{\partial}{\partial x^n}_{|p} \right) = \sqrt{g(p)}$$

where $g(p) := \det\left((g_{ij}(p))_{ij}\right)$. In particular, the Riemannian volume form ω_g is integrable.

Proof. The existence of the Riemannian volume form is a standard result in the literature. It is for example shown in [89, Proposition 15.29, Proposition 15.31] in the case of smooth metrics. The proof can be adapted to our situation. The integrability of ω_g follows from Proposition A.21. □

Proposition A.23 ($*$–operator)**.** *Let $(M,\mathcal{A},g)$ be an oriented Riemannian manifold of regularity C^k with $k \in \mathbb{N}_0$ and let ω_g be the Riemannian volume form. The function*

$$* : C^k(M) \to \Omega^n_{(k)}M\,, \qquad f \mapsto *f := f\omega_g$$

is bijective.

Proof. The function $*$ is well-defined. Let ω be any element in $\Omega^n_{(k)}(M)$ and $p \in M$. As the space $\Lambda^n\left((T_pM)^*\right)$ is one dimensional, there exists $f(p) \in \mathbb{R}$ such that $\omega_p = f(p)\,(\omega_g)_p$. This defines a function $f : M \to \mathbb{R}$ and the component of ω with respect to a chart $(U,\phi) \in \mathcal{A}$ is given by

$$\omega_{1,\ldots,n} = f\sqrt{g}\,.$$

As $\sqrt{g}$ is non zero on all of U, the function $f = \omega_{1,\dots,n}\sqrt{g}^{-1}$ is of regularity C^k on U. This implies $f \in C^k(M;\mathbb{R})$ and $\omega = *f$. Suppose that $f, g \in C^k(M)$ satisfy $*f = *g$. Then for all $p \in M$ it holds that $(f(p) - g(p))\,(\omega_g)_p = 0$ and hence $f(p) = g(p)$. □

Definition A.24 (Integrability of functions)**.** Let $(M, \mathcal{A}, g)$ be a compact oriented Riemannian manifold of regularity C^k and dimension n with $k \in \mathbb{N}_0$ and let ω_g be the Riemannian volume form. A function $a : M \to \mathbb{R}$ is said to be *integrable* over M if the n–form $a\omega$ is integrable. In this case we write

$$\int_M a\mathrm{d}V_g := \int_M a\omega_g \,,$$

and call $\mathrm{d}V_g$ the *canonical volume measure.*

In particular, every continuous function on a compact oriented Riemannian manifold is integrable. The following lemma can be seen as a version of the Fundamental Lemma of Calculus of Variations on manifolds.

Lemma A.25. *Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold of dimension $n \in \mathbb{N}$. If $\mathcal{G} \in C(M;\mathbb{R})$ is such that for all $\psi \in C^\infty(M)$ with $\psi \equiv 0$ on ∂M,*

$$\int_M \mathcal{G}(p)\psi(p)\,\mathrm{d}V_g = 0\,,$$

then $\mathcal{G} = 0$ on M.

Proof. As $\mathcal{G}$ is continuous, it is enough to prove that $\mathcal{G}$ is zero in the interior of the manifold. Assume that $p \in M \setminus \partial M$ is such that $\mathcal{G}(p) > 0$.[1] Then there exist a constant $\delta > 0$ and open neighbourhoods U and V of p such that $p \in U \subset \overline{U}^M \subset V \subset M \setminus \partial M$ and $\mathcal{G} \geq \delta$ on V where $\overline{U}^M$ denotes the closure of U in M. By [89, Proposition 2.25] there exists a bump function $\psi \in C^\infty(M)$ supported in V such that $0 \leq \psi \leq 1$ on M and $\psi \equiv 1$ on $\overline{U}^M$. In particular, $\psi \equiv 0$ on ∂M and

$$0 = \int_M \mathcal{G}\psi\,\mathrm{d}V_g = \int_V \mathcal{G}\psi\,\mathrm{d}V_g \geq \delta \int_V \psi\,\mathrm{d}V_g \geq \delta \int_U 1\,\mathrm{d}V_g > 0\,,$$

a contradiction. □

In the remaining part of the section we study a generalisation of the Fundamental Theorem of Calculus to manifolds with (possibly empty) boundary and differential forms, namely the *Divergence Theorem* which is a special case of the famous Stokes' Theorem. To this end we define a metric on the boundary that "fits" to the metric on the manifold.

Proposition A.26. *Let $(M, \mathcal{A}, g)$ be a Riemannian manifold with boundary of regularity C^k and dimension n with $k \in \mathbb{N}_0$, $n \in \mathbb{N}$. Then $(\partial M, \mathcal{A}_{\partial M}, \tilde{g})$ is a Riemannian manifold of regularity C^k and dimension $n-1$ with respect to the induced metric $\tilde{g} := (\mathrm{d}\iota)^*(g)$ on $T\partial M$.*

Proof. In Proposition A.15 we have shown that $(\partial M, \mathcal{A}_{\partial M})$ is a smooth manifold of dimension $n-1$. As the differential of the embedding ι is injective and linear at every point, $\tilde{g}$ defines a family of positive definite symmetric bilinear forms. For a boundary chart $(U, \phi) \in \mathcal{A}$ it remains to verify that for $i, j \in \{1, \dots, n-1\}$ the map

$$p \mapsto \widetilde{g}_{ij}(p) = \tilde{g}_p\left(\mathrm{d}\,(\pi\circ\phi\circ\iota)_p^{-1}\left(\frac{\partial}{\partial x^i}_{|\pi(\phi(\iota(p)))}\right), \mathrm{d}\,(\pi\circ\phi\circ\iota)_p^{-1}\left(\frac{\partial}{\partial x^i}_{|\pi(\phi(\iota(p)))}\right)\right)$$

[1] The case $\mathcal{G}(p) < 0$ for some $p \in M \setminus \partial M$ can be treated analogously.

is of regularity C^k on $\partial M \cap U$ where $\frac{\partial}{\partial x^i}_{|\pi(\phi(\iota(p)))}$ denotes the derivation $(De_i)_{|\pi(\phi(\iota(p)))}$ with respect to the i-th standard basis vector in $\mathbb{R}^{n-1}$. Inserting the definitions of $\tilde{g}$ and the differential, one obtains for all $i, j \in \{1, \dots, n-1\}$ that $\widetilde{g}_{ij} = g_{ij} \circ \iota$ on $U \cap \partial M$. Thus $\widetilde{g}_{ij}$ is of regularity C^k on $\partial M \cap U$ being the composition of $\iota \in C^\infty(\partial M \cap U; M)$ and $g_{ij} \in C^k(U; \mathbb{R})$. □

The Riemannian volume form on the boundary can be characterised via the volume form on the manifold and the *outer unit conormal* along the boundary.

Proposition A.27 (Existence of outer unit conormal)**.** *[89, Proposition 15.33] Let $(M, \mathcal{A}, g)$ be a Riemannian manifold with boundary of regularity C^k and dimension n with $k \in \mathbb{N}_0$, $n \in \mathbb{N}$. There is a unique outward-pointing vector field N along ∂M normal to $T_p\partial M$ of regularity C^k with $g(N, N) = 1$. The vector field N is called outer unit conormal along ∂M.*

Proof. This result is shown in [89, Proposition 15.33] in the smooth setting. It relies on the existence of a smooth function $f : M \to \mathbb{R}$ constructed in [89, Proposition 5.43] with the property that $f^{-1}(\{0\}) = \partial M$ and $\mathrm{d}f_p \neq 0$ for all $p \in \partial M$. The outward-pointing vector field N is of regularity C^k and given by $N(p) := -\mathrm{grad}_p f \left|\mathrm{grad}_p f\right|_g^{-1}$. □

Proposition A.28. *[89, Proposition 15.32, 15.34] Let $(M, \mathcal{A}, g)$ be an oriented Riemannian manifold of regularity C^k and dimension n with boundary, $k \in \mathbb{N}_0$, $n \in \mathbb{N}$, and volume form ω_g and let N be the outer unit conormal along ∂M with induced orientation on ∂M, see A.18. The Riemannian volume form of $(\partial M, \tilde{g})$ is given by $\omega_{\tilde{g}} = \iota^* (N \lrcorner\, \omega_g)$ where*

$$\iota^* (N \lrcorner\, \omega_g)_p (v_1, \dots, v_{n-1}) := (\omega_g)_{\iota(p)} (N(p), \mathrm{d}\iota_p(v_1), \dots, \mathrm{d}\iota_p(v_{n-1}))$$

for $p \in \partial M$ and $v_1, \dots, v_{n-1} \in T_p\partial M$.

Proof. Using that $\mathrm{d}\iota_p$ is linear and injective for every $p \in \partial M$ one verifies that $\iota^* (N \lrcorner\, \omega_g)$ defines a section in $\Lambda^{n-1} \left((T\partial M)^*\right)$. Let $(U, \phi) \in \mathcal{A}$ be a boundary chart. A computation shows that the component of $\iota^* (N \lrcorner\, \omega_g)$ with respect to $(U \cap \partial M, \pi \circ \phi \circ \iota)$ is given by

$$(\iota^* (N \lrcorner\, \omega_g))_{1,\dots,n-1} = (-1)^n N^n \omega_{1,\dots,n} \circ \iota\,.$$

Thus $\iota^* (N \lrcorner\, \omega_g)$ lies in $\Omega^{n-1}_{(k)}(\partial M)$ as ω_g and N are of regularity C^k. Let $v_1, \dots, v_{n-1}$ be a positively oriented orthonormal basis of $T_p\partial M$ for some $p \in \partial M$. Then by the Definition A.18 of the orientation on ∂M the vectors $(N(p), \mathrm{d}\iota_p(v_1), \dots, \mathrm{d}\iota_p(v_{n-1}))$ form a positively oriented basis of T_pM. It is orthonormal as $\delta_{ij} = \tilde{g}_p(v_i, v_j) = g_p (\mathrm{d}\iota_p(v_i), \mathrm{d}\iota_p(v_j))$ and as N is a unit normal vector field along ∂M. This implies $\omega_{\iota(p)} (N(p), \mathrm{d}\iota_p(v_1), \dots, \mathrm{d}\iota_p(v_{n-1})) = 1$. □

The Divergence Theorem says that the integral of a form ω over the boundary of some orientable manifold is equal to the integral of its *exterior derivative* over the whole manifold. For details on the proof and properties of the exterior derivative of differential forms we refer to [89, Chapters 12, 14, 16]. We simply state what is necessary to understand how the theorem is applied in the derivation of the Euler–Lagrange equations in Section 1.2.

Theorem A.29 (Exterior differentiation)**.** *[89, Theorem 14.24] Let $(M, \mathcal{A})$ be a smooth manifold of dimension $n \in \mathbb{N}$. There are unique operators $\mathrm{d} : \Omega^m_{(k)}(M) \to \Omega^{m+1}_{(k-1)}(M)$ for all $m \in \mathbb{N}_0$, $k \in \mathbb{N}$, called exterior differentiation, with the following properties:*

- *The operator d is linear over $\mathbb{R}$.*
- *If $\omega \in \Omega^m_{(k)}(M)$ and $\eta \in \Omega^j_{(k)}(M)$, then*

$$\mathrm{d}\,(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^m \omega \wedge (\mathrm{d}\eta)\,.$$

- *The operator* d *satisfies* $\mathrm{d} \circ \mathrm{d} = 0$.
- *For* $f \in C^1(M;\mathbb{R}) = \Omega^0_{(1)}(M)$, $\mathrm{d}f$ *is the differential of* f *as defined in Remark A.6.*

In any chart $(U,\phi) \in \mathcal{A}$ *the exterior derivative of* $\omega \in \Omega^m_{(k)}(M)$ *is given by*

$$\mathrm{d}\left(\sum_{1 \le i_1 < \dots < i_m \le n} \omega_{i_1,\dots,i_m} \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_m} \right) = \sum_{1 \le i_1 < \dots < i_m \le n} \mathrm{d}\omega_{i_1,\dots,i_m} \wedge \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_m} .$$

Definition A.30 (Divergence operator)**.** Let $(M,\mathcal{A},g)$ be an oriented Riemannian manifold of dimension n and regularity C^k, $k \in \mathbb{N}$, and with volume form ω_g. The *divergence* of a vector field $X \in \mathcal{X}^l(M)$, $l \in \mathbb{N}$, is the $C^{\min\{l-1,k\}}$–function on M defined by

$$\mathrm{div} X := *^{-1}\left(\mathrm{d}\beta(X)\right)$$

where $\beta : \mathcal{X}^l(M) \to \Omega^{n-1}_{(l)}(M)$ is defined by $\beta(X) := X \lrcorner\, \omega_g$.

The divergence operator allows to generalise the concept of the *Laplacian* of twice continuously differentiable functions in Euclidean space to Riemannian manifolds.

Definition A.31 (Laplace–Beltrami operator)**.** Let $(M,\mathcal{A},g)$ be a Riemannian manifold of regularity C^k and dimension n with $k \in \mathbb{N}$. The *Laplacian* of a function $f \in C^2(M;\mathbb{R})$ is defined by

$$\Delta_g f := \mathrm{div}\left(\mathrm{grad} f\right).$$

The operator Δ_g is often referred to as the *Laplace–Beltrami operator.* Given $f \in C^2\left(M;\mathbb{R}^d\right)$ with $d \in \mathbb{N}$, $d \ge 2$, with $f = (f_1,\dots,f_d)$, the expression $\Delta_g f$ is defined to be the vector-valued function $M \to \mathbb{R}^d$ with components $\left(\Delta_g f\right)_i := \Delta_g(f_i)$ for $i \in \{1,\dots,d\}$.

Proposition A.32 (Divergence and Laplacian in local coordinates)**.** *Let* $(M,\mathcal{A},g)$ *be an oriented Riemannian manifold of regularity* C^k *and dimension* n *with* $k \in \mathbb{N}$*, and let* $(U,\phi) \in \mathcal{A}$ *be a chart. The divergence of a* C^1*–vector field* $X = X^i \frac{\partial}{\partial x^i}$ *and the Laplacian applied to* $f \in C^2(M;\mathbb{R})$ *have the local representations*

$$\mathrm{div} X = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(X^j \sqrt{g}\right), \tag{A.6}$$

and

$$\Delta_g f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(g^{ij} \frac{\partial f}{\partial x^i} \sqrt{g}\right),$$

respectively.

Proof. Let ω_g be the Riemannian volume form. On the chart domain U,

$$\mathrm{div} X \omega_g \left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}\right) = \mathrm{div} X \sqrt{g} = \mathrm{d}\left(\beta(X)\right)\left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}\right).$$

Observe that the form $\beta(X)$ has the coordinate representation

$$\beta(X) = \sum_{i=1}^n (-1)^{i-1} X^i \left(\omega_g\right)_{1,\dots,n} \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \dots \wedge \mathrm{d}x^n .$$

Its exterior derivative is given by

$$d\beta(X) = \sum_{i=1}^n \mathrm{d}x^1 \wedge \cdots \mathrm{d}x^{i-1} \wedge \mathrm{d}\left(X^i \sqrt{g}\right) \wedge \mathrm{d}x^{i+1} \wedge \mathrm{d}x^n$$

with component

$$\mathrm{d}\beta(X)\left(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}\right)=\sum_{i=1}^{n}\mathrm{d}\left(X^i\sqrt{g}\right)\left(\frac{\partial}{\partial x^i}\right)=\frac{\partial}{\partial x^i}\left(X^i\sqrt{g}\right).$$

The formula for the Laplacian follows directly combining the formulas (A.5) and (A.6). □

Proposition A.33 (Product rule for the divergence operator)**.** *Let $(M,\mathcal{A},g)$ be a Riemannian manifold of dimension n and regularity C^k, $k\in\mathbb{N}$. For all $X\in\mathcal{X}^1(M)$ and $f\in C^1(M;\mathbb{R})$, fX lies in $\mathcal{X}^1(M)$ with divergence*

$$\mathrm{div}(fX)=f\mathrm{div}X+g\left(\mathrm{grad}f,X\right).$$

Proof. We use the formulas (A.5) and (A.6) in local coordinates. With respect to $(U,\phi)\in\mathcal{A}$, $X=X^i\frac{\partial}{\partial x^i}$, $\mathrm{grad}f=g^{mj}\frac{\partial f}{\partial x^m}\frac{\partial}{\partial x^j}$ and $fX=fX^i\frac{\partial}{\partial x^i}$. Thus,

$$g\left(\mathrm{grad}f,X\right)=g^{mj}\frac{\partial f}{\partial x^m}X^i g\left(\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^i}\right)=g^{mj}\frac{\partial f}{\partial x^m}X^i g_{ij}=\frac{\partial f}{\partial x^i}X^i,$$

$$\mathrm{div}(fX)=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(fX^j\sqrt{g}\right)=\frac{\partial f}{\partial x^j}X^j+f\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(X^j\sqrt{g}\right)=g\left(\mathrm{grad}f,X\right)+f\mathrm{div}X.$$

□

Theorem A.34 (Divergence Theorem)**.** *[89, Theorem 16.11, 16.32] Let (M,g) be a compact oriented Riemannian manifold with (possibly empty) boundary of regularity C^k, $k\in\mathbb{N}$, and dimension n. For any C^1–vector field X on M,*

$$\int_M \mathrm{div}X\,\mathrm{d}V_g=\int_{\partial M} g\left(X,N\right)\,\mathrm{d}V_{\tilde{g}},$$

where N is the outer unit conormal along ∂M and $\tilde{g}$ is the induced Riemannian metric on ∂M.

Proof. In the smooth setting this is shown in [89, Theorem 16.32] as a consequence of Stokes' Theorem [89, Theorem 16.11]. The arguments used are valid in the case that the metric and vector field are of class C^1. □

The Divergence Theorem also holds in the case of non-orientable compact Riemannian manifolds, see [89, Theorem 16.48]. In this case one uses the representation of the divergence of a vector field in local coordinates to define the divergence operator and integration of a function on M with respect to $\mathrm{d}V_g$ is defined with the help of the *Riemannian density*, see [89, Definition 16.45].

The remaining part of the section is devoted to the construction of a covering for a given compact Riemannian manifold, that is "well-behaved" in the sense that for all charts composing the covering, the respective local representations of the metric, its inverse and their derivatives are bounded by uniform constants. This is achieved with the help of *geodesics* that can be seen as "locally shortest paths" on the manifold and are characterised via covariant derivatives with respect to a certain connection on M that is in some sense compatible with the metric g.

Proposition A.35 (Fundamental Lemma of Riemannian Geometry)**.** *Let $(M,\mathcal{A},g)$ be a Riemannian manifold of regularity C^1 with or without boundary. There exists a unique linear connection D on M with the properties that the covariant derivative ∇g of the $(2,0)$-tensor field g is identically equal to zero and in every chart $(U,\phi)\in\mathcal{A}$,*

$$\nabla_i\left(\frac{\partial}{\partial x^j}\right)=\nabla_j\left(\frac{\partial}{\partial x^i}\right).$$

It is called the Levi-Civita connection of g. The Christoffel symbols at p are given by

$$\Gamma_{ij}^{l}(p) := \frac{1}{2}\, g^{ml}(p) \left(\frac{\partial g_{im}}{\partial x^j}_{|p} + \frac{\partial g_{jm}}{\partial x^i}_{|p} - \frac{\partial g_{ij}}{\partial x^m}_{|p} \right)$$

for $i, j, l \in \{1, \ldots, n\}$.

Proof. This is shown in [88, Theorem 5.4] for smooth metrics. The proof is also valid in the case that the metric is of class C^1. The regularity property required in Definition A.8 follows from the local representation of the connection in terms of the Christoffel symbols. □

It is worth mentioning that the local representation of the Laplace–Beltrami operator applied to a function $f \in C^2(M, \mathbb{R})$ in a chart (U, ϕ) can be expressed in terms of the Christoffel symbols via

$$\Delta_g f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(g^{ij} \frac{\partial f}{\partial x^i} \sqrt{g} \right) = g^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^{l} \frac{\partial f}{\partial x^l} \right). \tag{A.7}$$

The following concept of the covariant derivative of a vector field along a curve tracks how the values of the vector field change along the curve. Here the vector field does not have to be defined on an open subset of the manifold - one only needs to know its values along the curve.

We notice that given a smooth function $\gamma : I \to M$ from an open or half-open interval to a smooth manifold M, we obtain a smooth vector field $\gamma' : I \to TM$ via $\gamma'(t) := \mathrm{d}\gamma_t \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$. If t lies in the interior of J, the differential $\mathrm{d}\gamma_t : T_t J \to T_{\gamma(t)} M$ is acting on $f \in C^\infty(M; \mathbb{R})$ via

$$\mathrm{d}\gamma_t \left(\frac{\mathrm{d}}{\mathrm{d}t} \right) f = \frac{\mathrm{d}}{\mathrm{d}\tau} (f \circ \gamma)(t + \tau)_{|\tau=0} = (f \circ \gamma)'(t). \tag{A.8}$$

In the case that J is half–open, J is a smooth one–dimensional manifold with boundary and if t equals the boundary point of J, one observes that $\frac{\mathrm{d}}{\mathrm{d}t} : C^\infty(J; \mathbb{R}) \to \mathbb{R}$ defined by

$$g \mapsto \frac{\mathrm{d}}{\mathrm{d}t} g := \lim_{t_n \to t, t_n \in J} \frac{g(t_n) - g(t)}{t_n - t}$$

defines a derivation and thus (A.8) holds when understood as a one-sided derivative.

Proposition A.36 (Covariant derivative along a curve)**.** *Let $(M, \mathcal{A}, g)$ be a smooth Riemannian manifold of dimension $n \in \mathbb{N}$, $I = (a, b)$ or $I = [a, b)$ with $a, b \in \mathbb{R}$, $a < b$, $c : I \to M$ a smooth curve and $\eta : I \to TM$ a smooth vector field such that $\eta(t) \in T_{c(t)}M$ for all $t \in I$. Given a chart $(U, \phi) \in \mathcal{A}$ and $t \in I$ with $c(t) \in U$ we let $\eta^j(t)$ and $c^j(t)$ denote the components of $\eta(t)$ and $c'(t)$, respectively, with respect to the basis $\frac{\partial}{\partial x^1}_{|c(t)}, \ldots, \frac{\partial}{\partial x^n}_{|c(t)}$ of $T_{c(t)}M$. The covariant derivative of η along c at $t \in I$ defined by*

$$\frac{\nabla \eta}{\mathrm{d}t}(t) := \sum_{k=1}^{n} \left((\eta^k)'(t) + \sum_{i,j=1}^{n} c^i(t) \eta^j(t) \Gamma_{ij}^{k}(c(t)) \right) \frac{\partial}{\partial x^k}_{|c(t)}$$

is independent of the chosen chart (U, ϕ) with $c(t) \in U$. In the case $t = a$ the expression $(\eta^k)'(t)$ should be understood as a one-sided derivative.

Proof. Let $t \in I$ and (U, x), $(V, y) \in \mathcal{A}$ be two charts around $c(t) =: p$. The assertion follows from straightforward computations using the transformation rules of the components

$$\widetilde{\eta}^k(t) = \eta^i(t) \frac{\partial (y^k \circ x^{-1})}{\partial x_i}(x(c(t))),$$

the basis vectors

$$\frac{\partial}{\partial x^i}_{|c(t)} = \frac{\partial(y^j \circ x^{-1})}{\partial x_i}(x(c(t)))\frac{\partial}{\partial y^j}_{|c(t)}$$

and the Christoffel symbols

$$\widetilde{\Gamma}^k_{ij}(p) = \frac{\partial(y^k \circ x^{-1})}{\partial x_l}(x(p))\left(\Gamma^l_{rs}(p)\frac{\partial(x^r \circ y^{-1})}{\partial y_i}(y(p))\frac{\partial(x^s \circ y^{-1})}{\partial y_j}(y(p)) + \frac{\partial^2(x^l \circ y^{-1})}{\partial y_i \partial y_j}(y(p))\right).$$

□

We recall the well-known concept of the exponential map at points in the interior of a smooth Riemannian manifold. It allows to define coordinates that are well adapted to the geometry of the manifold.

Proposition A.37 (Geodesics starting at interior points)**.** *Let $(M, \mathcal{A}, g)$ be a smooth Riemannian manifold. For every interior point $p \in M \setminus \partial M$ and any $\xi \in T_pM$ there exists an open interval I containing 0 and a smooth curve $c : I \to M$ such that $c(0) = p$, $c'(0) = \xi$ and $\frac{\nabla c'}{\mathrm{d}t}(t) = 0$ for all $t \in I$. If $c : I \to M$ and $\widetilde{c} : \widetilde{I} \to M$ are two such curves, then $c = \widetilde{c}$ on $I \cap \widetilde{I}$. The curve c is called the geodesic starting at p in direction ξ.*

Proof. Let $(U, \phi) \in \mathcal{A}$ be an interior chart around $p \in U$ with associated basis vectors $\frac{\partial}{\partial x^i}_{|p}$, $i \in \{1, \dots, n\}$, of T_pM. As the manifold is smooth, the Christoffel symbols $\Gamma^m_{ij} \circ \phi^{-1}$ are Lipschitz continuous on any compact subset $K \subset \phi(U)$. The Theorem of Picard-Lindelöf implies the existence of a smooth function $\gamma = \gamma_\xi : (-\varepsilon, \varepsilon) \to \phi(U) \subset \mathbb{R}^n$, $\varepsilon > 0$, solving the second order ODE system

$$(\gamma^m)''(t) + \sum_{i,j=1}^{n} (\gamma^i)'(t)(\gamma^j)'(t)\Gamma^m_{ij}\left(\phi^{-1}(\gamma(t))\right) = 0\,, \qquad t \in (-\varepsilon, \varepsilon)\,,\ m \in \{1, \dots, n\}\,,$$

$$\gamma'(0) = (\mathrm{d}\phi)_p(\xi)\,,$$
$$\gamma(0) = \phi(p)\,.$$

The curve $c : I \to M$ defined by $c(t) := \phi^{-1}(\gamma(t))$ is smooth and satisfies $c(0) = p$, $c'(t) = \mathrm{d}\phi^{-1}_{\gamma(t)}(\gamma'(t)) = c^i(t)\frac{\partial}{\partial x^i}_{|c(t)}$, $c^i(t) = (\gamma^i)'(t)$, $c^i(0) = \xi^i$ and $\frac{\nabla c'}{\mathrm{d}t} = 0$. Uniqueness of c follows from the uniqueness assertion in the Theorem of Picard–Lindelöf. □

Definition A.38 (Exponential map in interior points)**.** Let $(M, \mathcal{A}, g)$ be a smooth Riemannian manifold. Given $p \in M \setminus \partial M$ and $\xi \in T_pM$ we let c_ξ be the geodesic starting at p in direction ξ and $I_{max}(p, \xi)$ its maximal interval of existence. Further, we define

$$\mathcal{D}_p := \{\xi \in T_pM \,:\, 1 \in I_{max}(p, \xi)\}\,.$$

The map $\exp_p : \mathcal{D}_p \to M$ defined by $\exp_p(\xi) := c_\xi(1)$ is called the *exponential map* in p.

Proposition A.39. *Let $(M, \mathcal{A}, g)$ be a smooth Riemannian manifold and $p \in M \setminus \partial M$. Then $\mathcal{D}_p$ is open,* $\exp_p$ *is smooth on $\mathcal{D}_p$ and for all $\xi \in T_pM$, $t \mapsto \exp_p(t\xi)$ is the geodesic starting in p in direction ξ for t small enough. Moreover, there exists a neighbourhood $\mathcal{V}_p \subset \mathcal{D}_p$ of 0 in T_pM such that*

$$(\exp_p)_{|\mathcal{V}_p} : \mathcal{V}_p \to \exp_p(\mathcal{V}_p) =: U_p$$

is a smooth diffeomorphism.

Proof. The facts that $\mathcal{D}_p$ is open and $\exp_p : \mathcal{D}_p \to M$ is smooth follow from the smooth dependence of solutions to ordinary differential equations on the initial value. Given $p \in M$ and $\xi \in T_pM$, there exists $\varepsilon > 0$ such that for all $t \in (-\varepsilon, \varepsilon)$, $t\xi \in \mathcal{D}_p$. It is straightforward to check that $\tau \mapsto c_\xi(t\tau)$ is the geodesic starting at p in direction $t\xi$ which yields $c_\xi(t\tau) = c_{t\xi}(\tau)$. Thus $\exp_p(t\xi) = c_{t\xi}(1) = c_\xi(t)$ which shows that $t \mapsto \exp_p(t\xi)$ is the geodesic starting at p in direction ξ. It remains to prove that $\exp_p$ is a local diffeomorphism around $0 \in \mathcal{D}_p$. By Proposition A.4 the mapping $\Phi : T_pM \to T_0T_pM$, $\xi \mapsto D\xi_{|0}$ is a linear isomorphism. Given $\xi \in T_pM$ and $f \in C^\infty(M;\mathbb{R})$ we observe

$$\begin{aligned}(\mathrm{d}\exp_p)_{|0}(\Phi(\xi))f &= \Phi(\xi)\left(f \circ \exp_p\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(f \circ \exp_p\right)(t\xi)_{|t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(f(\exp_p(t\xi))\right)_{|t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(f(c_\xi(t))\right)_{|t=0} = (\mathrm{d}c_\xi)_{|0}\left(\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\right) f = \xi f\end{aligned}$$

as $t \mapsto \exp_p(t\xi) = c_\xi(t)$ is the geodesic starting at p in direction ξ satisfying

$$\begin{aligned}(\mathrm{d}c_\xi)_{|0}\left(\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\right) &= \mathrm{d}\left(\phi^{-1} \circ \gamma_\xi\right)_{|0}\left(\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\right) = \mathrm{d}\phi^{-1}_{\phi(p)}\left((\mathrm{d}\gamma_\xi)_{|0}\left(\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\right)\right) \\ &= \mathrm{d}\phi^{-1}_{\phi(p)}\left(\gamma_\xi'(0)\right) = \xi\,.\end{aligned}$$

Here (U, ϕ) is a chart around p and $c_\xi = \phi^{-1} \circ \gamma_\xi$ with γ_ξ as in Proposition A.37. Moreover, we identified $\mathbb{R}^n$ and $T_{\phi(p)}\mathbb{R}^n$ via $v \mapsto (f \mapsto (Df)(\phi(p))v)$. This shows that $(\mathrm{d}\exp_p)_{|0} = \mathrm{id}_{T_pM} \circ \Phi^{-1}$ which implies that $(\mathrm{d}\exp_p)_{|0}$ is a linear isomorphism. The claim then follows from the Inverse Function Theorem. □

In the case of boundary points it is only possible to consider curves that point in the interior of the manifold.

Proposition A.40 (Geodesics starting at boundary points)**.** *Let $(M, \mathcal{A}, g)$ be a smooth Riemannian manifold. Given a boundary point $p \in \partial M$ and a tangent vector $\xi \in T_pM^+$ there exists an interval $I = [0, a)$ with $a > 0$ and a smooth curve $c : I \to M$ such that $c(0) = p$, $c'(0) = \xi$ and $\frac{\nabla c'}{\mathrm{d}t}(t) = 0$ for all $t \in I$. If $c : I \to M$ and $\widetilde{c} : \widetilde{I} \to M$ are two such curves, then $c = \widetilde{c}$ on $I \cap \widetilde{I}$. The curve c is called the geodesic starting at p in direction ξ.*

Proof. Let $(U, \phi) \in \mathcal{A}$ be a boundary chart with $p \in U$ and denote by $\frac{\partial}{\partial x^i}_{|q}$, $i \in \{1, \dots, n\}$, $q \in U$, the associated tangent vectors. As the manifold is smooth, the Christoffel symbols $\Gamma^k_{ij} \circ \phi^{-1}$ are smooth on the set $\phi(U)$ that is open in $\mathbb{H}^n$. Thus there exist smooth functions h^k_{ij} defined on an open ball $B_\delta(\phi(p))$ in $\mathbb{R}^n$ such that on $B_\delta(\phi(p)) \cap \mathbb{H}^n \subset \phi(U)$ it holds $\Gamma^k_{ij} \circ \phi^{-1} = h^k_{ij}$. We may choose the ball so small that the h^k_{ij} are Lipschitz continuous on $B_\delta(\phi(p))$. By the Theorem of Picard-Lindelöf there exists a smooth function $\gamma : (-\varepsilon, \varepsilon) \to B_\delta(\phi(p))$, $\varepsilon > 0$, solving the second order ODE system

$$\begin{aligned}(\gamma^k)''(t) + \sum_{i,j=1}^{n} (\gamma^i)'(t)(\gamma^j)'(t)h^k_{ij}(\gamma(t)) = 0\,, \qquad & t \in (-\varepsilon, \varepsilon)\,,\ k \in \{1, \dots, n\}\,, \\ \gamma'(0) = (\mathrm{d}\phi)_p(\xi)\,, & \\ \gamma(0) = \phi(p)\,. &\end{aligned}$$

As $\gamma(0) = \phi(p) \in B_\delta(\phi(p)) \cap \partial\mathbb{H}^n$ and

$$\lim_{t \searrow 0} \frac{\gamma^n(t)}{t} = \lim_{t \searrow 0} \frac{\gamma^n(t) - \gamma^n(0)}{t} = (\gamma^n)'(0) = (\mathrm{d}\phi_p(\xi))^n = \xi^n > 0\,,$$

there exists $\boldsymbol{\varepsilon} \in (0, \varepsilon]$ such that for all $t \in (0, \boldsymbol{\varepsilon})$, $\gamma^n(t) > 0$. This shows for all $t \in [0, \boldsymbol{\varepsilon})$ that $\gamma(t) \in B_\delta(\phi(p)) \cap \mathbb{H}^n \subset \phi(U)$. The curve $c : [0, \boldsymbol{\varepsilon}) \to M$ defined by $c(t) := \phi^{-1}(\gamma(t))$ is smooth and satisfies $c(0) = p$, $c'(t) = \mathrm{d}\phi^{-1}_{\gamma(t)}(\gamma'(t)) = c^i(t) \frac{\partial}{\partial x^i}{}_{|c(t)}$, $c^i(t) = (\gamma^i)'(t)$, $c^i(0) = \xi^i$ and $\frac{\nabla c'}{\mathrm{d}t} = 0$ for all $t \in [0, \boldsymbol{\varepsilon})$. To show (local) uniqueness we assume that $\widetilde{c} : [0, a) \to U$ is another curve with $\widetilde{c}(0) = p$, $\widetilde{c}'(0) = \xi$ and $\frac{\nabla \widetilde{c}'}{\mathrm{d}t}(t) = 0$. Then both $\gamma := \phi \circ c$ and $\widetilde{\gamma} := \phi \circ \widetilde{c}$ are solutions to a system of the form

$$\begin{pmatrix} \eta'(t) \\ \eta''(t) \end{pmatrix} = f(\eta(t), \eta'(t)), \qquad t \in I \cap \widetilde{I},$$
$$\begin{pmatrix} \eta(0) \\ \eta'(0) \end{pmatrix} = \begin{pmatrix} \phi(p) \\ \mathrm{d}\phi_p(\xi) \end{pmatrix},$$

with a Lipschitz continuous function f. An easy argument using the Gronwall–Lemma implies $\gamma = \widetilde{\gamma}$ on $I \cap \widetilde{I}$ and thus $c = \widetilde{c}$ on $I \cap \widetilde{I}$. Global uniqueness can be deduced from the preceding observation by considering overlapping chart domains. □

Given a boundary point $p \in \partial M$ of a Riemannian manifold, the tangent vector $0 \in T_pM$ is not inward-pointing. Nevertheless we may define the *geodesic starting at* p *in direction* 0 by $c(t) := p$ for all $t \in \mathbb{R}$.

Definition A.41 (Exponential map in boundary points)**.** Let $(M, \mathcal{A}, g)$ be a smooth Riemannian manifold. Let $p \in \partial M$ and consider a tangent vector ξ such that $\xi \in T_pM^+ \cup \{0\}$ for some chart $(U, \phi) \in \mathcal{A}$ around p. We let $c_\xi : I_{max}(p, \xi) \to M$ be the geodesic starting in p in direction ξ defined on its maximal time interval of existence $I_{max}(p, \xi)$ and

$$\mathcal{D}_p := \left\{ \xi \in T_pM^+ \cup \{0\} \,:\, 1 \in I_{max}(p, \xi) \right\}.$$

The *exponential map* in p is defined by $\exp_p : \mathcal{D}_p \to M$, $\xi \mapsto c_\xi(1)$.

In the following we adapt the definitions and arguments given in [12, 72, 131] to construct a "nice" covering for compact manifolds with boundary.

Proposition A.42 (Normal collar coordinates)**.** *Let $(M, \mathcal{A}, g)$ be a smooth Riemannian manifold with boundary. Let $\exp^M$ and $\exp^{\partial M}$ denote the exponential maps of M and ∂M, respectively, where ∂M is endowed with the metric induced by g. We denote by N the outward-pointing unit normal field along ∂M. Given $p \in \partial M$ there exists $r(p) > 0$ and a neighbourhood $\mathcal{W}_p$ of 0 in $T_p\partial M$ such that*

$$\kappa_p : \mathcal{W}_p \times [0, r(p)) \to M, \qquad (v, t) \mapsto \exp^M_{\exp_p^{\partial M}(v)}(-tN(\exp_p^{\partial M}(v)))$$

is a smooth diffeomorphism onto its image.

Proof. Let $p \in \partial M$ be given. The construction of the exponential map in Proposition A.37 and A.40 and the smooth dependence of solutions to ODEs on the initial values imply that there exist a neighbourhood $\mathcal{W}$ of 0 in $T_p\partial M$ and a radius $r > 0$ such that

$$\kappa_p : \mathcal{W} \times [0, r) \to M, \qquad (v, t) \mapsto \exp^M_{\exp_p^{\partial M}(v)}(-tN(\exp_p^{\partial M}(v)))$$

is well-defined and smooth. The claim follows from the Inverse Function Theorem once we have shown that

$$(\mathrm{d}\kappa_p)_{|(0,0)} : T_{(0,0)}\left(\mathcal{W} \times [0, r)\right) \to T_pM$$

is a linear isomorphism. Denoting by ι_1 and ι_2 the canonical embeddings of $\mathcal{W}$ and $[0,r)$ into $\mathcal{W} \times [0,r)$, respectively, one observes that

$$\Psi : T_0\mathcal{W} \oplus T_0[0,r) \to T_{(0,0)}\left(\mathcal{W} \times [0,r)\right), \quad \Psi((z_1,z_2)) := (\mathrm{d}\iota_1)_{|0}(z_1) + (\mathrm{d}\iota_2)_{|0}(z_2)$$

defines a linear isomorphism. Furthermore, we let $\Phi_1 : T_p\partial M \to T_0T_p\partial M$, $\xi \mapsto D\xi_{|0}$ and $\Phi_2 : \mathbb{R} \to T_0\mathbb{R}$, $t \mapsto Dt_{|0}$ be the canonical isomorphisms, see Proposition A.4. We identify $T_0\mathbb{R}$ with $T_0[0,r)$ and $T_0\mathcal{W}$ with $T_0T_p\partial M$, respectively. Given $\xi \in T_p\partial M$ and $t \in \mathbb{R}$ we obtain

$$(\mathrm{d}\kappa_p)_{|(0,0)}\left(\Psi((\Phi_1(\xi),\Phi_2(t)))\right) = (\mathrm{d}\kappa_p)_{|(0,0)}\left((\mathrm{d}\iota_1)_{|0}(\Phi_1(\xi))\right) + (\mathrm{d}\kappa_p)_{|(0,0)}\left((\mathrm{d}\iota_2)_{|0}(\Phi_2(t))\right).$$

These derivations act on $f \in C^\infty(M;\mathbb{R})$ via

$$\begin{aligned}(\mathrm{d}\kappa_p)_{|(0,0)}\left((\mathrm{d}\iota_1)_{|0}(\Phi_1(\xi))\right) f &= (\mathrm{d}\iota_1)_{|0}(\Phi_1(\xi))\,(f \circ \kappa_p) = \Phi_1(\xi)\,(f \circ \kappa_p \circ \iota_1) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\,(f \circ \kappa_p \circ \iota_1)\,(t\xi)_{|t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\big(f\big(\exp^M_{\exp_p^{\partial M}(t\xi)}(0)\big)\big)_{|t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(f\left(\exp_p^{\partial M}(t\xi)\right)\right)_{|t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f(c_\xi(t))_{|t=0}\end{aligned}$$

with $c_\xi : I(p,\xi) \to \partial M$ denoting the geodesic in ∂M starting in p in direction ξ. Using the canonical embeddings $\iota : \partial M \to M$, $\mathrm{d}\iota_p : T_p\partial M \to T_pM$ we observe

$$\frac{\mathrm{d}}{\mathrm{d}t} f(c_\xi(t))_{|t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f(\iota(c_\xi(t)))_{|t=0} = \mathrm{d}(\iota \circ c_\xi)_{|0}\left(\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\right) f = \mathrm{d}\iota_p\left((\mathrm{d}c_\xi)_{|0}\left(\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\right)\right) f = \mathrm{d}\iota_p(\xi) f\,.$$

Similarly, using Proposition A.39 we obtain

$$(\mathrm{d}\kappa_p)_{|(0,0)}\left((\mathrm{d}\iota_2)_{|0}(\Phi_2(t))\right) f = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(f\left(\exp_p^M(-\tau t N(p))\right)\right)_{|\tau=0} = -tN(p)f$$

which implies

$$(\mathrm{d}\kappa_p)_{|(0,0)}\left(\Psi(\Phi_1(\xi),\Phi_2(t))\right) = \mathrm{d}\iota_p(\xi) - tN(p)\,.$$

As $T_pM \cong \mathrm{d}\iota_p\left(T_p\partial M\right) \oplus \operatorname{span}\{N(p)\}$, this shows that $(\mathrm{d}\kappa_p)_{|(0,0)}$ is an isomorphism. □

Proposition A.43 (Normal coordinates)**.** *Let $(M,\mathcal{A},g)$ be a smooth Riemannian manifold, $Q > 1$ and $p \in M$ be given. There exists a chart $(U_p,\phi_p) \in \mathcal{A}$ such that in this chart, $\phi_p(p) = 0$, $g_{ij}(p) = g^{ij}(p) = \delta_{ij}$ and for all $v \in \mathbb{R}^n$ and all $q \in U_p$,*

$$Q^{-1}|v|^2 \le \sum_{i,j=1}^n g_{ij}(q)v^iv^j \le Q|v|^2\,, \tag{A.9}$$

$$Q^{-1}|v|^2 \le \sum_{i,j=1}^n g^{ij}(q)v^iv^j \le Q|v|^2\,. \tag{A.10}$$

Furthermore, all derivatives of $g_{ij} \circ \phi_p^{-1}$ and $g^{ij} \circ \phi_p^{-1}$ are uniformly bounded on $\phi_p(U_p)$. We refer to the chart (U_p,ϕ_p) as normal coordinate chart around p.

Proof. Given $p \in M$ we construct a chart $(U_p,\phi_p) \in \mathcal{A}$ with $\phi_p(p) = 0$ and $g_{ij}(p) = \delta_{ij}$ (which then implies $g^{ij}(p) = \delta_{ij}$). A continuity argument then yields the estimates (A.9) and (A.10) on a possibly smaller chart domain. Indeed, let $q \in U_p$ be given. As the matrix $G(q) := \left(g_{ij}(q)\right)_{ij}$ is symmetric and positive definite, the expressions $\max_{v\in\mathbb{R}^n,|v|=1}\langle G(q)v,v\rangle$ and $\min_{v\in\mathbb{R}^n,|v|=1}\langle G(q)v,v\rangle$ are precisely equal to its biggest and its smallest eigenvalues, $\lambda_{max}(q)$ and $\lambda_{min}(q)$, respectively. As the eigenvalues of $G(q)$ depend continuously on its entries, the mappings $q \mapsto \lambda_{max}(q)$ and

$q \mapsto \lambda_{min}(q)$ are continuous on U_p. Let $Q > 1$ be fixed. Since $\lambda_{max}(p) = \lambda_{min}(p) = 1$ there exists a neighbourhood $\mathfrak{U}_p \subset U_p$ of p such that for all $q \in \mathfrak{U}_p$,

$$Q^{-1} \leq \lambda_{min}(q) \leq \lambda_{max}(q) \leq Q$$

which implies for $v \in \mathbb{R}^n \setminus \{0\}$,

$$Q^{-1}|v|^2 \leq |v|^2\lambda_{min}(q) \leq \langle G(q)v, v\rangle = |v|^2 \left\langle G(q)\frac{v}{|v|}, \frac{v}{|v|}\right\rangle \leq |v|^2\lambda_{max}(q) \leq Q|v|^2 .$$

This shows the desired estimate. The estimate for G^{-1} follows analogously by possibly further decreasing the size of the chart domain. The derivatives of $g_{ij} \circ \phi_p^{-1}$ and $g^{ij} \circ \phi_p^{-1}$ are continuous on $\phi_p(U_p)$ and thus uniformly bounded on any compact subset of $\phi_p(U_p)$. We thus choose $\mathfrak{U}_p \subset U_p$ such that the closure of $\phi_p(\mathfrak{U}_p)$ is contained in $\phi_p(U_p)$.
It now remains to construct a chart $(U_p, \phi_p) \in \mathcal{A}$ with $g_{ij}(p) = \delta_{ij}$. In the case that p is an interior point, we let $E_1, \ldots, E_n$ be an orthonormal basis of T_pM and obtain a linear isomorphism

$$A_p : \mathbb{R}^n \to T_pM\,, \qquad (\alpha_1, \ldots, \alpha_n) \mapsto \sum_{i=1}^n \alpha^i E_i\,.$$

By Proposition A.39 there exists an open neighbourhood $\mathcal{V}_p$ of 0 in T_pM such that the exponential map $\exp_p : \mathcal{V}_p \to U_p := \exp_p(\mathcal{V}_p)$ is a smooth diffeomorphism. Thus $\phi_p := \left(\exp_p \circ A_p\right)^{-1}$ yields a chart $(U_p, \phi_p) \in \mathcal{A}$ with $\phi_p(p) = 0$ and, identifying $\mathbb{R}^n$ and $T_0\mathbb{R}^n$ via $v \mapsto Dv_{|0}$, we find

$$(\mathrm{d}\phi_p^{-1})_{|0} = (\mathrm{d}\exp_p)_{|0} \circ \Phi \circ A_p$$

where $\Phi : T_pM \to T_0T_pM$, $\xi \mapsto D\xi_{|0}$ is the canonical isomorphism. This implies

$$\begin{aligned} g_{ij}(p) &= g_p\left((\mathrm{d}\phi_p^{-1})_{|0}(e_i), (\mathrm{d}\phi_p^{-1})_{|0}(e_j)\right) = g_p\left((\mathrm{d}\exp_p)_{|0}(\Phi(E_i)), (\mathrm{d}\exp_p)_{|0}(\Phi(E_j))\right) \\ &= g_p(E_i, E_j) = \delta_{ij}\,, \end{aligned}$$

where we used the formula $(\mathrm{d}\exp_p)_{|0} \circ \Phi = \mathrm{id}_{T_pM}$ shown in the proof of Proposition A.39. In the case that p is a boundary point, let $E_1, \ldots, E_{n-1}$ be an orthonormal basis of $T_p\partial M$ and define a linear isomorphism

$$A_p : \mathbb{R}^{n-1} \times [0, \infty) \to T_p\partial M \times [0, \infty)\,, \quad (\alpha, t) \mapsto \left(\sum_{i=1}^{n-1} \alpha^i E_i, t\right).$$

Observe that for any vector $(v, t) \in \mathbb{R}^{n-1} \times [0, \infty) \cong T_{(0,0)}\left(\mathbb{R}^{n-1} \times [0, \infty)\right)$ it holds

$$(\mathrm{d}A_p)_{|0}(v', t) = (\mathrm{d}A_p)_{|0}\big(\left(D_{(v',t)}\right)_{|0}\big) = \left(D_{A_p(v',t)}\right)_{|0} = (\Phi \circ A_p)\,(v', t)$$

where we denote by Φ the canonical isomorphism

$$\Phi : T_p\partial M \times [0, \infty) \to T_{(0,0)}\left(T_p\partial M \times [0, \infty)\right), \qquad (\alpha, t) \mapsto \left(D_{(\alpha,t)}\right)_{|0}\,.$$

By Proposition A.42 there exist $r(p) > 0$ and a neighbourhood $\mathcal{W}_p$ of 0 in $T_p\partial M$ such that $\kappa_p : \mathcal{W}_p \times [0, r(p)) \to M$ is a smooth diffeomorphism onto its image $U_p := \kappa_p\left(\mathcal{W}_p \times [0, r(p))\right)$. Setting $\phi_p^{-1} := (\kappa_p \circ A_p)_{|A_p^{-1}(\mathcal{W}_p \times [0, r(p)))}$ we obtain a boundary chart $(U_p, \phi_p) \in \mathcal{A}$ with $\phi_p^{-1}(0) = \kappa_p(0, 0) = p$ and

$$(\mathrm{d}\phi_p^{-1})_{|0} = (\mathrm{d}\kappa_p)_{|(0,0)} \circ (\mathrm{d}A_p)_{|0} = (\mathrm{d}\kappa_p)_{|(0,0)} \circ \Phi \circ A_p\,.$$

As in the proof of Proposition A.42 we let ι_1 and ι_2 denote the embeddings of $\mathcal{W}_p$ and $[0, r(p))$ into $\mathcal{W}_p \times [0, r(p))$, respectively, and we let $\Phi_1 : T_p\partial M \to T_0 T_p\partial M$ and $\Phi_2 : \mathbb{R} \to T_0\mathbb{R}$ be the canonical isomorphisms. Then it is straightforward to check that

$$\Phi((E_i, 0)) = (\mathrm{d}\iota_1)_{|0} (\Phi_1(E_i)) , \qquad \Phi((0, 1)) = (\mathrm{d}\iota_2)_{|0} (\Phi_2(1)) .$$

Denoting by $\iota : \partial M \to M$ the natural embedding we obtain for $i \in \{1, \ldots, n-1\}$, using the properties of $(\mathrm{d}\kappa_p)_{|(0,0)}$ shown in the proof of Proposition A.42,

$$(\mathrm{d}\phi_p^{-1})_{|0}(e_i) = (\mathrm{d}\kappa_p)_{|(0,0)} (\Phi((E_i, 0))) = (\mathrm{d}\kappa_p)_{|(0,0)} \big((\mathrm{d}\iota_1)_{|0} (\Phi_1(E_i))\big) = \mathrm{d}\iota_p(E_i)$$

and similarly

$$(\mathrm{d}\phi_p^{-1})_{|0}(e_n) = (\mathrm{d}\kappa_p)_{|(0,0)} (\Phi((0, 1))) = (\mathrm{d}\kappa_p)_{|(0,0)} \big((\mathrm{d}\iota_2)_{|0} (\Phi_2(1))\big) = -N(p) .$$

This shows the claim. □

Proposition A.44 (Normal covering)**.** *Let $(M, \mathcal{A}, g)$ be a smooth compact Riemannian manifold. Given $Q > 1$ there exists a finite covering $\mathcal{T} \subset \mathcal{A}$ of M such that for all charts $(U, \phi) \in \mathcal{T}$, all derivatives of $g_{ij} \circ \phi^{-1}$ and $g^{ij} \circ \phi^{-1}$ are uniformly bounded on $\overline{\phi(U)}$ and for all $v \in \mathbb{R}^n$ and $q \in U$,*

$$Q^{-1}|v|^2 \le \sum_{i,j=1}^{n} g_{ij}(q)v^i v^j \le Q|v|^2 , \tag{A.11}$$

$$Q^{-1}|v|^2 \le \sum_{i,j=1}^{n} g^{ij}(q)v^i v^j \le Q|v|^2 . \tag{A.12}$$

A covering with these properties is called normal covering of the manifold.

The covering can be chosen in such a way that for all $f \in C^0(M; \mathbb{R})$ and $(U, \phi) \in \mathcal{T}$, the function $f \circ \phi^{-1}$ is uniformly bounded on the closure of $\phi(U)$ in the Euclidean topology of $\mathbb{R}^n$. In this case we call $\mathcal{T}$ a *bounded normal covering* of the manifold.

Proof. Let $Q > 1$ be fixed. Given $p \in M$ we let (U_p, ϕ_p) be a normal coordinate chart around p as constructed in Proposition A.43. We may assume that there exists a chart $(\mathcal{U}_p, \phi_p) \in \mathcal{A}$ such that U_p is compactly contained in $\mathcal{U}_p$, otherwise we decrease the size of U_p and define a new chart by restricting ϕ to the smaller set. As the manifold is compact, it can be covered by finitely many normal coordinate charts $(U_{p_\alpha}, \phi_{p_\alpha})$, $\alpha \in \{1, \ldots, N\}$, which yield the desired covering $\mathcal{T}$. For any continuous function f on M and $(U_p, \phi_p) \in \mathcal{T}$ we let $(\mathcal{U}_p, \phi_p) \in \mathcal{A}$ be such that U_p is compactly contained in $\mathcal{U}_p$. Then $f \circ \phi_p^{-1}$ is continuous on $\phi_p(\mathcal{U}_p)$ and thus bounded on the compact set $\phi_p\left(\overline{U}_p\right)$. □

It is readily checked that the covering can be chosen in such a way that in the case of an interior chart, $\phi(U) \subset \operatorname{int}\mathbb{H}^n$ is a domain with smooth boundary as defined in Definition B.16. The boundary charts (U, ϕ) in $\mathcal{T}$ can be constructed in such a way that $\phi(U) \cap \operatorname{int}\mathbb{H}^n$ is a domain with smooth boundary. This can be achieved by replacing U by a smaller set V with the property that $\phi(V) \cap \operatorname{int}\mathbb{H}^n$ is a smooth domain with $\partial\phi(V) \cap \partial\mathbb{H}^n = \partial\phi(U) \cap \partial\mathbb{H}^n \cap \overline{V}$ which is illustrated in Figure A.1. In this case we say that the covering has *smooth chart domains*. If M is an oriented manifold with dimension $n \ge 2$, one can modify the covering in such a way that it consists only of positively oriented charts. This can be achieved by changing the sign of one of the first $n-1$ coordinates of each chart in the covering.

In analogy to [72] we introduce the notion of uniform localisation systems.

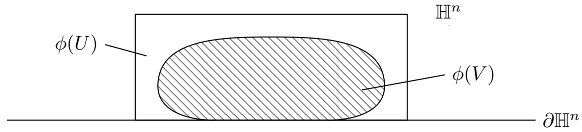

Figure A.1: The construction of smooth chart domains.

Definition A.45 (Uniform localisation system)**.** Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold with boundary. A collection

$$\mathcal{C} := \{(U_\alpha, \phi_\alpha, \psi_\alpha) \,:\, \alpha \in \{1, \ldots, N\}\}$$

is called *uniform localisation system* if the charts (U_α, ϕ_α), $\alpha \in \{1, \ldots, N\}$, are positively oriented and form a bounded normal covering $\mathcal{T}$ of the manifold and if $(\psi_\alpha)_{\alpha\in\{1,\ldots,N\}}$ is a smooth partition of unity subordinate to the covering.

Proposition A.46 (Normal covering of the boundary)**.** *Let $(M, \mathcal{A}, g)$ be a smooth compact Riemannian manifold with boundary. Given $Q > 1$ let $\mathcal{T} = \{(U_\alpha, \phi_\alpha) : \alpha \in \{1, \ldots, N\}\} \subset \mathcal{A}$ be a normal covering of the manifold satisfying the properties in Proposition A.44. Let $\mathcal{J} := \{\alpha \in \{1, \ldots, N\} \,:\, (U_\alpha, \phi_\alpha)$ is a boundary chart$\}$, $\iota : \partial M \to M$ the inclusion mapping and $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection onto the first $n-1$ components. Furthermore, we let $\mathcal{A}_{\partial M}$ and $\tilde{g}$ be the induced smooth structure and Riemannian metric on ∂M as constructed in Proposition A.15 and Proposition A.26, respectively. Then the charts $(U_\alpha \cap \partial M, \pi \circ \phi_\alpha \circ \iota) \in \mathcal{A}_{\partial M}$, $\alpha \in \mathcal{J}$, form a normal covering of ∂M.*

Proof. Proposition A.15 implies that the charts $(U_\alpha \cap \partial M, \pi \circ \phi_\alpha \circ \iota)$, $\alpha \in \mathcal{J}$, lie in the induced structure $\mathcal{A}_{\partial M}$. Let $\alpha \in \mathcal{J}$ be given and let $\widetilde{g}_{ij}$ and g_{ij} be the local representations of $\tilde{g}$ and g with respect to the charts $(U_\alpha \cap \partial M, \pi \circ \phi_\alpha \circ \iota)$ and (U_α, ϕ_α), respectively. By Proposition A.26 the induced metric $\tilde{g}$ satisfies for all $i, j \in \{1, \ldots, n-1\}$, $\widetilde{g}_{ij} = g_{ij} \circ \iota$. This implies in particular for all $q \in U_\alpha \cap \partial M$ and $\mathfrak{v} \in \mathbb{R}^{n-1}$ the estimates

$$Q^{-1}|\mathfrak{v}|^2 \leq \sum_{i,j=1}^{n-1} \widetilde{g}_{ij}(q)\mathfrak{v}^i\mathfrak{v}^j \leq Q|\mathfrak{v}|^2\,, \tag{A.13}$$

$$Q^{-1}|\mathfrak{v}|^2 \leq \sum_{i,j=1}^{n-1} \widetilde{g}^{ij}(q)\mathfrak{v}^i\mathfrak{v}^j \leq Q|\mathfrak{v}|^2 \tag{A.14}$$

using (A.11) and (A.12) for the manifold M with the choice $v := (\mathfrak{v}, 0) \in \mathbb{R}^n$. Furthermore, the functions $\widetilde{g}_{ij} \circ (\pi \circ \phi_\alpha \circ \iota)^{-1}$ and $\widetilde{g}^{ij} \circ (\pi \circ \phi_\alpha \circ \iota)^{-1}$ are uniformly bounded on the closure of $\pi\left(\phi_\alpha\left(U_\alpha \cap \partial M\right)\right)$. ☐

Appendix B

Function spaces on domains

This chapter provides the background knowledge on the function spaces that occur in this thesis. In Section B.1 we recall spaces of Banach space valued differentiable functions and Hölder spaces and some of their properties. Vector-valued Sobolev and Slobodeckij spaces and their relations to interpolation and Besov spaces are studied in Section B.2. In Section B.3 we define anisotropic Sobolev spaces as intersections of vector-valued Sobolev (Slobodeckij) spaces. We show among other important properties that this notion is equivalent to the definition given in [136, §20] where the author considers closures of smooth functions with respect to appropriate norms.

B.1 Hölder spaces and parabolic Hölder spaces

Let Ω be a domain in $\mathbb{R}^n$ and X be a Banach space. The space of continuous functions $f : \Omega \to X$ is denoted by $C^0(\Omega; X)$ or $C(\Omega; X)$. For $k \in \mathbb{N}$ we define

$$\begin{aligned} C^k(\Omega; X) &:= \{f : \Omega \to X : \partial^\alpha f \in C(\Omega; X) \text{ for all } |\alpha| \leq k\} , \\ C^k_b(\Omega; X) &:= \left\{f \in C^k(\Omega; X) : \partial^\alpha f \text{ is bounded on } \Omega \text{ for all } |\alpha| \leq k\right\} , \\ C^k_{buc}(\Omega; X) &:= \left\{f \in C^k_b(\Omega; X) : \partial^\alpha f \text{ is uniformly continuous on } \Omega \text{ for all } |\alpha| \leq k\right\} \end{aligned}$$

where

$$\partial^\alpha f := D^\alpha f := \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n} f := \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n}}$$

for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| := \sum_{i=1}^n \alpha_i$ and $\partial_{x_j} f : \Omega \to X$ is defined as $\partial_{x_j} f(x) := \lim\limits_{t \to 0} \frac{f(x+te_j)-f(x)}{t} \in X$. The spaces $C^k_b(\Omega; X)$ and $C^k_{buc}(\Omega; X)$ are Banach spaces in the norm

$$\|f\|_{C^k_b(\Omega;X)} := \max_{0 \leq |\alpha| \leq k} \max_{x \in \Omega} \|\partial^\alpha f(x)\|_X .$$

One easily sees that if a function f lies in $C^k_{buc}(\Omega; X)$, then $\partial^\alpha f$ possesses a unique bounded continuous extension to $\overline{\Omega}$ for all $|\alpha| \leq k$. In the case that Ω is bounded it holds that

$$C^k_{buc}(\Omega; X) = \left\{f \in C^k_b(\Omega; X) : \text{ all } \partial^\alpha f \text{ possess a unique continuous bounded extension to } \overline{\Omega}\right\} .$$

The latter space is denoted by $C^k(\overline{\Omega}; X)$. If Ω is a bounded domain we write $\|f\|_{C^k(\overline{\Omega};X)}$ instead of $\|f\|_{C^k_b(\Omega;X)}$ for all $f \in C^k(\overline{\Omega}; X)$.
A function $g : \Omega \to X$ is λ–*Hölder continuous* for $\lambda \in (0, 1]$ if there exists a constant K such that for all $x, y \in \Omega$,

$$\|g(x) - g(y)\|_X \leq K|x - y|^\lambda .$$

For $k \in \mathbb{N}_0$ the space

$$C^{k,\lambda}(\overline{\Omega}; X) := C^{k,\lambda}_{buc}(\Omega; X) := \left\{f \in C^k_{buc}(\Omega; X) : \partial^\alpha f \text{ is } \lambda\text{-Hölder continuous for all } |\alpha| \leq k\right\}$$

is a Banach space in the norm

$$\|f\|_{C^{k,\lambda}(\bar{\Omega};X)} := \|f\|_{C^{k,\lambda}_{buc}(\Omega;X)} := \|f\|_{C^k_b(\Omega;X)} + \max_{0\le|\alpha|\le k} \sup_{x,y\in\Omega, x\neq y} \frac{\|\partial^\alpha f(x) - \partial^\alpha f(y)\|_X}{|x-y|^\lambda} .$$

In the case $k=0$ we usually write $C^\lambda\left(\overline{\Omega};X\right) := C^{0,\lambda}\left(\overline{\Omega};X\right)$.
In the following we collect some results on Hölder spaces that are needed throughout the thesis.

Proposition B.1. *Let T be positive and X_0, X_1, Y be Banach spaces with $X_0 \cap X_1 \subset Y$. Suppose that there exist $\sigma\in(0,1)$ and a constant $C>0$ such that for all $y\in X_0\cap X_1$ there holds*

$$\|y\|_Y \le C\,\|y\|_{X_0}^{1-\sigma}\,\|y\|_{X_1}^{\sigma} .$$

Then for all $\beta\in(0,1)$ the continuous embedding

$$C([0,T];X_1)\cap C^\beta([0,T];X_0) \hookrightarrow C^{(1-\sigma)\beta}([0,T];Y)$$

is valid with embedding constant independent of T.

Proof. Young's inequality implies for all $y\in X_0\cap X_1$,

$$\|y\|_Y \le C\,\|y\|_{X_0}^{1-\sigma}\,\|y\|_{X_1}^{\sigma} \le C\left(\|y\|_{X_0}+\|y\|_{X_1}\right) = C\,\|y\|_{X_0\cap X_1}$$

which shows that the inclusion $\iota: X_0\cap X_1 \hookrightarrow Y$ is continuous. Given

$$f\in C([0,T];X_1)\cap C^\beta([0,T];X_0) \subset C([0,T];X_0\cap X_1)$$

there holds in particular $f\in C([0,T];Y)$ and

$$\sup_{t\in[0,T]}\|f(t)\|_Y \le C\left(\|f\|_{C([0,T];X_0)}+\|f\|_{C([0,T];X_1)}\right).$$

Furthermore, we find for all $s,t\in[0,T]$ with $s\neq t$,

$$\begin{aligned}\frac{\|f(t)-f(s)\|_Y}{|t-s|^{(1-\sigma)\beta}} &\le C\frac{\|f(t)-f(s)\|_{X_0}^{1-\sigma}}{|t-s|^{(1-\sigma)\beta}}\,\|f(t)-f(s)\|_{X_1}^{\sigma} \le C\,\|f\|_{C^\beta([0,T];X_0)}^{1-\sigma}\,\|f\|_{C([0,T];X_1)}^{\sigma}\\ &\le C\left(\|f\|_{C^\beta([0,T];X_0)}+\|f\|_{C([0,T];X_1)}\right).\end{aligned}$$

This shows $f\in C^\beta([0,T];Y)$ and the desired estimate. □

Proposition B.2. *Let T be positive, X be a Banach space and $\alpha,\beta\in(0,1)$ with $\alpha<\beta$ be given. Then*

$$C^\beta([0,T];X)\hookrightarrow C^\alpha([0,T];X)$$

with continuous embedding and for all $f\in C^\beta([0,T];X)$ it holds

$$\|f\|_{C^\alpha([0,T];X)} \le \max\left\{1,T^{\beta-\alpha}\right\}\|f\|_{C^\beta([0,T];X)} .$$

Moreover, for all $f\in C_0^\beta([0,T];X) := \left\{g\in C^\beta([0,T];X) : g(0)=0\right\}$ we have

$$\|f\|_{C^\alpha([0,T];X)} \le 2T^{\beta-\alpha}\max\{T^\alpha,1\}\,\|f\|_{C^\beta([0,T];X)} .$$

Proof. Given $s,t\in[0,T]$, $f\in C^\beta([0,T];X)$ we have

$$\|f(t)-f(s)\|_X \le |t-s|^\beta\,\|f\|_{C^\beta([0,T];X)} \le |t-s|^\alpha\,T^{\beta-\alpha}\,\|f\|_{C^\beta([0,T];X)} .$$

As further

$$\sup_{t\in[0,T]} \|f(t)\|_X \le \|f\|_{C^\beta([0,T];X)}$$

we obtain $f \in C^\alpha([0,T];X)$ and the desired estimate. Given $f \in C_0^\beta([0,T];X)$ we have for any $t \in [0,T]$,

$$\|f(t)\|_X = \|f(t) - f(0)\|_X \le t^\beta \|f\|_{C^\beta([0,T];X)} \le T^\beta \|f\|_{C^\beta([0,T];X)} .$$

This completes the proof. □

Proposition B.3. *Let $\alpha \in (0,1)$, $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded domain and T be positive. The Banach space $C^\alpha\left([0,T];C(\overline{\Omega})\right)$ is a Banach algebra. Moreover, given $d \in \mathbb{N}$, $U \subset \mathbb{R}^d$ an open set, $F \in C^2\left(U;C\left(\overline{\Omega};\mathbb{R}\right)\right)$ and $f \in C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}^d\right)\right)$ with $f\left([0,T]\times\overline{\Omega}\right) \subset K$ for some compact convex subset $K \subset U$, the composition $(t,x) \mapsto (F\circ f)(t,x) = F(f(t,x),x)$ lies in $C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}\right)\right)$ and satisfies the estimate*

$$[F\circ f]_{C^\alpha([0,T];C(\overline{\Omega}))} \le \|F\|_{C^1(K;C(\overline{\Omega};\mathbb{R}))} \|f\|_{C^\alpha([0,T];C(\overline{\Omega};\mathbb{R}^d))} .$$

If g is another function in $C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}^d\right)\right)$ with $g\left([0,T]\times\overline{\Omega}\right) \subset K$, it holds

$$\begin{aligned}&\|F\circ f - F\circ g\|_{C^\alpha([0,T];C(\overline{\Omega};\mathbb{R}))}\\ \le& 4\max\left\{1, [f]_{C^\alpha([0,T];C(\overline{\Omega};\mathbb{R}^d))} + [g]_{C^\alpha([0,T];C(\overline{\Omega};\mathbb{R}^d))}\right\} \|F\|_{C^2(K;C(\overline{\Omega};\mathbb{R}))} \|f-g\|_{C^\alpha([0,T];C(\overline{\Omega};\mathbb{R}^d))} .\end{aligned}$$

Proof. It is straightforward to check that $C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}\right)\right)$ is a Banach algebra. Given F and f as in the statement the function $(t,x) \mapsto F(f(t,x),x)$ is continuous on the compact set $[0,T]\times\overline{\Omega}$ and thus $F\circ f \in C\left([0,T];C\left(\overline{\Omega};\mathbb{R}\right)\right)$. Moreover, given $t,s \in [0,T]$, $x \in \overline{\Omega}$, there holds due to convexity and compactness of K,

$$|(F\circ f)(t,x) - (F\circ f)(s,x)| \le \|F\|_{C^1(K;C(\overline{\Omega};\mathbb{R}))} [f]_{C^\alpha([0,T];C(\overline{\Omega};\mathbb{R}^d))} |t-s|^\alpha$$

which yields $F\circ f \in C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}\right)\right)$. Given another function $g \in C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}^d\right)\right)$ with $g\left([0,T]\times\overline{\Omega}\right) \subset K$, we let for $t \in [0,T]$, $x \in \overline{\Omega}$,

$$H(t,x) := \int_0^1 (DF)\left(\tau f(t,x) + (1-\tau)g(t,x), x\right) \mathrm{d}\tau .$$

Then H lies in $C\left([0,T];C\left(\overline{\Omega};\mathbb{R}^{1\times d}\right)\right)$ with $\|H\|_{C([0,T];C(\overline{\Omega};\mathbb{R}^{1\times d}))} \le C(d)\|F\|_{C^1(K;C(\overline{\Omega};\mathbb{R}))}$. As the function $DF \in C^1\left(U;C\left(\overline{\Omega};\mathbb{R}^{1\times d}\right)\right)$ is Lipschitz continuous on the compact convex set K with Lipschitz constant $0 \le L \le \|F\|_{C^2(K;C(\overline{\Omega};\mathbb{R}))}$, we obtain

$$\begin{aligned}&\sup_{s,t\in[0,T]}\sup_{x\in\overline{\Omega}} |H(t,x) - H(s,x)|\,|t-s|^{-\alpha}\\ \le& \|F\|_{C^2(K;C(\overline{\Omega};\mathbb{R}))} \sup_{s,t\in[0,T]}\sup_{x\in\overline{\Omega}} |t-s|^{-\alpha} \int_0^1 \tau|f(t,x)-f(s,x)| + (1-\tau)|g(t,x)-g(s,x)|\mathrm{d}\tau\\ \le& \|F\|_{C^2(K;C(\overline{\Omega};\mathbb{R}))} \left([f]_{C^\alpha([0,T];C(\overline{\Omega};\mathbb{R}^d))} + [g]_{C^\alpha([0,T];C(\overline{\Omega};\mathbb{R}^d))}\right)\end{aligned}$$

and hence $H \in C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}^d\right)\right)$. In particular, using the identity

$$(F\circ f)(t,x) - (F\circ g)(t,x) = H(t,x)\left(f(t,x) - g(t,x)\right),$$

we conclude that

$$\|F\circ f - F\circ g\|_{C([0,T];C(\overline{\Omega}))} \le \|H\|_{C([0,T];C(\overline{\Omega};\mathbb{R}^{1\times d}))} \|f-g\|_{C([0,T];C(\overline{\Omega};\mathbb{R}^d))}$$

and

$$\begin{aligned}
&[F \circ f - F \circ g]_{C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}\right)\right)} \\
&= \sup_{s,t\in[0,T]} \sup_{x\in\overline{\Omega}} |t-s|^{-\alpha} \left| H(t,x)\left(f(t,x)-g(t,x)\right) - H(s,x)\left(f(s,x)-g(s,x)\right)\right| \\
&\leq \|H\|_{C\left([0,T];C\left(\overline{\Omega};\mathbb{R}^{1\times d}\right)\right)} [f-g]_{C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}^d\right)\right)} + [H]_{C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}^{1\times d}\right)\right)} \|f-g\|_{C\left([0,T];C\left(\overline{\Omega};\mathbb{R}^d\right)\right)} .
\end{aligned}$$

□

Proposition B.4. *Let $\alpha \in (0,1)$, $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded domain and T be positive. Suppose that $f = (f_1, \dots, f_d) \in C^\alpha\left([0,T]; C\left(\overline{\Omega};\mathbb{R}^d\right)\right)$ satisfies $\min_{t\in[0,T],x\in\overline{\Omega}} |f(t,x)| \geq \sigma$ for a constant $\sigma > 0$. Then the function*

$$(t,x) \mapsto \|f\|^{-1}(t,x) := \|f(t,x)\|^{-1} = \|f(t,x)\|_2^{-1} = \left(\sum_{i=1}^d |f_i(t,x)|^2\right)^{-\frac{1}{2}}$$

lies in $C^\alpha\left([0,T];C(\overline{\Omega})\right)$ with

$$\left\| \|f\|^{-1} \right\|_{C^\alpha\left([0,T];C\left(\overline{\Omega}\right)\right)} \leq C(\sigma) \|f\|_{C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}^d\right)\right)} .$$

Moreover, given $f, g \in C^\alpha\left([0,T]; C\left(\overline{\Omega};\mathbb{R}^d\right)\right)$ with $\min_{t\in[0,T],x\in\overline{\Omega}} \{|f(t,x)|, |g(t,x)|\} \geq \sigma$ for a constant $\sigma > 0$, there holds

$$\begin{aligned}
&\left\| \|f\|^{-1} - \|g\|^{-1} \right\|_{C^\alpha\left([0,T];C\left(\overline{\Omega}\right)\right)} \\
\leq &C\left(\sigma, \|f\|_{C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}^d\right)\right)}, \|g\|_{C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}^d\right)\right)}\right) \|f-g\|_{C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}^d\right)\right)} .
\end{aligned}$$

The analogous result holds true if $C\left(\overline{\Omega};\mathbb{R}^d\right)$ is replaced by $\mathbb{R}^d$.

Proof. The function $(t,x) \mapsto \|f(t,x)\|^{-1}$ is continuous on $[0,T] \times \overline{\Omega}$ being the composition of continuous functions, thus $\|f\|^{-1} \in C\left([0,T];C(\overline{\Omega})\right)$. Given $s,t \in [0,T]$ and $x \in \overline{\Omega}$, we obtain

$$\left| \frac{1}{\|f(t,x)\|} - \frac{1}{\|f(s,x)\|} \right| = \frac{\left| \|f(s,x)\| - \|f(t,x)\| \right|}{\|f(t,x)\| \, \|f(s,x)\|} \leq \frac{\|f(t,x) - f(s,x)\|}{\|f(t,x)\| \, \|f(s,x)\|} \leq C(\sigma) \|f(t,x) - f(s,x)\| ,$$

which implies in particular

$$\left\| \|f\|^{-1} \right\|_{C^\alpha\left([0,T];C\left(\overline{\Omega}\right)\right)} \leq C(\sigma) \|f\|_{C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}^d\right)\right)} .$$

Given $f, g \in C^\alpha\left([0,T]; C\left(\overline{\Omega}\right)\right)$ with $\min_{t\in[0,T],x\in\overline{\Omega}} \{|f(t,x)|, |g(t,x)|\} \geq \sigma$ for a constant $\sigma > 0$, we let $F : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that on $\mathbb{R}^d \setminus B_\sigma(0)$ there holds $F(y) = \|y\|^{-1}$. (One may consider $F(y) := \left(1 - \eta\left(\frac{\|y\|^2}{\sigma^2}\right)\right) \|y\|^{-1}$ for a function $\eta \in C^\infty(\mathbb{R};\mathbb{R})$ with $\eta \equiv 0$ on $[1,\infty)$ and $\eta \equiv 1$ on $(-\infty, 1/2]$). Let further $R > 0$ be such that $|f(t,x)|$, $|g(t,x)| \leq R$ for all $t \in [0,T]$ and $x \in \overline{\Omega}$. Proposition B.3 applied to $F \in C^\infty\left(\mathbb{R}^d;\mathbb{R}\right)$ and $K := \overline{B_R(0)}$ yields

$$\begin{aligned}
&\left\| \|f\|^{-1} - \|g\|^{-1} \right\|_{C^\alpha\left([0,T];C\left(\overline{\Omega}\right)\right)} = \|F(f) - F(g)\|_{C^\alpha\left([0,T];C\left(\overline{\Omega}\right)\right)} \\
&\leq C\left(\sigma, R, \|f\|_{C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}^d\right)\right)}, \|g\|_{C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}^d\right)\right)}\right) \|f-g\|_{C^\alpha\left([0,T];C\left(\overline{\Omega};\mathbb{R}^d\right)\right)} .
\end{aligned}$$

This shows the claim as one may choose $R := \max\{\|f\|_\infty, \|g\|_\infty\}$. □

Proposition B.5. *Let $\beta \in (0,1)$, $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded domain and T be positive. The Banach space $C\left([0,T];C^\beta(\overline{\Omega})\right)$ is a Banach algebra. Moreover, given $d \in \mathbb{N}$, $U \subset \mathbb{R}^d$ an open set, $F \in C^2(U;\mathbb{R})$ and $f \in C\left([0,T];C^\beta\left(\overline{\Omega};\mathbb{R}^d\right)\right)$ with $f\left([0,T]\times\overline{\Omega}\right) \subset K$ for some compact convex set $K \subset U$, the composition $(t,x) \mapsto (F\circ f)(t,x) = F(f(t,x))$ lies in $C\left([0,T];C^\beta\left(\overline{\Omega};\mathbb{R}\right)\right)$. In particular, given $f = (f_1,\dots,f_d) \in C\left([0,T];C^\beta\left(\overline{\Omega};\mathbb{R}^d\right)\right)$ with $\min_{t\in[0,T],x\in\overline{\Omega}}|f(t,x)| \geq \sigma$ for a constant $\sigma > 0$, the function*

$$(t,x) \mapsto \|f\|^{-1}(t,x) := \|f(t,x)\|^{-1} = \|f(t,x)\|_2^{-1} = \left(\sum_{i=1}^d |f_i(t,x)|^2\right)^{-\frac{1}{2}}$$

lies in $C\left([0,T];C^\beta(\overline{\Omega})\right)$.

Proof. One readily verifies that $C\left([0,T];C^\beta\left(\overline{\Omega};\mathbb{R}^d\right)\right)$ is a Banach algebra. Given F and f as in the statement, the composition $F\circ f$ is continuous on the compact set $[0,T]\times\overline{\Omega}$ which yields $F\circ f \in C\left([0,T];C(\overline{\Omega})\right)$. Given $t \in [0,T]$ and $x\,,y \in \overline{\Omega}$ we have due to convexity and compactness of the set K,

$$|F(f(t,x)) - F(f(t,y))| \leq \|DF\|_{C^1(K;\mathbb{R})}\,|f(t,x)-f(t,y)|$$

which yields in particular $(F\circ f)(t) \in C^\beta(\overline{\Omega})$. Furthermore, we observe for $x \in \overline{\Omega}$ and $t\,,s \in [0,T]$ the identity

$$F(f(t,x)) - F(f(s,x)) = H(t,s,x)\,(f(t,x)-f(s,x))$$

where the function H is given by

$$H(t,s,x) = \int_0^1 (DF)\,(\tau f(t,x) + (1-\tau)\,f(s,x))\,\mathrm{d}\tau\,.$$

As DF is Lipschitz continuous on K with constant $0 \leq L \leq \|F\|_{C^2(K;\mathbb{R})}$, we have for $t\,,s \in [0,T]$,

$$\begin{aligned}
&\sup_{x,y\in\overline{\Omega}} |H(t,s,x)-H(t,s,y)|\,|x-y|^{-\beta} \\
&\leq \|F\|_{C^2(K;\mathbb{R})}\int_0^1 \tau\,|f(t,x)-f(t,y)| + (1-\tau)\,|f(s,x)-f(s,y)|\,\mathrm{d}\tau\,|x-y|^{-\beta} \\
&\leq \|F\|_{C^2(K;\mathbb{R})}\left(\|f(t)\|_{C^\beta(\overline{\Omega};\mathbb{R}^d)} + \|f(s)\|_{C^\beta(\overline{\Omega};\mathbb{R}^d)}\right) \leq 2\,\|F\|_{C^2(K;\mathbb{R})}\,\|f\|_{C\left([0,T];C^\beta\left(\overline{\Omega};\mathbb{R}^d\right)\right)}\,.
\end{aligned}$$

This yields for $t\,,s \in [0,T]$,

$$\begin{aligned}
&\sup_{x,y\in\overline{\Omega}} |F(f(t,x)) - F(f(s,x)) - (F(f(t,y)) - F(f(s,y)))|\,|x-y|^{-\beta} \\
&= \sup_{x,y\in\overline{\Omega}} |H(t,s,x)\,(f(t,x)-f(s,x)) - H(t,s,y)\,(f(t,y)-f(s,y))|\,|x-y|^{-\beta} \\
&\leq \sup_{x,y\in\overline{\Omega}} |H(t,s,x)|\,|f(t,x)-f(s,x)-(f(t,y)-f(s,y))|\,|x-y|^{-\beta} \\
&\quad + \sup_{x,y\in\overline{\Omega}} |H(t,s,x)-H(t,s,y)|\,|x-y|^{-\beta}\,|f(t,y)-f(s,y)| \\
&\leq \|DF\|_{C^1(K;\mathbb{R})}\,\|f(t)-f(s)\|_{C^\beta(\overline{\Omega};\mathbb{R}^d)} + 2\,\|F\|_{C^2(K;\mathbb{R})}\,\|f\|_{C\left([0,T];C^\beta\left(\overline{\Omega};\mathbb{R}^d\right)\right)}\,\|f(t)-f(s)\|_{C(\overline{\Omega};\mathbb{R}^d)}\,.
\end{aligned}$$

As f lies in $C\left([0,T];C^\beta\left(\overline{\Omega};\mathbb{R}^d\right)\right)$, this shows $F\circ f \in C\left([0,T];C^\beta\left(\overline{\Omega};\mathbb{R}\right)\right)$.

The additional statement on $\|f\|^{-1}$ for a given function $f \in C\left([0,T];C^\beta\left(\overline{\Omega};\mathbb{R}^d\right)\right)$ uniformly bounded from below by a constant $\sigma > 0$ follows considering $F \in C^\infty\left(\mathbb{R}^d;\mathbb{R}\right)$ with $F(y) = \|y\|^{-1}$ on $\mathbb{R}^d \setminus B_\sigma(0)$ as in Proposition B.4. □

Lemma B.6 (Extension operator in Hölder spaces)**.** *Let $T_0 > 0$ be given, X be a Banach space and $\beta \in [0,1)$. Then there exists a constant $C(T_0) > 0$ such that for any $T \in (0, T_0/2]$,*

$$E_{T_0} : C^\beta([0,T];X) \to C^\beta([0,T_0];X)$$
$$(E_{T_0} f)(t) := \begin{cases} f(t), & t \in [0,T], \\ f\left(T\frac{T_0-t}{T_0-T}\right), & t \in (T,T_0], \end{cases}$$

defines a continuous linear operator with

$$\|E_{T_0} f\|_{C^\beta([0,T_0];X)} \le C(T_0) \|f\|_{C^\beta([0,T];X)} .$$

Proof. Let $T \in (0, T_0/2]$ and $f \in C^\beta([0,T];X)$ be given. Then one readily checks that $E_{T_0} f$ lies in $C([0,T_0];X)$. Suppose that $\beta \in (0,1)$ and let $t, s \in [0,T_0]$ be given. In the case $t, s \in [0,T]$ we have

$$\|(E_{T_0} f)(t) - (E_{T_0} f)(s)\|_X = \|f(t) - f(s)\|_X \le \|f\|_{C^\beta([0,T];X)} |t-s|^\beta .$$

If t, s lie in $(T, T_0]$ we obtain

$$\begin{aligned}\|(E_{T_0} f)(t) - (E_{T_0} f)(s)\|_X &= \left\| f\left(T\frac{T_0-t}{T_0-T}\right) - f\left(T\frac{T_0-s}{T_0-T}\right)\right\|_X \\ &\le \|f\|_{C^\beta([0,T];X)} \left(\frac{T}{T_0-T}\right)^\beta |t-s|^\beta \le \|f\|_{C^\beta([0,T];X)} |t-s|^\beta .\end{aligned}$$

as the function $T \mapsto \frac{T}{T_0-T}$ is increasing on $(0, T_0/2]$. In the case that $t \in [0,T]$ and $s \in (T, T_0]$ we have

$$\begin{aligned}&\|(E_{T_0} f)(t) - (E_{T_0} f)(s)\|_X = \left\| f(t) - f\left(T\frac{T_0-s}{T_0-T}\right)\right\|_X \le \|f\|_{C^\beta([0,T];X)} \left| t - T\frac{T_0-s}{T_0-T}\right|^\beta \\ &\le \|f\|_{C^\beta([0,T];X)} |T_0-T|^{-\beta} \left(T|t-s| + T_0|t-T|\right)^\beta \\ &\le \|f\|_{C^\beta([0,T];X)} (T_0/2)^{-\beta} (3T_0)^\beta |t-s|^\beta \le C(T_0) \|f\|_{C^\beta([0,T];X)} |t-s|^\beta .\end{aligned}$$

□

To solve (fourth order) parabolic problems in a Hölder setting we consider the *parabolic Hölder spaces* as solution spaces (see also [136, §11, §13]) which are defined as follows.

Definition B.7. (Parabolic Hölder spaces) Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded domain, T be positive, $d \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $\alpha \in (0,1)$. For $\rho \in (0,1)$ and a function $u : [0,T] \times \overline{\Omega} \to \mathbb{R}^d$ we define the semi-norms

$$[u]_{\rho,0} := \sup_{x \in \overline{\Omega}, \tau, t \in [0,T], \tau \neq t} \frac{|u(t,x) - u(\tau,x)|}{|t-\tau|^\rho},$$

and

$$[u]_{0,\rho} := \sup_{t \in [0,T], x, y \in \overline{\Omega}, x \neq y} \frac{|u(t,x) - u(t,y)|}{|x-y|^\rho}.$$

The *parabolic Hölder space*

$$C^{\frac{k+\alpha}{4}, k+\alpha}\left([0,T] \times \overline{\Omega}; \mathbb{R}^d\right)$$

is the space of all functions $u : [0,T] \times \overline{\Omega} \to \mathbb{R}^d$ that have continuous derivatives $\partial_t^i \partial_x^\beta u$, where $i \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0^n$ are such that $4i + |\beta| \le k$, for which the norm

$$\|u\|_{C^{\frac{k+\alpha}{4}, k+\alpha}} := \sum_{4i+|\beta|=0}^{k} \left\|\partial_t^i \partial_x^\beta u\right\|_\infty + \sum_{4i+|\beta|=k} \left[\partial_t^i \partial_x^\beta u\right]_{0,\alpha} + \sum_{0 < k+\alpha-4i-|\beta| < 4} \left[\partial_t^i \partial_x^\beta u\right]_{\frac{k+\alpha-4i-|\beta|}{4}, 0}$$

is finite where

$$\left\|\partial_t^i\partial_x^\beta u\right\|_\infty := \sup_{t\in[0,T],x\in\overline{\Omega}} \left|\partial_t^i\partial_x^\beta u(t,x)\right| .$$

We notice that

$$C^{\frac{\alpha}{4},\alpha}\left([0,T]\times\overline{\Omega};\mathbb{R}^d\right) = C^{\frac{\alpha}{4}}\left([0,T];C^0\left(\overline{\Omega};\mathbb{R}^d\right)\right)\cap C^0\left([0,T];C^\alpha\left(\overline{\Omega};\mathbb{R}^d\right)\right) \tag{B.1}$$

with equivalent norms.

If there is no ambiguity concerning the domain of definition, we often use the abbreviation

$$\|u\|_{\frac{k+\alpha}{4},k+\alpha} := \|u\|_{C^{\frac{k+\alpha}{4},k+\alpha}} .$$

Proposition B.8. *Let $n\in\mathbb{N}$, $\Omega\subset\mathbb{R}^n$ be a bounded domain, T be positive, $d\in\mathbb{N}$, $k\in\mathbb{N}_0$ and $\alpha\in(0,1)$. Given any function*

$$u\in C^{\frac{k+\alpha}{4},k+\alpha}\left([0,T]\times\overline{\Omega};\mathbb{R}^d\right)$$

there holds for all $4i+|\beta|\leq k$,

$$\partial_t^i\partial_x^\beta u\in C^{\frac{k-4i-|\beta|+\alpha}{4},k-4i-|\beta|+\alpha}\left([0,T]\times\overline{\Omega};\mathbb{R}^d\right)$$

with

$$\left\|\partial_t^i\partial_x^\beta u\right\|_{\frac{k-4i-|\beta|+\alpha}{4},k-4i-|\beta|+\alpha}\leq\|u\|_{\frac{k+\alpha}{4},k+\alpha} .$$

Moreover, given $l\in\mathbb{N}_0$ with $l\leq k$, there holds the continuous embedding

$$C^{\frac{k+\alpha}{4},k+\alpha}\left([0,T]\times\overline{\Omega};\mathbb{R}^d\right)\hookrightarrow C^{\frac{l+\alpha}{4},l+\alpha}\left([0,T]\times\overline{\Omega};\mathbb{R}^d\right)$$

Proof. This is a direct consequence of the definition of parabolic Hölder spaces and Proposition B.2. □

Proposition B.9 (Properties of parabolic Hölder spaces)**.** *Let $n\in\mathbb{N}$, $\Omega\subset\mathbb{R}^n$ be a bounded domain, T be positive, $d\in\mathbb{N}$, $k\in\mathbb{N}_0$, and $\alpha\in(0,1)$. The space $C^{\frac{k+\alpha}{4},k+\alpha}\left([0,T]\times\overline{\Omega};\mathbb{R}\right)$ is a Banach algebra. Given $U\subset\mathbb{R}^d$ an open set, $F\in C^{k+2}\left(U;\mathbb{R}\right)$ and $f\in C^{\frac{k+\alpha}{4},k+\alpha}\left([0,T]\times\overline{\Omega};\mathbb{R}^d\right)$ with $f\left([0,T]\times\overline{\Omega}\right)\subset K$ for some compact convex subset $K\subset U$, the composition $(t,x)\mapsto(F\circ f)(t,x)$ lies in $C^{\frac{k+\alpha}{4},k+\alpha}\left([0,T]\times\overline{\Omega};\mathbb{R}\right)$. In particular, given $f\in C^{\frac{k+\alpha}{4},k+\alpha}\left([0,T]\times\overline{\Omega};\mathbb{R}^d\right)$ with $\min_{t\in[0,T],x\in\overline{\Omega}}|f(t,x)|\geq\sigma$ for a constant $\sigma>0$, the function*

$$(t,x)\mapsto\|f\|^{-1}(t,x) := \left(\sum_{i=1}^d|f_i(t,x)|^2\right)^{-\frac{1}{2}}$$

lies in $C^{\frac{k+\alpha}{4},k+\alpha}\left([0,T]\times\overline{\Omega};\mathbb{R}\right)$.

Proof. The Banach algebra property of $C^{\frac{k+\alpha}{4},k+\alpha}\left([0,T]\times\overline{\Omega};\mathbb{R}\right)$ follows from the product rule for classical derivatives and the fact that $C^\beta\left([0,T];C(\overline{\Omega})\right)$ and $C\left([0,T];C^\beta(\overline{\Omega})\right)$ are Banach algebras for all $\beta\in(0,1)$. Given F and f as in the statement, the function $F\circ f$ possesses continuous derivatives $\partial_t^i\partial_x^\beta(F\circ f)$ for all $i\in\mathbb{N}_0$ and $\beta\in\mathbb{N}_0^n$ with $4i+|\beta|\leq k$. The Hölder regularity of the derivatives $\partial_t^i\partial_x^\beta(F\circ f)$ in time and space as required in Definition B.7 follows from the regularities of F and f and Proposition B.3 and B.5. The statement for $\|f\|^{-1}$ then follows considering $F\in C^\infty\left(\mathbb{R}^d;\mathbb{R}\right)$ with $F(y)=\|y\|^{-1}$ on $\mathbb{R}^d\setminus B_\sigma(0)$ as in Proposition B.4. □

Lemma B.10 (Extension operator in parabolic Hölder spaces)**.** *Let $T_0 > 0$ be given, $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ a bounded domain, $\alpha \in (0,1)$ and $d \in \mathbb{N}$. Then for any $T \in (0, T_0/2]$,*

$$\boldsymbol{E}_{T_0} : C^{\frac{\alpha}{4},\alpha}\left([0,T] \times \overline{\Omega}; \mathbb{R}^d\right) \to C^{\frac{\alpha}{4},\alpha}\left([0,T] \times \overline{\Omega}; \mathbb{R}^d\right)$$

$$\left(\boldsymbol{E}_{T_0} f\right)(t) := \begin{cases} f(t)\,, & t \in [0,T]\,, \\ f\left(T\frac{T_0 - t}{T_0 - T}\right), & t \in (T, T_0]\,, \end{cases}$$

defines a continuous linear operator with

$$\left\|\boldsymbol{E}_{T_0} f\right\|_{C^{\frac{\alpha}{4},\alpha}\left([0,T]\times\overline{\Omega};\mathbb{R}^d\right)} \le C\left(T_0\right) \|f\|_{C^{\frac{\alpha}{4},\alpha}\left([0,T]\times\overline{\Omega};\mathbb{R}^d\right)} .$$

Proof. This is a direct consequence of Lemma B.6 and the identity (B.1). □

B.2 Vector-valued Sobolev and Slobodeckij spaces

The notion of Sobolev functions defined on open subsets of Euclidean space with values in a Banach space relies on the concept of *Bochner integrability.* A detailed introduction can be found in [150]. Unless said otherwise we always consider the Lebesgue measure on the respective subsets of the Euclidean space.

Let $n \in \mathbb{N}$, $M \subset \mathbb{R}^n$ be Lebesgue measurable and X be a Banach space. A function $f : M \to X$ is called *simple* if there are Lebesgue measurable sets $B_k \subset \mathbb{R}^n$ with finite measure $\lambda^n(B_k)$ and elements $a_k \in X$, $k \in \{1, \ldots, N\}$, $N \in \mathbb{N}$, such that for all $x \in M$,

$$f(x) = \sum_{k=1}^{N} a_k \chi_{B_k}(x)\,,$$

where $\chi_{B_k}(x) = 1$ if $x \in B_k$ and 0 else. The *Bochner integral* of f is then given by

$$\int_M f(x)\,\mathrm{d}x := \sum_{k=1}^{N} a_k \lambda^n(B_k)\,.$$

A function $f : M \to X$ is called *strongly measurable* if there exist simple functions $f_n : M \to X$, $n \in \mathbb{N}$ such that for almost every $x \in M$,

$$\lim_{n\to\infty} \|f_n(x) - f(x)\|_X = 0\,.$$

A strongly measurable function $f : M \to X$ is called *Bochner-integrable*, if there exists a sequence of simple functions $f_n : M \to X$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \int_M \|f_n(x) - f(x)\|_X \,\mathrm{d}x = 0\,.$$

In this case the *Bochner integral* of f is defined by

$$\int_M f(x)\,\mathrm{d}x := \lim_{n\to\infty} \int_M f_n(x)\,\mathrm{d}x\,.$$

We notice that if $f : M \to X$ is strongly measurable, then $\|f(\cdot)\|_X : M \to \mathbb{R}$ is Lebesgue measurable. It is a well-known theorem that a strongly measurable function $f : M \to X$ is Bochner-integrable if and only if $\|f(\cdot)\|_X : M \to \mathbb{R}$ is Lebesgue integrable. We state some well-known properties.

Lemma B.11 (Elementary properties of the Bochner integral). *Let $n \in \mathbb{N}$, $M \subset \mathbb{R}^n$ be Lebesgue measurable and X be a Banach space. If $f, g : M \to X$ are Bochner-integrable, then $f+g : M \to X$ is Bochner-integrable and the integral is linear. Furthermore, $\|f(\cdot)\|_X : M \to \mathbb{R}$ is Lebesgue integrable and there holds*

$$\left\| \int_M f(x)\,\mathrm{d}x \right\|_X \leq \int_M \|f(x)\|_X \,\mathrm{d}x\,.$$

If Y is another Banach space and $A \in \mathcal{L}(X, Y)$ is a bounded linear operator, the function $Af : M \to Y$, $Af(x) := A(f(x))$ is Bochner-integrable and

$$A\left(\int_M f(x)\,\mathrm{d}x\right) = \int_M (A(f(x))\,\mathrm{d}x\,.$$

Proof. This follows easily from the definitions. □

For $p \in [1, \infty]$, we define

$$L_p(M; X) := \left\{ f : M \to X \text{ strongly measurable } : \|f\|_{L_p(M;X)} < \infty \right\}$$

where $\|f\|_{L_p(M;X)} := \|\|f(\cdot)\|_X\|_{L_p(M;\mathbb{R})}$. If $\Omega \subset \mathbb{R}^n$ is open, we let

$$L_{1,loc}(\Omega; X) := \left\{ f : \Omega \to X \text{ strongly measurable} : \text{ for all } K \subset \Omega \text{ compact, } f_{|K} \in L_1(K; X) \right\}\,.$$

Given $f \in L_{1,loc}(\Omega; X)$ and $\alpha \in \mathbb{N}^n$ a multi-index, the *α-th distributional derivative* $\partial^\alpha f$ of f is the functional on $C_0^\infty(\Omega; \mathbb{R})$ given by

$$\langle \partial^\alpha f\,, \phi \rangle := (-1)^{|\alpha|} \int_\Omega f(x) \partial^\alpha \phi(x) \mathrm{d}x \qquad \text{for all } \phi \in C_0^\infty(\Omega; \mathbb{R})\,.$$

The distribution $\partial^\alpha f$ is called *regular* if there exists $v \in L_{1,loc}(\Omega; X)$ such that

$$\langle \partial^\alpha f\,, \phi \rangle = \int_\Omega v(x)\phi(x)\mathrm{d}x \qquad \text{for all } \phi \in C_0^\infty(\Omega; \mathbb{R})\,.$$

In this case we write $\partial^\alpha f = v \in L_{1,loc}(\Omega; X)$. The *Sobolev space of order* $m \in \mathbb{N}$ is defined as

$$W_p^m(\Omega; X) := \{ f \in L_p(\Omega; X) : \partial^\alpha f \in L_p(\Omega; X) \text{ for all } \alpha \in \mathbb{N}^n, |\alpha| \leq m \}\,,$$

where $\partial^\alpha f$ is the distributional derivative. The space $W_p^m(\Omega; X)$ is a Banach space in the norm

$$\|f\|_{W_p^m(\Omega;X)} := \begin{cases} \left(\sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L_p(\Omega;X)}^p\right)^{1/p}, & p \in [1, \infty)\,, \\ \max_{|\alpha| \leq m} \|\partial^\alpha f\|_{L_\infty(\Omega;X)}\,, & p = \infty\,. \end{cases} \tag{B.2}$$

It is a well-known result in the literature that in the case $1 \leq p < \infty$, the set

$$W_p^m(\Omega; X) \cap C^m(\Omega; X)$$

is dense in $W_p^m(\Omega; X)$.

Given a non-integer $s > 0$ the *Sobolev-Slobodeckij space* (short: *Slobodeckij space*) $W_p^s(\mathbb{R}^n; X)$ is defined as

$$W_p^s(\mathbb{R}^n; X) := \left\{ f \in W_p^{\lfloor s \rfloor}(\mathbb{R}^n; X) : [\,f\,]_{W_p^s(\mathbb{R}^n;X)} < \infty \right\}$$

where

$$[\,f\,]_{W_p^s(\mathbb{R}^n;X)} := \sum_{|\alpha| = \lfloor s \rfloor} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\|\partial^\alpha f(x) - \partial^\alpha f(y)\|_X^p}{|x-y|^{n+(s-\lfloor s \rfloor)p}} \,\mathrm{d}x\,\mathrm{d}y \right)^{1/p}\,.$$

It is easy to see that the Slobodeckij space $W_p^s(\mathbb{R}^n;X)$ is a Banach space in the norm

$$\|\cdot\|_{W_p^s(\mathbb{R}^n,X)} := \|\cdot\|_{W_p^{\lfloor s\rfloor}(\mathbb{R}^n;X)} + [\,\cdot\,]_{W_p^s(\mathbb{R}^n;X)}\ .$$

In the literature there exist several approaches to define Sobolev spaces $W_p^s(\mathbb{R}^n;X)$ of non-integer order $s>0$. As we intend to use results from a broad range of references, we shall now draw a relation between these different notions.
Besov spaces are usually defined using dyadic spectral decomposition. A detailed characterisation of scalar-valued Besov spaces can be found in [140, Chapters 2,4]. Vector-valued Besov spaces are discussed in [6, 7].
To show that Slobodeckij, Besov and certain real interpolation spaces coincide with equivalent norms, assumptions on the Banach space X are needed in the vector-valued case. More precisely, the Banach space X needs to be a so-called *UMD space*. We refer to [5, Section III.4] and [82, Sections 3,4] and the references therein for the definition and properties of such spaces.
Details on interpolation theory can be found in [95, 140].

Theorem B.12. *Let $n\in\mathbb{N}$, $p\in(1,\infty)$, $s>0$ be non-integer and X be a UMD space. Then the Slobodeckij space $W_p^s(\mathbb{R}^n;X)$, the Besov space $B_{pp}^s(\mathbb{R}^n;X)$ and the real interpolation space*

$$\left(W_p^{\lfloor s\rfloor}(\mathbb{R}^n,X), W_p^{\lfloor s\rfloor+1}(\mathbb{R}^n;X)\right)_{s-\lfloor s\rfloor,p}$$

coincide with equivalent norms.

Proof. The fact that the real interpolation space $\left(W_p^{\lfloor s\rfloor}(\mathbb{R}^n;\mathbb{R}), W_p^{\lfloor s\rfloor+1}(\mathbb{R}^n;\mathbb{R})\right)_{s-\lfloor s\rfloor,p}$ coincides with $B_{pp}^s(\mathbb{R}^n;\mathbb{R})$ with equivalent norms follows combining [140, Theorem 2.3.3] and [140, Theorem 2.4.2/2]. The same property holds true in the vector-valued case which is shown in [8, Theorem 3.7.1]. In [140, Theorem 2.5.1] it is shown that $W_p^s(\mathbb{R}^n;\mathbb{R})$ and $B_{pp}^s(\mathbb{R}^n;\mathbb{R})$ coincide with equivalent norms. For the corresponding property in the vector-valued case we refer to [6, Eq. (5.8)]. □

Remark B.13. We give some further remarks in the context of Theorem B.12.

(i) Some authors define Besov spaces $B_{pp}^s(\mathbb{R}^n;X)$ for non-integer $s>0$ as real interpolation spaces $\left(H_p^{\lfloor s\rfloor}(\mathbb{R}^n,X), H_p^{\lfloor s\rfloor+1}(\mathbb{R}^n;X)\right)_{s-\lfloor s\rfloor,p}$ of *Bessel potential spaces* $H^k(\mathbb{R}^n;X)$, $k\in\mathbb{N}$. For a thorough introduction to Bessel potential spaces we refer to [140] in the scalar-valued case and to [8] in the vector-valued case. However, if X is a UMD space, the Bessel potential space $H_p^k(\mathbb{R}^n;X)$ and the Sobolev space $W_p^k(\mathbb{R}^n;X)$ coincide for $p\in(1,\infty)$ and $k\in\mathbb{N}$ with equivalent norms which follows e.g. from [152, Proposition 3]. In particular, Theorem B.12 yields that the two definitions of Besov spaces lead to the same spaces up to equivalence of norms provided that X is a UMD space.

(ii) In this thesis we use Theorem B.12 only in the cases $X=\mathbb{R}^d$ with $d\in\mathbb{N}$ and $X=W_p^s\left(\Omega;\mathbb{R}^d\right)$ with $d\in\mathbb{N}$, $p\in(1,\infty)$, $s>0$ and an open set $\Omega\subset\mathbb{R}^n$, $n\in\mathbb{N}$. These spaces are all examples of UMD spaces which follows from [5, Theorem 4.5.2] and [7, Example 7.1], respectively. Although some results in the following are true for general Banach spaces X, we will stick to the assumption that X is a UMD space.

Definition B.14 (Slobodeckij spaces on domains)**.** Let $n\in\mathbb{N}$, $p\in[1,\infty)$, $s>0$ be non-integer and X be a UMD space. The *Slobodeckij space* $W_p^s(\Omega;X)$ is defined as the quotient space

$$W_p^s(\Omega;X) := \left\{Rg : g\in W_p^s(\mathbb{R}^n;X)\right\}\ .$$

Here Rg denotes the restriction of the function $g\in W_p^s(\mathbb{R}^n;X)\subset L_p(\mathbb{R}^n;X)$ to the set Ω.

As $W_p^s(\Omega; X)$ is a quotient space, it is a Banach space in the quotient norm

$$\|f\|'_{W_p^s(\Omega;X)} := \inf\left\{\|g\|_{W_p^s(\mathbb{R}^n;X)} : g \in W_p^s(\mathbb{R}^n; X) \text{ such that } Rg = f\right\} .$$

In the case $X = \mathbb{R}$ we write $W_p^s(\Omega) := W_p^s(\Omega; \mathbb{R})$.
Besov spaces on domains are defined correspondingly via restriction of Besov functions on the full space, see [140, Definition 4.2.1]. To prove that Slobodeckij spaces on domains have the analogous properties as the corresponding spaces on the full space, certain regularity assumptions on the underlying domain are needed. These guarantee the existence of extension operators, see [2, Definition 5.17].

Definition B.15 (Total extension operator)**.** Let Ω be a domain in $\mathbb{R}^n$ and X be a Banach space. An operator E mapping functions defined almost everywhere in Ω to functions defined almost everywhere in $\mathbb{R}^n$ is called *total extension operator for* Ω if for every $m \in \mathbb{N}$ and $p \in [1, \infty)$ it holds

$$E \in \mathcal{L}\left(W_p^m(\Omega; X); W_p^m(\mathbb{R}^n; X)\right)$$

and $Eu(x) = u(x)$ for almost every $x \in \Omega$.

Definition B.16 (C^m–domain)**.** Let $m, n \in \mathbb{N}$ be given. A *C^m–domain* is a domain $\Omega \subset \mathbb{R}^n$ such that for each point $x \in \partial\Omega$ there exists an open set $U \subset \mathbb{R}^n$ with $x \in U$ and a function $g \in C^m(U; \mathbb{R}^n)$ mapping U bijectively onto the ball $B_1(0) \subset \mathbb{R}^n$ such that $g^{-1} \in C^m(B_1(0); \mathbb{R}^n)$ and

$$g(U \cap \Omega) = \{y \in B_1(0) : y_n > 0\} .$$

In this case we also say that Ω has *C^m–boundary*. A set $\Omega \subset \mathbb{R}^n$ that is a C^m–domain for all $m \in \mathbb{N}$ is referred to as *smooth domain* or *domain with smooth boundary*, respectively.

We remark that a bounded C^m–domain, $m \in \mathbb{N}$, satisfies the *strong local Lipschitz condition* stated in [2, Definition 4.9], the *uniform C^m–regularity condition* defined in [2, Definition 4.10] and, in the case $m \geq 2$, also the *ordinary Ljapunov conditions* listed in [136, §13].

Theorem B.17. *Let X be a Banach space and Ω a bounded smooth domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Then there exists a total extension operator E for Ω.*

Proof. This is shown in [2, Theorem 5.22]. □

The following result is the analogon to Theorem B.12 for functions defined on domains.

Theorem B.18. *Let $n \in \mathbb{N}$, $p \in (1, \infty)$, $s > 0$ be non-integer, X be a UMD space and $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Then the Slobodeckij space $W_p^s(\Omega; X)$ coincides with the Besov space $B_{pp}^s(\Omega; X)$ and with the real interpolation space*

$$\left(W_p^{\lfloor s \rfloor}(\Omega; X), W_p^{\lfloor s \rfloor+1}(\Omega; X)\right)_{s-\lfloor s \rfloor, p}$$

with equivalent norms.

Proof. Let E be a total extension operator for Ω. For $k \in \{\lfloor s \rfloor, \lfloor s \rfloor+1\}$ we consider the restriction operator $R : W_p^k(\mathbb{R}^n; X) \to W_p^k(\Omega; X)$ defined by $Rf(x) := f_{|\Omega}(x)$. Then R is well-defined, linear, bounded and left-inverse to $E : W_p^k(\Omega; X) \to W_p^k(\mathbb{R}^n; X)$ for $k = \lfloor s \rfloor$ and $k = \lfloor s \rfloor+1$, respectively. By [140, Theorem 1.2.4] and injectivity of E the real interpolation spaces satisfy

$$\left(W_p^{\lfloor s \rfloor}(\Omega; X), W_p^{\lfloor s \rfloor+1}(\Omega; X)\right)_{s-\lfloor s \rfloor, p} = R\left(\left(W_p^{\lfloor s \rfloor}(\mathbb{R}^n; X), W_p^{\lfloor s \rfloor+1}(\mathbb{R}^n; X)\right)_{s-\lfloor s \rfloor, p}\right)$$

with equivalent norms. Theorem B.12 implies that the latter space is equal to $R\left(B^s_{pp}(\mathbb{R}^n;X)\right)$ which is a Banach space in the quotient norm

$$\|g\| := \inf\left\{\|f\|_{B^s_{pp}(\mathbb{R}^n;X)} : f \in B^s_{pp}(\mathbb{R}^n;X) \text{ such that } Rf = g\right\}$$

and by [140, Definition 4.2.1] equal to the Besov space $B^s_{pp}(\Omega;X)$. By Theorem B.12 there further holds the identity

$$B^s_{pp}(\Omega;X) = R\left(B^s_{pp}(\mathbb{R}^n;X)\right) = R\left(W^s_p(\mathbb{R}^n;X)\right) = W^s_p(\Omega;X)$$

and there is an equivalent norm on $B^s_{pp}(\Omega;X)$ given by

$$\|g\|'_{W^s_p(\Omega;X)} := \inf\left\{\|f\|_{W^s_p(\mathbb{R}^n;X)} : f \in W^s_p(\mathbb{R}^n;X) \text{ such that } Rf = g\right\}.$$

This shows the claim. □

Proposition B.19 (Equivalent norms on Slobodeckij spaces)**.** *Let $n \in \mathbb{N}$, $p \in (1,\infty)$, $s > 0$ be non-integer, X be a UMD space and $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Then*

$$\|f\|_{W^s_p(\Omega;X)} := \|f\|_{W^{\lfloor s\rfloor}_p(\Omega;X)} + [\,f\,]_{W^s_p(\Omega;X)}$$

where

$$[\,f\,]_{W^s_p(\Omega;X)} := \sum_{|\alpha|=\lfloor s\rfloor}\left(\int_\Omega\int_\Omega \frac{\|\partial^\alpha f(x) - \partial^\alpha f(y)\|^p_X}{|x-y|^{n+(s-\lfloor s\rfloor)p}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}$$

defines a norm on $W^s_p(\Omega;X)$ that is equivalent to the quotient norm $\|\cdot\|'_{W^s_p(\Omega;X)}$. Furthermore, there holds the identity

$$W^s_p(\Omega,X) = \left\{f \in W^{\lfloor s\rfloor}_p(\Omega;X) : [f]_{W^s_p(\Omega;X)} < \infty\right\}.$$

Proof. In [7, Corollary 4.3] it is shown that the Besov space $B^s_{pp}(\Omega,X) = R\left(B^s_{pp}(\mathbb{R}^n;X)\right)$ coincides with

$$\left\{f \in W^{\lfloor s\rfloor}_p(\Omega;X) : [f]_{W^s_p(\Omega;X)} < \infty\right\}$$

and that $\|\cdot\|_{s,p,\Omega,X}$ defines a norm on $B^s_{pp}(\Omega;X)$ that is equivalent to the quotient norm on $B^s_{pp}(\Omega,X)$. The claim now follows from Theorem B.18. □

In the sequel we always consider the norm $\|\cdot\|_{W^s_p(\Omega;X)}$ on the space $W^s_p(\Omega;X)$. In the case $X = \mathbb{R}$ this norm is also denoted by $\|\cdot\|_{p,s}$ if there is no ambiguity concerning the domain Ω.

The embedding properties of Sobolev spaces $W^m_p(\Omega)$ depend on the regularity properties of the underlying domain $\Omega \subset \mathbb{R}^n$. We state the version of the Sobolev Embedding Theorem that is relevant for us. Details on the proof and sufficient regularity conditions on the domain can be found in [2, Chapter 4].

Theorem B.20 (Sobolev Embedding Theorem)**.** *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\alpha \in (0,1)$ and $s \geq 0$ such that*

$$s - \frac{n}{p} \geq k + \alpha\,.$$

Then, with continuous embedding,

$$W^s_p(\Omega) \hookrightarrow C^{k+\alpha}\left(\overline{\Omega}\right).$$

Proof. This follows from [140, Theorem 4.6.1.(e)]. □

Theorem B.21 (Characterisation of scalar-valued Slobodeckij spaces)**.** *Let $n \in \mathbb{N}$, $p \in (1,\infty)$, $s > 0$ be non-integer and $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Then the set $C^\infty\left(\overline{\Omega}\right)$ is a dense subset of $\left(W_p^s(\Omega), \|\cdot\|_{s,p,\Omega}\right)$.*

Proof. Let E be a total extension operator for Ω and let $C > 0$ be the norm equivalence constant such that for all $f \in W_p^s(\Omega)$,

$$\|f\|_{s,p,\Omega} \le C \|f\|_{W_p^s(\Omega)}^{'}$$

where $\|\cdot\|_{W_p^s(\Omega)}^{'}$ denotes the quotient norm on $W_p^s(\Omega)$. A function $f \in C^\infty\left(\overline{\Omega}\right)$ satisfies $f \in W_p^m(\Omega)$ for all $m \in \mathbb{N}$ and thus, by the property of the extension operator, $Ef \in W_p^m(\mathbb{R}^n)$ for all $m \in \mathbb{N}$. In particular, Theorem B.20 yields that Ef is smooth on $\mathbb{R}^n$. Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be such that $\eta \equiv 1$ on Ω. Then ηEf lies in $C_0^\infty(\mathbb{R}^n) \subset W_p^s(\mathbb{R}^n)$ and thus, by definition of $W_p^s(\Omega)$, the restriction $f = R(\eta Ef)$ is an element of $W_p^s(\Omega)$. This shows that $C^\infty\left(\overline{\Omega}\right)$ is a subset of $W_p^s(\Omega)$. By [140, Theorem 2.3.2(a)] the set $C_0^\infty(\mathbb{R}^n)$ is dense in $B_{pp}^s(\mathbb{R}^n) = W_p^s(\mathbb{R}^n)$. Let $f \in W_p^s(\Omega)$ and $\varepsilon > 0$ be given. Using the definition of $W_p^s(\Omega)$ and the results in Theorem B.12 we find a function $g \in W_p^s(\mathbb{R}^n)$ such that $Rg = f$ and a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\|\varphi - g\|_{W_p^s(\mathbb{R}^n)} < \varepsilon/C$. In particular, $R\varphi \in C^\infty\left(\overline{\Omega}\right) \subset W_p^s(\Omega)$ and

$$\|R\varphi - f\|_{s,p,\Omega} \le C \|R\varphi - f\|_{W_p^s(\Omega)}^{'} = C \|R(\varphi - g)\|_{W_p^s(\Omega)}^{'} \le C \|\varphi - g\|_{W_p^s(\mathbb{R}^n)} < C\varepsilon/C = \varepsilon\,.$$

□

In the remaining part of this section we collect some useful properties of Slobodeckij spaces on intervals.

Corollary B.22 (Equivalent norm for Slobodeckij spaces on intervals)**.** *Let $I \subset \mathbb{R}$ be a bounded open interval with $0 \in \overline{I}$, $p \in (1,\infty)$ and $s > \frac{1}{p}$. Then there is an equivalent norm on $W_p^s(I)$ given by*

$$|||u|||_{W_p^s(I)} := \|u\|_{W_p^s(I)} + |u(0)|\,,$$

where $\|\cdot\|_{W_p^s(I)}$ is the norm defined in Proposition B.19. The equivalence constants depend on the interval I.

Proof. Theorem B.20 implies $W_p^s(I) \hookrightarrow C\left(\overline{I}\right)$ and thus $u(0) \in \mathbb{R}$ with

$$|u(0)| \le \|u\|_{C(\overline{I})} \le C(I)\|u\|_{W_p^s(I)}\,.$$

Thus, $|||\cdot|||_{W_p^s(I)}$ defines a norm on $W_p^s(I)$ which is equivalent to $\|\cdot\|_{W_p^s(I)}$. □

Proposition B.23 (Temporal extension operator)**.** *Let $p \in (1,\infty)$, $T_0 > 0$, $T \in (0, T_0]$ and $s \in (1/p, 1)$. There exists a linear operator*

$$\boldsymbol{E} : W_p^s((0,T)) \to W_p^s((0,T_0))$$

such that for all $h \in W_p^s((0,T))$, $(\boldsymbol{E}h)_{|(0,T)} = h$ and

$$|||\boldsymbol{E}h|||_{W_p^s((0,T_0))} \le C(T_0)\,|||h|||_{W_p^s((0,T))}\,.$$

Proof. Let $T_0 > 0$ and $T \in (0, T_0]$ be given and

$${}_0W_p^s((0,T)) := \left\{f \in W_p^s((0,T)) : f(0) = 0\right\}.$$

One easily sees that

$$
{}_0\boldsymbol{E}f : (0,\infty) \to \mathbb{R}\,, \qquad t \mapsto \begin{cases} f(t)\,, & t \in (0,T)\,, \\ f(2T-t)\,, & t \in (T,2T)\,, \\ 0\,, & t \in (2T,\infty) \end{cases}
$$

defines a continuous linear operator ${}_0\boldsymbol{E} : {}_0W_p^s\left((0,T)\right) \to W_p^s\left((0,\infty)\right)$ with

$$
\|{}_0\boldsymbol{E}f\|_{W_p^s((0,\infty))} \leq 4\,\|f\|_{W_p^s((0,T))}\,.
$$

Given any $h \in W_p^s\left((0,T)\right)$ we observe that $h - h(0) \in {}_0W_p^s\left((0,T)\right)$ with

$$
\|h - h(0)\|_{W_p^s((0,T))} \leq \|h\|_{W_p^s((0,T))} + T^{1/p}\,|h(0)|\,.
$$

Then $\boldsymbol{E} : W_p^s\left((0,T)\right) \to W_p^s\left((0,T_0)\right)$ with $\boldsymbol{E}h := {}_0\boldsymbol{E}\left(h - h(0)\right) + h(0)$ is a continuous linear operator with $\left(\boldsymbol{E}h\right)_{|(0,T)} = h$ and

$$
\begin{aligned}
\|\boldsymbol{E}h\|_{W_p^s((0,T_0))} &\leq \|{}_0\boldsymbol{E}\left(h - h(0)\right)\|_{W_p^s((0,T_0))} + \|h(0)\|_{W_p^s((0,T_0))} \\
&\leq 4\,\|h - h(0)\|_{W_p^s((0,T))} + \left(1 + T_0^{1/p}\right)|h(0)| \\
&\leq 4\,\|h\|_{W_p^s((0,T))} + 4T^{1/p}\,|h(0)| + \left(1 + T_0^{1/p}\right)|h(0)| \leq C(T_0)\,\|h\|_{W_p^s((0,T))}\,.
\end{aligned}
$$

□

Proposition B.24. *Let $I \subset \mathbb{R}$ be a bounded open interval, $p \in (1,\infty)$ and $s \in (0,\infty)$ be such that $s - \frac{1}{p} > 0$. Then $W_p^s(I)$ is a Banach algebra, where the sub-multiplicativity of the norm holds up to a constant.*

Proof. Let $f, g \in W_p^s(I)$ be given. Theorem B.20 yields $W_p^s(I) \hookrightarrow C\left(\overline{I}\right)$. In particular, f and g lie in $C\left(\overline{I}\right)$ and

$$
\|fg\|_{L_p(I)} \leq \|f\|_{C(\overline{I})}\,\|g\|_{L_p(I)} \leq C\,\|f\|_{W_p^s(I)}\,\|g\|_{W_p^s(I)}\,.
$$

Furthermore,

$$
\begin{aligned}
[fg]_{W_p^s(I)}^p &= \int_I\int_I \frac{|f(t)g(t) - f(s)g(s)|^p}{|t-s|^{1+sp}}\,\mathrm{d}s\mathrm{d}t \\
&\leq \int_I\int_I |f(t)|^p\,\frac{|g(t) - g(s)|^p}{|t-s|^{1+sp}} + |g(t)|^p\,\frac{|f(t) - f(s)|^p}{|t-s|^{1+sp}}\,\mathrm{d}s\mathrm{d}t \\
&\leq \|f\|_{C(\overline{I})}^p\,[g]_{W_p^s(I)}^p + \|g\|_{C(\overline{I})}^p\,[f]_{W_p^s(I)}^p \leq 2C^p\,\|f\|_{W_p^s(I)}^p\,\|g\|_{W_p^s(I)}^p\,.
\end{aligned}
$$

□

An analogue to the Sobolev Embedding Theorem holds in the case of vector-valued Sobolev functions defined on an interval.

Theorem B.25. *Let X be a UMD space, $I \subset \mathbb{R}$ be a bounded interval, $s \in (0,1)$ and $p \in (1,\infty)$ such that $s > \frac{1}{p}$. Then the space $W_p^s(I;X)$ is continuously embedded in $C^{s-1/p}\left(\overline{I};X\right)$.*

Proof. This is shown in [133, Corollary 26]. □

B.3 Anisotropic Sobolev spaces

One natural class of function spaces associated with parabolic equations is given by the so called *anisotropic Sobolev spaces.*

Definition B.26 (Anisotropic Sobolev spaces)**.** Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $p \in (1,\infty)$ and $T \in (0,\infty)$. Given $r,s \in [0,\infty)$ the space $W_p^{r,s}((0,T)\times\Omega)$ is defined by the intersection of vector-valued Sobolev (Slobodeckij) spaces

$$W_p^{r,s}((0,T)\times\Omega) := W_p^r((0,T);L_p(\Omega)) \cap L_p\big((0,T);W_p^s(\Omega)\big)$$

with norm

$$\|\cdot\|_{p,r,s} := \|\cdot\|_{W_p^{r,s}((0,T)\times\Omega)} := \|\cdot\|_{W_p^r((0,T);L_p(\Omega))} + \|\cdot\|_{L_p((0,T);W_p^s(\Omega))} \tag{B.3}$$

where vector- and scalar-valued Slobodeckij spaces are always endowed with the norm defined in Proposition B.19.

We remark that the results on vector-valued spaces in the preceding section are valid due to Remark B.13 (ii).

A detailed investigation of interpolation and trace properties of anisotropic fractional Sobolev spaces with temporal weights depending on a parameter $\mu \in (1/p, 1]$ can be found in [107]. In the following, we state the results relevant in the context of this work without weights in the time variable which corresponds to the case $\mu = 1$. We also refer to [8, 11] for a profound study of anisotropic Sobolev spaces without temporal weights.

B.3.1 Density of smooth functions

This subsection is devoted to show that the anisotropic Sobolev spaces defined in [136, §20] coincide with the ones in Definition B.26 with equivalent norms.

We note that due to Remark B.13 (ii), Theorem B.18 and Proposition B.19 the anisotropic spaces defined in [107] coincide with the ones introduced in Definition B.26 with equivalent norms in the case $\mu = 1$.

Proposition B.27. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $p \in (1,\infty)$ and $T \in (0,\infty]$. Let further $\alpha \in (0,2)$ and $\beta \in (0,\infty)$ be non-integer and $\sigma \in [0,1]$ such that $\sigma\alpha$ is non-integer and $(1-\sigma)\beta \in \mathbb{N}_0$. Then the space*

$$W_p^{\alpha}((0,T);L_p(\Omega)) \cap L_p\big((0,T);W_p^{\beta}(\Omega)\big)$$

is continuously embedded into

$$W_p^{\sigma\alpha}\big((0,T);W_p^{(1-\sigma)\beta}(\Omega)\big).$$

In the case that the interval $(0,T)$ is finite, the embedding constant depends on the length T of the interval.

Proof. This is shown in [107, Proposition 3.2]. □

Proposition B.28. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $p \in (1,\infty)$, $T \in (0,\infty]$ and $s \in (0,4)$ be non-integer. Given*

$$u \in W_p^{s/4}((0,T);L_p(\Omega)) \cap L_p\big((0,T);W_p^s(\Omega)\big)$$

and $\nu \in \mathbb{N}_0^n$ *with* $s - |\nu| > 0$*, we have*

$$\partial_x^\nu u \in W_p^{(s-|\nu|)/4}\left((0,T); L_p(\Omega)\right)$$

where $(\partial_x^\nu u)(t)(x) := \left(\partial_x^\nu (u(t))\right)(x)$*. Furthermore, there exists a constant* $C(T) > 0$ *such that*

$$\|\partial_x^\nu u\|_{W_p^{(s-|\nu|)/4}((0,T);L_p(\Omega))} \leq C(T)\,\|u\|_{p,\frac{s}{4},s}\,.$$

Proof. Let $\nu \in \mathbb{N}_0^n$ with $s-|\nu| > 0$ be given. We apply Proposition B.27 in the case $\alpha = s/4 \in (0,2)$, $\beta = s$ and $\sigma = \frac{s-|\nu|}{s}$. The assumptions $s \in (0,4)$ and $s-|\nu| > 0$ imply that α, β are non-integer and $\sigma \in (0,1]$. Furthermore, we have $\sigma\alpha = (s-|\nu|)/4 \in (0,1)$ and $\beta(1-\sigma) = |\nu| \in \mathbb{N}_0$. Thus, given

$$u \in W_p^{s/4}\left((0,T); L_p(\Omega)\right) \cap L_p\left((0,T); W_p^s(\Omega)\right),$$

there holds $u \in W_p^{(s-|\nu|)/4}\big((0,T); W_p^{|\nu|}(\Omega)\big)$ with

$$\|u\|_{W_p^{(s-|\nu|)/4}\left((0,T);W_p^{|\nu|}(\Omega)\right)} \leq C(T)\,\|u\|_{p,\frac{s}{4},s}\,.$$

It is readily verified that $(\partial_x^\nu u)(t)(x) := \left(\partial_x^\nu(u(t))\right)(x)$ defines a function in $W_p^{(s-|\nu|)/4}\left((0,T); L_p(\Omega)\right)$ with

$$\|\partial_x^\nu u\|_{W_p^{(s-|\nu|)/4}((0,T);L_p(\Omega))} \leq \|u\|_{W_p^{(s-|\nu|)/4}\left((0,T);W_p^{|\nu|}(\Omega)\right)}\,.$$

This shows the claim. □

Corollary B.29. *Let* $n \in \mathbb{N}$*,* $\Omega \subset \mathbb{R}^n$ *be a bounded smooth domain,* $p \in (1,\infty)$*,* $T \in (0,\infty]$ *and* $s \in (0,4)$ *be non-integer. Then*

$$\|u\|'_{p,s/4,s} := \|u\|_{W_p^{s/4}((0,T);L_p(\Omega))} + \|u\|_{L_p((0,T);W_p^s(\Omega))} + \sum_{\nu\in\mathbb{N}_0^n,\, s-|\nu|>0} \|\partial_x^\nu u\|_{W_p^{(s-|\nu|)/4}((0,T);L_p(\Omega))}$$

defines a norm on $W_p^{s/4,s}\left((0,T)\times\Omega\right)$ *that is equivalent to* $\|\cdot\|_{p,s/4,s}$ *with equivalence constant depending on* T*.*

Proof. This is a direct consequence of Proposition B.28. □

We make use of the following result for vector-valued Sobolev (Slobodeckij) spaces.

Proposition B.30. *Let* $p \in (1,\infty)$*,* $r \in [0,1]$*,* $T \in (0,\infty)$*,* X_0 *be a UMD space and* X_1 *be a Banach space such that* X_1 *is embedded in* X_0*. Then* $C^\infty\left([0,T]; X_1\right)$ *is dense in the space* $L_p\left((0,T); X_1\right) \cap W_p^r\left((0,T); X_0\right)$*.*

Proof. Let $u \in L_p\left((0,T); X_1\right) \cap W_p^r\left((0,T); X_0\right)$ with $\operatorname{supp} u \subset (0,T]$ be given. In the following we extend u by 0 to $u \in L_p\left((-\infty,T); X_1\right) \cap W_p^r\left((-\infty,T); X_0\right)$. Consider a function $\varphi \in C_0^\infty\left((0,1)\right)$ with $\varphi \geq 0$ and $\int_{\mathbb{R}} \varphi \,\mathrm{d}t = 1$ and set $\varphi_\varepsilon(t) := \frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right)$. Studying the difference quotient $\frac{1}{h}\left(u_\varepsilon(t+h) - u_\varepsilon(t)\right)$ and higher order analogues for $h > 0$ so small that $\operatorname{dist}\left(\operatorname{supp}\varphi_\varepsilon, \{0,\varepsilon\}\right) > h$, it is readily checked that the function

$$t \mapsto u_\varepsilon(t) := \int_0^\varepsilon u(t-s)\varphi_\varepsilon(s)\,\mathrm{d}s$$

lies in $C^\infty\left([0,T]; X_1\right)$ with

$$u'_\varepsilon(t) = \int_{t-\varepsilon}^t u(\tau)\left(\varphi'_\varepsilon\right)(t-\tau)\,\mathrm{d}\tau \;\in X_1\,.$$

In the case that u is in fact continuous on $[0,T]$, straightforward estimates show the convergence $\|u_\varepsilon - u\|_{L_p((0,T);X_1)} \to 0$ as $\varepsilon \to 0$. Using that $C^\infty([0,T];X_1)$ is dense in $L_p((0,T);X_1)$ one proves that $(u_\varepsilon)_{\varepsilon>0}$ converges to u in $L_p((0,T);X_1)$ for general $u \in L_p((0,T);X_1)$ with $\operatorname{supp} u \subset (0,T]$, as the latter satisfy

$$\|u_\varepsilon\|^p_{L_p((0,T);X_1)} \le \int_0^T \int_0^\varepsilon \|u(t-\tau)\|^p_{X_1} \varphi_\varepsilon(\tau)\,\mathrm{d}\tau\,\mathrm{d}t \le \|u\|_{L_p((0,T);X_1)}\,.$$

As X_1 is embedded in X_0, we further have $\|u - u_\varepsilon\|_{L_p((0,T);X_0)} \to 0$ as $\varepsilon \to 0$. In the case $r = 1$ the function u has a derivative $\partial_t u \in L_p((-\infty,T);X_0)$ in the distributional sense and in the space X_0 we thus obtain for all $\varepsilon > 0$ and $t \in [0,T]$ the identity

$$u'_\varepsilon(t) = \int_{t-\varepsilon}^t u(\tau)(\varphi'_\varepsilon)(t-\tau)\,\mathrm{d}\tau = \int_0^\varepsilon (\partial_t u)(t-s)\varphi_\varepsilon(s)\,\mathrm{d}s \ \in X_0\,.$$

Using analogous arguments as for $\|u - u_\varepsilon\|_{L_p((0,T);X_0)}$ we conclude $\|u'_\varepsilon - \partial_t u\|_{L_p((0,T);X_0)} \to 0$ as $\varepsilon \to 0$. It remains to consider the case $r \in (0,1)$. By Proposition B.18 and B.19 the space $W^r_p((0,T);X_0)$ coincides with the interpolation space $\left(L_p((0,T);X_0), W^1_p((0,T);X_0)\right)_{r,p}$ with equivalent norms. Thus, [140, Theorem 1.6.2] yields that $W^1_p((0,T);X_0)$ is densely embedded in $W^r_p((0,T);X_0)$ which implies in particular that $C^\infty([0,T];X_0)$ is dense in $W^r_p((0,T);X_0)$. Let us suppose for the moment that the considered function u lies in $C^\infty([0,T];X_0)$. Then it is readily checked that $\|u - u_\varepsilon\|_{C^1([0,T];X_0)} \to 0$ as $\varepsilon \to 0$ and using the identity

$$\|u_\varepsilon(t) - u(t) - (u_\varepsilon(s) - u(s))\|_{X_0} \le |t-s|\,\|u_\varepsilon - u\|_{C^1([0,T];X_0)}$$

we obtain

$$[u_\varepsilon - u]^p_{W^r_p((0,T);X_0)} \le \int_0^T \int_0^T |t-s|^{-1+(1-r)p}\mathrm{d}t\mathrm{d}s\,\|u_\varepsilon - u\|^p_{C^1([0,T];X_0)} \le C(T)\,\|u_\varepsilon - u\|^p_{C^1([0,T];X_0)}$$

which converges to 0 as $\varepsilon \to 0$. Given a general function u in $L_p((0,T);X_1) \cap W^r_p((0,T);X_0)$ with $\operatorname{supp} u \subset (0,T]$ and $0 < \varepsilon < \operatorname{dist}(\operatorname{supp} u, \{0\}) =: \delta$, there holds

$$\begin{aligned}[u_\varepsilon]^p_{W^r_p((0,T);X_0)} &\le \int_0^\varepsilon \varphi_\varepsilon(\tau) \int_0^T \int_0^T \frac{\|u(t-\tau) - u(s-\tau)\|^p_{X_0}}{|t-s|^{1+rp}}\,\mathrm{d}t\mathrm{d}s\mathrm{d}\tau \\ &\le C(\delta)\tau\,\|u\|^p_{L_p((0,T);X_0)} + [u]^p_{W^r_p((0,T);X_0)} \le C\,\|u\|^p_{W^r_p((0,T);X_0)}\,.\end{aligned}$$

Thus for any $\boldsymbol{\varepsilon} > 0$ we find $v \in C^\infty([0,T];X_0)$ with $\|u - v\|_{W^r_p((0,T);X_0)} < \boldsymbol{\varepsilon}$ and $\varepsilon > 0$ with $\|v - v_\varepsilon\|_{W^r_p((0,T);X_0)} < \boldsymbol{\varepsilon}$, which yields

$$\begin{aligned}\|u - u_\varepsilon\|_{W^r_p((0,T);X_0)} &\le \|u - v\|_{W^r_p((0,T);X_0)} + \|v - v_\varepsilon\|_{W^r_p((0,T);X_0)} + \|v_\varepsilon - u_\varepsilon\|_{W^r_p((0,T);X_0)} \\ &\le \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} + C\|v - u\|_{W^r_p((0,T);X_0)} \le C\boldsymbol{\varepsilon}\,.\end{aligned}$$

This shows the claim in the case $r \in (0,1)$. In the case that the support of $u \in L_p((0,T);X_1) \cap W^r_p((0,T);X_0)$ lies within $[0,T)$, one argues analogously using the approximations

$$t \mapsto u_\varepsilon(t) := \int_0^\varepsilon u(t+s)\varphi_\varepsilon(s)\,\mathrm{d}s\,.$$

Given a general function u in the space $L_p((0,T);X_1) \cap W^r_p((0,T);X_0)$ we consider the decomposition $u = \eta u + (1-\eta)u$ where $\eta \in C^\infty(\mathbb{R})$ is a function that satisfies $\operatorname{supp}\eta \subset (0,\infty)$ and $\eta \equiv 1$ on $[T/2, T]$. □

The following proposition shows that the anisotropic spaces defined in [136, §20] coincide with the ones in Definition B.26 with equivalent norms.

Proposition B.31. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $p \in (1,\infty)$, $T \in (0,\infty)$ and $s \in [0,4]$. The space $W_p^{s/4,s}((0,T)\times\Omega)$ coincides with the closure of $C^\infty\left([0,T];C^\infty\left(\overline{\Omega}\right)\right)$ with respect to the norm $\|\cdot\|_{p,s/4,s}$.*

Proof. Lemma B.30 with the choice $X_1 := W_p^s(\Omega) \hookrightarrow L_p(\Omega) =: X_0$ implies that the space $C^\infty\left([0,T];W_p^s(\Omega)\right)$ is dense in $W_p^{s/4,s}((0,T)\times\Omega)$ with respect to the norm $\|\cdot\|_{p,s/4,s}$. As a consequence, $W_p^1\left((0,T);W_p^s(\Omega)\right)$ is dense in $W_p^{s/4,s}((0,T)\times\Omega)$ with respect to the norm $\|\cdot\|_{p,s/4,s}$. As $C^\infty\left(\overline{\Omega}\right)$ is separable and dense in $W_p^s(\Omega)$, there exist $x_n \in C^\infty\left(\overline{\Omega}\right)$, $n \in \mathbb{N}$, such that $\{x_n : n \in \mathbb{N}\}$ is dense in $W_p^s(\Omega)$ with respect to the norm $\|\cdot\|_{p,s}$. Using properties of vector-valued Sobolev spaces on intervals one shows that

$$\left\{\sum_{n=1}^N x_n\varphi_n \;:\; \varphi_n \in C^\infty([0,T])\,, N \in \mathbb{N}\right\}$$

is dense in $W_p^1\left((0,T);W_p^s(\Omega)\right)$ with respect to the usual norm $\|\cdot\|_{W_p^1\left((0,T);W_p^s(\Omega)\right)}$ defined in (B.2). This yields the claim. □

Corollary B.32. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $p \in (1,\infty)$ and $T \in (0,\infty)$. Given functions $\eta \in C_0^\infty((0,\infty);\mathbb{R})$ and $f \in W_p^{1,4}((0,T)\times\Omega)$, the product ηf lies in $W_p^{1,4}((0,T)\times\Omega)$. Moreover, for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le 4$ and almost every $t \in (0,T)$ there holds $\partial_x^\alpha(\eta(t)f(t)) = \eta(t)\partial_x^\alpha f(t)$ and $\partial_t(\eta(t)f(t)) = \eta'(t)f(t) + \eta(t)\partial_t f(t)$.*

Proof. Proposition B.31 yields that there exists a sequence $f_n \in C^\infty\left([0,T];C^\infty\left(\overline{\Omega}\right)\right)$, $n \in \mathbb{N}$, such that $\|f_n - f\|_{p,1,4} \to 0$ as $n \to \infty$. For every $n \in \mathbb{N}$ there holds $\eta f_n \in C^\infty\left([0,T];C^\infty\left(\overline{\Omega}\right)\right)$ and $(\eta f_n)_{n\in\mathbb{N}}$ is Cauchy with respect to $\|\cdot\|_{p,1,4}$ as for $n\,, m \to \infty$,

$$\|\eta f_n - \eta f_m\|^p_{L_p\left((0,T);W_p^4(\Omega)\right)} \le \|\eta\|^p_{C_0^\infty((0,T))}\|f_n - f_m\|^p_{L_p\left((0,T);W_p^4(\Omega)\right)} \to 0$$

and

$$\begin{aligned}&\|\partial_t(\eta f_n) - \partial_t(\eta f_m)\|^p_{L_p((0,T);L_p(\Omega))}\\&\le \|\eta\|^p_{C_0^\infty((0,T))}\|\partial_t f_n - \partial_t f_m\|^p_{L_p((0,T);L_p(\Omega))} + \|\eta\|^p_{C_0^\infty((0,T))}\|f_n - f_m\|^p_{L_p((0,T);L_p(\Omega))} \to 0\,,\end{aligned}$$

where all derivatives exist in the classical sense. Thus there exists $g \in W_p^{1,4}((0,T)\times\Omega)$ with $\|\eta f_n - g\|_{p,1,4} \to 0$ as $n \to \infty$. Since

$$\|\eta f_n - \eta f\|_{L_p((0,T);L_p(\Omega))} \le \|\eta\|_{C_0^\infty((0,T);\mathbb{R})}\|f_n - f\|_{L_p((0,T);L_p(\Omega))} \to 0$$

as $n \to \infty$, uniqueness of limits in $L_p((0,T);L_p(\Omega))$ yields $g = \eta f$. The differentiation rules follow by taking the limit in the respective formulas for $\eta f_n \in C^\infty\left([0,T];C\left(\overline{\Omega}\right)\right)$. □

Corollary B.33. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $p \in (1,\infty)$ and $T_2 > T_1 > 0$. Given $f \in W_p^{1,4}((0,T_1)\times\Omega)$ and $g \in W_p^{1,4}((T_1,T_2)\times\Omega)$ with $f(T_1) = g(T_2)$ in $L_p(\Omega)$, the function $h(t) := f(t)\chi_{(0,T_1)}(t) + g(t)\chi_{(0,T_2)}(t)$ lies in $W_p^{1,4}((0,T_2)\times\Omega)$.*

Proof. It is straightforward to check that h lies in $L_p\left((0,T_2);W_p^4(\Omega)\right)$. In the following we show $h \in W_p^1((0,T_2);L_p(\Omega))$ with distributional derivative

$$\mathfrak{h} := \partial_t f\chi_{(0,T_1)} + \partial_t g\chi_{(T_1,T_2)} \in L_p((0,T_2);L_p(\Omega))\,.$$

Proposition B.30 yields the existence of $f^n \in C^\infty([0,T_1];L_p(\Omega))$, $g^n \in C^\infty([T_1,T_2];L_p(\Omega))$, $n \in \mathbb{N}$, with $f^n \to f$ in $W_p^1((0,T_1);L_p(\Omega))$ and $g^n \to g \in W_p^1((T_1,T_2);L_p(\Omega))$. Given a test function $\psi \in C_0^\infty((0,T_2))$ we obtain the following identity in $L_p(\Omega)$,

$$\int_0^{T_2} h(t)\psi'(t) - \mathfrak{h}(t)\psi(t)\,\mathrm{d}t = \int_0^{T_1} f(t)\psi'(t) - \partial_t f(t)\psi(t)\,\mathrm{d}t + \int_{T_1}^{T_2} g(t)\psi'(t) - \partial_t g(t)\psi(t)\,\mathrm{d}t$$
$$= \lim_{n\to\infty}\int_0^{T_1} f^n(t)\psi'(t) - (f^n)'(t)\psi(t)\,\mathrm{d}t + \lim_{n\to\infty}\int_{T_1}^{T_2} g^n(t)\psi'(t) - (g^n)'(t)\psi(t)\,\mathrm{d}t$$
$$= \lim_{n\to\infty} f^n(T_1)\psi(T_1) - \lim_{n\to\infty} g^n(T_1)\psi(T_1) = 0\,,$$

where we used that $W_p^1((0,T);L_p(\Omega)) \hookrightarrow C([0,T];L_p(\Omega))$ due to Theorem B.25. □

B.3.2 Extension results and uniform embeddings

In the first part of this subsection we generalise the embedding result [45, Lemma 4.4] for anisotropic spaces on the full space to anisotropic spaces on smooth domains.
However, the corresponding embedding constants depend on the considered time interval and blow up when considering smaller and smaller time intervals.
The contraction estimates in Part I and II crucially rely on embeddings with constants that are independent of the considered time interval. The second part of this subsection shows that this can be achieved by using a certain equivalent norm on the anisotropic spaces.

In the following, we make use of the results shown in [45] which is justified by Theorem B.12 and Remark B.13. To this end, the following extension lemma is needed.

Lemma B.34. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and let E be a total extension operator for Ω. Given $T \in (0,\infty)$, $s \in [0,1]$, $r \in [0,\infty)$ and $p \in [1,\infty)$,*

$$E : W_p^s\left((0,T);W_p^r(\Omega)\right) \to W_p^s\left((0,T);W_p^r(\mathbb{R}^n)\right), \quad (Ef)(t) := E(f(t))$$

defines a continuous linear operator.

Proof. Let $f \in W_p^s\left((0,T);W_p^r(\Omega)\right)$ be given. In particular, $f : (0,T) \to W_p^r(\Omega)$ is Bochner-integrable and as $E \in \mathcal{L}\left(W_p^r(\Omega);W_p^r(\mathbb{R}^n)\right)$, Lemma B.11 implies that $Ef : (0,T) \to W_p^r(\mathbb{R}^n)$ is Bochner-integrable with

$$E\left(\int_0^T f(t)\,\mathrm{d}t\right) = \int_0^T E(f(t))\,\mathrm{d}t = \int_0^T (Ef)(t)\,\mathrm{d}t\,.$$

Furthermore, the estimate $\|E(f(t))\|_{W_p^r(\mathbb{R}^n)} \le C\|f(t)\|_{W_p^r(\Omega)}$ yields $Ef \in L_p\left((0,T);W_p^r(\mathbb{R}^n)\right)$ and if $s \in [0,1)$ even $Ef \in W_p^s\left((0,T);W_p^r(\mathbb{R}^n)\right)$ with

$$\|Ef\|_{W_p^s\left((0,T);W_p^r(\mathbb{R}^n)\right)} \le C\|f\|_{W_p^s\left((0,T);W_p^r(\Omega)\right)}\,.$$

It remains to verify the claim in the case $s = 1$. Suppose that $f \in W_p^1\left((0,T);W_p^r(\Omega)\right)$ and $g \in L_p\left((0,T);W_p^r(\Omega)\right)$ are such that for all $\psi \in C_0^\infty((0,T))$,

$$\int_0^T f(t)\psi'(t)\,\mathrm{d}t = -\int_0^T g(t)\psi(t)\,\mathrm{d}t\,.$$

Applying the extension operator E to this identity we obtain using Lemma B.11 that Eg lies in $L_p\left((0,T);W_p^r(\mathbb{R}^n)\right)$ with $\|Eg\|_{L_p\left((0,T);W_p^r(\mathbb{R}^n)\right)} \le C\|g\|_{L_p\left((0,T);W_p^r(\Omega)\right)}$ and

$$\int_0^T (Ef)\,\psi'(t)\,\mathrm{d}t = -\int_0^T (Eg)(t)\psi(t)\,\mathrm{d}t\,.$$

This shows that E is well-defined. Linearity of E follows from linearity of E on $W_p^r(\Omega)$ and linearity of the integral. The above estimates imply that E is continuous. □

The following proposition generalises [45, Lemma 4.4] to smooth domains.

Proposition B.35. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, T be positive, $p \in [1, \infty)$, $r_1, r_2 \in [0, \infty)$ with $r_2 < r_1$ and $s_1, s_2 \in [0, 1]$ such that*

$$s_1 < 1/p < s_2$$

and let $\gamma := (r_1 - r_2)/(s_2 - s_1)$. Then

$$W_p^{s_2}((0,T); W_p^{r_2}(\Omega)) \cap W_p^{s_1}((0,T); W_p^{r_1}(\Omega)) \hookrightarrow C([0,T]; W_p^{r_1 - \gamma(1/p - s_1)}(\Omega)) .$$

In particular, for $p \in (4+n, \infty)$, there holds

$$W_p^1\left((0,T); L_p(\Omega)\right) \cap L_p\left((0,T); W_p^4(\Omega)\right) \hookrightarrow C([0,T]; W_p^{4-4/p}(\Omega)) \hookrightarrow C\left([0,T]; C^3\left(\overline{\Omega}\right)\right) .$$

The operator norms of the embeddings depend on T.

Proof. In [45, Lemma 4.4] the statement is shown in the case $\Omega = \mathbb{R}^n$. Let E be a total extension operator for Ω. By Lemma B.34, E defines a continuous linear operator

$$E : W_p^{s_2}\left(0,T; W_p^{r_2}(\Omega)\right) \cap W_p^{s_1}\left(0,T; W_p^{r_1}(\Omega)\right) \to W_p^{s_2}\left(0,T; W_p^{r_2}(\mathbb{R}^n)\right) \cap W_p^{s_1}\left(0,T; W_p^{r_1}(\mathbb{R}^n)\right) .$$

By [45, Lemma 4.4] the latter space embeds continuously into $C([0,T]; W_p^{r_1-\gamma(1/p-s_1)}(\mathbb{R}^n))$ via a mapping ι. The restriction operator $R : W_p^{r_1-\gamma(1/p-s_1)}(\mathbb{R}^n) \to W_p^{r_1-\gamma(1/p-s_1)}(\Omega)$ defined by $(Rf)(x) := f(x)$ for $x \in \Omega$, is well-defined, linear and continuous with operator norm bounded by 1 and yields a continuous linear operator

$$R : C([0,T]; W_p^{r_1-\gamma(1/p-s_1)}(\mathbb{R}^n)) \to C([0,T]; W_p^{r_1-\gamma(1/p-s_1)}(\Omega)) , \quad (Rf)(t) := R(f(t)) .$$

The composition $R \circ \iota \circ E$ defines a continuous linear operator

$$R \circ \iota \circ E : W_p^{s_2}\left(0,T; W_p^{r_2}(\Omega)\right) \cap W_p^{s_1}\left(0,T; W_p^{r_1}(\Omega)\right) \to C([0,T]; W_p^{r_1-\gamma(1/p-s_1)}(\Omega))$$

and it is straightforward to check that $(R \circ \iota \circ E)(f) = f$. With $p \in (1, \infty)$ and the choice $s_1 = 0$, $r_1 = 4$, $s_2 = 1$, $r_2 = 0$ we obtain

$$W_p^1\left((0,T); L_p(\Omega)\right) \cap L_p\left((0,T); W_p^4(\Omega)\right) \hookrightarrow C([0,T]; W_p^{4-4/p}(\Omega)) .$$

In the case $p \in (4+n, \infty)$ Theorem B.20 implies the continuous embedding $W_p^{4-4/p}(\Omega) \hookrightarrow C^3(\overline{\Omega})$ which yields the desired result. □

Proposition B.36 (Equivalent norms on anisotropic spaces)**.** *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, T be positive, $p \in (1, \infty)$ and $\alpha \in (1/p, 1]$. There is an equivalent norm on $W_p^{\alpha}\left((0,T); L_p(\Omega)\right) \cap L_p\left((0,T); W_p^{4\alpha}(\Omega)\right)$ given by*

$$|||u|||_{W_p^{\alpha,4\alpha}((0,T)\times\Omega)} := \|u\|_{W_p^{\alpha,4\alpha}((0,T)\times\Omega)} + \|u(0)\|_{W_p^{4\alpha-4/p}(\Omega)} \tag{B.4}$$

with equivalence constants depending on T.

Proof. With $s_1 = 0$, $r_1 = 4\alpha$, $s_2 = \alpha$ and $r_2 = 0$, Proposition B.35 yields the embedding

$$W_p^{\alpha}((0,T);L_p(\Omega)) \cap L_p\big((0,T);W_p^{4\alpha}(\Omega)\big) \hookrightarrow C\big([0,T];W_p^{4\alpha-4/p}(\Omega)\big).$$

In particular, given $u \in W_p^{\alpha,4\alpha}((0,T)\times\Omega)$, $u(0)$ lies in $W_p^{4\alpha-4/p}(\Omega)$ with

$$\|u(0)\|_{W_p^{4\alpha-4/p}(\Omega)} \le C(T)\,\|u\|_{W_p^{\alpha,4\alpha}((0,T)\times\Omega)}\,.$$

This shows that $|\!|\!|u|\!|\!|_{W_p^{\alpha,4\alpha}((0,T)\times\Omega)}$ defines a norm on $W_p^{\alpha,4\alpha}((0,T)\times\Omega)$ that is equivalent to $\|\cdot\|_{W_p^{\alpha,4\alpha}((0,T)\times\Omega)}$. □

To obtain embeddings with constants independent of the considered time interval we make use of the following proposition.

Proposition B.37 (Temporal extension operator)**.** *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, T_0 be positive, $T \in (0,T_0)$, $p \in (1,\infty)$ and $\alpha \in (1/p, 1]$. Then for every*

$$g \in W_p^{\alpha,4\alpha}((0,T)\times\Omega) = W_p^{\alpha}((0,T);L_p(\Omega)) \cap L_p\big((0,T);W_p^4(\Omega)\big)$$

there exists an extension

$$\boldsymbol{E}g \in W_p^{\alpha,4\alpha}((0,T_0)\times\Omega) = W_p^{\alpha}((0,T_0);L_p(\Omega)) \cap L_p\big((0,T_0);W_p^4(\Omega)\big)$$

such that $(\boldsymbol{E}g)_{|(0,T)} = g$ and

$$|\!|\!|\boldsymbol{E}g|\!|\!|_{W_p^{\alpha,4\alpha}((0,T_0)\times\Omega)} \le C(T_0,p)\,|\!|\!|g|\!|\!|_{W_p^{\alpha,4\alpha}((0,T)\times\Omega)}\,.$$

The hereby induced extension operator is linear.

Proof. Let $p \in (1,\infty)$, $\alpha \in (1/p, 1]$ and $T > 0$ be fixed. Given $s \ge 0$ and a function $f \in L_p\big((0,T);W_p^s(\Omega)\big)$, the function

$$Ef : (0,\infty) \to W_p^s(\Omega)\,, \qquad t \mapsto \begin{cases} f(t)\,, & t \in (0,T)\,,\\ f(2T-t)\,, & t \in (T,2T)\,,\\ 0\,, & t \in (2T,\infty)\,,\end{cases}$$

lies in $L_p\big((0,\infty);W_p^s(\Omega)\big)$ with $\|Ef\|_{L_p((0,\infty);W_p^s(\Omega))} \le 2\,\|f\|_{L_p((0,T);W_p^s(\Omega))}$ which yields

$$E \in \mathcal{L}\left(L_p\big((0,T);W_p^s(\Omega)\big), L_p\big((0,\infty);W_p^s(\Omega)\big)\right).$$

For an interval $J \subset \mathbb{R}$ we consider the space

$${}_0W_p^{\alpha}(J;L_p(\Omega)) := \big\{h \in W_p^{\alpha}(J;L_p(\Omega)) : h(0) = 0\big\}\,.$$

It is readily checked that the operator E restricts to

$$E \in \mathcal{L}\left({}_0W_p^{\alpha}((0,T);L_p(\Omega))\,,\, {}_0W_p^{\alpha}((0,\infty);L_p(\Omega))\right)$$

with $\|E\|_{\mathcal{L}({}_0W_p^{\alpha}((0,T);L_p(\Omega)),\,{}_0W_p^{\alpha}((0,\infty);L_p(\Omega)))} \le 4$. Given $g \in W_p^{\alpha,4\alpha}((0,T)\times\Omega)$ Proposition B.35 implies $g(0) \in W_p^{4\alpha-4/p}(\Omega)$. By [126, Proposition 3.4.3] the initial value problem

$$\partial_t u + \Delta^2 u = 0\,,$$
$$u_{|t=0} = g(0)\,,$$

admits a solution $u \in W_p^{\alpha,4\alpha}\left((0,T_0)\times\Omega\right)$ with

$$\|u\|_{W_p^{\alpha,4\alpha}((0,T_0)\times\Omega)} \le C(T_0,p)\,\|g(0)\|_{W_p^{4\alpha-4/p}(\Omega)}\,.$$

The constant depends on T_0 as the equivalence constants for the norms on $W_p^{\alpha}\left((0,T_0);L_p(\Omega)\right)$ and $W_p^{\alpha,4\alpha}\left((0,T_0)\times\Omega\right)$ depend on the interval $(0,T_0)$. This is due to the fact that the norm of the extension operator in Theorem B.17 depends on the underlying domain. The function $f := u_{|(0,T)} - g$ lies in ${}_0W_p^{\alpha}\left((0,T);L_p(\Omega)\right)\cap L_p\left((0,T);W_p^{4\alpha}(\Omega)\right)$ and thus, the extension operator constructed above yields the existence of $Ef \in {}_0W_p^{\alpha}\left((0,\infty);L_p(\Omega)\right)\cap L_p\left((0,\infty);W_p^{4\alpha}(\Omega)\right)$ with $(Ef)_{|(0,T)} = f$. Now we define

$$\boldsymbol{E}g := u - Ef \in W_p^{\alpha}\left((0,T_0);L_p(\Omega)\right)\cap L_p\left((0,T_0);W_p^{4\alpha}(\Omega)\right).$$

Then $\boldsymbol{E}g$ satisfies $(\boldsymbol{E}g)_{|(0,T)} = u_{|(0,T)} - f = g$ and

$$\begin{aligned}
|||\boldsymbol{E}g|||_{W_p^{\alpha,4\alpha}((0,T_0)\times\Omega)} &\le \|u\|_{W_p^{\alpha,4\alpha}((0,T_0)\times\Omega)} + \|Ef\|_{W_p^{\alpha,4\alpha}((0,T_0)\times\Omega)} + \|(\boldsymbol{E}g)(0)\|_{W_p^{4\alpha-4/p}(\Omega)}\\
&\le C(T_0,p)\,\|g(0)\|_{W_p^{4\alpha-4/p}(\Omega)} + 4\,\|f\|_{W_p^{\alpha,4\alpha}((0,T)\times\Omega)}\\
&\le C(T_0,p)\,\|g(0)\|_{W_p^{4\alpha-4/p}(\Omega)} + 4\,\|g\|_{W_p^{\alpha,4\alpha}((0,T)\times\Omega)}\,.
\end{aligned}$$

The operator $\boldsymbol{E}$ is linear as E is and as the solution u to the linear initial value problem depends linearly on the initial value. □

Corollary B.38 (Uniform embeddings). *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, T_0 be positive, $p \in (1,\infty)$ and $\alpha \in (1/p, 1]$. Then there exists a constant $C(T_0)$ such that for all $T \in (0,T_0)$ the embedding ι_T given by*

$$\left(W_p^{\alpha,4\alpha}\left((0,T)\times\Omega\right),\,|||\cdot|||_{W_p^{\alpha,4\alpha}((0,T)\times\Omega)}\right) \overset{\iota_T}{\hookrightarrow} \left(C\left([0,T];W_p^{4\alpha-4/p}(\Omega)\right), \|\cdot\|_{C\left([0,T];W_p^{4\alpha-4/p}(\Omega)\right)}\right)$$

holds with operator norm

$$\sup\left\{\|\iota_T(u)\|_{C\left([0,T];W_p^{4\alpha-4/p}(\Omega)\right)} : u \in W_p^{\alpha,4\alpha}\left((0,T)\times\Omega\right),\, |||u|||_{W_p^{\alpha,4\alpha}((0,T)\times\Omega)} \le 1\right\} \le C(T_0)\,.$$

Proof. This follows by considering the decomposition $\iota_T = R\circ\iota_{T_0}\circ\boldsymbol{E}$ with

$$\boldsymbol{E} : \left(W_p^{\alpha,4\alpha}\left((0,T)\times\Omega\right),\,|||\cdot|||_{W_p^{\alpha,4\alpha}((0,T)\times\Omega)}\right) \to \left(W_p^{\alpha,4\alpha}\left((0,T_0)\times\Omega\right),\|\cdot\|_{W_p^{\alpha,4\alpha}((0,T_0)\times\Omega)}\right)$$

the extension operator defined in Proposition B.37 and

$$R : C\left([0,T_0];W_p^{4\alpha-4/p}(\Omega)\right) \to C\left([0,T];W_p^{4\alpha-4/p}(\Omega)\right)$$

the restriction operator. □

Corollary B.39 (Uniform embeddings II). *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, T_0 be positive, $T \in (0,T_0]$ and $p \in \left(\frac{4+n}{3},\infty\right)$. Then for all $\theta \in \left(\frac{n/p+1}{4-4/p},1\right)$ and all $\delta \in (0,1-1/p)$ there holds the embedding*

$$i_T : W_p^{1,4}\left((0,T)\times\Omega\right) \hookrightarrow C^{(1-\theta)(1-1/p-\delta)}\left([0,T];C^1\left(\overline{\Omega}\right)\right)$$

and there exists a constant $C\left(T_0,\theta,\delta\right)$ such that for all $u \in W_p^{1,4}\left((0,T)\times\Omega\right)$,

$$\|i_T(u)\|_{C^{(1-\theta)(1-1/p-\delta)}\left([0,T];C^1(\overline{\Omega})\right)} \le C\left(T_0,\theta,\delta\right)\,|||u|||_{W_p^{1,4}((0,T)\times\Omega)}\,.$$

Proof. Theorem B.25 yields for any $\delta \in (0, 1 - 1/p)$ the continuous embedding

$$W_p^{1,4}((0,T_0)\times\Omega) \hookrightarrow C^{1-1/p-\delta}([0,T_0];L_p(\Omega))$$

with operator norm depending on T_0. Furthermore, Corollary B.38 gives

$$W_p^{1,4}((0,T_0)\times\Omega) \hookrightarrow C\big([0,T_0];W_p^{4-4/p}(\Omega)\big).$$

Theorem B.18, Proposition B.19 and [140, Theorem 2.4] imply

$$W_p^{\theta(4-4/p)}(\Omega) = \big(L_p(\Omega), W_p^{4-4/p}(\Omega)\big)_{\theta,p}$$

with equivalent norms. In particular, for all $f \in W_p^{\theta(4-4/p)}(\Omega)$ there holds the estimate

$$\|f\|_{W_p^{\theta(4-4/p)}(\Omega)} \leq C\,\|f\|_{L_p(\Omega)}^{1-\theta}\,\|f\|_{W_p^{4-4/p}(\Omega)}^{\theta}.$$

As $\theta(4 - 4/p) > 1 + \frac{n}{p}$, the Sobolev Embedding Theorem B.20 yields $W_p^{\theta(4-4/p)}(\Omega) \hookrightarrow C^1\big(\overline{\Omega}\big)$. Applying Proposition B.1 with the choice $X_0 = L_p(\Omega)$ and $X_1 = W_p^{4-4/p}(\Omega)$ we obtain for all $\alpha \in (0,1)$,

$$\begin{aligned} C\big([0,T_0];W_p^{4-4/p}(\Omega)\big) \cap C^\alpha([0,T_0];L_p(\Omega)) &\hookrightarrow C^{(1-\theta)\alpha}\big([0,T_0];W_p^{\theta(4-4/p)}(\Omega)\big) \\ &\hookrightarrow C^{(1-\theta)\alpha}\big([0,T_0];C^1\big(\overline{\Omega}\big)\big) \end{aligned}$$

which yields for all $\delta \in (0, 1 - 1/p)$ the continuous embedding

$$W_p^{1,4}((0,T_0)\times\Omega) \hookrightarrow C^{(1-\theta)(1-1/p-\delta)}\big([0,T_0];C^1\big(\overline{\Omega}\big)\big).$$

Given $T \in (0, T_0]$ we let

$$\boldsymbol{E} : \Big(W_p^{1,4}((0,T)\times\Omega), |||\cdot|||_{W_p^{1,4}((0,T)\times\Omega)}\Big) \to \Big(W_p^{1,4}((0,T_0)\times\Omega), \|\cdot\|_{W_p^{1,4}((0,T_0)\times\Omega)}\Big)$$

be the extension operator defined in Proposition B.37 and

$$R : C^{(1-\theta)(1-1/p-\delta)}\big([0,T_0];C^1\big(\overline{\Omega}\big)\big) \to C^{(1-\theta)(1-1/p-\delta)}\big([0,T];C^1\big(\overline{\Omega}\big)\big)$$

the restriction operator. The claim now follows considering the decomposition $i_T = R \circ i_T \circ \boldsymbol{E}$. □

B.3.3 A composition property

This subsection is devoted to prove the composition result in Lemma B.45 which is needed to prove the geometric uniqueness of strong solutions to the elastic flow of networks in Subsection 4.2.1.

Lemma B.40. *Let T be positive, $p \in (1,\infty)$, $f \in L_p((0,T);L_p((0,1)))$, $\varphi \in C([0,T];C([0,1]))$ and $g \in C\big([0,T];C^1([0,1])\big)$ be such that for every $t \in [0,T]$, $g(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism. Then for every $\psi \in L_q((0,1))$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$t \mapsto \int_0^1 \psi(x)\,(f(t)\circ g(t))(x)\varphi(t,x)\,\mathrm{d}x \in \mathbb{R}$$

defines a Lebesgue measurable function on $(0,T)$.

Proof. For almost every $t \in (0,T)$ the function $f(t) \circ g(t)$ lies in $L_p((0,1))$ due to [2, Theorem 3.41]. As $\varphi(t)$ is continuous on $[0,1]$, the function $h : (0,T) \mapsto L_p((0,1))$, $t \mapsto (f(t) \circ g(t))\,\varphi(t)$ is well-defined. Given any $\psi \in L_q((0,1))$ with $\frac{1}{p} + \frac{1}{q} = 1$, the integral

$$F_\psi(t) := \int_0^1 \psi(x) h(t)(x)\,\mathrm{d}x = \int_0^1 \psi(x)\,(f(t) \circ g(t))\,(x)\varphi(t,x)\,\mathrm{d}x$$

exists for almost every $t \in (0,T)$. Moreover, [4, A 6.10] yields that $(t,y) \mapsto f(t,y) := f(t)(y)$ is measurable on $(0,T) \times (0,1)$. As $(t,x) \mapsto (t, g(t,x))$ is continuous and hence Lebesgue measurable on $(0,T) \times (0,1)$, we obtain that the composition $(t,x) \mapsto (f(t) \circ g(t))\,(x) = f(t)\,(g(t,x)) = f\,(t, g(t,x))$ is Lebesgue measurable on $(0,T) \times (0,1)$. As further $(t,x) \mapsto \psi(x)$ and $(t,x) \mapsto \varphi(t,x)$ are Lebesgue measurable on $(0,T) \times (0,1)$, we conclude that $(t,x) \mapsto \psi(x)\,(f(t) \circ g(t))\,(x)\varphi(t,x)$ is Lebesgue measurable on $(0,T) \times (0,1)$ being a product of measurable functions on $(0,T) \times (0,1)$. The measurability of $t \mapsto F_\psi(t)$ on $(0,T)$ is now a consequence of Fubini's Theorem (see for example [24, Theorem 4.5]). □

Lemma B.41. *Let T be positive and $g \in C\left([0,T]; C^1([0,1])\right)$ be such that for every $t \in [0,T]$, $g(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism with inverse $g(t)^{-1} =: g^{-1}(t)$. Then there holds $g^{-1} \in C\left([0,T]; C^1([0,1])\right)$.*

Proof. We first show that $(t,y) \mapsto g^{-1}(t,y)$ and $(t,y) \mapsto \partial_y g^{-1}(t,y)$ lie in $C([0,T] \times [0,1])$. To prove that g^{-1} is continuous we let $(t_n, y_n) \subset [0,T] \times [0,1]$, $n \in \mathbb{N}$, be a sequence with $(t_n, y_n) \to (t,y) \in [0,1]$. We intend to prove that $g^{-1}(t_n, y_n) \to g^{-1}(t,y)$. There exists $x \in [0,1]$ such that (after passing to a subsequence) $g^{-1}(t_n, y_n) \to x$. In particular, $g(t)\left(g^{-1}(t_n, y_n)\right) \to g(t)(x)$ due to continuity of $g(t)$. As $g(t)$ is bijective, it is enough to show that $g(t)\left(g^{-1}(t_n, y_n)\right) \to y$ which follows from

$$\begin{aligned} \left|g(t)\left(g^{-1}(t_n, y_n)\right) - y\right| &\leq \left|g(t)\left(g^{-1}(t_n, y_n)\right) - g(t_n)\left(g^{-1}(t_n, y_n)\right)\right| + \left|g(t_n)\left(g^{-1}(t_n, y_n)\right) - y\right| \\ &\leq \|g(t) - g(t_n)\|_{C([0,1])} + |y_n - y| \to 0\,. \end{aligned}$$

Furthermore, [151, Corollary 4.37] yields for every $t \in [0,T]$ and $x \in [0,1]$ the differentiation rule $\partial_y g(t)^{-1}(y) = \left(\partial_x g(t)\left(g(t)^{-1}(y)\right)\right)^{-1}$. Hence $(t,y) \mapsto \partial_y g(t)^{-1}(y)$ is continuous on $[0,T] \times [0,1]$ being a composition of continuous functions. An application of the Theorem by Ascoli–Arzelà (see for example [24, Theorem 4.25]) yields $g^{-1} \in C\left([0,T]; C^1([0,1])\right)$. □

Lemma B.42. *Let T be positive. The space $C([0,T]; C([0,1]))$ is a Banach algebra. Given $f, g \in C([0,T]; C([0,1]))$ the composition $(t,x) \mapsto h(t,x) := f(t, g(t,x))$ lies in $C([0,T]; C([0,1]))$.*

Proof. The Banach algebra property is straight forward to verify. Given $f, g \in C([0,T]; C([0,1]))$ the composition h given by $h(t,x) := f(t, g(t,x))$ satisfies $h(t) \in C([0,1])$ for every $t \in [0,T]$ and is continuous on $[0,T] \times [0,1]$. In particular, given $t \in [0,T]$ and a sequence $(t_n)_{n \in \mathbb{N}} \subset [0,T]$ with $t_n \to t$, the sequence $h(t_n)$, $n \in \mathbb{N}$, converges to $h(t)$ pointwise in $[0,1]$. The uniform convergence $\|h(t) - h(t_n)\|_{C([0,1])} \to 0$ follows from [24, Theorem 4.25] once we have shown that $\{h(t_n) : n \in \mathbb{N}\}$ is uniformly bounded and uniformly equicontinuous. Given $x \in [0,1]$ and $n \in \mathbb{N}$ we have

$$|h(t_n, x)| \leq \|f\|_{C([0,T];C([0,1]))}\,.$$

Let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that for all $x, y \in [0,1]$, $|x - y| < \delta$, $|f(t,x) - f(t,y)| < \varepsilon/5$ and $|h(t,x) - h(t,y)| < \varepsilon/5$. Furthermore, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\|g(t_n) - g(t)\|_{C([0,1])} < \delta$ and $\|f(t_n) - f(t)\|_{C([0,1])} < \frac{\varepsilon}{5}$. This allows us to conclude for all $n \geq N$ and $x, y \in [0,1]$ with $|x - y| < \delta$,

$$|h(t_n, x) - h(t_n, y)|$$

$$\begin{aligned}&\leq |f(t_n, g(t_n, x)) - f(t, g(t_n, x))| + |f(t, g(t_n, x)) - f(t, g(t, x))| \\ &+ |f(t, g(t, x)) - f(t, g(t, y))| + |f(t, g(t, y)) - f(t, g(t_n, y))| + |f(t, g(t_n, y)) - f(t_n, g(t_n, y))| \\ &< \|f(t) - f(t_n)\|_{C([0,1])} + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \|f(t) - f(t_n)\|_{C([0,1])} < \varepsilon\,.\end{aligned}$$

□

Lemma B.43 (Difference quotients I)**.** *Let $p \in (1,\infty)$, T be positive, $u \in W_p^1((0,T); L_p((0,1)))$, $[a,b] \subset (0,T)$ and $|\varepsilon| < \frac{1}{2}\operatorname{dist}([a,b], \partial(0,T))$. Then as $\varepsilon \to 0$, the difference quotients*

$$\partial^\varepsilon u(t) := \frac{1}{\varepsilon}(u(t+\varepsilon) - u(t)) \tag{B.5}$$

converge weakly to $\partial_t u$ in $L_p((a,b); L_p((0,1)))$.

Proof. As the space $L_p((0,T); L_p((0,1)))$ is reflexive for $p \in (1,\infty)$, see [48, Chapter IV, Corollary 2], the proof of [57, Chapter 5.8, Theorem 3] stays valid in the case that the functions take values in $L_p((0,1))$. The statement then follows combining the arguments in the proofs of [57, Chapter 5.8, Theorem 3(i), (ii)]. □

Lemma B.44 (Difference quotients II)**.** *Let $p \in (1,\infty)$, T be positive, $u \in L_p((0,T); L_p((0,1)))$, $[a,b] \subset (0,T)$ and $|\varepsilon| < \frac{1}{2}\operatorname{dist}([a,b], \partial(0,T))$. Suppose that the difference quotients $\partial^\varepsilon u \in L_p((a,b); L_p((0,1)))$ defined in* (B.5) *converge weakly in $L_p((a,b); L_p((0,1)))$ to a function $v \in L_p((a,b); L_p((0,1)))$ as $\varepsilon \to 0$. Then u lies in $W_p^1((a,b); L_p((0,1)))$ with $\partial_t u = v$.*

Proof. This can be shown using the arguments in the proof of [57, Chapter 5.8, Theorem 3(ii)] and reflexivity of the space $L_p((0,T); L_p((0,1)))$ for $p \in (1,\infty)$, see [48, Chapter IV, Corollary 2]. □

Lemma B.45. *Let T be positive, $p \in (5,\infty)$ and $f, g \in W_p^{1,4}((0,T) \times (0,1))$ be such that for every $t \in [0,T]$, $g(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism. Then the function $h(t) := f(t) \circ g(t)$ lies in $W_p^{1,4}((0,T) \times (0,1))$ and all derivatives can be calculated with chain and product rule.*

Proof. Proposition B.35 yields that both f and g lie in $C([0,T]; C^3([0,1]))$ and thus $h(t) \in C^3([0,1])$ for every $t \in [0,T]$. Lemma B.42 implies

$$h \in C\big([0,T]; C^3([0,1])\big) \hookrightarrow L_p\big((0,T); W_p^3((0,1))\big)$$

and for every $t \in [0,T]$ and $x \in [0,1]$ it holds

$$\begin{aligned}\partial_x^3(h(t))(x) &= \big(\partial_x^3 f(t)\big)(g(t,x))\,(\partial_x g(t)(x))^3 + 3\big(\partial_x^2 f(t)\big)(g(t,x))\,\partial_x^2 g(t)(x)\partial_x g(t)(x) \\ &\quad + \partial_x^3 g(t)(x)\,(\partial_x f(t))(g(t,x))\,.\end{aligned}$$

There exists a set $N \subset (0,T)$ of measure zero such that for every $t \in (0,T) \setminus N$, the functions $x \mapsto \partial_x^3 f(t)(x)$ and $x \mapsto \partial_x^3 g(t)(x)$ lie in $W_p^1((0,1))$. Given $t \in (0,T) \setminus N$ the map $g(t)$ is a C^1–diffeomorphism of $[0,1]$ and thus [2, Theorem 3.41] implies $\partial_x^3 f(t) \circ g(t) \in W_p^1((0,1))$ with

$$\partial_x\big(\partial_x^3 f(t) \circ g(t)\big)(x) = \big(\partial_x^4 f(t)\big)(g(t,x))\,\partial_x g(t)(x)\,.$$

As $x \mapsto (\partial_x g(t)(x))^3$ lies in $C^1([0,1]) \subset W_p^1((0,1))$, the Banach algebra property of $W_p^1((0,1))$ for $p \in (1,\infty)$ shown in [2, Theorem 4.39] yields $\big(\partial_x^3 f(t) \circ g(t)\big)(\partial_x g(t))^3 \in W_p^1((0,1))$. All remaining terms in the formula for $\partial_x^3 h(t)$ lie in $W_p^1((0,1))$ which allows us to conclude $\partial_x^3 h(t) \in W_p^1((0,1))$ and thus $h(t) \in W_p^4((0,1))$ for all $t \in (0,T) \setminus N$. To prove that $h : (0,T) \to W_p^4((0,1))$ is strongly measurable, the characterisation of the dual of $W_p^4((0,1))$ in [2, Theorem 3.9] and Pettis' Theorem (see for example [129, Satz 1.8]) yield that it is enough to show that for all $\psi^i \in L_q((0,1))$, $i \in \{0,1,2,3,4\}$, $\frac{1}{p}+\frac{1}{q}=1$,

$$t \mapsto \sum_{i=0}^{4} \int_0^1 \psi^i(x)\partial_x^i h(t)(x)\,\mathrm{d}x$$

is Lebesgue measurable on $(0,T)$. This is a direct consequence of Lemma B.40. Directly estimating the terms in the formula for $(t,x) \mapsto \partial_x^4 h(t)(x)$ with the help of Lemma B.41 and [2, Theorem 3.41], one obtains $h \in L_p\left((0,T); W_p^4((0,1))\right)$. In the next step we show that h lies in $W_p^1((0,T); L_p((0,1)))$ with distributional derivative

$$\mathfrak{h}(t) := \partial_t f(t) \circ g(t) + (\partial_x f(t) \circ g(t))\,(\partial_t g(t))\,.$$

First, we verify that $\mathfrak{h}$ lies in $L_p((0,T); L_p((0,1)))$. There exists a set $\mathcal{N} \subset (0,T)$ of measure zero such that for all $t \in (0,T) \setminus \mathcal{N}$, $\partial_t f(t)$ and $\partial_t g(t)$ lie in $L_p((0,1))$, and by [2, Theorem 3.41] also $\partial_t f(t) \circ g(t) \in L_p((0,1))$. As further $\partial_x f(t) \circ g(t) \in C([0,1])$ for all $t \in [0,T]$, we obtain $\mathfrak{h}(t) \in L_p((0,1))$ for all $t \in (0,T) \setminus \mathcal{N}$. Lemma B.40 yields for every $\psi \in L_q((0,1))$ with $\frac{1}{p}+\frac{1}{q}=1$ that

$$t \mapsto \int_0^1 \psi(x)\mathfrak{h}(t)(x)\,\mathrm{d}x$$

is Lebesgue measurable on $(0,T)$ which implies strong measurability of $\mathfrak{h} : (0,T) \to L_p((0,1))$ by Pettis' Theorem. Using Lemma B.41 we obtain $\mathfrak{h} \in L_p((0,T); L_p((0,1)))$.
Given a fixed test function $\psi \in C_0^\infty((0,T))$ we need to prove

$$\int_0^T h(t)\psi'(t)\,\mathrm{d}t = -\int_0^T \mathfrak{h}(t)\psi(t)\,\mathrm{d}t \tag{B.6}$$

as an identity in $L_p((0,1))$. Let $[a,b] \subset (0,T)$ be such that $\operatorname{supp}\psi \subset (a,b)$. Given $\varepsilon > 0$ with $|\varepsilon| < \frac{1}{2}\operatorname{dist}([a,b], \partial(0,T)) =: \varepsilon_0$ and $t \in [a,b]$ we consider the difference quotients

$$\begin{aligned}\partial^\varepsilon h(t) &= \frac{1}{\varepsilon}(h(t+\varepsilon) - h(t))\\ &= \frac{1}{\varepsilon}(f(t+\varepsilon) \circ g(t+\varepsilon) - f(t+\varepsilon) \circ g(t)) + \frac{1}{\varepsilon}(f(t+\varepsilon) - f(t)) \circ g(t)\,.\end{aligned}$$

Notice that $\partial^\varepsilon h$ lies in $L_p((a,b); L_p((0,1)))$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. To prove (B.6) Lemma B.44 yields that it is enough to show that $\partial^\varepsilon h$ converges weakly to $\mathfrak{h}$ in $L_p((a,b); L_p((0,1)))$. Due to [48, Chapter IV, Corollary 2] the dual of $L_p((a,b); L_p((0,1)))$ is given by $L_q((a,b); L_q((0,1)))$ with $q = \frac{p}{p-1}$. Thus it is enough to show for $\alpha \in C_0^\infty((0,1))$ and $\phi \in C_0^\infty((a,b))$,

$$\lim_{\varepsilon \to 0} \int_a^b \int_0^1 \partial^\varepsilon h(t)(x)\alpha(x)\phi(t)\,\mathrm{d}x\,\mathrm{d}t = \int_a^b \int_0^1 \mathfrak{h}(t)(x)\alpha(x)\phi(t)\,\mathrm{d}x\,\mathrm{d}t\,.$$

As $\partial^\varepsilon f$ converges weakly to $\partial_t f$ in $L_p((a,b); L_p((0,1)))$ by Lemma B.43 and

$$(t,y) \mapsto \phi(t)\alpha\left(g^{-1}(t,y)\right)\left|\partial_y g(t)^{-1}(y)\right| \in L_q((a,b); L_q((0,1)))$$

due to Lemma B.41, we obtain that

$$\begin{aligned}&\int_a^b \int_0^1 \frac{1}{\varepsilon}(f(t+\varepsilon) - f(t))\,(g_t(x))\,\phi(t)\alpha(x)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_a^b \int_0^1 \frac{1}{\varepsilon}(f(t+\varepsilon) - f(t))\,(y)\alpha\left(g^{-1}(t,y)\right)\phi(t)\left|\partial_y g(t)^{-1}(y)\right|\mathrm{d}y\,\mathrm{d}t\end{aligned}$$

converges as $\varepsilon \to 0$ to

$$\int_a^b \int_0^1 \partial_t f(t)(y)\,\alpha\left(g^{-1}(t,y)\right)\phi(t)\left|\partial_y g(t)^{-1}(y)\right| \mathrm{d}y\,\mathrm{d}t = \int_a^b \int_0^1 \partial_t f(t)\left(g(t,x)\right)\alpha(x)\phi(t)\,\mathrm{d}x\,\mathrm{d}t\,.$$

Furthermore, for every $t \in [a,b]$, $|\varepsilon| < \varepsilon_0$ and $x \in [0,1]$, differentiability of $f(t+\varepsilon)$ yields

$$\begin{aligned}
&\frac{1}{\varepsilon}\left(f(t+\varepsilon)\left(g(t+\varepsilon,x)\right) - f(t+\varepsilon)\left(g(t,x)\right)\right)\\
&= \int_0^1 \partial_x f(t+\varepsilon)\left(\tau g(t+\varepsilon,x) + (1-\tau)g(t,x)\right)\mathrm{d}\tau \frac{1}{\varepsilon}\left(g(t+\varepsilon,x) - g(t,x)\right).
\end{aligned}$$

Uniform continuity of $\partial_x f$ and g on $[0,T]\times[0,1]$ yield

$$\lim_{\varepsilon\to 0} \sup_{t\in[a,b],x\in[0,1]} \left|\int_0^1 \partial_x f(t+\varepsilon)\left(\tau g(t+\varepsilon,x)+(1-\tau)g(t,x)\right)\mathrm{d}\tau - \partial_x f(t)(g(t,x))\right| = 0\,.$$

As $\partial^\varepsilon g$ converges weakly to $\partial_t g$ in $L_p\left((a,b);L_p\left((0,1)\right)\right)$ as $\varepsilon \to 0$ by Lemma B.43, we conclude that

$$\int_a^b \int_0^1 \frac{1}{\varepsilon}\left(f(t+\varepsilon)\circ g(t+\varepsilon) - f(t+\varepsilon)\circ g(t)\right)\alpha(x)\phi(t)\,\mathrm{d}x\,\mathrm{d}t$$

tends to

$$\int_a^b \int_0^1 \partial_x f(t)(g(t,x))\partial_t g(t,x)\alpha(x)\phi(t)\,\mathrm{d}x\,\mathrm{d}t$$

as $\varepsilon \to 0$ using that the weakly convergent sequence $(\partial^\varepsilon g)$ is bounded in $L_p\left((a,b);L_p\left((0,1)\right)\right)$. □

Lemma B.46. *Let $p \in (5,\infty)$, T be positive and $g \in W_p^{1,4}\left((0,T)\times(0,1)\right)$ be such that for every $t \in [0,T]$, $g(t) : [0,1] \to [0,1]$ is a C^1–diffeomorphism with inverse $g^{-1}(t) := g(t)^{-1}$. Then the function g^{-1} lies in $W_p^{1,4}\left((0,T)\times(0,1)\right)$ with $\partial_y g^{-1}(t,y) = \left(\partial_x g\left(t, g^{-1}(t,y)\right)\right)^{-1}$, $\partial_t g^{-1}(t,y) = -\left(\partial_t g\right)\left(t,g^{-1}(t,y)\right)\partial_y g^{-1}(t,y)$ and all higher order spacial derivatives can be calculated by chain and product rule.*

Proof. Proposition B.35 implies $g \in C\left([0,T];C^3\left([0,1]\right)\right)$ and thus $g^{-1} \in C\left([0,T];C^1\left([0,1]\right)\right)$ by Lemma B.41. For every $t \in [0,T]$, $g(t) \in C^3\left([0,1]\right)$ is a diffeomorphism with inverse $g^{-1}(t)$ and hence [151, Corollary 4.37] yields $g^{-1}(t) \in C^3\left([0,1]\right)$ with

$$\begin{aligned}
\partial_y g^{-1}(t)(y) &= \left(\left(\partial_x g(t)\right)\left(g^{-1}(t,y)\right)\right)^{-1}\,,\\
\partial_y^2 g^{-1}(t)(y) &= -\left(\left(\partial_x g(t)\right)\left(g^{-1}(t,y)\right)\right)^{-3}\left(\partial_x^2 g(t)\right)\left(g^{-1}(t,y)\right)\\
\partial_y^3 g(t)^{-1}(y) &= 3\left(\left(\partial_x g(t)\right)\left(g^{-1}(t,y)\right)\right)^{-5}\left(\left(\partial_x^2 g(t)\right)\left(g^{-1}(t,y)\right)\right)^2\\
&\quad - \left(\left(\partial_x g(t)\right)\left(g^{-1}(t,y)\right)\right)^{-4}\left(\partial_x^3 g(t)\right)\left(g^{-1}(t,y)\right).
\end{aligned}$$

Lemma B.42 gives $g^{-1} \in C\left([0,T];C^3\left([0,1]\right)\right) \hookrightarrow L_p\left((0,T);W_p^3\left((0,1)\right)\right)$. There exists a set $N \subset (0,T)$ of measure zero such that for all $t \in (0,T)\setminus N$, $\partial_x^3 g(t) \in W_p^1\left((0,1)\right)$, and thus, using [2, Theorem 3.41], in particular $\partial_x^3 g(t)\circ g^{-1}(t) \in W_p^1\left((0,1)\right)$ with

$$\partial_y\left(\partial_x^3 g(t)\circ g^{-1}(t)\right)(y) = \partial_x^4 g(t)\left(g^{-1}(t,y)\right)\partial_y g^{-1}(t)(y)\,.$$

The Banach algebra property of $W_p^1\left((0,1)\right)$ in [2, Theorem 4.39] then yields $\partial_y^3 g^{-1}(t) \in W_p^1\left((0,1)\right)$ and thus $g^{-1}(t) \in W_p^4\left((0,1)\right)$ for all $t \in (0,T)\setminus N$. The strong measurability of $g^{-1} : (0,T) \to W_p^4\left((0,1)\right)$ follows from [129, Satz 1.8] and Lemma B.40 as in the proof of Lemma B.45. Explicitly computing the expression $\partial_y^4 g^{-1}(t)(y)$ using [2, Theorem 3.41], product and chain rule, and g, $g^{-1} \in C\left([0,T]; C^3([0,1])\right)$, one directly obtains $\partial_y^4 g^{-1} \in L_p\left((0,T); W_p^4((0,1))\right)$. It remains to show that g^{-1} lies in $W_p^1((0,T); L_p((0,1)))$ with distributional derivative

$$h(t) := -\left(\partial_t g(t) \circ g^{-1}(t)\right) \partial_y g^{-1}(t) \,.$$

Observe that for almost every $t \in (0,T)$, $\partial_t g(t)$ lies in $L_p((0,1))$ and thus, [2, Theorem 3.41] and $\partial_y g^{-1}(t) \in C([0,1])$ yield $h(t) \in L_p((0,1))$ for almost every $t \in (0,T)$. Furthermore, Lemma B.40, [129, Satz 1.8] and Lemma B.41 yield $h \in L_p((0,T); L_p((0,1)))$. To prove that h is the distributional derivative of g^{-1}, the arguments in Lemma B.45 yield that it is enough to show that for any $[a,b] \subset (0,T)$, $\varepsilon > 0$ with $|\varepsilon| < \frac{1}{2}\mathrm{dist}([a,b], \partial(0,T)) =: \varepsilon_0$, the difference quotients $\partial^\varepsilon g^{-1} \in L_p((a,b); L_p((0,1)))$ defined by

$$\partial^\varepsilon g^{-1}(t) = \frac{1}{\varepsilon}\left(g^{-1}(t+\varepsilon) - g^{-1}(t)\right)$$

converge weakly to h in $L_p((a,b); L_p((0,1)))$ as $\varepsilon \to 0$. Observe that for any $t \in [a,b]$, $|\varepsilon| < \varepsilon_0$, $x \in [0,1]$,

$$\begin{aligned}
&g^{-1}(t+\varepsilon)(g(t,x)) - g^{-1}(t+\varepsilon)(g(t+\varepsilon,x)) \\
&= -\int_0^1 \partial_y g^{-1}(t+\varepsilon)(\tau g(t,x) + (1-\tau) g(t+\varepsilon,x))\, \mathrm{d}\tau\, (g(t+\varepsilon,x) - g(t,x)) \,.
\end{aligned}$$

The integral in the above expression converges uniformly on $[a,b] \times [0,1]$ to $\left(\partial_y g^{-1}(t)\right)(g(t,x))$. As $\partial^\varepsilon g$ converges weakly to $\partial_t g$ in $L_p((a,b); L_p((0,1)))$ due to Lemma B.43, we obtain for any $\alpha \in C_0^\infty((0,1))$ and $\phi \in C_0^\infty((a,b))$ as $\varepsilon \to 0$,

$$\begin{aligned}
&\int_a^b \int_0^1 \frac{1}{\varepsilon}\left(g^{-1}(t+\varepsilon,y) - g^{-1}(t,y)\right) \alpha(y)\phi(t)\, \mathrm{d}y\, \mathrm{d}t \\
&= \int_a^b \int_0^1 \frac{1}{\varepsilon}\left(g^{-1}(t+\varepsilon)(g(t,x)) - g^{-1}(t+\varepsilon)(g(t+\varepsilon,x))\right) \alpha(g(t,x))\, \partial_x g(t)(x)\, \mathrm{d}x\, \phi(t)\, \mathrm{d}t \\
&\to -\int_a^b \int_0^1 \left(\partial_y g^{-1}(t)\right)(g(t,x))\, \partial_t g(t,x) \alpha(g(t,x))\, \partial_x g(t)(x)\, \mathrm{d}x\, \phi(t)\, \mathrm{d}t \\
&= -\int_a^b \int_0^1 \partial_y g^{-1}(t)(y)\, (\partial_t g(t))\left(g^{-1}(t,y)\right) \alpha(y)\phi(t)\, \mathrm{d}y\, \mathrm{d}t \,.
\end{aligned}$$

□

Appendix C

Function spaces on compact manifolds

This chapter introduces the function spaces on compact manifolds required to study the higher dimensional problem considered in Chapter 2 in a Sobolev setting. In Chapter 2 we use a localisation procedure to translate boundary value problems on the manifold to problems on the chart domains. To this end, it is convenient to characterise the respective function spaces on the manifold in terms of a uniform localisation system as constructed in Section A.4.
In Subsection C.1.1 we use the normal coordinates constructed in Section A.4 to give an equivalent characterisation of Sobolev spaces on compact manifolds, usually defined in a coordinate-free way, in terms of the respective Sobolev norms of the localised functions on the chart domains. Such characterisations are well-known in the literature. We refer to [12, 72] and the references therein where the authors treat broad classes of manifolds. For the convenience of the reader we give a detailed proof in the (easy) case of compact manifolds with boundary. We hereby follow the sketch given in [96] and make use of [97, Lemma 1.3]. [1] The statement of [97, Lemma 1.3] is given in the identity (A.2) in Proposition A.11.
This result is then used to define Slobodeckij spaces of non-integer order on manifolds in Subsection C.1.2 and to give characterisations of anisotropic spaces on manifolds in Section C.2. In the case that the manifold is given by the boundary of a smooth domain in Euclidean space, our notion of Sobolev (Slobodeckij) spaces on manifolds is consistent with the one in [136, 141] which allows us to use the *trace theorems* stated therein. This is studied in Section C.3.
We give some remarks on the notation used in this chapter. The considered manifold is endowed with a normal covering $\mathcal{T}$ and an associated uniform localisation system $\mathcal{C}$ as defined in Section A.4. The "localised" norms are denoted by superscript $\mathcal{T}$. To indicate constants that depend on the choice of $\mathcal{T}$ we use the notation $C(Q)$ where Q is the quantity appearing in Proposition A.44.

C.1 Sobolev spaces on compact manifolds

The Sobolev spaces $W_p^m(M)$ on a Riemannian manifold M are usually defined via completion of smooth functions with respect to the norm

$$\|f\|_{W_p^m(M)} := \sum_{l=0}^{m} \left(\int_M |\nabla^l f|^p \, \mathrm{d}V_g \right)^{1/p} . \tag{C.1}$$

In this way the spaces inherit plenty of properties of their counterparts in Euclidean space, as for example, the Sobolev Embedding Theorem. Such results can for instance be found in [13, 74, 75] in the case of manifolds with bounded geometry and positive injectivity radius. In [14] the author shows analogous properties for compact manifolds with boundary. A profound study of a

[1] At this point I would like to express my gratitude to the MathOverflow user macbeth for the helpful answers to my question `https://mathoverflow.net/q/327475` and for drawing my attention to [97, Lemma 1.3].

broader class of function spaces on singular manifolds can be found in [10, 11]. Here the author considers (possibly) non-compact, non-complete manifolds with or without boundary with singular behaviour described by a singularity function ρ including the non-singular case that corresponds to $\rho \equiv 1$.

C.1.1 Characterisation of Sobolev spaces via localisation

This subsection is devoted to prove the following result.

Proposition C.1 (Characterisation of $W_p^m(M)$ via localisation)**.** *Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n \in \mathbb{N}$ with Levi-Civita connection D and uniform localisation system $(U_\alpha, \phi_\alpha, \psi_\alpha)$, $p \in [1, \infty)$ and $m \in \mathbb{N}_0$. Given an element f of $W_p^m(M)$, the function $f \circ \phi_\alpha^{-1}$ lies in $W_p^m(\phi_\alpha(U_\alpha))$ for all $\alpha \in \{1, \ldots, N\}$ with the estimate*

$$\left\| f \circ \phi_\alpha^{-1} \right\|_{W_p^m(\phi_\alpha(U_\alpha))} \leq C(Q) \, \|f\|_{W_p^m(M)} \, . \tag{C.2}$$

On the other hand, if $f : M \to \mathbb{R}$ is such that $(\psi_\alpha f) \circ \phi_\alpha^{-1} \in W_p^m(\phi_\alpha(U_\alpha))$ for all $\alpha \in \{1, \ldots, N\}$, then it holds $f \in W_p^m(M)$. Furthermore,

$$\|f\|^{\mathcal{T}}_{W_p^m(M)} := \sum_{\alpha=1}^{N} \left\| (\psi_\alpha f) \circ \phi_\alpha^{-1} \right\|_{W_p^m(\phi_\alpha(U_\alpha))}$$

defines an equivalent norm on $W_p^m(M)$ with equivalence constants depending on Q and the bounds on the derivatives of g and g^{-1} in the normal coordinates.

Its proof is given at the end of this subsection and uses the hints given in [96].

We remark that the convention regarding Sobolev (Slobodeckij) spaces on open subsets of the closed upper half-space $\mathbb{H}^n$ is explained in the paragraph on *Notation* in the introductory part of the thesis.

In the first part of this section we study the classical notion of Sobolev spaces on manifolds via completion with respect to the norm (C.1). The second part is then devoted to the proof of Proposition C.1.

Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold with (or without) boundary of dimension $n \in \mathbb{N}$ (or $n \in \mathbb{N}_0$) with associated Levi-Civita connection D, see Proposition A.35, and canonical volume measure $\mathrm{d}V_g$ as defined in Definition A.24.
A function $f : M \to \mathbb{R}$ is called *measurable* if $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is Lebesgue measurable for all $(U, \phi) \in \mathcal{A}$. Two measurable functions $f, g : M \to \mathbb{R}$ are said to be *equal almost everywhere* if $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are equal Lebesgue-almost everywhere in $\phi(U)$ for all $(U, \phi) \in \mathcal{A}$.
Identifying functions that are equal almost everywhere, we define for $p \in [1, \infty)$ the *L_p–space* $L_p(M)$ by

$$L_p(M) := \left\{ f : M \to \mathbb{R} \, : \, f \text{ is measurable and } \int_M |f|^p \, \mathrm{d}V_g < \infty \right\} .$$

Proposition C.2. *Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold of dimension $n \in \mathbb{N}$ with or without boundary. Given $p \in [1, \infty)$ the space $L_p(M)$ is a Banach space in the norm*

$$\|f\|_{L_p(M)} := \left(\int_M |f|^p \, \mathrm{d}V_g \right)^{1/p} .$$

Proof. The space $L_p(M)$ is a vector space and it is straightforward to verify that $\|\cdot\|_{L_p(M)}$ defines a norm on $L_p(M)$. Let $(f_j)_{j\in\mathbb{N}}$ be a Cauchy sequence in $\left(L_p(M), \|\cdot\|_{L_p(M)}\right)$ and $\mathcal{T}$ be a normal covering of M composed of (U_α, ϕ_α), $\alpha \in \{1, \dots, N\}$, with smooth partition of unity (ψ_α) subordinate to the covering. For all $\alpha \in \{1, \dots, N\}$ the properties of the normal coordinate charts imply that the metric satisfies

$$Q^{-n} \le \det\left(g_{ij} \circ \phi_\alpha^{-1}\right) \le Q^n \qquad \text{in } \phi_\alpha(U_\alpha)\,.$$

The function $(\psi_\alpha f_j) \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \mathbb{R}$ is measurable and

$$\int_{\phi_\alpha(U_\alpha)} |(\psi_\alpha f_j) \circ \phi_\alpha^{-1}|^p \, \mathrm{d}x \le C(Q) \int_{\phi_\alpha(U_\alpha)} |\psi_\alpha \circ \phi_\alpha^{-1}| |f_j \circ \phi_\alpha^{-1}|^p \sqrt{\det(g_{ij})} \, \mathrm{d}x \le C(Q) \, \|f_j\|^p_{L_p(M)}\,.$$

The analogous estimate for $(f_j - f_l)$ instead of f_j shows that $\left((\psi_\alpha f_j) \circ \phi_\alpha^{-1}\right)_{j\in\mathbb{N}}$ forms a Cauchy sequence in $L_p(\phi_\alpha(U_\alpha))$ converging to a function $f_\alpha \in L_p(\phi_\alpha(U_\alpha))$. The function given by $f := \sum_{\alpha=1}^N (f_\alpha \circ \phi_\alpha)$ is measurable as for $(U, \phi) \in \mathcal{A}$, $f \circ \phi^{-1} = \sum_{\alpha=1}^N f_\alpha \circ (\phi_\alpha \circ \phi^{-1})$, with f_α measurable on $\phi(U \cap U_\alpha)$ and $\phi_\alpha \circ \phi^{-1}$ continuous on $\phi(U \cap U_\alpha)$. Furthermore, with $I_\alpha := \{\beta \in \{1, \dots, N\} : U_\alpha \cap U_\beta \neq \emptyset\}$ for $\alpha \in \{1, \dots, N\}$, we obtain

$$\begin{aligned}\int_M |f|^p \, \mathrm{d}V_g &= \sum_{\alpha=1}^N \int_{\phi_\alpha(U_\alpha)} (\psi_\alpha \circ \phi_\alpha^{-1}) |f \circ \phi_\alpha^{-1}|^p \sqrt{\det(g_{ij})} \, \mathrm{d}x \\ &\le C(Q) \sum_{\alpha=1}^N \sum_{\beta \in I_\alpha} \int_{\phi_\alpha(U_\alpha \cap U_\beta)} |f_\beta \circ \phi_\beta \circ \phi_\alpha^{-1}|^p \, \mathrm{d}x \le C(Q) \sum_{\beta=1}^N \|f_\beta\|^p_{L_p(\phi_\beta(U_\beta))} < \infty\,.\end{aligned}$$

Observing for $\alpha \in \{1, \dots, N\}$, $j \in \mathbb{N}$ the identity

$$\begin{aligned}(f - f_j) \circ \phi_\alpha^{-1} &= f \circ \phi_\alpha^{-1} - \sum_{\beta \in I_\alpha} (\psi_\beta f_j) \circ \phi_\alpha^{-1} = \sum_{\beta \in I_\alpha} f_\beta \circ \phi_\beta \circ \phi_\alpha^{-1} - \sum_{\beta \in I_\alpha} (\psi_\beta f_j) \circ \phi_\alpha^{-1} \\ &= \sum_{\beta \in I_\alpha} \left(f_\beta - (\psi_\beta f_j) \circ \phi_\beta^{-1}\right) \circ \phi_\beta \circ \phi_\alpha^{-1}\,,\end{aligned}$$

we conclude as $j \to \infty$,

$$\begin{aligned}\|f - f_j\|^p_{L_p(M)} &\le C(Q) \sum_{\alpha=1}^N \int_{\phi_\alpha(U_\alpha)} (\psi_\alpha \circ \phi_\alpha^{-1}) \left|(f - f_j) \circ \phi_\alpha^{-1}\right|^p \, \mathrm{d}x \\ &\le C(Q) \sum_{\alpha=1}^N \sum_{\beta \in I_\alpha} \int_{\phi_\alpha(U_\alpha \cap U_\beta)} \left|f_\beta - (\psi_\beta f_j) \circ \phi_\beta^{-1}\right|^p \circ \phi_\beta \circ \phi_\alpha^{-1} \, \mathrm{d}x \\ &\le C(Q) \sum_{\alpha=1}^N \sum_{\beta \in I_\alpha} \int_{\phi_\beta(U_\alpha \cap U_\beta)} \left|f_\beta - (\psi_\beta f_j) \circ \phi_\beta^{-1}\right|^p \left|\det D(\phi_\alpha \circ \phi_\beta^{-1})\right| \, \mathrm{d}x \\ &\le C(Q) \sum_{\beta=1}^N \left\|f_\beta - (\psi_\beta f_j) \circ \phi_\beta^{-1}\right\|^p_{L_p(\phi_\beta(U_\beta))} \to 0\,.\end{aligned}$$

□

Given $m \in \mathbb{N}_0$ and a function $f \in C^m(M; \mathbb{R})$ we denote its k-th covariant derivative with respect to the Levi-Civita connection by $\nabla^k f$, $0 \le k \le m$, see Definition A.10. The norm $|\nabla^k f|$ of the $(k, 0)$-tensor field $\nabla^k f$ with respect to the metric g is defined via its representation in a local chart

$$|\nabla^k f|^2 := g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k f)_{i_1 \cdots i_k} (\nabla^k f)_{j_1 \cdots j_k}\,. \tag{C.3}$$

For $m \in \mathbb{N}_0$ we define $C^m(\mathrm{int} M)$ to be the space of functions $f : M \setminus \partial M \to \mathbb{R}$ such that for every interior point $p \in M \setminus \partial M$ there exists an interior chart $(U, \phi) \in \mathcal{A}$ with $f \circ \phi^{-1} \in C^m(\phi(U))$.

Proposition C.3. *Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n \in \mathbb{N}$ with Levi-Civita connection D and uniform localisation system $(U_\alpha, \phi_\alpha, \psi_\alpha)$. Given $p \in [1, \infty)$, $m \in \mathbb{N}_0$ and $f \in C^m(\mathrm{int}M)$ we let*

$$\|f\|_{W_p^m(M)} := \sum_{l=0}^{m} \left(\int_M |\nabla^l f|^p \, \mathrm{d}V_g \right)^{1/p} . \tag{C.4}$$

Then the space

$$\mathcal{C}_p^m(M) := \left\{ f \in C^m(\mathrm{int}(M)) \, : \, \|f\|_{W_p^m(M)} < \infty \right\}$$

is a normed vector space with respect to $\|\cdot\|_{W_p^m(M)}$. Given $f \in \mathcal{C}_p^m(M)$ and $\alpha \in \{1, \ldots, N\}$, the function $f \circ \phi_\alpha^{-1}$ lies in $W_p^m(\phi_\alpha(U_\alpha))$ and satisfies

$$\left\| f \circ \phi_\alpha^{-1} \right\|_{W_p^m(\phi_\alpha(U_\alpha))} \le C(Q) \, \|f\|_{W_p^m(M)} . \tag{C.5}$$

Furthermore,

$$\|f\|_{W_p^m(M)}^{\mathcal{T}} := \sum_{\alpha=1}^{N} \left\| (\psi_\alpha f) \circ \phi_\alpha^{-1} \right\|_{W_p^m(\phi_\alpha(U_\alpha))}$$

defines a norm on $\mathcal{C}_p^m(M)$ equivalent to $\|\cdot\|_{W_p^m(M)}$ with equivalence constants depending on Q and the bounds on the derivatives of g and g^{-1} in the normal coordinates.

Proof. The set of functions $\mathcal{C}_p^m(M)$ is a vector space with respect to pointwise scalar multiplication and addition and it is straightforward to verify that $\|\cdot\|_{W_p^m(M)}$ defines a norm on it. We show by induction with respect to the variable $m \in \mathbb{N}_0$ that for $f \in \mathcal{C}_p^m(M)$ the function $f \circ \phi_\alpha^{-1}$ lies in $W_p^m(\phi_\alpha(U_\alpha))$ with

$$\left\| f \circ \phi_\alpha^{-1} \right\|_{W_p^m(\phi_\alpha(U_\alpha))} \le C(Q) \, \|f\|_{W_p^m(M)}$$

and that $\|\cdot\|_{W_p^m(M)}^{\mathcal{T}}$ defines an equivalent norm on $\mathcal{C}_p^m(M)$. This follows essentially from the identities (A.11) and (A.12) which yield in particular

$$Q^{-n} \le \det\left(g_{ij} \circ \phi_\alpha^{-1} \right) \le Q^n \qquad \text{in } \phi_\alpha(U_\alpha) .$$

In the following we write $\det(g_\alpha) := \det\left((g_{ij} \circ \phi_\alpha^{-1}) \right)$. In the case $m = 0$ we obtain for $f \in \mathcal{C}_p^0(M)$ that $f \circ \phi_\alpha^{-1} : \mathrm{int}\phi_\alpha(U_\alpha) \to \mathbb{R}$ is continuous and thus Lebesgue measurable and

$$\begin{aligned} \int_{\mathrm{int}\phi_\alpha(U_\alpha)} \left| f \circ \phi_\alpha^{-1} \right|^p \mathrm{d}x &\le Q^{n/2} \int_{\phi_\alpha(U_\alpha)} \left| f \circ \phi_\alpha^{-1} \right|^p \sqrt{\det(g_\alpha)} \, \mathrm{d}x = Q^{n/2} \int_{U_\alpha} |f|^p \mathrm{d}V_g \\ &\le Q^{n/2} \int_M |f|^p \mathrm{d}V_g < \infty . \end{aligned}$$

This shows $f \circ \phi_\alpha^{-1} \in L_p(\phi_\alpha(U_\alpha))$ and the estimate (C.5). Furthermore, the function $(\psi_\alpha f) \circ \phi_\alpha^{-1} : \mathrm{int}\phi_\alpha(U_\alpha) \to \mathbb{R}$ is continuous and thus measurable. As $\left| \psi_\alpha \circ \phi_\alpha^{-1} \right| \le 1$ on $\phi_\alpha(U_\alpha)$, we obtain

$$\int_{\mathrm{int}\phi_\alpha(U_\alpha)} \left| (\psi_\alpha f) \circ \phi_\alpha^{-1} \right|^p \mathrm{d}x \le \int_{\mathrm{int}\phi_\alpha(U_\alpha)} \left| f \circ \phi_\alpha^{-1} \right|^p \mathrm{d}x \le C(Q) \, \|f\|_{L_p(M)}^p < \infty$$

which shows $\|f\|_{L_p(M)}^{\mathcal{T}} \le C(Q) \, \|f\|_{L_p(M)}$. In particular, $\|f\|_{L_p(M)}^{\mathcal{T}}$ is well-defined for all $f \in \mathcal{C}_p^0(M)$ and it is straightforward to check that it defines a norm.
On the other hand we have with $I_\alpha := \{\beta \in \{1, \ldots, N\} : U_\alpha \cap U_\beta \neq \emptyset\}$,

$$\|f\|_{L_p(M)}^p = \sum_{\alpha=1}^{N} \int_{\phi_\alpha(U_\alpha)} \left(\psi_\alpha \circ \phi_\alpha^{-1} \right) \left| f \circ \phi_\alpha^{-1} \right|^p \sqrt{\det(g_\alpha)} \, \mathrm{d}x$$

$$\leq C(Q) \sum_{\alpha=1}^{N} \sum_{\beta \in I_\alpha} \int_{\phi_\alpha(U_\alpha \cap U_\beta)} \left|(\psi_\beta f) \circ \phi_\alpha^{-1}\right|^p \mathrm{d}x$$

$$\leq C(Q) \sum_{\alpha=1}^{N} \sum_{\beta \in I_\alpha} \int_{\phi_\beta(U_\alpha \cap U_\beta)} \left|(\psi_\beta f) \circ \phi_\beta^{-1}\right|^p \mathrm{d}x \leq C(Q)\big(\|f\|_{L_p(M)}^{\mathcal{T}}\big)^p .$$

This shows the claim in the case $m = 0$. Suppose that the claim is shown for $m \in \mathbb{N}_0$. Let $f \in \mathcal{C}_p^{m+1}(M)$ and $\alpha \in \{1, \ldots, N\}$ be given. We show that $f \circ \phi_\alpha^{-1}$ lies in $W_p^{m+1}(\phi_\alpha(U_\alpha))$. By induction hypothesis we know that $f \circ \phi_\alpha^{-1}$ lies in $W_p^m(\phi_\alpha(U_\alpha))$ and $f \in \mathcal{C}_p^{m+1}(M)$ implies that $f \circ \phi_\alpha^{-1}$ lies in $C^{m+1}(\mathrm{int}\phi_\alpha(U_\alpha))$. It remains to show that for any indices $i_1, \ldots, i_{m+1} \in \{1, \ldots, n\}$ the function $\frac{\partial(f \circ \phi_\alpha^{-1})}{\partial x_{i_1} \cdots \partial x_{i_{m+1}}}$ lies in $L_p(\phi_\alpha(U_\alpha))$. Lebesgue measurability follows from continuity of the function on $\mathrm{int}\phi_\alpha(U_\alpha)$. We observe that the estimate (A.12) implies

$$\sum_{i_1, \ldots, i_{m+1}=1}^{n} \big(\big(\nabla^{m+1} f\big)_{i_1 \ldots i_{m+1}}\big)^2 \circ \phi_\alpha^{-1} \leq C(Q)\big(\left|\nabla^{m+1} f\right|^2 \circ \phi_\alpha^{-1}\big),$$

where the component $\big(\nabla^{m+1} f\big)_{i_1 \ldots i_{m+1}} \circ \phi_\alpha^{-1}$ in the chart (U_α, ϕ_α) is given by formula (A.2), namely,

$$\big(\nabla^{m+1} f\big)_{i_1 \ldots i_{m+1}} \circ \phi_\alpha^{-1} = \frac{\partial^{m+1}(f \circ \phi_\alpha^{-1})}{\partial x_{i_1} \cdots \partial x_{i_{m+1}}} + \sum_{|\gamma| \leq m} \big(S_\gamma \circ \phi_\alpha^{-1}\big) \frac{\partial^{|\gamma|}(f \circ \phi_\alpha^{-1})}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}}$$

with coefficients $S_\gamma \circ \phi_\alpha^{-1}$ that are uniformly bounded on $\phi_\alpha(U_\alpha)$. As $\frac{\partial^{|\gamma|}(f \circ \phi_\alpha^{-1})}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}}$ lies in $L_p(\phi_\alpha(U_\alpha))$ for all $|\gamma| \leq m$ by induction hypothesis, it is enough to show $\big(\nabla^{m+1} f\big)_{i_1 \ldots i_{m+1}} \circ \phi_\alpha^{-1} \in L_p(\phi_\alpha(U_\alpha))$. This follows from

$$\begin{aligned}\int_{\phi_\alpha(U_\alpha)} \left|\big(\nabla^{m+1} f\big)_{i_1 \ldots i_{m+1}} \circ \phi_\alpha^{-1}\right|^p \mathrm{d}x &\leq C(Q) \int_{\phi_\alpha(U_\alpha)} \left|\nabla^{m+1} f\right|^p \circ \phi_\alpha^{-1} \,\mathrm{d}x \\ &\leq C(Q) \int_{\phi_\alpha(U_\alpha)} \left|\nabla^{m+1} f\right|^p \circ \phi_\alpha^{-1} \sqrt{\det(g_\alpha)}\, \mathrm{d}x \\ &\leq C(Q) \int_{U_\alpha} \left|\nabla^{m+1} f\right|^p \mathrm{d}V_g \leq C(Q) \int_M \left|\nabla^{m+1} f\right|^p \mathrm{d}V_g < \infty .\end{aligned}$$

Combined with the induction hypothesis this shows $f \circ \phi_\alpha^{-1} \in W_p^{m+1}(\phi_\alpha(U_\alpha))$ and the estimate (C.5). As $\psi_\alpha \circ \phi_\alpha^{-1}$ is smooth and compactly supported in $\phi_\alpha(U_\alpha)$, the product rule on $W_p^{m+1}(\phi_\alpha(U_\alpha))$ implies $(\psi_\alpha f) \circ \phi_\alpha^{-1} \in W_p^{m+1}(\phi_\alpha(U_\alpha))$. The induction hypothesis implies that

$$\left\|(\psi_\alpha f) \circ \phi_\alpha^{-1}\right\|_{W_p^m(\phi_\alpha(U_\alpha))} \leq C(Q) \left\|f\right\|_{W_p^m(M)} .$$

To prove the estimate $\|f\|_{W_p^{m+1}(M)}^{\mathcal{T}} \leq C(Q) \|f\|_{W_p^{m+1}(M)}$ it remains to show that the derivatives of $(\psi_\alpha f) \circ \phi_\alpha^{-1}$ of order $m+1$ can be bounded in $\|\cdot\|_{L_p(\phi_\alpha(U_\alpha))}$ by $\|f\|_{W_p^m(M)}$. Given multi-indices α, $\beta \in \mathbb{N}_0$ with $|\alpha| + |\beta| = m+1$ there holds

$$\begin{aligned}&\left\|\big(\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}(\psi_\alpha \circ \phi_\alpha^{-1})\big)\big(\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}(f \circ \phi_\alpha^{-1})\big)\right\|_{L_p(\phi_\alpha(U_\alpha))} \\ &\leq C \left\|\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}(f \circ \phi_\alpha^{-1})\right\|_{L_p(\phi_\alpha(U_\alpha))} \leq C \left\|f \circ \phi_\alpha^{-1}\right\|_{W_p^{m+1}(\phi_\alpha(U_\alpha))} \leq C(Q) \left\|f\right\|_{W_p^{m+1}(M)} .\end{aligned}$$

In particular, the expression $\|f\|_{W_p^{m+1}(M)}^{\mathcal{T}}$ is well-defined for $f \in \mathcal{C}_p^{m+1}(M)$ and defines a norm on $\mathcal{C}_p^{m+1}(M)$. To prove the estimate $\|f\|_{W_p^{m+1}(M)} \leq C(Q) \|f\|_{W_p^{m+1}(M)}^{\mathcal{T}}$ for $f \in \mathcal{C}_p^{m+1}(M)$, using the induction hypothesis, it remains to show

$$\int_M \left|\nabla^{m+1} f\right|^p \mathrm{d}V_g \leq C(Q)\big(\|f\|_{W_p^{m+1}(M)}^{\mathcal{T}}\big)^p . \tag{C.6}$$

To show (C.6) we observe that the estimate (A.12) implies

$$\begin{aligned}\left|\nabla^{m+1} f\right|^2 &= g^{i_1 j_1} \cdots g^{i_{m+1} j_{m+1}} \left(\nabla^{m+1} f\right)_{i_1 \dots i_{m+1}} \left(\nabla^{m+1} f\right)_{j_1 \dots j_{m+1}} \\ &\le C(Q) \sum_{i_1, \dots, i_{m+1}=1}^{n} \left(\left(\nabla^{m+1} f\right)_{i_1 \dots i_{m+1}} \right)^2 ,\end{aligned}$$

where the component $\left(\nabla^{m+1} f\right)_{i_1 \dots i_{m+1}}$ in the chart (U_α, ϕ_α) is given by formula (A.2), namely,

$$\left(\nabla^{m+1} f\right)_{i_1 \dots i_{m+1}} \circ \phi_\alpha^{-1} = \frac{\partial^{m+1}(f \circ \phi_\alpha^{-1})}{\partial x_{i_1} \cdots \partial x_{i_{m+1}}} + \sum_{|\gamma| \le m} \left(S_\gamma \circ \phi_\alpha^{-1}\right) \frac{\partial^{|\gamma|}(f \circ \phi_\alpha^{-1})}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}}$$

with coefficients $S_\gamma \circ \phi_\alpha^{-1}$ that are uniformly bounded on $\phi_\alpha(U_\alpha)$. Given any $\beta \in \{1, \dots, N\}$ the chain rule implies for every point x in the interior of $\phi_\alpha(U_\alpha \cap U_\beta)$,

$$\partial_{x_1}^{\gamma_1} \cdots \partial_{x_n}^{\gamma_n} \left((\psi_\beta f) \circ \phi_\alpha^{-1} \right) = \sum_{\delta \le \gamma} M_{\gamma\delta}(x) \big(\partial_{x_1}^{\delta_1} \cdots \partial_{x_n}^{\delta_n} \big((\psi_\beta f) \circ \phi_\beta^{-1} \big) \big) (\phi_\beta \circ \phi_\alpha^{-1})(x) \tag{C.7}$$

where $M_{\gamma\delta}$ is a polynomial of degree at most $|\delta|$ in derivatives of order at most $|\gamma|$ of the components of $\phi_\beta \circ \phi_\alpha^{-1}$. Using estimates like

$$\int_{\phi_\alpha(U_\alpha)} \left(\psi_\alpha \circ \phi_\alpha^{-1}\right) \left| \frac{\partial^{|\gamma|}(f \circ \phi_\alpha^{-1})}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}} \right|^p \mathrm{d}x \le C \sum_{\beta \in I_\alpha} \int_{\phi_\alpha(U_\alpha \cap U_\beta)} \left| \frac{\partial^{|\gamma|}((\psi_\beta f) \circ \phi_\alpha^{-1})}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}} \right|^p \mathrm{d}x \,,$$

the induction hypothesis, the formula (C.7) and the fact that all derivatives of $\phi_\beta \circ \phi_\alpha^{-1}$ are uniformly bounded on $\phi_\alpha(U_\alpha \cap U_\beta)$, we obtain, using the transformation formula to pass from $\phi_\alpha(U_\alpha \cap U_\beta)$ to $\phi_\beta(U_\alpha \cap U_\beta)$,

$$\begin{aligned}\left\| \left(\nabla^{m+1} f\right)_{i_1 \dots i_{m+1}} \right\|_{L_p(M)}^p &\le C(Q) \sum_{\alpha=1}^{N} \int_{\phi_\alpha(U_\alpha)} (\psi_\alpha \circ \phi_\alpha^{-1}) \left| \frac{\partial^{m+1}(f \circ \phi_\alpha^{-1})}{\partial x_{i_1} \cdots \partial x_{i_{m+1}}} \right|^p \mathrm{d}x \\ &\quad + C(Q) \sum_{\alpha=1}^{N} \sum_{|\gamma| \le m} \int_{\phi_\alpha(U_\alpha)} \left(\psi_\alpha \circ \phi_\alpha^{-1}\right) \left| \frac{\partial^{|\gamma|}(f \circ \phi_\alpha^{-1})}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}} \right|^p \mathrm{d}x \\ &\le C(Q) \sum_{\alpha=1}^{N} \sum_{\beta \in I_\alpha} \int_{\phi_\beta(U_\alpha \cap U_\beta)} \left| \frac{\partial^{m+1}((\psi_\beta f) \circ \phi_\beta^{-1})}{\partial x_{i_1} \cdots \partial x_{i_{m+1}}} \right|^p \mathrm{d}x \\ &\quad + C(Q) \sum_{\alpha=1}^{N} \sum_{\beta \in I_\alpha} \sum_{|\gamma| \le m} \int_{\phi_\beta(U_\alpha \cap U_\beta)} \left| \frac{\partial^{|\gamma|}((\psi_\beta f) \circ \phi_\beta^{-1})}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}} \right|^p \mathrm{d}x \\ &\le C(Q) \sum_{\beta=1}^{N} \left\| (\psi_\beta f) \circ \phi_\beta^{-1} \right\|_{W_p^{m+1}(\phi_\beta(U_\beta))}^p \le C(Q) \big(\|f\|_{W_p^{m+1}(M)}^{\mathcal{T}} \big)^p .\end{aligned}$$

This shows (C.6) since

$$\left\| |\nabla^{m+1} f|^p \right\|_{L_1(M)}^p \le C(Q) \sum_{i_1, \dots, i_{m+1}=1}^{n} \left\| \left(\nabla^{m+1} f\right)_{i_1 \dots i_{m+1}} \right\|_{L_p(M)}^p .$$

□

Definition C.4 (The classical definition of the Sobolev space $W_p^m(M)$)**.** Let $p \in [1, \infty)$ and $m \in \mathbb{N}$ be given. The *Sobolev space* $W_p^m(M)$ is defined to be the completion of the normed space $\big(\mathcal{C}_p^m(M), \|\cdot\|_{W_p^m(M)}\big)$. We set $W_p^0(M) := L_p(M)$.

It is worth mentioning that this notion is consistent with the Sobolev spaces in Euclidean space.

Corollary C.5. *Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n \in \mathbb{N}$ with Levi-Civita connection D, $p \in [1, \infty)$, $m \in \mathbb{N}$. Every Cauchy sequence $(f_j)_{j\in\mathbb{N}}$ in $\left(\mathcal{C}_p^m(M), \|\cdot\|_{W_p^m(M)}\right)$ converging to 0 in $\|\cdot\|_{L_p(M)}$ also converges to 0 in $\|\cdot\|_{W_p^m(M)}$.*

Proof. Let $(U_\alpha, \phi_\alpha, \psi_\alpha)$ be a uniform localisation system. The equivalence of norms on $\mathcal{C}_p^m(M)$ implies for all $\alpha \in \{1, \dots, N\}$,

$$\left\|(\psi_\alpha(f_j - f_l) \circ \phi_\alpha^{-1}\right\|_{W_p^m(\phi_\alpha(U_\alpha))} \leq C(Q)\, \|f_j - f_l\|_{W_p^m(M)} \to 0 \qquad \text{as } j\,, l \to \infty\,.$$

By completeness of $W_p^m(\phi_\alpha(U_\alpha))$ there exists a unique function $f^\alpha \in W_p^m(\phi_\alpha(U_\alpha))$ such that $\left((\psi_\alpha f_j) \circ \phi_\alpha^{-1}\right)_{j\in\mathbb{N}}$ converges to f^α in $\|\cdot\|_{W_p^m(\phi_\alpha(U_\alpha))}$. In particular,

$$\left\|(\psi_\alpha f_j) \circ \phi_\alpha^{-1} - f^\alpha\right\|_{L_p(\phi_\alpha(U_\alpha))} \to 0 \qquad \text{as } j \to \infty\,.$$

The equivalence of norms shown in Proposition C.2 in the case $m = 0$ implies

$$\left\|(\psi_\alpha f_j) \circ \phi_\alpha^{-1}\right\|_{L_p(\phi_\alpha(U_\alpha))} \leq C(Q)\, \|f_j\|_{L_p(M)}^{\mathcal{T}} \leq C(Q)\, \|f_j\|_{L_p(M)} \to 0 \qquad \text{as } j \to \infty$$

by hypothesis which implies $f^\alpha = 0$ by uniqueness of the limit in $L_p(\phi_\alpha(U_\alpha))$. Thus in the limit $j \to \infty$ we obtain

$$\|f_j\|_{W_p^m(M)} \leq C(Q)\, \|f_j\|_{W_p^m(M)}^{\mathcal{T}} = C(Q) \sum_{\alpha=1}^{N} \left\|(\psi_\alpha f_j) \circ \phi_\alpha^{-1}\right\|_{W_p^m(\phi_\alpha(U_\alpha))} \to 0\,.$$

□

Proposition C.6 (Characterisation of Sobolev spaces on compact manifolds)**.** *Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n \in \mathbb{N}$ with Levi-Civita connection D, $p \in [1, \infty)$ and $m \in \mathbb{N}$. The Sobolev space $W_p^m(M)$ can be identified with*

$$\left\{f \in L_p(M) \,:\, \exists \text{ sequence } (f_j)_{j\in\mathbb{N}} \subset \mathcal{C}_p^m(M) \text{ with } \|f - f_j\|_{L_p(M)} \to 0\,, \|f_j - f_l\|_{W_p^m(M)} \to 0\right\}$$

which is a linear subspace of $L_p(M)$. It is a Banach space with respect to $\|\cdot\|_{W_p^m(M)}$ where for $f \in W_p^m(M)$ and $(f_j)_{j\in\mathbb{N}} \subset \mathcal{C}_p^m(M)$ any approximating sequence for f,

$$\|f\|_{W_p^m(M)} := \lim_{j\to\infty} \|f_j\|_{W_p^m(M)}\ .$$

Proof. The completion of the metric space $\left(\mathcal{C}_p^m(M), d\right)$ with $d(f, g) := \|f - g\|_{W_p^m(M)}$ is given by the complete metric space $\left(\widetilde{X}, \widetilde{d}\right)$ where $\widetilde{X}$ is the space of sequences $(f_j)_{j\in\mathbb{N}} \subset \mathcal{C}_p^m(M)$ that are Cauchy in $\left(\mathcal{C}_p^m(M), d\right)$ with the equivalence relation

$$(f_j)_{j\in\mathbb{N}} = (g_j)_{j\in\mathbb{N}} \text{ in } \widetilde{X} \text{ if and only if } \lim_{j\to\infty} \|f_j - g_j\|_{W_p^m(M)} = 0$$

and with $\widetilde{d}\left((f_j)_{j\in\mathbb{N}}, (g_j)_{j\in\mathbb{N}}\right) := \lim_{j\to\infty} d(f_j, g_j)$. Component wise addition and scalar multiplication define a vector space structure on $\left(\widetilde{X}, \widetilde{d}\right)$ and

$$\|(f_j)_{j\in\mathbb{N}}\| := \lim_{j\to\infty} \|f_j\|_{W_p^m(M)}$$

is a norm on $\widetilde{X}$ with respect to which it is complete. Let $\widetilde{f} \in \widetilde{X}$ be represented by a Cauchy sequence $(f_j)_{j\in\mathbb{N}}$ in $\left(\mathcal{C}_p^m(M), \|\cdot\|_{W_p^m(M)}\right)$. Then $(f_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $\left(\mathcal{C}_p^m(M), \|\cdot\|_{L_p(M)}\right)$ converging in $\|\cdot\|_{L_p(M)}$ to a function $f \in L_p(M)$ by completeness of $L_p(M)$. If $(g_j)_{j\in\mathbb{N}}$ is another sequence representing $\widetilde{f}$, then as $j \to \infty$,

$$\|f - g_j\|_{L_p(M)} \le \|f - f_j\|_{L_p(M)} + \|f_j - g_j\|_{L_p(M)} \le \|f - f_j\|_{L_p(M)} + \|f_j - g_j\|_{W_p^m(M)} \to 0\,.$$

We denote the limit function f by $\overset{L_p}{\lim\limits_{j\to\infty}} f_j$. Thus we obtain a well-defined linear mapping $T : \widetilde{X} \to L_p(M)$ by $\widetilde{f} = (f_j)_{j\in\mathbb{N}} \mapsto \overset{L_p}{\lim\limits_{j\to\infty}} f_j$. The mapping T is injective. Indeed, suppose that $\widetilde{f} \in \widetilde{X}$ satisfies $T\widetilde{f} = 0$ and let $(f_j)_{j\in\mathbb{N}}$ be any representative of $\widetilde{f}$. Thus, $(f_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $\left(\mathcal{C}_p^m(M), \|\cdot\|_{W_p^m(M)}\right)$ with $\overset{L_p}{\lim\limits_{j\to\infty}} f_j = 0$. By Corollary C.5 the sequence $(f_j)_{j\in\mathbb{N}}$ converges to 0 in $\|\cdot\|_{W_p^m(M)}$ which means precisely that $(0)_{j\in\mathbb{N}}$ is a representative of $\widetilde{f}$. This shows $\widetilde{f} = 0$ and we may identify $\widetilde{X} = W_p^m(M)$ with its image $T\left(W_p^m(M)\right)$ in $L_p(M)$. □

In the sequel we identify $W_p^m(M)$ with the linear subspace of $L_p(M)$.

Lemma C.7. *Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n \in \mathbb{N}$ with Levi-Civita connection D and uniform localisation system $(U_\alpha, \phi_\alpha, \psi_\alpha)$. Let $p \in [1, \infty)$, $m \in \mathbb{N}_0$. Given $f \in W_p^m(M)$ and $(f_l)_{l\in\mathbb{N}} \subset \mathcal{C}_p^m(M)$ an approximating sequence for f, $1 \le r \le m$, $i_1, \dots, i_r \in \{1, \dots, n\}$, the components $(\nabla^r f_l)_{i_1\cdots i_r}$ of $\nabla^r f_l$ with respect to (U_α, ϕ_α), $l \in \mathbb{N}$, form a Cauchy sequence in $L_p(U_\alpha)$. The limit function is denoted by $(\nabla^r f)_{i_1\cdots i_r}$ and is independent of the choice of approximating sequence for f. If $\alpha, \beta \in \{1, \dots, N\}$ satisfy $U_\alpha \cap U_\beta \neq \emptyset$, then the L_p-limits $(\nabla^r f)^\alpha_{i_1\cdots i_r}$ and $(\nabla^r f)^\beta_{i_1\cdots i_r}$ of the components $(\nabla^r f_l)^\alpha_{i_1\dots i_r}$ and $(\nabla^r f_l)^\beta_{i_1\dots i_r}$ of $\nabla^r f_l$, $l \in \mathbb{N}$, with respect to the charts (U_α, ϕ_α) and (U_β, ϕ_β), respectively, satisfy the following identity almost everywhere on $U_\alpha \cap U_\beta$:*

$$(\nabla^r f)^\alpha_{i_1\cdots i_r}\, \mathrm{d}x^{\alpha,i_1} \otimes \cdots \otimes \mathrm{d}x^{\alpha,i_r} = (\nabla^r f)^\beta_{i_1\cdots i_r}\, \mathrm{d}x^{\beta,i_1} \otimes \cdots \otimes \mathrm{d}x^{\beta,i_r}\,.$$

Proof. We observe that the uniform estimate (A.12) on the inverse metric implies

$$|\nabla^r f_l|^2 = g^{i_1j_1}\cdots g^{i_rj_r}\,(\nabla^r f_l)_{i_1\cdots i_r}\,(\nabla^r f_l)_{j_1\cdots j_r} \ge Q^{-r} \sum_{i_1,\dots,i_r=1}^{n} \left((\nabla^r f_l)_{i_1\cdots i_r}\right)^2$$

and, by linearity of the covariant derivative,

$$|\nabla^r (f_l - f_{\tilde{l}})|^2 \ge C(Q) \sum_{i_1,\dots,i_r=1}^{n} \left((\nabla^r (f_l - f_{\tilde{l}}))_{i_1\cdots i_r}\right)^2.$$

As integration on U_α with respect to the induced volume measure is monotone, we obtain

$$\begin{aligned}\int_{U_\alpha} \left|(\nabla^r (f_l - f_{\tilde{l}}))_{i_1\cdots i_r}\right|^p \mathrm{d}V_g &\le C(Q)\int_{U_\alpha} |\nabla^r (f_l - f_{\tilde{l}})|^p\, \mathrm{d}V_g \\ &\le C(Q)\int_{M} |\nabla^r (f_l - f_{\tilde{l}})|^p\, \mathrm{d}V_g \to 0\end{aligned}$$

as $l\,, \tilde{l} \to \infty$. By completeness of $L_p(U_\alpha)$ there exists a limit function in $L_p(U_\alpha)$, denoted by $(\nabla^r f)_{i_1\dots i_r}$. If $(g_l)_{l\in\mathbb{N}} \subset \mathcal{C}_p^m(M)$ is another approximating sequence for f in $W_p^m(M)$, then as $l \to \infty$,

$$\left\|(\nabla^r g_l)_{i_1\cdots i_r} - (\nabla^r f)_{i_1\cdots i_r}\right\|^p_{L_p(U_\alpha)}$$

$$\leq C\left(\left\|(\nabla^r(g_l - f_l))_{i_1\cdots i_r}\right\|^p_{L_p(U_\alpha)} + \left\|(\nabla^r f_l)_{i_1\cdots i_r} - (\nabla^r f)_{i_1\cdots i_r}\right\|^p_{L_p(U_\alpha)}\right)$$
$$\leq C(Q)\int_M |\nabla^r(g_l - f_l)|^p \, \mathrm{d}V_g + C\left\|(\nabla^r f_l)_{i_1\cdots i_r} - (\nabla^r f)_{i_1\cdots i_r}\right\|^p_{L_p(U_\alpha)}$$
$$\leq C(Q)\,\|g_l - f_l\|^p_{W_p^m(M)} + C\left\|(\nabla^r f_l)_{i_1\cdots i_r} - (\nabla^r f)_{i_1\cdots i_r}\right\|^p_{L_p(U_\alpha)} \to 0\,.$$

This shows that the limit function $(\nabla^r f)_{i_1\cdots i_r} \in L_p(U_\alpha)$ is independent of the choice of approximating sequence for f. For every $p \in U_\alpha \cap U_\beta$ and all $l \in \mathbb{N}$ we have the following identity in $T_r^0(T_pM)$:

$$(\nabla^r f_l)^\alpha_{i_1\cdots i_r}\,\mathrm{d}x^{\alpha,i_1}\otimes\cdots\otimes \mathrm{d}x^{\alpha,i_r} = (\nabla^r f_l)^\beta_{i_1\cdots i_r}\,\mathrm{d}x^{\beta,i_1}\otimes\cdots\otimes \mathrm{d}x^{\beta,i_r}\,.$$

Passing to the limit $l \to \infty$ implies that for almost every $p \in U_\alpha \cap U_\beta$ it holds in $T_r^0(T_pM)$:

$$(\nabla^r f)^\alpha_{i_1\cdots i_r}\,\mathrm{d}x^{\alpha,i_1}\otimes\cdots\otimes \mathrm{d}x^{\alpha,i_r} = (\nabla^r f)^\beta_{i_1\cdots i_r}\,\mathrm{d}x^{\beta,i_1}\otimes\cdots\otimes \mathrm{d}x^{\beta,i_r}\,.$$

□

Definition C.8 (Covariant derivative on $W_p^m(M)$)**.** Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n \in \mathbb{N}$ with Levi-Civita connection D and uniform localisation system $(U_\alpha, \phi_\alpha, \psi_\alpha)$. Let $p \in [1,\infty)$, $m \in \mathbb{N}_0$. Given $f \in W_p^m(M)$ and $0 \leq r \leq m$ the *covariant derivative* $\nabla^r f$ is defined to be the $(r,0)$-tensor field on M with components in the chart (U_α, ϕ_α) given by $(\nabla^r f)_{i_1\cdots i_r} \in L_p(U_\alpha)$.

In view of Lemma C.7 the object $\nabla^r f$ is a $(r,0)$-tensor field that is defined almost everywhere on M. It is uniquely determined by its components $(\nabla^r f)_{i_1\cdots i_r}$ with respect to the charts (U_α, ϕ_α) that are themselves determined up to equality almost everywhere.

Proposition C.9. *Let $(M, \mathcal{A}, g)$ be a smooth compact Riemannian manifold with or without boundary of dimension $n \in \mathbb{N}$ with Levi-Civita connection D and uniform localisation system $(U_\alpha, \phi_\alpha, \psi_\alpha)$. Let $p \in [1,\infty)$, $m \in \mathbb{N}_0$ and $f \in W_p^m(M)$. Then for all $0 \leq r \leq m$ the expression $|\nabla^r f|^p$, in local coordinates given by $g^{i_1j_1}\cdots g^{i_rj_r}(\nabla^r f)_{i_1\cdots i_r}(\nabla^r f)_{j_1\cdots j_r}$, is integrable on M and we have the identity*

$$\|f\|_{W_p^m(M)} = \sum_{r=0}^m \left(\int_M |\nabla^r f|^p \,\mathrm{d}V_g\right)^{1/p}\,.$$

Proof. In local coordinates the estimate (A.12) implies

$$|\nabla^r f|^p \leq C(Q)\sum_{i_1,\ldots,i_r=1}^n \left|(\nabla^r f)_{i_1\cdots i_r}\right|^p$$

which shows that $|\nabla^r f|^p$ is integrable on M. Let $(f_l)_{l\in\mathbb{N}} \subset \mathcal{C}_p^m(M)$ be any approximating sequence for f in $W_p^m(M)$. Then for any $0 \leq r \leq m$ we have as $l \to \infty$,

$$\int_M |\nabla^r f - \nabla^r f_l|^p \,\mathrm{d}V_g = \sum_{\alpha=1}^N \int_{\phi_\alpha(U_\alpha)} (\psi_\alpha |\nabla^r(f - f_l)|^p)\circ\phi_\alpha^{-1}\sqrt{\det(g_{ij}\circ\phi_\alpha^{-1})}\,\mathrm{d}x$$
$$\leq C(Q)\sum_{\alpha=1}^N\sum_{i_1,\ldots,i_r=1}^n \int_{\phi_\alpha(U_\alpha)} \left|(\nabla^r(f - f_l))_{i_1\cdots i_r}\right|^p\circ\phi_\alpha^{-1}\sqrt{\det(g_{ij}\circ\phi_\alpha^{-1})}\,\mathrm{d}x$$
$$\leq C(Q)\sum_{\alpha=1}^N\sum_{i_1,\ldots,i_r=1}^n \left\|(\nabla^r f)_{i_1\cdots i_r} - (\nabla^r f_l)_{i_1\cdots i_r}\right\|^p_{L_p(U_\alpha)} \to 0\,.$$

This shows

$$\|f\|_{W_p^m(M)} = \lim_{l\to\infty} \|f_l\|_{W_p^m(M)} = \lim_{l\to\infty} \sum_{r=0}^m \left(\int_M |\nabla^r f_l|^p \, \mathrm{d}V_g \right)^{1/p} = \sum_{r=0}^m \left(\int_M |\nabla^r f|^p \, \mathrm{d}V_g \right)^{1/p} .$$

□

We finally give a proof of Proposition C.1.

Proof of Proposition C.1. The proof follows the instructions given in [96]. We prove the claim by induction with respect to the variable $m \in \mathbb{N}_0$. Suppose that f lies in $L_p(M)$. Then $f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \mathbb{R}$ is Lebesgue measurable for all $\alpha \in \{1, \dots, N\}$ and so is $(\psi_\alpha f) \circ \phi_\alpha^{-1}$. For all $\alpha \in \{1, \dots, N\}$ the metric satisfies

$$Q^{-n} \le \det\left(g_{ij} \circ \phi_\alpha^{-1}\right) \le Q^n \qquad \text{in } \phi_\alpha(U_\alpha) .$$

This implies $f \circ \phi_\alpha^{-1} \in L_p(\phi_\alpha(U_\alpha))$ as

$$\begin{aligned}
\int_{\phi_\alpha(U_\alpha)} \left|f \circ \phi_\alpha^{-1}\right|^p \mathrm{d}x &\le Q^{n/2} \int_{\phi_\alpha(U_\alpha)} \left|f \circ \phi_\alpha^{-1}\right|^p Q^{-n/2} \mathrm{d}x \\
&\le Q^{n/2} \int_{\phi_\alpha(U_\alpha)} \left|f \circ \phi_\alpha^{-1}\right|^p \det\left(g_{ij} \circ \phi_\alpha^{-1}\right)^{1/2} \mathrm{d}x \\
&= Q^{n/2} \int_{U_\alpha} |f|^p \mathrm{d}V_g \le Q^{n/2} \int_M |f|^p \mathrm{d}V_g < \infty
\end{aligned}$$

which shows the estimate (C.2). Furthermore, $(\psi_\alpha f) \circ \phi_\alpha^{-1}$ lies in $L_p\left(\phi_\alpha(U_\alpha)\right)$ and as $|\psi_\alpha \circ \phi_\alpha^{-1}| \le 1$ on $\phi_\alpha(U_\alpha)$, the above estimate implies

$$\int_{\phi_\alpha(U_\alpha)} |(\psi_\alpha f) \circ \phi_\alpha^{-1}|^p \mathrm{d}x \le \int_{\phi_\alpha(U_\alpha)} |f \circ \phi_\alpha^{-1}|^p \mathrm{d}x \le C(Q) \int_M |f|^p \mathrm{d}V_g .$$

Suppose on the other hand that $(\psi_\alpha f) \circ \phi_\alpha^{-1} \in L_p\left(\phi_\alpha(U_\alpha)\right)$ for all $\alpha \in \{1, \dots, N\}$. Then we obtain for any chart $(U, \phi) \in \mathcal{A}$ and $I_\phi := \{\alpha \in \{1, \dots, N\} \,:\, U_\alpha \cap U \neq \emptyset\}$,

$$f \circ \phi^{-1} = \sum_{\alpha \in I_\phi} (\psi_\alpha f) \circ \phi^{-1} = \sum_{\alpha \in I_\phi} \left((\psi_\alpha f) \circ \phi_\alpha^{-1}\right) \circ \left(\phi_\alpha \circ \phi^{-1}\right) ,$$

which shows that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is measurable.
Furthermore, with $I_\alpha := \{\beta \in \{1, \dots, N\} : U_\alpha \cap U_\beta \neq \emptyset\}$ we obtain,

$$\begin{aligned}
\int_M |f|^p \mathrm{d}V_g &= \sum_{\alpha=1}^N \int_{\phi_\alpha(U_\alpha)} \left(\psi_\alpha \circ \phi_\alpha^{-1}\right) |f \circ \phi_\alpha^{-1}|^p \det\left(g_{ij} \circ \phi_\alpha^{-1}\right)^{1/2} \mathrm{d}x \\
&\le C(Q) \sum_{\alpha=1}^N \sum_{\beta \in I_\alpha} \int_{\phi_\alpha(U_\alpha \cap U_\beta)} \left(\psi_\alpha \circ \phi_\alpha^{-1}\right) \left|(\psi_\beta f) \circ \phi_\alpha^{-1}\right|^p \mathrm{d}x \\
&\le C(Q) \sum_{\alpha=1}^N \sum_{\beta \in I_\alpha} \int_{\phi_\alpha(U_\alpha \cap U_\beta)} \left|(\psi_\beta f) \circ \phi_\beta^{-1}\right|^p \circ \left(\phi_\beta \circ \phi_\alpha^{-1}\right) \mathrm{d}x \\
&\le C(Q) \sum_{\alpha=1}^N \sum_{\beta \in I_\alpha} \int_{\phi_\beta(U_\alpha \cap U_\beta)} \left|(\psi_\beta f) \circ \phi_\beta^{-1}\right|^p |\det D(\phi_\alpha \circ \phi_\beta^{-1})| \,\mathrm{d}x \\
&\le C(Q) \sum_{\beta=1}^N \left\|(\psi_\beta f) \circ \phi_\beta^{-1}\right\|_{L_p(\phi_\beta(U_\beta))}^p .
\end{aligned}$$

This shows the claim in the case $m = 0$. To prove the statement in the case $m = 1$ we let $f \in W_p^1(M)$ be given. Then by Proposition C.6 there exists a sequence $(f_l)_{l\in\mathbb{N}}$ in $\mathcal{C}_p^1(M)$ such that $\|f - f_l\|_{L_p(M)} \to 0$ as $l \to \infty$ and with the property that $\|f_l - f_{\tilde{l}}\|_{W_p^1(M)} \to 0$ for $l, \tilde{l} \to \infty$. The definition of the covariant derivative and the uniform bound (A.12) on the inverse metric imply for all $\alpha \in \{1, \dots, N\}$ on $\phi_\alpha(U_\alpha)$,

$$\begin{aligned}
|\nabla(f_l - f_{\tilde{l}})|^p \circ \phi_\alpha^{-1} &= \left(g^{ij} \left(\nabla(f_l - f_{\tilde{l}})\right)_i \left(\nabla(f_l - f_{\tilde{l}})\right)_j\right)^{p/2} \circ \phi_\alpha^{-1} \\
&\geq Q^{-p/2}\Big(\sum_{i=1}^n \left(\left(\nabla(f_l - f_{\tilde{l}})\right)_i\right)^2\Big)^{p/2} \circ \phi_\alpha^{-1} \\
&= Q^{-p/2}\Big(\sum_{i=1}^n \left(\partial_{x^i}\left((f_l - \tilde{f}_l) \circ \phi_\alpha^{-1}\right)\right)^2\Big)^{p/2} \\
&= Q^{-p/2}\left(\left|\nabla\left((f_l - f_{\tilde{l}}) \circ \phi_\alpha^{-1}\right)\right|^2\right)^{p/2} .
\end{aligned}$$

This implies in particular

$$\begin{aligned}
\left\|\nabla((f_l - f_{\tilde{l}}) \circ \phi_\alpha^{-1})\right\|^p_{L_p(\phi_\alpha(U_\alpha))} &= \int_{\phi_\alpha(U_\alpha)} \left|\nabla((f_l - f_{\tilde{l}}) \circ \phi_\alpha^{-1})\right|^p \mathrm{d}x \\
&\leq C(Q) \int_{\phi_\alpha(U_\alpha)} |\nabla(f_l - f_{\tilde{l}})|^p \circ \phi_\alpha^{-1} \,\mathrm{d}x \\
&\leq C(Q) \int_{\phi_\alpha(U_\alpha)} |\nabla(f_l - f_{\tilde{l}})|^p \circ \phi_\alpha^{-1} \sqrt{\det(g_{ij} \circ \phi_\alpha^{-1})}\,\mathrm{d}x \\
&= C(Q) \int_{U_\alpha} |\nabla(f_l - f_{\tilde{l}})|^p \,\mathrm{d}V_g \leq C(Q) \int_M |\nabla(f_l - f_{\tilde{l}})|^p \,\mathrm{d}V_g .
\end{aligned}$$

As further

$$\left\|(f_l - f_{\tilde{l}}) \circ \phi_\alpha^{-1}\right\|^p_{L_p(\phi_\alpha(U_\alpha))} \leq C(Q) \, \|f_l - f_{\tilde{l}}\|^p_{L_p(M)} ,$$

we conclude that the family of C^1–functions $\left(f_l \circ \phi_\alpha^{-1}\right)_{l\in\mathbb{N}}$ forms a Cauchy sequence in $W_p^1(\phi_\alpha(U_\alpha))$ converging to a function $g \in W_p^1(\phi_\alpha(U_\alpha))$. On the other hand we observe for $l \to \infty$,

$$\left\|(f - f_l) \circ \phi_\alpha^{-1}\right\|^p_{L_p(\phi_\alpha(U_\alpha))} \leq C(Q) \, \|f - f_l\|^p_{L_p(M)} \to 0 .$$

By uniqueness of the limits in $L_p(\phi_\alpha(U_\alpha))$ we conclude that $g = f \circ \phi_\alpha^{-1} \in W_p^1(\phi_\alpha(U_\alpha))$. The estimate (C.2) follows from (C.5) as

$$\left\|f \circ \phi_\alpha^{-1}\right\|_{W_p^1(\phi_\alpha(U_\alpha))} = \lim_{l\to\infty} \left\|f_l \circ \phi_\alpha^{-1}\right\|_{W_p^1(\phi_\alpha(U_\alpha))} \leq C(Q) \lim_{l\to\infty} \|f_l\|_{W_p^1(M)} = C(Q) \, \|f\|_{W_p^1(M)} .$$

The equivalence of the norms $\|\cdot\|_{W_p^1(M)}$ and $\|\cdot\|^{\mathcal{T}}_{W_p^1(M)}$ on $\mathcal{C}_p^1(M)$ shown in Proposition C.3 gives the desired estimate

$$\begin{aligned}
\left\|(\psi_\alpha f) \circ \phi_\alpha^{-1}\right\|_{W_p^1(\phi_\alpha(U_\alpha))} &= \lim_{l\to\infty} \left\|(\psi_\alpha f_l) \circ \phi_\alpha^{-1}\right\|_{W_p^1(\phi_\alpha(U_\alpha))} \leq \lim_{l\to\infty} \|f_l\|^{\mathcal{T}}_{W_p^1(M)} \\
&\leq C(Q) \lim_{l\to\infty} \|f_l\|_{W_p^1(M)} = C(Q) \, \|f\|_{W_p^1(M)} .
\end{aligned}$$

Suppose on the other hand that $(\psi_\alpha f) \circ \phi_\alpha^{-1}$ lies in $W_p^1(\phi_\alpha(U_\alpha))$ for all $\alpha \in \{1, \dots, N\}$. The case $m = 0$ implies $f \in L_p(M)$ and there exist sequences $(g_l^\alpha)_{l\in\mathbb{N}}$ with $g_l^\alpha \in C^1(\operatorname{int}\phi_\alpha(U_\alpha)) \cap W_p^1(\phi_\alpha(U_\alpha))$ such that $(g_l^\alpha)_{l\in\mathbb{N}}$ converges in $W_p^1(\phi_\alpha(U_\alpha))$ to $(\psi_\alpha f) \circ \phi_\alpha^{-1}$. For $\alpha \in \{1, \dots, N\}$ let $\xi_\alpha \in C^\infty(M)$ be such that $\operatorname{supp}\xi_\alpha \subset U_\alpha$, $0 \leq \xi_\alpha \leq 1$ on M and $\xi_\alpha \equiv 1$ on $\operatorname{supp}\psi_\alpha$. The existence of such a bump function follows from [89, Proposition 2.2.5]. We extend the function $g_l^\alpha \circ \phi_\alpha$ by 0 to the entire manifold M. Then the family of functions $(g_l)_{l\in\mathbb{N}}$ with

$$g_l := \sum_{\alpha=1}^{N} \xi_\alpha \left(g_l^\alpha \circ \phi_\alpha\right)$$

lies in $C^1(\mathrm{int} M) \cap L_p(M)$ and converges to f in $L_p(M)$. Indeed, with $I_\alpha := \{\beta \in \{1,\dots,N\} : U_\alpha \cap U_\beta \neq \emptyset\}$ we observe

$$\begin{aligned}
\|g_l - f\|_{L_p(M)}^p &= \sum_{\alpha=1}^{N} \int_{\phi_\alpha(U_\alpha)} (\psi_\alpha |g_l - f|^p) \circ \phi_\alpha^{-1} \sqrt{\det(g_{ij} \circ \phi_\alpha^{-1})}\,\mathrm{d}x \\
&\leq C(Q) \sum_{\alpha=1}^{N} \int_{\phi_\alpha(U_\alpha)} (\psi_\alpha \circ \phi_\alpha^{-1}) \left|g_l \circ \phi_\alpha^{-1} - f \circ \phi_\alpha^{-1}\right|^p \mathrm{d}x \\
&\leq C(Q) \sum_{\alpha=1}^{N} \int_{\phi_\alpha(U_\alpha)} \left|g_l \circ \phi_\alpha^{-1} - f \circ \phi_\alpha^{-1}\right|^p \mathrm{d}x \\
&\leq C(Q) \sum_{\alpha=1}^{N} \sum_{\beta\in I_\alpha} \int_{\phi_\alpha(U_\alpha\cap U_\beta)} \left|(\xi_\beta \circ \phi_\alpha^{-1})\big(g_l^\beta \circ \phi_\beta \circ \phi_\alpha^{-1}\big) - (\psi_\beta f) \circ \phi_\alpha^{-1}\right|^p \mathrm{d}x \\
&= C(Q) \sum_{\alpha=1}^{N} \sum_{\beta\in I_\alpha} \int_{\phi_\alpha(U_\alpha\cap U_\beta)} \left|\xi_\beta \circ \phi_\alpha^{-1}\right|^p \left|g_l^\beta \circ \phi_\beta \circ \phi_\alpha^{-1} - (\psi_\beta f) \circ \phi_\alpha^{-1}\right|^p \mathrm{d}x \\
&\leq C(Q) \sum_{\alpha=1}^{N} \sum_{\beta\in I_\alpha} \int_{\phi_\alpha(U_\alpha\cap U_\beta)} \left|g_l^\beta \circ \phi_\beta \circ \phi_\alpha^{-1} - (\psi_\beta f) \circ \phi_\alpha^{-1}\right|^p \mathrm{d}x \\
&= C(Q) \sum_{\alpha=1}^{N} \sum_{\beta\in I_\alpha} \int_{\phi_\beta(U_\alpha\cap U_\beta)} \left|g_l^\beta - (\psi_\beta f) \circ \phi_\beta^{-1}\right|^p \left|\det D(\phi_\alpha \circ \phi_\beta^{-1})\right| \mathrm{d}x \\
&\leq C(Q) \sum_{\alpha=1}^{N} \sum_{\beta\in I_\alpha} \int_{\phi_\beta(U_\alpha\cap U_\beta)} \left|g_l^\beta - (\psi_\beta f) \circ \phi_\beta^{-1}\right|^p \mathrm{d}x \\
&\leq C(Q) \sum_{\beta=1}^{N} \left\|g_l^\beta - (\psi_\beta f) \circ \phi_\beta^{-1}\right\|_{L_p(\phi_\beta(U_\beta))} \to 0\,.
\end{aligned}$$

To prove that the sequence $(g_l)_{l\in\mathbb{N}}$ is Cauchy in $W_p^1(M)$, we observe that the estimate (A.12) implies

$$|\nabla(g_l - g_{\tilde{l}})|^p \circ \phi_\alpha^{-1} \leq Q^{p/2} \left|\nabla\left((g_l - g_{\tilde{l}}) \circ \phi_\alpha^{-1}\right)\right|^p .$$

Hence we obtain for $l, \tilde{l} \to \infty$,

$$\begin{aligned}
&\int_M |\nabla(g_l - g_{\tilde{l}})|^p \,\mathrm{d}V_g \leq C(Q) \sum_{\alpha=1}^{N} \int_{\phi_\alpha(U_\alpha)} \left(\psi_\alpha \circ \phi_\alpha^{-1}\right) |\nabla(g_l - g_{\tilde{l}})|^p \circ \phi_\alpha^{-1}\,\mathrm{d}x \\
&\leq C(Q) \sum_{\alpha=1}^{N} \int_{\phi_\alpha(U_\alpha)} \left(\psi_\alpha \circ \phi_\alpha^{-1}\right) \left|\nabla\left((g_l - g_{\tilde{l}}) \circ \phi_\alpha^{-1}\right)\right|^p \mathrm{d}x \\
&\leq C(Q) \sum_{\alpha=1}^{N} \sum_{\beta\in I_\alpha} \int_{\phi_\alpha(U_\alpha\cap U_\beta)} \left|\nabla\big((\xi_\beta \circ \phi_\alpha^{-1})\big((g_l^\beta - g_{\tilde{l}}^\beta) \circ \phi_\beta \circ \phi_\alpha^{-1}\big)\big)\right|^p \mathrm{d}x \\
&\leq C(Q) \sum_{\alpha=1}^{N} \sum_{\beta\in I_\alpha} \int_{\phi_\alpha(U_\alpha\cap U_\beta)} \left|\nabla\big((g_l^\beta - g_{\tilde{l}}^\beta) \circ \phi_\beta \circ \phi_\alpha^{-1}\big)\right|^p + \left|(g_l^\beta - g_{\tilde{l}}^\beta) \circ \phi_\beta \circ \phi_\alpha^{-1}\right|^p \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\le C(Q)\sum_{\alpha=1}^{N}\sum_{\beta\in I_\alpha}\int_{\phi_\alpha(U_\alpha\cap U_\beta)}\left(\left|\nabla\left(g_l^\beta-g_{\tilde l}^\beta\right)\right|^p+\left|g_l^\beta-g_{\tilde l}^\beta\right|^p\right)\circ\phi_\beta\circ\phi_\alpha^{-1}\,\mathrm{d}x\\
&\le C(Q)\sum_{\alpha=1}^{N}\sum_{\beta\in I_\alpha}\int_{\phi_\beta(U_\alpha\cap U_\beta)}\left|\nabla\left(g_l^\beta-g_{\tilde l}^\beta\right)\right|^p+\left|g_l^\beta-g_{\tilde l}^\beta\right|^p\mathrm{d}x\\
&\le C(Q)\sum_{\beta=1}^{N}\left\|g_l^\beta-g_{\tilde l}^\beta\right\|^p_{W_p^1(\phi_\beta(U_\beta))}\to 0\,.
\end{aligned}
$$

This shows $g_l\in\mathcal{C}_p^1(M)$, $f\in W_p^1(M)$ and the above estimates imply

$$
\begin{aligned}
\|f\|_{W_p^1(M)}=\lim_{l\to\infty}\|g_l\|_{W_p^1(M)}&\le C(Q)\sum_{\beta=1}^{N}\lim_{l\to\infty}\|g_l^\beta\|_{W_p^1(\phi_\beta(U_\beta))}\\
&=C(Q)\sum_{\beta=1}^{N}\left\|(\psi_\beta f)\circ\phi_\beta^{-1}\right\|_{W_p^1(\phi_\beta(U_\beta))}\,.
\end{aligned}
$$

Suppose that the claim is shown for $m\in\mathbb{N}_0$. Given $f\in W_p^{m+1}(M)$ we aim to show that $f\circ\phi_\alpha^{-1}\in W_p^{m+1}(\phi_\alpha(U_\alpha))$. As we have $f\circ\phi_\alpha^{-1}\in W_p^{m}(\phi_\alpha(U_\alpha))$ by induction hypothesis, it is enough to show for any $i_1,\dots,i_m\in\{1,\dots,n\}$,

$$
\frac{\partial^m\left(f\circ\phi_\alpha^{-1}\right)}{\partial x_{i_1}\cdots\partial x_{i_m}}\in W_p^1\left(\phi_\alpha(U_\alpha)\right).
$$

As the Christoffel symbols and its derivatives are uniformly bounded on the chart domains and as $\frac{\partial^l(f\circ\phi_\alpha^{-1})}{\partial x_{j_1}\cdots\partial x_{j_l}}$ lies in $W_p^1(\phi_\alpha(U_\alpha))$ for all $0\le l\le m-1$, $j_1,\dots,j_l\in\{1,\dots,n\}$, formula (A.2) implies that it is enough to prove

$$
(\nabla^m f)_{i_1\dots i_m}\circ\phi_\alpha^{-1}\in W_p^1\left(\phi_\alpha(U_\alpha)\right).
$$

Let $(f_l)_{l\in\mathbb{N}}\subset\mathcal{C}_p^{m+1}(M)$ be any approximating sequence for f in $W_p^{m+1}(M)$. Then by Lemma C.7 we have $(\nabla^m f)_{i_1\cdots i_m}\circ\phi_\alpha^{-1}\in L_p\left(\phi_\alpha(U_\alpha)\right)$ and as $l\to\infty$,

$$
\left\|\left((\nabla^m f)_{i_1\cdots i_m}-(\nabla^m f_l)_{i_1\cdots i_m}\right)\circ\phi_\alpha^{-1}\right\|_{L_p(\phi_\alpha(U_\alpha))}\to 0\,.
$$

Furthermore, for all $l\in\mathbb{N}$, $(\nabla^m f_l)_{i_1\cdots i_m}\circ\phi_\alpha^{-1}$ lies in $C^1\left(\mathrm{int}\phi_\alpha(U_\alpha)\right)$ and by formula (A.2)

$$
\begin{aligned}
&\left\|\partial_{x^i}\left(\left(\nabla^m(f_l-f_{\tilde l})\right)_{i_1\cdots i_m}\circ\phi_\alpha^{-1}\right)\right\|_{L_p(\phi_\alpha(U_\alpha))}\\
&\le C(Q)\left\|\left(\nabla^{m+1}(f_l-f_{\tilde l})\right)_{ii_1\cdots i_m}\right\|_{L_p(U_\alpha)}+C(Q)\sum_{j_1,\dots,j_m=1}^{n}\left\|\left(\nabla^m(f_l-f_{\tilde l})\right)_{j_1\cdots j_m}\right\|_{L_p(U_\alpha)}\,.
\end{aligned}
$$

Lemma C.7 implies that the right hand side tends to zero as $l,\tilde l\to\infty$. This shows that the functions $(\nabla^m f_l)_{i_1\dots i_m}\circ\phi_\alpha^{-1}$, $l\in\mathbb{N}$ form a Cauchy sequence in $W_p^1(\phi_\alpha(U_\alpha))$ converging to $(\nabla^m f)_{i_1\dots i_m}\circ\phi_\alpha^{-1}$ in $L_p\left(\phi_\alpha(U_\alpha)\right)$. By completeness of $W_p^1\left(\phi_\alpha(U_\alpha)\right)$ and uniqueness of limits in $L_p\left(\phi_\alpha(U_\alpha)\right)$ we conclude $(\nabla^m f)_{i_1\dots i_m}\circ\phi_\alpha^{-1}\in W_p^1\left(\phi_\alpha(U_\alpha)\right)$. The above estimates combined with (A.12) and (C.5) imply

$$
\begin{aligned}
\left\|(\nabla^m f)_{i_1\dots i_m}\circ\phi_\alpha^{-1}\right\|_{W_p^1(\phi_\alpha(U_\alpha))}&=\lim_{l\to\infty}\left\|(\nabla^m f_l)_{i_1\dots i_m}\circ\phi_\alpha^{-1}\right\|_{W_p^1(\phi_\alpha(U_\alpha))}\\
&\le C(Q)\lim_{l\to\infty}\|f_l\|_{W_p^{m+1}(M)}=C(Q)\,\|f\|_{W_p^{m+1}(M)}\,.
\end{aligned}
$$

Combined with the induction hypothesis $\left\|f\circ\phi_\alpha^{-1}\right\|_{W_p^m(\phi_\alpha(U_\alpha))}\le C(Q)\,\|f\|_{W_p^m(M)}$ we obtain the desired estimate

$$
\left\|f\circ\phi_\alpha^{-1}\right\|_{W_p^{m+1}(\phi_\alpha(U_\alpha))}\le C(Q)\,\|f\|_{W_p^{m+1}(M)}\,. \tag{C.8}
$$

The estimate

$$\left\|(\psi_\alpha f)\circ\phi_\alpha^{-1}\right\|_{W_p^{m+1}(\phi_\alpha(U_\alpha))} \le C(Q)\,\|f\|_{W_p^{m+1}(M)}$$

follows from the induction hypothesis and the estimate (C.8) using similar arguments as before. Suppose on the other hand that for all $\alpha \in \{1,\dots,N\}$, $(\psi_\alpha f)\circ\phi_\alpha^{-1} \in W_p^{m+1}(\phi_\alpha(U_\alpha))$. Then there exist sequences $(g_l^\alpha)_{l\in\mathbb{N}} \subset C^{m+1}(\mathrm{int}\phi_\alpha(U_\alpha)) \cap W_p^{m+1}(\phi_\alpha(U_\alpha))$ such that

$$\left\|(\psi_\alpha f)\circ\phi_\alpha^{-1} - g_l^\alpha\right\|_{W_p^{m+1}(\phi_\alpha(U_\alpha))} \to 0$$

as $l\to\infty$. We have already seen that the family of functions $g_l = \sum_{\alpha=1}^N \xi_\alpha\left(g_l^\alpha\circ\phi_\alpha\right)$ lies in $\mathcal{C}_p^1(M)$ and converges to f in $L_p(M)$. To show $f\in W_p^{m+1}(M)$ we prove that for all $1\le r\le m+1$, the functions g_l, $l\in\mathbb{N}$ lie in $\mathcal{C}_p^r(M)$ and satisfy the estimate

$$\|g_l\|_{W_p^r(M)} \le C(Q)\sum_{\alpha=1}^N \|g_l^\alpha\|_{W_p^r(\phi_\alpha(U_\alpha))}\,. \tag{C.9}$$

The same reasoning applied to differences $g_l - g_{\tilde{l}}$ then shows that $(g_l)\subset\mathcal{C}_p^{m+1}(M)$ is a Cauchy sequence in the norm $\|\cdot\|_{W_p^{m+1}(M)}$ which then implies $f\in W_p^{m+1}(M)$ and

$$\begin{aligned}\|f\|_{W_p^{m+1}(M)} = \lim_{l\to\infty}\|g_l\|_{W_p^{m+1}(M)} &\le C(Q)\sum_{\alpha=1}^N \lim_{l\to\infty}\|g_l^\alpha\|_{W_p^{m+1}(\phi_\alpha(U_\alpha))}\\ &= C(Q)\sum_{\alpha=1}^N \left\|(\psi_\alpha f)\circ\phi_\alpha^{-1}\right\|_{W_p^{m+1}(\phi_\alpha(U_\alpha))}\,.\end{aligned}$$

The case $r=1$ has already been shown. Suppose that for some integer $1\le r\le m$ the functions g_l lie in $\mathcal{C}_p^r(M)$ and satisfy the estimate (C.9). We observe that g_l lies in $C^{m+1}(\mathrm{int}(M))$. The uniform bound (A.12) on the inverse metric implies

$$\left|\nabla^{r+1}g_l\right|^2\circ\phi_\alpha^{-1} \le C(Q)\sum_{i_1,\dots,i_{r+1}=1}^n \left(\left(\nabla^{r+1}g_l\right)_{i_1\cdots i_{r+1}}\circ\phi_\alpha^{-1}\right)^2$$

and using formula (A.2) we observe that there exist functions $c_{\alpha,\beta}$, uniformly bounded on $\phi_\alpha(U_\alpha)$, such that

$$\left(\nabla^{r+1}g_l\right)_{i_1\cdots i_{r+1}}\circ\phi_\alpha^{-1} = \frac{\partial^{r+1}\left(g_l\circ\phi_\alpha^{-1}\right)}{\partial x_{i_1}\cdots\partial x_{i_{r+1}}} + \sum_{|\beta|\le r} c_{\alpha,\beta}\partial x_1^{\beta_1}\cdots\partial x_n^{\beta_n}\left(g_l\circ\phi_\alpha^{-1}\right).$$

Then [2, Theorem 3.41] implies for every $\beta\in\mathbb{N}_0^n$ with $|\beta|\le r+1$, $\gamma\in\{1,\dots,N\}$ and $x\in\phi_\alpha(U_\alpha\cap U_\gamma)$,

$$\partial_{x_1}^{\beta_1}\cdots\partial_{x_n}^{\beta_n}\left(g_l^\gamma\circ\phi_\gamma\circ\phi_\alpha^{-1}\right)(x) = \sum_{\delta\le\beta} M_{\beta\delta}(x)\left(\partial_{x_1}^{\delta_1}\cdots\partial_{x_n}^{\delta_n}(g_l^\gamma)\right)\left(\phi_\gamma\circ\phi_\alpha^{-1}(x)\right)$$

where $M_{\beta\delta}$ is a polynomial of degree at most $|\delta|$ in derivatives of the components of $\phi_\gamma\circ\phi_\alpha^{-1}$ of order at most $|\beta|$. Using uniform boundedness of $\phi_\gamma\circ\phi_\alpha^{-1}$, $\xi_\gamma\circ\phi_\alpha^{-1}$ and all their derivatives on $\phi_\alpha(U_\alpha\cap U_\gamma)$, we obtain with a change of variables,

$$\begin{aligned}&\int_M \left|\nabla^{r+1}g_l\right|^p \,\mathrm{d}V_g \le C(Q)\sum_{\alpha=1}^N\int_{\phi_\alpha(U_\alpha)}\left(\psi_\alpha|\nabla^{r+1}g_l|^p\right)\circ\phi_\alpha^{-1}\,\mathrm{d}x\\ &\le C(Q)\sum_{\alpha=1}^N\sum_{|\beta|\le r+1}\int_{\phi_\alpha(U_\alpha)}\left(\psi_\alpha\circ\phi_\alpha^{-1}\right)\left|\partial_{x_1}^{\beta_1}\cdots\partial_{x_n}^{\beta_n}\left(g_l\circ\phi_\alpha^{-1}\right)\right|^p\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\le C(Q)\sum_{\alpha=1}^{N}\sum_{|\beta|\le r+1}\int_{\phi_\alpha(U_\alpha)}\left|\partial_{x_1}^{\beta_1}\cdots\partial_{x_n}^{\beta_n}\left(g_l\circ\phi_\alpha^{-1}\right)\right|^p\,\mathrm{d}x\\
&\le C(Q)\sum_{|\beta|\le r+1}\sum_{\alpha=1}^{N}\sum_{\gamma\in I_\alpha}\int_{\phi_\alpha(U_\alpha\cap U_\gamma)}\left|\partial_{x_1}^{\beta_1}\cdots\partial_{x_n}^{\beta_n}\left(\left(\xi_\gamma\circ\phi_\alpha^{-1}\right)\left(g_l^\gamma\circ\phi_\gamma\circ\phi_\alpha^{-1}\right)\right)(x)\right|^p\,\mathrm{d}x\\
&\le C(Q)\sum_{|\beta|\le r+1}\sum_{\alpha=1}^{N}\sum_{\gamma\in I_\alpha}\int_{\phi_\alpha(U_\alpha\cap U_\gamma)}\left|\partial_{x_1}^{\beta_1}\cdots\partial_{x_n}^{\beta_n}\left(g_l^\gamma\circ\phi_\gamma\circ\phi_\alpha^{-1}\right)(x)\right|^p\,\mathrm{d}x\\
&\le C(Q)\sum_{|\beta|\le r+1}\sum_{\alpha=1}^{N}\sum_{\gamma\in I_\alpha}\int_{\phi_\alpha(U_\alpha\cap U_\gamma)}\left|\partial_{x_1}^{\beta_1}\cdots\partial_{x_n}^{\beta_n}\left(g_l^\gamma\right)\right|^p\left((\phi_\gamma\circ\phi_\alpha^{-1})(x)\right)\,\mathrm{d}x\\
&\le C(Q)\sum_{|\beta|\le r+1}\sum_{\alpha=1}^{N}\sum_{\gamma\in I_\alpha}\int_{\phi_\gamma(U_\alpha\cap U_\gamma)}\left|\partial_{x_1}^{\beta_1}\cdots\partial_{x_n}^{\beta_n}\left(g_l^\gamma\right)\right|^p\,\mathrm{d}x\\
&\le C(Q)\sum_{|\beta|\le r+1}\sum_{\alpha=1}^{N}\|g_l^\alpha\|_{W_p^{|\beta|}(\phi_\alpha(U_\alpha))}\le C(Q)\sum_{\alpha=1}^{N}\|g_l^\alpha\|_{W_p^{r+1}(\phi_\alpha(U_\alpha))}\,.
\end{aligned}
$$

Combining this estimate with the induction hypothesis shows the desired result. □

C.1.2 A local definition of Slobodeckij spaces

The characterisation of Sobolev spaces in Proposition C.1 suggests that the Slobodeckij spaces should comprise all functions on the manifold with the property that their representations in normal coordinates have the corresponding regularity in Euclidean space. We use this description to *define* the space $W_p^s(M)$ with non-integer order $s > 0$.

Definition C.10 (Slobodeckij spaces on compact manifolds)**.** Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold of dimension $n \in \mathbb{N}$ with Levi-Civita connection D, normal covering $\mathcal{T} = \{(U_\alpha, \phi_\alpha)\}$ and uniform localisation system $(U_\alpha, \phi_\alpha, \psi_\alpha)$. Given $p \in (1, \infty)$ and $s \geq 0$, the *Slobodeckij space* $W_p^s(M)$ is defined to be the space of functions $f \in L_p(M)$ that satisfy for all $\alpha \in \{1, \dots, N\}$, $(\psi_\alpha f) \circ \phi_\alpha^{-1} \in W_p^s(\phi_\alpha(U_\alpha))$.

There do exist different notions of fractional order Sobolev spaces on manifolds, see for example [10, 11, 12, 72, 140]. In [10, 11] the author defines Slobodeckij spaces via real interpolation of Sobolev spaces on the respective manifold which is one of their characterisations in Euclidean space as shown in Proposition B.18. It is an interesting question under what assumptions on the manifold these different approaches actually yield the same function space. An answer to this is beyond the purpose of this work.

We always work with the “localised” Definition C.10. All Sobolev-Slobodeckij spaces on domains Ω in Euclidean space shall be endowed with the norm

$$\|f\|_{W_p^s(\Omega)} := \|f\|_{W_p^{\lfloor s\rfloor}(\Omega)} + [\,f\,]_{W_p^s(\Omega)}$$

as defined in Proposition B.19.

Proposition C.11 (A norm on $W_p^s(M)$)**.** *Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n \in \mathbb{N}$ with Levi-Civita connection D, normal covering $\mathcal{T} = \{(U_\alpha, \phi_\alpha)\}$ and uniform localisation system $(U_\alpha, \phi_\alpha, \psi_\alpha)$. Let $p \in (1, \infty)$ and $s \geq 0$.*

The space $W_p^s(M)$ is a Banach space in the norm

$$\|f\|_{W_p^s(M)} := \|f\|^{\mathcal{T}}_{W_p^s(M)} := \sum_{\alpha=1}^{N} \left\|(\psi_\alpha f) \circ \phi_\alpha^{-1}\right\|_{W_p^s(\phi_\alpha(U_\alpha))} .$$

Furthermore, given any $f \in W_p^s(M)$, then for all $\alpha \in \{1, \dots, N\}$ the function $f \circ \phi_\alpha^{-1}$ lies in $W_p^s(\phi_\alpha(U_\alpha))$ and satisfies

$$\left\|f \circ \phi_\alpha^{-1}\right\|_{W_p^s(\phi_\alpha(U_\alpha))} \leq C(Q) \|f\|_{W_p^s(M)} .$$

Proof. The space $W_p^s(M)$ is a subspace of $W_p^{\lfloor s \rfloor}(M)$ and it is straightforward to check that $\|\cdot\|_{W_p^s(M)}$ defines a norm on it. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $W_p^s(M)$ that satisfies $\|f_n - f_m\|_{W_p^s(M)} \to 0$ as $n, m \to \infty$. By Proposition C.1 the functions f_n, $n \in \mathbb{N}$, form a Cauchy sequence in $\left(W_p^{\lfloor s \rfloor}(M), \|\cdot\|_{W_p^{\lfloor s \rfloor}(M)}\right)$ converging to a function $f \in W_p^{\lfloor s \rfloor}(M)$. In particular, for every $\alpha \in \{1, \dots, N\}$ the sequence $\left((\psi_\alpha f_n) \circ \phi_\alpha^{-1}\right)_{n\in\mathbb{N}}$ tends to $(\psi_\alpha f) \circ \phi_\alpha^{-1}$ in $W_p^{\lfloor s \rfloor}(\phi_\alpha(U_\alpha))$. On the other hand, the functions $(\psi_\alpha f_n) \circ \phi_\alpha^{-1}$, $n \in \mathbb{N}$, form a Cauchy sequence in the Banach space $W_p^s(\phi_\alpha(U_\alpha))$ converging to a limit function $h_\alpha \in W_p^s(\phi_\alpha(U_\alpha))$. Uniqueness of limits in $W_p^{\lfloor s \rfloor}(\phi_\alpha(U_\alpha))$ implies $h_\alpha = (\psi_\alpha f) \circ \phi_\alpha^{-1}$ and in particular $f \in W_p^s(M)$. Furthermore,

$$\|f_n - f\|_{W_p^s(M)} = \sum_{\alpha=1}^{N} \left\|(\psi_\alpha (f_n - f)) \circ \phi_\alpha^{-1}\right\|_{W_p^s(\phi_\alpha(U_\alpha))} = \sum_{\alpha=1}^{N} \left\|(\psi_\alpha f_n) \circ \phi_\alpha^{-1} - h_\alpha\right\|_{W_p^s(\phi_\alpha(U_\alpha))} \to 0$$

as $n \to \infty$. This shows that $\left(W_p^s(M), \|\cdot\|_{W_p^s(M)}\right)$ is a Banach space.

Let $\alpha \in \{1, \dots, N\}$ and $f \in W_p^s(M)$ be given. As $W_p^s(M)$ is contained in $W_p^{\lfloor s \rfloor}(M)$, Proposition C.1 implies $f \circ \phi_\alpha^{-1} \in W_p^{\lfloor s \rfloor}(\phi_\alpha(U_\alpha))$ and

$$\left\|f \circ \phi_\alpha^{-1}\right\|_{W_p^{\lfloor s \rfloor}(\phi_\alpha(U_\alpha))} \leq C(Q) \|f\|_{W_p^{\lfloor s \rfloor}(M)} \leq C(Q) \|f\|_{W_p^s(M)} .$$

It remains to estimate the semi-norm part of the norm on $W_p^{s-\lfloor s \rfloor}(\phi_\alpha(U_\alpha))$ applied to derivatives of order $\lfloor s \rfloor$ of the function $f \circ \phi_\alpha^{-1}$. Let $I_\alpha := \{\gamma \in \{1, \dots, N\} : U_\alpha \cap U_\gamma \neq \emptyset\}$ and $\beta \in \mathbb{N}_0^n$ with $|\beta| = \lfloor s \rfloor$ be given. For every $\gamma \in \{1, \dots, N\}$ the function $(\psi_\gamma f) \circ \phi_\alpha^{-1}$ lies in $W_p^{\lfloor s \rfloor}(\phi_\alpha(U_\alpha))$ and by [2, Theorem 3.41] it holds for $\gamma \in I_\alpha$, $x \in \phi_\alpha(U_\alpha \cap U_\gamma)$ and $\delta \in \mathbb{N}_0^n$ with $|\delta| \leq \lfloor s \rfloor$,

$$\partial_{x_1}^{\delta_1} \cdots \partial_{x_n}^{\delta_n} \left((\psi_\gamma f) \circ \phi_\alpha^{-1}\right)(x) = \sum_{\varepsilon \leq \delta} M_{\delta\varepsilon}(x) \left(\partial_{x_1}^{\varepsilon_1} \cdots \partial_{x_n}^{\varepsilon_n} \left((\psi_\gamma f) \circ \phi_\gamma^{-1}\right)\right)\left((\phi_\gamma \circ \phi_\alpha^{-1})(x)\right) ,$$

where $M_{\delta\varepsilon}$ denotes a polynomial of degree at most $|\varepsilon|$ in derivatives of order at most $|\delta|$ of the components of $\phi_\gamma \circ \phi_\alpha^{-1}$. As all derivatives of $\phi_\gamma \circ \phi_\alpha^{-1}$ are uniformly bounded on $\overline{\phi_\alpha(U_\alpha \cap U_\gamma)}$, we have in particular for all $|\varepsilon| \leq |\beta|$ that $M_{\beta\varepsilon} \in C^{0,1}\left(\overline{\phi_\alpha(U_\alpha \cap U_\gamma)}\right)$. Using further the transformation formula, linearity of the weak derivative on $W_p^{\lfloor s \rfloor}(\phi_\alpha(U_\alpha))$, uniform boundedness of $\phi_\gamma \circ \phi_\alpha^{-1}$ and its inverse and all their derivatives, we obtain

$$\begin{aligned}
&\int_{\phi_\alpha(U_\alpha)} \int_{\phi_\alpha(U_\alpha)} \frac{|\partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n} \left(f \circ \phi_\alpha^{-1}\right)(x) - \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n} \left(f \circ \phi_\alpha^{-1}\right)(y)|^p}{|x-y|^{n+(s-\lfloor s \rfloor)p}} \mathrm{d}x\mathrm{d}y \\
&\leq C \sum_{\gamma \in I_\alpha} \int \int_{\phi_\alpha(U_\alpha \cap U_\gamma)} \frac{|\partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n} \left((\psi_\gamma f) \circ \phi_\alpha^{-1}\right)(x) - \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n} \left((\psi_\gamma f) \circ \phi_\alpha^{-1}\right)(y)|^p}{|x-y|^{n+(s-\lfloor s \rfloor)p}} \mathrm{d}x\mathrm{d}y \\
&\leq C \sum_{\gamma \in I_\alpha, \varepsilon \leq \beta} \|M_{\beta\varepsilon}\|_{C^{0,1}\left(\overline{\phi_\alpha(U_\alpha \cap U_\gamma)}\right)} \left\|\partial_{x_1}^{\varepsilon_1} \cdots \partial_{x_n}^{\varepsilon_n} \left((\psi_\gamma f) \circ \phi_\gamma^{-1}\right)\right\|_{L_p(\phi_\gamma(U_\alpha \cap U_\gamma))}
\end{aligned}$$

$$+\|M_{\beta\varepsilon}\|_\infty \int\int_{\phi_\gamma(U_\alpha\cap U_\gamma)} \frac{|\partial_{x_1}^{\varepsilon_1}\cdots\partial_{x_n}^{\varepsilon_n}\left((\psi_\gamma f)\circ\phi_\gamma^{-1}\right)(x)-\partial_{x_1}^{\varepsilon_1}\cdots\partial_{x_n}^{\varepsilon_n}\left((\psi_\gamma f)\circ\phi_\gamma^{-1}\right)(y)|^p}{|x-y|^{n+(s-\lfloor s\rfloor)p}}\mathrm{d}x\mathrm{d}y$$

$$\leq C\sum_{\gamma=1}^{N}\left\|(\psi_\gamma f)\circ\phi_\gamma^{-1}\right\|_{W_p^s(\phi_\gamma(U_\gamma))}.$$

This shows the claim. □

The following result shows that on a compact Riemannian manifold any finite covering can be used to characterise the space $W_p^s(M)$ independently of the particular charts. Furthermore, the induced norm is equivalent to the one corresponding to the normal covering. This observation is in analogy to [72, Theorem 14] where the authors show that on a complete non-compact Riemannian manifold with bounded geometry fractional Sobolev spaces can be equivalently characterised using an *admissible trivialisation* instead of geodesic normal coordinates. In this context an admissible trivialisation is a locally finite covering of the manifold that is compatible with the normal covering in the sense that all derivatives of the transition maps and their inverses are uniformly bounded, see [72, Definition 12]. This condition appears also in the following result although it is somewhat redundant in the case of a compact manifold. Indeed, given any covering of M consisting of charts in the smooth atlas, one can decrease the size of the chart domains around every point to guarantee that the transition maps are defined on the closure of the chart domain and thus uniformly bounded. Using compactness of the manifold, one hereby still obtains a finite cover of M. Finally, we remark that the result in [72, Theorem 14] is *not* applicable to our situation as manifolds of bounded geometry are required to have a positive injectivity radius. This condition is violated if the manifold has a boundary.
As before the solution is to use the normal collar coordinates introduced in Section A.4.

Proposition C.12 (Norms with respect to arbitrary coverings)**.** *Let $(M,\mathcal{A},g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n\in\mathbb{N}$ with Levi-Civita connection D, normal covering $\mathcal{T}=\{(U_\alpha,\phi_\alpha)\}$ and uniform localisation system $(U_\alpha,\phi_\alpha,\psi_\alpha)$, $p\in(1,\infty)$ and $s\geq 0$. Let $M\in\mathbb{N}$ and*

$$\mathcal{S}:=\{(V_\beta,\varrho_\beta):\beta\in\{1,\dots,M\}\}\subset\mathcal{A},$$

be any covering of M such that for all α, β with $U_\alpha\cap V_\beta\neq\emptyset$ all derivatives of $\phi_\alpha\circ\varrho_\beta^{-1}$ and its inverse are uniformly bounded by a constant $C(\mathcal{S})$. Let further $(\xi_\beta)_{\beta\in\{1,\dots,M\}}$ be a smooth partition of unity subordinate to the covering $\mathcal{S}$. Given a function $f\in W_p^s(M)$ it holds $f\circ\varrho_\beta^{-1}\in W_p^s(\varrho_\beta(V_\beta))$ for all $\beta\in\{1,\dots,M\}$. If conversely $f:M\to\mathbb{R}$ is such that $(\xi_\beta f)\circ\varrho_\beta^{-1}$ lies in $W_p^s(\varrho_\beta(V_\beta))$ for all $\beta\in\{1,\dots,M\}$, then f lies in $W_p^s(M)$. Furthermore,

$$\|f\|_{W_p^s(M)}^{\mathcal{S}}:=\sum_{\beta=1}^{M}\left\|(\xi_\beta f)\circ\varrho_\beta^{-1}\right\|_{W_p^s(\varrho_\beta(V_\beta))}$$

defines a norm on $W_p^s(M)$ that is equivalent to $\|\cdot\|_{W_p^s(M)}^{\mathcal{T}}$ with equivalence constant depending on $C(\mathcal{S})$.

Proof. Let $f\in W_p^s(M)$ be given and suppose that $s\in\mathbb{N}_0$ is an integer. Then $(\psi_\alpha f)\circ\phi_\alpha^{-1}$ lies in $W_p^s(\phi_\alpha(U_\alpha))$ which yields in particular $(\psi_\alpha f)\circ\varrho_\beta^{-1}\in W_p^s(\varrho_\beta(U_\alpha\cap V_\beta))$ using [2, Theorem 3.41]. As

$$\operatorname{dist}\big(\operatorname{supp}((\psi_\alpha f)\circ\varrho_\beta^{-1}),\partial(\operatorname{int}\varrho_\beta(U_\alpha\cap V_\beta))\cap\operatorname{int}\varrho_\beta(V_\beta)\big)>0,$$

Lemma C.13 implies $(\psi_\alpha f)\circ\varrho_\beta^{-1}\in W_p^s\left(\varrho_\beta(V_\beta)\right)$ which shows that $f\circ\varrho_\beta^{-1}=\sum_{\alpha=1}^N(\psi_\alpha f)\circ\varrho_\beta^{-1}$ lies in $W_p^s\left(\varrho_\beta(V_\beta)\right)$. The estimate

$$\left\|f\circ\varrho_\beta^{-1}\right\|_{W_p^s(\varrho_\beta(V_\beta))}\le C(\mathcal{S})\,\|f\|_{W_p^s(M)}^{\mathcal{T}}$$

follows from the estimates given in [2, Theorem 3.41] and Lemma C.13, respectively, and the boundedness of the coordinate changes between $\mathcal{T}$ and $\mathcal{S}$ and their derivatives. If f is a given function in $W_p^s(M)$ with $s>0$ non-integer, then by definition $(\psi_\alpha f)\circ\phi_\alpha^{-1}\in W_p^s(\phi_\alpha(U_\alpha))$ and the above arguments imply $f\circ\varrho_\beta^{-1}\in W_p^{\lfloor s\rfloor}(\varrho_\beta(V_\beta))$. It thus remains to estimate the semi-norm part of the norm on $W_p^{s-\lfloor s\rfloor}\left(\varrho_\beta(V_\beta)\right)$ applied to derivatives of order $\lfloor s\rfloor$ of $f\circ\varrho_\beta^{-1}=\sum_{\alpha=1}^N(\psi_\alpha f)\circ\varrho_\beta^{-1}$. As $(\psi_\alpha f)\circ\varrho_\beta^{-1}$ lies in $W_p^{\lfloor s\rfloor}(\varrho_\beta(V_\beta))$, we may use the arguments done in the proof of Proposition C.11. By [2, Theorem 3.41] it holds for all $\varepsilon\in\mathbb{N}_0^n$, $|\varepsilon|=\lfloor s\rfloor$,

$$\left\|\partial_{x_1}^{\varepsilon_1}\cdots\partial_{x_n}^{\varepsilon_n}\left((\psi_\alpha f)\circ\varrho_\beta^{-1}\right)\right\|_{W_p^{s-\lfloor s\rfloor}(\varrho_\beta(V_\beta))}\le C(\mathcal{S})\sum_{\gamma=1}^N\left\|(\psi_\gamma f)\circ\phi_\gamma^{-1}\right\|_{W_p^s(\phi_\gamma(U_\gamma))}$$

with constant $C(\mathcal{S})$ depending on the uniform bounds on the (derivatives of) coordinate transformations between $\mathcal{T}$ and $\mathcal{S}$. This shows $f\circ\varrho_\beta^{-1}\in W_p^s(\varrho_\beta(V_\beta))$ with the estimate

$$\left\|f\circ\varrho_\beta^{-1}\right\|_{W_p^s(\varrho_\beta(V_\beta))}\le C(\mathcal{S})\,\|f\|_{W_p^s(M)}^{\mathcal{T}}\,.$$

As the functions ξ_β and their derivatives are uniformly bounded, we obtain the desired estimate

$$\|f\|_{W_p^s(M)}^{\mathcal{S}}\le C(\mathcal{S})\,\|f\|_{W_p^s(M)}^{\mathcal{T}}\,.$$

Suppose conversely that $f:M\to\mathbb{R}$ is such that for all $\beta\in\{1,\dots,N\}$, $(\xi_\beta f)\circ\varrho_\beta^{-1}$ lies in $W_p^s(\varrho_\beta(V_\beta))$. The precisely same reasoning as above yields $f\circ\phi_\alpha^{-1}\in W_p^s(\phi_\alpha(U_\alpha))$ for all $\alpha\in\{1,\dots,N\}$ with

$$\left\|f\circ\phi_\alpha^{-1}\right\|_{W_p^s(\phi_\alpha(U_\alpha))}\le C(\mathcal{S})\sum_{\beta=1}^M\left\|(\xi_\beta f)\circ\varrho_\beta^{-1}\right\|_{W_p^s(\varrho_\beta(V_\beta))}=C(\mathcal{S})\,\|f\|_{W_p^s(M)}^{\mathcal{S}}\,.$$

In particular, $(\psi_\alpha f)\circ\phi_\alpha^{-1}$ lies in $W_p^s(\phi_\alpha(U_\alpha))$ with

$$\left\|(\psi_\alpha f)\circ\phi_\alpha^{-1}\right\|_{W_p^s(\phi_\alpha(U_\alpha))}\le C(\mathcal{S})\,\|f\|_{W_p^s(M)}^{\mathcal{S}}$$

which shows $f\in W_p^s(M)$ and the desired estimate. □

C.1.3 An extension result

In this subsection we prove a result that allows the extension of Sobolev functions defined on a subset of the manifold to Sobolev functions on the entire manifold. This property is needed in the localisation procedure in Chapter 2.

Lemma C.13. *Let $n\in\mathbb{N}$, $p\in(1,\infty)$ and $s\ge 0$. Suppose that Ω_1 and Ω_2 are open sets in $\mathbb{R}^n$ with $\Omega_1\subset\Omega_2$ and Ω_1 bounded. Given any function $g:\Omega_1\to\mathbb{R}$ we let $Eg:\Omega_2\to\mathbb{R}$ be the function that satisfies $Eg=g$ on Ω_1 and $Eg=0$ on $\Omega_2\setminus\Omega_1$. If $f\in W_p^s\left(\Omega_1\right)$ satisfies*

$$\operatorname{dist}\left(\operatorname{supp}(f)\cap\overline{\Omega}_1,\partial\Omega_1\cap\Omega_2\right)>0\,,$$

then the trivial extension Ef of f lies in $W_p^s(\Omega_2)$ and $\partial^\alpha(Ef)=E(\partial^\alpha f)$ for all $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\le\lfloor s\rfloor$ and $\|Ef\|_{W_p^s(\Omega_2)}\le\|f\|_{W_p^s(\Omega_1)}$. In particular, $E:W_p^s(\Omega_1)\to W_p^s(\Omega_2)$ is a continuous linear operator.

Proof. Suppose first that $s = m \in \mathbb{N}_0$ is an integer. We prove the claim by induction with respect to $m \in \mathbb{N}_0$. The case $m = 0$ is trivial. Suppose that the claim is shown for $m - 1 \in \mathbb{N}_0$. Let $f \in W_p^m(\Omega_1)$ be given and $\delta > 0$ with $\operatorname{dist}\big(\operatorname{supp}(f) \cap \overline{\Omega}_1, \partial\Omega_1 \cap \Omega_2\big) > \delta$. There exists a sequence $(\widetilde{f}_n)_{n\in\mathbb{N}}$ in $C^m(\Omega_1) \cap W_p^m(\Omega_1)$ that converges to f in $W_p^m(\Omega_1)$. Let $\xi \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \le \xi \le 1$, $\xi \equiv 1$ on $\operatorname{supp} f \cap \overline{\Omega}_1$ and $\operatorname{supp}\xi \subset \{x \in \mathbb{R}^n : \operatorname{dist}(x, \operatorname{supp} f \cap \overline{\Omega}_1) < \delta/2\}$. Then the sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n := \xi\widetilde{f}_n$ lies in $C^m(\Omega_1) \cap W_p^m(\Omega_1)$ and satisfies $\|f_n - f\|_{W_p^m(\Omega_1)} \to 0$ as $n \to \infty$ and

$$\inf_{n\in\mathbb{N}} \operatorname{dist}\big(\operatorname{supp}(f_n) \cap \overline{\Omega}_1, \partial\Omega_1 \cap \Omega_2\big) > \delta/2\,.$$

In particular, all derivatives of f_n vanish on $\partial\Omega_1 \cap \Omega_2$. By induction hypothesis we have $Ef \in W_p^{m-1}(\Omega_2)$. Let $\alpha \in \mathbb{N}_0^n$ be any multi-index with $|\alpha| = m-1$. We observe that $(\partial^\alpha f_n)_{n\in\mathbb{N}}$ forms a sequence in $C^1(\Omega_1) \cap W_p^1(\Omega_1)$ converging to $\partial^\alpha f$ in $W_p^1(\Omega_1)$. Given any test function $\phi \in C_0^\infty(\Omega_2)$ and any $i \in \{1, \ldots, n\}$ we have, denoting by ν the outward unit normal to Ω_1,

$$\begin{aligned}
\int_{\Omega_2} \partial^\alpha(Ef)\partial_{x_i}\phi \,\mathrm{d}x &= \int_{\Omega_2} E(\partial^\alpha f)\partial_{x_i}\phi \,\mathrm{d}x = \int_{\Omega_1} \partial^\alpha f \partial_{x_i}\phi \,\mathrm{d}x = \lim_{n\to\infty}\int_{\Omega_1} \partial^\alpha f_n \partial_{x_i}\phi \,\mathrm{d}x \\
&= \lim_{n\to\infty}\left(-\int_{\Omega_1} \partial^{\alpha+e_i} f_n \phi \,\mathrm{d}x + \int_{\partial\Omega_1} \partial^\alpha f_n \phi \nu^i \,\mathrm{d}\mathcal{H}^{n-1}\right) \\
&= -\int_{\Omega_1} \partial^{\alpha+e_i} f \phi \,\mathrm{d}x + \lim_{n\to\infty}\int_{\partial\Omega_1\cap\Omega_2} \partial^\alpha f_n \phi\nu^i \,\mathrm{d}\mathcal{H}^{n-1} \\
&= -\int_{\Omega_1} \partial^{\alpha+e_i} f\phi \,\mathrm{d}x = -\int_{\Omega_2} E(\partial^{\alpha+e_i} f)\phi \,\mathrm{d}x\,,
\end{aligned}$$

where we used that $\phi = 0$ on $\partial\Omega_1 \cap \partial\Omega_2$ and $\partial^\alpha f_n = 0$ on $\partial\Omega_1 \cap \Omega_2$. Thus $\partial^\alpha(Ef)$ lies in $W_p^1(\Omega_2)$ with $\partial^{\alpha+e_i}(Ef) = E(\partial^{\alpha+e_i} f)$. The estimate on the norm is a direct consequence of the formula $\partial^\alpha(Ef) = E(\partial^\alpha f)$. This completes the proof in the case that $s = m \in \mathbb{N}_0$ is an integer. Now let $s \ge 0$ and $f \in W_p^s(\Omega_1)$ with $\operatorname{dist}\big(\operatorname{supp}(f) \cap \overline{\Omega}_1, \partial\Omega_1 \cap \Omega_2\big) > 0$ be given. The previous case implies $f \in W_p^{\lfloor s\rfloor}(\Omega_2)$ with $\partial^\alpha(Ef) = E(\partial^\alpha f)$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le \lfloor s\rfloor$. Given any $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = \lfloor s\rfloor$ we conclude that

$$[\partial^\alpha(Ef)]_{W_p^{s-\lfloor s\rfloor}(\Omega_2)} = [\partial^\alpha f]_{W_p^{s-\lfloor s\rfloor}(\Omega_1)} < \infty$$

which shows $Ef \in W_p^s(\Omega_2)$ and the desired estimate. □

Lemma C.14. *Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n \in \mathbb{N}$ with Levi-Civita connection D and uniform localisation system $(U_\alpha, \phi_\alpha, \psi_\alpha)$. Let $p \in (1, \infty)$, $s \ge 0$ and $(U, \phi) \in \mathcal{A}$. Suppose we are given a function $f \in W_p^s(\operatorname{int}\phi(U))$ with the property that*

$$\operatorname{dist}\big(\operatorname{supp} f \cap \overline{\phi(U)}, \partial\phi(U) \cap \operatorname{int}\mathbb{H}^n\big) > 0\,.$$

Then the trivial extension $E(f \circ \phi)$ of $f \circ \phi$ to the entire manifold M lies in $W_p^s(M)$.

Proof. By Proposition C.1 it is enough to show for all $(U_\alpha, \phi_\alpha) \in \mathcal{T}$ that $E(f \circ \phi) \circ \phi_\alpha^{-1}$ lies in $W_p^s(\operatorname{int}\phi_\alpha(U_\alpha))$. Given $(U_\alpha, \phi_\alpha) \in \mathcal{T}$, we observe that

$$\operatorname{dist}\big(\operatorname{supp} f \cap \overline{\phi(U \cap U_\alpha)}, \partial\phi(U) \cap \operatorname{int}\mathbb{H}^n \cap \phi(U \cap U_\alpha)\big) > 0\,.$$

As $\phi_\alpha \circ \phi^{-1}$ is a homeomorphism, this implies in particular

$$\operatorname{dist}\big(\operatorname{supp}\big(f \circ \phi \circ \phi_\alpha^{-1}\big) \cap \overline{\phi_\alpha(U \cap U_\alpha)}, \phi_\alpha(\partial U \cap U_\alpha) \cap \operatorname{int}\phi_\alpha(U_\alpha)\big) > 0\,.$$

We observe that $\phi_\alpha(\partial U \cap U_\alpha) \cap \mathrm{int}\phi_\alpha(U_\alpha) = \partial\phi_\alpha(U \cap U_\alpha) \cap \mathrm{int}\phi_\alpha(U_\alpha)$ where $\partial\phi_\alpha(U \cap U_\alpha)$ is the boundary of $\phi_\alpha(U \cap U_\alpha)$ in the topology on $\mathbb{R}^n$. As $f \circ \phi \circ \phi_\alpha^{-1}$ lies in $W_p^s\left(\mathrm{int}\phi_\alpha(U \cap U_\alpha)\right)$, Lemma C.13 implies

$$E\left(f \circ \phi\right) \circ \phi_\alpha^{-1} = E\left(f \circ \phi \circ \phi_\alpha^{-1}\right) \in W_p^s\left(\mathrm{int}\phi_\alpha(U_\alpha)\right).$$

□

C.2 Anisotropic Sobolev spaces on compact manifolds

To give a definition of anisotropic Sobolev spaces on manifolds we use the notion of vector-valued Sobolev spaces $W_p^m((0,T);X)$, $m \in \mathbb{N}$, $p \in (1,\infty)$, as introduced in the beginning of Section B.2 with norm $\|\cdot\|_{W_p^m((0,T);X)}$ as in (B.2). Here, no restrictions on the Banach space X are imposed. Given $r \in (0,1)$ and a Banach space X we let

$$W_p^r((0,T);X) := \left\{ f \in L_p((0,T);X) : \ [f]_{W_p^r((0,T);X)} < \infty \right\} \tag{C.10}$$

where

$$[f]^p_{W_p^r((0,T);X)} := \int_0^T \int_0^T \frac{\|f(x)-f(y)\|_X^p}{|x-y|^{1+rp}} \,\mathrm{d}x\,\mathrm{d}y\,. \tag{C.11}$$

If X happens to be a UMD space, this notion is consistent with Definition B.14 due to Proposition B.19.

Definition C.15 (Anisotropic Sobolev spaces)**.** Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n \in \mathbb{N}$ with Levi-Civita connection D, $p \in (1,\infty)$ and $T \in (0,\infty)$. In the case $s \in \mathbb{N}_0$ we consider the geometric norm (C.1) on $W_p^s(M)$. If $s > 0$ is non-integer, the Slobodeckij space $W_p^s(M)$ as defined in Definition C.1.2 is endowed with the localised norm. Given $r \in [0,1]$ and $s \geq 0$ the space $W_p^{r,s}\left((0,T)\times M\right)$ is defined by

$$W_p^{r,s}\left((0,T)\times M\right) := W_p^r\left((0,T);L_p\left(M\right)\right) \cap L_p\left((0,T);W_p^s\left(M\right)\right)$$

with norm

$$\|\cdot\|_{p,r,s} := \|\cdot\|_{W_p^{r,s}((0,T)\times M)} := \|\cdot\|_{W_p^r((0,T);L_p(M))} + \|\cdot\|_{L_p((0,T);W_p^s(M))}$$

according to (B.2) and (C.11).

In [9] the author gives a profound investigation of a broad class of anisotropic function spaces on possibly non-compact and non-complete Riemannian manifolds with singularities described by a singularity function ρ. These results include in particular the case of compact manifolds when setting $\rho \equiv 1$. The function spaces appearing in [9] depend on several parameters that are needed to impose a suitable decay on the function ρ characterising the singular behaviour of the manifold. In our case, one can choose $\mu = \lambda = 0$. Then the anisotropic Sobolev space defined in [9, (8.1)] yields the same space as Definition C.15 with equivalent norms in the case that the time and space regularities r and s are integer-valued. In analogy to the definition of fractional order Sobolev spaces in his previous work [10], the author defines anisotropic Sobolev spaces of fractional order in [9] via interpolation of integer-ordered spaces.
We remark that at least the solution space

$$W_p^1\left((0,T);L_p(M)\right) \cap L_p\left((0,T);W_p^4(M)\right)$$

used in Part I is defined unambiguously. We conjecture that the spaces in Definition C.15 coincide with the ones introduced in [9] also in the fractional order case. A proof of this is however beyond the purpose of this work.

C.2.1 Characterisation via localisation

Motivated by the localisation techniques applied in the existence proof in Chapter 2 we show that the anisotropic Sobolev spaces defined in Definition C.15 can be characterised via a uniform localisation system. Hereby, the anisotropic spaces in Euclidean space are endowed with the norm in Definition B.26 where the Slobodeckij spaces are always considered with the norm given in Proposition B.19.
This subsection is devoted to prove the following proposition.

Proposition C.16 (Characterisation of $W_p^{r,s}((0,T)\times M)$)**.** *Let $(M,\mathcal{A},g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n\in\mathbb{N}$ with Levi-Civita connection D and uniform localisation system $(U_\alpha,\phi_\alpha,\psi_\alpha)$. Let $p\in(1,\infty)$, $r\in[0,1]$, $s\geq 0$ and $T>0$. Given an element f of $W_p^{r,s}((0,T)\times M)$, then for all $\alpha\in\{1,\dots,N\}$ the function $t\mapsto f(t)\circ\phi_\alpha^{-1}$ lies in $W_p^{r,s}((0,T)\times\phi_\alpha(U_\alpha))$ satisfying the estimate*

$$\left\|f\circ\phi_\alpha^{-1}\right\|_{W_p^{r,s}((0,T)\times\phi_\alpha(U_\alpha))}\leq C(Q)\left\|f\right\|_{W_p^{r,s}((0,T)\times M)}\,.$$

If on the other hand, $f:(0,T)\times M\to\mathbb{R}$ is such that for all $\alpha\in\{1,\dots,N\}$ the function $t\mapsto(\psi_\alpha f(t))\circ\phi_\alpha^{-1}$ lies in $W_p^{r,s}((0,T)\times\phi_\alpha(U_\alpha))$, then it holds $f\in W_p^{r,s}((0,T)\times M)$. Furthermore, there is an equivalent norm on $W_p^{r,s}((0,T)\times M)$ given by

$$\|f\|^{\mathcal{T}}_{W_p^{r,s}((0,T)\times M)}:=\sum_{\alpha=1}^{N}\left\|(\psi_\alpha f)\circ\phi_\alpha^{-1}\right\|_{W_p^{r,s}((0,T)\times\phi_\alpha(U_\alpha))}$$

with equivalence constant depending on the uniform bounds on the metric, the inverse metric and the Christoffel symbols.

Proof. This follows combining the results shown in Lemma C.17, C.18, C.19, and C.20. ☐

Lemma C.17. *Let $(M,\mathcal{A},g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n\in\mathbb{N}$ with Levi-Civita connection D and uniform localisation system $(U_\alpha,\phi_\alpha,\psi_\alpha)$. Let $p\in(1,\infty)$, $s\geq 0$ and $T>0$. Given $f\in L_p\big((0,T);W_p^s(M)\big)$ then for all $\alpha\in\{1,\dots,N\}$ the functions $f_\alpha,\big(\psi_\alpha\circ\phi_\alpha^{-1}\big)f_\alpha$, where $f_\alpha(t,x):=f\big(t,\phi_\alpha^{-1}(x)\big)$, lie in $L_p\big((0,T);W_p^s(\phi_\alpha(U_\alpha))\big)$ with*

$$\|f_\alpha\|_{L_p\left((0,T);W_p^s(\phi_\alpha(U_\alpha))\right)}+\left\|\big(\psi_\alpha\circ\phi_\alpha^{-1}\big)f_\alpha\right\|_{L_p\left((0,T);W_p^s(\phi_\alpha(U_\alpha))\right)}\leq C(Q)\|f\|_{L_p\left((0,T);W_p^s(M)\right)}\,.$$

Proof. Let $f\in L_p\big((0,T);W_p^s(M)\big)$ be given and suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence of simple functions in $L_p\big((0,T);W_p^s(M)\big)$ such that for almost every $t\in(0,T)$, $\|f_n(t)-f(t)\|_{W_p^s(M)}\to 0$ as $n\to\infty$. In the case that $s=m\in\mathbb{N}_0$ is an integer, Proposition C.1 implies for almost every $t\in(0,T)$ that $f_\alpha(t)=f(t)\circ\phi_\alpha^{-1}$, $f_{\alpha,n}(t)=f_n(t)\circ\phi_\alpha^{-1}$ lie in $W_p^m(\phi_\alpha(U_\alpha))$ and as $n\to\infty$,

$$\|f_\alpha(t)-f_{n,\alpha}(t)\|_{W_p^m(\phi_\alpha(U_\alpha))}\leq C(Q)\|f(t)-f_n(t)\|_{W_p^m(M)}\to 0\,.$$

As $f_{n,\alpha}:(0,T)\to W_p^m(\phi_\alpha(U_\alpha))$, $n\in\mathbb{N}$, form a sequence of simple functions, this shows that $f_\alpha:(0,T)\to W_p^m(\phi_\alpha(U_\alpha))$ is strongly measurable. Finally, we have

$$\int_0^T\|f_\alpha(t)\|^p_{W_p^m(\phi_\alpha(U_\alpha))}\,\mathrm{d}t\leq C(Q)\int_0^T\|f(t)\|^p_{W_p^m(M)}\,\mathrm{d}t=C(Q)\|f\|^p_{L_p\left((0,T);W_p^m(M)\right)}\,.$$

The product rule implies that $\big(\psi_\alpha\circ\phi_\alpha^{-1}\big)f_\alpha(t)$ and $\big(\psi_\alpha\circ\phi_\alpha^{-1}\big)f_{n,\alpha}(t)$ lie in $W_p^m(\phi_\alpha(U_\alpha))$ for Lebesgue-almost every $t\in(0,T)$. The functions $\big(\psi_\alpha\circ\phi_\alpha^{-1}\big)f_{n,\alpha}:(0,T)\to W_p^m(\phi_\alpha(U_\alpha))$, $n\in\mathbb{N}$, form a sequence of simple functions and the equivalence of norms shown in Proposition C.1 implies for every $t \in (0,T)$ as $n \to \infty$,

$$\left\| \left(\psi_\alpha \circ \phi_\alpha^{-1}\right) f_\alpha(t) - \left(\psi_\alpha \circ \phi_\alpha^{-1}\right) f_{\alpha,n}(t) \right\|_{W_p^m(\phi_\alpha(U_\alpha))} \leq C(Q) \left\| f(t) - f_n(t) \right\|_{W_p^m(M)} \to 0$$

which shows that $\left(\psi_\alpha \circ \phi_\alpha^{-1}\right) f_\alpha : (0,T) \to W_p^m\left(\phi_\alpha(U_\alpha)\right)$ is strongly measurable. The estimate follows using the equivalence of norms on $W_p^m(M)$.
In the case that $s > 0$ is non-integer, Definition C.10 applies and yields for all $\alpha \in \{1, \ldots, N\}$ and almost every $t \in (0,T)$ that the functions $\widetilde{f}_\alpha(t) := (\psi_\alpha f(t)) \circ \phi_\alpha^{-1}$, $\widetilde{f}_{n,\alpha}(t) := (\psi_\alpha f_n(t)) \circ \phi_\alpha^{-1}$ lie in $W_p^s\left(\phi_\alpha(U_\alpha)\right)$. As $\widetilde{f}_{n,\alpha} : (0,T) \to W_p^s\left(\phi_\alpha(U_\alpha)\right)$, $n \in \mathbb{N}$, form a sequence of simple functions such that for almost every $t \in (0,T)$,

$$\left\| \widetilde{f}_{n,\alpha}(t) - \widetilde{f}_\alpha(t) \right\|_{W_p^s(\phi_\alpha(U_\alpha))} \leq \left\| f_n(t) - f(t) \right\|_{W_p^s(M)} \to 0$$

as $n \to \infty$, we conclude that $\widetilde{f}_\alpha : (0,T) \to W_p^s\left(\phi_\alpha(U_\alpha)\right)$ is strongly measurable. Finally, $\left\| \widetilde{f}_\alpha(t) \right\|_{W_p^s(\phi_\alpha(U_\alpha))} \leq \| f(t) \|_{W_p^s(M)}$ implies $\widetilde{f}_\alpha \in L_p\left((0,T); W_p^s\left(\phi_\alpha(U_\alpha)\right)\right)$ with the desired estimate. By Proposition C.11 the functions $f_\alpha(t) = f(t) \circ \phi_\alpha^{-1}$, $f_{n,\alpha}(t) := f_n(t) \circ \phi_\alpha^{-1}$ lie in $W_p^s\left(\phi_\alpha(U_\alpha)\right)$ for almost every $t \in (0,T)$ with

$$\left\| f_{n,\alpha}(t) - f_\alpha(t) \right\|_{W_p^s(\phi_\alpha(U_\alpha))} \leq C(Q) \left\| f_n(t) - f(t) \right\|_{W_p^s(M)} \to 0$$

as $n \to \infty$. As $f_{n,\alpha} : (0,T) \to W_p^s\left(\phi_\alpha(U_\alpha)\right)$, $n \in \mathbb{N}$, form a family of simple functions, this shows that $f_\alpha : (0,T) \to W_p^s\left(\phi_\alpha(U_\alpha)\right)$ is strongly measurable, and the estimate in Proposition C.11 implies

$$\| f_\alpha \|_{L_p\left((0,T); W_p^s(\phi_\alpha(U_\alpha))\right)} \leq C(Q) \| f \|_{L_p\left((0,T); W_p^s(M)\right)} .$$

□

Lemma C.18. *Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n \in \mathbb{N}$ with Levi-Civita connection D and uniform localisation system $(U_\alpha, \phi_\alpha, \psi_\alpha)$. Let $p \in (1, \infty)$, $T > 0$ and $r \in (0,1]$. Given $f \in W_p^r\left((0,T); L_p(M)\right)$ then for all $\alpha \in \{1, \ldots, N\}$ the functions f_α, $(\psi_\alpha \circ \phi_\alpha^{-1}) f_\alpha$, where $f_\alpha(t,x) := f\left(t, \phi_\alpha^{-1}(x)\right)$, lie in $W_p^r\left((0,T); L_p\left(\phi_\alpha(U_\alpha)\right)\right)$ with*

$$\| f_\alpha \|_{W_p^r((0,T); L_p(\phi_\alpha(U_\alpha)))} + \left\| (\psi_\alpha \circ \phi_\alpha^{-1}) f_\alpha \right\|_{W_p^r((0,T); L_p(\phi_\alpha(U_\alpha)))} \leq C(Q) \| f \|_{W_p^r((0,T); L_p(M))} .$$

Proof. Lemma C.17 implies f_α, $(\psi_\alpha \circ \phi_\alpha^{-1}) f_\alpha \in L_p\left((0,T); L_p(\phi_\alpha(U_\alpha))\right)$. In the case $r \in (0,1)$ the claim immediately follows from

$$\| f_\alpha(t) - f_\alpha(s) \|_{L_p(\phi_\alpha(U_\alpha))} + \left\| \left(\psi_\alpha \circ \phi_\alpha^{-1}\right) \left(f_\alpha(t) - f_\alpha(s)\right) \right\|_{L_p(\phi_\alpha(U_\alpha))} \leq C(Q) \| f(t) - f(s) \|_{L_p(M)}$$

for $s, t \in (0,T)$ as shown in Proposition C.1. Suppose that $r = 1$. Let $g \in L_p\left((0,T); L_p(M)\right)$ be such that for all $\psi \in C_0^\infty\left((0,T)\right)$ the identity

$$\int_0^T f(t) \psi'(t) \, \mathrm{d}t = - \int_0^T g(t) \psi(t) \, \mathrm{d}t$$

is satisfied in $L_p(M)$. In the following we let $\varphi \equiv 1$ on M or $\varphi = \psi_\alpha$. In particular, for Lebesgue-almost every $x \in \phi_\alpha(U_\alpha)$,

$$\left(\varphi \int_0^T f(t) \psi'(t) \, \mathrm{d}t \right) \left(\phi_\alpha^{-1}(x)\right) = - \left(\varphi \int_0^T g(t) \psi(t) \, \mathrm{d}t \right) \left(\phi_\alpha^{-1}(x)\right) . \tag{C.12}$$

Approximating the functions f and g in $L_p((0,T);L_p(M))$ by a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$, and $(g_n)_{n\in\mathbb{N}}$, respectively, one sees that the identity (C.12) can equivalently be written as

$$\left(\int_0^T \left((\varphi f(t))\circ\phi_\alpha^{-1}\right)\psi'(t)\,\mathrm{d}t\right)(x) = -\left(\int_0^T \left((\varphi g(t))\circ\phi_\alpha^{-1}\right)\psi(t)\,\mathrm{d}t\right)(x) \tag{C.13}$$

for almost every $x\in\phi_\alpha(U_\alpha)$. Indeed, let $(f_n)_{n\in\mathbb{N}}$ be a sequence of simple functions converging to f in $L_p((0,T);L_p(M))$. Then it is straightforward to see that for $x\in\phi_\alpha(U_\alpha)$,

$$\left(\varphi\int_0^T f_n(t)\psi'(t)\,\mathrm{d}t\right)\left(\phi_\alpha^{-1}(x)\right) = \left(\int_0^T \left((\varphi f_n(t))\circ\phi_\alpha^{-1}\right)\psi'(t)\,\mathrm{d}t\right)(x)\,. \tag{C.14}$$

Using Proposition C.1 we obtain as $n\to\infty$,

$$\begin{aligned}
&\left\|\varphi\left(\int_0^T (f(t)-f_n(t))\,\psi'(t)\mathrm{d}t\right)\circ\phi_\alpha^{-1}\right\|_{L_p(\phi_\alpha(U_\alpha))} \le C(Q)\left\|\int_0^T (f(t)-f_n(t))\,\psi'(t)\mathrm{d}t\right\|_{L_p(M)}\\
\le& C(Q)\int_0^T \|f(t)-f_n(t)\|_{L_p(M)}\,|\psi'(t)|\,\mathrm{d}t \le C(Q)\,\|f-f_n\|_{L_p(0,T;L_p(M))}\to 0\,,
\end{aligned}$$

and on the other hand

$$\begin{aligned}
&\left\|\int_0^T \left((\varphi(f_n(t)-f(t)))\circ\phi_\alpha^{-1}\right)\psi'(t)\,\mathrm{d}t\right\|_{L_p(\phi_\alpha(U_\alpha))}\\
\le& \int_0^T \left\|(\varphi(f_n(t)-f(t)))\circ\phi_\alpha^{-1}\right\|_{L_p(\phi_\alpha(U_\alpha))}|\psi'(t)|\,\mathrm{d}t \le C(Q)\int_0^T \|f_n(t)-f(t)\|_{L_p(M)}\,|\psi'(t)|\,\mathrm{d}t\\
\le& C(Q)\,\|f_n-f\|_{L_p((0,T);L_p(M))}\to 0\,.
\end{aligned}$$

The identity (C.14) and uniqueness of the limit in $L_p(\phi_\alpha(U_\alpha))$ imply

$$\left(\varphi\int_0^T f(t)\psi'(t)\,\mathrm{d}t\right)\left(\phi_\alpha^{-1}(x)\right) = \left(\int_0^T \left((\varphi f(t))\circ\phi_\alpha^{-1}\right)\psi'(t)\,\mathrm{d}t\right)(x)$$

for almost every $x\in\phi_\alpha(U_\alpha)$. The same argument for g instead of f shows the desired identity (C.13). This implies the claim as the function $t\mapsto(\varphi g(t))\circ\phi_\alpha^{-1}$ lies in $L_p((0,T);L_p(\phi_\alpha(U_\alpha)))$ by Lemma C.17 with

$$\left\|t\mapsto(\varphi g(t))\circ\phi_\alpha^{-1}\right\|_{L_p((0,T);L_p(\phi_\alpha(U_\alpha))))} \le C(Q)\,\|g\|_{L_p((0,T);L_p(M))}\,.$$

□

To prove the converse of Lemma C.17 and C.18 we make use of the extension results C.13 and C.14.

Lemma C.19. *Let $(M,\mathcal{A},g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n\in\mathbb{N}$ with Levi-Civita connection D and uniform localisation system $(U_\alpha,\phi_\alpha,\psi_\alpha)$. Let $p\in(1,\infty)$, $s\ge 0$ and $T>0$. Suppose that $f:(0,T)\times M\to\mathbb{R}$ is such that for all $\alpha\in\{1,\dots,N\}$ it holds $\left(\psi_\alpha\circ\phi_\alpha^{-1}\right)f_\alpha\in L_p\left((0,T);W_p^s(\phi_\alpha(U_\alpha))\right)$ where $f_\alpha(t,x):=f\left(t,\phi_\alpha^{-1}(x)\right)$. Then f lies in $L_p\left((0,T);W_p^s(M)\right)$ and satisfies the estimate*

$$\|f\|_{L_p\left((0,T);W_p^s(M)\right)} \le C(Q)\sum_{\alpha=1}^N \left\|\left(\psi_\alpha\circ\phi_\alpha^{-1}\right)f_\alpha\right\|_{L_p\left((0,T);W_p^s(\phi_\alpha(U_\alpha))\right)}\,.$$

Proof. For all $\alpha \in \{1,\dots,N\}$ and almost every $t \in (0,T)$ the functions $\widetilde{f}_\alpha(t) := \left(\psi_\alpha \circ \phi_\alpha^{-1}\right) f_\alpha(t)$ lie in $W^s_p\left(\phi_\alpha(U_\alpha)\right)$ which implies $f(t) \in W^s_p(M)$ for almost every $t \in (0,T)$. In the case that $s = m \in \mathbb{N}_0$ is an integer, this is a consequence of Proposition C.1, for non-integer $s > 0$ it follows from Definition C.10. As $\widetilde{f}_\alpha : (0,T) \to W^s_p\left(\phi_\alpha(U_\alpha)\right)$ is strongly measurable for $\alpha \in \{1,\dots,N\}$, there exist sequences of simple functions $\widetilde{f}_{n,\alpha} : (0,T) \to W^s_p\left(\phi_\alpha(U_\alpha)\right)$, $n \in \mathbb{N}$, such that for almost every $t \in (0,T)$,

$$\left\|\widetilde{f}_\alpha(t) - \widetilde{f}_{n,\alpha}(t)\right\|_{W^s_p(\phi_\alpha(U_\alpha))} \to 0$$

as $n \to \infty$. For every $\alpha \in \{1,\dots,N\}$ we let $\xi_\alpha : M \to [0,1]$ be a smooth bump function with the properties $\xi_\alpha \equiv 1$ on $\operatorname{supp}\psi_\alpha$ and $\operatorname{supp}\xi_\alpha \subset U_\alpha$. The existence of such a function follows from [89, Proposition 2.25]. Given $\alpha \in \{1,\dots,N\}$ we observe that for all $n \in \mathbb{N}$ the function

$$f_{n,\alpha}(t) := \widetilde{f}_{n,\alpha}(t)\left(\xi_\alpha \circ \phi_\alpha^{-1}\right)$$

lies in $W^s_p\left(\phi_\alpha(U_\alpha)\right)$. The properties of ξ_α imply further for almost every $t \in (0,T)$,

$$\left\|f_{n,\alpha}(t) - (\psi_\alpha f(t)) \circ \phi_\alpha^{-1}\right\|_{W^s_p(\phi_\alpha(U_\alpha))} \to 0$$

as $n \to \infty$. As

$$\operatorname{dist}\left(\operatorname{supp} f_{n,\alpha}(t) \cap \overline{\phi_\alpha(U_\alpha)}, \partial\phi_\alpha(U_\alpha) \cap \operatorname{int}\mathbb{H}^n\right) > 0\,,$$

Lemma C.14 implies that $E\left(f_{n,\alpha}(t) \circ \phi_\alpha\right)$ lies in $W^s_p(M)$ and as a consequence, we obtain $f_n(t) := \sum_{\alpha=1}^N E\left(f_{n,\alpha}(t) \circ \phi_\alpha\right) \in W^s_p(M)$. To show that the sequence $(f_n(t))_{n\in\mathbb{N}}$ converges to $f(t)$ in $W^s_p(M)$ for almost every $t \in (0,T)$ we observe that

$$(\psi_\gamma f_n(t)) \circ \phi_\gamma^{-1} = \sum_{\alpha=1}^N \left(\psi_\gamma \circ \phi_\gamma^{-1}\right) E\left(f_{n,\alpha}(t) \circ \phi_\alpha \circ \phi_\gamma^{-1}\right)$$

with each summand being an element of $W^s_p\left(\phi_\gamma(U_\gamma)\right)$. Indeed, given $\alpha, \gamma \in \{1,\dots,N\}$, the transformation formula [2, Theorem 3.41] implies $f_{n,\alpha}(t) \circ \phi_\alpha \circ \phi_\gamma^{-1} \in W^{\lfloor s \rfloor}_p\left(\phi_\gamma(U_\alpha \cap U_\gamma)\right)$ and in the case that $s > 0$ is non-integer, similar arguments as in the proof of Proposition C.11 imply $f_{n,\alpha}(t) \circ \phi_\alpha \circ \phi_\gamma^{-1} \in W^s_p\left(\phi_\gamma(U_\alpha \cap U_\gamma)\right)$. As further

$$\operatorname{dist}\left(\operatorname{supp}\left(f_{n,\alpha}(t) \circ \phi_\alpha \circ \phi_\gamma^{-1}\right), \partial\left(\operatorname{int}\phi_\gamma(U_\alpha \cap U_\gamma)\right) \cap \operatorname{int}\phi_\gamma(U_\gamma)\right) > 0\,,$$

Lemma C.13 implies that the trivial extension $E\left(f_{n,\alpha}(t) \circ \phi_\alpha \circ \phi_\gamma^{-1}\right)$ lies in $W^s_p\left(\phi_\gamma(U_\gamma)\right)$. The same reasoning implies $(\psi_\gamma\psi_\alpha f(t)) \circ \phi_\gamma^{-1} \in W^s_p(\phi_\gamma(U_\gamma))$ for all $\alpha \in \{1,\dots,N\}$. Hence we may estimate

$$\begin{aligned}
&\left\|\psi_\gamma(f_n(t) - f(t)) \circ \phi_\gamma^{-1}\right\|_{W^s_p(\phi_\gamma(U_\gamma))} \\
&\le \sum_{\alpha=1}^N \left\|\left(\psi_\gamma \circ \phi_\gamma^{-1}\right)\left(E\left(f_{n,\alpha}(t) \circ \phi_\alpha \circ \phi_\gamma^{-1}\right) - (\psi_\alpha f(t)) \circ \phi_\gamma^{-1}\right)\right\|_{W^s_p(\phi_\gamma(U_\gamma))} \\
&\le \sum_{\alpha=1}^N \left\|\left(\psi_\gamma \circ \phi_\gamma^{-1}\right)\left(f_{n,\alpha}(t) \circ \phi_\alpha \circ \phi_\gamma^{-1} - (\psi_\alpha f(t)) \circ \phi_\gamma^{-1}\right)\right\|_{W^s_p(\phi_\gamma(U_\alpha\cap U_\gamma))} \\
&\le C\sum_{\alpha=1}^N \left\|\left(\psi_\gamma \circ \phi_\alpha^{-1}\right)\left(f_{n,\alpha}(t) - (\psi_\alpha f(t)) \circ \phi_\alpha^{-1}\right)\right\|_{W^s_p(\phi_\alpha(U_\alpha\cap U_\gamma))} \to 0
\end{aligned}$$

as $n \to \infty$. Since $f_n : (0,T) \to W^s_p(M)$, $n \in \mathbb{N}$, are simple functions, we conclude that $f : (0,T) \to W^s_p(M)$ is strongly measurable. Furthermore, we have

$$\int_0^T \|f(t)\|^p_{W^s_p(M)}\,\mathrm{d}t \le C(Q)\sum_{\alpha=1}^N \int_0^T \left\|\left(\psi_\alpha \circ \phi_\alpha^{-1}\right) f_\alpha(t)\right\|^p_{W^s_p(\phi_\alpha(U_\alpha))}\,\mathrm{d}t$$

$$\leq C(Q) \sum_{\alpha=1}^{N} \left\| \left(\psi_\alpha \circ \phi_\alpha^{-1}\right) f_\alpha \right\|_{L_p\left((0,T);W_p^s(\phi_\alpha(U_\alpha))\right)}^p .$$

In the case that $s = m \in \mathbb{N}_0$ is an integer, this is a consequence of Proposition C.1, for non-integer $s > 0$ it follows from the definition of the norm on $W_p^s(M)$ and the estimate even holds with constant $C = 1$. □

Lemma C.20. *Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n \in \mathbb{N}$ with Levi-Civita connection D and uniform localisation system $(U_\alpha, \phi_\alpha, \psi_\alpha)$. Let $p \in (1,\infty)$, $T > 0$ and $r \in (0,1]$. Suppose that $f : (0,T) \times M \to \mathbb{R}$ is such that for all $\alpha \in \{1,\dots,N\}$ it holds $\left(\psi_\alpha \circ \phi_\alpha^{-1}\right) f_\alpha \in W_p^r\left((0,T); L_p\left(\phi_\alpha(U_\alpha)\right)\right)$ where $f_\alpha(t,x) := f\left(t, \phi_\alpha^{-1}(x)\right)$. Then f lies in $W_p^r\left((0,T); L_p(M)\right)$ and satisfies the estimate*

$$\|f\|_{W_p^r((0,T);L_p(M))} \leq C(Q) \sum_{\alpha=1}^{N} \left\| \left(\psi_\alpha \circ \phi_\alpha^{-1}\right) f_\alpha \right\|_{W_p^r((0,T);L_p(\phi_\alpha(U_\alpha)))} .$$

Proof. Lemma C.19 implies that f lies in $L_p\left((0,T); L_p(M)\right)$. In the case $r \in (0,1)$ the claim immediately follows from the equivalence of norms on $L_p(M)$ shown in Proposition C.1. Suppose that $r = 1$. Given $\alpha \in \{1,\dots,N\}$ let $g_\alpha \in L_p\left((0,T); L_p\left(\phi_\alpha(U_\alpha)\right)\right)$ be such that for all $\psi \in C_0^\infty((0,T))$ the identity

$$\int_0^T \left(\psi_\alpha \circ \phi_\alpha^{-1}\right) f_\alpha(t)\psi'(t)\,\mathrm{d}t = -\int_0^T g_\alpha(t)\psi(t)\,\mathrm{d}t$$

holds in $L_p\left(\phi_\alpha(U_\alpha)\right)$. Similar arguments as in the proof of Proposition C.1 show that for $t \in (0,T)$ the function $g(t) := \sum_{\alpha=1}^N E\left(g_\alpha(t) \circ \phi_\alpha\right)$ lies in $L_p(M)$ satisfying the estimate

$$\|g(t)\|_{L_p(M)} \leq C(Q) \sum_{\alpha=1}^{N} \|g_\alpha(t)\|_{L_p(\phi_\alpha(U_\alpha))} \tag{C.15}$$

where $E(f)$ denotes the trivial extension of a function f defined on a subset of M to the entire manifold. Let $(g_{n,\alpha})_{n\in\mathbb{N}}$ be a sequence of simple functions in $L_p\left((0,T); L_p(\phi_\alpha(U_\alpha))\right)$ such that for Lebesgue-almost every $t \in (0,T)$, $\|g_{n,\alpha}(t) - g_\alpha(t)\|_{L_p(\phi_\alpha(U_\alpha))} \to 0$ as $n \to \infty$. Then the function $g_n(t) := \sum_{\alpha=1}^N E\left(g_{n,\alpha}(t) \circ \phi_\alpha\right)$ lies in $L_p(M)$ satisfying as $n \to \infty$,

$$\|g_n(t) - g(t)\|_{L_p(M)} \leq C(Q) \sum_{\alpha=1}^{N} \|g_{n,\alpha}(t) - g_\alpha(t)\|_{L_p(\phi_\alpha(U_\alpha))} \to 0 .$$

As $g_n : (0,T) \to L_p(M)$ is a simple function, this allows us to conclude that g lies in the space $L_p\left((0,T); L_p(M)\right)$. Approximating f_α by a sequence of simple functions shows the identity

$$\begin{aligned}\sum_{\alpha=1}^{N} \left(\int_0^T \left(\psi_\alpha \circ \phi_\alpha^{-1}\right) f_\alpha(t)\psi'(t)\,\mathrm{d}t \right) \circ \phi_\alpha &= \int_0^T \left(\sum_{\alpha=1}^{N} \left(\psi_\alpha \circ \phi_\alpha^{-1}\right) f_\alpha(t) \circ \phi_\alpha \right) \psi'(t)\,\mathrm{d}t \\ &= \int_0^T f(t)\psi'(t)\,\mathrm{d}t\end{aligned}$$

in $L_p(M)$. Using the analogous identity for g and g_α, we obtain for all $\psi \in C_0^\infty\left((0,T)\right)$ the identity

$$\int_0^T f(t)\psi'(t)\,\mathrm{d}t = -\int_0^T g(t)\psi(t)\,\mathrm{d}t$$

in $L_p(M)$. This shows $f \in W_p^1\left((0,T); L_p(M)\right)$ and the desired estimate follows from (C.15). □

C.2.2 An extension result

In this subsection we prove the analogous result to the one given in Subsection C.1.3 in the case of anisotropic spaces.

Lemma C.21. *Let $n \in \mathbb{N}$, $p \in (1,\infty)$, $T > 0$, $r \in [0,1]$ and $s \geq 0$. Suppose that Ω_1 and Ω_2 are open sets in $\mathbb{R}^n$ with $\Omega_1 \subset \Omega_2$ and Ω_1 bounded. Given any function $g : \Omega_1 \to \mathbb{R}$ we let $Eg : \Omega_2 \to \mathbb{R}$ be the function that satisfies $Eg = g$ on Ω_1 and $Eg = 0$ on $\Omega_2 \setminus \Omega_1$. If $f \in W_p^{r,s}\left((0,T)\times\Omega_1\right)$ satisfies*

$$\operatorname{dist}\left(\operatorname{supp}(f(t)) \cap \overline{\Omega_1}, \partial\Omega_1 \cap \Omega_2\right) \geq \delta > 0$$

for almost every $t \in (0,T)$ and a constant δ independent of t, then the trivial extension Ef of f given by $(Ef)(t) := E(f(t))$ lies in $W_p^{r,s}\left((0,T)\times\Omega_2\right)$. Furthermore, for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq \lfloor s \rfloor$ and almost every $t \in (0,T)$ it holds $\partial_x^\alpha((Ef)(t)) = E\left(\partial_x^\alpha f(t)\right)$ and, if $s = 4$, $\partial_t(Ef)(t) = E(\partial_t f(t))$. The hereby induced operator $E : W_p^{r,s}\left((0,T)\times\Omega_1\right) \to W_p^{r,s}\left((0,T)\times\Omega_2\right)$ is linear and continuous.

Proof. Given $t \in (0,T)$ it holds $f(t) \in W_p^s(\Omega_1)$ and Lemma C.13 implies $E(f(t)) \in W_p^s(\Omega_2)$ with $\partial_x^\alpha(E(f(t))) = E(\partial_x^\alpha f(t))$. Thus the mapping $Ef : (0,T) \to W_p^s(\Omega_2)$, $t \mapsto (Ef)(t) = E(f(t))$ is well-defined. As $f : (0,T) \to W_p^s(\Omega_1)$ is strongly measurable, there exist sequences of simple functions $\widetilde{f}_n : (0,T) \to W_p^s(\Omega_1)$ such that for almost every $t \in (0,T)$,

$$\left\| f(t) - \widetilde{f}_n(t) \right\|_{W_p^s(\Omega_1)} \to 0$$

as $n \to \infty$. Let $\xi \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \leq \xi \leq 1$, $\xi \equiv 1$ on $\left\{x \in \overline{\Omega}_1 : \operatorname{dist}(x, \partial\Omega_1 \cap \Omega_2) \geq \frac{\delta}{2}\right\}$ and $\operatorname{dist}\left(\operatorname{supp}\xi \cap \overline{\Omega}_1, \partial\Omega_1 \cap \Omega_2\right) \geq \frac{\delta}{4}$. Then for every $t \in (0,T)$, $f_n(t) := \xi \widetilde{f}_n(t)$ lies in $W_p^s(\Omega_1)$ with $\|f(t) - f_n(t)\|_{W_p^s(\Omega_1)} \to 0$ as $n \to \infty$. Lemma C.13 implies for all $n \in \mathbb{N}$ and $t \in (0,T)$, $E(f_n(t)) \in W_p^s(\Omega_2)$ with $\partial_x^\alpha E(f_n(t)) = E(\partial_x^\alpha f_n(t))$. Thus $Ef_n : (0,T) \to W_p^s(\Omega_2)$, $(Ef_n)(t) := E(f_n(t))$, $n \in \mathbb{N}$, forms a sequence of simple functions and Lemma C.13 implies for almost every $t \in (0,T)$,

$$\|(Ef_n)(t) - (Ef)(t)\|_{W_p^s(\Omega_2)} = \|E(f_n(t)) - E(f(t))\|_{W_p^s(\Omega_2)} \leq \|f_n(t) - f(t)\|_{W_p^s(\Omega_1)} \to 0$$

as $n \to \infty$. This shows $Ef \in L_p\left((0,T); W_p^s(\Omega_2)\right)$ and $\partial_x^\alpha(Ef(t)) = E(\partial_x^\alpha f(t))$. In the case $r \in [0,1)$ the claim then follows from the identity $\|Ef(t) - Ef(s)\|_{L_p(\Omega_2)} = \|f(t) - f(s)\|_{L_p(\Omega_1)}$. In the case $r = 1$ the arguments before imply that $E(\partial_t f)$ defined by $E(\partial_t f)(t) := E(\partial_t f(t))$ lies in $L_p((0,T); L_p(\Omega_2))$. Given $\psi \in C_0^\infty((0,T))$ and $x \in \Omega_1$, we observe

$$\int_0^T (Ef)(t,x)\psi(t)\,\mathrm{d}t = \int_0^T f(t,x)\psi(t)\,\mathrm{d}t = -\int_0^T \partial_t f(t,x)\psi(t)\,\mathrm{d}t = -\int_0^T (E\partial_t f)(t,x)\psi(t)\,\mathrm{d}t\,,$$

while in the case $x \in \Omega_2 \setminus \Omega_1$,

$$\int_0^T (Ef)(t,x)\psi(t)\,\mathrm{d}t = 0 = -\int_0^T (E\partial_t f)(t,x)\psi(t)\,\mathrm{d}t\,.$$

This shows $Ef \in W_p^1\left((0,T); L_p(\Omega_2)\right)$ with $\partial_t(Ef)(t) = E(\partial_t f(t)) = E(\partial_t f)(t)$. □

Lemma C.22. *Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n \in \mathbb{N}$ with Levi-Civita connection D and uniform localisation system $(U_\alpha, \phi_\alpha, \psi_\alpha)$. Let $p \in (1,\infty)$, $T > 0$, $r \in [0,1]$, $s \geq 0$ and $(U,\phi) \in \mathcal{A}$. Suppose we are given a function $f \in W_p^{r,s}\left((0,T)\times \operatorname{int}\phi(U)\right)$ with the property that*

$$\operatorname{dist}\left(\operatorname{supp} f(t) \cap \overline{\phi(U)}, \partial\phi(U) \cap \operatorname{int}\mathbb{H}^n\right) \geq \delta > 0$$

for almost every $t \in (0,T)$ *and a constant* δ *independent of* t. *Then the trivial extension* $E(f \circ \phi)$ *of* $f \circ \phi$ *given by* $E(f\circ\phi)(t) = E(f(t)\circ\phi)$ *to the entire manifold* M *lies in* $W_p^{r,s}((0,T)\times M)$ *with*

$$\|E(f\circ\phi)\|_{W_p^{r,s}((0,T)\times M)} \leq C(Q)\,\|f\|_{W_p^{r,s}((0,T)\times \mathrm{int}\phi(U))}\,.$$

Proof. We show that for all $\alpha \in \{1,\dots,N\}$ the trivial extension of

$$(t,x) \mapsto f_\alpha(t,x) := f\left(t, \phi(\phi_\alpha^{-1}(x))\right)$$

lies in $W_p^{r,s}((0,T)\times \mathrm{int}\phi_\alpha(U_\alpha))$. The transformation formula [2, Theorem 3.41] implies

$$(t,x) \mapsto f\left(t,\phi(\phi_\alpha^{-1}(x))\right) \in W_p^{r,s}\left((0,T)\times \mathrm{int}\phi_\alpha(U\cap U_\alpha)\right)$$

with

$$\left\|(t,x)\mapsto f\left(t,\phi(\phi_\alpha^{-1}(x))\right)\right\|_{W_p^{r,s}((0,T)\times \mathrm{int}\phi_\alpha(U\cap U_\alpha))} \leq C\,\|f\|_{W_p^{r,s}((0,T)\times \mathrm{int}\phi(U))}$$

with constant C depending on the uniform bounds on (derivatives of) the coordinate transformations. By hypothesis we have

$$\mathrm{dist}\left(\mathrm{supp}\, f(t) \cap \overline{\phi(U\cap U_\alpha)}, \partial\phi(U)\cap \mathrm{int}\mathbb{H}^n \cap \phi(U\cap U_\alpha)\right) \geq \delta > 0$$

for almost every $t \in (0,T)$ and a constant δ independent of t. As ϕ_α is a homeomorphism, this yields for $f_\alpha(t) := f(t)\circ\phi\circ\phi_\alpha^{-1}$,

$$\mathrm{dist}\left(\mathrm{supp}\, f_\alpha(t) \cap \overline{\phi_\alpha(U\cap U_\alpha)}, \partial\phi_\alpha(U\cap U_\alpha)\cap \mathrm{int}\phi_\alpha(U_\alpha)\right) \geq \boldsymbol{\delta} > 0$$

for almost every $t \in (0,T)$ and a constant $\boldsymbol{\delta}$ independent of t. Lemma C.21 implies that the trivial extension Ef_α lies in $W_p^{r,s}((0,T)\times \mathrm{int}\phi_\alpha(U_\alpha))$ with

$$\|Ef_\alpha\|_{W_p^{r,s}((0,T)\times \mathrm{int}\phi_\alpha(U_\alpha))} \leq C\,\|f\|_{W_p^{r,s}((0,T)\times \mathrm{int}\phi(U))}\,.$$

As $(E(f\circ\phi))\circ\phi_\alpha^{-1} = Ef_\alpha$ on $\phi_\alpha(U_\alpha)$, Proposition C.16 implies $E(f\circ\phi) \in W_p^{r,s}((0,T)\times M)$ with

$$\begin{aligned}\|E(f\circ\phi)\|_{W_p^{r,s}((0,T)\times M)} &\leq C(Q)\sum_{\alpha=1}^{N}\left\|(E(f\circ\phi))\circ\phi_\alpha^{-1}\right\|_{W_p^{r,s}((0,T)\times \mathrm{int}\phi_\alpha(U_\alpha))}\\ &\leq C(Q)\sum_{\alpha=1}^{N}\|Ef_\alpha\|_{W_p^{r,s}((0,T)\times \mathrm{int}\phi_\alpha(U_\alpha))} \leq C(Q)\,\|f\|_{W_p^{r,s}((0,T)\times \mathrm{int}\phi(U))}\,.\end{aligned}$$

□

C.2.3 Further equivalent norms

In this subsection we derive further equivalent norms on anisotropic spaces on compact manifolds.

Proposition C.23 (Norms with respect to arbitrary coverings)**.** *Let* $(M,\mathcal{A},g)$ *be a smooth compact oriented Riemannian manifold with or without boundary of dimension* $n \in \mathbb{N}$ *with Levi-Civita connection* D, *normal covering* $\mathcal{T} = \{(U_\alpha,\phi_\alpha)\}$ *and uniform localisation system* $(U_\alpha,\phi_\alpha,\psi_\alpha)$, $p \in (1,\infty)$ *and* $s \geq 0$, $r \in [0,1]$. *Let* $M \in \mathbb{N}$ *and*

$$\mathcal{S} := \{(V_\beta,\varrho_\beta) : \beta \in \{1,\dots,M\}\} \subset \mathcal{A}\,,$$

be any covering of M *such that for all* α, β *with* $U_\alpha \cap V_\beta \neq \emptyset$ *all derivatives of* $\phi_\alpha \circ \varrho_\beta^{-1}$ *and its inverse are uniformly bounded by a constant* $C(\mathcal{S})$. *Let further* $(\xi_\beta)_{\beta\in\{1,\dots,M\}}$ *be a smooth* *partition of unity subordinate to the covering $\mathcal{S}$. Given a function $f \in W_p^{r,s}((0,T) \times M)$, then for all $\beta \in \{1, \ldots, M\}$ the function $t \mapsto f(t) \circ \varrho_\beta^{-1}$ lies in $W_p^{r,s}((0,T) \times \varrho_\beta(V_\beta))$. If conversely $f : (0,T) \times M \to \mathbb{R}$ is such that for all $\beta \in \{1, \ldots, M\}$ the function $t \mapsto (\xi_\beta f(t)) \circ \varrho_\beta^{-1}$ lies in $W_p^{r,s}((0,T) \times \varrho_\beta(V_\beta))$, then it holds $f \in W_p^{r,s}((0,T) \times M)$. Furthermore,*

$$\|f\|^{\mathcal{S}}_{W_p^{r,s}((0,T)\times M)} := \sum_{\beta=1}^{M} \left\| (\xi_\beta f) \circ \varrho_\beta^{-1} \right\|_{W_p^{r,s}((0,T)\times \varrho_\beta(V_\beta))}$$

defines a norm on $W_p^{r,s}((0,T) \times M)$ equivalent to $\|\cdot\|^{\mathcal{T}}_{W_p^{r,s}((0,T)\times M)}$ with equivalence constants depending on $C(\mathcal{S})$.

Proof. This follows from Proposition C.16 using similar arguments as in the proof of Proposition C.12. □

In analogy to Corollary B.29 we may consider an equivalent norm on $W_p^{\beta,4\beta}((0,T) \times M)$ with $\beta \in (1/p, 1]$.

Proposition C.24. *Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n \in \mathbb{N}$ with Levi-Civita connection D and uniform localisation system $(U_\alpha, \phi_\alpha, \psi_\alpha)$. Let T be positive, $p \in (1, \infty)$ and $\beta \in (1/p, 1]$. Then*

$$\begin{aligned} |||f|||_{W_p^{\beta,4\beta}((0,T)\times M)} &:= \sum_{\alpha=1}^{N} |||(\psi_\alpha f) \circ \phi_\alpha^{-1}|||_{W_p^{\beta,4\beta}((0,T)\times \phi_\alpha(U_\alpha))} \\ &= \|f\|^{\mathcal{T}}_{W_p^{\beta,4\beta}((0,T)\times M)} + \|f(0)\|^{\mathcal{T}}_{W_p^{4\beta-4/p}(M)} \end{aligned}$$

defines a norm on $W_p^{\beta,4\beta}((0,T) \times M)$ that is equivalent to $\|\cdot\|_{W_p^{\beta,4\beta}((0,T)\times M)}$ with equivalence constants depending on T.

Proof. This is a direct consequence of Propositions B.36, C.11 and C.16. □

We prove an analogous result to Proposition B.37.

Proposition C.25 (Temporal extension operator on manifolds)**.** *Let $(M, \mathcal{A}, g)$ be a smooth compact oriented Riemannian manifold with or without boundary of dimension $n \in \mathbb{N}$ with Levi-Civita connection D and uniform localisation system $(U_\alpha, \phi_\alpha, \psi_\alpha)$. Let T_0 be positive, $T \in (0, T_0)$, $p \in (1, \infty)$ and $\alpha \in (1/p, 1]$. Then for every $g \in W_p^{\alpha,4\alpha}((0,T) \times M)$ there exists an extension $\boldsymbol{E}g \in W_p^{\alpha,4\alpha}((0,T_0) \times M)$ such that $(\boldsymbol{E}g)_{|(0,T)} = g$ and*

$$|||\boldsymbol{E}g|||_{W_p^{\alpha,4\alpha}((0,T_0)\times M)} \le C(T_0, Q)\, |||g|||_{W_p^{\alpha,4\alpha}((0,T)\times M)}\,.$$

The hereby induced extension operator is linear.

Proof. Let $\xi_\alpha \in C^\infty(M)$ satisfy $0 \le \xi_\alpha \le 1$, $\operatorname{supp} \xi_\alpha \subset U_\alpha$ and $\xi_\alpha \equiv 1$ on $\operatorname{supp} \psi_\alpha$. Proposition C.16 implies that

$$(t, x) \mapsto g_\alpha(t, x) := \psi_\alpha\left(\phi_\alpha^{-1}(x)\right) g\left(t, \phi_\alpha^{-1}(x)\right)$$

lies in $W_p^{\alpha,4\alpha}((0,T) \times \phi_\alpha(U_\alpha))$ and by Proposition B.37 there exists

$$\boldsymbol{E}_\alpha g_\alpha \in W_p^{\alpha,4\alpha}((0,T_0) \times \phi_\alpha(U_\alpha))$$

with $(\boldsymbol{E}_\alpha g_\alpha)_{|(0,T)} = g_\alpha$ and

$$\|\boldsymbol{E}_\alpha g_\alpha\|_{W_p^{\alpha,4\alpha}((0,T_0)\times\phi_\alpha(U_\alpha))} \le C(T_0)\, |||g_\alpha|||_{W_p^{\alpha,4\alpha}((0,T)\times\phi_\alpha(U_\alpha))}\,. \tag{C.16}$$

The extension $\boldsymbol{E}_\alpha g_\alpha$ is linear in g_α. As

$$\operatorname{dist}\left(\operatorname{supp}\left(\xi_\alpha\circ\phi_\alpha^{-1}\right)\cap\overline{\phi_\alpha(U_\alpha)}, \partial\phi_\alpha(U_\alpha)\cap\operatorname{int}\mathbb{H}^n\right) > 0\,,$$

Lemma C.22 implies that the trivial extension of the function $(t,q)\mapsto \xi_\alpha(q)\boldsymbol{E}_\alpha g_\alpha(t,\phi_\alpha(q))$ to the entire manifold (denoted by $\xi_\alpha\left((\boldsymbol{E}_\alpha g_\alpha)\circ\phi_\alpha\right)$) lies in $W_p^{\alpha,4\alpha}\left((0,T_0)\times M\right)$. In particular,

$$\boldsymbol{E}g(t,q) := \sum_{\alpha=1}^N \xi_\alpha(q)\boldsymbol{E}_\alpha g_\alpha(t,\phi_\alpha(q))$$

lies in $W_p^{\alpha,4\alpha}\left((0,T_0)\times M\right)$ with

$$(\boldsymbol{E}g)_{|(0,T)} = \sum_{\alpha=1}^N \xi_\alpha\left(\boldsymbol{E}_\alpha g_\alpha\right)_{|(0,T)}\circ\phi_\alpha = \sum_{\alpha=1}^N \xi_\alpha\left(g_\alpha\circ\phi_\alpha\right) = g\,.$$

The operator $\boldsymbol{E}$ is linear in g as the operators $\boldsymbol{E}_\alpha$ are linear in g_α. Finally, the estimates shown in Lemma C.22 and (C.16) imply

$$\begin{aligned}
\|\boldsymbol{E}g\|_{W_p^{\alpha,4\alpha}((0,T_0)\times M)} &\le \sum_{\alpha=1}^N \|\xi_\alpha\left((\boldsymbol{E}_\alpha g_\alpha)\circ\phi_\alpha\right)\|_{W_p^{\alpha,4\alpha}((0,T_0)\times M)}\\
&\le C(Q)\sum_{\alpha=1}^N \|\boldsymbol{E}_\alpha g_\alpha\|_{W_p^{\alpha,4\alpha}((0,T_0)\times\phi_\alpha(U_\alpha))}\\
&\le C(T_0,Q)\sum_{\alpha=1}^N |||g_\alpha|||_{W_p^{\alpha,4\alpha}((0,T)\times\phi_\alpha(U_\alpha))} \le C\left(T_0,Q\right)\, |||g|||_{W_p^{\alpha,4\alpha}((0,T)\times M)}\,.
\end{aligned}$$

As $(\boldsymbol{E}g)(0) = g(0)$, we obtain the desired estimate

$$|||\boldsymbol{E}g|||_{W_p^{\alpha,4\alpha}((0,T_0)\times M)} \le C\left(T_0,Q\right)\, |||g|||_{W_p^{\alpha,4\alpha}((0,T)\times M)}\,.$$

□

C.3 Sobolev spaces on the boundary of smooth domains and trace theorems

Given a function of certain regularity defined on a bounded smooth domain $\Omega\subset\mathbb{R}^n$, or on the cylinder $(0,T)\times\Omega$, the so called *trace theorems* characterise its regularity on the boundary $\partial\Omega$ or on $(0,T)\times\partial\Omega$, respectively. In the isotropic case the relevant results are given in [140], while the theory in [136] provides suitable theorems in the anisotropic case. To make use of these results we verify that the considered spaces in [136, 141] coincide with the spaces introduced in the preceding sections when viewing $\partial\Omega$ as a smooth Riemannian manifold of dimension $n-1$. For better readability we do not always differentiate between a function f defined on Ω, or $(0,T)\times\Omega$, and its restriction to $\partial\Omega$, or $(0,T)\times\partial\Omega$, respectively.

Lemma C.26. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with smooth boundary. Then, with the subspace topology inherited from the ambient space $\mathbb{R}^n$, $\partial\Omega$ is a smooth compact orientable Riemannian manifold of dimension $n-1$.*

Proof. Definition B.16 implies that for each $x \in \partial\Omega$ there exists an open set $V \subset \mathbb{R}^n$ with $x \in V$ and a function $g \in C^\infty(V;\mathbb{R}^n)$ mapping V bijectively onto the ball $B_1(0) \subset \mathbb{R}^n$ such that $g^{-1} \in C^\infty(B_1(0);\mathbb{R}^n)$ and $g(V \cap \Omega) = \{y \in B_1(0) : y_n > 0\}$. As $\partial\Omega$ is compact, we find finitely many V_α, g_α, $\alpha \in \{1,\ldots,N\}$, such that every $x \in \partial\Omega$ lies in at least one set V_α. By definition of the subspace topology, the set $U_\alpha := V_\alpha \cap \partial\Omega$ is open in $\partial\Omega$ and we observe that $g_\alpha(U_\alpha) = \{y \in B_1(0) : y_n = 0\}$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection onto the first $n-1$ components and set $\phi_\alpha := \pi \circ (g_\alpha)_{|U_\alpha}$. Then $\phi_\alpha : U_\alpha \to B_1^{n-1}(0) := \{y' \in \mathbb{R}^{n-1} : |y'| < 1\}$ is a homeomorphism and the coordinate change $\phi_\beta \circ \phi_\alpha^{-1} = \pi \circ g_\beta \circ g_\alpha^{-1} \circ i$ is smooth on $\phi_\alpha(U_\alpha \cap U_\beta)$ where $i : \mathbb{R}^{n-1} \to \mathbb{R}^n$ denotes the canonical embedding. Let $\mathcal{A}$ be the maximal smooth structure containing (U_α, ϕ_α), $\alpha \in \{1,\ldots,N\}$. Then $(\partial\Omega, \mathcal{A})$ is a smooth compact manifold of dimension $n-1$. It is orientable as the smooth domain Ω admits a smooth unit normal field $\nu : \partial\Omega \to \mathbb{R}^n$ that defines an orientation on $\partial\Omega$. The pull-back of the Euclidean scalar product under the embedding $\partial\Omega \hookrightarrow \mathbb{R}^n$ defines a smooth Riemannian metric on $(\partial\Omega, \mathcal{A})$. □

It is a direct consequence of Proposition C.12 that the spaces $W_p^s(\partial\Omega)$ defined in [141, Definition 3.6.1.] coincide with our notion of $W_p^s(\partial\Omega)$ as introduced in Definition C.10 with equivalent norms. In particular, we may use the trace theorem shown in [141, Theorem 4.7.1.] that characterises the regularity of functions $f \in W_p^s(\Omega)$ on the boundary $\partial\Omega$.

Theorem C.27 (Trace Theorem I)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a domain with smooth boundary and let $p \in (1,\infty)$, $s \in (1/p,\infty)$ and $u \in W_p^s(\Omega)$ be given. Then for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le s - \frac{1}{p}$ it holds*

$$(\partial_x^\alpha u)_{|\partial\Omega} \in W_p^{s-|\alpha|-1/p}(\partial\Omega)$$

and

$$\|\partial_x^\alpha u\|_{W_p^{s-|\alpha|-1/p}(\partial\Omega)} \le C\,\|u\|_{W_p^s(\Omega)} .$$

Proof. This is shown in [141, Theorem 4.7.1]. □

Corollary C.28. *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a domain with smooth boundary and outer unit normal $\nu : \partial\Omega \to \mathbb{R}^n$ and let $p \in (1,\infty)$, $s \in (1/p,\infty)$ and $u \in W_p^s(\Omega)$ be given. Then for all $j \in \mathbb{N}_0$ with $j \le s - \frac{1}{p}$ it holds*

$$\frac{\partial^j u}{\partial\nu^j} := \sum_{i_1,\ldots,i_j=1}^{n} \left(\partial_{x_{i_1}} \cdots \partial_{x_{i_j}} u\right)_{|\partial\Omega} \nu_{i_1} \cdots \nu_{i_j} \in W_p^{s-j-1/p}(\partial\Omega)$$

and

$$\left\|\frac{\partial^j u}{\partial\nu^j}\right\|_{W_p^{s-j-1/p}(\partial\Omega)} \le C\,\|u\|_{W_p^s(\Omega)} .$$

Proof. This is an easy consequence of Theorem C.27. □

In [136] the author considers possibly unbounded domains that satisfy the so called *ordinary Ljapunov conditions* listed in [136, §13]. It is readily checked that every smooth compact domain Ω satisfies these conditions. It is a direct consequence of Proposition B.31, Proposition B.36 and Proposition C.23 that the anisotropic Sobolev spaces defined in [136, §20, page 130] coincide with the spaces introduced in Definition C.15 in the case $M = \partial\Omega$ with equivalent norms where the equivalence constants depend on the length of the time interval. The trace theorem given in [136, Theorem 5.1] for functions in $W_p^{r,s}((0,T)\times\Omega)$ characterises their regularity on $(0,T)\times\partial\Omega$. We state the result only in the special case required in this work. The norms $\|\cdot\|_{p,r,s}$, $r \in [0,1]$, $s \ge 0$, are defined in Definition B.26 and Definition C.15, respectively.

Theorem C.29 (Trace Theorem II). *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $p \in (1,\infty)$, $T > 0$ and*

$$u \in W_p^1\left((0,T); L_p(\Omega)\right) \cap L_p\left((0,T); W_p^4(\Omega)\right)$$

be given. If $\alpha \in \mathbb{N}_0^n$ with $|\alpha| < 4 - 1/p$, then

$$\left(\partial_x^\alpha u\right)_{\big|(0,T)\times\partial\Omega} \in W_p^{\frac{4-|\alpha|-1/p}{4}, 4-|\alpha|-\frac{1}{p}}\left((0,T)\times(\partial\Omega)\right)$$

and

$$\left\|\partial_x^\alpha u\right\|_{p,\frac{4-|\alpha|-1/p}{4}, 4-|\alpha|-\frac{1}{p}} \le C(T)\left\|u\right\|_{p,1,4}.$$

Proof. This is shown in [136, Theorem 5.1]. □

Using the extension operator one obtains the estimates in the above theorem with constants independent of T.

Theorem C.30 (Trace Theorem III). *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $p \in (1,\infty)$, $T_0 > 0$, $T \in (0, T_0]$ and*

$$u \in W_p^1\left((0,T); L_p(\Omega)\right) \cap L_p\left((0,T); W_p^4(\Omega)\right)$$

be given. Then for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| < 4 - 1/p$, there holds

$$\left(\partial_x^\alpha u\right)_{\big|(0,T)\times\partial\Omega} \in W_p^{\frac{4-|\alpha|-1/p}{4}, 4-|\alpha|-\frac{1}{p}}\left((0,T)\times(\partial\Omega)\right)$$

and

$$|||\partial_x^\alpha u|||_{W_p^{\frac{4-|\alpha|-1/p}{4}, 4-|\alpha|-\frac{1}{p}}((0,T)\times(\partial\Omega))} \le C\left(T_0\right)\ |||u|||_{W_p^{1,4}((0,T)\times\Omega)}$$

with a constant $C\left(T_0\right)$ independent of T.

Proof. By Proposition B.36 and Proposition C.24 the $|||\cdot|||$–norms are equivalent to the usual ones on the respective spaces. Thus, the previous theorem implies

$$\left(\partial_x^\alpha u\right)_{\big|(0,T)\times\partial\Omega} \in W_p^{\frac{4-|\alpha|-1/p}{4}, 4-|\alpha|-\frac{1}{p}}\left((0,T)\times(\partial\Omega)\right)$$

with

$$|||\partial_x^\alpha u|||_{W_p^{\frac{4-|\alpha|-1/p}{4}, 4-|\alpha|-\frac{1}{p}}((0,T)\times(\partial\Omega))} \le C(T)\ |||u|||_{W_p^{1,4}((0,T)\times\Omega)}.$$

Proposition B.37 implies the existence of an extension $\boldsymbol{E}u \in W_p^{1,4}\left((0,T_0)\times\Omega\right)$ that satisfies $(\boldsymbol{E}u)_{|(0,T)} = u$ and

$$|||\boldsymbol{E}u|||_{W_p^{1,4}((0,T_0)\times\Omega)} \le C\left(T_0\right)\ |||u|||_{W_p^{1,4}((0,T)\times\Omega)}.$$

In particular,

$$\left(\partial_x^\alpha(\boldsymbol{E}u)\right)_{\big|(0,T_0)\times\partial\Omega} \in W_p^{\frac{4-|\alpha|-1/p}{4}, 4-|\alpha|-\frac{1}{p}}\left((0,T_0)\times(\partial\Omega)\right)$$

with

$$\begin{aligned}|||\partial_x^\alpha(\boldsymbol{E}u)|||_{W_p^{\frac{4-|\alpha|-1/p}{4}, 4-|\alpha|-\frac{1}{p}}((0,T_0)\times(\partial\Omega))} &\le C\left(T_0\right)\ |||\boldsymbol{E}u|||_{W_p^{1,4}((0,T_0)\times\Omega)} \\ &\le C\left(T_0\right)\ |||u|||_{W_p^{1,4}((0,T)\times\Omega)}.\end{aligned}$$

As $\partial_x^\alpha(\boldsymbol{E}u)\big|_{(0,T)\times\partial\Omega} = \partial_x^\alpha u$, there holds

$$\|\!|\partial_x^\alpha u\|\!|_{W_p^{\frac{4-|\alpha|-1/p}{4},4-|\alpha|-\frac{1}{p}}((0,T)\times(\partial\Omega))} \leq \|\!|\partial_x^\alpha(\boldsymbol{E}u)\|\!|_{W_p^{\frac{4-|\alpha|-1/p}{4},4-|\alpha|-\frac{1}{p}}((0,T_0)\times(\partial\Omega))} .$$

This shows the claim. □

In the case that $\Omega \subset \mathbb{R}$ is a bounded smooth domain, the boundary $\partial\Omega$ is a compact zero-dimensional manifold and thus consists of finitely many points $y_1, \ldots, y_N \in \mathbb{R}$. In this case we obtain for $p \in (1,\infty)$ that

$$L_p(\partial\Omega) = \left\{ f : \partial\Omega \to \mathbb{R} : \sum_{i=1}^N |f(y_i)|^p < \infty \right\} = \{f : \partial\Omega \to \mathbb{R}\} \tag{C.17}$$

can be identified with $\mathbb{R}^N$ via the evaluation mapping

$$f \mapsto (f(y_i))_{i=1}^N .$$

The same holds true for all of the spaces $W_p^s(\partial\Omega)$ with $s > 0$ and $p \in (1,\infty)$. This is used in Part II.
We conclude the section with some further useful results.

Proposition C.31. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $\tau, \beta \in (0,1)$ with $\tau > \beta$, $p \in (1,\infty)$, T_0 be positive and $T \in (0,T_0)$. Given $f \in W_p^\beta((0,T); L_p(\partial\Omega))$ and $g \in C^\tau\left([0,T]; C(\overline{\Omega})\right)$ there holds $fg \in W_p^\beta((0,T); L_p(\partial\Omega))$ and there exists a constant $C(T_0) > 0$ such that*

$$\|fg\|_{W_p^\beta((0,T);L_p(\partial\Omega))} \leq C\, \|f\|_{W_p^\beta((0,T);L_p(\partial\Omega))} \|g\|_{C^\tau([0,T];C(\overline{\Omega}))} .$$

Proof. For almost every $t \in (0,T)$ there holds $f(t) \in L_p(\partial\Omega)$ and $g(t) \in C(\overline{\Omega})$. As $\overline{\Omega}$ is compact, one easily verifies that $f(t)g(t) \in L_p(\partial\Omega)$ with $\|f(t)g(t)\|_{L_p(\partial\Omega)} \leq \|f(t)\|_{L_p(\partial\Omega)} \|g(t)\|_{C(\overline{\Omega})}$. It is straightforward to show that $fg : (0,T) \to L_p(\partial\Omega)$ is strongly measurable. Moreover, we have

$$\int_0^T \|f(t)g(t)\|_{L_p(\partial\Omega)}^p \,\mathrm{d}t \leq \|f\|_{L_p((0,T);L_p(\partial\Omega))} \|g\|_{C([0,T];C(\overline{\Omega}))} .$$

and

$$\begin{aligned} &\int_0^T \int_0^T \|f(t)g(t) - f(s)g(s)\|_{L_p(\partial\Omega)}^p |t-s|^{-\beta p-1} \mathrm{d}t\,\mathrm{d}s \\ \leq &\int_0^T \int_0^T \left(\|f(t)\|_{L_p(\partial\Omega)}^p \|g(t)-g(s)\|_{C(\overline{\Omega})}^p + \|f(t)-f(s)\|_{L_p(\partial\Omega)}^p \|g(s)\|_{C(\overline{\Omega})}^p \right) |t-s|^{-\beta p-1} \mathrm{d}t\,\mathrm{d}s . \end{aligned}$$

The second term is controlled by the quantity $\|f\|_{W_p^\beta((0,T);L_p(\partial\Omega))} \|g\|_{C([0,T];C(\overline{\Omega}))}$ and the first term can be estimated as follows:

$$\begin{aligned} &\int_0^T \int_0^T \|f(t)\|_{L_p(\partial\Omega)}^p \|g(t)-g(s)\|_{C(\overline{\Omega})}^p |t-s|^{-\beta p-1} \mathrm{d}t\,\mathrm{d}s \\ \leq &\|g\|_{C^\tau([0,T];C(\overline{\Omega}))}^p \int_0^T \|f(t)\|_{L_p(\partial\Omega)}^p \int_0^T |t-s|^{(\tau-\beta)p-1} \,\mathrm{d}s\,\mathrm{d}t \\ = &\|g\|_{C^\tau([0,T];C(\overline{\Omega}))}^p \int_0^T \|f(t)\|_{L_p(\partial\Omega)}^p \frac{1}{(\tau-\beta)p} \left(t^{(\tau-\beta)p} + (T-t)^{(\tau-\beta)p} \right) \mathrm{d}t \\ \leq &C(\tau,\beta,p) T^{(\tau-\beta)p} \|g\|_{C^\tau([0,T];C(\overline{\Omega}))}^p \|f\|_{L_p((0,T);L_p(\partial\Omega))}^p . \end{aligned}$$

□

Proposition C.32. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain $\alpha, \beta \in (0,1)$ with $\alpha > \beta$, $p \in (1, \infty)$, T_0 be positive and $T \in (0, T_0]$. Then*

$$C^\alpha([0,T]; L_p(\partial\Omega)) \hookrightarrow W_p^\beta((0,T); L_p(\partial\Omega))$$

and there exist constants $\sigma = \sigma(\alpha, \beta, p) \in (0,1)$ and $C = C(\alpha, \beta, p, T_0) > 0$ such that for all $f \in C^\alpha([0,T]; L_p(\partial\Omega))$,

$$\|f\|_{W_p^\beta((0,T);L_p(\partial\Omega))} \leq C(\alpha, \beta, p, T_0)\, T^\sigma \|f\|_{C^\alpha([0,T];L_p(\partial\Omega))}\,.$$

Proof. Let $T \in (0, T_0]$ and $f \in C^\alpha([0,T]; L_p(\partial\Omega))$ be given. Then $f : (0,T) \to L_p(\partial\Omega)$ is strongly measurable and

$$\|f\|_{L_p((0,T);L_p(\partial\Omega))} \leq T^{1/p} \|f\|_{C([0,T];L_p(\partial\Omega))} \leq T^{1/p} \|f\|_{C^\alpha([0,T];L_p(\partial\Omega))}\,.$$

Furthermore, the semi-norm part can be estimated by

$$\begin{aligned}
[f]^p_{W_p^\beta((0,T);L_p(\partial\Omega))} &\leq \|f\|^p_{C^\alpha([0,T];L_p(\partial\Omega))} \int_0^T \int_0^T |t-s|^{(\alpha-\beta)p-1} \mathrm{d}t\mathrm{d}s \\
&\leq \frac{2T_0}{(\alpha-\beta)p} \|f\|^p_{C^\alpha([0,T];L_p(\partial\Omega))} T^{(\alpha-\beta)p}
\end{aligned}$$

and hence the claim follows with $\sigma = \min\{1/p, \alpha - \beta\} \in (0,1)$. □

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