



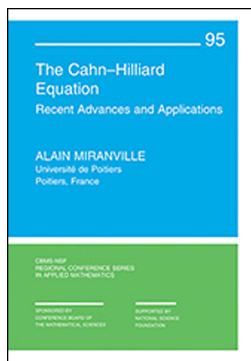
## Alain Miranville: “The Cahn–Hilliard Equation: Recent Advances and Applications”

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The Cahn–Hilliard equation is a basic partial differential equation in the context of so-called *phase field models*, which are also called *diffuse interface models*. It is used to describe the mixture of two conserved components, e.g. two different kinds of atoms in a binary alloy or two different fluids. The equation can be written in the form of a second order system

$$\partial_t u(t, x) = \kappa \Delta \mu(t, x) \quad (1)$$

$$\mu(t, x) = -\alpha \Delta u(t, x) + f(u(t, x)), \quad (2)$$

for all  $t \in [0, T)$ ,  $x \in \Omega$  for a certain domain  $\Omega \subseteq \mathbb{R}^n$  and  $T > 0$ . Here  $u: [0, T) \times \Omega \rightarrow \mathbb{R}$  usually describes the difference of the two concentrations of the conserved quantities. Therefore the physically reasonable values of  $u$  are in  $[-1, 1]$ . Moreover,  $\mu: [0, T) \times \Omega \rightarrow \mathbb{R}$  is the so-called *chemical potential* of the mixture and is the  $L^2(\Omega)$ -gradient of the functional

$$E(u) = \int_{\Omega} \left( \frac{\alpha}{2} |\nabla u|^2 + F(u) \right) dx$$

that describes the energy of the mixture at a given time. Here  $f = F'$  and  $F: \mathbb{R} \rightarrow \mathbb{R}$  is a homogeneous free energy density, which is usually a so-called *double-well po-*

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tential, i.e.,  $F$  has precisely two strict local minima. A standard choice is

$$f(s) = \frac{1}{4}(s^2 - 1)^2 \quad \text{for all } s \in \mathbb{R}. \quad (3)$$

Finally,  $\alpha, \kappa$  are positive constants and  $\Delta = \partial_{x_1}^2 + \dots + \partial_{x_n}^2$  denotes the Laplace operator acting only on  $x \in \mathbb{R}^n$ . In order to obtain a well-posed equation, the Cahn-Hilliard equations (1)-(2) have to be complemented with an initial condition

$$u|_{t=0} = u_0$$

for some given initial value  $u_0: \Omega \rightarrow \mathbb{R}$  and suitable boundary conditions for  $u$  and  $\mu$ , which we will describe below. We note that (1)-(2) is equivalent to the fourth order parabolic equation

$$\partial_t u + \alpha \kappa \Delta^2 u - \alpha \Delta f(u) = 0.$$

Although this equation looks simpler at first sight, writing it as a second order system (1)-(2) is advantageous for many purposes and reflects the physical origin better.

Originally, the equation was derived by Cahn & Hilliard [1] to model the phase separation in a binary alloy. In this situation after a short time regions form, where  $u(t, x)$  is either close to  $+1$  or  $-1$ . These regions are separated by a transition layer of positive but usually small thickness, where  $u$  varies smoothly between  $-1$  and  $+1$ . This layer is also called *diffuse interface*. Its thickness depends on the parameters and is of order  $\varepsilon > 0$  if  $\kappa = \frac{1}{\varepsilon}$ ,  $\alpha = \varepsilon^2$ . During the last decades such diffuse interface models and different variants of the Cahn-Hilliard equation were applied very successfully in different areas, e.g. in fluid mechanics to describe two-phase flows, in mathematical biology to describe tumor growth or phase separation on lipid membranes, or even for image inpainting and shape optimization. One reason for their success and growing popularity is that they can be used to approximate so-called sharp interface models, where the regions of the two components are separated by an  $(n - 1)$ -dimensional manifold. An advantage of diffuse interface models compared to sharp interface models is that their mathematical analysis and numerical treatment is easier. Moreover, the treatment of singularities in the interface due to coalescence of different regions or pinch off can be easily described in diffuse interface models, while sharp interface models break down in such situations.

## 1 Content of the Book

In the first chapter an introduction and overview to the Cahn-Hilliard equation and a lot of variants are given. Afterwards in the second chapter some preliminary material is collected. It provides a summary of results on linear operators, in particular self-adjoint unbounded operators with compact inverse, fractional powers of these and the associated abstract evolution equations of first order in time. Moreover, basic tools like the Lemma of Aubin-Lions and many useful inequalities are recalled. Finally, abstract dynamical systems are considered. In this setting results on the existence of global attractors and a condition for its finite dimensionality are discussed. The

results are applied in the preceding chapter to obtain corresponding results for the Cahn-Hilliard type equations. The main part of the book consists of six chapters. In each chapter a different version of the Cahn-Hilliard equation is treated.

The simplest case of the Cahn-Hilliard equation is treated in Chap. 3, namely the Cahn-Hilliard equation with regular free energy density as e.g. given in (3) and Neumann boundary conditions

$$n \cdot \nabla u|_{\partial\Omega} = 0, \quad (4)$$

$$n \cdot \nabla \mu|_{\partial\Omega} = 0. \quad (5)$$

Here  $n$  denotes the exterior normal of  $\partial\Omega$  and the boundary condition for  $\mu$  describes that there is no flux of the components through the boundary. The Neumann boundary condition for  $u$  can be interpreted as a  $90^\circ$ -angle condition for the diffuse interface since it implies that level sets  $\{x \in \Omega : u(x, t) = c\}$  for  $c \in \mathbb{R}$  and  $t \in [0, T)$  are orthogonal to the boundary. In this case existence of weak and strong solutions of the system can be proved with the aid of a standard finite dimensional approximation/Galerkin scheme under different assumptions on the initial value  $u_0$ . Actually, it can even be shown that the solution becomes more regular for  $t > 0$  since this is a parabolic equation. To this end the main point is to obtain suitable a priori estimates for solutions of the equation. This is done by multiplying the equation by  $u$  or suitable terms depending on  $u$  and careful estimates of certain nonlinear terms. These arguments are explained in great detail. In a similar manner uniqueness of solutions can be proved. Finally, these a priori estimates and the regularity for  $t > 0$  is used to prove the existence of a bounded absorbing set, a global attractor, and to show that the global attractor has finite dimension.

One disadvantage of the choice of a regular potential as in (3) is that in general the solution  $u$  can attain values outside the physical reasonable interval  $[-1, 1]$ , which is shown in a simple example. To overcome this defect one solution is to use a singular free energy density as e.g.

$$F(s) = \frac{\theta_c}{2}(1-s^2) + \frac{\theta}{2} \left[ (1-s) \ln \left( \frac{1-s}{2} \right) + (1+s) \ln \left( \frac{1+s}{2} \right) \right], \quad s \in [-1, 1],$$

where  $0 < \theta < \theta_c$ .  $F$  is only differentiable in  $(-1, 1)$  and leads to

$$f(s) = -\theta_c s + \frac{\theta}{2} \ln \left( \frac{1+s}{1-s} \right), \quad s \in (-1, 1).$$

This free energy density is physically relevant and leads mathematically to solutions  $u$  such that  $u(x, t) \in (-1, 1)$  for almost every  $x \in \Omega$ ,  $t \in (0, T)$ . In Chap. 4 the Cahn-Hilliard equation is studied for this free energy density in the case of Neumann boundary conditions (4)-(5). Of course the singularities of  $f(s)$  as  $s \rightarrow \pm 1$  cause difficulties in the mathematical analysis of this system. In order to prove the existence of weak solutions  $f$  is approximated by suitable regular free energies  $f_N$ ,  $N \in \mathbb{N}$ . The existence of weak solutions for the Cahn-Hilliard equation with  $f_N$  instead of  $f$  follows from the results in the chapter before. In order to pass to the limit  $N \rightarrow \infty$ , suitable a priori estimates uniformly in  $N \in \mathbb{N}$  are needed. To this end it is essential

that  $f'$  is bounded below. More precisely,  $f'(s) \geq -\theta_c$  for all  $s \in (-1, 1)$  since the logarithmic terms in  $F$  define convex functions. Using this property it is also possible to show higher regularity of the solutions for positive times. Interestingly, in the case  $n = 2$  it can even be shown, that for every  $r > 0$  there is some  $\delta > 0$  such that

$$|u(t, x)| \leq 1 - \delta \quad \text{for all } t \geq r, x \in \Omega.$$

This is known as the strict separation property.

In general the contact angle of the (diffuse) interface does not have to be  $90^\circ$ . Depending on the chemical properties of the components and the boundary of the domain, different contact angles can occur. In dynamic situations the angle can even vary. In order to describe such dynamic contact angles one replaces (4)-(5) by a suitable dynamic boundary condition for  $u$ . In Chap. 5 the author discusses the Cahn-Hilliard equation (1)-(2) coupled with

$$\begin{aligned} \partial_t u &= \eta \Delta_\Gamma \mu - n \cdot \nabla \mu & \text{on } \Gamma = \partial\Omega, \\ \mu &= -\sigma \Delta_\Gamma \mu + n \cdot \nabla u|_\Gamma + g(u) & \text{on } \Gamma = \partial\Omega, \end{aligned}$$

where  $\sigma, \eta > 0$ ,  $\Delta_\Gamma$  denotes the Laplace-Beltrami operator on  $\Gamma$ ,  $g = G'$ , and  $G$  is a free energy density on  $\Gamma$ . In this case the analysis is more involved and can be done by considering the Cahn-Hilliard equation as a system, where a parabolic equation in  $\Omega$  is coupled to a parabolic equation on  $\Gamma$ . By a careful choice of the function spaces and suitable definitions of operators and bilinear forms on certain product spaces the analysis can finally be done in a similar manner as before. Also in this case existence and uniqueness of solutions as well as existence of a global attractor can be shown. Finally, some results on numerical simulations for these dynamic boundary conditions are shown.

In Chap. 6, the author studies the Cahn-Hilliard-Oono equation

$$\partial_t u + \beta u + \Delta^2 u - \Delta f(u) = 0$$

together with the Neumann boundary conditions (4)-(5), where  $\beta > 0$  and  $\alpha = \kappa = 1$  for simplicity. This equation was proposed in order to describe long-range interaction between the components in a phase separation. Mathematically, the main new aspect is that the spatial mean value of  $u$  is no longer preserved. More precisely, we have

$$\frac{d}{dt} \frac{1}{|\Omega|} \int_\Omega u(t, x) dx = -\beta \frac{1}{|\Omega|} \int_\Omega u(t, x) dx \quad \text{for } t \in [0, T].$$

This causes new problems in the mathematical analysis, in particular in the case that  $f$  is a logarithmic free energy density as before. Nevertheless similar results on existence, uniqueness and regularity of solutions are shown in the case of a regular and a logarithmic free energy density  $f$ . In the case of a regular free energy density existence of a finite-dimensional global attractor is shown. In the case of a logarithmic free energy density higher regularity for positive times and the strict separation property in the two-dimensional case are shown.

In Chap. 7, an application of the Cahn–Hilliard equation for image inpainting is discussed. To this end the following modification of the Cahn–Hilliard is used:

$$\partial_t u + \varepsilon \Delta^2 u - \frac{1}{\varepsilon} \Delta f(u) + \lambda_0 \chi_{\Omega \setminus D}(x)(u - h) = 0,$$

where  $\varepsilon, \lambda_0 > 0$  are suitable parameters,  $h: \Omega \rightarrow \mathbb{R}$  describes the (damaged) image and  $D \subseteq \Omega$  is the region where the image shall be inpainted. Here  $\varepsilon > 0$  controls the width of the diffuse interface and  $\lambda_0$  controls the difference between the inpainted image, described by  $u$  at fixed time  $t$ , and the image described by  $h$  outside of  $D$ . In the case of Neumann boundary conditions for  $u$  and  $\Delta u$  existence, uniqueness and regularity of solutions is shown in the case of a regular and a logarithmic free energy density. One of the main difficulties is to control the mean value of  $u$  in time and to estimate the new source term. Finally, numerical results for some inpaintings of images are shown.

A variant of the Cahn–Hilliard equation with a proliferation term

$$\partial_t u + \Delta^2 u - \Delta f(u) + g(u) = 0$$

together with Neumann boundary conditions for  $u$  and  $\Delta u$  is studied in Chap. 8. Such equations have been suggested for models for wound healing or tumor growth. Here  $g(s) = s(s - 1)$  is assumed. In the case of a regular free energy density  $f(s) = s^3 - s$  the existence of unique solutions on some time interval  $[0, T)$  is proved. But in contrast to the previous chapters the solution can “blow-up” in finite time and does not need to exist for all times, which is shown in some cases. Interestingly, this does not happen in the case of a logarithmic free energy density. In this case the existence of a global solution, which tends to 1 as time tends to infinity is shown. Finally, some results on numerical simulations are shown.

Finally, in Chap. 9, other variants of the Cahn–Hilliard equation and its derivation based on a theory due to Gurtin are presented. Moreover, a hyperbolic relaxation of the Cahn–Hilliard equation and a coupled Navier–Stokes/Cahn–Hilliard system is discussed briefly.

## 2 Readership

The book under review is the first that is solely devoted to the mathematical analysis of the Cahn–Hilliard equation and several variants. It provides an overview on analytic results on well-posedness and the qualitative behaviour of these equations. Therefore it fills an important gap in the literature and gives a good introduction to the Cahn–Hilliard equation and several variants from an analytic point of view. Moreover, the book gives a good overview on recent mathematical analysis since a lot of references are given and every chapter ends with a section “Further reading and comments”. The reader is assumed to have a basic knowledge in functional analysis and the mathematical analysis of partial differential equation.

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