



# On a degenerate parabolic system describing the mean curvature flow of rotationally symmetric closed surfaces

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*Abstract.* We show that the mean curvature flow for a closed and rotationally symmetric surface can be formulated as an evolution problem consisting of an evolution equation for the square of the function whose graph is rotated and two ODEs describing the evolution of the points of the evolving surface that lie on the rotation axis. For the fully nonlinear and degenerate parabolic problem we establish the well-posedness property in the setting of classical solutions. Besides we prove that the problem features the effect of parabolic smoothing.

## 1. Introduction

Mean curvature flow is the most efficient way to decrease the surface area of a hypersurface. It hence has been of great interest in geometry as well as in materials science and image analysis, see [8, 9, 14, 18, 20, 24, 29]. Since the pioneering work of Brakke [8] and Huisken [20] many results have been shown for mean curvature flow and we refer to [24] and the references therein for more information about the subject. The case of rotationally symmetric evolutions lead to spatially one-dimensional problems and due to the reduced complexity this situation has been studied by several authors analytically [1, 13, 21, 22, 25] as well as numerically [7, 27].

In particular, rotationally symmetric mean curvature flow has been helpful to understand singularity formation in curvature flows, see [1, 13, 15, 21, 25]. Most of the analytical results have been restricted to the case of surfaces with boundary or periodic unbounded situations. The situation becomes analytically far more involved if one considers closed surfaces, i.e., compact surfaces without boundaries. In this context the governing equation can be recast, provided that the points on the rotation axis have positive curvature, as a free boundary problem which involves both degenerate and singular terms. This paper gives first well-posedness and parabolic smoothing results for the free boundary problem describing compact rotationally symmetric surfaces evolving by mean curvature derived herein.

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Let us now precisely formulate the analytic problem. We study the evolution of a family of rotationally symmetric surfaces  $\{\Gamma(t)\}_{t \geq 0}$  by the mean curvature flow. Given  $t \geq 0$ , we assume that

$$\Gamma(t) := \{(x, u(t, x) \cos \vartheta, u(t, x) \sin \vartheta) : x \in [a(t), b(t)], \vartheta \in [0, 2\pi]\} \subset \mathbb{R}^3$$

is the surface obtained by rotating the graph of the map  $u(t) : [a(t), b(t)] \rightarrow \mathbb{R}$  around the  $x$ -axis. Moreover, we consider herein the case when the surfaces  $\Gamma(t)$  are closed, meaning in particular that also the domain of definition  $[a(t), b(t)]$  of  $u(t)$  is unknown. Since the motion of the surfaces is governed by the equation

$$V(t) = H(t) \quad \text{on } \Gamma(t), \quad t \geq 0, \tag{1.1}$$

where  $V(t)$  is the normal velocity of  $\Gamma(t)$  and  $H(t) = k_1(t) + k_2(t)$  the mean curvature of  $\Gamma(t)$ , with  $k_i(t)$ ,  $i = 1, 2$ , denoting the principle curvatures of  $\Gamma(t)$ , we obtain the following evolution equation for the unknown function  $u$ :

$$u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{1}{u}, \quad t \geq 0, \quad x \in (a(t), b(t)). \tag{1.2a}$$

We assumed that

$$u(t, x) > 0, \quad t \geq 0, \quad x \in (a(t), b(t)). \tag{1.2b}$$

This equation cannot be realized at  $x \in \{a(t), b(t)\}$  as we impose the following boundary conditions

$$\begin{cases} u(t, a(t)) = u(t, b(t)) = 0, & t \geq 0, \\ \lim_{x \searrow a(t)} u_x(t, x) = \infty, \quad \lim_{x \nearrow b(t)} u_x(t, x) = -\infty, & t \geq 0, \end{cases} \tag{1.2c}$$

which express the fact that  $\Gamma(t)$  is a closed surface without boundary.

*The evolution of the boundaries: the first approach* As the functions  $a$  and  $b$  are unknown, we have to derive equations describing the evolution of these two boundaries. If we want to evaluate the normal velocity at  $(a(t), 0, 0) \in \Gamma(t)$ , it follows from (1.2c) that  $V(t)|_{(a(t), 0, 0)} = a'(t)$ . The next goal is to express  $H(t)$  at  $(a(t), 0, 0)$  in terms of  $u(t)$ . To this end we assume, for some  $\varepsilon > 0$ , that

$$u(t) : [a(t), a(t) + \varepsilon] \rightarrow [0, u(t, a(t) + \varepsilon)]$$

is invertible with the inverse function  $w(t) \in C^2([0, u(t, a(t) + \varepsilon)])$ , so that in particular  $\Gamma(t)$  is a  $C^2$ -surface close to  $(a(t), 0, 0)$ . Then due to the fact that  $u_x(t, x) \rightarrow \infty$  for  $x \rightarrow a(t)$  we have  $w_y(t, 0) = 0$  and

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$$\begin{aligned}\lim_{x \rightarrow a(t)} k_1(t, x, \vartheta) &= \lim_{x \rightarrow a(t)} \left( -\frac{u_{xx}}{(1 + u_x^2)^{3/2}} \right) (t, x) \\ &= \lim_{y \rightarrow 0} \frac{w_{yy}}{(1 + w_y^2)^{3/2}} (t, y) = w_{yy}(t, 0), \\ \lim_{x \rightarrow a(t)} k_2(t, x, \vartheta) &= \lim_{x \rightarrow a(t)} \frac{1}{u(1 + u_x^2)^{1/2}} (t, x) \\ &= \lim_{y \rightarrow 0} \frac{w_y}{y(1 + w_y^2)^{1/2}} (t, y) = w_{yy}(t, 0),\end{aligned}$$

hence  $H(t)|_{(a(t), 0, 0)} = 2w_{yy}(t, 0)$ . Noticing that

$$\lim_{x \rightarrow a(t)} (uu_x)(t, x) = \lim_{y \rightarrow 0} \frac{y}{w_y(t, y)} = \frac{1}{w_{yy}(t, 0)},$$

in the case when  $H(t)|_{(a(t), 0, 0)} = 2w_{yy}(t, 0) > 0$  we obtain the following relation

$$a'(t) = H(t)|_{(a(t), 0, 0)} = \frac{2}{\lim_{x \rightarrow a(t)} (u^2/2)_x(t, x)}. \quad (1.2d)$$

Similarly, assuming that, for some  $\varepsilon > 0$ ,

$$u(t) : [b(t) - \varepsilon, b(t)] \rightarrow [0, u(t, b(t) - \varepsilon)]$$

is invertible and  $H(t)|_{(b(t), 0, 0)} > 0$ , we find for  $b$  the evolution equation

$$b'(t) = H(t)|_{(b(t), 0, 0)} = \frac{2}{\lim_{x \rightarrow b(t)} (u^2/2)_x(t, x)}. \quad (1.2e)$$

*The evolution of the boundaries: the second approach* A major drawback of the (formally) quasilinear parabolic equation (1.2a) is that the boundary conditions (1.2c) make the equation highly degenerate as:

- (I) The diffusion coefficient vanishes in the limit  $x \rightarrow a(t)$  and  $x \rightarrow b(t)$ ;
- (II) The term  $1/u$  becomes unbounded for  $x \rightarrow a(t)$  and  $x \rightarrow b(t)$ .

In order to overcome (II) we introduce, motivated also by (1.2d)–(1.2e), a new unknown  $v$  via

$$v(t, x) := \frac{u^2(t, x)}{2} \quad \text{for } t \geq 0 \text{ and } x \in [a(t), b(t)].$$

Then  $v(t)$  also vanishes at the boundary points  $a(t)$  and  $b(t)$  and (1.2a) can be expressed as

$$v_t = \frac{2vv_{xx}}{2v + v_x^2} - \frac{v_x^2}{2v + v_x^2} - 1, \quad t \geq 0, x \in (a(t), b(t)). \quad (1.3)$$

This equation is also (formally) quasilinear parabolic and also degenerate—as the diffusion coefficient vanishes for  $x \rightarrow a(t)$  and  $x \rightarrow b(t)$ , cf. (1.4) below—but now

none of the coefficients is singular. In order to obtain an evolution equation also for the functions describing the boundaries, we assume that

$$v_x(t, a(t)) > 0 \quad \text{and} \quad v_x(t, b(t)) < 0, \quad t \geq 0, \tag{1.4}$$

and

$$\lim_{x \rightarrow a(t)} (vv_{xx})(t, x) = \lim_{x \rightarrow b(t)} (vv_{xx})(t, x) = 0, \quad t \geq 0. \tag{1.5}$$

Note that (1.4) implies in particular that the corresponding function  $u$  satisfies (1.2c). Furthermore, (1.5) is a nonlinear boundary condition for  $v$  which is equivalent to our former assumption that  $\Gamma(t)$  is a  $C^2$ -surface, cf. Lemma A.1. Differentiating now the relation  $v(t, a(t)) = 0, t \geq 0$ , with respect to time, it follows in virtue of (1.3) and (1.5), that

$$a'(t) = \frac{2}{v_x(t, a(t))} \quad \text{and} \quad b'(t) = \frac{2}{v_x(t, b(t))}, \quad t \geq 0.$$

These are the very same relations as in (1.2d)–(1.2e). It is not difficult to see, cf. Lemma A.1, that the two approaches are equivalent.

Summarizing, we may formulate the problem by using  $v$  as an unknown and we arrive at the evolution problem

$$\left\{ \begin{array}{l} v_t = \frac{2vv_{xx}}{2v + v_x^2} - \frac{v_x^2}{2v + v_x^2} - 1, \quad t \geq 0, x \in (a(t), b(t)), \\ a'(t) = \frac{2}{v_x(t, a(t))}, \quad t \geq 0, \\ b'(t) = \frac{2}{v_x(t, b(t))}, \quad t \geq 0, \\ v(t, a(t)) = v(t, b(t)) = 0, \quad t \geq 0, \\ \lim_{x \rightarrow a(t)} (vv_{xx})(t, x) = \lim_{x \rightarrow b(t)} (vv_{xx})(t, x) = 0, \quad t \geq 0, \\ v(t, x) > 0, \quad t \geq 0, x \in (a(t), b(t)), \\ v_x(t, a(t)) > 0, v_x(t, b(t)) < 0, \quad t \geq 0, \end{array} \right. \tag{1.6a}$$

with initial conditions

$$v(0) = v_0, \quad a(0) = a_0, \quad b(0) = b_0. \tag{1.6b}$$

In the following we use the formulation (1.6) in order to investigate the mean curvature flow (1.1). We are interested here to prove the existence and uniqueness of solutions which satisfy the equations in a classical sense (a weak formulation of (1.6) is not available yet). The formulation (1.6) has two advantages compared to the classical approach followed in [12, 17, 24] for example. Firstly, the equations are explicit (we do not need to work with local charts) and, secondly because the maximal solutions to (1.6) are defined in general on a larger time interval compared to the ones in [12, 17, 24]

(the solutions in [12, 17, 24] exist only in a small neighborhood of a fixed reference manifold over which they are parameterized). A disadvantage of our approach is that herein the initial surfaces are necessarily of class  $C^2$  while in [17] only  $h^{1+\alpha}$ -regularity, for some fixed  $\alpha \in (0, 1)$ , is required. An interesting research topic which we next plan to follow is to determine initial data  $u_0 : [a_0, b_0] \rightarrow \mathbb{R}$  for which the closed rotationally symmetric surface  $\Gamma(0)$  evolves such that neck pinching at the origin occurs in finite time. This topic has been already studied in the context of (1.2a), but in the special setting when  $a(t)$  and  $b(t)$  are kept fixed, the function  $u$  is strictly positive, and suitable boundary conditions (either of Neumann or Dirichlet type) are imposed at these two fixed boundary points, cf. [13, 15, 21, 25, 28]. In the context of closed surfaces without boundary considered herein there are several results establishing the convergence of initially convex surfaces towards a round point in finite time, cf. e.g. [3, 20, 28], but to the best of our knowledge no result establishing neck pinching at the origin is available. It is worthwhile to emphasize that in this context however, by using maximum principles and some explicit solutions to the mean curvature flow, such as spheres, hyperboloids, or shrinking donuts, there are several examples of dumbbell shaped surfaces which develop singularities in finite time, cf. [6, 14, 19].

*Remark 1.1.* (i) If  $u(t, x) = \sqrt{a^2(t) - x^2}$ ,  $|x| \leq a(t)$ , then the surfaces under consideration are spheres and the radius  $a(t) > 0$  solves the ODE

$$a'(t) = -\frac{2}{a(t)} = \frac{2}{v_x(t, a(t))}, \quad t \geq 0.$$

(ii) The conditions (1.4)–(1.5) impose some restrictions on the initial data. Lemma A.1 shows that any rotationally symmetric surface of class  $C^2$  with mean curvature that does not vanish at the points on the rotation axis satisfies (1.4)–(1.5). These properties are then preserved by the flow.

We will solve the degenerate parabolic system in the setting of small Hölder spaces. The small Hölder space  $h^{k+\alpha}(\mathbb{S})$ ,  $k \in \mathbb{N}$ ,  $\alpha \in (0, 1)$ , is defined as the closure of the smooth periodic functions  $C^\infty(\mathbb{S})$  (or equivalently of  $C^{k+\alpha'}(\mathbb{S})$ ,  $\alpha' > \alpha$ ) in the classical Hölder space  $C^{k+\alpha}(\mathbb{S})$  of  $2\pi$ -periodic functions on the line with  $\alpha$ -Hölder continuous  $k$ -th derivatives. Besides,  $h_e^{k+\alpha}(\mathbb{S})$  denotes the subspace of  $h^{k+\alpha}(\mathbb{S})$  consisting only of even functions. By definition, the embedding  $h_e^r(\mathbb{S}) \hookrightarrow h_e^s(\mathbb{S})$ ,  $r > s$ , is dense and moreover it holds

$$(h_e^r(\mathbb{S}), h_e^s(\mathbb{S}))_\theta = h_e^{(1-\theta)r + \theta s}(\mathbb{S}) \quad \text{for } \theta \in (0, 1) \text{ and } (1-\theta)r + \theta s \notin \mathbb{N}. \quad (1.7)$$

Here  $(\cdot, \cdot)_\theta = (\cdot, \cdot)_{\theta, \infty}^0$  denotes the continuous interpolation functor introduced by Da Prato and Grisvard [11].

The main result of this paper is the following theorem.

**Theorem 1.2.** *Let  $\alpha \in (0, 1)$  be fixed,  $a_0 < b_0 \in \mathbb{R}$ , and let  $v_0 \in C^1([a_0, b_0])$  be positive in  $(a_0, b_0)$  such that  $v_0(a_0) = v_0(b_0) = 0$ ,  $v_0'(a) > 0 > v_0'(b)$ , and*

$$v_0((a_0 + b_0)/2 - (b_0 - a_0) \cos(\cdot)/2) \in h_e^{2+\alpha}(\mathbb{S}).$$

Then (1.6) has a unique maximal solution  $(v, a, b) := (v, a, b)(\cdot; (v_0, a_0, b_0))$  such that

$$\begin{aligned} h &\in C^1([0, t^+), h_e^\alpha(\mathbb{S})) \cap C([0, t^+), h_e^{2+\alpha}(\mathbb{S})), \\ a, b &\in C^1([0, t^+), \mathbb{R}), \\ a(t) &< b(t) \quad \text{for all } t \in [0, t^+), \end{aligned}$$

where

$$h(t, x) := v(t, (a(t) + b(t))/2 - (b(t) - a(t)) \cos(x)/2), \quad t \in [0, t^+), x \in \mathbb{R},$$

and  $t^+ := t^+(v_0, a_0, b_0) \in (0, \infty]$ . Moreover, it holds that

$$a, b \in C^\omega((0, t^+)), \quad v \in C^\omega(\{(t, x) : 0 < t < t^+, a(t) < x < b(t)\}, (0, \infty)).$$

*Remark 1.3.* (i) The choice of the small Hölder spaces is essential. Indeed, using a singular transformation from [4], we may recast the evolution problem (1.6) as a fully nonlinear evolution equation with the leading order term in (1.6a)<sub>1</sub> having in the linearisation—when working within this class of functions—a positive and bounded coefficient. Besides, the setting of small Hölder spaces is a smart choice when dealing with fully nonlinear parabolic equations, cf. e.g. [11, 23]. A further departure of these spaces from the classical Hölder spaces is illustrated in Lemma A.2.

(ii) The problem considered in [4] is general enough to include also (1.6). However, the technical details, see Sect. 3, are different from those in [4] and also simpler. Besides, the parabolic smoothing property for  $v$  in Theorem 1.2 is a new result in this degenerate parabolic setting and it extends also to the general problem considered in [4]. In particular, this proves in the context of the porous medium equation (which is the equation that motivates the analysis in [4]) that the interface separating a fluid blob, that expands under the effect of gravity, from air is real-analytic in the positivity set, see [31] for more references on this topic.

(iii) If  $v_0 \in h^{2+\alpha}([a_0, b_0])$  satisfies  $v_0(a_0) = v_0(b_0)$ ,  $v_0 > 0$  in  $(a_0, b_0)$ , and

$$v_0'(a_0) > 0 > v_0'(b_0),$$

then  $v_0$  can be chosen as an initial condition in (1.6). However, the initial data  $v_0$  in Theorem 1.2 are not required to be twice differentiable at  $x = a_0$  and  $x = b_0$ . For example if  $b_0 = -a_0 = 1$ , then one can choose

$$v_0(x) := 1 - x^2 + (1 - x^2)^{3/2}, \quad x \in [-1, 1],$$

as an initial condition in (1.6) as  $v_0 \circ (-\cos) \in W_\infty^3(\mathbb{S})$ , but  $v_0 \notin C^2([-1, 1])$ .

(iv) The assumption that  $h_0 := v_0((a_0 + b_0)/2 - (b_0 - a_0) \cos / 2) \in h_e^{2+\alpha}(\mathbb{S})$  guarantees that the nonlinear boundary condition (1.6a)<sub>5</sub> holds at  $t = 0$ . Indeed, since

$$(v_0 v_0'') \circ ((a_0 + b_0)/2 - (b_0 - a_0) \cos / 2) = \frac{4}{(b - a)^2} \frac{h_0}{\sin^2} \left( h_0'' - \frac{h_0'}{\tan} \right),$$

l'Hospital's rule shows that  $\lim_{x \rightarrow a_0} (v_0 v_0''(x)) = \lim_{x \rightarrow b_0} (v_0 v_0''(x)) = 0$ .

## 2. The transformed problem

In order to study (1.6) we use an idea from [4] and transform the evolution problem (1.6) into a system defined in the setting of periodic functions by using a diffeomorphism that has a first derivative which is singular at the points  $a(t)$  and  $b(t)$ . More precisely, we introduce the new unknown

$$h(t, x) := v(t, c(t) - d(t) \cos(x)), \quad x \in \mathbb{R}, t \geq 0,$$

where

$$c(t) := \frac{a(t) + b(t)}{2} \quad \text{and} \quad d(t) := \frac{b(t) - a(t)}{2} > 0.$$

Given  $t \geq 0$ ,  $h(t)$  is a  $2\pi$ -periodic function on  $\mathbb{R}$  which is even and merely the continuous differentiability of  $v(t)$  implies that

$$h_{xx}(t, 0) = \frac{b(t) - a(t)}{a'(t)} > 0 \quad \text{and} \quad h_{xx}(t, \pi) = -\frac{b(t) - a(t)}{b'(t)} > 0.$$

In terms of the new variable  $(h, c, d)$  the problem (1.6) can be recast as follows

$$\left\{ \begin{array}{l} h_t = \frac{2}{2d^2h + h_x^2/\sin^2} \frac{h}{\sin^2} \left( h_{xx} - \frac{h_x}{\tan} \right) - \frac{h_x^2/\sin^2}{2d^2h + h_x^2/\sin^2} - 1 \\ \quad + \left( \frac{1 + \cos}{h_{xx}(t, 0)} - \frac{1 - \cos}{h_{xx}(t, \pi)} \right) \frac{h_x}{\sin}, \quad t \geq 0, x \in \mathbb{R}, \\ c' = d \left( \frac{1}{h_{xx}(t, 0)} - \frac{1}{h_{xx}(t, \pi)} \right), \quad t \geq 0, \\ d' = -d \left( \frac{1}{h_{xx}(t, 0)} + \frac{1}{h_{xx}(t, \pi)} \right), \quad t \geq 0, \\ d(t) > 0, h(t, 0) = h(t, \pi) = 0, \quad t \geq 0, \\ h(t, x) > 0, \quad t \geq 0, x \in (0, \pi), \\ h_{xx}(t, 0) > 0, h_{xx}(t, \pi) > 0, \quad t \geq 0, \end{array} \right. \quad (2.1a)$$

with initial conditions

$$h(0) = h_0 := v_0(c_0 - d_0 \cos), \quad c(0) = \frac{a_0 + b_0}{2}, \quad d(0) = \frac{b_0 - a_0}{2}. \quad (2.1b)$$

We point out that nonlinear boundary condition (1.6a)<sub>5</sub> has not been taken into account in the transformed system (2.1). This is due to the choice of the function spaces below as, similarly as in Remark 1.3 (iv), requiring that  $h(t) \in \mathbb{E}_1$  ensures that (1.6a)<sub>5</sub> holds at time  $t \geq 0$ .

In order to study (2.1) we choose as an appropriate framework the setting of periodic small Hölder spaces. For a fixed  $\alpha \in (0, 1)$  we define the Banach spaces

$$\begin{aligned} \mathbb{E}_0 &:= \{h \in h_\varepsilon^\alpha(\mathbb{S}) : h(0) = h(\pi) = 0\}, \\ \mathbb{E}_1 &:= \{h \in h_\varepsilon^{2+\alpha}(\mathbb{S}) : h(0) = h(\pi) = 0\}, \end{aligned}$$

with the corresponding norms  $\|\cdot\|_i = \|\cdot\|_{C^{2i+\alpha}(\mathbb{S})}$ ,  $i \in \{0, 1\}$ . It is important to point out that the embedding  $\mathbb{E}_1 \hookrightarrow \mathbb{E}_0$  is dense. Though at formal level the equation (2.1a)<sub>1</sub> has a quasilinear structure, our analysis below shows that the problem (2.1) is actually (as a result of the boundary conditions) fully nonlinear (see Lemma 2.1 and the subsequent discussion). This loss of linearity is however compensated by the fact that none of the terms on the right hand side of (2.1a)<sub>1</sub> is singular when choosing  $h \in \mathbb{E}_1$ . Moreover the function multiplying  $h_{xx}$  in (2.1a)<sub>1</sub> is now  $\alpha$ -Hölder continuous and positive.

**Lemma 2.1.** *The operators*

$$\left[ h \mapsto \frac{h}{\sin^2} \right], \left[ h \mapsto \frac{h'}{\sin} \right] : \mathbb{E}_1 \rightarrow h_\varepsilon^\alpha(\mathbb{S})$$

are bounded.

*Proof.* See [4, Lemma 2.1]. □

We emphasize that it is not possible to choose in Lemma 2.1 as target space a small Hölder space  $h_\varepsilon^{\alpha'}(\mathbb{S})$  with  $\alpha' > \alpha$ . In particular, the terms  $h/\sin^2$  and  $h_x/\tan$  on the right-hand side of (2.1a)<sub>1</sub> have the same importance as  $h_{xx}$  when linearizing this expression.

We now set

$$\mathcal{O} := \{h \in \mathbb{E}_1 : h''(0) > 0, h''(\pi) > 0 \text{ and } h > 0 \text{ in } (0, \pi)\}.$$

Then,  $\mathcal{O}$  is an open subset of  $\mathbb{E}_1$ . Let further

$$\Phi := (\Phi_1, \Phi_2, \Phi_3) : \mathcal{O} \times \mathbb{R} \times (0, \infty) \subset \mathbb{E}_1 \times \mathbb{R}^2 \rightarrow \mathbb{E}_0 \times \mathbb{R}^2$$

be the operator defined by

$$\begin{aligned} \Phi_1(h, c, d) &:= \frac{2}{2d^2h + h'^2/\sin^2} \frac{h}{\sin^2} \left( h'' - \frac{h'}{\tan} \right) - \frac{h'^2/\sin^2}{2d^2h + h'^2/\sin^2} - 1 \\ &\quad + \left( \frac{1 + \cos}{h''(0)} - \frac{1 - \cos}{h''(\pi)} \right) \frac{h'}{\sin}, \\ \Phi_2(h, c, d) &:= d \left( \frac{1}{h''(0)} - \frac{1}{h''(\pi)} \right), \\ \Phi_3(h, c, d) &:= -d \left( \frac{1}{h''(0)} + \frac{1}{h''(\pi)} \right). \end{aligned}$$

It is not difficult to check that  $\Phi_1(h, c, d)|_{x=0} = \Phi_1(h, c, d)|_{x=\pi} = 0$ , so that  $\Phi$  is well-defined. In virtue of Lemma 2.1 it further holds that

$$\Phi \in C^\omega(\mathcal{O} \times \mathbb{R} \times (0, \infty), \mathbb{E}_0 \times \mathbb{R}^2). \quad (2.2)$$

Hence, we are led to the fully nonlinear evolution problem

$$(\dot{h}, \dot{c}, \dot{d}) = \Phi(h, c, d), \quad t \geq 0, \quad (h(0), c(0), d(0)) = (h_0, c_0, d_0), \quad (2.3)$$

with  $(h_0, c_0, d_0) \in \mathcal{O} \times \mathbb{R} \times (0, \infty)$ . We shall establish the existence and uniqueness of strict solutions (in the sense of [23]) to (2.3) by using the fully nonlinear parabolic theory presented in [23]. To this end we next identify the Fréchet derivative  $\partial\Phi(h_0, c_0, d_0)$  and we prove that it generates, for each  $(h_0, c_0, d_0) \in \mathcal{O} \times \mathbb{R} \times (0, \infty)$ , a strongly continuous and analytic semigroup. In the notation of Amann [2] this means by definition

$$-\partial\Phi(h_0, c_0, d_0) \in \mathcal{H}(\mathbb{E}_1 \times \mathbb{R}^2, \mathbb{E}_0 \times \mathbb{R}^2).$$

In fact, in view of [2, Corollary I.1.6.3], we only need to show that the partial derivative  $\partial_h\Phi_1(h_0, c_0, d_0)$  generates a strongly continuous analytic semigroup in  $\mathcal{L}(\mathbb{E}_0)$ . Given  $h \in \mathbb{E}_1$ , it holds that

$$\begin{aligned} \partial_h\Phi_1(h_0, c_0, d_0)[h] &= A_1\left(h'' - \frac{h'}{\tan}\right) + A_2\frac{h'}{\sin} + A_3\frac{h}{\sin^2} \\ &\quad + A_4h + A_5h''(0) + A_6h''(\pi) \end{aligned}$$

where

$$\begin{aligned} A_1 &:= \frac{2}{2d_0^2h_0 + h_0'^2/\sin^2} \frac{h_0}{\sin^2}, \\ A_2 &:= \frac{2}{(2d_0^2h_0 + h_0'^2/\sin^2)^2} \left[ \frac{h_0'^2}{\sin^2} - \frac{2h_0}{\sin^2} \left( h_0'' - \frac{h_0'}{\tan} \right) \right] \frac{h_0'}{\sin} \\ &\quad - \frac{2}{2d_0^2h_0 + h_0'^2/\sin^2} \frac{h_0'}{\sin} + \frac{1 + \cos}{h_0''(0)} - \frac{1 - \cos}{h_0''(\pi)}, \\ A_3 &:= \frac{2}{2d_0^2h_0 + h_0'^2/\sin^2} \left( h_0'' - \frac{h_0'}{\tan} \right), \\ A_4 &:= \frac{2d_0^2}{(2d_0^2h_0 + h_0'^2/\sin^2)^2} \left[ \frac{h_0'^2}{\sin^2} - \frac{2h_0}{\sin^2} \left( h_0'' - \frac{h_0'}{\tan} \right) \right], \\ A_5 &:= -\frac{h_0'}{\sin(h_0''(0))^2}, \quad A_6 := \frac{h_0'}{\sin(h_0''(\pi))^2}. \end{aligned}$$

We note that  $A_i \in \mathfrak{h}_\varepsilon^\alpha(\mathbb{S})$ ,  $1 \leq i \leq 6$ , with  $A_1$  being positive. Moreover, it is straightforward to verify that  $A_3 \in \mathbb{E}_0$ .

Since

$$\|A_4 h\|_0 \leq \|A_4\|_0 \|h\|_0 \quad \text{for all } h \in \mathbb{E}_1,$$

the operator  $[h \mapsto A_4 h] : \mathbb{E}_1 \rightarrow \mathbb{E}_0$  may be viewed as being a lower order perturbation of  $\partial_h \Phi_1(h_0, c_0, d_0)$ , cf. [2, Theorem I.1.3.1 (ii)]. The following result enables us to regard also other terms of  $\partial_h \Phi_1(h_0, c_0, d_0)$  as being lower order perturbations.

**Lemma 2.2.** *Let  $A \in \mathbb{E}_0$ . Then, given  $\varepsilon > 0$ , there exists a constant  $C(\varepsilon) > 0$  such that*

$$\left\| A \frac{h}{\sin^2} \right\|_0 + \left\| A \frac{h'}{\sin} \right\|_0 \leq \varepsilon \|h\|_1 + C(\varepsilon) \|h\|_0 \quad \text{for all } h \in \mathbb{E}_1. \quad (2.4)$$

*Proof.* Letting  $I := [-2\pi/3, 5\pi/3]$ , it is not difficult to verify that

$$\|h\|_{C^{k+\alpha}(I)} \leq \|h\|_{C^{k+\alpha}(\mathbb{S})} \leq 5 \|h\|_{C^{k+\alpha}(I)}, \quad k \in \mathbb{N}, \alpha \in (0, 1), h \in C^{k+\alpha}(\mathbb{S}). \quad (2.5)$$

In view of this equivalence, the claim for  $[h \mapsto Ah'/\sin]$  follows from the observation that

$$\begin{aligned} \frac{h'(x)}{\sin(x)} &= \varphi_1(x) \int_0^1 h''(sx) \, ds, \quad x \in [-2\pi/3, \pi/2], \\ \frac{h'(x)}{\sin(x)} &= \varphi_2(x) \int_0^1 h''((1-s)\pi + sx) \, ds, \quad x \in [\pi/2, 5\pi/3], \end{aligned}$$

where the functions  $\varphi_1(x) := x/\sin(x)$  and  $\varphi_2(x) := (x - \pi)/\sin(x)$  belong to  $C^\infty([-2\pi/3, \pi/2])$  and  $C^\infty([\pi/2, 5\pi/3])$ , respectively. The proof of the second claim follows by similar arguments.  $\square$

Recalling that  $A_3 \in \mathbb{E}_0$ , Lemma 2.2 implies that also  $[h \mapsto A_3 h/\sin^2] : \mathbb{E}_1 \rightarrow \mathbb{E}_0$  can be viewed as being a perturbation. Let us now notice that

$$A_2(0) = 2/h_0''(0) = 2A_1(0) \quad \text{and} \quad A_2(\pi) = -2/h_0''(\pi) = -2A_1(\pi).$$

Observing that

$$A_2 \frac{h'}{\sin} = (A_2 - 2A_1 \cos) \frac{h'}{\sin} + 2A_1 \frac{h'}{\tan},$$

where  $A_2 - 2A_1 \cos \in \mathbb{E}_0$ , we may regard in view of Lemma 2.2 also the operator

$$[h \mapsto (A_2 - 2A_1 \cos)h'/\sin] : \mathbb{E}_1 \rightarrow \mathbb{E}_0$$

as being a perturbation and we are left to prove the generator property for

$$\tilde{\mathbb{A}} := \left[ h \mapsto A_1 \left( h'' + \frac{h'}{\tan} \right) + A_5 h''(0) + A_6 h''(\pi) \right] : \mathbb{E}_1 \rightarrow \mathbb{E}_0.$$

In fact, it suffices to establish the generator property for the operator

$$\mathbb{A} := \left[ h \mapsto A_1 \left( h'' + \frac{h'}{\tan} \right) \right] : \mathfrak{h}_e^{2+\alpha}(\mathbb{S}) \rightarrow \mathfrak{h}_e^\alpha(\mathbb{S}) \quad (2.6)$$

where we have dropped the lower order term  $[h \mapsto A_5 h''(0) + A_6 h''(\pi)]$ . Indeed, assuming that  $-\mathbb{A} \in \mathcal{H}(\mathfrak{h}_e^{2+\alpha}(\mathbb{S}), \mathfrak{h}_e^\alpha(\mathbb{S}))$ , it follows  $-\tilde{\mathbb{A}} \in \mathcal{H}(\mathfrak{h}_e^{2+\alpha}(\mathbb{S}), \mathfrak{h}_e^\alpha(\mathbb{S}))$ . This latter property is equivalent to the existence of constants  $\kappa \geq 1$  and  $\omega > 0$  such that

- (1)  $\omega - \tilde{\mathbb{A}} : \mathfrak{h}_e^{2+\alpha}(\mathbb{S}) \rightarrow \mathfrak{h}_e^\alpha(\mathbb{S})$  is an isomorphism, and
- (2)  $\kappa^{-1} \leq \frac{\|(\lambda - \tilde{\mathbb{A}})[h]\|_0}{|\lambda| \cdot \|h\|_0 + \|h\|_1} \leq \kappa$  for all  $\operatorname{Re} \lambda \geq \omega$  and  $0 \neq h \in \mathfrak{h}_e^{2+\alpha}(\mathbb{S})$ ,

cf. [2, Chapter I]. The relation (2) holds in particular for  $0 \neq h \in \mathbb{E}_1$ . In order to conclude that  $-\tilde{\mathbb{A}} \in \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0)$ , we are thus left to show that  $\omega - \tilde{\mathbb{A}} : \mathbb{E}_1 \rightarrow \mathbb{E}_0$  is an isomorphism too. Hence, given  $f \in \mathbb{E}_0$ , for  $h \in \mathfrak{h}_e^{2+\alpha}(\mathbb{S})$  with  $(\omega - \tilde{\mathbb{A}})[h] = f$  we set

$$\tilde{h} := \frac{1 + \cos}{2} h(0) + \frac{1 - \cos}{2} h(\pi).$$

Taking into account that  $h - \tilde{h} \in \mathbb{E}_1$ , it follows that  $(\omega - \tilde{\mathbb{A}})[\tilde{h}] \in \mathbb{E}_0$ . A simple computation shows that  $\tilde{\mathbb{A}}[\tilde{h}] \in \mathbb{E}_0$ , so that also  $\tilde{h} \in \mathbb{E}_0$ . We may thus conclude that  $h \in \mathbb{E}_1$ , so that (1) holds also when replacing  $\mathfrak{h}_e^{2i+\alpha}(\mathbb{S})$  with  $\mathbb{E}_i$ ,  $i \in \{0, 1\}$ . The nontrivial property  $-\mathbb{A} \in \mathcal{H}(\mathfrak{h}_e^{2+\alpha}(\mathbb{S}), \mathfrak{h}_e^\alpha(\mathbb{S}))$  is established in detail in Sect. 3 below, cf. Theorem 3.1.

### 3. The generator property

The first goal of this section is to establish Theorem 3.1, which is a main ingredient in the proof of the main result.

**Theorem 3.1.** *Given  $h_0 \in \mathcal{O}$ , it holds that  $-\mathbb{A} \in \mathcal{H}(\mathfrak{h}_e^{2+\alpha}(\mathbb{S}), \mathfrak{h}_e^\alpha(\mathbb{S}))$ .*

We consider for  $\varepsilon \in (0, \varepsilon_0]$ , with  $\varepsilon_0 > 0$  sufficiently small, partitions  $\{\pi_1^\varepsilon, \pi_2^\varepsilon, \pi_3^\varepsilon\} \subset C^\infty(I, [0, 1])$  of the interval  $I = [-2\pi/3, 5\pi/3]$  and corresponding families  $\{\chi_1^\varepsilon, \chi_2^\varepsilon, \chi_3^\varepsilon\} \subset C^\infty(I, [0, 1])$  with the following properties

- $\pi_1^\varepsilon + \pi_2^\varepsilon + \pi_3^\varepsilon = 1$  in  $C^\infty(I)$ ;
- $\operatorname{supp}(\pi_1^\varepsilon) = [-3\varepsilon, 3\varepsilon]$ ,  $\operatorname{supp}(\pi_2^\varepsilon) = [\pi - 3\varepsilon, \pi + 3\varepsilon]$ ,  
 $\operatorname{supp}(\pi_3^\varepsilon) = I \setminus ([-2\varepsilon, 2\varepsilon] \cup [\pi - 2\varepsilon, \pi + 2\varepsilon])$ ;
- $\chi_i^\varepsilon = 1$  on  $\operatorname{supp}(\pi_i^\varepsilon)$ ,  $1 \leq i \leq 3$ ;
- $\operatorname{supp}(\chi_1^\varepsilon) = [-4\varepsilon, 4\varepsilon]$ ,  $\operatorname{supp}(\chi_2^\varepsilon) = [\pi - 4\varepsilon, \pi + 4\varepsilon]$ ,  
 $\operatorname{supp}(\chi_3^\varepsilon) = I \setminus ([-\varepsilon, \varepsilon] \cup [\pi - \varepsilon, \pi + \varepsilon])$ ;
- $\pi_1^\varepsilon$  and  $\pi_2^\varepsilon(\pi + \cdot)$  are even on  $[-3\varepsilon, 3\varepsilon]$ ;
- $\pi_3^\varepsilon$  has an even and periodic extension in  $C^\infty(\mathbb{S})$ .

Extending  $\pi_1^\varepsilon$  and  $\pi_2^\varepsilon(\pi + \cdot)$  by zero in  $\mathbb{R} \setminus [-3\varepsilon, 3\varepsilon]$ , we may view these functions as being smooth and even functions on  $\mathbb{R}$ .

As a first step towards proving Theorem 3.1 we approximate  $\mathbb{A}$  locally by certain operators which are simpler to analyze.

**Lemma 3.2.** *Let  $\mu > 0$  be given. Then, there exists  $\varepsilon > 0$ , a constant  $K = K(\varepsilon) > 0$ , and a partition  $\{\pi_1^\varepsilon, \pi_2^\varepsilon, \pi_3^\varepsilon\}$  such that the operator  $\mathbb{A}$  introduced in (2.6) satisfies*

$$\|\pi_i^\varepsilon \mathbb{A}[h] - \mathbb{A}_i[\pi_i^\varepsilon h]\|_{C^\alpha(I)} \leq \mu \|\pi_i^\varepsilon h\|_{C^{2+\alpha}(I)} + K \|h\|_{C^2(I)} \tag{3.1}$$

for  $1 \leq i \leq 3$  and  $h \in \mathfrak{h}_e^{2+\alpha}(\mathbb{S})$ , where

$$\mathbb{A}_1 = A_1(0) \left( \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} \right), \tag{3.2}$$

$$\mathbb{A}_2 = A_1(\pi) \left( \frac{d^2}{dx^2} + \frac{1}{x - \pi} \frac{d}{dx} \right), \tag{3.3}$$

$$\mathbb{A}_3 = A_1 \frac{d^2}{dx^2}. \tag{3.4}$$

*Proof.* Observing that  $\pi_3^\varepsilon / \tan \in C^\infty(I)$ , it follows that

$$\begin{aligned} \|\pi_3^\varepsilon \mathbb{A}[h] - \mathbb{A}_3[\pi_3^\varepsilon h]\|_{C^\alpha(I)} &\leq \|A_1[(\pi_3^\varepsilon)'' h + 2(\pi_3^\varepsilon)' h']\|_{C^\alpha(I)} \\ &\quad + \|A_1 h' \pi_3 / \tan\|_{C^\alpha(I)} \leq K \|h\|_{C^{1+\alpha}(I)}, \end{aligned}$$

which proves (3.1) for  $i = 3$ .

Furthermore, it holds that

$$\pi_1^\varepsilon \mathbb{A}[h] - \mathbb{A}_1[\pi_1^\varepsilon h] = T_1[h] + T_2[h],$$

where

$$\begin{aligned} T_1[h] &:= A_1 \pi_1^\varepsilon h'' - A_1(0) (\pi_1^\varepsilon h)'', \\ T_2[h] &:= A_1 \frac{1}{\tan} \pi_1^\varepsilon h' - A_1(0) \frac{1}{x} (\pi_1^\varepsilon h)'. \end{aligned}$$

Using  $\chi_1^\varepsilon \pi_1^\varepsilon = \pi_1^\varepsilon$ , we now obtain

$$\begin{aligned} \|T_1[h]\|_{C^\alpha(I)} &\leq \|(A_1 - A_1(0)) \chi_1^\varepsilon\|_{C(I)} \|\pi_1^\varepsilon h\|_{C^{2+\alpha}(I)} + K \|h\|_{C^2(I)} \\ &\leq \frac{\mu}{2} \|\pi_1^\varepsilon h\|_{C^{2+\alpha}(I)} + \|h\|_{C^2(I)}, \end{aligned}$$

provided that  $\varepsilon$  is sufficiently small.

Concerning the second term we write

$$T_2[h] = T_{2a}[h] + T_{2b}[h] - T_{2c}[h],$$

where

$$\begin{aligned} T_{2a}[h] &= (A_1 - A_1(0))\chi_1^\varepsilon \frac{1}{\tan} (\pi_1^\varepsilon h)', \\ T_{2b}[h] &= A_1(0) \left( \frac{1}{\tan} - \frac{1}{x} \right) (\pi_1^\varepsilon h)', \\ T_{2c}[h] &= A_1 \frac{1}{\tan} (\pi_1^\varepsilon h)'. \end{aligned}$$

The arguments in the proof of Lemma 2.2 yield

$$\begin{aligned} \|T_{2a}[h]\|_{C^\alpha(I)} &\leq \|(A_1 - A_1(0))\chi_1^\varepsilon\|_{C(I)} \|(\pi_1^\varepsilon h)' / \tan\|_{C^\alpha(I)} + K \|(\pi_1^\varepsilon h)' / \tan\|_{C(I)} \\ &\leq \frac{\mu}{2} \|\pi_1^\varepsilon h\|_{C^{2+\alpha}(I)} + K \|h\|_{C^2(I)}. \end{aligned}$$

Besides, since  $(\pi_1^\varepsilon)' / \tan \in C^\infty(I)$ , we get

$$\|T_{2c}[h]\|_{C^\alpha(I)} \leq K \|h\|_{C^\alpha(I)} \leq K \|h\|_{C^2(I)}.$$

Finally, it is not difficult to see that the function

$$\phi(x) := \frac{1}{\tan} - \frac{1}{x}$$

satisfies  $\chi_1^\varepsilon \phi \in C^\infty(I)$ . Therewith we have

$$\|T_{2b}[h]\|_{C^\alpha(I)} \leq C \|\chi_1^\varepsilon \phi\|_{C^\alpha(I)} \|(\pi_1^\varepsilon h)'\|_{C^\alpha(I)} \leq K \|h\|_{C^{1+\alpha}(I)},$$

and we conclude that

$$\|T_2[h]\|_{C^\alpha(I)} \leq \frac{\mu}{2} \|\pi_1^\varepsilon h\|_{C^{2+\alpha}(I)} + K \|h\|_{C^2(I)},$$

provided that  $\varepsilon$  is sufficiently small. This proves (3.1) for  $i = 1$ . The proof of the claim for  $i = 2$  is similar and we therefore omit it.  $\square$

We now consider the operators  $\mathbb{A}_i$ ,  $1 \leq i \leq 3$ , found in Lemma 3.2 in suitable functional analytic settings. Regarding  $\mathbb{A}_3$  as an element of  $\mathcal{L}(\mathfrak{h}_e^{2+\alpha}(\mathbb{S}), \mathfrak{h}_e^\alpha(\mathbb{S}))$ , it is well-known that  $\mathbb{A}_3$  generates an analytic semigroup in  $\mathcal{L}(\mathfrak{h}_e^\alpha(\mathbb{S}))$ . In particular, there exist constants  $\kappa_3 \geq 1$  and  $\omega_3 > 0$  such that

$$\kappa_3 \|(\lambda - \mathbb{A}_3)[h]\|_0 \geq |\lambda| \cdot \|h\|_0 + \|h\|_1, \quad h \in \mathfrak{h}_e^{2+\alpha}(\mathbb{S}), \operatorname{Re} \lambda \geq \omega_3, \quad (3.5)$$

cf. [2, Theorem I.1.2.2]. The operator  $\mathbb{A}_1$  can be viewed as an element of the Banach space  $\mathcal{L}(\mathfrak{h}_e^{2+\alpha}(\mathbb{R}), \mathfrak{h}_e^\alpha(\mathbb{R}))$ .<sup>1</sup> Furthermore, in this context  $\mathbb{A}_1$  appears as the restriction of  $A_1(0)\Delta \in \mathcal{L}(\mathfrak{h}^{2+\alpha}(\mathbb{R}^2), \mathfrak{h}^\alpha(\mathbb{R}^2))$  to the subset of rotationally symmetric functions. Indeed, given  $h \in \mathfrak{h}_e^{k+\alpha}(\mathbb{R})$ ,  $k \in \{0, 2\}$ , let

$$u(z) := (h \circ |\cdot|)(z) = h(\sqrt{x^2 + y^2}), \quad z = (x, y) \in \mathbb{R}^2.$$

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<sup>1</sup>For a definition of  $\mathfrak{h}^{k+\alpha}(\mathbb{R}^n)$ ,  $k, n \in \mathbb{N}$ , see [23]. Again,  $\mathfrak{h}_e^{k+\alpha}(\mathbb{R})$ ,  $k \in \mathbb{N}$ , denotes the closed subspace of  $\mathfrak{h}^{k+\alpha}(\mathbb{R})$  consisting of even functions.

One can show that the radially symmetric function  $u$  belongs to  $h^{k+\alpha}(\mathbb{R}^2)$  and that

$$\begin{aligned} \|h\|_{C^\alpha(\mathbb{R})} &= \|u\|_{C^\alpha(\mathbb{R}^2)}, \\ \|h\|_{C^{2+\alpha}(\mathbb{R})} &\leq \|u\|_{C^{2+\alpha}(\mathbb{R}^2)} \leq C \|h\|_{C^{2+\alpha}(\mathbb{R}^2)}, \end{aligned}$$

with  $C \geq 1$  independent of  $h$ . Recalling that  $-A_1(0)\Delta \in \mathcal{H}(h^{2+\alpha}(\mathbb{R}^2), h^\alpha(\mathbb{R}^2))$ , cf. [23, Theorem 3.1.14 and Corollary 3.1.16], there exist constants  $\kappa_1 \geq 1$  and  $\omega_1 > 0$  such that

$$\kappa_1 \|(\lambda - A_1(0)\Delta)[u]\|_{C^\alpha(\mathbb{R}^2)} \geq |\lambda| \cdot \|u\|_{C^\alpha(\mathbb{R}^2)} + \|u\|_{C^{2+\alpha}(\mathbb{R}^2)}.$$

for all  $u \in h^{2+\alpha}(\mathbb{R}^2)$ ,  $\operatorname{Re} \lambda \geq \omega_1$ . In particular it holds that

$$\begin{aligned} \kappa_1 \|(\lambda - A_1(0)\Delta)[h \circ |\cdot|]\|_{C^\alpha(\mathbb{R}^2)} &\geq |\lambda| \cdot \|h \circ |\cdot|\|_{C^\alpha(\mathbb{R}^2)} + \|h \circ |\cdot|\|_{C^{2+\alpha}(\mathbb{R}^2)} \\ &\geq |\lambda| \cdot \|h\|_{C^\alpha(\mathbb{R})} + \|h\|_{C^{2+\alpha}(\mathbb{R})} \end{aligned}$$

for  $h \in h_e^{2+\alpha}(\mathbb{R})$  and  $\operatorname{Re} \lambda \geq \omega_1$ . Moreover, in virtue of

$$\begin{aligned} \|(\lambda - A_1(0)\Delta)[h \circ |\cdot|]\|_{C^\alpha(\mathbb{R}^2)} &= \|((\lambda - \mathbb{A}_1)[h]) \circ |\cdot|\|_{C^\alpha(\mathbb{R}^2)} \\ &= \|(\lambda - \mathbb{A}_1)[h]\|_{C^\alpha(\mathbb{R})} \end{aligned}$$

we conclude that

$$\kappa_1 \|(\lambda - \mathbb{A}_1)[h]\|_{C^\alpha(\mathbb{R})} \geq |\lambda| \cdot \|h\|_{C^\alpha(\mathbb{R})} + \|h\|_{C^{2+\alpha}(\mathbb{R})}. \quad (3.6)$$

for all  $h \in h_e^{2+\alpha}(\mathbb{R})$ ,  $\operatorname{Re} \lambda \geq \omega_1$ . The constants  $\kappa_1$  and  $\omega_1$  can be chosen such that (3.6) holds true also when replacing  $\mathbb{A}_1$  by  $(A_1(\pi)/A_1(0))\mathbb{A}_1 = \tau_{-\pi}\mathbb{A}_2\tau_\pi$ , where  $\tau_a$ ,  $a \in \mathbb{R}$ , denotes the right translation by  $a$ .

In particular (2.5), (3.5), and (3.6) ensure there exists  $\kappa' \geq 1$  and  $\omega' > 0$  such that

$$\kappa' \|(\lambda - \mathbb{A}_i)[\pi_i^\varepsilon h]\|_{C^\alpha(I)} \geq |\lambda| \cdot \|\pi_i^\varepsilon h\|_{C^\alpha(I)} + \|\pi_i^\varepsilon h\|_{C^{2+\alpha}(I)} \quad (3.7)$$

for all  $h \in h_e^{2+\alpha}(\mathbb{S})$ ,  $\operatorname{Re} \lambda \geq \omega'$ ,  $1 \leq i \leq 3$ , and all  $\varepsilon \in (0, \varepsilon_0]$ . The estimate (3.7) together with the observation that the map

$$\left[ h \mapsto \sum_{i=1}^3 \|\pi_i^\varepsilon h\|_{C^{k+\alpha}(I)} \right] : C^{k+\alpha}(I) \rightarrow \mathbb{R}, \quad k \in \mathbb{N}, \quad (3.8)$$

defines a norm on  $C^{k+\alpha}(I)$  which is equivalent to the standard Hölder norm are essential for establishing the following result.

**Lemma 3.3.** *There exist  $\kappa \geq 1$  and  $\omega > 0$  such that*

$$\kappa \|(\lambda - \mathbb{A})[h]\|_0 \geq |\lambda| \cdot \|h\|_0 + \|h\|_1 \quad (3.9)$$

for all  $h \in h_e^{2+\alpha}(\mathbb{S})$  and all  $\operatorname{Re} \lambda \geq \omega$ .

*Proof.* Letting  $\kappa' \geq 1$  and  $\omega' > 0$  denote the constants in (3.7), we chose  $\mu := (2\kappa')^{-1}$  in Lemma 3.2. Lemma 3.2 together with (3.7) yields

$$\begin{aligned} \kappa' \|\pi_i^\varepsilon(\lambda - \mathbb{A})[h]\|_{C^\alpha(I)} &\geq \kappa' \|(\lambda - \mathbb{A}_i)[\pi_i^\varepsilon h]\|_{C^\alpha(I)} - \kappa' \|\pi_i^\varepsilon \mathbb{A}[h] - \mathbb{A}_i[\pi_i^\varepsilon h]\|_{C^\alpha(I)} \\ &\geq \frac{1}{2} \|\pi_i^\varepsilon h\|_{C^{2+\alpha}(I)} + |\lambda| \cdot \|\pi_i^\varepsilon h\|_{C^\alpha(I)} - \kappa' K \|h\|_{C^2(I)} \end{aligned}$$

for  $1 \leq i \leq 3$ ,  $h \in h_e^{2+\alpha}(\mathbb{S})$ , and  $\operatorname{Re} \lambda \geq \omega'$ . In virtue of (3.8) and of (2.5) it now follows that there exists a constant  $\kappa'' \geq 1$  such that

$$\kappa'' (\|h\|_{C^{2+\alpha/2}(\mathbb{S})} + \|(\lambda - \mathbb{A})[h]\|_{C^\alpha(\mathbb{S})}) \geq \|h\|_{C^{2+\alpha}(\mathbb{S})} + |\lambda| \cdot \|h\|_{C^\alpha(\mathbb{S})}$$

for  $h \in h_e^{2+\alpha}(\mathbb{S})$  and  $\operatorname{Re} \lambda \geq \omega'$ . Finally, the interpolation property (1.7), the latter estimate, and Young's inequality ensure that there exist constants  $\kappa \geq 1$  and  $\omega > 0$  such that (3.9) is satisfied.  $\square$

In order to derive the desired generation result it remains to prove that  $\omega - \mathbb{A} \in \operatorname{Isom}(h_e^{2+\alpha}(\mathbb{S}), h_e^\alpha(\mathbb{S}))$ . To this end we infer from (3.9) that  $\omega - \mathbb{A}$  is one-to-one. Having shown that  $\mathbb{A} \in \mathcal{L}(h_e^{2+\alpha}(\mathbb{S}), h_e^\alpha(\mathbb{S}))$  is a Fredholm operator of index zero, the isomorphism property follows then in view of the compactness of the embedding  $h_e^{2+\alpha}(\mathbb{S}) \hookrightarrow h_e^\alpha(\mathbb{S})$ .

**Lemma 3.4.**  $\mathbb{A} \in \mathcal{L}(h_e^{2+\alpha}(\mathbb{S}), h_e^\alpha(\mathbb{S}))$  is a Fredholm operator of index zero.

*Proof.* Since  $A_1 > 0$ , the equation  $\mathbb{A}[h] = 0$  is equivalent to

$$h'' + \frac{h'}{\tan} = 0,$$

hence  $(h' \sin)' = 0$ . The kernel of  $\mathbb{A}$  consists thus only of constant functions.

It is easy to see that the range of  $\mathbb{A}$  is contained in

$$Y := \left\{ f \in h_e^\alpha(\mathbb{S}) : \int_0^\pi \frac{f \sin}{A_1} dx = 0 \right\},$$

which is a closed subspace of  $h_e^\alpha(\mathbb{S})$  of codimension 1. To show that the range of  $\mathbb{A}$  coincides with  $Y$  we associate to  $f \in Y$  the function

$$h(x) := \int_0^x \frac{1}{\sin(t)} \int_0^t \frac{f \sin}{A_1}(s) ds dt, \quad x \in [0, 2\pi].$$

Using the property defining  $Y$ , it is not difficult to check that  $h$  is twice continuously differentiable with

$$\begin{aligned} h(2\pi) &= \int_0^{2\pi} \frac{1}{\sin(t)} \int_0^t \frac{f \sin}{A_1}(s) ds dt = - \int_0^{2\pi} \frac{1}{\sin(t)} \int_t^\pi \frac{f \sin}{A_1}(s) ds dt \\ &= - \int_0^\pi \frac{1}{\sin(t)} \int_t^\pi \frac{f \sin}{A_1}(s) ds dt - \int_\pi^{2\pi} \frac{1}{\sin(t)} \int_t^\pi \frac{f \sin}{A_1}(s) ds dt \\ &= 0 = h(0). \end{aligned}$$

The second last identity above follows by using appropriate substitutions in the second integral. Moreover, it holds that  $h'(0) = h'(2\pi) = 0$ ,  $h''(0) = h''(2\pi)$ , and

$$A_1\left(h'' + \frac{h'}{\tan}\right) = f \quad \text{in } \mathbb{R},$$

as we may extend  $h$  by periodicity to  $\mathbb{R}$ . Some standard (but lengthy) arguments show that  $h'/\sin$  lies in  $h_e^\alpha(\mathbb{S})$ , which implies that  $h \in h_e^{2+\alpha}(\mathbb{S})$ . Thus,  $f$  belongs to the range of  $\mathbb{A}$  and the claim follows.  $\square$

*Proof of Theorem 3.1.* In view of Lemma 3.3 it remains to show that  $\omega - \mathbb{A} : h_e^{2+\alpha}(\mathbb{S}) \rightarrow h_e^\alpha(\mathbb{S})$  is an isomorphism. This property is an immediate consequence of the estimate (3.9), which implies in particular that  $\omega - \mathbb{A}$  is injective, and of the fact that  $\omega - \mathbb{A}$  is a Fredholm operator of index zero, cf. Lemma 3.4 (we recall at this point that the embedding  $h_e^{2+\alpha}(\mathbb{S}) \hookrightarrow h_e^\alpha(\mathbb{S})$  is compact).  $\square$

We conclude this section with the proof of the well-posedness result stated in Theorem 1.2. The proof of the parabolic smoothing property for the function  $v$  is postponed to Sect. 4.

*Proof of Theorem 1.2.* We first address the solvability of (2.1). As a direct consequence of Theorem 3.1 we have that

$$-\partial\Phi(h_0, c_0, d_0) \in \mathcal{H}(\mathbb{E}_1 \times \mathbb{R}^2, \mathbb{E}_0 \times \mathbb{R}^2)$$

for all  $(h_0, c_0, d_0) \in \mathcal{O} \times \mathbb{R} \times (0, \infty)$ . Recalling also (2.2) and the interpolation property of the small Hölder spaces (1.7), the assumptions of [23, Theorem 8.4.1] are all satisfied in the context of (2.1). Hence, for each  $(h_0, c_0, d_0) \in \mathcal{O} \times \mathbb{R} \times (0, \infty)$ , the problem (2.1) possesses a unique maximal strict solution

$$(h, c, d) := (h, c, d)(\cdot; (h_0, c_0, d_0))$$

such that

$$\begin{aligned} h &\in C^1([0, t^+), \mathbb{E}_0) \cap C([0, t^+), \mathcal{O}), \\ c &\in C^1([0, t^+), \mathbb{R}), \\ d &\in C^1([0, t^+), (0, \infty)), \end{aligned}$$

where  $t^+ := t^+(h_0, c_0, d_0) \in (0, \infty]$ . Since by assumption  $h_0 := v_0(c_0 - d_0 \cos)$  belongs to  $\mathcal{O}$ , the existence and uniqueness claim in Theorem 1.2 follows. Moreover, Lunardi [23, Corollary 8.4.6] yields  $c, d \in C^\omega((0, t^+))$  and this shows that  $a, b \in C^\omega((0, t^+))$ . The real-analyticity property for  $v$  (or  $h$ ) is however more subtle and is established in Sect. 4 below.  $\square$

#### 4. Parabolic smoothing

In the following we consider a solution  $(v, a, b)$  to (1.6) with maximal existence time  $t^+$  as found in Theorem 1.2. and we prove that the associated function

$$[(t, x) \mapsto h(t, x)] : (0, t^+) \times (0, \pi) \rightarrow \mathbb{R}$$

is real-analytic. In this way we establish the parabolic smoothing property for the function  $v$  as stated in Theorem 1.2. The proof below exploits a parameter trick which has been used in other variants also in [5, 10, 16, 26, 30] to improve the regularity of solutions to parabolic or elliptic equations. The degenerate parabolic setting considered herein raises new difficulties, in particular due to the fact that the solutions  $h$  vanish at 0 and  $\pi$ , which hinder us to establish real-analyticity of  $h$  in a neighborhood of these points.

To start, we fix an arbitrary constant  $T$  such that  $0 < T < t^+$ . Given  $\lambda \in \mathbb{R}$  with

$$T|\lambda| < \min_{[0, T]} \frac{d(t)}{2} =: \vartheta_0 \quad (4.1)$$

and  $t \in [0, T]$ , we introduce the function  $\phi_\lambda(t) : \mathbb{R} \rightarrow (a(t), b(t))$  with

$$\phi_\lambda(t, x) := c(t) - d(t) \cos(x) + t\lambda \sin^2(x), \quad t \in [0, T], x \in (0, \pi).$$

The smallness condition (4.1) ensures that  $\phi_\lambda(t) : (0, \pi) \rightarrow (a(t), b(t))$  is a real-analytic diffeomorphism. We associate to  $v$  the function  $h(t, x, \lambda) := v(t, \phi_\lambda(t, x))$ , with  $x \in \mathbb{R}$ ,  $t \in [0, T]$ ,  $|\lambda| < T^{-1}\vartheta_0$ . Let further  $h(\lambda) := h(\cdot, \cdot, \lambda)$ . Taking into account that  $h(t, x, 0) = h(t, x)$  for  $t \in [0, T]$  and  $x \in \mathbb{R}$ , Theorem 1.2 yields

$$h(0) \in C^1([0, T], \mathfrak{h}_e^\alpha(\mathbb{S})) \cap C([0, T], \mathfrak{h}_e^{2+\alpha}(\mathbb{S})).$$

Clearly,  $h(\lambda)$  is  $2\pi$ -periodic and even with respect to  $x$ . Observing that

$$h(t, x, \lambda) = h\left(t, \arccos\left(\cos x - \frac{t\lambda \sin^2(x)}{d(t)}\right), 0\right), \quad t \in [0, T], x \in \mathbb{R},$$

tedious computations show that

$$h(\lambda) \in C^1([0, T], \mathfrak{h}_e^\alpha(\mathbb{S})) \cap C([0, T], \mathfrak{h}_e^{2+\alpha}(\mathbb{S})) \quad \text{for all } |\lambda| < T^{-1}\vartheta_0. \quad (4.2)$$

We emphasize that Lemma A.2 (ii) plays a key role in the proof of (4.2). Moreover, given  $t \in [0, T]$ , it holds that

$$\partial_x^2 h(t, 0, \lambda) > h(t, 0, \lambda) = 0 = h(t, \pi, \lambda) < \partial_x^2 h(t, \pi, \lambda)$$

together with

$$h(t, x, \lambda) > 0, \quad x \in (0, \pi).$$

Furthermore, the pair  $(h(\lambda), c, d)$  solves the parameter dependent evolution problem

$$(\dot{h}, \dot{c}, \dot{d}) = \Psi(t, h, c, d, \lambda), \quad t \in [0, T], \quad (h, c, d)(0) = (h_0, c_0, d_0), \quad (4.3)$$

where  $\Psi := (\Psi_1, \Psi_2, \Psi_3) : [0, T] \times \mathcal{O} \times \mathbb{R} \times (0, \infty) \times (-T^{-1}\vartheta_0, T^{-1}\vartheta_0) \rightarrow \mathbb{E}_0 \times \mathbb{R}^2$  is defined by

$$\begin{aligned} &\Psi_1(t, h, c, d, \lambda) \\ &:= \frac{2(d + t\lambda \cos)^2}{(d + t\lambda \cos)^2 - t^2\lambda^2} \cdot \frac{1}{2(d + 2t\lambda \cos)^2 h + h^2/\sin^2} \cdot \frac{h}{\sin^2} \\ &\quad \times \left[ \frac{(d + t\lambda \cos)^2 - t^2\lambda^2}{(d + 2t\lambda \cos)^2} h'' - \frac{d}{d + 2t\lambda \cos} \frac{h'}{\tan} \right. \\ &\quad \left. - \frac{t\lambda(2t^2\lambda^2 - 2d^2 - 3dt\lambda \cos + 4t^2\lambda^2 \cos^2)}{(d + 2t\lambda \cos)^3} h' \sin \right] \\ &\quad - \frac{h^2/\sin^2}{2(d + 2t\lambda \cos)^2 h + h^2/\sin^2} - 1 \\ &\quad + \left[ \frac{(d + 2t\lambda)(d + d \cos - t\lambda \sin^2)}{d(d + 2t\lambda \cos)h''(0)} - \frac{(d - 2t\lambda)(d - d \cos + t\lambda \sin^2)}{d(d + 2t\lambda \cos)h''(\pi)} \right] \frac{h'}{\sin} \\ &\quad + \frac{\lambda}{d + 2t\lambda \cos} \left[ 1 + \frac{t}{d} \left( \frac{d + 2t\lambda}{h''(0)} + \frac{d - 2t\lambda}{h''(\pi)} \right) \right] h' \sin \end{aligned}$$

and

$$\begin{aligned} \Psi_2(t, h, c, d, \lambda) &:= \frac{d + 2t\lambda}{h''(0)} - \frac{d - 2t\lambda}{h''(\pi)}, \\ \Psi_3(t, h, c, d, \lambda) &:= -\frac{d + 2t\lambda}{h''(0)} - \frac{d - 2t\lambda}{h''(\pi)}. \end{aligned}$$

Recalling (4.1), it then follows that

$$\Psi \in C^\omega([0, T] \times \mathcal{O} \times \mathbb{R} \times (0, \infty) \times (-T^{-1}\vartheta_0, T^{-1}\vartheta_0), \mathbb{E}_0 \times \mathbb{R}^2).$$

Observing that  $[h \mapsto h' \sin] : \mathbb{E}_1 \rightarrow \mathbb{E}_0$  is a bounded operator which can be estimated in a similar way as the operators in Lemma 2.2, we may repeat the arguments in Sects. 2–3 to conclude that

$$-\partial_{(h,c,d)}\Psi(t, h_0, c_0, d_0, \lambda) \in \mathcal{H}(\mathbb{E}_1 \times \mathbb{R}^2, \mathbb{E}_0 \times \mathbb{R}^2)$$

for all  $(t, h_0, c_0, d_0, \lambda) \in [0, T] \times \mathcal{O} \times \mathbb{R} \times (0, \infty) \times (-T^{-1}\vartheta_0, T^{-1}\vartheta_0)$ . Applying [23, Theorem 8.4.1], it follows that (4.3) possesses for each  $(h_0, c_0, d_0, \lambda)$  that belongs to  $\mathcal{O} \times \mathbb{R} \times (0, \infty) \times (-T^{-1}\vartheta_0, T^{-1}\vartheta_0)$  a unique maximal strict solution  $(h, c, d) = (h, c, d)(\cdot; (h_0, c_0, d_0, \lambda))$  with

$$(h, c, d) \in C^1([0, t^+), \mathbb{E}_0 \times \mathbb{R}^2) \cap C([0, t^+), \mathcal{O} \times \mathbb{R} \times (0, \infty)),$$

where  $t^+ = t^+(h_0, c_0, d_0, \lambda) \in (0, T]$  is the maximal existence time. In view of [23, Corollary 8.4.6] we may conclude that the mapping

$$[(t, h_0, c_0, d_0, \lambda) \mapsto h(t; (h_0, c_0, d_0, \lambda))] : \Omega \rightarrow \mathbb{E}_1,$$

where

$$\Omega := \left\{ (t, h_0, c_0, d_0, \lambda) : \begin{array}{l} (h_0, c_0, d_0, \lambda) \in \mathcal{O} \times \mathbb{R} \times (0, \infty), \\ \lambda \in (-T^{-1}\vartheta_0, T^{-1}\vartheta_0), t \in (0, t^+) \end{array} \right\}$$

is real-analytic. Let now  $x_0 \in (0, \pi)$  be fixed. Since  $[a \mapsto a(x_0)] : \mathbb{E}_1 \rightarrow \mathbb{R}$  is a real-analytic map, we obtain for the function  $h$  determined by the solution  $(v, a, b)$  considered above, in particular that

$$\left[ (t, \lambda) \mapsto h\left(t, \arccos\left(\cos x_0 - \frac{t\lambda \sin^2(x_0)}{d(t)}\right)\right) \right] : (0, T) \times (-T^{-1}\vartheta_0, T^{-1}\vartheta_0) \rightarrow \mathbb{R} \quad (4.4)$$

is real-analytic too. Additionally, given  $\tau \in (0, T)$ , for sufficiently small  $\delta > 0$  it holds that  $g : (\tau, T) \times (x_0 - \delta, x_0 + \delta) \rightarrow (0, T) \times (-T^{-1}\vartheta_0, T^{-1}\vartheta_0)$  with

$$g(t, x) := \left( t, \frac{(\cos(x_0) - \cos(x))d(t)}{t \sin^2(x_0)} \right) \quad (4.5)$$

is well-defined and real-analytic. Here we use the real-analyticity of  $d$  in  $(0, T)$  which we already established. Composing the mappings (4.4) and (4.5), it follows in view of the fact that  $x_0 \in (0, \pi)$  is arbitrary that

$$[(t, x) \mapsto h(t, x)] : (0, T) \times (0, \pi) \rightarrow \mathbb{R}$$

is real-analytic. Recalling that  $h(t, x) = v(t, c(t) - d(t) \cos(x))$ , the property

$$v \in C^\omega(\{(t, x) : 0 < t < t^+, a(t) < x < b(t)\}, (0, \infty))$$

follows at once.

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**Appendix A.**

The next result shows that the two approaches used in the Introduction to derive evolution equations for the functions  $a$  and  $b$  require the same assumptions. In particular, it shows that the solutions to the problem (1.6) describe closed  $C^2$ -surfaces without boundary and with positive curvature at the points on the rotation axis.

**Lemma A.1.** *Let  $0 < a$  and let  $u \in C([0, a]) \cap C^2((0, a))$  satisfy  $u(x) > 0$  for all  $x \in [0, a)$  and  $u(a) = 0$ . Then, the following are equivalent:*

- (i)  $\lim_{x \rightarrow a} u'(x) = -\infty$ ,  $\lim_{x \rightarrow a} (uu')(x) < 0$ , and

$$\exists \lim_{x \rightarrow a} \frac{u''}{u^3}(x) \in \mathbb{R} \cup \{\pm\infty\}.$$

- (ii) *There exists  $\varepsilon > 0$  such that  $u : [a - \varepsilon, a] \rightarrow [0, u(a - \varepsilon)]$  is invertible and the inverse  $w : [0, u(a - \varepsilon)] \rightarrow [a - \varepsilon, a]$  satisfies  $w \in C^2([0, u(a - \varepsilon)])$ ,  $w'(0) = 0$ , and  $w''(0) < 0$ .*
- (iii) *The function  $v := u^2/2$  satisfies  $v \in C^1([0, a])$ ,  $v'(a) < 0$ , and  $\lim_{x \rightarrow a} (vv'')(x) = 0$ .*

*Proof.* We first prove the implication (i)  $\implies$  (ii). It is obvious that if  $\varepsilon > 0$  is sufficiently small, the map  $u : [a - \varepsilon, a] \rightarrow [0, u(a - \varepsilon)]$  has an inverse function  $w : [0, u(a - \varepsilon)] \rightarrow [a - \varepsilon, a]$  and  $w \in C([0, u(a - \varepsilon)]) \cap C^2((0, u(a - \varepsilon)))$ . Furthermore, it holds that  $w'(0) = 0$  and

$$\lim_{y \rightarrow 0} \frac{w'(y)}{y} = \frac{1}{\lim_{x \rightarrow a} (uu')(x)}. \tag{A.1}$$

Hence,  $w$  is twice differentiable in 0 and  $w''(0) < 0$ . Furthermore, the mean value theorem yields the existence of a sequence  $y_n \rightarrow 0+$  such that  $w''(y_n) \rightarrow w''(0)$ . Since

$$\lim_{y \rightarrow 0} w''(y) = \lim_{x \rightarrow a} w''(u(x)) = - \lim_{x \rightarrow a} \frac{u''}{u^3}(x),$$

we obtain the following relation

$$\lim_{x \rightarrow a} \frac{u''}{u'^3}(x) = -w''(0),$$

and therewith we get that  $w \in C^2([0, u(a - \varepsilon)])$ .

We now prove the implication (ii)  $\implies$  (iii). We may assume that  $w'(y) < 0$  for  $y > 0$ . Invoking (A.1) we get that  $v \in C^1([0, a])$  and  $v'(a) = 1/w''(0) < 0$ . Moreover it holds that

$$2 \lim_{x \rightarrow a} (vv'')(x) = \lim_{x \rightarrow a} (u^3 u'' + u^2 u'^2)(x) = \lim_{y \rightarrow 0} \left( -\frac{y^3 w''(y)}{w'^3(y)} + \frac{y^2}{w'^2(y)} \right) = 0,$$

and this proves (iii).

We conclude with the proof of (iii)  $\implies$  (i). The relations  $\lim_{x \rightarrow a} u'(x) = \infty$  and  $\lim_{x \rightarrow a} (uu')(x) < 0$  are immediate and together with

$$\lim_{x \rightarrow a} \frac{u''}{u'^3}(x) = \lim_{x \rightarrow a} \left( \frac{2vv''}{v'^3}(x) - \frac{1}{v'(x)} \right) = -\frac{1}{v'(a)}$$

we have completed the proof.  $\square$

Lemma A.2 provides a continuity result which is used to establish (4.2). This lemma also exemplifies why the small Hölder spaces are to be preferred in certain applications to the classical ones.

**Lemma A.2.** *Let  $\alpha \in (0, 1)$ .*

(i) *Given  $a \in C^\alpha(\mathbb{S})$ , the mapping*

$$[b \mapsto a \circ b] : W_\infty^1(\mathbb{S}) \rightarrow C^\alpha(\mathbb{S})$$

*is in general not continuous.*

(ii) *Given  $a \in h^\alpha(\mathbb{S})$ , the mapping*

$$[b \mapsto a \circ b] : W_\infty^1(\mathbb{S}) \rightarrow h^\alpha(\mathbb{S})$$

*is continuous.*

*Proof.* It is easy to verify that  $[b \mapsto a \circ b] : W_\infty^1(\mathbb{S}) \rightarrow C^\alpha(\mathbb{S})$  is well-defined. The following example shows that this (nonlinear) mapping is in general not continuous. Indeed, let  $\phi \in C_0^\infty(\mathbb{R})$  be a function which satisfies  $\phi = 1$  on  $[-1, 1]$  and  $\phi = 0$  in  $\mathbb{R} \setminus [-2, 2]$ . The  $2\pi$ -periodic extension  $a$  of

$$[x \mapsto |x|^\alpha \phi(x)] : [-\pi, \pi] \rightarrow \mathbb{R}$$

satisfies  $a \in C^\alpha(\mathbb{S})$ . Given  $1 \leq n \in \mathbb{N}$ , let  $b_n$  and  $b$  denote the  $2\pi$ -periodic extensions of

$$[x \mapsto (x + 1/n)\phi(x)] : [-\pi, \pi] \rightarrow \mathbb{R} \quad \text{and} \quad [x \mapsto x\phi(x)] : [-\pi, \pi] \rightarrow \mathbb{R}.$$

It then holds  $b, b_n \in W_\infty^1(\mathbb{S})$  and  $b_n \rightarrow b$  in  $W_\infty^1(\mathbb{S})$ . Since

$$|a(b_n(-1/n)) - a(b(-1/n)) - (a(b_n(0)) - a(b(0)))| = \frac{2}{n^\alpha}$$

it follows that

$$\|a \circ b_n - a \circ b\|_\alpha \geq [a \circ b_n - a \circ b]_\alpha \geq 2 \quad \text{for all } n \in \mathbb{N}, n \geq 1,$$

which proves (i).

We now prove (ii). Let thus  $b_n, b \in W_\infty^1(\mathbb{S})$  with  $b_n \rightarrow b$  as  $n \rightarrow \infty$  in  $W_\infty^1(\mathbb{S})$  and  $(a_m) \subset C^\infty(\mathbb{S})$  be a sequence with  $a_m \rightarrow a$  in  $C^\alpha(\mathbb{S})$ . It follows that  $a_m \circ b_n \in \mathfrak{h}^\alpha(\mathbb{S})$  for  $n, m \in \mathbb{N}$ . Moreover, it holds

$$\begin{aligned} \|a \circ b_n - a \circ b\|_\alpha &\leq \|a \circ b_n - a_m \circ b_n\|_\alpha + \|a_m \circ b_n - a_m \circ b\|_\alpha \\ &\quad + \|a_m \circ b - a \circ b\|_\alpha, \end{aligned}$$

where

$$\begin{aligned} \|a \circ b_n - a_m \circ b_n\|_\alpha &\leq (1 + \|b'_n\|_0^\alpha) \|a_m - a\|_\alpha, \\ \|a_m \circ b - a \circ b\|_\alpha &\leq (1 + \|b'\|_0^\alpha) \|a_m - a\|_\alpha, \\ \|a_m \circ b_n - a_m \circ b\|_\alpha &\leq \|a_m\|_\alpha \|b_n - b\|_0^\alpha \\ &\quad + \|a'_m\|_\alpha \|b_n b\|_\alpha (1 + \|b'\|_0^\alpha + \|b'_n\|_0^\alpha). \end{aligned}$$

These estimates show that  $a \circ b_n \rightarrow a \circ b$  in  $C^\alpha(\mathbb{S})$  and that each ball in  $C^\alpha(\mathbb{S})$  centered in  $a \circ b$  contains a function  $a_m \circ b_n$  with  $n, m \in \mathbb{N}$  suitably large. Since  $a_m \circ b_n \in \mathfrak{h}^\alpha(\mathbb{S})$ , it follows that also  $a \circ b \in \mathfrak{h}^\alpha(\mathbb{S})$ , and this completes the proof.  $\square$

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