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Untwisting 3-strand torus knots

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Abstract

We prove that the signature bound for the topological 4-genus of 3-strand torus knots is sharp, using McCoy's twisting method. We also show that the bound is off by at most 1 for 4-strand and 6-strand torus knots, and improve the upper bound on the asymptotic ratio between the topological 4-genus and the Seifert genus of torus knots from 2/3 to 14/27.

1. *Introduction*

The braid group on 3-strand B_3 is generated by two elements a, b satisfying the braid relation $aba = bab$. In this note, we are interested in the natural closure of the positive braid $(ab)^n$ in S^3, known as torus link of type $T(3,n)$. Whenever $n \in \mathbb{N}$ is a multiple of 3, the link $T(3,n)$ has three components; otherwise it is a knot. The topological 4-genus $g_t(K)$ of a knot $K \subset S^3$ is defined to be the minimal genus among all surfaces $\Sigma \subset D^4$, embedded in a locally flat way, with boundary $\partial\Sigma = K$. As with the smooth version of the 4-genus invariant, the topological 4-genus of knots K is bounded below by the signature invariant [11]: $g_t(K) \geqslant |\sigma(K)|/2$. The same lower bound holds with the signature invariant replaced by the maximum value of the Levine–Tristram signature function outside of the set of roots of the Alexander polynomial $\Delta_K(t)$ of K

$$\widehat{\sigma}(K) := \max_{\omega \in S^1 \setminus \Delta_K^{-1}(0)} |\sigma_\omega(K)|.$$

Theorem 1. *Let $n \geqslant 4$ be a natural number not divisible by 3. Then*

$$g_t(T(3,n)) = \frac{\widehat{\sigma}(T(3,n))}{2} = \left\lceil \frac{2n}{3} \right\rceil.$$

We believe that the equality $g_t = \widehat{\sigma}/2$ holds for a much larger class of torus knots, possibly for all. This can be seen as a topological counterpart of the local Thom conjecture, which states that the smooth 4-genus g_s of torus knots coincides with their Seifert genus [6, 12, 14]. Unlike in the smooth case, where the hard part is finding suitable lower bounds, the difficulty in the topological case is figuring out genus-minimising surfaces (see [2, 13] for first attempts in this direction). We will not see any of these surfaces. Rather, we will find a precise upper bound for the topological 4-genus via an operation called null-homologous twisting, which has recently received some attention [5, 8–10]. A null-homologous twist is an operation on oriented links that inserts a full twist into an even number $2m$ of parallel strands, m of which point upwards, and m of which point downwards (see, for example, Figure 3). Throughout this paper,

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we will use the term *twist* for a null-homologous twist. The case of two strands corresponds to a simple crossing change. For a knot K, we define the *untwisting number* $t(K)$ to be the minimal number of twists needed to transform K into the trivial knot, as in [5]. Relying on Freedman's disc theorem [3], McCoy proved that the untwisting number is an upper bound for the topological 4-genus of knots [9]. This is the tool we use to construct the genus-minimising surfaces in Theorem 1.

Let us take another look at the resemblance of the smooth and topological setting. Writing s and u for the Rasmussen invariant and unknotting number, respectively, it follows from the (smooth) local Thom conjecture that the inequalities

$$s(K)/2 \leqslant g_s(K) \leqslant u(K),$$

which hold for all knots K, become equalities for all torus knots:

$$s(T(p,q))/2 = g_s(T(p,q)) = u(T(p,q)).$$

We show that in the topological setting, in striking analogy, the inequalities

$$\widehat{\sigma}(K)/2 \leqslant g_t(K) \leqslant t(K),$$

which hold for all knots K, become equalities for all 3-strand torus knots:

$$\widehat{\sigma}(T(3,n))/2 = t(T(3,n)) = g_t(T(3,n)).$$

Thus in the topological setting, the untwisting number apparently takes the place that the unknotting number has in the smooth setting.

Untwisting might very well lead to the equality $g_t = \widehat{\sigma}/2 = t$ for all torus knots. For the time being, we show that the equality is off by at most 1 for torus knots with four and six strands.

Proposition 2. *For all odd natural numbers $n \geqslant 3$,*

$$n \leqslant \frac{\widehat{\sigma}(T(4,n))}{2} \leqslant g_t(T(4,n)) \leqslant t(T(4,n)) \leqslant \frac{2}{3} g(T(4,n)) + 2 = n+1.$$

Moreover, for all natural numbers $n \geqslant 5$ coprime to 6,

$$\frac{3n+1}{2} \leqslant \frac{\widehat{\sigma}(T(6,n))}{2} \leqslant g_t(T(6,n)) \leqslant t(T(6,n)) \leqslant \frac{3}{5} g(T(6,n)) + 3 = \frac{3n+3}{2}.$$

McCoy also developed an induction scheme that allows him to estimate the asymptotic ratio between the topological 4-genus and the Seifert genus of torus knots [9]:

$$\limsup_{p,q\to\infty} \frac{g_t(T(p,q))}{g(T(p,q))} \leqslant \frac{2}{3}.$$

Theorem 3.

$$\limsup_{p,q\to\infty} \frac{g_t(T(p,q))}{g(T(p,q))} \leqslant \frac{14}{27} \approx 0.519.$$

The proof of Theorem 1 uses a calculus for positive 3-braids introduced in [1], which we present in the next section. Sections 3 and 4 contain the proofs of Theorems 1 and 2. The latter follows from the former by untwisting torus knots on four and six strands via torus knots on three strands. Theorem 3 follows from McCoy's induction scheme, which we briefly review in the last section.

2. *A calculus for positive* 3*-braids*

Let $k_1, k_2, \ldots, k_n$ be strictly positive integers. The positive braid

$$[k_1, k_2, \ldots, k_n] := a^{k_1} b a^{k_2} b \cdots a^{k_n} b \in B_3$$

defines a link $L[k_1, k_2, \ldots, k_n]$, via its closure. For example, the torus link of type $T(3, n)$ can be written as $L[1, 1, \ldots, 1]$, where the number 1 appears n times. This notation is far from unique. The full twist on three strands can be written as

$$[1, 1, 1] = ababab = aabaab = [2, 2].$$

The double full twist can be written as

$$aab(abaaba)aab = aaabaaabaaab = [3, 3, 3].$$

In the first equality, we used the fact that the full twist $abaaba \in B_3$ commutes with all 3-braids. Adding another full twist to this, we obtain the following representative for the triple full twist:

$$aaabaaab(abaaba)aaab = aaabaaaabaaabaaaab = [3, 4, 3, 4].$$

From here, we see that the operation

$$[\ldots, x, y, \ldots] \to [\ldots, x + 1, 3, y + 1, \ldots]$$

corresponds to adding a full twist to a given positive braid on three strands. With this combinatorial calculus, we obtain the following family of positive braid presentations for iterated full twists on three strands.

Lemma 4. *For all $k \in \mathbb{N}$:*

(1) $T(3, 6k + 9) = L[3, 5^k, 4, 3, 5^k, 4]$,
(2) $T(3, 6k + 12) = L[3, 5^{k+1}, 3, 4, 5^k, 4]$,

where 5^k stands for a sequence $5, \ldots, 5$ of length k.

The proof is by induction on k, starting at zero. A repeated application of the above move yields the desired sequence of presentations for increasing powers of the full twist:

$$[3, 5, 3, 4, 4], \;\; [3, 5, 4, 3, 5, 4], \;\; [3, 5, 5, 3, 4, 5, 4], \;\; [3, 5, 5, 4, 3, 5, 5, 4], \; \ldots$$

3. *Untwisting torus knots with three strands*

Lemma 5. *For all $k \in \mathbb{N}$:*

(1) $t(T(3, 3k + 4)) \leqslant 2k + 3$,
(2) $t(T(3, 3k + 5)) \leqslant 2k + 4$.

Proof. The two statements are obviously true for $k = 0$, since the knots $T(3, 4)$ and $T(3, 5)$ can be unknotted by three and four crossing changes, respectively. Moreover, for all $k \in \mathbb{N}$, the two knots $T(3, 3k + 4)$ and $T(3k + 5)$ are related by a single crossing change, so we only need to prove (1). We will do so by considering the three special cases $T(3, 7)$, $T(3, 10)$, $T(3, 13)$ separately, and then the two families $T(3, 6k + 16)$, $T(3, 6k + 19)$.

The key observation is that the 2-braids $abbaabba$ and bb are related by a sequence of two twists, as shown in Figure 1. Here the first arrow stands for a twist on four strands, while the second arrow is a simple crossing change.

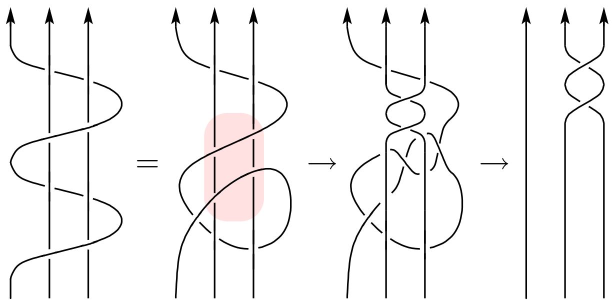

FIGURE 1 (colour online). *Sequence of two twists. The first twist is on four strands, marked in red. The second twist is a crossing changes, untying the trefoil summand in the third drawing.*

As a consequence, the double full twist on three strands,

$$(ab)^6 = abbaabbabbbb,$$

is related to the braid b^6 (and also to a^6) by a sequence of two twists. For the first knot, $T(3,7)$, we turn the braid $(ab)^7$ into a^7b by two twists, and into ab by another three crossing changes, thus showing $t(T(3,7)) \leqslant 5$. In order to deal with the other two knots, we use the notation $A = a^{-1}, B = b^{-1}$. We write

$$(ab)^{10} = (ab)^{12}BABA = (ab)^{12}ABAA = AB(ab)^{12}AA,$$

which transforms into $ABb^6a^6AA = Ab^5a^4$ by a sequence of four twists, and into Aba^2 by another three twists. The closure of the last braid is the trivial knot; this shows $t(T(3,10)) \leqslant 7$. For the knot $T(3,13)$, we observe that the braid

$$(AB)^5 = (AB)^6ba = A^3BA^3BA^3Bba = A^3BA^3BA^2$$

represents the torus knot $T(3,-5)$. Therefore, we can write

$$(ab)^{13} = (ab)^{18}A^3BA^3BA^2 = A^3(ab)^{12}BA^3(ab)^6BA^2,$$

which transforms into $A^3a^6b^6BA^3a^6BA^2 = a^3b^5a^3BA^2$ by a sequence of six twists, and into $a^3baBA^2 = a^2bA$ by another three twists. This shows $t(T(3,13)) \leqslant 9$.

We now turn to the family of torus knots $T(3,6k+16)$. Using again $A = a^{-1}, B = b^{-1}$, we write

$$T(3,6k+16) = L[ab(ab)^{6(2k+4)}(BA)^{6k+9}].$$

By Lemma 4, we have

$$(BA)^{6k+9} = A^3B(A^5B)^kA^4BA^3B(A^5B)^kA^4B,$$

which contains precisely $2k+4$ pure powers of A. We slide one double full twist $(ab)^6$ to the right of each power of A and transform it into a^6 by a sequence of $2(2k+4)$ twists, in total. This leaves us with the braid

$$aba^3B(aB)^ka^2Ba^3B(aB)^ka^2B.$$

Sliding the half-twist aba from the left to the middle yields

$$b^2A(bA)^kb^2abaBa^3B(aB)^ka^2B.$$

Then we transform the middle part $b^2abaBa^3B = b^3a^4B$ into the empty braid by three crossing changes. What remains is the braid b^2Aa^2B, whose closure is the trivial knot. Therefore $t(T(3,6k+16)) \leqslant 2(2k+4)+3 = 4k+11$, in accordance with statement (1).

The second family, $T(3,6k+19)$, works in complete analogy, using the expression

$$T(3,6k+19) = L[ab(ab)^{6(2k+5)}(BA)^{6k+12}]$$

and

$$(BA)^{6k+12} = A^3B(A^5B)^{k+1}A^3BA^4B(A^5B)^kA^4B.$$

The resulting intermediate braid, after a sequence of $2(2k+5)$ twists, is

$$aba^3B(aB)^{k+1}a^3Ba^2B(aB)^ka^2B = b^2A(bA)^{k+1}b^3Ab^2abaB(aB)^ka^2B.$$

Again, the middle part b^3Ab^2ab transform into the empty braid by three crossing changes. The remaining braid is b^2aB, whose closure is the trivial knot. This shows $t(T(3,6k+19)) \leqslant 2(2k+5)+3 = 4k+13$, in accordance with statement (1). $\square$

As mentioned before, the inequalities

$$\widehat{\sigma}(K)/2 \leqslant g_t(K) \leqslant t(K)$$

hold for all knots K. To complete the proof of Theorem 1, we estimate the maximal Levine–Tristram signature $\widehat{\sigma}$ for the 3-strand torus knots. Let K be a knot and consider the jump function

$$\delta_K(x) = \lim_{s\to x^+} \sigma_{e^{2\pi is}}(K) - \lim_{s\to x^-} \sigma_{e^{2\pi is}}(K).$$

Litherland (see the comments after [7, Proposition 1]) notes that the discontinuities of the Levine–Tristram signature of the $T(p,q)$ torus knot occur precisely at the $x \in (0,1)$ satisfying $pqx \in \mathbb{Z}$ but $px \notin \mathbb{Z}$ and $qx \notin \mathbb{Z}$. For x a discontinuity, let $pqx = pa + qb$ with $0 < a < q$. Litherland also shows that $\delta(x) = +2$ if $b < 0$ and $\delta(x) = -2$ if $b > 0$. If x is the minimal discontinuity satisfying $x \geqslant \frac{1}{2}$, then $\widehat{\sigma}(K) \geqslant \sigma(K) + \delta_K(x)$. We distinguish the following cases.

$T(3,3k+4)$ for k even: The periodicity of the ordinary signature, see [4, Theorem 5.2], implies that $\widehat{\sigma}(T(3,3k+4)) \geqslant \sigma(T(3,3k+4)) = \sigma(T(3,4)) + 4k = 6+4k$.

$T(3,3k+5)$ for k even: Similarly to the previous case, $\widehat{\sigma}(T(3,3k+5)) \geqslant 8+4k$.

$T(3,3k+4) = T(3,6l+7)$ for $k = 2l+1$ odd: The minimal $x \geqslant \frac{1}{2}$ satisfying $pqx \in \mathbb{Z}, px \notin \mathbb{Z}, qx \notin \mathbb{Z}$ is $x = (9l+11)/(3(6l+7))$. Note that $pqx = 9l+11 = 3(5l+6) + (6l+7)(-1)$, which implies that $\delta(x) = +2$. The estimate $\widehat{\sigma} \geqslant \sigma + \delta(x)$ and the periodicity of the ordinary signature imply $\widehat{\sigma}(T(3,3k+4)) \geqslant \sigma(T(3,3k+4)+2 = \sigma(T(3,3(k+1)+1) = 4(k+1)+2 = 4k+6$.

$T(3,3k+5) = T(3,6l+8)$ for $k = 2l+1$ odd: Similar to the previous case, the minimal discontinuity $x \geqslant \frac{1}{2}$ is $x = (9l+13)/(3(6l+8))$. From $pqx = 9l+13 = 3(5l+7) + (6l+8)(-1)$ it follows that $\delta(x) = 2$ and then $\widehat{\sigma}(T(3,3k+4) \geqslant 4k+8$.

Therefore, for all $k \in \mathbb{N}$,

$$\widehat{\sigma}(T(3,3k+4)) \geqslant 4k+6, \ \ \widehat{\sigma}(T(3,3k+5)) \geqslant 4k+8,$$

and Theorem 1 now follows.

4. *Untwisting torus knots with four and six strands*

To prove Proposition 2, we essentially untwist torus knots with four and six strands to torus knots on three strands, and conclude using Theorem 1.

By a similar calculation as was presented at the end of the previous section for 3-strand torus knots, one may prove that $\widehat{\sigma}(T(4,n)) \geqslant 2n$ and $\widehat{\sigma}(T(6,n)) \geqslant 3n+1$. For this, consider the signature (for $T(4,4k+3)$ and $T(6,k+5)$), or the Levine–Tristram signature after the

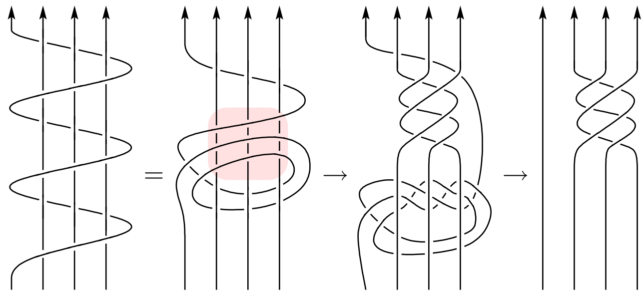

FIGURE 2 (colour online). *Sequence of four twists. The first twist is on six strands, marked in red. The other three twists are crossing changes, untying the $T(3,4)$ summand in the third drawing.*

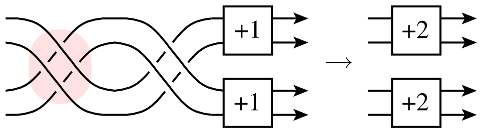

FIGURE 3 (colour online). *Untwisting a full twist on four strands, marked in red. The numbers $+1(+2)$ stand for a (double) positive full twist.*

first jump after 1/2 (for $T(4,4k+1)$), or after the second jump (for $T(6,6k+1)$). We note (without proof) that the stated inequalities for $\widehat{\sigma}$ are in fact equalities.

Now, let us show that $t(T(4,n)) \leqslant n+1$. Denote the standard Artin generators of the braid groups B_4 by a,b,c. The crucial move is to transform three full twists on four strands, $(abc)^{12}$, into four full twists on three strands, $(bc)^{12}$ (or $(ab)^{12}$), by four twist operations. This can be seen by composing the braids in Figure 2 with $(bc)^9$.

Hence, for $n = 12k+\varepsilon$, with $\varepsilon \in \{\pm1,\pm3,\pm5\}$, one may change $T(4,n) = L[(abc)^n]$ by $4k$ twists into $L[(ab)^{12k}(abc)^{\varepsilon}] =: K_\varepsilon$. We now consider the possible values of ε one by one, showing $t(T(4,n)) \leqslant n+1$ in each case, thus completing the proof of the first part of Proposition 2.

- K_1 is in fact $T(3,12k+1)$, which may be untwisted by $8k+1$ twists, as established previously. In total, $t(T(4,n)) = n$.
- Similarly $K_{-1} = T(3,12k-1)$, which may be untwisted by $8k$ twists, resulting in $t(T(4,n)) \leqslant n+1$.
- $K_3 = L[(ab)^{12k}(abc)^3] = L[(ab)^{12k}aba^2b^2ab] = T(3,12k+4)$, which may be untwisted by $8k+3$ twists. In total $t(T(4,n)) = n$.
- Similarly, $K_{-3} = T(3,12k-4)$, which may be untwisted by $8k-2$ twists, giving a total of $t(T(4,n)) \leqslant n+1$.
- K_5 can be transformed into K_1 by a twist on two strands and four crossing changes (cf. Figure 3), in total $t(T(4,n)) \leqslant n+1$.
- Similarly, K_{-5} can be transformed into $T(3,12(k-1)+1)$ by five twists, resulting in $t(T(4,n)) \leqslant n+1$.

This concludes the proof of the first half of Proposition 2.

To show $t(T(6,n)) \leqslant (3n+3)/2$, denote the standard Artin generators of B_6 by a,b,c,d,e, respectively. The full twist $(abcde)^6$ on six strands may be transformed by a single twist into $(ab)^6(de)^6$, see Figure 3 for the analogous operation on four instead of six strands. Applying this k times to $T(6,6k\pm1)$ yields the connected sum of two copies of $T(3,6k\pm1)$, which is finished off using Theorem 1. Summing up, the second half of Proposition 2 follows.

5. *Asymptotic genus ratio*

The key point in McCoy's induction scheme is that a positive full twist in a braid with $2n$ strands can be transformed into two parallel copies of positive double full twists in n strands, with a single twist operation (see [**9**, Lemma 13]). This is shown in Figure 3, for $n = 2$, and was used in the previous section for $n = 2$ and $n = 3$. Similarly, a single twist operation transforms the torus knot $T(2n, 2n + 1)$ into the connected sum of two copies of the torus knot $T(n, 2n + 1)$. This is seen by adding the word $\sigma_1\sigma_2 \ldots \sigma_{2n-1}$ to both braids in the same figure.

When iterating this operation on successive powers of two, one gets $2/3$ as an upper bound for the asymptotic ratio g_t/g for torus knots with increasing parameters. We will apply the same procedure, starting from braids with three strands, successively multiplying the strand number by 2:

(1) $T(6, 7)$ transforms into the disjoint union of two copies of $T(3, 7)$ by one twist;
(2) $T(12, 13)$ transforms into the disjoint union of two copies of $T(6, 13)$ by one twist, then into the disjoint union of four copies of $T(3, 13)$ by 4 more twists;
(3) $T(24, 25)$ transforms into the disjoint union of two copies of $T(12, 25)$ by one twist, then into the disjoint union of eight copies of $T(3, 25)$ by $4(1 + 4) = 20$ more twists,
(4) $T(3 \cdot 2^k, 3 \cdot 2^k + 1)$ transforms into the disjoint union of 2^k copies of $T(3, 3 \cdot 2^k + 1)$ by a total number of $1 + 4 + 16 + \cdots + 4^{k-1} = 1/3 \cdot (4^k - 1)$ twists.

By Theorem 1, the untwisting number of $T(3, 3 \cdot 2^k + 1)$ is of the order

$$2/3 \cdot (3 \cdot 2^k) = 2^{k+1}.$$

We conclude that the untwisting number of $T(3 \cdot 2^k, 3 \cdot 2^k + 1)$ is bounded above by an expression of the order

$$1/3 \cdot (4^k - 1) + 2^k \cdot 2^{k+1} \approx (1/3 + 2) \cdot 4^k,$$

while its Seifert genus is of the order

$$1/2 \cdot (3 \cdot 2^k)^2 = 9/2 \cdot 4^k,$$

by the well-known genus formula $g(T(p, q)) = 1/2 \cdot (p - 1)(q - 1)$. In summary,

$$\limsup_{k\to\infty} \frac{g_t(T(3 \cdot 2^k, 3 \cdot 2^k + 1))}{g(T(3 \cdot 2^k, 3 \cdot 2^k + 1))} \leqslant \frac{1/3 + 2}{9/2} = \frac{14}{27}.$$

The existence of the more general upper bound,

$$\limsup_{p,q\to\infty} \frac{g_t(T(p, q))}{g(T(p, q))} \leqslant \frac{14}{27},$$

follows from a general principle on subadditive functions, see the proof of [**9**, Theorem 5], or the paragraph preceding [**2**, Theorem 2]. We are left with the strong belief that the ratio tends to $1/2$, in accordance with the asymptotic behaviour of the signature invariant:

$$\lim_{p,q\to\infty} \frac{\sigma(T(p, q))}{2g(T(p, q))} = \frac{1}{2}.$$

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