Semigroup Forum
https://doi.org/10.1007/s00233-020-10102-9

RESEARCH ARTICLE

Numerical semigroups generated by primes

Michael Hellus[1] · Anton Rechenauer[2] · Rolf Waldi[1]

Received: 25 October 2019 / Accepted: 4 March 2020

Abstract

Let $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ be the consecutive prime numbers, S_n the numerical semigroup generated by the primes not less than p_n and u_n the largest irredundant generator of S_n. We will show, that $u_n \sim 3p_n$. Similarly, for the largest integer f_n not contained in S_n, by computational evidence (https://www.uni-regensburg.de/Fakultaeten/nat_Fak_I/Hellus/table_1.pdf) we suspect that (1) f_n is an odd number for $n \geq 5$ and (2) $f_n \sim 3p_n$; further (3) $4p_n > f_{n+1}$ for $n \geq 1$. If f_n is odd for large n, then $f_n \sim 3p_n$. In case $f_n \sim 3p_n$ every large even integer x is the sum of two primes. If $4p_n > f_{n+1}$ for $n \geq 1$, then the Goldbach conjecture holds true. Further, Wilf's question in Wilf (Am Math Mon 85:562–565, 1978) has a positive answer for the semigroups S_n.

Keywords Numerical semigroup · Diophantine Frobenius problem · Goldbach conjecture · Wilf's conjecture on numerical semigroups

1 Introduction

A *numerical semigroup* is an additively closed subset S of $\mathbb{N}$ with $0 \in S$ and only finitely many positive integers outside from S, the so-called *gaps* of S. The *genus* g of S is the number of its gaps. The set $E = S^* \setminus (S^* + S^*)$, where $S^* = S \setminus \{0\}$, is the (unique) minimal system of generators of S. Its elements are called the *atoms*

Communicated by Fernando Torres.

✉ Michael Hellus
michael.hellus@mathematik.uni-regensburg.de

Anton Rechenauer
antonrechenauer@gmail.com

Rolf Waldi
rolf.waldi@mathematik.uni-regensburg.de

1 Fakultät für Mathematik, Universität Regensburg, Regensburg, Germany

2 Regensburg, Germany

Published online: 28 May 2020

of S; their number e is the *embedding dimension* of S. The *multiplicity* of S is the smallest element p of S^*.

From now on we assume that $S \neq \mathbb{N}$. Then the greatest gap f is the *Frobenius number* of S. Since $(f+1)+\mathbb{N} \subseteq S^*$ we have $(p+f+1)+\mathbb{N} \subseteq p+S^*$, hence the atoms of S are contained in the interval $[p, p+f]$.

For our investigation of certain numerical semigroups S generated by prime numbers, the fractions

$$\frac{f}{p}, \frac{1+f}{p}, \frac{g}{1+f} \text{ and } \frac{e-1}{e}$$

will play a role. For general S, what is known about these fractions?

First of all it is well known and easily seen that

$$\frac{1}{2} \leq \frac{g}{1+f} \leq \frac{p-1}{p},$$

and both bounds for $\frac{g}{1+f}$ are attained.

However, the following is still open:

Wilf's question [17]: Is it (even) true that

$$\frac{g}{1+f} \leq \frac{e-1}{e} \tag{1}$$

for every numerical semigroup?

A partial answer is given by the following result of Eliahou:

[4, Corollary 6.5] If $\frac{1+f}{p} \leq 3$, then $\frac{g}{1+f} \leq \frac{e-1}{e}$.

In [18], Zhai has shown that $\frac{1+f}{p} \leq 3$ holds for almost all numerical semigroups of genus g (as g goes to infinity).

Therefore, for randomly chosen S, one has $\frac{g}{1+f} \leq \frac{e-1}{e}$ almost surely.

We shall consider the following semigroups: Let $p_1 = 2,\ p_2 = 3,\ p_3 = 5, \ldots$ be the sequence of prime numbers in natural order and let S_n, for $n \geq 1$, be the numerical semigroup generated by all prime numbers not less than p_n; the multiplicity of S_n is p_n and we denote the aforementioned invariants of S_n by g_n, f_n, e_n and E_n. Since S_{n+1} is a subsemigroup of S_n it is clear that $f_n \leq f_{n+1}$ for all $n \geq 1$. The atoms of S_n are contained in the interval $[p_n, p_n + f_n]$; conversely, each odd integer from $S_n \cap [p_n, 3p_n[$ is an atom of S_n.

As a major result we will see that Wilf's question has a positive answer for S_n. Further g_n/p_n converges to 5/2 for $n \to \infty$.

The prime number theorem suggests that there should be—like for the sequence (p_n)—some asymptotic behavior of (g_n), (f_n) and (e_n).

Based on the list $f_1, f_2, \ldots, f_{2000}$ from [8], extensive calculations (cf. our table 1 in [9]) gave evidence for the following three conjectures:

(C1) $f_n \sim 3p_n$, i. e. $\lim_{n\to\infty} \frac{f_n}{p_n} = 3$, as already observed by Kløve [12], see also the comments in [6, p. 56]; note that Kløve works with *distinct* primes, therefore his conjecture is formally stronger than ours, however see also [10, comment by user "Emil Jeřábek", Apr 4 '12].

In Proposition 1, we will show that

$$3p_n - 6 \le f_n. \tag{2}$$

(C2) $f_{n+1} < 4p_n$ for all $n \ge 1$.

and

$3p_n < f_{n+1}$ for $n \ge 3$.

It is immediate from (2) that at least

$$3p_n \le 3(p_{n+1} - 2) \le f_{n+1} \text{ for } n \ge 2.$$

As already noticed in [12] and in [10, answer by user "Woett", Apr 3 '12], both conjectures (C1) and (C2) are closely related to Goldbach's conjecture. As we will see in Proposition 4, (C1) would be a consequence of conjecture

(C3) f_n is odd for $n \ge 5$.

Notice again, that a conjecture similar to (C3) was already formulated in [12], however for the (related) notion 'threshold of completeness' for the sequence of all prime numbers, in the sense of [6].

Figure 1 indicates, that $\lim_{n\to\infty} \frac{f_n}{p_n} = 3$ should be true.

As for (C2), by Figs. 1 and 2, evidently $4p_n - f_{n+1}$ should stay positive for all time.

Observations Numerical experiments suggest that similiar conjectures can be made if one restricts the generating sequence to prime numbers in a fixed arithmetic progression $a + kd$ for $(a, d) = 1$. In such a case the limit of $\frac{f_n}{p_n}$ would apparently be $d + 1$ (d even) or $2d + 1$ (d odd), see Fig. 3, and table 2 in [11].

The following version of Vinogradov's theorem is due to Matomäki, Maynard and Shao. It is fundamental for the considerations in this paper.

[13, Theorem 1.1] Let $\theta > \frac{11}{20}$. Every sufficiently large odd integer n can be written as the sum $n = q_1 + q_2 + q_3$ of three primes with the restriction

$$\left| q_i - \frac{n}{3} \right| \le n^\theta \text{ for } i = 1, 2, 3.$$

Of course we could have used just as well one of the predecessors of this theorem, see the references in [13].

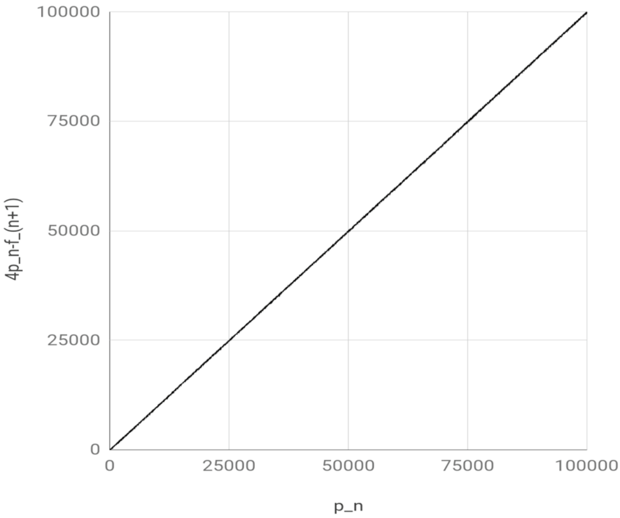

Fig. 1 $4p_n - f_{n+1}$ versus p_n

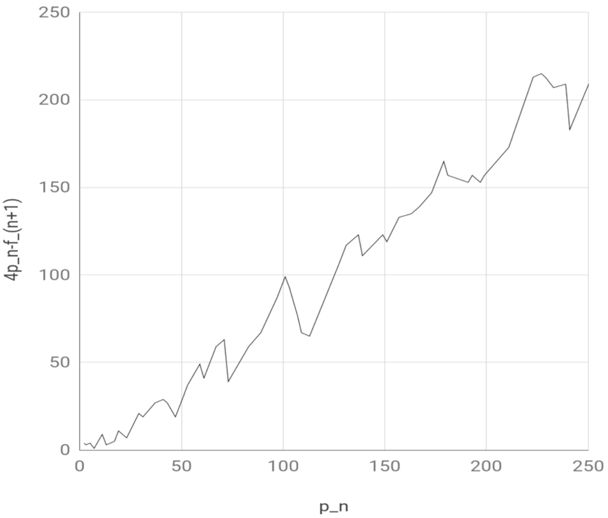

Fig. 2 $4p_n - f_{n+1}$ versus p_n

2 Variants of Goldbach's conjecture

For $x, y \in \mathbb{Q}$, $x \leq y$ we denote by $[x, y]$ the 'integral interval'

$$\{n \in \mathbb{Z} | x \leq n \leq y\},$$

accordingly we define $[x, y[$, $]x, y]$, $]x, y[$, $[x, \infty[$.

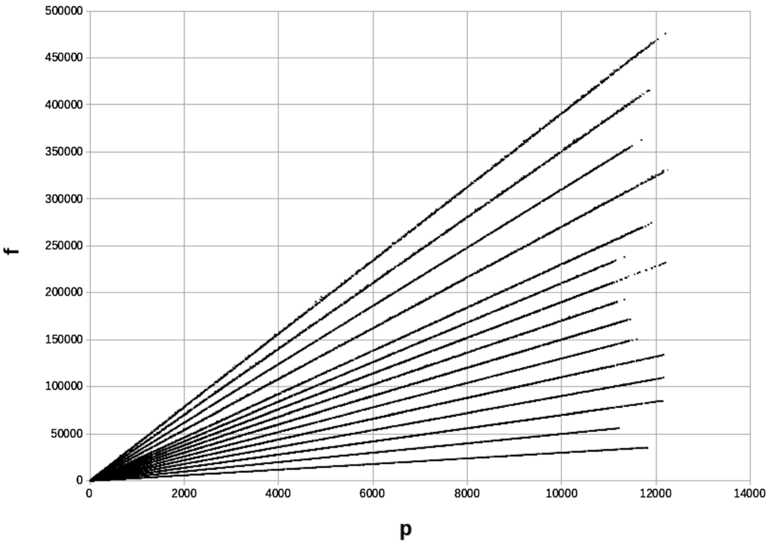

Fig. 3 f versus p for some series of semigroups as in the 'Observations'

For $x \geq 2$ we define S_n^x to be the numerical semigroup generated by the primes in the interval $I_n^x := [p_n, x \cdot p_n[$ and f_n^x its Frobenius number.

A minor step towards a proof of conjecture (C1) is

Proposition 1

$$f_n \geq 3p_n - 6.$$

In particular for the null sequence $r(n) := 6/p_n$ *we have*

$$\frac{f_n}{p_n} \geq 3 - r(n) \text{ for every } n \geq 1.$$

Proof For $n \geq 3$, obviously, the odd number $3p_n - 6$ is neither a prime nor the sum of primes greater than or equal to p_n, hence $3p_n - 6$ is not contained in S_n. □

Remark 1 A final (major) step on the way to (C1) would be to find a null sequence $l(n)$ such that

$$3 + l(n) \geq \frac{f_n}{p_n}.$$

Proposition 2 *If (C1) is true then every sufficiently large even number x can be written as the sum* $x = p + q$ *of prime numbers* p, q.

Addendum The prime number p can be chosen from the interval $]\frac{x}{4}, \frac{x}{2}]$.

Proof By the prime number theorem, we have $p_{n+1} \sim p_n$. (C1) implies

$$f_{n+1} \sim 3p_{n+1} \sim 3p_n,$$

i. e.

$$\lim_{n\to\infty} \frac{f_{n+1}}{p_n} = 3.$$

In particular, there exists $n_0 \geq 1$ such that $\frac{f_{n+1}}{p_n} < 4$ for all $n \geq n_0$.

It remains to show:

Lemma 1 *If $n_0 \geq 1$ is such that $\frac{f_{n+1}}{p_n} < 4$ for all $n \geq n_0$ then every even number $x > 2$ with $x > f_{n_0}$ can be written as the sum*

$$x = p + q \text{ with prime numbers } p \leq q \text{ and such that } \frac{x}{4} < p \leq \frac{x}{2}. \tag{1}$$

Proof By our hypothesis,

$$f_n \leq f_{n+1} < 4p_n < 4p_{n+1} \text{ for all } n \geq n_0$$

and hence, for $I_n := [1 + f_n, 4p_n[$ $(n \geq n_0)$,

$$[1 + f_{n_0}, \infty[= \bigcup_{n\geq n_0} I_n.$$

Therefore it suffices to prove (1) for all even numbers $x > 2$ from the interval I_n, for $n \geq n_0$.

By definition of f_n, every $x \in I_n$ can be written as the sum of primes $p \geq p_n$.

If in addition $x > 2$ is even, then, because of $f_n < x < 4p_n$, the number x is the sum of precisely two prime numbers $p \leq q$ with

$$p_n \leq p \leq q = x - p < 4p_n - p \leq 3p,$$

hence

$$\frac{x}{4} < p \leq \frac{x}{2}.$$

□

The special case $n_0 = 1$ of Lemma 1 gives

Proposition 3 *If (C2) is true then every even number $x > 2$ can be written as the sum $x = p + q$ of prime numbers $p \leq q$ as described in the Addendum above. In particular for each $n \geq 1$, $4p_n = p + q$ with primes $p_{n+1} \leq p \leq q$, implying Bertrand's postulate.* □

Lemma 2 *Let $\varepsilon > 0$. For odd N large enough, there are prime numbers q_1, q_2, q_3 with*

$$N = q_1 + q_2 + q_3$$

and such that

$$\frac{1}{3+\varepsilon} \cdot N < q_i < \frac{3+2\varepsilon}{9+3\varepsilon} \cdot N, \text{ i. e. } \left| q_i - \frac{N}{3} \right| < \frac{\varepsilon}{9+3\varepsilon} \cdot N \text{ for } i = 1, 2, 3.$$

Proof The claim follows immediately from [13, Theorem 1.1], since $\theta := \frac{3}{5} > \frac{11}{20}$ and, for large N, $N^{\frac{3}{5}} < \frac{\varepsilon}{9+3\varepsilon} \cdot N$. □

Lemma 3 *Let $\varepsilon > 0$. Then for large n, each odd integer $N \geq (3+\varepsilon)p_n$ is contained in S_{n+1}. In particular, for large n*

$$\begin{aligned} f_{n+1} &< (3+\varepsilon)p_n \text{ if } f_{n+1} \text{ is odd, and} \\ f_{n+1} &< (3+\varepsilon)p_n + p_{n+1} \text{ if } f_{n+1} \text{ is even,} \end{aligned}$$

since then $f_{n+1} - p_{n+1}$ is odd and not in S_{n+1}.

Proof Since N is odd and large for large n, by Lemma 2 there exist prime numbers q_1, q_2, q_3 with

$$N = q_1 + q_2 + q_3$$

and such that

$$\frac{N}{3+\varepsilon} < q_i \text{ for } i = 1, 2, 3.$$

By assumption, $\frac{N}{3+\varepsilon} \geq p_n$, hence

$$q_i > p_n, \text{ i. e. } q_i \geq p_{n+1}$$

for the prime numbers q_i. This implies $N = q_1 + q_2 + q_3 \in S_{n+1}$. □

Proposition 4 *If the Frobenius number f_n is odd for all large n, then $f_n \sim 3p_n$. In particular, conjecture (C3) implies conjecture (C1).*

Proof This is immediate from Proposition 1 and Lemma 3. □

For a similar argument, see [10, answer by user "Anonymous", Apr 5'12].

Remark 2

(a) It is immediate from Lemma 3 that

$$\limsup_{n\to\infty} \frac{f_n}{p_n} \leq 4.$$

As a consequence, a proof of $\limsup_{n\to\infty} \frac{f_n}{p_n} \neq 4$ would imply the binary Goldbach conjecture for large x with the Addendum from above—see Lemma 1 and the proof of Proposition 2.

(b) The estimate $\limsup_{n\to\infty} \frac{f_n}{p_n} \leq 4$ together with a sketch of proof was already formulated in [10, comment by user "François Brunault" (Apr 6 '12) to answer by user "Anonymous" (Apr 5 '12)]. Our proof is essentially an elaboration of this sketch.

(c) Lemma 3 shows that

$$f_{n+1} < 5p_{n+1} \text{ for large } n.$$

Because of $p_{n+1} < 2p_n$ (Bertrand's postulate) this implies also that there exists a constant C with

$$f_{n+1} < Cp_n \text{ for all } n. \tag{2}$$

Conjecture (C2) says that in (2) one can actually take $C = 4$.

Notice that (2) already follows from [1, Lemma 1].

Problem Find an explicit pair (n_0, C_0) of numbers such that

$$f_{n+1} < C_0 \cdot p_n \text{ for every } n \geq n_0.$$

Next we shall study the asymptotic behavior of the set of atoms of S_n.

Lemma 2 implies

Lemma 4 *Let $\varepsilon > 0$. Then $S_n = S_n^{3+\varepsilon}$ for large n.*

Proof It suffices to prove the claim for arbitrarily small values of ε:

First we show that, if $\varepsilon < 3$, then

$$S_{n+1}^{3+\varepsilon} \subseteq S_n^{3+\varepsilon}$$

for large n. For this it suffices to show that every prime number p on the interval $[p_{n+1}, (3+\varepsilon)p_{n+1}[$ is in $S_n^{3+\varepsilon}$:

Firstly, $p \geq p_{n+1} > p_n$.

Now we distinguish two cases:

I $p < (3+\varepsilon)p_n$: Then $p \in I_n^{3+\varepsilon}$, hence $p \in S_n^{3+\varepsilon}$.

II $p \geq (3+\varepsilon)p_n$: For n large enough, by Lemma 2 there exist prime numbers q_1, q_2, q_3 with

$$p = q_1 + q_2 + q_3$$

and such that

$$p_n \overset{\text{II}}{\leq} \frac{p}{3+\varepsilon} < q_i < \frac{3+2\varepsilon}{9+3\varepsilon} p \text{ for } i = 1, 2, 3.$$

By Chebyshev, Bertrand's postulate $p_{n+1} < 2p_n$ holds. Therefore,

$$p \overset{\text{hypothesis}}{<} (3+\varepsilon)p_{n+1} < (6+2\varepsilon)p_n$$

and hence

$$q_i < \frac{3+2\varepsilon}{9+3\varepsilon} p < \frac{3+2\varepsilon}{9+3\varepsilon}(6+2\varepsilon)p_n < (3+\varepsilon)p_n,$$

if $\varepsilon < 3$. It follows that

$$q_i \in [p_n, (3+\varepsilon)p_n[\text{ for } i = 1, 2, 3 \text{ and hence}$$
$$p = q_1 + q_2 + q_3 \in S_n^{3+\varepsilon},$$

which proves the above claim.

Recursively, we get from $S_{n+1}^{3+\varepsilon} \subseteq S_n^{3+\varepsilon}$ that

$$p_k \in S_k^{3+\varepsilon} \subseteq S_n^{3+\varepsilon} \text{ for all } k \geq n.$$

Therefore,

$$S_n = S_n^{3+\varepsilon}.$$

□

For $x \geq 0$ let $\pi(x)$ be the number of primes less than or equal to x. Applying Lemma 4, the prime number theorem (PNT) yields

Theorem 1 *Let u_n be the largest atom of S_n. Then*

$$\pi(u_n) \sim 3n, u_n \sim 3p_n \text{ and } e_n \sim 2n.$$

Proof By Lemma 4, for each $\varepsilon > 0$ there is an $n(\varepsilon) > 0$ such that $E_n \subseteq [p_n, (3+\varepsilon)p_n]$ for all $n \geq n(\varepsilon)$. On the other hand, the primes from $[p_n, 3p_n]$ are atoms of S_n. Hence

$$\pi(3p_n) \leq \pi(u_n) \leq \pi((3+\varepsilon)p_n) \text{ for } n \geq n(\varepsilon). \tag{3}$$

By the PNT, $\pi(3p_n) \sim 3n$ and $\pi((3+\varepsilon)p_n) \sim (3+\varepsilon)n$ for $\varepsilon > 0$. Hence (3) implies $\pi(u_n) \sim 3n$.

From $p_n \leq u_n \leq (3+\varepsilon)p_n$ we get $\log u_n \sim \log p_n$, hence again by the PNT,

$$u_n \sim \pi(u_n) \cdot \log u_n \sim 3n \cdot \log p_n \sim 3p_n.$$

Finally, $e_n = \pi(u_n) - n + 1 \sim 3n - n + 1 \sim 2n$. □

By [4, Cor. 6.5], for arbitrary numerical semigroups S, Wilf's inequality $\frac{g}{1+f} \leq \frac{e-1}{e}$ holds, whenever $f < 3 \cdot p$. Further by [18], the latter is true for almost every numerical semigroup of genus g (as g goes to infinity).

In contrast, according to table 1 in [9], for the semigroups S_n, the relation $f_n < 3 \cdot p_n$ seems to occur extremely seldom, but over and over again (see Fig. 4).

The following considerations are related to [10, answer by user "Aaron Meyerowitz", Apr 3 '12]:

Let $f_n < 3 \cdot p_n$. Then the odd number $3 \cdot p_n + 6$ is in S_n, but not a prime; hence $p_{n+1} \leq p_n + 6$.

1. If $p_{n+1} = p_n + 4$, since $3 \cdot p_n + 6 \in S_n$ is not a prime, $p_n + 6$ must be prime.
2. If $p_{n+1} = p_n + 6$, then the odd numbers $3p_n + 2$ and $3p_n + 4$ must be atoms in S_n, hence primes.

In any case:

Nota bene If $f_n < 3p_n$, then there is a twin prime pair within $[p_n, 3p_n + 4]$.

So we cannot expect to prove, that $f_n < 3p_n$ happens infinitely often, since this would prove the *twin prime conjecture*, that there are infinitely many twin prime pairs. Another consequence would be that

$$\liminf_{n\to\infty} \frac{f_n}{p_n} = 3,$$

since one always has that this limit inferior is ≥ 3, by Proposition 1.

The next section is attended to Wilf's question mentioned above.

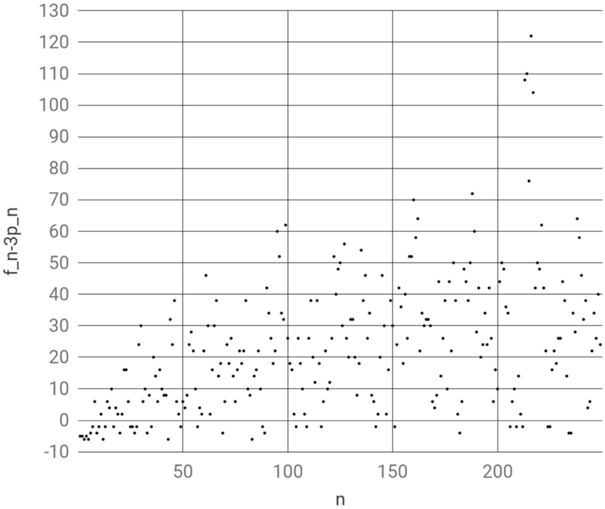

Fig. 4 $f_n - 3p_n$ versus n

3 The question of Wilf for the semigroups S_n

Proposition 5 *For the semigroups S_n, Wilf's (proposed) inequality*

$$\frac{g_n}{1+f_n} \le \frac{e_n - 1}{e_n} \tag{1}$$

holds.

Proof For $n < 429$, have a look at table 1 in [9]. Now let $n \ge 429$.

Instead of (1), we would rather prove the equivalent relation

$$e_n(1+f_n-g_n) \ge 1+f_n. \tag{2}$$

According to [4, Cor. 6.5] we may assume, that $3p_n < 1+f_n$. Hence the primes in the interval $[p_n, 3p_n[$ are elements of S_n lying below $1+f_n$, and in fact, they are atoms of S_n as well. This implies for the *prime-counting function* π

$$e_n(1+f_n-g_n) \ge (\pi(3p_n)-n+1)^2. \tag{3}$$

By Rosser and Schoenfeld [15, Theorem 2] we have

$$\pi(x) < \frac{x}{\log x - \frac{3}{2}} \text{ for } x > e^{\frac{3}{2}}, \text{ and} \tag{4}$$

$$\pi(x) > \frac{x}{\log x - \frac{1}{2}} \text{ for } x \ge 67. \tag{5}$$

From (4) and (5) we will get in a moment:

$$2n < \pi(3p_n) < 3n \text{ for } n \ge 429. \tag{6}$$

Proof of (6) Since the function $\lambda(x) := 3 \cdot \frac{\log x - \frac{3}{2}}{\log(3x) - \frac{1}{2}}$ is strictly increasing for $x > 1$, we get for $n \ge 429$, i. e. $p_n \ge 2971$

$$\pi(3p_n) \overset{(5)}{>} \frac{3p_n}{\log(3p_n) - \frac{1}{2}} \overset{(4)}{>} \pi(p_n) \cdot \lambda(p_n) \ge n \cdot \lambda(2971) > 2n, \text{ and}$$

$$\pi(3p_n) \overset{(4)}{<} \frac{3p_n}{\log p_n + \log 3 - \frac{3}{2}} < \frac{3p_n}{\log p_n - \frac{1}{2}} \overset{(5)}{<} 3n.$$

In particular, by (3) and (6)

$$e_n(1+f_n-g_n) \overset{(3)}{\ge} (\pi(3p_n)-n+1)^2 \overset{(6)}{\ge} (n+2)^2.$$

It remains to prove

Lemma 5 *If* $n \geq 429$, *then*

$$f_n < n^2.$$

Proof Let $N \leq a_1 < \cdots < a_N$ be positive integers with $(a_1, \ldots, a_N) = 1$, $S = \langle a_1, \ldots, a_N \rangle$ the numerical semigroup generated by these numbers and f its Frobenius number. Then, by Selmer [16] we have the following theorem (see the book [14] of Ramírez Alfonsín). It is an improvement of a former result [5, Theorem 1] of Erdős and Graham.

[14, Theorem 3.1.11]

$$f \leq 2 \cdot a_N \left\lfloor \frac{a_1}{N} \right\rfloor - a_1. \tag{7}$$

We will apply this to the semigroup $S_n^3 \subseteq S_n$ generated by the primes

$$p_n = a_1 < p_{n+1} = a_2 < \ldots < p_{N+n-1} = a_N$$

in the interval $I_n^3 = [p_n, 3p_n[$, with Frobenius number f_n^3, hence

$$N = \pi(3p_n) - n + 1, a_N = p_{\pi(3p_n)} = \text{the largest prime in } I_n^3.$$

By [15, Theorem 3, Corollary, (3.12)] we have

$$p_n > n \log n \geq n \log 429 > 6n \overset{(6)}{>} N,$$

hence the above theorem can be applied.

By (6) and (7), $p_{\pi(3p_n)} \overset{(6)}{<} p_{3n}$ and

$$f_n \leq f_n^3 \overset{(7)}{<} 2 \cdot p_{\pi(3p_n)} \cdot \frac{p_n}{\pi(3p_n) - n + 1} \overset{(6)}{<} 2 \cdot p_{3n} \cdot \frac{p_n}{n+2}.$$

It remains to show, that $2 \cdot p_{3n} \cdot \frac{p_n}{n+2} < n^2$ for $n \geq 429$:

By [15, Theorem 3, Corollary, (3.13)], we have

$$p_k < k(\log k + \log \log k) \text{ for } k \geq 6. \tag{8}$$

We consider the function

$$\lambda_2(x) := 6 \cdot (\log(3x) + \log \log(3x)) \cdot (\log x + \log \log x).$$

Since $\frac{\log(3x)}{x}$ is decreasing and $\lambda_2'(x) < 48 \cdot \frac{\log(3x)}{x}$ for $x \geq 3$, we obtain

$$\lambda_2'(x) < 48 \cdot \frac{\log 1287}{429} < 1 = (x+2)' \text{ for } x \geq 429; \text{further } \lambda_2(429) < 431.$$

Elementary calculus yields

$$\lambda_2(x) < x + 2 \text{ for } x \geq 429. \tag{9}$$

Hence

$$2 \cdot p_{3n} \cdot p_n \overset{(8)}{<} n^2 \cdot \lambda_2(n) \overset{(9)}{<} n^2 \cdot (n+2) \text{ for } n \geq 429.$$

□

See also Dusart's thèse [3] for more estimates like (4), (5) and (8).

Remark 3 Looking at table 3 in [7] we see, that even

$$\pi(3p_n) > 2n \text{ for } n > 8 \text{ and } \pi(3p_n) < 3n \text{ for n} > 1$$

(which may be found elsewhere), and

$$f_n \leq n^2 \text{ for } n \neq 5.$$

At last we will see that, apparently, the quotient $\frac{g_n}{1+f_n}$ should converge to $\frac{5}{6}$ (whereas $\lim_{n\to\infty} \frac{e_n-1}{e_n} = 1$, since $e_n \sim 2n$ by our Theorem).

Proposition 6 *The quotient* $\frac{g_n}{p_n}$ *converges, and*

$$\lim_{n\to\infty} \frac{g_n}{p_n} = \frac{5}{2}.$$

Proof For that, we consider the proportion $\alpha_k(n)$ of gaps of S_n among the integers in $[k \cdot p_n, (k+1) \cdot p_n]$, $(k, n \geq 1)$. Besides [13, Theorem 1.1], we shall need the following similar result about the representation of *even* numbers as the sum of two primes:

[2, Theorem 1, Corollary] Let $\varepsilon > 0$ and $A > 0$ be real constants. For $N > 0$ let $E(N)$ be the set of even numbers $2m \in [N, 2N]$, which cannot be written as the sum $2m = q_1 + q_2$ of primes q_1 and q_2 with the restriction

$$|q_j - m| \leq m^{\frac{5}{8}+\varepsilon} \text{ for } j = 1, 2.$$

Then there is a constant $D > 0$ such that $\#E(N) < D \cdot N/(\log N)^A$.

From these two facts together with the prime number theorem, we conclude the following asymptotic behavior of the numbers $\alpha_k(n)$, as n goes to infinity:

$$\alpha_0(n) \to 1, \alpha_1(n) \to 1, \alpha_2(n) \to \frac{1}{2} \text{ and } \alpha_k(n) \to 0 \text{ for } k \geq 3.$$

Hence

$$\lim_{n\to\infty} \frac{g_n}{p_n} = 1 + 1 + \frac{1}{2} = \frac{5}{2}.$$

(Notice that for large n, by Lemma 3 we have $f_n < 5p_n$, hence $\alpha_k(n) = 0$ for $k \geq 5$.)

□

Remark 4 Under the assumption $\lim_{n\to\infty} \frac{p_n}{f_n} = \frac{1}{3}$ (C1) (which should be true by computational evidence), by Proposition 6,

$$\lim_{n\to\infty} \frac{g_n}{1+f_n} = \frac{5}{6}.$$

Remark 5 Let $f_{n,e}$ be the largest even gap of S_n. Our computations (see table 1 in [9]) suggest that $f_{n,e} \sim 2p_n$. In this case, by Proposition 1 and Proposition 4, f_n is odd for large n and conjecture (C1) holds.

Acknowledgements Open Access funding enabled and organized by Projekt DEAL. We thank F. Brunault, O. Forster and K. Matomäki for valuable hints.

References

1. Benkoski, S.J., Erdős, P.: On Weird and pseudoperfect numbers. Math. Comput. **28**, 617–623 (1974)
2. Coppola, G., Laporta, M.B.S.: On the representation of even integers as the sum of two almost equal primes. Rend. Sem. Mat. Univ. Pol. Torino **53**, 245–252 (1995)
3. Dusart, P.: Autour de la fonction qui compte le nombre de nombres premiers, thèse, Université de Limoges, p. 171 (1998)
4. Eliahou, S.: Wilf's conjecture and Macaulay's theorem. J. Eur. Math. Soc. **20**, 2105–2129 (2018)
5. Erdős, P., Graham, R.L.: On a linear diophantine problem of Frobenius. Acta Arithm. **21**, 399–408 (1972)
6. Erdős, P., Graham, R.L.: Old and new problems and results in combinatorial number theory. Monographies de L'Enseignement Mathématique **28**, 53–60 (1980)
7. https://www.uni-regensburg.de/Fakultaeten/nat_Fak_I/Hellus/table_3.pdf. Accessed 21 Oct 2019
8. https://oeis.org/A180306. Accessed 27 May 2019
9. https://www.uni-regensburg.de/Fakultaeten/nat_Fak_I/Hellus/table_1.pdf. Accessed 21 Oct 2019
10. https://mathoverflow.net/questions/93002/finite-sums-of-prime-numbers-geq-x. Accessed 27 May 2019
11. https://www.uni-regensburg.de/Fakultaeten/nat_Fak_I/Hellus/table_2.pdf. Accessed 21 Oct 2019
12. Kløve, T.: Sums of distinct primes. Nordisk Mat. Tidskr. **21**, 138–140 (1974)
13. Matomäki, K., Maynard, J., Shao, X.: Vinogradov's theorem with almost equal summands. Proc. Lond. Math. Soc. **115**, 327–347 (2017)
14. Ramírez Alfonsín, J.L.: The Diophantine Frobenius Problem. Oxford Lecture Series in Mathematics and Its Applications. Oxford University Press, New York (2005)
15. Rosser, J .B., Schoenfeld, L.: Approximate formulas for some functions of prime numbers. Ill. J. Math. **6**(1), 64–94 (1962)
16. Selmer, E.S.: On the linear diophantine problem of Frobenius. J. Reine Angew. Math. **293/294**(1), 1–17 (1977)
17. Wilf, H.S.: A circle-of-lights algorithm for the "money-changing problem". Am. Math. Mon. **85**, 562–565 (1978)

18. Zhai, A.: Fibonacci-like growth of numerical semigroups of a given genus. Semigroup Forum **86**, 634–662 (2013)