The Jones R&D Growth Model: Comment on Stability

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Abstract

The dynamics of most prominent endogenous growth models are well understood. One notable exception is the Jones (1995) R&D growth model. This paper provides an analytical treatment of this model's transitional dynamics. It is shown that, given constant returns to labor in R&D (as conventionally assumed in R&D growth models), a unique trajectory converging to the balanced growth path exists. The equilibrium growth path can be monotonic or oscillatory. Moreover, applying a theorem from Arnold (2004), this result can be used to characterize the dynamic behavior of the multi-country open-economy version of the model.

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1 Introduction

New growth theory has greatly improved our understanding of the causes of technical change and of its consequences for economic growth. Three direct extensions of the Solow (1956) model are among the most important ones: Lucas' (1988) human capital model, Romer's (1990) model of endogenous technological change, and Jones' (1995) R&D growth model with diminishing returns to knowledge in R&D. While Lucas (1988) emphasizes the role of human capital accumulation and human capital externalities in the growth process, Romer (1990) focuses on innovation via R&D. Jones (1995) makes the Romer approach compatible with the observations that rising employment in R&D in the postwar period did not cause an acceleration of economic growth and that the influence of policy measures on long-term growth appears to be weaker than the Romer model indicates.

From a technical point of view, a common feature of these and many other growth models is that the original papers focused on the models' balanced-growth path, and, later on, attention turned to the models' dynamics. Caballé and Santos (1993), Faig (1995), Barro and Sala-i-Martin (1995, Subsection 5.2.2), and Arnold (1997) showed that the equilibrium growth path is uniquely determined in the Lucas (1988) model without externalities emanating from human capital.[1] Benhabib and Perli (1994) demonstrated that equilibrium indeterminacy may arise if, as in Lucas (1988), one reintroduces human capital externalities. As for the Romer (1990) model, Arnold (2000a) proved the market equilibrium to be a saddle point. Exploiting a close analogy to the Lucas (1988) model without externalities, Arnold (2000b) showed that the optimal growth path is also uniquely determined.[2] By contrast, the dynamics of the Jones (1995) model are not yet fully understood. Jones (1995, Section V, 2002) considers the dynamics for a given allocation of factors of production. A complete analysis is much more demanding because the system of differential equations one has to analyze is of order four so that one has to find the roots' signs for a quartic polynomial. Eicher and Turnovsky (2001) provide some analytical and simulation results for the optimal growth path. As for the market equilibrium with optimizing behavior, Williams (1995) and Steger (2003) analyze calibrated versions of an extended model with a focus on the speed of convergence.[3] Analytical proof that the equilibrium growth path in the Jones (1995) model is uniquely determined is missing, however. The purpose of this paper is to fill this

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[1] Mulligan and Sala-i-Martin (1993), Bond, Wang, and Yip (1996), Mino (1996), and Ladrón-de-Guevara, Ortigueira, and Santos (1997) consider the model dynamics with physical capital as an input in human capital accumulation.

[2] Benhabib, Perli, and Xie (1994) demonstrated that the equilibrium may become indeterminate if one adds complementarities between intermediate goods to the Romer (1990) model. Using numerical examples, Asada, Nowak, and Semmler (1998) demonstrated that other kinds of dynamic behavior can than arise.

[3] Eicher and Turnovsky (2001) make less restrictive assumptions about the production elasticities than Jones (1995). Steger (2003) also allows for physical capital in the R&D technology.

gap in the literature. In accordance the bulk of new growth theory (see, for example, Romer, 1990, and Grossman and Helpman, 1991), we assume constant returns to labor in R&D. We show that there is, then (without any further parameter restrictions), a unique growth path converging to the balanced growth path described by Jones (1995). This equilibrium growth path may be monotonic or oscillatory. Applying a theorem from Arnold (2004), this finding can be adapted to the multi-country open economy version of the model with international trade, financial capital mobility, and international knowledge spillovers.

Section 2 briefly recapitulates the assumptions of the Jones (1995) model. Section 3 describes the steady state. In Section 4, we prove that a unique equilibrium growth path exists. Section 5 concludes.

2 Model

Consider a closed economy in continuous time populated by a continuum of mass one of identical households. The number of household members at time t, $L(t)$, is initially positive ($L(0) > 0$) and subsequently grows at a rate $\dot{L}(t)/L(t) = n$ (> 0). Each household member inelastically supplies one unit of labor and receives the same amount of consumption, $c(t)$. The households maximize the intertemporal utility function $\int_t^\infty e^{-\rho(\tau-t)}[c(\tau)^{1-\sigma} - 1]/(1-\sigma)d\tau$, where ρ (> 0) is the subjective discount rate, and σ (≥ 0) is the inverse of the intertemporal elasticity of substitution in consumption.[4] There is a single final good, which is used for consumption or investment. Capital does not depreciate. So $\dot{K} = Y - Lc$, where Y is the aggregate production of the homogeneous final good and K is the capital stock. Output is produced using labor L_Y and a set of intermediates j, which are obtained one-to-one from capital. The production function is $Y = L_Y^{1-\alpha} \int_0^A x(j)^\alpha dj$, where $x(j)$ is input of intermediate j and A is the "number" of available intermediates. α is in the interval $(0, 1)$, and $1/(1-\alpha)$ is the elasticity of substitution for any pair of intermediates. R&D enables firms to produce new intermediates. The R&D technology is $\dot{A} = A^{1-\chi}L_A/a$ ($a > 0$, $\chi > 0$), where L_A is employment in R&D. Notice that, in accordance with the bulk of new growth theory, we assume that the returns to labor in R&D are constant (see, for example, Romer, 1990, and Grossman and Helpman, 1991). Jones (1995, equations (4) and (5), p. 765) takes this R&D technology as the starting point for his analysis, but then allows for a non-unitary production elasticity. $1 - \chi$ captures the externalities emanating from R&D. We allow for "fishing-out" effects, i.e. $\chi > 1$ (see the discussion in Jones, 1995, p. 765). Innovators are protected from imitation by infinitely-lived patents, so there is monopolistic

[4]Cf. Jones (1995, eqn. (A.12), p. 782). Alternatively, one could take the sum of the household members' utilities, so that the factor e^{nt} would appear in the utility function. It is understood that instantaneous utility is logarithmic if $\sigma = 1$. In what follows, we suppress the time argument, t, wherever this does not cause confusion.

competition in the intermediate-goods sector. The other markets (for labor, final goods, and financial capital) are perfectly competitive. There is free entry into R&D. All markets clear.[5]

3 Steady state

In this section, we describe the economy's steady state. Let the homogeneous final good be the numeraire. Utility maximization yields the Ramsey rule $\dot{c}/c = (r - \rho - n)/\sigma$. Profit maximization in the final goods sector yields the inverse demand curves $w = (1-\alpha)Y/L_Y$ and $p(j) = \alpha L_Y^{1-\alpha} x(j)^{\alpha-1}$, where r is the interest rate, w is the wage rate, and $p(j)$ is the price of intermediate j. Intermediate goods producers maximize $\pi(j) \equiv [p(j) - r]x(j)$ given the final goods sector's demand for the intermediates. The monopoly price is the markup price $p(j) = r/\alpha \equiv p$ ($j \in [0, A]$). Letting x denote the corresponding demand, monopoly profits equal $\pi(j) = (1-\alpha)px \equiv \pi$. Because of symmetry with respect to the different intermediates, we have $K = Ax$, $Y = K^\alpha (AL_Y)^{1-\alpha}$, and $w = (1-\alpha)A[K/(AL_Y)]^\alpha$. Let $z \equiv Y/K$ and $\gamma \equiv cL/K$. From zero profits in the final goods sector ($Apx = Y - wL_Y = \alpha Y$), it follows that $r = \alpha p = \alpha^2 Y/(Ax) = \alpha^2 Y/K = \alpha^2 z$. Moreover, $\dot{K}/K = Y/K - Lc/K = z - \gamma$. Using the Ramsey rule, it follows that

$$\dot{\gamma} = \gamma \left[-\left(1 - \frac{\alpha^2}{\sigma}\right) z + \gamma + \frac{\sigma - 1}{\sigma} n - \frac{\rho}{\sigma} \right]. \tag{1}$$

Monopoly profits equal $\pi = (1-\alpha)px = \alpha(1-\alpha)Y/A$. Let $v(t) \equiv \int_t^\infty e^{-\int_t^\tau r(\theta)d\theta} \pi(\tau) d\tau$ denote the value of an innovation, $\nu \equiv (1-\alpha)Y/(Av)$, and $l \equiv L/(aA^\chi)$. Free entry into R&D implies $wL_A = vA^{1-\chi}L_A/a$. Together with the final goods sector's demand for labor ($w = (1-\alpha)Y/L_Y$) and the labor market clearing condition ($L = L_Y + L_A$), this can be used to rewrite the R&D technology as $\dot{A}/A = (L - L_Y)/(aA^\chi) = L/(aA^\chi) - (1-\alpha)Y/(waA^\chi) = L/(aA^\chi) - (1-\alpha)Y/(Av) = l - \nu$. Hence,

$$\dot{l} = l[-\chi(l - \nu) + n]. \tag{2}$$

Differentiating $v(t)$ with respect to time gives $\dot{v}/v = r - \pi/v = r - \alpha(1-\alpha)Y/(Av) = \alpha^2 z - \alpha\nu$. From the final goods sector's production function and demand for labor, $z \equiv Y/K = [(AL_Y)/K]^{1-\alpha} = [z(1-\alpha)A/w]^{1-\alpha}$. Solving for z and using the condition for free entry into R&D, it follows that

[5] The model is identical to Romer's (1990) except for two modifications: Romer assumes that population is constant ($n = 0$) and that the R&D technology displays constant returns to existing knowledge ($\chi = 0$). The consequences of these seemingly harmless differences are striking. In Romer's model, everything that has an impact on R&D employment, L_A, also has an impact on the rate of technical change, $\dot{A}/A = L_A/a$. In Jones' model, if a constant fraction of the labor force is employed in R&D ($\dot{L}_A/L_A = \dot{L}/L = n$), the rate of technical change converges to the exogenous level $\dot{A}/A = n/\chi$.

$z = [(1-\alpha)A/w]^{(1-\alpha)/\alpha} = [(1-\alpha)aA^{\chi}/v]^{(1-\alpha)/\alpha}$. Differentiating with respect to time shows $\dot{z}/z = [(1-\alpha)/\alpha](\chi\dot{A}/A - \dot{v}/v)$. Inserting $\dot{A}/A = l - \nu$ and $\dot{v}/v = \alpha^2 z - \alpha\nu$ yields

$$\dot{z} = \frac{1-\alpha}{\alpha} z[\chi l - (\chi - \alpha)\nu - \alpha^2 z]. \tag{3}$$

Finally, we derive an equation for $\dot{\nu}/\nu = \dot{Y}/Y - \dot{A}/A - \dot{v}/v$. From the definition of z and $\dot{K}/K = z - \gamma$, it follows that $\dot{Y}/Y = \dot{z}/z + z - \gamma$. Together with (3), $\dot{A}/A = l - \nu$, and $\dot{v}/v = \alpha^2 z - \alpha\nu$, one obtains

$$\dot{\nu} = \nu\left[-\left(1 - \frac{1-\alpha}{\alpha}\chi\right) l + \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu + (1-\alpha)z - \gamma\right]. \tag{4}$$

Equations (1)-(4) comprise a system of four autonomous differential equations in $\mathbf{x} \equiv (l, \nu, z, \gamma)'$. $\mathbf{x}^* \equiv (l^*, \nu^*, z^*, \gamma^*)'$ is a steady state of the system if $\dot{\mathbf{x}} = \mathbf{0}$, $\mathbf{x} = \mathbf{x}^*$ solves these equations, $\mathbf{x}^* > \mathbf{0}$, the utility integral is bounded, and the transversality condition for the households' utility maximization problem is satisfied.

Theorem 1: *A (unique) steady state exists if, and only if,*

$$\Delta \equiv \frac{\sigma - 1}{\chi} n + \rho > 0. \tag{5}$$

Proof:[6] Using the definition of Δ in (5), we have, from(1)-(4),

$$l^* = \frac{1}{\alpha}\left(\Delta + \frac{1+\alpha}{\chi} n\right) \tag{6}$$

$$\nu^* = \frac{1}{\alpha}\left(\Delta + \frac{1}{\chi} n\right) \tag{7}$$

$$z^* = \frac{1}{\alpha^2}\left(\Delta + \frac{1+\chi}{\chi} n\right) \tag{8}$$

$$\gamma^* = \frac{1}{\alpha^2}\left[\Delta + (1-\alpha^2)\frac{1+\chi}{\chi} n\right]. \tag{9}$$

Clearly, (5) ensures $\mathbf{x}^* > \mathbf{0}$. The utility integral is bounded if $(\sigma - 1)\dot{c}/c + \rho > 0$. In a steady state $\gamma \equiv cL/K$, $z \equiv Y/K = (AL_Y/K)^{1-\alpha}$, L_Y/L, and $l \equiv L/(aA^{\chi})$ are constant, so $\dot{c}/c = \dot{K}/K - n = \dot{A}/A = n/\chi$, and (5) is necessary and sufficient for the utility integral to be bounded. It is also necessary and sufficient for the transversality condition for the households' utility maximization to hold.

[6] Derivations of many of the formulae reported subsequently are delegated to a technical appendix.

4 Dynamics

In this section, we show that a unique growth path converges to the steady state described in the previous section. To do so, we linearize system (1)-(4) about the steady state:

$$\begin{pmatrix} \dot{l} \\ \dot{\nu} \\ \dot{z} \\ \dot{\gamma} \end{pmatrix} = \begin{pmatrix} -\chi l^* & \chi l^* & 0 & 0 \\ -\left(1-\frac{1-\alpha}{\alpha}\chi\right)\nu^* & \left(2-\frac{1-\alpha}{\alpha}\chi\right)\nu^* & (1-\alpha)\nu^* & -\nu^* \\ \frac{1-\alpha}{\alpha}\chi z^* & \left(1-\alpha-\frac{1-\alpha}{\alpha}\chi\right)z^* & -\alpha(1-\alpha)z^* & 0 \\ 0 & 0 & -\left(1-\frac{\alpha^2}{\sigma}\right)\gamma^* & \gamma^* \end{pmatrix} \begin{pmatrix} l-l^* \\ \nu-\nu^* \\ z-z^* \\ \gamma-\gamma^* \end{pmatrix} \tag{10}$$

or $\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x}-\mathbf{x}^*)$, where $\mathbf{J}$ is the Jacobian in (10). The eigenvalues q satisfy the characteristic equation $f(q) \equiv |\mathbf{J}-q\mathbf{I}| = 0$, where $\mathbf{I}$ is the 4×4 identity matrix. Let δ_i $(i=0,\ldots,3)$ denote the coefficient of q^i in the characteristic equation: $f(q) = \delta_0 + \delta_1 q + \delta_2 q^2 + \delta_3 q^3 + q^4$.[7] Developing the determinant in the characteristic equation with respect to the fourth row gives:

$$\delta_0 = (1-\alpha)\frac{\alpha^2}{\sigma}\chi l^*\nu^* z^*\gamma^* \tag{11}$$

$$\begin{aligned} \delta_1 &= \alpha(1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\nu^* z^*\gamma^* \\ &\quad -\chi\left[(1-\alpha)l^*\nu^* z^* - l^*\nu^*\gamma^* + \alpha(1-\alpha)l^* z^*\gamma^* - \alpha(1-\alpha)\left(1-\frac{1}{\sigma}\right)\nu^* z^*\gamma^*\right] \end{aligned} \tag{12}$$

$$\begin{aligned} \delta_2 &= 2\nu^*\gamma^* - (1-\alpha^2)\nu^* z^* - \alpha(1-\alpha)z^*\gamma^* \\ &\quad -\chi\left[l^*\nu^* - \alpha(1-\alpha)l^* z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*\right] \end{aligned} \tag{13}$$

$$\delta_3 = -2\nu^* + \alpha(1-\alpha)z^* - \gamma^* + \chi\left(l^* + \frac{1-\alpha}{\alpha}\nu^*\right). \tag{14}$$

Letting $q_1,\ldots,q_4$ denote the characteristic roots, the characteristic equation can also be written as $f(q) = \prod_{i=1}^4 (q_i - q)$. This yields Vièta's formulae: $\delta_0 = q_1q_2q_3q_4$, $\delta_1 = -(q_1q_2q_3 + q_1q_2q_4 + q_1q_3q_4 + q_2q_3q_4)$, $\delta_2 = q_1q_2 + q_1q_3 + q_1q_4 + q_2q_3 + q_2q_4 + q_3q_4$, and $\delta_3 = -(q_1+q_2+q_3+q_4)$. Our main result is:

Theorem 2: *If a steady state exists, then there exists a unique path converging to the steady state.*

The proof makes use of three lemmas, which establish that the stable manifold is two-dimensional. The initial conditions determine the starting point in this stable manifold.

Lemma 1: *The number of negative eigenvalues is even.*

[7] As is well known, $\delta_0 = \mathrm{Det}\,\mathbf{J}$ and $\delta_3 = -\mathrm{Tr}\,\mathbf{J}$.

Proof: Vièta's formulae imply that $\delta_0 < 0$ if the number of negative eigenvalues is odd.[8] However, by (11), $\delta_0 > 0$.

Remark: According to Lemma 1, the number of negative eigenvalues is zero (complete instability), two (determinacy), or four (indeterminacy). Our next task is to rule out instability.[9] From the Vièta formulae, $\delta_1 < 0$ and $\delta_2 > 0$ if all eigenvalues are positive.[10] So if either $\delta_1 > 0$ or $\delta_2 < 0$, instability can be ruled out. However, neither $\delta_1 > 0$ nor $\delta_2 < 0$ holds true generally. To see this, consider the following two examples:

Example 1: Let $\alpha = 0.2$, $\sigma = 0.5$, $\chi = 0.75$, $n = 0.03$, and $\rho = 0.03$. Condition (5) ($0.01 > 0$) is satisfied. The characteristic polynomial reads $f(q) = 0.0134 - 0.0160q - 0.5447q^2 - 1.1425q^3 + q^4$, with $\delta_1 = -0.0160 < 0$.

Example 2: Let $\alpha = 0.33$, $\sigma = 1$, $\chi = 0.1$, $n = 0.015$, and $\rho = 0.02$. Condition (5) ($0.02 > 0$) is satisfied. The characteristic polynomial is $f(q) = 0.0065 + 0.3700q + 0.1109q^2 - 2.0174q^3 + q^4$, with $\delta_2 = 0.1109 > 0$.

So it is not possible to prove in general either that $\delta_1 > 0$ or that $\delta_2 < 0$. Fortunately, there is an escape route: we can show that for each set of parameter values *either* $\delta_1 > 0$ *or* $\delta_2 < 0$.

Lemma 2: *The number of negative eigenvalues is not equal to zero.*

Proof: We assume that $\delta_1 < 0$ and $\delta_2 > 0$ and produce a contradiction. The first term in (12) is positive. So $\delta_1 < 0$ requires that the term in square brackets is positive too, and

$$\chi > \frac{\alpha(1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\nu^* z^* \gamma^*}{(1-\alpha)l^*\nu^* z^* - l^*\nu^*\gamma^* + \alpha(1-\alpha)l^* z^* \gamma^* - \alpha(1-\alpha)\left(1-\frac{1}{\sigma}\right)\nu^* z^* \gamma^*}. \tag{15}$$

From (6)-(9), the term in square brackets in (13) is positive. So $\delta_2 > 0$ implies

$$\chi < \frac{2\nu^*\gamma^* - (1-\alpha^2)\nu^* z^* - \alpha(1-\alpha)z^*\gamma^*}{l^*\nu^* - \alpha(1-\alpha)l^* z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*}. \tag{16}$$

[8] This makes use of the fact that the product of two eigenvalues, q_j and q_k, which form complex conjugates ($q_{j/k} = \beta \pm \eta i$ with $\beta < 0$) is positive ($q_j q_k = \beta^2 + \eta^2 > 0$). We refer to both negative real eigenvalues and complex eigenvalues with negative real part as negative eigenvalues, and analogously for positive eigenvalues.

[9] In their analysis of the optimal growth path, Eicher and Turnovsky (2001, fn. 12, p. 95) state that they could not find a general condition to rule out this case. They do state simple sufficient conditions: $\sigma \geq 1$ and $\alpha \leq 1/(1+\sigma)$ (in our notation).

[10] Since the sum of two negative eigenvalues, q_j and q_k, which form complex conjugates ($q_{j/k} = \beta \pm \eta i$ with $\beta < 0$) is negative ($q_j + q_k = 2\beta < 0$).

In order for these inequalities to hold simultaneously, the expression on the right-hand side of (15) must be less than the expression on the right-hand side of (16). This condition can be written as

$$\begin{aligned} 0 \;<\; & -\frac{\alpha^2}{\sigma}\left[l^*\nu^* - \alpha(1-\alpha)l^*z^* + l^*\gamma^* + (1-\alpha^2)\nu^*z^* - \frac{1+\alpha}{\alpha}\nu^*\gamma^* + (1-\alpha)z^*\gamma^*\right] \\ & -\Big[-3l^*\nu^* + 2\alpha(1-\alpha)l^*z^* - 2\alpha l^*\gamma^* - (1-\alpha^2)\nu^*z^* + (1+\alpha)\nu^*\gamma^* \\ & -\alpha^2(1-\alpha)z^*\gamma^* + (1-\alpha^2)\frac{l^*\nu^*z^*}{\gamma^*} + \frac{2}{1-\alpha}\frac{l^*\nu^*\gamma^*}{z^*} + \alpha^2(1-\alpha)\frac{l^*z^*\gamma^*}{\nu^*}\Big]. \end{aligned} \tag{17}$$

Using (5)-(9), we find that the first term in brackets in (17) is positive if a steady state exists:

$$\frac{n}{\alpha^2\chi}\Delta\left[1+\alpha+\alpha^2+(1-\alpha)\frac{1-\alpha^2}{\alpha^2}\chi\right] + \left(\frac{n}{\alpha\chi}\right)^2(1+\alpha)\left[1+\left(\frac{1-\alpha}{\alpha}\right)^2\chi(1+\chi)\right] > 0.$$

So the second term in brackets in (17) must be negative. Using (6)-(9), this implies:

$$\Delta^2\frac{1+\alpha-\alpha^2}{\alpha^2} + \Delta\frac{n}{\alpha^2\chi}\left[2+\alpha-\alpha^2-\alpha^3+\chi(1-\alpha)^2\right] + \left(\frac{n}{\alpha\chi}\right)^2\left[1+\alpha^3+\chi(1-\alpha^2)(1-2\alpha)\right] < 0.$$

Because of (5), the first two terms in the sum are positive if a steady state exists. So the third term must be negative. This requires $\alpha > 1/2$[11] and

$$\chi > \frac{1+\alpha^3}{(1-\alpha^2)(2\alpha-1)}. \tag{18}$$

The fraction on the right-hand side of (18) is greater than 4 for all $\alpha > 1/2$.[12] But $\chi > 4$ contradicts (16). This proves Lemma 2.

Lemma 3: *The number of negative eigenvalues is not equal to four.*

Proof: The Vièta formulae imply that $\delta_2 > 0$ and $\delta_3 > 0$ if all four eigenvalues are negative.[13] From (13) and (14), it follows that (16) holds and that

$$\chi > \frac{2\nu^* - \alpha(1-\alpha)z^* + \gamma^*}{l^* + \frac{1-\alpha}{\alpha}\nu^*}, \tag{19}$$

respectively. This presupposes that the expression on the right-hand side of (19) must be less than the expression on the right-hand side of (16). This condition can be written as

$$\begin{aligned} 0 \;<\; & -\Big[2l^*(\nu^*)^2 + (1-\alpha)(1-2\alpha)l^*\nu^*z^* + l^*\nu^*\gamma^* + \alpha^2(1-\alpha)^2l^*(z^*)^2 - \alpha(1-\alpha)l^*z^*\gamma^* + l^*(\gamma^*)^2 \\ & -(1-\alpha^2)\frac{1-\alpha}{\alpha}(\nu^*)^2z^* + (1-\alpha^2)(1-\alpha)^2\nu^*(z^*)^2 - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^*z^*\gamma^* + \frac{1-\alpha}{\alpha}\nu^*(\gamma^*)^2\Big]. \end{aligned}$$

[11] Hence, $\alpha \le 1/2$ is a simple sufficient condition for the validity of Lemma 2.

[12] $\chi < 4$ is thus another simple sufficient condition for the validity of Lemma 2.

[13] The latter observation alone allows it to derive a simple sufficient condition for determinacy: using (14), (6)-(9), and (5), $\chi < \min\{1+\alpha(1-\alpha),\, 1/\alpha\}$ implies $\delta_3 = \Delta[-(1+\alpha)/\alpha - (1-\chi)/\alpha^2] + (1+\alpha)[1-1/(\alpha\chi)]n/\alpha < 0$.

We have a contradiction if the term in square brackets is positive. This is implied by (6)-(9).

Remark: Experimentation with numerical values shows that the stable eigenvalues are real for a broad range of parameters (see Example 1 below). It is possible to construct counterexamples, however (see Example 2 below).

Example 1: The characteristic polynomial attains a minimum at $q = -0.2403$ with $f(-0.2403) = 0.0050 > 0$. This proves that no negative real roots exist.

Example 2: The characteristic polynomial attains a minimum at $q = -0.2141$ with $f(-0.2141) = -0.0457 < 0$. This implies the existence of two negative real roots.

Proof of Theorem 2: It remains for us to show that $A(0)$, $K(0)$, and $L(0)$ uniquely determine the starting point in the stable manifold. The initial values $\mathbf{x}(0)$ satisfy

$$\mathbf{x}(0) - \mathbf{x}^* = \sum_{i=1}^{2} B_i \mathbf{b}_i, \tag{20}$$

where the $\mathbf{b}_i$'s are the eigenvectors ($\mathbf{b}_i = (b_{li}, b_{\nu i}, b_{zi}, b_{\gamma i})'$) corresponding to the two negative eigenvalues, q_i, and the B_i's are constants to be determined below. The definitions of ν and z yield

$$\nu(0) = \frac{z(0)^{\frac{1}{1-\alpha}} K(0)}{aA(0)^{1+\chi}}. \tag{21}$$

(20) and (21) comprise a system of five equations in the five unknowns $\nu(0)$, $z(0)$, $\gamma(0)$, B_1, and B_2. The first three equations in (20) determine a relation between $\nu(0)$ and $z(0)$:

$$\nu(0) - \nu^* = \frac{(b_{z1}b_{\nu 2} - b_{\nu 1}b_{z2})[l(0) - l^*] + (b_{\nu 1}b_{l2} - b_{l1}b_{\nu 2})[z(0) - z^*]}{b_{z1}b_{l2} - b_{l1}b_{z2}}. \tag{22}$$

If $K(0)/A(0)^{1+\chi} = (K/A^{1+\chi})^*$ and $l(0) = l^*$, then $\nu(0) = \nu^*$, $z(0) = z^*$, $\gamma(0) = \gamma^*$, $B_1 = 0$, and $B_2 = 0$ solve (20)-(22), and the economy is in its steady state immediately. If $K(0)/A(0)^{1+\chi}$ and $l(0)$ differ slightly from their balanced growth levels (so that our local analysis applies), then (21) and (22) determine $\nu(0)$ and $z(0)$, and (20) then determines $\gamma(0)$, B_1, and B_2. This completes the proof of Theorem 2.

5 Multi-country open economy

In Arnold (2004), we analyzed a class of growth models in which the "Dixit-Norman property" (cf. Dixit and Norman, 1980, Chapter 4), that a world economy made up of several countries replicates the equilibrium of a hypothetical integrated economy without restrictions on factor movements, holds true. The Jones model belongs to this class of growth models. So we can immediately infer from the analysis in Arnold (2004) the conditions under which a world economy made up of several countries of the type described in Section 2 (with labor immobility) behaves just like the hypothetical integrated world economy (with complete labor mobility):

Theorem 3: *Suppose the world economy is made up of M (≥ 2) countries with identical tastes and technologies in each country. Suppose further that there is free trade in the final good and the intermediates, financial capital is perfectly mobile, and knowledge spillovers are international in scope. Then, the M-country world economy replicates the equilibrium of the hypothetical integrated world economy if, and only if, physical capital is mobile internationally and/or multinational firms or international patent licensing are allowed for.*

The proof can be sketched as follows. There are three productive activities: final goods production, intermediate goods production, and R&D. Final goods production and R&D are internationally mobile, in that nothing pins down their location. Intermediate goods production is also mobile if domestically invented goods can be produced abroad, either within multinational firms or with the help of international patent licensing agreements. Otherwise intermediate goods have to be produced where they have been invented. There are two primary factors of production: labor and physical capital. While labor is immobile by assumption, physical capital, once installed, can be assumed to be mobile or not. We use lower-case letters with a superscript m to denote the country-m ($m = 1, \ldots, M$) levels of variables denoted by upper-case letters so far. Let k^m denote physical capital *owned by* country-m residents and k'^m capital *used in* country m. Similarly, let a^m denote the number of intermediates *invented by* firms from country m and a'^m the number of intermediates *produced in* country m. Since intermediate goods production is the only use of physical capital,

$$k'^m = a'^m x, \quad m = 1, \ldots, M, \tag{23}$$

where x is the uniform quantity produced of each intermediate. If physical capital is immobile (i.e., $k'^m = k^m$) and intermediates have to be produced where they were invented ($a'^m = a^m$), then (23) is only satisfied by coincidence. With physical capital mobility, k'^m is free to adjust so that (23) is satisfied. With multinationals or patent licensing, a'^m adjusts so that (23) holds. In both cases, the allocation of labor l^m to final goods production or R&D is indeterminate. If both physical capital and intermediate goods production are internationally mobile, there is another degree of freedom.

6 Conclusion

The Jones (1995) R&D model with non-diminishing returns to labor in R&D has well-behaved dynamics. Whenever the parameters are such that a steady state exists, the steady state is unique, and a unique convergent trajectory exists. If physical capital is mobile and/or multinational firms exist or patent licensing is possible, this result carries over to the multi-country open economy version of the model. For one thing, this reassures us that the steady state analyses of the Jones (1995) and related models are in fact concerned with the long-run equilibrium growth paths of the models. For another, it potentially opens up new routes for numerical analyses of the models, especially their multi-country open economy variants.

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Technical appendix

Steady state

From (2) and (3),

$$\chi l = \chi\nu + n \tag{A.1}$$

$$\chi l = (\chi - \alpha)\nu + \alpha^2 z.$$

Hence,

$$\alpha^2 z = \alpha\nu + n. \tag{A.2}$$

From (4) and (A.1):

$$\left(1 - \frac{1-\alpha}{\alpha}\chi\right) l + \gamma = \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu + (1-\alpha)z$$

$$\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\left(\nu + \frac{n}{\chi}\right) + \gamma = \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu + (1-\alpha)z$$

$$\left(\frac{1}{\chi} - \frac{1-\alpha}{\alpha}\right) n + \gamma = \nu + (1-\alpha)z.$$

From (1),

$$\frac{\sigma-1}{\sigma} n - \frac{\rho}{\sigma} + \gamma = \left(1 - \frac{\alpha^2}{\sigma}\right) z. \tag{A.3}$$

Hence,

$$\left(\frac{\sigma-1}{\sigma} - \frac{1}{\chi} + \frac{1-\alpha}{\alpha}\right) n - \frac{\rho}{\sigma} = -\nu + \left(\frac{1}{\alpha} - \frac{1}{\sigma}\right)\alpha^2 z. \tag{A.4}$$

(A.2) and (A.4) can be solved for ν and z:

$$\left(\frac{\sigma-1}{\sigma} - \frac{1}{\chi} + \frac{1-\alpha}{\alpha}\right) n - \frac{\rho}{\sigma} = -\nu + \left(\frac{1}{\alpha} - \frac{1}{\sigma}\right)(\alpha\nu + n)$$

$$\begin{aligned} \nu &= \frac{1}{\alpha}\left(\frac{\sigma-1}{\chi} n + \rho + \frac{1}{\chi} n\right) \\ &= \frac{1}{\alpha}\left(\Delta + \frac{1}{\chi} n\right) \end{aligned}$$

$$\begin{aligned} z &= \frac{\nu}{\alpha} + \frac{n}{\alpha^2} \\ &= \frac{1}{\alpha^2}\left(\Delta + \frac{1+\chi}{\chi} n\right). \end{aligned}$$

These are (7) and (8) in the main text. (6) is obtained from (A.1):

$$\begin{aligned} l &= \nu + \frac{n}{\chi} \\ &= \frac{1}{\alpha}\left(\Delta + \frac{n}{\chi}\right) + \frac{n}{\chi} \\ &= \frac{1}{\alpha}\left(\Delta + \frac{1+\alpha}{\chi} n\right). \end{aligned}$$

Finally, (9) is obtained from (A.3):

$$\begin{aligned}
\gamma &= \left(1-\frac{\alpha^2}{\sigma}\right)z - \frac{\sigma-1}{\sigma}n + \frac{\rho}{\sigma} \\
&= \left(1-\frac{\alpha^2}{\sigma}\right)\frac{1}{\alpha^2}\left(\Delta + \frac{1+\chi}{\chi}n\right) - \frac{\sigma-1}{\sigma}n + \frac{\rho}{\sigma} \\
&= \frac{1}{\alpha^2}\left(\Delta + \frac{1+\chi}{\chi}n\right) - \frac{1}{\sigma}\left(\Delta + \frac{1+\chi}{\chi}n\right) - \frac{\sigma-1}{\sigma}n + \frac{\rho}{\sigma} \\
&= \frac{1}{\alpha^2}\left(\Delta + \frac{1+\chi}{\chi}n\right) - \frac{1}{\sigma}\left(\frac{\sigma-1}{\chi}n + \rho + \frac{1+\chi}{\chi}n\right) - \frac{\sigma-1}{\sigma}n + \frac{\rho}{\sigma} \\
&= \frac{1}{\alpha^2}\left(\Delta + \frac{1+\chi}{\chi}n\right) - \frac{1+\chi}{\chi}n \\
&= \frac{1}{\alpha^2}\left[\Delta + (1-\alpha^2)\frac{1+\chi}{\chi}n\right].
\end{aligned}$$

For future reference, notice that l^*, ν^*, z^*, and γ^* satisfy the following relations:

$$l^* > \nu^*, \quad z^* > \frac{\nu^*}{\alpha}, \quad z^* > \gamma^*, \quad \nu^* > \frac{l^*}{1+\alpha}, \quad \gamma^* > (1-\alpha^2)z^*. \tag{A.5}$$

The first three inequalities are obvious. The latter two are easily proved as follows:

$$\nu^* = \frac{1}{\alpha}\left(\Delta + \frac{1}{\chi}n\right) = \frac{1}{\alpha}\frac{(1+\alpha)\Delta + \frac{1+\alpha}{\chi}n}{1+\alpha} > \frac{1}{\alpha}\frac{\Delta + \frac{1+\alpha}{\chi}n}{1+\alpha} = \frac{l^*}{1+\alpha}$$

$$\gamma^* = \frac{1}{\alpha^2}\left[\Delta + (1-\alpha^2)\frac{1+\chi}{\chi}n\right] = \frac{1-\alpha^2}{\alpha^2}\left(\frac{\Delta}{1-\alpha^2} + \frac{1+\chi}{\chi}n\right) > \frac{1-\alpha^2}{\alpha^2}\left(\Delta + \frac{1+\chi}{\chi}n\right) = (1-\alpha^2)z^*.$$

Transversality condition

The transversality condition for the households' utility maximization problem is

$$\lim_{t\to\infty} e^{-\rho t}[K(t) + A(t)v(t)]\lambda(t) = 0,$$

where λ is the co-state variable and $K + Av$ is financial wealth. According to the first-order condition for an optimal consumption profile,

$$\lambda = \frac{c^{-\sigma}}{L}.$$

In a steady state, $\dot{A}/A = n/\chi$, $\dot{K}/K = \dot{A}/A + n = n/\chi + n$, $\dot{v}/v = \dot{Y}/Y - \dot{A}/A = n$, $\dot{c}/c = n/\chi$, and $\dot{\lambda}/\lambda = -(\dot{c}/c)/\sigma - n = -\sigma n/\chi - n$. Hence, the transversality condition requires

$$-\rho + \left(\frac{n}{\chi} + n\right) + \left(-\sigma\frac{n}{\chi} - n\right) < 0.$$

Rearranging terms yields (5):

$$\Delta \equiv \frac{\sigma-1}{\chi}n + \rho > 0.$$

The linearized system (10):

From (2):

$$\frac{\partial \dot{l}}{\partial l} = \underbrace{[-\chi(l-\nu)+n]}_{=0} - \chi l = -\chi l.$$

$$\frac{\partial \dot{l}}{\partial \nu} = \chi l$$

$$\frac{\partial \dot{l}}{\partial z} = 0$$

$$\frac{\partial \dot{l}}{\partial \gamma} = 0.$$

Analogously, from (4):

$$\frac{\partial \dot{\nu}}{\partial l} = -\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\nu$$

$$\frac{\partial \dot{\nu}}{\partial \nu} = \underbrace{\left[-\left(1-\frac{1-\alpha}{\alpha}\chi\right) l + \left(2-\frac{1-\alpha}{\alpha}\chi\right)\nu + (1-\alpha)z - \gamma\right]}_{=0} + \left(2-\frac{1-\alpha}{\alpha}\chi\right)\nu = \left(2-\frac{1-\alpha}{\alpha}\chi\right)\nu$$

$$\frac{\partial \dot{\nu}}{\partial z} = (1-\alpha)\nu$$

$$\frac{\partial \dot{\nu}}{\partial \gamma} = -\nu.$$

From (3):

$$\frac{\partial \dot{z}}{\partial l} = \frac{1-\alpha}{\alpha}\chi z$$

$$\frac{\partial \dot{z}}{\partial \nu} = \frac{1-\alpha}{\alpha}(\alpha-\chi)z = \left(1-\alpha-\frac{1-\alpha}{\alpha}\chi\right) z$$

$$\frac{\partial \dot{z}}{\partial z} = \frac{1-\alpha}{\alpha}\underbrace{[\chi l - (\chi-\alpha)\nu - \alpha^2 z]}_{=0} - \alpha(1-\alpha)z = -\alpha(1-\alpha)z$$

$$\frac{\partial \dot{z}}{\partial \gamma} = 0.$$

From (1)

$$\frac{\partial \dot{\gamma}}{\partial l} = 0$$

$$\frac{\partial \dot{\gamma}}{\partial \nu} = 0$$

$$\frac{\partial \dot{\gamma}}{\partial z} = -\left(1-\frac{\alpha^2}{\sigma}\right)\gamma$$

$$\frac{\partial \dot{\gamma}}{\partial \gamma} = \underbrace{\left[-\left(1-\frac{\alpha^2}{\sigma}\right)z + \gamma + \frac{\sigma-1}{\sigma}n - \frac{\rho}{\sigma}\right]}_{=0} + \gamma = \gamma.$$

The characteristic equation

The characteristic equation is:

$$f(q) = \begin{vmatrix} -\chi l^* - q & \chi l^* & 0 & 0 \\ -\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* & \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - q & (1-\alpha)\nu^* & -\nu^* \\ \frac{1-\alpha}{\alpha}\chi z^* & \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right) z^* & -\alpha(1-\alpha)z^* - q & 0 \\ 0 & 0 & -\left(1 - \frac{\alpha^2}{\sigma}\right)\gamma^* & \gamma^* - q \end{vmatrix} = 0.$$

Developing the determinant in the characteristic equation with respect to the fourth row gives:

$$f(q) = \left(1 - \frac{\alpha^2}{\sigma}\right)\gamma^* \Delta_1(q) + (\gamma^* - q)\Delta_2(q), \tag{A.6}$$

where

$$\Delta_1(q) = \begin{vmatrix} -\chi l^* - q & \chi l^* & 0 \\ -\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* & \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - q & -\nu^* \\ \frac{1-\alpha}{\alpha}\chi z^* & \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right) z^* & 0 \end{vmatrix}$$

and

$$\Delta_2(q) = \begin{vmatrix} -\chi l^* - q & \chi l^* & 0 \\ -\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* & \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - q & (1-\alpha)\nu^* \\ \frac{1-\alpha}{\alpha}\chi z^* & \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right) z^* & -\alpha(1-\alpha)z^* - q \end{vmatrix}.$$

$\Delta_1(q)$ can be written as:

$$\begin{aligned} \Delta_1(q) &= \begin{vmatrix} -\chi l^* - q & \chi l^* & 0 \\ -\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* & \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - q & -\nu^* \\ \frac{1-\alpha}{\alpha}\chi z^* & \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right) z^* & 0 \end{vmatrix} \\ &= -\frac{1-\alpha}{\alpha}\chi^2 l^* \nu^* z^* - \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\chi l^* \nu^* z^* - \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^* q \\ &= -(1-\alpha)\chi l^* \nu^* z^* - \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^* q. \end{aligned} \tag{A.7}$$

As for $\Delta_2(q)$:

$$\begin{aligned} \Delta_2(q) &= \begin{vmatrix} -\chi l^* - q & \chi l^* & 0 \\ -\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* & \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - q & (1-\alpha)\nu^* \\ \frac{1-\alpha}{\alpha}\chi z^* & \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right) z^* & -\alpha(1-\alpha)z^* - q \end{vmatrix} \\ &= (-\chi l^* - q)\left[\left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - q\right][-\alpha(1-\alpha)z^* - q] \\ &\quad + \frac{(1-\alpha)^2}{\alpha}\chi^2 l^* \nu^* z^* + \left(1 - \frac{1-\alpha}{\alpha}\chi\right)\chi l^* \nu^* [-\alpha(1-\alpha)z^* - q] \\ &\quad -(1-\alpha)\left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^* (-\chi l^* - q). \end{aligned}$$

The first term in this sum can be rewritten as:

$$
\begin{aligned}
& (-\chi l^* - q)\left[\left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - q\right][-\alpha(1-\alpha)z^* - q] \\
= & \left[\left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - q\right]\{\alpha(1-\alpha)\chi l^* z^* + [\chi l^* + \alpha(1-\alpha)z^*]q + q^2\} \\
= & \alpha(1-\alpha)\left(2 - \frac{1-\alpha}{\alpha}\chi\right)\chi l^* \nu^* z^* \\
& + \left[\left(2 - \frac{1-\alpha}{\alpha}\chi\right)\chi l^* \nu^* - \alpha(1-\alpha)\chi l^* z^* + \alpha(1-\alpha)\left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^*\right] q \\
& + \left[-\chi l^* + \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - \alpha(1-\alpha)z^*\right] q^2 \\
& - q^3.
\end{aligned}
$$

The remaining terms can be rewritten as:

$$
\begin{aligned}
& \frac{(1-\alpha)^2}{\alpha}\chi^2 l^* \nu^* z^* + \left(1 - \frac{1-\alpha}{\alpha}\chi\right)\chi l^* \nu^* [-\alpha(1-\alpha)z^* - q] \\
& -(1-\alpha)\left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^* (-\chi l^* - q) \\
= & (1-\alpha)\chi l^* \nu^* z^* \left[\frac{1-\alpha}{\alpha}\chi - \alpha\left(1 - \frac{1-\alpha}{\alpha}\chi\right) + \left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\right] \\
& + \left[-\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\chi l^* \nu^* + (1-\alpha)\left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^*\right] q \\
= & (1-\alpha)\left[-\alpha\left(1 - \frac{1-\alpha}{\alpha}\chi\right) + 1 - \alpha\right]\chi l^* \nu^* z^* \\
& + \left[-\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\chi l^* \nu^* + (1-\alpha)\left(1 - \alpha - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^*\right] q.
\end{aligned}
$$

Taken together, it follows that

$$
\begin{aligned}
\Delta_2(q) = \; & (1-\alpha)\chi l^* \nu^* z^* \\
& + \left[\chi l^* \nu^* - \alpha(1-\alpha)\chi l^* z^* + (1-\alpha^2)\left(1 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* z^*\right] q \\
& + \left[-\chi l^* + \left(2 - \frac{1-\alpha}{\alpha}\chi\right)\nu^* - \alpha(1-\alpha)z^*\right] q^2 \\
& - q^3.
\end{aligned} \tag{A.8}
$$

Equations (11)-(14):

From (A.6)-(A.8), one can calculate the expressions for δ_i $(i = 0, 1, 2, 3)$ in (11)-(14):

$$
\begin{aligned}
\delta_0 &= -(1-\alpha)\left(1 - \frac{\alpha^2}{\sigma}\right)\chi l^* \nu^* z^* \gamma^* + (1-\alpha)\chi l^* \nu^* z^* \gamma^* \\
&= (1-\alpha)\frac{\alpha^2}{\sigma}\chi l^* \nu^* z^* \gamma^*
\end{aligned}
$$

$$
\begin{aligned}
\delta_1 &= -\left(1-\frac{\alpha^2}{\sigma}\right)\left(1-\alpha-\frac{1-\alpha}{\alpha}\chi\right)\nu^* z^* \gamma^* \\
&\quad +\gamma^*\left[\chi l^*\nu^* - \alpha(1-\alpha)\chi l^* z^* + (1-\alpha^2)\left(1-\frac{1-\alpha}{\alpha}\chi\right)\nu^* z^*\right] - (1-\alpha)\chi l^*\nu^* z^* \\
&= \left[-\left(1-\frac{\alpha^2}{\sigma}\right)\left(1-\alpha-\frac{1-\alpha}{\alpha}\chi\right) + (1-\alpha^2)\left(1-\frac{1-\alpha}{\alpha}\chi\right)\right]\nu^* z^* \gamma^* \\
&\quad -(1-\alpha)\chi l^*\nu^* z^* + \chi l^*\nu^*\gamma^* - \alpha(1-\alpha)\chi l^* z^*\gamma^* \\
&= \alpha(1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\nu^* z^*\gamma^* \\
&\quad -\chi\left[(1-\alpha)l^*\nu^* z^* - l^*\nu^*\gamma^* + \alpha(1-\alpha)l^* z^*\gamma^* - \alpha(1-\alpha)\left(1-\frac{1}{\sigma}\right)\nu^* z^*\gamma^*\right]
\end{aligned}
$$

$$
\begin{aligned}
\delta_2 &= \gamma^*\left[-\chi l^* + \left(2-\frac{1-\alpha}{\alpha}\chi\right)\nu^* - \alpha(1-\alpha)z^*\right] \\
&\quad -\left[\chi l^*\nu^* - \alpha(1-\alpha)\chi l^* z^* + (1-\alpha^2)\left(1-\frac{1-\alpha}{\alpha}\chi\right)\nu^* z^*\right] \\
&= 2\nu^*\gamma^* - (1-\alpha^2)\nu^* z^* - \alpha(1-\alpha)z^*\gamma^* \\
&\quad -\chi\left[l^*\nu^* - \alpha(1-\alpha)l^* z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*\right]
\end{aligned}
$$

$$
\begin{aligned}
\delta_3 &= -\gamma^* - \left[-\chi l^* + \left(2-\frac{1-\alpha}{\alpha}\chi\right)\nu^* - \alpha(1-\alpha)z^*\right] \\
&= \chi l^* - \left(2-\frac{1-\alpha}{\alpha}\chi\right)\nu^* + \alpha(1-\alpha)z^* - \gamma^* \\
&= -2\nu^* + \alpha(1-\alpha)z^* - \gamma^* + \chi\left(l^* + \frac{1-\alpha}{\alpha}\nu^*\right).
\end{aligned}
$$

The term in square brackets in (13) is positive.

From (A.5) and $\mathbf{x}^* > \mathbf{0}$,

$$
\begin{aligned}
& l^*\nu^* - \underbrace{\alpha(1-\alpha)}_{=(1-\alpha^2)-(1-\alpha)} l^* z^* + l^* \underbrace{\gamma^*}_{>(1-\alpha^2)z^*} -(1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* + \frac{1-\alpha}{\alpha}\nu^* \underbrace{\gamma^*}_{>(1-\alpha^2)z^*} \\
> \; & l^*\nu^* - (1-\alpha^2)l^* z^* + (1-\alpha)l^* z^* + (1-\alpha^2)l^* z^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* + (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* \\
= \; & \underbrace{l^*\nu^*}_{>0} + \underbrace{(1-\alpha)l^* z^*}_{>0} \qquad \text{(A.9)} \\
> \; & 0.
\end{aligned}
$$

Equation (17):

The expression on the right-hand side of (15) is less than the expression on the right-hand side of (16) if

$$
\frac{\alpha(1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\nu^* z^*\gamma^*}{(1-\alpha)l^*\nu^* z^* - l^*\nu^*\gamma^* + \alpha(1-\alpha)l^* z^*\gamma^* - \alpha(1-\alpha)\left(1-\frac{1}{\sigma}\right)\nu^* z^*\gamma^*}
$$

$$< \quad \frac{2\nu^*\gamma^* - (1-\alpha^2)\nu^* z^* - \alpha(1-\alpha)z^*\gamma^*}{l^*\nu^* - \alpha(1-\alpha)l^* z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*}.$$

Since the denominators on both sides of the inequality are positive, this can be rewritten as:

$$\begin{aligned}
0 \quad < \quad & [2\nu^*\gamma^* - (1-\alpha^2)\nu^* z^* - \alpha(1-\alpha)z^*\gamma^*] \\
& \times \left[(1-\alpha)l^*\nu^* z^* - l^*\nu^*\gamma^* + \alpha(1-\alpha)l^* z^*\gamma^* - \alpha(1-\alpha)\left(1-\frac{1}{\sigma}\right)\nu^* z^*\gamma^*\right] \\
& -\alpha(1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\nu^* z^*\gamma^* \\
& \times \left[l^*\nu^* - \alpha(1-\alpha)l^* z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*\right]
\end{aligned}$$

or, after division by $(1-\alpha)\nu^* z^*\gamma^*$, as

$$\begin{aligned}
0 \quad < \quad & [2\nu^*\gamma^* - (1-\alpha^2)\nu^* z^* - \alpha(1-\alpha)z^*\gamma^*] \\
& \times \left[\frac{l^*}{\gamma^*} - \frac{1}{1-\alpha}\frac{l^*}{z^*} + \alpha\frac{l^*}{\nu^*} - \alpha\left(1-\frac{1}{\sigma}\right)\right] \\
& -\alpha\left(1+\frac{\alpha}{\sigma}\right) \\
& \times \left[l^*\nu^* - \alpha(1-\alpha)l^* z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*\right].
\end{aligned}$$

Multiplying out yields:

$$\begin{aligned}
0 \quad < \quad & 2l^*\nu^* - \frac{2}{1-\alpha}\frac{l^*\nu^*\gamma^*}{z^*} + 2\alpha l^*\gamma^* - 2\alpha\left(1-\frac{1}{\sigma}\right)\nu^*\gamma^* \\
& -(1-\alpha^2)\frac{l^*\nu^* z^*}{\gamma^*} + (1+\alpha)l^*\nu^* - \alpha(1-\alpha^2)l^* z^* + \alpha(1-\alpha^2)\left(1-\frac{1}{\sigma}\right)\nu^* z^* \\
& -\alpha(1-\alpha)l^* z^* + \alpha l^*\gamma^* - \alpha^2(1-\alpha)\frac{l^* z^*\gamma^*}{\nu^*} + \alpha^2(1-\alpha)\left(1-\frac{1}{\sigma}\right)z^*\gamma^* \\
& -\alpha\left(1+\frac{\alpha}{\sigma}\right)l^*\nu^* + \alpha^2(1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)l^* z^* - \alpha\left(1+\frac{\alpha}{\sigma}\right)l^*\gamma^* \\
& +(1-\alpha^2)(1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\nu^* z^* - (1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\nu^*\gamma^*.
\end{aligned}$$

Collecting terms, we have:

$$\begin{aligned}
0 \quad < \quad & \left[2 + (1+\alpha) - \alpha\left(1+\frac{\alpha}{\sigma}\right)\right] l^*\nu^* \\
& + \left[-\alpha(1-\alpha^2) - \alpha(1-\alpha) + \alpha^2(1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\right] l^* z^* \\
& + \left[2\alpha + \alpha - \alpha\left(1+\frac{\alpha}{\sigma}\right)\right] l^*\gamma^* \\
& + \left[\alpha(1-\alpha^2)\left(1-\frac{1}{\sigma}\right) + (1-\alpha^2)(1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\right] \nu^* z^* \\
& + \left[-2\alpha\left(1-\frac{1}{\sigma}\right) - (1-\alpha)\left(1+\frac{\alpha}{\sigma}\right)\right] \nu^*\gamma^*
\end{aligned}$$

$$+\alpha^2(1-\alpha)\left(1-\frac{1}{\sigma}\right)z^*\gamma^*$$
$$-(1-\alpha^2)\frac{l^*\nu^*z^*}{\gamma^*}-\frac{2}{1-\alpha}\frac{l^*\nu^*\gamma^*}{z^*}-\alpha^2(1-\alpha)\frac{l^*z^*\gamma^*}{\nu^*}.$$

Simplifying terms, we obtain:

$$\begin{aligned}0 \quad < \quad & \left(3-\frac{\alpha^2}{\sigma}\right)l^*\nu^* \\ & +\alpha(1-\alpha)\left(-2+\frac{\alpha^2}{\sigma}\right)l^*z^* \\ & +\left(2\alpha-\frac{\alpha^2}{\sigma}\right)l^*\gamma^* \\ & +(1-\alpha^2)\left(1-\frac{\alpha^2}{\sigma}\right)\nu^*z^* \\ & +(1+\alpha)\left(-1+\frac{1}{\alpha}\frac{\alpha^2}{\sigma}\right)\nu^*\gamma^* \\ & +(1-\alpha)\left(\alpha^2-\frac{\alpha^2}{\sigma}\right)z^*\gamma^* \\ & -(1-\alpha^2)\frac{l^*\nu^*z^*}{\gamma^*}-\frac{2}{1-\alpha}\frac{l^*\nu^*\gamma^*}{z^*}-\alpha^2(1-\alpha)\frac{l^*z^*\gamma^*}{\nu^*}.\end{aligned}$$

Collecting terms yields (17):

$$\begin{aligned}0 \quad < \quad & -\frac{\alpha^2}{\sigma}\left[l^*\nu^*-\alpha(1-\alpha)l^*z^*+l^*\gamma^*+(1-\alpha^2)\nu^*z^*-\frac{1+\alpha}{\alpha}\nu^*\gamma^*+(1-\alpha)z^*\gamma^*\right] \\ & -\Big[-3l^*\nu^*+2\alpha(1-\alpha)l^*z^*-2\alpha l^*\gamma^*-(1-\alpha^2)\nu^*z^*+(1+\alpha)\nu^*\gamma^* \\ & -\alpha^2(1-\alpha)z^*\gamma^*+(1-\alpha^2)\frac{l^*\nu^*z^*}{\gamma^*}+\frac{2}{1-\alpha}\frac{l^*\nu^*\gamma^*}{z^*}+\alpha^2(1-\alpha)\frac{l^*z^*\gamma^*}{\nu^*}\Big].\end{aligned}$$

The first term in square brackets in (17) is positive:

$$\begin{aligned} & l^*\nu^*-\alpha(1-\alpha)l^*z^*+l^*\gamma^*+(1-\alpha^2)\nu^*z^*-\frac{1+\alpha}{\alpha}\nu^*\gamma^*+(1-\alpha)z^*\gamma^* \\ = \quad & \frac{1}{\alpha^2}\left(\Delta+\frac{1+\alpha}{\chi}n\right)\left(\Delta+\frac{1}{\chi}n\right) \\ & -\frac{1-\alpha}{\alpha^2}\left(\Delta+\frac{1+\alpha}{\chi}n\right)\left(\Delta+\frac{1+\chi}{\chi}n\right) \\ & +\frac{1}{\alpha^3}\left(\Delta+\frac{1+\alpha}{\chi}n\right)\left[\Delta+(1-\alpha^2)\frac{1+\chi}{\chi}n\right] \\ & +\frac{1-\alpha^2}{\alpha^3}\left(\Delta+\frac{1}{\chi}n\right)\left(\Delta+\frac{1+\chi}{\chi}n\right) \\ & -\frac{1+\alpha}{\alpha^4}\left(\Delta+\frac{1}{\chi}n\right)\left[\Delta+(1-\alpha^2)\frac{1+\chi}{\chi}n\right] \\ & +\frac{1-\alpha}{\alpha^4}\left(\Delta+\frac{1+\chi}{\chi}n\right)\left[\Delta+(1-\alpha^2)\frac{1+\chi}{\chi}n\right]\end{aligned}$$

$$
\begin{aligned}
= \quad & \Delta^2 \frac{1}{\alpha^2}\left[1-(1-\alpha)+\frac{1}{\alpha}+\frac{1-\alpha^2}{\alpha}-\frac{1+\alpha}{\alpha^2}+\frac{1-\alpha}{\alpha^2}\right] \\
& +\Delta \frac{n}{\alpha^2\chi}\Bigg\{1+\alpha+1 \\
& -(1-\alpha)(1+\alpha+1+\chi) \\
& +\frac{1}{\alpha}\Big[1+\alpha+(1-\alpha^2)(1+\chi)\Big] \\
& +\frac{1-\alpha^2}{\alpha}(1+1+\chi) \\
& -\frac{1+\alpha}{\alpha^2}\Big[1+(1-\alpha^2)(1+\chi)\Big] \\
& +\frac{1-\alpha}{\alpha^2}[1+\chi+(1-\alpha^2)(1+\chi)]\Bigg\} \\
& +\left(\frac{n}{\alpha\chi}\right)^2\Bigg[1+\alpha \\
& -(1-\alpha)(1+\alpha)(1+\chi) \\
& +\frac{1}{\alpha}(1+\alpha)(1-\alpha^2)(1+\chi) \\
& +\frac{1-\alpha^2}{\alpha}(1+\chi) \\
& -\frac{1+\alpha}{\alpha^2}(1-\alpha^2)(1+\chi) \\
& +\frac{1-\alpha}{\alpha^2}(1-\alpha^2)(1+\chi)+\frac{1-\alpha}{\alpha^2}(1-\alpha^2)\chi(1+\chi)\Bigg] \\
= \quad & \Delta^2 \frac{1}{\alpha^2}\left(1-1+\alpha+\frac{1}{\alpha}+\frac{1}{\alpha}-\alpha-\frac{1}{\alpha^2}-\frac{1}{\alpha}+\frac{1}{\alpha^2}-\frac{1}{\alpha}\right) \\
& +\Delta \frac{n}{\alpha^2\chi}\Bigg\{2+\alpha \\
& -2+\alpha+\alpha^2-(1-\alpha)\chi \\
& +\frac{2}{\alpha}+1-\alpha+(1-\alpha)\left(\frac{1}{\alpha}+1\right)\chi \\
& +\frac{2}{\alpha}-2\alpha+(1-\alpha)\left(\frac{1}{\alpha}+1\right)\chi \\
& -\frac{2}{\alpha^2}-\frac{2}{\alpha}+1+\alpha+(1-\alpha)\left(-\frac{1}{\alpha^2}-\frac{2}{\alpha}-1\right)\chi \\
& +\frac{2}{\alpha^2}-\frac{2}{\alpha}-1+\alpha+(1-\alpha)\left(\frac{2}{\alpha^2}-1\right)\chi\Bigg\} \\
& +\left(\frac{n}{\alpha\chi}\right)^2\Bigg[1+\alpha \\
& +(1-\alpha^2)(1+\chi)\left(-1+\frac{1}{\alpha}+1+\frac{1}{\alpha}-\frac{1}{\alpha^2}-\frac{1}{\alpha}+\frac{1}{\alpha^2}-\frac{1}{\alpha}\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1-\alpha}{\alpha^2}(1-\alpha^2)\chi(1+\chi)\Bigg] \\
= \; & \Delta\frac{n}{\alpha^2\chi}\left[1+\alpha+\alpha^2+(1-\alpha)\frac{1-\alpha^2}{\alpha^2}\chi\right] \\
& +\left(\frac{n}{\alpha\chi}\right)^2(1+\alpha)\left[1+\left(\frac{1-\alpha}{\alpha}\right)^2\chi(1+\chi)\right] \\
> \; & 0.
\end{aligned}
$$

Second term in square brackets in (17):

$$
\begin{aligned}
& -3l^*\nu^*+2\alpha(1-\alpha)l^*z^*-2\alpha l^*\gamma^*-(1-\alpha^2)\nu^*z^*+(1+\alpha)\nu^*\gamma^* \\
& -\alpha^2(1-\alpha)z^*\gamma^*+\underbrace{(1-\alpha^2)\frac{l^*\nu^*z^*}{\gamma^*}}_{>(1-\alpha^2)l^*\nu^*}+\underbrace{\frac{2}{1-\alpha}\frac{l^*\nu^*\gamma^*}{z^*}}_{>2(1+\alpha)l^*\nu^*}+\underbrace{\alpha^2(1-\alpha)\frac{l^*z^*\gamma^*}{\nu^*}}_{>\alpha^2(1-\alpha)z^*\gamma^*} \\
> \; & \alpha(2-\alpha)l^*\nu^*+2\alpha(1-\alpha)l^*z^*-2\alpha l^*\gamma^*-(1-\alpha^2)\nu^*z^*+(1+\alpha)\nu^*\gamma^* \\
= \; & \frac{2-\alpha}{\alpha}\left(\Delta+\frac{1+\alpha}{\chi}n\right)\left(\Delta+\frac{1}{\chi}n\right) \\
& +\frac{2(1-\alpha)}{\alpha^2}\left(\Delta+\frac{1+\alpha}{\chi}n\right)\left(\Delta+\frac{1+\chi}{\chi}n\right) \\
& -\frac{2}{\alpha^2}\left(\Delta+\frac{1+\alpha}{\chi}n\right)\left[\Delta+(1-\alpha^2)\frac{1+\chi}{\chi}n\right] \\
& -\frac{1-\alpha^2}{\alpha^3}\left(\Delta+\frac{1}{\chi}n\right)\left(\Delta+\frac{1+\chi}{\chi}n\right) \\
& +\frac{1+\alpha}{\alpha^3}\left(\Delta+\frac{1}{\chi}n\right)\left[\Delta+(1-\alpha^2)\frac{1+\chi}{\chi}n\right] \\
= \; & \Delta^2\frac{1}{\alpha^2}\left[\alpha(2-\alpha)+2(1-\alpha)-2-\frac{1-\alpha^2}{\alpha}+\frac{1+\alpha}{\alpha}\right] \\
& +\Delta\frac{n}{\alpha^2\chi}\Bigg\{\alpha(2-\alpha)(1+\alpha+1) \\
& +2(1-\alpha)(1+\alpha+1+\chi) \\
& -2[1+\alpha+(1-\alpha^2)(1+\chi)] \\
& -\frac{1-\alpha^2}{\alpha}(1+1+\chi) \\
& +\frac{1+\alpha}{\alpha}\left[1+(1-\alpha^2)(1+\chi)\right]\Bigg\} \\
+ \; & \left(\frac{n}{\alpha\chi}\right)^2\Bigg[\alpha(2-\alpha)(1+\alpha) \\
& +2(1-\alpha)(1+\alpha)(1+\chi) \\
& -2(1+\alpha)(1-\alpha^2)(1+\chi)
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1-\alpha^2}{\alpha}(1+\chi) \\
& +\frac{1+\alpha}{\alpha}(1-\alpha^2)(1+\chi)\Big] \\
= \; & \Delta^2 \frac{1}{\alpha^2}\left(2\alpha-\alpha^2+2-2\alpha-2-\frac{1}{\alpha}+\alpha+\frac{1}{\alpha}+1\right) \\
& +\Delta\frac{n}{\alpha^2\chi}\Big[4\alpha-\alpha^3 \\
& +2-2\alpha^2+2(1-\alpha)(1+\chi) \\
& -2-2\alpha+(-2-2\alpha)(1-\alpha)(1+\chi) \\
& -\frac{1}{\alpha}+\alpha+\left(-\frac{1}{\alpha}-1\right)(1-\alpha)(1+\chi) \\
& +\frac{1}{\alpha}+1+\left(\frac{1}{\alpha}+2+\alpha\right)(1-\alpha)(1+\chi)\Big] \\
& +\left(\frac{n}{\alpha\chi}\right)^2\Big[2\alpha+\alpha^2-\alpha^3 \\
& +\left(2-2-2\alpha-\frac{1}{\alpha}+\frac{1}{\alpha}+1\right)(1-\alpha^2)(1+\chi)\Big] \\
= \; & \Delta^2\frac{1+\alpha-\alpha^2}{\alpha^2} \\
& +\Delta\frac{n}{\alpha^2\chi}\left[1+3\alpha-2\alpha^2-\alpha^3+(1-\alpha)^2(1+\chi)\right] \\
& +\left(\frac{n}{\alpha\chi}\right)^2\left[2\alpha+\alpha^2-\alpha^3+(1-\alpha^2)(1-2\alpha)(1+\chi)\right] \\
= \; & \Delta^2\frac{1+\alpha-\alpha^2}{\alpha^2} \\
& +\Delta\frac{n}{\alpha^2\chi}\left[2+\alpha-\alpha^2-\alpha^3+\chi(1-\alpha)^2\right] \\
& +\left(\frac{n}{\alpha\chi}\right)^2\left[1+\alpha^3+\chi(1-\alpha^2)(1-2\alpha)\right].
\end{aligned}
$$

The fraction in (18) is greater than 4:

Denote the denominator of the fraction on the right-hand side of (18) as $g(\alpha) \equiv (1-\alpha^2)(2\alpha-1) = -2\alpha^3+\alpha^2+2\alpha-1$. The first and second derivatives are

$$
g'(\alpha) = -6\left(\alpha^2-\frac{1}{3}\alpha-\frac{1}{3}\right)
$$

$$
g''(\alpha) = -6\left(2\alpha-\frac{1}{3}\right).
$$

$g(\alpha)$ assumes a local maximum at $\alpha = (1+\sqrt{13})/6$ with

$$
g\left(\frac{1+\sqrt{13}}{6}\right) = \frac{13\sqrt{13}-35}{54}.
$$

Since $g(0) = -1 < 0$, this is a global maximum, given $\alpha > 0$. So the denominator is less than $(13\sqrt{13} - 35)/54$. For $\alpha > 1/2$, the numerator is greater than $9/8$, and, consequently, the fraction is greater than $243/(52\sqrt{13} - 140) = 5.117 > 4$. (Actually, the fraction is greater than 6.464.)

$\chi > 4$ is inconsistent with (16):

Suppose $\chi > 4$ and (16) holds true. Then:

$$4 < \frac{2\nu^*\gamma^* - (1-\alpha^2)\nu^* z^* - \alpha(1-\alpha) z^*\gamma^*}{l^*\nu^* - \alpha(1-\alpha)l^* z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*}$$

Using the fact that $\mathbf{x}^* > \mathbf{0}$, that the denominator is greater than $l^*\nu^* + (1-\alpha)l^*z^*$ (cf. (A.9)), that $2/\alpha < 4$ for $\alpha > 1/2$, and that $2(1+\alpha) < 4$, it follows that

$$\begin{aligned} 4 &< \frac{2\nu^*\gamma^*}{l^*\nu^* - \alpha(1-\alpha)l^* z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*} \\ &< \frac{2\nu^*\gamma^*}{l^*\nu^* + (1-\alpha)l^* z^*} \end{aligned}$$

$$\begin{aligned} 4l^*\nu^* + 4(1-\alpha)l^*z^* - 2\nu^*\gamma^* &< 0 \\ \frac{2}{\alpha}l^*\nu^* + 2(1+\alpha)(1-\alpha)l^*z^* - 2\nu^*\gamma^* &< 0 \\ \frac{2}{\alpha}l^*\nu^* + 2(1-\alpha^2)l^*z^* - 2\nu^*\gamma^* &< 0 \\ \frac{1}{\alpha}l^*\nu^* + (1-\alpha^2)l^*z^* - \nu^*\gamma^* &< 0 \end{aligned}$$

Using (6)-(9), it follows that

$$\begin{aligned} 0 &> \frac{1}{\alpha^3}\left(\Delta + \frac{1+\alpha}{\chi}n\right)\left(\Delta + \frac{1}{\chi}n\right) \\ &\quad + \frac{1-\alpha^2}{\alpha^3}\left(\Delta + \frac{1+\alpha}{\chi}n\right)\left(\Delta + \frac{1+\chi}{\chi}n\right) \\ &\quad - \frac{1}{\alpha^3}\left(\Delta + \frac{1}{\chi}n\right)\left[\Delta + (1-\alpha^2)\frac{1+\chi}{\chi}n\right] \\ &= \Delta^2\frac{1}{\alpha^3}\left[1 + (1-\alpha^2) - 1\right] \\ &\quad + \Delta\frac{n}{\alpha^3\chi}\left\{1 + \alpha + 1 + (1-\alpha^2)(1+\alpha+1+\chi) - \left[1 + (1-\alpha^2)(1+\chi)\right]\right\} \\ &\quad + \left(\frac{n}{\chi}\right)^2\frac{1}{\alpha^3}\left[1 + \alpha + (1-\alpha^2)(1+\alpha)(1+\chi) - (1-\alpha^2)(1+\chi)\right] \\ &= \Delta^2\frac{1}{\alpha^3}(1-\alpha^2) \\ &\quad + \Delta\frac{n}{\alpha^3\chi}(1+\alpha)(2-\alpha^2) \\ &\quad + \left(\frac{n}{\chi}\right)^2\frac{1}{\alpha^3}(1+\alpha)\left[1 + \alpha(1-\alpha)(1+\chi)\right] \\ &> 0, \end{aligned}$$

a contradiction.

Lemma 3:

The expression on the right-hand side of (19) is less than the expression on the right-hand side of (16) if

$$\frac{2\nu^* - \alpha(1-\alpha)z^* + \gamma^*}{l^* + \frac{1-\alpha}{\alpha}\nu^*} < \frac{2\nu^*\gamma^* - (1-\alpha^2)\nu^* z^* - \alpha(1-\alpha)z^*\gamma^*}{l^*\nu^* - \alpha(1-\alpha)l^* z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*}.$$

It has been shown above that the denominator on the right-hand side (i.e., the term in square brackets in (13)) is positive. So this condition can be manipulated as follows:

$$\begin{aligned}
0 \;<\;& \left(l^* + \frac{1-\alpha}{\alpha}\nu^*\right)\left[2\nu^*\gamma^* - (1-\alpha^2)\nu^* z^* - \alpha(1-\alpha)z^*\gamma^*\right] \\
& - [2\nu^* - \alpha(1-\alpha)z^* + \gamma^*]\left[l^*\nu^* - \alpha(1-\alpha)l^* z^* + l^*\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^* + \frac{1-\alpha}{\alpha}\nu^*\gamma^*\right] \\
=\;& 2l^*\nu^*\gamma^* - (1-\alpha^2)l^*\nu^* z^* - \alpha(1-\alpha)l^* z^*\gamma^* \\
& + 2\frac{1-\alpha}{\alpha}(\nu^*)^2\gamma^* - (1-\alpha^2)\frac{1-\alpha}{\alpha}(\nu^*)^2 z^* - (1-\alpha)^2\nu^* z^*\gamma^* \\
& - 2l^*(\nu^*)^2 + 2\alpha(1-\alpha)l^*\nu^* z^* - 2l^*\nu^*\gamma^* + 2(1-\alpha^2)\frac{1-\alpha}{\alpha}(\nu^*)^2 z^* - 2\frac{1-\alpha}{\alpha}(\nu^*)^2\gamma^* \\
& + \alpha(1-\alpha)l^*\nu^* z^* - \alpha^2(1-\alpha)^2 l^*(z^*)^2 + \alpha(1-\alpha)l^* z^*\gamma^* - (1-\alpha^2)(1-\alpha)^2\nu^*(z^*)^2 + (1-\alpha)^2\nu^* z^*\gamma^* \\
& - l^*\nu^*\gamma^* + \alpha(1-\alpha)l^* z^*\gamma^* - l^*(\gamma^*)^2 + (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^*\gamma^* - \frac{1-\alpha}{\alpha}\nu^*(\gamma^*)^2 \\
=\;& -2l^*(\nu^*)^2 \\
& + [-(1-\alpha^2) + 2\alpha(1-\alpha) + \alpha(1-\alpha)]l^*\nu^* z^* \\
& + (2-2-1)l^*\nu^*\gamma^* \\
& + [-\alpha^2(1-\alpha)^2]l^*(z^*)^2 \\
& + [-\alpha(1-\alpha) + \alpha(1-\alpha) + \alpha(1-\alpha)]l^* z^*\gamma^* \\
& + (-1)l^*(\gamma^*)^2 \\
& + \left[-(1-\alpha^2)\frac{1-\alpha}{\alpha} + 2(1-\alpha^2)\frac{1-\alpha}{\alpha}\right](\nu^*)^2 z^* \\
& + \left(2\frac{1-\alpha}{\alpha} - 2\frac{1-\alpha}{\alpha}\right)(\nu^*)^2\gamma^* \\
& + [-(1-\alpha^2)(1-\alpha)^2]\nu^*(z^*)^2 \\
& + \left[-(1-\alpha)^2 + (1-\alpha)^2 + (1-\alpha^2)\frac{1-\alpha}{\alpha}\right]\nu^* z^*\gamma^* \\
& + \left(-\frac{1-\alpha}{\alpha}\right)\nu^*(\gamma^*)^2 \\
=\;& -\Big[2l^*(\nu^*)^2 + (1-\alpha)(1-2\alpha)l^*\nu^* z^* + l^*\nu^*\gamma^* + \alpha^2(1-\alpha)^2 l^*(z^*)^2 - \alpha(1-\alpha)l^* z^*\gamma^* + l^*(\gamma^*)^2 \\
& - (1-\alpha^2)\frac{1-\alpha}{\alpha}(\nu^*)^2 z^* + (1-\alpha^2)(1-\alpha)^2\nu^*(z^*)^2 - (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^* z^*\gamma^* + \frac{1-\alpha}{\alpha}\nu^*(\gamma^*)^2\Big].
\end{aligned}$$

We have to show that the term in square brackets is positive. To do so, we use five preliminary results (obtained using (A.5)):

$$(1-\alpha)(1-2\alpha)l^*\nu^*z^* = (1-\alpha)^2 l^*\nu^*z^* - \alpha(1-\alpha)l^*\nu^*z^* > -\alpha(1-\alpha)l^*\nu^*z^*$$

$$\underbrace{l^*}_{>\nu^*}\nu^*\underbrace{\gamma^*}_{>(1-\alpha^2)z^*} > (1-\alpha^2)(\nu^*)^2z^*$$

$$l^*(\gamma^*)^2 = l^*\underbrace{\gamma^*}_{>(1-\alpha^2)z^*}\gamma^* > (1-\alpha^2)l^*z^*\gamma^* = (1+\alpha)(1-\alpha)l^*z^*\gamma^* > \alpha(1-\alpha)l^*z^*\gamma^*$$

$$\begin{aligned}
& -(1-\alpha^2)\frac{1-\alpha}{\alpha}(\nu^*)^2z^* \\
= \; & -(1-\alpha+\alpha)(1-\alpha^2)\frac{1-\alpha}{\alpha}(\nu^*)^2z^* \\
= \; & -(1-\alpha^2)\frac{(1-\alpha)^2}{\alpha}(\nu^*)^2z^* - (1-\alpha^2)(1-\alpha)(\nu^*)^2z^* \\
= \; & -(1-\alpha^2)\frac{(1-\alpha)^2}{\alpha}(\nu^*)^2z^* - (1-\alpha^2)(\nu^*)^2z^* + \alpha(1-\alpha^2)\underbrace{\nu^*}_{>\frac{l^*}{1+\alpha}}\nu^*z^* \\
> \; & -(1-\alpha^2)\frac{(1-\alpha)^2}{\alpha}(\nu^*)^2z^* - (1-\alpha^2)(\nu^*)^2z^* + \alpha(1-\alpha)l^*\nu^*z^*
\end{aligned}$$

$$(1-\alpha^2)(1-\alpha)^2\nu^*(z^*)^2 = (1-\alpha^2)(1-\alpha)^2\nu^*\underbrace{z^*}_{>\frac{\nu^*}{\alpha}}z^* > (1-\alpha^2)\frac{(1-\alpha)^2}{\alpha}(\nu^*)^2z^*$$

$$\frac{1-\alpha}{\alpha}\nu^*(\gamma^*)^2 = \frac{1-\alpha}{\alpha}\nu^*\underbrace{\gamma^*}_{>(1-\alpha^2)z^*}\gamma^* > (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^*z^*\gamma^*.$$

Using these results and $\mathbf{x}^* > \mathbf{0}$, we find:

$$\begin{aligned}
& \overbrace{2l^*(\nu^*)^2}^{>0}\overbrace{+(1-\alpha)(1-2\alpha)l^*\nu^*z^*}^{>-\alpha(1-\alpha)l^*\nu^*z^*}\overbrace{+l^*\nu^*\gamma^*}^{>(1-\alpha^2)(\nu^*)^2z^*}\overbrace{+\alpha^2(1-\alpha)^2l^*(z^*)^2}^{>0}-\alpha(1-\alpha)l^*z^*\gamma^*\overbrace{+l^*(\gamma^*)^2}^{>\alpha(1-\alpha)l^*z^*\gamma^*} \\
& \underbrace{-(1-\alpha^2)\frac{1-\alpha}{\alpha}(\nu^*)^2z^*}_{\substack{> -(1-\alpha^2)\frac{(1-\alpha)^2}{\alpha}(\nu^*)^2z^* \\ -(1-\alpha^2)(\nu^*)^2z^* \\ +\alpha(1-\alpha)l^*\nu^*z^*}}\underbrace{+(1-\alpha^2)(1-\alpha)^2\nu^*(z^*)^2}_{>(1-\alpha^2)\frac{(1-\alpha)^2}{\alpha}(\nu^*)^2z^*}-(1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^*z^*\gamma^*\underbrace{+\frac{1-\alpha}{\alpha}\nu^*(\gamma^*)^2}_{>(1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^*z^*\gamma^*} \\
> \; & -\alpha(1-\alpha)l^*\nu^*z^* + (1-\alpha^2)(\nu^*)^2z^* - \alpha(1-\alpha)l^*z^*\gamma^* + \alpha(1-\alpha)l^*z^*\gamma^* \\
& -(1-\alpha^2)\frac{(1-\alpha)^2}{\alpha}(\nu^*)^2z^* - (1-\alpha^2)(\nu^*)^2z^* + \alpha(1-\alpha)l^*\nu^*z^* + (1-\alpha^2)\frac{(1-\alpha)^2}{\alpha}(\nu^*)^2z^* \\
& -(1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^*z^*\gamma^* + (1-\alpha^2)\frac{1-\alpha}{\alpha}\nu^*z^*\gamma^* \\
= \; & 0.
\end{aligned}$$

Equation (22):

$\mathbf{b}_i e^{q_i t}$ are particular solutions to (10) ($\mathbf{b}_i = (b_{li}, b_{\nu i}, b_{zi}, b_{\gamma i})'$, $i = 1, \ldots, 4$). From $\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x} - \mathbf{x}^*)$, it follows that $q_i \mathbf{b}_i e^{q_i t} = \mathbf{J}\mathbf{b}_i e^{q_i t}$, that is $(\mathbf{J} - q_i \mathbf{I})\mathbf{b}_i = \mathbf{0}$. The characteristic equation, $f(q) = |\mathbf{J} - q\mathbf{I}| = 0$, ensures that non-zero solutions $\mathbf{b}_i$ exist. $\mathbf{b}_i$ is the eigenvector corresponding to eigenvalue q_i ($i = 1, \ldots, 4$). The general solution of system (10) is $\mathbf{x}(t) - \mathbf{x}^* = \sum_{i=1}^4 B_i \mathbf{b}_i e^{q_i t}$, where the B_i's are constants to be determined below ($i = 1, \ldots, 4$). Since two eigenvalues, q_3 and q_4, say, are positive, we must have $\mathbf{x}(t) - \mathbf{x}^* = \sum_{i=1}^2 B_i \mathbf{b}_i e^{q_i t}$. Evaluating this equation at $t = 0$ yields (20):

$$\begin{aligned} \mathbf{x}(0) - \mathbf{x}^* &= \sum_{i=1}^{2} B_i \mathbf{b}_i \\ \begin{pmatrix} l(0) - l^* \\ \nu(0) - \nu^* \\ z(0) - z^* \\ \gamma(0) - \gamma^* \end{pmatrix} &= B_1 \begin{pmatrix} b_{l1} \\ b_{\nu 1} \\ b_{z1} \\ b_{\gamma 1} \end{pmatrix} + B_2 \begin{pmatrix} b_{l2} \\ b_{\nu 2} \\ b_{z2} \\ b_{\gamma 2} \end{pmatrix}. \end{aligned} \tag{A.10}$$

From the first the line in (A.10),

$$B_2 = \frac{l(0) - l^* - b_{l1} B_1}{b_{l2}}. \tag{A.11}$$

Inserting this into the third line in (A.10) yields:

$$\begin{aligned} z(0) - z^* &= B_1 b_{z1} + \frac{l(0) - l^* - b_{l1} B_1}{b_{l2}} b_{z2} \\ &= \frac{b_{z2}}{b_{l2}}[l(0) - l^*] + B_1 \left(b_{z1} - \frac{b_{l1} b_{z2}}{b_{l2}} \right) \\ B_1 &= \frac{z(0) - z^* - \frac{b_{z2}}{b_{l2}}[l(0) - l^*]}{b_{z1} - \frac{b_{l1} b_{z2}}{b_{l2}}}. \end{aligned} \tag{A.12}$$

Using (A.11) and (A.12) in the second equation in (A.10) yields (22):

$$\begin{aligned} \nu(0) - \nu^* &= B_1 b_{\nu 1} + \frac{l(0) - l^* - b_{l1} B_1}{b_{l2}} b_{\nu 2} \\ &= \frac{b_{\nu 2}}{b_{l2}}[l(0) - l^*] + B_1 \left(b_{\nu 1} - \frac{b_{l1} b_{\nu 2}}{b_{l2}} \right) \\ &= \frac{b_{\nu 2}}{b_{l2}}[l(0) - l^*] + \frac{z(0) - z^* - \frac{b_{z2}}{b_{l2}}[l(0) - l^*]}{b_{z1} - \frac{b_{l1} b_{z2}}{b_{l2}}} \left(b_{\nu 1} - \frac{b_{l1} b_{\nu 2}}{b_{l2}} \right) \\ &= [l(0) - l^*] \left[\frac{b_{\nu 2}}{b_{l2}} - \frac{\frac{b_{z2}}{b_{l2}}}{b_{z_1} - \frac{b_{l1} b_{z2}}{b_{l2}}} \left(b_{\nu 1} - \frac{b_{l1} b_{\nu 2}}{b_{l2}} \right) \right] + [z(0) - z^*] \frac{b_{\nu 1} - \frac{b_{l1} b_{\nu 2}}{b_{l2}}}{b_{z_1} - \frac{b_{l1} b_{z2}}{b_{l2}}} \\ &= [l(0) - l^*] \left[\frac{b_{\nu 2}}{b_{l2}} - \frac{\frac{b_{z2}}{b_{l2}}(b_{\nu 1} b_{l2} - b_{l1} b_{\nu 2})}{b_{z1} b_{l2} - b_{l1} b_{z2}} \right] + [z(0) - z^*] \frac{b_{\nu 1} b_{l2} - b_{l1} b_{\nu 2}}{b_{z1} b_{l2} - b_{l1} b_{z2}} \\ &= [l(0) - l^*] \frac{\frac{b_{\nu 2}}{b_{l2}}(b_{z1} b_{l2} - b_{l1} b_{z2}) - \frac{b_{z2}}{b_{l2}}(b_{\nu 1} b_{l2} - b_{l1} b_{\nu 2})}{b_{z1} b_{l2} - b_{l1} b_{z2}} + [z(0) - z^*] \frac{b_{\nu 1} b_{l2} - b_{l1} b_{\nu 2}}{b_{z1} b_{l2} - b_{l1} b_{z2}} \end{aligned}$$

$$
\begin{aligned}
&= [l(0)-l^*]\frac{b_{z1}b_{\nu 2}-b_{\nu 1}b_{z2}}{b_{z1}b_{l2}-b_{l1}b_{z2}}+[z(0)-z^*]\frac{b_{\nu 1}b_{l2}-b_{l1}b_{\nu 2}}{b_{z1}b_{l2}-b_{l1}b_{z2}} \\
&= \frac{[l(0)-l^*](b_{z1}b_{\nu 2}-b_{\nu 1}b_{z2})+[z(0)-z^*](b_{\nu 1}b_{l2}-b_{l1}b_{\nu 2})}{b_{z1}b_{l2}-b_{l1}b_{z2}}.
\end{aligned}
$$