Thin Vibrating Rods: Γ-Convergence, Large Time Existence and First Order Asymptotics

Dissertation zur Erlangung des Doktorgrades der Naturwissenschaften (Dr. rer. nat.) der Fakultät für Mathematik der Universität Regensburg

vorgelegt von

Tobias Ameismeier

aus Regensburg

im Jahr 2021

Promotionsgesuch eingereicht am: 20.1.2021

Die Arbeit wurde angeleitet von: Prof. Dr. Helmut Abels (Universität Regensburg, Erstbetreuer)
Prof. Dr. Georg Dolzmann (Universität Regensburg, Zweitbetreuer)

Prüfungsausschuss: Vorsitzender: Prof. Dr. Walter Gubler
Erst-Gutachter: Prof. Dr. Helmut Abels
Zweit-Gutachterin: Prof. Ph.D. Maria Giovanna Mora
weiterer Prüfer: Prof. Dr. Georg Dolzmann
Ersatzprüfer: Prof. Dr. Harald Garcke

Abstract

We consider a thin rod $\Omega_h := (0, L) \times hS$ for some smooth domain $S \subset \mathbb{R}^2$ and study the limiting behaviour of a scaled elastic energy $\mathcal{E}^{(h)}$, which enforces periodic boundary conditions on the end faces, by means of Γ-convergence. The limiting energies are of von Kármán and linear type, respectively. We took into account the presence of external forces with zero tangential component. Furthermore, assuming that the elastic energy density satisfies $|DW(F)| \leq C(|F| + 1)$ for all $F \in \mathbb{R}^{3\times 3}$, we proved that local minimizers, for which $\mathcal{E}^{(h)}$ is bounded by Ch^4, converge (for a subsequence) to stationary points of the limiting energy.
Subsequently, we regarded the dynamical evolution of the thin rod described by an appropriately scaled, nonlinear wave equation. Under the assumption of well prepared initial data and external forces, we proved that a solution exists for arbitrarily large times, if the diameter of the cross section is chosen sufficiently small. The scaling regime is such that the limiting equations are linear.
Finally, for a specific scaling, we constructed an approximation of the solution, using a suitable asymptotic expansion ansatz based upon solutions to the one-dimensional beam equation. Following this, we derive the existence of appropriately scaled initial data and can bound the difference between the analytical solution and the approximating sequence.

Zusammenfassung

Wir betrachten für einen dünnen Stab $\Omega_h := (0, L) \times hS$, wobei $S \subset \mathbb{R}^2$ ein glattes Gebiet sei, das Grenzverhalten einer skalierten, elastischen Energie $\mathcal{E}^{(h)}$, welche periodische Randbedingungen an den Enden vorschreibt. Die Grenzfunktionale sind von Kármán beziehungsweise lineare Energien. Hierbei haben wir externe Kräfte berücksichtigt, welche in tangentialer Richtung verschwinden. Des Weiteren zeigen wir unter der Bedingung, dass die elastische Energiedichte $|DW(F)| \leq C(|F| + 1)$ für alle $F \in \mathbb{R}^{3\times 3}$ erfüllt, die Konvergenz von (einer Teilfolge von) lokalen Minimieren, für welche $\mathcal{E}^{(h)}$ durch Ch^4 beschränkt ist, gegen stationäre Punkte der Grenzenergien.
Anschließend betrachten wir die dynamische Evolution von dünnen Stäben, welche durch eine geeignet skalierte Wellengleichung beschrieben werden. Unter der Annahme von passend skalierenden Anfangsdaten und externen Kräften haben wir gezeigt, dass die Lösung für beliebig große Zeiten existiert, wenn der Durchmesser des Querschnitts klein genug gewählt wird. Hierbei betrachten wir das Skalierungsregime, bei welchem die Grenzgleichungen linear sind.
Abschließend konstruieren wir für eine spezielle Skalierungswahl eine Näherungslösung mittels eines asymptotischen Expansionsansatzes, welcher auf der Lösung von einer eindimensionalen Balkengleichung beruht. Im Anschluss daran beweisen wir die Existenz von geeigneten Anfangswerten und schätzen die Differenz zwischen der analytischen Lösung und der Näherungsfolge ab.

Contents

1 Introduction

> Si habeatur annulus *ADBEA* elasticus, eique alicubi impetus imprimatur, mutabit is formam circularem, sed rursus, ob elasticitatem, se restituet, versum nimis, et ita oscillationes peraget. Hae dissertatione constitui oscillationes hasce persequi, et tempora earum ex legibus mechanicis determinare.
>
> — Leonhard Euler, [Eul27]

The problems of three-dimensional elasticity theory have a long history, starting with Bernoulli and Euler in the seventeenth and eighteenth century. They already introduced fundamental quantities, including strain and formulated crucial laws such as the Law of Angular Momentum. Although the mathematical tools and perspectives changed over the decades, the core question remained the same: How does a body of elastic material deform under external forces?

The models trying to answer this question are nowadays often stated in terms of a minimization problem of integral form, for example, given an energy density $W\colon \mathbb{R}^{3\times 3} \to [0,\infty]$ suitably chosen one considers

$$\min_{y\in W_2^1(\Omega;\mathbb{R}^3)} \int_\Omega \Big(W(\nabla y) - f\cdot y\Big)dx.$$

However, due to the character of the physical situation the energy density cannot be convex and hence neither is the functional making an analysis in the general situation extremely complex. One possible workaround is to consider lower dimensional objects - such as plates, shells or rods - for which most situations are much simpler from an analytical point of view. The downside of this approach is that most often this was done by imposing *a priori* assumptions on the distribution of stress, the relation between stress and strain, or the properties of the studied deformations. A more in-depth introduction to continuum mechanics can be found for instance in [Gur81; EGK17] and for a survey about lower dimensional theories and their derivation we refer to [Ant05]. One example of such an assumption introduced by Euler can be found in [Eul27]. He examined the vibration elastic rings, which he then applied to bells. In his work he assumed that the inner edge does not change in length, which later turned out to be false. Such approaches led to several lower dimensional theories which were not consistently derived from the general three-dimensional one.

However, such a theory would be desirable, due to the following reasons:

- First, the mathematical description of the three-dimensional process arises from general balance laws.

- Second, the lower dimensional problems are much better understood analytically. For example large time existence results often exist, whereas these are quite rare for the higher dimensional equations.
- Lastly, it is simpler to numerically approximate lower dimensional equations. Thus if convergence can be proven, one could justify that the numerical solution to the reduced dimension problem is an approximation of its three-dimensional counterpart.

Thus the question arises, which lower dimensional theory originates from the three-dimensional one, if we - for instance - reduce the thickness parameter?
One major method for connecting both is via the notion of Γ-convergence introduced by De Giorgi and Franzoni in the mid 70's, cf. [DG75; DGF75]. It is a definition of a variational convergence for functionals with the property that accumulation points of quasi minimizing sequences are minimizers of the limiting functional (see [DM93; Bra02] for an introduction). Which lower dimensional theory one obtains, largely depends on how the energy per unit volume behaves. In the case of rods this would correspond to $\mathcal{E}^h/h^2 \sim h^\beta$ for some $\beta \in [0, \infty)$, where $\mathcal{E}^{(h)}$ is the elastic energy and h is the diameter.
The first derivation using no *a priori* assumption is done by Acerbi, Buttazzo and Percivale in [ABP91], where they successfully deduced a string theory using Γ-convergence. A sharp tool for this approach was given by the result of Friesecke, James and Müller in [FJM02]. They were able to prove a strong geometric rigidity result, which can be summarised as a nonlinear variant of the Korn inequality. This led to a major breakthrough as it made it possible to deduce quantitative convergence results with respect to the thickness or diameter h. This was not possible with former results, e.g. Rešetnjak [Re67]. In subsequent papers many authors derived convergence results under various assumptions and energies based on this geometric rigidity. Examples include [FJM06; MM03; MM04; Sca09], where the geometries of plates and (curved) rods are considered with and without external forces, respectively. Moreover, additional analysis works on nonlinear constrained models, such as incompressibility or traction forces (e.g. [EK20b; EK20a] and [MM20]).
The downside of an approach via Γ-convergence is that, roughly speaking, only *global* minimizers of the three-dimensional theory converge to *global* minimizers of the limiting lower dimensional theory. Therefore, the natural next step was to study the behaviour of critical points of the energy functional, i.e. solutions to appropriate Euler-Lagrange equations. The formulation of such is in the context of elasticity a non-trivial problem, as physically coherent elastic energy densities W should satisfy the assumptions

$$W(A) \to \infty, \quad \text{if } \det(A) \to 0, \quad \text{and } W(A) = \infty, \quad \text{if } \det(A) \leq 0$$

in order to prevent self-penetration. But it can happen that for a minimizer $y^{min} \in W^1_2(\Omega; \mathbb{R}^3)$ the elastic energy density $W(\nabla y^{min})$ is unbounded (see [BM85]), as it might be energetically preferable. Thus the difference quotient of the energy functional can not be bounded in a suitable way, such that the limit can be obtained. Hence, in several works additional assumptions on the elastic energy density are introduced. Pakzad and Müller used

$$|DW(A)| \leq C(|A| + 1) \quad \text{for all } A \in \mathbb{R}^{3\times 3}$$

in order to rigorously justify the Euler-Lagrange equations considered in [MP08]. In [MMS07] a global Lipschitz property of DW is utilized. Furthermore, in [Bal02] a variant of Euler-Lagrange equations is derived under the subtle condition

$$|DW(A)A^T| \leq C(W(A) + 1) \quad \text{for all } A \in \mathbb{R}^{3\times 3},$$

which was later used in the work by Mora and Scardia, [MS12].
However, these results are restricted to the static case where the solution is not time dependent. The first of few dynamic considerations can be found in the works by Abels, Mora and Müller, [AMM11a; AMM11b]. The biggest problem when considering the convergence of local minimizers in a dynamic setting is, as mentioned, the lack of large time existence results for solutions of

the three-dimensional problem. Hence, it is *a priori* not clear on which time interval the two solutions can be compared. In the first paper the authors therefore prove the existence of a solution for large times, if the thickness of the plate is small enough. This holds true for suitably small forces and well prepared initial data. With this, it is then possible to compare the solutions of the von Kármán equations, which exist for all times, with the ones of the three-dimensional nonlinear problem, as published in their second contribution. Similarly, Qin and Yao deduced a convergence result for shells in [QY20], but under the assumption of large time existence.

The starting point of the analysis of thin rods, provided in this thesis, is the elastic energy

$$\tilde{\mathcal{E}}^{(h)}(z) := \begin{cases} \dfrac{1}{h^2}\displaystyle\int_{\Omega_h} W(\nabla z(x)) - (z(x)-x)\cdot f^{(h)}(x)dx, & \text{if } z - \mathrm{Id} \in H^1_{per}(\Omega_h;\mathbb{R}^3), \\ +\infty, & \text{otherwise,} \end{cases}$$

where $z \in W^1_2(\Omega_h;\mathbb{R}^3)$ denotes the deformation, $\Omega_h := (0,L)\times hS$ is the reference configuration of the thin rod and $f^{(h)}$ describes external loads. The respective boundary conditions on the ends of the rod are already incorporated in the formulation of the energy. This will later become essential in the analysis of the dynamical process. For the mathematical treatment it is convenient to rescale the domain Ω_h to $\Omega := \Omega_1$. This leads to the scaled energy

$$\mathcal{E}^{(h)}(y) := \begin{cases} \displaystyle\int_{\Omega} W(\nabla_h y(x)) - (y(x)-x^{(h)})\cdot f^{(h)}(x)dx, & \text{if } y - \mathrm{Id}_h \in H^1_{per}(\Omega;\mathbb{R}^3), \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\mathrm{Id}_h(x) = x^{(h)} := (x_1, hx_2, hx_3)^T$ for all $x\in\Omega$ and $\nabla_h := (\partial_{x_1}, \frac{1}{h}\partial_{x_2}, \frac{1}{h}\partial_{x_3})$. The limiting functional depends on the scaling properties of $f^{(h)}$ and thus on the scaling of $\mathcal{E}^{(h)}$ with respect to $h\to 0$. In this thesis we will regard energies of order $h^{2\alpha-2}$ for $\alpha \geq 3$, corresponding to $f^{(h)}$ being of order h^α. As in the literature, the choice of $\alpha = 3$ and $\alpha > 3$ leads to a von Kármán limiting energy and a linearised theory, respectively. Deformations of this scaling behaviour are close to a rigid motion. The limiting energies for $\frac{1}{h^{2\alpha-2}}\mathcal{E}^{(h)}$ are derived as

$$\mathcal{E}_\alpha(u,v_2,v_3,w,R') := \mathcal{I}_\alpha(u,v_2,v_3,w) - \int_0^L (R')^T \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \cdot \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} dx_1,$$

where

$$\mathcal{I}_\alpha(u,v_2,v_3,w) := \begin{cases} \dfrac{1}{2}\displaystyle\int_0^L Q^0(u_{,1} + \tfrac{1}{2}(v_{2,1}^2 + v_{3,1}^2), A_{,1})dx_1, & \text{if } \alpha = 3, \\ \dfrac{1}{2}\displaystyle\int_0^L Q^0(u_{,1}, A_{,1})dx_1, & \text{if } \alpha > 3 \end{cases}$$

with $u,\ w \in H^1_{per}(0,L)$, $v \in H^2_{per}(0,L;\mathbb{R}^2)$ are the limits of appropriately scaled means of $y^{(h)}$ and R' is the 2×2-lower left submatrix of the limit of an approximating rotation derived from the geometrical rigidity of [FJM02]. The matrix $A \in H^1_{per}(0,L;\mathbb{R}^{3\times 3})$ is given by

$$A = \begin{pmatrix} 0 & -v_{2,1} & -v_{3,1} \\ v_{2,1} & 0 & -w \\ v_{3,1} & w & 0 \end{pmatrix}.$$

Moreover, $Q^0 \colon \mathbb{R}\times\mathbb{R}^{3\times 3}_{skew} \to [0,\infty)$ is defined by

$$Q^0(t,F) := \min_{\varphi\in H^1(S,\mathbb{R}^3)} \int_S Q_3\Big(te_1 + F\begin{pmatrix} 0 \\ x' \end{pmatrix}\Big|\varphi_{,2}\Big|\varphi_{,3}\Big)dx'$$

with $Q_3(G) := D^2W(Id)[G,G]$, the quadratic form of linearised elasticity.

Next, we show that the limits of local minimizers of the three-dimensional energy converge to solutions of the Euler-Lagrange equations for the limiting functionals in the von Kármán case, i.e. $\alpha = 3$. For this we introduce the assumption on W of Pakzad and Müller, namely

$$|DW(A)| \leq C(|A| + 1) \quad \text{for all } A \in \mathbb{R}^{3\times 3},$$

which leads to a rigorous derivation of the classical Euler-Lagrange equations in the three-dimensional case. We subsequently show that for a sequence of local minimizers $y^{(h)}$ of $\mathcal{E}^{(h)}$ satisfying

$$\int_\Omega W(\nabla_h y^{(h)})dx \leq Ch^4$$

the limit of $(u^{(h)}, v_2^{(h)}, v_3^{(h)}, w^{(h)}, \bar{R}^{(h)})$ exists (for a suitable subsequence) and $(u, v_2, v_3, w, \bar{R})$ is a stationary point of the limiting functional $\mathcal{E}_3$.
In the dynamical setting we investigate the linear limiting case $\alpha \geq 4$ in the pointwise formulation. Formally, the system can be derived by the Fréchet derivative of the total energy

$$\int_\Omega \left(\frac{|\partial_t y|^2}{2} + W(\nabla_h y) - f^h \cdot y \right) dx.$$

The scaling behaviour of the total energy is assumed to be $h^{2\alpha-2}$, which in turn implies that $f^{(h)} \sim h^\alpha$. In order to balance the kinetic and elastic part of the total energy we rescale the time via $\tau = ht$. Hence we obtain

$$E^{(h)}(y) = h^{2\alpha-2} \int_\Omega \frac{1}{h^{2\alpha-4}} \frac{|\partial_\tau y|^2}{2} + \frac{1}{h^{2\alpha-2}} W(\nabla_h y) - \frac{1}{h^\alpha} f^h \cdot \frac{1}{h^{\alpha-2}} y dx.$$

This leads to the scaled evolution equation with $g^h := \frac{1}{h^\alpha} f^h$

$$\partial_\tau^2 y - \frac{1}{h^2} \operatorname{div}_h(DW(\nabla_h y)) = h^{\alpha-2} g^h$$

where $g^h \sim 1$ for $h \to 0$. Moreover, we assume homogeneous Neumann boundary conditions on the outer surface and periodicity on the end faces of Ω. For well prepared initial data we are able to show that for any $T > 0$ there exists an $h_0 > 0$ such that strong solutions exist on $(0, T)$ for all $h \in (0, h_0]$.
In the case $\alpha = 4$ and $W(F) = \operatorname{dist}(F, SO(3))^2$ for all $F \in \mathbb{R}^{3\times 3}$ we are even able to construct an approximation of the solution. In an initial step, we solve a suitable one-dimensional beam equation. Using a formal asymptotic expansion ansatz we derive equations which have to be solved by the prefactors of lower order. Following this ansatz, we prove the existence of initial data satisfying the conditions of our large time existence result. Finally, we are able to bound the difference between the analytical solution, which exists on a fixed time interval, and the approximation solution.

Overview The second chapter is devoted to the notational convention and auxiliary results needed throughout the thesis. Most importantly, we derive relevant properties of the elastic energy density in Section 2.3 and prove Korn's inequality for thin rods in Section 2.4. In the third chapter we prove a Γ-convergence result for relative periodic deformations with external force. Subsequently, the convergence of local minimizers for the three-dimensional energy to solutions of the Euler-Lagrange equations of the one dimensional energy is provided in Chapter 4. In Chapter 5 the main result of the thesis is proven: large time existence for thin vibrating rods modelled by nonlinear elasticity. Finally Chapter 6 is devoted to the construction of the asymptotic expansion.

2

Mathematical Foundations

2.1 Preliminaries and Auxiliary Results

2.1.1 Notation

This section summarises the notation used throughout the thesis. The natural numbers without zero are denoted by $\mathbb{N}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the rational numbers with plus and minus infinity are denoted by $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. For any $n \in \mathbb{N}$ we denote the norm on $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ and the absolute value by $|.|$. With $L^p(M)$, $W_p^k(M)$ and $H^k(M) := W_2^k(M)$ we denote the classical Lebesgue and Sobolev spaces for some measurable, open set $M \subset \mathbb{R}^n$, with the embedding properties

$$W_p^m(M) \hookrightarrow W_p^k(M) \quad \text{if} \quad m - \frac{n}{p} \geq k - \frac{n}{q} \tag{2.1}$$

$$W_p^m(M) \hookrightarrow C^{k,\gamma}(\overline{M}) \quad \text{if} \quad m - \frac{n}{p} \geq k + \gamma \tag{2.2}$$

where $C^{k,\gamma}$ are the Hölder spaces. These embeddings are compact if M is a C^1-domain, $k \in \mathbb{N}_0$, $k < m$ and the respective inequalities are strict.
For the whole thesis we denote by $S \subset \mathbb{R}^2$ a smooth domain and $\Omega_h := (0, L) \times hS \subset \mathbb{R}^3$ for $h \in (0, 1]$ and $L > 0$ some length in $\mathbb{R}$. As an abbreviation we will write $\Omega := \Omega_1$. We assume that S satisfies

$$\int_S x_2 x_3 dx' = 0 \tag{2.3}$$

and

$$\int_S x_2 dx' = \int_S x_3 dx' = 0 \tag{2.4}$$

where $x' := (x_2, x_3) \subset \mathbb{R}^2$. This can always be achieved via a translation and rotation. Moreover, we assume $|S| = 1$, as with a scaling argument this is no loss of generality. Furthermore, we denote with ∇_h the scaled gradient defined as

$$\nabla_h = \Big(\partial_{x_1}, \frac{1}{h}\partial_{x_2}, \frac{1}{h}\partial_{x_3}\Big)^T. \tag{2.5}$$

The group of special orthogonal matrices is denoted by $SO(n)$. The definition is as follows

$$SO(n) := \{Q \in \mathbb{R}^{n\times n} \ : \ Q^T Q = QQ^T = Id \ \wedge \ \det Q = 1\}.$$

Remark 2.1.1. For the special case of three dimensions we want to note that $Q = (u|w|z) \in SO(3)$ for u, w and $z \in \mathbb{R}^3$ is equivalent to

$$|u| = |w| = |z| = 1, \quad w \cdot z = 0 \quad \text{ and } u = w \times z.$$

The subset of all rotations around the x_1-axes is denoted by $\mathcal{U}$, i.e.

$$\mathcal{U} := \{R \in SO(3) \ : \ R_{11} = 1\} \subset SO(3).$$

Remark 2.1.2. For $R \in \mathcal{U}$ it holds

$$R = \begin{pmatrix} 1 & 0 \\ 0 & R' \end{pmatrix}$$

where R' denotes the 2×2 submatrix of R consisting of the second and third rows and columns. This follows because for the first column it holds

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = R^T R e_1 = \begin{pmatrix} 1 + R_{21}^2 + R_{31}^2 \\ R_{12} + R_{22}R_{21} + R_{32}R_{31} \\ R_{13} + R_{23}R_{21} + R_{33}R_{31} \end{pmatrix}$$

Thus the first row implies $R_{21} = R_{31} = 0$ and with this the second and third row lead to $R_{12} = R_{13} = 0$. Thus the claim holds.

Throughout the thesis $\mathcal{L}^n(V)$, $n \in \mathbb{N}$ denotes the space of all n-linear mappings $G\colon V^n \to \mathbb{R}$ for a vector space V. As common we will use the classical identification of $\mathcal{L}^1(\mathbb{R}^{n\times n}) = (\mathbb{R}^{n\times n})'$ with $\mathbb{R}^{n\times n}$, i.e. $G \in \mathcal{L}^1(\mathbb{R}^{n\times n})$ is identified with $A \in \mathbb{R}^{n\times n}$ such that

$$G(X) = A : X \text{ for all } X \in \mathbb{R}^{n\times n}$$

where $A : X = \sum_{i,j=1}^n a_{ij}x_{ij}$ is the standard inner product on $\mathbb{R}^{n\times n}$. Equivalently, $G \in \mathcal{L}^2(\mathbb{R}^{n\times n})$ is identified with $\tilde{G}\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by

$$\tilde{G}X : Y = G(X, Y) \text{ for all } X, Y \in \mathbb{R}^{n\times n}. \tag{2.6}$$

In anticipation of some scaled Korn inequality stated in Section 2.4 we introduce a scaled inner product on $\mathbb{R}^{n\times n}$

$$A :_h B := \frac{1}{h^2} \operatorname{sym} A : \operatorname{sym} B + \operatorname{skew} A : \operatorname{skew} B$$

for all A, $B \in \mathbb{R}^{n\times n}$ and $h > 0$. The corresponding norm is denoted by $|A|_h := \sqrt{A :_h A}$. For $W \in \mathcal{L}^d(\mathbb{R}^{n\times n})$ we define the induced scaled norm by

$$|W|_h := \sup_{|A_j|_h \le 1, j=\{1,\ldots,d\}} |W(A_1, \ldots, W_d)|$$

As $|A|_h \ge |A|_1 =: |A|$ for all $A \in \mathbb{R}^{n\times n}$ it follows that $|W|_h \le |W|_1 =: |W|$ for all $W \in \mathcal{L}^d(\mathbb{R}^{n\times n})$ and $0 < h \le 1$. The scaled L^p-spaces we define as follows

$$\|W\|_{L_h^p(U,\mathcal{L}^d(\mathbb{R}^{n\times n}))} = \|W\|_{L_h^p(U)} = \left(\int_U |W(x)|_h^p dx \right)^{\frac{1}{p}}$$

if $p \in [1, \infty)$, where $U \subset \mathbb{R}^d$ is measurable. Thus $\|W\|_{L_h^p(U;\mathcal{L}^d(\mathbb{R}^{n\times n}))} \le \|W\|_{L^p(U;\mathcal{L}^d(\mathbb{R}^{n\times n}))}$.

The scaled norm for $f \in L^p(U, \mathbb{R}^{n\times n})$ is defined in the same way

$$\|f\|_{L_h^p(U,\mathbb{R}^{n\times n})} = \|f\|_{L_h^p(U)} = \left(\int_U |f(x)|_h^p dx \right)^{\frac{1}{p}}$$

and the inequality holds the other way round

$$\|f\|_{L^p_h(U;\mathbb{R}^{n\times n})} \geq \|f\|_{L^p(U;\mathbb{R}^{n\times n})}.$$

The standard notation $H^k(\Omega)$ and $H^k(\Omega;X)$ is used for L^2-Sobolev spaces with values in $\mathbb{R}$ and some space X, respectively. Moreover, we denote for $m \in \mathbb{N}$

$$H^m_{per}(\Omega) := \Big\{ f \in H^m(\Omega) \ : \ \partial^\alpha_x f|_{x_1=0} = \partial^\alpha_x f|_{x_1=L},\ |\alpha| \leq m-1 \Big\}.$$

A subscript (0) on a function space will always indicate that elements have zero mean value, i.e. for $g \in H^1_{(0)}(U)$ we have

$$\int_U g(x)dx = 0 \tag{2.7}$$

where $U \subset \mathbb{R}^n$ is open and bounded. In various estimates we will use an anisotropic variant of H^k, as we will have more regularity in lateral direction. Therefore we define

$$\begin{aligned} H^{m_1,m_2}(\Omega) := \Big\{ u \in L^2(\Omega) \ : & \partial^l_{x_1}\nabla^k_x u \in L^2(\Omega), k = 0,\dots,m_1, l = 0,\dots,m_2 \\ & \partial^q_{x_1}\partial^\alpha_x u\Big|_{x_1=0} = \partial^q_{x_1}\partial^\alpha_x u\Big|_{x_1=L}, q = 0,\dots,m_1, |\alpha| \leq m_2 \\ & \text{and } q + |\alpha| \leq m_1 + m_2 - 1\Big\} \end{aligned}$$

where $m_1,\ m_2 \in \mathbb{N}$, the inner product is given by

$$(f,g)_{H^{m_1,m_2}(\Omega)} = \sum_{k=0,\dots,m_1; l=0,\dots m_2} \left(\partial^l_{x_1}\nabla^k_x f, \partial^l_{x_1}\nabla^k_x g\right)_{L^2(\Omega)}.$$

Furthermore we will use the scaled norms

$$\begin{aligned} \|A\|_{H^m_h(\Omega)} &:= \left(\sum_{|\alpha|\leq m} \|\partial^\alpha_x A\|^2_{L^2_h(\Omega)} \right)^{\frac{1}{2}} \\ \|B\|_{H^{m_1,m_2}_h(\Omega)} &:= \left(\sum_{k=0,\dots,m_1; l=0,\dots,m_2} \|\partial^l_{x_1}\nabla^k_x B\|^2_{L^2_h(\Omega)} \right)^{\frac{1}{2}}. \end{aligned}$$

for $A \in H^m(\Omega;\mathbb{R}^{n\times n})$ and $B \in H^{m_1,m_2}(\Omega;\mathbb{R}^{n\times n})$ and $n \in \mathbb{N}$. As an abbreviation we denote for $u \in H^k(\Omega;\mathbb{R}^3)$ the symmetric scaled gradient by $\varepsilon_h(u) := \operatorname{sym}(\nabla_h u)$ and $\varepsilon(u) = \varepsilon_1(u) = \operatorname{sym}(\nabla u)$.
The space of periodic functions can be defined in an equivalent way, which is in some situations more convenient

$$\tilde{H}^m_{per}(\Omega) := \Big\{ f \in H^m_{loc}(\mathbb{R}\times\bar{S}) \ : \ f(x_1,x') = f(x_1+L,x') \text{ almost everywhere} \Big\}$$

equipped with the standard $H^m(\Omega)$-norm. As the maps $f \mapsto f|_\Omega$ and $f \mapsto f_{per}$ are isomorphisms, we identify $\tilde{H}^m_{per}(\Omega)$ with $H^m_{per}(\Omega)$. With this definition we obtain immediately that $C^\infty_{per}(\Omega)$ dense in $H^m_{per}(\Omega)$, because, as S is smooth there exists an appropriate extension operator and thus we can use a convolution argument. The following lemma provides the possibility to take traces for $u \in H^{0,1}(\Omega)$, more prices

Lemma 2.1.3. *The operator* $\operatorname{tr}_a\colon H^{0,1}(\Omega) \to L^2(S)$, $u \mapsto u|_{x_1=a}$ *is well defined and bounded.*

Proof: This is an immediate consequence of the embedding

$$H^{0,1}(\Omega) = H^1(0,L;L^2(S)) \hookrightarrow BUC([0,L];L^2(S))$$

where $BUC([0,L];X)$ is the space of all uniformly continuous functions $f\colon [0,L] \to X$ for some Banach space X. $\square$

In the discussion on uniform bounds for the linearised system we often rely on a Banach algebra structure of the respective space. For this we define the h-dependent norms

$$\|u\|_{2,h} := \|(u, \nabla_h u)\|_{H^1(\Omega;\mathbb{R}^3\times\mathbb{R}^{3\times 3})} \quad \text{and} \quad \|u\|_{1,h} := \|(u, \nabla_h u)\|_{L^2(\Omega;\mathbb{R}^3\times\mathbb{R}^{3\times 3})},$$

and set $V_h(\Omega) := H^2_{per}(\Omega;\mathbb{R}^3)$ equipped with the $\|\cdot\|_{2,h}$ norm. Then $V_h(\Omega) \hookrightarrow L^\infty(\Omega;\mathbb{R}^3)$ and fulfils the following corollary.

Corollary 2.1.4. *Let $F \in C^2_b(\overline{U})$ for some open $U \in \mathbb{R}^N$, $N \in \mathbb{N}$ and $u \in H^2_{per}(\Omega;\mathbb{R}^N)$, then for every $R > 0$ there is some $C(R)$ independent of u and $h \in (0,1]$ such that*

$$\|(F(u), \nabla_h F(u))\|_{H^1(\Omega)} \le C(R)$$

if $\|(u, \nabla_h u)\|_{H^1(\Omega)} \le R$ and $u(x) \in \overline{U}$ for all $x \in \overline{\Omega}$.

Proof: The proof can be done analogously to the proof of Corollary 2.5 in [AMM11a]. We have

$$\begin{aligned}
\partial_{x_j} F(u) &= DF(u)[\partial_{x_j} u] \\
\partial_{x_i}\partial_{x_j} F(u) &= DF(u)[\partial_{x_i}\partial_{x_j} u] + D^2F(u)[\partial_{x_j} u, \partial_{x_i} u]
\end{aligned}$$

for $i,\ j \in \{1,2,3\}$. As $F \in C^2_b(\bar{U})$ and $H^2(\Omega;\mathbb{R}^N) \hookrightarrow C^0(\bar{\Omega};\mathbb{R}^N)$ holds, we conclude that $DF(u)$ and $D^2F(u)$ are uniformly bounded. Therefore

$$\|(F(u), \nabla_h F(u))\|_{L^2(\Omega)} \le C(1 + \|\nabla_h u\|_{L^2(\Omega)}) \le C(R).$$

The last summand of the second derivatives can be bounded via

$$\begin{aligned}
\left\| D^2F(u)[\partial_{x_j} u, \frac{1}{h}\partial_{x_i} u] \right\|_{L^2(\Omega)} &\le C(R)\|\partial_{x_j} u\|_{L^4(\Omega)} \left\| \frac{1}{h}\partial_{x_i} u \right\|_{L^4(\Omega)} \\
&\le C(R)\|\nabla_h u\|_{H^1(\Omega)} = C(R)
\end{aligned}$$

for all $j = 1,2,3$ and $i = 2,3$. This holds due to the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$. Similarly one obtains the bound

$$\left\| D^2F(u)[\partial_{x_j} u, \partial_{x_1} u] \right\|_{L^2(\Omega)} \le C(R)$$

for $j = 1,2,3$. For the first part, we use an analogous argument as for the first derivatives. $\square$

2.1.2 Γ-convergence

As we want to deduce a limiting functional from the energy sequence $(\mathcal{E}^{(h)})_{h>0}$, a suitable variational convergence is needed. The integral structure suggests to use the framework of Γ-convergence, introduced by De Giorgi and Franzoni [DGF75; DG75]. The core of Γ-convergence is, that sequences of minimizers converge to minimizers of the limiting functional. Some in depth analysis of the notion of Γ-convergence can be found in [DM93; Bra02].

Definition 2.1.5 (Γ-convergence in topological spaces, Definition 4.1, [DM93])**.** Let X be a topological space and the set of all open neighbourhoods of $x \in X$ is denoted by $\mathcal{N}(x)$. Then the Γ-*lower limit* and Γ-*upper limit* of a sequence of functions $F_j\colon X \to \bar{\mathbb{R}}$, $j \in \mathbb{N}$ is defined by

$$\begin{aligned}
(\Gamma - \liminf_{j\to\infty} F_j)(x) &:= \sup_{U \in \mathcal{N}(x)} \liminf_{j\to\infty} \inf_{y \in U} F_j(y) \\
(\Gamma - \limsup_{j\to\infty} F_j)(x) &:= \sup_{U \in \mathcal{N}(x)} \limsup_{j\to\infty} \inf_{y \in U} F_j(y),
\end{aligned}$$

respectively. If there exists a function $F\colon X \to \bar{\mathbb{R}}$ such that

$$F \equiv \Gamma - \liminf_{j\to\infty} F_j \equiv \Gamma - \limsup_{j\to\infty} F_j$$

we say that F_j *Γ-converges to F in X*. The function F is called *Γ-limit* of F_j.

In the later applications it is more convenient to work with the sequential description of Γ-convergence. Therefore we state.

Proposition 2.1.6 (Γ-convergence in metric spaces; Proposition 8.1, [DM93])**.** *Let X be a metric space. A sequence $(F_j)_{j\in\mathbb{N}}$ of functions $F_j\colon X \to \bar{\mathbb{R}}$ Γ-converge in X to $F\colon X \to \bar{\mathbb{R}}$ if for all $x \in X$ it holds*

(i) for every sequence $(x_j)_{j\in\mathbb{N}}$ converging to x

$$F(x) \le \liminf_{j\to\infty} F_j(x_j) \qquad (\liminf \textit{-inequality}) \tag{2.8}$$

(ii) there exists a sequence $(x_j)_{j\in\mathbb{N}}$ converging to x such that

$$F(x) \ge \limsup_{j\to\infty} F_j(x_j). \qquad (\limsup \textit{-inequality}) \tag{2.9}$$

The properties (i) and (ii) are denoted by $\liminf$ *and* $\limsup$ *inequality.*

Remark 2.1.7. The $\limsup$ inequality can be refined by (2.8). Let $(x_j)_{j\in\mathbb{N}} \subset X$ be a to $x \in X$ convergent sequence such that (2.9) holds. Then we obtain

$$F(x) \ge \limsup_{j\to\infty} F_j(x_j) \ge \liminf_{j\to\infty} F_j(x_j) \ge F(x).$$

Thus for $(x_j)_{j\in\mathbb{N}}$ it holds indeed $F(x) = \lim_{j\to\infty} F_j(x_j)$, which is why one can equivalently to (ii) require

(ii) there exists a sequence $(x_j)_{j\in\mathbb{N}}$ converging to x such that

$$F(x) = \lim_{j\to\infty} F_j(x_j). \qquad \text{(recovery sequence)} \tag{2.10}$$

Example 2.1.8. Let $X = \mathbb{R}$ endowed with the standard metric induced by the norm on $\mathbb{R}$ and $F_j\colon \mathbb{R} \to \overline{\mathbb{R}}$ be given by

$$F_j(t) = t^2 - \cos(jt).$$

Then the Γ-limit is given by $F\colon \mathbb{R} \to \overline{\mathbb{R}}$, $F(t) = t^2 - 1$. To see this we have to establish the $\liminf$ inequality and find some appropriate recovery sequence. First, we notice that if $(t_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ converges to $t \in \mathbb{R}$ it holds

$$\liminf_{j\to\infty} F_j(t_j) = \lim_{j\to\infty} t_j^2 + \liminf_{j\to\infty}(-\cos(jt_j)) \ge t^2 - 1$$

due to $\cos(t) \in [-1, 1]$. Hence the $\liminf$ inequality (i) holds. Second, let $t \in \mathbb{R}$ be arbitrary, then $t_j = \frac{2\pi}{j}\lfloor \frac{jt}{2\pi} \rfloor \to t$ and

$$F(t) = t^2 - 1 = \lim_{j\to\infty} \left(t_j^2 - \cos(jt_j)\right) = \limsup_{j\to\infty} \left(t_j^2 - \cos(jt_j)\right)$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

Sometimes the assumption X to be a metric space of Proposition 2.1.6 is too restrictive. Therefore we extend the above result to Banach spaces equipped with the weak topology. For this we need

Definition 2.1.9 (Definition 7.6, [DM93]). A sequence $(F_j)_{j\in\mathbb{N}}$ is called *equi-coercive* on X, if for every $t \in \mathbb{R}$ there exists a compact subset K_t of X such that $\{F_j \le t\} \subset K_t$ for all $j \in \mathbb{N}$.

Proposition 2.1.10 (Proposition 8.16, [DM93]). *Assume that X is a reflexive Banach space endowed with its weak topology and that the sequence $(F_j)_{j\in\mathbb{N}}$ is equi-coercive in the weak topology of X. Then, if F_j satisfies (i) and (ii) of Definition 2.1.6, where convergence means now weak convergence in X, F_j Γ-converges to F.*

Theorem 2.1.11 (Theorem 7.8, [DM93]). *Suppose that X is a reflexive Banach space, F_j is equi-coercive on X (in weak or strong topology on X) and F_j Γ-converges to F in X, then*

$$\exists \min_{x\in X} f(x) = \lim_{j\to\infty} \inf_{x\in X} F_j(x).$$

Remark 2.1.12. The latter theorem shows that if every F_j admits a minimizer x_j, then up to a subsequence x_j converges to a minimizer of F. This is one key feature of Γ-convergence. Unfortunately, this does not hold for local minimizer of F_j, as the Example 2.1.8 shows.

2.1.3 Strongly elliptic systems

In this paragraph we want to investigate the solvability and regularity theory of elliptic systems satisfying the Legendre-Hadarmad and Legendre condition. For this purpose we introduce a general second order elliptic operator and summarize classical existence and regularity results. Suppose U is a bounded Lipschitz domain in $\mathbb{R}^n$ and define a second order elliptic operator

$$\mathcal{L}u := -\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \partial_\alpha(A^{\alpha\beta}\partial_\beta u) \tag{2.11}$$

where for α, $\beta \in \{1,\dots,n\}$

$$A^{\alpha\beta} = (a_{ij}^{\alpha\beta})_{i,j=1,\dots,m}$$

such that $A^{\alpha\beta} \in L^\infty(U;\mathbb{R}^{n\times n})$. If $A^{\alpha\beta}$ are sufficiently regular, for example $A^{\alpha\beta} \in C_b^1(U;\mathbb{R}^{n\times n})$, then the operator $\mathcal{L}$ maps then a suitable function $u\colon U \to \mathbb{R}^m$ to a vector-valued function $\mathcal{L}u\colon U \to \mathbb{R}^m$ with the following components

$$(\mathcal{L}u)_i = -\sum_{j=1}^{m}\sum_{\alpha,\beta=1}^{n} \partial_\alpha(a_{ij}^{\alpha\beta}\partial_\beta u_j).$$

Definition 2.1.13. A second order elliptic operator with coefficients $(a_{ij}^{\alpha\beta})_{i,j=1,\dots,m}^{\alpha,\beta=1,\dots,n}$ is said to fulfil

(i) the *Legendre* condition, if $\lambda > 0$ exists such that

$$\sum_{i,j=1}^{m}\sum_{\alpha,\beta=1}^{n} a_{ij}^{\alpha\beta}\xi_{i\alpha}\xi_{j\beta} \ge \lambda|\xi|^2 \tag{2.12}$$

for all $\xi \in \mathbb{R}^{m\times n}$.

(ii) the *Legendre-Hadamard* condition, if $\lambda > 0$ exits such that

$$\sum_{i,j=1}^{m}\sum_{\alpha,\beta=1}^{n} a_{ij}^{\alpha\beta}\xi_\alpha\xi_\beta\eta_i\eta_j \ge \lambda|\xi|^2|\eta|^2 \tag{2.13}$$

for all $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^m$.

Let ν be the outer unit normal to U and define

$$\mathfrak{B}_j u = \sum_{k=1}^{n} A_{jk} \partial_k u.$$

Then we call $\mathfrak{B}_\nu u$ the *conormal derivative* defined by

$$\mathfrak{B}_\nu u = \sum_{j=1}^{n} \operatorname{tr}_{\partial U}(\mathfrak{B}_j u)\nu_j \qquad \text{on } \partial U. \tag{2.14}$$

In the later applications we will use the two cases $\Gamma_D = \partial U$, $\Gamma_N = \emptyset$ and $\Gamma_D = \emptyset$, $\Gamma_N = \partial U$. We consider the elliptic boundary value problem

$$\begin{aligned} \mathcal{L}u &= f && \text{in } U \\ \mathfrak{B}_\nu u &= g_N && \text{on } \Gamma_N \\ u &= g_D && \text{on } \Gamma_D \end{aligned} \tag{2.15}$$

for given f, g_N and g_D. The operator $\mathcal{L}$ is naturally joint to a bilinear form, defined by

$$a(u,v) := \int_U \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} a_{ij}^{\alpha\beta}(x) \partial_\beta u_j \partial_\alpha v_i dx$$

for all u, $v \in H^1(\Omega;\mathbb{R}^m)$. From the boundedness of $A^{\alpha\beta}$ it follows that a is bounded on $H^1(U;\mathbb{R}^m)$, i.e.

$$|a(u,v)| \leq C\|u\|_{H^1(U)}\|v\|_{H^1(U)}$$

for all u, $v \in H^1(\Omega;\mathbb{R}^m)$.

Definition 2.1.14. Let $\mathcal{L}$, $\mathfrak{B}_\nu$, Γ_D, Γ_N and a as above. Moreover let $f \in L^2(U;\mathbb{R}^m)$, $g_N \in L^2(\Gamma_N;\mathbb{R}^m)$ and $g_D \in H^{\frac{1}{2}}(\Gamma_D)$. Then $u \in H^1(U;\mathbb{R}^m)$ is called a *weak solution* of the system (2.15) if $\operatorname{tr}_{\partial U} u = g_D$ on Γ_D and

$$a(u,v) = (f,v)_{L^2(\Omega)} + (g_N, \operatorname{tr}_{\partial U} v)_{L^2(\Gamma_N)}$$

for all $v \in H^1_{\Gamma_D}(U;\mathbb{R}^m)$.

With this we can state the first existence result for pure Dirichlet boundary conditions, which is a immediate concequence of the Lax-Milgramm Lemma.

Theorem 2.1.15 (cf. [GM12], Theorem 3.39)**.** *Let $a_{ij}^{\alpha\beta} \in L^\infty(U)$ and satisfy the Legendre condition for some $\lambda > 0$. Moreover we assume $\Gamma_D = \partial U$, $\Gamma_N = \emptyset$. Then there exists for every $g_D \in H^{\frac{1}{2}}(\Gamma_D;\mathbb{R}^m)$ and $f \in L^2(U;\mathbb{R}^m)$ a unique weak solution $u \in H^1(U;\mathbb{R}^m)$ to the Dirichlet problem* (2.15)*. The solution satisfies*

$$\|u\|_{H^1(U)} \leq C\Big(\|f\|_{L^2(U)} + \|g_D\|_{H^{\frac{1}{2}}(U)}\Big). \tag{2.16}$$

Remark 2.1.16. (i) The inequality (2.16) is not explicitly proven in Theorem 3.29 of [GM12]. But it follows directly from the structure of the proof provided there or the open mapping theorem. One uses that we reduce the problem to finding an auxiliary weak solution $\tilde{u}$ to a homogeneous Dirichlet problem with a slightly modified right hand side given by f and an extension of g_D. Then the Lax-Milgram Lemma implies

$$\|u\|_{H^1(U)} \leq C\Big(\|f\|_{L^2(U)} + \|E(g_N)\|_{H^1(U)}\Big) \leq C\Big(\|f\|_{L^2(U)} + \|g_D\|_{H^{\frac{1}{2}}(\Gamma_D)}\Big).$$

Here we used the boundedness of the extension operator $E\colon H^{\frac{1}{2}}(\Gamma_D) \to H^1(U)$.

(ii) We can extend Theorem 2.1.15 to the case that $a_{ij}^{\alpha\beta}$ are constant and satisfy the Legendre-Hadamard condition [GM12, Chapter 3.4.3]. In case the coefficient matrix is not constant only weak coercivity of a, see definition below, follows.

(iii) The latter result can be shown as well in the context of a Fredholm alternative, which yields an even more general result. A more in depth analysis can be found in [McL00, Chapter 4].

At the core of our proof for large times existence of solutions we have to bound higher norms of the solution for the linearised system. Therefore we need a regularity result for solutions of (2.15) up to the boundary.

Definition 2.1.17. Let $\mathcal{L}$ and a be given as above and $V \subset H^1(U;\mathbb{R}^m)$ a closed subspace, such that V is dense in $L^2(U;\mathbb{R}^m)$. Then we say that $\mathcal{L}$ and a are *weakly coercive* on V if there exist $\lambda_0 > 0$ and $\lambda_1 \geq 0$ such that

$$a(u,u) \geq \lambda_0 \|u\|_{H^1(U)} - \lambda_1 \|u\|_{L^2(U)}$$

for all $u \in V$ holds.

Theorem 2.1.18 ([McL00], Theorem 4.18)**.** *Let U be a $C^{r+1,1}$ domain, for some $r \geq 0$ and $\Gamma_D = \emptyset$, $\Gamma_N = \partial U$. Moreover assume that $a_{ij}^{\alpha\beta}$ are constant and $\mathcal{L}$ is weakly coercive on $H^1(U;\mathbb{R}^m)$. Let $u \in H^1(U,\mathbb{R}^m)$ and $f \in H^r(U;\mathbb{R}^m)$ satisfy*

$$\mathcal{L}u = f \qquad \text{on } U$$

and $\mathfrak{B}_\nu u \in H^{r+\frac{1}{2}}(\partial U)$. Then $u \in H^{2+r}(U;\mathbb{R}^m)$ and

$$\|u\|_{H^{r+2}(U)} \leq C\|u\|_{H^1(U)} + C\|\mathfrak{B}_\nu u\|_{H^{r+\frac{1}{2}}(U)} + C\|f\|_{H^r(U)} \tag{2.17}$$

holds for some $C > 0$ independent of u and f.

2.2 Introduction to Non-linear Elasticity

In this section we investigate the classical theory of continuum mechanics with special interest in non-linear elasticity. The introduction is based on a book of Gurtin on the very same subject, cf. [Gur81].

In elasticity theory we are interested in the deformational behaviour of solid bodies, which have the physical property to occupy regions in space. As a body can deform over time, we have to make a choice for a reference configuration $\mathcal{B}$. A *deformation* is then given via some C^3 mapping $\mathsf{x}\colon \mathcal{B} \to \mathbb{R}^n$ from the reference configuration $\mathcal{B}$ to a deformed state in $\mathbb{R}^n$, where material points p are mapped to a spacial point $\mathsf{x}(p)$. The physical property that it should be impossible for a body to self penetrate is expressed by the assumption that x is injective. Locally the volume after the deformation per unit original volume can be expressed by $\det F$ for $F = \nabla \mathsf{x}$. Thus a natural assumption is $\det F > 0$, cf. [Gur81, Chapter 6].

If we add a time dependency to the deformation, we obtain a *motion* $\mathsf{x}\colon \mathcal{B} \times \mathbb{R} \to \mathbb{R}^n$, such that for fixed t: $\mathsf{x}(\cdot,t)\colon \mathcal{B} \to \mathcal{B}_t$ is a deformation, where $\mathcal{B}_t := \mathsf{x}(\mathcal{B},t)$. The *trajectory* of the motion is defined as

$$\mathcal{T} := \{(x,t) \ : \ x \in \mathcal{B}_t, t \in \mathbb{R}\}.$$

Using the bijectivity for each time we can define the *velocity* $v\colon \mathcal{T} \to \mathbb{R}^n$ by $v(x,t) = \partial_t \mathsf{x}(\mathsf{x}^{-1}(x,t),t)$. In the following we call $\mathcal{P}$ a *part* of $\mathcal{B}$, if $\mathcal{P}$ is a bounded subdomain of $\mathcal{B}$ and the boundary of $\mathcal{P}$ is sufficiently regular. More details on how to define parts see [Gur81] and [Kel67]. Using this we define for some part $\mathcal{P}$ of $\mathcal{B}$ the *linear momentum* and *angular momentum* by

$$l(\mathcal{P},t) = \int_{\mathcal{P}_t} v\rho dx \quad \text{ and } \quad a(\mathcal{P},t) = \int_{\mathcal{P}_t} x \times v\rho dx,$$

respectively. Here ρ is a given density of the material and $\mathcal{P}_t := \mathsf{x}(\mathcal{P}, t)$. During the motion forces are exerted within the body and from the surrounding environment upon the body. A core question is now how these forces can be described. *Cauchy's hypothesis* states that there exists a surface force density $s(\nu, x, t)$ defined for all unit normals ν and every $(x, t) \in \mathcal{T}$. With this we can describe the *force* $f(\mathcal{P}, t)$ and *momentum* $m(\mathcal{P}, t)$ by

$$\begin{aligned} f(\mathcal{P}, t) &= \int_{\partial \mathcal{P}_t} s(\nu, x, t) d\sigma(x) + \int_{\mathcal{P}_t} b(x, t) dx \\ m(\mathcal{P}, t) &= \int_{\partial \mathcal{P}_t} x \times s(\nu, x, t) d\sigma(x) + \int_{\mathcal{P}_t} x \times b(x, t) dx \end{aligned}$$

where $\nu \equiv \nu(x)$ is the outward unit normal to $\partial \mathcal{P}_t$ and $b \colon \mathcal{T} \to \mathbb{R}^n$ is the *body force density*. The basic axiom, the *momentum balance laws*, conects the motion x and the system of forces (s, b) by claiming that for any part $\mathcal{P}$ and time t it holds

$$f(\mathcal{P}, t) = \partial_t l(\mathcal{P}, t) \quad \text{and} \quad m(\mathcal{P}, t) = \partial_t a(\mathcal{P}, t)$$

which is equivalent to

$$\begin{aligned} \int_{\partial \mathcal{P}_t} s(\nu, x, t) d\sigma(x) + \int_{\mathcal{P}_t} b(x, t) dx &= \int_{\mathcal{P}_t} \partial_t v(x, t) \rho dx \\ \int_{\partial \mathcal{P}_t} x \times s(\nu, x, t) d\sigma(x) + \int_{\mathcal{P}_t} x \times b(x, t) dx &= \int_{\mathcal{P}_t} x \times \partial_t v \rho dx. \end{aligned}$$

A fundamental result in continuum mechanics is Cauchy's Theorem, stating that (s, b) satisfying the momentum balance laws is equivalent to the existence of a unique, symmetric tensor field $T(x, t)$ such that $s(\nu, x, t) = T(x, t)\nu$ and T satisfies the equation of motion

$$\operatorname{div} T(x, t) + b = \rho \partial_t v(x, t).$$

Hence the forces (s, b) which act during a motion can be fully described by the stress T and the motion x, cf. [Gur81, Chapter 13, 14 and 15].

In order to model an elastic body we have to introduce appropriate constitutive assumptions on how T depends on x. As F measures local length changes the assumption

$$T(x, t) = \hat{T}(F(p, t), t)$$

for $x = \mathsf{x}(p, t)$ arises naturally, cf. [Gur81, Chapter 25]. Moreover we assume that the material respond to a deformation should not depend on the observer, which in turn is equivalent to

$$Q \hat{T}(F) Q^T = \hat{T}(QF)$$

for all $F \in \mathbb{R}^{n \times n}$ with $\det F > 0$ and $Q \in SO(n)$.

Note that the Cauchy's stress tensor is defined on the deformed configuration, which in many problems of deforming solids is inconvenient. Therefore we want to introduce a stress tensor which is defined on the reference configuration $\mathcal{B}$, cf. [Gur81, Chapter 27]

Let (x, T) be a given motion and Cauchy stress. Then for $\mathcal{P}$ a part of $\mathcal{B}$ it holds

$$\int_{\partial \mathcal{P}_t} T(x, t) \mu d\sigma(x) = \int_{\partial \mathcal{P}} \det(F) T(\mathsf{x}(p, t), t) F^{-T} \nu d\sigma(p)$$

where μ and ν are the respective outward unit normals. Thus we define the *Piola-Kirchhoff stress* $S \colon \mathcal{B} \times \mathbb{R} \to \mathbb{R}^{n \times n}$ by

$$S(p, t) := \det(F) T(\mathsf{x}(p, t), t) F^{-T}.$$

If b is the body force associated to (x, T) it follows

$$\int_{\mathcal{P}_t} b dV = \int_{\mathcal{P}} b_0 dV$$

for $b_0(p,t) = \det(F) b(\mathsf{x}(p,t),t)$. Hence, S satisfying the balance equations

$$\begin{aligned} \int_{\partial\mathcal{P}} S\mu d\sigma(p) + \int_{\mathcal{P}} b_0 dp &= \int_{\mathcal{P}} \rho_0 \partial_t^2 \mathsf{x} dp \\ \int_{\partial\mathcal{P}} \mathsf{x} \times S\mu d\sigma(p) + \int_{\mathcal{P}} \mathsf{x} \times b_0 dp &= \int_{\mathcal{P}} \mathsf{x} \times \rho_0 \partial_t^2 \mathsf{x} dp \end{aligned} \tag{2.18}$$

for every part $\mathcal{P}$ is equivalent to

$$\begin{aligned} s(\nu, \mathsf{x}(p,t), t) &= (\det F)^{-1} S(p,t) F^T \nu \\ \operatorname{div} S + b_0 &= \rho_0 \partial_t^2 \mathsf{x} \\ SF^T &= FS^T. \end{aligned} \tag{2.19}$$

If the body is elastic and we assume that the material behaviour is independent of the observer it follows $S = \hat{S}(F)$ and

$$\hat{S}(QF) = Q\hat{S}(F)$$

for all $F \in \mathbb{R}^{n\times n}$ with $\det F > 0$ and $Q \in SO(n)$, cf [Gur81, Chapter 27].

We now use a standard axiom of thermodynamic, the assumption of non-negative work in closed processes. This leads for any time interval and part $\mathcal{P}$ to

$$\int_{t_0}^{t_1} \int_{\mathcal{P}} S : \partial_t F dx dt \geq 0$$

if $\mathsf{x}(p,t_0) = \mathsf{x}(p,t_1)$ and $\partial_t \mathsf{x}(p,t_0) = \partial_t \mathsf{x}(p,t_1)$ for all $p \in \mathcal{B}$. Following from this we can show [Gur81, Chapter 28] the existence of a *strain-energy density* $W(F,p)$ such that

$$\hat{S}(F,p) = \frac{\partial}{\partial F_{ij}} W(F,p).$$

An elastic body such that the Piola-Kirchhoff stress $\hat{S}$ is given as above is called *hyperelastic*. The strain-energy of a part $\mathcal{P}$ for a hyperelastic body is given via

$$\int_{\mathcal{P}} W(F(p,t),t) dp.$$

Deploying the independence of the observer we can deduce

$$W(QF) = W(F)$$

for all $F \in \mathbb{R}^{n\times n}$ with $\det F > 0$ and $Q \in SO(n)$. An example for a hyperelastic energy density is

$$W(F) = \operatorname{dist}^2(F, SO(n))$$

and the so called St. Vernant-Kirchhoff materials, described by

$$W(F) = \frac{\lambda}{2}(\operatorname{tr} G)^2 + \mu \operatorname{tr}(G^2) \quad \text{ for } G = \frac{1}{2}(F^T F - Id)$$

where λ and μ are the so called Lamé constants [EGK17, Chapter 5]. Remarkable is that in both cases the strain-energy density does not depend on p. Hence we say that a body $\mathcal{B}$ is *homogeneous* if $\rho_0(p)$ and $\hat{S}(F,p)$ are independent of the material point p [Gur81, Chapter 25]. Hyperelastic materials have several interesting properties, cf [Gur81, Chapter 28]. The most important one for the following is the balance of energy. For each part $\mathcal{P}$ of $\mathcal{B}$ the following equality holds

$$\int_{\partial\mathcal{P}} Sn \cdot \partial_t \mathsf{x} d\sigma(p) + \int_{\mathcal{P}} b_0 \cdot \partial_t \mathsf{x} dp = \frac{d}{dt}\int_{\mathcal{P}} \frac{|\partial_t \mathsf{x}|^2}{2} + W(p, \nabla \mathsf{x}) dx. \tag{2.20}$$

where S is the Piola-Kirchhoff stress and b_0 an external force.

2.3 The Strain Energy Density W

In the preceding section we carefully distinguished between material points p and spacial points x. As in the thesis we will always work on a fixed reference domain we will use the standard spacial variable x.
Now we investigate the mathematical assumptions and resulting properties of the strain-energy density W in three dimensions. Assume $W \colon \mathbb{R}^{3\times3} \to [0,\infty]$ satisfies the following conditions:

(i) W is C^2 on $B_\delta(Id) \subset \mathbb{R}^{3\times3}$ for some $\delta > 0$;

(ii) W is frame-invariant, i.e. $W(RF) = W(F)$ for all $F \in \mathbb{R}^{3\times3}$ and $R \in SO(3)$;

(iii) there exists $c_0 > 0$ such that $W(F) \geq c_0 \operatorname{dist}^2(F, SO(3))$ for all $F \in \mathbb{R}^{3\times3}$ and $W(R) = 0$ for every $R \in SO(3)$.

The upcoming remarks summaries the important properties of the elastic energy density, which can be deduced from the properties (i)–(iii).

Remark 2.3.1. First of all we note that W has a minimum point at the identity, as $W(Id) = 0$ and $W(F) \geq 0$ for all $F \in \mathbb{R}^{3\times3}$. Hence, we have $DW(Id)[G] = 0$ for all $G \in \mathbb{R}^{3\times3}$.
Using $R(t) = \exp(tS)$ with $S \in \mathbb{R}^{3\times3}_{skew}$ and $t \in \mathbb{R}$, it follows with the frame invariance

$$\begin{aligned} 0 = \frac{d}{dt} W(R(t)F)|_{t=0} &= DW(R(t)F)[R'(t)F]\Big|_{t=0} = DW(F)[SF] \\ &= DW(F) : SF = DW(F)F^T : S \end{aligned} \tag{2.21}$$

Thus we obtain all $F \in \mathbb{R}^{3\times3}$: $DW(F)F^T = F DW(F)^T$, which corresponds to the symmetry property of the Piola-Kirchhoff stress.

Remark 2.3.2. We assert that there exits a constant $c_1 > 0$ such that

$$D^2W(Id)[G,G] = D^2W(Id)[\varepsilon(G), \varepsilon(G)] \geq c_1 |\varepsilon(G)|^2 \tag{2.22}$$

for all $G \in \mathbb{R}^{3\times3}$. This can be seen as follows: frame invariant leads by differentiation to

$$DW(RF)[RG] = \frac{d}{d\tau} W(R(F+\tau G))\Big|_{\tau=0} = \frac{d}{d\tau} W(F+\tau G)\Big|_{\tau=0} = DW(F)[G] \tag{2.23}$$

for all $R \in SO(3)$, $F,\ G \in \mathbb{R}^{3\times3}$. Using the choice $F = Id$ and $R(t) = \exp(tS)$ with $S \in \mathbb{R}^{3\times3}_{skew}$, it follows

$$0 = \frac{d}{dt} DW(R(t))[R(t)G]|_{t=0} = D^2W(Id)[G,S] + DW(Id)[GS] = D^2W(Id)[G,S] \tag{2.24}$$

Bilinearity and symmetry of $D^2W(Id)$ leads to $D^2W(Id)[G,G] = D^2W(Id)[\varepsilon(G), \varepsilon(G)]$.
To show the lower bound we deploy property (iii) and a Taylor expansion around the identity in $\mathbb{R}^{3\times3}$. As for $\det(F) > 0$, it holds due to the Lemma 2.3.3 below

$$\operatorname{dist}^2(F; SO(3)) = \left|(F^TF)^{\frac{1}{2}} - Id\right|^2 \tag{2.25}$$

and with a Taylor expansion

$$\begin{aligned}\operatorname{dist}^2(Id+G;SO(3)) &= 2\left|\varepsilon(G)\right|^2 + O(|G|^3)\\ W(Id+G) &= \frac{1}{2}D^2W(Id)[G,G] + O(|G|^3)\end{aligned}$$

for all $G \in B_r(0)$ with $r > 0$ sufficiently small. For t small enough, we can choose $G(t) = \exp(tA) - Id$ with $A \in \mathbb{R}^{3\times 3}$. Then it follows

$$D^2W(Id)\left[\frac{\exp(tA)-Id}{t}, \frac{\exp(tA)-Id}{t}\right] \geq C\left|\varepsilon\Big(\frac{\exp(tA)-Id}{t}\Big)\right|^2 + O(t).$$

Passing to the limit $t \to 0$, leads to

$$D^2W(Id)[A,A] \geq c_1|\varepsilon(A)|^2.$$

From (2.22) it follows that $D^2W(Id)$ is elliptic in the sense of Legendre-Hadamard, i.e.

$$D^2W(Id)[a\otimes b, a\otimes b] \geq c|a|^2|b|^2 \quad \text{for all } a,\ b \in \mathbb{R}^3.$$

We obtain this by

$$\begin{aligned}D^2W(Id)[a\otimes b, a\otimes b] \geq c_1|\varepsilon(a\otimes b)|^2 &= \frac{c_1}{2}\Big(ab^T : ab^T + ba^T : ba^T\Big)\\ &= \frac{c_1}{2}\Big(|a|^2|b|^2 + (a,b)^2\Big)\end{aligned}$$

which leads, because of $(a,b)^2 \geq 0$, to the desired inequality.

Lemma 2.3.3. *Let $F \in \mathbb{R}^{3\times 3}$ with $\det(F) > 0$ and $F = RU$ the polar decomposition of F, i.e. $R \in SO(3)$ and $U \in \mathbb{R}^{3\times 3}$ is positive definite and symmetric. Then it holds*

$$|F - R| < |F - Q|$$

for all $Q \in SO(3)$ such that $Q \neq R$.

Proof: Using the polar decomposition theorem, cf. [Koc93, Chapter 2], we have the existence of $R \in SO(3)$ and $U \in \mathbb{R}^{3\times 3}$ positive definite and symmetric, such that $F = RU$. Let $Q \in SO(3)$ with $Q \neq R$. Then we can compute

$$|F-Q|^2 = (F-Q):(F-Q) = |F|^2 - 2F : Q + 3.$$

Hence it follows

$$|F-Q|^2 - |F-R|^2 = 2U : (Id - R^TQ)$$

where $R^TQ \neq Id$, due to the assumption $R \neq Q$. The definition of the matrix skalar product leads to

$$2U : (Id - R^TQ) = U : (R^TQ - Id)(R^TQ - Id)^T = \operatorname{tr}((R^TQ - Id)^T U (R^TQ - Id)).$$

Using that U is positive definite and symmetric it follows $(R^TQ - Id)^T U (R^TQ - Id)$ is positive semi definite and symmetric, hence all eigenvalues are nonnegative. Moreover, because $R^TQ - Id \neq 0$, there exists $v \in \mathbb{R}^3$ such that $(R^TQ - Id)v = w \neq 0$. Thus with $w \cdot Uw > 0$, we obtain that at least one eigenvalue is positive. Therefore it holds

$$2U : (Id - R^TQ) > 0$$

and the claim follows. □

Remark 2.3.4. For more details on the minimality of R see also [MPG79]. Moreover due to the equality

$$R = (F^T F)^{\frac{1}{2}}$$

from [Koc93, Chapter 2, Polar Decompositon Theorem] the equality (2.25) follows.

Remark 2.3.5. Using the identification (2.6), we can find $B^{\alpha\beta} = (b_{ij}^{\alpha\beta})_{i,j=1,2,3} \in \mathbb{R}^{3\times 3}$, $\alpha,\ \beta \in \{1,2,3\}$ such that

$$D^2W(Id)[X,Y] = D^2W(Id)X : Y = \sum_{\alpha,\beta,i,j=1}^{3} b_{ij}^{\alpha\beta} x_{i\alpha} y_{j\beta}$$

for all $X,\ Y \in \mathbb{R}^{3\times 3}$. Therefore

$$D^2W(Id)[X]\nu = (D^2W(Id)X)\nu = \left(\sum_{\alpha,\beta,i,j=1}^{3} b_{ij}^{\alpha} x_{i\alpha} \nu_j \right)_{\beta=1,2,3}$$

for $X \in \mathbb{R}^{3\times 3}$ and $\nu \in \mathbb{R}^3$. Hence, we obtain with (2.24) that $b_{ij}^{\alpha\beta} = b_{\beta\alpha}^{ji}$. In order to see this we choose $X = e_i \otimes e_\alpha - e_\alpha \otimes e_i$, $Y = e_j \otimes e_\beta$ and $X = e_i \otimes e_\alpha$, $Y = e_j \otimes e_\beta - e_\beta \otimes e_j$, respectively. Thus either $\operatorname{sym}(X) = 0$ or $\operatorname{sym}(Y) = 0$ and with (2.24) it follows

$$0 = D^2W(Id)[X,Y] = b_{ij}^{\alpha\beta} - b_{\alpha j}^{\beta i} = b_{ij}^{\beta\alpha} - b_{i\beta}^{j\alpha}. \tag{2.26}$$

For later use we introduce

$$(D^2W(Id))^{\approx} := (B^{\alpha\beta})_{\alpha=2,3}^{\beta=2,3}.$$

For convenience we will work in the chapter on large time existence for the non-linear problem with relative displacement. For this we introduce $\tilde{W} \colon \mathbb{R}^{3\times 3} \to [0,\infty]$ by $\tilde{W}(F) := W(Id + F)$. The results of Remark 2.3.2 therefore hold for $\tilde{W}$ as well, i.e.

$$D^2\tilde{W}(0)[G,G] = D^2\tilde{W}(0)[\varepsilon(G), \varepsilon(G)] \geq c_1 |\varepsilon(G)|^2$$

and

$$D^2\tilde{W}(0)[a \otimes b, a \otimes b] \geq c|a|^2|b|^2 \quad \text{for all } a,\ b \in \mathbb{R}^3.$$

The following lemma provides an essential decomposition of $D^3\tilde{W}$.

Lemma 2.3.6 ([AMM11a, Lemma 2.6])**.** *There is some constant $C > 0$, $\varepsilon > 0$ and $A \in C^\infty(\overline{B_\varepsilon(0)}; \mathcal{L}^3(\mathbb{R}^{n\times n}))$ such that for all $G \in \mathbb{R}^{n\times n}$ with $|G| \leq \varepsilon$ we have*

$$D^3\tilde{W}(G) = D^3\tilde{W}(0) + A(G)$$

where

$$|D^3\tilde{W}(0)|_h \leq Ch \text{ for all } 0 < h \leq 1, \tag{2.27}$$

$$|A(G)| \leq C|G| \text{ for all } |G| \leq \varepsilon. \tag{2.28}$$

With this we can prove the following bound for $D^3\tilde{W}$.

Corollary 2.3.7. *There exist $C,\ \varepsilon > 0$ such that*

$$\|D^3\tilde{W}(Z)(Y_1,Y_2,Y_3)\|_{L^1(\Omega)} \leq Ch\|Y_1\|_{H_h^2(\Omega)}\|Y_2\|_{L_h^2(\Omega)}\|Y_3\|_{L_h^2(\Omega)} \tag{2.29}$$

for all $Y_1 \in H^2(\Omega, \mathbb{R}^{n\times n})$, $Y_2,\ Y_3 \in L^2(\Omega; \mathbb{R}^{n\times n})$, $0 < h \leq 1$ and $\|Z\|_{L^\infty(\Omega} \leq \min\{\varepsilon, h\}$ and

$$\|D^3\tilde{W}(Z)(Y_1,Y_2,Y_3)\|_{L^1(\Omega)} \leq Ch\|Y_1\|_{H_h^1(\Omega)}\|Y_2\|_{H_h^1(\Omega)}\|Y_3\|_{L_h^2(\Omega)} \tag{2.30}$$

for all Y_1, $Y_2 \in H^1(\Omega, \mathbb{R}^{n\times n})$, $Y_3 \in L^2(\Omega;\mathbb{R}^{n\times n})$, $0 < h \leq 1$ *and* $\|Z\|_{L^\infty(\Omega} \leq \min\{\varepsilon, h\}$ *and*

$$\|D^3\tilde{W}(Z)(Y_1,Y_2,Y_3)\|_{L^1(\Omega)} \leq Ch\left\|\left(Y_1, \frac{1}{h}\operatorname{sym}(Y_1)\right)\right\|_{L^\infty(\Omega)} \|Y_2\|_{H^1_h(\Omega)}\|Y_3\|_{L^2_h(\Omega)} \tag{2.31}$$

for all $Y_1 \in L^\infty(\Omega, \mathbb{R}^{n\times n})$, Y_2, $Y_3 \in L^2(\Omega;\mathbb{R}^{n\times n})$, $0 < h \leq 1$ *and* $\|Z\|_{L^\infty(\Omega} \leq \min\{\varepsilon, h\}$.

Proof: The main ingredient of the proof is Lemma 2.3.6 and Hölder's inequality. We first notice

$$\begin{aligned}|D^3\tilde{W}(Z)(Y_1,Y_2,Y_3)| &\leq |D^3\tilde{W}(0)(Y_1,Y_2,Y_3)| + |A(Z)(Y_1,Y_2,Y_3)| \\ &\leq |D^3\tilde{W}(0)|_h|Y_1|_h|Y_2|_h|Y_3|_h + |A(Z)||Y_1||Y_2||Y_3| \\ &\leq Ch|Y_1|_h|Y_2|_h|Y_3|_h\end{aligned}$$

where we have used that $|A|_h \geq |A|$, (2.27) and (2.28). Thus

$$\begin{aligned}\|D^3\tilde{W}(Z)(Y_1,Y_2,Y_3)\|_{L^1(\Omega)} &= \int_\Omega |D^3\tilde{W}(Z)(Y_1,Y_2,Y_3)|dx \\ &\leq Ch\int_\Omega |Y_1|_h|Y_2|_h|Y_3|_h dx \\ &\leq Ch\||Y_1|_h\|_{C^0(\Omega)}\||Y_2|_h|Y_3|_h\|_{L^1(\Omega)} \\ &\leq Ch\|Y_1\|_{H^2_h(\Omega)}\|Y_2\|_{L^2_h(\Omega)}\|Y_3\|_{L^2_h(\Omega)}\end{aligned}$$

as $H^2(\Omega) \hookrightarrow C^0(\Omega)$. And for (2.30)

$$\begin{aligned}\|D^3\tilde{W}(Z)(Y_1,Y_2,Y_3)\|_{L^1(\Omega)} &= \int_\Omega |D^3\tilde{W}(Z)(Y_1,Y_2,Y_3)|dx \\ &\leq Ch\||Y_1|_h\|_{L^4(\Omega)}\||Y_2|_h\|_{L^4(\Omega)}\|Y_3\|_{L^2_h(\Omega)} \\ &\leq Ch\|Y_1\|_{H^1_h(\Omega)}\|Y_2\|_{H^1_h(\Omega)}\|Y_3\|_{L^2_h(\Omega)}\end{aligned}$$

using $H^1(\Omega) \hookrightarrow L^4(\Omega)$. Finally (2.31) follows immediately from the following inequality

$$|Y_1|_h \leq \left|\left(Y_1, \frac{1}{h}\operatorname{sym}(Y_1)\right)\right|. \qquad \square$$

2.4 Korn's Inequality in Thin Rods

In order to derive sharp estimates based on the linearised system, we need a good understanding in how the scaled gradient $\nabla_h g$ of a function $g \in H^1_{per}(U)$ can be bounded by the scaled symmetric gradient $\varepsilon_h(g)$. As rigid motions $x \mapsto \alpha x^\perp$ for $\alpha \in \mathbb{R}$ arbitrary are admissible functions in $H^1_{per}(U)$ we can not expect that the full scaled gradient is bounded by $\varepsilon_h(g)$. Moreover, a quantitative, sharp understanding of the dependency of a possible pre factor from the small parameter h is essential. This section is devoted to such generalisations of Korn's inequality. The idea is to apply the classical Korn inequality on cubes of size with edge length h. Therefore we recap one possible formulation and proof following [Sch13, §25.2, Theorem 25.4].

Theorem 2.4.1. *Let $U \subset \mathbb{R}^n$ a bounded, Lipschitz domain. Then there exists $C(U) > 0$ such that for all $u \in H^1(U)$ there exists some $B \in \mathbb{R}^{n\times n}_{skew}$ such that*

$$\|\nabla u - B\|_{L^2(U)} \leq C(U)\|\varepsilon(u)\|_{L^2(U)}. \tag{2.32}$$

Moreover the constant $C(U)$ is independent with respect to the mapping $x \mapsto \lambda x + b$, for $\lambda \in \mathbb{R}$ and $b \in \mathbb{R}^n$ arbitrary, i.e. one can choose

$$C(\lambda U + b) = C(U).$$

For the proof of the later theorem we use a slightly different formulation, which is more convenient to proof.

Theorem 2.4.2. *Let $U \subset \mathbb{R}^n$ be a bounded, Lipschitz domain. Then there exists $C(U) > 0$ such that for all $u \in H^1(U)$*

$$\|\nabla u\|_{L^2(U)} \leq C(U)\big(\|\varepsilon(u)\|_{L^2(U)} + \|u\|_{L^2(U)}\big)$$

holds.

Proof: Let $u \in H^1(U)$. Then it holds $\nabla u \in L^2(U)$ and for $\varepsilon(u)$ it holds

$$\partial_{x_j}\partial_{x_k} u_i = \partial_{x_j}\varepsilon(u)_{ik} + \partial_{x_k}\varepsilon(u)_{ij} - \partial_{x_i}\varepsilon(u)_{jk}$$

in the sense of distributions. Thus using the boundedness of the derivative operator $\nabla\colon L^2(U) \to H^{-1}(U)$ and Lions Lemma it follows

$$\begin{aligned}
\|\nabla u\|_{L^2(U)} &\leq C\big(\|\nabla^2 u\|_{H^{-1}(U)} + \|\nabla u\|_{H^{-1}(U)}\big)\\
&\leq C\big(\|\nabla\varepsilon(u)\|_{H^{-1}(U)} + \|\nabla u\|_{H^{-1}(U)}\big)\\
&\leq C\big(\|\varepsilon(u)\|_{L^2(U)} + \|u\|_{L^2(U)}\big)
\end{aligned}$$

where the constant C is independent of u. □

In the later proof, we used the following Lions Lemma, cf. [Neč67, Chaper 3, Lemma 7.1]

Lemma 2.4.3. *Let $U \in \mathbb{R}^n$ be a bounded Lipschitz domain. Then there exists a constant $C > 0$ such that for all $u \in L^2(U)$*

$$\|u\|_{L^2(U)} \leq C\big(\|\nabla u\|_{H^{-1}(U)} + \|u\|_{H^{-1}(U)}\big)$$

holds.

Proof of Theorem 2.4.1: STEP 1: (reduction to mean symmetric gradient). First we want to prove that there exists $C > 0$ such that if

$$\int_U \nabla u dx \in \mathbb{R}^{n\times n}_{sym}, \tag{2.33}$$

then it follows

$$\|\nabla u\|_{L^2(U)} \leq C\|\varepsilon(u)\|_{L^2(U)}. \tag{2.34}$$

We want to prove the inequality via a proof by contradiction. Assume therefore that $(u_k)_{k\in\mathbb{N}} \subset H^1(U)$ such that (2.33) holds for ∇u_k and

$$1 = \|\nabla u_k\|_{L^2(U)} > \frac{1}{k}\|\varepsilon(u_k)\|_{L^2(U)}. \tag{2.35}$$

Without loss of generality we can assume that u_k has zero mean value, as one can subtract the mean value. Poincaré's inequality with mean value implies then that the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in $H^1(U)$. Hence, with the compact embedding $H^1(U) \hookrightarrow L^2(U)$ it follows for some subsequence, again denoted by u_k,

$$u_k \rightharpoonup u \quad \text{in } H^1(U), \quad u_k \to u \quad \text{in } L^2(U) \quad \text{as } k \to \infty$$

for some $u \in H^1(U)$. Using (2.35) and the uniqueness of weak limits it follows $\varepsilon(u) = 0$, as $\varepsilon(u_k) \to 0$ in $L^2(U)$ and $\varepsilon(u_k) \rightharpoonup \varepsilon(u)$ in $L^2(U)$. This implies that u is given via a rigid motion, i.e. there exists $B \in \mathbb{R}^{n\times n}_{skew}$ and $c \in \mathbb{R}^n$ such that

$$u(x) = Bx + c \quad \text{for all } x \in U.$$

Using now that ∇u_k has symmetric mean, we obtain $B = 0$, as ∇u is constant. Zero mean of u_k for all $k \in \mathbb{N}$ leads to $u \equiv 0$. But with this it follows a contradiction

$$1 = \|u\|_{L^2(U)} \to \|u\|_{L^2(U)} = 0.$$

Hence the claim follows.
STEP 2: (General gradients). Let $u \in H^1(U)$ and define

$$B := \int_U (\nabla u - \varepsilon(u))dx.$$

Then we consider the function $\tilde{u}(x) = u(x) - Bx \in H^1(U)$, which satisfies the condition of the first step

$$\int_U \nabla \tilde{u} dx \in \mathbb{R}^{n\times n}_{skew}.$$

Thus using (2.34) and $\varepsilon(\tilde{u}) = \varepsilon(u)$ it follows (2.32).
STEP 3: (Independence of scaling and translation). Let $\lambda \in \mathbb{R}$ and $c \in \mathbb{R}^n$ be arbitrary, $V := \lambda U + c$ and $v \in H^1(V)$. Thus for $u \in H^1(U)$ defined by $u(x) := v(\lambda x + c)$ it holds

$$\|\nabla u - B\|_{L^2(U)} \leq C(U)\|\varepsilon(u)\|_{L^2(U)}.$$

Using integration by substitution it follows

$$\begin{aligned}\int_V \lambda^{1-n}|\nabla v(y) - B|^2 dy = \int_U |\nabla u(x) - B|^2 dx &\leq C(U) \int_U |\varepsilon(u(x))|^2 dx \\ &= \int \lambda^{1-n}|\varepsilon(v(y))|^2 dy.\end{aligned}$$

Thus we can choose $C(V) = C(U)$. □

The classical Korn inequality of Theorem 2.4.1 allows us now to prove a similar looking inequality for the special domain Ω of thin rods. The following Lemma is then used to derive a crucial estimate of the scaled gradient with respect to the symmetric scaled gradient and the mean rotation around the x_1 axis.

Lemma 2.4.4 (Korn inequality in rotational form)**.** *For all $0 < h \leq 1$ and $u \in H^1_{per}(\Omega;\mathbb{R}^3)$, there exists a constant $C = C(\Omega) > 0$ and*

$$B(u) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a(u) \\ 0 & -a(u) & 0 \end{pmatrix} \tag{2.36}$$

with $a(u) = \frac{1}{|\Omega|}\int_\Omega \partial_{x_3} u_2(x) - \partial_{x_2} u_3(x) dx$, such that

$$\left\| \nabla_h u - \frac{1}{h} B(u) \right\|_{L^2(\Omega)} \leq C \left\| \frac{1}{h}\varepsilon_h(u) \right\|_{L^2(\Omega)}. \tag{2.37}$$

Proof: Using the scaling $\phi\colon \Omega_h \to \Omega$, $x \mapsto \left(x_1, \frac{1}{h}x_2, \frac{1}{h}x_3\right)$, we obtain

$$\begin{aligned} a(u) =& \frac{h^2}{|\Omega_h|}\int_\Omega \partial_{x_3} u_2(x) - \partial_{x_2} u_3(x) dx = \frac{1}{|\Omega_h|}\int_{\Omega_h} (\partial_{x_3} u_2)(\phi(x)) - (\partial_{x_2} u_3)(\phi(x)) dx \\ =& h\frac{1}{|\Omega_h|}\int_{\Omega_h} \partial_{x_3}(u_2 \circ \phi)(x) - \partial_{x_2}(u_3\circ\phi)(x) dx \end{aligned}$$

With this and another use of the transformation formula with ϕ, it follows, that (2.37) is equivalent to

$$\left\|\nabla u - \tilde{B}(u)\right\|_{L^2(\Omega_h)} \le C\left\|\frac{1}{h}\varepsilon(u)\right\|_{L^2(\Omega_h)} \qquad \text{for all } u \in H^1_{per}(\Omega_h;\mathbb{R}^3)$$

where $\tilde{B}$ has the same structure as B, with $a(u)$ replaced by

$$\tilde{a}(u) = \frac{1}{|\Omega_h|}\int_{\Omega_h} \partial_{x_3}u_2(x) - \partial_{x_2}u_3(x)dx.$$

Now we set $N_h := \lfloor \frac{L}{h} \rfloor$, $l_h = \frac{L}{N_h}$ and

$$J_h := \{kl_h \ : \ k = 0,\dots,N_h - 1\}.$$

We apply Korn inequality on every cube $(a, a + l_h) \times hS$ with $a \in J_h$ and obtain therefore a piecewise constant function $A\colon (0,L) \to \mathbb{R}^{3\times 3}$ such that $A(s)$ is skew symmetric and

$$\int_{\Omega_h} |\nabla u - A|^2 dx \le C\int_{\Omega_h} |\varepsilon(u)|^2 dx.$$

As $0 < h \le 1$, we obtain that $\frac{l_h}{h}$ is bounded because of $\frac{l_h}{h} \sim 1$ in h. Thus we can assume that C is independent of h.
Claim: For $A_0 := \lim_{\tau\searrow 0} A(\tau)$ it holds

$$\int_{\Omega_h} |A(x_1) - A_0|^2 dx \le \frac{C}{h^2}\int_{\Omega_h} |\varepsilon(u)|^2 dx.$$

Proof: We notice that because A is constant on $(0, l_h)$ the limit in the definition of A_0 exists. Fix some $a \in J_h$ and set $b_\lambda := a + \lambda l_h$, $\lambda \in \{0,1\}$. By applying Korn's inequality on $(a, a+2l_h)\times hS$ it follows that there exists $\tilde{A} \in \mathbb{R}^{3\times 3}$ such that

$$\int_{(a,a+2l_h)\times hS} |\nabla u - \tilde{A}|^2 dx \le C\int_{(a,a+2_h)\times hS} |\varepsilon(u)|^2 dx.$$

Thus, as $|S| = 1$

$$\begin{aligned} h^2 l_h|A(b_\lambda) - \tilde{A}|^2 &= \int_{(b_\lambda,b_\lambda+l_h)\times hS} |A(b_\lambda) - \tilde{A}|^2 dx \\ &\le 2\int_{(b_\lambda,b_\lambda+l_h)\times hS} |\nabla u - A(b)|^2 dx + 2\int_{(b_\lambda,b_\lambda+l_h)\times hS} |\nabla u - \tilde{A}|^2 dx \\ &\le C\int_{(b_\lambda,b_\lambda+l_h)\times hS} |\varepsilon(u)|^2 dx \end{aligned}$$

for $\lambda \in \{0,1\}$. Therefore,

$$\begin{aligned} h^2 l_h|A(a) - A(a+l_h)|^2 &\le 2h^2 l_h(|A(a) - \tilde{A}|^2 + |A(a+l_h) - \tilde{A}|^2) \\ &\le C\int_{(a,a+2l_h)\times hS} |\varepsilon(u)|^2 dx. \end{aligned}$$

As A is constant on the interval $(a, a + l_h)$, it follows

$$\begin{aligned} \int_{(a,a+l_h)\times hS} |A(x_1) - A(x_1 + l_h)|^2 dx &= h^2 l_h|A(a) - A(a+l_h)| \qquad (2.38) \\ &\le C\int_{(a,a+2l_h)\times hS} |\varepsilon(u)|^2 dx. \end{aligned}$$

In order to obtain the full domain, we consider $I_{k,j} := l_h(k, k+j)$ for $k \in \{0, \dots, N_h - 1\}$, $j \in \{1, 2\}$ and obtain

$$
\begin{aligned}
\int_{\Omega_h} |A(x_1) - A_0|^2 dx &= h^2 \sum_{k=0}^{N_h-1} \int_{I_{k,1}} |A(x_1) - A_0|^2 dx_1 \\
&= h^2 \sum_{k=0}^{N_h-1} \int_{I_{k,1}} \left| \sum_{m=0}^{k-1} A(x_1 - ml_h) - A(x_1 - (m+1)l_h) \right|^2 dx_1 \\
&\leq h^2 \sum_{k=0}^{N_h-1} k \sum_{m=0}^{k-1} \int_{I_{k,1}} |A(x_1 - ml_h) - A(x_1 - (m+1)l_h)|^2 dx_1
\end{aligned}
$$

where we used the Cauchy-Schwarz inequality in $\mathbb{R}^k$. Due to definition of $I_{k,j}$ we have the equality

$$
\int_{I_{k,1}} |A(x_1 - ml_h) - A(x_1 - (m+1)l_h)|^2 dx_1 = \int_{I_{k-m-1,1}} |A(x_1 + l_h) - A(x_1)|^2 dx_1
$$

for $m = 0, \dots, k-1$. Thus, it follows with (2.38)

$$
\int_{\Omega_h} |A(x_1) - A_0|^2 dx \leq C \sum_{k=0}^{N_h-1} k \sum_{m=0}^{k-1} \int_{I_{k-m-1,2} \times hS} |\varepsilon(u)|^2 dx.
$$

Using that $I_{k-m-1,2} = ((k-m-1)l_h, (k-m)l_h] \cup ((k-m)l_h, (k-m+1)l_h)$ and $|\varepsilon(u)|^2 \geq 0$ it follows

$$
\sum_{m=0}^{k-1} \int_{I_{k-m-1,2} \times hS} |\varepsilon(u)|^2 dx \leq 2 \int_{\Omega_h} |\varepsilon(u)|^2 dx.
$$

Hence, we can conclude

$$
\begin{aligned}
\int_{\Omega_h} |A(x_1) - A_0|^2 dx &\leq 2C \int_{\Omega_h} |\varepsilon(u)|^2 dx \cdot \sum_{k=0}^{N_h-1} k \\
&\leq C N_h^2 \int_{\Omega_h} |\varepsilon(u)|^2 dx \leq \frac{C}{h^2} \int_{\Omega_h} |\varepsilon(u)|^2 dx.
\end{aligned}
$$

As $0 < h \leq 1$ we obtain that for every $u \in H^1(\Omega_h, \mathbb{R}^3)$ there exists a constant skew symmetric matrix $A_0 \in \mathbb{R}^{3\times 3}$ such that

$$
\int_{\Omega_h} |\nabla u - A_0|^2 dx \leq \frac{C}{h^2} \int_{\Omega_h} |\varepsilon(u)|^2 dx. \tag{2.39}
$$

Using the Hölder Inequality for 1 and $\operatorname{skw}(\nabla u) - A_0$ we obtain

$$
\int_{\Omega_h} \left| \frac{1}{|\Omega_h|} \int_{\Omega_h} (\operatorname{skw} \nabla u) dx - A_0 \right|^2 dx \leq \int_{\Omega_h} |\operatorname{skw} \nabla u - A_0|^2 dx
$$

and thus

$$
\int_{\Omega_h} \left| \nabla u - \frac{1}{|\Omega_h|} \int_{\Omega_h} \operatorname{skw}(\nabla u) dx \right|^2 dx \leq \frac{C}{h^2} \int_{\Omega_h} |\varepsilon(u)|^2 dx \tag{2.40}
$$

for all $u \in H^1(\Omega_h; \mathbb{R}^3)$.

If we have in addition that u is periodic in x_1 direction it follows

$$
\int_{\Omega_h} \partial_{x_1} u dx = \int_S \int_0^L \partial_{x_1} u dx_1 dx' = 0.
$$

Hence, we conclude

$$\int_{\Omega_h} \operatorname{skw} \nabla u - \tilde{B}(u) dx$$
$$= \begin{pmatrix} 0 & \int_{\Omega_h} \partial_{x_2} u_1 dx & \int_{\Omega_h} \partial_{x_3} u_1 dx \\ -\int_{\Omega_h} \partial_{x_2} u_1 dx & 0 & \int_{\Omega_h} \partial_{x_3} u_2 - \partial_{x_2} u_3 - \tilde{B}(u)_{23} dx \\ -\int_{\Omega_h} \partial_{x_3} u_1 dx & -\int_{\Omega_h} \partial_{x_3} u_2 - \partial_{x_2} u_3 + \tilde{B}(u)_{32} dx & 0 \end{pmatrix}.$$

Using the definition of $\tilde{B}(u)$ it follows

$$\begin{aligned} \int_{\Omega_h} \frac{1}{|\Omega_h|} \left| \int_{\Omega_h} \operatorname{skw}(\nabla u) dx - \tilde{B}(u) \right|^2 dx & \\ &= \frac{1}{2|\Omega_h|} \int_{\Omega_h} \left(\int_{\Omega_h} \partial_{x_2} u_1 dx \right)^2 + \left(\int_{\Omega_h} \partial_{x_3} u_1 dx \right)^2 dx \\ &\leq \frac{1}{2} \left(\int_{\Omega_h} (\partial_{x_2} u_1 + \partial_{x_1} u_2)^2 dx + \int_{\Omega_h} (\partial_{x_3} u_1 + \partial_{x_1} u_3)^2 dx \right) \\ &\leq C \int_{\Omega_h} |\varepsilon(u)|^2 dx. \end{aligned}$$

Combining the last estimate with (2.40) the statement follows. □

Theorem 2.4.5. *Let $U \subset \mathbb{R}^3$ open and $v \in H^1(U, \mathbb{R}^3)$. Furthermore let $D \subset U$ be a two dimensional submanifold with normal ν. Then one obtains*

$$\int_D \operatorname{rot} v \cdot \nu dx = \int_{\partial D} v \cdot \tau d\sigma(x).$$

Proof: Use the classical Stokes Theorem and a density argument, see for instance [Tay11]. □

Lemma 2.4.6. *Let $0 < h \leq 1$, $u \in H^1_{per}(\Omega; \mathbb{R}^3)$ and $B(u)$ as in Lemma 2.37. Then it holds for all $\varepsilon > 0$*

$$\|S(u)\|_{L^2(\Omega)} \leq C\varepsilon \|u\|_{L^2(0,L;H^1(S))} + \frac{C}{\varepsilon} \|u\|_{L^2(\Omega)} \tag{2.41}$$

Proof: Using the definition of $S(u)$ it follows

$$\begin{aligned} \|B(u)\|^2_{L^2(\Omega)} &= \int_\Omega B(u) : B(u) dx = 2 \int_\Omega a(u)^2 dx = 2 \int_\Omega \frac{1}{|\Omega|^2} \left(\int_\Omega \partial_3 u_2 - \partial_2 u_3 dx \right)^2 dx \\ &= \frac{2}{|\Omega|} \left(\int_\Omega \partial_3 u_2 - \partial_2 u_3 dx \right)^2 \end{aligned}$$

Stokes Theorem applied to u, $U := (0, L) \times S$ and $D = x_1 \times S$ for some $x_1 \in (0, L)$ yields

$$\begin{aligned} \int_\Omega \partial_3 u_2 - \partial_2 u_3 dx &= \int_0^L \int_S \partial_3 u_2(x) - \partial_2 u_3(x) dx' dx_1 \\ &= \int_0^L \int_S \operatorname{rot} u \cdot \nu dx' dx_1 = \int_0^L \int_{\partial S} u \cdot \tau d\sigma dx_1 \end{aligned} \tag{2.42}$$

Hence

$$\begin{aligned} \|B(u)\|^2_{L^2(\Omega)} &\leq \int_0^L \int_{\partial S} |u| d\sigma dx_1 = \|u\|_{L^1(0,L;L^1(\partial S))} \leq C \| \|u(\cdot)\|^{\frac{1}{2}}_{L^2(S)} \|u(.)\|^{\frac{1}{2}}_{H^1(S)} \|_{L^2(0,L)} \\ &\leq C\varepsilon \|u\|_{L^2(0,L;H^1(S))} + C_\varepsilon \|u\|_{L^2(\Omega)} \end{aligned}$$

where we have used the interpolation space $(L^2(S), H^1(S))_{\frac{1}{2},2} = H^{\frac{1}{2}}(S)$ and scaled Young's inequality. $\square$

Lemma 2.4.7. *For all* $0 < h \leq 1$ *and* $u \in H^1_{per}(\Omega;\mathbb{R}^3)$*, there exists a constant* $C = C(\Omega)$*, such that*

$$\|\nabla_h u\|_{L^2(\Omega)} \leq \frac{C}{h}\Big(\|\varepsilon_h(u)\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\Big). \tag{2.43}$$

Proof: From Lemma 2.4.4 it follows that

$$\Big\|\nabla_h u - \frac{1}{h}B(u)\Big\|_{L^2(\Omega)} \leq \frac{C}{h}\|\varepsilon_h(u)\|_{L^2(\Omega)} \tag{2.44}$$

holds. With Lemma 2.4.6 we get

$$\begin{aligned}
\|\nabla_h u\|_{L^2(\Omega)} &\leq C\Big\|\frac{1}{h}\varepsilon_h(u)\Big\|_{L^2(\Omega)} + \Big\|\frac{1}{h}B(u)\Big\|_{L^2(\Omega)} \\
&\leq C\Big\|\frac{1}{h}\varepsilon_h(u)\Big\|_{L^2(\Omega)} + \frac{C}{\varepsilon h}\|u\|_{L^2(\Omega)} + \frac{C\varepsilon}{h}\|u\|_{L^2(0,L;H^1(\Omega))}
\end{aligned}$$

for all $\varepsilon > 0$. Furthermore, we have due to definition

$$\frac{1}{h}\|u\|_{L^2(0,L;H^1(\Omega))} \leq \frac{C}{h}\|u\|_{L^2(\Omega)} + C\|\nabla'_h u\|_{L^2(\Omega)}.$$

Choosing $\varepsilon = \frac{1}{2}$ and absorbing $\|\nabla'_h u\|_{L^2(\Omega)} \leq \|\nabla_h u\|_{L^2(\Omega)}$ it follows

$$\|\nabla_h u\|_{L^2(\Omega)} \leq \frac{C}{h}\|\varepsilon_h(u)\|_{L^2(\Omega)} + \frac{C}{h}\|u\|_{L^2(\Omega)}. \qquad \square$$

Lemma 2.4.8 (Korn inequality in integral form)**.** *For all* $0 < h \leq 1$ *and* $u \in H^1_{per}(\Omega;\mathbb{R}^3)$*, there exists a constant* $C_K = C_K(\Omega)$*, such that*

$$\|\nabla_h u\|_{L^2(\Omega)} \leq \frac{C_K}{h}\left(\|\varepsilon_h(u)\|_{L^2(\Omega)} + \left|\int_\Omega u\cdot x^\perp dx\right|\right) \tag{2.45}$$

where $x^\perp = (0, -x_3, x_2)^T$.

Proof: First we notice that we can reduce to the case of mean value free functions. If (2.45) holds for mean value free functions and $\int_\Omega u \neq 0$, we can regard $v := u - \int_\Omega u$. Then it follows

$$\begin{aligned}
\|\nabla_h u\|_{L^2(\Omega)} = \|\nabla_h v\|_{L^2(\Omega)} &\leq \frac{C_K}{h}\left(\|\varepsilon_h(v)\|_{L^2(\Omega)} + \left|\int_\Omega v\cdot x^\perp dx\right|\right) \\
&\leq \frac{C_K}{h}\left(\|\varepsilon_h(u)\|_{L^2(\Omega)} + \left|\int_\Omega u\cdot x^\perp dx\right| + \left|\int_\Omega u dx\cdot \int_\Omega x^\perp dx\right|\right)
\end{aligned}$$

and thus (2.45) holds for u, as $\int_\Omega x^\perp dx = 0$.

In the following we will argue by contradiction and therefore assume that (2.45) does not hold. Thus we can find a monotone sequence $h_k \to 0$ for $k \to \infty$ and $(u^{h_k})_{k\in\mathbb{N}} \subset H^1_{per,(0)}(\Omega,\mathbb{R}^3)$ such that

$$1 = \|\nabla_{h_k} u^{h_k}\|_{L^2(\Omega)} \geq \frac{k}{h_k}\left(\|\varepsilon_{h_k}(u^{h_k})\|_{L^2(\Omega)} + \left|\int_\Omega u^{h_k}\cdot x^\perp dx\right|\right). \tag{2.46}$$

For sake of readability, we will drop the k in forthcoming calculations. From (2.46) it follows

$$\frac{1}{h}\|\varepsilon_h(u^h)\|_{L^2(\Omega)} \leq \frac{1}{k} \to 0$$

which implies, using Lemma 2.4.4

$$\left\|\nabla_h u^h - \frac{1}{h}B(u^h)\right\|_{L^2(\Omega)} \leq C\left\|\frac{1}{h}\varepsilon_h(u^h)\right\|_{L^2(\Omega)} \to 0.$$

Thus $\frac{1}{h}B(u^h)$ is bounded in $L^2(\Omega)$ and therefore bounded in $\mathbb{R}^{n\times n}_{skew}$. Using a subsequence, also denoted by u^h, it follows $\frac{1}{h}B(u^h) \to \bar{B}$ for $h \to 0$. As a consequence of (2.36) the structure of $\bar{B}$ is given by

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\bar{a} \\ 0 & \bar{a} & 0 \end{pmatrix}$$

where $\bar{a} \neq 0$ as, $\nabla_h u^h \to \bar{B}$ in $L^2(\Omega)$ and $\|\nabla_h u^h\|_{L^2(\Omega)} = 1$

Define now

$$w_l^h(x') := \frac{1}{L}\int_0^L \frac{u_l^h(x)}{h}dx_1$$

with $x' \in S$ and $l = 2, 3$. Then

$$\nabla' w^h \to \begin{pmatrix} 0 & -\bar{a} \\ \bar{a} & 0 \end{pmatrix}$$

in $L^2(S)$ and $\int_S w^h dx' = 0$. Thus using the Poincaré inequality it follows

$$\|w^h\|_{H^1(S)} \leq C\|\nabla' w^h\|_{L^2(S)} \leq C\|\nabla_h u^h\|_{L^2(\Omega)} \leq C$$

Thus, there exists a subsequence $w^h \rightharpoonup w$ in $H^1(S)$ and $w^h \to w$ in $L^2(S)$. Choose $S' \subset \bar{S}' \subset S$ and $\delta > 0$ such that $\delta \leq \operatorname{dist}(S', \partial S)$. Then

$$\frac{w_2^h(x_2+\delta, x_3) - w_2^h(x_2, x_3)}{\delta} = \frac{1}{\delta}\int_0^\delta \partial_2 w_2^h(x_2+\tau, x_3)d\tau.$$

From the above we know

$$\frac{w_2^h(x_2+\delta, x_3) - w_2^h(x_2, x_3)}{\delta} \to \frac{w_2(x_2+\delta, x_3) - w_2(x_2, x_3)}{\delta} \qquad \text{for } h \to 0$$

in $L^2(S')$ and thus a subsequence converges point wise almost everywhere. For the right hand side we know that $\partial_{x_2} w_2^h \to 0$ for $h \to 0$ in $L^2(S)$ and thus the mean value $\frac{1}{\delta}\int_0^\delta \partial_2 w_2^h(x_2+\tau, x_3)d\tau \to 0$ in $L^2(S')$. This can be seen by

$$\begin{aligned}
\left\|\frac{1}{\delta}\int_0^\delta \partial_2 w_2^h(x_2+\tau, x_3)d\tau\right\|^2_{L^2(S')} &= \int_{S'}\left|\frac{1}{\delta}\int_0^\delta \partial_2 w_2^h(x_2+\tau, x_3)d\tau\right|^2 dx' \\
&\leq \int_{S'}\frac{1}{\delta}\int_0^\delta |\partial_2 w_2^h(x_2+\tau, x_3)|^2 d\tau dx \\
&= \frac{1}{\delta}\int_0^\delta \|\partial_2 w_2^h(\cdot + \tau e_2)\|_{L^2(S')}d\tau \to 0
\end{aligned}$$

where we used Hölder's inequality and the dominated convergence theorem for

$$\|\partial_2 w_2^h(\cdot + \tau e_2)\|_{L^2(S')} \leq \|\partial_2 w_2^h\|_{L^2(S)} \to 0 \qquad \text{for } h \to 0.$$

Hence, as S is open, w_2 is independent of x_2. Similarly one can show that w_3 is independent of x_3.

Furthermore

$$\frac{w_2^h(x_2, x_3+\delta) - w_2^h(x_2, x_3)}{\delta} = \frac{1}{\delta}\int_0^\delta \partial_3 w_2^h(x_2, x_3+\tau)d\tau$$

where the right-hand side converges to the constant $-\bar{a}$ in $L^2(S')$. Since $x' \in S'$ was chosen arbitrarily it follows

$$w_2(x') = w_2^0 - \bar{a}x_3.$$

where $w_2^0 \in \mathbb{R}$ is constant. Applying the same argument to w_3 it follows that

$$w(x) = \begin{pmatrix} w_2^0 \\ w_3^0 \end{pmatrix} + \bar{a} \begin{pmatrix} -x_3 \\ x_2 \end{pmatrix}$$

Hence

$$\begin{aligned} \left| \int_\Omega \frac{u^h}{h} \cdot x^\perp dx \right| &= L \left| \frac{1}{L} \int_0^L \int_S \frac{u^h}{h} \cdot x^\perp dx \right| \\ &= L \left| \int_S w^h \cdot x^\perp dx \right| \to L\bar{a} \left| \int_S x_2^2 + x_3^2 dx \right| \neq 0 \end{aligned}$$

as $\bar{a} \neq 0$, but this contradicts (2.46). □

Later we will need the Korn inequality in two dimensions without scaling while analysing a stationary problem associated with the linearised equation.

Corollary 2.4.9 (Korn inequality in two dimensions). *For all $u \in H^1(S;\mathbb{R}^2)$, there exists a constant $C = C(\Omega) > 0$, such that*

$$\|\nabla u\|_{L^2(S)} \leq C\Big(\|\varepsilon(u)\|_{L^2(S)} + \Big| \int_S u \cdot x^\perp dx \Big| \Big) \tag{2.47}$$

where in this situation $x^\perp := (-x_3, x_2)^T$.

Proof: We can deduce (2.47) form Lemma 2.4.8. Let $u \in H^1(S;\mathbb{R}^2)$ and define $\tilde{u} \in H^1_{per}(\Omega;\mathbb{R}^3)$ by

$$\tilde{u}(x) := \frac{1}{\sqrt{L}} \begin{pmatrix} 0 \\ u(x') \end{pmatrix}.$$

Then it follows from (2.45) for $h = 1$ applied to $\tilde{u}$

$$\begin{aligned} \|\nabla u\|_{L^2(S)} = \|\nabla \tilde{u}\|_{L^2(\Omega)} &\leq C\Big(\|\varepsilon(\tilde{u})\|_{L^2(\Omega)} + \Big| \int_\Omega \tilde{u} \cdot x^\perp dx \Big| \Big) \\ &= C\Big(\|\varepsilon(u)\|_{L^2(S)} + \Big| \int_S u \cdot x^\perp dx \Big| \Big). \end{aligned}$$

□

3

Γ-Convergence for Loaded Periodic Rods

The chapter's aim is to prove a dimension reduction result via the notion of Γ-convergence. More precisely we compute the variational limit of the energy of non-linear three dimensional elasticity. We start with an introduction of the energy series. Then we use the rigidity result by Friesecke, James and Müller [FJM02] to deduce a scaled version in thin rods for deformations which have an energy comparable to $h^{2\alpha-2}$. Up to a fixed rotation this leads to an approximation of $\nabla_h y$ by a smooth map $R^{(h)}$ which converges to the Id in L^∞. From these we derive the scaling properties for essential intrinsic quantities on which the limiting energy will depend. Having this we can deduce the convergence of an approximated non-linear strain $G^{(h)}$, which in turn leads us to the $\liminf$-inequality in the proof of Γ-convergence. For the $\limsup$-inequality we use a classical ansatz function which reflects the respective scaling properties.
Finally using a revisited approximation theorem, where no conditions on the deformation is made, we can show the Γ-convergence for models which exhibit an external force.
The results of this chapter are based on ideas from the work of Lucia Scardia [Sca09].

3.1 Rigorous Formulation of the Energies

We shortly revisit the most important notation and all assumptions made for this chapter. Let $S \subset \mathbb{R}^2$ be a smooth domain such that

$$\int_S x_2 x_3 dx' = 0 \tag{3.1}$$

and

$$\int_S x_2 dx' = \int_S x_3 dx' = 0 \tag{3.2}$$

where $x' := (x_2, x_3) \subset \mathbb{R}^2$. Moreover we assume $|S| = 1$ and set $\Omega_h := (0, L) \times hS \subset \mathbb{R}^3$ for $h \in (0, 1]$ and $L > 0$ some length in $\mathbb{R}$. As abbreviation we will write $\Omega := \Omega_1$ and for differentiation we use the abbreviation $\partial_{x_k} v = v_{,k}$ for all $v \in H^1(\Omega)$ and $k = 1, 2, 3$.
We assume that the strain-energy density $W\colon \mathbb{R}^{3\times3} \to [0, \infty]$ satisfies the following conditions:

(i) W is C^2 on $\{F \in \mathbb{R}^{3\times3} \;:\; \operatorname{dist}(F; SO(3)) < \delta\}$ for some $\delta > 0$;

(ii) there exists $c_1 > 0$ such that the second derivative D^2W satisfies

$$\left| D^2 W(F)[G, G] \right| \le c_1 |G|^2 \quad \text{for } \operatorname{dist}(F, SO(3)) < \delta \text{ and } G \in \mathbb{R}^{3\times3}_{sym};$$

(iii) W is frame-invariant, i.e. $W(RF) = W(F)$ for all $F \in \mathbb{R}^{3\times 3}$ and $R \in SO(3)$;

(iv) there exists $c_2 > 0$ such that $W(F) \geq c_2 \operatorname{dist}^2(F, SO(3))$ for all $F \in \mathbb{R}^{3\times 3}$ and $W(R) = 0$ for every $R \in SO(3)$.

We want to mention that at this point no growth condition from above is needed. Therefore we are able to treat the physical relevant case in which $W = \infty$ for $\det(F) < 0$ and $W \to +\infty$ for $\det(F) \to 0^+$ is necessary.
The scaled gradient of $z \in W_2^1(\Omega;\mathbb{R}^3)$ is given via

$$\nabla_h z := \left(\partial_{x_1} z \middle| \frac{1}{h}\partial_{x_2} z \middle| \frac{1}{h}\partial_{x_3} z\right).$$

Moreover, we set $x^{(h)} := (x_1, hx_2, hx_3)$ and define $\mathrm{Id}_h \colon \Omega \to \mathbb{R}^3$ via $x \mapsto x^{(h)}$. The subset of all rotations around the x_1-axes is denoted by $\mathcal{U}$, i.e.

$$\mathcal{U} := \{R \in SO(3) \ : \ R_{11} = 1\} \subset SO(3).$$

Let a sequence of external loads $f^{(h)} \in W_2^1(0, L;\mathbb{R}^3)$ and a limiting force $f \in W_2^1(0, L;\mathbb{R}^3)$ be given such that

$$\frac{1}{h^\alpha} f^{(h)} \rightharpoonup f \quad \text{ in } L^2(0, L;\mathbb{R}^3) \quad \text{for } h \to 0 \tag{3.3}$$

and we assume that the force does not act in e_1-direction, i.e. $f_1^{(h)} \equiv 0$. Moreover, we assume that the total force on the rod is zero, more precisely

$$\int_0^L f^{(h)}(x_1)dx_1 = 0.$$

The later assumption excludes the case of no lower bound due to the invariance $y \mapsto y + c$ for some constant $c \in \mathbb{R}^3$.
The total energy for relative periodic deformations is given by $\mathcal{E}^{(h)} \colon H^1(\Omega;\mathbb{R}^3) \to [0, \infty]$

$$\mathcal{E}^{(h)}(y) := \begin{cases} \displaystyle\int_\Omega W(\nabla_h y(x)) - (y(x) - x^{(h)}) \cdot f^{(h)}(x)dx, & \text{if } y - \mathrm{Id}_h \in H^1_{per}(\Omega;\mathbb{R}^3), \\ +\infty, & \text{otherwise} \end{cases} \tag{3.4}$$

and the scaled elastic energy $\mathcal{I}^{(h)} \colon H^1(\Omega;\mathbb{R}^3) \to [0, \infty]$

$$\mathcal{I}^{(h)}(y) := \begin{cases} \displaystyle\int_\Omega W(\nabla_h y(x))dx, & \text{if } y - \mathrm{Id}_h \in H^1_{per}(\Omega;\mathbb{R}^3), \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.5}$$

3.2 Compactness in von Karmán Regime

The necessary compactness results of sequences of deformations whose elastic energies are of order $h^{2\alpha-2}$, $\alpha \geq 3$, are proven in this section. We show that the set $\{\frac{1}{h^{2\alpha-2}}\mathcal{I}^{(h)} \leq C\}$ is compact with respect to the particular topology. This allows us later to use the sequential definition of Γ-convergence estabilshed in Proposition 2.1.6 and 2.1.10.
The main ingredient for the proof is the fundamental regidity theorem proven by Friesecke, James and Müller in [FJM02].

Theorem 3.2.1. *Let U be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Then there exists a constant $C(U)$ with the following property: for every $u \in W_2^1(U;\mathbb{R}^n)$ there is an associated rotation $R \in SO(n)$ such that*

$$\|\nabla u - R\|_{L^2(U)} \leq C(U)\|\operatorname{dist}(\nabla u, SO(n))\|_{L^2(U)}. \tag{3.6}$$

Remark 3.2.2. The constant $C(U)$ can be chosen invariant under uniform scaling and translation of the domain, i.e. $C(U)$ can be used as a constant for all $\lambda U + c$. This can be shown in the same way as it was shown for the Korn inequality in Theorem 2.4.1.

Definition 3.2.3 (Scaling of sequences)**.** For some sequence $Y^{(h)} \subset H^1(\Omega; \mathbb{R}^3)$ we define

$$u^{(h)}(x_1) := \frac{1}{h^{\alpha-1}} \int_S (Y_1^{(h)}(x_1, x') - x_1) dx' \tag{3.7}$$

$$v_k^{(h)}(x_1) := \frac{1}{h^{\alpha-2}} \int_S Y_k^{(h)}(x_1, x') dx' \quad \text{for } k = 2,\ 3 \tag{3.8}$$

$$w^{(h)}(x_1) := \frac{1}{h^{\alpha-1}} \int_S \frac{x_2 Y_3^{(h)}(x_1, x') - x_2 Y_2^{(h)}(x_1, x')}{\mu(S)} dx' \tag{3.9}$$

where $\mu(S) := \int_S x_2^2 + x_3^2 dx'$.

The main result is now the following theorem.

Theorem 3.2.4. *Let $y^{(h)} \subset H^1(\Omega; \mathbb{R}^3)$ be such that*

$$\frac{1}{h^{2\alpha-2}} \mathcal{I}^{(h)}(y^{(h)}) \leq C < +\infty \tag{3.10}$$

for every $h > 0$. Then there exist associated maps $R^{(h)} \in C^\infty(0, L; \mathbb{R}^{3\times3})$ and constants $\bar{R}^{(h)} \in SO(3)$, $c^{(h)} \in \mathbb{R}^3$ such that (up to subsequences) $\bar{R}^{(h)} \to \bar{R} \in \mathcal{U}$ and for

$$\tilde{y}^{(h)} := (\bar{R}^{(h)})^T y^{(h)} - c^{(h)}$$

we have

$$R^{(h)}(s) \in SO(3) \text{ for every } s \in (0, L), \tag{3.11}$$

$$\|R^{(h)} - Id\|_{L^\infty(0,L)} \leq Ch^{\alpha-2}, \quad \|(R^{(h)})'\|_{L^2(0,L)} \leq Ch^{\alpha-2} \tag{3.12}$$

$$\|\nabla_h \tilde{y}^{(h)} - R^{(h)}\|_{L^2(\Omega)} \leq Ch^{\alpha-1} \tag{3.13}$$

$$|R^{(h)}(0) - R^{(h)}(L)| \leq Ch^{\alpha-\frac{3}{2}}. \tag{3.14}$$

Moreover for $v_k^{(h)}$, $w^{(h)}$ and $u^{(h)}$ defined as in (3.7)–(3.9) for $Y^{(h)} = \tilde{y}^{(h)}$ it follows

(a) $u^{(h)} \rightharpoonup u$ weakly in $H^1_{per}(0, L)$;

(b) $v_k^{(h)} \to v_k$ strongly in $H^1_{per}(0, L)$, where $v_k \in H^2_{per}(0, L)$ for $k = 2,\ 3$;

(c) $w^{(h)} \rightharpoonup w$ weakly in $H^1_{per}(0, L)$

(d) $(\nabla_h \tilde{y}^{(h)} - Id)/h^{\alpha-2} \to A$ strongly in $L^2(\Omega; \mathbb{R}^{3\times3})$, where $A \in H^1_{per}(0, L; \mathbb{R}^{3\times3})$ is given by

$$A = \begin{pmatrix} 0 & -v_{2,1} & -v_{3,1} \\ v_{2,1} & 0 & -w \\ v_{3,1} & w & 0 \end{pmatrix};$$

(e) $(R^{(h)} - Id)/h^{\alpha-2} \rightharpoonup A$ weakly in $H^1(0, L; \mathbb{R}^{3\times3})$;

(f) $\mathrm{sym}(R^{(h)} - Id)/h^{2(\alpha-2)} \to A^2/2$ uniformly on $(0, L)$.

Proof: The proof follows the classical ideas as e.g. in [MM03; MM04; Sca09]. Key ingredient is to use the geometric rigidity result of Friesecke et al [FJM02]. We will apply it to a division of the domain Ω in small cubes of length h and an appropriately scaled function $r^{(h)}$. With this construction we obtain a piecewise constant function $Q^{(h)}\colon [0,L] \to SO(3)$ satisfying appropriate uniform bounds. The constants $\bar{R}$ and rotations $R^{(h)}$ originate then from $Q^{(h)}$ via mollification and projection. A finely tuned Poincaré like inequality leads then to (3.14). We use $R^{(h)}$ to define an approximation $A^{(h)}$ to show the convergence results (d)–(f). Lastly using the definitions of $u^{(h)}$, $v_k^{(h)}$ and $w^{(h)}$ one can identify the limit A and obtain (a)–(c).

Let $y^{(h)} \subset H^1(\Omega;\mathbb{R}^3)$ be given such that (3.10) is fulfilled. The coercivity assumption (iv) of W implies then

$$\frac{1}{h^{2\alpha-2}} \int_\Omega \operatorname{dist}^2(\nabla_h y^{(h)}, SO(3))dx \le C.$$

Step 1: (construction of approximating sequences). For $h > 0$ small enough, let $K_h \in \mathbb{N}$ be, such that $h \le l_h < 2h$ where $l_h := L/K_h$ and set $J_h := [0,L) \cap L/K_h\mathbb{N}$. Define for $a \in J_h$

$$I_{a,K_h} := \begin{cases} (a, a+2h), & \text{if } a < L-2h, \\ (L-2h, L), & \text{otherwise.} \end{cases}$$

Applying now Theorem 3.2.1 to $r^{(h)}(z) := y(z_1, \frac{z_2}{h}, \frac{z_3}{h})$ on the set $I_{a,K_h} \times hS$, we obtain a constant $Q_a^{(h)} \in SO(3)$ such that

$$\int_{I_{a,K_h}\times S} |\nabla_h y^{(h)}(x) - Q_a^{(h)}|^2 dx \le C \int_{I_{a,K_h}\times S} \operatorname{dist}^2(\nabla_h y^{(h)}(x), SO(3))dx. \tag{3.15}$$

Note that the constant $C(I_{a,K_h} \times hS) = C((0,2)\times S)$ for all $a \in J_h$, due to Remark 3.2.2 and $h > 0$. Therefore $C > 0$ does not depend on h. With this we can define the piecewise constant map $Q^{(h)}\colon [0,L) \to SO(3)$ via

$$Q^{(h)}(s) := Q_a^{(h)} \quad \text{for } s \in \Big[a, a + \frac{L}{K_h}\Big),\ a \in J_h.$$

Summing over all $a \in J_h$, we obtain

$$\int_\Omega |\nabla_h y^{(h)}(x) - Q^{(h)}|^2 dx \le C \int_\Omega \operatorname{dist}^2(\nabla_h y(x), SO(3))dx \le Ch^{2\alpha-2}. \tag{3.16}$$

Let now $a \in J_h$ be such that $(a, a+4h) \subset (0,L)$ and $b = a + l_h$. Applying Theorem 3.2.1 for $r^{(h)}$ and $(a, a+2l_h) \times hS$, leads to the existence of $\tilde{Q} \in SO(3)$ such that

$$\int_{(a,a+2l_h)\times S} |\nabla_h y^{(h)}(x) - \tilde{Q}|^2 dx \le C \int_{(a,a+2l_h)\times S} \operatorname{dist}^2(\nabla_h y(x), SO(3))dx \tag{3.17}$$

holds. Thus because $(a, a+l_h)$, $(b, b+l_h)$ are contained in $(a, a+2l_h)$,(3.17) and (3.15), we have

$$h|Q^{(h)}(a) - Q^{(h)}(b)|^2 \le C \int_{(a,a+4h)\times S} \operatorname{dist}^2(\nabla_h y(x), SO(3))dx.$$

Since Q is piecewise constant, we deduce

$$\int_{(a,a+l_h)} |Q^{(h)}(x_1) - Q^{(h)}(x_1 + l_h)|^2 dx_1 \le C \int_{(a,a+4h)\times S} \operatorname{dist}^2(\nabla_h y(x), SO(3))dx,$$

hence for $0 \le \xi \le h$

$$\int_{(a,a+l_h)} |Q^{(h)}(x_1) - Q^{(h)}(x_1 + \xi)|^2 dx_1 \le C \int_{(a,a+4h)\times S} \operatorname{dist}^2(\nabla_h y(x), SO(3))dx. \tag{3.18}$$

The same construction can be made for every $a \in J_h$ such that $(a-2h, a+2h) \subset (0,L)$. Then we obtain for $-h \leq \xi \leq 0$

$$\int_{(a,a+l_h)} |Q^{(h)}(x_1+\xi) - Q^{(h)}(x_1)|^2 dx_1 \leq C \int_{(a-2h,a+2h)\times S} \operatorname{dist}^2(\nabla_h y(x), SO(3)) dx. \quad (3.19)$$

Thus we obtain for $x_1 \in (2h, L-2h)$ and every $|\xi| < h$

$$|Q^{(h)}(x_1+\xi) - Q(x_1)|^2 \leq Ch^{2\alpha-3}. \quad (3.20)$$

Now for $I' \subset (0,L)$ compactly contained and $|\zeta| + 4h < \operatorname{dist}(I', \{0,L\})$ we deduce iteratively

$$\int_{I'} |Q^{(h)}(x_1+\zeta) - Q(x_1)|^2 dx_1 \leq Ch^{2(\alpha-2)}(|\zeta|+h)^2 \quad (3.21)$$

as follows: Define $N := |[\frac{\zeta}{h}]|$, where $[\cdot]$ denotes the integer part, and choose $\zeta_0, \dots, \zeta_{N+1}$ such that $\zeta_0 = 0$, $\zeta_{N+1} = \zeta$ and $|\zeta_{k+1} - \zeta_k| \leq h$. Then it follows

$$|Q^{(h)}(x_1+\zeta) - Q^{(h)}(x_1)|^2 \leq (N+1) \sum_{k=0}^{N} |Q^{(h)}(x_1+\zeta_{k+1}) - Q^{(h)}(x_1+\zeta_k)|^2$$

and hence

$$\int_{(a,a+l_h)} |Q^{(h)}(x_1+\zeta) - Q^{(h)}(x_1)|^2 dx \leq (N+1) \sum_{k=0}^{N} \int_{(a+\zeta_k, a+\zeta_k+4h)\times S} \operatorname{dist}^2(\nabla_h y^{(h)}, SO(3)) dx$$

Summing over all $a \in I_h := \{q \in J_h \ : \ (a, a+l_h) \cap I' \neq \emptyset\}$ it follows

$$\begin{aligned}
\int_{I'} |Q^{(h)}(x_1+\zeta) - Q^{(h)}(x_1)|^2 dx &\leq (N+1) \sum_{k=0}^{N} \sum_{a\in I_h} \int_{(a+\zeta_k, a+\zeta_k+4h)\times S} \operatorname{dist}^2(\nabla_h y^{(h)}, SO(3)) dx \\
&\leq C(N+1)^2 \int_{\Omega} \operatorname{dist}^2(\nabla_h y^{(h)}, SO(3)) dx \\
&\leq C\left(\frac{|\zeta|}{h} + 1\right)^2 \int_{\Omega} \operatorname{dist}^2(\nabla_h y^{(h)}, SO(3)) dx \\
&\leq Ch^{2(\alpha-2)}(|\zeta|+h)^2
\end{aligned}$$

Let $\eta \in C_0^\infty(0,1)$ be such that $\eta \geq 0$ and $\int_0^1 \eta(s) ds = 1$ and set

$$\eta^{(h)}(s) := \frac{1}{h} \eta\left(\frac{s}{h}\right). \quad (3.22)$$

We extend $Q^{(h)}$ constantly outside of $(0,L)$, i.e. $Q^{(h)}(s) := Q^{(h)}(0)$ for $s \leq 0$ and $Q^{(h)}(s) := Q^{(h)}(L)$ for $s \geq L$. Then we can define

$$\tilde{Q}^{(h)}(s) := \int_0^h \eta^{(h)}(t) Q^{(h)}(s-t) dt$$

for all $s \in [0,L]$. Applying (3.20), (3.21) and Jensen's inequality, it follows

$$\|\tilde{Q}^{(h)} - Q^{(h)}\|_{L^2(0,L)} \leq Ch^{\alpha-1}, \quad (3.23)$$

$$\|(\tilde{Q}^{(h)})'\|_{L^2(0,L)} \leq Ch^{\alpha-2} \quad (3.24)$$

$$\|\tilde{Q}^{(h)} - Q^{(h)}\|^2_{L^\infty(0,L)} \leq Ch^{2\alpha-3}. \quad (3.25)$$

Combining (3.16) and (3.23) leads to

$$\|\nabla_h y^{(h)} - \tilde{Q}^{(h)}\|_{L^2(\Omega)} \le Ch^{\alpha-1}. \tag{3.26}$$

As $SO(3)$ is a smooth manifold, there exists a neighbourhood U of $SO(3)$ and a smooth orthogonal projection $\pi\colon U \to SO(3)$, cf. [Lee13, Proposition 6.25]. Due to (3.25) we deduce that $\tilde{Q}^{(h)}$ takes values in U for sufficiently small h. Thus we can define $\tilde{R}^{(h)} := \pi(\tilde{Q}^{(h)})$. Since $\|(\tilde{R}^{(h)})'\|_{L^2(0,L)} \le Ch^{\alpha-2}$ by (3.24), we deduce with the Poincaré Inequality we obtain with $P^{(h)} := \int_0^L \tilde{R}^{(h)}(s)ds$

$$\|\tilde{R}^{(h)} - P^{(h)}\|_{L^\infty(0,L)} \le C\|(\tilde{R}^{(h)})'\|_{L^2(0,L)} \le Ch^{\alpha-2}. \tag{3.27}$$

This implies that

$$\operatorname{dist}(P^{(h)}, SO(3)) \le Ch^{\alpha-2},$$

and thus there exit $\bar{R}^{(h)} \in SO(3)$ such that $|P^{(h)} - \bar{R}^{(h)}| \le Ch^{\alpha-2}$. Using this and (3.26) it follows

$$\|\tilde{R}^{(h)} - \bar{R}^{(h)}\|_{L^\infty(0,L)} \le Ch^{\alpha-2}.$$

Now $R^{(h)} := (\bar{R}^{(h)})^T \tilde{R}^{(h)} \in C^\infty(0,L;\mathbb{R}^{3\times 3})$ and fulfils (3.11) and (3.12).

By choosing $c^{(h)}$ accordingly we can assume

$$\int_\Omega \tilde{y}_1^{(h)} - x_1 dx = 0, \qquad \int_\Omega \tilde{y}_k^{(h)} dx = 0 \text{ for } k \in \{2,3\}$$

and by (3.26) and (3.23) it holds

$$\|\nabla_h \tilde{y}^{(h)} - R^{(h)}\|_{L^2(\Omega)} \le C\|\nabla_h y - \tilde{Q}^{(h)} + \tilde{Q}^{(h)} - \tilde{R}^{(h)}\|_{L^2(\Omega)} \le Ch^{\alpha-1}. \tag{3.28}$$

Step 2: (structure of $\bar{R}$). The definition of $\mathcal{I}^{(h)}$ and (3.10) leads to $y^{(h)} - \mathrm{Id}^{(h)} \in H^1_{per}(\Omega;\mathbb{R}^3)$. Therefore, we obtain

$$\int_\Omega (\partial_{x_1} y_1^{(h)}(x) - 1)dx = \int_S \int_0^L \partial_{x_1}\Big(y_1^{(h)}(x_1,x') - x_1\Big)dx_1 dx' = \int_S \Big[y_1^{(h)}(x_1,x') - x_1\Big]_0^L dx' = 0.$$

Using this we obtain

$$\begin{aligned} 0 \le |\bar{R}_{11} - 1| = \left|\frac{1}{L}\int_\Omega (\bar{R}_{11} - 1)dx\right| &= \left|\frac{1}{L}\int_\Omega ()\bar{R}_{11} - \partial_{x_1} y_1^{(h)})dx + \frac{1}{L}\int_\Omega (\partial_{x_1} y_1^{(h)} - 1)dx\right| \\ &\le \frac{1}{L}\int_\Omega |\partial_{x_1} y_1^{(h)} - \bar{R}_{11}|dx \end{aligned}$$

for all $h > 0$. Thus as $L^2(\Omega) \hookrightarrow L^1(\Omega)$ and $\|\nabla_h y^{(h)} - \bar{R}\|_{L^2(\Omega)} \to 0$ we can conclude with Remark 2.1.2

$$\bar{R} = \begin{pmatrix} 1 & 0 \\ 0 & \bar{R}' \end{pmatrix}.$$

Step 3: (periodicity inequality). Due to the definitions of $R^{(h)}$, $\tilde{R}^{(h)}$ and $\tilde{Q}^{(h)}$ it is sufficient to show (3.14) for $Q^{(h)}$. We know from (3.15)

$$\int_{(0,L/K_h)\times S} |\nabla_h y^{(h)}(x) - Q_0^{(h)}|^2 dx \le C \int_{(0,2h)\times S} \operatorname{dist}^2(\nabla_h y^{(h)}(x), SO(3))dx.$$

We use the inequality

$$\int_S |v(0,x') - \bar{v}|^2 dx' \le c \int_{(0,l)\times S} |\nabla v|^2 dx$$

where $\bar{v} := \int_S v(0,x')dx'$ and $1 \leq l \leq 2$. Applying this estimate for

$$v(x) := \frac{1}{h}\Big(y^{(h)}(hx_1,x') - hQ_0^{(h)}x\Big)$$

leads to

$$\begin{aligned}\int_S \Big| y^{(h)}(0,x') - & \int_S y^{(h)}(0,x')dx' - hQ_0^{(h)}(0,x')\Big|^2 dx' \\ &\leq Ch^2 \int_{(0,l)\times S} |\nabla_h (y^{(h)} - Q_0^{(h)})(hx_1,x')|^2 dx \\ &\leq Ch \int_{(0,hl)\times S} |\nabla_h y^{(h)}(x) - Q_0^{(h)}|^2 dx' \leq Ch^{2\alpha-1}.\end{aligned}$$

When we use

$$v(x) := \frac{1}{h}\Big(y^{(h)}(hx_1,x') - hQ_L^{(h)}x\Big)$$

it follows due to $y^{(h)} - \mathrm{Id}_h \in H^1_{per}(\Omega)$ for every $h > 0$ analogously

$$\begin{aligned}\int_S \Big| y^{(h)}(L,x') - & \int_S y^{(h)}(L,x')dx' - hQ_L^{(h)}(0,x')\Big|^2 dx' \\ &\leq Ch \int_{(L-hl,L)\times S} |\nabla_h y^{(h)}(x) - Q_L^{(h)}|^2 dx' \leq Ch^{2\alpha-1}\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\int_S \Big|[Q_0^{(h)} - Q_L^{(h)}](0,x')\Big|^2 dx' = \frac{1}{h^2}\int_S \Big| & hQ_0^{(h)}(0,x') - y^{(h)}(0,x') + \int_S y^{(h)}(0,x')dx' \\ & + y^{(h)}(L,x') - \int_S y^{(h)}(L,x')dx' - hQ_L^{(h)}(0,x')\Big|^2 dx'\end{aligned}$$

Hence, it follows

$$\int_S \Big|[Q_0^{(h)} - Q_L^{(h)}](0,x')\Big|^2 dx' \leq Ch^{2\alpha-3}.$$

Remark 2.1.1 leads to

$$|Q^{(h)}(0) - Q^{(h)}(L)| \leq Ch^{\alpha-\frac{3}{2}}.$$

STEP 4: (definition of A). Define the sequence $A^{(h)}$ pointwise as

$$A^{(h)}(s) := \frac{1}{h^{\alpha-2}}(R^{(h)}(s) - Id)$$

for $s \in [0,L]$. By (3.12) and the embedding $L^\infty(0,L) \hookrightarrow L^2(0,L)$ there exists some $A \in H^1(0,L;\mathbb{R}^{3\times 3})$ such that, up to subsequences,

$$A^{(h)} \rightharpoonup A \quad \text{weakly in } H^1(0,L;\mathbb{R}^{3\times 3}).$$

Hence, with the compact embedding $H^1(0,L;\mathbb{R}^{3\times 3}) \hookrightarrow C^0([0,L];\mathbb{R}^{3\times 3})$, we deduce $A^{(h)} \to A$ uniformly. Since $R^{(h)} \in SO(3)$, it follows

$$A^{(h)} + (A^{(h)})^T = -h^{\alpha-2}(A^{(h)})^T A^{(h)}$$

and therefore $A \in \mathbb{R}^{3\times 3}_{skew}$. Dividing the latter equation by $2h^{\alpha-2}$, we get

$$\frac{1}{h^{2(\alpha-2)}}\,\mathrm{sym}(R^{(h)} - Id) \to \frac{A^2}{2}$$

uniformly. Now, due to (3.13) and the convergence properties of $A^{(h)}$, we know

$$\frac{1}{h^{\alpha-2}}(\nabla_h \tilde{y}^{(h)} - Id) \to A \tag{3.29}$$

strongly in $L^2(\Omega;\mathbb{R}^{3\times 3})$. Lastly, we use the improved scaling in (3.14) to obtain periodicity of A, more precisely we have

$$|A^{(h)}(0) - A^{(h)}(L)| = \frac{1}{h^{\alpha-2}}|R^{(h)}(0) - R^{(h)}(L)| \leq Ch^{\frac{1}{2}}.$$

Uniform convergence then leads to periodicity of A.
Step 5: (identification of A). Let $v_k^{(h)}$ be as in (3.8) for $Y^{(h)}$ replaced by $\tilde{y}^{(h)}$. By the choice of $c^{(h)}$ the mean on $[0,L]$ is zero for $k=2,\ 3$ and due to (3.29) we know that the derivative converges strongly in $L^2(0,L)$. Hence, by Poincaré's inequality, there exists $v_k \in H^1(0,L)$ such that

$$v_k^{(h)} \to v \quad \text{strongly in } H^1(0,L).$$

Moreover, because $\tilde{y}_k^{(h)}$ is L-periodic in x_1 direction, $v \in H^1_{per}(0,L)$. Again (3.29) implies $v_{k,1} = A_{k1}$, $k=2,\ 3$ and by $A \in H^1_{per}(0,L;\mathbb{R}^{3\times 3})$ it follows $v_k \in H^2_{per}(0,L)$.
The second and third columns of (3.29) lead us to

$$\frac{1}{h^{\alpha-1}}\partial_{x_k}(\tilde{y}^{(h)} - \mathrm{Id}_h) \to Ae_k \tag{3.30}$$

strongly in $L^2(\Omega;\mathbb{R}^3)$ for $k=2,\ 3$. Poincaré's inequality on S implies

$$\|\tilde{y}^{(h)} - \mathrm{Id}_h - \langle\tilde{y}^{(h)}\rangle_S\|^2_{L^2(S)} \leq C\Big(\|\partial_{x_2}(\tilde{y} - \mathrm{Id}_h)\|^2_{L^2(S)} + \|\partial_{x_3}(\tilde{y} - \mathrm{Id}_h)\|^2_{L^2(S)}\Big) \tag{3.31}$$

for almost every $x_1 \in (0,L)$, where we used (2.4). Integration with respect to x_1 over $(0,L)$ yields that

$$\frac{1}{h^{\alpha-1}}\Big(\tilde{y}^{(h)} - \mathrm{Id}_h - \langle\tilde{y}^{(h)}\rangle_S\Big)$$

is bounded in $L^2(\Omega;\mathbb{R}^3)$. With a similar argument as in the proof of Lemma 2.4.8 and (3.30) we deduce, that there exits $q \in L^2(0,L;\mathbb{R}^3)$ such that

$$\frac{1}{h^{\alpha-1}}\Big(\tilde{y}^{(h)} - \mathrm{Id}_h - \langle\tilde{y}^{(h)}\rangle_S\Big) \to x_2Ae_2 + x_3Ae_3 + q \tag{3.32}$$

strongly in $L^2(\Omega;\mathbb{R}^3)$. From the definition of $w^{(h)}$ it follows now

$$\begin{aligned}
w^{(h)}(x_1) &= \frac{1}{h^{\alpha-1}}\int_S \frac{x_2\tilde{y}_3^{(h)}(x) - x_2\tilde{y}_2^{(h)}(x)}{\mu(S)}dx' \\
&= \frac{1}{h^{\alpha-1}}\frac{1}{\mu(S)}\bigg(\int_S x_2\Big(\tilde{y}_3^{(h)} - hx_3 - \langle\tilde{y}_3^{(h)}\rangle_S\Big)dx' - \int_S x_3\Big(\tilde{y}_2^{(h)} - hx_2 - \langle\tilde{y}_2^{(h)}\rangle_S\Big)dx'\bigg) \\
&= \frac{1}{h^{\alpha-1}}\frac{1}{\mu(S)}\int_S \Big(\tilde{y}^{(h)}(x) - x^{(h)} - \langle\tilde{y}^{(h)}\rangle_S\Big)\cdot(x_2e_3 - x_3e_2)dx' \\
&\to w = \frac{1}{\mu(S)}\int_S (x_2Ae_2 + x_3Ae_3)\cdot(x_2e_3 - x_3e_2)dx' = A_{32}
\end{aligned}$$

strongly in $L^2(0,L)$, where we used (2.3) and the fact that A is skew-symmetric. Computing the derivative of w gives

$$\begin{aligned}
w_{,1}^{(h)} &= \frac{1}{h^{\alpha-1}}\frac{1}{\mu(S)}\int_S \partial_{x_1}(\tilde{y}^{(h)}(x) - x^{(h)})\cdot(x_2e_3 + x_3e_2)dx' \\
&= \frac{1}{h^{\alpha-1}}\frac{1}{\mu(S)}\int_S (\partial_{x_1}\tilde{y}^{(h)}(x) - R^{(h)}e_1)\cdot(x_2e_3 + x_3e_2)dx'
\end{aligned}$$

$$+\frac{1}{h^{\alpha-1}}\frac{1}{\mu(S)}\int_S (R^{(h)}-\partial_{x_1}x^{(h)})\cdot(x_2e_3+x_3e_2)dx'.$$

Hence with (2.4) it follows that the last term is zero. Using (3.13) we obtain that $w^{(h)}$ is bounded in $H^1(0,L)$ and fulfils the periodicity conditions due to $A\in H^1_{per}(\Omega;\mathbb{R}^{3\times 3})$. As A is skew symmetric we obtain

$$A=\begin{pmatrix}0 & -v_{2,1} & -v_{3,1}\\ v_{2,1} & 0 & -w\\ v_{3,1} & w & 0\end{pmatrix}$$

as claimed. The derivative of $u^{(h)}$ can be written as

$$u^{(h)}_{,1}=\frac{1}{h^{\alpha-1}}\int_S(\nabla_h\tilde{y}-R^{(h)})_{11}dx'+\frac{1}{h^{\alpha-1}}(\operatorname{sym}R^{(h)}-Id)_{11}.$$

The first summand is bounded in $L^2(0,L)$ due to (3.28), while the second part converges to zero because of property (f). Lastly, due to definition of $u^{(h)}$ and because $\tilde{y}^{(h)}-\mathrm{Id}_h\in H^1_{per}(\Omega;\mathbb{R}^3)$, we conclude $u^{(h)}$ and $u\in H^1_{per}(0,L;\mathbb{R}^3)$. □

3.3 Limsup and Liminf Inequalities

Now we are in a position to start proving the Γ-convergence result without external force. We split the result in the common two parts. First we prove the Liminf inequality and in the second statement we construct an appropriate recovery sequence.
Let $Q_3\colon\mathbb{R}^{3\times 3}\to[0,\infty)$ be the quadratic form of linearised elasticity

$$Q_3(G):=D^2W(Id)[G,G] \tag{3.33}$$

and let $Q^0\colon\mathbb{R}\times\mathbb{R}^{3\times 3}_{skew}\to[0,\infty)$ be defined by

$$Q^0(t,F):=\min_{\varphi\in H^1(S,\mathbb{R}^3)}\int_S Q_3\Big(te_1+F\begin{pmatrix}0\\ x'\end{pmatrix}\Big|\varphi_{,2}\Big|\varphi_{,3}\Big)dx'. \tag{3.34}$$

For u, $w\in H^1_{per}(0,L)$ and v_2, $v_3\in H^2_{per}(0,L)$ we introduce the functional

$$\mathcal{I}_\alpha(u,v_2,v_3,w):=\begin{cases}\dfrac{1}{2}\displaystyle\int_0^L Q^0(u_{,1}+\tfrac{1}{2}(v_{2,1}^2+v_{3,1}^2),A_{,1})dx_1, & \text{if } \alpha=3,\\ \dfrac{1}{2}\displaystyle\int_0^L Q^0(u_{,1},A_{,1})dx_1, & \text{if } \alpha>3,\end{cases}$$

where $A\in H^1_{per}(0,L;\mathbb{R}^{3\times 3})$ is given by

$$A=\begin{pmatrix}0 & -v_{2,1} & -v_{3,1}\\ v_{2,1} & 0 & -w\\ v_{3,1} & w & 0\end{pmatrix}.$$

Remark 3.3.1. The minimum defining Q^0 is attained and even unique in the space

$$\mathcal{V}:=\Big\{\varphi\in H^1(S;\mathbb{R}^3)\ :\ \int_S\varphi dx'=\int_S\varphi\cdot x^\perp dx'=0\Big\}.$$

Moreover, the minimizer φ depends linearly on t and F. Hence if $t\in L^2(0,L)$ and $F\in L^2(0,L;\mathbb{R}^{3\times 3})$ then $\varphi(t,F)\in L^2(\Omega;\mathbb{R}^3)$ and $\varphi_{,k}(t,F)\in L^2(\Omega;\mathbb{R}^3)$ for $k=2$, 3. In order to see this we first note that Q_3 depends only on the symmetric part of G, due to (2.22). Hence the functional in (3.34) is independent under the transformation $\alpha\mapsto\alpha+c_0+c_1x^\perp$. Therefore, the minimum can be computed on the subspace $\mathcal{V}$ of $H^1(\Omega;\mathbb{R}^3)$. Using $Q_3(F)\geq c_1|\operatorname{sym}F|^2$ for every $F\in\mathbb{R}^{3\times 3}$ it follows with Korn's inequality, Lemma 2.4.8 for $h=1$, that minimizing sequences $\alpha_n \subset \mathcal{V}$ are bounded in $H^1(S;\mathbb{R}^3)$ norm. Hence they are weakly compact. Moreover, the functional to minimize is weakly lower semicontinuous on $H^1(S;\mathbb{R}^3)$. Thus with the direct method of calculus we obtain the existence of a minimizer. As Q^3 is strictly convex on the set of symmetric matrices the uniqueness of the minimizer follows.

For the linear dependency we proceed as in [MM04, Remark 4.2]. First we fix $t \in \mathbb{R}$, $F \in \mathbb{R}^{3\times 3}_{skew}$ and let $\kappa^{min} \in \mathcal{V}$ be the unique minimizer for (3.34). Define now

$$g(x_2, x_3) := F \begin{pmatrix} 0 \\ x' \end{pmatrix} + te_1 \qquad b^{\alpha\beta}_{ij} := \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}}(Id).$$

Then κ^{min} satisfies the Euler-Lagrange equation

$$\int_S \sum_{\alpha,\beta=2,3} \Big(B^{\alpha\beta}\kappa^{min}_{,\beta}, \varphi_{,\alpha}\Big) dx' = -\int_S \sum_{\alpha=2,3} (B^{\alpha 1} g, \varphi_{,\alpha}) dx' \tag{3.35}$$

for every $\varphi \in H^1(S;\mathbb{R}^3)$ and $(B^{\alpha\beta})_{ij} := b^{\alpha\beta}_{ij}$. Exploiting $Q^3(F) \geq c_1 |\operatorname{sym} F|^2$ we deduce

$$\frac{1}{C} \sum_{\alpha,\beta=2,3} (B^{\alpha\beta}\varphi_{,\beta}, \varphi_{,\alpha}) \geq \sum_{\beta=2,3} |\varphi_{1,\beta}|^2 + \sum_{\alpha,\beta=2,3} |\varepsilon_{\alpha\beta}(\varphi)|^2$$

for all $\varphi \in H^1(S;\mathbb{R}^3)$. Testing the Euler-Lagrange equation (3.35) with κ^{min}, we obtain

$$\begin{aligned} \int_S \sum_{\beta=2,3} |\kappa^{min}_{1,\beta}|^2 dx' + \int_S \sum_{\alpha,\beta=2,3} |\varepsilon_{\alpha\beta}(\kappa^{min})|^2 dx' &\leq -\frac{1}{C}\int_S (B^{\alpha 1} g, \kappa^{min}_{,\alpha}) dx' \\ &\leq \sum_{\alpha=2,3} \|B^{\alpha 1} g\|_{L^2(S)} \|\kappa^{min}_{,\alpha}\|_{L^2(S)} \end{aligned}$$

By the Korn inequality and the fact that κ^{min} is mean value free we obtain

$$\int_S \sum_{\alpha,\beta=2,3} |\kappa^{min}_{\alpha,\beta}|^2 dx' \leq C \int_S \sum_{\alpha,\beta=2,3} |\varepsilon_{\alpha\beta}(\kappa^{min})|^2 dx'.$$

Hence, by the later inequality it follows

$$\|\nabla\kappa^{min}\|_{L^2(S)} \leq C \sum_{\alpha,\beta=2,3} \|B^{\alpha 1} g\|_{L^2(S)} \|\kappa^{min}_{,\alpha}\|_{L^2(S)} \leq C \|g\|_{L^2(S)} \|\nabla\kappa^{min}\|_{L^2(S)}.$$

By the Poincaré inequality we obtain

$$\|\kappa^{min}\|_{H^1(S)} \leq C\|g\|_{L^2(S)}. \tag{3.36}$$

Using (3.35) we deduce that κ^{min} depends linearly on t and F. This can be seen in the following way: Assume

$$g_k(x_2, x_3) := F_k \begin{pmatrix} 0 \\ x' \end{pmatrix} + t_k e_1$$

for $k = 1,\ 2$ and denote by $\kappa^{min}_k \in \mathcal{V}$ the unique minimizer of (3.34) for t_k, F_k. Moreover, we denote by $\theta^{min} \in \mathcal{V}$ the minimizer for $t_1 + t_2$ and $F_1 + F_2$. Hence with (3.35) it follows

$$\int_S \sum_{\alpha,\beta=2,3} \Big(B^{\alpha\beta}\Big(\kappa^{min}_{1,\beta} + \kappa^{min}_{2,\beta} - \theta^{min}\Big), \varphi_{,\alpha}\Big) dx' = 0.$$

Choosing $\varphi = \kappa^{min}_{1,\beta} + \kappa^{min}_{2,\beta} - \theta^{min}$ and using an analogous calculation as above we obtain

$$\kappa^{min}_1 + \kappa^{min}_2 = \theta^{min}.$$

Remark 3.3.2. For sake of completeness we want to mention that as in our case Q^3 does not depend on x it is possible to find a more explicit formula for Q^0, done as in [MM04, Remark 4.4.]. We claim that Q^0 can be decomposed as

$$Q^0(t,F) = Q^1(t) + Q^2(F) \tag{3.37}$$

for

$$Q^1(t) := \min_{a,b\in\mathbb{R}^3} Q^3(te_1|a|b) \quad \text{and} \quad Q^2(F) = Q^0(0,F).$$

For this let $t \in \mathbb{R}$, $F \in \mathbb{R}^{3\times3}_{skew}$ and $\alpha \in \mathcal{V}$ be given. We define

$$a := \int_S \alpha_{,2} dx', \quad b := \int_S \alpha_{,3} dx'$$
$$\beta(x') := \alpha(x') - ax_2 - bx_3.$$

Hence it follows

$$\int_S Q^3\left(F\begin{pmatrix}0\\x'\end{pmatrix} + te_1\Big|\alpha_{,2}\Big|\alpha_{,3}\right)dx' = Q^3(te_1|a|b) + \int_S Q^3\left(F\begin{pmatrix}0\\x'\end{pmatrix}\Big|\beta_{,2}\Big|\beta_{,3}\right)dx' \tag{3.38}$$

as $(te_1|a|b)$ does not depend on x and the coupling term vanished because of

$$\begin{aligned}\int_S D^2W(Id)&\left[(te_1|a|b), F\begin{pmatrix}0\\x'\end{pmatrix}\Big|\beta_{,2}\Big|\beta_{,3}\right]\\ &= D^2W(Id)\left[(te_1|a|b), \int_S F\begin{pmatrix}0\\x'\end{pmatrix}\Big|\beta_{,2}\Big|\beta_{,3}\right] = 0.\end{aligned}$$

Here we used for the last equality (2.4) and the definition of β. This implies then $Q^0(t,F) \geq Q^1(t) + Q^2(F)$.
Conversely let $\beta \in H^1(\Omega;\mathbb{R}^3)$ be a minimizer of (3.34) for $t = 0$ and define

$$\bar{\beta}_k = \int_S \beta_{,k} dx' \quad \text{for } k = 2,\ 3. \tag{3.39}$$

Then it holds

$$\begin{aligned}\int_S Q^3\left(F\begin{pmatrix}0\\x'\end{pmatrix}\Big|\beta_{,2}\Big|\beta_{,3}\right)dx' &\leq \int_S Q^3\left(F\begin{pmatrix}0\\x'\end{pmatrix}\Big|\beta_{,2}-\bar{\beta}_2\Big|\beta_{,3}-\bar{\beta}_3\right)dx'\\ &= \int_S Q^3\left(F\begin{pmatrix}0\\x'\end{pmatrix}\Big|\beta_{,2}\Big|\beta_{,3}\right)dx' - Q^3\left(0\Big|\bar{\beta}_2\Big|\bar{\beta}_3\right)\end{aligned}$$

where we used (2.4). Due to the nonnegativity of Q^3, it follows then $(0|\bar{\beta}_2|\bar{\beta}_3) \in \mathbb{R}^{3\times3}_{skew}$. With this we can define

$$\tilde{\beta}(x') = \beta(x') - \bar{\beta}_2 x_2 - \bar{\beta}_3 x_3$$

and obtain that $\tilde{\beta} \in H^1(\Omega;\mathbb{R}^3)$ is another minimizer defining (3.34). Moreover it holds

$$\int_S \tilde{\beta}_{,2} dx' = \int_S \tilde{\beta}_{,3} dx' = 0.$$

Let $(a,b) \in \mathbb{R}^3 \times \mathbb{R}^3$ be the minimizer defining Q^1 and set

$$\alpha(x') = \tilde{\beta}(x') + x_2 a + x_3 b.$$

Then it follows with (3.38)

$$Q^0(t,F) \leq \int_S Q^3\left(F\begin{pmatrix}0\\x'\end{pmatrix} + te_1\Big|\alpha_{,2}\Big|\alpha_{,3}\right)dx' = Q^3(te_1|a|b) + \int_S Q^3\left(F\begin{pmatrix}0\\x'\end{pmatrix}\Big|\beta_{,2}\Big|\beta_{,3}\right)dx'$$

$$= Q^1(t) + Q^2(F).$$

With this (3.37) is proven.

Remark 3.3.3. Lastly, we want to show that Q^0 is uniformly positive definite, i.e. there exists some $C > 0$ such that

$$Q^0(t,F) \geq C(t^2 + |F|^2) \quad \text{for all } t \in \mathbb{R} \text{ and } F \in \mathbb{R}^{3\times 3}_{skew}. \tag{3.40}$$

It is sufficient to show the bound only for the special case that $Q^3(F) = |\operatorname{sym} F|^2$, because of $Q^3(F) \geq c_1|\operatorname{sym} F|^2$. Then we can use the decomposition of proven in the preceding remark and split

$$Q^0(t,F) = Q^1(t) + Q^2(F) \quad \text{for all } t \in \mathbb{R} \text{ and } F \in \mathbb{R}^{3\times 3}_{skew}.$$

We can see $Q^1(t) \geq t^2$ easily, by calculating

$$Q^1(t) = \min_{a,b\in\mathbb{R}^3} Q^3(te_1|a|b) = \min_{a,b\in\mathbb{R}^3} \operatorname{sym}(te_1|a|b) \geq t^2.$$

Hence, it remains to show $Q^2(F) \geq C|F|^2$, where we argue towards a contradiction. Assume that the claimed inequality does not hold. Then there exists a sequence $F_n \in \mathbb{R}^{3\times 3}_{skew}$ such that

$$Q^2(F_n) \leq \frac{1}{n}|F_n|^2.$$

This implies

$$\begin{aligned}
Q^2\left(\frac{F_n}{|F_n|}\right) = Q^0\left(\frac{F_n}{|F_n|}\right) &= \min_{\varphi\in H^1(S,\mathbb{R}^3)} \int_S \left|\operatorname{sym}\left(\frac{1}{|F_n|}F_n\begin{pmatrix}0\\x'\end{pmatrix}\Big|\varphi_{,2}\Big|\varphi_{,3}\right)\right|^2 dx' \\
&= \frac{1}{|F_n|^2}\min_{\varphi\in H^1(S,\mathbb{R}^3)} \int_S \left|\operatorname{sym}\left(F_n\begin{pmatrix}0\\x'\end{pmatrix}\Big|\varphi_{,2}\Big|\varphi_{,3}\right)\right|^2 dx' \leq \frac{1}{n}.
\end{aligned}$$

Hence, there exists $F \neq 0$, a cluster point of $F_n/|F_n|$, and $\kappa^{min} \in \mathcal{V}$ such that for all $x' \in S$

$$\operatorname{sym}\left(F\begin{pmatrix}0\\x'\end{pmatrix}\Big|\kappa^{min}_{,2}\Big|\kappa^{min}_{,3}\right) = 0.$$

The first column provides then for all $x' \in S$

$$\begin{aligned}
F_{12}x_2 + F_{13}x_3 &= 0 \\
F_{23}x_3 + \kappa^{min}_{1,2} &= 0 \\
-F_{23}x_2 + \kappa^{min}_{1,3} &= 0.
\end{aligned}$$

From the first line, we obtain $F_{12} = F_{13} = 0$. Deriving the second and third line by x_3 and x_2, respectively, it follows $F_{23} = 0$. Thus $(t,F) = (0,0)$, which is a contradiction.

Theorem 3.3.4 (Liminf inequality)**.** *Let u, $w \in H^1_{per}(0,L)$ and $v_k \in H^2_{per}(0,L)$ for $k = 2$, 3. Then, for every sequence $(y^{(h)})_{h>0} \subset H^1(\Omega;\mathbb{R}^3)$ such that $\tilde{y}^{(h)} := (\bar{R}^{(h)})^T y^{(h)} - c^{(h)}$ satisfies the properties (a)–(f) of Theorem 3.2.4, it holds*

$$\liminf_{h\to 0} \frac{1}{h^{2\alpha-2}}\mathcal{I}^{(h)}(y^{(h)}) \geq \mathcal{I}_\alpha(u, v_2, v_3, w). \tag{3.41}$$

Proof: Step 1: (lower bound). We can assume

$$\liminf_{h\to 0} \frac{1}{h^{2\alpha-2}}\mathcal{I}^{(h)}(y^{(h)}) \leq C < \infty$$

as otherwise (3.41) is trivially fulfilled. Hence, by passing to a subsequence, we can assume that

$$\frac{1}{h^{2\alpha-2}}\mathcal{I}^{(h)}(y^{(h)}) \leq C < +\infty$$

holds. Using Theorem 3.2.4 there exists $R^{(h)}\colon [0,L] \to SO(3)$, $\bar{R}^{(h)} \in SO(3)$ and $c^{(h)} \in \mathbb{R}^3$ such that

$$\|\nabla_h \tilde{y}^{(h)} - R^{(h)}\|_{L^2(\Omega)} \leq Ch^{\alpha-1} \tag{3.42}$$

and $R^{(h)} \to \mathrm{Id}$ uniformly. Define the sequence $G^{(h)}\colon \Omega \to \mathbb{R}^{3\times 3}$ by

$$G^{(h)} := \frac{1}{h^{\alpha-1}}((R^{(h)})^T \nabla_h \tilde{y}^{(h)} - \mathrm{Id}). \tag{3.43}$$

Due to (3.42) the sequence $G^{(h)}$ is bounded in $L^2(\Omega;\mathbb{R}^{3\times 3})$. Hence, there exists some $G \in L^2(\Omega;\mathbb{R}^{3\times 3})$ such that $G^{(h)} \rightharpoonup G$ weakly in $L^2(\Omega;\mathbb{R}^{3\times 3})$. As an intermediate result we now want to prove

$$\liminf_{h\to 0} \frac{1}{h^{2\alpha-2}}\int_\Omega W(\nabla_h y^{(h)})dx \geq \frac{1}{2}\int_\Omega Q^3(G)dx. \tag{3.44}$$

For this we introduce $\chi_h\colon \Omega \to \mathbb{R}$ defined by

$$\chi^{(h)}(x) := \begin{cases} 1, & \text{if } |G^{(h)}(x)| \leq h^{2-\alpha}, \\ 0, & \text{otherwise.} \end{cases}$$

From the boundedness of $G^{(h)}$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$ one can deduce that $\chi^{(h)} \to 1$ in measure and by

$$\lim_{h\to 0}\int_\Omega |\chi^{(h)}(x)-1|^2 dx = \lim_{h\to 0}\int_{\{x\in\Omega\,:\,|\chi^{(h)}(x)-1|\geq\frac{1}{2}\}} 1dx = \left|\{x\in\Omega \;:\; |\chi^{(h)}(x)-1| \geq \tfrac{1}{2}\}\right| \to 0$$

we obtain $\chi^{(h)} \to 1$ strongly in $L^2(\Omega)$. Thus

$$\chi^{(h)}G^{(h)} \rightharpoonup G \quad \text{weakly in } L^2(\Omega;\mathbb{R}^{3\times 3}). \tag{3.45}$$

By a Taylor expansion of W around the identity, it follows for $A \in \mathbb{R}^{3\times 3}$

$$W(\mathrm{Id}+A) = \frac{1}{2}D^2W(Id+tA)[A,A] \tag{3.46}$$

where $t \in (0,1)$ depends on A. Frame invariants, (3.46) and the definition of $G^{(h)}$ leads to

$$\begin{aligned}\frac{1}{h^{2\alpha-2}}\int_\Omega W(\nabla_h y^{(h)}) &\geq \frac{1}{h^{2\alpha-2}}\int_\Omega \chi^{(h)}W(\nabla_h y^{(h)})dx = \frac{1}{h^{2\alpha-2}}\int_\Omega \chi^{(h)}W((R^{(h)})^T\nabla_h y^{(h)})dx \\ &= \int_\Omega \frac{1}{2}\chi^{(h)}D^2W(\mathrm{Id}+t_h(x)h^{\alpha-1}G^{(h)})[G^{(h)},G^{(h)}]dx\end{aligned}$$

where $t_h(x) \in (0,1)$ depends on $G^{(h)}$. Rewriting the last integral by

$$\begin{aligned}&\int_\Omega \frac{1}{2}\chi^{(h)}D^2W(\mathrm{Id}+t_h(x)h^{\alpha-1}G^{(h)})[G^{(h)},G^{(h)}]dx \\ &\quad= \int_\Omega \frac{1}{2}\Big(\chi^{(h)}D^2W(\mathrm{Id}+t_h(x)h^{\alpha-1}\chi^{(h)}G^{(h)})[G^{(h)},G^{(h)}] - Q^3(\chi^{(h)}G^{(h)})\Big)dx \\ &\qquad+ \int_\Omega \frac{1}{2}Q^3(\chi^{(h)}G^{(h)})dx.\end{aligned} \tag{3.47}$$

Since D^2W is continuous on $\overline{B_\delta(Id)}$ for $\delta > 0$ sufficiently small, we conclude that D^2W is uniformly continuous on $\overline{B_\delta(Id)}$. Using that $h^{\alpha-1}t_h\chi^{(h)}G^{(h)}$ is uniformly small for h small enough, we have for every $\varepsilon > 0$

$$\int_\Omega \frac{1}{2}\Big(\chi^{(h)} D^2 W(\mathrm{Id} + t_h(x) h^{\alpha-1}\chi^{(h)} G^{(h)})[G^{(h)}, G^{(h)}] - Q^3(G^{(h)})\Big)dx \\ \geq -\varepsilon \int_\Omega \chi^{(h)} |G^{(h)}|^2 dx = -C\varepsilon.$$

The second integral on the right hand side of (3.47) is lower semi-continuous with respect to weak convergence in $L^2(\Omega;\mathbb{R}^{3\times 3})$, because Q^3 is convex and lower semi-continuous with respect to the strong topology. As $\varepsilon > 0$ is arbitrary it follows that (3.44) holds. Since Q^3 depends only on the symmetric part of G, we obtain

$$\liminf_{h\to 0} \frac{1}{h^{2\alpha-2}} \int_\Omega W(\nabla_h y^{(h)}) dx \geq \frac{1}{2}\int_\Omega Q^3(\tilde{G}) dx. \tag{3.48}$$

where $\tilde{G} := \varepsilon(G)$.

Step 2: (characterisation of $\tilde{G}$). Note that, because of $R^{(h)} \to \mathrm{Id}$ uniformly, it follows

$$R^{(h)} G^{(h)} = \frac{1}{h^{\alpha-1}}(\nabla_h \tilde{y}^{(h)} - R^{(h)}) \rightharpoonup G$$

weakly in $L^2(\Omega;\mathbb{R}^{3\times 3})$. For the second and third columns of the matrix we get

$$R^{(h)} G^{(h)} e_k = \frac{1}{h^\alpha}(\tilde{y}^{(h)}_{,k} - hR^{(h)} e_k) \rightharpoonup G e_k \tag{3.49}$$

weakly in $L^2(\Omega;\mathbb{R}^{3\times 3})$ for $k = 2,\ 3$. Define the the functions $\tilde{\beta}^{(h)} \colon \Omega \to \mathbb{R}^3$ by

$$\tilde{\beta}^{(h)}(x) := \frac{1}{h^\alpha}(\tilde{y}^{(h)} - hx_2 R^{(h)} e_2 - hx_3 R^{(h)} e_3)$$

Then

$$\tilde{\beta}^{(h)}_{,k} = R^{(h)} G^{(h)} e_k \quad \text{for } k = 2,\ 3. \tag{3.50}$$

Thus $\tilde{\beta}^{(h)}_{,k}$ is bounded in $L^2(\Omega)$ and by the Poincaré Inequality on S, it follows

$$\|\tilde{\beta}^{(h)} - \langle\tilde{\beta}^{(h)}\rangle_S\|^2_{L^2(\Omega)} \leq C(\|\tilde{\beta}^{(h)}_{,2}\|^2_{L^2(\Omega)} + \|\tilde{\beta}^{(h)}_{,3}\|^2_{L^2(\Omega)}).$$

So, there exists some $\beta \in L^2(\Omega;\mathbb{R}^3)$ such that

$$\beta^{(h)} := \tilde{\beta}^{(h)} - \langle\tilde{\beta}^{(h)}\rangle_S \rightharpoonup \beta \quad \text{weakly in } L^2(\Omega;\mathbb{R}^3). \tag{3.51}$$

Hence, we obtain with (3.50) for $h \to 0$

$$\beta_{,k} = G e_k \quad \text{for } k = 2,\ 3. \tag{3.52}$$

For the first column we have

$$R^{(h)} G^{(h)} e_1 = \frac{1}{h^{\alpha-1}}(\tilde{y}^{(h)}_{,1} - R^{(h)} e_1) \rightharpoonup G e_1$$

weakly in $L^2(\Omega;\mathbb{R}^3)$. Using the definition of β and (2.4), we can rewrite the right hand side to

$$\begin{aligned} R^{(h)} G^{(h)} e_1 &= \frac{1}{h^{\alpha-1}}(\tilde{y}^{(h)}_{,1} - R^{(h)} e_1) \\ &= h\beta^{(h)}_{,1} + \frac{1}{h^{\alpha-2}}(R^{(h)})'(x_2 e_2 + x_3 e_3) + \frac{1}{h^{\alpha-1}}\int_S \tilde{y}^{(h)}_{,1} - R^{(h)} e_1 dx'. \end{aligned}$$

By (3.51) it follows
$$h\beta^{(h)}_{,1} \rightharpoonup 0 \text{ weakly in } H^{-1}(\Omega;\mathbb{R}^3).$$
For the last summand it follows due to (3.42)
$$\frac{1}{h^{\alpha-1}}\int_S \tilde{y}^{(h)}_{,1} - R^{(h)}e_1 dx' \rightharpoonup g \tag{3.53}$$
weakly in $L^2(0,L;\mathbb{R}^3)$ for some $g \in L^2(0,L;\mathbb{R}^3)$. Passing to the limit it follows with property (e) of Theorem 3.2.4
$$Ge_1 = A_{,1}\begin{pmatrix} 0 \\ x' \end{pmatrix} + g.$$
Putting everything together we obtain
$$\tilde{G} = \operatorname{sym} G = \operatorname{sym}\left(A_{,1}\begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} + g \middle| \beta_{,2} \middle| \beta_{,3} \right) = \operatorname{sym}\left(A_{,1}\begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} + g_1 e_1 \middle| \varphi_{,2} \middle| \varphi_{,3} \right)$$
for $\varphi := \beta + (x_2 g_2 + x_3 g_3)e_1$. From now on we want to distinguish two cases.

Case 1: ($\alpha = 3$). The definition of $u^{(h)}$ leads to
$$\frac{1}{h^{\alpha-1}}\int_S \tilde{y}^{(h)}_{1,1} - R^{(h)}_{11} dx' = u^{(h)}_{,1} - \frac{1}{h^{\alpha-1}}\int_S (R^{(h)} - \mathrm{Id})_{11} dx'. \tag{3.54}$$
Passing to the limit in h implies then due to property (f) of Theorem (3.2.4), $2(\alpha-2) = \alpha - 1$ and (3.53)
$$g_1 = u_{,1} - \frac{1}{2}(A^2)_{11} = u_{,1} + \frac{1}{2}(v_{2,1}^2 + v_{3,1}^2).$$
Hence $\tilde{G}$ is given via
$$\tilde{G} = \operatorname{sym}\left(A_{,1}\begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} + \left(u_{,1} + \tfrac{1}{2}(v_{2,1}^2 + v_{3,1}^2)\right) e_1 \middle| \varphi_{,2} \middle| \varphi_{,3} \right).$$
Using $\varphi(x_1,\cdot) \in H^1(S;\mathbb{R}^3)$ and the definition of Q^0 we obtain with (3.48)
$$\liminf_{h\to 0} \frac{1}{h^4}\int_\Omega W(\nabla_h y^{(h)})dx \geq \mathcal{I}_3(u, v_2, v_3, w).$$

Case 2: ($\alpha > 3$). In this case we have $2(\alpha-2) > \alpha - 1$ and hence due to (f) of Theorem 3.2.4
$$\frac{1}{h^{\alpha-1}} \operatorname{sym}(R^{(h)} - \mathrm{Id}) \to 0$$
uniformly on $(0,L)$. Thus, with (3.54), it follows analogously to the first case $g_1 = u_{,1}$ and therefore
$$\tilde{G} = \operatorname{sym}\left(A_{,1}\begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} + u_{,1} e_1 \middle| \varphi_{,2} \middle| \varphi_{,3} \right).$$
As in the first case we obtain with (3.48)
$$\liminf_{h\to 0} \frac{1}{h^{2\alpha-2}}\int_\Omega W(\nabla_h y^{(h)})dx \geq \mathcal{I}_\alpha(u, v_2, v_3, w). \qquad \square$$

Theorem 3.3.5 (Recovery sequence)**.** *Let u, $w \in H^1_{per}(0,L)$ and $v_k \in H^2_{per}(0,L)$ for $k = 2$, 3. Then there exists a sequence $(\check{y}^{(h)})_{h>0} \subset H^1(\Omega;\mathbb{R}^3)$ such that (a)–(d) of Theorem 3.2.4 are* *satisfied and*

$$\limsup_{h\to 0} \frac{1}{h^{2\alpha-2}} \mathcal{I}^{(h)}(\check{y}^{(h)}) \leq \mathcal{I}_\alpha(u, v_2, v_3, w). \tag{3.55}$$

Proof: We start with more regularity off the functions, i.e. suppose $u,\ w \in C^1([0,L])$ and $v_k \in C^2([0,L])$, $k = 2,\ 3$, such that

$$u(0) = u(L), \quad w(0) = w(L), \quad v_k(0) = v_k(L) \ \text{ and } \ v_k'(0) = v_k'(L). \tag{3.56}$$

We define $\gamma_2,\ \gamma_3 \colon (0,L) \to \mathbb{R}^3$ by

$$\begin{aligned}
\gamma_2(x_1) &:= 2v_{3,1}we_1 + (w^2 + v_{2,1}^2)e_2 + v_{2,1}v_{3,1}e_3 \\
\gamma_3(x_1) &:= -2v_{2,1}we_1 + v_{2,1}v_{3,1}e_2 + (w^2 + v_{3,1}^2)e_3
\end{aligned}$$

and let $\beta \in C^1(\Omega;\mathbb{R}^3)$. Set $\varphi \colon \Omega \to \mathbb{R}^3$ by

$$\varphi(x) := \begin{cases} \beta(x) - \frac{1}{2}x_2\gamma_2(x_1) - \frac{1}{2}x_3\gamma_3(x_1), & \text{if } \alpha = 3, \\ \beta(x), & \text{if } \alpha > 3 \end{cases}$$

and consider for $h > 0$ a cut off function $\rho^{(h)} \in C^1(0,L)$ such that $\operatorname{supp}(\rho^{(h)}) \subset [L - \sqrt{h}, L]$, $\rho^{(h)}(L) = 1$ and $|(\rho^{(h)})'| \leq C/\sqrt{h}$ with C independent of h. Then we define for $h > 0$ the function $\check{y}^{(h)} \colon \Omega \to \mathbb{R}^3$ by

$$\check{y}^{(h)}(x) := \begin{pmatrix} x_1 \\ hx_2 \\ hx_3 \end{pmatrix} + h^{\alpha-2}\begin{pmatrix} 0 \\ v_2 \\ v_3 \end{pmatrix} + h^{\alpha-1}\begin{pmatrix} u - x_2v_{2,1} - x_3v_{3,1} \\ -x_3w \\ x_2w \end{pmatrix} + h^\alpha\varphi^{(h)} \tag{3.57}$$

where $\varphi^{(h)}(x) := \varphi(x) + \rho^{(h)}(x_1)(\beta(0,x') - \beta(L,x'))$. With this definition we obtain $(\check{y}^{(h)})_{h>0} \subset H^1(\Omega;\mathbb{R}^3)$ and $(\check{y}^{(h)} - \mathrm{Id}_h)_{h>0} \subset H^1_{per}(\Omega;\mathbb{R}^3)$ due to (3.56). The preceding properties imply

$$\frac{1}{h^{2\alpha-2}}\mathcal{I}^{(h)}(\check{y}^{(h)}) = \frac{1}{h^{2\alpha-2}}\int_\Omega W(\nabla_h\check{y}^{(h)})dx. \tag{3.58}$$

To prove the convergence results (a)–(d) we compute the scaled gradient

$$\nabla_h\check{y}^{(h)} = \mathrm{Id} + h^{\alpha-2}A + h^{\alpha-1}\left(A_{,1}\begin{pmatrix} 0 \\ x' \end{pmatrix} + u_{,1}e_1 \middle| \varphi^{(h)}_{,2} \middle| \varphi^{(h)}_{,3}\right) + O\left(h^{\alpha-\frac{1}{2}}\right) \tag{3.59}$$

and thus

$$\frac{1}{h^{\alpha-2}}(\nabla_h\check{y}^{(h)} - \mathrm{Id}) = A + O(h)$$

which proves (d). Using the definition of $v_k^{(h)}$ and (2.4), one can deduce

$$v_k^{(h)} = v_k + h^2\int_S \varphi_k^{(h)}dx' \quad \text{ for } k = 2,\ 3.$$

Thus it follows with $\varphi^{(h)} \to \varphi$ strongly in $L^2(\Omega;\mathbb{R}^3)$ and $|(\rho^{(h)})'| \leq C/\sqrt{h}$ that statement (b) holds. By (2.4) we have $u^{(h)} = u + h\varphi_1^{(h)}$ providing (a). Lastly, because (2.3) and (2.4)

$$w^{(h)} = \frac{1}{\mu(S)}\int_S (x_2^2 + x_3^2)w + h(x_2\varphi_3^{(h)} + x_3\varphi_2^{(h)})dx' = w + O(h)$$

leading to (c). Using now the identity $(\mathrm{Id} + B^T)(\mathrm{Id} + B) = \mathrm{Id} + 2\operatorname{sym}B + B^TB$ for the nonlinear strain and that A is skew symmetric, it follows

$$(\nabla_h \check{y}^{(h)})^T \nabla_h \check{y}^{(h)} = \mathrm{Id} + 2h^{\alpha-1} \operatorname{sym}\left(A_{,1} \begin{pmatrix} 0 \\ x' \end{pmatrix} + u_{,1} e_1 \middle| \varphi_{,2}^{(h)} \middle| \varphi_{,3}^{(h)} \right) + h^{2(\alpha-2)} A^T A + O(h^{2\alpha-3}) + O\Big(h^{\alpha-\frac{1}{2}}\Big). \tag{3.60}$$

Now we distinguish the case $\alpha = 3$ and $\alpha > 3$.

Case 1. ($\alpha = 3$) In this case it follows by (3.60)

$$(\nabla_h \check{y}^{(h)})^T \nabla_h \check{y}^{(h)} = \mathrm{Id} + 2h^2 \operatorname{sym}\left(A_{,1} \begin{pmatrix} 0 \\ x' \end{pmatrix} + u_{,1} e_1 \middle| \varphi_{,2}^{(h)} \middle| \varphi_{,3}^{(h)} \right) + h^2 A^T A + O\Big(h^{\frac{5}{2}}\Big).$$

Hence, the square root is given by

$$[(\nabla_h \check{y}^{(h)})^T \nabla_h \check{y}^{(h)}]^{\frac{1}{2}} = \mathrm{Id} + h^2 \tilde{G}^{(h)} + O\Big(h^{\frac{5}{2}}\Big), \tag{3.61}$$

where

$$\begin{aligned} \tilde{G}^{(h)} &:= \operatorname{sym}\left(A_{,1} \begin{pmatrix} 0 \\ x' \end{pmatrix} + u_{,1} e_1 \middle| \varphi_{,2}^{(h)} \middle| \varphi_{,3}^{(h)} \right) - \frac{A^2}{2} \\ &= \operatorname{sym}\left(A_{,1} \begin{pmatrix} 0 \\ x' \end{pmatrix} + (u_{,1} + \frac{1}{2}(v_{2,1}^2 + v_{3,1}^2)) e_1 \middle| \varphi_{,2}^{(h)} \middle| \varphi_{,3}^{(h)} \right) \\ &\quad + \frac{1}{2} \begin{pmatrix} 0 & -v_{3,1} w & v_{2,1} w \\ -v_{3,1} w & -(v_{2,1}^2 + w) & -v_{2,1} v_{3,1} \\ v_{2,1} w & -v_{2,1} v_{3,1} & -(v_{3,1}^2 + w^2) \end{pmatrix}. \end{aligned}$$

Using the definition of $\varphi^{(h)}$, β and γ_k, we can express $\tilde{G}$ via

$$\tilde{G}^{(h)} = \operatorname{sym}\left(A_{,1} \begin{pmatrix} 0 \\ x' \end{pmatrix} + \Big(u_{,1} + \tfrac{1}{2}(v_{2,1}^2 + v_{3,1}^2)\Big) e_1 \middle| \beta_{,2}^{(h)} \middle| \beta_{,3}^{(h)} \right). \tag{3.62}$$

where $\beta^{(h)}(x) := \beta(x) + \rho^{(h)}(x_1)(\beta(0, x') - \beta(L, x'))$. Thus

$$\tilde{G}^{(h)} \to \tilde{G} := \operatorname{sym}\left(A_{,1} \begin{pmatrix} 0 \\ x' \end{pmatrix} + \Big(u_{,1} + \tfrac{1}{2}(v_{2,1}^2 + v_{3,1}^2)\Big) e_1 \middle| \beta_{,2} \middle| \beta_{,3} \right) \quad \text{as } h \to 0 \tag{3.63}$$

pointwise for all $x \in \Omega$.

By construction of $\check{y}^{(h)}$ it holds $\det(\nabla_h \check{y}^{(h)}) > 0$ for all $h \in (0, h_0]$ and h_0 sufficiently small. Hence we can use the polar decomposition and frame invariance of W to obtain

$$W(\nabla_h \check{y}^{(h)}) = W([(\nabla_h \check{y}^{(h)})^T \nabla_h \check{y}^{(h)}]^{\frac{1}{2}}).$$

Thus, by a Taylor expansion, (3.61) and (3.63)

$$\frac{1}{h^4} W(\nabla_h \check{y}^{(h)}) \to \frac{1}{2} Q_3(\tilde{G}) \quad \text{as } h \to 0$$

almost everywhere in Ω and

$$\begin{aligned} \frac{1}{h^4} W(\nabla_h \check{y}^{(h)}) &\le \frac{1}{2} c_1 |\tilde{G}^{(h)}|^2 + Ch \\ &\le C\Big(|A|^4 + |A_{,1}|^2 + |u_{,1}|^2 + |\beta_{,2}|^2 + |\beta_{,3}|^2 + 1\Big) \in L^1(\Omega) \end{aligned}$$

because of $|v_{2,1}^2 + v_{3,1}^2| \le |A^2|$ and $|\beta(\xi, x')| \le C$ for all $x' \in S$ and $\xi \in \{0, L\}$. The dominated convergence Theorem and (3.58) yields

$$\limsup_{h\to 0} \frac{1}{h^4}\mathcal{I}^{(h)}(\check{y}^{(h)}) = \limsup_{h\to 0} \frac{1}{h^4}\int_\Omega W(\nabla_h \check{y}^{(h)})dx = \frac{1}{2}\int_\Omega Q_3(\tilde{G})dx. \tag{3.64}$$

For the general case, let $u,\ w \in H^1_{per}(0,L)$ and $v_k \in H^2_{per}(0,L)$, $k = 2,\, 3$. Define $\beta(x_1,\cdot) \in \mathcal{V}$ as the unique solution of the minimum problem (3.34), with $F := A_{,1}$ and $t := u_{,1} + \frac{1}{2}(v_{2,1}^2 + v_{3,1}^2)$. Hence $\beta \in L^2(\Omega)$ and $\beta_{,k} \in L^2(\Omega)$, due to Remark 3.3.1.

We can smoothly approximate u, w in H^1_{per}, v_k in H^2_{per} and β, $\beta_{,k}$ in L^2. Hence using a diagonal sequence argument, the continuity of the right hand side of (3.64) with respect to the particular convergences, it follows that (3.64) holds in the general case. Using the definition of $\beta \in \mathcal{V}$ and the structure of $\tilde{G}$ we obtain

$$\limsup_{h\to 0} \frac{1}{h^4}\mathcal{I}^{(h)}(\check{y}^{(h)}) = \limsup_{h\to 0} \frac{1}{h^4}\int_\Omega W(\nabla_h \check{y}^{(h)})dx \le \mathcal{I}_\alpha(u, v_2, v_3, w).$$

Case 2. $(\alpha > 3)$ In this case, the $A^T A$ term in (3.60) is, because $2(\alpha - 2) > \alpha - 1$, of order $\alpha - 1$. Thus it follows by (3.60)

$$(\nabla_h \check{y}^{(h)})^T \nabla_h \check{y}^{(h)} = \mathrm{Id} + 2h^{\alpha-1}\,\mathrm{sym}\left(A_{,1}\begin{pmatrix}0\\ x'\end{pmatrix} + u_{,1}e_1\middle|\varphi^{(h)}_{,2}\middle|\varphi^{(h)}_{,3}\right) + O(h^{\alpha-1}).$$

Taking again the square root, it follows

$$[(\nabla_h \check{y}^{(h)})^T \nabla_h \check{y}^{(h)}]^{\frac{1}{2}} = \mathrm{Id} + h^{\alpha-1}\tilde{G}^{(h)} + o(h^{\alpha-1}), \tag{3.65}$$

where

$$\tilde{G}^{(h)} = \mathrm{sym}\left(A_{,1}\begin{pmatrix}0\\ x'\end{pmatrix} + u_{,1}e_1\middle|\beta^{(h)}_{,2}\middle|\beta^{(h)}_{,3}\right).$$

as in this case $\beta^{(h)} = \varphi^{(h)}$. Thus

$$\tilde{G}^{(h)} \to \tilde{G} := \mathrm{sym}\left(A_{,1}\begin{pmatrix}0\\ x'\end{pmatrix} + u_{,1}e_1\middle|\beta_{,2}\middle|\beta_{,3}\right) \tag{3.66}$$

pointwise for all $x \in \Omega$. With frame indifference of W and $\det \nabla_h \check{y}^{(h)} > 0$ for h small enough it follows

$$W(\nabla_h \check{y}^{(h)}) = W([(\nabla_h \check{y}^{(h)})^T \nabla_h \check{y}^{(h)}]^{\frac{1}{2}}).$$

Thus by Taylor expansion, (3.65) and (3.66)

$$\frac{1}{h^{2(\alpha-1)}} W(\nabla_h \check{y}^{(h)}) \to \frac{1}{2}Q^3(\tilde{G})$$

almost everywhere in Ω and

$$\frac{1}{h^4}W(\nabla_h \check{y}^{(h)}) \le \frac{1}{2}c_1|\tilde{G}|^2 + Ch \le C\Big(|A_{,1}|^2 + |u_{,1}|^2 + |\beta_{,2}|^2 + |\beta_{,3}|^2 + 1\Big) \in L^1(\Omega).$$

Hence, dominated convergence theorem implies

$$\limsup_{h\to 0} \frac{1}{h^{2\alpha-2}}\mathcal{I}^{(h)}(\check{y}^{(h)}) = \limsup_{h\to 0} \frac{1}{h^{2\alpha-2)}}\int_\Omega W(\nabla_h \check{y}^{(h)})dx = \frac{1}{2}\int_\Omega Q_3(\tilde{G})dx.$$

Analogously to the first case, for $u,\ w \in H^1_{per}(0,L)$ and $v_k \in H^2_{per}(0,L)$, let $\varphi(x_1,\cdot) \in \mathcal{V}$ be the solution of the minimum problem (3.34) for $t := u_{,1}$ and $F := A_{,1}$. The same argument as above leads then to

$$\limsup_{h\to 0} \frac{1}{h^{2\alpha-2}} \mathcal{I}^{(h)}(\check{y}^{(h)}) = \limsup_{h\to 0} \frac{1}{h^{2\alpha-2}} \int_\Omega W(\nabla_h \check{y}^{(h)})dx \leq \mathcal{I}_\alpha(u, v_2, v_3, w)$$

due to the minimality of $\varphi \in \mathcal{V}$. $\square$

Remark 3.3.6. If we additionaly assume that W is isotropic, i.e.

$$W(F) = W(FR) \quad \text{for all } R \in SO(3),$$

then the quadratic form Q^3 is given by

$$Q^3(G) = 2\mu|\varepsilon(G)|^2 + \lambda(\operatorname{tr} G)^2 \tag{3.67}$$

for some $\mu > 0$, $\lambda \geq 0$. The existence of such μ and λ follows from Theorem B.5. Positivity of λ can be deduced from $G^3(G) \geq c_1|\operatorname{sym}(G)|^2$. Thus it follows for the coefficients $b_{ij}^{\alpha\beta}$

$$b_{ij}^{\alpha\beta} := \frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}(Id) = \begin{cases} 2\mu + \lambda, & \text{if } \alpha = \beta = i = j, \\ \mu, & \text{if } \alpha = \beta \ \wedge \ i = j \ \wedge \ \alpha \neq i \\ & \text{ and } \alpha = j \ \wedge \ \beta = i \ \wedge \ \alpha \neq k, \\ \lambda, & \text{if } \alpha = i \ \wedge \ \beta = j \ \wedge \ \alpha \neq k. \\ 0, & \text{otherwise} \end{cases}$$

With this the Euler-Lagrange equations (3.35) simplify to

$$\int_S \left[\begin{pmatrix} \mu\alpha_{1,2} & \mu\alpha_{1,3} \\ (2\mu+\lambda)\alpha_{2,2} + \lambda\alpha_{3,3} & \mu\alpha_{3,2} + \mu\alpha_{2,3} \\ \mu\alpha_{3,2} + \mu\alpha_{2,3} & \lambda\alpha_{2,2} + (2\mu+\lambda)\alpha_{3,3} \end{pmatrix} + \begin{pmatrix} \mu g_2 & \mu g_3 \\ \lambda g_1 & 0 \\ 0 & \lambda g_1 \end{pmatrix} \right] : \nabla_{x'}\varphi dx' = 0$$

for all $\varphi \in H^1(S;\mathbb{R}^3)$, where $\nabla_{x'}\varphi = (\partial_{x_2}\varphi|\partial_{x_3}\varphi)$ and

$$g(x_2, x_3) := F\begin{pmatrix} 0 \\ x' \end{pmatrix} + te_1.$$

Thus the system decouples to

$$\begin{cases} \Delta\alpha_1 = 0 & \text{in } S \\ \partial_\nu\alpha_1 = F_{23}(-x_3, x_2)\cdot\nu & \text{on } \partial S \end{cases}$$

and

$$\begin{cases} \operatorname{div}\big((2\mu+\lambda)\alpha_{2,2} + \lambda\alpha_{3,3}, \mu\alpha_{3,2} + \mu\alpha_{2,3}\big) = -\lambda F_{12} & \text{in } S \\ \operatorname{div}\big(\mu\alpha_{3,2} + \mu\alpha_{2,3}, \lambda\alpha_{2,2} + (2\mu+\lambda)\alpha_{3,3}\big) = -\lambda F_{13} & \text{in } S \\ \big((2\mu+\lambda)\alpha_{2,2} + \lambda\alpha_{3,3}, \mu\alpha_{3,2} + \mu\alpha_{2,3}\big)\cdot\nu = -\lambda(t + F_{12}x_2 + F_{13}x_3)\nu_2 & \text{on } \partial S \\ \big(\mu\alpha_{3,2} + \mu\alpha_{2,3}, \lambda\alpha_{2,2} + (2\mu+\lambda)\alpha_{3,3}\big)\cdot\nu = -\lambda(t + F_{12}x_2 + F_{13}x_3)\nu_3 & \text{on } \partial S \end{cases}$$

We denote the *torsion function* by ψ, more precisely ψ solves

$$\begin{cases} \Delta\psi = 0 & \text{in } S \\ \partial_\nu\psi = (-x_3, x_2)\cdot\nu & \text{on } \partial S \end{cases}$$

Then a short computation shows that α defined by $\alpha_1(x') := F_{23}\psi(x')$ and

$$\alpha_2(x') := -\frac{\lambda}{4(\lambda+\mu)}\Big(F_{12}x_2^2 - F_{12}x_3^2 + 2F_{13}x_2x_3 + 2tx_2\Big)$$

$$\alpha_3(x') := -\frac{\lambda}{4(\lambda+\mu)}\Big(-F_{13}x_2^2 + F_{13}x_3^2 + 2F_{12}x_2x_3 + 2tx_3\Big)$$

solves the related systems. Plugging this minimum point in the definition of $Q^0(t,F)$, we compute with (3.67)

$$\begin{aligned} Q^0(t,F) &= \int_S Q_3\left(F\begin{pmatrix}0\\x'\end{pmatrix} + te_1\Big|\alpha_{,2}\Big|\alpha_{,3}\right)dx' \\ &= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\Big(F_{12}^2 I_2 + F_{13}^2 I_3 + t^2\Big) + \mu\tau F_{23}^2 \end{aligned}$$

where $I_2 := \int_S x_2^2 dx'$, $I_3 := \int_S x_3^2 dx'$ and

$$\tau := \int_S x_3^2 + x_2^2 + x_3\psi_{,2} - x_2\psi_{,3} + \psi_{,2}^2 + \psi_{,3}^2 dx'.$$

3.4 Asymptotic Models for Loaded Rods

The Γ-convergence of the total energy follows now from Theorems 3.3.4 and 3.3.5, more precise we will prove an approximation theorem like in Theorem 3.2.4 dropping the condition of bounded scaled elastic energy. This is then used to prove that, if a sequence has bounded scaled total energy, it follows that the scaled elastic energy is bounded as well.

Theorem 3.4.1. *Let $h > 0$, $y \in H^1(\Omega;\mathbb{R}^3)$ and*

$$E = \int_\Omega \operatorname{dist}^2(\nabla_h y, SO(3))dx.$$

Then there exit a constant rotation $\bar{Q}^{(h)} \in SO(3)$ such that

$$\|\nabla_h y - \bar{Q}^{(h)}\|^2_{L^2(\Omega)} \le \frac{C}{h^2}E. \tag{3.68}$$

Proof: The idea of the proof is similar to the one of Theorem 3.2.4, but in this case we can not expect $\tilde{Q}^{(h)}$ to be in a tubular neighbourhood of $SO(3)$. Hence there might not be a well defined projection to $SO(3)$. Nevertheless it is possible to use a part of the construction to obtain (3.68).
Using the same construction as in the proof of Theorem 3.2.4 we obtain $Q^{(h)}\colon [0,L] \to SO(3)$, $\tilde{Q}^{(h)}\colon [0,L] \to \mathbb{R}^{3\times 3}$ such that

$$\|\nabla_h y - Q^{(h)}\|^2_{L^2(\Omega)} \le CE, \quad \|\tilde{Q}^{(h)} - Q^{(h)}\|^2_{L^2(0,L)} \le CE, \tag{3.69}$$

$$\|(\tilde{Q}^{(h)})'\|^2_{L^2(0,L)} \le \frac{C}{h^2}E, \quad \|\tilde{Q}^{(h)} - Q^{(h)}\|^2_{L^\infty(0,L)} \le \frac{C}{h}E. \tag{3.70}$$

Hence we can use the Poincaré inequality to deduce

$$\|\tilde{Q}^{(h)} - \bar{Q}^{(h)}\|^2_{L^2(0,L)} \le \|(\tilde{Q}^{(h)})'\|^2_{L^2(0,L)} \le \frac{C}{h^2}E$$

where $\bar{Q}^{(h)} = \int_0^L \tilde{Q}^{(h)}(x_1)dx_1$. As $SO(3)$ is compact, there exists $\bar{Q}^{(h)}_\pi \in SO(3)$ such that

$$|\bar{Q}^{(h)} - \bar{Q}^{(h)}_\pi| = \operatorname{dist}(\bar{Q}^{(h)}, SO(3)) \le |\nabla_h y - \bar{Q}^{(h)}| + \operatorname{dist}(\nabla_h y, SO(3)).$$

Hence with the definition of E, we deduce

$$\|\nabla_h y - \bar{Q}^{(h)}_\pi\|^2_{L^2(\Omega)} \le 2\|\nabla_h y - \bar{Q}^{(h)}\|^2_{L^2(\Omega)} + 2\|\bar{Q}^{(h)} - \bar{Q}^{(h)}_\pi\|^2_{L^2(\Omega)}$$

$$\le 2\|\nabla_h y - \bar{Q}^{(h)}\|^2_{L^2(\Omega)} + 2\int_\Omega \operatorname{dist}^2(\nabla_h y, SO(3))dx$$
$$\le \frac{C}{h^2}E. \qquad \square$$

Remark 3.4.2. The result is similar to Theorem 6 in [FJM06], which is formulated for thin plates. The proof however differ in the sense that in [FJM06] first $\tilde{Q}$ is constructed, first in the interior and later near the boundary. Using this $\tilde{Q}$ one can then define Q via projection near $SO(3)$ and constantly as Id otherwise.

Now we can prove the convergence result.

Theorem 3.4.3. *Let $\mathcal{E}^{(h)}$ be given as in* (3.4) *and $f^{(h)} \in W^1_2(0, L; \mathbb{R}^3)$ shall satisfy* (3.3)*. Then for $h \to 0$ the functionals $\frac{1}{h^{2\alpha-2}}\mathcal{E}^{(h)}$ Γ-converges to the functional $\mathcal{E}_\alpha$ given by*

$$\mathcal{E}_\alpha(u, v_2, v_3, w, R) := \mathcal{I}_\alpha(u, v_2, v_3, w) - \int_0^L (R')^T \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \cdot \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} dx_1.$$

In detail it holds

(i) if $\liminf_{h\to 0} \frac{1}{h^{2\alpha-2}}\mathcal{E}^{(h)}(y^{(h)}) < \infty$, there exists $\bar{R}^{(h)} \in SO(3)$, $c^{(h)} \in \mathbb{R}^3$ such that $\bar{R}^{(h)} \to \bar{R} \in \mathcal{U}$ and for

$$\tilde{y}^{(h)} := (\bar{R}^{(h)})^T y^{(h)} - c^{(h)}$$

(a)–(f) of Theorem 3.2.4 is satisfied. Moreover, we have

$$\liminf_{h\to 0} \frac{1}{h^{2\alpha-2}}\mathcal{E}^{(h)}(y^{(h)}) \ge \mathcal{E}_\alpha(u, v_2, v_3, w, \bar{R})$$

(ii) for all u, $w \in H^1_{per}(0, L)$, $v_k \in H^2_{per}(0, L)$ for $k = 2, 3$ and $\bar{R} \in \mathcal{U}$ there exists a sequence $(\check{y}^{(h)})_{h>0} \subset H^1(\Omega; \mathbb{R}^3)$ such that for $Y^{(h)} := \bar{R}^T\check{y}^{(h)}$ the properties (a)–(d) of Theorem 3.2.4 are satisfied and

$$\limsup_{h\to 0} \frac{1}{h^{2\alpha-2}}\mathcal{E}^{(h)}(\check{y}^{(h)}) \le \mathcal{E}_\alpha(u, v_2, v_3, w, \bar{R}).$$

Proof: Step 1: (Boundedness of elastic energy). Assume $(y^{(h)})_{h>0} \subset H^1(\Omega; \mathbb{R}^3)$ such that

$$\liminf_{h\to 0} \frac{1}{h^{2\alpha-2}}\mathcal{E}^{(h)}(y^{(h)}) \le C < \infty$$

holds and by passing to a subsequence if necessary we obtain

$$\frac{1}{h^{2\alpha-2}}\mathcal{E}^{(h)}(y^{(h)}) \le C < \infty.$$

Then $(y^{(h)} - \mathrm{Id}_h)_{h>0} \subset H^1_{per}(\Omega; \mathbb{R}^3)$ and due to Theorem 3.4.1 and Poincaré's inequality there exist $(\bar{Q}^{(h)})_{h>0} \subset SO(3)$ and $c^{(h)} \in \mathbb{R}^3$ such that

$$Y^{(h)} := \left(\bar{Q}^{(h)}\right)^T y^{(h)} - c^{(h)} - \begin{pmatrix} x_1 \\ hx' \end{pmatrix}$$

satisfies

$$\|Y^{(h)}\|^2_{L^2(\Omega)} + \|\nabla_h Y^{(h)}\|^2_{L^2(\Omega)} \le \frac{C}{h^2}E.$$

Using a subsequence we can assume $\bar{Q}^{(h)} \to \bar{Q} \in SO(3)$. Hence, it follows

$$\frac{1}{h^{2\alpha-2}}\mathcal{I}^{(h)}(y^{(h)}) = \frac{1}{h^{2\alpha-2}}\mathcal{E}^{(h)}(y^{(h)}) + \frac{1}{h^{2\alpha-2}}\int_\Omega f^{(h)} \cdot \left(y^{(h)} - x^{(h)}\right)dx$$

$$
\begin{aligned}
&= \frac{1}{h^{2\alpha-2}}\mathcal{E}^{(h)}(y^{(h)}) + \int_\Omega \frac{1}{h^\alpha}\bar{Q}^{(h)} f^{(h)} \cdot \frac{1}{h^{\alpha-2}} Y^{(h)} dx \\
&\le \frac{1}{h^{2\alpha-2}}\mathcal{E}^{(h)}(y^{(h)}) + C\left\|\frac{1}{h^\alpha} f^{(h)}\right\|_{L^2(0,L)} \left\|\frac{1}{h^{\alpha-2}}\nabla_h Y^{(h)}\right\|_{L^2(\Omega)} \\
&\le \frac{1}{h^{2\alpha-2}}\mathcal{E}^{(h)}(y^{(h)}) + \frac{C}{h^{\alpha-1}} E^{\frac{1}{2}} \\
&\le \frac{1}{h^{2\alpha-2}}\mathcal{E}^{(h)}(y^{(h)}) + \frac{C}{h^{\alpha-1}} \left(\mathcal{I}^{(h)}(y^{(h)})\right)^{\frac{1}{2}}.
\end{aligned}
$$

and therefore

$$
\frac{1}{h^{2\alpha-2}}\mathcal{I}^{(h)}(y^{(h)}) \le C < \infty.
$$

Step 2: (Liminf inequality). Theorem 3.2.4 implies now the existence of $\bar{R}^{(h)} \in SO(3)$, $c^{(h)} \in \mathbb{R}^3$ such that $\bar{R}^{(h)} \to \bar{R} \in \mathcal{U}$ and for

$$
\tilde{y}^{(h)} := (\bar{R}^{(h)})^T y^{(h)} - c^{(h)}
$$

it holds (3.11)–(3.14) and (a)–(f) are satisfied. Hence, Theorem 3.3.4 leads to

$$
\liminf_{h\to 0} \frac{1}{h^{2\alpha-2}}\mathcal{I}^{(h)}(y^{(h)}) \ge \mathcal{I}_\alpha(u, v_2, v_3, w). \tag{3.71}
$$

Therefore we only have to establish convergence of the external force part. Using the assumptions on $f^{(h)}$ and property (b) we obtain

$$
\begin{aligned}
\frac{1}{h^{2\alpha-2}} \int_\Omega f^{(h)} \cdot \left(y^{(h)} - x^{(h)}\right) dx &= \frac{1}{h^{2\alpha-2}} \int_\Omega (\bar{R}^{(h)})^T f^{(h)} \cdot \tilde{y}^{(h)} dx \\
&= \int_0^L (\bar{R}^{(h)})^T \frac{1}{h^\alpha} \begin{pmatrix} 0 \\ f_2^{(h)} \\ f_3^{(h)} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v_2^{(h)} \\ v_3^{(h)} \end{pmatrix} dx_1 \to \int_0^L (\bar{R}')^T \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \cdot \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} dx_1
\end{aligned}
$$

Putting this together with (3.71), we obtain the desired result

$$
\liminf_{h\to 0} \frac{1}{h^{2\alpha-2}}\mathcal{E}^{(h)}(y^{(h)}) \ge \mathcal{E}_\alpha(u, v_2, v_3, w, \bar{R}).
$$

Step 3: (Recovery sequence). As in the proof of Theorem 3.3.5 we first suppose $u,\ w \in C^1([0,L])$ and $v_k \in C^2([0,L])$, $k = 2,\ 3$, such that

$$
u(0) = u(L), \quad w(0) = w(L), \quad v_k(0) = v_k(L) \ \text{ and } \ v_k'(0) = v_k'(L)
$$

and define the sequence $(\check{y}^{(h)})_{h>0}$ similarly as in (3.57) via

$$
\bar{R}^T \check{y}^{(h)}(x) := \begin{pmatrix} x_1 \\ hx_2 \\ hx_3 \end{pmatrix} + h^{\alpha-2} \begin{pmatrix} 0 \\ v_2 \\ v_3 \end{pmatrix} + h^{\alpha-1} \begin{pmatrix} u - x_2 v_{2,1} - x_3 v_{3,1} \\ -x_3 w \\ x_2 w \end{pmatrix} + h^\alpha \varphi^{(h)}.
$$

Where $\gamma_2,\ \gamma_3,\ \beta,\ \varphi$, the cut of function $\rho^{(h)}$ and $\varphi^{(h)}(x) := \varphi(x) + \rho^{(h)}(x_1)(\beta(0,x') - \beta(L,x'))$ are as above.
With this definition we obtain $(\check{y}^{(h)})_{h>0} \subset H^1(\Omega;\mathbb{R}^3)$ and $(\check{y}^{(h)} - \mathrm{Id}_h)_{h>0} \subset H^1_{per}(\Omega;\mathbb{R}^3)$ due to $\bar{R} \in \mathcal{U}$. Moreover from Theorem 3.3.5 we obtain that for $u^{(h)}$, $v_k^{(h)}$ and $w^{(h)}$ defined as in (3.7)–(3.9) assertions (a)–(d) of Theorem 3.2.4 hold.
Hence, we only have to show that (3.57) holds. As in the proof of Theorem 3.3.5 we obtain due to $\bar{R}^T\bar{R} = \mathrm{Id}$ the following formula for the nonlinear strain

$$
(\nabla_h \check{y}^{(h)})^T \nabla_h \check{y}^{(h)} = \mathrm{Id} + 2h^{\alpha-1} \operatorname{sym}\left(A_{,1} \begin{pmatrix} 0 \\ x' \end{pmatrix} + u_{,1} e_1 \middle| \varphi_{,2}^{(h)} \middle| \varphi_{,3}^{(h)}\right)
$$

$$+ h^{2(\alpha-2)}A^TA + O(h^{2\alpha-3}) + O\Big(h^{\alpha-\frac{1}{2}}\Big).$$

Thus we can use the same strategy for the elastic energy part of $\frac{1}{h^{2\alpha-2}}\mathcal{E}^{(h)}$ and obtain

$$\lim_{h\to 0}\frac{1}{h^{2\alpha-2}}\mathcal{I}^{(h)}(\check{y}^{(h)}) = \mathcal{I}_\alpha(u, v_2, v_3, w).$$

Hence, we only have to establish the convergence of the external force part of $\mathcal{E}^{(h)}(\check{y}^{(h)})$. The definition of $\check{y}^{(h)}$ leads to

$$\begin{aligned}\frac{1}{h^{2\alpha-2}}\int_\Omega f^{(h)}\cdot\Big(\check{y}^{(h)} - x^{(h)}\Big)dx &= \int_0^L \frac{1}{h^\alpha}\bar{R}^T f^{(h)}\cdot\left(\begin{pmatrix}0\\ v_2\\ v_3\end{pmatrix} + O(h)\right)dx_1\\ &\quad + \int_\Omega \frac{1}{h^\alpha}f^{(h)}\cdot\frac{1}{h^{\alpha-2}}\left(\bar{R}\begin{pmatrix}x_1\\ hx'\end{pmatrix} - \begin{pmatrix}x_1\\ hx'\end{pmatrix}\right)dx.\end{aligned}$$

The structure of $\bar{R}\in\mathcal{U}$ then implies

$$\bar{R} - \mathrm{Id} = \begin{pmatrix}0 & 0 & 0\\ 0 & \bar{R}_{22}-1 & \bar{R}_{23}\\ 0 & \bar{R}_{32} & \bar{R}_{33}-1\end{pmatrix}.$$

Hence, we obtain with (2.4), the fact that $f^{(h)}$ does not depend on x' and (3.3)

$$\begin{aligned}\frac{1}{h^{2\alpha-2}}\int_\Omega f^{(h)}\cdot\Big(\check{y}^{(h)} - x^{(h)}\Big)dx &= \int_0^L \frac{1}{h^\alpha}(\bar{R}')^T\begin{pmatrix}f_2^{(h)}\\ f_3^{(h)}\end{pmatrix}\cdot\begin{pmatrix}v_2\\ v_3\end{pmatrix}dx_1 + O(h)\\ &\to \int_0^L (\bar{R}')^T\begin{pmatrix}f_2\\ f_3\end{pmatrix}\cdot\begin{pmatrix}v_2\\ v_3\end{pmatrix}dx_1\end{aligned}$$

Putting this together it follows

$$\lim_{h\to 0}\frac{1}{h^{2\alpha-2}}\mathcal{E}^{(h)}(\check{y}^{(h)}) \le \mathcal{E}_\alpha(u, v_2, v_3, w, \bar{R}).$$

An approximation argument, as in Theorem 3.3.5 for u, $w\in H^1_{per}(0,L)$ and $v_k\in H^2_{per}(0,L)$ leads to the desired result. □

4

Convergence of Equilibria for Thin Elastic Rods for the von Kármán Regime

In Example 2.1.8 we have seen that Γ-convergence is not capable of catching convergence of local minimizers, in contrast to global ones. We are therefore interested in the convergence behaviour of solutions to the Euler-Lagrange equations for the total energy. More precisely we will show that for every sequence $y^{(h)}$ of stationary points of $\mathcal{E}^{(h)}$ the limits of $(u^{(h)}, v_2^{(h)}, v_3^{(h)}, w^{(h)}, \bar{R}^{(h)})$ exist and solve the Euler-Lagrange equations for $\mathcal{E}_\alpha$.
In addition to the assumptions (i)–(iv) made for the elastic energy density, we assume in this chapter that $DW(F)$ grows at most linear at infinity. More precisely we assume

(v) there exists $C > 0$ such that

$$|DW(F)| \leq C(|F| + 1) \quad \text{for all } F \in \mathbb{R}^{3\times 3}. \tag{4.1}$$

Using that W is C^2 in a ball around the identity, a Taylor expansion and $DW(Id) = 0$ we deduce

$$|DW(Id + A)| \leq C|A| \quad \text{for all } A \in \mathbb{R}^{3\times 3}. \tag{4.2}$$

This assumption ensures that the Euler-Lagrange equations in the classical form are satisfied by local minimizers, see also the discussion in [Bal02]. Unfortunately the assumption prohibits that the strain-energy density satisfies the physically justified condition $W \to \infty$ for $\det(F) \to 0$. As we will regard the von Kármán case, the deformation is close to a rigid motion, defending the choice of the assumption (v).
This result fits in the context of papers by Mora and Müller [MM08] and Müller and Pakzad [MP08] on similar problems. The main novelty is the incorporation of an external force in the energy. Moreover, the scaling of the total energy differs from the one in [MM08] and in [MP08] a thin plate is considered. In the work by Mora and Scardia [MS12] the same scaling regime with external force was considered, but with a different growth condition and in the plate case. We restrict to the case $\alpha = 3$, in order to retain the scaling for the von Kármán regime. For sake of convenience we will denote $D^2W(Id)[G, G] =: \mathcal{L}G : G = Q_3(G)$.
In order to derive the Euler-Lagrange equations for the limiting problem, we recap the definition of $\mathcal{E} := \mathcal{E}_3$

$$\mathcal{E}\colon H^1_{per}(0, L) \times H^2_{per}(0, L)^2 \times H^1_{per}(0, L) \times \mathcal{U} \to \mathbb{R}_+$$
$$\mathcal{E}(u, v_2, v_3, w, \bar{R}) := \mathcal{I}_3(u, v_2, v_3, w) - \int_0^L (\bar{R}')^T \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \cdot \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} dx_1.$$

In Remark 3.3.1 we derived the Euler-Lagrange equations, which minimizers $\alpha \in \mathcal{V}$ have to satisfy. This can be used to prove the following lemma.

Lemma 4.0.1. *Let $t \in \mathbb{R}$, $F \in \mathbb{R}^{3\times 3}_{skew}$ and $G_{t,F} \colon H^1(S;\mathbb{R}^3) \to [0,\infty)$ be defined by*

$$G_{t,F}(\varphi) := \int_S Q_3\Big(te_1 + F\begin{pmatrix}0\\x'\end{pmatrix}\Big|\varphi_{,2}\Big|\varphi_{,3}\Big)dx'.$$

Then $\alpha \in \mathcal{V}$ is the unique minimizer of $G_{t,F}$ if and only if

$$E := \mathcal{L}(te_1 + F(0,x')^T|\alpha_{,2}|\alpha_{,3})$$

solves the boundary-value problem

$$\begin{cases} \operatorname{div}_{x'}\Big(Ee_2|Ee_3\Big) = 0 & \text{in } S,\\ (Ee_2|Ee_3)\nu_{\partial S} = 0 & \text{on } \partial S \end{cases}$$

in the weak form, where ν is the outer normal on ∂S. Moreover, α depends linearly on t and F.

Proof: The functional $G_{t,F}$ is convex and thus $\alpha \in \mathcal{V}$ minimizes $G_{t,F}$ if and only if for all $\varphi \in H^1(S,\mathbb{R}^3)$

$$\frac{d}{d\varepsilon}\int_S Q_3\Big(te_1 + F(0,x')^T|\alpha_{,2}+\varepsilon\varphi_{,2}|\alpha_{,3}+\varepsilon\varphi_{,3}\Big)dx'\Big|_{\varepsilon=0} = 0$$

Differentiation of parameter integrals leads then to

$$\int_S E : (0|\varphi_{,2}|\varphi_{,3})dx' = \int_S \mathcal{L}(te_1 + F(0,x')^T|\alpha_{,2}|\alpha_{,3}) : (0|\varphi_{,2}|\varphi_{,3})dx' = 0.$$

The linear dependency follows then in the same way as in Remark 3.3.1. □

Next we want to derive the structure of the Euler-Lagrange equations for the limiting problem. For this we assume u, $w \in H^1_{per}(0,L)$, $v_k \in H^2_{per}(0,L)$ for $k = 2, 3$ and $\bar{R} \in \mathcal{U}$. Moreover we denote the solution to the minimum problem defining (3.34) by $\alpha \in L^2(0,L;\mathcal{V})$ for $t = u_{,1} + \frac{1}{2}(v^2_{2,1} + v^2_{3,1})$ and $F = A_{,1}$. The stress $E\colon \Omega \to \mathbb{R}^{3\times 3}$ is defined by

$$E(x) := \mathcal{L}\Big(u_{,1}(x_1) + \frac{1}{2}(v^2_{2,1}(x_1) + v^2_{3,1}(x_1)) + A_{,1}(x_1)(0,x')^T|\alpha_{,2}(x)|\alpha_{,3}(x)\Big). \tag{4.3}$$

The Euler-Lagrange equations will be stated in terms of the zeroth and first moment of E. We define $E^0\colon (0,L) \to \mathbb{R}^{3\times 3}$, $E^2\colon (0,L) \to \mathbb{R}^{3\times 3}$ and $E^3\colon (0,L) \to \mathbb{R}^{3\times 3}$ by

$$E^0(x_1) := \int_S E(x)dx', \quad E^k(x_1) := \int_S x_k E(x)dx' \text{ for } k = 2,3. \tag{4.4}$$

Lemma 4.0.2. *Let u, $w \in H^1_{per}(0,L)$, $v_k \in H^2_{per}(0,L)$ for $k = 2$, 3 and $\bar{R} \in \mathcal{U}$. Then $(u, v_2, v_3, w, \bar{R})$ is a stationary point of $\mathcal{E}$ if and only if*

$$\int_0^L \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\bar{R}'\begin{pmatrix}f_2\\f_3\end{pmatrix}\cdot\begin{pmatrix}v_2\\v_3\end{pmatrix} = 0 \tag{4.5}$$

and

$$\begin{aligned}(E^0_{11})' &= 0\\ v_2''E^0_{11} + (E^2_{11})'' + \bar{R}_{22}f_2 + \bar{R}_{23}f_3 &= 0\\ v_3''E^0_{11} + (E^3_{11})'' + \bar{R}_{32}f_2 + \bar{R}_{33}f_3 &= 0\\ (E^2_{31})' - (E^3_{21})' &= 0\end{aligned}$$

holds in a weak sense.

Proof: We compute the Gâteaux derivative with respect to the arguments of $\mathcal{E}$. Hence assume $\phi \in H^1_{per}(0,L)$ and regard

$$\frac{d}{d\varepsilon}\mathcal{E}(u+\varepsilon\phi, v_2, v_3, w, \bar{R})\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\frac{1}{2}\int_\Omega Q_3\Big((u_{,1}+\varepsilon\phi_{,1}+\frac{1}{2}(v_{2,1}^2+v_{3,1}^2))e_1 + A_{,1}\begin{pmatrix}0\\x'\end{pmatrix}\Big|\alpha^\varepsilon_{,2}\Big|\alpha^\varepsilon_{,3}\Big)dx\Big|_{\varepsilon=0}$$

where α^ε is the solution of the minimum problem for (3.34). From Remark 3.3.1 we get that α^ε depends linearly on ε, i.e.

$$\alpha^\varepsilon = \alpha^{(t,F)} + \varepsilon\alpha^{(\phi_{,1},0)}$$

where $t = u_{,1} + \frac{1}{2}(v_{2,1}^2 + v_{3,1}^2)$, $F = A_{,1}$. Therefore we obtain

$$\begin{aligned}
&\frac{d}{d\varepsilon}\mathcal{E}(u+\varepsilon\phi, v_2, v_3, w, \bar{R})\Big|_{\varepsilon=0} \\
&= \int_\Omega \mathcal{L}\Big((u_{,1}+\frac{1}{2}(v_{2,1}^2+v_{3,1}^2))e_1 + A_{,1}\begin{pmatrix}0\\x'\end{pmatrix}\Big|\alpha^{(t,F)}_{,2}\Big|\alpha^{(t,F)}_{,3}\Big) : \Big(\phi_{,1}e_1\Big|\alpha^{(\phi_{,1},0)}_{,2}\Big|\alpha^{(\phi_{,1},0)}_{,3}\Big)dx \\
&= \int_\Omega E : \Big(\phi_{,1}e_1\Big|\alpha^{(\phi_{,1},0)}_{,2}\Big|\alpha^{(\phi_{,1},0)}_{,3}\Big)dx = \int_\Omega Ee_1\cdot\phi_{,1}e_1dx + \int_\Omega Ee_2\cdot\alpha^{(\phi_{,1},0)}_{,2} + Ee_3\cdot\alpha^{(\phi_{,1},0)}_{,3}dx \\
&= \int_0^L E^0e_1\cdot\phi_{,1}e_1dx_1 = \int_0^L E^0_{11}\phi_{,1}dx_1.
\end{aligned}$$

where the penultimate equality comes from Lemma 4.0.1.
We use the same idea for the derivative in v_2 direction. Let $\varphi \in H^2_{per}(0,L)$ and consider

$$\begin{aligned}
&\frac{d}{d\varepsilon}\mathcal{E}(u, v_2+\varepsilon\varphi, v_3, w, \bar{R})\Big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\frac{1}{2}\int_\Omega Q_3\Big((u_{,1}+\frac{1}{2}((v_{2,1}+\varepsilon\varphi_{,1})^2+v_{3,1}^2))e_1 + A^\varepsilon_{,1}\begin{pmatrix}0\\x'\end{pmatrix}\Big|\alpha^\varepsilon_{,2}\Big|\alpha^\varepsilon_{,3}\Big)dx \\
&\quad - \int_0^L(\bar{R}')^T\begin{pmatrix}f_2\\f_3\end{pmatrix}\cdot\begin{pmatrix}v_2+\varepsilon\varphi\\v_3\end{pmatrix}dx_1\Big|_{\varepsilon=0},
\end{aligned}$$

where

$$A^\varepsilon = \begin{pmatrix}0 & -v_{2,1}-\varepsilon\varphi_{,1} & -v_{3,1}\\ v_{2,1}+\varepsilon\varphi_{,1} & 0 & -w\\ v_{3,1} & w & 0\end{pmatrix} = A + \varepsilon\begin{pmatrix}0 & -\varphi_{,1} & 0\\ \varphi_{,1} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

We first represent the solution of the minimum problem, again denoted by α^ε, as a polynomial in ε. Using the linearity in t and F it follows

$$\alpha^\varepsilon = \alpha^{(t,F)} + \varepsilon\alpha^{(\varphi_{,1}v_{2,1},\Phi_{,1})} + \varepsilon^2\alpha^{(\varphi_{,1}^2,0)}$$

where

$$\Phi = \begin{pmatrix}0 & -\varphi_{,1} & 0\\ \varphi_{,1} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

With this it follows with $\beta = \alpha^{(\varphi_{,1}v_{2,1},\Phi_{,1})}$ and Lemma 4.0.1

$$\begin{aligned}
&\frac{d}{d\varepsilon}\mathcal{E}(u, v_2+\varepsilon\varphi, v_3, w, \bar{R})\Big|_{\varepsilon=0} \\
&= \int_\Omega E : \Big(\Phi\begin{pmatrix}0\\x'\end{pmatrix} + \varphi_{,1}v_{2,1}e_1\Big|\beta_{,2}\Big|\beta_{,3}\Big)dx' - \int_0^L(\bar{R}')^T\begin{pmatrix}f_2\\f_3\end{pmatrix}\cdot\begin{pmatrix}\varphi\\0\end{pmatrix}dx_1
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega Ee_1 \cdot \big(v_{2,1}\varphi_{,1} - \varphi_{,11}x_2\big)e_1 dx - \int_0^L \Big(\bar R_{22} f_2 + \bar R_{23} f_3\Big)\varphi dx_1 \\
&= \int_0^L v_{2,1}E^0_{11}\varphi_{,1} - E^2_{11}\varphi_{,11} - \Big(\bar R_{22} f_2 + \bar R_{23} f_3\Big)\varphi dx_1.
\end{aligned}
$$

Analogously we obtain

$$
\frac{d}{d\varepsilon}\mathcal{E}(u, v_2, v_3 + \varepsilon\varphi, w, \bar R)\Big|_{\varepsilon=0} = \int_0^L v_{3,1}E^0_{11}\varphi_{,1} - E^3_{11}\varphi_{,11} - \Big(\bar R_{32} f_2 + \bar R_{33} f_3\Big)\varphi dx_1.
$$

Now let $\sigma \in H^1_{per}(0, L)$, then

$$
\begin{aligned}
\frac{d}{d\varepsilon}\mathcal{E}(u, v_2, v_3, w + \varepsilon\sigma, \bar R)\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\frac{1}{2}\int_\Omega Q_3\Big(\big(u_{,1} + \frac{1}{2}(v_{2,1}^2 + v_{3,1}^2)\big)e_1 + B^\varepsilon_{,1}\begin{pmatrix}0\\x'\end{pmatrix}\Big|\alpha^\varepsilon_{,2}\Big|\alpha^\varepsilon_{,3}\Big)dx \\
&\quad - \int_0^L (\bar R')^T\begin{pmatrix}f_2\\f_3\end{pmatrix}\cdot\begin{pmatrix}v_2\\v_3\end{pmatrix} dx_1\Big|_{\varepsilon=0} \\
&= \int_\Omega E : \Big(\sigma_{,1}(e_3\otimes e_2 - e_2\otimes e_3)\begin{pmatrix}0\\x'\end{pmatrix}\Big|\alpha^{(0,\Psi_{,1})}_{,2}\Big|\alpha^{(0,\Psi_{,1})}_{,3}\Big) \\
&= \int_\Omega Ee_1\cdot\sigma_{,1}(e_3\otimes e_2 - e_2\otimes e_3)\begin{pmatrix}0\\x'\end{pmatrix} \\
&= \int_0^L \int_S x_2 E dx' e_1 \cdot \sigma_{,1} e_3 - \int_S x_3 E dx' e_1\cdot\sigma_{,1} e_2 dx_1 \\
&= \int_0^L (E^2_{31} - E^3_{21})\sigma_{,1} dx_1
\end{aligned}
$$

where

$$
\Psi := \sigma(e_3\otimes e_2 - e_2\otimes e_3) \quad \text{ and } \quad B^\varepsilon := A + \varepsilon\Psi.
$$

Finally the tangent space of $\mathcal{U}$ in R is given by $T_R\mathcal{U} = RB$ where

$$
B \in \operatorname{span}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}.
$$

Let $\gamma\colon (-\varepsilon,\varepsilon)\to\mathcal{U}$ be such that $\gamma(0) = \bar R$ and $\gamma'(0) = \bar R B$, then it follows

$$
\begin{aligned}
d\mathcal{E}(u, v_2, v_3, w, \bar R)[RB] &= \frac{d}{dt}\mathcal{E}(u, v_2, v_3, w, \gamma(t))\Big|_{t=0} = -\frac{d}{dt}\int_0^L \gamma(t)^T\begin{pmatrix}f_2\\f_3\end{pmatrix}\cdot\begin{pmatrix}v_2\\v_3\end{pmatrix}dx_1\Big|_{t=0} \\
&= -\tau\int_0^L \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\bar R^T\begin{pmatrix}f_2\\f_3\end{pmatrix}\cdot\begin{pmatrix}v_2\\v_3\end{pmatrix}dx_1
\end{aligned}
$$

for some $\tau\in\mathbb{R}$. □

In order to establish the convergence of the stress, if some given strain converges weakly, we use the following proposition.

Proposition 4.0.3. *Let $1 < p < \infty$, U a bounded domain in $\mathbb{R}^n$ and $f\colon\mathbb{R}^d\to\mathbb{R}$ such that f is differentiable at zero and $|f(A)| \le C|A|$ for all $A\in\mathbb{R}^d$. If*

$$
z^\delta \rightharpoonup z \quad \text{for } \delta\to 0 \text{ in } L^p(U;\mathbb{R}^d),
$$

then

$$
\frac{1}{\delta}f(\delta z^\delta) \rightharpoonup Df(0)z \quad \text{for } \delta\to 0 \text{ in } L^p(U). \tag{4.6}
$$

Proof: A proof can be found in [MP08, Proposition 2.3.]. For the convenience of the reader we give a detailed proof.

Without loss of generality we can assume $Df(0) \equiv 0$, because if not one can regard $f(A) - Df(0)A$ and use that $Df(0)$ defines a continuous, linear function. Set

$$\omega(\delta) := \sup_{|A| \leq \sqrt{\delta}} \frac{|f(A)|}{|A|}.$$

Using the bound $|f(A)| \leq C|A|$ and the assumption $Df(0) = 0$ it follows $\omega(\delta) \to 0$ for $\delta \to 0$. Set $A_\delta := \{x \in E \ : \ |z^\delta| \geq \delta^{-\frac{1}{2}}\}$, then it holds for the measure $|A_\delta| \to 0$ as $\delta \to 0$. For $g \in L^q(E; \mathbb{R}^d)$ such that $\frac{1}{q} + \frac{1}{p} = 1$ it follows

$$\left| \int_E g \cdot \frac{1}{\delta} f(\delta z^\delta) \right| \leq \omega(\delta) \|g\|_{L^q} \sup_{\delta > 0} \|z^\delta\|_{L^p} + M \left(\int_{A_\delta} |g|^q dx \right)^{\frac{1}{q}} \sup_{\delta > 0} \|z^\delta\|_{L^p}.$$

As $\sup_{\delta > 0} \|z^\delta\|_{L^p}$ is bounded, $\omega(\delta) \to 0$ and

$$\int_{A_\delta} |g|^q dx \to 0$$

we can conclude $\frac{1}{\delta} f(\delta z^\delta) \rightharpoonup 0$ in $L^p(E; \mathbb{R}^d)$. $\square$

The main result of this subsection is the following.

Theorem 4.0.4. *Assume that W satisfies the assumptions (i)–(v) and let $(y^{(h)})_{h>0} \subset H^1(\Omega; \mathbb{R}^3)$ be a sequence of local minimizers of $\mathcal{E}^{(h)}$ such that*

$$\int_\Omega W(\nabla_h y^{(h)}) dx \leq Ch^4.$$

The, up to a subsequence, respective limit of $(u^{(h)}, v_2^{(h)}, v_3^{(h)}, w^{(h)}, \bar{R}^{(h)})$ exists and $(u, v_2, v_3, w, \bar{R})$ is a stationary point of the limiting functional $\mathcal{E}$.

Proof: Let $y^{(h)}$ be a sequence of local minimizers of $\mathcal{E}^{(h)}$. For convenience we define $g^{(h)} := \frac{1}{h^3} f^{(h)}$ and have due to (3.3)

$$g^{(h)} \rightharpoonup g := f \quad \text{ in } L^2(0, L; \mathbb{R}^3).$$

Theorem 3.2.4 implies now, that there exist maps $R^{(h)} \in C^\infty(0, L; \mathbb{R}^{3\times 3})$ and constants $\bar{R}^{(h)} \in SO(3)$, $c^{(h)} \in \mathbb{R}^3$ such that (up to subsequences) $\bar{R}^{(h)} \to \bar{R} \in \mathcal{U}$ and for

$$\tilde{y}^{(h)} := (\bar{R}^{(h)})^T y^{(h)} - c^{(h)}$$

we have

$$R^{(h)}(s) \in SO(3) \text{ for every } s \in (0, L), \tag{4.7}$$

$$\|R^{(h)} - Id\|_{L^\infty(0,L)} \leq Ch, \quad \|(R^{(h)})'\|_{L^2(0,L)} \leq Ch \tag{4.8}$$

$$\|\nabla_h \tilde{y}^{(h)} - R^{(h)}\|_{L^2(\Omega)} \leq Ch^2 \tag{4.9}$$

$$|R^{(h)}(0) - R^{(h)}(L)| \leq Ch^{\frac{3}{2}}. \tag{4.10}$$

and $(u^{(h)}, v_2^{(h)}, v_3^{(h)}, w^{(h)}, \bar{R}^{(h)})$ converge in the sense that (a)-(c) of Theorem 3.2.4 is satisfied. From (4.8) and (4.9) it follows $\nabla_h \tilde{y}^{(h)} \to \mathrm{Id}$ strongly in $L^2(\Omega, \mathbb{R}^{3\times 3})$ and $\tilde{y}^{(h)}_{,2}$, $\tilde{y}^{(h)}_{,3} \to 0$, which implies $\nabla \tilde{y}^{(h)} \to \mathrm{diag}(1, 0, 0)$ strongly in $L^2(\Omega, \mathbb{R}^{3\times 3})$. From the choice of $c^{(h)}$ it follows

$$\int_\Omega \tilde{y}^{(h)} - x^{(h)} dx = 0$$

and hence, Poincaré's inequality implies $\tilde{y}^{(h)} \to (x_1, 0, 0)^T$ strongly in $H^1(\Omega; \mathbb{R}^3)$. As in the proof of Theorem 3.3.4 we define $G^{(h)}\colon \Omega \to \mathbb{R}^{3\times 3}$ by

$$G^{(h)} := \frac{1}{h^2}\big((R^{(h)})^T \nabla_h \tilde{y}^{(h)} - \mathrm{Id}\big).$$

Due to the arguments in the proof of Theorem 3.3.4 in the von Kármán case we obtain

$$G^{(h)} \rightharpoonup G \text{ weakly in } L^2(\Omega;\mathbb{R}^{3\times 3}) \tag{4.11}$$

where

$$\tilde{G} := \operatorname{sym} G = \left(A_{,1}\begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} + \Big(u_{,1} + \tfrac{1}{2}(v_{2,1}^2 + v_{3,1}^2)\Big) e_1 \Big| \beta_{,2} \Big| \beta_{,3} \right). \tag{4.12}$$

for some $\beta \in L^2(0,L;H^1(S;\mathbb{R}^3))$.

Step 1: (Euler-Lagrange equations). Using the definition of $\mathcal{E}^{(h)}$ and assumption (v) it follows

$$\frac{d}{d\tau}\mathcal{E}^{(h)}(y^{(h)} + \tau\varphi)\Big|_{\tau=0} = \int_\Omega DW(\nabla_h y^{(h)}) : \nabla_h \varphi - f^{(h)} \cdot \varphi dx$$

for all $\varphi \in H^1_{per}(\Omega;\mathbb{R}^3)$. For this we apply the dominated convergence theorem, which is applicable as

$$|\partial_\tau W(\nabla_h y(h) + \tau \nabla_h \varphi)| \leq C\big(|\nabla_h y^{(h)}| + |\nabla_h \varphi|\big)|\nabla_h \varphi| \in L^1(\Omega)$$

and $W(\nabla_h y^{(h)}(x) + \cdot \nabla_h \varphi(x)) \in C^1((-\varepsilon,\varepsilon),\mathbb{R})$ for almost every $x \in \Omega$. Thus as $y^{(h)}$ is a sequence of local minima we obtain the Euler-Lagrange equations

$$\int_\Omega DW(\nabla_h y^{(h)}) : \nabla_h \varphi - f^{(h)} \cdot \varphi dx = 0$$

for all $\varphi \in H^1_{per}(\Omega;\mathbb{R}^3)$.

Step 2: (convergence of the scaled stress). We define the scaled stress by

$$E^{(h)} := \frac{1}{h^2} DW(Id + h^2 G^{(h)}).$$

Hence by Proposition 4.0.3 and (4.11) it follows $E^{(h)} \rightharpoonup \tilde{E} := \mathcal{L}G$ weakly in $L^2(\Omega;\mathbb{R}^{3\times 3})$. Therefore $\tilde{E}$ is symmetric, due to the frame invariance of W, see Remark 2.3.2. Transforming the definition of $G^{(h)}$ we are led to

$$\nabla_h y^{(h)} = \bar{R}^{(h)} R^{(h)}\big(Id + h^2 G^{(h)}\big).$$

This implies

$$DW(\nabla_h y^{(h)}) = \bar{R}^{(h)} R^{(h)} DW(Id + h^2 G(h)) = h^2 \bar{R}^{(h)} R^{(h)} E^{(h)}$$

where we used the frame invariance of W. Plugging this in to the Euler-Lagrange equations and using the definition of $g^{(h)}$ it follows

$$\int_\Omega \bar{R}^{(h)} R^{(h)} E^{(h)} : \nabla_h \varphi - h g^{(h)} \cdot \varphi dx = 0 \tag{4.13}$$

for all $\varphi \in H^1_{per}(\Omega;\mathbb{R}^3)$. Using $\varphi = \bar{R}\psi$ for $\psi \in H^1_{per}(\Omega;\mathbb{R}^3)$ we obtain

$$\int_\Omega R^{(h)} E^{(h)} : (\bar{R}^{(h)})^T \bar{R} \nabla_h \psi - h \bar{R}^T g^{(h)} \cdot \psi dx = 0 \tag{4.14}$$

for all $\psi \in H^1(\Omega;\mathbb{R}^3)$. Multiplying this equation with h and deploying $(\bar{R}^{(h)})^T \bar{R} \to Id$,

$R^{(h)} \to Id$ strongly in $H^1(0,L)$ and $E^{(h)} \rightharpoonup \tilde{E}$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$ we obtain

$$\int_\Omega \tilde{E}e_2 \cdot \psi_{,2} + \tilde{E}e_3 \cdot \psi_{,3} dx = 0.$$

Hence $\tilde{E}$ solves for almost every $x_1 \in (0,L)$ the system

$$\begin{cases} \operatorname{div}_{x'}\left(\tilde{E}e_2|\tilde{E}e_3\right) = 0 & \text{in } S, \\ (\tilde{E}e_2|\tilde{E}e_3)\nu_{\partial S} = 0 & \text{on } \partial S \end{cases} \tag{4.15}$$

in the weak sense. Using again the symmetry properties of $D^2W(Id)$, we obtain that $\mathcal{L}F \equiv 0$ for all $F \in \mathbb{R}^{3\times 3}_{skew}$. Thus it follows for $\gamma\colon \Omega \to \mathbb{R}^3$

$$\gamma(x) := \beta(x) - \frac{1}{\mu(S)}(\beta, x^\perp)_{L^2(S)}x^\perp - \frac{1}{|\Omega|}(\beta,1)_{L^2(S)}$$

where β is as in (4.12), that $\gamma \in L^2(0,L;\mathcal{V})$ holds and

$$\tilde{E} = \mathcal{L}G = \mathcal{L}\tilde{G} = \mathcal{L}\left(A_{,1}\begin{pmatrix}0\\x_2\\x_3\end{pmatrix} + \left(u_{,1} + \tfrac{1}{2}(v_{2,1}^2 + v_{3,1}^2)\right)e_1\middle|\gamma_{,2}\middle|\gamma_{,3}\right).$$

Lemma 4.0.1 and equation (4.15) guarantee that $\gamma(x_1,\cdot) \in \mathcal{V}$ is the unique solution to the minimum problem defining $Q^0(t,F)$ for $t = u_{,1} + \frac{1}{2}(v_{2,1}^2 + v_{3,1}^2)$ and $F = A_{,1}$ for almost every $x_1 \in (0,L)$. Hence we have in particular

$$\tilde{E} = \mathcal{L}G = E$$

where E is defined as in (4.3).
We define now analogue to the definition of the moments of E, those for $E^{(h)}$, i.e.

$${}^0E^{(h)}(x_1) := \int_S E^{(h)}dx', \quad {}^2E^{(h)}(x_1) := \int_S x_2E^{(h)}dx_1 \quad \text{and} \quad {}^3E^{(h)}(x_1) := \int_S x_3E^{(h)}dx_1.$$

Step 3: (Euler-Lagrange equations for zeroth moment of E). Using $\psi \in H^1_{per}(0,L;\mathbb{R}^3)$ as a test function in (4.14) we obtain

$$\begin{aligned} 0 &= \int_\Omega R^{(h)}E^{(h)} : (\bar{R}^{(h)})^T\bar{R}\nabla_h\psi - h\bar{R}^Tg^{(h)} \cdot \psi dx \\ &= \int_\Omega R^{(h)}E^{(h)} : (\bar{R}^{(h)})^T\bar{R}\psi_{,1} \otimes e_1 - h\bar{R}^Tg^{(h)} \cdot \psi dx. \end{aligned}$$

As $R^{(h)}$, $\bar{R}^{(h)}$, $g^{(h)}$ and ψ depend only on x_1 we can integrate over S and deduce

$$\int_0^L R^{(h)}\,{}^0E^{(h)} : (\bar{R}^{(h)})^T\bar{R}\psi_{,1} \otimes e_1 - h\bar{R}^Tg^{(h)} \cdot \psi dx_1 = 0$$

Since $R^{(h)} \to Id$ in $H^1(0,L;\mathbb{R}^3)$, $(\bar{R}^{(h)})^T\bar{R} \to Id$ in $\mathbb{R}^{3\times 3}$ and because ${}^0E^{(h)}$ weakly converges in $L^2(0,L;\mathbb{R}^{3\times 3})$, we obtain in the limit $h \to 0$

$$\int_0^L E^0e_1 \cdot \psi_{,1}dx_1 = \int_0^L E^0 : \psi_{,1} \otimes e_1 dx_1 = 0.$$

The second integral vanishes as $g^{(h)}$ is bounded in $L^2(0,L;\mathbb{R}^3)$.

Moreover, we obtain with (4.15) that

$$\int_S \big(E(x_1,x')e_2 \big| E(x_1,x')e_3\big) : \nabla_{x'}\sigma(x')dx' = 0$$

for all $\sigma \in H^1(S;\mathbb{R}^3)$. Hence choosing $\sigma(x') := x_k e_l$ for $k = 2, 3$ and $l = 1, \dots, 3$ we obtain

$$\big(E^0(x_1)e_2 \big| E^0(x_1)e_3\big) = 0 \quad \text{ for a.e. } x_1 \in (0, L).$$

The symmetry property of E implies that E^0 is symmetric and hence

$$E^0(x_1) = \begin{pmatrix} E^0_{11}(x_1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.16}$$

Step 4: (Symmetry inequality of $E^{(h)}$). From frame invariance of W we obtain that $DW(F)F^T$ is symmetric. Hence

$$\begin{aligned} E^{(h)} - (E^{(h)})^T &= \frac{1}{h^2}DW(Id + h^2G^{(h)}) - \frac{1}{h^2}DW(Id + h^2G^{(h)})^T \\ &= -DW(Id + h^2G^{(h)})(G^{(h)})^T + G^{(h)}DW(Id + h^2G^{(h)})^T \\ &= -h^2\big(E^{(h)}(G^{(h)})^T - G^{(h)}(E^{(h)})^T\big) \end{aligned}$$

leading to

$$\|E^{(h)} - (E^{(h)})^T\|_{L^1(\Omega)} \leq Ch^2. \tag{4.17}$$

Step 5: (Euler-Lagrange equations for first moments of E). We test (4.13) with $\varphi \in H^1_{per}(\Omega;\mathbb{R}^3)$, where

$$\varphi(x) := \bar{R}^{(h)}\phi(x_1)e_k$$

for some $\phi \in H^1_{per}(0, L)$ and $k = 2, 3$. This leads to

$$\int_\Omega R^{(h)}E^{(h)} : \phi_{,1}e_k \otimes e_1 - h(\bar{R}^{(h)})^T g^{(h)} \cdot \phi e_k dx = 0.$$

From the definition of $A^{(h)}$, namely

$$A^{(h)}(s) = \frac{1}{h^{\alpha-2}}(R^{(h)}(s) - Id) \quad \text{for all } s \in [0, L]$$

we can deduce the identity $R^{(h)}E^{(h)} = hA^{(h)}E^{(h)} + E^{(h)}$. Inserting the identity, dividing by h and integration over S leads to

$$\int_0^L \left(\frac{1}{h}\,{}^0E^{(h)} : \phi_{,1}e_k \otimes e_1 + A^{(h)}\,{}^0E^{(h)} : \phi_{,1}e_k \otimes e_1 - (\bar{R}^{(h)})^T g^{(h)} \cdot \phi e_k\right)dx = 0.$$

Hence, as $A^{(h)} \rightharpoonup A$ weakly in $H^1(0, L;\mathbb{R}^{3\times3})$, due to property (e) of Theorem 3.2.4, we have $A^{(h)}\,{}^0E^{(h)} \rightharpoonup AE^0$ weakly in $L^2(\Omega,\mathbb{R}^{3\times3})$. Moreover, $(\bar{R}^{(h)})^T g^{(h)} \to \bar{R}^T g$ strongly in $L^2(0, L;\mathbb{R}^3)$. Therefore we obtain

$$\int_0^L \frac{1}{h}\,{}^0E^{(h)}_{k1}\phi_{,1}dx_1 \to \int_0^L \Big(-(AE^0)_{k1}\phi_{,1} + (\bar{R}_{k2}g_2 + \bar{R}_{k3}g_3)\phi\Big)dx_1.$$

Hence, with the structure of A and E^0

$$(AE^0)_{21} = v_{2,1}E^0_{11} \quad \text{and} \quad (AE^0)_{31} = v_{3,1}E^0_{11}$$

we arrive at

$$\int_0^L \frac{1}{h}\,{}^0E^{(h)}_{21}\phi_{,1}dx_1 \to \int_0^L \Big(-v_{2,1}E^0_{11}\phi_{,1} + (\bar{R}_{22}g_2 + \bar{R}_{23}g_3)\phi\Big)dx_1 \tag{4.18}$$

$$\int_0^L \frac{1}{h}\,{}^0E^{(h)}_{31}\phi_{,1}dx_1 \to \int_0^L \Big(-v_{3,1}E^0_{11}\phi_{,1} + (\bar{R}_{22}g_2 + \bar{R}_{23}g_3)\phi\Big)dx_1. \tag{4.19}$$

for all $\phi \in H^1_{per}(0,L)$. Choose now $\rho^{(h)} \in C^1([0,L])$ such that $\mathrm{supp}(\rho^{(h)}) \subset [L-\sqrt{h}, L]$, $\rho(L) = 1$ and $|\rho^{(h)}_{,1}| \le C/\sqrt{h}$. Then we can use as a test function $\varphi_k \in H^1_{per}(\Omega;\mathbb{R}^3)$ defined by

$$\varphi_k(x) := x_k\bar{R}^{(h)}\Phi^{(h)}(x_1)\psi(x_1)e_1$$

for $k = 2,\ 3$, $\psi \in H^1_{per}(0,L)$ and $\Phi^{(h)} := R^{(h)} + \rho^{(h)}(R^{(h)}(0) - R^{(h)}(L)) \in H^1_{per}(0,L;\mathbb{R}^{3\times 3})$. Then (4.13) reduces to

$$\int_\Omega R^{(h)}E^{(h)} : \nabla_h\Big(x_k\Phi^{(h)}(x_1)\psi(x_1)\Big)dx_1 = 0$$

as the second summand is zero due to (2.4). Calculating the scaled gradient leads to

$$\begin{aligned}\int_\Omega \Big(\frac{1}{h}E^{(h)} : \psi e_1\otimes e_k + x_kE^{(h)} : \psi_{,1}e_1\otimes e_1 + x_kR^{(h)}E^{(h)} : \Phi^{(h)}_{,1}\psi e_1\otimes e_1\Big)dx \\ + \int_\Omega R^{(h)}E^{(h)} : \frac{1}{h}\rho^{(h)}\psi(R^{(h)}(0) - R^{(h)}(L))e_1\otimes e_k dx \\ + \int_\Omega R^{(h)}E^{(h)} : \frac{1}{h}\rho^{(h)}\psi_{,1}(R^{(h)}(0) - R^{(h)}(L))e_1\otimes e_1 dx = 0\end{aligned}$$

The definition of $\Phi^{(h)}$ and integration over S leads to

$$\begin{aligned}\int_\Omega \Big(\frac{1}{h}\,{}^0E^{(h)}_{1k}\psi + {}^kE^{(h)}_{11}\psi_{,1} + R^{(h)}\,{}^kE^{(h)} : R^{(h)}_{,1}\psi e_1\otimes e_1 \\ + R^{(h)}\,{}^kE^{(h)} : \rho^{(h)}_{,1}(R^{(h)}(0) - R^{(h)}(L))\psi e_1\otimes e_1\Big)dx_1 \\ + \int_\Omega R^{(h)}E^{(h)} : \frac{1}{h}\rho^{(h)}\psi(R^{(h)}(0) - R^{(h)}(L))e_1\otimes e_k dx \\ + \int_\Omega R^{(h)}E^{(h)} : \frac{1}{h}\rho^{(h)}\psi_{,1}(R^{(h)}(0) - R^{(h)}(L))e_1\otimes e_1 dx = 0\end{aligned}$$

Then it follows for the right hand side

$$\begin{aligned}\int_\Omega \Big(R^{(h)}E^{(h)} : \frac{1}{h}\rho^{(h)}\psi(R^{(h)}(0) - R^{(h)}(L))e_1\otimes e_k \\ + R^{(h)}E^{(h)} : \frac{1}{h}\rho^{(h)}\psi_{,1}(R^{(h)}(0) - R^{(h)}(L))e_1\otimes e_1\Big)dx \to 0\end{aligned}$$

as $\rho^{(h)}$ is bounded and $|R^{(h)}(0) - R^{(h)}(L)| \le Ch^{\frac{3}{2}}$. With $\|R^{(h)}_{,1}\|_{L^2(0,L)} \le Ch$ and the boundedness of $R^{(h)}\,{}^kE^{(h)}$ in $L^2(0,L)$ we obtain

$$\int_\Omega R^{(h)}\,{}^kE^{(h)} : R^{(h)}_{,1}\psi e_1\otimes e_1 dx_1 \to 0.$$

Moreover, because of $|\rho^{(h)}_{,1}| \leq C/\sqrt{h}$ and $|R^{(h)}(0) - R^{(h)}| \leq Ch^{\frac{3}{2}}$

$$\int_0^L R^{(h)}\,{}^kE^{(h)} : \rho^{(h)}_{,1}(R^{(h)}(0) - R^{(h)}(L))\psi e_1 \otimes e_1 dx_1 \to 0.$$

This leads to

$$\int_0^L \frac{1}{h}\,{}^0E^{(h)}_{1k}\psi dx_1 \to -\int_0^L E^k_{11}\psi_{,1} dx_1. \tag{4.20}$$

Using (4.17) it follows

$$\int_0^L \Big(\frac{1}{h}\,{}^0E^{(h)}_{k1} - \frac{1}{h}\,{}^0E^{(h)}_{1k}\Big)\psi_{,1} dx \to 0 \quad \text{for } k = 2,\ 3.$$

for all $\psi \in H^1_{per}(\Omega)$. Hence combing (4.18), (4.19) and (4.20) leads to

$$\int_0^L \Big(\frac{1}{h}\,{}^0E^{(h)}_{21} - \frac{1}{h}\,{}^0E^{(h)}_{12}\Big)\phi_{,1} dx \to \int_0^L \Big(- v_{2,1}E^0_{11}\phi_{,1} + (\bar{R}_{22}g_2 + \bar{R}_{23}g_3)\phi + E^2_{11}\phi_{,11}\Big) dx_1 = 0$$

and

$$\int_0^L \Big(\frac{1}{h}\,{}^0E^{(h)}_{31} - \frac{1}{h}\,{}^0E^{(h)}_{13}\Big)\phi_{,1} dx \to \int_0^L \Big(- v_{3,1}E^0_{11}\phi_{,1} + (\bar{R}_{32}g_2 + \bar{R}_{33}g_3)\phi + E^3_{11}\phi_{,11}\Big) dx_1 = 0$$

for $h \to 0$ and all $\phi \in H^1_{per}(0, L)$. Consider again the Euler-Lagrange equations, in a slightly modified but equivalent form:

$$\int_\Omega R^{(h)}E^{(h)} : \nabla_h\phi - h\bar{R}^{(h)}g^{(h)} \cdot \phi dx = 0 \quad \text{for all } \phi \in H^1_{per}(\Omega; \mathbb{R}^3).$$

We now plug

$$\phi(x) = x_2\Phi^{(h)}(x_1)\psi(x_1)e_3 \quad \text{and} \quad \phi(x) = x_3\Phi^{(h)}(x_1)\psi(x_1)e_2$$

in the equation, respectively. This leads to

$$\begin{aligned} 0 &= \int_\Omega R^{(h)}E^{(h)} : \nabla_h\big(x_2\Phi^{(h)}(x_1)\psi(x_1)e_3\big) dx \\ &= \int_\Omega \frac{1}{h}E^{(h)} : \psi(x_1)e_3 \otimes e_2 + x_2E^{(h)} : \psi_{,1}e_3 \otimes e_1 + x_2R^{(h)}E^{(h)} : \Phi^{(h)}_{,1}\psi e_3 \otimes e_1 dx \\ &= \int_0^L \frac{1}{h}\,{}^0E^{(h)}_{32}\psi + {}^2E^{(h)}_{31}\psi_{,1} + R^{(h)}\,{}^2E^{(h)} : \Phi^{(h)}_{,1}\psi e_3 \otimes e_1 dx_1. \end{aligned}$$

The force term vanishes as $\Psi^{(h)}$, $\psi^{(h)}$ and $g^{(h)}$ do not depend on x' and

$$\int_S x_k dx' = 0 \quad \text{for } k = 2,\ 3.$$

As above we obtain that the last summand tends to zero and we are led to

$$\int_0^L \frac{1}{h}\,{}^0E^{(h)}_{32}\psi dx_1 \to \int_0^L E^2_{31}\psi_{,1} dx_1.$$

Analogously we obtain

$$\int_0^L \frac{1}{h}\,{}^0E^{(h)}_{23}\psi dx_1 \to \int_0^L E^3_{21}\psi_{,1} dx_1.$$

Hence with (4.17)

$$\int_0^L \big(E_{31}^2 - E_{21}^3\big)\psi_{,1} dx_1 = 0 \qquad \text{for all } \psi \in H^1_{per}(0,L).$$

STEP 6: (Optimality condition for $\bar{R}$). Lastly we want to derive (4.5), by exploiting the assumption that $y^{(h)}$ is a sequence of local minimizers. This implies that there exists $\delta_h > 0$ such that $\mathcal{E}^{(h)}(y^{(h)}) \leq \mathcal{E}^{(h)}(\varphi)$ for all $\varphi \in B_{\delta_h}(y^{(h)}) \subset H^1(\Omega;\mathbb{R}^3)$. Hence for all $\varphi \in B_{\delta_h}(y^{(h)})$ such that $\varphi - \mathrm{Id}_h \in H^1_{per}(\Omega;\mathbb{R}^3)$ it holds

$$\int_\Omega W(y^{(h)}) - f^{(h)} \cdot (y^{(h)} - x^{(h)}) dx \leq \int_\Omega W(\nabla_h \varphi) - f^{(h)} \cdot (\varphi - x^{(h)}) dx. \tag{4.21}$$

Now we can choose for the testfunction φ the following

$$\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix} y^{(h)}$$

for $t \in (-\varepsilon_h, \varepsilon_h)$ and $\varepsilon_h > 0$ such that

$$2\sqrt{1-\cos(t)} < \delta_h \qquad \text{for } t \in (-\varepsilon_h, \varepsilon_h).$$

Then we obtain due to frame invariance of W and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix} \in SO(3)$$

that

$$-\int_\Omega f^{(h)} \cdot (y^{(h)} - x^{(h)}) dx \leq -\int_\Omega f^{(h)} \cdot (\varphi^{(h)} - x^{(h)})$$

for all $t \in (\varepsilon_t, \varepsilon_h)$. Taking the derivative with respect to t at 0 leads to

$$0 = -\int_\Omega f^{(h)} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} (y^{(h)} - x^{(h)}) dx.$$

as the right hand side of (4.21) exhibits a local minimum at $t = 0$. Dividing the equality by h^4 and using that $f^{(h)}$ depends only on x_1, we deduce

$$\begin{aligned} 0 &= -\int_0^L g^{(h)} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \bar{R}^{(h)} \frac{1}{h} \int_S \tilde{y}^{(h)} - x^{(h)} dx' dx_1 \\ &= -\int_0^L g^{(h)} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \bar{R}^{(h)} \begin{pmatrix} hu^{(h)} \\ v_2^{(h)} \\ v_3^{(h)} \end{pmatrix} dx_1 \to \int_0^L \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{R}' \begin{pmatrix} f_2 \\ f_3 \end{pmatrix} \cdot \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} dx_1 \end{aligned}$$

where we have used the definition of $(u^{(h)}, v_2^{(h)}, v_3^{(h)})$ and (2.4). □

Remark 4.0.5. The statement of Theorem 4.0.4 holds true if the assumption $y^{(h)}$ to be a sequence of local minimizers is substituted by local maxima. The proof does not change except of step 6 where one obtains reversed inequalities during the testing.

5

Large Time Existence for Non-linear Problem

This chapter is devoted to the study of large time existence for solutions to a non-linear wave equation originating from finite elasticity theory. We show that for sufficiently small diameter of the rod Ω_h the strong solution for a naturally scaled wave equation with periodic boundary conditions admits a solution for large times. For the latter to hold we use well prepared initial data.
In this chapter we consider the case $\alpha \geq 4$, now in the dynamical setting. The basic equations from continuum mechanics emerge from the balance of linear and angular momentum and the balance of energy. As we have seen in Section 2.2 the evolution preserves the total energy

$$\int_\Omega \left(\frac{|\partial_t y|^2}{2} + W(\nabla_h y) - f^h \cdot y \right) dx$$

if f^h is independent of t and the Piola-Kirchhoff stress satisfies appropriate boundary conditions. According to the scaling behaviour of $\tilde{y}^{(h)}$ in Chapter 3 we expect

$$y_1 - x_1 \sim h^{\alpha-1}, \quad \begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \sim h^{\alpha-2} \qquad \text{for } \alpha \geq 4.$$

Moreover, Chapter 3 suggests to assume $f^h \sim h^\alpha$ and $f_1 \equiv 0$. In order to balance the kinetic and elastic part of the total energy we rescale the time via $\tau = ht$. Hence, we obtain

$$E^{(h)}(y) = h^{2\alpha-2} \int_\Omega \frac{1}{h^{2\alpha-4}} \frac{|\partial_\tau y|^2}{2} + \frac{1}{h^{2\alpha-2}} W(\nabla_h y) - \frac{1}{h^\alpha} f^h \cdot \frac{1}{h^{\alpha-2}} y dx.$$

This leads to the scaled evolution equation with $g^h := \frac{1}{h^\alpha} f^h$

$$\partial_\tau^2 y - \frac{1}{h^2} \operatorname{div}_h(DW(\nabla_h y)) = h^{\alpha-2} g^h \quad \text{ in } \Omega \times [0, T)$$

where $g^h \sim 1$ for $h \to 0$. Furthermore, we assume homogeneous Neumann boundary conditions on the outer surface, periodicity on the end faces of Ω and suitable initial conditions.

5.1 Main Result

In the following we will present the main result of the thesis, large time existence for the non-linear wave equation. For this we introduce the precise problem and notation. We will work in terms of relative deformations, therefore the system of equations we consider is

$$\partial_t^2 u_h - \frac{1}{h^2}\operatorname{div}_h\left(D\tilde{W}(\nabla_h u_h)\right) = h^{1+\theta} f_h \quad \text{in } \Omega\times[0,T) \tag{5.1}$$

$$D\tilde{W}(\nabla_h u_h)\nu|_{(0,L)\times\partial S} = 0 \tag{5.2}$$

$$u_h \text{ is } L\text{-periodic w.r.t. } x_1 \tag{5.3}$$

$$(u_h, \partial_t u_h)|_{t=0} = (u_{0,h}, u_{1,h}) \tag{5.4}$$

where $\tilde{W}(F) = W(Id+F)$ for all $F\in\mathbb{R}^{3\times3}$, $T>0$ and for convenience $\theta = \alpha - 3 \geq 1$. As an abbreviation we denote in the following $z=(t,x_1)$. We assume $W\colon \mathbb{R}^{3\times3}\to[0,\infty]$ to satisfy the conditions:

(i) $W\in C^\infty(B_\delta(Id);[0,\infty))$ for some $\delta>0$;

(ii) W is frame-invariant, i.e. $W(RF)=W(F)$ for all $F\in\mathbb{R}^{3\times3}$ and $R\in SO(3)$;

(iii) there exists $c_0>0$ such that $W(F)\geq c_0 \operatorname{dist}^2(F,SO(3))$ for all $F\in\mathbb{R}^{3\times3}$ and $W(R)=0$ for every $R\in SO(3)$.

Then for $D^2W(Id)$ the Legendre-Hadamard condition and Lemma 2.3.6 holds. The assumptions on W are slightly stronger than in Section 2.3. The main result of the thesis is:

Theorem 5.1.1. *Let $\theta\geq 1$, $0<T<\infty$, $f_h\in W_1^3(0,T;L^2(\Omega))\cap W_1^1(0,T;H^2_{per}(\Omega))$, $h\in(0,1]$ and $u_{0,h}\in H^4_{per}(\Omega)$, $u_{1,h}\in H^3_{per}(\Omega)$ such that*

$$D\tilde{W}(\nabla_h u_{0,h})\nu|_{(0,L)\times\partial S} = D^2\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{1,h}]\nu|_{(0,L)\times\partial S} = 0,$$
$$(D^2\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{2,h}] + D^3\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{1,h},\nabla_h u_{1,h}])\nu|_{(0,L)\times\partial S} = 0,$$

where

$$\begin{aligned}
u_{2,h} &= h^{1+\theta} f_h|_{t=0} + \frac{1}{h^2}\operatorname{div}_h(D\tilde{W}(\nabla_h u_{0,h}))\\
u_{3,h} &= h^{1+\theta}\partial_t f_h|_{t=0} + \frac{1}{h^2}\operatorname{div}_h(D^2\tilde{W}(\nabla_h u_{0,h})\nabla_h u_{1,h})\\
u_{4,h} &= h^{1+\theta}\partial_t^2 f_h|_{t=0} + \frac{1}{h^2}\operatorname{div}_h(D^2\tilde{W}(\nabla_h u_{0,h})\nabla_h u_{2,h})\\
&\quad + \frac{1}{h^2}\operatorname{div}_h(D^3\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{1,h},\nabla_h u_{1,h}]).
\end{aligned}$$

Moreover we assume for the initial data

$$\left\|\frac{1}{h}\varepsilon_h(u_{0,h})\right\|_{H^2} + \max_{k=0,1,2}\left\|\left(\frac{1}{h}\varepsilon_h(u_{1+k,h}), \partial_{x_1}\frac{1}{h}\varepsilon_h(u_{k,h}), u_{2+k,h}\right)\right\|_{H^{2-k}} \leq Mh^{1+\theta} \tag{5.5}$$

$$\left\|\nabla_h^2 u_{0,h}\right\|_{H^1} + \max_{k=0,1}\left\|\left(\nabla_h^2 u_{1+k,h}, \partial_{x_1}\nabla_h^2 u_{k,h}\right)\right\|_{H^{1-k}} \leq Mh^{1+\theta} \tag{5.6}$$

$$\max_{k=0,1,2,3}\left|\frac{1}{h}\int_\Omega u_{k,h}\cdot x^\perp dx\right| \leq Mh^{1+\theta}. \tag{5.7}$$

and for the right hand side

$$\max_{|\alpha|\leq 1}\left(\|\partial_z^\alpha f_h\|_{W_1^2(L^2)} + \|\partial_z^\alpha f_h\|_{W_\infty^1(L^2)\cap W_1^1(H^{0,1})} + \|\partial_z^\alpha f_h\|_{L^\infty(H^1)}\right) \leq M \tag{5.8}$$

$$\max_{\sigma=0,1,2}\left\|\frac{1}{h}\int_\Omega \partial_t^\sigma f_h\cdot x^\perp dx\right\|_{C^0(0,T)} \leq M \tag{5.9}$$

uniformly in $0<h\leq 1$.

Then there exists $h_0 \in (0,1]$ *and* $C > 0$ *depending only on* M *and* T *such that for every* $h \in (0, h_0]$ *there is a unique solution* $u_h \in \bigcap_{k=0}^4 C^k(0,T; H^{4-k}_{per})$ *of* (5.1)–(5.4) *satisfying*

$$\max_{\substack{|\alpha|\le 1, |\beta|\le 2, |\gamma|\le 1\\ \sigma=0,1,2}} \left(\left\| \left(\partial_t^2 \partial_t^\sigma u_h, \nabla_{x,t}^\beta \frac{1}{h} \varepsilon_h(\partial_z^\alpha u_h), \nabla_{x,t}^\gamma \nabla_h^2 \partial_z^\alpha u_h \right) \right\|_{C^0(0,T,L^2)} + \left\| \frac{1}{h} \int_\Omega \partial_z^{\alpha+\beta} u_h \cdot x^\perp dx \right\|_{C^0(0,T)} \right) \le Ch^{1+\theta} \tag{5.10}$$

uniformly in $0 < h \le h_0$.

For fixed $h > 0$ short time existence is already known via methods the of [Koc93], if the fixed time of existence is replaced by some h dependent maximal time $T(h) > 0$. Hence only the uniform estimates for u_h and that T does not depend on h has to be shown. In detail, one obtains from [Koc93]:

Theorem 5.1.2. *Assume the assumption of Theorem 5.1.1 are valid. Then for any* $0 < h \le 1$ *there exists a neighbourhood* $U_h \subset \mathbb{R}^{3\times 3}$ *of* 0 *and some* $T_{max}(h) > 0$ *such that* (5.1)–(5.4) *has a unique solution* $u_h \in \bigcap_{k=0}^4 C^k([0, T_{max}(h)); H^{4-k}_{per})$. *If* $T_{max}(h) < \infty$, *then either* $\{\nabla_h u_h(x,t) : x \in \overline{\Omega},\ t \in [0, T_{max}(h))\}$ *is not precompact in* U_h *or*

$$\lim_{t \to T_{max}(h)} \int_0^t \|\nabla_{x,t}^2 u_h(s)\|_{L^\infty(\Omega)} ds = \infty.$$

Remark 5.1.3. We will give a more precise explanation on how the results of [Koc93] are applied to our situtation in the appendix. At this point however we want to mention that the neighbourhood U_h can be chosen as

$$U_h := \left\{ A \in \mathbb{R}^{3\times 3} \ : \ \left| \left(A, \frac{1}{h} \operatorname{sym}(A) \right) \right| \le \varepsilon h \right\}$$

where $\varepsilon > 0$ is sufficiently small. With this it follows that as long as $\nabla_h u_h(x,t) \in U_h$ is satisfied the necessary weak coercivity holds, cf. Section 5.2.

The strategy for proving the main result is as follows. In a first step we will derive precise estimates for solutions of the linearisation of (5.1)–(5.4) under the assumption that u_h is small in appropriate norms. To this end we use using the natural boundary conditions, differentiate tangentially and utilize the central estimate

$$\frac{1}{h^2} \left(D^2 \tilde{W}(\nabla_h u_h) \nabla_h w, \nabla_h w \right)_{L^2(\Omega)} \ge \frac{c_0}{2} \left\| \frac{1}{h} \varepsilon_h(w) \right\|^2_{L^2(\Omega)} - CR \left| \frac{1}{h} \int_\Omega w \cdot x^\perp dx \right|^2, \tag{5.11}$$

proven below. By differentiating (5.1) in time and x_1 we obtain that the respective derivative of u_h solves now the linearised system. Applying then the results of the first step we can deduce with the balance of angular momentum, that the solutions are uniformly bounded in h if the initial values and external force are sufficiently small.

5.2 Uniform Estimates for Linearised System

The linearised system for (5.1)–(5.4) is given by

$$\partial_t^2 w - \frac{1}{h^2} \operatorname{div}_h (D^2 \tilde{W}(\nabla_h u_h) \nabla_h w) = f \quad \text{in } \Omega \times [0,T) \tag{5.12}$$

$$D^2 \tilde{W}(\nabla_h u)[\nabla_h w]\nu = 0 \quad \text{on } (0,L) \times \partial S \times [0,T) \tag{5.13}$$

$$w \text{ is } L\text{-periodic in } x_1 \text{ coordinate} \tag{5.14}$$

$$(w, \partial_t w)|_{t=0} = (w_0, w_1). \tag{5.15}$$

We want to show h-independent estimates for solutions of the linearised system. For this we assume that u_h satisfies for $0 < h \le 1$

$$\sup_{|\alpha|\le 1, k=0,1,2} \left(\left\| \left(\nabla^k_{x,t} \frac{1}{h} \varepsilon_h(\partial_z^\alpha u_h), \nabla^k_{x,t} \nabla_h \partial_z^\alpha u_h \right) \right\|_{C^0([0,T];L^2(\Omega))} \right.$$
$$\left. + \left\| \frac{1}{h} \int_\Omega \partial_z^{\alpha+\beta} u_h \cdot x^\perp dx \right\|_{C^0([0,T])} \right) \le Rh \tag{5.16}$$

where $R \in (0, R_0]$, with R_0 chosen later appropriately small. With this it follows that u_h satisfies

$$\|\nabla_h u_h\|_{C^0([0,T];H^2_h(\Omega))} + \left\| \left(\frac{1}{h} \varepsilon_h(u_h), \nabla_h u_h \right) \right\|_{C^0([0,T];L^\infty(\Omega))} \le CRh \tag{5.17}$$

and

$$\sup_{|\alpha|\le 2} \left(\left\| \left(\frac{1}{h} \varepsilon_h(\partial_z^\alpha u_h), \nabla_h \partial_z^\alpha u_h \right) \right\|_{C^0([0,T];H^1(\Omega))} + \left\| \frac{1}{h} \int_\Omega \partial_z^\alpha u_h \cdot x^\perp dx \right\|_{C^0([0,T])} \right) \le CRh. \tag{5.18}$$

Here $C > 0$ is independent of h, R and R_0. In the following we assume that R_0 is chosen so small such that we can evaluate $D^2\tilde{W}$ at $\nabla_h u_h$, i.e. $\tilde{W} \in C^\infty(\overline{B_{CR_0}(0)})$ and Lemma 2.3.6 is applicable.
Using Corollary 2.3.7, we obtain

$$\begin{aligned} \left| \frac{1}{h^2} \int_0^1 \left(D^3\tilde{W}(\tau\nabla_h u_h)[\nabla_h u_h, \nabla_h v], \nabla_h w \right)_{L^2(\Omega)} d\tau \right| &\le \frac{C}{h} \|\nabla_h u_h\|_{H^2_h(\Omega)} \|\nabla_h v\|_{L^2_h(\Omega)} \|\nabla_h w\|_{L^2_h(\Omega)} \\ &\le CR \|\nabla_h v\|_{L^2_h(\Omega)} \|\nabla_h w\|_{L^2_h(\Omega)}. \end{aligned}$$

uniformly in $v,\ w$, $0 \le t \le T$ and h. Thus it follows

$$\begin{aligned} &\frac{1}{h^2} \left(D^2\tilde{W}(\nabla_h u_h)\nabla_h v, \nabla_h v \right)_{L^2(\Omega)} \\ &= \frac{1}{h^2} (D^2\tilde{W}(0)\nabla_h v, \nabla_h v)_{L^2(\Omega)} + \frac{1}{h^2} \int_0^1 (D^3\tilde{W}(\tau\nabla_h u_h)[\nabla_h u_h, \nabla_h v], \nabla_h v)_{L^2(\Omega)} d\tau \\ &\ge \frac{c_0}{h^2} \|\varepsilon_h(v)\|^2_{L^2(\Omega)} - \left| \frac{1}{h^2} \int_0^1 (D^3\tilde{W}(\tau\nabla_h u_h)[\nabla_h u_h, \nabla_h v], \nabla_h v)_{L^2(\Omega)} d\tau \right| \\ &\ge c_0 \left\| \frac{1}{h} \varepsilon_h(v) \right\|^2_{L^2(\Omega)} - CR \|\nabla_h v\|^2_{L^2_h(\Omega)}. \end{aligned} \tag{5.19}$$

5.2.1 Existence Theory for Linear Problem with Fixed h

In order to solve the linearised problem in a weak sense we use the general theory of abstract hyperbolic equations. We define the Gelfand triple $V \hookrightarrow H \hookrightarrow V'$ by $V := H^1_{per}(\Omega; \mathbb{R}^3)$ and $H := L^2(\Omega; \mathbb{R}^3)$. Furthermore

$$L(t) \colon V \to V' \tag{5.20}$$

is defined by

$$(L(t)\phi, \psi)_{V',V} = \frac{1}{h^2} \int_\Omega D^2\tilde{W}(\nabla_h u_h(x,t))\nabla_h \phi : \nabla_h \psi dx \tag{5.21}$$

for all ϕ, $\psi \in V$.

Remark 5.2.1. Multiplying the equation (5.12) with $v \in V$ and integration over Ω we obtain

$$-\frac{1}{h^2} \int_\Omega \operatorname{div}_h(D^2\tilde{W}(\nabla_h u_h)\nabla_h w) \cdot v dx = \frac{1}{h^2} \int_\Omega D^2\tilde{W}(\nabla_h u_h)\nabla_h w : \nabla_h v dx$$

$$- \frac{1}{h^2} \int_{\partial\Omega} v \cdot D^2 \tilde{W}(\nabla_h u_h)[\nabla_h w, \nu] d\sigma(x)$$

and by (5.13) and (5.14)

$$\begin{aligned}
\frac{1}{h^2} \int_{\partial\Omega} v \cdot D^2 \tilde{W}(\nabla_h u_h)[\nabla_h w, \nu] d\sigma(x) &= \int_S \nabla_h v \cdot D^2 \tilde{W}(\nabla_h u_h)[\nabla_h w, \nu] d\sigma(x) \Big|_{x_1 = \{0, L\}} \\
&= \int_S (\nabla_h v \cdot D^2 \tilde{W}(\nabla_h u_h)[\nabla_h w, \nu_0]) \Big|_{x_1 = 0} - \int_S (\nabla_h v \cdot D^2 \tilde{W}(\nabla_h u_h)[\nabla_h w, \nu_0]) \Big|_{x_1 = L} = 0
\end{aligned}$$

where $\nu_0 = (-1, 0, 0)^T$ and because $\nu|_{x_1 = L} = (1, 0, 0) = -\nu_0$ holds.

The linear operator $L(t)$ defines a bilinear form $a(t; \phi, \psi) := (L(t)\phi, \psi)_H$ for which it follows

$$\begin{aligned}
|a(t, w, v)| &= \frac{1}{h^2} \left| \int_\Omega D^2 \tilde{W}(\nabla_h u_h(x, t)) \nabla_h w : \nabla_h v dx \right| \\
&\leq \frac{1}{h^2} \sup_{(x,t) \in \Omega \times [0,T]} |D^2 \tilde{W}(\nabla_h u_h(x, t))| \int_\Omega |\nabla_h v| \times |\nabla_h w| dx \\
&\leq \frac{C}{h^2} \|\nabla_h v\|_{L^2(\Omega)} \|\nabla_h w\|_{L^2(\Omega)} \leq \frac{C}{h^4} \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}
\end{aligned}$$

uniformly in $0 \leq t < T$, as $\tilde{W} \in C^2$ and u_h satisfies (5.17). Moreover as $D^2 \tilde{W}(\nabla_h u_h)$ is due to Schwarz's Theorem symmetric, it follows that

$$a(t; v, w) = a(t; w, v).$$

In order to show that $a(t; \cdot)$ is coercive we use Korn inequality

$$\begin{aligned}
(L(t)v, v)_H &\geq c_0 \left\| \frac{1}{h} \varepsilon_h(v) \right\|^2_{L^2(\Omega)} - CR \|\nabla_h v\|^2_{L^2_h(\Omega)} \\
&\geq (c_0 - CR) \left\| \frac{1}{h} \varepsilon_h(v) \right\|^2_{L^2(\Omega)} - CR \|\nabla_h v\|_{L^2(\Omega)} \\
&\geq (c_0 - CR) \|\nabla_h v\|^2_{L^2(\Omega)} - \frac{C}{h^2} \left| \int_\Omega v \cdot x^\perp dx \right|^2 \\
&\geq \frac{c_0}{2} \|\nabla_h v\|^2_{L^2(\Omega)} - \frac{C}{h^2} \|v\|^2_{L^2(\Omega)}
\end{aligned}$$

for R_0 sufficiently small. Since $\tilde{W}$ is smooth, $\nabla_h u_h \in C^1([0, T]; H^2(\Omega; \mathbb{R}^{3\times3}))$ and $\phi, \psi \in H^1_{per}(\Omega)$ it follows for

$$h(t, x) := D^2 \tilde{W}(\nabla_h u_h(x, t)) \nabla_h \phi : \nabla_h \psi$$

that $h(t, \cdot) \in L^1(\Omega)$ and $h(\cdot, x) \in C^1([0, T]; \mathbb{R})$ for all $x \in \Omega$ holds. To obtain the upper bound, we calculate $\partial_t h$:

$$\begin{aligned}
|\partial_t h(t, x)| &= |D^3 \tilde{W}(\nabla_h u_h)[\partial_t \nabla_h u_h, \partial_h \phi, \nabla_h \psi]| \\
&\leq \Big(\sup_{(t,x) \in [0,T] \times \overline{\Omega}} |D^3 \tilde{W}(\nabla_h u_h)| \Big) \Big(\sup_{(t,x) \in [0,T] \times \overline{\Omega}} |\partial_t \nabla_h u_h| \Big) |\nabla_h \phi(x)| |\nabla_h \psi(x)| \\
&\leq C |\nabla_h \phi(x)| |\nabla_h \psi(x)| \in L^1(\Omega)
\end{aligned}$$

With this it follows that $a(t; \phi, \psi) \in C^1([0, T]; \mathbb{R})$ and

$$\left| \frac{d}{dt} a(t; \phi, \psi) \right| \leq \frac{C}{h^2} \|\nabla_h \phi\|_{L^2(\Omega)} \|\nabla_h \psi\|_{L^2(\Omega)} \leq \frac{C}{h^4} \|\phi\|_{H^1(\Omega)} \|\psi\|_{H^1(\Omega)}$$

for all ϕ, $\psi \in H^1(\Omega)$. Using Theorem 29.1 in [Wlo87] for all $f \in L^2(0, T; H)$, $w_0 \in V$ and

$w_1 \in H$ there exists a solution $w \in L^2(0,T;V) \cap W_2^1(0,T;H) \cap W_2^2(0,T;V')$ in the following sense: w satisfies

$$\begin{aligned} \partial_t^2 w + L(t)w &= f(t) && \text{in } \mathcal{D}'(0,T;V') \\ (w, \partial_t w)|_{t=0} &= (w_0, w_1) && \text{in } V \times H. \end{aligned}$$

By Theorem 8.2 of [LM72] we deduce that

$$w \in C^0([0,T];V) \cap C^1([0,T];H)$$

holds if $f \in L^2(0,T;H)$, $w_0 \in V$ and $w_1 \in H$.

5.2.2 Uniform Estimates for the Linearised System

The structure of this technical subsection is that we will start with a general lemma providing a bound for derivatives in z of $D^2\tilde{W}(\nabla_h u_h)$ in the case that u_h satisfies (5.16). To obtain bounds on higher derivatives we investigate the statical problem and apply these subsequently to the dynamical equation. This approach leads to uniform estimates for the solution of the linearised system.

Lemma 5.2.2. *Assume that* (5.16) *holds,* $t \in [0,T]$ *and* $0 < R \le R_0$. *Then*

$$\left| \frac{1}{h^2} \left(\partial_z^\beta D^2\tilde{W}(\nabla_h u_h(t)) \nabla_h w, \nabla_h v \right)_{L^2(\Omega)} \right| \le CR \|\nabla_h w\|_{H_h^{|\beta|-1}(\Omega)} \|\nabla_h v\|_{L_h^2(\Omega)} \tag{5.22}$$

for $1 \le |\beta| \le 3$.

Proof: If $|\beta| = 1$, we obtain by (5.16) and (2.29)

$$\begin{aligned} \left| \frac{1}{h^2} \Big((\partial_z^\beta D^2\tilde{W}(\nabla_h u_h(t))) \nabla_h w, \nabla_h v \Big)_{L^2(\Omega)} \right| &= \left| \frac{1}{h^2} \Big(D^3\tilde{W}(\nabla_h u_h(t))[\partial_z^\beta \nabla_h u, \nabla_h w], \nabla_h v \Big)_{L^2(\Omega)} \right| \\ &\le \frac{C}{h} \|\nabla_h \partial_z^\beta u_h\|_{H_h^2(\Omega)} \|\nabla_h w\|_{L_h^2(\Omega)} \|\nabla_h v\|_{L_h^2(\Omega)} \\ &\le CR \|\nabla_h w\|_{L_h^2(\Omega)} \|\nabla_h v\|_{L_h^2(\Omega)} \end{aligned}$$

If $|\beta| = 2$, it follows for j, $k \in \{0,1\}$ chosen correctly

$$\partial_z^\beta D^2\tilde{W}(\nabla_h u_h) = D^3\tilde{W}[\partial_z^\beta \nabla_h u_h] + D^4\tilde{W}(\nabla_h u_h)[\partial_{z_j} \nabla_h u_h, \partial_{z_k} \nabla_h u_h]. \tag{5.23}$$

For the first term we use (2.30)

$$\left| \frac{1}{h^2} \Big(D^3\tilde{W}(\nabla_h u_h)[\partial_z^\beta \nabla_h u_h, \nabla_h w], \nabla_h v \Big)_{L^2(\Omega)} \right| \le CR \|\nabla_h w\|_{H_h^1(\Omega)} \|\nabla_h v\|_{L_h^2(\Omega)}$$

and as $D^4\tilde{W}(\nabla_h u_h) \in C^0(\overline{\Omega}, \mathcal{L}^4(\mathbb{R}^{3\times 3}))$

$$\begin{aligned} \frac{1}{h^2} \int_\Omega \big| D^4\tilde{W}(\nabla_h u_h)[\partial_{z_j} \nabla_h u_h, \partial_{z_k} \nabla_h u_h, \nabla_h w, \nabla_h v] \big| dx \\ \le \frac{C}{h^2} \int_\Omega |\partial_{z_j} \nabla_h u_h| |\partial_{z_k} \nabla_h u_h| |\nabla_h w| |\nabla_h v| dx \\ \le CR \int_\Omega |\nabla_h w| |\nabla_h v| dx \le CR \|\nabla_h w\|_{L^2(\Omega)} \|\nabla_h v\|_{L^2(\Omega)} \end{aligned}$$

Finally for $|\beta| = 3$ and j, k and $l \in \{0,1\}$ such that $\partial_z^\beta = \partial_{z_j} \partial_{z_k} \partial_{z_l}$ chosen

$$\begin{aligned} \partial_z^\beta D^2\tilde{W}(\nabla_h u_h) = D^3\tilde{W}(\nabla_h u_h)[\partial_z^\beta \nabla_h u_h] &+ D^4\tilde{W}(\nabla_h u_h)[\partial_{z_l} \partial_{z_j} \nabla_h u_h, \partial_{z_k} \nabla_h u_h] \\ &+ D^4\tilde{W}(\nabla_h u_h)[\partial_{z_j} \nabla_h u_h, \partial_{z_l} \partial_{z_k} \nabla_h u_h] \end{aligned}$$

$$+ D^4\tilde{W}(\nabla_h u_h)[\partial_{z_k}\partial_{z_j}\nabla_h u_h, \partial_{z_l}\nabla_h u_h]$$
$$+ D^5\tilde{W}(\nabla_h u_h)[\partial_{z_j}\nabla_h u_h, \partial_{z_k}\nabla_h u_h, \partial_{z_l}\nabla_h u_h]$$

The fifth order term can be done in the same way as in the case of $|\beta| = 2$. As $\partial_{z_l}\partial_{z_j}\nabla_h u_h \in H^1(\Omega) \hookrightarrow L^4(\Omega)$ and $\partial_{z_l}\nabla_h u \in H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, it follows with Hölder inequality that

$$\begin{aligned}
&\frac{1}{h^2}\int_\Omega \left|D^4\tilde{W}(\nabla_h u_h)[\partial_{z_l}\partial_{z_j}\nabla_h u_h, \partial_{z_l}\nabla_h u_h, \nabla_h w, \nabla_h v]\right|dx \\
&\qquad\leq \frac{1}{h^2}\int_\Omega |\partial_{z_l}\partial_{z_j}\nabla_h u_h||\partial_{z_l}\nabla_h u_h||\nabla_h w||\nabla_h v|dx \\
&\qquad\leq \frac{C}{h^2}\|\partial_{z_l}\partial_{z_j}\nabla_h u_h\|_{L^4(\Omega)}\|\partial_{z_l}\nabla_h u_h\|_{L^\infty(\Omega)}\|\nabla_h w\|_{L^4(\Omega)}\|\nabla_h v\|_{L^2(\Omega)} \\
&\qquad\leq CR\|\nabla_h w\|_{H^1(\Omega)}\|\nabla_h v\|_{L^2(\Omega)}.
\end{aligned}$$

For the last term we use (2.29)

$$\left|\frac{1}{h^2}\Big(D^3\tilde{W}(\nabla_h u_h)[\partial_z^\beta\nabla_h u_h]\nabla_h w, \nabla_h v\Big)_{L^2(\Omega)}\right| \leq CR\|\nabla_h w\|_{H^2_h(\Omega)}\|\nabla_h v\|_{L^2_h(\Omega)}. \qquad \square$$

Lemma 5.2.3 (Basic Inequality)**.** *Let $0 < T < \infty$, $h \in (0,1]$, $0 < R \leq R_0$ be given, where R_0 is chosen small enough, but independent of h. Furthermore, assume that u_h satisfies* (5.16)*. For every $f \in L^1(0,T;L^2(\Omega))$, $w_0 \in H^1_{per}(\Omega)$ and $w_1 \in L^2(\Omega)$ there exists a unique solution $w \in C^0(0,T;H^1_{per}(\Omega)) \cap C^1(0,T;L^2(\Omega))$ of the system* (5.12)–(5.15) *satisfying*

$$\begin{aligned}
\left\|\Big(\partial_t w, \frac{1}{h}\varepsilon_h(w)\Big)\right\|^2_{C^0(0,T;L^2)} \leq C_L\Bigg(&\|w_1\|^2_{L^2(\Omega)} + |A_0| + \|f\|^2_{L^1(0,T;L^2(\Omega))} \\
&+ (1+T)R\left\|\frac{1}{h}\int_\Omega w\cdot x^\perp dx\right\|^2_{C^0(0,T)}\Bigg)
\end{aligned}$$

where $C_L > 0$ is independent of h and

$$A_0 = \frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_h|_{t=0})\nabla_h w_0, \nabla_h w_0\Big)_{L^2(\Omega)}.$$

Proof: The existence of a unique solution follows directly from Section 5.2.1. Hence only the uniform estimate has to be shown. For R_0 sufficiently small we deduce from (5.19)

$$\begin{aligned}
\frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h w\Big)_{L^2(\Omega)} &\geq c_0\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2_{L^2(\Omega)} - CR\|\nabla_h w\|^2_{L^2_h(\Omega)} \\
&\geq (c_0 - CR)\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2_{L^2(\Omega)} - CR\left|\frac{1}{h}\int_\Omega w\cdot x^\perp dx\right|^2 \\
&\geq \frac{c_0}{2}\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2_{L^2(\Omega)} - CR\left|\frac{1}{h}\int_\Omega w\cdot x^\perp dx\right|^2
\end{aligned}$$

Testing the linear equation with $\partial_t w$ and using the boundary conditions

$$\int_{\partial\Omega} w\cdot D^2\tilde{W}(\nabla_h u_h)[\nabla_h w]\nu d\sigma(x) = 0$$

we obtain

$$\frac{d}{dt}\frac{1}{2}\Big(\|\partial_t w\|^2_{L^2(\Omega)} + \frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h w\Big)_{L^2(\Omega)}\Big)$$

$$= (f, \partial_t w)_{L^2(\Omega)} + \frac{1}{2h^2}\Big(\partial_t D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h w\Big)_{L^2(\Omega)}.$$

Hence,

$$\begin{aligned}
&\sup_{t\in[0,T]}\left\|\left(\partial_t w, \frac{1}{h}\varepsilon_h(w)\right)\right\|^2_{L^2(\Omega)} \\
&\leq \sup_{t\in[0,T]}\left(\frac{1}{2}\|\partial_t w\|^2_{L^2(\Omega)} + \frac{1}{h^2}\big(D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h w\big)\right) + CR\sup_{t\in[0,T]}\left|\frac{1}{h}\int_\Omega w\cdot x^\perp dx\right|^2 \\
&\leq \frac{1}{2}\|w_1\|^2_{L^2(\Omega)} + |A_0| + \int_0^T |(f,\partial_t w)_{L^2(\Omega)}|d\tau + CR\sup_{t\in[0,T]}\left|\frac{1}{h}\int_\Omega w\cdot x^\perp dx\right|^2 \\
&\qquad + \int_0^T\left|\frac{1}{h^2}\Big(\partial_t D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h w\Big)_{L^2(\Omega)}\right|d\tau
\end{aligned}$$

For the last term on the right hand side, we apply Lemma 5.2.2 and Korn's inequality

$$\begin{aligned}
&\int_0^T\left|\frac{1}{h^2}\Big(\partial_t D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h w\Big)_{L^2(\Omega)}\right|d\tau \\
&\qquad\leq CR\|\nabla_h w\|^2_{L^1(0,T;L^2_h)} \leq CR\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2_{L^1(0,T;L^2)} + \left\|\frac{1}{h}\int_\Omega w\cdot x^\perp dx\right\|^2_{L^1(0,T)} \\
&\qquad\leq CRT\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2_{C^0(0,T;L^2)} + CRT\left\|\frac{1}{h}\int_\Omega w\cdot x^\perp dx\right\|^2_{C^0(0,T)}.
\end{aligned}$$

Applying Cauchy-Schwarz's, Hölder's and Young's inequality we obtain

$$\begin{aligned}
\int_0^T |(f,\partial_t w)_{L^2(\Omega)}|dt &\leq C\|f\|_{L^1(0,T;L^2(\Omega))}\|\partial_t w\|_{L^\infty(0,T;L^2(\Omega))} \\
&\leq C(\varepsilon)\|f\|^2_{L^1(0,T;L^2(\Omega))} + C\varepsilon^2\|\partial_t w\|_{L^\infty(0,T;L^2(\Omega))}.
\end{aligned}$$

Choosing $\varepsilon > 0$ and $R_0 > 0$ sufficiently small and using an absorption argument the claim follows. ☐

The first step to obtain higher regularity is done in the following theorem.

Theorem 5.2.4. *Assume u_h satisfies* (5.16). *Then there exist $C_0 > 0$ and $R_0 \in (0,1]$ such that if $\varphi \in H^2_{per}(\Omega)$ solves for some $g \in L^2(\Omega)$ and $g_N \in L^2(0,L;H^{\frac{1}{2}}(\partial S))$*

$$\begin{cases}
-\frac{1}{h^2}\operatorname{div}_h(D^2\tilde{W}(\nabla_h u_h)\nabla_h\varphi) = g & \text{in } \Omega \\
D^2\tilde{W}(\nabla_h u_h)[\nabla_h\varphi]\nu\Big|_{(0,L)\times\partial S} = g_N & \text{on } \partial\Omega
\end{cases}$$

then

$$\begin{aligned}
\left\|\left(\nabla\frac{1}{h}\varepsilon_h(\varphi), \nabla_h^2\varphi\right)\right\|_{L^2(\Omega)} \leq C_0\bigg(& h^2\|g\|_{L^2(\Omega)} + \left\|\frac{1}{h}g_N\right\|_{L^2(0,L;H^{\frac{1}{2}}(\partial S))} \\
&+ \left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{H^{0,1}(\Omega)} + R\left|\frac{1}{h}\int_\Omega \varphi\cdot x^\perp dx\right|\bigg).
\end{aligned} \tag{5.24}$$

Proof: Using the fundamental theorem of calculus it follows

$$\operatorname{div}_h(D^2\tilde{W}(\nabla_h u_h)\nabla_h\varphi) = \operatorname{div}_h(D^2\tilde{W}(0)\nabla_h\varphi) + \int_0^1 \operatorname{div}_h(D^3\tilde{W}(\tau\nabla_h u_h)[\nabla_h u_h, \nabla_h\varphi])d\tau$$

and

$$\operatorname{div}_h(D^2\tilde{W}(0)\nabla_h\varphi) = \partial_{x_1}(D^2\tilde{W}(0)\nabla_h\varphi)_1 + \frac{1}{h}\operatorname{div}_{x'}(D^2\tilde{W}(0)^{\sim}\partial_{x_1}\varphi\otimes e_1) \\ + \frac{1}{h^2}\operatorname{div}_{x'}(D^2\tilde{W}(0)^{\approx}\nabla_{x'}\varphi)$$

where $(A)_k$ denotes the kth column of $A\in\mathbb{R}^{3\times 3}$. Moreover

$$\begin{aligned} g_N &= D^2\tilde{W}(\nabla_h u_h)\nabla_h\varphi\nu\Big|_{(0,L)\times\partial S} = \operatorname{tr}_{\partial S}(D^2\tilde{W}(\nabla_h u_h)\nabla_h\varphi)\nu \\ &= \operatorname{tr}_{\partial S}\left(D^2\tilde{W}(0)\nabla_h\varphi + \int_0^1 D^3\tilde{W}(\tau\nabla_h u_h)[\nabla_h u,\nabla_h\varphi]d\tau\right)\nu \\ &= \operatorname{tr}_{\partial S}\left(\frac{1}{h}D^2\tilde{W}(0)^{\approx}\nabla_{x'}\varphi\right)\nu_{\partial S} \\ &\quad + \underbrace{\operatorname{tr}_{\partial S}\left(D^2\tilde{W}(0)(\partial_{x_1}\varphi\otimes e_1) + \int_0^1 D^3\tilde{W}(\tau\nabla_h u_h)[\nabla_h u_h,\nabla_h\varphi]d\tau\right)}_{=:\,r_N}\begin{pmatrix}0\\ \nu_{\partial S}\end{pmatrix} \end{aligned} \tag{5.25}$$

where we have used that $\nu = (0,\nu_{\partial S}(x_2,x_3))^T$ and $\nu_{\partial S}$ is the outer unit normal on ∂S. Hence, we know that

$$\varphi_{(0)}(x_1,\cdot) := \varphi(x_1,\cdot) - \frac{1}{\mu(S)}(\varphi,x^\perp)_{L^2(S)}x^\perp - \frac{1}{|S|}(\varphi,1)_{L^2(S)} \tag{5.26}$$

solves for almost all $x_1\in(0,L)$ the system

$$\begin{cases} -\dfrac{1}{h^2}\operatorname{div}_{x'}\left(D^2\tilde{W}(0)^{\approx}\nabla_{x'}\varphi(x_1,\cdot)\right) = \tilde{g}(x_1,\cdot) & \text{in } S \\ \dfrac{1}{h}\Big(D^2\tilde{W}(0)^{\approx}\nabla_{x'}\varphi(x_1,\cdot)\Big)\nu_{\partial S}\Big|_{\partial S} = g_N(x_1,\cdot) - a_N(x_1,\cdot) & \text{on } \partial S \end{cases} \tag{5.27}$$

with $a_N := \operatorname{tr}_{\partial S}(r_N)(0,\nu_{\partial S})^T$,

$$\tilde{g} := h^2 g + \mathcal{R}(\varphi) + \int_0^1 \operatorname{div}_h(D^3\tilde{W}(\tau\nabla_h u_h)[\nabla_h u_h,\nabla_h\varphi])d\tau \tag{5.28}$$

and

$$\mathcal{R}(\varphi) := \partial_{x_1}(D^2\tilde{W}(0)\nabla_h\varphi)_1 + \frac{1}{h}\operatorname{div}_{x'}\left(D^2\tilde{W}(0)^{\sim}\partial_{x_1}\varphi\otimes e_1\right)$$

satisfying

$$\int_S \varphi_{(0)}(x_1,x')dx' = 0 \quad\text{and}\quad \int_S \varphi_{(0)}(x_1,x')\cdot x^\perp dx' = 0$$

for almost all $x_1\in(0,L)$. Then due to the inequality of Lemma 5.2.5 below it follows

$$\begin{aligned} \frac{1}{h^2}\|\nabla^2_{x'}\varphi(x_1,\cdot)\|_{L^2(S)} &= \|\nabla^2_{h,x'}\varphi_{(0)}(x_1,\cdot)\|_{L^2(S)} \\ &\le C\bigg(\|\tilde{g}(x_1,\cdot)\|_{L^2(S)} + \frac{1}{h}\|g_N(x_1,\cdot)-a_N(x_1,\cdot)\|_{H^{\frac{1}{2}}(\partial S)}\bigg) \end{aligned}$$

for a.e. $x_1\in(0,L)$. Using a generalised Poincaré's inequality on S

$$\|q\|_{L^2(S)} \le C\bigg(\|\nabla_{x'}q\|_{L^2(S)} + \left|\int_{\partial S} q\,d\sigma(x')\right|\bigg) \quad \text{for all } q\in H^1(S)$$

we obtain with the boundedness of $\mathrm{tr}_{\partial S}\colon H^1(S) \to H^{\frac{1}{2}}(\partial S)$

$$\|q\|_{H^{\frac{1}{2}}(\partial S)} \leq C\bigg(\|\nabla_{x'} a(q)\|_{L^2(S)} + \bigg| \int_{\partial S} q d\sigma(x') \bigg| \bigg) \quad \text{for all } q \in H^{\frac{1}{2}}(\partial S)$$

for $a(q) \in H^1(S)$ such that $\mathrm{tr}_{\partial S}(a(q)) = q$ in $H^{\frac{1}{2}}(\partial S)$. Such an $a(q)$ exists using a classical extension operator $E\colon H^{\frac{1}{2}}(\partial S) \to H^1(S)$, which is right inverse to $\mathrm{tr}_{\partial S}$. Applying the preceding inequality on $(g_N - a_N)$ we deduce

$$\begin{aligned}\|\nabla^2_{h,x'} \varphi_{(0)}(x_1,\cdot)\|_{L^2(S)} \leq C\bigg(&\|\tilde{g}(x_1,\cdot)\|_{L^2(S)} + \frac{1}{h}\|\nabla_{x'}(a(g_N)(x_1,\cdot) - r_N(x_1,\cdot))\|_{L^2(S)} \\ &+ \frac{1}{h}\bigg| \int_{\partial S} (g_N - a_N)(x_1,x')d\sigma(x') \bigg| \bigg).\end{aligned}$$

Using Gauss's theorem and (5.27) leads to

$$\int_{\partial S} (g_N - a_N)(x_1,x')d\sigma(x') = h\int_S \tilde{g}(x_1,x')dx'.$$

Integration over x_1 yields

$$\|\nabla^2_{h,x'}\varphi\|_{L^2(\Omega)} \leq C\bigg(\|\tilde{g}\|_{L^2(\Omega)} + \frac{1}{h}\|\nabla_{x'} a(g_N)\|_{L^2(\Omega)} + \frac{1}{h}\|\nabla_{x'} r_N\|_{L^2(\Omega)} \bigg).$$

We can now estimate each term separately, beginning with $\tilde{g}$. As $\mathcal{R}(\varphi)$ is a linear combination of terms involving only $\partial_{x_1}\nabla_h\varphi$ it follows

$$\begin{aligned}\|\mathcal{R}(\varphi)\|_{L^2(\Omega)} \leq C\|\partial_{x_1}\nabla_h\varphi\|_{L^2(\Omega)} &\leq C\bigg(\bigg\| \frac{1}{h}\varepsilon_h(\partial_{x_1}\varphi) \bigg\|_{L^2(\Omega)} + \frac{1}{h}\bigg| \int_\Omega \partial_{x_1}\varphi \cdot x^\perp dx \bigg| \bigg) \\ &= C\bigg\| \frac{1}{h}\varepsilon_h(\partial_{x_1}\varphi) \bigg\|_{L^2(\Omega)}\end{aligned}$$

where we used the fact that φ is L-periodic in x_1 direction. Next we have

$$\begin{aligned}\int_0^1 \mathrm{div}_h(D^3\tilde{W}(\tau\nabla_h u_h)[\nabla_h u_h, \nabla_h\varphi])d\tau &= \mathrm{div}_h\left(\int_0^1 D^3\tilde{W}(\tau\nabla_h u_h)d\tau[\nabla_h u_h, \nabla_h\varphi] \right) \\ &= \mathrm{div}_h\Big(G(\nabla_h u_h)[\nabla_h u_h, \nabla_h\varphi] \Big) \end{aligned} \tag{5.29}$$

where $G \in C^\infty(\overline{B_\varepsilon(0)}, \mathcal{L}^3(\mathbb{R}^{3\times 3}))$ for some suitable $\varepsilon > 0$. Thus it follows with the identification of $\mathcal{L}^1(\mathbb{R}^{3\times 3})$ with $\mathbb{R}^{3\times 3}$ via the standard scalar product

$$\begin{aligned}\|\,\mathrm{div}_h(G(\nabla_h u_h)[\nabla_h u_h, \nabla_h\varphi])\|_{L^2(\Omega)} &\leq \frac{1}{h}\sup_{|\alpha|=1} \|\partial_x^\alpha(G(\nabla_h u_h)[\nabla_h u_h, \nabla_h\varphi])\|_{L^2(\Omega;\mathcal{L}^1(\mathbb{R}^{3\times 3}))} \\ &\leq \frac{1}{h}\sup_{|\alpha|=1} \Big(\|G(\nabla_h u_h)[\nabla_h u_n, \nabla_h\partial_x^\alpha\varphi]\|_{L^2(\Omega;\mathcal{L}^1(\mathbb{R}^{3\times 3}))} \\ &\quad + \|G(\nabla_h u_h)[\nabla_h\partial_x^\alpha u_n, \nabla_h\varphi]\|_{L^2(\Omega;\mathcal{L}^1(\mathbb{R}^{3\times 3}))} \\ &\quad + \|DG(\nabla_h u_h)[\partial_x^\alpha\nabla_h u_h, \nabla_h u_h, \nabla_h\varphi]\|_{L^2(\Omega;\mathcal{L}^1(\mathbb{R}^{3\times 3}))} \Big)\end{aligned}$$

As $G \in C^\infty(\overline{B_\varepsilon(0)}; \mathcal{L}^3(\mathbb{R}^{3\times 3}))$ and $\nabla_h u_h \in C^0(\overline{\Omega}; \mathbb{R}^3)$ it follows

$$\|G(\nabla_h u_h)[\nabla_h\partial_x^\alpha u_n, \nabla_h\varphi]\|_{L^2(\Omega;\mathcal{L}^1(\mathbb{R}^{3\times 3}))} = \sup_{\substack{B\in\mathbb{R}^{3\times 3}\\ |B|\leq 1}} \left(\int_\Omega |G(\nabla_h u_h)[\nabla_h\partial_x^\alpha u_h, \nabla_h\varphi, B]|^2 dx \right)^{\frac{1}{2}}$$

$$\leq C\|\nabla_h \partial_x^\alpha u_h\|_{L^4(\Omega)}\|\nabla_h \varphi\|_{L^4(\Omega)} \leq CRh\|\nabla_h \varphi\|_{H^1(\Omega)}$$

where we used Hölder inequality, the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and $\|\nabla_h \partial_x^\alpha u_h\|_{H^1(\Omega)} \leq CRh$ due to (5.16). Analogously using $H^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$ and $\|\nabla_h u_h\|_{H^2(\Omega)} \leq CRh$ it follows

$$\|G(\nabla_h u_h)[\nabla_h u_h, \nabla_h \partial_x^\alpha \varphi]\|_{L^2(\Omega, \mathcal{L}^1(\mathbb{R}^{3\times 3}))} \leq CRh\|\nabla_h w\|_{H^1(\Omega)}.$$

Finally as $DG \in C^\infty(\overline{B_\varepsilon(0)}; \mathcal{L}^4(\mathbb{R}^{3\times 3}))$ is bounded, we obtain

$$\|DG(\nabla_h u_h)[\nabla_h \partial_x^\alpha u_h, \nabla_h u_h, \nabla_h \varphi]\|_{L^2(\Omega; \mathcal{L}^1(\mathbb{R}^{3\times 3}))} \leq CRh\|\nabla_h \varphi\|_{H^1(\Omega)}.$$

All in all we can conclude

$$\|\operatorname{div}_h(G(\nabla_h u_h)[\nabla_h u_h, \nabla_h \varphi])\|_{L^2(\Omega)} \leq CR\|\nabla_h \varphi\|_{H^1(\Omega)}. \tag{5.30}$$

From the definition of r_N it follows

$$\begin{aligned}
&\frac{1}{h}\|\nabla_{x'} r_N\|_{L^2(\Omega)} \\
&\quad= \left\|\nabla_{h,x'}(D^2\tilde{W}(0)(\partial_{x_1}\varphi \otimes e_1)) + \int_0^1 \nabla_{h,x'}\big(D^3\tilde{W}(\tau\nabla_h u_h)[\nabla_h u_h, \nabla_h \varphi]\big)d\tau\right\|_{L^2(\Omega)} \\
&\quad\leq \|\nabla_{h,x'}(D^2\tilde{W}(0)(\partial_{x_1}\varphi \otimes e_1))\|_{L^2(\Omega)} + \|\nabla_{h,x'}(G(\nabla_h u_h)[\nabla_h u_h, \nabla_h \varphi])\|_{L^2(\Omega)}
\end{aligned}$$

The first term on the right hand side is a linear combination of $\nabla_h \partial_{x_1}\varphi$. Hence

$$\|\nabla_{h,x'} D^2\tilde{W}(0)(\partial_{x_1}\varphi \otimes e_1)\|_{L^2(\Omega)} \leq C\|\nabla_h \partial_{x_1}\varphi\|_{L^2(\Omega)} \leq C\left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{H^{0,1}(\Omega)}$$

The second term can be bounded analogously to (5.30)

$$\|\nabla_{h,x'} G(\nabla_h u_h)[\nabla_h u_h, \nabla_h \varphi]\|_{L^2(\Omega)} \leq CR\|\nabla_h \varphi\|_{H^1(\Omega)}$$

Hence as $\|\nabla_h u\|_{H^1(\Omega)} \leq \|(\nabla_h u, \nabla_h^2 u)\|_{L^2(\Omega)}$ we have

$$\begin{aligned}
\|\nabla_{h,x'}^2 \varphi\|_{L^2(\Omega)} \leq C\bigg(& h^2\|g\|_{L^2(\Omega)} + \left\|\frac{1}{h} g_N\right\|_{L^2(0,L;H^{\frac{1}{2}}(\partial S))} + \left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{H^{0,1}(\Omega)} \\
&+ R\|(\nabla_h u, \nabla_h^2 u)\|_{L^2(\Omega)}\bigg).
\end{aligned} \tag{5.31}$$

Due to the structure of $\nabla_h^2 \varphi$,

$$\nabla_h^2 \varphi_j = \begin{pmatrix} \partial_{x_1}^2 \varphi_j & (\nabla_{h,x'}\partial_{x_1}\varphi_j)^T \\ \nabla_{h,x'}\partial_{x_1}\varphi_j & \nabla_{h,x'}^2 \varphi_j \end{pmatrix} \tag{5.32}$$

for $j \in \{1, 2, 3\}$, it follows

$$\begin{aligned}
\|\nabla_h^2 \varphi\|_{L^2(\Omega)} \leq C\bigg(&\left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{L^2(\Omega)} + \left|\frac{1}{h}\int_\Omega \varphi \cdot x^\perp dx\right|\bigg) \\
&+ \left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{H^{0,1}(\Omega)} + \|\nabla_{h,x'}^2 \varphi\|_{L^2(\Omega)}.
\end{aligned}$$

Hence, by plugging all inequalities into (5.31) and applying Korn's inequality

$$\|\nabla_{h,x'}^2 \varphi\|_{L^2(\Omega)} \leq C\bigg(h^2\|g\|_{L^2(\Omega)} + \left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{H^{0,1}(\Omega)} + \left\|\frac{1}{h} g_N\right\|_{L^2(0,L;H^{\frac{1}{2}}(\partial S))}$$

$$+ R\left|\frac{1}{h}\int_\Omega \varphi\cdot x^\perp dx\right|\Bigg) + CR\|(\nabla^2_{h,x'}\varphi)\|_{L^2(\Omega)}$$

Using an absorption argument for $R_0 \in (0,1]$ sufficiently small and the structure in (5.32) it follows

$$\begin{aligned}\|\nabla_h^2\varphi\|_{L^2(\Omega)} &\leq \|\partial^2_{x_1}\varphi\|_{L^2(\Omega)} + 2\|\nabla_{h,x'}\partial_{x_1}\varphi\|_{L^2(\Omega)} + \|\nabla^2_{h,x'}\varphi\|_{L^2(\Omega)} \\ &\leq C\Bigg(h^2\|g\|_{L^2(\Omega)} + \left\|\frac{1}{h}g_N\right\|_{L^2(0,L;H^{\frac12}(\partial S))} + \left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{H^{0,1}(\Omega)} + R\left|\frac{1}{h}\int_\Omega\varphi\cdot x^\perp dx\right|\Bigg)\end{aligned}$$

as

$$\|\nabla_{h,x'}\partial_{x_1}\varphi\|_{L^2(\Omega)} \leq \|\nabla_h\partial_{x_1}\varphi\|_{L^2(\Omega)} \leq C\left\|\frac{1}{h}\varepsilon_h(\partial_{x_1}\varphi)\right\|_{L^2(\Omega)} \leq C\left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{H^{0,1}(\Omega)}$$

and $\|\partial^2_{x_1}\varphi\|_{L^2(\Omega)} \leq \left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{H^{0,1}(\Omega)}$ holds. Finally (5.24) follows from

$$\left\|\nabla\frac{1}{h}\varepsilon_h(\varphi)\right\|_{L^2(\Omega)} \leq \|\nabla_h^2\varphi\|_{L^2(\Omega)} + \left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{H^{0,1}(\Omega)}. \qquad \square$$

Lemma 5.2.5. *Assume that the assumptions of Theorem 5.2.4 are satisfied and $\varphi_{(0)}$ as in* (5.26) *solution of* (5.27). *Then for almost all $x_1 \in (0,L)$ it holds*

$$\frac{1}{h^2}\|\varphi_{(0)}\|_{H^2(S)} \leq C\Bigg(\|\tilde g(x_1,\cdot)\|_{L^2(S)} + \frac{1}{h}\|(g_N - a_N)(x_1,\cdot)\|_{H^{\frac12}(\partial S)}\Bigg). \tag{5.33}$$

Proof: As we have $\varphi_{(0)}(x_1,\cdot) \in H^2(S)$, one can test the equation (5.27) with $\varphi_{(0)}(x_1,\cdot)$ to obtain

$$\frac{1}{h^2}\Big(D^2\tilde W(0)^{\approx}\nabla_{x'}\varphi_{(0)}, \nabla_{x'}\varphi_{(0)}\Big)_{L^2(S)} = (\tilde g, \varphi_{(0)})_{L^2(S)} + \frac{1}{h}(g_N - a_N, \varphi_{(0)})_{L^2(\partial S)}$$

for almost all $x_1 \in (0,L)$. Using now the Legendre-Hadamard condition, Korn's inequality in two dimensions and Poincaré's inequality it follows, due to the fact that $\varphi_{(0)}$ is mean value free

$$\begin{aligned}\Big(D^2\tilde W(0)^{\approx}\nabla_{x'}\varphi_{(0)}, \nabla_{x'}\varphi_{(0)}\Big)_{L^2(S)} &= \Bigg(D^2\tilde W(0)\begin{pmatrix}0\\ \nabla_{x'}\end{pmatrix}\varphi_{(0)}, \begin{pmatrix}0\\ \nabla_{x'}\end{pmatrix}\varphi_{(0)}\Bigg)_{L^2(S)} \\ &\geq c_0\left\|\operatorname{sym}\left(\begin{pmatrix}0\\ \nabla_{x'}\end{pmatrix}\varphi_{(0)}\right)\right\|^2_{L^2(S)} \\ &\geq C\|\nabla_{x'}\varphi_{(0)}\|^2_{L^2(S)} - C\left|\int_S \varphi_{(0)}\cdot x^\perp dx'\right|^2 \\ &\geq C\|\varphi_{(0)}\|^2_{H^1(S)}.\end{aligned}$$

Here we used that $\int_S \varphi_{(0)}\cdot x^\perp dx' = 0$. Thus applying Young's inequality and an absorption argument we are led to

$$\frac{1}{h^2}\|\varphi_{(0)}\|_{H^1(S)} \leq C\Bigg(\|\tilde g\|_{L^2(S)} + \frac{1}{h}\|g_N - a_N\|_{H^{\frac12}(\partial S)}\Bigg).$$

For higher regularity we want to apply Theorem 2.1.18. Hence, we define $U := S$, which has a smooth boundary and $n = 2$, $m = 3$, as well as

$$\mathcal{L}u := -\sum_{\alpha,\beta=1}^2 \partial_\alpha(B^{\alpha\beta}\partial_\beta u) := -\operatorname{div}_{x'}\Big(D^2\tilde W(0)^{\approx}\nabla_{x'}u\Big)$$

where $u\colon S \to \mathbb{R}^3$ and for α, $\beta = 1,2$: $B^{\alpha\beta} = (b_{ij}^{\alpha\beta})_{i,j=1,2,3}$. With this it follows

$$\mathfrak{B}_\nu u = \mathrm{tr}_{\partial S}\left(D^2\tilde{W}(0)^{\approx}\nabla_{x'}u\right)\nu_{\partial S}.$$

When considering the data one notices that $\tilde{g}(x_1,\cdot) \in L^2(S;\mathbb{R}^3)$ and $(g_N - a_N)(x_1,\cdot) \in H^{\frac{1}{2}}(\partial S;\mathbb{R}^3)$ for almost every $x_1 \in (0,L)$ holds. This comes on the one hand from the assumptions on $g \in L^2(\Omega)$ and $g_N \in L^2(0,L;H^{\frac{1}{2}}(S))$. On the other hand we know that $\mathcal{R}(\varphi)$ is a linear combination of terms involving only $\nabla_h\partial_{x_1}\varphi \in L^2(\Omega)$, as well as $\nabla_h u \in H^2(\Omega)$ and $\nabla_h\varphi \in H^1(\Omega)$ for $\tilde{g}$. Moreover, because $\mathrm{tr}_{\partial S}\colon H^1(S) \to H^{\frac{1}{2}}(\partial S)$ it follows $a_N \in H^{\frac{1}{2}}(\partial S)$. The Legendre-Hadamard condition of $D^2\tilde{W}(0)^{\approx}$ is inherited from $D^2\tilde{W}(0)$. Finally due to Korn's inequality in two dimensions and Poincaré's inequality with mean value we can conclude that $\mathcal{L}$ is weakly coercive on $H^1(S)$. Applying now (2.17) of Theorem 2.1.18 we obtain

$$\frac{1}{h^2}\|\varphi_{(0)}\|_{H^2(S;\mathbb{R}^3)} \leq C\left(\frac{1}{h^2}\|\varphi_{(0)}\|_{H^1(S)} + \frac{1}{h}\|(g_N - a_N)(x_1,\cdot)\|_{H^{\frac{1}{2}}(\partial S)} + \|\tilde{g}\|_{L^2(S)}\right).$$

Putting the above inequalities together leads to the desired result. □

In case we have homogeneous Neumann boundary conditions, we can refine Theorem 5.2.4 in the following way.

Corollary 5.2.6. *Assume u_h fulfils* (5.16). *If $w \in H^2_{per}(\Omega)$ satisfies*

$$\begin{cases} -\dfrac{1}{h^2}\operatorname{div}_h(D^2\tilde{W}(\nabla_h u_h)\nabla_h w) = f & \text{in } \Omega \\ D^2\tilde{W}(\nabla_h u_h)[\nabla_h w]\nu\Big|_{(0,L)\times\partial S} = 0 & \text{on } \partial\Omega \end{cases} \tag{5.34}$$

for some $f \in L^2(\Omega)$, then it holds

$$\left\|\left(\nabla\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\right)\right\|_{L^2(\Omega)} \leq C_0\left(\|f\|_{L^2(\Omega)} + R\left|\frac{1}{h}\int_\Omega w\cdot x^\perp dx\right|\right). \tag{5.35}$$

Proof: Applying Theorem 5.2.4 with $\varphi := w$, $g := f$ and $g_N := 0$ we are lead to

$$\left\|\left(\nabla\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\right)\right\|_{L^2(\Omega)} \leq C\left(h^2\|f\|_{L^2(\Omega)} + \left\|\frac{1}{h}\varepsilon_h(w)\right\|_{H^{0,1}(\Omega)} + R\left|\frac{1}{h}\int_\Omega w\cdot x^\perp dx\right|\right). \tag{5.36}$$

Consequently we want to eliminate the $\|\frac{1}{h}\varepsilon_h(w)\|_{H^{0,1}}$ term on the right hand side. From (5.34) it follows

$$\frac{1}{h^2}\left(D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h\varphi\right)_{L^2(\Omega)} = (f,\varphi)_{L^2(\Omega)} \tag{5.37}$$

for $\varphi \in H^1_{per}(\Omega)$. Now we want to choose $\varphi = \partial_{x_1}^{2k} w_{(0)}$ where $w_{(0)} := w - \frac{1}{\mu(\Omega)}(w,x^\perp)_{L^2(\Omega)}x^\perp - \frac{1}{|\Omega|}(w,1)_{L^2(\Omega)}$ and $k = 0$, 1. First we start with $k = 0$. Periodicity of $\nabla_h u_h$, w and $w_{(0)}$, a Taylor expansion and creation of a $-\frac{1}{\mu(\Omega)}(w,x^\perp)_{L^2(\Omega)}x^\perp$ part lead to

$$\begin{aligned}\frac{1}{h^2}\left(D^2\tilde{W}(0)\nabla_h w_{(0)},\nabla_h w_{(0)}\right)_{L^2(\Omega)} &+ \frac{1}{h^2}\int_0^1\left(D^3\tilde{W}(\tau\nabla_h u_h)[\nabla_h u_h, \nabla_h w_{(0)}], \nabla_h w_{(0)}\right)_{L^2(\Omega)} d\tau \\ &+ \frac{1}{h^2}\int_0^1\left(D^3\tilde{W}(\tau\nabla_h u_h)\left[\nabla_h u_h, \nabla_h\frac{1}{\mu(\Omega)}(w,x^\perp)_{L^2}x^\perp\right], \nabla_h w_{(0)}\right)_{L^2(\Omega)} d\tau \\ &= (f, w_{(0)})_{L^2(\Omega)}.\end{aligned}$$

Here we used that, due to (2.24)

$$\left(D^2\tilde{W}(0)\nabla_h x^\perp, \nabla_h w_{(0)}\right)_{L^2(\Omega)} = 0$$

holds. Moreover, because of the specific construction of $w_{(0)}$ we can deduce

$$\left\|\frac{1}{h}\varepsilon_h(w_{(0)})\right\|_{L^2} = \left\|\frac{1}{h}\varepsilon_h(w)\right\|_{L^2} \quad \text{and} \quad \int_\Omega w_{(0)} \cdot x^\perp dx = 0.$$

Hence,

$$\begin{aligned}\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2_{L^2(\Omega)} &\leq |(f, w_{(0)})_{L^2(\Omega)}| + \left|\frac{1}{h^2}\int_0^1 \Big(D^3\tilde{W}(\tau\nabla_h u_h)[\nabla_h u_h, \nabla_h w_{(0)}], \nabla_h w_{(0)}\Big)_{L^2(\Omega)} d\tau\right| \\ &+ \left|\frac{1}{h^2}\int_0^1 \Big(D^3\tilde{W}(\tau\nabla_h u_h)\Big[\nabla_h u_h, \nabla_h \frac{1}{\mu(\Omega)}(w, x^\perp)_{L^2} x^\perp\Big], \nabla_h w_{(0)}\Big)_{L^2(\Omega)} d\tau\right|.\end{aligned}$$

Now, by the Hölder, Young and Poincaré Inequality

$$\begin{aligned}|(f, w_{(0)})_{L^2(\Omega)}| &\leq C(\epsilon)\|f\|^2_{L^2(\Omega)} + \epsilon\|w_{(0)}\|^2_{L^2} \leq C(\epsilon)\|f\|^2_{L^2(\Omega)} + \epsilon\|\nabla w_{(0)}\|^2_{L^2} \\ &\leq C(\epsilon)\|f\|^2_{L^2(\Omega)} + \epsilon\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2_{L^2}.\end{aligned}$$

for any $\epsilon > 0$. Secondly with Corollary 2.3.7, (5.17) and $\|\nabla_h w_{(0)}\|_{L^2(\Omega)} \leq C\left\|\frac{1}{h}\varepsilon_h(w_{(0)})\right\|_{L^2(\Omega)}$, we obtain

$$\left|\frac{1}{h^2}\int_0^1 \Big(D^3\tilde{W}(\tau\nabla_h u_h)[\nabla_h u_h, \nabla_h w_{(0)}], \nabla_h w_{(0)}\Big)_{L^2(\Omega)} d\tau\right| \leq CR\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2_{L^2(\Omega)}$$

and

$$\begin{aligned}&\left|\frac{1}{h^2}\int_0^1 \Big(D^3\tilde{W}(\tau\nabla_h u_h)\Big[\nabla_h u_h, \nabla_h \frac{1}{\mu(\Omega)}(w, x^\perp)_{L^2} x^\perp\Big], \nabla_h w_{(0)}\Big)_{L^2(\Omega)} d\tau\right| \\ &\qquad\qquad \leq \frac{CR}{h}\|(w, x^\perp)_{L^2(\Omega)}\|_{L^2(\Omega)}\|\nabla_h w_{(0)}\|_{L^2(\Omega)} \\ &\qquad\qquad \leq CR\left(\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2_{L^2(\Omega)} + \left|\frac{1}{h}\int_\Omega w \cdot x^\perp dx\right|^2\right).\end{aligned}$$

For sufficiently small $\epsilon > 0$ and $R_0 \in (0, 1]$ it follows

$$\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2_{L^2(\Omega)} \leq C\|f\|^2_{L^2(\Omega)} + CR\left|\frac{1}{h}\int_\Omega w \cdot x^\perp dx\right|^2. \tag{5.38}$$

In order to use $\varphi = \partial^2_{x_1} w_{(0)}$, we exploit the density of $C^\infty_{per}(\Omega)$ in $H^1_{per}(\Omega)$. For $\psi \in C^\infty_{per}(\Omega)$ we can use $\varphi = \partial_{x_1}\psi$ as a testfunction and obtain

$$\frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h \partial_{x_1}\psi\Big)_{L^2(\Omega)} = (f, \partial_{x_1}\psi)_{L^2(\Omega)}.$$

Integration by parts, the periodicity and the homogeneous Neumann boundary conditions lead to

$$\frac{1}{h^2}\Big(\partial_{x_1}(D^2\tilde{W}(\nabla_h u_h)\nabla_h w), \nabla_h\psi\Big)_{L^2(\Omega)} = (f, \partial_{x_1}\psi)_{L^2(\Omega)}.$$

The density of $C^\infty_{per}(\Omega)$ implies now that the latter equation holds for $\partial_{x_1} w_{(0)} \in H^1_{per}(\Omega)$ as well. Thus

$$\frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_h)\nabla_h \partial_{x_1} w_{(0)}, \nabla_h \partial_{x_1} w_{(0)}\Big)_{L^2(\Omega)}$$

$$= (f, \partial_{x_1}\varphi)_{L^2(\Omega)} - \frac{1}{h^2}\Big((\partial_{x_1} D^2\tilde{W}(\nabla_h u_h))\nabla_h w, \nabla_h \partial_{x_1} w_{(0)}\Big)_{L^2(\Omega)}$$

as $\partial_{x_1} w_{(0)} = \partial_{x_1} w$. Using $\int_\Omega \partial_{x_1} w \cdot x^\perp dx = 0$, we can conclude

$$\begin{aligned}\left\|\frac{1}{h}\varepsilon_h(\partial_{x_1} w)\right\|^2_{L^2(\Omega)} &\leq |(f, \partial^2_{x_1} w)_{L^2(\Omega)}| + \left|\frac{1}{h^2}\Big(\partial_{x_1} D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h \partial_{x_1} w\Big)_{L^2(\Omega)}\right| \\ &\leq C\|f\|^2_{L^2(\Omega)} + \epsilon\left\|\frac{1}{h}\varepsilon_h(\partial_{x_1} w)\right\|^2_{L^2(\Omega)} + CR\|\nabla_h w\|_{L^2_h(\Omega)}\|\nabla_h \partial_{x_1} w\|_{L^2_h(\Omega)}.\end{aligned}$$

In the preceding calculation Hölder's and Young's inequality are used as well as Lemma 5.2.2. Hence, by Korn's inequality and (5.38)

$$\begin{aligned}\left\|\frac{1}{h}\varepsilon_h(\partial_{x_1} w)\right\|^2_{L^2(\Omega)} &\leq C\|f\|^2_{L^2(\Omega)} + CR\Big(\|\nabla_h w\|^2_{L^2_h(\Omega)} + \|\nabla_h \partial_{x_1} w\|^2_{L^2_h(\Omega)}\Big) \\ &\leq C\left(\|f\|^2_{L^2(\Omega)} + R\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2 + R\left|\frac{1}{h}\int_\Omega w \cdot x^\perp dx\right|^2\right) \\ &\quad + CR\left\|\frac{1}{h}\varepsilon_h(\partial_{x_1} w)\right\|^2\end{aligned}$$

Choosing $R_0 \in (0,1]$ sufficiently small leads to

$$\left\|\frac{1}{h}\varepsilon_h(\partial_{x_1} w)\right\|^2_{L^2(\Omega)} \leq C\left(\|f\|^2_{L^2(\Omega)} + R\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2 + R\left|\frac{1}{h}\int_\Omega w \cdot x^\perp dx\right|^2\right). \tag{5.39}$$

Combining (5.38) and (5.39) with (5.24) the desired inequality follows. □

The preceding Corollary 5.2.6 yields higher regularity bounds for the dynamical system (5.12)–(5.15), proven in the next theorem.

Theorem 5.2.7 (Second Order Inequality). *Let $0 < T < \infty$, $h \in (0,1]$, $0 < R \leq R_0$ be given, where R_0 is chosen small enough, but independent of h. Furthermore, let u_h satisfies* (5.16). *Assume $w \in \bigcap_{k=0}^2 C^k([0,T]; H^{2-k}(\Omega))$ is the unique solution of the system* (5.12)–(5.15) *for some $f \in W^1_1(0,T; L^2(\Omega))$, $w_0 \in H^2_{per}(\Omega)$ and $w_1 \in H^1_{per}(\Omega)$, then there exist constants C_{L1}, $C_1 > 0$ such that*

$$\begin{aligned}\left\|\left(\partial_t^2 w, \nabla_{x,t}\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\right)\right\|^2_{C^0([0,T];L^2(\Omega))} &\leq C_{L1}e^{C_1 TR}\Big(\|f\|^2_{W^1_1(0,T;L^2)} + \|(w_1, w_2, f|_{t=0})\|^2_{L^2} \\ &\quad + |(A_0, A_1)| + (1+T)R\max_{\sigma=0,1}\left\|\frac{1}{h}\int_\Omega \partial_t^\sigma w \cdot x^\perp dx\right\|^2_{C^0(0,T)}\Big)\end{aligned}$$

holds, where

$$A_1 := \frac{1}{h^2}\big(D^2\tilde{W}(\nabla_h u_h|_{t=0})\nabla_h w_1, \nabla_h w_1\big)_{L^2(\Omega)}$$

and

$$w_2 := \frac{1}{h^2}\operatorname{div}_h\big(D^2\tilde{W}(\nabla_h u_h|_{t=0})\nabla_h w_0\big) + f|_{t=0}.$$

Proof: Differentiating (5.12) with respect to t and testing the result with $\partial_t^2 w$ yields

$$\begin{aligned}&\Big(\partial_t^3 w, \partial_t^2 w\Big)_{L^2(\Omega)} + \frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_h)\nabla_h \partial_t w, \nabla_h \partial_t^2 w\Big)_{L^2(\Omega)} \\ &\quad = \Big(\partial_t f, \partial_t^2 w\Big)_{L^2(\Omega)} - \frac{1}{h^2}\Big(\partial_t D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h \partial_t^2 w\Big)_{L^2(\Omega)}\end{aligned} \tag{5.40}$$

as

$$\frac{d}{dt}\Big(D^2\tilde{W}(\nabla_h u_h)\nabla_h\partial_t w, \nabla_h\partial_t w\Big)_{L^2(\Omega)} = 2\Big(D^2\tilde{W}(\nabla_h u_h)\nabla_h\partial_t w, \nabla_h\partial_t^2 w\Big)_{L^2(\Omega)} + \Big(\partial_t D^2\tilde{W}(\nabla_h u_h)\nabla_h\partial_t w, \nabla_h\partial_t w\Big)_{L^2(\Omega)}$$

and

$$\frac{d}{dt}\Big(\partial_t D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h\partial_t w\Big)_{L^2(\Omega)} = \Big(\partial_t D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h\partial_t^2 w\Big)_{L^2(\Omega)} + \Big(\partial_t D^2\tilde{W}(\nabla_h u_h)\nabla_h\partial_t w, \nabla_h\partial_t w\Big)_{L^2(\Omega)} + \Big(\partial_t^2 D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h\partial_t w\Big)_{L^2(\Omega)}$$

it follows

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\left[\|\partial_t^2 w\|^2_{L^2(\Omega)} + \frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_h)\nabla_h\partial_t w, \nabla_h\partial_t w\Big)_{L^2(\Omega)}\right] \\
&\quad\le |(\partial_t f, \partial_t^2 w)_{L^2(\Omega)}| + \frac{3}{2}\left|\frac{1}{h^2}\Big(\partial_t D^2\tilde{W}(\nabla_h u_h)\nabla_h\partial_t w, \nabla_h\partial_t w\Big)_{L^2(\Omega)}\right| \\
&\qquad + \left|\frac{1}{h^2}\Big(\partial_t^2 D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h\partial_t w\Big)_{L^2(\Omega)}\right| - \frac{d}{dt}\frac{1}{h^2}\Big(\partial_t D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h\partial_t w\Big)_{L^2(\Omega)}.
\end{aligned}$$

Due to (5.22) and Korn's inequality

$$\begin{aligned}
\left|\frac{1}{h^2}\Big(\partial_t D^2\tilde{W}(\nabla_h u_h)\nabla_h\partial_t w, \nabla_h\partial_t w\Big)_{L^2(\Omega)}\right| &\le CR\|\nabla_h\partial_t w\|^2_{L^2_h(\Omega)} \\
&\le CR\left(\left\|\frac{1}{h}\varepsilon_h(\partial_t w)\right\|^2_{L^2(\Omega)} + \frac{1}{h^2}\left|\int_\Omega \partial_t w\cdot x^\perp\right|^2\right) \\
\left|\frac{1}{h^2}\Big(\partial_t^2 D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h\partial_t w\Big)_{L^2(\Omega)}\right| &\le CR\|\nabla_h w\|_{H^1_h(\Omega)}\|\nabla_h\partial_t w\|_{L^2_h(\Omega)} \\
&\le CR\|\nabla_h w\|^2_{H^1_h(\Omega)} + CR\|\nabla_h\partial_t w\|^2_{L^2_h(\Omega)}.
\end{aligned}$$

Using the definition of $H^1_h(\Omega)$ and Korn inequality, it follows

$$\begin{aligned}
\|\nabla_h w\|^2_{H^1_h(\Omega)} &\le C\|\nabla_h w\|^2_{L^2_h(\Omega)} + C\sum_{i=1}^{3}\|\partial_{x_i}\nabla_h w\|^2_{L^2_h(\Omega)} \\
&\le \left\|\frac{1}{h}\varepsilon_h(w)\right\|^2_{L^2(\Omega)} + \frac{1}{h^2}\left|\int_\Omega w\cdot x^\perp dx\right|^2 + C\left\|\Big(\nabla\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\Big)\right\|^2_{L^2(\Omega)}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\sup_{\tau\in[0,t]}\left|\frac{1}{h^2}\Big(\partial_t D^2\tilde{W}(\nabla_h u_h(\tau))\nabla_h w(\tau)), \nabla_h\partial_t w(\tau)\Big)_{L^2(\Omega)}\right| \\
&\qquad\le CR\sup_{\tau\in[0,t]}\Big(\|\nabla_h w(\tau)\|_{L^2_h(\Omega)}\|\nabla_h\partial_t w(\tau)\|_{L^2_h(\Omega)}\Big) \\
&\qquad\le CR\left(\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2_{L^\infty(0,t;L^2(\Omega))} + \left\|\frac{1}{h}\int_\Omega w\cdot x^\perp dx\right\|^2_{L^\infty(0,t)}\right) \\
&\qquad\quad + CR\left(\left\|\frac{1}{h}\varepsilon_h(\partial_t w)\right\|^2_{L^\infty(0,t;L^2(\Omega))} + \left\|\frac{1}{h}\int_\Omega \partial_t w\cdot x^\perp dx\right\|^2_{L^\infty(0,t)}\right).
\end{aligned}$$

Putting everything together, using coercivity (5.19) of $D^2\tilde{W}(\nabla_h u_h)$ and Young's inequality we obtain

$$
\begin{aligned}
&\sup_{\tau\in[0,t]}\left\|\Big(\partial_t^2 w(\tau), \frac{1}{h}\varepsilon_h(\partial_t w(\tau))\Big)\right\|^2_{L^2(\Omega)} \\
&\quad\le \|w_2\|^2_{L^2(\Omega)} + |A_1| + CR\sup_{\tau\in[0,t]}\left|\frac{1}{h}\int_\Omega \partial_t w\cdot x^\perp dx\right|^2 + C\|\partial_t f\|^2_{L^1(0,T;L^2(\Omega))} \\
&\qquad + \frac{1}{2}\|\partial_t^2 w\|^2_{L^\infty(0,t;L^2(\Omega))} + CR\Big(\left\|\frac{1}{h}\varepsilon_h(\partial_t w)\right\|^2_{L^2(0,t;L^2(\Omega))} + \left\|\frac{1}{h}\int_\Omega \partial_t w\cdot x^\perp dx\right\|^2_{L^2(0,t)}\Big) \\
&\qquad + CR\Big(\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2_{L^2(0,t;L^2(\Omega))} + \left\|\frac{1}{h}\int_\Omega w\cdot x^\perp dx\right\|^2_{L^2(0,t)}\Big) \\
&\qquad + CR\Big(\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2_{L^\infty(0,t;L^2(\Omega))} + \left\|\frac{1}{h}\int_\Omega w\cdot x^\perp dx\right\|^2_{L^\infty(0,t)}\Big) \\
&\qquad + CR\Big(\left\|\frac{1}{h}\varepsilon_h(\partial_t w)\right\|^2_{L^\infty(0,t;L^2(\Omega))} + \left\|\frac{1}{h}\int_\Omega \partial_t w\cdot x^\perp dx\right\|^2_{L^\infty(0,t)}\Big) \\
&\qquad + CR\left\|\Big(\nabla\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\Big)\right\|^2_{L^2(0,t;L^2(\Omega))}
\end{aligned}
$$

uniformly in $0 \le t \le T$. We use an absorption argument for

$$
\frac{1}{2}\|\partial_t^2 w\|^2_{L^\infty(0,t;L^2(\Omega))} \quad\text{and}\quad CR\left\|\frac{1}{h}\varepsilon_h(\partial_t w)\right\|^2_{L^\infty(0,t;L^2(\Omega))},
$$

and the fact that $L^\infty(0,t) \hookrightarrow L^2(0,t)$ with $\|g\|_{L^2(0,t)} \le \sqrt{t}\|g\|_{L^\infty(0,t)}$. Now, due to (5.35)

$$
\begin{aligned}
\sup_{\tau\in[0,t]}\left\|\Big(\nabla\frac{1}{h}\varepsilon_h(w(\tau)), \nabla_h^2 w(\tau)\Big)\right\|^2_{L^2(\Omega)} &\le C\|f\|^2_{C^0(0,t;L^2)} + C\|\partial_t^2 w\|^2_{C^0(0,t;L^2(\Omega))} \\
&\quad + CRt\left\|\frac{1}{h}\int_\Omega w\cdot x^\perp dx\right\|^2_{L^\infty(0,t)}.
\end{aligned}
$$

Applying

$$
\|g\|_{C^0([0,t];L^2(\Omega))} \le C\Big(\|g\|_{W^1_1(0,t;L^2(\Omega))} + \|g(0)\|_{L^2(\Omega)}\Big) \text{ for all } g \in W^1_1(0,t;L^2(\Omega))
$$

for f, we arrive at

$$
\begin{aligned}
&\sup_{\tau\in[0,t]}\left\|\Big(\partial_t^2 w(\tau), \nabla_{x,t}\frac{1}{h}\varepsilon_h(w(\tau)), \nabla_h^2 w(\tau)\Big)\right\|^2_{L^2(\Omega)} \\
&\quad\le \sup_{\tau\in[0,t]}\left\|\Big(\partial_t^2 w(\tau), \frac{1}{h}\varepsilon_h(\partial_t w(\tau))\Big)\right\|^2_{L^2(\Omega)} + \sup_{\tau\in[0,t]}\left\|\Big(\nabla\frac{1}{h}\varepsilon_h(w(\tau)), \nabla_h^2 w(\tau)\Big)\right\|_{L^2(\Omega)} \\
&\quad\le \|w_2\|^2_{L^2(\Omega)} + |A_1| + C\|\partial_t f\|^2_{L^1(0,T;L^2)} + C\|f\|^2_{W^1_1(0,t;L^2(\Omega))} + C\|f|^2_{t=0}\|_{L^2(\Omega)} \\
&\qquad + CR(1+t)\left\|\frac{1}{h}\varepsilon_h(w)\right\|^2_{L^\infty(0,t;L^2(\Omega))} + CR(1+t)\max_{\sigma=0,1}\left\|\frac{1}{h}\int_\Omega \partial_t^\sigma w\cdot x^\perp dx\right\|^2_{L^\infty(0,t)} \\
&\qquad + CR\left\|\Big(\frac{1}{h}\varepsilon_h(\partial_t w), \nabla\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\Big)\right\|^2_{L^2(0,t;L^2(\Omega))}
\end{aligned}
$$

where we used $L^\infty(0,t) \hookrightarrow L^2(0,t)$ again. Hence, due to the basic inequality (5.2.3) and choose

R_0 small

$$
\begin{aligned}
\sup_{\tau\in[0,t]}&\left\|\left(\partial_t^2 w(\tau), \nabla_{x,t}\frac{1}{h}\varepsilon_h(w(\tau)), \nabla_h^2 w(\tau)\right)\right\|_{L^2(\Omega)}^2 \\
&\leq \|w_2\|_{L^2(\Omega)}^2 + |A_1| + C\|f|_{t=0}\|_{L^2(\Omega)}^2 + CR\lambda_t\Big(\|w_1\|_{L^2}^2 + |A_0| + \|f\|_{L^1(0,t;L^2)}^2\Big) \\
&\quad + C\|\partial_t f\|_{L^1(0,T;L^2)}^2 + C\|f\|_{W_1^1(0,t;L^2)}^2 + CR(1+t)\max_{\sigma=0,1}\left\|\frac{1}{h}\int_\Omega \partial_t^\sigma w\cdot x^\perp dx\right\|_{L^\infty(0,t)}^2 \\
&\quad + CR\left\|\left(\partial_t^2 w, \nabla_{t,x}\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\right)\right\|_{L^2(0,t;L^2)}^2
\end{aligned}
$$

where $\lambda_t = \max\{1,t\}$. As $0<t<T<\infty$, there exists $R_0\in(0,1]$, such that

$$
C\|\partial_t f\|_{L^1(0,T;L^2)}^2 + C\|f\|_{W_1^1(0,t;L^2)}^2 + CR\lambda_t\|f\|_{L^1(0,t;L^2)}^2 \leq C\|f\|_{W_1^1(0,T;L^2)}^2
$$

because of $W_1^1(0,t)\hookrightarrow L^1(0,t)$ with $\|g\|_{L^1(0,t)}\leq\|g\|_{W_1^1(0,t)}$. The Lemma of Gronwall yields then

$$
\begin{aligned}
\left\|\left(\partial_t^2 w,\nabla_{x,t}\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\right)\right\|_{C([0,T];L^2(\Omega))}^2 &\leq C_{L1}e^{C_1(1+T)R}\Big(\|f\|_{W_1^1(0,T;L^2)}^2 + |(A_0,A_1)| \\
&\quad + \|(w_0,w_1,f|_{t=0})\|_{L^2}^2 + (1+T)R\max_{\sigma=0,1}\left\|\frac{1}{h}\int_\Omega\partial_t^\sigma w\cdot x^\perp dx\right\|_{C^0([0,T])}^2\Big). \qquad \square
\end{aligned}
$$

Remark 5.2.8. The existence of a unique solution $w\in C^0(0,T;H_{per}^1(\Omega))\cap C^1(0,T;L^2(\Omega))$ for (5.12)–(5.15) under the conditions of Theorem 5.2.7 follows from Section 5.2.1. For higher regularity one would then apply classical hyperbolic regularity theory, cf. [Wlo87] and [LM72, Chapter 5]. Necessarily we need at this point that suitable compatibility conditions hold. As in the proof of the main result solutions of the linearised system are obtained via differentiation. We will not show the details here.

Theorem 5.2.9. *Assume that u_h satisfies* (5.16). *Then there exists $C>0$ and $R_0\in(0,1]$ such that if $\psi\in H_{per}^3(\Omega)$ solves*

$$
\begin{cases}
-\dfrac{1}{h^2}\operatorname{div}_h(D^2\tilde{W}(\nabla_h u_h)\nabla_h\psi) = q & \text{in } L^2(\Omega)\\
\qquad D^2\tilde{W}(\nabla_h u_h)[\nabla_h\psi]\nu\Big|_{\partial S} = q_N & \text{in } H^{\frac12}(S)
\end{cases}
\tag{5.41}
$$

for some $q\in H_{per}^1(\Omega)$ and $q_N\in L^2(0,L;H^{\frac32}(\partial S))\cap H^1(0,L;H^{\frac12}(\partial S))$, then

$$
\begin{aligned}
\left\|\left(\nabla\frac{1}{h}\varepsilon_h(\psi),\nabla_h^2\psi\right)\right\|_{H^1(\Omega)} &\leq C\Big(h^2\|q\|_{H^1(\Omega)} + \left\|\frac{1}{h}q_N\right\|_{L^2(0,L;H^{\frac32}(\partial S))\cap H^1(0,L;H^{\frac12}(\partial S))} \\
&\quad + \left\|\frac{1}{h}\varepsilon_h(\psi)\right\|_{H^{0,2}(\Omega)} + R\left|\frac{1}{h}\int_\Omega\psi\cdot x^\perp dx\right|\Big)
\end{aligned}
\tag{5.42}
$$

Proof: We prove the theorem in two steps. In the first step we differentiate (5.41) in direction of x_1 and will obtain by Theorem 5.2.4

$$
\begin{aligned}
\|\nabla_h^2\partial_{x_1}\psi\|_{L^2(\Omega)} &\leq C\Big(h^2\|q\|_{H^{0,1}(\Omega)} + \left\|\frac{1}{h}\varepsilon_h(\psi)\right\|_{H^{0,2}(\Omega)} + \left\|\frac{1}{h}q_N\right\|_{H^1(0,L;H^{\frac12}(\partial S))} \\
&\quad + R\left|\frac{1}{h}\int_\Omega\psi\cdot x^\perp dx\right| + Rh\|\nabla_h^2\psi\|_{H^1(\Omega)}\Big).
\end{aligned}
$$

The second step consists of an application of Theorem 2.1.18 with $r = 1$ and analogous inequalities as in Theorem 5.2.4.

Step 1: Differentiation in x_1 direction of (5.41) leads to

$$\begin{cases} -\frac{1}{h^2}\operatorname{div}_h\left(D^2\tilde{W}(\nabla_h u_h)\nabla_h\partial_{x_1}\psi\right) = \partial_{x_1}q + \frac{1}{h^2}\operatorname{div}_h\left(\partial_{x_1}D^2\tilde{W}(\nabla_h u_h)\nabla_h\psi\right) & \text{in } \Omega \\ D^2\tilde{W}(\nabla_h u_h)\nabla_h\partial_{x_1}\psi\nu\Big|_{(0,L)\times\partial S} = \partial_{x_1}q_N - \partial_{x_1}D^2\tilde{W}(\nabla_h u_h)\nabla_h\psi\nu\Big|_{(0,L)\times\partial S} & \text{on } \partial\Omega. \end{cases}$$

Consequently we can apply Theorem 5.2.4 with

$$\begin{aligned} \varphi &:= \partial_{x_1}\psi \\ g &:= \partial_{x_1}q + \frac{1}{h^2}\operatorname{div}_h\left((\partial_{x_1}D^2\tilde{W}(\nabla_h u_h))\nabla_h\psi\right) \\ g_N &:= \partial_{x_1}q_N - \partial_{x_1}D^2\tilde{W}(\nabla_h u_h)\nabla_h\psi\Big|_{(0,L)\times\partial S}. \end{aligned}$$

This leads, because of $\int_\Omega \partial_{x_1}\psi\cdot x^\perp dx = 0$, to

$$\begin{aligned} &\left\|\left(\nabla\frac{1}{h}\varepsilon_h(\partial_{x_1}\psi), \nabla_h^2\partial_{x_1}\psi\right)\right\|_{L^2(\Omega)} \\ &\qquad \le C\left(h^2\|g\|_{L^2(\Omega)} + \left\|\frac{1}{h}g_N\right\|_{L^2(0,L;H^{\frac{1}{2}}(\partial S))} + \left\|\frac{1}{h}\varepsilon_h(\partial_{x_1}\psi)\right\|_{H^{0,1}(\Omega)}\right). \end{aligned} \tag{5.43}$$

Starting with the second part of the right hand side, we notice

$$\|g_N\|_{L^2(0,L;H^{\frac{1}{2}}(\partial S))} \le C\big(\|q_N\|_{H^1(0,L;H^{\frac{1}{2}}(\partial S))} + \|(\partial_{x_1}D^2\tilde{W}(\nabla_h u_h))\nabla_h\psi\|_{L^2(0,L;H^1(S))}\big)$$

due to boundedness of $\operatorname{tr}_{\partial S}\colon H^1(S)\to H^{\frac{1}{2}}(\partial S)$ and the definition of g_N. Using that $\nabla_h\partial_{x_1}u_h \in H^2(\Omega)$ is of order Rh we are led to

$$\begin{aligned} \|(\partial_{x_1}D^2\tilde{W}(\nabla_h u_h))\nabla_h\psi\|_{L^2(0,L;H^1(S))} &\le \|D^3\tilde{W}(\nabla_h u_h)[\nabla_h\partial_{x_1}u_h, \nabla_h\psi]\|_{H^1(\Omega)} \\ &\le CRh\|\nabla_h\psi\|_{H^1(\Omega)}. \end{aligned}$$

where we used that $D^3\tilde{W}\in C^\infty(B_\delta(0);\mathcal{L}^3(\mathbb{R}^{3\times 3}))$ and $\nabla_h u_h\in C^0(\overline{\Omega})$. Hence it follows

$$\|g_N\|_{L^2(0,L;H^{\frac{1}{2}}(S))} \le C\big(\|q_N\|_{H^1(0,L;H^{\frac{1}{2}}(\partial S))} + Rh\|\nabla_h\psi\|_{H^1(\Omega)}\big).$$

Now we start to estimate the second part of g in $L^2(\Omega)$. First it holds

$$\begin{aligned} \left\|\operatorname{div}_h\left((\partial_{x_1}D^2\tilde{W}(\nabla_h u_h))\nabla_h\psi\right)\right\|_{L^2(\Omega)} &\le \frac{1}{h}\sum_{j=1}^3\left\|\partial_{x_j}\left((\partial_{x_1}D^2\tilde{W}(\nabla_h u_h))\nabla_h\psi\right)\right\|_{L^2(\Omega)} \\ &\le \|(\partial_{x_j}\partial_{x_1}D^2\tilde{W}(\nabla_h u_h))\nabla_h\psi\|_{L^2(\Omega)} + \|(\partial_{x_1}D^2\tilde{W}(\nabla_h u_h))\nabla_h\partial_{x_j}\psi\|_{L^2(\Omega)} \end{aligned}$$

Dealing with each term separately, it follows for $j = 1, 2, 3$

$$\begin{aligned} \|(\partial_{x_j}\partial_{x_1}D^2\tilde{W}(\nabla_h u_h))\nabla_h\psi\|_{L^2(\Omega)} &\le \|D^3\tilde{W}(\nabla_h u_h)[\nabla_h\partial_{x_j}\partial_{x_1}u_h, \nabla_h\psi]\|_{L^2(\Omega)} \\ &\quad + \|D^4\tilde{W}(\nabla_h u_h)[\nabla_h\partial_{x_1}u_h, \nabla_h\partial_{x_j}u_h, \nabla_h\psi]\|_{L^2(\Omega)}. \end{aligned}$$

Using twice Hölder's inequality with $p = \frac{3}{2}$, $q = 3$ and $p = q = 2$, respectively, and $\nabla_h u_h \in H^1(\Omega)\hookrightarrow L^6(\Omega)$ yields

$$\begin{aligned} &\|D^4\tilde{W}(\nabla_h u_h)[\nabla_h\partial_{x_1}u_h, \nabla_h\partial_{x_j}u_h, \nabla_h\psi]\|_{L^2(\Omega)} \\ &\qquad \le C\|\nabla_h\partial_{x_1}u_h\|_{L^6(\Omega)}\|\nabla_h\partial_{x_j}u_h\|_{L^6(\Omega)}\|\nabla_h\psi\|_{L^6(\Omega)} \le CRh\|\nabla_h\psi\|_{H^1(\Omega)}. \end{aligned}$$

Analogously with $p = q = 2$ and $\nabla_h \partial_{x_j} \partial_{x_1} u_h \in H^1(\Omega) \hookrightarrow L^4(\Omega)$ we obtain

$$\begin{aligned}\|D^3\tilde{W}(\nabla_h u_h)[\nabla_h \partial_{x_j}\partial_{x_1} u_h, \nabla_h \psi]\|_{L^2(\Omega)} &\leq C\|\nabla_h \partial_{x_j}\partial_{x_1} u_h\|_{L^4(\Omega)} \|\nabla_h \psi\|_{L^4(\Omega)} \\ &\leq CRh\|\nabla_h \psi\|_{H^1(\Omega)}\end{aligned}$$

and

$$\begin{aligned}\|(\partial_{x_1} D^2\tilde{W}(\nabla_h u_h))\nabla_h \partial_{x_j}\psi\|_{L^2(\Omega)} &\leq C\|\nabla_h \partial_{x_1} u_h\|_{L^4(\Omega)} \|\nabla_h \partial_{x_j}\psi\|_{L^4(\Omega)} \\ &\leq CRh\|\nabla_h \psi\|_{H^2(\Omega)}.\end{aligned}$$

Summarizing we are led to

$$\left\| \operatorname{div}_h \Big((\partial_{x_1} D^2\tilde{W}(\nabla_h u_h))\nabla_h \psi \Big) \right\|_{L^2(\Omega)} \leq CRh\|\nabla_h \psi\|_{H^2(\Omega)}$$

Hence

$$\|g\|_{L^2(\Omega)} \leq \|\partial_{x_1} q\|_{L^2(\Omega)} + \frac{CR}{h}\|\nabla_h \psi\|_{H^2(\Omega)}$$

Consequently, plugging all inequalities in (5.43)

$$\begin{aligned}\left\| \Big(\nabla \frac{1}{h}\varepsilon_h(\partial_{x_1}\psi), \nabla_h^2 \partial_{x_1}\psi\Big)\right\|_{L^2(\Omega)} &\leq C\bigg(h^2\|\partial_{x_1} q\|_{L^2(\Omega)} + \left\|\frac{1}{h} q_N\right\|_{H^1(0,L;H^{\frac{1}{2}}(\partial S))} \\ &\quad + Rh\|\nabla_h \psi\|_{H^2(\Omega)} + \left\|\frac{1}{h}\varepsilon_h(\psi)\right\|_{H^{0,2}(\Omega)}\bigg).\end{aligned}$$

Applying now (5.24) for $\|\nabla_h \psi\|_{H^1(\Omega)}$ it follows

$$\begin{aligned}\left\| \Big(\nabla \frac{1}{h}\varepsilon_h(\partial_{x_1}\psi), \nabla_h^2 \partial_{x_1}\psi\Big)\right\|_{L^2(\Omega)} &\leq C\bigg(h^2\|q\|_{H^{0,1}(\Omega)} + \left\|\frac{1}{h} q_N\right\|_{H^1(0,L;H^{\frac{1}{2}})} + \left\|\frac{1}{h}\varepsilon_h(\psi)\right\|_{H^{0,2}(\Omega)} \\ &\quad + Rh\left|\frac{1}{h}\int_\Omega \psi \cdot x^\perp dx\right| + Rh\|\nabla_h^2 \psi\|_{H^1(\Omega)}\bigg). \qquad (5.44)\end{aligned}$$

STEP 2: As $\psi \in H^3_{per}(\Omega)$ solves (5.41), we obtain as in Theorem 5.2.4 that

$$\psi_{(0)}(x_1, \cdot) = \psi(x_1, \cdot) - \frac{1}{\mu(S)}(\psi, x^\perp)_{L^2(S)} x^\perp - \frac{1}{|S|}(\psi, 1)_{L^2(S)}$$

solves for almost every $x_1 \in (0, L)$

$$\begin{cases} -\operatorname{div}_{x'}\Big(D^2\tilde{W}(0)^{\approx}\nabla_{x'}\frac{1}{h^2}\psi(x_1,\cdot)\Big) = \tilde{f}(x_1,\cdot) & \text{in } S \\ D^2\tilde{W}(0)^{\approx}\nabla_{x'}\frac{1}{h}\psi(x_1,\cdot)\nu_{\partial S}\Big|_{\partial S} = q_N(x_1,\cdot) - a_N(x_1,\cdot) & \text{on } \partial S \end{cases}$$

where $\tilde{f}$ and a_N is as in (5.28) and (5.25), respectively, with φ replaced by ψ. We want to apply Theorem 2.1.18 for higher regularity. Due to the assumptions it follows, because of $\psi \in H^3_{per}(\Omega)$: $D^2\tilde{W}(0)\partial_{x_1}\psi \otimes e_1 \in H^2_{per}(\Omega)$. Moreover for G defined in (5.29) we obtain

$$G(\nabla_h u_h)[\nabla_h u_h, \nabla_h \psi] = \int_0^1 D^3\tilde{W}(\tau \nabla_h u_h)[\nabla_h u_h, \nabla_h \psi] d\tau \in H^2_{per}(\Omega).$$

Here we used, for $|\beta| = 2$ and $k,\ l \in \{1,2,3\}$ such that $\partial_x^\beta = \partial_{x_k}\partial_{x_l}$

$$\partial_x^\beta \Big(G(\nabla_h u_h)[\nabla_h u_h]\nabla_h \psi \Big) = D^2 G(\nabla_h u_h)[\nabla_h \partial_{x_l} u_h, \nabla_h \partial_{x_k} u_h, \nabla_h u_h]\nabla_h \psi$$

$$
\begin{aligned}
&+ DG(\nabla_h u_h)[\nabla_h \partial_x^\beta u_h, \nabla_h u]\nabla_h \psi + 2DG(\nabla_h u_h)[\nabla_h \partial_{x_k} u_h, \nabla_h \partial_{x_l} u]\nabla_h \psi \\
&+ DG(\nabla_h u_h)[\nabla_h u_h, \nabla_h \partial_{x_l} u]\nabla_h \partial_{x_k}\psi + DG(\nabla_h u_h)[\nabla_h \partial_{x_k} u_h, \nabla_h u]\nabla_h \partial_{x_l}\psi \\
&+ G(\nabla_h u_h)[\nabla_h \partial_x^\beta u_h]\nabla_h \psi \;+ G(\nabla_h u_h)[\nabla_h \partial_{x_k} u_h]\nabla_h \partial_{x_l}\psi \\
&+ G(\nabla_h u_h)[\nabla_h \partial_{x_l} u_h]\nabla_h \partial_{x_k}\psi + G(\nabla_h u_h)[\nabla_h u_h]\nabla_h \partial_x^\beta \psi
\end{aligned}
$$

holds and belongs to $L^2(\Omega)$, where we exploit $\nabla_h u_h \in H^2_{per}(\Omega) \hookrightarrow C^0(\Omega)$, $\nabla_h \partial_x^\beta u_h \in H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $\nabla_h \partial_x^\beta \psi \in L^2(\Omega)$.
In conclusion it follows $\tilde{f} \in H^1_{per}(\Omega)$, as $f \in H^1_{per}(\Omega)$. Moreover, because S is a smooth domain and $\mathrm{tr}_{\partial S}\colon H^2(S) \to H^{\frac{3}{2}}(\partial S)$ is a bounded, linear operator it follows for almost all $x_1 \in (0, L)$: $a_N(x_1, \cdot) \in H^{\frac{3}{2}}(\partial S)$. Therefore $(q_N(x_1, \cdot) - a_N(x_1, \cdot)) \in H^{\frac{3}{2}}(\partial S)$ due to the assumptions on q_N. Thus we can apply Theorem 2.1.18 with $r = 1$ and obtain for almost all $x_1 \in (0, L)$

$$
\begin{aligned}
\|\nabla^2_{h,x'}\psi_{(0)}(x_1,\cdot)\|_{H^1(S)} &= \|\nabla^2_{h,x'}\psi(x_1,\cdot)\|_{H^1(S)} \\
&\le C\biggl(\Bigl\|\frac{1}{h^2}\psi_{(0)}(x_1,\cdot)\Bigr\|_{H^1(S)} + \Bigl\|\frac{1}{h}\bigl(q_N(x_1,\cdot) - a_N(x_1,\cdot)\bigr)\Bigr\|_{H^{\frac{3}{2}}(\partial S)} + \|\tilde{f}(x_1,\cdot)\|_{H^1(S)}\biggr).
\end{aligned}
$$

Hence with Theorem 5.2.5

$$
\|\nabla^2_{h,x'}\psi\|_{L^2(0,L;H^1(S))} \le C\biggl(\|\tilde{f}\|_{L^2(0,L;H^1(S))} + \Bigl\|\frac{1}{h}(q_N - a_N)\Bigr\|_{L^2(0,L;H^{\frac{3}{2}}(\partial S))}\biggr).
$$

Using, as in the proof of Theorem 5.2.4, Poincaré's inequality and

$$
\int_{\partial S}(g_N - a_N)(x_1, x')dx' = h\int_S \tilde{f}(x_1, x')dx'
$$

it follows

$$
\begin{aligned}
\|\nabla_{h,x'}\psi\|_{L^2(0,L;H^1(S))} \le C\biggl(&\|\tilde{f}\|_{L^2(0,L;H^1(S))} + \Bigl\|\frac{1}{h}q_N\Bigr\|_{L^2(0,L;H^{\frac{3}{2}}(\partial S))} \\
&+ \Bigl\|\frac{1}{h}\nabla_{x'}r_N\Bigr\|_{L^2(0,L;H^1(S))}\biggr)
\end{aligned}
$$

With the definition of r_N and the bound (5.16) we obtain

$$
\Bigl\|\frac{1}{h}\nabla_{x'}r_N\Bigr\|_{L^2(0,L;H^1(S))} \le C\Bigl\|\frac{1}{h}\varepsilon_h(\psi)\Bigr\|_{H^{1,1}(\Omega)} + CRh\|\nabla_h^2\psi\|_{H^1(\Omega)}
$$

As $\mathcal{R}(\partial_{x_1}\psi)$ is a linear combination of entries of $\nabla_h \partial_{x_1}\psi$ we have

$$
\|\mathcal{R}(\partial_{x_1}\psi)\|_{L^2(0,L;H^1(S))} \le C\|\nabla_h \partial_{x_1}\psi\|_{H^1(\Omega)} \le C\Bigl\|\frac{1}{h}\varepsilon_h(\psi)\Bigr\|_{H^{1,1}(\Omega)}.
$$

Moreover, similar to (5.30) we can bound

$$
\begin{aligned}
&\Bigl\|\operatorname{div}_h\Bigl(G(\nabla_h u_h)[\nabla_h u_h, \nabla_h \psi]\Bigr)\Bigr\|_{L^2(0,L;H^1(S))} \\
&\qquad\le \frac{1}{h}\sup_{|\alpha|=1}\|\partial_x^\alpha(G(\nabla_h u_h)[\nabla_h u_h, \nabla_h \psi])\|_{L^2(0,L;H^1(S))}.
\end{aligned}
$$

First we notice that

$$
\sup_{|\alpha|=1}\|\partial_x^\alpha(G(\nabla_h u_h)[\nabla_h u_h, \nabla_h \psi])\|_{L^2(0,L;H^1(S))}
$$

$$\leq \max_{k=2,3} \sup_{|\alpha|=1} \|\partial_{x_k} \partial_x^\alpha (G(\nabla_h u_h)[\nabla_h u_h, \nabla_h \psi])\|_{L^2(\Omega)}$$
$$+ \sup_{|\alpha|=1} \|\partial_x^\alpha (G(\nabla_h u_h)[\nabla_h u_h, \nabla_h \psi])\|_{L^2(\Omega)}$$

and thus we only have to bound the first part on the right hand side, as the second summand is bounded in (5.30). For the first summand we use $G \in C^\infty(\overline{B_\varepsilon(0)}; \mathcal{L}^3(\mathbb{R}^{3\times 3}))$, the regularity of u_h and assumption (5.5) to obtain

$$\left\| \operatorname{div}_h \Big(G(\nabla_h u_h)[\nabla_h u_h, \nabla_h \psi] \Big) \right\|_{L^2(0,L;H^1(S))} \leq CR \|\nabla_h \psi\|_{H^2(\Omega)}$$

Summing up the inequalities of the second step leads to

$$\begin{aligned}
&\|\nabla^2_{h,x'} \psi\|_{L^2(0,L;H^1(S))} \\
&\quad \leq C\bigg(h^2 \|q\|_{H^1(\Omega)} + \left\| \frac{1}{h} q_N \right\|_{L^2(0,L;H^{\frac{3}{2}}(\partial S))} + \left\| \frac{1}{h} \varepsilon_h(\psi) \right\|_{H^{1,1}(\Omega)} + R \|(\nabla_h \psi, \nabla_h^2 \psi)\|_{H^1(\Omega)} \bigg) \\
&\quad \leq C\bigg(h^2 \|q\|_{H^1(\Omega)} + \left\| \frac{1}{h} q_N \right\|_{L^2(0,L;H^{\frac{3}{2}}(\partial S))} + \left\| \frac{1}{h} \varepsilon_h(\psi) \right\|_{H^{1,1}(\Omega)} \bigg) \\
&\qquad + CR\bigg(\|\nabla^2_{h,x'} \psi\|_{H^1(\Omega)} + \left| \frac{1}{h} \int_\Omega \psi \cdot x^\perp dx \right| \bigg)
\end{aligned}$$

where we used Korn inequality for $\|\nabla_h \psi\|_{L^2(\Omega)}$, Theorem 5.2.4 and the structure of $\nabla_h^2 \psi$. Hence with

$$\|\nabla^2_{h,x'} \psi\|_{H^1(\Omega)} \leq C \|\nabla^2_{h,x'} \psi\|_{L^2(0,L;H^1(S))} + C \|\nabla_{h,x'} \partial_{x_1} \psi\|_{L^2(\Omega)}$$

we can absorbe $\|\nabla^2_{h,x'} \psi\|_{L^2(0,L;H^1(S))}$ choosing R_0 sufficiently small. With this we arrive at

$$\begin{aligned}
\|\nabla^2_{h,x'} \psi\|_{L^2(0,L;H^1(S))} \leq C\bigg(& h^2 \|q\|_{H^1(\Omega)} + \left\| \frac{1}{h} q_N \right\|_{L^2(0,L;H^{\frac{3}{2}}(\partial S))} + \left\| \frac{1}{h} \varepsilon_h(\psi) \right\|_{H^{1,1}(\Omega)} \\
& + R \left| \frac{1}{h} \int_\Omega \psi \cdot x^\perp dx \right| + \|\nabla^2_{h,x'} \partial_{x_1} \psi\|_{L^2(\Omega)} \bigg).
\end{aligned} \tag{5.45}$$

Putting step one and two together it follows, using (5.44), (5.45) and (5.24) of Theorem 5.2.6

$$\begin{aligned}
\|\nabla_h^2 \psi\|_{H^1(\Omega)} &\leq \|\nabla_h^2 \psi\|_{L^2(\Omega)} + \|\nabla_h^2 \partial_{x_1} \psi\|_{L^2(\Omega)} + \|\nabla^2_{h,x'} \psi\|_{L^2(0,L;H^1(S))} \\
&\leq C\bigg(h^2 \|q\|_{H^1(\Omega)} + \left\| \frac{1}{h} q_N \right\|_{L^2(0,L;H^{\frac{3}{2}}(\partial S)) \cap H^1(0,L;H^{\frac{1}{2}}(\partial S))} + \left\| \frac{1}{h} \varepsilon_h(\psi) \right\|_{H^{0,2}(\Omega)} \\
&\qquad + R \left| \frac{1}{h} \int_\Omega \psi \cdot x^\perp dx \right| + R \|\nabla_h^2 \psi\|_{H^1(\Omega)} + \left\| \left(\nabla \frac{1}{h} \varepsilon_h(\partial_{x_1} \psi), \nabla^2_{h,x'} \partial_{x_1} \psi \right) \right\|_{L^2(\Omega)} \bigg)
\end{aligned}$$

Another application of (5.44) for the last term of the later inequality leads to

$$\begin{aligned}
\|\nabla_h^2 \psi\|_{H^1(\Omega)} \leq C\bigg(& h^2 \|q\|_{H^1(\Omega)} + \left\| \frac{1}{h} q_N \right\|_{L^2(0,L;H^{\frac{3}{2}}(\partial S)) \cap H^1(0,L;H^{\frac{1}{2}}(\partial S))} + \left\| \frac{1}{h} \varepsilon_h(\psi) \right\|_{H^{0,2}(\Omega)} \\
& + R \left| \frac{1}{h} \int_\Omega \psi \cdot x^\perp dx \right| + R \|\nabla_h^2 \psi\|_{H^1(\Omega)} \bigg).
\end{aligned}$$

With an appropriate choice of $R_0 \in (0, 1]$, we can absorb $\|\nabla_h^2 \psi\|_{H^1(\Omega)}$ in the left hand side and arrive at the claimed inequality. □

Corollary 5.2.10. *Assume that u_h satisfies* (5.16). *If $w \in H^3_{per}(\Omega)$ solves*

$$\begin{cases} -\dfrac{1}{h^2}\operatorname{div}_h(D^2\tilde{W}(\nabla_h u_h)\nabla_h w) = f & \text{in } L^2(\Omega) \\ D^2\tilde{W}(\nabla_h u_h)[\nabla_h w]\nu\Big|_{\partial S} = 0 & \text{in } H^{\frac{1}{2}}(S) \end{cases} \tag{5.46}$$

for some $f \in H^1_{per}(\Omega)$, then

$$\left\| \left(\nabla \frac{1}{h}\varepsilon_h(w), \nabla_h^2 w \right) \right\|_{H^1(\Omega)} \le C\left(\|f\|_{H^1(\Omega)} + R\left| \frac{1}{h}\int_\Omega w \cdot x^\perp dx \right| \right). \tag{5.47}$$

Proof: We can apply Theorem 5.2.9 for $\psi := w$, $q := f$ and $q_N := 0$ and obtain with (5.42)

$$\left\| \left(\nabla \frac{1}{h}\varepsilon_h(w), \nabla_h^2 w \right) \right\|_{H^1(\Omega)} \le C\left(h^2\|f\|_{H^1(\Omega)} + \left\| \frac{1}{h}\varepsilon_h(w) \right\|_{H^{0,2}(\Omega)} + R\left| \frac{1}{h}\int_\Omega w \cdot x^\perp dx \right| \right).$$

From the proof of Corollary 5.2.6 we have already a bound for $\|\frac{1}{h}\varepsilon_h(w)\|_{H^{0,1}(\Omega)}$. Hence, it remains to estimate $\|\frac{1}{h}\varepsilon_h(\partial^2_{x_1} w)\|_{L^2(\Omega)}$. The bound can be seen as follows: Integration by parts leads for all $\varphi \in H^1_{per}(\Omega)$ to

$$\frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h \varphi \Big)_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)}.$$

Analogously to Corollary 5.2.6 we can choose first $\varphi = \partial^2_{x_1}\psi$ for $\psi \in C^\infty_{per}(\Omega)$. Thus, twice integration by parts leads to

$$\frac{1}{h^2}\Big(\partial^2_{x_1}(D^2\tilde{W}(\nabla_h u_h)\nabla_h w), \nabla_h \psi \Big)_{L^2(\Omega)} = (\partial_{x_1} f, \partial_{x_1}\psi)_{L^2(\Omega)}.$$

Then the density of $C^\infty_{per}(\Omega) \subset H^1_{per}(\Omega)$ implies that the latter equaltiy holds for $\psi = \partial^2_{x_1} w$. Using

$$\begin{aligned} \Big(\partial^2_{x_1}(D^2\tilde{W}(\nabla_h u_h)\nabla_h w), \nabla_h \partial^2_{x_1} w \Big)_{L^2(\Omega)} &= \Big(D^2\tilde{W}(\nabla_h u_h)\nabla_h \partial^2_{x_1} w, \nabla_h \partial^2_{x_1} w \Big)_{L^2(\Omega)} \\ + 2\Big((\partial_{x_1} D^2\tilde{W}(\nabla_h u_h))\nabla_h \partial_{x_1} w, \nabla_h \partial^2_{x_1} w \Big)_{L^2(\Omega)} &+ \Big((\partial^2_{x_1} D^2\tilde{W}(\nabla_h u_h))\nabla_h w, \nabla_h \partial^2_{x_1} w \Big)_{L^2(\Omega)}. \end{aligned}$$

By virtue of Lemma 5.2.2, we obtain

$$\begin{aligned} \left\| \frac{1}{h}\varepsilon_h(\partial^2_{x_1} w) \right\|^2_{L^2(\Omega)} &\le |(\partial_{x_1} f, \partial^3_{x_1} w)_{L^2(\Omega)}| + CR\|\nabla_h \partial_{x_1} w\|_{L^2_h(\Omega)} \|\nabla_h \partial^2_{x_1} w\|_{L^2_h(\Omega)} \\ &\quad + CR\|\nabla_h w\|_{H^1_h(\Omega)} \|\nabla_h \partial^2_{x_1} w\|_{L^2_h(\Omega)} \\ &\le C\|f\|^2_{H^{0,1}(\Omega)} + \epsilon\left\| \frac{1}{h}\varepsilon_h(\partial^2_{x_1} w) \right\|^2_{L^2(\Omega)} + CR\|\nabla_h \partial_{x_1} w\|^2_{L^2(\Omega)} \\ &\quad + CR\|\nabla_h w\|^2_{H^1_h(\Omega)} + CR\|\nabla_h \partial^2_{x_1} w\|^2_{L^2_h(\Omega)} \\ &\le C\|f\|_{H^{0,1}(\Omega)} + (\epsilon + CR)\left\| \frac{1}{h}\varepsilon_h(\partial^2_{x_1} w) \right\|^2_{L^2(\Omega)} \\ &\quad + CR\left\| \frac{1}{h}\varepsilon_h(w) \right\|_{H^{0,1}(\Omega)} + CR\left| \int_\Omega w \cdot x^\perp dx \right| \end{aligned}$$

Finally, choosing ϵ and R_0 small, using an absorption argument and applying (5.38) and (5.39) of the proof of Corollary 5.2.6, leads to the desired inequality. □

Theorem 5.2.11 (Third Order Inequality). *Let $0 < T < \infty$, $h \in (0,1]$, $0 < R \leq R_0$ be given, where R_0 is chosen small enough, but independent of h and let u_h satisfy (5.16). Assume $w \in \bigcap_{k=0}^{3} C^k([0,T]; H^{3-k}_{per}(\Omega))$ to be the unique solution of the linearised system (5.12)–(5.15) for some $f \in W^2_1(0,T; L^2(\Omega)) \cap W^1_1(0,T; H^1_{per}(\Omega))$, $w_0 \in H^3_{per}(\Omega)$ and $w_1 \in H^2_{per}(\Omega)$. Then there exist constants C_{L2}, $C_2 > 0$ depending on T such that*

$$\begin{aligned}
&\left\| \left(\partial_t^3 w, \nabla^2_{t,x} \frac{1}{h}\varepsilon_h(w), \nabla_{x,t}\nabla_h^2 w \right) \right\|^2_{C^0(0,T;L^2(\Omega))} \\
&\leq C_{L2} e^{C_2 R(1+\sqrt{T})} \Bigg(\|\partial_t^2 f\|^2_{L^1(0,T;L^2)} + \|\partial_t f\|^2_{L^\infty(0,T;L^2)\cap L^1(0,T;H^{0,1})} + \|f\|^2_{L^\infty(0,T;H^1)} \\
&\quad + \|(w_1, w_2, w_3, f|_{t=0})\|^2_{L^2} + |(A_0, A_1, A_2)| + \|\partial_{x_1} w_2\|^2_{L^2} \\
&\quad + \frac{1}{h^2} \left| \left(D^2 \tilde{W}(\nabla_h u_h|_{t=0}) \nabla_h \partial_{x_1} w_1, \nabla_h \partial_{x_1} w_1 \right)_{L^2(\Omega)} \right| + \frac{1}{h^2} \left\| \int_\Omega \partial_t^2 w \cdot x^\perp dx \right\|^2_{L^\infty(0,T)} \\
&\quad + (1+T) R \max_{\sigma=0,1,2} \left\| \frac{1}{h} \int_\Omega \partial_t^\sigma w \cdot x^\perp dx \right\|^2_{C^0([0,T])} \Bigg)
\end{aligned}$$

where

$$A_2 := \frac{1}{h^2} \Big(D^2 \tilde{W}(\nabla_h u_h|_{t=0}) \nabla_h w_2, \nabla_h w_2 \Big)_{L^2(\Omega)}$$

and

$$w_3 := \frac{1}{h^2} \operatorname{div}_h \left(D^2 \tilde{W}(\nabla_h u_h|_{t=0}) \nabla_h w_1 \right) + \partial_t f|_{t=0}$$

Proof: Differentiation with respect to z_j of the system (5.12)–(5.13) leads to

$$\begin{aligned}
\partial_t^2 \tilde{w}_j - \frac{1}{h^2} \operatorname{div}_h \left(D^2 \tilde{W}(\nabla_h u_h) \nabla_h \tilde{w}_j \right) &= \partial_{z_j} f + \frac{1}{h^2} \operatorname{div}_h \left(\partial_{z_j} D^2 \tilde{W}(\nabla_h u_h) \nabla_h w \right) \\
D^2 \tilde{W}(\nabla_h u_h) \nabla_h \tilde{w}_j \nu \Big|_{(0,L)\times\partial S} &= -\partial_{z_j} D^2 \tilde{W}(\nabla_h u_h) \nabla_h w \nu \Big|_{(0,L)\times\partial S} \\
&\tilde{w} \text{ is } L\text{-periodic with respect to } x_1
\end{aligned}$$

where $\tilde{w}_j = \partial_{z_j} w$. First we want to apply Theorem 5.2.4 with $\varphi := \tilde{w}_j$ and

$$\begin{aligned}
g &:= \tilde{f}_j + \frac{1}{h^2} \operatorname{div}_h \left(\partial_{z_j} D^2 \tilde{W}(\nabla_h u_h) \nabla_h w \right) - \partial_t^2 \tilde{w}_j \\
g_N &:= -\partial_{z_j} D^2 \tilde{W}(\nabla_h u_h) \nabla_h w \nu \Big|_{(0,L)\times\partial S}
\end{aligned}$$

where we used the convention $\tilde{f}_j := \partial_{z_j} f$. Then we obtain

$$\begin{aligned}
\left\| \left(\nabla \frac{1}{h} \varepsilon_h(\tilde{w}_j), \nabla_h^2 \tilde{w}_j \right) \right\|_{L^2(\Omega)} &\leq C h^2 \left\| \tilde{f}_j - \partial_t^2 \tilde{w}_j + \frac{1}{h^2} \operatorname{div}_h \left(\partial_{z_j} D^2 \tilde{W}(\nabla_h u_h) \nabla_h w \right) \right\|_{L^2(\Omega)} \\
&\quad + C \left\| \frac{1}{h} \operatorname{tr}_{\partial S} \left(\partial_{z_j} D^2 \tilde{W}(\nabla_h u_h) \partial_h w \right) \right\|_{L^2(0,L;H^{\frac{1}{2}}(\partial S))} \\
&\quad + C \left\| \frac{1}{h} \varepsilon_h(\tilde{w}_j) \right\|_{H^{0,1}(\Omega)} + CR \left| \frac{1}{h} \int_\Omega \tilde{w}_j \cdot x^\perp dx \right|.
\end{aligned}$$

Moreover, as $D^3 \tilde{W}(\nabla_h u_h)$ is uniformly bounded and u_h satisfies (5.16)

$$\begin{aligned}
\left\| \operatorname{div}_h \left(\partial_{z_j} D^2 \tilde{W}(\nabla_h u_h) \nabla_h w \right) \right\|_{L^2(\Omega)} &\leq \left\| \operatorname{div}_h \left(D^3 \tilde{W}(\nabla_h u_h)[\nabla_h \partial_{z_j} u_h] \nabla_h w \right) \right\|_{L^2(\Omega)} \\
&\leq CR \left\| \left(\nabla_h w, \nabla_h^2 w \right) \right\|_{L^2(\Omega)}
\end{aligned}$$

and

$$
\begin{aligned}
&\left\| \frac{1}{h} \operatorname{tr}_{\partial S} \left(\partial_{z_j} D^2 \tilde{W}(\nabla_h u_h) \nabla_h w \right) \right\|_{L^2(0,L;H^{\frac{1}{2}}(\partial S))} \\
&\qquad \leq C \left\| \frac{1}{h} D^3 \tilde{W}(\nabla_h u_h)[\nabla_h \partial_{z_j} u_h] \nabla_h w \right) \right\|_{L^2(0,L;H^1(S))} \\
&\qquad \leq CR \|\nabla_h w\|_{L^2(\Omega)} + CRh \|\nabla_h^2 w\|_{L^2(\Omega)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left\| \left(\nabla \frac{1}{h} \varepsilon_h(\tilde{w}_j), \nabla_h^2 \tilde{w}_j \right) \right\|_{L^2(\Omega)} \\
&\leq Ch^2 \left(\|\tilde{f}_j\|_{L^2(\Omega)} + \|\partial_t^2 \tilde{w}_j\|_{L^2(\Omega)} \right) + C \left\| \frac{1}{h} \varepsilon_h(\tilde{w}_j) \right\|_{H^{0,1}(\Omega)} \\
&\quad + CR \frac{1}{h} \left| \int_\Omega \tilde{w}_j \cdot x^\perp dx \right| + CR \left\| \left(\nabla_h w, \nabla_h^2 w \right) \right\|_{L^2(\Omega)}
\end{aligned}
\tag{5.48}
$$

for almost every $t \in (0,T)$. With this we can follow a similar argument as in the proof of Theorem 5.2.7. For this we differentiate the equation for $\tilde{w}_j$ in time and test with $\partial_t^2 \tilde{w}_j$. Then it follows

$$
\begin{aligned}
&(\partial_t^3 \tilde{w}_j, \partial_t^2 \tilde{w}_j)_{L^2(\Omega)} + \frac{1}{h^2} \left(D^2 \tilde{W}(\nabla_h u_h) \nabla_h \partial_t \tilde{w}_j, \nabla_h \partial_t^2 \tilde{w}_j \right)_{L^2(\Omega)} \\
&\quad = (\partial_t \tilde{f}_j, \partial_t^2 \tilde{w}_j)_{L^2(\Omega)} - \frac{1}{h^2} \left(\partial_t \big(\partial_{z_j} D^2 \tilde{W}(\nabla_h u_h) \nabla_h w \big), \nabla_h \partial_t^2 \tilde{w}_j \right)_{L^2(\Omega)} \\
&\qquad - \frac{1}{h^2} \left(\partial_t D^2 \tilde{W}(\nabla_h u_h) \nabla_h \tilde{w}_j, \nabla_h \partial_t^2 \tilde{w}_j \right)_{L^2(\Omega)}.
\end{aligned}
$$

because all boundary integrals disappear due to periodicity of $\tilde{w}_j$ and u_h, respectively, and the Neumann boundary conditions. Now as

$$
\begin{aligned}
&\frac{d}{dt} \left(\frac{1}{2} \frac{1}{h^2} \left(D^2 \tilde{W}(\nabla_h u_h) \nabla_h \partial_t \tilde{w}_j, \nabla_h \partial_t \tilde{w}_j \right)_{L^2} + \frac{1}{h^2} \left(\partial_t (\partial_{z_j} D^2 \tilde{W}(\nabla_h u_h) \nabla_h w), \nabla_h \partial_t \tilde{w}_j \right)_{L^2} \right) \\
&= \frac{1}{h^2} \left(D^2 \tilde{W}(\nabla_h u_h) \nabla_h \partial_t \tilde{w}_j, \nabla_h \partial_t^2 \tilde{w}_j \right)_{L^2} + \frac{1}{2} \frac{1}{h^2} \left(\partial_t D^2 \tilde{W}(\nabla_h u_h) \nabla_h \partial_t \tilde{w}_j, \nabla_h \partial_t \tilde{w}_j \right)_{L^2} \\
&\quad + \frac{1}{h^2} \left(\partial_t^2 (\partial_{z_j} D^2 \tilde{W}(\nabla_h u_h) \nabla_h w), \nabla_h \partial_t \tilde{w}_j \right)_{L^2} + \frac{1}{h^2} \left(\partial_t (\partial_{z_j} D^2 \tilde{W}(\nabla_h u_h) \nabla_h w), \nabla_h \partial_t^2 \tilde{w}_j \right)_{L^2}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\frac{d}{dt} \frac{1}{2} \left(\|\partial_t^2 \tilde{w}_j\|_{L^2}^2 + \frac{1}{h^2} \left(D^2 \tilde{W}(\nabla_h u_h) \nabla_h \partial_t \tilde{w}_j, \nabla_h \partial_t \tilde{w}_j \right)_{L^2} \right) = (\partial_t \tilde{f}_j, \partial_t^2 \tilde{w}_j)_{L^2} \\
&\quad + \frac{3}{2} \frac{1}{h^2} \left(\partial_t D^2 \tilde{W}(\nabla_h u_h) \nabla_h \partial_t \tilde{w}_j, \nabla_h \partial_t \tilde{w}_j \right)_{L^2} + \frac{1}{h^2} \left(\partial_t^2 (\partial_{z_j} D^2 \tilde{W}(\nabla_h u_h) \nabla_h w), \nabla_h \partial_t \tilde{w}_j \right)_{L^2} \\
&\quad - \frac{1}{h^2} \left(\partial_t^2 D^2 \tilde{W}(\nabla_h u_h) \nabla_h \tilde{w}_j, \nabla_h \partial_t \tilde{w}_j \right)_{L^2} - \frac{d}{dt} \frac{1}{h^2} \left(\partial_t (\partial_{z_j} D^2 \tilde{W}(\nabla_h u_h) \nabla_h w), \nabla_h \partial_t \tilde{w}_j \right)_{L^2}
\end{aligned}
$$

So, as

$$
\frac{1}{h^2} \left(D^2 \tilde{W}(\nabla_h u_h) \nabla_h \partial_t \tilde{w}_j, \nabla_h \partial_t \tilde{w}_j \right)_{L^2(\Omega)} + CR \left| \frac{1}{h} \int_\Omega \partial_t \tilde{w}_j \cdot x^\perp dx \right|^2 \geq \frac{c_0}{2} \left\| \frac{1}{h} \varepsilon_h(\partial_t \tilde{w}_j) \right\|_{L^2(\Omega)}^2
$$

we are led to

$$\begin{aligned}
&\sup_{\tau\in[0,t]}\left(\frac{1}{2}\|\partial_t^2\tilde{w}_j\|^2_{L^2(\Omega)}+\frac{c_0}{2}\left\|\frac{1}{h}\varepsilon_h(\partial_t\tilde{w}_j)\right\|^2_{L^2(\Omega)}\right)\\
&\qquad\le\int_0^t|(\partial_t\tilde{f}_j,\partial_t^2\tilde{w}_j)_{L^2}|d\tau+CR\sup_{\tau\in[0,t]}\frac{1}{h^2}\left|\int_\Omega\partial_t\tilde{w}_j\cdot x^\perp dx\right|^2 \qquad (5.49)\\
&\qquad\quad+\left\|\partial_t^2\tilde{w}_j\Big|_{t=0}\right\|_{L^2}+\frac{1}{h^2}\left|\Big(D^2\tilde{W}(\nabla_h u_h|_{t=0})\nabla_h\partial_t\tilde{w}_j|_{t=0},\nabla_h\partial_t\tilde{w}_j|_{t=0}\Big)_{L^2}\right|\\
&\qquad\quad+\frac{3}{2}\frac{1}{h^2}\int_0^t\left|\Big(\partial_t D^2\tilde{W}(\nabla_h u_h)\nabla_h\partial_t\tilde{w}_j,\nabla_h\partial_t\tilde{w}_j\Big)_{L^2}\right|d\tau\\
&\qquad\quad+\frac{1}{h^2}\int_0^t\left|\Big(\partial_t^2(\partial_{z_j}D^2\tilde{W}(\nabla_h u_h)\nabla_h w),\nabla_h\partial_t\tilde{w}_j\Big)_{L^2}\right|d\tau\\
&\qquad\quad+\frac{1}{h^2}\int_0^t\left|\Big(\partial_t^2 D^2\tilde{W}(\nabla_h u_h)\nabla_h\tilde{w}_j,\nabla_h\partial_t\tilde{w}_j\Big)_{L^2(\Omega)}\right|d\tau\\
&\qquad\quad+\left|\frac{1}{h^2}\Big(\partial_t(\partial_{z_j}D^2\tilde{W}(\nabla_h u_h)\nabla_h w),\nabla_h\partial_t\tilde{w}_j\Big)_{L^2}\Big|_{\tau=0}^t\right|
\end{aligned}$$

Thus, because of Lemma 5.2.2 it follows

$$\left|\frac{1}{h^2}\Big(\partial_t D^2\tilde{W}(\nabla_h u_h)\nabla_h\partial_t\tilde{w}_j,\nabla_h\partial_t\tilde{w}_j\Big)_{L^2}\right|\le CR\left(\left\|\frac{1}{h}\varepsilon_h(\partial_t\tilde{w}_j)\right\|^2_{L^2(\Omega)}+\frac{1}{h^2}\left|\int_\Omega\partial_t\tilde{w}_j\cdot x^\perp dx\right|^2\right)$$

and with Young's and Korn's inequalities

$$\begin{aligned}
&\left|\frac{1}{h^2}\Big(\partial_t^2 D^2\tilde{W}(\nabla_h u_h)\nabla_h\tilde{w}_j,\nabla_h\partial_t\tilde{w}_j\Big)_{L^2(\Omega)}\right|\le CR\|\nabla_h\tilde{w}_j\|_{H^1_h(\Omega)}\|\nabla_h\partial_t\tilde{w}_j\|_{L^2_h(\Omega)}\\
&\quad\le CR\|\nabla_h\tilde{w}_j\|^2_{H^1_h(\Omega)}+CR\|\nabla_h\partial_t\tilde{w}_j\|^2_{L^2_h(\Omega)}\\
&\quad\le CR\left(\left\|\frac{1}{h}\varepsilon_h(\tilde{w}_j)\right\|^2_{L^2(\Omega)}+\left|\frac{1}{h}\int_\Omega\tilde{w}_j\cdot x^\perp dx\right|^2+\left\|\left(\nabla\frac{1}{h}\varepsilon_h(\tilde{w}_j),\nabla_h^2\tilde{w}_j\right)\right\|^2_{L^2(\Omega)}\right)\\
&\qquad+CR\left(\left\|\frac{1}{h}\varepsilon_h(\partial_t\tilde{w}_j)\right\|^2_{L^2(\Omega)}+\left|\frac{1}{h}\int_\Omega\partial_t\tilde{w}_j\cdot x^\perp dx\right|^2\right).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\frac{1}{h^2}\left|\Big(\partial_t^2(\partial_{z_j}D^2\tilde{W}(\nabla_h u_h)\nabla_h w),\nabla_h\partial_t\tilde{w}_j\Big)_{L^2}\right|\\
&\quad=\frac{1}{h^2}\Big|\Big(\partial_t^2\partial_{z_j}D^2\tilde{W}(\nabla_h u_h)\nabla_h w+2\partial_t\partial_{z_j}D^2\tilde{W}(\nabla_h u_h)\nabla_h\partial_t w\\
&\qquad\qquad+\partial_{z_j}D^2\tilde{W}(\nabla_h u_h)\nabla_h\partial_t^2 w,\nabla_h\partial_t\tilde{w}_j\Big)_{L^2(\Omega)}\Big|\\
&\quad\le CR\Big(\|\nabla_h w\|_{H^2_h(\Omega)}+\|\nabla_h\partial_t w\|_{H^1_h(\Omega)}+\|\nabla_h\partial_t^2 w\|_{L^2_h(\Omega)}\Big)\cdot\|\nabla_h\partial_t\tilde{w}_j\|_{L^2_h(\Omega)}.
\end{aligned}$$

Hence, with Hölder's and Young's inequality

$$\begin{aligned}
&\frac{1}{h^2}\int_0^t\left|\Big(\partial_t^2(\partial_{z_j}D^2\tilde{W}(\nabla_h u_h)\nabla_h w),\nabla_h\partial_t\tilde{w}_j\Big)_{L^2}\right|d\tau\\
&\quad\le CR\Big(\|\nabla_h w\|^2_{L^2(0,t;H^2_h)}+\|\nabla_h\partial_t w\|^2_{L^2(0,t;H^1_h)}+\|\nabla_h\partial_t^2 w\|^2_{L^2(0,t;L^2_h)}\Big)\\
&\qquad+CR\|\nabla_h\partial_t\tilde{w}_j\|^2_{L^2(0,t;L^2_h)}
\end{aligned}$$

Dealing with each part on its own we can use Korn's inequality for $\nabla_h\partial_t\tilde{w}_j$ and $\nabla_h\partial_t^2 w$ in their respective norms. For $\nabla_h w$ and $\nabla_h \partial_t w$ we apply for $k = 1,\ 2$, respectively

$$
\begin{aligned}
\|\nabla_h \phi\|^2_{L^2(0,t;H^{1+k}_h)} \le C\Big\|\frac{1}{h}\varepsilon_h(\phi)\Big\|^2_{L^2(0,t;L^2)} &+ \frac{C}{h^2}\Big\|\int_\Omega \phi\cdot x^\perp dx\Big\|^2_{L^2(0,t)} \\
&+ \Big\|\Big(\nabla\frac{1}{h}\varepsilon_h(\phi), \nabla_h^2\phi\Big)\Big\|^2_{L^2(0,t;H^k)}.
\end{aligned}
$$

where we used Korn inequality and the convention $H^0(\Omega) := L^2(\Omega)$. Next, we notice

$$
\begin{aligned}
&\Big|\frac{1}{h^2}\Big(\partial_t(\partial_{z_j}D^2\tilde{W}(\nabla_h u_h)\nabla_h w), \nabla_h\partial_t\tilde{w}_j\Big)_{L^2(\Omega)}\Big| \\
&\quad\le CR\|\nabla_h w\|_{H^1_h(\Omega)}\|\nabla_h\partial_t\tilde{w}_j\|_{L^2_h(\Omega)} + CR\|\nabla_h\partial_t w\|_{L^2_h(\Omega)}\|\nabla_h\partial_t\tilde{w}_j\|_{L^2_h(\Omega)} \\
&\quad\le CR\Big(\Big\|\frac{1}{h}\varepsilon_h(w)\Big\|_{L^2(\Omega)} + \Big|\frac{1}{h}\int_\Omega w\cdot x^\perp dx\Big| + \Big\|\Big(\nabla\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\Big)\Big\|_{L^2(\Omega)} \\
&\qquad + \Big\|\frac{1}{h}\varepsilon_h(\partial_t w)\Big\|_{L^2(\Omega)} + \frac{1}{h}\Big|\int_\Omega \partial_t w\cdot x^\perp dx\Big|\Big)\cdot\|\nabla_h\partial_t\tilde{w}_j\|_{L^2_h(\Omega)}.
\end{aligned}
$$

Hence, with $|g(\tau)|^t_{\tau=0}| \le 2\,\mathrm{essup}_{\tau\in[0,t]}|g(\tau)|$ for all $g \in L^\infty(0,t)$ it follows

$$
\begin{aligned}
&\Big|\frac{1}{h^2}\Big(\partial_t(\partial_{z_j}D^2\tilde{W}(\nabla_h u_h)\nabla_h w), \nabla_h\partial_t\tilde{w}_j\Big)_{L^2(\Omega)}\Big|^t_{\tau=0}\Big| \\
&\quad\le CR\Big(\Big\|\frac{1}{h}\varepsilon_h(w)\Big\|_{W^1_\infty(0,t;L^2)} + \frac{1}{h}\Big\|\int_\Omega w\cdot x^\perp dx\Big\|_{W^1_\infty(0,t)} + \Big\|\Big(\nabla\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\Big)\Big\|_{L^\infty(0,t;L^2)}\Big) \\
&\quad\times\|\nabla_h\partial_t\tilde{w}_j\|_{L^\infty(0,t;L^2_h(\Omega))}.
\end{aligned}
$$

Plugging everything in (5.49), we are led to

$$
\begin{aligned}
&\sup_{\tau\in[0,t]}\Big(\frac{1}{2}\|\partial_t^2\tilde{w}_j\|^2_{L^2(\Omega)} + \frac{c_0}{2}\Big\|\frac{1}{h}\varepsilon_h(\partial_t\tilde{w}_j)\Big\|^2_{L^2(\Omega)}\Big) \\
&\quad\le \int_0^t |(\partial_t\tilde{f}_j, \partial_t^2\tilde{w}_j)_{L^2}|d\tau + CR\sup_{\tau\in[0,t]}\Big|\frac{1}{h}\int_\Omega \partial_t\tilde{w}_j\cdot x^\perp dx\Big|^2 \\
&\qquad + CR\Big(\Big\|\frac{1}{h}\varepsilon_h(\partial_t\tilde{w}_j)\Big\|^2_{L^1(0,t;L^2)} + \frac{1}{h^2}\Big\|\int_\Omega \partial_t\tilde{w}_j\cdot x^\perp dx\Big\|^2_{L^1(0,t)}\Big) \\
&\qquad + CR\Big(\Big\|\frac{1}{h}\varepsilon_h(\partial_t\tilde{w}_j)\Big\|^2_{L^2(0,t;L^2)} + \frac{1}{h^2}\Big\|\int_\Omega \partial_t\tilde{w}_j\cdot x^\perp dx\Big\|^2_{L^2(0,t)}\Big) \\
&\qquad + \Big\|\partial_t^2\tilde{w}_j\Big|_{t=0}\Big\|_{L^2} + \frac{1}{h^2}\Big|\Big(D^2\tilde{W}(\nabla_h u_h|_{t=0})\nabla_h\partial_t\tilde{w}_j|_{t=0}, \nabla_h\partial_t\tilde{w}_j|_{t=0}\Big)_{L^2(\Omega)}\Big| \\
&\qquad + CR\Big(\Big\|\frac{1}{h}\varepsilon_h(\partial_t^2 w)\Big\|^2_{L^2(0,t;L^2)} + \frac{1}{h^2}\Big\|\int_\Omega \partial_t^2 w\cdot x^\perp dx\Big\|^2_{L^2(0,t)}\Big) \\
&\qquad + CR\Big(\Big\|\frac{1}{h}\varepsilon_h(w)\Big\|^2_{W^1_2(0,t;L^2)} + \frac{1}{h^2}\Big\|\int_\Omega w\cdot x^\perp dx\Big\|^2_{W^1_2(0,t)}\Big) \\
&\qquad + CR\Big(\Big\|\frac{1}{h}\varepsilon_h(w)\Big\|^2_{W^1_\infty(0,t;L^2)} + \frac{1}{h^2}\Big\|\int_\Omega w\cdot x^\perp dx\Big\|^2_{W^1_\infty(0,t)}\Big) \\
&\qquad + CR\Big\|\Big(\nabla\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\Big)\Big\|^2_{L^\infty(0,t;L^2)}
\end{aligned}
$$

$$+ CR\Big(\Big\|\Big(\nabla\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\Big)\Big\|^2_{L^2(0,t;H^1)} + \Big\|\Big(\nabla\frac{1}{h}\varepsilon_h(\partial_t w), \nabla_h^2\partial_t w\Big)\Big\|^2_{L^2(0,t;L^2)}\Big)$$

Using Cauchy-Schwarz and Young Inequality it follows

$$\begin{aligned}\int_0^t |(\partial_t\tilde{f}_j, \partial_t^2\tilde{w}_j)_{L^2}|d\tau &\leq \|\partial_t\tilde{f}_j\|_{L^1(0,T;L^2)}\|\partial_t^2\tilde{w}_j\|_{L^\infty(0,t;L^2)} \\ &\leq C(\epsilon)\|\partial_t\tilde{f}_j\|_{L^1(0,T;L^2)} + \epsilon\|\partial_t^2\tilde{w}_j\|_{L^\infty(0,t;L^2)}.\end{aligned}$$

In the case $j = 1$, integration by parts in x_1 direction and periodicity leads to

$$\int_\Omega \partial_t\tilde{w}_j \cdot x^\perp dx = 0.$$

For $\epsilon > 0$ sufficiently small we can absorb $\|\partial_t^2\tilde{w}_j\|_{L^\infty(0,t;L^2)}$ in the left hand side. Furthermore, we have $\|g\|_{L^2(0,t;L^2)} \leq \sqrt{t}\|g\|_{L^\infty(0,t;L^2)}$ for all $g \in L^\infty(0,t;L^2)$ and $\|l\|_{L^1(0,t;L^2)} \leq \sqrt{t}\|l\|_{L^2(0,t;L^2)}$ for all $l \in L^2(0,t;L^2)$. Hence we have with $\sqrt{T} \leq \frac{1}{2}(1+T)$

$$\begin{aligned}&\sup_{\tau\in[0,t]}\left(\frac{1}{2}\|\partial_t^2\tilde{w}_j\|^2_{L^2(\Omega)} + \frac{c_0}{2}\Big\|\frac{1}{h}\varepsilon_h(\partial_t\tilde{w}_j)\Big\|^2_{L^2(\Omega)}\right) \\ &\quad\leq C\|\partial_t\tilde{f}_j\|^2_{L^1(0,T;L^2)} + \Big\|\partial_t^2\tilde{w}_j\Big|_{t=0}\Big\|^2_{L^2} + \frac{1}{h^2}\Big|\Big(D^2\tilde{W}(\nabla_h u_h)\nabla_h\partial_t\tilde{w}_j, \nabla_h\partial_t\tilde{w}_j\Big)_{L^2}\Big|_{t=0}\Big| \\ &\quad + CR\sqrt{T}\Big\|\frac{1}{h}\varepsilon_h(\partial_t\tilde{w}_j)\Big\|^2_{L^2(0,t;L^2)} + CR\Big\|\frac{1}{h}\varepsilon_h(\partial_t^2 w)\Big\|^2_{L^2(0,t;L^2)} \\ &\quad + CR(1+T)\Big\|\Big(\frac{1}{h}\varepsilon_h(w), \nabla_{x,t}\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\Big)\Big\|^2_{L^\infty(0,t;L^2)} \\ &\quad + CR(1+T)\max_{\sigma=0,1,2}\Big\|\frac{1}{h}\int_\Omega \partial_t^\sigma w\cdot x^\perp dx\Big\|^2_{L^\infty(0,t)} \\ &\quad + CR\Big(\Big\|\Big(\nabla\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\Big)\Big\|^2_{L^2(0,t;H^1)} + \Big\|\Big(\nabla\frac{1}{h}\varepsilon_h(\partial_t w), \nabla_h^2\partial_t w\Big)\Big\|^2_{L^2(0,t;L^2)}\Big)\end{aligned} \tag{5.50}$$

Using the assumption that w solves (5.12)–(5.15), we can apply Corollary 5.2.10 to

$$\begin{cases} -\frac{1}{h^2}\operatorname{div}_h\Big(D^2\tilde{W}(\nabla_h u_h)\nabla_h w\Big) = f - \partial_t^2 w \\ D^2\tilde{W}(\nabla_h u_h)\nabla_h w\nu\Big|_{(0,L)\times\partial S} = 0\end{cases} \tag{5.51}$$

for almost every $t \in [0,T)$. Hence with (5.47)

$$\begin{aligned}\Big\|\Big(\nabla\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\Big)\Big\|_{H^1(\Omega)} &\leq C\|f\|_{H^1(\Omega)} + \|\partial_t^2 w\|_{H^1(\Omega)} + CR\Big|\frac{1}{h}\int_\Omega w\cdot x^\perp dx\Big| \\ &\leq C\|f\|_{H^1(\Omega)} + \|\partial_t^2 w\|_{L^2(\Omega)} + \Big\|\frac{1}{h}\varepsilon_h(\partial_t^2 w)\Big\|_{L^2(\Omega)} \\ &\quad + C\Big|\frac{1}{h}\int_\Omega \partial_t^2 w\cdot x^\perp dx\Big| + CR\Big|\frac{1}{h}\int_\Omega w\cdot x^\perp dx\Big|\end{aligned} \tag{5.52}$$

For $j = 0$, we deduce from (5.48)

$$\Big\|\Big(\nabla\frac{1}{h}\varepsilon_h(\partial_t w), \nabla_h^2\partial_t w\Big)\Big\|_{L^2(\Omega)} \leq Ch^2\Big(\|\tilde{f}_0\|_{L^2(\Omega)} + \|\partial_t^3 w\|_{L^2(\Omega)}\Big) \tag{5.53}$$

$$
\begin{aligned}
&+ C\left\| \frac{1}{h}\varepsilon_h(\partial_t w) \right\|_{H^{0,1}(\Omega)} + CR\left| \frac{1}{h} \int_\Omega \partial_t w \cdot x^\perp dx \right| \\
&+ CR\left\| \Big(\nabla_h w, \nabla_h^2 w \Big) \right\|_{L^2(\Omega)}.
\end{aligned}
$$

By virtue of (5.50) for $j = 0$, (5.52) and (5.53) we obtain

$$
\begin{aligned}
&\sup_{\tau \in [0,t]} \left\| \Big(\partial_t^3 w, \nabla_{t,x}^2 \frac{1}{h}\varepsilon_h(w), \nabla_{x,t} \nabla_h^2 w \Big) \right\|_{L^2(\Omega)} \\
&\quad \leq \sup_{\tau \in [0,t]} \left\| \Big(\partial_t^3 w, \frac{1}{h}\varepsilon_h(\partial_t^2 w) \Big) \right\| + \sup_{\tau \in [0,t]} \left\| \Big(\nabla \frac{1}{h}\varepsilon_h(w), \nabla_h^2 w \Big) \right\|_{H^1(\Omega)}^2 \\
&\qquad + \sup_{\tau \in [0,t]} \left\| \Big(\nabla \frac{1}{h}\varepsilon_h(\partial_t w), \nabla_h^2 \partial_t w \Big) \right\|_{L^2(\Omega)}^2 \\
&\quad \leq \|w_3\|_{L^2}^2 + |A_2| + C\|\partial_t^2 f\|_{L^1(0,T;L^2)}^2 + C\|f\|_{L^\infty(0,t;H^1)}^2 + C\|\partial_t f\|_{L^\infty(0,t;L^2)}^2 \\
&\quad + CR(1+\sqrt{T}) \left\| \frac{1}{h}\varepsilon_h(\partial_t^2 w) \right\|_{L^2(0,t;L^2)}^2 \\
&\quad + CR\left(\left\| \Big(\nabla \frac{1}{h}\varepsilon_h(w), \nabla_h^2 w \Big) \right\|_{L^2(0,t;H^1)}^2 + \left\| \Big(\nabla \frac{1}{h}\varepsilon_h(\partial_t w), \nabla_h^2 \partial_t w \Big) \right\|_{L^2(0,t;L^2)}^2 \right) \\
&\quad + C\|\partial_t^3 w\|_{L^\infty(0,t;L^2)}^2 + C\left\| \Big(\frac{1}{h}\varepsilon_h(\partial_t^2 w), \partial_{x_1} \frac{1}{h}\varepsilon_h(\partial_t w) \Big) \right\|_{L^\infty(0,t;L^2)}^2 \\
&\quad + C\left\| \Big(\partial_t^2 w, \frac{1}{h}\varepsilon_h(\partial_t w) \Big) \right\|_{L^\infty(0,t;L^2)}^2 + \frac{C}{h^2} \left\| \int_\Omega \partial_t^2 w \cdot x^\perp dx \right\|_{L^\infty(0,t)}^2 \\
&\quad + CR(1+T) \left\| \Big(\frac{1}{h}\varepsilon_h(w), \nabla_{x,t} \frac{1}{h}\varepsilon_h(w), \nabla_h^2 w \Big) \right\|_{L^\infty(0,t;L^2)}^2 \\
&\quad + CR(1+T) \max_{\sigma=0,1,2} \left\| \frac{1}{h} \int_\Omega \partial_t^\sigma w \cdot x^\perp dx \right\|_{L^\infty(0,t)}^2
\end{aligned}
$$

We use (5.50) for $j = 0$ in order to bound $\|\partial_t^3 w\|_{L^\infty(0,t;L^2)}^2$ and with $j = 1$ we can estimate $\|(\frac{1}{h}\varepsilon_h(\partial_t^2 w), \partial_{x_1} \frac{1}{h}\varepsilon_h(\partial_t w))\|_{L^\infty(0,t;L^2)}^2$. This leads to

$$
\begin{aligned}
&\sup_{\tau \in [0,t]} \left\| \Big(\partial_t^3 w, \nabla_{t,x}^2 \frac{1}{h}\varepsilon_h(w), \nabla_{x,t} \nabla_h^2 w \Big) \right\|_{L^2(\Omega)} \\
&\quad \leq C\|\partial_t^2 f\|_{L^1(0,T;L^2)}^2 + C\|\partial_t \partial_{x_1} f\|_{L^1(0,T;L^2)}^2 + C\|f\|_{L^\infty(0,t;H^1)}^2 + C\|\partial_t f\|_{L^\infty(0,t;L^2)}^2 \\
&\quad + \|w_3\|_{L^2}^2 + |A_2| + \|\partial_{x_1} w_2\|_{L^2}^2 + \frac{1}{h^2} \left| \Big(D^2 \tilde{W}(\nabla_h u_h|_{t=0}) \nabla_h \partial_{x_1} w_1, \nabla_h \partial_{x_1} w_1 \Big)_{L^2(\Omega)} \right| \\
&\quad + CR(1+\sqrt{T}) \sum_{j=0}^{1} \left\| \frac{1}{h}\varepsilon_h(\partial_t \tilde{w}_j) \right\|_{L^2(0,t;L^2)}^2 + CR(1+T) \max_{\sigma=0,1,2} \left\| \frac{1}{h} \int_\Omega \partial_t^\sigma w \cdot x^\perp dx \right\|_{L^\infty(0,t)}^2 \\
&\quad + CR\left(\left\| \Big(\nabla \frac{1}{h}\varepsilon_h(w), \nabla_h^2 w \Big) \right\|_{L^2(0,t;H^1)}^2 + \left\| \Big(\nabla \frac{1}{h}\varepsilon_h(\partial_t w), \nabla_h^2 \partial_t w \Big) \right\|_{L^2(0,t;L^2)}^2 \right) \\
&\quad + C\left\| \Big(\partial_t^2 w, \frac{1}{h}\varepsilon_h(\partial_t w) \Big) \right\|_{L^\infty(0,t;L^2)}^2 + \frac{C}{h^2} \left\| \int_\Omega \partial_t^2 w \cdot x^\perp dx \right\|_{L^\infty(0,t)}^2 \\
&\quad + CR(1+T) \left\| \Big(\frac{1}{h}\varepsilon_h(w), \nabla_{x,t} \frac{1}{h}\varepsilon_h(w), \nabla_h^2 w \Big) \right\|_{L^\infty(0,t;L^2)}^2
\end{aligned}
$$

where we used $\int_\Omega \partial_t\partial_{x_1} w\cdot x^\perp dx = 0$. The Lemma of Gronwall yields then

$$
\begin{aligned}
&\sup_{\tau\in[0,t]}\left\|\left(\partial_t^3 w, \nabla_{t,x}^2 \frac{1}{h}\varepsilon_h(w), \nabla_{x,t}\nabla_h^2 w\right)\right\|_{L^2(\Omega)}\\
&\le \tilde{C} e^{C_2 R(1+\sqrt{T})}\Bigg(\|\partial_t^2 f\|^2_{L^1(0,T;L^2)} + \|\partial_t\partial_{x_1} f\|^2_{L^1(0,T;L^2)} + \|f\|^2_{L^\infty(0,t;H^1)}\\
&\quad + \|\partial_t f\|^2_{L^\infty(0,t;L^2)} + \|w_3\|^2_{L^2} + |A_2| + \|\partial_{x_1} w_2\|^2_{L^2}\\
&\quad + \frac{1}{h^2}\left|\left(D^2\tilde{W}(\nabla_h u_h|_{t=0})\nabla_h\partial_{x_1} w_1, \nabla_h\partial_{x_1} w_1\right)_{L^2(\Omega)}\right| + \left\|\left(\partial_t^2 w, \frac{1}{h}\varepsilon_h(\partial_t w)\right)\right\|^2_{L^\infty(0,t;L^2)}\\
&\quad + (1+T)R\left\|\left(\frac{1}{h}\varepsilon_h(w), \nabla_{x,t}\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\right)\right\|^2_{L^\infty(0,t;L^2)}\\
&\quad + \left\|\frac{1}{h}\int_\Omega \partial_t^2 w\cdot x^\perp dx\right\|^2_{L^\infty(0,t)} + (1+T)R\max_{\sigma=0,1,2}\left\|\frac{1}{h}\int_\Omega \partial_t^\sigma w\cdot x^\perp dx\right\|^2_{L^\infty(0,t)}\Bigg)
\end{aligned}
$$

Now due to Theorem 5.2.3 and 5.2.7 we have

$$
\begin{aligned}
&\left\|\left(\partial_t^2 w, \frac{1}{h}\varepsilon_h(\partial_t w)\right)\right\|^2_{L^\infty(0,t;L^2)} + R(1+T)\left\|\left(\frac{1}{h}\varepsilon_h(w), \nabla_{x,t}\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\right)\right\|^2_{L^\infty(0,t;L^2)}\\
&\quad\le C_{L1}(1+T)e^{C_1 TR}\Bigg(\|f\|^2_{W_1^1(0,T;L^2)} + \|(w_1,w_2,f|_{t=0})\|^2_{L^2} + |(A_0,A_1)|\\
&\qquad + (1+T)R\max_{\sigma=0,1}\left\|\frac{1}{h}\int_\Omega \partial_t^\sigma w\cdot x^\perp dx\right\|^2_{C(0,t)}\Bigg)
\end{aligned}
$$

Since $\|f\|_{W_1^1(0,T;L^2)} \le T\|f\|_{L^\infty(0,T;H^1)} + \|\partial_t f\|_{L^1(0,T;H^{0,1})}$ the claimed inequality

$$
\begin{aligned}
&\sup_{\tau\in[0,t]}\left\|\left(\partial_t^3 w, \nabla_{t,x}^2 \frac{1}{h}\varepsilon_h(w), \nabla_{x,t}\nabla_h^2 w\right)\right\|_{L^2(\Omega)}\\
&\quad\le C_{L2} e^{C_2 R(1+\sqrt{T})}\Bigg(\|\partial_t^2 f\|^2_{L^1(0,T;L^2)} + \|\partial_t f\|^2_{L^\infty(0,T;L^2)\cap L^1(0,T;H^{0,1})} + \|f\|^2_{L^\infty(0,T;H^1)}\\
&\qquad + \|(w_1,w_2,w_3,f|_{t=0})\|^2_{L^2} + |(A_0,A_1,A_2)| + \|\partial_{x_1} w_2\|^2_{L^2}\\
&\qquad + \frac{1}{h^2}\left|\left(D^2\tilde{W}(\nabla_h u_h|_{t=0})\nabla_h\partial_{x_1} w_1, \nabla_h\partial_{x_1} w_1\right)_{L^2(\Omega)}\right|\\
&\qquad + \left\|\frac{1}{h}\int_\Omega \partial_t^2 w\cdot x^\perp dx\right\|^2_{L^\infty(0,t)} + (1+T)R\max_{\sigma=0,1,2}\left\|\frac{1}{h}\int_\Omega \partial_t^\sigma w\cdot x^\perp dx\right\|^2_{L^\infty(0,t)}\Bigg)
\end{aligned}
$$

follows. □

With the following corollary we summarizes the bounds for solutions of the linear system in a convenient way for the proof of large time existence.

Corollary 5.2.12. *Assume the assumptions of Theorem 5.2.11 are satisfied. Then there exist constants $C_{max} \ge 1$, $C' > 0$ depending on T such that*

$$
\begin{aligned}
&\max_{\sigma=0,1,2}\left\|\left(\partial_t^{1+\sigma} w, \nabla_{t,x}^\sigma \frac{1}{h}\varepsilon_h(w), \nabla_{x,t}^{\max\{\sigma-1,0\}}\nabla_h^2 w\right)\right\|^2_{C^0(0,T;L^2(\Omega))}\\
&\le C_{max} e^{C'(1+T)R}\Bigg(\|\partial_t^2 f\|^2_{L^1(0,T;L^2)} + \|\partial_t f\|^2_{L^\infty(0,T;L^2)\cap L^1(0,T;H^{0,1})} + \|f\|^2_{L^\infty(0,T;H^1)}\\
&\quad + \|(w_1,w_2,w_3,f|_{t=0})\|^2_{L^2} + \left\|\left(\frac{1}{h}\varepsilon_h(\partial_{x_1} w_1), \partial_{x_1} w_2\right)\right\|^2_{L^2}
\end{aligned}
$$

$$+\max_{k=0,1,2}\left(\left\|\frac{1}{h}\varepsilon_h(w_k)\right\|_{L^2}^2+\left|\frac{1}{h}\int_\Omega w_k\cdot x^\perp dx\right|^2\right)$$
$$+\frac{1}{h^2}\left\|\int_\Omega \partial_t^2 w\cdot x^\perp dx\right\|_{L^\infty(0,T)}^2+(1+T)R\max_{\sigma=0,1,2}\left\|\frac{1}{h}\int_\Omega \partial_t^\sigma w\cdot x^\perp dx\right\|_{C^0([0,T])}^2\Bigg)$$

Proof: The assumptions allow to apply Lemma 5.2.3 and Theorem 5.2.7 and 5.2.11. Hence we obtain

$$\begin{aligned}
&\max_{\sigma=0,1,2}\left\|\left(\partial_t^{1+\sigma}w,\nabla_{t,x}^\sigma\frac{1}{h}\varepsilon_h(w),\nabla_{x,t}^{\max\{\sigma-1,0\}}\nabla_h^2 w\right)\right\|_{C^0(0,T;L^2(\Omega))}^2\\
&\le\left(C_L+C_{L1}e^{C_1TR}+C_{L2}e^{C_2(1+\sqrt{T})R}\right)\\
&\quad\times\Bigg(\|f\|_{W_2^2(0,T;L^2)\cap W_2^1(0,T;H^1)}^2+\|(w_1,w_2,w_3,f|_{t=0})\|_{L^2}^2+|(A_0,A_1,A_2)|\\
&\qquad+\|\partial_{x_1}w_2\|_{L^2}^2+\frac{1}{h^2}\left|\left(D^2\tilde W(\nabla_h u_h|_{t=0})\nabla_h\partial_{x_1}w_1,\nabla_h\partial_{x_1}w_1\right)_{L^2(\Omega)}\right|\\
&\qquad+\frac{1}{h^2}\left\|\int_\Omega\partial_t^2 w\cdot x^\perp dx\right\|_{L^\infty(0,T)}^2+(1+T)R\max_{\sigma=0,1,2}\left\|\frac{1}{h}\int_\Omega\partial_t^\sigma w\cdot x^\perp dx\right\|_{C^0([0,T])}^2\Bigg).
\end{aligned}$$

We can find $C>0$ and $C'>0$ such that

$$C_L+C_{L1}e^{C_1TR}+C_{L2}e^{C_2(1+\sqrt{T})R}\le Ce^{C'(1+T)R}$$

holds. For $|(A_0,A_1,A_2)|$ we use a Taylor expansion in the following way

$$\begin{aligned}
|A_k|&=\left|\frac{1}{h}\left(D^2\tilde W(\nabla_h u_{0,h})\nabla_h w_k,\nabla_h w_k\right)_{L^2}\right|\\
&\le\left|\frac{1}{h^2}\left(D^2\tilde W(0)\nabla_h w_k,\nabla_h w_k\right)_{L^2}\right|\\
&\quad+\left|\frac{1}{h^2}\int_0^1\left(D^3\tilde W(\tau\nabla_h u_{0,h})[\nabla_h u_{0,h}]\nabla_h w_k,\nabla_h w_k\right)_{L^2}\right|\\
&\le C\left\|\frac{1}{h}\varepsilon_h(w_k)\right\|_{L^2}^2+\frac{C}{h}\|\nabla_h u_{0,h}\|_{H_h^2}\|\nabla_h w_k\|_{L_h^2}^2\\
&\le C\left\|\frac{1}{h}\varepsilon_h(w_k)\right\|_{L^2}^2+C\left|\frac{1}{h}\int_\Omega w_k\cdot x^\perp dx\right|^2,
\end{aligned}$$

where we used Korn's inequality and (5.5). With the same kind of calculations it is possible to bound

$$\left|\frac{1}{h^2}\left(D^2\tilde W(\nabla_h u_{0,h})\nabla_h\partial_{x_1}w_1,\nabla_h\partial_{x_1}w_1\right)_{L^2}\right|\le C\left\|\frac{1}{h}\varepsilon_h(\partial_{x_1}w_1)\right\|_{L^2}^2.$$

Composing all inequalities the statement follows. □

5.3 Proof of Theorem 5.1.1

Before we start the proof of the main theorem, we will prepare some bounds on the rotation of the solution around the x_1-axis. More precisely we need to bound the following quantity

$$\max_{\sigma=1,2,3}\left\|\frac{1}{h}\int_\Omega\partial_t^\sigma u_h\cdot x^\perp dx\right\|_{C^0([0,T(h)])}.$$

We can transform the system (5.1)–(5.4) via $\phi_h\colon \Omega_h \to \Omega$, $x \mapsto (x_1, \frac{1}{h}x_2, \frac{1}{h}x_3)$. Hence $y_h := u_h \circ \phi_h$ solve the equation

$$\begin{aligned}\partial_t^2 y_h - \frac{1}{h^2}\operatorname{div} D\tilde{W}(\nabla y_h) = \hat{f}_h := h^{1+\theta} f_h \circ \phi_h \quad \text{in } \Omega_h \times [0,T_*)\\ D\tilde{W}(\nabla y_h)\nu|_{(0,L)\times h\partial S} = 0\\ y_h \text{ is } L\text{-periodic w.r.t. } x_1\\ (y_h, \partial_t y_h)|_{t=0} = (y_{0,h}, y_{1,h})\end{aligned}$$

with $(y_{0,h}, y_{1,h}) := (u_{0,h}\circ\phi_h, u_{1,h}\circ\phi_h)$. Due to the frame invariance we have with (2.21)

$$DW(F)F^T = FDW(F)^T$$

and with $\tilde{W}(F) = W(I+F)$, it follows $D\tilde{W}(Id+F)(Id+F)^T = (Id+F)D\tilde{W}(Id+F)^T$. Thus the Piola-Kirchhoff stress $D\tilde{W}(\nabla_h y_h)$ fulfils the symmetry condition (2.19) and one can apply the balance law of angular momentum

$$\begin{aligned}\int_{\partial\Omega_h}(x + u_h\circ\phi_h)\times\frac{1}{h^2}D\tilde{W}(\nabla(u_h\circ\phi_h))\nu d\sigma(x) + \int_{\Omega_h}(x+u_h\circ\phi_h)\times\hat{f}dx\\ = \int_{\Omega_h}(x+u_h\circ\phi_h)\times\partial_t^2 u_h\circ\phi_h dx\end{aligned}$$

With the transformation formula applied for ϕ^{-1} we conclude with $g^h := h^{1+\theta}f^h$

$$\begin{aligned}h\int_{\partial\Omega}(\phi_h^{-1}(x)+u_h)\times\frac{1}{h^2}D\tilde{W}(\nabla_h u_h)\nu d\sigma(x) + h^2\int_\Omega(\phi_h^{-1}(x)+u_h)\times g^h dx\\ = h^2\int_\Omega(\phi_h^{-1}(x)+u_h)\times\partial_t^2 u_h dx\end{aligned}$$

We can restrict to just the first component, as only rotations around x_1 axis have to be controlled for the use of Korn's inequality. For the first component we have

$$\begin{aligned}&\left(\int_{\partial\Omega}(\phi_h^{-1}(x)+u_h)\times\frac{1}{h^2}D\tilde{W}(\nabla_h u_h)\nu d\sigma(x)\right)_1\\ &= \int_{\partial\Omega}(hx^\perp + u_h^\perp)\cdot\frac{1}{h^2}D\tilde{W}(\nabla_h u_h)\nu d\sigma(x) = \int_S\left[(hx^\perp+u_h)\cdot\frac{1}{h^2}D\tilde{W}(\nabla_h u_h)\nu\right]_0^L dx'\end{aligned}$$

because $D\tilde{W}(\nabla_h u_h)\nu = 0$ on $(0,L)\times\partial S$ and as $x^\perp$ does not depend on x_1 it follows that $hx^\perp + u_h$ is L-periodic in x_1 direction. Using this and $\nu(0,x') = -\nu(L,x')$ for all $x'\in S$ we deduce

$$\left(\int_{\partial\Omega}(\phi_h^{-1}(x)+u_h)\times\frac{1}{h^2}D\tilde{W}(\nabla_h u_h)\nu d\sigma(x)\right)_1 = 0.$$

Thus we have

$$\partial_t^2\int_\Omega x^\perp\cdot u_h dx = \int_\Omega x^\perp\cdot g^h dx + \frac{1}{h}\int_\Omega u_h^\perp\cdot g^h dx - \frac{1}{h}\int_\Omega u_h^\perp\cdot\partial_t^2 u_h dx \tag{5.54}$$

for almost all $t\in[0,T_{max}(h))$. With this we can later bound $\|\frac{1}{h}\int_\Omega\partial_t^\sigma u_h\cdot x^\perp dx\|_{C^0([0,T(h)])}$ for $\sigma = 1,2,3$ uniformly in $h\in(0,1]$. We note that with (5.54) it follows

$$\begin{aligned}\left\|\frac{1}{h}(\partial_t u_h, x^\perp)_{L^2(\Omega)}\right\|_{C^0([0,T(h)])} \le \left|\frac{1}{h}(u_{1,h},x^\perp)_{L^2(\Omega)}\right| + \frac{1}{h}\int_0^{T(h)}|(x^\perp,g^h)_{L^2}|d\tau \quad (5.55)\\ + \frac{1}{h^2}\int_0^{T(h)}|(u_h^\perp,g^h)_{L^2}|d\tau + \frac{1}{h^2}\int_0^{T(h)}|(u_h^\perp,\partial_t^2 u_h)_{L^2}|d\tau\end{aligned}$$

$$\left\|\frac{1}{h}(\partial_t^2 u_h, x^\perp)_{L^2(\Omega)}\right\|_{C^0([0,T(h)])} \leq \left\|\frac{1}{h}(g^h, x^\perp)_{L^2(\Omega)}\right\|_{C^0([0,T(h)])} + \left\|\frac{1}{h^2}(u_h^\perp, g^h)_{L^2(\Omega)}\right\|_{C^0([0,T(h)])} + \left\|\frac{1}{h^2}(u_h^\perp, \partial_t^2 u_h)_{L^2(\Omega)}\right\|_{C^0([0,T(h)])} \tag{5.56}$$

$$\left\|\frac{1}{h}(\partial_t^3 u_h, x^\perp)_{L^2(\Omega)}\right\|_{C^0([0,T(h)])} \leq \left\|\frac{1}{h}(x^\perp, \partial_t g^h)_{L^2(\Omega)}\right\|_{C^0([0,T(h)])} + \left\|\frac{1}{h^2}(\partial_t u_h^\perp, g^h)_{L^2(\Omega)}\right\|_{C^0([0,T(h)])} + \left\|\frac{1}{h^2}(u_h^\perp, \partial_t g^h)\right\|_{C^0([0,T(h)])} + \left\|\frac{1}{h^2}(\partial_t u_h^\perp, \partial_t^2 u_h)_{L^2(\Omega)}\right\|_{C^0([0,T(h)])} + \left\|\frac{1}{h^2}(u_h^\perp, \partial_t^3 u_h)_{L^2(\Omega)}\right\|_{C^0([0,T(h)])} \tag{5.57}$$

For convenience we revise the theorem once more.

Theorem 5.3.1. *Let $\theta \geq 1$, $0 < T < \infty$, $f_h \in W_1^3(0,T; L^2(\Omega;\mathbb{R}^3)) \cap W_1^1(0,T; H^2_{per}(\Omega;\mathbb{R}^3))$, $h \in (0,1]$ and $u_{0,h} \in H^4_{per}(\Omega;\mathbb{R}^3)$, $u_{1,h} \in H^3_{per}(\Omega;\mathbb{R}^3)$ such that*

$$D\tilde{W}(\nabla_h u_{0,h})\nu|_{(0,L)\times\partial S} = D^2\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{1,h}]\nu|_{(0,L)\times\partial S} = 0,$$
$$(D^2\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{2,h}] + D^3\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{1,h}, \nabla_h u_{1,h}])\nu|_{(0,L)\times\partial S} = 0,$$

where

$$\begin{aligned} u_{2,h} &= h^{1+\theta} f|_{t=0} + \frac{1}{h^2}\operatorname{div}_h(D\tilde{W}(\nabla_h u_{0,h})) \\ u_{3,h} &= h^{1+\theta}\partial_t f|_{t=0} + \frac{1}{h^2}\operatorname{div}_h(D^2\tilde{W}(\nabla_h u_{0,h})\nabla_h u_{1,h}) \\ u_{4,h} &= h^{1+\theta}\partial_t^2 f|_{t=0} + \frac{1}{h^2}\operatorname{div}_h(D^2\tilde{W}(\nabla_h u_{0,h})\nabla_h u_{2,h}) \\ &\quad + \frac{1}{h^2}\operatorname{div}_h(D^3\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{1,h}, \nabla_h u_{1,h}]). \end{aligned}$$

Moreover we assume for the initial data

$$\left\|\frac{1}{h}\varepsilon_h(u_{0,h})\right\|_{H^2} + \max_{k=0,1,2}\left\|\left(\frac{1}{h}\varepsilon_h(u_{1+k,h}), \partial_{x_1}\frac{1}{h}\varepsilon_h(u_{k,h}), u_{2+k,h}\right)\right\|_{H^{2-k}} \leq Mh^{1+\theta} \tag{5.5}$$

$$\left\|\nabla_h^2 u_{0,h}\right\|_{H^1} + \max_{k=0,1}\left\|\left(\nabla_h^2 u_{1+k,h}, \partial_{x_1}\nabla_h^2 u_{k,h}\right)\right\|_{H^{1-k}} \leq Mh^{1+\theta} \tag{5.6}$$

$$\max_{k=0,1,2,3}\left|\frac{1}{h}\int_\Omega u_{k,h}\cdot x^\perp dx\right| \leq Mh^{1+\theta}. \tag{5.7}$$

and for the right hand side

$$\max_{|\alpha|\leq 1}\left(\|\partial_z^\alpha f\|_{W_1^2(L^2)} + \|\partial_z^\alpha f\|_{W^1_\infty(L^2)\cap W_1^1(H^{0,1})} + \|\partial_z^\alpha f\|_{L^\infty(H^1)}\right) \leq M \tag{5.8}$$

$$\max_{\sigma=0,1,2}\left\|\frac{1}{h}\int_\Omega \partial_t^\sigma f\cdot x^\perp dx\right\|_{C^0([0,T])} \leq M \tag{5.9}$$

uniformly in $0 < h \leq 1$.

Then there exists $h_0 \in (0,1]$ and C depending only on M and T such that for every $h \in (0,h_0]$ there is a unique solution $u_h \in \bigcap_{k=0}^4 C^k([0,T]; H^{4-k}_{per})$ of (5.1)–(5.4) satisfying

$$\max_{\substack{|\alpha|\leq 1, |\beta|\leq 2, |\gamma|\leq 1\\ \sigma=0,1,2}}\left(\left\|\left(\partial_t^2\partial_t^\sigma u_h, \nabla_{x,t}^\beta\frac{1}{h}\varepsilon_h(\partial_z^\alpha u_h), \nabla_{x,t}^\gamma\nabla_h^2\partial_z^\alpha u_h\right)\right\|_{C^0([0,T],L^2)}\right.$$

$$+\left\|\frac{1}{h}\int_\Omega \partial_z^{\alpha+\beta}u_h\cdot x^\perp dx\right\|_{C^0([0,T])}\Bigg)\leq Ch^{1+\theta} \tag{5.10}$$

uniformly in $0<h\leq h_0$.

Proof: Without loss of generality we will assume that $0<T\leq 1$. This is possible as we can perform a rescaling in h and t by T^{-1}, changing only M by a T-depending factor. Furthermore we assume that R_0 is sufficiently small, such that all results form Section 5.2.2 are applicable.

The assumptions (5.5)–(5.7) and (5.8) imply

$$\begin{aligned}
&\max_{|\alpha|=1} h^{1+\theta}\Big(\|\partial_t^2\partial_z^\alpha f\|_{L^1(0,T;L^2)}+\|\partial_t\partial_z^\alpha f\|_{L^\infty(0,T;L^2)\cap L^1(0,T;H^{0,1})}+\|\partial_z^\alpha f\|_{L^\infty(0,T;H^1)}\Big)\\
&+\max_{k=0,1,2}\left(\left\|\left(\frac{1}{h}\varepsilon_h(u_{1+k,h}),\partial_{x_1}\frac{1}{h}\varepsilon_h(u_{k,h})\right)\right\|_{L^2}+\left|\frac{1}{h}\int_\Omega u_{1+k,h}\cdot x^\perp dx\right|\right)\\
&+\max_{|\alpha|=1} h^{1+\theta}\|\partial_z^\alpha f|_{t=0}\|_{L^2}+\left\|\left(\frac{1}{h}\varepsilon_h(\partial_{x_1}u_{2,h}),\frac{1}{h}\varepsilon_h(\partial_{x_1}^2u_{1,h}),\partial_{x_1}u_{3,h},\partial_{x_1}^2u_{2,h}\right)\right\|_{L^2}^2\\
&+\left\|u_{4,h}-\frac{1}{h^2}\operatorname{div}_h\left(D^2\tilde W(\nabla_h u_{0,h})[\nabla_h u_{1,h},\nabla_h u_{1,h}]\right)\right\|_{L^2}\\
&+\left\|\partial_{x_1}u_{3,h}-\frac{1}{h^2}\operatorname{div}_h\left(D^2\tilde W(\nabla_h u_{0,h})[\nabla_h u_{1,h},\nabla_h\partial_{x_1}u_{0,h}]\right)\right\|_{L^2}\\
&+\max_{k=1,2}\|(u_{1+k,h},\partial_{x_1}u_{k,h})\|\leq \tilde M h^{1+\theta}
\end{aligned}\tag{5.58}$$

for $\tilde M = CM$ with some universal constant $C\geq 1$. We choose $h_0>0$ small enough such that $R:=106C_{max}\tilde M h_0^\theta\leq R_0$ holds, where $C_{max}\geq 1$ is chosen as in Corollary 5.2.12. Let $u_h\in\bigcap_{k=0}^4 C^k([0,T_{max}(h));H^{4-k}_{per})$ be the solution of (5.1)–(5.4) from Theorem 5.1.2. Then there exists some maximal $0<T'=T'(h)\leq\min\{T_{max}(h),T\}$ such that

$$\begin{aligned}
\max_{\substack{|\alpha|\leq 1,|\beta|\leq 2,|\gamma|\leq 1\\ \sigma=0,1,2}}&\left(\left\|\left(\partial_t^{2+\sigma}u_h,\nabla_{x,t}^\beta\frac{1}{h}\varepsilon_h(\partial_z^\alpha u_h),\nabla_{x,t}^\gamma\nabla_h^2\partial_z^\alpha u_h\right)\right\|_{C^0([0,T'(h)];L^2(\Omega))}\right.\\
&\left.+\left\|\frac{1}{h}\int_\Omega\partial_z^{\alpha+\beta}u_h\cdot x^\perp dx\right\|_{C^0([0,T'(h)])}\right)\leq 106C_{max}\tilde M h^{1+\theta}.
\end{aligned}\tag{5.59}$$

This maximum exists since, as if (5.59) holds the set $\{\nabla_h u_h(x,t)\ :\ x\in\overline{\Omega},\ t\in[0,T'(h)]\}$ is precompact in U_h, cf. Appendix A. Moreover it holds

$$\int_0^{T'(h)}\|\nabla_{x,t}^2 u_h\|_{L^\infty(\Omega)}dt<\infty. \tag{5.60}$$

This can be seen by using (5.59), as it follows

$$\nabla_h^2 u_h(t,\cdot)\in H^{1,1}(\Omega)\hookrightarrow H^1(0,L;H^1(S))\hookrightarrow \mathrm{BUC}([0,L];L^\infty(S))$$

and

$$\partial_t^2 u(t,\cdot)\in H^2(\Omega)\hookrightarrow C^0(\Omega)$$

for all $t\in[0,T'(h))$. Moreover as long as (5.59) is valid, u_h satisfies (5.16) and all the results of Section 5.2, especially Corollary 5.2.12, are applicable. We want to reduce to the case that u_h is mean value free. Hence we assume in a first step that

$$\int_\Omega u_h dx=\int_\Omega u_{0,h}dx=\int_\Omega u_{1,h}dx=\int_\Omega f^h dx=0$$

for all $t \in [0, T'(h)]$. Using (5.1)–(5.4), we obtain that $w_h^j := \partial_{z_j} u_h$, $j = 0, 1$ solves

$$\begin{gathered}
\partial_t^2 w_h^j - \frac{1}{h^2} \operatorname{div}_h (D^2 \tilde{W}(\nabla_h u_h) \nabla_h w_h^j) = h^{1+\theta} \partial_{z_j} f_h \quad \text{in } \Omega \times (0, T'(h)) \\
D^2 \tilde{W}(\nabla_h u_h)[\nabla_h w_h^j] \nu|_{(0,L)\times\partial S} = 0 \\
w_h^j \text{ is } L\text{-periodic in } x_1 \\
(w_h^j, \partial_t w_h^j)\Big|_{t=0} = (w_{0,h}^j, w_{1,h}^j)
\end{gathered}$$

with $w_{k,h}^0 = u_{1+k,h}$ and $w_{k,h}^1 = \partial_{x_1} u_{k,h}$. Hence with Corollary 5.2.12 and (5.58) it follows

$$\begin{aligned}
&\max_{\substack{|\alpha|=1, |\beta|\le 2, |\gamma|\le 1 \\ \sigma=0,1,2}} \left(\left\| \left(\partial_t^{2+\sigma} u_h, \nabla_{x,t}^\beta \frac{1}{h} \varepsilon_h(\partial_z^\alpha u_h), \nabla_{x,t}^\gamma \nabla_h^2 \partial_z^\alpha u_h \right) \right\|_{C^0([0,T'(h)],L^2)} \right. \\
&\qquad \left. + \left\| \frac{1}{h} \int_\Omega \partial_z^{\alpha+\beta} u_h \cdot x^\perp dx \right\|_{C^0([0,T'(h)])} \right) \\
&\le C_{max} e^{C\tilde{M} h_0^\theta} \left(\tilde{M} h^{1+\theta} + \frac{1}{h} \left\| \int_\Omega \partial_t^3 u_h \cdot x^\perp dx \right\|_{L^\infty(0,T'(h))} \right. \\
&\quad \left. + 2R \max_{\sigma=0,1,2} \left\| \frac{1}{h} \int_\Omega \partial_t^{1+\sigma} u_h \cdot x^\perp dx \right\|_{L^\infty(0,T'(h))} \right) + \max_{\sigma=0,1,2} \left\| \frac{1}{h} \int_\Omega \partial_t^{1+\sigma} u_h \cdot x^\perp dx \right\|_{C^0([0,T'(h)])} \\
&\le C_{max} e^{C\tilde{M} h_0^\theta} \left(\tilde{M} h^{1+\theta} + 4 \max_{\sigma=0,1,2} \left\| \frac{1}{h} \int_\Omega \partial_t^{1+\sigma} u_h \cdot x^\perp dx \right\|_{C^0([0,T'(h)])} \right)
\end{aligned}$$

where we used $(\partial_{x_1} w, x^\perp)_{L^2(\Omega)} = 0$ and note that $\partial_t^2 \partial_z^\eta u_h$ for $|\eta| \le 2$ is given via, e.g.

$$\|\partial_t^2 \partial_{x_1} u_h\|_{L^2(\Omega)} \le \|\nabla_h \partial_t^2 u_h\|_{L^2(\Omega)} \le \left\| \frac{1}{h} \varepsilon_h(\partial_t^2 u_h) \right\|_{L^2(\Omega)} + \left| \frac{1}{h} \int_\Omega \partial_t^2 u_h \cdot x^\perp dx \right|$$

due to Korn's inequality. Now we want to apply (5.55)–(5.57) in order to bound the rotational part of u_h. It follows for $\sigma \in \{0, 1, 2\}$

$$\begin{aligned}
&\left\| \frac{1}{h} (\partial_t^{1+\sigma} u_h, x^\perp)_{L^2} \right\|_{C^0([0,T'(h)])} \le \left\| \frac{1}{h} (u_{1,h}, x^\perp)_{L^2} \right\|_{C^0([0,T'(h)])} + \left\| \frac{1}{h^2} (\partial_t u_h^\perp, g^h)_{L^2} \right\|_{C^0([0,T'(h)])} \\
&\quad + \left\| \frac{1}{h} (\partial_t u_h^\perp, \partial_t^2 u_h)_{L^2} \right\|_{C^0([0,T'(h)])} + \max_{k=0,1} \left(\left\| \frac{1}{h} (x^\perp, \partial_t^k g^h)_{L^2} \right\|_{C^0([0,T'(h)])} \right. \\
&\quad \left. + \left\| \frac{1}{h^2} (u_h^\perp, \partial_t^k g^h)_{L^2} \right\|_{C^0([0,T'(h)])} + \left\| \frac{1}{h^2} (u_h^\perp, \partial_t^{2+k} u_h)_{L^2} \right\|_{C^0([0,T'(h)])} \right)
\end{aligned}$$

Due to the assumptions on initial values $u_{1,h}$ and the external force f^h, we obtain that

$$\left| \frac{1}{h} (u_{1,h}, x^\perp)_{L^2(\Omega)} \right| \le \tilde{M} h^{1+\theta}$$

and

$$\max_{\sigma=0,1} \left\| \frac{1}{h} (x^\perp, \partial_t^\sigma g^h)_{L^2} \right\|_{C^0([0,T'(h)])} \le \tilde{M} h^{1+\theta}$$

holds. Moreover for $\sigma \in \{0, 1\}$ we deduce with Cauchy-Schwarz and Poincaré inequality

$$\begin{aligned}
\left\| \frac{1}{h^2} (u_h^\perp, \partial_t^\sigma g^h)_{L^2} \right\|_{C^0([0,T'(h)])} &\le \frac{1}{h^2} \|u_h\|_{C^0([0,T'(h)];L^2)} \|\partial_t^\sigma g^h\|_{C^0([0,T'(h)];L^2)} \\
&\le \frac{C_p}{h^2} \|\nabla_h u_h\|_{C^0([0,T'(h)];L^2)} \|\partial_t^\sigma g^h\|_{C^0([0,T'(h)];L^2)} \le C_p \tilde{M} h^{1+\theta}
\end{aligned}$$

as $h^{\theta-1} \leq 1$ for $h \in (0,1]$. Similarly we obtain

$$\left\| \frac{1}{h^2}(\partial_t u_h^\perp, \partial_t^\sigma g^h)_{L^2} \right\|_{C^0([0,T'(h)])} \leq C_p \tilde{M} h^{1+\theta}.$$

and

$$\begin{aligned}\left\| \frac{1}{h^2}(u_h^\perp, \partial_t^{2+\sigma} u_h)_{L^2} \right\|_{C^0([0,T'(h)])} &\leq \frac{C_p^2}{h^2} \|\nabla_h u_h\|_{C^0([0,T'(h)];L^2)} \|\nabla_h \partial_t^{2+\sigma} u_h\|_{C^0([0,T'(h)];L^2)} \\ &\leq C_p^2 \tilde{M} h^{1+\theta}.\end{aligned}$$

Alltogether this leads to

$$\max_{\sigma=0,1,2} \left\| \frac{1}{h}(\partial_t^{1+\sigma} u_h, x^\perp)_{L^2} \right\|_{C^0([0,T'(h)])} \leq 6\tilde{M} h^{1+\theta}$$

Thus we have shown

$$\begin{aligned}\max_{\substack{|\alpha|=1,|\beta|\leq 2,|\gamma|\leq 1 \\ \sigma=0,1,2}} \Bigg(\left\| \left(\partial_t^2 \partial_t^\sigma u_h, \nabla_{x,t}^\beta \frac{1}{h} \varepsilon_h(\partial_z^\alpha u_h), \nabla_{x,t}^\gamma \nabla_h^2 \partial_z^\alpha u_h \right) \right\|_{C^0([0,T'(h)];L^2)} \\ + \left\| \frac{1}{h} \int_\Omega \partial_t^{1+\sigma} u_h \cdot x^\perp dx \right\|_{C^0([0,T'(h)])} \Bigg) \leq 25 C_{max} e^{C\tilde{M} h_0^\theta} \tilde{M} h^{1+\theta}\end{aligned} \tag{5.61}$$

Moreover, exploting

$$u_h = u_{0,h} + \int_0^t \partial_t u_h ds \quad \text{and} \quad \nabla_h u_h = \nabla_h u_{0,h} + \int_0^t \nabla_h w_h^0 ds$$

we obtain with (5.6)

$$\begin{aligned}\|\nabla_h^2 u_h\|_{C^0([0,T'(h)];L^2)} &\leq \|\nabla_h^2 u_{0,h}\|_{L^2)} + T \|\nabla_h^2 \partial_t u_h\|_{C^0([0,T'(h)];L^2)} \\ &\leq M h^{1+\theta} + 25 C_{max} e^{C\tilde{M} h_0^\theta} \tilde{M} h^{1+\theta}\end{aligned}$$

and

$$\begin{aligned}&\left\| \frac{1}{h} \varepsilon_h(u_h) \right\|_{C^0([0,T'(h)],L^2)} + \left\| \frac{1}{h} \int_\Omega u_h \cdot x^\perp dx \right\|_{C^0([0,T'(h)])} \\ &\leq \left\| \frac{1}{h} \varepsilon_h(u_{0,h}) \right\|_{L^2} + T \left\| \frac{1}{h} \varepsilon_h(\partial_t u_h) \right\|_{C^0([0,T'(h)],L^2)} \\ &\quad + \left| \frac{1}{h} \int_\Omega u_{0,h} \cdot x^\perp dx \right| + T \left\| \frac{1}{h} \int_\Omega \partial_t u_{0,h} \cdot x^\perp dx \right\|_{C^0([0,T'(h)])} \\ &\leq 2M h^{1+\theta} + 50 C_{max} e^{C\tilde{M} h_0^\theta} \tilde{M} h^{1+\theta}\end{aligned}$$

due to (5.5), (5.7) and (5.61). Hence we deduce

$$\begin{aligned}\max_{\substack{|\alpha|\leq 1,|\beta|\leq 2,|\gamma|\leq 1 \\ \sigma=0,1,2}} \Bigg(\left\| \left(\partial_t^2 \partial_t^\sigma u_h, \nabla_{x,t}^\beta \frac{1}{h} \varepsilon_h(\partial_z^\alpha u_h), \nabla_{x,t}^\gamma \nabla_h^2 \partial_z^\alpha u_h \right) \right\|_{C^0([0,T'(h)];L^2)} \\ + \left\| \frac{1}{h} \int_\Omega \partial_z^{\alpha+\beta} u_h \cdot x^\perp dx \right\|_{C^0([0,T'(h)])} \Bigg) \leq 103 C_{max} e^{C\tilde{M} h_0^\theta} \tilde{M} h^{1+\theta}.\end{aligned} \tag{5.62}$$

As $\theta \geq 1$ we can find $h_0 > 0$ such that

$$\max_{\substack{|\alpha|\leq 1, |\beta|\leq 2, |\gamma|\leq 1\\ \sigma=0,1,2}} \left(\left\| \left(\partial_t^2 \partial_t^\sigma u_h, \nabla_{x,t}^\beta \frac{1}{h} \varepsilon_h(\partial_z^\alpha u_h), \nabla_{x,t}^\gamma \nabla_h^2 \partial_z^\alpha u_h \right) \right\|_{C^0([0,T'(h)];L^2)} + \left\| \frac{1}{h} \int_\Omega \partial_z^{\alpha+\beta} u_h \cdot x^\perp dx \right\|_{C^0([0,T'(h)])} \right) \leq 104 C_{max} \tilde{M} h^{1+\theta}.$$

uniformly in $h \in (0, h_0]$.

Now we have to consider the case that the force or the initial data is not mean value free. In this case we define

$$a(t) := \fint_\Omega u_{0,h}(t)dx - t \fint_\Omega u_{1,h}(t) - \int_0^t (t-s) \fint_\Omega f^h(s) dx ds. \tag{5.63}$$

Then $\tilde{u}_h(t) := u_h(t) - a(t)$ solves

$$\begin{aligned} \partial_t^2 \tilde{u}_h - \frac{1}{h^2} \operatorname{div}_h D\tilde{W}(\nabla_h \tilde{u}_h) &= h^{1+\theta} \tilde{f}_h \quad \text{in } \Omega \times [0,T) \\ D\tilde{W}(\nabla_h \tilde{u}_h)\nu|_{(0,L)\times\partial S} &= 0 \\ \tilde{u}_h \text{ is } L\text{-periodic w.r.t. } x_1 & \\ (\tilde{u}_h, \partial_t \tilde{u}_h)|_{t=0} &= (\tilde{u}_{0,h}, \tilde{u}_{1,h}) \end{aligned}$$

where we subtracted from $(f, u_{0,h}, u_{1,h})$ their mean values to obtain $(\tilde{f}, \tilde{u}_{0,h}, \tilde{u}_{1,h})$.

Then it holds for $\tilde{u}$

$$\int_\Omega \tilde{u}_h(t) dx = \int_\Omega u_h(t) - a(t) dx = 0.$$

as, integration of the nonlinear equation (5.1) implies with the boundary and periodicity condition, (5.2) and (5.3), respectively

$$\partial_t^2 \int_\Omega u(x,t) dx = \int_\Omega f^h(x,t) dx.$$

Moreover (5.7) and (5.9) is fulfilled for $\tilde{u}_{k,h}$, $k \in \{0,1,2,3\}$ and $\tilde{f}$, because $\int_\Omega x^\perp dx = 0$. Deploying the fact that the initial data is only changed by a constant, (5.6) holds for the new initial values. With $L^2(\Omega) \hookrightarrow L^1(\Omega)$ and triangle inequality it follows (5.8) with $\tilde{C}M$ instead of M, for some $\tilde{C} \geq 1$ independent of h_0, h and M. In the same way one can deal with $\tilde{u}_{2+k,h}$ in (5.16). Hence, as for (5.6), we obtain that (5.5) holds with M replaced by $\tilde{C}M$. Thus we can apply (5.62) for $\tilde{u}_h$ and get

$$\max_{\substack{|\alpha|\leq 1, |\beta|\leq 2, |\gamma|\leq 1\\ \sigma=0,1,2}} \left(\left\| \left(\partial_t^2 \partial_t^\sigma \tilde{u}_h, \nabla_{x,t}^\beta \frac{1}{h} \varepsilon_h(\partial_z^\alpha \tilde{u}_h), \nabla_{x,t}^\gamma \nabla_h^2 \partial_z^\alpha \tilde{u}_h \right) \right\|_{C^0([0,T'(h)];L^2)} + \left\| \frac{1}{h} \int_\Omega \partial_z^{\alpha+\beta} \tilde{u}_h \cdot x^\perp dx \right\|_{C^0([0,T'(h)])} \right) \leq 103 C_{max} \tilde{C} e^{C\tilde{C}\tilde{M}h_0^\theta} \tilde{M} h^{1+\theta}.$$

From the definition of $\tilde{u}_h$ it follows

$$\nabla_{x,t}^\beta \frac{1}{h} \varepsilon_h(\partial_z^\alpha \tilde{u}_h) = \nabla_{x,t}^\beta \frac{1}{h} \varepsilon_h(\partial_z^\alpha u_h) \quad \text{and} \quad \nabla_{x,t}^\gamma \nabla_h^2 \partial_z^\alpha \tilde{u}_h = \nabla_{x,t}^\gamma \nabla_h^2 \partial_z^\alpha u_h.$$

Because of $\int_\Omega x^\perp dx = 0$, we have

$$\frac{1}{h} \int_\Omega \partial_z^{\alpha+\beta} \tilde{u}_h \cdot x^\perp dx = \frac{1}{h} \int_\Omega \partial_z^{\alpha+\beta} u_h \cdot x^\perp dx.$$

Lastly for $\sigma \in \{0,1,2\}$ we deduce from $\partial_t^{2+\sigma} a(t) = h^{1+\theta} \int_\Omega \partial_t^\sigma f^h dx$ and (5.9) that

$$
\begin{aligned}
&\max_{\sigma=0,1,2} \|\partial_t^{2+\theta} u_h\|_{C^0([0,T'(h)];L^2)} \\
&\qquad \leq \max_{\sigma=0,1,2} \|\partial_t^{2+\sigma} \tilde{u}_h\|_{C^0([0,T'(h)];L^2)} + \max_{\sigma=0,1,2} \|\partial_t^{2+\sigma} a(t)\|_{C^0([0,T'(h)];L^2)} \\
&\qquad \leq \max_{\sigma=0,1,2} \|\partial_t^{2+\sigma} \tilde{u}_h\|_{C^0([0,T'(h)];L^2)} + M h^{1+\theta}
\end{aligned}
$$

Thus it follows

$$
\begin{aligned}
&\max_{\substack{|\alpha|\leq 1, |\beta|\leq 2, |\gamma|\leq 1 \\ \sigma=0,1,2}} \left(\left\| \left(\partial_t^2 \partial_t^\sigma u_h, \nabla_{x,t}^\beta \frac{1}{h} \varepsilon_h(\partial_z^\alpha u_h), \nabla_{x,t}^\gamma \nabla_h^2 \partial_z^\alpha u_h \right) \right\|_{C^0([0,T'(h)];L^2)} \right. \\
&\qquad\qquad \left. + \left\| \frac{1}{h} \int_\Omega \partial_z^{\alpha+\beta} u_h \cdot x^\perp dx \right\|_{C^0([0,T'(h)])} \right) \leq 104 C_{max} \tilde{C} e^{C\tilde{C}\tilde{M}h_0^\theta} \tilde{M} h^{1+\theta} \\
&\quad \leq 105 C_{max} \tilde{M} h^{1+\theta}
\end{aligned}
$$

for h_0 small. □

6

Asymptotic First Order Expansion in a Linearised Regime

In this chapter we construct an approximation to the unique solution of the non-linear system (5.1)–(5.4). To this end we use a solution to an appropriate one dimensional system of plate equations, which stems from the limiting energy $\mathcal{I}_\alpha$.

The ansatz is inspired by the recovery sequence in the proof of Γ-convergence, i.e.

$$\tilde{u}_h(x,t) = h^{1+\theta}\begin{pmatrix} 0 \\ v \end{pmatrix} + h^{2+\theta}\begin{pmatrix} -x_2\partial_{x_1}v_2 - x_3\partial_{x_1}v_3 \\ 0 \\ 0 \end{pmatrix} + O(h^{3+\theta})$$

where v is a function and $\theta \geq 0$. Hence it holds

$$\varepsilon_h(\tilde{u}_h) = h^{2+\theta}\begin{pmatrix} -x_2\partial_{x_1}^2 v_2 - x_3\partial_{x_1}^2 v_3 & 0 \\ 0 & 0 \end{pmatrix} + O(h^{2+\theta}).$$

We see that such an ansatz solves (5.1) in highest order. To obtain a solution of (5.1)–(5.4) we need higher correction terms. We will construct suitable coefficient functions defined on S as solutions to some boundary value problems. Then we construct well prepared initial values $(u_{0,h}, u_{1,h})$, which meet the conditions of Theorem 5.1.1. Then the existence of a solution to the non-linear system is guaranteed by Theorem 5.1.1 and we can bound the difference $u_h - \tilde{u}_h$ in a suitable norm by $h^{1+2\theta}$.

In this chapter we will impose some simplifications. First we assume that

$$f^h(x,t) = \begin{pmatrix} 0 \\ g(x_1,t) \end{pmatrix}$$

for some $g \in \bigcap_{k=0}^{3} W_1^k(0,T; H_{per}^{10-2k}(0,L;\mathbb{R}^2))$, which implies

$$\int_S f^h(x,t)x_k dx' = 0$$

for $k = 2,\ 3$. Moreover we assume that

$$\max_{\sigma=0,1,2} \|\partial_t^\sigma g|_{t=0}\|_{H^{2-2\sigma}(0,L)} \leq M, \tag{6.1}$$

where $M > 0$ is chosen later. Without loss of generality we can assume $\int_0^L g dx_1 = 0$, using (5.63) analogously as in the proof of Theorem 5.1.1. Moreover let

$$W(F) = \frac{1}{2}\operatorname{dist}^2(F, SO(3))$$

for all $F \in \mathbb{R}^{3\times 3}$. This implies then $D^2\tilde{W}(0)F = \operatorname{sym} F$ and for $P \in \mathbb{R}^{3\times 3}_{skew}$, $A,\ B \in \mathbb{R}^{3\times 3}$ it holds

$$D^3\tilde{W}(0)[A, B, P] = \Big((A^T - A)^T \operatorname{sym}(B) + (B^T - B)^T \operatorname{sym}(A)\Big) : P.$$

Moreover we restrict to the case of $\theta = 1$. For the ansatz function we regard the following system of $1d$-beam equations

$$\partial_t^2 v + \begin{pmatrix} I_2 & 0 \\ 0 & I_3 \end{pmatrix} \partial_{x_1}^4 v = g$$
$$v \text{ is } L\text{-periodic in } x_1$$
$$(v, \partial_t v)|_{t=0} = (\tilde{v}_0, \tilde{v}_1)$$

where $\tilde{v}_0 \in H^{12}_{per}(0, L; \mathbb{R}^2)$, $\tilde{v}_1 \in H^{10}_{per}(0, L; \mathbb{R}^2)$, such that

$$\|\tilde{v}_0\|_{H^8(0,L)} \le M \quad \text{and} \quad \|\tilde{v}_1\|_{H^5(0,L)} \le M \tag{6.2}$$

and

$$I_k := \int_S x_k^2 dx'$$

for $k = 2,\ 3$. Then we obtain with standard methods, as e.g. in [RR04, Theorem 11.8], the existence of a unique solution

$$v \in \bigcap_{j=0}^{4} C^j([0, T]; H^{12-2j}_{per}(0, L; \mathbb{R}^2)).$$

Moreover, due to the assumptions for g and the periodicity of v it follows

$$\partial_t^2 \int_0^L v dx_1 = 0.$$

Remark 6.0.1. We want to give a short explanation on how one can formally derive the system for v. Assume for this remark that the modelled material is just isotropic. Hence there exists $\lambda \ge 0$ and $\mu > 0$ such that

$$D^2\tilde{W}(0)F = 2\mu \operatorname{sym} F + \lambda \operatorname{tr}(F) Id$$

due to Theorem B.5. Then we can compute the minimum defining Q^0 as in Remark 3.3.6 and obtain

$$Q^0(t, F) = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}\Big(F_{12}^2 I_2 + F_{13}^2 I_3 + t^2\Big) + \mu\tau F_{23}^2$$

for

$$\tau := \int_S \big(x_3^2 + x_2^2 + x_3\partial_{x_2}\psi - x_2\partial_{x_3}\psi + (\partial_{x_2}\psi)^2 + (\partial_{x_3}\psi)^2\big)dx'.$$

Hence for $\alpha \in (3, 4]$ the limiting energy is given via

$$\begin{aligned}
\mathcal{I}_\alpha(u, v, w) &= \frac{1}{2}\int_0^L Q^0(\partial_{x_1} u, \partial_{x_1} A) dx_1 \\
&= \frac{1}{2}\int_0^L \frac{\mu(3\lambda + \mu)}{\lambda + \mu}\Big((\partial_{x_1}^2 v_2)^2 I_2 + (\partial_{x_1}^2 v_3)^2 I_3 + (\partial_{x_1} u)^2\Big) + \mu\tau(\partial_{x_1} w)^2 dx_1
\end{aligned}$$

Thus in the dynamic case this could lead to the Euler-Lagrange equations

$$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\partial_{x_1}^2 u = 0$$
$$\partial_t^2 v + \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\begin{pmatrix} I_2 & 0 \\ 0 & I_3 \end{pmatrix}\partial_{x_1}^4 v = (\bar{R}')^T g$$
$$\mu\tau\partial_{x_1}^2 w = 0$$
$$\int_0^L \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\bar{R}' g\cdot v dx_1 = 0.$$

Here the heuristic argument uses the scaling properties of $u^{(h)}$, $v^{(h)}$ and $w^{(h)}$, which suggest that

$$\int_S \partial_t y_1^{(h)} dx' \sim O(h^3) \quad \text{and} \quad \int_S \partial_t y_k^{(h)} dx' \sim O(h^2), \qquad \text{for } k=2,\ 3$$

and

$$\int_S \frac{x_2\partial_t y_3^{(h)}(x_1,x') - x_2\partial_t y_2^{(h)}(x_1,x')}{\mu(S)}dx' \sim O(h^3).$$

As u and w have to be affine linear functions satisfying periodic boundary conditions, it follows $u\equiv\bar{u}$ and $w\equiv\bar{w}$.

In the following we construct the refined ansatz function. For this we define

$$\begin{aligned}\tilde{u}_h(x,t) = {}& h^2\begin{pmatrix} 0 \\ v_2 \\ v_3 \end{pmatrix} + h^3\begin{pmatrix} -x_2\partial_{x_1}v_2 - x_3\partial_{x_1}v_3 \\ 0 \\ 0 \end{pmatrix} + h^5\begin{pmatrix} a_2(x')\partial_{x_1}^3 v_2 + a_3(x')\partial_{x_1}^3 v_3 \\ 0 \\ 0 \end{pmatrix} \\ & + h^6\begin{pmatrix} 0 \\ b_2(x')\partial_{x_1}^4 v_2 + c_3(x')\partial_{x_1}^4 v_3 \\ b_3(x')\partial_{x_1}^4 v_3 + c_2(x')\partial_{x_1}^4 v_2 \end{pmatrix}\end{aligned} \tag{6.3}$$

where a, b, $c\colon S\to\mathbb{R}^2$ are chosen later. Then

$$\begin{aligned}\nabla_h\tilde{u}_h(x,t) = {}& h^2\begin{pmatrix} 0 & -\partial_{x_1}v_2 & -\partial_{x_1}v_3 \\ \partial_{x_1}v_2 & 0 & 0 \\ \partial_{x_1}v_3 & 0 & 0 \end{pmatrix} + h^3\begin{pmatrix} -x_2\partial_{x_1}^2 v_2 - x_3\partial_{x_1}^2 v_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ & + h^4\begin{pmatrix} 0 & \partial_{x_2}a_2\partial_{x_1}^3 v_2 + \partial_{x_2}a_3\partial_{x_1}^3 v_3 & \partial_{x_3}a_2\partial_{x_1}^3 v_2 + \partial_{x_3}a_3\partial_{x_1}^3 v_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ & + h^5\begin{pmatrix} a_2\partial_{x_1}^4 v_2 + a_3\partial_{x_1}^4 v_3 & 0 & 0 \\ 0 & \partial_{x_2}b_2\partial_{x_1}^4 v_2 + \partial_{x_2}c_3\partial_{x_1}^4 v_3 & \partial_{x_3}b_2\partial_{x_1}^4 v_2 + \partial_{x_3}c_3\partial_{x_1}^4 v_3 \\ 0 & \partial_{x_2}b_3\partial_{x_1}^4 v_3 + \partial_{x_2}c_2\partial_{x_1}^4 v_2 & \partial_{x_3}b_3\partial_{x_1}^4 v_3 + \partial_{x_3}c_2\partial_{x_1}^4 v_2 \end{pmatrix} \\ & + h^6\begin{pmatrix} 0 & 0 & 0 \\ b_2\partial_{x_1}^5 v_2 + c_3\partial_{x_1}^5 v_3 & 0 & 0 \\ b_3\partial_{x_1}^5 v_3 + c_2\partial_{x_1}^5 v_2 & 0 & 0 \end{pmatrix}\end{aligned}$$

Thus with $D^2W(Id)F = \operatorname{sym} F$ we can derive

$$\begin{aligned}\frac{1}{h^2}\operatorname{div}_h(D^2W(Id)\nabla_h\tilde{u}_h) = {}& h\begin{pmatrix} \left(\frac{1}{2}\Delta a - (x_2,x_3)^T\right)\cdot\partial_{x_1}^3 v \\ 0 \\ 0 \end{pmatrix} \\ & + h^2\begin{pmatrix} 0 \\ \nabla_{x'}a(x')^T\partial_{x_1}^4 v + \begin{pmatrix} \partial_{x_2}^2 b_2 & \partial_{x_2}^2 c_3 \\ \partial_{x_3}^2 c_2 & \partial_{x_3}^2 b_3 \end{pmatrix}\partial_{x_1}^4 v \end{pmatrix}\end{aligned}$$

$$+\frac{h^2}{2}\left(\begin{pmatrix} & 0 & \\ \begin{pmatrix}\partial_{x_3}\partial_{x_2}c_2+\partial_{x_3}^2 b_2 & \partial_{x_2}\partial_{x_3}b_3+\partial_{x_3}^2 c_3\\ \partial_{x_2}^2 c_2+\partial_{x_2}\partial_{x_3}b_2 & \partial_{x_2}^2 b_3+\partial_{x_2}\partial_{x_3}c_3\end{pmatrix}\partial_{x_1}^4 v\end{pmatrix}\right)+r_h(x,t)$$

for

$$r_h(x,t)=O(h^3).$$

Moreover for the boundary condition it holds

$$\begin{aligned}
D^2W(Id)[\nabla_h\tilde{u}_h]\nu &= h^4\begin{pmatrix}\frac{1}{2}(\nabla_{x'}a\nu_{\partial S})\cdot\partial_{x_1}^3 v\\ 0\\ 0\end{pmatrix}\\
&+h^5\begin{pmatrix}0\\ \left(\partial_{x_2}b_2\nu_2+\frac{1}{2}(\partial_{x_2}c_2+\partial_{x_3}b_2)\nu_3\right)\partial_{x_1}^4 v_2+\left(\partial_{x_2}c_3\nu_2+\frac{1}{2}(\partial_{x_2}b_3+\partial_{x_3}c_3)\nu_3\right)\partial_{x_1}^4 v_3\\ \left(\frac{1}{2}(\partial_{x_2}c_2+\partial_{x_3}b_2)\nu_2+\partial_{x_3}c_2\nu_3\right)\partial_{x_1}^4 v_2+\left(\frac{1}{2}(\partial_{x_2}b_3+\partial_{x_3}c_3)\nu_2+\partial_{x_3}b_3\nu_3\right)\partial_{x_1}^4 v_3\end{pmatrix}\\
&+\frac{h^6}{2}\begin{pmatrix}\nu^T\begin{pmatrix}b_2 & c_2\\ b_3 & c_3\end{pmatrix}\partial_{x_1}^5 v\\ 0\\ 0\end{pmatrix}\\
&= h^4\begin{pmatrix}(\nabla_{x'}a\nu_{\partial S})\cdot\partial_{x_1}^3 v\\ 0\\ 0\end{pmatrix}+h^5\begin{pmatrix}0\\ \nu^T\begin{pmatrix}\partial_{x_2}b_2 & \frac{1}{2}(\partial_{x_2}c_2+\partial_{x_3}b_2)\\ \partial_{x_2}c_3 & \frac{1}{2}(\partial_{x_2}b_3+\partial_{x_3}c_3)\end{pmatrix}\partial_{x_1}^4 v\\ \nu^T\begin{pmatrix}\frac{1}{2}(\partial_{x_2}c_2+\partial_{x_3}b_2) & \partial_{x_3}c_2\\ \frac{1}{2}(\partial_{x_2}b_3+\partial_{x_3}c_3) & \partial_{x_3}b_3\end{pmatrix}\partial_{x_1}^4 v\end{pmatrix}\\
&+\frac{h^6}{2}\begin{pmatrix}\nu^T\begin{pmatrix}b_2 & c_2\\ b_3 & c_3\end{pmatrix}\partial_{x_1}^5 v\\ 0\\ 0\end{pmatrix}.
\end{aligned}$$

We choose now $a\colon S\to\mathbb{R}^2$ as the solution of the following system

$$\begin{cases}-\Delta a=-2\begin{pmatrix}x_2\\ x_3\end{pmatrix} & \text{in } S\\ \nabla_{x'}a\nu=0 & \text{on } \partial S\end{cases}$$

with

$$\int_S a(x')dx'=0.$$

Such a solution exists, because we can apply the Lax-Milgram Lemma for the weak Laplacian on $H^1_{(0)}(\overline{S};\mathbb{R}^2)$. Thereby, the coercivity follows from Poincaré's inequality. With the regularity result, Theorem 2.1.18, we obtain $a\in C^\infty(S,\mathbb{R}^2)$. The systems for b and c decouple to

$$\begin{cases}\partial_{x_2}^2 b_2+\frac{1}{2}\partial_{x_3}^2 b_2+\frac{1}{2}\partial_{x_3}\partial_{x_2}c_2=I_1-\partial_{x_2}a_2 & \text{in } S\\ \frac{1}{2}\partial_{x_2}^2 c_2+\partial_{x_3}^2 c_2+\frac{1}{2}\partial_{x_2}\partial_{x_3}b_2=-\partial_{x_3}a_2 & \text{in } S\end{cases}\tag{6.4}$$

and

$$\begin{cases}\partial_{x_2}^2 c_3+\frac{1}{2}\partial_{x_3}^2 c_3+\frac{1}{2}\partial_{x_2}\partial_{x_3}b_3=-\partial_{x_2}a_3 & \text{in } S\\ \frac{1}{2}\partial_{x_2}^2 b_3+\partial_{x_3}^2 b_3+\frac{1}{2}\partial_{x_2}\partial_{x_3}c_3=I_2-\partial_{x_3}a_3 & \text{in } S\end{cases}\tag{6.5}$$

Defining the matrix of coefficients $(\mathfrak{p}_{ij}^{\alpha\beta})_{i,j=1,2}^{\alpha,\beta=2,3}$ in the following way

$$\begin{gathered}\mathfrak{p}_{11}^{22}=1 \quad \mathfrak{p}_{11}^{33}=\frac{1}{2} \quad \mathfrak{p}_{12}^{32}=\frac{1}{4} \quad \mathfrak{p}_{12}^{23}=\frac{1}{4}\\ \mathfrak{p}_{22}^{22}=\frac{1}{2} \quad \mathfrak{p}_{22}^{33}=1 \quad \mathfrak{p}_{21}^{23}=\frac{1}{4} \quad \mathfrak{p}_{21}^{32}=\frac{1}{4}\\ \mathfrak{p}_{ij}^{\alpha\beta}=0 \text{ otherwise}\end{gathered}$$

and defining $w=(b_2,c_2)^T$ and $f=(-I_1-\partial_{x_2}a_2,-\partial_{x_3}a_2)^T$. Then (6.4) is equivalent to

$$\sum_{\alpha,\beta=2}^{3}\sum_{j=1}^{2}-\partial_\beta\left(\mathfrak{p}_{ij}^{\alpha\beta}\partial_\alpha w_j\right)=f_i$$

for $i=1$, 2. Let now

$$\xi:=\begin{pmatrix}\xi_{12} & \xi_{13}\\ \xi_{22} & \xi_{23}\end{pmatrix}\in\mathbb{R}^{2\times2}.$$

be arbitrary. Then it holds

$$\begin{aligned}\sum_{\alpha,\beta=2}^{3}\sum_{i,j=1}^{2}\mathfrak{p}_{ij}^{\alpha\beta}\xi_{i\alpha}\xi_{j\beta}&=\xi_{12}^2+\frac{1}{2}\xi_{13}^2+\frac{1}{2}\xi_{22}^2+\xi_{23}^2+\frac{1}{2}\xi_{22}\xi_{13}+\frac{1}{2}\xi_{12}\xi_{23}\\&=\frac{3}{4}(\xi_{12}^2+\xi_{23}^2)+\frac{1}{4}(\xi_{12}+\xi_{23})^2+\frac{1}{4}(\xi_{13}^2+\xi_{22}^2)+\frac{1}{4}(\xi_{13}+\xi_{22})^2\\&\geq\frac{1}{4}(\xi_{12}^2+\xi_{13}^2+\xi_{22}^2+\xi_{23}^2)=\frac{1}{4}|\xi|^2\end{aligned}$$

and thus $a_{ij}^{\alpha\beta}$ satisfies the Legendre condition for $\lambda=\frac{1}{4}$. Thus we can solve (6.4) and (6.5) with homogeneous Dirichlet boundary condition

$$\begin{pmatrix}b_2\\c_2\end{pmatrix}=0 \quad \text{and} \quad \begin{pmatrix}b_3\\c_3\end{pmatrix}=0 \quad \text{on } \partial S$$

as the system (6.5) can be treated in the same manner. The regularity of a implies now with that $b=(b_2,b_3)$ and $c=(c_2,c_3)$ are $C^\infty(\overline{S};\mathbb{R}^2)$.
The approximating solution $\tilde{u}_h$ solves then the following system

$$\begin{aligned}\partial_t^2\tilde{u}_h-\frac{1}{h^2}\operatorname{div}_h\left(D^2\tilde{W}(0)\nabla_h\tilde{u}_h\right)&=h^{1+\theta}f_h-r_h &&\text{in } \Omega\times(0,T)\\ D^2\tilde{W}(0)[\nabla_h\tilde{u}_h]\nu\Big|_{(0,L)\times\partial S}&=\operatorname{tr}_{\partial\Omega}(r_{N,h})\nu &&\text{on } \partial\Omega\times(0,T)\\ &\tilde{u}_h \text{ is } L\text{-periodic in } x_1\text{-direction}\\ &(\tilde{u}_h,\partial_t\tilde{u}_h)|_{t=0}=(\tilde{u}_{0,h},\tilde{u}_{1,h})\end{aligned}$$

where r_h is chosen as above,

$$r_{N,h}:=h^5\begin{pmatrix}0\\ \nu^T\begin{pmatrix}\partial_{x_2}b_2 & \frac{1}{2}(\partial_{x_2}c_2+\partial_{x_3}b_2)\\ \partial_{x_2}c_3 & \frac{1}{2}(\partial_{x_2}b_3+\partial_{x_3}c_3)\end{pmatrix}\partial_{x_1}^4v\\ \nu^T\begin{pmatrix}\frac{1}{2}(\partial_{x_2}c_2+\partial_{x_3}b_2) & \partial_{x_3}c_2\\ \frac{1}{2}(\partial_{x_2}b_3+\partial_{x_3}c_3) & \partial_{x_3}b_3\end{pmatrix}\partial_{x_1}^4v\end{pmatrix},$$

and the initial data is given by

$$\tilde{u}_{j,h}(x,t)=h^2\begin{pmatrix}0\\v_2^j\\v_3^j\end{pmatrix}+h^3\begin{pmatrix}-x_2\partial_{x_1}v_2^j-x_3\partial_{x_1}v_3^j\\0\\0\end{pmatrix}+h^5\begin{pmatrix}a_2(x')\partial_{x_1}^3v_2^j+a_3(x')\partial_{x_1}^3v_3^j\\0\\0\end{pmatrix}$$

$$+ h^6 \begin{pmatrix} 0 \\ b_2(x')\partial_{x_1}^4 v_2^j + c_3(x')\partial_{x_1}^4 v_3^j \\ b_3(x')\partial_{x_1}^4 v_3^j + c_2(x')\partial_{x_1}^4 v_2^j \end{pmatrix} \tag{6.6}$$

with $v^j := \partial_t^j v|_{t=0}$ and $j = 0, \dots, 4$. For the remainder it holds

$$\|r_h\|_{C^0(0,T;L^2)} \leq Ch^3$$
$$\|r_{N,h}\|_{C^2(0,T;H^1)} \leq Ch^5.$$

In order to bound differences between the approximation $\tilde{u}_h$ and the analytic solution u_h we regard the following weak form of the linearised system (5.12)–(5.15):

$$\begin{aligned} -\big(\partial_t w, \partial_t \varphi\big)_{L^2(Q_T)} + \frac{1}{h^2}\big(D^2\tilde{W}(\nabla_h u_h)\nabla_h w, \nabla_h \varphi\big)_{L^2(Q_T)} &= \big(f_1, \nabla_h \varphi\big)_{L^2(Q_T)} + \big(f_2, \varphi\big)_{L^2(Q_T)} \\ &+\langle w_1, \varphi|_{t=0}\rangle_{X_h', X_h} + \frac{1}{h^2}(\mathrm{tr}_{\partial\Omega}(a_N), \mathrm{tr}_{\partial\Omega}(\varphi))_{L^2(0,T;L^2(\partial\Omega))} \end{aligned} \tag{6.7}$$
$$w \text{ is } L\text{-periodic in } x_1 \text{ direction}$$
$$w|_{t=0} = w_0$$

for all $\varphi \in C^1([0,T]; H^1_{per,(0)}(\Omega;\mathbb{R}^3))$ with $\varphi|_{t=T} = 0$. Here we denote $Q_T := \Omega \times (0,T)$ and

$$X_h := H^1_{per,(0)}(\Omega;\mathbb{R}^3) := H^1_{per}(\Omega;\mathbb{R}^3) \cap \left\{ u \in L^1(\Omega;\mathbb{R}^3) \ : \ \int_\Omega u(x)dx = 0 \right\}$$

equipped with the h dependent norm

$$\|u\|_{X_h} := \|\nabla_h u\|_{L^2_h(\Omega)}.$$

Lemma 6.0.2. *Assume that u_h satisfies* (5.16) *with $R \in (0, R_0]$ and $h \in (0,1]$. Let R_0 be sufficiently small and $w \in C^0([0,T]; X_h) \cap C^1([0,T]; L^2(\Omega;\mathbb{R}^3))$ be a solution of* (6.7) *for $f_1 \in L^1(0,T; L^2(\Omega, \mathbb{R}^{3\times 3}))$, $f_2 \in L^1(0,T;L^2(\Omega;\mathbb{R}^3))$, $a_N \in L^1(0,T;H^1(\Omega;\mathbb{R}^3))$ $w_0 \in L^2(\Omega;\mathbb{R}^3)$ and $w_1 \in X_h'$. Then there are C_0, $C > 0$ independent of w and T such that*

$$\begin{aligned} \left\| \left(w, \frac{1}{h}\varepsilon_h(u) \right) \right\|_{C^0(0,T;L^2)} \leq & C_0 e^{CRT} \Bigg(\|f_1\|_{L^1(0,T;(L^2_h)')} + \|f_2\|_{L^1(0,T;L^2)} + \|w_0\|_{L^2(\Omega)} + \|w_1\|_{X_h'} \\ & + \frac{1}{h^2}\|a_N\|_{L^1(0,T;H^1)} + (1+T)\left\| \frac{1}{h}\int_\Omega u \cdot x^\perp dx \right\|_{C^0([0,T])} \Bigg) \end{aligned} \tag{6.8}$$

where $u(t) := \int_0^t w(\tau)d\tau$ and $(L^2_h)'$ is an abbreviation for $(L^2_h(\Omega;\mathbb{R}^{3\times 3}))'$.

Proof: Let $0 \leq T' \leq T$ and define $\tilde{u}_{T'}(t) = -\int_t^{T'} w(\tau)d\tau$. We use, after smooth approximation, $\varphi = \tilde{u}_{T'}\chi_{[0,T']}$. Then it follows

$$\begin{aligned} &\frac{1}{2}\|w(T')\|^2_{L^2} + \frac{1}{2h^2}\Big(D^2\tilde{W}(\nabla_h u_h|_{t=0})\nabla_h \tilde{u}_{T'}(0), \nabla_h \tilde{u}_{T'}(0)\Big)_{L^2(\Omega)} \\ &\quad = -\frac{1}{2h^2}\Big(\partial_t D^2\tilde{W}(\nabla_h u_h)\nabla_h \tilde{u}_{T'}, \nabla_h \tilde{u}_{T'}\Big)_{L^2(Q_{T'})} - (f_1, \nabla_h \tilde{u}_{T'})_{L^2(Q_{T'})} - (f_2, \tilde{u}_{T'})_{L^2(Q_{T'})} \\ &\quad + \langle w_1, \tilde{u}_{T'}(0)\rangle_{X_h', X_h} - \frac{1}{h^2}(a_N, \mathrm{tr}_{\partial\Omega}(\tilde{u}_{T'}))_{L^2(0,T';L^2(\partial\Omega))} + \frac{1}{2}\|w(0)\|^2_{L^2}. \end{aligned}$$

Using

$$\frac{1}{2h^2}\Big(D^2\tilde{W}(\nabla_h u_h|_{t=0})\nabla_h \tilde{u}_{T'}(0), \nabla_h \tilde{u}_{T'}(0)\Big)_{L^2(\Omega)}$$

$$\geq \frac{c_0}{2}\left\|\frac{1}{h}\varepsilon_h(\tilde{u}_{T'}(0))\right\|^2_{L^2(\Omega)} - CR\left|\frac{1}{h}\int_\Omega \tilde{u}_{T'}(0)\cdot x^\perp dx\right|^2$$

it follows with $\tilde{u}_{T'}(0) = -u(T')$

$$\begin{aligned}
\|w(T')\|^2_{L^2} + \left\|\frac{1}{h}\varepsilon_h(u(T'))\right\|^2_{L^2} &\leq CR\int_0^{T'} \left\|\frac{1}{h}\varepsilon_h(\tilde{u}_{T'})\right\|^2_{L^2} + \left|\frac{1}{h}\int_\Omega \tilde{u}_{T'}\cdot x^\perp dx\right|^2 dt \\
&+ C\Big(\|f_1\|_{L^1(0,T;(L^2_h)')} + \|f_2\|_{L^1(0,T;L^2)} + \|w_1\|_{X'_h} + \frac{1}{h^2}\|a_N\|_{L^1(0,T;H^1)}\Big)\|\nabla_h\tilde{u}_{T'}\|_{C^0([0,T];L^2_h)} \\
&+ C\|w_0\|^2_{L^2} + CR\left|\frac{1}{h}\int_\Omega \tilde{u}_{T'}(0)\cdot x^\perp dx\right|^2
\end{aligned}$$

where we used Lemma 5.2.2 and Korn inequality, as well as the subsequent inequalities

$$\begin{aligned}
&|\langle w_1, \tilde{u}_{T'}(0)\rangle_{X'_h,X_h}| \leq \|w_1\|_{X'_h}\|\tilde{u}_{T'}(0)\|_{X_h} \leq \|w_1\|_{X'_h}\|\nabla_h\tilde{u}_{T'}\|_{C^0([0,T'];L^2_h)} \\
&|(f_1, \nabla_h\tilde{u}_{T'})_{Q_{T'}}| \leq \int_0^{T'} \|f_1(t)\|_{(L^2_h)'}\|\nabla_h\tilde{u}_{T'}\|_{L^2}dt \leq \|f_1(t)\|_{L^1(0,T;(L^2_h)')}\|\nabla_h\tilde{u}_{T'}\|_{C^0([0,T'];L^2_h)}.
\end{aligned}$$

Now we can use $\tilde{u}_{T'}(0) = -u(T')$ and $\tilde{u}_{T'}(t) = -u(T') + u(t)$ to deduce

$$\begin{aligned}
\left|\int_0^{T'} \left\|\frac{1}{h}\varepsilon_h(\tilde{u}_{T'}(t))\right\|^2_{L^2} dt\right| &\leq \left\|\frac{1}{h}\varepsilon_h(u)\right\|^2_{L^2(Q_T)} + T'\left\|\frac{1}{h}\varepsilon_h(u(T'))\right\|^2_{L^2(\Omega)} \\
\|\nabla_h\tilde{u}_{T'}\|_{C^0([0,T'];L^2_h(\Omega))} &\leq C\left\|\frac{1}{h}\varepsilon_h(u)\right\|_{C^0([0,T'];L^2(\Omega))} + C\left\|\frac{1}{h}\int_\Omega u\cdot x^\perp dx\right\|_{C^0([0,T'])} \\
\left|\int_0^{T'}\int_\Omega \tilde{u}_{T'}\cdot x^\perp dxdt\right| &\leq T'\left|\int_\Omega u(T')\cdot x^\perp dx\right| + T'\left\|\int_\Omega u\cdot x^\perp dx\right\|_{C^0([0,T'])}.
\end{aligned}$$

Using the later inequalities and applying the supremum over $T' \in [0, \bar{T}]$ such that $R\bar{T} \leq \kappa$, $\kappa \in (0,1]$ it follows

$$\begin{aligned}
\|w\|^2_{C^0([0,\bar{T}],L^2)} + \left\|\frac{1}{h}\varepsilon_h(u)\right\|^2_{C^0([0,\bar{T}];L^2)} &\leq CR\left\|\frac{1}{h}\varepsilon_h(u)\right\|^2_{L^2(Q_{\bar{T}})} + C\kappa\left\|\frac{1}{h}\varepsilon_h(u)\right\|^2_{C^0([0,\bar{T}];L^2)} \\
&+ C\Big(\|f_1\|_{L^1(0,T;(L^2_h)')} + \|f_2\|_{L^1(0,T;L^2)} + \|w_1\|_{X'_h} + \|w_1\|_{X'_h} + \frac{1}{h^2}\|a_N\|_{L^1(0,T;H^1)}\Big) \\
&\times\left(\left\|\frac{1}{h}\varepsilon_h(u)\right\|_{C^0([0,T'];L^2)} + C\left\|\frac{1}{h}\int_\Omega u\cdot x^\perp dx\right\|_{C^0([0,T'])}\right) \\
&+ C\|w_0\|^2_{L^2} + CR(1+\bar{T})\left\|\frac{1}{h}\int_\Omega u\cdot x^\perp dx\right\|^2_{C^0([0,\bar{T}])}
\end{aligned}$$

Hence, with Young's inequality and κ, thus $\bar{T}$, small enough, we can conclude with an absorption argument that

$$\begin{aligned}
\|w\|^2_{C^0([0,\bar{T}],L^2)} + \left\|\frac{1}{h}\varepsilon_h(u)\right\|^2_{C^0([0,\bar{T}];L^2)} &\leq CR\left\|\frac{1}{h}\varepsilon_h(u)\right\|^2_{L^2(Q_{\bar{T}})} + C_0\Big(\|f_1\|^2_{L^1(0,T;(L^2_h)')} \\
&+ \|f_2\|^2_{L^1(0,T;L^2)} + \|w_1\|^2_{X'_h} + \frac{1}{h^4}\|a_N\|^2_{L^1(0,T;H^1)} + (1+T)\left\|\frac{1}{h}\int_\Omega u\cdot x^\perp dx\right\|^2_{C^0([0,T])}\Big).
\end{aligned}$$

Applying now the Lemma of Gronwall we obtain (6.8) for all $0 < T < \infty$ such that $RT \leq \kappa$ holds.
For an arbitrary $0 < T < \infty$, we choose $0 = T_0 < T_1 < \ldots < T_{N-1} < T_N = T$ such that $\frac{1}{2}\kappa \leq R(T_{j+1} - T_j) \leq \kappa$ for $j = 0, \ldots N-1$. Then we use $\varphi = \tilde{u}_{T_{j+1}}\chi_{[T_j,T_{j+1}]}$ and obtain via

analogous arguments as above, because of $R(T_{j+1}-T_j)\le\kappa$

$$\begin{aligned}\left\|\left(w,\frac{1}{h}\varepsilon_h(u)\right)\right\|_{C^0([T_j,T_{j+1}];L^2)} &\le C_0e^{CR(T_{j+1}-T_j)}\Bigg(\left\|\left(w(T_j),\frac{1}{h}\varepsilon_h(u(T_j))\right)\right\|_{L^2}\\ &\quad+\|f_1\|_{L^1(0,T;(L^2_h)')}+\|f_2\|_{L^1(0,T;L^2)}+\|w_1\|_{X_h'}\\ &\quad+\frac{1}{h}\|a_N\|_{L^1(0,T;H^1)}+(1+T)\left\|\frac{1}{h}\int_\Omega u\cdot x^\perp dx\right\|_{C^0([0,T])}\Bigg).\end{aligned}$$

Hence an iterative application leads to

$$\begin{aligned}\left\|\left(w,\frac{1}{h}\varepsilon_h(u)\right)\right\|_{C^0(0,T;L^2)} &\le (C_0)^Ne^{CRT}\Bigg(\|f_1\|_{L^1(0,T;X_h')}+\|f_2\|_{L^1(0,T;L^2)}+\|w_0\|_{L^2(\Omega)}\\ &\quad+\|w_1\|_{X_h'}+\frac{1}{h}\|a_N\|_{L^1(0,T;H^1)}+(1+T)\left\|\frac{1}{h}\int_\Omega u\cdot x^\perp dx\right\|_{C^0(0,T)}\Bigg).\end{aligned}$$

Finally due to $\frac{1}{2}\kappa\le R(T_{j+1}-T_j)$, we obtain $N\le 2\kappa^{-1}RT$ and thus

$$(C_0)^N=\exp(N\ln C_0)\le\exp(2\kappa^{-1}RT\ln C_0)\le\exp(C_0'RT).$$

Hence (6.8) holds for some C_0, $C>0$ independent of $R\in(0,R_0]$, $h\in(0,1]$ and $0<T<\infty$. $\square$

Define now

$$\mathcal{B}:=H^1_{per}(\Omega;\mathbb{R}^3)\cap\left\{u\in L^2(\Omega;\mathbb{R}^3)\ :\ \int_\Omega udx=\int_\Omega u\cdot x^\perp dx=0\right\}$$

equipped with the norm

$$\|u\|_{\mathcal{B}_h}:=\left\|\frac{1}{h}\varepsilon_h(u)\right\|_{L^2(\Omega)}.$$

Lemma 6.0.3. *For $0<h\le 1$ there exists constants $C_0>0$ and $M_0\in(0,1]$ such that for $f\in H^{1,1}_{per}(\Omega;\mathbb{R}^3)$ with $\|f\|_{H^{1,1}(\Omega)}\le M_0h$ and $\int_\Omega fdx=0$ there exists a unique solution $w\in H^3_{per}(\Omega;\mathbb{R}^3)\cap\mathcal{B}$ with $\partial_{x_1}w\in H^3_{per}(\Omega;\mathbb{R}^3)$ of*

$$\frac{1}{h^2}\Big(D\tilde{W}(\nabla_hw),\nabla_h\varphi\Big)_{L^2(\Omega)}=(f,\varphi)_{L^2(\Omega)}\tag{6.9}$$

for all $\varphi\in\mathcal{B}$. Moreover

$$\left\|\left(\frac{1}{h}\varepsilon_h(w),\nabla_h\frac{1}{h}\varepsilon(w),\nabla_h^2w\right)\right\|_{H^{1,1}(\Omega)}\le C_0\|f\|_{H^{1,1}(\Omega)}\tag{6.10}$$

holds. If $w'\in H^3_{per}(\Omega;\mathbb{R}^3)\cap\mathcal{B}$ with $\partial_{x_1}w\in H^3_{per}(\Omega;\mathbb{R}^3)$ is the solution to $f'\in H^{1,1}_{per}(\Omega;\mathbb{R}^3)$ with $\|f'\|_{H^{1,1}(\Omega)}\le M_0h$ and $\int_\Omega f'dx=0$ then it holds

$$\left\|\left(\frac{1}{h}\varepsilon_h(w-w'),\nabla_h\frac{1}{h}\varepsilon(w-w'),\nabla_h^2(w-w')\right)\right\|_{H^{1,1}(\Omega)}\le C_0\|f-f'\|_{H^{1,1}(\Omega)}.\tag{6.11}$$

Proof: Using a Taylor series for $D\tilde{W}(\nabla_hw)$ we obtain

$$\begin{aligned}D\tilde{W}(\nabla_hw)&=D\tilde{W}(0)+D^2\tilde{W}(0)[\nabla_hw]+\int_0^1(1-\tau)D^3\tilde{W}(\tau\nabla_hw)[\nabla_hw,\nabla_hw]d\tau\\ &=:D^2\tilde{W}(0)\nabla_hw+G(\nabla_hw)\end{aligned}\tag{6.12}$$

Thus (6.9) is equivalent to

$$\begin{aligned}\langle L_h w, \varphi\rangle_{\mathcal{B}',\mathcal{B}} :=& \frac{1}{h^2}\Big(D^2\tilde{W}(0)\nabla_h w, \nabla_h\varphi\Big)_{L^2(\Omega)}\\ =&(f,\varphi)_{L^2(\Omega)} - \frac{1}{h^2}(G(\nabla_h w), \nabla_h \varphi)_{L^2(\Omega)}.\end{aligned}$$

The idea is now to use the contraction mapping principle in order prove the existence of a solution for (6.9), i.e. with the later equivalence

$$w = \mathcal{G}_{h,f}(w) := L_h^{-1}\Big(f, G_h(w)\Big)$$

with $G_h(w) := \frac{1}{h^2}G(\nabla_h w)$. Consequently we investigate the mapping properties of L_h and G_h.

For $f \in L^2(\Omega;\mathbb{R}^3)$ and $F \in L^2(\Omega;\mathbb{R}^{3\times3})$ we obtain with the Lemma of Lax-Milgram the existence of a unique solution $w \in \mathcal{B}$ for

$$\langle L_h w, \varphi\rangle_{\mathcal{B}',\mathcal{B}} = (f,\varphi)_{L^2(\Omega)} - (F, \nabla_h\varphi)_{L^2(\Omega)} \tag{6.13}$$

for all $\varphi \in \mathcal{B}$. The solution satisfies

$$\|w\|_{\mathcal{B}} = \left\|\frac{1}{h}\varepsilon_h(w)\right\|_{L^2(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|F\|_{(L^2_h)'}\big).$$

If now $f \in H^{0,k}(\Omega;\mathbb{R}^3)$ and $F \in H^{0,k}(\Omega;\mathbb{R}^{3\times3})$ for $k = 1,2$, it follows by a different quotient argument that $w \in H^{0,k}(\Omega;\mathbb{R}^3)$ holds and

$$\left\|\frac{1}{h}\varepsilon_h(w)\right\|_{H^{0,k}(\Omega)} \leq C\Big(\|f\|_{H^{0,k-1}(\Omega)} + \max_{j=0,\dots,k}\|\partial^j_{x_1}F\|_{(L^2_h)'}\Big). \tag{6.14}$$

Using the decomposition $\mathcal{B} \oplus \operatorname{span}\{x \mapsto x^\perp\} = H^1_{(0),per}(\Omega;\mathbb{R}^3)$ it follows that for

$$\alpha := (F, \nabla_h x^\perp)_{L^2(\Omega)} - (f, x^\perp)_{L^2(\Omega)}$$

we have

$$\frac{1}{h^2}\Big(D^2\tilde{W}(0)\nabla_h w, \nabla_h\varphi\Big)_{L^2(\Omega)} = (f + \alpha x^\perp, \varphi)_{L^2(\Omega)} - (F, \nabla_h\varphi)_{L^2(\Omega)}$$

for all $\varphi \in H^1_{(0),per}(\Omega;\mathbb{R}^3)$. Hence, if $f \in H^1_{per}(\Omega;\mathbb{R}^3)$ and $F \in H^2_{per}(\Omega;\mathbb{R}^{3\times3})$, then w solves the system

$$\begin{cases} -\dfrac{1}{h^2}\operatorname{div}_h(D^2\tilde{W}(0)\nabla_h w) = f + \alpha x^\perp - \operatorname{div}_h F & \text{in } \Omega\\ \qquad D^2\tilde{W}(0)[\nabla_h w]\nu\Big|_{\partial S} = h^2 \operatorname{tr}_{\partial\Omega}(F)\nu\Big|_{\partial S} & \text{in } \partial\Omega\end{cases}$$

in a weak sense . Thus with elliptic regularity theory it follows $w \in H^3_{per}(\Omega;\mathbb{R}^3) \cap \mathcal{B}$. Theorem 5.2.9 and the later inequalities imply

$$\begin{aligned}\left\|\left(\frac{1}{h}\varepsilon_h(w), \nabla_h\frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\right)\right\|_{H^1(\Omega)} \leq C\Big(& h^2\|(f, \operatorname{div}_h F)\|_{H^1(\Omega)} + \|f\|_{H^{0,1}(\Omega)}\\ &+ \max_{j=0,1,2}\|\partial^j_{x_1}F\|_{(L^2_h)'} + \Big\|h\operatorname{tr}_{\partial\Omega}(F)\Big\|_{L^2(0,L;H^{\frac{3}{2}}(\partial S))\cap H^1(0,L;H^{\frac{1}{2}}(\partial S))}\Big).\end{aligned}$$

where we have exploit

$$h^2|\alpha| \leq Ch^2\|f\|_{L^2(\Omega)} + Ch\|F\|_{(L^2_h)'}.$$

Using that $\mathrm{tr}_{\partial S}\colon H^2(S) \to H^{\frac{3}{2}}(\partial S)$ is a bounded operator we obtain

$$\begin{aligned} h\Big\| \mathrm{tr}_{\partial\Omega}(F)\Big\|_{L^2(0,L;H^{\frac{3}{2}}(\partial S))\cap H^1(0,L;H^{\frac{1}{2}}(\partial S))} &\leq Ch\Big(\|F\|_{H^{1,1}(\Omega)} + \max_{k=0,1,2} \|\nabla_{x'}^k F\|_{L^2(\Omega)}\Big) \\ &\leq C\Big(\max_{j=0,1,2} \|\partial_{x_1}^j F\|_{(L_h^2)'} + h^2\|\nabla_h F\|_{H^1(\Omega)}\Big) \end{aligned}$$

because of

$$\|F\|_{H^1(\Omega)} \leq \|F\|_{H^{0,1}(\Omega)} + \|\nabla_{x'}F\|_{L^2(\Omega)} \quad \text{and} \quad \|F\|_{L^2(\Omega)} \leq \frac{1}{h}\|F\|_{(L_h^2)'}.$$

Thus we deduce for some $C_L > 0$

$$\begin{aligned} &\Big\|\Big(\frac{1}{h}\varepsilon_h(w), \nabla_h \frac{1}{h}\varepsilon_h(w), \nabla_h^2 w\Big)\Big\|_{H^1(\Omega)} \\ &\qquad \leq C_L\Big(h^2\|(f,\nabla_h F)\|_{H^1(\Omega)} + \|f\|_{H^{0,1}(\Omega)} + \max_{j=0,1,2}\|\partial_{x_1}^j F\|_{(L_h^2)'}\Big). \end{aligned} \tag{6.15}$$

We define $\mathcal{X}_h := H^3_{per}(\Omega;\mathbb{R}^3) \cap \mathcal{B}$ and $\mathcal{Y}_h := H^1_{per}(\Omega;\mathbb{R}^3) \times H^2_{per}(\Omega;\mathbb{R}^{3\times 3})$ normed via

$$\begin{aligned} \|g\|_{\mathcal{X}_h} &:= \Big\|\Big(\frac{1}{h}\varepsilon_h(g), \nabla\frac{1}{h}\varepsilon_h(g), \nabla_h^2 g\Big)\Big\|_{H^1(\Omega)} \\ \|(f,F)\|_{\mathcal{Y}_h} &:= h^2\|(f,\nabla_h F)\|_{H^1(\Omega)} + \|f\|_{H^{0,1}(\Omega)} + \max_{j=0,1,2}\|\partial_{x_1}^j F\|_{(L_h^2)'}. \end{aligned}$$

With this $L_h^{-1}\colon \mathcal{Y}_h \to \mathcal{X}_h$ is a bilinear, bijective and bounded operator, mapping a tuple $(f,F) \in \mathcal{Y}_h$ to the corresponding solution $w \in \mathcal{X}_h$ of (6.13). In order to close the proof we have to show that G_h fulfils contraction mapping properties with respect to the relevant norms.
In a first step we assume that $w_i \in \mathcal{X}_h$ with

$$\|w_i\|_{\mathcal{X}_h} \leq C_0 M_1 h$$

for $i = 1,2$ and $M_1 > 0$ chosen later. Then

$$\begin{aligned} \|G_h(w_1) - G_h(w_2)\|_{(L_h^2)'} &= \Big\|\frac{1}{h^2}\int_0^1 (1-\tau)\Big(D^3\tilde{W}(\tau\nabla_h w_1)[\nabla_h w_1 - \nabla_h w_2, \nabla_h w_1] \\ &\quad + D^3\tilde{W}(\tau\nabla_h w_2)[\nabla_h w_1 - \nabla_h w_2, \nabla_h w_2] \\ &\quad + \big(D^3\tilde{W}(\tau\nabla_h w_1) - D^3\tilde{W}(\tau\nabla_h w_2)\big)[\nabla_h w_1, \nabla_h w_2]\Big)d\tau\Big\|_{(L_h^2)'} \\ &\leq CM_1\|\nabla_h(w_1 - w_2)\|_{H_h^1(\Omega)} \\ &\quad + \Big\|\frac{1}{h^2}\int_0^1(1-\tau)\int_0^1 Q(\tau,t,w_1,w_2)dt[\tau(\nabla_h w_1 - \nabla_h w_2), \nabla_h w_1, \nabla_h w_2]d\tau\Big\|_{(L_h^2)'} \\ &\leq CM_1\Big\|\frac{1}{h}\varepsilon_h(w_1 - w_2)\Big\|_{\mathcal{X}_h} \end{aligned}$$

where we used Corollary 2.3.7, $\|\nabla_h w_j\|_{H_h^1(\Omega)} \leq C\|w_j\|_{\mathcal{X}_h}$ and the boundedness of

$$Q(\tau,t,w_1,w_2) := D^4\tilde{W}(t\tau\nabla_h w_1 + (1-t)\tau\nabla_h w_2).$$

The definition of G implies that for $k = 1,2,3$ it holds

$$\begin{aligned} \partial_{x_k} G(\nabla_h w) &= D^2\tilde{W}(\nabla_h w)[\nabla_h \partial_{x_k} w] - D^2\tilde{W}(0)[\nabla_h\partial_{x_k} w] \\ &= \int_0^1 D^3\tilde{W}(\tau\nabla_h w)[\nabla_h w, \nabla_h\partial_{x_k} w]d\tau. \end{aligned} \tag{6.16}$$

Hence, analogously as above

$$
\begin{aligned}
\|\partial_{x_k}(G_h(w_1)-G_h(w_2)\|_{(L^2_h)'} &\le \left\| \frac{1}{h^2}\int_0^1 D^3\tilde{W}(\tau\nabla_h w_1)[\nabla_h(w_1-w_2),\nabla_h\partial_{x_k}w_1]d\tau\right\|_{(L^2_h)'}\\
&+\left\| \frac{1}{h^2}\int_0^1 D^3\tilde{W}(\tau\nabla_h w_2)[\nabla_h w_2,\nabla_h\partial_{x_k}(w_1-w_2)]d\tau\right\|_{(L^2_h)'}\\
&+\left\| \frac{1}{h^2}\int_0^1 \big(D^3\tilde{W}(\tau\nabla_h w_1)-D^3\tilde{W}(\tau\nabla_h w_2)\big)[\nabla_h w_2,\nabla_h\partial_{x_k}w_1]d\tau\right\|_{(L^2_h)'}\\
&\le \frac{C}{h}\|\nabla_h(w_1-w_2)\|_{H^2_h(\Omega)}\|\nabla_h\partial_{x_k}w_1\|_{L^2_h(\Omega)}+\frac{C}{h}\|\nabla_h w_2\|_{H^2_h(\Omega)}\|\nabla_h\partial_{x_k}(w_1-w_2)\|_{(L^2_h)'}\\
&+CM_1\left\|\frac{1}{h}\varepsilon_h(w_1-w_2)\right\|_{H^1_h(\Omega)}\\
&\le CM_1\|w_1-w_2\|_{\mathcal{X}_h}
\end{aligned}
$$

as

$$
\|\nabla_h\partial_{x_k}\varphi\|_{L^2_h(\Omega)}\le\|\nabla_h\partial_{x_k}\varphi\|_{L^2(\Omega)}+\left\|\frac{1}{h}\varepsilon_h(\partial_{x_k}\varphi)\right\|_{L^2(\Omega)}\le\left\|\left(\nabla\frac{1}{h}\varepsilon_h(\varphi),\nabla_h^2\varphi\right)\right\|_{L^2(\Omega)}\le\|\varphi\|_{\mathcal{X}_h}
$$

for $\varphi=w_1$ and $\varphi=w_1-w_2$. Deploying (6.16) it follows for $j,\,k=1,2,3$

$$
\begin{aligned}
\partial_{x_j}\partial_{x_k}G(\nabla_h w) &= D^2\tilde{W}(\nabla_h u_h)[\nabla_h\partial_{x_j}\partial_{x_k}w]-D^2\tilde{W}(0)[\nabla_h\partial_{x_j}\partial_{x_k}w]\\
&\quad+D^3\tilde{W}(\nabla_h w)[\nabla_h\partial_{x_j}w,\nabla_h\partial_{x_k}w]\\
&=\int_0^1 D^3\tilde{W}(\tau\nabla_h w)[\nabla_h w,\nabla_h\partial_{x_j}\partial_{x_k}w]d\tau+D^3\tilde{W}(\nabla_h w)[\nabla_h\partial_{x_j}w,\nabla_h\partial_{x_k}w].
\end{aligned}
$$

Thus we obtain in the same manner as above

$$
\begin{aligned}
&\|\partial_{x_j}\partial_{x_k}(G_h(w_1)-G_h(w_2))\|_{(L^2_h)'}\\
&\le\left\|\frac{1}{h^2}\int_0^1 D^3\tilde{W}(\tau\nabla_h w_1)[\nabla_h(w_1-w_2),\nabla_h\partial_{x_j}\partial_{x_k}w_1]d\tau\right\|_{(L^2_h)'}\\
&+\left\|\frac{1}{h^2}\int_0^1 D^3\tilde{W}(\tau\nabla_h w_2)[\nabla_h w_2,\nabla_h\partial_{x_k}\partial_{x_j}(w_1-w_2)]d\tau\right\|_{(L^2_h)'}\\
&+\left\|\frac{1}{h^2}\int_0^1 \big(D^3\tilde{W}(\tau\nabla_h w_1)-D^3\tilde{W}(\tau\nabla_h w_2)\big)[\nabla_h w_2,\nabla_h\partial_{x_k}\partial_{x_j}w_1]d\tau\right\|_{(L^2_h)'}\\
&+\left\|\frac{1}{h^2}D^3\tilde{W}(\nabla_h w_1)[\nabla_h\partial_{x_j}(w_1-w_2),\nabla_h\partial_{x_k}w_1]\right\|_{(L^2_h)'}\\
&+\left\|\frac{1}{h^2}D^3\tilde{W}(\nabla_h w_2)[\nabla_h\partial_{x_j}w_2,\nabla_h\partial_{x_k}(w_1-w_2)]\right\|_{(L^2_h)'}\\
&+\left\|\frac{1}{h^2}\int_0^1 \big(D^3\tilde{W}(\tau\nabla_h w_1)-D^3\tilde{W}(\tau\nabla_h w_2)\big)[\nabla_h\partial_{x_j}w_2,\nabla_h\partial_{x_k}w_1]\right\|_{(L^2_h)'}\\
&\le\frac{C}{h}\|\nabla_h(w_1-w_2)\|_{H^2_h(\Omega)}\|\nabla_h\partial_{x_j}\partial_{x_k}w_1\|_{L^2_h(\Omega)}\\
&+\frac{C}{h}\|\nabla_h w_2\|_{H^2_h(\Omega)}\|\nabla_h\partial_{x_k}\partial_{x_j}(w_1-w_2)\|_{L^2_h(\Omega)}\\
&+\frac{C}{h}\max_{i=1,2}\|\nabla_h\partial_{x_j}(w_1-w_2)\|_{H^1_h(\Omega)}\|\nabla_h\partial_{x_k}w_i\|_{H^1_h(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{C}{h^2}\|\nabla_h(w_1-w_2)\|_{H^2_h(\Omega)}\|\nabla_h\partial_{x_j}w_2\|_{H^1_h(\Omega)}\|\nabla_h\partial_{x_k}w_1\|_{H^1_h(\Omega)}\\
&\leq CM_1\|w_1-w_2\|_{\mathcal{X}_h}.
\end{aligned}
$$

The fact that $h^2\|\nabla_h F\|_{H^1(\Omega)} \leq h\|\nabla F\|_{H^1(\Omega)}$ and $\|F\|_{L^2(\Omega)} \leq \frac{1}{h}\|F\|_{(L^2_h)'}$ implies with the later estimates that for $M_1 \in (0,1]$ small enough

$$\mathcal{G}_{h,f}\colon \overline{B_{CM_1h}(0)} \subset \mathcal{X}_h \to \mathcal{X}_h$$

is a $\frac{1}{2}$-contraction. The self mapping property of $\mathcal{G}_{h,f}$ follows, because of

$$\|\mathcal{G}_{h,f}(0)\|_{\mathcal{X}_h} = \|L^{-1}(f,0)\|_{\mathcal{X}_h} \leq C_L\|(f,0)\|_{\mathcal{Y}_h} \leq C_L\|f\|_{H^{1,1}(\Omega)} \leq C_L M_0 h.$$

Thus we can choose $M_0 > 0$ sufficiently small, such that $C_L M_0 h \leq \frac{CM_1}{2}$. Then we obtain with the $\frac{1}{2}$-contraction property of $\mathcal{G}_{h,f}$ for $w \in \overline{B_{CM_1h}(0)}$

$$\|\mathcal{G}_{h,f}(w)\|_{\mathcal{X}_h} \leq \|\mathcal{G}_{h,f}(w)-\mathcal{G}_{h,f}(0)\|_{\mathcal{X}_h} + \|\mathcal{G}_{h,f}(0)\|_{\mathcal{X}_h} \leq \frac{1}{2}\|w\|_{\mathcal{X}_h} + C_L M_0 h \leq CM_1 h.$$

Therefore (6.10) and (6.11) hold with the $H^{1,1}(\Omega)$-norm on the left hand side replaced by the $\mathcal{X}_h$-norm.
Using the decomposition $\mathcal{B} \oplus \operatorname{span}\{x \mapsto x^\perp\} = H^1_{(0),per}(\Omega;\mathbb{R}^3)$ it follows that for

$$\rho := \frac{1}{\mu(S)h^2}\Big(D\tilde{W}(\nabla_h w), \nabla_h x^\perp\Big)_{L^2(\Omega)}$$

we have

$$\frac{1}{h^2}\Big(D\tilde{W}(\nabla_h w), \nabla_h \varphi\Big)_{L^2(\Omega)} = (f-\rho x^\perp, \varphi)_{L^2(\Omega)}$$

for all $\varphi \in H^1_{(0),per}(\Omega\mathbb{R}3)$. If now $f \in H^{1,1}_{per}(\Omega;\mathbb{R}^3)$ we obtain, with an difference quotient argument that $w \in H^3_{per}(\Omega;\mathbb{R}^3) \cap \mathcal{B}$ satisfies

$$\frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h w)\nabla_h\partial_{x_1}w, \nabla_h\varphi\Big)_{L^2(\Omega)} = (\partial_{x_1}f,\varphi)_{L^2(\Omega)}.$$

for all $\varphi \in H^1_{(0),per}(\Omega;\mathbb{R}^3)$. Thus with Theorem 5.2.9 the claimed inequalities follow. $\square$

Define the initial values for the analytical problem as

$$u_{2+j,h} := h^2\begin{pmatrix}0\\ v_2^{2+j}\\ v_3^{2+j}\end{pmatrix} + h^3\begin{pmatrix}-x_2\partial_{x_1}v_2^{2+j} - x_3\partial_{x_1}v_3^{2+j}\\ 0\\ 0\end{pmatrix}.$$

for $j = 1,\ 2$ and $v^{2+j} = \partial_t^{2+j}v|_{t=0}$ as above.

Lemma 6.0.4. *Let $\tilde{u}_h$ as in (6.3), $\tilde{u}_{j,h}$ for $j = 0,1,2$ as in (6.6), $u_{3,h}$, $u_{4,h}$ and f^h as above. Then for sufficiently small $h_0 \in (0,1]$ and $M > 0$ there exist solutions $(u_{0,h}, u_{1,h}, u_{2,h})$ of*

$$\frac{1}{h^2}\Big(D\tilde{W}(\nabla_h u_{0,h}), \nabla_h\varphi\Big)_{L^2(\Omega)} = (h^2 f_h|_{t=0},\varphi)_{L^2(\Omega)} - (u_{2,h},\varphi)_{L^2(\Omega)} \tag{6.17}$$

$$\frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_{0,h})\nabla_h u_{1,h}, \nabla_h\varphi\Big)_{L^2(\Omega)} = (h^2\partial_t f|_{t=0},\varphi)_{L^2(\Omega)} - (u_{3,h},\varphi)_{L^2(\Omega)} \tag{6.18}$$

and

$$\frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_{0,h})\nabla_h u_{2,h}, \nabla_h\varphi\Big)_{L^2(\Omega)} = (h^2\partial_t^2 f|_{t=0} - u_{4,h})_{L^2(\Omega)} \tag{6.19}$$

$$-\frac{1}{h^2}\Big(D^3\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{1,h},\nabla_h u_{1,h}],\nabla_h\varphi\Big)_{L^2(\Omega)}-\frac{\gamma_h}{h^3}\Big(D^2\tilde{W}(\nabla_h u_{0,h})P,\nabla_h\varphi\Big)_{L^2(\Omega)}$$

for all $\varphi\in\mathcal{B}$, where

$$\gamma_h(u_{0,h}):=\frac{1}{\mu(S)h^3}\Big(D\tilde{W}(\nabla_h u_{0,h}),P\Big)_{L^2(\Omega)}$$

and

$$P:=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}.$$

The solution satisfies

$$\left\|\left(\frac{1}{h}\varepsilon_h(u_{0,h}),\nabla\frac{1}{h}\varepsilon_h(u_{0,h}),\nabla_h^2u_{0,h}\right)\right\|_{H^{1,1}(\Omega)}\le Ch^2 \tag{6.20}$$

$$\max_{j=1,2}\left\|\left(\frac{1}{h}\varepsilon_h(u_{j,h}),\nabla\frac{1}{h}\varepsilon_h(u_{j,h}),\nabla_h^2u_{j,h}\right)\right\|_{H^{2-j}(\Omega)}\le Ch^2 \tag{6.21}$$

and $u_{k,h}\in\mathcal{B}$ for $k=0,1,2$. Moreover we have

$$\max_{j=0,1,2}\left\|\left(\frac{1}{h}\varepsilon_h(u_{j,h})-\frac{1}{h}\varepsilon_h(\tilde{u}_{j,h})\right)\right\|_{L^2(\Omega)}\le\begin{cases}Ch^3, & \text{if } j=0,1,\\ Ch^2, & \text{if } j=2,\end{cases} \tag{6.22}$$

for all $h\in(0,h_0]$ and $C>0$ independent of h.

Proof: We can equivalently formulate (6.17)–(6.19) via

$$\frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_{0,h})\nabla_h u_{1,h},\nabla_h\varphi\Big)_{L^2(\Omega)}=(h^2\partial_t f_h|_{t=0},\varphi)_{L^2(\Omega)}-(u_{3,h},\varphi)_{L^2(\Omega)} \tag{6.23}$$

and

$$\begin{aligned}\frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_{0,h})\nabla_h u_{2,h},\nabla_h\varphi\Big)_{L^2(\Omega)}&=(h^2\partial_t^2 f_h|_{t=0}-u_{4,h},\varphi)_{L^2(\Omega)}\\&\quad-\frac{1}{h^2}\Big(D^3\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{1,h},\nabla_h u_{1,h}],\nabla_h\varphi\Big)_{L^2(\Omega)}\\&\quad-\frac{\gamma_h(u_{0,h})}{h^3}\Big(D^2\tilde{W}(\nabla_h u_{0,h})P,\nabla_h\varphi\Big)_{L^2(\Omega)}\end{aligned} \tag{6.24}$$

for all $\varphi\in\mathcal{B}$, where $u_{0,h}=\mathcal{G}_{h,f}(u_{0,h})$ is the solution of (6.17) with $f=h^2f^h-u_{2,h}$. Defining

$$\mathcal{G}_{0,h}(u_{2,h}):=\mathcal{G}_{h,f}(u_{0,h})$$

and deploying (6.11) we obtain for $u_{2,h},\ u'_{2,h}\in H^{1,1}(\Omega;\mathbb{R}^3)$

$$\max_{k=0,1}\left\|\partial_{x_1}^k\big(\mathcal{G}_{0,h}(u_{2,h})-\mathcal{G}_{0,h}(u'_{2,h})\big)\right\|_{\mathcal{X}_h}\le C_0\|u_{2,h}-u'_{2,h}\|_{H^{1,1}(\Omega)} \tag{6.25}$$

if $\|u_{2,h}\|_{H^{1,1}(\Omega)}\le\frac{1}{2}M_0h$, $\|u'_{2,h}\|_{H^{1,1}(\Omega)}\le\frac{1}{2}M_0h$ and $h^2\|f^h\|_{H^{1,1}(\Omega)}\le\frac{1}{2}M_0h$. This can always be achieved if $h_0\in(0,1]$ is small enough and $u_{2,h},\ u'_{2,h}$ are of order h^2.
Using the definition of L_h it follows that (6.23)–(6.24) are equivalent to

$$\langle L_hu_{1,h},\varphi\rangle_{\mathcal{B}',\mathcal{B}}=(h^2\partial_tf_h|_{t=0}-u_{3,h},\varphi)_{L^2(\Omega)}-\frac{1}{h^2}\Big(DG(\nabla_hu_{0,h})\nabla_hu_{1,h},\nabla_h\varphi\Big)_{L^2(\Omega)}$$

and

$$\langle L_hu_{2,h},\varphi\rangle_{\mathcal{B}',\mathcal{B}}=(h^2\partial_t^2f_h|_{t=0}-u_{4,h},\varphi)_{L^2(\Omega)}-\frac{1}{h^2}\Big(DG(\nabla_hu_{0,h})\nabla_hu_{2,h},\nabla_h\varphi\Big)_{L^2(\Omega)}$$

$$-\frac{1}{h^2}\Big(D^3\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{1,h},\nabla_h u_{1,h}],\nabla_h\varphi\Big)_{L^2(\Omega)} - \frac{\gamma_h(u_{0,h})}{h^3}\Big(D^2\tilde{W}(\nabla_h u_{0,h})P,\nabla_h\varphi\Big)_{L^2(\Omega)}$$

for all $\varphi \in \mathcal{B}$. Defining now the relevant function spaces by

$$\mathcal{D}_h := H^2_{per}(\Omega;\mathbb{R}^{3\times3}) \times H^1_{per}(\Omega;\mathbb{R}^{3\times3})$$
$$\mathcal{Z}_h := H^1_{per}(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R}^3) \times \mathcal{D}_h$$

and

$$\mathcal{W}_h := \mathcal{X}_h \times \Big(H^2_{per}(\Omega;\mathbb{R}^3) \cap \mathcal{B}\Big)$$

with the respective norms defined by

$$\|(F_1,F_2)\|_{\mathcal{D}_h} := \max_{i=1,2}\Big(h^2\|\nabla_h F_i\|_{H^{2-i}(\Omega)} + \max_{\sigma=0,\dots,3-i}\|\partial^\sigma_{x_1}F_i\|_{(L^2_h)'}\Big)$$
$$\|(f_1,f_2,F_1,F_2)\|_{\mathcal{Z}_h} := \max_{i=1,2}\Big(h^2\|(f_i,\nabla_h F_i)\|_{H^{2-i}(\Omega)} + \|f_i\|_{H^{0,2-i}(\Omega)} + \max_{\sigma=0,\dots,3-i}\|\partial^\sigma_{x_1}F_i\|_{(L^2_h)'}\Big)$$

and

$$\|(g_1,g_2)\|_{\mathcal{W}_h} := \max_{i=1,2}\left\|\left(\frac{1}{h}\varepsilon_h(g_i),\nabla\frac{1}{h}\varepsilon_h(g_i),\nabla_h^2 g_i\right)\right\|_{H^{2-i}(\Omega)}.$$

With this we define the linear operator $\mathcal{L}_h^{-1}\colon \mathcal{Z}_h \to \mathcal{W}_h$ by mapping (f_1,f_2,F_1,F_2) to the solution (w_1,w_2) of

$$\langle L_h w_i,\varphi\rangle_{\mathcal{B}',\mathcal{B}} = (f_i,\varphi)_{L^2(\Omega)} - (F_i,\nabla_h\varphi)_{L^2(\Omega)} \tag{6.26}$$

for $i=1,2$. Then due to (6.15), Theorem 5.2.4 and (6.14) we obtain

$$\|(w_1,w_2)\|_{\mathcal{W}_h} \le C\|(f_1,f_2,F_1,F_2)\|_{\mathcal{Z}_h}. \tag{6.27}$$

Hence $\mathcal{L}_h^{-1}$ is a bijective, linear and bounded operator. For the nonlinearity we define

$$\mathcal{Q}_h\colon \mathcal{W}_h \to \mathcal{D}_h$$

via

$$\begin{pmatrix}u_1\\u_2\end{pmatrix} \mapsto \begin{pmatrix}-\frac{1}{h^2}DG(\nabla_h u_0)\nabla_h u_1\\ -\frac{1}{h^2}DG(\nabla_h u_0)\nabla_h u_2 - \frac{1}{h^2}D^3\tilde{W}(\nabla_h u_0)[\nabla_h u_1,\nabla_h u_1] - \frac{\gamma_h(u_{0,h})}{h^3}D^2\tilde{W}(\nabla_h u_{0,h})[P]\end{pmatrix}$$
$$=: \begin{pmatrix}\mathcal{Q}_{1,h}(u_1,u_2)\\ \mathcal{Q}_{2,h}(u_1,u_2)\end{pmatrix}$$

where $u_0 := \mathcal{G}_{h,f-u_2}(u_{0,h})$ for some fixed $f \in H^{1,1}_{per}(\Omega)$ with $\|f\|_{H^{1,1}(\Omega)} \le Mh^2$ and $\int_\Omega f dx = 0$ and G is defined as in (6.12).

We deduce the contraction properties of $\mathcal{Q}_h$ similar as in the proof of Lemma 6.0.3. For this we assume that $\|(u_1,u_2)\|_{\mathcal{W}_h}$ and $\|(u_1',u_2')\|_{\mathcal{W}_h} \le CM_2h^2$. Starting with $\mathcal{Q}_{1,h}$ we obtain

$$\begin{aligned}
&\|\mathcal{Q}_{1,h}(u_1,u_2) - \mathcal{Q}_{1,h}(u_1',u_2')\|_{(L^2_h)'}\\
&\quad = \frac{1}{h^2}\left\|\int_0^1 D^3\tilde{W}(\tau\nabla_h u_0)[\nabla_h u_0,\nabla_h u_1]d\tau - \int_0^1 D^3\tilde{W}(\tau\nabla_h u_0')[\nabla_h u_0',\nabla_h u_1']d\tau\right\|_{(L^2_h)'}\\
&\quad \le \frac{C}{h}\|\nabla_h(u_0-u_0')\|_{H^2_h(\Omega)}\|\nabla_h u_1\|_{L^2_h(\Omega)} + \frac{C}{h}\|\nabla_h u_0'\|_{H^2_h(\Omega)}\|\nabla_h(u_1-u_1')\|_{L^2_h(\Omega)}\\
&\qquad + CM_2\left\|\frac{1}{h}\varepsilon_h(u_0-u_0')\right\|_{H^1_h(\Omega)}\\
&\quad \le CM_2\|u_2-u_2'\|_{H^{1,1}(\Omega)} + CM_2\left\|\frac{1}{h}\varepsilon_h(u_1-u_1')\right\|_{L^2(\Omega)} \le CM_2\|(u_1-u_1',u_2-u_2')\|_{\mathcal{W}_h}
\end{aligned}$$

where we used (6.25). Similarly one deduces that

$$\begin{aligned}\|\partial_{x_j}(\mathcal{Q}_{1,h}(u_1,u_2)-\mathcal{Q}_{1,h}(u_1',u_2'))\|_{L^2(\Omega)}&\leq CM_2\|(u_1-u_1',u_2-u_2')\|_{\mathcal{W}_h}\\ \|\partial_{x_k}\partial_{x_j}(\mathcal{Q}_{1,h}(u_1,u_2)-\mathcal{Q}_{1,h}(u_1',u_2'))\|_{L^2(\Omega)}&\leq CM_2\|(u_1-u_1',u_2-u_2')\|_{\mathcal{W}_h}\end{aligned}$$

for $j,\ k=1,2,3$. Analogously we deduce for $\mathcal{Q}_{2,h}$

$$\begin{aligned}&\|\mathcal{Q}_{2,h}(u_1,u_2)-\mathcal{Q}_{2,h}(u_1',u_2')\|_{(L^2_h)'}\\ &\leq\frac{1}{h^2}\left\|\int_0^1 D^3\tilde{W}(\tau\nabla_h u_0)[\nabla_h u_0,\nabla_h u_2]-D^3\tilde{W}(\tau\nabla_h u_0')[\nabla_h u_0',\nabla_h u_2']d\tau\right\|_{(L^2_h)'}\\ &\quad+\frac{|\gamma_h(u_{0,h})|}{h^3}\left\|\int_0^1 D^3\tilde{W}(\tau\nabla_h u_0)[\nabla_h u_0,P]-D^3\tilde{W}(\tau\nabla_h u_0')[\nabla_h u_0',P]d\tau\right\|_{(L^2_h)'}\\ &\quad+\frac{|\gamma_h(u_{0,h})-\gamma_h(u_{0,h}')|}{h^3}\left\|\int_0^1 D^3\tilde{W}(\tau\nabla_h u_0')[\nabla_h u_0',P]d\tau\right\|_{(L^2_h)'}\\ &\quad+\frac{1}{h^2}\left\|D^3\tilde{W}(\nabla_h u_0)[\nabla_h u_1,\nabla_h u_1]-D^3\tilde{W}(\nabla_h u_0')[\nabla_h u_1',\nabla_h u_1']\right\|_{(L^2_h)'}\\ &\leq\frac{C}{h}\|\nabla_h(u_0-u_0')\|_{H^2_h(\Omega)}\|\nabla_h u_2\|_{L^2_h(\Omega)}+\frac{C}{h}\|\nabla_h u_0'\|_{H^2_h(\Omega)}\|\nabla_h(u_2-u_2')\|_{L^2_h(\Omega)}\\ &\quad+\frac{C}{h}\|\nabla_h(u_1-u_1')\|_{H^2_h(\Omega)}\|\nabla_h u_1\|_{L^2_h(\Omega)}+\frac{C}{h}\|\nabla_h u_1'\|_{H^2_h(\Omega)}\|\nabla_h(u_1-u_1')\|_{L^2_h(\Omega)}\\ &\quad+\frac{C}{h^2}\|\nabla_h(u_0-u_0')\|_{H^2_h(\Omega)}\|\nabla_h u_1\|_{H^1_h(\Omega)}\|\nabla_h u_1'\|_{H^1_h(\Omega)}\\ &\quad+\frac{C}{h^2}\|\nabla_h(u_0-u_0')\|_{H^2_h(\Omega)}\|\nabla_h u_0'\|_{H^2_h(\Omega)}\|\nabla_h u_2\|_{L^2_h(\Omega)}\\ &\quad+\frac{|\gamma_h(u_{0,h})|}{h^2}\|\nabla_h(u_0-u_0')\|_{H^2_h(\Omega)}+|\gamma_h(u_{0,h})-\gamma_h(u_{0,h}')|\\ &\leq CM_2\|(u_1-u_1',u_2-u_2')\|_{\mathcal{W}_h}\end{aligned}$$

where we used again Corollary 2.3.7, $|P|_h=|P|$, $|\gamma_h(u_{0,h})|\leq Ch^2$ and

$$\begin{aligned}|\gamma_h(u_{0,h})-\gamma_h(u_{0,h}')|&\leq\frac{1}{h^3}\left\|\int_0^1(1-\tau)\Big(D^3\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{0,h},\nabla_h u_{0,h}]\right.\\ &\qquad\left.-D^3\tilde{W}(\nabla_h u_{0,h}')[\nabla_h u_{0,h}',\nabla_h u_{0,h}']\Big)\right\|_{(L^2_h)'}\\ &\leq CM_2\|\nabla_h(u_0-u_0')\|_{H^2_h(\Omega)}\leq CM_2\|(u_1-u_1',u_2-u_2')\|_{\mathcal{W}_h}.\end{aligned}$$

Finally via

$$\begin{aligned}\partial_{x_j}\mathcal{Q}_{2,h}(u_1,u_2)&=\frac{1}{h^2}\int_0^1 D^3\tilde{W}(\tau\nabla_h u_0)[\nabla_h u_0,\nabla_h\partial_{x_j}u_2]d\tau+\frac{1}{h^2}D^3\tilde{W}(\nabla_h u_0)[\nabla_h\partial_{x_j}u_0,\partial_h u_2]\\ &\quad+\frac{2}{h^2}D^3\tilde{W}(\nabla_h u_0)[\nabla_h\partial_{x_j}u_1,\nabla_h u_1]+\frac{1}{h^2}D^4\tilde{W}(\nabla_h u_0)[\nabla_h\partial_{x_j}u_0,\nabla_h u_1,\nabla_h u_1]\\ &\quad-\frac{\gamma_h}{h^3}D^3\tilde{W}(\nabla_h u_{0,h})[\nabla_h\partial_{x_j}u_0,P]\end{aligned}$$

it follows

$$\|\partial_{x_j}(\mathcal{Q}_{2,h}(u_1,u_2)-\mathcal{Q}_{2,h}(u_1',u_2'))\|_{(L^2_h)'}\leq CM_2\|(u_1-u_1',u_2-u_2')\|_{\mathcal{W}_h}.$$

Choosing now $M_2\in(0,1]$ small enough we obtain that

$$\mathcal{F}_{h,f_0,f_1,f_2}\colon\overline{B_{CM_2h^2}(0)}\subset\mathcal{X}_h\times\mathcal{W}_h\to\mathcal{X}_h\times\mathcal{W}_h$$

defined by

$$\begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{G}_{h,f_0-u_2}(u_0) \\ \mathcal{L}_h^{-1}\left(\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \mathcal{Q}_h(u_1,u_2)\right) \end{pmatrix}$$

is a $\frac{1}{2}$-contraction, where $f_0 := h^2 f^h|_{t=0}$, $f_1 := h^2\partial_t f^h|_{t=0} - u_{3,h}$ and $f_2 := h^2\partial_t^2 f^h|_{t=0} - u_{4,h}$. We can use an analogous argument as in Lemma 6.0.3. First it holds, due to (6.2) and (6.1), for $M > 0$ sufficiently small

$$\|\mathcal{F}_{h,f_0,f_1,f_2}(0)\|_{\mathcal{X}_h\times\mathcal{W}_h} \leq \tilde{C}Mh^2 \leq \frac{CM_2h^2}{2}$$

and with the $\frac{1}{2}$-contraction property we obtain the self mapping $\mathcal{F}_{h,f_0,f_1,f_2}$. Moreover due to the norm on $\mathcal{X}_h$ and $\mathcal{W}_h$ we obtain (6.20) and (6.21), respectively.
Finally, the construction of $\tilde{u}_h$ implies that $\tilde{u}_{j,h}$ satisfies

$$\begin{aligned} \frac{1}{h^2}\left(D^2\tilde{W}(0)\nabla_h\tilde{u}_{j,h}, \nabla_h\varphi\right)_{L^2(\Omega)} &= \left(h^2\partial_t^j f^h|_{t=0} - \tilde{u}_{2+j,h}, \varphi\right)_{L^2(\Omega)} + (\partial_t^j r_h, \varphi)_{L^2(\Omega)} \\ &\quad - \frac{1}{h^2}\int_0^L \left(\operatorname{tr}_{\partial S}(\partial_t^j r_{N,h}(x_1,\cdot)), \operatorname{tr}_{\partial S}(\varphi(x_1,\cdot))\right)_{L^2(\partial S)} dx_1 \end{aligned}$$

for $j = 0,1,2$ and all $\varphi \in \mathcal{B}$. This implies with (6.17)–(6.19)

$$\begin{aligned} \frac{1}{h^2}\left(\varepsilon_h(u_{1,h} - \tilde{u}_{1,h}), \varepsilon_h(\varphi)\right)_{L^2(\Omega)} &= -\frac{1}{h^2}\left((D^2\tilde{W}(\nabla_h u_{0,h}) - D^2\tilde{W}(0))\nabla_h u_{1,h}, \nabla_h\varphi\right)_{L^2(\Omega)} \\ + (r_{1,h},\varphi)_{L^2(\Omega)} &- \frac{1}{h^2}\int_0^L \left(\operatorname{tr}_{\partial S}(\partial_t r_{N,h}(x_1,\cdot)), \operatorname{tr}_{\partial S}(\varphi(x_1,\cdot))\right)_{L^2(\partial S)} dx_1 \\ \frac{1}{h^2}\left(\varepsilon_h(u_{2,h} - \tilde{u}_{2,h}), \varepsilon_h(\varphi)\right)_{L^2(\Omega)} &= -\frac{1}{h^2}\left((D^2\tilde{W}(\nabla_h u_{0,h}) - D^2\tilde{W}(0))\nabla_h u_{2,h}, \nabla_h\varphi\right)_{L^2(\Omega)} \\ +(r_{2,h},\varphi)_{L^2(\Omega)} &- \frac{1}{h^2}\int_0^L \left(\operatorname{tr}_{\partial S}(\partial_t^2 r_{N,h}(x_1,\cdot)), \operatorname{tr}_{\partial S}(\varphi(x_1,\cdot))\right)_{L^2(\partial S)} dx_1 \\ -\frac{1}{h^2}\left(D^3\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{1,h}, \nabla_h u_{1,h}], \nabla_h\varphi\right)_{L^2(\Omega)} &- \frac{\gamma_h}{h^3}\left(D^2\tilde{W}(\nabla_h u_{0,h})P, \nabla_h\varphi\right)_{L^2(\Omega)} \end{aligned}$$

for all $\varphi \in \mathcal{B}$, where we defined

$$r_{j,h} := u_{2+j,h} - \tilde{u}_{2+j,h} - \partial_t^j r_h.$$

With this it follows $\max_{j=1,2}\|r_{j,h}\|_{C^0(0,T;L^2(\Omega))} \leq Ch^3$, because of the definition of $u_{2+j,h}$ and the bound on $\partial_t r_h$. Additionally we have due to Lemma 2.3.6 and Corollary 2.3.7, the bounds on $(u_{0,h}, u_{1,h}, u_{2,h})$ and $\varphi \in \mathcal{B}$

$$\begin{aligned} &\left|\frac{1}{h^2}\left((D^2\tilde{W}(\nabla_h u_{0,h}) - D^2\tilde{W}(0))\nabla_h u_{j,h}, \nabla_h\varphi\right)_{L^2(\Omega)}\right| \\ &\quad = \left|\frac{1}{h^2}\int_0^1 \left(D^3\tilde{W}(\tau\nabla_h u_{0,h})[\nabla_h u_{0,h}, \nabla_h u_{j,h}], \nabla_h\varphi\right)_{L^2(\Omega)} d\tau\right| \leq Ch^3\left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{L^2(\Omega)} \end{aligned}$$

as well as

$$\left|\frac{1}{h^2}\left(D^3\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{1,h}, \nabla_h u_{1,h}], \nabla_h\varphi\right)_{L^2(\Omega)}\right| \leq Ch^3\left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{L^2(\Omega)}.$$

and

$$\left|\frac{\gamma_h}{h^3}\left(D^2\tilde{W}(\nabla_h u_{0,h})P, \nabla_h\varphi\right)_{L^2(\Omega)}\right| \leq Ch^2\left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{L^2(\Omega)}$$

Regarding the boundary terms we use that $\mathrm{tr}_{\partial S}\colon H^1(S) \to H^{\frac{1}{2}}(\partial S)$ is linear and bounded. Hence for $j = 0, 1, 2$

$$\begin{aligned}\left|\frac{1}{h^2}\int_0^L \Big(\mathrm{tr}_{\partial S}(\partial_t^j r_{N,h}(x_1,\cdot)), \mathrm{tr}_{\partial S}(\varphi(x_1,\cdot))\Big)_{L^2(\partial S)} dx_1\right| \\ \leq \frac{1}{h^2}\|\partial_t^j r_{N,h}\|_{L^2(0,L;H^1(S))}\|\varphi\|_{L^2(0,L;H^1(S))} \leq Ch^3\left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{L^2(\Omega)}\end{aligned} \tag{6.28}$$

where we used that $\|r_{N,h}\|_{C^2(0,T;H^1(\Omega))} \leq Ch^5$ and Poincaré and Korn inequality for φ. Choosing now $\varphi = u_{j,h} - \tilde{u}_{j,h}$ it follows with an absorption argument

$$\max_{j=1,2}\left\|\frac{1}{h}\varepsilon_h(u_{j,h}) - \frac{1}{h}\varepsilon_h(\tilde{u}_{j,h})\right\|_{L^2(\Omega)} \leq \begin{cases} Ch^3, & \text{if } j = 1, \\ Ch^2, & \text{if } j = 2. \end{cases}$$

Now, for $u_{0,h} - \tilde{u}_{0,h}$ it holds

$$\begin{aligned}\frac{1}{h^2}(\varepsilon_h(u_{0,h} - \tilde{u}_{0,h}),\varepsilon_h(\varphi))_{L^2(\Omega)} &= -\frac{1}{h^2}(G(\nabla_h u_{0,h}), \nabla_h\varphi)_{L^2(\Omega)} \\ &\quad + (r_{0,h}, \varphi)_{L^2(\Omega)} - \frac{1}{h^2}\int_0^L \Big(\mathrm{tr}_{\partial S}(r_{N,h}(x_1,\cdot)), \mathrm{tr}_{\partial S}(\varphi(x_1,\cdot))\Big)_{L^2(\partial S)} dx_1.\end{aligned}$$

The definition of G implies now

$$\begin{aligned}\left|\frac{1}{h^2}(G(\nabla_h u_{0,h}), \nabla_h\varphi)_{L^2(\Omega)}\right| &= \left|\frac{1}{h^2}\int_0^1 (1-\tau)\Big(D^3\tilde{W}(\tau\nabla_h u_{0,h})[\nabla_h u_{0,h}, \nabla_h u_{0,h}], \nabla_h\varphi\Big)_{L^2(\Omega)} d\tau\right| \\ &\leq Ch^3\left\|\frac{1}{h}\varepsilon_h(\varphi)\right\|_{L^2(\Omega)}\end{aligned}$$

because of the bounds for $u_{0,h}$ and Corollary 2.3.7. Using (6.28) it follows

$$\left\|\frac{1}{h}\varepsilon_h(u_{0,h}) - \frac{1}{h}\varepsilon_h(\tilde{u}_{0,h})\right\|_{L^2(\Omega)} \leq Ch^3$$

□

Theorem 6.0.5. *Let f_h, $\tilde{v}_0$, $\tilde{v}_1$, $\tilde{u}_{j,h}$, $j = 0, 1, 2$ and $\tilde{u}_h$ be given as above. Then there exists some $h_0 \in (0, 1]$ such that for $h \in (0, h_0]$ there are initial values $(u_{0,h}, u_{1,h})$ satisfying* (5.5)–(5.7) *and such that*

$$\max_{j=0,1}\left\|\frac{1}{h}\varepsilon_h(u_{j,h}) - \frac{1}{h}\varepsilon_h(\tilde{u}_{j,h})\right\|_{L^2(\Omega)} \leq Ch^3.$$

Moreover, if u_h solves (5.1)–(5.4)*, then*

$$\left\|\left((u_h - \tilde{u}_h), \frac{1}{h}\int_0^t \varepsilon_h\big(u_h(\tau) - \tilde{u}_h(\tau)\big)d\tau\right)\right\|_{L^\infty(0,L;L^2(\Omega))} \leq Ch^3$$

for all $0 < h \leq h_0$.

Proof: Given $(u_{3,h}, u_{4,h})$ we construct $(u_{0,h}, u_{1,h}, u_{2,h})$ such that (5.5)–(5.7) holds. First we notice that $\|u_{4,h}\|_{L^2(\Omega)}$ is of order h^2 as $\partial_{x_1}^l v^4$ is bounded in $L^2(0, L)$ for $l = 0, 1$. Moreover we have

$$\int_\Omega u_{2+j,h} dx = 0$$

for $j = 1, 2$ and

$$\int_\Omega u_{3,h} \cdot x^\perp dx = 0.$$

Using the structure of $u_{3,h}$ we obtain

$$\frac{1}{h}\varepsilon_h(u_{3,h}) = h^2 \begin{pmatrix} -x_2\partial_{x_1}^2 v_2^3 - x_3\partial_{x_1}^2 v_3^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

All together we obtain that $u_{3,h}$ and $u_{4,h}$ satisfy (5.5)–(5.7). The assumptions on g and the structure of f_h imply that (5.8) and (5.9) are fulfilled. Applying Lemma 6.0.3 and 6.0.4 we obtain for h_0 sufficiently small, the existence of $(u_{0,h}, u_{1,h}, \bar{u}_{2,h})$ such that

$$\begin{aligned} \frac{1}{h^2}\Big(D\tilde{W}(\nabla_h u_{0,h}), \nabla_h\varphi\Big)_{L^2(\Omega)} &= (h^2 f_h|_{t=0}, \varphi)_{L^2(\Omega)} - (\bar{u}_{2,h}, \varphi)_{L^2(\Omega)} \\ \frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_{0,h})\nabla_h u_{1,h}, \nabla_h\varphi\Big)_{L^2(\Omega)} &= (h^2\partial_t f_h|_{t=0}, \varphi)_{L^2(\Omega)} - (u_{3,h}, \varphi)_{L^2(\Omega)} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_{0,h})\nabla_h \bar{u}_{2,h}, \nabla_h\varphi\Big)_{L^2(\Omega)} &= (h^2\partial_t^2 f_h|_{t=0} - u_{4,h}, \varphi)_{L^2(\Omega)} \\ &-\frac{1}{h^2}\Big(D^3\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{1,h}, \nabla_h u_{1,h}], \nabla_h\varphi\Big)_{L^2(\Omega)} - \frac{\gamma_h}{h^3}\Big(D^2\tilde{W}(\nabla_h u_{0,h})P, \nabla_h\varphi\Big)_{L^2(\Omega)} \end{aligned}$$

for all $\varphi \in \mathcal{B}$. We use the ansatz $u_{2,h} = \bar{u}_{2,h} + \gamma_2^h x^\perp$ and $\bar{u}_{2+j,h} = u_{2+j,h} + \gamma_{2+j}^h x^\perp$ for $j = 1, 2$. Choosing

$$\gamma_2^h := -\frac{1}{\mu(S)h^2}\Big(D\tilde{W}(\nabla_h u_{0,h}), \nabla_h x^\perp\Big)_{L^2}$$

it follows

$$\frac{1}{h^2}\Big(D\tilde{W}(\nabla_h u_{0,h}), \nabla_h\varphi\Big)_{L^2(\Omega)} = (h^2 f_h|_{t=0}, \varphi)_{L^2(\Omega)} - (u_{2,h}, \varphi)_{L^2(\Omega)} \tag{6.29}$$

for all $\varphi \in H^1_{per}(\Omega; \mathbb{R}^3)$. Moreover, for

$$\gamma_3^h := \frac{1}{\mu(S)h^2}\Big(D^2\tilde{W}(\nabla_h u_{0,h})\nabla_h u_{1,h}, \nabla_h x^\perp\Big)_{L^2}$$

we deduce

$$\frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_{0,h})\nabla_h u_{1,h}, \nabla_h\varphi\Big)_{L^2(\Omega)} = (h^2\partial_t f_h|_{t=0}, \varphi)_{L^2(\Omega)} - (\bar{u}_{3,h}, \varphi)_{L^2(\Omega)} \tag{6.30}$$

for all $\varphi \in H^1_{per}(\Omega; \mathbb{R}^3)$. Then it holds $|\gamma_2^h| \le Ch^2$ as

$$D\tilde{W}(\nabla_h u_{0,h}) = D^2\tilde{W}(0)[\nabla_h u_{0,h}] + \int_0^1 (1-\tau)D^3\tilde{W}(\tau\nabla_h u_{0,h})[\nabla_h u_{0,h}, \nabla_h u_{0,h}]d\tau$$

and $|\gamma_3^h| \le Ch^2$ with a similar calculation. Lastly, we need to find γ_4^h such that

$$\begin{aligned} \frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_{0,h})\nabla_h u_{2,h}, \nabla_h\varphi\Big)_{L^2(\Omega)} &= (h^2\partial_t^2 f_h|_{t=0} - \bar{u}_{4,h}, \varphi)_{L^2(\Omega)} \\ &-\frac{1}{h^2}\Big(D^3\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{1,h}, \nabla_h u_{1,h}], \nabla_h\varphi\Big)_{L^2(\Omega)} \end{aligned} \tag{6.31}$$

for all $\varphi \in H^1_{per}(\Omega; \mathbb{R}^3)$. Choose therefore

$$\begin{aligned} \gamma_4^h := &-\frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_{0,h})\nabla_h u_{2,h}, \nabla_h x^\perp\Big)_{L^2} - \frac{\gamma_2^h}{h^2}\Big(D^2\tilde{W}(\nabla_h u_{0,h})\nabla_h x^\perp, \nabla_h x^\perp\Big)_{L^2} \\ &+ \frac{1}{h^2}\Big(D^3\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{1,h}, \nabla_h u_{1,h}], \nabla_h x^\perp\Big)_{L^2}. \end{aligned}$$

The first and last term can be bounded easily, using Lemma 2.3.7

$$\left|\frac{1}{h^2}\Big(D^2\tilde{W}(\nabla_h u_{0,h})\nabla_h \bar{u}_{0,h}, \nabla_h x^\perp\Big)_{L^2}\right| = \left|\frac{1}{h^3}\int_0^1 \Big(D^3\tilde{W}(\nabla_h u_{0,h})[\nabla_h u_{0,h}, \nabla_h \bar{u}_{2,h}], P\Big)_{L^2}\right|$$
$$\leq \frac{C}{h^2}\|\nabla_h u_{0,h}\|_{H^1_h}\|\nabla_h \bar{u}_{2,h}\|_{H^1_h} \leq Ch^2$$

and

$$\left|\frac{1}{h^2}\Big(D^3\tilde{W}(\nabla_h u_{2,h})[\nabla_h u_{1,h}, \nabla_h u_{1,h}], \nabla_h x^\perp\Big)_{L^2}\right| \leq Ch^2.$$

For the second part of γ_4^h we use the following equality

$$\Big(D^2\tilde{W}(\nabla_h u_{0,h})P, P\Big)_{L^2(\Omega)} = \Big(D^3\tilde{W}(0)[\nabla_h u_{0,h}, P], P\Big)_{L^2(\Omega)}$$
$$+ \int_0^1 (1-\tau)\Big(D^4\tilde{W}(\tau\nabla_h u_{0,h})[\nabla_h u_{0,h}, \nabla_h u_{0,h}, P], P\Big)_{L^2(\Omega)} d\tau$$

where

$$\left|\Big(D^4\tilde{W}(\tau\nabla_h u_{0,h})[\nabla_h u_{0,h}, \nabla_h u_{0,h}, P], P\Big)_{L^2(\Omega)}\right| \leq Ch^4 \quad \text{for all } \tau \in [0,1] \tag{6.32}$$

as $\|u_{0,h}\|_{H^1_h(\Omega)} \leq Ch^2$ and $|P|_h = |P|$, because $P \in \mathbb{R}^{3\times 3}_{skew}$. Furthermore, we obtain with

$$D^3\tilde{W}(0)[A, B, P] = \Big((A^T - A)^T \operatorname{sym}(B) + (B^T - B)^T \operatorname{sym}(A)\Big) : P. \tag{6.33}$$

for all A, $B \in \mathbb{R}^{3\times 3}$ and $P \in \mathbb{R}^{3\times 3}_{skew}$, that

$$\Big(D^3\tilde{W}(0)[\nabla_h u_{0,h}, P], P\Big)_{L^2(\Omega)} = h\left(D^3\tilde{W}(0)\left[\frac{1}{h}\varepsilon_h(u_{0,h}) - \frac{1}{h}\varepsilon_h(\tilde{u}_{0,h}), P\right], P\right)_{L^2(\Omega)}$$
$$+ \Big(D^3\tilde{W}(0)[\nabla_h \tilde{u}_{0,h}, P], P\Big)_{L^2(\Omega)}$$

holds. Utilizing the inequality for the initial values (6.22), we deduce

$$\left|h\left(D^3\tilde{W}(0)\left[\frac{1}{h}\varepsilon_h(u_{0,h}) - \frac{1}{h}\varepsilon_h(\tilde{u}_{0,h}), P\right], P\right)_{L^2}\right| \leq Ch^4.$$

Lastly due to the symmetrie properites of $D^3\tilde{W}$, the structure of $\nabla_h \tilde{u}_{0,h}$ and (6.33) it follows

$$\left|\Big(D^3\tilde{W}(0)[\operatorname{sym}(\nabla_h \tilde{u}_{0,h}), P], P\Big)_{L^2}\right| = \left|\Big(D^3\tilde{W}(0)[h^3 Q, P], P\Big)_{L^2} + \Big(D^3\tilde{W}(0)[R, P], P\Big)_{L^2}\right|$$

where

$$Q = \begin{pmatrix} -x_2\partial_{x_1}^2 v_2 - x_3\partial_{x_1}^2 v_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$R = h^4 \operatorname{sym}\begin{pmatrix} 0 & \partial_{x_2}a_2\partial_{x_1}^3 v_2 + \partial_{x_2}a_3\partial_{x_1}^3 v_3 & \partial_{x_3}a_2\partial_{x_1}^3 v_2 + \partial_{x_3}a_3\partial_{x_1}^3 v_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$+ h^5 \operatorname{sym}\begin{pmatrix} a_2\partial_{x_1}^4 v_2 + a_3\partial_{x_1}^4 v_3 & 0 & 0 \\ 0 & \partial_{x_2}b_2\partial_{x_1}^4 v_2 + \partial_{x_2}c_3\partial_{x_1}^4 v_3 & \partial_{x_3}b_2\partial_{x_1}^4 v_2 + \partial_{x_3}c_3\partial_{x_1}^4 v_3 \\ 0 & \partial_{x_2}b_3\partial_{x_1}^4 v_3 + \partial_{x_2}c_2\partial_{x_1}^4 v_2 & \partial_{x_3}b_3\partial_{x_1}^4 v_3 + \partial_{x_3}c_2\partial_{x_1}^4 v_2 \end{pmatrix}$$

$$+ h^6 \operatorname{sym} \begin{pmatrix} 0 & 0 & 0 \\ b_2 \partial_{x_1}^5 v_2 + c_3 \partial_{x_1}^5 v_3 & 0 & 0 \\ b_3 \partial_{x_1}^5 v_3 + c_2 \partial_{x_1}^5 v_2 & 0 & 0 \end{pmatrix}.$$

Due to the structure of Q and $P = \nabla x^\perp$ it follows

$$D^3 \tilde{W}(0)[Q, P, P] = \Big((Q^T - Q)^T \operatorname{sym}(P) + (P^T - P)^T \operatorname{sym}(Q) \Big) : P = 0.$$

Hence, with $R = O(h^4)$ we obtain

$$\left| \Big(D^3 \tilde{W}(0)[\operatorname{sym}(\nabla_h \tilde{u}_{0,h}), P], P \Big)_{L^2} \right| \le C h^4.$$

Thus, altogether, it follows with $|\gamma_2^h| \le C h^2$

$$\left| \frac{\gamma_2^h}{h^2} \Big(D^2 \tilde{W}(\nabla_h u_{0,h}) \nabla_h x^\perp, \nabla_h x^\perp \Big)_{L^2} \right| \le C h^2.$$

We obtain for h_0 sufficiently small, the existence of $(u_{0,h}, u_{1,h}, u_{2,h})$ such that (5.5)–(5.7) are satisfied and

$$\max_{j=0,1} \left\| \left(\frac{1}{h} \varepsilon_h(u_{j,h}) - \frac{1}{h} \varepsilon_h(\tilde{u}_{j,h}) \right) \right\|_{L^2(\Omega)} \le C h^3$$

holds.

Due to Theorem 5.1.1 there exits a solution $u_h \in \bigcap_{k=0}^4 C^k([0,T]; H^{4-k}_{per}(\Omega; \mathbb{R}^3))$ of (5.1)–(5.4). Thus $w_h := u_h - \tilde{u}_h$ solves the following system

$$\begin{aligned} -\big(\partial_t w_h, \partial_t \varphi\big)_{L^2(Q_T)} &+ \frac{1}{h^2} \big(D^2 \tilde{W}(0) \nabla_h w_h, \nabla_h \varphi \big)_{L^2(Q_T)} - (w_1, \varphi|_{t=0})_{L^2(\Omega)} \\ &= \frac{1}{h^2} \int_0^1 \big((D^2 \tilde{W}(\tau \nabla_h u_h) - D^2 \tilde{W}(0)) \nabla_h \tilde{u}_h, \nabla_h \varphi \big)_{L^2(Q_T)} d\tau - \big(r_h, \varphi \big)_{L^2(Q_T)} \\ &\quad - \frac{1}{h^2} (\operatorname{tr}_{\partial\Omega}(r_{N,h}), \operatorname{tr}_{\partial\Omega}(\varphi))_{L^2(0,T;L^2(\partial\Omega))} \\ &w_h \text{ is } L\text{-periodic in } x_1 \text{ direction} \\ &w_h|_{t=0} = w_{0,h} \end{aligned}$$

for all $\varphi \in C^1([0,T]; H^1_{per,(0)}(\Omega; \mathbb{R}^3)$ with $\varphi|_{t=T} = 0$ and with $w_{j,h} := u_{j,h} - \tilde{u}_{j,h}$, $j = 0,1$. Hence with (6.8) we obtain an upper bound for w. For this we use that, due to the structure of r_h and $r_{N,h}$ it follows

$$\frac{1}{h^2} \| r_{N,h} \|_{L^1(0,T;H^1)} \le C h^3$$
$$\| r_h \|_{L^1(0,T;L^2)} \le C h^3$$

as a, b, c and v are sufficiently regular. Moreover, using (6.22)

$$\| w_{k,h} \|_{L^2(\Omega)} \le \max_{j=0,1} \left\| \left(\frac{1}{h} \varepsilon_h(u_{j,h}) - \frac{1}{h} \varepsilon_h(\tilde{u}_{j,h}) \right) \right\|_{L^2(\Omega)} \le C h^3$$

for $k = 0,1$, where we used Poincaré's and Korn's inequality, as well as the fact that $w_{k,h} \in \mathcal{B}$ holds for $k = 0,1$. With the fundamental theorem of calculus and Corollary 2.3.7 we deduce

$$\sup_{\varphi \in X_h, \|\varphi\|_{X_h} = 1} \left| \frac{1}{h^2} \int_0^1 \Big((D^2 \tilde{W}(\tau \nabla_h u_h) - D^2 \tilde{W}(0)) \nabla_h \partial_t \tilde{u}_h, \nabla_h \varphi \Big)_{L^2(\Omega)} d\tau \right|$$

$$\leq \sup_{\varphi \in X_h, \|\varphi\|_{X_h}=1} \left| \frac{1}{h^2} \int_0^1 \int_0^1 \Big(D^3 \tilde{W}(s\tau \nabla_h u_h)[\nabla_h u_h, \nabla_h \partial_t \tilde{u}_h], \nabla_h \varphi \Big)_{L^2(\Omega)} ds d\tau \right|$$
$$\leq \sup_{\varphi \in X_h, \|\varphi\|_{X_h}=1} \frac{C}{h} \|\nabla_h u_h\|_{H^2_h(\Omega)} \|\nabla_h \partial_t \tilde{u}_h\|_{L^2_h(\Omega)} \|\nabla_h \varphi\|_{L^2_h(\Omega)} \leq CRh^3.$$

Lastly we have to deal with the rotational term. Using (5.54), $u_{0,h}$, $u_{1,h} \in \mathcal{B}$ and the structure of g, we obtain with $q^h := h^{1+\theta} f^h$

$$\begin{aligned} \int_0^t \int_\Omega u_h \cdot x^\perp dx d\tau =& t \int_\Omega u_{0,h} \cdot x^\perp dx + \frac{1}{2} t^2 \int_\Omega u_{1,h} \cdot x^\perp dx + \int_0^t (t-s) \int_\Omega q^h \cdot x^\perp dx ds \\ &+ \frac{1}{h} \int_0^t \int_0^\tau (\tau - s) \int_\Omega q^h \cdot u_h^\perp - u_h^\perp \cdot \partial_t^2 u_h dx ds d\tau \\ =& \frac{1}{h} \int_0^t \int_0^\tau (\tau - s) \int_\Omega q^h \cdot u_h^\perp - u_h^\perp \cdot \partial_t^2 u_h dx ds d\tau. \end{aligned}$$

Hence it follows analogously as in the proof of Theorem 5.1.1

$$\begin{aligned} \left\| \int_0^t \frac{1}{h} \int_\Omega u_h \cdot x^\perp dx d\tau \right\|_{C^0(0,T)} &\leq C \left(\left\| \frac{1}{h^2} \int_\Omega q^h \cdot u_h^\perp dx \right\|_{C^0(0,T)} + \left\| \frac{1}{h^2} \int_\Omega \partial_t^2 u_h \cdot u_h^\perp dx \right\|_{C^0(0,T)} \right) \\ &\leq Ch^3 \end{aligned}$$

as due to (5.10)

$$\left\| \frac{1}{h} \varepsilon(\partial_t^\delta u_h) \right\|_{L^\infty(0,T;L^2)} + \left\| \frac{1}{h} \int_\Omega \partial_t^\delta u_h \cdot x^\perp dx \right\|_{L^\infty(0,T)} \leq Ch^2$$

for $\delta = 0, 2$. Thus with (6.8) it follows

$$\left\| \left((u_h - \tilde{u}_h), \frac{1}{h} \int_0^t \varepsilon_h \big(u_h(\tau) - \tilde{u}_h(\tau) \big) d\tau \right) \right\|_{C^0(0,T;L^2)} \leq Ch^3. \qquad \square$$

A Existence of classical solutions for fixed $h > 0$

In this appendix we want to give a more in depth analysis on how the existence result of [Koc93] is applied in the regarded situation. First we shortly summaries the assumptions and equation considered in [Koc93] and the main result [Koc93, Theorem 1.1], which we want to apply. Second we give some remarks on how our system is obtained and why the assumptions assumed in Theorem 5.1.1 are sufficient.
In [Koc93] a quasi-linear hyperbolic equation of the form

$$\sum_{i=0}^{n} \partial_{x_i} F_j^i(t,x,u,Du) = w_j(t,x,u,Du) \qquad \text{in } \Omega \times (0,T) \tag{A.1}$$

$$\sum_{i=0}^{n} \nu_i F_j^i(t,x,u,Du) = g_j \qquad \text{on } \partial\Omega \times (0,T) \tag{A.2}$$

$$(u|_{t=0}, \partial_t u|_{t=0}) = (u_0, u_1) \qquad \text{in } \Omega \tag{A.3}$$

is considered, where $1 \leq j \leq N$, $x_0 = t$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary of class C^{s+2}, $s > n/2 + 1$ and ν is the outer normal. Moreover $u\colon \Omega \times [0,T) \to \mathbb{R}^n$ and $Du = (\partial_t u, \partial_{x_1} u, \ldots, \partial_{x_n} u)$, with $0 < T \leq \infty$. For convenience will state the assumptions made in [Koc93] in a slightly simplified version.

A 1 We assume that $u_0 \in H^{s+1}(\Omega)$, $u_1 \in H^s(\Omega)$ and let U be an open neighbourhood of $\{0\} \times \operatorname{graph}((u_0, u_1, D_x u_0))$ in $[0,T) \times \bar{\Omega} \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$. Moreover assume F, $g \in C^{s+1}(U)$, $w \in C^s(U)$ and define

$$a_{jl}^{ik} = \frac{\partial F_j^i}{\partial(\partial_{x_k} u^l)}(t,x,u,Du) \text{ for all } 0 \leq i,\ k \leq n,\ 1 \leq l,\ k \leq N.$$

A 2 $a_{jl}^{ik} = a_{lj}^{ki}$ in U for all $0 \leq i,\ k \leq n$, $1 \leq l,\ k \leq N$.

A 3 For any $t \in [0,T)$, $v_0 \in C^1(\bar{\Omega})$ and $v_1 \in C(\bar{\Omega})$ with $\{t\} \times \operatorname{graph}((v_0, v_1, D_x v_0)) \subset U$ there exists $\kappa_0 > 0$ and $\mu \geq 0$ such that for all $\psi \in C^\infty(\bar{\Omega})$ the inequality

$$\sum_{\alpha,\beta=1}^{n} \sum_{j,l=1}^{N} (a_{jl}^{\alpha\beta} \partial_{x_\beta} \psi_j, \partial_{x_\alpha} \psi_l)_{L^2(\Omega)} \geq \kappa_0 \|\psi\|_{H^1(\Omega)}^2 - \mu \|\psi\|_{L^2(\Omega)}^2.$$

A 4 For any $\xi \in U$ there exists a $\kappa > 0$ such that for all $\eta \in \mathbb{R}^N$ the inequality

$$a_{jl}^{00}(\xi)\eta_j\eta_l \leq -\kappa_1 |\eta|^2$$

holds.

A 5 We suppose that the compatibility condition holds up to order s.

Under these assumptions the following theorem holds:

Theorem A.1 (Theorem 1.1, [Koc93])**.** *There exists a unique $0 < t_0 \leq T$ and a unique classical solution $u \in C^2([0,t_0) \times \bar{\Omega})$ of* (A.1)–(A.3) *such that $D^\sigma u(t) \in L^2(\Omega)$ for $0 \leq \sigma \leq s+1$.*

Moreover t_0 *is characterised by the two alternatives: either the graph of* (u, Du) *is not precompact in* U *or*

$$\int_0^t \|D^2u(\tau)\|_{L^\infty(\Omega)} d\tau \to \infty \quad \text{for } t \to t_0.$$

In the situation of Chapter 5 the considered domain Ω is not sufficiently smooth, but due to the periodic boundary condition on the end faces of Ω the equations (5.1)–(5.4) are equivalent to the equations on the manifold $M := (\mathbb{R}/L\mathbb{Z}) \times S$. This is a bounded manifold with smooth boundary, as S is a C^∞ domain. The ideas of [Koc93] are similar as in Chapter 5, using differentiation in time and applying results from the elliptic theory.
Choosing $n,\ N = 3$, $g_j \equiv 0$, $w_j(t, x, u, Du) = -h^{1+\theta}(f_h)_j$ and

$$F_j^0(t, x, u, Du) = -\partial_t u_j, \quad F_j^i(t, x, u, Du) = \frac{1}{h^2}(D\tilde{W}(\nabla_h u))_{j,i} = \frac{1}{h^2}\left(\frac{\partial W}{\partial(\partial_{x_i} u_j)}\right)(\nabla_h u)$$

for $i, j = 1, 2, 3$. Then we obtain the symmetry condition A 2. As $(a_{jl}^{00})_{j,l=1,2,3} = -Id \in \mathbb{R}^{3\times 3}$ the assumption A 4 is fulfilled, with $\kappa_1 = 1$. Moreover the compatability assumptions of Theorem 5.1.1 imply A 5. For the first assumption we choose $s = 3$. Then the initial data is sufficiently regular and as f^h does not depend on (u, Du) the prescribed regularity is sufficient. Lastly we can choose U as

$$U = [0, T) \times \bar{\Omega} \times \mathbb{R}^3 \times \mathbb{R}^3 \times U_h \quad \text{where } U_h := \left\{ A \in \mathbb{R}^{3\times 3} \quad \left| \left(A, \frac{1}{h}\operatorname{sym}(A) \right) \right| \le \varepsilon h \right\}$$

for some sufficiently small ε. This is indeed a applicable neighbourhood as for small $h > 0$, it holds $\nabla_h u_{0,h}(x) \in U_h$ for all $x \in \bar{\Omega}$, as $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$. Finally due to Corollary 2.3.6, (5.11) and Korn's inequality we obtain that the coerciveness assumption A 3 is satisfied.

Remark A.2. We want to give a short remark, why the graph of the solution u_h is precompact in U_h for all $h > 0$ as long as (5.59) holds. First we notice that the neighbourhood U lies in a finite dimensional space. Thus $\mathcal{G}(u, Du) := \{(x, t, u, \partial_t, \nabla u) \ :\ (x, t) \in \overline{\Omega} \times [0, T'(h)]\}$ is precompact if and only if $\{(x, t, u, \partial_t, \nabla u) \ :\ (x, t) \in \overline{\Omega} \times [0, T'(h)]\}$ is bounded and the closure satisfies $\overline{\mathcal{G}(u, Du)} \subset U$. Due to the regularity of u_h, it follows that $\mathcal{G}(u_h, Du_h)$ is bounded and for $h_0 > 0$ sufficiently small we have

$$\operatorname{dist}(\{\nabla_h u_h(x, t) \ :\ (x, t) \in \bar{\Omega} \times [0, T'(h)]\}, \partial U_h) \ge \varepsilon > 0$$

for some uniformly $\varepsilon > 0$. Hence, the graph of (u_h, Du_h) is precompact in U.

B Isotropic Functions

Let $n \in \mathbb{N}$ be given. For sake of completeness we cite the well known spectral decomposition theorem for symmetric matrices:

Theorem B.1 (Spectral Theorem, [Göl17])**.** *Let $S \in \mathbb{R}^{n\times n}$ be symmetric. Then there is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of S. For any such basis $\{e_i \mid i = 1, \dots, n\}$ the corresponding eigenvalues $\{\lambda_i \mid i = 1, \dots, n\}$ form the entire spectrum of S and*

$$S = \sum_i \lambda_i e_i \otimes e_i.$$

Moreover, S has exactly two distinct eigenvalues if and only if S admits a decomposition of the following from

$$S = \lambda_1 e \otimes e + \lambda_2(\operatorname{Id} - e \otimes e)$$

for $\lambda_1 \neq \lambda_2$. Conversely, if S has the two eigenspaces $\operatorname{span}\{e\}$ and $\{e\}^\perp$, the later form has to be satisfied.

Let $\mathcal{G} \subset O(n)$. We say that a set $\mathcal{A} \subset \mathbb{R}^{n\times n}$ is invariant under $\mathcal{G}$ if $QAQ^T \in \mathcal{A}$ for all $A \in \mathcal{A}$ and $Q \in \mathcal{G}$.

Proposition B.2. *The sets $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times n}_+$, $O(n)$, $SO(n)$, $\mathbb{R}^{n\times n}_{sym}$ and $\mathbb{R}^{n\times n}_{skew}$ are invariant under $O(n)$, where $\mathbb{R}^{n\times n}_+ := \{A \in \mathbb{R}^{n\times n} \mid \det(A) > 0\}$.*

We say that a scalar function $\varphi\colon \mathcal{A} \to \mathbb{R}$ is invariant under $\mathcal{G}$ if

$$\varphi(A) = \varphi(QAQ^T) \tag{B.1}$$

for all $Q \in \mathcal{G}$ and $\mathcal{A}$ is invariant under $\mathcal{G}$. Similarly, a matrix valued function $G\colon \mathcal{A} \to \mathbb{R}^{n\times n}$ is called invariant under $\mathcal{G}$ if

$$QG(A)G^T = G(QAQ^T) \tag{B.2}$$

and $\mathcal{A}$ is invariant under $\mathcal{G}$. Functions invariant under $O(n)$ are called isotropic.

Proposition B.3. *Let Θ be a scalar or matrix valued function with domain $\mathbb{R}^{n\times n}$. Then Θ is isotropic if* (B.1) *and* (B.2) *hold for $Q \in SO(n)$, respectively.*

Proof: The statement follows from the identity

$$(-Q)A(-Q)^T = QAQ^T$$

for all $Q \in O(n)$ and the fact that for $Q \in O(n)$ either $-Q$ or $Q \in SO(n)$ holds. □

Lemma B.4. *Let $G\colon \mathcal{A} \subset \mathbb{R}^{n\times n}_{sym} \to \mathbb{R}^{n\times n}$ be isotropic. Then every eigenvector of $A \in \mathcal{A}$ is an eigenvector of $G(A)$.*

Proof: Let e be an eigenvector of $A \in \mathcal{A}$. Then there exists $Q \in O(n)$ such that

$$Qe = -e, \qquad Qf = f \quad \text{for all } f \in \{e\}^\perp.$$

Thus Q leaves all the eigenspaces of A invariant and hence A commutes with Q, which yields

$$QAQ^T = A.$$

Thus, using the isotropy of G it follows

$$QG(A)Q^T = Q(QAQ^T) = G(A).$$

Thus Q commutes with $G(A)$ and we have therefore

$$QG(A)e = G(A)Qe = -Q(A)e.$$

Thus by the properties of Q, this implies that $G(A)e \in \operatorname{span}\{e\}$. Thus e is an eigenvector of $G(A)$. $\square$

Now we can proof the main theorem of this paragraph:

Theorem B.5 (Representation of Isotropic Linear Functions)**.** *A linear function $G\colon \mathbb{R}^{n\times n}_{sym} \to \mathbb{R}^{n\times n}_{sym}$ is isotropic if and only if there exist λ, $\mu \in \mathbb{R}$ such that*

$$G(A) = 2\mu A + \lambda \operatorname{tr}(A) \operatorname{Id}$$

for all $A \in \mathbb{R}^{n\times n}_{sym}$.

Proof: Let $\mathcal{N}$ be the set of all unit vectors. Let $e \in \mathcal{N}$. Then $e \otimes e$ has the two distinct eigenvalues 0 and 1 with corresponding eigenspaces $\{e\}^\perp$ and $\operatorname{span}\{e\}$, respectively. Then due to Lemma B.4 the two subspaces must be contained in an eigenspace for $G(A)$. Hence either $G(A)$ has eigenspaces $\{e\}^\perp$ and $\operatorname{span}\{e\}$ or the only eigenspace of $G(A)$ is $\mathbb{R}^{n\times n}$. With the spectral theorem it follows now that there exists functions λ, $\mu\colon \mathcal{N} \to \mathbb{R}$ such that

$$G(e \otimes e) = 2\mu(e)e \times e + \lambda(e) \operatorname{Id}.$$

for every $e \in \mathcal{N}$. Choose now e, $f \in \mathcal{N}$ and $Q \in O(n)$ such that $Qe = f$. Then with $Q(e \otimes e)Q^T = f \otimes f$ and the isotropy of G it follows

$$0 = QG(e \otimes e)Q^T - G(f \otimes f) = 2[\mu(e) - \mu(f)]f \otimes f + [\lambda(e) - \lambda(f)] \operatorname{Id}.$$

But $\{\operatorname{Id}, f \otimes f\}$ is linearly independent, thus $\mu(e) = \mu(f)$ and $\lambda(e) = \lambda(f)$. Therefore λ and μ must be constants.

If $A \in \mathbb{R}^{n\times n}_{sym}$ is arbitrary, the spectral theorem B.1 implies

$$S = \sum_i \lambda_i e_i \otimes e_i.$$

with $e_i \in \mathcal{N}$. The linearity of G admits now

$$G(A) = \sum_i \lambda G(e_i \otimes e_i) = 2\mu A + \lambda \sum_i \lambda_i \operatorname{Id}.$$

The converse follows immediately, via the fact that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all A, $B \in \mathbb{R}^{n\times n}$. $\square$

Bibliography

[ABP91] Emilio Acerbi, Giuseppe Buttazzo, and Danilo Percivale. “A variational definition of the strain energy for an elastic string”. In: *J. Elasticity* 25.2 (1991), pp. 137–148. ISSN: 0374-3535. DOI: 10.1007/BF00042462. URL: https://doi.org/10.1007/BF00042462.

[AMM11a] Helmut Abels, Maria Giovanna Mora, and Stefan Müller. “Large time existence for thin vibrating plates”. In: *Comm. Partial Differential Equations* 36.12 (2011), pp. 2062–2102. ISSN: 0360-5302. DOI: 10.1080/03605302.2011.618209. URL: https://doi.org/10.1080/03605302.2011.618209.

[AMM11b] Helmut Abels, Maria Giovanna Mora, and Stefan Müller. “The time-dependent von Kármán plate equation as a limit of 3d nonlinear elasticity”. In: *Calc. Var. Partial Differential Equations* 41.1-2 (2011), pp. 241–259. ISSN: 0944-2669. DOI: 10.1007/s00526-010-0360-0. URL: https://doi.org/10.1007/s00526-010-0360-0.

[Ant05] Stuart S. Antman. *Nonlinear Problems of Elasticity.* Second. Vol. 107. Springer-Verlag, Berlin-New York, 2005. ISBN: 0-387-20880-1.

[Bal02] John M. Ball. “Some open problems in elasticity”. In: (2002), pp. 3–59. DOI: 10.1007/0-387-21791-6_1. URL: https://doi.org/10.1007/0-387-21791-6_1.

[BM85] J. M. Ball and V. J. Mizel. “One-dimensional variational problems whose minimizers do not satisfy the Euler-Lagrange equation”. In: *Arch. Rational Mech. Anal.* 90.4 (1985), pp. 325–388. ISSN: 0003-9527. DOI: 10.1007/BF00276295. URL: https://doi.org/10.1007/BF00276295.

[Bra02] Andrea Braides. *Γ-convergence for beginners.* Vol. 22. Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford, 2002, pp. xii+218. ISBN: 0-19-850784-4. DOI: 10.1093/acprof:oso/9780198507840.001.0001. URL: https://doi.org/10.1093/acprof:oso/9780198507840.001.0001.

[DG75] Ennio De Giorgi. “Sulla convergenza di alcune successioni d’integrali del tipo dell’area”. In: *Rend. Mat. (6)* 8 (1975), pp. 277–294. ISSN: 0034-4427.

[DGF75] Ennio De Giorgi and Tullio Franzoni. “Su un tipo di convergenza variazionale”. In: *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)* 58.6 (1975), pp. 842–850. ISSN: 0392-7881.

[DM93] Gianni Dal Maso. *An introduction to Γ-convergence.* Vol. 8. Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Boston Inc., Boston, MA, 1993, pp. xiv+340. ISBN: 0-8176-3679-X. DOI: 10.1007/978-1-4612-0327-8. URL: https://doi.org/10.1007/978-1-4612-0327-8.

[EGK17] Christof Eck, Harald Garcke, and Peter Knabner. *Mathematical modeling.* Springer Undergraduate Mathematics Series. Springer, Cham, 2017, pp. xv+509. ISBN: 978-3-319-55160-9; 978-3-319-55161-6. DOI: 10.1007/978-3-319-55161-6. URL: https://doi.org/10.1007/978-3-319-55161-6.

[EK20a] Dominik Engl and Carolin Kreisbeck. "Asymptotic variational analysis of incompressible elastic strings". In: *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* (2020), 1–28. ISSN: 1473-7124. DOI: 10.1017/prm.2020.70. URL: http://dx.doi.org/10.1017/prm.2020.70.

[EK20b] Dominik Engl and Carolin Kreisbeck. *Theories for incompressible rods: a rigorous derivation via Γ-convergence*. 2020. arXiv: 2002.09886 [math.AP].

[Eul27] Leonhard Euler. "De oscillationibus annulorum elasticorum". In: *Opera Omina II* 11 (1727), pp. 378 –382.

[FJM02] Gero Friesecke, Richard D. James, and Stefan Müller. "A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity". In: *Comm. Pure Appl. Math.* 55.11 (2002), pp. 1461–1506. ISSN: 0010-3640. DOI: 10.1002/cpa.10048. URL: https://doi.org/10.1002/cpa.10048.

[FJM06] Gero Friesecke, Richard D. James, and Stefan Müller. "A hierarchy of plate models derived from nonlinear elasticity by gamma-convergence". In: *Arch. Ration. Mech. Anal.* 180.2 (2006), pp. 183–236. ISSN: 0003-9527. DOI: 10.1007/s00205-005-0400-7. URL: https://doi.org/10.1007/s00205-005-0400-7.

[GM12] Mariano Giaquinta and Luca Martinazzi. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*. Second. Vol. 11. Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]. Edizioni della Normale, Pisa, 2012, pp. xiv+366. ISBN: 978-88-7642-442-7; 978-88-7642-443-4. DOI: 10.1007/978-88-7642-443-4. URL: https://doi.org/10.1007/978-88-7642-443-4.

[Göl17] Laurenz Göllmann. *Lineare Algebra. im algebraischen Kontext*. Springer Spektrum, 2017. ISBN: 978-3-662-54342-9. DOI: 10.1007/978-3-662-54343-6.

[Gur81] Morton E. Gurtin. *An introduction to continuum mechanics*. Vol. 158. Mathematics in Science and Engineering. Academic Press, New York-London, 1981, pp. xi+265. ISBN: 0-12-309750-9.

[Kel67] Oliver Dimon Kellogg. *Foundations of potential theory*. Reprint from the first edition of 1929. Die Grundlehren der Mathematischen Wissenschaften, Band 31. Springer-Verlag, Berlin-New York, 1967, pp. ix+384.

[Koc93] Herbert Koch. "Mixed problems for fully nonlinear hyperbolic equations". In: *Math. Z.* 214.1 (1993), pp. 9–42. ISSN: 0025-5874. DOI: 10.1007/BF02572388. URL: https://doi.org/10.1007/BF02572388.

[Lee13] John M. Lee. *Introduction to smooth manifolds*. Second. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708. ISBN: 978-1-4419-9981-8.

[LM72] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I-III.* Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181. Springer-Verlag, New York-Heidelberg, 1972, pp. xvi+357.

[McL00] William McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000, pp. xiv+357. ISBN: 0-521-66332-6; 0-521-66375-X.

[MM03] Maria Giovanna Mora and Stefan Müller. "Derivation of the nonlinear bending-torsion theory for inextensible rods by Γ-convergence". In: *Calc. Var. Partial Differential Equations* 18.3 (2003), pp. 287–305. ISSN: 0944-2669. DOI: 10.1007/s00526-003-0204-2. URL: https://doi.org/10.1007/s00526-003-0204-2.

[MM04] Maria Giovanna Mora and Stefan Müller. "A nonlinear model for inextensible rods as a low energy Γ-limit of three-dimensional nonlinear elasticity". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 21.3 (2004), pp. 271–293. ISSN: 0294-1449. DOI: 10.1016/S0294-1449(03)00044-1. URL: https://doi.org/10.1016/S0294-1449(03)00044-1.

[MM08] M. G. Mora and S. Müller. "Convergence of equilibria of three-dimensional thin elastic beams". In: *Proc. Roy. Soc. Edinburgh Sect. A* 138.4 (2008), pp. 873–896. ISSN: 0308-2105. DOI: 10.1017/S0308210506001120. URL: https://doi.org/10.1017/S0308210506001120.

[MM20] Cy Maor and Maria Giovanna Mora. *Reference configurations vs. optimal rotations: a derivation of linear elasticity from finite elasticity for all traction forces*. 2020.

[MMS07] M. G. Mora, S. Müller, and M. G. Schultz. "Convergence of equilibria of planar thin elastic beams". In: *Indiana Univ. Math. J.* 56.5 (2007), pp. 2413–2438. ISSN: 0022-2518. DOI: 10.1512/iumj.2007.56.3023. URL: https://doi.org/10.1512/iumj.2007.56.3023.

[MP08] S. Müller and M. R. Pakzad. "Convergence of equilibria of thin elastic plates—the von Kármán case". In: *Comm. Partial Differential Equations* 33.4-6 (2008), pp. 1018–1032. ISSN: 0360-5302. DOI: 10.1080/03605300701629443. URL: https://doi.org/10.1080/03605300701629443.

[MPG79] Luiz Carlos Martins and Paolo Podio-Guidugli. "A variational approach to the polar decomposition theorem". In: *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)* 66.6 (1979), pp. 487–493. ISSN: 0392-7881.

[MS12] Maria Giovanna Mora and Lucia Scardia. "Convergence of equilibria of thin elastic plates under physical growth conditions for the energy density". In: *Journal of Differential Equations* 252.1 (2012), pp. 35 –55. ISSN: 0022-0396. DOI: https://doi.org/10.1016/j.jde.2011.09.009. URL: http://www.sciencedirect.com/science/article/pii/S0022039611003871.

[Neč67] Jindřich Nečas. *Les méthodes directes en théorie des équations elliptiques*. Masson et Cie, Éditeurs, Paris, 1967, p. 351.

[QY20] Yizhao Qin and Peng-Fei Yao. "The Time-Dependent Von Kármán Shell Equation as a Limit of Three-Dimensional Nonlinear Elasticity". In: *Journal of Systems Science and Complexity* (2020). DOI: 10.1007/s11424-020-9146-4. URL: https://doi.org/10.1007/s11424-020-9146-4.

[Re67] Ju. G. Rešetnjak. "Liouville's conformal mapping theorem under minimal regularity hypotheses". In: *Sibirsk. Mat. Ž.* 8 (1967), pp. 835–840. ISSN: 0037-4474.

[RR04] Michael Renardy and Robert C. Rogers. *An introduction to partial differential equations*. Second. Vol. 13. Texts in Applied Mathematics. Springer-Verlag, New York, 2004, pp. xiv+434. ISBN: 0-387-00444-0.

[Sca09] Lucia Scardia. "Asymptotic models for curved rods derived from nonlinear elasticity by Γ-convergence". In: *Proc. Roy. Soc. Edinburgh Sect. A* 139.5 (2009), pp. 1037–1070. ISSN: 0308-2105. DOI: 10.1017/S0308210507000194. URL: https://doi.org/10.1017/S0308210507000194.

[Sch13] Ben Schweizer. *Partielle Differentialgleichungen*. Eine anwendungsorientierte Einführung. [An application-oriented introduction]. Springer-Verlag, Berlin, 2013, p. 599. ISBN: 978-3-642-40637-9; 978-3-642-40638-6. DOI: 10.1007/978-3-642-40638-6. URL: https://doi.org/10.1007/978-3-642-40638-6.

[Tay11] Michael E. Taylor. *Partial differential equations I. Basic theory*. Second. Vol. 115. Applied Mathematical Sciences. Springer, New York, 2011, pp. xxii+654. ISBN: 978-1-4419-7054-1. DOI: 10.1007/978-1-4419-7055-8. URL: https://doi.org/10.1007/978-1-4419-7055-8.

[Wlo87] Joseph Wloka. *Partial differential equations*. Translated from the German by C. B. Thomas and M. J. Thomas. Cambridge University Press, Cambridge, 1987, pp. xii+518. ISBN: 0-521-25914-2; 0-521-27759-0. DOI: 10.1017/CBO9781139171755. URL: https://doi.org/10.1017/CBO9781139171755.

Acknowledgements

Writing such a thesis after years of work would not have been possible without the aid of a lot of great people, whom I want to thank.

First and foremost, I want to thank my supervisor Prof. Dr. Helmut Abels, who gave me the opportunity to work in the wonderful analysis group of Regensburg. I appreciated your calm confidence in times of no mathematical progress and thank you for all the - sometimes on short notice - discussions on problems and obstacles.

I would also like to thank my co-advisor Prof. Dr. Georg Dolzmann for the fruitful discussions, sometimes even over lunch, and valuable advice for and feedback after presentations.

A special thanks goes out to my colleagues, starting with the ”Keller-WG”, namely Max, Michi, Johannes and Feli. I do miss the coffee breaks, ”Weißwurst” breakfast and billiard games. You made the start of my Ph.D. time wonderful, with lots of fun and help. Especially I want to thank Max for the 8 years of studying together, helping each other out during difficult times and friendly mocking each other. Moreover, I want to thank Kira for her cheerful nature, hours of discussions (I am still sorry for how many counterexamples I found) and the fun on conferences and retreats. Of course a big thank you goes to Jonas, our third skat player, for the relaxed breaks and discussions on sports. During my time I had lots of office mates; I want to thank Maximilian and Matthias for the time together. Lastly, a special thanks goes to the analysis group of Erlangen - I always enjoyed the times we met.

I gratefully acknowledge the financial support by the DFG graduate school GRK 2339 ”Interfaces, Complex Structures, and Singular Limits” in Regensburg.

Lastly, I would like to thank my family, friends and my wife, for all their support over the years.