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Toroidal b-divisors and Monge–Ampère measures

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Abstract
We generalize the intersection theory of nef toric (Weil) b-divisors on smooth and complete toric varieties to the case of nef b-divisors on complete varieties which are toroidal with respect to a snc divisor. As a key ingredient we show the existence of a limit measure, supported on a balanced rational conical polyhedral space attached to the toroidal embedding, which arises as a limit of discrete measures defined via tropical intersection theory on the polyhedral space. We prove that the intersection theory of nef Cartier b-divisors can be extended continuously to nef toroidal Weil b-divisors and that their degree can be computed as an integral with respect to this limit measure. As an application, we show that a Hilbert–Samuel type formula holds for big and nef toroidal Weil b-divisors.

Keywords b-divisors · Convex analysis · Polyhedral spaces · Tropical geometry · Monge–Ampère measures

Mathematics Subject Classification Primary 14C17; Secondary 14T05 · 26B25 · 52B70

Contents

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Introduction

The theory of b-divisors (where b stands for "birational") was introduced by Shokurov [34] in the context of Mori's minimal model program. Since then b-divisors have appeared in many contexts. For instance in the work by Fujino [19] on base point free theorems, in the work of Küronya and Maclean [29] on the Zariski decomposition of divisors, in the proof of the differentiability of the volumes of divisors by Boucksom, Favre and Jonsson [9], and in the work of Aluffi on Chow groups of Riemann–Zariski spaces [2]. In [26], Kaveh and Khovanskii give an isomorphism between the group of Cartier b-divisors and the Groethendieck group associated to the semigroup of subspaces of rational functions preserving the top intersection index, and although stated in another language, in [18], Fulton and Sturmfels give an isomorphism between the Cartier b-Chow group of a toric variety and the polytope algebra which preserves the intersection product.

Moreover, b-divisors have been associated to dynamical systems in [7] and to psh functions in [8]. In the last paper, b-divisors whose support is a single point are studied. In particular, a top intersection product, that can be $-\infty$, among (relatively) nef b-divisors is defined and it is proved that such top intersection products can be computed by means of a Monge–Ampère type measure in a valuation space. In [6], this top intersection product is generalized from the smooth case to case of isolated singularities. In the paper [12], a b-divisor is associated to the invariant metric on the line bundle of Jacobi forms, and it is shown that such a b-divisor is integrable, in the sense that its top self intersection product is well defined and finite. Moreover, it is proved that, considering this b-divisor, one recovers a Chern–Weil formula, that says that the top self intersection product of the b-divisor is computed as the integral on an open subset of a power of the first Chern form of the metrized line bundle of Jacobi forms, and a Hilbert–Samuel formula that states that the asymptotic growth of the dimension of the space of Jacobi forms is governed by the top self intersection product of the associated b-divisor. In addition it is shown that in this case the associated b-divisor is toroidal and that its top self intersection product can be computed using toric methods.

It is expected that the results of [12] can be extended to the invariant metrics on automorphic line bundles on mixed Shimura varieties, i.e. that the associated b-divisors are toroidal and their degrees computable using toric techniques. In this spirit, in [14], the first author studied the theory of toric b-divisors on toric varieties and showed that much of the theory of ordinary divisors on toric varieties can be extended to the setting of b-divisors.

The aim of the present paper is to study toroidal b-divisors. In particular, we generalize two results of [8] from the local case to the global toroidal case. Namely, that there is a well defined top intersection product between (global i.e. not only supported on a single point) nef b-divisors and that this top intersection product is given by a Monge–Ampère type measure on a rational polyhedral complex. We moreover prove a Hilbert–Samuel formula for nef and big toroidal b-divisors and a Brunn–Minkowski type inequality.

We have chosen to restrict ourselves to toroidal b-divisors because they appear naturally in the applications to mixed Shimura varieties and they are technically simpler than arbitrary b-divisors.

After the present paper was made public, the preprint [17] appeared. In this preprint a general theory of intersection of nef b-divisors over countable fields is developed. Since any toroidal situation can be reduced to a situation defined over a countable field, the existence of an intersection product for toroidal b-divisors can also be deduced from [17, Theorem 6]. Moreover, since the intersection product defined in [17] and the one introduced here are both continuous extensions of the usual intersection theory, they agree. We think that the present paper is still valuable as the technique of proof is different and it gives a very concrete interpretation of the intersection product in the toroidal case by means of a Monge–Ampère type measure. It would be interesting to know if, in general, the intersection product of nef b-divisors can also be interpreted in terms of a Monge–Ampère type measure.

Toroidal b-divisors come in two flavors: Cartier and Weil. We explain this briefly. Let $U \hookrightarrow X$ be a fixed smooth and complete toroidal embedding without self-intersections of dimension n over an algebraically closed field k. To (X, U) one can associate naturally a weakly embedded, smooth conical rational complex $\Pi = \Pi_{(X,U)}$ (Proposition 3.22, Definition 3.35). This is the usual cone complex associated to a simple normal crossings divisor $X \setminus U$ endowed with a natural weak embedding.

An important feature of the weakly embbeded conical complex Π is that it comes equipped with a balancing condition (Definition 4.4) i.e. it is a tropical cycle. This balancing condition will play an important role in the definition of the Monge–Ampère measure and in the strong continuity properties of concave functions on balanceable polyhedral complexes [5, Section 6].

In general the weakly embedded rational conical complex Π is not useful to compute arbitrary intersections between toroidal divisors. We need to impose a further condition: that of being quasi-embedded (Definition 1.10). This condition implies that the algebraic *moving lemma* from algebraic geometry for toroidal divisors is captured by the combinatorics of the dual complex. Hence it is a necessary condition which permits to relate algebraic intersection numbers with combinatorial intersection numbers. In the case of characteristic zero this condition can always be achieved after shrinking U (Proposition 3.39).

Let $|\Pi|$ denote the support of Π which carries a structure of a conical rational polyhedral space (Definition 3.2). Let $R_{sm}(\Pi)$ be the directed system of all smooth rational conical subdivisions of Π. This is a directed set under the relation

$$\Pi'' \geqslant \Pi' \quad \text{iff} \quad \Pi'' \text{ is a smooth rational subdivision of } \Pi'.$$

An element Π' in $R_{sm}(\Pi)$ corresponds to a smooth and complete toroidal embedding $U \hookrightarrow X_{\Pi'}$ together with a proper toroidal birational morphism $X_{\Pi'} \to X$ (Theorem 3.33).

Now, for $\Pi' \in \mathsf{R}_{\mathrm{sm}}(\Pi)$, we denote by $\mathrm{Div}\,(X_{\Pi'}, U)_{\mathbb{R}}$ the $\mathbb{R}$-vector space of toroidal $\mathbb{R}$-divisors on $X_{\Pi'}$, i.e. the group of divisors on $X_{\Pi'}$ supported on the boundary $X_{\Pi'} \setminus U$ with real coefficients. For $\Pi'' \geqslant \Pi'$ there are linear maps

$$\mathrm{Div}\,(X_{\Pi'}, U)_{\mathbb{R}} \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \mathrm{Div}\,(X_{\Pi''}, U)_{\mathbb{R}}\,.$$

Then the spaces of Weil and Cartier toroidal b-divisors are defined as the projective and injective limits

$$\mathrm{bDiv}(X, U)_{\mathbb{R}} := \varprojlim_{\Pi' \in \mathsf{R}_{\mathrm{sm}}(\Pi)} \mathrm{Div}\,(X_{\Pi'}, U)_{\mathbb{R}}\,,$$
$$\mathrm{CbDiv}(X, U)_{\mathbb{R}} := \varinjlim_{\Pi' \in \mathsf{R}_{\mathrm{sm}}(\Pi)} \mathrm{Div}\,(X_{\Pi'}, U)_{\mathbb{R}}\,,$$

respectively, with maps given by proper push-forward of divisors in the first case and pull-back in the second (Definition 4.15). In other words, a Weil toroidal b-divisor is given by a net

$$\mathbf{D} = (D_{\Pi'})_{\Pi' \in \mathsf{R}_{\mathrm{sm}}(\Pi)}\,,$$

where for each $\Pi' \in \mathsf{R}_{\mathrm{sm}}(\Pi)$, the element $D_{\Pi'}$ is a toroidal $\mathbb{R}$-divisor on $X_{\Pi'}$, and all these elements are compatible under push-forward. For $\Pi' \in \mathsf{R}(\Pi)$, we say that $D_{\Pi'}$ is the incarnation of $\mathbf{D}$ on $X_{\Pi'}$. On the other hand, a Cartier toroidal b-divisor on (X, U) is determined by a single $\Pi' \in \mathsf{R}(\Pi)$ and a divisor $D_{\Pi'} \in \mathrm{Div}(X_{\Pi'}, U)_{\mathbb{R}}$. There is a natural inclusion $\mathrm{CbDiv}(X, U)_{\mathbb{R}} \subseteq \mathrm{bDiv}(X, U)_{\mathbb{R}}$. Roughly speaking, Cartier b-divisors are b-divisors that stabilize after a birational map, while Weil b-divisors may keep changing for all blow ups. To simplify notation, we will usually omit the coefficient ring $\mathbb{R}$ from the notation, real coefficients being always implicit.

As in toric geometry, for any $\Pi' \in \mathsf{R}_{\mathrm{sm}}(\Pi)$, we may view toroidal divisors on $X_{\Pi'}$ as piecewise linear functions on $|\Pi|$ which are linear on each cone of Π'. Hence, it is easy to see that a Cartier toroidal b-divisor corresponds to a real valued piecewise linear function whose locus of linearity is rational, while a Weil toroidal b-divisor $\mathbf{D}$ corresponds to a (not necessarily piecewise linear) conical function

$$\phi_{\mathbf{D}}\colon |\Pi|\,(\mathbb{Q}) \longrightarrow \mathbb{R},$$

where $|\Pi|(\mathbb{Q})$ denotes the set of rational points of $|\Pi|$. Note that the only condition required from the function $\phi_{\mathbf{D}}$ is that it is conical, which shows that the space of Weil b-divisors is very wild. Nevertheless, it turns out that if we impose the nefness condition to $\mathbf{D}$ (Definition 4.26), then the function $\phi_{\mathbf{D}}$ extends to a continuous (weakly concave) function

$$\phi_{\mathbf{D}}\colon |\Pi| \longrightarrow \mathbb{R},$$

whose restriction to each cone $\sigma \in \Pi$ is concave (see Theorem 4.29).

Since Cartier toroidal b-divisors are determined on a concrete birational model, the intersection theory of divisors gives immediately an intersection theory of Cartier toroidal b-divisors. The main result of this paper is that the intersection product of nef Cartier toroidal b-divisors can be extended continuously to nef Weil toroidal b-divisors and that this product can be computed as the integral of a Monge–Ampère type measure.

Theorem A *Let* (U, X) *be a toroidal embedding defined over an algebraically closed field* k*, with* X *smooth and projective and* $X \setminus U$ *the support of an effective snc ample divisor*

B. *Assume that the associated weakly embedded conical rational polyhedral space* $|\Pi|$ *is quasi embedded. Then the top intersection product of nef Cartier toroidal b-divisors on* (X, U) *can be extended continuously to a top intersection product of nef Weil toroidal b-divisors on* (X, U). *Moreover, to a family* $\mathbf{D}_2, \dots, \mathbf{D}_n$ *of nef Weil toroidal b-divisors on* (X, U) *we associate a Monge–Ampère type measure* $\mu_{\mathbf{D}_2,\dots,\mathbf{D}_n}$, *supported on a compact subset* $\mathbb{S}^{|\widehat{\Pi}|} \subseteq |\Pi|$, *in such a way that*

$$\mathbf{D}_1 \cdots \mathbf{D}_n = \int_{\mathbb{S}^{|\widehat{\Pi}|}} \phi_{\mathbf{D}_1} \mu_{\mathbf{D}_2,\dots,\mathbf{D}_n}.$$

for any nef Weil toroidal b-divisor $\mathbf{D}_1$ *on* X.

If the field has characteristic zero, the set of birational toroidal structures (see Definition 3.37) forms a directed set (see Remark 4.21) and we can define toroidal b-divisor as an element of the direct limit with respect to all possible birational toric structures of X. The next result means that the extra assumptions of Theorem A can always be achieved after shrinking U if necessary. It is just a consequence of resolution of singularities and Proposition 3.39.

Theorem B *Let* X *be a smooth and projective variety over a field of characteristic zero. Let* $D_1, \dots, D_n$ *be toroidal Weil b-divisors. Then there is a birational toroidal structure* $(\pi, \widetilde{X}, U)$ *such that the b-divisors* $D_1, \dots, D_n$ *are toroidal with respect to* π *and* $B := \widetilde{X} \setminus U$ *is the support of an ample divisor,*

In consequence, if the field has characteristic zero, we can apply Theorem A to any family of toroidal nef b-divisors coming from different birational toroidal structures.

In Theorem A, the subspace $\mathbb{S}^{|\widehat{\Pi}|} \subseteq |\Pi|$ and the measure $\mu_{\mathbf{D}_2,\dots,\mathbf{D}_n}$ depend on the choice of an Euclidean metric, but the integral does not. A more canonical representation can be obtained using Corollary 3.17 where the top intersection product is computed as an integral over the lattice unit sphere of Definition 3.14. Finally, note that nowadays, the existence of the product can also be deduced from [17].

As an application, following [14, Section 5], we define the space of non zero global sections of a toroidal b-divisor $\mathbf{D}$ as the space of rational functions f such that $\text{b-div}(f) + \mathbf{D}$ is effective. Then the volume of a Weil b-divisor is defined in analogy to the volume of divisors by the asymptotic growth of the spaces of global sections of multiples of the b-divisor and a Weil b-divisor is called big if it has positive volume. Moreover, to a Weil toroidal b-divisor $\mathbf{D}$ we can associate an Okounkov body $\Delta_{\mathbf{D}}$ (Definition 5.6). Then we obtain the following extension of the Hilbert–Samuel theorem to the b-case (Theorem 5.14).

Theorem C *Let* $\mathbf{D}$ *be a big and nef Weil toroidal b-divisor on* (X, U). *Then*

$$\text{vol}(\mathbf{D}) = n! \text{vol}(\Delta_{\mathbf{D}}) = \mathbf{D}^n.$$

As a corollary, we obtain the continuity of the volume function on the space of nef and big toroidal b-divisors (Corollary 5.16) and a Brunn–Minkowski type inequality (Corollary 5.17).

In Theorem C, the hypothesis $\mathbf{D}$ toroidal is necessary. In fact, in a forthcoming paper with R. de Jong and D. Holmes we will show with an example that the volume function is not continuous even for big and nef b-divisors defined over a countable field and that it does not necessarily agree with the degree.

One of the key ingredients to prove the two stated theorems is the combinatorial machinery developed in Sections 1 and 2. The existence of the limit measure $\mu_{\mathbf{D}}$ associated to a nef toroidal b-divisor $\mathbf{D}$ follows directly from Theorem 2.24 and the existence of the mixed limit

measure $\mu_{\mathbf{D}_2,\ldots,\mathbf{D}_n}$ is a direct consequence of Corollary 2.28. These results are based on the convex analysis on polyhedral spaces developed by M. Sombra and the authors in [5], that, in turn, use techniques from [11] and [10].

Another key ingredient is a result in [21] relating tropical and algebraic intersection numbers on complete toroidal embeddings (Theorem 4.6).

It is important to note that for the applications to algebraic geometry it is convenient to work with conical complexes provided with an integral structure. Nevertheless, when studying convex analysis on polyhedral complexes as in [5], the integral structure plays no role, only the affine structure does. Moreover, to write down explicit estimates it is handy to choose a Euclidean structure. Therefore, to study Monge–Ampère measures associated to nef toroidal b-divisors it is convenient to shift the focus from rational conical complexes to Euclidean ones.

As has been noted in [12] and [14], a nef toroidal b-divisor encodes the singularities of the invariant metric on an automorphic line bundle over a mixed Shimura variety of non-compact type along any toroidal compactification. This article together with the above mentioned ones lays the ground of a geometric intersection theory with singular metrics, satisfying Chern–Weil theory and a Hilbert–Samuel formula, to be applied to mixed Shimura varieties of non-compact type.

The article is organized as follows. In Section 1 we recall the tropical intersection theory on Euclidean conical polyhedral spaces as in [5]. This is a Euclidean version of the tropical intersection theory on weakly embedded rational conical polyhedral complexes developed in [21].

In Section 2 we show the combinatorial version of our main result stated in Theorem A. For this, we define the space of conical functions on Euclidean conical spaces and introduce a concavity notion for them. We show that the top intersection product of such concave functions exists, is finite and is given by the total mass of a week limit of discrete Monge–Ampère measures. This is done by introducing the notion of the *size* of a tropical cycle. This allows us to prove a Chern–Levine–Nirenberg type inequality (Lemma 2.18) from which we conclude the weak convergence of the discrete measures (Theorem 2.24).

In Section 3, we define quasi-embedded *rational* conical polyhedral spaces. In short, these are conical polyhedral spaces endowed with a lattice structure together with a quasi embedding which is compatible with the lattice structure. Following [21], there is a *rational tropical intersection product* on quasi-embedded rational conical spaces. We compare the rational tropical intersection with the Euclidean one from Section 1 by means of the *normalization* of cycles. We further show that the total mass of the Monge–Ampère measures from Section 2.4 are independent of the Euclidean metric and only depend on the integral structure and on the choice of a smooth subdivision (Corollary 3.17). We then recall the definition of a toroidal embedding and describe a natural rational conical polyhedral space associated to it (see [27] or [3] for further details). We describe the proper toroidal birational modifications of a toroidal embedding which, on the combinatorial side, correspond to subdivisions of rational conical complexes on this rational conical space. Finally, following [21], we give a natural weak embedding of this space and we show that by adding boundary components one can modify the toroidal structure of a toroidal embedding in such a way that this natural weak embedding becomes a quasi embedding.

In Section 4 we state and prove our main results. We show that nef toroidal b-divisors have well defined top intersection products (Definitions 4.20 and 4.26 and Theorem 4.32). For this, we first relate the geometric intersection product of toroidal divisors with the *rational tropical intersection product* on quasi-embedded rational conical spaces (Theorem 4.6) (see [21]). Then we use the convergence results of Section 2 in order to extend the top intersection product

to nef toroidal b-divisors. However, note that the Monge–Ampère measures of Section 2 are defined in a Euclidean setting (no integral structure). Therefore we will use the comparison in Section 3.2 to relate the rational tropical intersection product with the Euclidean one.

Finally, in Section 5, as an application, we give a Hilbert–Samuel type formula for nef and big toroidal b-divisors. This relates the degree of a nef toroidal b-divisor both with the volume of the b-divisor and with the volume of the associated convex Okounkov body (Definitions 5.3 and 5.6 and Theorem 5.14). As a corollary, we obtain a Brunn–Minkowski type inequality (Corollary 5.17).

1 Euclidean tropical intersection theory

In this section we recall the tropical intersection theory on Euclidean conical polyhedral spaces as in [5]. This is an adapted Euclidean version of the tropical intersection theory on weakly embedded rational conical polyhedral complexes developed in [21].

1.1 Euclidean conical polyhedral spaces

We give the definition of a quasi-embedded conical polyhedral space endowed with a Euclidean structure. We also discuss morphisms and subdivisions of such spaces.

Definition 1.1 Let X be a topological space. A *conical polyhedral structure* on X is a pair

$$\Pi = (\{\sigma^\alpha\}_{\alpha\in\Lambda}, \{M^\alpha\}_{\alpha\in\Lambda})$$

consisting of a finite covering by closed subsets $\sigma^\alpha \subseteq X$ and for each σ^α, a finite dimensional $\mathbb{R}$-vector space M^α of continuous, $\mathbb{R}$-valued functions on σ^α satisfying the following conditions. Let $N^\alpha := \mathrm{Hom}(M^\alpha, \mathbb{R})$ denote the dual vector space.

(1) For each $\alpha \in \Lambda$, the evaluation map $\phi^\alpha \colon \sigma^\alpha \to N^\alpha$ given by the assignment

$$v \longmapsto (u \mapsto u(v)) \quad (u \in M^\alpha),$$

maps σ^α homeomorphically to a strictly convex, full-dimensional, polyhedral cone in N^α.

(2) The preimage under ϕ^α of each face of $\phi^\alpha(\sigma^\alpha)$ is a cone $\sigma^{\alpha'}$ for some index $\alpha' \in \Lambda$, and we have that $M^{\alpha'} = \left\{u|_{\sigma^{\alpha'}} \,\middle|\, u \in M^\alpha\right\}$.

(3) The intersection of two cones is a union of common faces.

The following notations will be used.

(1) By abuse of notation we will think of Π as the set of cones $\{\sigma^\alpha\}_{\alpha\in\Lambda}$. For every integer $k \geqslant 0$ we write $\Pi(k)$ for the set of cones of dimension k.

(2) Given a cone $\sigma \in \Pi$, we will write M^σ, N^σ and ϕ^σ for the corresponding $\mathbb{R}$-vector space, dual vector space and evaluation map, respectively. We denote by $\langle\,,\,\rangle_\sigma$ the pairing induced by the dual vector spaces M^σ and N^σ. We will usually omit the index "σ" from the pairing.

(3) We will identify a cone σ with its image in N^σ. The linear structure of N^σ induces a linear structure in σ. Therefore we can talk of linear maps between cones.

(4) If τ is a face of σ we will write $\tau \prec \sigma$ or $\sigma \succ \tau$.

(5) We will denote by 0_σ the zero for the linear structure of N_σ. Since σ is strictly convex, the set $\{0_\sigma\}$ is a face of σ. By abuse of notation we will denote this face also as 0_σ.

(6) By the *relative interior* of a cone σ, denoted $\operatorname{relint}(\sigma)$ we mean the preimage under ϕ^σ of the interior of the cone $\phi^\sigma(\sigma) \subseteq N^\sigma$.
(7) Π is called *simplicial* if every cone $\phi^\sigma(\sigma)$ is generated by an $\mathbb{R}$-basis of N^σ.
(8) The space of linear functions on Π, denoted $L(\Pi)$ is the space of all continuous functions on X whose restriction to each cone of Π is linear.

Definition 1.2 Let X be a topological space and Π, Π' two conical polyhedral structures on X. Then Π' is a *subdivision* of Π, denoted by $\Pi' \geqslant \Pi$, if for every $\sigma' \in \Pi'$ there exists a $\sigma \in \Pi$ with $\sigma' \subseteq \sigma$, the inclusion being a linear map. Two conical structures on X are *equivalent* if they admit a common subdivision.

The following follows as in [5, Proposition 2.4].

Proposition 1.3 *Let X be a topological space. Then*

(1) the relation $\geqslant$ is a partial order on the set of conical polyhedral structures on X,
(2) the subdivisions of a given conical polyhedral structure on X form a directed set,
(3) "being equivalent" is an equivalence relation between conical polyhedral structures on X.

Definition 1.4 A *conical polyhedral space* X is a topological space equipped with an equivalence class of conical polyhedral structures. A *conical polyhedral complex on* X is the choice of a representative of the class of conical polyhedral structures on X.

Definition 1.5 If X is a conical polyhedral space, we let $S(X)$ denote the set of conical polyhedral complexes on it. By Proposition 1.3, this is a directed set ordered by subdivision. We further denote by $S_{sp}(X)$ the subset of all *simplicial* conical polyhedral complexes on X, endowed with the induced directed set structure.

We will usually refer to a *conical polyhedral space* just as a *conical space* and to a *conical polyhedral complex* just as a *conical complex*.

Definition 1.6 The *dimension* of a conical space X is defined as as

$$\dim(X) = \sup_{\sigma \in \Pi} \dim(M_\sigma)$$

for any conical complex Π on X. We say that X has *pure dimension* n if every cone of Π that is maximal (with respect to the inclusion) has dimension n. These notions do not depend on the choice of Π.

The following remark follows from [31, Remark 2.6].

Remark 1.7 Let X be a conical space and let Π be any conical complex on X. The connected components of X are in one to one correspondence with the zero dimensional cones of Π. The points belonging to the zero dimensional cones are called *vertices* of Π. In particular, if X is connected, then Π has a unique vertex.

Definition 1.8 Let X and X' be conical spaces. Given conical complexes Π on X and Π' on X', a *morphism of conical spaces between* Π *and* Π' is a continuous map $f\colon X \to X'$ such that for every cone $\sigma \in \Pi$ there is $\sigma' \in \Pi'$ with $f(\sigma) \subseteq \sigma'$, and the restriction $f|_\sigma\colon \sigma \to \sigma'$ is a linear map.

The following definition is the Euclidean version of [21, Definition 2.1].

Definition 1.9 A *weakly embedded conical space* is a triple (X, N, ι) where X is a conical space, N is a finite dimensional $\mathbb{R}$-vector space, and $\iota\colon X \to N$ is a map such that there is a conical complex Π on X for which the restriction of ι to every cone σ of Π is linear. The map ι is called the weak embedding of X in N. A *conical complex* Π on the weakly embedded conical space (X, N, ι) is a conical complex Π on X satisfying the above condition, namely that ι is linear on each of its cones.

We will usually denote a weakly embedded conical space by the underlying conical space X and, in this case, we denote the corresponding weak embedding, vector space and dual vector space by ι_X and N^X and M^X, respectively. Given a conical complex Π on the weakly embedded conical space X, for every cone $\sigma \in \Pi$, we write N^X_σ for $N^X \cap \mathrm{Span}\,(\iota_X(\sigma))$ and $M^X_\sigma = \mathrm{Hom}\left(N^X_\sigma, \mathbb{R}\right)$ for its dual.

The following notion is stronger than that of a weakly embedded conical space.

Definition 1.10 A weakly embedded conical space X is said to be *quasi-embedded* if there is a conical complex Π on X such that the restriction $\iota_X|_\sigma$ is injective in each cone $\sigma \in \Pi$. In this case, we identify each vector space N^σ with its image N^X_σ in N^X. As before, a *conical complex* Π on the quasi-embedded conical space X is a conical complex Π on X satisfying the above condition, namely that ι is linear and injective on each of its cones.

Example 1.11 Let X be a conical space and Π a conical structure on X. We may consider the space of linear functions $L(\Pi)$ on Π. Then the evaluation map $X \to \mathrm{Hom}\,(L(\Pi), \mathbb{R})$ defines a canonical quasi-embedding of X. This quasi-embedding however does not serve our purposes for the following two reasons. First, it not necessarily *balanceable* (see Definition 1.22) and hence we can not use the strong continuity properties of convex and concave functions on balanceable polyhedral complexes studied in [5]. Second, if the conical space arises from a geometric object (i.e. a toric variety) it does not necessarily contain enough information to recover the geometric intersection theory (see Example 3.34) .

The following is a useful property of quasi-embedded conical spaces that is not true in general for weakly embedded ones.

Lemma 1.12 *Let X be a quasi-embedded conical space. Then the map $\iota_X\colon X \to N^X$ is proper.*

Proof Let Π be a conical complex on X. Since X is a finite union of closed cones $\sigma \in \Pi$, it is enough to show that $\iota_X|_\sigma$ is proper. Since X is quasi-embedded, we have that the map $\iota_X|_\sigma \circ \phi^{\sigma -1}\colon N^\sigma \to N^X$ is an injective linear map, hence proper. By definition, the map $\phi^\sigma\colon \sigma \to N^\sigma$ is proper. Hence $\iota_X|_\sigma = \iota_X|_\sigma \circ \phi^{\sigma -1} \circ \phi^\sigma$ is proper. □

Definition 1.13 A *Euclidean conical space* is a quasi-embedded conical space X together with a Euclidean product on the real vector space N^X. Given any conical complex Π on X, this Euclidean structure induces compatible Euclidean structures in each vector space N^σ for $\sigma \in \Pi$.

Definition 1.14 A *morphism of weakly embedded conical spaces* consists of a morphism of conical spaces $f\colon X \to X'$ together with a morphism of finite-dimensional $\mathbb{R}$-vector spaces $f'\colon N^X \to N^{X'}$ forming a commutative square with the weak embeddings.

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle \iota_X}\downarrow & & \downarrow{\scriptstyle \iota_{X'}} \\ N^X & \xrightarrow{f'} & N^{X'} \end{array}$$

A *morphism of quasi-embedded or of Euclidean conical spaces* is a morphism of weakly embedded conical spaces.

1.2 The Euclidean tropical intersection product

Throughout this section X will denote a Euclidean conical space of pure dimension n with quasi-embedding given by $\iota_X \colon X \to N^X$. The goal of this section is to define the Euclidean tropical intersection product between Euclidean tropical cycles and piecewise linear functions on X. This is a Euclidean version of the tropical intersection product given in [21]. For the interested reader, the articles [4], [18] and [24] constitute a more thorough reference for tropical intersection theory on globally embedded conical polyhedral complexes with an integral structure.

We start with some definitions. These are the Euclidean adaptations of [21, Section 3.1].

Definition 1.15 Let $k \geqslant 1$ be an integer and let $\tau \in \Pi(k-1)$ be a cone. For every cone $\sigma \in \Pi(k)$ with $\tau \prec \sigma$ we define the *Euclidean normal vector* $\hat{v}_{\sigma/\tau}$ *of* σ *relative to* τ to be the unique unitary vector of N^σ that is orthogonal to N^τ and points in the direction of σ. By abuse of notation $\hat{v}_{\sigma/\tau}$ will also denote its image in N^X. If $k = 1$, we write $\hat{v}_\sigma := \hat{v}_{\sigma/\{0_\sigma\}}$.

Recall that, given a conical complex Π on X, we are denoting by Π both the complex and its set of cones.

Definition 1.16 A weight on Π is a map

$$c \colon \Pi \longrightarrow \mathbb{R}.$$

It is called a k-dimensional weight if $c(\sigma) = 0$ for all $\sigma \notin \Pi(k)$. A k-dimensional weight is called a k-*dimensional Euclidean weight* on Π if, for every cone $\tau \in \Pi(k-1)$, the Euclidean balancing condition

$$\sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} c(\sigma)\, \hat{v}_{\sigma/\tau} = 0 \tag{1.1}$$

holds true in N^X.

The set of weights on Π is a real graded vector space denoted by $W_*(\Pi)$. The k-dimensional Euclidean weights form an abelian group, which is denoted by $E_k(\Pi)$.

We can now define the pull-back of a Euclidean weight along a subdivision.

Definition 1.17 Let Π' be a subdivision of Π with its induced structure of Euclidean conical complex and denote by $f \colon \Pi' \to \Pi$ the corresponding morphism of Euclidean conical complexes. Let $c \in E_k(\Pi)$ be a Euclidean weight. Then the pull-back of c by f is the Euclidean weight

$$f^*(c)(\sigma') = \begin{cases} c(\sigma) & \text{if } \dim \sigma = \dim \sigma', \\ 0 & \text{otherwise,} \end{cases}$$

where $\sigma' \in \Pi'(k)$ and σ is the minimal cone of Π that contains σ'. This construction defines a group homomorphism

$$E_k(\Pi) \longrightarrow E_k(\Pi').$$

The fact that if c is a Euclidean weight then $f^*(c)$ is also a Euclidean weight can be argued as in [20, Example 2.11 (4)].

More generally, *Euclidean tropical cycles* on the Euclidean conical space X are defined as direct limits of Euclidean weights over all conical structures on X. Recall that $S(X)$ denotes the set of all conical complexes on X. It has the structure of a directed set ordered by inclusion.

Definition 1.18 The group of *Euclidean tropical* k*-cycles* on X is defined as the direct limit

$$EZ_k(X) := \varinjlim_{\Pi' \in S(X)} E_k(\Pi'),$$

with maps given by the pull-back maps of Definition 1.17. If c is a k-dimensional Euclidean weight on a Euclidean conical complex Π in $S(X)$, we denote by $[c]$ its image in $EZ_k(X)$.

Definition 1.19 The *degree* of a zero cycle $[c] \in Z_0(X)$ determined on Π, is defined as

$$\deg([c]) = \deg(c) = \sum_{v \in \Pi(0)} c(v).$$

Definition 1.20 A Euclidean weight is called *positive* if it has non-negative values. A Euclidean tropical cycle is called *positive* if it is represented by a positive Euclidean weight. The sub-semigroups of positive Euclidean weights on Π and of positive Euclidean tropical cycles of dimension k on X are denoted by $E_k^+(\Pi)$ and by $EZ_k^+(X)$, respectively.

Remark 1.21 It would seem natural to call positive tropical cycles *effective* in order to mimic the usual terminology for algebraic cycles. This would be however misleading since, as we will see later in Sect. 4.2, it is not true that the tropicalization of effective cycles are positive.

We now define the objects where we want to compute top intersection numbers, namely *balanced Euclidean conical spaces*.

Definition 1.22 The Euclidean conical space X is said to be *balanced* if it is provided with an n-dimensional Euclidean tropical cycle $[X] \in EZ_n(X)$ represented by a Euclidean weight $b \in E_n(\Pi)$, for some conical complex Π in $S(X)$, satisfying

$$b(\sigma) > 0, \ \forall \sigma \in \Pi(n).$$

In this case, we say that the conical complex Π is *balanced*.

Remark 1.23 If X is a balanced Euclidean conical space, then its is in particular a balanced Euclidean polyhedral space in the sense of [5, Definition 3.27]. Thus we have at our disposal the theory of concave functions on polyhedral spaces developed in that paper. See section 2.1 for a short recap.

We now define piecewise linear functions on the conical space X.

Definition 1.24 A *piecewise linear function* on X is a function $\phi\colon X \to \mathbb{R}$ for which there is a conical complex Π on X such that $\phi \in L(\Pi)$. That is, ϕ is linear on each cone of Π. In this situation, we say that ϕ is *defined* on Π. We denote by $PL(X)$ the real vector space of piecewise linear functions on X,

If ϕ is defined on Π, for each $\sigma \in \Pi$, we denote by ϕ_σ a linear function on N^X satisfying

$$\phi|_\sigma = \phi_\sigma \circ \iota_{X,\sigma}.$$

By abuse of notation ϕ_σ will also denote this linear function restricted to N^σ.

Remark 1.25 Let $\Pi' \geqslant \Pi$ in $S(X)$ and $f\colon \Pi' \to \Pi$ the corresponding morphism of conical complexes. There is a natural pull-back map $L(\Pi) \to L(\Pi')$ given by $\phi \mapsto f^*\phi := \phi \circ f$. The vector space of piecewise linear functions $PL(X)$ can be seen as the direct limit in the category of real vector spaces

$$PL(X) = \varinjlim_{\Pi \in S(X)} L(\Pi)$$

with respect to these pull-back morphisms.

A piecewise linear function on X is always continuous because the restrictions to the components of a finite closed covering are continuous. Moreover it is conical in the sense that, for all real $\lambda > 0$, $\phi(\lambda x) = \lambda\phi(x)$.

For any $\Pi \in S(X)$ and $k \geqslant 1$ we now construct a *Euclidean tropical intersection product*

$$L(\Pi) \times E_k(\Pi) \longrightarrow E_{k-1}(\Pi).$$

Definition 1.26 Let $\Pi \in S(X)$ be a Euclidean conical complex on X, ϕ a piecewise linear function defined on Π and $c \in E_k(\Pi)$ a k-dimensional Euclidean weight. Then the *Euclidean tropical intersection product* is the $(k-1)$-dimensional Euclidean weight $\phi \cdot c\colon E_{k-1}(\Pi) \to \mathbb{R}$ given by

$$\phi \cdot c(\tau) := \sum_{\substack{\sigma \in \Pi(k) \\ \sigma \succ \tau}} -\phi_\sigma\left(\hat{v}_{\sigma/\tau}\right) c(\sigma).$$

The fact that $\phi \cdot c$ is indeed a Euclidean weight is proven in [5, Proposition 3.19] adapting the standard proof in tropical geometry.

We now see that this intersection product extends to an intersection product between $PL(X)$ and Euclidean tropical cycles. To this end we see that the Euclidean tropical intersection product is compatible with the restriction to subdivisions.

Lemma 1.27 *Let $\Pi' \geqslant \Pi$ in $S(X)$ and $f\colon \Pi' \to \Pi$ the corresponding morphism of conical complexes. Let ϕ be a piecewise linear function defined on Π and $c \in E_k(\Pi)$ a Euclidean weight. Then*

$$f^*(\phi \cdot c) = f^*\phi \cdot f^*c.$$

Proof This is proved in [5, Proposition 3.12]. □

We can now define the intersection product between $PL(X)$ and Euclidean tropical cycles.

Definition 1.28 Let $\phi \in PL(X)$ be a piecewise linear function on X and let $[c] \in EZ_k(X)$ be a Euclidean tropical cycle. Let $\Pi \in S(X)$ be any conical complex on X such that ϕ is determined on Π and such that $[c]$ is represented by $c \in E_k(\Pi)$. Then the bilinear pairing

$$PL(X) \times EZ_k(X) \longrightarrow EZ_{k-1}(X).$$

given by

$$\phi \cdot [c] := [\phi \cdot c]$$

is well defined by Lemma 1.27. We call this pairing the *Euclidean tropical intersection product* as well.

The Euclidean tropical intersection product satisfies the following symmetry property.

Proposition 1.29 *Let ϕ_1 and ϕ_2 be two piecewise linear functions on X and c a Euclidean tropical cycle. Then*

$$\phi_1 \cdot (\phi_2 \cdot c) = \phi_2 \cdot (\phi_1 \cdot c).$$

Proof This is proved in [5, Proposition 3.15]. □

We define Euclidean tropical top intersection numbers of piecewise linear functions on X.

Definition 1.30 Assume that X is balanced with balancing condition $[X]$. Let $\phi_1, \dots, \phi_n \in \mathrm{PL}(X)$. The *Euclidean tropical top intersection number* $\langle \phi_1 \cdots \phi_n \rangle$ is defined by

$$\langle \phi_1 \cdots \phi_n \rangle := \deg\left(\phi_1 \cdots \phi_{n-1} \cdot (\phi_n \cdot [X])\right).$$

This defines a multilinear map

$$\underbrace{\mathrm{PL}(X) \times \cdots \times \mathrm{PL}(X)}_{n\text{-times}} \longrightarrow \mathbb{R}.$$

It is symmetric by Proposition 1.29.

Remark 1.31 In [21], the author works with weakly embedded conical complexes with an integral structure. As a consequence of working with a weakly embedded conical complex, only a tropical intersection product between tropical cycles and so called *combinatorially principal* piecewise linear functions (which are called *Cartier divisors*) can be defined. In our setting, we assume that the complex is quasi-embedded. It follows that every piecewise linear function is combinatorially principal, hence arbitrary products between piecewise linear functions and tropical cycles can be defined. As we will see later, the price to pay for this in the algebro-geometric setting of Section 3.5 is that we will have to modify the toroidal structure by adding more components at the boundary to be sure that the conical complex corresponding to a toroidal embedding is quasi-embedded. Moreover, in the study of Monge–Ampère measures it is more natural to replace the integral structure by a Euclidean structure.

1.3 Conical functions on Euclidean conical spaces

As before, X denotes a Euclidean conical space of pure dimension n with quasi-embedding given by $\iota_X : X \to N^X$.

Definition 1.32 Let Π be a conical complex on X. We denote by $1\text{-}\Pi$ the 1-skeleton of Π. That is,

$$1\text{-}\Pi = \bigcup_{\sigma \in \Pi(1)} \sigma.$$

A *conical function* ψ on Π is a function

$$\psi : 1\text{-}\Pi \longrightarrow \mathbb{R}.$$

satisfying $\psi(\lambda v) = \lambda \psi(v)$ for all $\lambda \geqslant 0$. The space of conical functions on Π is denoted by $\mathrm{Conic}(\Pi)$.

If $\Pi' \geqslant \Pi$ in $S(X)$, then there is an inclusion of 1-skeletons $j : 1\text{-}\Pi \subseteq 1\text{-}\Pi'$. Let $f : \Pi' \to \Pi$ be the corresponding morphism of conical complexes, and ψ' a conical function on Π', then the *push-forward* $f_*\psi'$ *of* ψ' *by* f is the conical function on Π given by restriction

$$f_*\psi' = \psi'|_{1\text{-}\Pi}.$$

In other words, $f_*\psi' = \psi' \circ j$, so f_* is just "to forget" the rays of Π' that are not in Π.

Remark 1.33 If ϕ is a piecewise linear function on Π, then to it one can associate a conical function which is also denoted by ϕ by restricting to the 1-skeleton.

The following result follows as in the classical case of fans.

Lemma 1.34 *Let $\Pi \in S(X)$ be a conical complex on X. If Π is simplicial then the map*

$$L(\Pi) \longrightarrow \mathrm{Conic}(\Pi)$$

is an isomorphism.

Proof This amounts to the well known fact that affine functions on a simplex are in bijective correspondence with tuples of values on the vertices of the simplex. □

In view of Lemma 1.34 we can define the push-forward map of piecewise linear functions on simplicial subdivisions.

Definition 1.35 Let $\Pi'' \geqslant \Pi' \in S(X)$ with Π' simplicial. Let $f\colon \Pi'' \to \Pi'$ be the corresponding morphism of conical complexes. Then the push-forward of piecewise linear functions $L(\Pi'') \to L(\Pi')$ is defined as the composition

$$L(\Pi'') \longrightarrow \mathrm{Conic}(\Pi'') \xrightarrow{f_*} \mathrm{Conic}(\Pi') \xrightarrow{\cong} L(\Pi').$$

More concretely, if ϕ is a piecewise linear function on Π'' then $f_*\phi$ is the unique piecewise linear function on Π' that agrees with ϕ in the 1-skeleton of Π'.

We now define the space of conical functions on X and see it as an inverse limit over all conical complex structures.

Definition 1.36 The space of conical functions on X is the space of all functions $\phi\colon X \to \mathbb{R}$ such that $\phi(\lambda x) = \lambda\phi(x)$ for all $x \in X$ and $\lambda \in \mathbb{R}_{\geqslant 0}$ with the topology of pointwise convergence.

Remark 1.37 The real vector spaces $L(\Pi)$ and $\mathrm{Conic}(\Pi)$ are finite dimensional. Hence they have a canonical topology. It is easy to verify that there are canonical identifications

$$\mathrm{Conic}(X) = \varprojlim_{\Pi \in S(X)} \mathrm{Conic}(\Pi) = \varprojlim_{\Pi \in S_{sp}(X)} L(\Pi), \tag{1.2}$$

where the limits are taken in the category of topological vector spaces with respect to the push-forward maps. The second identification follows from Lemma 1.34 and the fact that simplicial subdivisions are cofinal.

Given an element $\psi \in \mathrm{Conic}(X)$, using (1.2), we can write $\psi = (\psi_\Pi)_{\Pi \in S_{sp}(X)}$ for $\psi_\Pi \in L(\Pi)$.

We can extend the Euclidean tropical top intersection number of Definition 1.30 to the case where there is at most one conical function involved.

Lemma 1.38 *Let $z \in EZ_k(X)$ be a Euclidean tropical cycle of dimension k, $\phi_1, \ldots, \phi_{k-1} \in PL(X)$ piecewise linear functions on X and $\psi = (\psi_\Pi)_{\Pi \in S_{sp}(\Pi)} \in \mathrm{Conic}(X)$ a conical function. Choose a simplicial subdivision $\Pi \in S_{sp}(X)$ where z can be represented by a Euclidean weight c and such that all of the ϕ_i's are defined on Π. Then the product*

$$\psi_\Pi \cdot \phi_1 \cdots \phi_{k-1} \cdot c.$$

is independent of the choice of Π.

Proof In view of Lemma 1.27 we are reduced to prove the following projection formula. Let $\Pi' \geqslant \Pi$ be simplicial conical complexes on X and $f\colon \Pi' \to \Pi$ the corresponding morphism. Moreover, let $c_1 \in E_1(\Pi)$ be a Euclidean weight of dimension one on Π and ϕ a piecewise linear function on Π'. Then

$$f^*(f_*\phi \cdot c_1) = \phi \cdot f^*c_1. \tag{1.3}$$

Again by Lemma 1.27, we have that

$$f^*(f_*\phi \cdot c_1) = f^*f_*\phi \cdot f^*c_1. \tag{1.4}$$

The piecewise linear function $\phi - f^*f_*\phi$ satisfies

$$(\phi - f^*f_*\phi)|_\rho = 0,\ \forall \rho \in \Pi'(1),$$

while the Euclidean weight f^*c_1 satisfies

$$f^*c_1(\rho) = 0,\ \forall \rho \in \Pi'(1) \setminus \Pi(1).$$

From the explicit description of the product in Definition 1.26 we deduce

$$(\phi - f^*f_*\phi) \cdot f^*c_1 = 0. \tag{1.5}$$

Equations (1.5) and (1.4) imply (1.3), which proves the lemma. □

Definition 1.39 Let $z, \phi_1, \dots, \phi_{k-1}$ and $\psi = (\psi_\Pi)_{\Pi \in S_{sp}(X)}$, Π and c be as in Lemma 1.38. Then the *Euclidean top intersection number* of $z, \phi_1, \dots, \phi_{k-1}$ and ψ is defined by

$$\langle \psi \cdot \phi_1 \cdots \phi_{k-1} \cdot z \rangle := \deg(\psi_\Pi \cdot \phi_1 \cdots \phi_{k-1} \cdot c).$$

One of the main motivations of this article is to extend Definition 1.39 to certain cases where all the functions involved are (not necessarily piecewise linear) conical functions and not just one of them.

2 Monge–Ampère measures

Throughout this section X will denote an n-dimensional balanced Euclidean conical space with quasi-embedding given by $\iota_X\colon X \to N^X$ and balancing condition $[X]$.

The goal of this section is to prove that given $\mathcal{C}$, an admissible family of concave functions on X (Definition 2.6), for any $\mathcal{C}$-concave conical function ϕ on X (Definition 2.12), its top intersection number exists, is finite, and is given by the integral of ϕ with respect to a weak limit of discrete Monge–Ampère measures associated to the elements of the given admissible family (Corollary 2.26). This is done by introducing the notion of the *size* of a Euclidean tropical cycle. This allows us to prove a Chern–Levine–Nirenberg type inequality (Lemma 2.18) from which we conclude the weak convergence of the discrete measures (Theorem 2.24).

2.1 Concave functions on balanced conical spaces

For the convenience of the reader, we gather here some definitions and results form [5] but translated to conical spaces instead of polyhedral ones.

Definition 2.1 A piecewise linear function ϕ on X is called *strongly concave* if it is the restriction of a concave function on N^X. It is called *concave* if, for every positive Euclidean tropical cycle w on X, the product $\phi \cdot w$ is positive. It is called *weakly concave* if the product with the balancing condition $\phi \cdot [X]$ is positive.

By [5, Proposition 4.9], a strongly concave function is concave and a concave function is weakly concave.

There are also notions of strong concavity, concavity and weak concavity for arbitrary functions $f\colon X \to \mathbb{R}$ which are not necessarily piecewise linear ([5, Definition 5.5]). These are based on different notions of *convex conbinations of points* on polyhedral spaces ([5, Definition 5.1].

The main results we will use from [5] are the following:

Theorem 2.2 ([5, Theorem 6.2]) *Let ϕ be a (not necessarily piecewise linear) weakly concave function on X. Then ϕ is continuous.*

Theorem 2.3 ([5, Theorem 6.23]) *Let $(f_i)_{i\in\mathbb{N}}$ a sequence of (not necessarily piecewise linear) weakly concave functions on X such that there exists a dense subset $C \subseteq X$ and for every $x \in C$ the sequence $(f_i(x))_{i\in\mathbb{N}}$ has a finite limit. Then the sequence f_i converges pointwise everywhere to a function $f\colon X \to \mathbb{R}$. The function f is weakly concave, hence continuous, and the convergence is uniform on compacts.*

Remark 2.4 After Theorem 2.3, for the purpose of this paper one can think that the space of weakly concave functions is the closure with respect to uniform convergence on compacts of the space of piecewise linear weakly concave functions. Then Theorem 2.3 implies that, on the space of weakly concave functions, pointwise convergence in a dense subset is equivalent to uniform convergence on compacts.

2.2 $\mathcal{C}$-Concave functions

Let $|\cdot|$ be the Euclidean norm on N^X and let

$$\mathbb{S}^X := \left\{ v \in |\Pi| \,\middle|\, |\iota_\Pi(v)| = 1 \right\}.$$

The set $\mathbb{S}^X$ is compact since it is the inverse image of a compact space under a proper map by Lemma 1.12. Note that the Euclidean normal vectors $\hat{v}_{\sigma/\tau}$ from Definition 1.15 are elements in $\mathbb{S}^X$.

We can view the space of conical function on X as a space of functions on $\mathbb{S}^X$. In fact $\mathbb{S}^X$ inherits from X a structure of compact polyhedral space. Then $\mathrm{PL}(X)$ can be identified with the space of $\mathbb{R}$-valued piecewise linear functions on $\mathbb{S}^X$, while $\mathrm{Conic}(X)$ can be identified with the space of all $\mathbb{R}$-valued functions on $\mathbb{S}^X$. We will use freely these identifications.

We will denote by $C^0\left(\mathbb{S}^X\right)$ the space of continuous functions on $\mathbb{S}^X$ with the topology of uniform convergence.

Remark 2.5 By the lattice version of the Stone-Weierstrass Theorem, the subset $\mathrm{PL}(X) \subseteq C^0\left(\mathbb{S}^X\right)$ is dense ([23, Theorem 7.29]).

Definition 2.6 Let $\mathcal{C} \subseteq \mathrm{PL}(X)$ be a collection of piecewise linear functions on X. We say that $\mathcal{C}$ is an *admissible family of concave functions* on X if the following properties are satisfied:

(1) If $\phi_1, \ldots, \phi_r \in \mathcal{C}$, then the Euclidean tropical cycle $\phi_1 \cdots \phi_r \cdot [X]$ is positive, i.e. if it belongs to $\mathrm{EZ}^+_{n-r}(X)$.
(2) $\mathcal{C}$ is a convex cone.
(3) The set $\mathcal{C} - \mathcal{C}$ is dense in $C^0\left(\mathbb{S}^X\right)$.

If $\mathcal{C}$ is an admissible family of concave functions on X, an element $\phi \in \mathcal{C}$ will be called $\mathcal{C}$-concave.

Remark 2.7 By the first condition, every element ϕ of $\mathcal{C}$ is weakly concave in the sense of Definition 2.1.

Remark 2.8 In Definition 2.6 only conditions 1 and 3 are essential. In fact, if $\mathcal{C}$ is a set satisfying only 1 and 3 then the convex cone generated by $\mathcal{C}$ satisfies the three conditions.

Example 2.9 The collection of piecewise linear concave functions on X in the sense of [5, Definition 4.6] (see Definition 2.1) is an example of an admissible family of piecewise linear functions on X by [5, Remark 4.36].

Example 2.10 We will see in the next section that if $X = |\Pi_Y|$ comes from the geometry of a smooth and complete toroidal embedding $U \hookrightarrow Y$ satisfying certain mild hypothesis, then there is a canonical admissible family $\mathcal{C}$ of concave functions on X induced by the collection of nef toroidal divisors on smooth toroidal modifications of Y.

Remark 2.11 As we will see in Example 4.12, the families in Examples 2.9 and 2.10 may be different. In fact it is conceivable that there are two toroidal embeddings giving rise to isomorphic balanced conical spaces but such that the spaces of functions coming from nef divisors are different. This is the reason why it is useful to be able to choose an admissible family of concave functions $\mathcal{C}$ and to just identify the needed properties instead of choosing a particular family.

From now on we fix an admissible family $\mathcal{C}$ of concave functions on X.

Definition 2.12 The *space of $\mathcal{C}$-concave conical functions on* X, denoted by $\mathrm{Conic}(X)_{\mathcal{C}}$, is the closure of $\mathcal{C}$ in $\mathrm{Conic}(X)$ with respect to pointwise convergence.

The following is a key result. Before stating it, recall that for a subset A of a topological space T, the sequential closure $|A|_{\mathrm{seq}}$ of A is the set of all points that are limits of sequences in A. Then $|A|_{\mathrm{seq}} \subseteq \overline{A}$, its topological closure. The space T is called a *Fréchet-Urysohn* space if, for all $A \subseteq T$, the condition $|A|_{\mathrm{seq}} = \overline{A}$ holds. A Fréchet-Urysohn space is sequential, hence the topology of such spaces is determined by the convergent sequences.

Theorem 2.13 *The space* $\mathrm{Conic}(X)_{\mathcal{C}}$ *of $\mathcal{C}$-concave conical functions on* X *is contained in* $C^0(\mathbb{S}^X)$. *Moreover the topologies induced in this space by the one of* $\mathrm{Conic}(X)$ *and the one of* $C^0(\mathbb{S}^X)$ *agree. That is, in* $\mathrm{Conic}(X)_{\mathcal{C}}$ *the topology of pointwise convergence and that of uniform convergence are the same. In particular* $\mathrm{Conic}(X)_{\mathcal{C}}$ *is metrizable.*

Proof Let $(f_\alpha)_{\alpha \in I}$ be a net of piecewise linear functions in $\mathcal{C}$ that converge to a conical function f. Choose a countable dense collection of points $x_1, x_2, \ldots$ of X. Since the topology of the space of conical functions is that of pointwise convergence, for any $i > 0$ there is an α_i such that, for all $\alpha \geqslant \alpha_i$ and all $j \leqslant i$, the condition

$$|f_\alpha(x_j) - f(x_j)| < \frac{1}{i}$$

is satisfied. Hence the sequence $(f_{\alpha_i})_{i>0}$ converges to f in a dense subset of X. By Remark 2.7 the functions f_{α_i} are weakly concave. Therefore, by Theorems 2.3 and 2.2 , the sequence $(f_{\alpha_i})_{i>0}$ converges to a weakly concave (hence continuous) function g, that agrees with f on the points x_i, $i > 0$. Let now y be another point of X. Repeating the argument with the sequence of points $y, x_1, x_2, \ldots$, we obtain a new continuous function g_1, that agrees with f in the point y and agrees with g in a dense subset. Hence $g(y) = f(y)$. Since y is arbitrary, we deduce that $f = g$. Therefore f is weakly concave and is a continuous conical function. Moreover, f is the limit of the sequence $(f_{\alpha_i})_{i>0}$. We conclude that the space of $\mathcal{C}$-concave conical functions is Fréchet-Urysohn. Hence the topology is determined by the convergent sequences. Using again Theorem 2.3 a sequence in $\mathrm{Conic}(X)_{\mathcal{C}}$ converges if and only if it converges uniformly in S^X. This concludes the proof. □

2.3 The size of a Euclidean tropical cycle

We have the following monotonicity lemma which we will use later on.

Lemma 2.14 *Let ϕ_1, ϕ_2 be piecewise linear functions on X and $\Pi \in S(X)$ a conical complex where they are defined. Assume that $\phi_1(x) \geqslant \phi_2(x)$ for all $x \in X$. Then for all positive Euclidean weights $c \in E_1^+(\Pi)$ and every vertex v of Π, the inequality*

$$(\phi_1 \cdot c)(v) \leqslant (\phi_2 \cdot c)(v)$$

is satisfied.

Proof We have

$$(\phi_1 \cdot c)(v) = \sum_{\substack{\sigma \in \Pi(1) \\ v \prec \sigma}} -\phi_1(\hat{v}_\sigma) c(\sigma) \leqslant \sum_{\substack{\sigma \in \Pi(1) \\ v \prec \sigma}} -\phi_2(\hat{v}_\sigma) c(\sigma) = (\phi_2 \cdot c)(v),$$

as we wanted to show. □

To define the size of a positive Euclidean tropical cycle we choose an auxiliary function.

Definition 2.15 Let $\varphi_0 \colon N^X \to \mathbb{R}$ be a concave piecewise linear function satisfying

$$\varphi_0(v) \leqslant -\|v\|, \tag{2.1}$$

write $\varphi = \varphi_0 \circ \iota_X$ and let $z \in EZ_k^+(X)$ be a k-dimensional positive Euclidean tropical cycle. Then *the size of* z (*with respect to* φ) is defined as

$$|z|_\varphi := \deg((\varphi \cdot)^k z) \in \mathbb{R}.$$

Remark 2.16

(1) It is clear that such a function φ_0 exists. For instance to construct one we can choose an orthonormal basis of N^X, denote $u_1, \ldots, u_r$ the corresponding coordinates, and write

$$\varphi_0(u_1, \ldots, u_r) = 2r \min(0, u_1, \ldots, u_r) - (u_1 + \cdots + u_r).$$

(2) Since φ_0 is concave on N^X, the function φ is strongly concave (Definition 2.1). Therefore, by [5, Proposition 4.9] it is concave. Hence, for every k dimensional positive cycle z and $j \leqslant k$, the cycle $(\varphi \cdot)^j \cdot z$ is positive. In particular the size of z is positive.

Lemma 2.17 *(1) If $z \in \mathrm{EZ}_0^+(X) \simeq \mathbb{R}_{\geqslant 0}$ is 0-dimensional, then*

$$|z|_\varphi = \sum_{v \in \Pi(0)} c(v) \in \mathbb{R}_{\geqslant 0}, \tag{2.2}$$

where $\Pi \in S(X)$ is any conical complex where z is represented by a Euclidean weight c.

(2) Let $z \in \mathrm{EZ}_1^+(X)$ be a positive 1-dimensional Euclidean tropical cycle and let Π be as above. Then

$$|z|_\varphi \geqslant \sum_{\tau \in \Pi(1)} c(\tau). \tag{2.3}$$

Proof The first statement follows directly from the definition. Let $\widetilde{\Pi}$ be a subdivision of Π such that φ is piecewise linear on $\widetilde{\Pi}$. By the definition of φ, for every $\tau \in \widetilde{\Pi}(1)$, the inequality $\varphi(\hat{v}_\tau) \leqslant -1$ is satisfied. Therefore

$$|z|_\varphi = \deg(\varphi \cdot z) = \sum_{\tau \in \widetilde{\Pi}(1)} -\varphi(\hat{v}_\tau) c(\tau) \geqslant \sum_{\tau \in \widetilde{\Pi}(1)} c(\tau) = \sum_{\tau \in \Pi(1)} c(\tau).$$

□

The following estimate is a Chern–Levine–Nirenberg type inequality (see [15, Section 3], also [16, Theorem 2.2]). Both in complex and real pluripotential theory these kind of estimates play a key role when proving existence of Monge–Ampère measures.

Lemma 2.18 *Let $z \in \mathrm{EZ}_k^+(X)$ and $\phi \in \mathrm{PL}(X)$ be a k-dimensional positive Euclidean tropical cycle and a piecewise linear function, respectively. Assume that the Euclidean tropical intersection product $\phi \cdot z$ is a positive Euclidean tropical cycle. Then the inequality*

$$|\phi \cdot z|_\varphi \leqslant \left(\sup_{\hat{v} \in \mathbb{S}^X} |\phi(\hat{v})| \right) \cdot |z|_\varphi$$

is satisfied.

Proof Let $\widetilde{\Pi} \geqslant \Pi$ be a subdivision of Π such that φ is piecewise linear on $\widetilde{\Pi}$ and such that the cycle z and the function ϕ are defined in $\widetilde{\Pi}$. We define the positive real constant B by

$$B := \sup_{\tau \in \widetilde{\Pi}(1)} |\phi(\hat{v}_\tau)| \leqslant \sup_{\hat{v} \in \mathbb{S}^X} |\phi(\hat{v})|.$$

Then for every $\tau \in \widetilde{\Pi}(1)$ we have that

$$\phi(\hat{v}_\tau) \geqslant -B \geqslant B\varphi(\hat{v}_\tau).$$

Hence, since both ϕ and $B\varphi$ are piecewise linear on $\widetilde{\Pi}$, we conclude that

$$\phi \geqslant B\varphi.$$

Therefore, using Lemma 2.14, the positivity of $(\varphi \cdot)^{k-1} z$ (Remark 2.16 (2)), and the commutativity of the Euclidean tropical intersection product, we get

$$|\phi \cdot z|_\varphi = \deg\left((\varphi \cdot)^{k-1} \phi \cdot z\right) = \deg\left(\phi \cdot (\varphi \cdot)^{k-1} z\right) \leqslant B \deg\left((\varphi \cdot)^k z\right) = B|z|_\varphi,$$

as we wanted to show.

□

2.4 Weak convergence of Monge–Ampère measures

We recall the definition of the total variation norm (see e.g. [1, Definition 4.2.5 and Proposition 4.2.5]).

Definition 2.19 Let Y be a locally compact topological space and let $\mathcal{M}(Y)$ be the space of finite Radon measures on Y, i.e. the space of continuous linear forms on the space $C^0(Y)$ of continuous real-valued functions on Y with respect to its weak topology. The *total variation norm* $\|\cdot\|$ on $\mathcal{M}(Y)$ is given by

$$\|\mu\| := \sup\left\{\sum_{i=1}^{\infty}|\mu(A_i)| \;\middle|\; \{A_i\}_{i\geqslant 1}\subseteq Y \text{ measurable}, A_i\cap A_j=\emptyset, i\neq j, \bigcup_{i\geqslant 1}A_i = Y\right\}$$

for any $\mu\in\mathcal{M}(Y)$. In case that Y is compact, the total variation is just the norm of the measure as a continuous linear form on $C^0(Y)$.

In order to prove the main result of this section (Theorem 2.24), we use the following version of Prokhorov's theorem which can be derived from [13, Proposition 8.6.2].

Theorem 2.20 *Let Y be a compact metrized space and let $\mathcal{M}\subseteq\mathcal{M}(X)$ be a family of Radon measures. The following are equivalent.*

(1) Every sequence $\{\mu_n\}\subseteq\mathcal{M}$ contains a weakly convergent subsequence.
(2) The family $\mathcal{M}$ has bounded total variation.

Definition 2.21

(1) Let $z\in \mathrm{E}Z_1(X)$ be a 1-dimensional Euclidean tropical cycle and let $\Pi\in S(X)$ such that z is represented by a Euclidean weight c in $\mathrm{E}_1(\Pi)$. We define the discrete measure μ_z on $\mathbb{S}^X$ by

$$\mu_z := \sum_{\tau\in\Pi(1)} c(\tau)\cdot\delta_{\hat{v}_\tau},$$

where $\delta_{\hat{v}_\tau}$ denotes the Dirac delta measure supported on $\hat{v}_\tau\in\mathbb{S}^X$. (This does not depend on the choice of Π.)

(2) Let $\phi\in \mathrm{PL}(X)$. The discrete Monge–Ampère measure μ_ϕ on $\mathbb{S}^X$ is defined by

$$\mu_\phi := \mu_{\phi^{n-1}\cdot[X]}$$

The total variation of a discrete measure with finite support is given by the sum of the absolute value of the measures of the points in the support. Therefore, for z and Π as in Definition 2.21 we have

$$\|\mu_z\| = \sum_{\tau\in\Pi(1)}|c(\tau)|. \tag{2.4}$$

Remark 2.22 Although defined in a different setting, we note the similarity between the discrete measure μ_ϕ and the Monge–Ampère measure $\mathcal{M}(g)$ given in [8, Section 4.2], defined with respect to a piecewise-affine plurisubharmonic function g (see [8, Proposition 4.9]).

The following proposition is a consequence of Lemma 2.18.

Proposition 2.23 *Let ψ be a $\mathcal{C}$-concave conical function on X, and let $(\phi_j)_{j\in\mathbb{N}}$ be a sequence of $\mathcal{C}$-concave piecewise linear functions converging to ψ. Moreover, fix a collection $\gamma_1,\dots,\gamma_{n-1-k} \in \mathcal{C}-\mathcal{C}$ for $k \in \{0,\dots,n-1\}$. Then the set*

$$\left\{\mu_{\gamma_1\cdots\gamma_{n-1-k}\cdot\phi_j^k\cdot[X]}\right\}$$

of measures on $\mathbb{S}^X$ has bounded total variation.

Proof Since, by Theorem 2.13, the convergence

$$\lim_{j\in\mathbb{N}} \phi_j|_{\mathbb{S}X} = \psi|_{\mathbb{S}X}$$

is uniform and $\psi|_{\mathbb{S}X}$ is a continuous function on a compact set, there exists a positive real number B such that

$$\sup_{j\in\mathbb{N}} \sup_{x\in\mathbb{S}^X} \left|\phi_j(x)\right| \leqslant B.$$

By assumption, for each $\ell = 1,\dots,n-1$, there exist elements β_ℓ^0 and β_ℓ^1 in $\mathcal{C}$ such that $\gamma_\ell = \beta_\ell^0 - \beta_\ell^1$.

Since there are finitely many, we may choose a positive real number C such that

$$\sup_{\ell,i} \sup_{x\in\mathbb{S}^X} \left|\beta_\ell^i(x)\right| \leqslant C.$$

Since the involved piecewise linear functions are $\mathcal{C}$-concave, we have that for every $j \in \mathbb{N}$ and for every tuple $(i_1,\dots,i_{n-1-k}) \in \{0,1\}^{n-1-k}$, the 1-dimensional Euclidean tropical cycle

$$\beta_1^{i_1}\cdots\beta_{n-1-k}^{i_{n-1-k}}\cdot\left(\phi_j\right)^k\cdot[X]$$

is positive.

Fix $j \in \mathbb{N}$ and let $\Pi \in S(X)$ be a conical complex on X where the ϕ_j and the functions γ_i, $i = 1,\dots,n-1-k$, are defined. Then, using equation (2.4), Lemma 2.18 and the estimate (2.3), we get

$$\begin{aligned}
\|\mu_{\gamma_1\cdots\gamma_{n-1-k}\cdot\phi_j^k\cdot[X]}\| &= \sum_{\tau\in\Pi(1)} \left|\gamma_1\cdots\gamma_{n-1-k}\cdot\phi_j^k\cdot[X](\tau)\right| \\
&= \sum_{\tau\in\Pi(1)} \left|(\beta_1^0-\beta_1^1)\cdots(\beta_{n-1-k}^0-\beta_{n-1-k}^1)\cdot\phi_j^k\cdot[X](\tau)\right| \\
&\leqslant \sum_{\{0,1\}^{n-1-k}} \sum_{\tau\in\Pi(1)} \beta_1^{i_1}\cdots\beta_{n-1-k}^{i_{n-1-k}}\cdot\phi_j^k\cdot[X](\tau) \\
&\leqslant \sum_{\{0,1\}^{n-1-k}} \left|\beta_1^{i_1}\cdots\beta_{n-1-k}^{i_{n-1-k}}\cdot\phi_j^k\cdot[X]\right|_\varphi \\
&\leqslant 2^{n-1-k}\cdot C^{n-1-k}\cdot B^k\cdot|[X]|_\varphi,
\end{aligned}$$

proving the proposition. □

The following is the main result of this section. The proof is inspired in the classical proof of the existence of Monge–Ampère measures of [32, Proposition 3.1].

Theorem 2.24 *Let ψ be a $\mathcal{C}$-concave conical function on X, $k \in \{0, \dots, n-1\}$ and $\gamma_1, \dots, \gamma_{n-1-k} \in \mathcal{C} - \mathcal{C}$. We view ψ as a function on $\mathbb{S}^X$. Then the following holds true.*

(1) Let $(\phi_i)_{i\in\mathbb{N}}$ and $(\phi'_j)_{j\in\mathbb{N}}$ be sequences of $\mathcal{C}$-concave piecewise linear functions both converging to ψ. Assume that

$$\lim_{i\in\mathbb{N}} \mu_{\gamma_1\cdots\gamma_{n-1-k}\cdot\phi_i^k\cdot[X]} = \mu, \qquad \lim_{j\in\mathbb{N}} \mu_{\gamma_1\cdots\gamma_{n-1-k}\cdot\phi_j'^k\cdot[X]} = \nu,$$

for some Radon measures μ and ν (with respect to the weak-$*$ topology). Then

$$\mu = \nu.$$

(2) The map from $\mathcal{C}$ to Radon measures on $\mathbb{S}^X$ given by

$$\phi \longmapsto \mu_{\gamma_1\cdots\gamma_{n-1-k}\cdot\phi^k\cdot[X]} \tag{2.5}$$

extends to a continuous operator from $\mathrm{Conic}(X)_{\mathcal{C}}$ to Radon measures on $\mathbb{S}^X$. This operator is also denoted as in (2.5).

Proof The fact that statement (1) implies statement (2) is a standard consequence of Theorem 2.13, Proposition 2.23 and Theorem 2.20.

We prove the theorem by induction on k. If $k = 0$ there is nothing to prove. So we can assume that both statements of the Theorem are true for $k-1$. By part (3) of Definition 2.6, in order to prove that $\mu = \nu$, it is enough to prove that $\mu(\eta) = \nu(\eta)$ for $\eta \in \mathcal{C} - \mathcal{C}$.

By Proposition 1.29 we have that

$$\begin{aligned}\mu(\eta) &= \lim_{i\in\mathbb{N}} \deg(\eta\cdot\gamma_1\cdots\gamma_{n-1-k}\cdot\phi_i^k\cdot[X]) \\ &= \lim_{i\in\mathbb{N}} \deg(\phi_i\cdot\eta\cdot\gamma_1\cdots\gamma_{n-1-k}\cdot\phi_i^{k-1}\cdot[X]) \\ &= \lim_{i\in\mathbb{N}} \mu_{\eta\cdot\gamma_1\cdots\gamma_{n-1-k}\cdot\phi_i^{k-1}\cdot[\Pi]}(\phi_i).\end{aligned}$$

By induction hypothesis, the sequence of measures $\mu_{\eta\cdot\gamma_1\cdots\gamma_{n-1-k}\cdot\phi_i^{k-1}\cdot[X]}$, $i \in \mathbb{N}$, converges to the measure $\mu_{\eta\cdot\gamma_1\cdots\gamma_{n-1-k}\cdot\psi^{k-1}\cdot[X]}$. Moreover, by Theorem 2.13 the sequence of functions $\phi_j, j \in \mathbb{N}$, converge uniformly to the continuous function ψ. Therefore, the double limit

$$\lim_{(i,j)\in\mathbb{N}\times\mathbb{N}} \mu_{\eta\cdot\gamma_1\cdots\gamma_{n-1-k}\cdot\phi_i^{k-1}\cdot[X]}(\phi_j)$$

exists and agrees with the diagonal limit $i = j$. Therefore

$$\mu(\eta) = \mu_{\eta\cdot\gamma_1\cdots\gamma_{n-1-k}\cdot\psi^{k-1}\cdot[X]}(\psi).$$

Similarly,

$$\nu(\eta) = \mu_{\eta\cdot\gamma_1\cdots\gamma_{n-1-k}\cdot\psi^{k-1}\cdot[X]}(\psi).$$

Hence, we get that $\mu(\eta) = \nu(\eta)$. This concludes the proof of the theorem. □

Definition 2.25 Let ψ be a $\mathcal{C}$-concave conical function on X. The associated *Monge–Ampère measure* is defined by

$$\mu_\psi := \mu_{\psi^{n-1}\cdot[X]}.$$

We obtain the following corollary.

Corollary 2.26 *Let ψ be a $\mathcal{C}$-concave conical function on X and let $(\phi_i)_{i\in\mathbb{N}}$ be a sequence of $\mathcal{C}$-concave piecewise linear functions on X converging to ψ. Then the limit*

$$\deg(\psi) := \lim_{i\in\mathbb{N}} \deg\left(\phi_i^n \cdot [X]\right)$$

exists, is finite, and is given by

$$\deg(\psi) = \int_{\mathbb{S}^X} \psi(u)\, d\mu_\psi.$$

It is called the degree *of the $\mathcal{C}$-concave conical function ψ.*

Proposition/Definition 2.27 *There is a symmetric map from the space of $(n-1)$-tuples of $\mathcal{C}$-concave conical functions on X to the space of finite measures on $\mathbb{S}^X$, called the* mixed Monge–Ampère measure*, and denoted by*

$$(\psi_{i_1}, \dots, \psi_{i_{n-1}}) \longmapsto \mu_{\psi_{i_1},\dots,\psi_{i_{n-1}}},$$

such that for every natural number ℓ and for every choice of non-negative real numbers $\lambda_1, \dots, \lambda_\ell$, the equality

$$\mu_{\lambda_1\psi_1+\dots+\lambda_\ell\psi_\ell} = \sum_{i_1,\dots,i_{n-1}=1}^{\ell} \lambda_{i_1} \dots \lambda_{i_\ell} \mu_{\psi_{i_1},\dots,\psi_{i_{n-1}}}$$

is satisfied for every collection $\psi_1, \dots, \psi_\ell$ of $\mathcal{C}$-concave conical functions on X.

Proof The argument is the same as the one given in the proof of [33, Theorem 5.17]. □

The following corollary follows from the definition of the mixed Monge–Ampère measure and Corollary 2.26.

Corollary 2.28 *Let $\psi_1, \dots, \psi_n$ be a collection of $\mathcal{C}$-concave conical functions on X, and let $(\phi_{i,j})_{j\in\mathbb{N}}$, $i = 1, \dots, n$ be sequences of $\mathcal{C}$-concave piecewise linear functions converging respectively to ψ_i. Then the limit*

$$\deg\left(\psi_1 \cdots \psi_n\right) := \lim_j \deg\left(\phi_{1,j} \cdots \phi_{n,j} \cdot [X]\right)$$

exists, is finite and is given by

$$\deg\left(\psi_1 \cdots \psi_n\right) = \int_{\mathbb{S}^X} \psi_1(u)\, d\mu_{\psi_2,\dots,\psi_n}.$$

Moreover, for any $1 \leqslant i \leqslant n$, we have integral formulae

$$\int_{\mathbb{S}^X} \psi_1(u)\, d\mu_{\psi_2,\dots,\psi_n} = \int_{\mathbb{S}^X} \psi_i(u)\, d\mu_{\psi_1,\dots,\widehat{\psi}_i,\dots,\psi_n}.$$

It is called the mixed degree *of the $\mathcal{C}$-concave conical functions $\psi_1, \dots, \psi_n$.*

Remark 2.29 By multilinearity, we can extend the definition of Monge–Ampère measures and degrees to functions of the space $\mathrm{Conic}(X)_{\mathcal{C}} - \mathrm{Conic}(X)_{\mathcal{C}}$. Then the corollaries 2.26 and 2.28 extend to this setting.

3 Toroidal embeddings and rational conical polyhedral spaces

In this section, we define quasi-embedded *rational* conical polyhedral spaces. In short, these are conical polyhedral spaces endowed with a lattice structure together with a quasi embedding which is compatible with the lattice structire. Following [21], there is a *rational tropical intersection product* on quasi-embedded rational conical spaces. We compare the rational tropical intersection with the Euclidean one from Section 1 by means of the *normalization* of cycles. We further show that the total mass of the Monge–Ampère measures from Section 2.4 are independent of the Euclidean metric and only depend on the integral structure and on the choice of a smooth subdivision (Corollary 3.17). We then recall the definition of a toroidal embedding and describe a natural rational conical polyhedral space associated to it (see [27] or [3] for further details). We describe the proper toroidal birational modifications of a toroidal embedding which, on the combinatorial side, correspond to subdivisions of rational conical complexes on this rational conical space. Finally, following [21], we give a natural weak embedding of this space and we show that by adding boundary components one can modify the toroidal structure of a toroidal embedding in such a way that this natural weak embedding becomes a quasi embedding.

3.1 Rational conical polyhedral spaces

Definition 3.1 Let X be a topological space. A *rational conical polyhedral structure* on X is a pair

$$\Pi = (\{\sigma^\alpha\}_{\alpha\in\Lambda}, \{M^\alpha\}_{\alpha\in\Lambda})$$

consisting of a finite covering by closed subsets $\sigma^\alpha \subseteq X$ and for each σ^α, a finitely generated $\mathbb{Z}$-module M^α of continuous, $\mathbb{R}$-valued functions on σ^α satisfying the following conditions. Let $N^\alpha := \mathrm{Hom}(M^\alpha, \mathbb{R})$ denote the dual lattice.

(1) For each $\alpha \in \Lambda$, the evaluation map $\phi^\alpha \colon \sigma^\alpha \to N^\alpha$ given by the assignment

$$v \longmapsto (u \mapsto u(v)) \quad (u \in M^\alpha),$$

maps σ^α homeomorphically to a strictly convex, full-dimensional, rational polyhedral cone in $N^\alpha_{\mathbb{R}}$. We call the sets σ^α *cones*.

(2) The preimage under ϕ^α of each face of $\phi^\alpha(\sigma^\alpha)$ is a cone $\sigma^{\alpha'}$ for some index $\alpha' \in \Lambda$, and we have that $M^{\alpha'} = \left\{u|_{\sigma^{\alpha'}} \,\middle|\, u \in M^\alpha\right\}$.

(3) The intersection of two cones is a union of common faces.

The $\mathbb{Z}$ modules M^α give X a so called *integral structure*.

A subdivision of a rational conical polyhedral structure is defined as in Section 1 but with the condition that it has to be rational as well. And we say that two rational conical polyhedral structures are *equivalent* if they admit a common subdivision.

Definition 3.2 A *rational conical polyhedral space* X is a topological space equipped with an equivalence class of rational conical polyhedral structures. A *rational conical polyhedral complex on* X is the choice of a representative of the class of rational conical polyhedral structures on X.

Most of the notations and terminology of Section 1 carry over to the case of rational conical polyhedral complexes, by taking into account the integral structure.

(1) Rational conical polyhedral complexes and rational conical polyhedral spaces will be referred as *rational conical complexes* and *rational conical spaces*, respectively.
(2) Given a rational conical space X, the set of all *rational* conical complexes on X ordered by inclusion is denoted by $R(X)$. This has the structure of a directed set.
(3) As in Remark 1.7, if X is a rational conical space, then the set of cones of dimension zero in any rational conical complex on X is in bijection with the set of connected components of X.
(4) The terminology concerning cones, faces, interior, support and dimension is the same as in the non-rational case keeping in mind the compatibility between the integral structures.
(5) The notion of a simplicial rational conical complex is the same. However, in the rational case we also have a notion of smoothness. A rational conical complex is called *smooth* if every cone $\sigma \in \Pi$ is unimodular, i.e. if $\phi^{\sigma}(\sigma)$ is generated by a $\mathbb{Z}$-basis of N^{σ}. Clearly, a smooth rational conical complex is automatically simplicial. We denote the set of simplicial and smooth complexes on a rational conical space X with their directed set structures by $R_{sp}(X)$ and $R_{sm}(X)$, respectively.
(6) The notion of a morphism between rational conical spaces is the same except that we require the restriction to each cone to be integral.
(7) The notions of weakly-embedded and quasi-embedded rational conical spaces are the same except that the co-domain of the weak- (respectively quasi-) embedding is an $\mathbb{R}$-vector space $N^X_{\mathbb{R}}$ with an integral structure N^X and the restriction of the weak (respectively quasi-) embedding to each cone is required to be integral.

3.2 A bridge between Euclidean and integral structures

Following [21], there is a *rational tropical intersection product* on quasi-embedded rational conical spaces. We compare the rational tropical intersection with the Euclidean one from Section 1 by means of the *normalization* of cycles.

The following definitions are adapted from [21, Section 3.1] and are small modifications of standard concepts in tropical geometry. See for instance the articles [4], [18] and [24].

Definition 3.3 Let X be weakly-embedded rational conical space with weak-embedding given by $\iota_X : X \to N^X_{\mathbb{R}}$. Let Π be a rational conical complex on X. Let $k \geqslant 0$ be an integer and let $\tau \in \Pi(k-1)$ be a cone. For every cone $\sigma \in \Pi(k)$ with $\tau \prec \sigma$. We define the *lattice normal vector* $v_{\sigma/\tau}$ *of* σ *relative to* τ to be the image in the quotient $N^X_{\mathbb{R}}/N^{\tau}_{\mathbb{R}}$ of the unique generator of N^{σ}/N^{τ} that points in the direction of σ. For every pair of cones σ and τ as before we will chose a lifting $\tilde{v}_{\sigma/\tau} \in N^{\sigma}_{\mathbb{R}}$ of $v_{\sigma/\tau}$. If $k = 1$, we write $v_{\sigma} := v_{\sigma/\{0_\sigma\}} = \tilde{v}_{\sigma/\{0_\sigma\}}$.

Definition 3.4 Let X and Π be as in Definition 3.3. A k-dimensional weight on Π is called a k-*dimensional Minkowski weight* on Π if, for every cone $\tau \in \Pi(k-1)$, the relation

$$\sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} c(\sigma)\, v_{\sigma/\tau} = 0 \tag{3.1}$$

holds true in $N^X_{\mathbb{R}}/N^{\tau}_{\mathbb{R}}$. Equivalently, c satisfies the relation

$$\sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} c(\sigma)\, \tilde{v}_{\sigma/\tau} \in N^{\tau}_{\mathbb{R}}. \tag{3.2}$$

The k-dimensional Minkowski weights on Π form a real vector subspace, which is denoted by $M_k(\Pi)$.

The condition (3.1) is called the *(lattice) balancing condition* around τ, while the condition (1.1) is called the *Euclidean balancing condition*.

Remark 3.5 Note that the balancing condition (3.1) is defined for any weakly-embedded (not necessarily quasi-embedded) rational conical space.

The following notions carry over from the Euclidean to the lattice case directly. Consider X a weakly-embedded rational conical space.

(1) The definition of the pull-back along a subdivision is the same.
(2) The definition of the group of *(lattice) tropical cycles* is analogous. This group is denoted by $Z_k(X)$.
(3) The definition of balanced rational conical space is the same as in the Euclidean case only that in the lattice case, we just ask X to be weakly-embedded and not necessarily quasi-embedded.
(4) The definition of the space $\mathrm{PL}(X)$ of piecewise linear functions on X is the same. We must have in mind that now we only allow rational subdivisions although we are working with real coefficients.
(5) Since X has a rational structure, we can define $X(\mathbb{Q})$ as the union of the subsets of rational points on each rational cone.
(6) The space of conical functions $\mathrm{Conic}(X)$ is defined as the space of functions f on $X(\mathbb{Q}) = X \cap \iota_X^{-1}(N_{\mathbb{Q}}^X)$ with real values, satisfying

$$f(\lambda x) = \lambda f(x), \qquad \lambda \in \mathbb{Q}_{\geqslant 0},$$

with the topology of pointwise convergence. Then

$$\mathrm{Conic}(X) = \varprojlim_{\Pi \in R_{\mathrm{sm}}(X)} \mathrm{PL}(\Pi) = \varprojlim_{\Pi \in R_{\mathrm{sp}}(X)} \mathrm{PL}(\Pi).$$

A difference with the Euclidean case is that now the limit is taken over a countable set, so every convergent net of conic functions has a converging subsequence.

Remark 3.6 Assume now that X is quasi-embedded. Let $\widehat{X}$ be the Euclidean conical space obtained by choosing a metric on $N_{\mathbb{R}}^X$ and forgetting the integral structure. Since we allow only rational conical complexes Π on X, the set $S(\widehat{X})$ is much bigger than $R(X)$ and hence the spaces of functions are different. Nevertheless, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{PL}(X) & \longrightarrow & \mathrm{Conic}(X) \\ \downarrow & & \uparrow \\ \mathrm{PL}(\widehat{X}) & \longrightarrow & \mathrm{Conic}(\widehat{X}) \end{array} \tag{3.3}$$

The space $\mathrm{PL}(\widehat{X})$ is the space of all piecewise linear functions on X, while $\mathrm{PL}(X)$ is the space of piecewise linear functions whose linearity locus is defined over $\mathbb{Q}$. The space $\mathrm{Conic}(\widehat{X})$ is the space of conical functions on X, while the space $\mathrm{Conic}(X)$ is the space of real valued conical functions on $X(\mathbb{Q})$. The arrows in diagram (3.3) are the obvious ones. In particular the upward arrow on the right of the diagram sends a conical function on X to its restriction to $X(\mathbb{Q})$.

The definition of the intersection product in the lattice case is different from the Euclidean case, because of the difference between Euclidean weights and Minkowski weights. Note

also that even though Minkowski weights and normal vectors are defined already for weakly embedded rational conical spaces, in order to define an intersection product in general, we must ask the space to be quasi-embedded (see Remark 1.31). To avoid confusion between the Euclidean and lattice intersection product, we will use a different symbol.

Definition 3.7 Let X be a quasi-embedded rational conical space. Let $\Pi \in R(X)$ a rational conical complex on X. Let $\phi \in L(\Pi)$ be a piecewise linear function and $c \in M_k(\Pi)$ a Minkowski weight. Then the *(lattice) tropical intersection product* $\phi \odot c \in M_{k-1}(\Pi)$ is the Minkowski weight given, for $\tau \in \Pi(k-1)$, by

$$(\phi \odot c)(\tau) := \sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} -\phi_\sigma\left(\tilde{v}_{\sigma/\tau}\right) c(\sigma) + \phi_\tau\left(\sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} c(\sigma)\tilde{v}_{\sigma/\tau}\right).$$

Note that this is well defined since $c \in M_k(\Pi)$ is a k-dimensional Minkowski weight and hence

$$\sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} c(\sigma)\tilde{v}_{\sigma/\tau} \in N^\tau_{\mathbb{R}}.$$

Moreover, if $\tilde{v}'_{\sigma/\tau}$ is another choice of liftings, then $w_{\sigma/\tau} := \tilde{v}_{\sigma/\tau} - \tilde{v}'_{\sigma/\tau} \in N^\tau_{\mathbb{R}}$ and therefore

$$\sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} -\phi_\sigma\left(w_{\sigma/\tau}\right) c(\sigma) + \phi_\tau\left(\sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} c(\sigma) w_{\sigma/\tau}\right) = 0,$$

so the intersection product is independent of the choice of liftings.

Lattice tropical cycles will be called just tropical cycles and lattice tropical intersection will be called tropical intersection.

As in the Euclidean case, the tropical intersection product extends to a bilinear pairing between piecewise linear functions and tropical cycles.

Definition 3.8 Let X be a quasi-embedded rational conical space. Let $z \in Z_k(X)$ be a k-dimensional tropical cycle and let $\phi \in PL(X)$. Let $\Pi \in R(X)$ be such that z is represented by a k-dimensional Minkowski weight $c \in M_k(\Pi)$ and such that ϕ is defined on Π. Then the tropical intersection product $\phi \odot z \in Z_{k-1}(\Pi)$ given by

$$\phi \odot z := [\phi \odot c]$$

is well defined.

Remark 3.9 If X is a balanced quasi-embedded rational conical space, then the definition of the *tropical top intersection numbers* is the analogue of the Euclidean case (Definition 1.30) but using the (lattice) tropical product.

We are now ready to relate Euclidean and lattice structures. As before, we let X be a quasi-embedded rational conical space and we denote by $\widehat{X}$ the Euclidean conical space induced by X by forgetting the rational structure and by choosing a Euclidean metric $\langle\ ,\ \rangle$ on $N^X_{\mathbb{R}}$. Given $\Pi \in R(X)$ we denote by $\widehat{\Pi}$ the induced Euclidean conical complex on $\widehat{X}$. Note that if Π is smooth, then $\widehat{\Pi}$ is simplicial.

We introduce the following notation. For $\Pi \in \mathsf{R}(X)$ and for a cone $\sigma \in \Pi$ we let

$$\mathrm{vol}(\sigma) := \mathrm{vol}_{\langle\,,\,\rangle}(\mathsf{N}^\sigma_{\mathbb{R}}/\mathsf{N}^\sigma) = \sqrt{\det\left(\langle v_i, v_j\rangle\right)_{i,j}}$$

where $\{v_1, \ldots, v_k\}$ is an integral basis of N^σ. Note that $\mathrm{vol}(\sigma)$ depends on both, the rational structure and the Euclidean one.

Recall that $W_*(\Pi)$ denotes the space of weights of Π. We define a map $\widehat{\ }\colon W_*(\Pi) \to W_*(\Pi)$ given by

$$\widehat{c}(\sigma) := \mathrm{vol}(\sigma)c(\sigma),$$

Lemma 3.10 *WIth notation as above, if* $c \in \mathsf{M}_k(\Pi)$ *is a* k*-dimensional Minkowski weight on* Π *then* $\widehat{c}$ *is a Euclidean weight* $\widehat{c} \in \mathsf{E}_k(\widehat{\Pi})$.

Proof Let $\tau \in \Pi(k-1)$. We have to show that $\widehat{c}$ is a Euclidean weight. For any $\sigma \in \Pi(k)$ containing τ let $\tilde{v}_{\sigma/\tau} \in \mathsf{N}^\sigma_{\mathbb{R}}$ be a lifting of the lattice normal vector as in Definition 3.3 and let

$$\tilde{v}_{\sigma/\tau} = v_{\sigma,\tau} + v_{\sigma,\tau^\perp}$$

be an orthogonal decomposition of $\tilde{v}_{\sigma/\tau}$ with $v_{\sigma,\tau} \in \mathsf{N}^\tau_{\mathbb{R}}$ and $v_{\sigma,\tau^\perp}$ orthogonal to $\mathsf{N}^\tau_{\mathbb{R}}$. The Euclidean normal vector $\hat{v}_{\sigma/\tau}$ of Definition 1.15 is just the normalization of $v_{\sigma,\tau^\perp}$, i.e. we have $\hat{v}_{\sigma/\tau} = v_{\sigma,\tau^\perp}/\|v_{\sigma,\tau^\perp}\|$. If $\{v_1, \ldots, v_{k-1}\}$ is an integral basis of N^τ, then $\{v_1, \ldots, v_{k-1}, \tilde{v}_{\sigma/\tau}\}$ is a basis on N^σ. Therefore,

$$\|v_{\sigma,\tau^\perp}\| = \frac{\mathrm{vol}(\sigma)}{\mathrm{vol}(\tau)}. \tag{3.4}$$

We compute

$$\begin{aligned}
\sum_{\substack{\sigma\in\Pi(k)\\ \tau\prec\sigma}} \widehat{c}\,(\sigma)\,\hat{v}_{\sigma/\tau} &= \sum_{\substack{\sigma\in\Pi(k)\\ \tau\prec\sigma}} c\,(\sigma)\,\mathrm{vol}(\sigma)\hat{v}_{\sigma/\tau} \\
&= \mathrm{vol}(\tau) \sum_{\substack{\sigma\in\Pi(k)\\ \tau\prec\sigma}} c\,(\sigma)\,\tilde{v}_{\sigma,\tau^\perp} \\
&= \mathrm{vol}(\tau) \sum_{\substack{\sigma\in\Pi(k)\\ \tau\prec\sigma}} c\,(\sigma)\left(\tilde{v}_{\sigma/\tau} - v_{\sigma,\tau}\right) = 0,
\end{aligned}$$

In the last equation we have used that, since c is a Minkowski weight then $\sum c(\sigma)\tilde{v}_{\sigma/\tau}$ belongs to N^τ, hence agrees with its orthogonal projection to $\mathsf{N}^\tau_{\mathbb{R}}$ which is $\sum c(\sigma)v_{\sigma,\tau}$. We deduce that $\widehat{c}$ is a Euclidean weight. □

Definition 3.11 Let X be a quasi-embedded rational conical space. Let $\Pi \in \mathsf{R}(X)$ and let $c \in \mathsf{M}_k(\Pi)_{\mathbb{R}}$ be a k-dimensional Minkowski weight. Then the Euclidean weight $\widehat{c} \in \mathsf{E}_k(\widehat{\Pi})$ is called the *normalization* of c. The normalization $\widehat{z}$ of a tropical cycle $z \in \mathsf{Z}_k(X)$ is defined to be the class $[\widehat{c}] \in \mathsf{EZ}_k(\widehat{X})$ of the normalization of any representative Minkowski weight $c \in \mathsf{M}_k(\Pi)$ of z.

Remark 3.12 If $c \in \mathsf{M}_0(\Pi)$ is 0-dimensional, then $\widehat{c} = c$.

The following proposition shows the compatibility between the tropical intersection product and the Euclidean one, allowing us to replace the integral structure by the Euclidean one in computations.

Proposition 3.13 *Let X be a quasi-embedded rational conical space and let $\Pi \in R(X)$ be a rational conical complex on X. Let $\phi \in PL(X)$ be a piecewise linear function defined on Π and let $c \in M_k(\Pi)$. Then*

$$\widehat{\phi \odot c} = \phi \cdot \widehat{c}.$$

Hence, also for a k-dimensional Minkowski cycle $z \in Z_k(X)$ we have

$$\widehat{\phi \odot z} = \phi \cdot \widehat{z}.$$

Proof Let $\tau \in \Pi(k-1)$. We use the same notation as in the proof of Lemma 3.10. Since c is a Minkowski weight, we have that

$$\sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} c(\sigma)\tilde{v}_{\sigma/\tau} = \sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} c(\sigma)v_{\sigma,\tau}. \tag{3.5}$$

We compute, using equation (3.4),

$$\begin{aligned}
\widehat{\phi \odot c}(\tau) &= \left(\sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} -\phi_\sigma\left(\tilde{v}_{\sigma/\tau}\right) c(\sigma) + \phi_\tau \left(\sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} c(\sigma)\tilde{v}_{\sigma/\tau} \right) \right) \cdot \mathrm{vol}(\tau) \\
&= \left(\sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} -\phi_\sigma\left(\tilde{v}_{\sigma/\tau}\right) c(\sigma) + \phi_\tau \left(\sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} c(\sigma)v_{\sigma,\tau} \right) \right) \cdot \mathrm{vol}(\tau) \\
&= \left(\sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} -c(\sigma)\phi_\sigma\left(v_{\sigma,\tau^\perp}\right) \right) \cdot \mathrm{vol}(\tau) \\
&= \sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} -c(\sigma)\mathrm{vol}(\sigma)\phi_\sigma\left(v_{\sigma,\tau^\perp}\right) \cdot \|v_{\sigma,\tau^\perp}\|^{-1} \\
&= \sum_{\substack{\sigma \in \Pi(k) \\ \tau \prec \sigma}} -\widehat{c}(\sigma)\phi_\sigma\left(\hat{v}_{\sigma/\tau}\right) \\
&= \phi \cdot \widehat{c}(\tau),
\end{aligned}$$

hence the first statement of the proposition follows. The second statement clearly follows from the first. □

Let Π be a quasi-embedded smooth rational complex and let $\widehat{\Pi}$ be the Euclidean one obtained by choosing a Euclidean metric on $N^{|\Pi|}$. Given an admissible family $\mathcal{C}$ of concave functions on $|\widehat{\Pi}|$ and a $\mathcal{C}$-concave conical function ψ on $|\widehat{\Pi}|$ we show that the total mass of its associated Monge–Ampère measure μ_ψ from Definition 2.25 is independent of the Euclidean metric and only depends on the integral structure and on the choice of the smooth subdivision.

Definition 3.14 For each cone $\sigma \in \Pi$ let Δ_σ be the simplex spanned by the lattice generators of the one-dimensional faces of σ. Then the *lattice unit sphere* $\mathbb{S}^\Pi$ is defined as

$$\mathbb{S}^\Pi = \bigcup_{\sigma \in \Pi} \Delta_\sigma \subseteq |\Pi|.$$

There is a homeomorphism $h_{\Pi,\widehat{\Pi}}\colon \mathbb{S}^{\Pi} \to \mathbb{S}^{|\widehat{\Pi}|}$ given by the radial projection and a function $n_{\Pi}\colon \mathbb{S}^{\Pi} \to \mathbb{R}_{>0}$ given by $n_{\Pi}(x) = \|\iota(x)\|$, so that

$$h_{\Pi,\widehat{\Pi}}(x) = \frac{1}{n_{\Pi}(x)}x.$$

Given a measure μ on $\mathbb{S}^{|\widehat{\Pi}|}$ we define the measure $\eta_{\Pi}(\mu)$ on $\mathbb{S}^{\Pi}$ by

$$d\eta_{\Pi}(\mu) = \frac{1}{n_{\Pi}}\, d(h^{-1}_{\Pi,\widehat{\Pi}})_*(\mu).$$

The notation will become clear shortly as we will see that $\eta_{\Pi}(\mu)$ is independent of the choice of the Euclidean structure.

Lemma 3.15 *Let f be a continuous conic function in $|\Pi|$. Then*

$$\int_{\mathbb{S}^{\pi}} f\, d\eta_{\Pi}(\mu) = \int_{\mathbb{S}^{|\widehat{\Pi}|}} f\, d\mu.$$

Proof Using that f is conic and the definition of the pushforward of a measure we obtain

$$\int_{\mathbb{S}^{\Pi}} f\, d\eta_{\Pi}(\mu) = \int_{\mathbb{S}^{\Pi}} \frac{1}{n_{\Pi}} f\, d(h^{-1}_{\Pi,\widehat{\Pi}})_*(\mu) = \int_{\mathbb{S}^{\Pi}} f \circ h^{-1}_{\Pi,\widehat{\Pi}}\, d(h^{-1}_{\Pi,\widehat{\Pi}})_*(\mu) = \int_{\mathbb{S}^{|\widehat{\Pi}|}} f\, d\mu.$$

□

Definition 3.16 Consider the admissible family $\mathcal{C}$ of concave functions on $|\widehat{\Pi}|$, the $\mathcal{C}$-concave conical function ψ on $|\widehat{\Pi}|$ and its associated Monge–Ampère measure μ_{ψ}. We denote by

$$\eta_{\Pi,\psi} := \eta_{\Pi}\left(\mu_{\psi}\right)$$

the induced Monge–Ampère measure on $\mathbb{S}^{\Pi}$ defined above. Similarly, for any collection $\psi_1, \ldots, \psi_{n-1}$ of $\mathcal{C}$-concave conical functions, we denote by

$$\eta_{\Pi,\psi_1,\ldots,\psi_{n-1}} := \eta_{\Pi}\left(\mu_{\psi_1,\ldots,\psi_{n-1}}\right)$$

the induced mixed Monge–Ampère measure on $\mathbb{S}^{\Pi}$.

Lemma 3.15 and Corollary 2.28 have the following immediate consequence.

Corollary 3.17 *Let $\psi_1, \ldots, \psi_n$ be a collection of $\mathcal{C}$-concave conical functions on $|\Pi|$. Then for any $i \in \{1, \ldots, n\}$ the measure $\eta_{\Pi,\psi_1,\ldots,\widehat{\psi}_i,\ldots,\psi_n}$ is independent of the choice of Euclidean metric and*

$$\deg\left(\psi_1 \cdots \psi_n\right) = \int_{\mathbb{S}^{\Pi}} \psi_i(u)\, d\eta_{\Pi,\psi_1,\ldots,\widehat{\psi}_i,\ldots,\psi_n}.$$

3.3 The rational conical space attached to a toroidal embedding

Throughout this section k will denote an algebraically closed field. All of the varieties appearing in this section will be defined over k even if not stated explicitly. We recall the definition of a toroidal embedding and describe its associated rational conical space. The following definition is taken from [27, Definition 1, pg. 54].

Definition 3.18 Let X be an n-dimensional normal, algebraic variety over k and let U be a smooth Zariski open subset of X. An open immersion $U \hookrightarrow X$ is a *toroidal embedding* if for every closed point $x \in X$ there exists an n-dimensional torus $\mathbb{T}$, an affine toric variety $X_\sigma \supseteq \mathbb{T}$, a point $x' \in X_\sigma$ and an isomorphism of k-local algebras

$$\widehat{\mathcal{O}}_{X,x} \xrightarrow{\simeq} \widehat{\mathcal{O}}_{X_\sigma,x'} \tag{3.6}$$

such that the ideal in $\widehat{\mathcal{O}}_{X,x}$ generated by the ideal of $X \setminus U$ corresponds under this isomorphism to the ideal in $\widehat{\mathcal{O}}_{X_\sigma,x'}$ generated by the ideal of $X_\sigma \setminus \mathbb{T}$. Here, the hat "$\widehat{\ }$" denotes the completion of the local ring at a point. Such an isomorphism is called a *chart* at x and the pair (X_σ, x') is called a *local model* at x.

If all the irreducible components of the boundary divisor $X \setminus U$ of a toroidal embedding are normal, then it is called a *toroidal embedding without self intersection*.

Definition 3.19 Let $U \hookrightarrow X$ be a toroidal embedding (defined over k) without self intersection and let $\{B_i \mid i \in I\}$ be the irreducible components of the boundary divisor $B = X \setminus U$. For every subset $J \subseteq I$, write $B_J := \bigcap_{i\in J} B_i \neq \emptyset$. The *strata* of the toroidal embedding are the irreducible components of the sets of the form $B_J \setminus \bigcup_{i\notin J} B_i$. The strata will be denoted $\{S_\alpha\}_{\alpha\in\Lambda}$ where Λ is a finite set. The maximal strata correspond to the irreducible components of the open set U.

The following lemma is [27, Proposition-Definition 2, pg. 57].

Lemma 3.20 *Let notations be as above and consider a subset $J \subseteq I$ such that $B_J \neq \emptyset$ and let S_{α_0} be an irreducible component of $B_J \setminus \bigcup_{i\notin J} B_i$. Then the following holds true:*

(1) B_J is normal.
(2) S_{α_0} is non-singular.

Moreover, the sets S_α, $\alpha \in \Lambda$ define a stratification of X, i.e. every point of X is in exactly one stratum and the closure of a stratum is a union of strata. Furthermore, if $x \in X$ and (X_σ, x') is a local model at x, then the closures $\overline{S_\alpha}$ of the strata S_α such that $x \in \overline{S_\alpha}$ correspond formally to the closure of the torus orbits in X_σ containing x'. In particular, if $x \in S_\alpha$, then S_α corresponds formally to the torus orbit $O(x')$ itself.

The following Proposition/Definition is adapted from [27, Definition 3, pg. 59]. See also Corollary 1 in page 61 of [27].

Proposition/Definition 3.21 *Let notations be as in Definition 3.19. For any non-empty stratum S_α of the toroidal embedding $U \hookrightarrow X$, the combinatorial open set $\mathrm{Star}(S_\alpha) \subseteq X$ is defined by*

$$\mathrm{Star}(S_\alpha) := \bigcup_{\beta\,:\, S_\alpha \subseteq \overline{S}_\beta} S_\beta = X \setminus \bigcup_{\gamma\,:\, \overline{S}_\gamma \cap S_\alpha = \emptyset} S_\gamma.$$

Moreover, let

$$M^{S_\alpha} := \left\{ B \in \text{Ca-Div}\left(\mathrm{Star}\left(S_\alpha\right)\right) \,\middle|\, \mathrm{supp}(B) \subseteq \mathrm{Star}\left(S_\alpha\right) \setminus U \right\},$$
$$M_+^{S_\alpha} := \left\{ B \in M^{S_\alpha} \,\middle|\, B \textit{ effective} \right\}.$$

Then M^{S_α} is a free abelian group (a lattice) while $M_+^{S_\alpha}$ has the structure of a sub-semigroup. For each stratum S_α we denote by $N^{S_\alpha} = \left(M^{S_\alpha}\right)^\vee$ the dual lattice of M^{S_α} and by $\langle\,,\,\rangle_{S_\alpha}$

the induced pairing. Finally, let

$$\sigma^{S_\alpha} := \left\{ v \in N_{\mathbb{R}}^{S_\alpha} \,\middle|\, \langle m, v\rangle_{S_\alpha} \geqslant 0,\ \forall m \in M_+^{S_\alpha} \right\} \subseteq N_{\mathbb{R}}^{S_\alpha}.$$

Then $\sigma^{S_\alpha} \subseteq N_{\mathbb{R}}^{S_\alpha}$ *is a strongly convex rational polyhedral cone of maximal dimension.*

The idea behind Proposition/Definition 3.21 is that given a stratum S, we have produced a maximal dimensional cone σ^S in the finite-dimensional real vector space $N_{\mathbb{R}}^S$ which comes equipped with a canonical lattice N^S.

We now see that these cones can be glued together into a rational conical complex. For a toroidal embedding $U \hookrightarrow X$ without self intersection, let $|\Pi_{(X,U)}|$ be the quotient topological space defined by

$$\left|\Pi_{(X,U)}\right| := \bigsqcup_{S_\alpha \text{ stratum}} \sigma^{S_\alpha} / \sim$$

where $\sim$ is the equivalence relation generated by isomorphisms

$$\beta^{\alpha,\alpha'} \colon \sigma^{S_\alpha} \xrightarrow{\simeq} \text{face of } \sigma^{S_{\alpha'}}$$

whenever $S_\alpha \subseteq \text{Star}\,(S_{\alpha'})$. Here, the map $\beta^{\alpha,\alpha'}$ is the restriction of the map $N_{\mathbb{R}}^{S_\alpha} \to N_{\mathbb{R}}^{S_{\alpha'}}$ defined as the dual of the map $M_{\mathbb{R}}^{S_{\alpha'}} \to M_{\mathbb{R}}^{S_\alpha}$, which in turn is induced by the map $M^{S_{\alpha'}} \to M^{S_\alpha}$ given by restricting divisors from $\text{Star}\,(S_{\alpha'})$ to $\text{Star}\,(S_\alpha)$ (see [27, Chapter II, Section 1]). We have the following proposition.

Proposition 3.22 *If* $U \hookrightarrow X$ *is a toroidal embedding without self intersection, then the pair*

$$\Pi_{(X,U)} = \left(\left\{ \sigma^{S_\alpha} \right\}_{S_\alpha \text{ stratum}}, \left\{ M^{S_\alpha} \right\}_{S_\alpha \text{ stratum}} \right)$$

defines a rational conical structure on $\left|\Pi_{(X,U)}\right|$ *in the sense of Definition 3.1. Hence,* $\left|\Pi_{(X,U)}\right|$ *is a rational conical space in the sense of Definition 3.2 with rational conical structure given by* $\Pi_{(X,U)}$.

Proof The proof can be found in [27, Chapter II, pg. 71]. □

The collection of lattices $\left\{M^{S_\alpha}\right\}$ in the above proposition gives the *integral structure* of the toroidal embedding.

The following lemma follows from [27, Chapter II, Corollary 1].

Lemma 3.23 *Let* $U \hookrightarrow X$ *be a toroidal embedding without self intersection and let* $x \in X$ *belonging to a stratum* S. *If* (X_σ, x') *is a local model at* x *then*

$$M^S \simeq M(\mathbb{T}) / \left(M(\mathbb{T}) \cap \sigma^\perp \right) \quad \text{and} \quad \sigma^S \simeq \sigma\,,$$

where $M(\mathbb{T})$ *refers to the lattice of characters of the torus* $\mathbb{T} \subseteq X_\sigma$ *and* $\sigma^\perp$ *is the set defined by*

$$\sigma^\perp := \{m \in M(\mathbb{T}) \,|\, \langle m, v\rangle = 0,\ \forall v \in \sigma\}.$$

In particular, the local model (X_σ, x') *is determined up to isomorphism by the stratum* S.

Given a cone σ in $\Pi_{(X,U)}$, we will denote by S^σ the stratum corresponding to σ and by $\overline{S^\sigma}$ its closure in X.

Example 3.24 Let Σ be a fan in $N_{\mathbb{R}}$ for some lattice N and let $M := N^{\vee}$ be its dual lattice. Furthermore, let X_{Σ} be its associated normal toric variety over k with dense torus $\mathbb{T} = \mathrm{Spec}(k[M])$. Clearly, the inclusion $\mathbb{T} \hookrightarrow X_{\Sigma}$ defines a toroidal embedding. The components of the boundary divisor $B = X_{\Sigma} \setminus \mathbb{T}$ are the $\mathbb{T}$-invariant prime divisors B_{τ} corresponding to the rays $\tau \in \Sigma(1)$, and the strata of X are the $\mathbb{T}$-orbits $O(\sigma)$ corresponding to the cones $\sigma \in \Sigma$. The combinatorial open sets of X_{Σ} are precisely its $\mathbb{T}$-invariant affine open subsets.

The isomorphism

$$M/\left(M \cap \sigma^{\perp}\right) \simeq M^{O(\sigma)}$$

given by the assignment

$$[m] \longmapsto \mathrm{div}\left(\chi^{m}\right),$$

where χ^{m} denotes the character of the torus associated to $m \in M$, induces an identification of lattices

$$N^{O(\sigma)} \simeq N_{\sigma} = N \cap \mathrm{Span}(\sigma)$$

and of cones

$$\sigma^{O(\sigma)} \simeq \sigma.$$

Remark 3.25 As in the toric case, the set of rays $\Pi_{(X,U)}(1)$ of the rational conical complex associated to a toroidal embedding $U \hookrightarrow X$ is in bijection with the set of irreducible components of the boundary divisor $B = X \setminus U$. Indeed for every irreducible component B_i, the corresponding ray in $\Pi_{(X,U)}(1)$, which we will denote by τ_{B_i} is the linear function $\tau_{B_i} : M^{S_{\{i\}}} \to \mathbb{Z}$ given by $nB_i \mapsto n$. Conversely, one can show that any ray $\tau \in \Pi_{(X,U)}(1)$ arises in this way (see [27, pg. 63]). For any such ray τ, we will denote by B_{τ} the corresponding irreducible boundary component.

Before giving a more general class of examples of toroidal embeddings, we recall some definitions.

Definition 3.26 Let $B \subseteq X$ be a divisor on a smooth variety X. We say that B is a *normal crossing divisor* (abbreviated nc) if the following condition holds:

(1) For all $x \in X$ we can choose local coordinates $x_1, \ldots, x_n$ and natural numbers $\ell_1, \ldots, \ell_n$ such that $B = \left\{\prod_i x_i^{\ell_i} = 0\right\}$ in a neighborhood of x.

We further say that B is a *simple normal crossing divisor* (abbreviated snc) if furthermore

(2) Every irreducible component of B is smooth.

We can now give a large class of examples of toroidal varieties.

Example 3.27 Let (X, B) be a pair consisting of a smooth projective variety X of dimension n together with a snc divisor $B \subseteq X$. We denote by $\{B_i\}_{i \in I}$ the irreducible components of B. Set $U := X \setminus B$. Then $U \hookrightarrow X$ is a toroidal embedding. The rational conical complex associated to the toroidal embedding $U \hookrightarrow X$ is smooth and is constructed by adding a k-dimensional cone for each subset $J \subseteq I$ with $\#J = k$ and each irreducible component of $\bigcap_{j \in J} B_j$. In particular, the zero dimensional cones correspond to the irreducible components of $X = \bigcap_{j \in \emptyset} B_j$.

Remark 3.28 It follows from the definition of a toroidal embedding $U \hookrightarrow X$ (without self intersection) that the boundary $X \setminus U$ is a divisor, however, it may not be snc. It is snc as soon as X is smooth. Hence, if $\mathrm{char}(k) = 0$, by Hironaka's resolution of singularities [22], we can always find an allowable modification of $X' \to X$ (Definition 3.32) such that the boundary divisor $X' \setminus U$ is snc.

3.4 Toroidal modifications and subdivisions of rational conical complexes

Recall from the classical theory of toric varieties that given a toric variety X_Σ corresponding to a fan Σ, there is a bijective correspondence between proper birational toric morphisms to X_Σ and subdivisions of the fan Σ. Following [27, Chapter 2, Section 2], a similar phenomenon occurs in the toroidal case. In this section we describe the proper toroidal birational modifications of a toroidal embedding which, on the combinatorial side, correspond to subdivisions of the associated rational conical complex.

Definition 3.29 Let $U_{X_1} \hookrightarrow X_1$ and $U_{X_2} \hookrightarrow X_2$ be two toroidal embeddings and let $f\colon X_1 \to X_2$ be a birational morphism mapping U_{X_1} to U_{X_2}. Then f is called *toroidal* if for every closed point $x_1 \in X_1$ there exist local models (X_{σ_1}, x_1') at $x_1 \in X_1$ and (X_{σ_2}, x_2') at $f(x_1) \in X_2$, and a toric morphism $g\colon X_{\sigma_1} \to X_{\sigma_2}$, with $f(x_1') = x_2'$, such that the following diagram commutes.

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X_1,x_1} & \xrightarrow{\simeq} & \widehat{\mathcal{O}}_{X_{\sigma_1},x_1'} \\ \hat{f}^{\#}\uparrow & & \uparrow\hat{g}^{\#} \\ \widehat{\mathcal{O}}_{X_2,x_2} & \xrightarrow{\simeq} & \widehat{\mathcal{O}}_{X_{\sigma_2},x_2'} \end{array}$$

Here, $\hat{f}^{\#}$ and $\hat{g}^{\#}$ are the ring homomorphisms induced by f and g, respectively.

Remark 3.30 The following two properties are satisfied.

(1) The composition of two birational toroidal morphisms is again a birational toroidal morphism.
(2) A toroidal morphism $f\colon (U_1 \hookrightarrow X_1) \to (U_2 \hookrightarrow X_2)$ induces a morphism

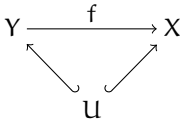
$$f_{|\Pi|}\colon |\Pi_{(X_1,U_1)}| \to |\Pi_{(X_2,U_2)}|$$

of rational conical spaces. The restrictions of $f_{|\Pi|}$ to the cones of $\Pi_{(X_1,U_1)}$ are dual to pulling back Cartier divisors. From this, we see that $f_{|\Pi|}$ can also be considered as a morphism between weakly embedded rational conical spaces by adding to it the data of the linear map $N^{|\Pi_{(X_1,U_1)}|} \to N^{|\Pi_{(X_2,U_2)}|}$ dual to the pullback $\Gamma\left(U_2, \mathcal{O}_{X_2}^{\times}\right) \to \Gamma\left(U_1, \mathcal{O}_{X_1}^{\times}\right)$.

The following definition is taken from [27, Definition 1, pg. 73].

Definition 3.31 A toroidal birational morphism $f\colon (U \hookrightarrow Y) \to (U \hookrightarrow X)$ between two toroidal embeddings of the same open subset U is called *canonical over* X if the following conditions hold true:

(1) The diagram



is commutative.

(2) For all $x_1, x_2 \in X$ in the same stratum S and for all morphisms

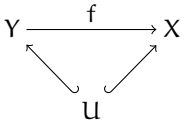
$$\xi\colon \hat{\mathcal{O}}_{X,x_1} \longrightarrow \hat{\mathcal{O}}_{X,x_2} \tag{3.7}$$

which preserve the strata (i.e. if $S \subseteq \overline{S'}$ for some stratum S' then ξ takes the ideal of $\overline{S'}$ at x_1 to the ideal of $\overline{S'}$ at x_2), we have that $\mathrm{Spec}(\xi)$ can be lifted to give an isomorphism $Y \times_X \mathrm{Spec}(\hat{\mathcal{O}}_{X,x_2}) \simeq Y \times_X \mathrm{Spec}(\hat{\mathcal{O}}_{X,x_1})$ preserving the strata, i.e. such that the following diagram commutes.

$$\begin{array}{ccc} Y \times_X \mathrm{Spec}(\hat{\mathcal{O}}_{X,x_2}) & \xrightarrow{\simeq} & Y \times_X \mathrm{Spec}(\hat{\mathcal{O}}_{X,x_1}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\hat{\mathcal{O}}_{X,x_2}) & \xrightarrow{\mathrm{Spec}(\xi)} & \mathrm{Spec}(\hat{\mathcal{O}}_{X,x_1}) \end{array}$$

We can now define the class of toroidal birational morphisms which correspond to subdivisions of rational conical complexes. The following is [27, Definition 3, pg. 87].

Definition 3.32 Consider a toroidal birational morphism $f\colon (U \hookrightarrow Y) \to (U \hookrightarrow X)$ forming a commutative diagram



and satisfying the following two conditions:

(1) Y has an open covering $\{V_i\}$ such that $U \subseteq V_i$, $f(V_i) \subseteq \mathrm{Star}(S_i)$ for some stratum S_i of X and V_i is affine and canonical over $\mathrm{Star}(S_i)$.
(2) Y is normal.

Toroidal embeddings $U \hookrightarrow Y$ as above are called *allowable modifications* of the toroidal embedding $U \hookrightarrow X$.

The following important theorem follows from [27, Theorem 6*, 8*].

Theorem 3.33 *Given a toroidal embedding without self intersection $U \hookrightarrow X$, there is a bijective correspondence between subdivisions of the rational conical complex $\Pi_{X,U}$ and isomorphism classes of proper allowable modifications of X.*

3.5 Weak embeddings and quasi-embeddings

Given a toroidal embedding (U, X) the conical complex $\Pi = \Pi_{(X,U)}$ comes equipped with an obvious quasi-embedding $|\Pi| \to \mathrm{Hom}(L(\Pi), \mathbb{R})$ as in Example 1.11. Nevertheless the following example shows that this quasi-embedding does not have enough information to encode the intersection product of toroidal divisors.

Example 3.34 Let a be a positive integer. The Hirzebruch surface H_a is defined as the projectivization of $\mathcal{O} \oplus \mathcal{O}(a)$ over $\mathbb{P}^1$. This is the toric variety associated to the two-dimensional smooth fan defined by the vectors $(1,0), (0,1), (-1,a)$ and $(0,-1)$. We then consider the toroidal (in fact toric) embedding $U = \mathbb{G}_m^2 \subseteq H_a$. Then $H_a \setminus U$ has four components. Two fibers, the zero section and the infinity section. The quasi-embedding of Example 1.11 embeds the fan in $\mathbb{R}^4$ and does not depend on a. However, the self-intersection number of the zero section is a, hence this embedding does not have enough information to extract the self-intersection numbers of the components of the boundary.

Following [21], when the ambient variety is proper, there is a canonical weak embedding of the rational conical complex associated to a toroidal embedding without self intersection, that has more information than the obvious quasi-embedding.

Definition 3.35 Let $U \hookrightarrow X$ be a toroidal embedding with X proper and let $|\Pi| = |\Pi_{(X,U)}|$ be the corresponding rational conical space. The group M^U is defined to be the set of classes of invertible regular functions on the open set U, modulo locally constant functions, i.e.

$$M^U := \Gamma\left(U, \mathcal{O}_X^\times\right) / \Gamma\left(U, k^\times\right).$$

Since X is proper, this is a torsion free finitely generated abelian group. That is, it is a lattice. Let $N^U := \left(M^U\right)^\vee$ be its dual lattice. For every stratum S of X we have a morphism of lattices $M^U \to M^S$ given by

$$f \longmapsto \operatorname{div}(f)|_{\operatorname{Star}(S)}.$$

Dualizing, we get a linear map $\sigma^S \to N^U_{\mathbb{R}}$. These maps glue to give a continuous function

$$\iota_\Pi \colon |\Pi| \longrightarrow N^U_{\mathbb{R}},$$

which is integral linear on the cones of Π, i.e. $|\Pi|$ has structure of a weakly embedded rational conical space. Moreover, we will see in Remark 4.5 that $|\Pi|$ is also naturally balanced.

Two of the following examples are taken from [21, Example 2.2].

Example 3.36 (1) Consider the toric setting $X = X_\Sigma$ from Example 3.24, and write $\Pi = \Pi_{(X_\Sigma, \mathbb{T})}$. Here, we have the lattice $M^{\mathbb{T}} = \Gamma\left(\mathbb{T}, \mathcal{O}_{X_\Sigma}^\times\right) / k^\times$, which we can identify with M via the isomorphism $M \simeq M^{\mathbb{T}}$ given by the assignment

$$m \longmapsto \chi^m.$$

We see that the image of $\sigma^{O(\sigma)}$ in $N_{\mathbb{R}}$ under the weak embedding ι_Π is precisely σ. Hence, Π is a weakly embedded rational conical complex, naturally isomorphic to Σ. Note that in this case, the weak embedding is globally injective. In particular, in the Example 3.34 we recover the usual embedding in $\mathbb{R}^2$ that depends on a and has enough information to recover the self-intersection of the boundary components.

(2) For a non-toric example, consider $X = \mathbb{P}^2$ with homogeneous coordinates $(x_0 : x_1 : x_2)$ but with open part U given by

$$U = X \setminus (H_1 \cup H_2),$$

where H_i is the hyperplane given by $\{x_i = 0\}$. This is a toroidal embedding with snc boundary divisor and we see that the rational conical complex $\Pi = \Pi_{(X,U)}$ is naturally

identified with the non-negative orthant $\mathbb{R}^2_{\geqslant 0}$, whose rays $\mathbb{R}_{\geqslant 0}(1,0)$ and $\mathbb{R}_{\geqslant 0}(0,1)$ correspond to the divisors H_1 and H_2, respectively. The lattice $M^{|\Pi|}$ is generated by x_1/x_2, and using that generator to identify $M^{|\Pi|}$ with $\mathbb{Z}$, we see that the weak embedding ι_Π sends $(1,0)$ to 1 and $(0,1)$ to -1. Note that in this case ι_Π is not a quasi-embedding since for example the cone $\mathbb{R}^2_{\geqslant 0}$ is two-dimensional while $N^{|\Pi|}_{\mathbb{R}}$ has dimension one.

(3) Consider again $\mathbb{P}^2$ with the same homogeneous coordinates, D_1 the line $x_0 = 0$ and D_2 the conic $x_0^2 + x_1^2 + x_2^2 = 0$. then $D_1 \cup D_2$ is a snc divisor and the corresponding conical complex consist of two copies of the non-negative orthant glued together by the axes. The description of the weak embedding is similar to the previous one.

Still, the weak embedding of Definition 3.35 is not enough for our purposes as it only allows intersecting with the so called combinatorially principal divisors (see Remark 1.31). In order to have an intersection theory with arbitrary toroidal divisors we need a quasi-embedding. This can be achieved by modifying the toroidal structure. Before that we need to compare the combinatorial structures of different toroidal structures.

Definition 3.37 Let X be a k-variety. A *birational toroidal structure* on X is a smooth proper modification $\pi\colon \widetilde{X} \to X$ and an snc divisor D on $\widetilde{X}$ such that π induces an isomorphism $\widetilde{X} \setminus D \to X \setminus \pi(D)$.

We say that a birational toroidal structure $(\pi', \widetilde{X}', D')$ *dominates* a second one $(\pi, \widetilde{X}, D)$ if there is a factorization

$$\begin{array}{ccc} \widetilde{X}' & \xrightarrow{\varphi} & \widetilde{X} \\ & \searrow^{\pi'} & \downarrow \pi \\ & & X \end{array}$$

such that $\varphi^{-1}(D) \subseteq D'$.

We say that two birational toroidal structures $(\pi, \widetilde{X}, D)$ and $(\pi', \widetilde{X}', D')$ are equivalent if there is a third one $(\pi'', \widetilde{X}'', D'')$ dominating both, with maps φ and φ' such that $\varphi^{-1}(D) = \varphi'^{-1}(D') = D''$ and φ and φ' are toroidal birational maps .

If $(\pi', \widetilde{X}', D')$ and $(\pi, \widetilde{X}, D)$ are birational toroidal structures on X with π' dominating π through a birational map φ. Then φ induces a map of weakly embedded rational conical complexes $\Pi_{(\widetilde{X}', D')} \to \Pi_{(\widetilde{X}, D)}$ that we now describe.

Write $U = \widetilde{X}' \setminus D'$ and $V = \widetilde{X} \setminus D$. Both U and V can be identified with open subsets of X and after this identification $U \subseteq V$.

Let S_α be a stratum of the toroidal embedding $U \hookrightarrow \widetilde{X}'$. Let S_β be the stratum of $V \hookrightarrow \widetilde{X}$ containing the image of the generic point of S_α. Then there is an open set $W \subset \mathrm{Star}(S_\alpha)$ that intersects S_α and such that $\varphi(W) \subset \mathrm{Star}(S_\beta)$. Then we can identify

$$M^{S_\alpha} = \left\{ B \in \mathrm{Ca\text{-}Div}\,(W) \;\middle|\; \mathrm{supp}(B) \subseteq W \setminus U \right\},$$

so, there is an induced map $\varphi^*\colon M^{S_\beta} \to M^{S_\alpha}$ that sends $M^{S_\beta}_+$ to $M^{S_\alpha}_+$. By duality we obtain a map $N^{S_\alpha} \to N^{S_\beta}$ that sends σ^{S_α} to S^{S_β}. These maps glue together to give a map

$$\varphi_{|\Pi|}\colon |\Pi_{(\widetilde{X}', U)}| \longrightarrow |\Pi_{(\widetilde{X}, V)}|.$$

Moreover the inclusion $U \subset V$ provides a map $M^V \to M^U$ and, by duality, a map $N^U \to N^V$ that fits in a commutative diagram

$$\begin{array}{ccc} |\Pi_{(\widetilde{X}',U)}| & \xrightarrow{\varphi_{|\Pi|}} & |\Pi_{(\widetilde{X},V)}| \\ \downarrow & & \downarrow \\ N^U & \longrightarrow & N^V \end{array}$$

providing the map of weakly embedded rational conical complexes. For easy treference we summarize the result of this discussion in the following proposition.

Proposition 3.38 *Let $(\pi', \widetilde{X}', D')$ and $(\pi, \widetilde{X}, D)$ be birational toroidal structures on X with π' dominating π. If $U = \widetilde{X} \setminus D$ and $U' = \widetilde{X}' \setminus D'$, then $U' \subset U$ and the restriction morphism $\Gamma(U, \mathcal{O}_X^\times) \to \Gamma(U', \mathcal{O}_X^\times)$ induces a map of weakly embedded rational conical complexes $\Pi_{(\widetilde{X}',D')} \to \Pi_{(\widetilde{X},D)}$.*

The following key proposition says that given a toroidal embedding with a snc boundary divisor, we can always modify the toroidal structure by a dominating one, in such a way that the associated weakly embedded rational conical complex becomes quasi-embedded.

Proposition 3.39 *Let $U \hookrightarrow X$ be a toroidal embedding with X smooth and projective, such that the boundary divisor $B = X \setminus U$ is snc and let $|\Pi| = |\Pi_{(X,U)}|$ be its associated weakly embedded rational conical space. Then there exists a snc divisor B' with $|B| \subseteq |B'|$ so (X, B') dominates (X, B) and such that, writing $U' = X \setminus B'$, the weakly embedded rational conical space $|\Pi'| = |\Pi_{(X,U')}|$ is quasi-embedded, i.e. the restriction of the weak embedding*

$$\iota_{|\Pi'|}|_{\sigma'} \colon |\sigma'| \longrightarrow N_{\mathbb{R}}^{|\Pi'|}$$

to any cone $\sigma' \in \Pi'$ is injective.

Proof Recall that n denotes the dimension of X. If $n = 1$ then $B = \{p_1, \ldots, p_k\}$ is a finite set of points. Choose rational functions f_i such that $\operatorname{ord}_{p_i}(f_i) \neq 0$ and write

$$B' = \bigcup |\operatorname{div}(f_i)| = \{q_1, \ldots, q_r\}$$

The corresponding polyhedral complex is given by the finite set of rays $\{\tau_{q_j}\}$, such that τ_{q_j} is joined with $\tau_{q_{j'}}$ at zero if and only if q_j and $q_{j'}$ are in the same irreducible component. Let v_j denote the primitive vector of τ_{q_j} and let x_i denote the point of $M^{|\Pi'|}$ corresponding to f_i. By construction, for each j there is an i such that $\langle \iota(v_j), x_i \rangle = \operatorname{ord}_{q_j}(f_i) \neq 0$. Therefore $\iota_{\Pi'}(v_j) \neq 0$ and Π' is quasi-embedded.

Assume now that $n \geqslant 2$. Write $B = B_1 \cup \cdots \cup B_r$ for the decomposition of B into irreducible components. There is a hypersurface C such that $B_i + C$ is very ample for $i = 1, \ldots, r$. Moreover we can find hypersurfaces $A_{i,j}$,

$$B_i \sim A_{i,j} - C, \quad i = 1, \ldots r, \; j = 1, \ldots, n,$$

and a second hypersurface $C_1 \neq C$ such that $C_1 \sim C$. Here the symbol $\sim$ means linear equivalence. Finally by Bertini's theorem we can assume that all the hypersurfaces C, C_1 and $A_{i,j}$ are different, smooth and irreducible and

$$B' := B \cup C \cup C_1 \cup \bigcup_{i,j} A_{i,j}$$

is a snc. Then there are rational functions $f_{i,j}$ and g such that

$$\begin{aligned} \operatorname{div}(f_{i,j}) &= B_i + C - A_{i,j} \\ \operatorname{div}(g) &= C - C_1. \end{aligned}$$

As in the statement of the theorem, write $U' = X \setminus B'$ and $|\Pi'| = |\Pi_{(X,U')}|$. Let $x_{i,j}$ be the point of $M^{|\Pi'|}$ corresponding to $f_{i,j}$ and y the point corresponding to g. For an irreducible component E of B', write v_E for the primitive generator of the ray corresponding to the the divisor E. By construction we have

$$\begin{aligned} \langle v_{B_i}, x_{k,j} \rangle &= \delta_{i,k} \\ \langle v_{A_{i,j}}, x_{k,\ell} \rangle &= -\delta_{i,k}\delta_{j,\ell} \\ \langle v_{B_i}, y \rangle = \langle v_{A_{i,j}}, y \rangle &= 0 \\ \langle v_C, x_{k,j} \rangle &= 1 \\ \langle v_{C_1}, x_{k,j} \rangle &= 0 \\ \langle v_C, y \rangle &= 1 \\ \langle v_{C_1}, x_{k,j} \rangle &= -1 \end{aligned}$$

From the above identities, it follows that any subset of n vectors contained in $\{v_{B_i}, v_{A_{k,j}}, v_C, v_{C_1}\}$ is linearly independent. This implies that the weak embedding $\iota_{\Pi'}$ is a quasi-embedding. □

Remark 3.40 Let $U \hookrightarrow X$ be a toroidal embedding with snc boundary divisor $B = X \setminus U$ and X projective. Then, by modifying the toroidal structure in a similar way as we did in the proof of Proposition 3.39, we may assume that there exists an ample and effective divisor with support contained in B. Moreover, by adding a small multiple of all components of B, we may assume that there is an ample $\mathbb{R}$-divisor A with positive multiplicity along all components of B. This will be useful for the monotone approximation lemma in Section 5.2.

4 Intersection theory of toroidal b-divisors

In this section we fix an algebraically closed field k. The goal of this section is to show that nef toroidal b-divisors have well defined top intersection products (Definitions 4.20 and 4.26 and Theorem 4.32) and that these intersection products can be computed as the integral of a function associated to one of the b-divisors with respect to a Monge–Ampère like measure associated to the remaining b-divisors. The existence of the product is also proved in [17] in a more general setting.

The idea of the construction is, first, following [21], to relate the geometric intersection product of toroidal divisors with the *rational tropical intersection product* on quasi-embedded rational conical complexes (Theorem 4.6). Second to use the convergence results of Sect. 2 in order to extend the top intersection product to nef b-divisors. However, note that the Monge–Ampère measures of Section 2 are defined in a Euclidean setting (no integral structure). Therefore we will use the comparison in Sect. 3.2 to relate the rational tropical intersection product with the Euclidean one.

4.1 Intersection products of toroidal divisors

Let $U \hookrightarrow X$ be a toroidal embedding with X smooth and $B = X \setminus U$ a snc divisor and such that the associated weakly embedded conical complex $\Pi = \Pi_{(X,U)}$ is quasi-embedded. Recall that thanks to Proposition 3.39 this can always be achieved by enlarging B. We will give the definition of $\mathbb{R}$-toroidal divisors and give a bijection between the set of $\mathbb{R}$-toroidal divisors on (X, U) and the set of piecewise linear functions on Π. Moreover, following [21], we recall the tropicalization of an algebraic cycle and relate algebraic and tropical intersection numbers. We end this section by showing that one can compute combinatorially the top intersection numbers of divisors.

Definition 4.1 Let $\mathrm{Div}(X)_{\mathbb{R}}$ be the vector space of $\mathbb{R}$-Cartier divisors on X. We define the subspace $\mathrm{Div}(X, U)_{\mathbb{R}} \subseteq \mathrm{Div}(X)_{\mathbb{R}}$ consisting of $\mathbb{R}$-Cartier divisors which are supported on the boundary B. It is a finite dimensional $\mathbb{R}$-vector space and it is endowed with a canonical topology. Elements in $\mathrm{Div}(X, U)_{\mathbb{R}}$ are called $\mathbb{R}$-*toroidal Cartier divisors (of* (X, U)*)*. From now on we will only consider $\mathbb{R}$-divisors. Thus to simplify notation, we will omit the coefficient ring $\mathbb{R}$ from the notation and call $\mathbb{R}$-toroidal Cartier divisors simply toroidal Cartier divisors.

Recall from Remark 3.25, that we have a bijective correspondence between the set of rays of Π and the set of irreducible components of the boundary divisor B. For a ray $\tau \in \Pi(1)$ we denote by B_τ the corresponding component and by $v_\tau = v_{\tau/0_\tau}$ the primitive lattice normal vector spanning the ray τ.

Definition 4.2 Let $D \in \mathrm{Div}(X, U)$ be a toroidal Cartier divisor on X. The corresponding piecewise linear function

$$\phi_D : |\Pi| \longrightarrow \mathbb{R}$$

defined on Π is given by

$$\phi_D|_\sigma = -D|_{\mathrm{Star}(S^\sigma)} \in M^\sigma = (N^\sigma)^\vee,$$

where, for a cone $\sigma \in \Pi$, we denote by S^σ the corresponding stratum of X. Since we are assuming that the toroidal embedding is smooth, we can give an alternative description going through conical functions. The function ϕ_D is linear on each cone and, for $\tau \in \Pi(1)$,

$$\phi_D(v_\tau) = -\mathrm{ord}_{B_\tau}(D),$$

where $\mathrm{ord}_{B_\tau}(D)$ denotes the coefficient of B_τ in D.

By Remark 3.25, any piecewise linear function defined on ϕ on Π induces a toroidal Cartier divisor D_ϕ by setting $D|_{B_\tau} = -\phi(v_\tau)$ for any $\tau \in \Pi(1)$. These constructions are clearly inverses of each other. We summarize the above in the following proposition, which can be seen as a special case of [27, Theorem 9*].

Proposition 4.3 *The map*

$$\mathrm{Div}(X, U) \longrightarrow \mathrm{L}(\Pi) \tag{4.1}$$

given by the assignment

$$D \longmapsto \phi_D$$

is an isomorphism of finite dimensional real vector spaces.

We recall the definition of the tropicalization of an algebraic cycle class on X as is explained in [21, Section 4.2].

For $0 \leqslant k \leqslant n$ we denote by $Z_k(X) = Z_k(X)_{\mathbb{R}}$ the group of algebraic k-cycles on X with real coefficients. For any $C \in Z_k(X)$, the assignment

$$\mathrm{trop}(C)\colon \Pi(k) \longrightarrow \mathbb{R},$$

given by

$$\sigma \longmapsto \deg\left(C \cdot \left[\overline{S^\sigma}\right]\right),$$

is a k-dimensional Minkowski weight on Π. Moreover, this Minkowski weight is compatible with taking refinements and thus the following definition makes sense.

Definition 4.4 Let notations be as above. The map

$$\mathrm{trop}\colon Z_k(X) \longrightarrow Z_k(|\Pi|)$$

given by

$$C \longmapsto [\mathrm{trop}(C)]$$

is called the *tropicalization map*. In particular, if $[X]$ is the fundamental cycle of X, then $\mathrm{trop}([X])$ is the tropical cycle that is represented by the Minkowski weight in $M_n(\Pi)$ given by assigning weight one to all n-dimensional cones of Π. We set

$$[|\Pi|] := \mathrm{trop}([X]).$$

In consequence $\Pi_{(X,U)}$ is canonically a balanced complex.

Remark 4.5 (1) The tropicalization map is well defined even if $|\Pi|$ is only weakly embedded. Thus to any toroidal embedding (X, U) with X smooth and $B \setminus U$ a snc, there is a canonically associated *balanced weakly-embedded rational conical complex*. The change of toroidal structure to make it quasi-embedded is needed to be able to intersect arbitrary Cartier divisors with tropical cycles.

(2) The tropicalization map factors through the group of numerical classes of k-cycles on X (with real coefficients), which is denoted by $N_k(X)_{\mathbb{R}} = N_k(X)$, and hence we get a well defined tropicalization map

$$N_k(X) \longrightarrow Z_k(|\Pi|)$$

which we also denote by trop.

The following theorem relates algebraic intersection numbers and tropical intersection numbers. Recall that the rational conical space $|\Pi|$ is assumed to be quasi-embedded.

Theorem 4.6 *Let $D \in \mathrm{Div}(X, U)$ be a toroidal divisor. Then for every k-dimensional cycle class $[C]$ in $N_k(X)$ the following tropical cycle classes agree*

$$\mathrm{trop}\,(D \cdot [C]) = [\phi_D] \odot \mathrm{trop}([C]) \in Z_{k-1}(|\Pi|),$$

where on the right hand side, the class $[\phi_D]$ is seen as an element in $\mathrm{PL}(|\Pi|)$.

Proof Using Remark 1.31, this follows from [21, Proposition 4.17]. □

Hence, in the case of characteristic zero, we can compute top intersection numbers of arbitrary divisors using the tropical intersection product.

Corollary 4.7 *Let X_0 be a proper variety over k and $D_1, \dots, D_n \in \mathrm{Div}(X_0)$ a collection of divisors on X_0. Write $B_0 = |D_1| \cup \cdots \cup |D_n|$. Assume that there exists a proper birational morphism $\pi\colon X_1 \to X_0$ and B_1 a snc divisor of X_1 satisfying the following two properties:*

(1) $\pi^{-1}(B_0) \subseteq B_1$.
(2) $X_1 \setminus B_1 \hookrightarrow X_1$ *is a toroidal embedding such that* $|\Pi_1| := |\Pi_{(X_1, X_1 \setminus B_1)}|$ *is a quasi-embedded rational conical space.*

This assumption is always satisfied if the field has characteristic zero. Then the algebraic top intersection number $\deg(D_1 \cdots D_n)$ *can be computed tropically on the quasi-embedded, balanced rational conical complex* Π_1 *as*

$$\deg(D_1 \cdots D_n) = \deg\left(\phi_{\pi^* D_1} \odot \cdots \odot \phi_{\pi^* D_n} \odot [|\Pi_1|]\right).$$

If we choose a Euclidean norm in $\mathrm{N}_{\mathbb{R}}^{|\Pi_1|}$ *and denote by "$\widehat{\ }$" the map between Minkowski cycles and Euclidean cycles, then the algebraic top intersection number can also be computed as*

$$\deg(D_1 \cdots D_n) = \deg\left(\phi_{\pi^* D_1} \cdot \cdots \cdot \phi_{\pi^* D_n} \cdot \widehat{[|\Pi_1|]}\right).$$

Proof If the field has characteristic zero, the pair (X_1, B_1) exists thanks to resolution of singularities [22], Proposition 3.39 and Theorem 3.33. By Theorem 4.6 and the functoriality of the intersection product, we get

$$\deg(D_1 \cdots D_n) = \deg(\pi^* D_1 \cdots \pi^* D_n) = \deg\left(\phi_{\pi^* D_1} \odot \cdots \odot \phi_{\pi^* D_n} \odot [|\Pi_1|]\right),$$

proving the first statement. By Proposition 3.13 and Remark 3.12, we get

$$\phi_{\pi^* D_1} \odot \cdots \odot \phi_{\pi^* D_n} \odot [|\Pi_1|] = \phi_{\pi^* D_1} \cdot \cdots \cdot \phi_{\pi^* D_n} \cdot \widehat{[|\Pi_1|]},$$

which proves the second one. □

4.2 Positivity properties of cycles

From now on we assume we have chosen a Euclidean norm on $\mathrm{N}_{\mathbb{R}}^{|\Pi|}$ and denote by $\widehat{|\Pi|}$ the induced Euclidean conical complex. We also denote by $\widehat{[|\Pi|]}$ the balancing condition obtained by normalizing $[|\Pi|] = \mathrm{trop}([X])$.

Since the map "trop" between algebraic cycles on X and Minkowski cycles on $|\Pi|$ is neither surjective nor injective, the positivity notions in the algebraic and tropical worlds do not correspond exactly. Recall that $\mathrm{N}^k(X)$ denotes the space of $\mathbb{R}$-cycles of codimension k up to numerical equivalence. The cone $\mathrm{Peff}_{n-k}(X) = \mathrm{Peff}^k(X)$ of pseudo-effective cycles is the closure of the cone generated by classes of effective cycles. The dual cone is the cone of numerically effective cycles

$$\mathrm{Nef}^k(X) = \{\beta \in \mathrm{N}^k(X) \mid \beta \cdot \alpha \geqslant 0,\ \forall \alpha \in \mathrm{Peff}_k(X)\}$$

The space of nef toroidal Cartier divisors is denoted by $\mathrm{Div}^+(X, U)$.

The first relation between positivity in the algebraic and the tropical worlds is the following.

Lemma 4.8 *(1) Let $\alpha \in \mathrm{Nef}^{n-k}(X)$, then $\mathrm{trop}(\alpha) \in Z_k(|\Pi|)$ is a positive cycle. Therefore $\widehat{\mathrm{trop}(\alpha)} \in \mathrm{E}Z_k(\widehat{|\Pi|})$ is also positive.*
(2) If $D \in \mathrm{Div}^+(X, U)$ is a nef toroidal Cartier divisor then the corresponding piecewise linear function ϕ_D defined on $\widehat{\Pi}$ is weakly concave in the sense of Definition 2.1 ([5, Definition 4.6]).

Proof For any cone $\sigma \in \Pi(k)$,

$$\mathrm{trop}(\alpha)(\sigma) = \alpha \cdot \overline{S^\sigma} \geqslant 0,$$

because α is nef and $\overline{S^\sigma}$ is an effective cycle. The second statement follows from the first and the equality

$$\phi_D \odot [\Pi] = \mathrm{trop}(D).$$

□

We next see several examples that show that the above lemma is almost all we can expect.

Example 4.9 Let X be the blow up of $\mathbb{P}^2$ at a point. Let B be a snc divisor such that the exceptional divisor E is contained in the support of B and such that the associated complex is quasi-embedded. Then $[E]$ is an effective cycle but $\mathrm{trop}([E])$ is not positive. So the statement (1) of the above lemma can not be extended to pseudo-effective cycles.

Example 4.10 Let again X be the blow up of $\mathbb{P}^2$ at a point p. Let r_1, r_2 be the strict transforms of two different lines passing through p and ℓ_i, $i = 1, 2, 3$ be the strict transforms of three general lines on $\mathbb{P}^2$. Put $B = \ell_1 \cup \ell_2 \cup \ell_3 \cup r_1 \cup r_2$. There are rational functions f_1, f_2 and g with

$$\mathrm{div}(f_1) = \ell_1 - \ell_2, \quad \mathrm{div}(f_2) = \ell_2 - \ell_3, \quad \mathrm{div}(g) = r_1 - r_2.$$

This easily implies that the complex associated to the toroidal embedding $X \setminus B \hookrightarrow X$ is quasi-embedded. Let E denote again the exceptional divisor. Then $\mathrm{trop}([E]) \geqslant 0$ because E is not contained in B. Nevertheless E is not nef. Therefore the converse of statement (1) is not true. Put $D = \ell_1 - r_1$. Since $D \sim E$, we see that ϕ_D is weakly concave but D is not nef. Therefore the converse of statement (2) does not hold.

Example 4.11 We put ourselves in the situation of Example 4.10 and let $B' = \ell_1 \cup \ell_2 \cup \ell_3$. Then the obtained conical complex is still quasi-embedded, but the map trop satisfies $\mathrm{trop}(\pm[E]) = 0$. Therefore $\mathrm{trop}(\alpha) \geqslant 0$ does not even imply that α is effective.

Example 4.12 Let X be an elliptic curve, O the marked point i.e. the neutral element for the group law, P a non torsion point and $Q = -P$. Put $B = \{O, P, Q\}$. There is a rational function f on X such that

$$\mathrm{div}(g) = 2O - P - Q.$$

Therefore the conical complex Π associated to $X \setminus B \hookrightarrow X$ consists of three rays τ_O, τ_P and τ_Q with lattice generators ν_O, ν_P and ν_Q. The lattice $N^{|\Pi|}$ is one dimensional and can be identified with $\mathbb{Z}$. Then the quasi-embedding is given by

$$\iota(\nu_O) = 2, \qquad \iota(\nu_P) = \iota(\nu_Q) = -1.$$

For simplicity a Minkowski weight $c \in M_1(\Pi)$ will be denoted as a triple of real numbers (c_O, c_P, c_Q). The balancing condition $[|\Pi|]$ is the Minkowski weight $(1, 1, 1)$. The Minkowski weight $(1, 2, 0)$ is also positive but it does not come from the geometry of X. Consider the divisor $D = -P + 2Q$. This is a nef divisor because it has positive degree. Nevertheless

$$\phi_D \odot (1, 2, 0) = -2 < 0.$$

Hence it is not true that nef divisors always give rise to concave functions in the sense of Definition 2.1.

We end this section showing that the set of nef toroidal Cartier divisors in allowable modifications of X provides an admissible family of concave functions on $\widehat{|\Pi|}$ in the sense of Definition 2.6.

Lemma 4.13 *Let $U \hookrightarrow X$ be a toroidal embedding with X smooth, $B = X \setminus U$ a snc divisor such that $\Pi_{(X,U)}$ is quasi-embedded and there is an ample divisor with support contained in B. Then the set $\mathcal{C}$ of piecewise linear functions on $|\Pi|$ given by*

$$\mathcal{C} = \left\{ \phi_D \,\middle|\, D \in \mathrm{Div}^+(X', U),\ \pi\colon X' \to X \text{ allowable modification} \right\}$$

forms an admissible family of concave functions on $\widehat{|\Pi|}$ in the sense of Definition 2.6.

Proof We have to show that $\mathcal{C}$ satisfies the three properties given in Definition 2.6. To show property (1), let $D_1, \ldots, D_r$ be nef Cartier toroidal divisors on the allowable modifications $X_{\Pi_1}, \ldots, X_{\Pi_r}$, respectively. Let Π' be a smooth common refinement of $\Pi_1, \ldots, \Pi_r$ and denote by $\pi_i\colon X_{\Pi'} \to X_{\Pi_i}$ the corresponding allowable modification. Then, by Theorem 4.6 and Proposition 3.13, we get

$$\deg(\phi_{D_1} \cdots \phi_{D_r} \cdot \widehat{[|\Pi|]}) = \deg\left(\pi_1^*[D_1] \cdots \pi_r^*[D_r] \cdot [X_{\Pi'}]\right) \geqslant 0,$$

where the last inequality uses the fact that the pullback of a nef divisor under a proper map is nef and Kleiman's criterion for nefness.

Property (2) is clear.

To prove (3) we first recall that the set of piecewise linear functions on $|\Pi|$ with rational slopes is dense in the set of continuous conical functions on $|\Pi|$ with the topology of uniform convergence on compacts. Thus it is enough to show that any piecewise linear function on $|\Pi|$ with rational slopes belongs to $\mathcal{C} - \mathcal{C}$.

A piecewise linear function ϕ with rational slopes on $|\Pi|$ defines a Cartier toroidal divisor D_ϕ on an allowable modification X' of X. Since we are assuming that there is an ample divisor on X whose support is contained in the boundary divisor B, there is an allowable modification $\pi\colon X'' \to X'$ and an ample toroidal divisor A on X''. We can choose an integer $r > 0$ such that $C = \pi^* D_\phi + rA$ is also ample. Therefore

$$\phi = \phi_C - \phi_{rA}, \quad \phi_C, \phi_{rA} \in \mathcal{C},$$

completing the proof. □

4.3 Toroidal b-divisors

We give the definition of Cartier and Weil toroidal b-divisors on a toroidal embedding. Extending the bijection between toroidal divisors and linear functions (Proposition 4.3), we give a bijection between Cartier (respectively Weil) toroidal b-divisors and piecewise linear (respectively conical) functions on a conical rational polyhedral space. Moreover, extending the results in Section 4.1 we show that the top intersection product of Cartier toroidal b-divisors can be computed tropically on the rational conical space.

We start by recalling the definition of Cartier and Weil b-divisors on a variety over k.

Definition 4.14 Let X be a variety over k. Then $B(X)$ is the category of proper birational modifications $\pi\colon X_\pi \to X$. The spaces of Cartier and Weil b-divisors on X are defined respectively as

$$\mathrm{CbDiv}(X) := \varinjlim_{\pi \in B(X)} \mathrm{Div}\,(X_\pi)_{\mathbb{R}},$$

$$\mathrm{bDiv}(X) := \varprojlim_{\pi \in \mathrm{B}(X)} \mathrm{WDiv}\,(X_\pi)_{\mathbb{R}}\,,$$

where the direct limit is defined by using the pullback of Cartier divisors and the inverse limit by using the push-forward of Weil divisors. Both limits are taken in the category of topological vector spaces.

We will write Cartier and Weil b-divisors in bold notation $\mathbf{D}$ to distinguish them from classical divisors denoted by D. Given a toroidal embedding (X, U) the definition of toroidal b-divisors is similar but restricting to allowable modifications.

Definition 4.15 Let (X, U) be a toroidal embedding with associated rational conical complex $\Pi = \Pi_{(X,U)}$. Then the spaces of toroidal Cartier and Weil b-divisors on (X, U) are defined as

$$\mathrm{CbDiv}(X, U) := \varinjlim_{\Pi' \in \mathrm{R}_{\mathrm{sm}}(\Pi)} \mathrm{Div}\,(X_{\Pi'}, U)_{\mathbb{R}}\,,$$
$$\mathrm{bDiv}(X, U) := \varprojlim_{\Pi' \in \mathrm{R}_{\mathrm{sm}}(\Pi)} \mathrm{Div}\,(X_{\Pi'}, U)_{\mathbb{R}}\,,$$

where the direct limit is defined by using the pullback of toroidal divisors and the inverse limit by using the push-forward of toroidal divisors. Both limits are taken in the category of topological vector spaces. Note that the $X_{\Pi'}$'s are smooth hence we may identify Cartier and Weil toroidal divisors. More generally, if $(\pi, \widetilde{X}, D)$ is a birational toroidal structure, we have at our disposal the groups $\mathrm{CbDiv}(\widetilde{X}, U)$ and $\mathrm{bDiv}(\widetilde{X}, U)$.

We make the following remarks.

Remark 4.16 (1) If (X, U) is a toroidal embedding with rational conical complex Π, we can view a Weil toroidal b-divisor $\mathbf{D} \in \mathrm{bDiv}(X, U)$ as a family

$$\mathbf{D} = (D_{\Pi'})_{\Pi' \in \mathrm{R}_{\mathrm{sm}}(\Pi)}\,,$$

where for each $\Pi' \in \mathrm{R}_{\mathrm{sm}}(\Pi)$, we have that $D_{\Pi'} \in \mathrm{Div}(X_{\Pi'}, U)$, and these elements are compatible under push-forward. A similar description is true for general Weil b-divisors.

(2) Similarly, we can view a Cartier toroidal b-divisor $\mathbf{D} \in \mathrm{CbDiv}(X, U)$ as a Weil toroidal b-divisor

$$\mathbf{E} = (E_{\Pi'})_{\Pi' \in \mathrm{R}_{\mathrm{sm}}(\Pi)}\,,$$

for which there is a model $X_{\widetilde{\Pi}}$ for some $\widetilde{\Pi} \in \mathrm{R}_{\mathrm{sm}}(\Pi)$ such that for every other model $X_{\Pi''}$ with $\Pi'' \geqslant \widetilde{\Pi}$ in $\mathrm{R}_{\mathrm{sm}}(\Pi)$, the incarnation $E_{\Pi''}$ is the pull-back of $E_{\widetilde{\Pi}}$ on $X_{\widetilde{\Pi}}$. Hence, we have the inclusion

$$\mathrm{CbDiv}(X, U) \subseteq \mathrm{bDiv}(X, U),$$

and we may refer to a Weil toroidal b-divisor just as a toroidal b-divisor.

(3) A net $(\mathbf{Z}_i)_{i \in I}$ converges to a b-divisor $\mathbf{Z}$ in $\mathrm{bDiv}(X, U)$ if and only if for each $\Pi' \in \mathrm{R}_{\mathrm{sm}}(\Pi)$ we have that $\left(Z_{i,\Pi'}\right)_{i \in I}$ converges to $Z_{\Pi'}$ coefficient-wise.

(4) By the following Proposition 4.17 we have that the spaces $\mathrm{bDiv}(X, U)$ and $\mathrm{CbDiv}(X, U)$ agree with the spaces of piecewise linear and conical functions on $|\Pi|$, respectively. However, they are different from the ones on $\widehat{|\Pi|}$, because the allowed subdivisions are different (see Remark 3.6).

We have the following combinatorial characterization of toroidal b-divisors. Recall that $|\Pi|(\mathbb{Q})$ denotes the set of points of $|\Pi|$ with rational coordinates in any cone of Π.

Proposition 4.17 *Let (X, U) be a toroidal embedding with associated rational conical complex Π. The map $\mathbf{D} \mapsto \phi_{\mathbf{D}}$ from Proposition 4.3 can be extended continuously to homeomorphisms*

$$\mathrm{CbDiv}(X, U) \simeq \mathrm{PL}\,(|\Pi|)$$

and

$$\mathrm{bDiv}(X, U) \simeq \mathrm{Conic}\,(|\Pi|)$$

between toroidal b-divisors on (X, U) and functions on $|\Pi|$. In particular, the space $\mathrm{bDiv}(X, U)$ is homeomorphic to the space of conical functions $|\Pi|(\mathbb{Q}) \to \mathbb{R}$ with the topology of pointwise convergence.

Proof This follows from the definition of conical functions on a rational conical complex (Remark 3.6) and Proposition 4.3. □

Lemma 4.18 *Let $(\pi, \widetilde{X}, D)$ be a birational toroidal structure on X with $U = \widetilde{X} \setminus D$ and $(\pi', \widetilde{X}', D')$ a second birational toroidal structure on X dominating π and $U' = \widetilde{X}' \setminus D'$, then there is a canonical commutative diagram*

$$\begin{array}{ccc} \mathrm{CbDiv}(\widetilde{X}, U) & \longrightarrow & \mathrm{bDiv}(\widetilde{X}, U) \\ \downarrow & & \downarrow \\ \mathrm{CbDiv}(\widetilde{X}', U') & \longrightarrow & \mathrm{bDiv}(\widetilde{X}', U'). \end{array}$$

with all the arrows monomorphisms.

Proof Let Π and Π') be the quasi-embedded polyhedral complexes corresponding to π and π', respectively. Since π' dominates π, by Proposition 3.38 there is a map $r\colon |\Pi| \to |\Pi|'$ of weakly embedded rational conical spaces. Then the diagram in the statement of the lemma is by Proposition 4.17 the translation of the diagram

$$\begin{array}{ccc} \mathrm{PL}(|\Pi|) & \longrightarrow & \mathrm{Conic}(|\Pi|) \\ \downarrow{\scriptstyle r^*} & & \downarrow{\scriptstyle r^*} \\ \mathrm{PL}(|\Pi'|) & \longrightarrow & \mathrm{Conic}(|\Pi'|) \end{array}$$

The injectivity of the vertical arrows is a consequence of the map r being surjective. □

Remark 4.19 If $(\pi', \widetilde{X}', D')$ is a equivalent to $(\pi, \widetilde{X}, D)$ then we have an equality of b-divisors

$$\mathrm{CbDiv}(\widetilde{X}, U) = \mathrm{CbDiv}(\widetilde{X}', U'), \qquad \mathrm{bDiv}(\widetilde{X}, U) = \mathrm{bDiv}(\widetilde{X}', U').$$

Definition 4.20 A *Cartier (resp. Weil) toroidal b-divisor* on X is a Cartier (resp. Weil) toroidal b-divisor in a birational toroidal structure. Two toroidal b-divisors are equivalent if there is a birrational toroidal structure dominating the structures of definition of both divisors such that the pullback of both divisors agree. The sets of equivalence classes of toroidal b-divisors will be denoted as

$$\mathrm{bDiv}(X)^{\mathrm{tor}} \quad \text{and} \quad \mathrm{CbDiv}(X)^{\mathrm{tor}},$$

respectively.

Remark 4.21 If the field k is of characteristic zero, by resolution of singularities, the set of birational toroidal structures is a directed set. In this case we can define toroidal b-divisors as

$$\mathrm{bDiv}(X)^{\mathrm{tor}} := \varinjlim_{(\widetilde{X},U,\pi)} \mathrm{bDiv}(\widetilde{X},U) \quad \text{and} \quad \mathrm{CbDiv}(X)^{\mathrm{tor}} := \varinjlim_{(\widetilde{X},U,\pi)} \mathrm{CbDiv}(\widetilde{X},U),$$

with the limit taken in the category of topological vector spaces, so it has more structure. By contrast, in positive characteristic it is not clear that the sum of two toroidal divisors with respect to different toroidal structures is again toroidal. In particular, when in positive characteristic, we will work with a fixed toroidal structure.

Remark 4.22 Again if the field has characteristic zero, every Cartier b-divisor is toroidal so we can identify $\mathrm{CbDiv}(X)^{\mathrm{tor}}$ with $\mathrm{CbDiv}(X)$. By contrast, for Weil divisors, being toroidal can be thought as a kind of finiteness condition.

Proposition 3.39 has the following consequence.

Lemma 4.23 *Let $\mathbf{D}$ be a toroidal b-divisor on X. If the field k has characteristic zero, then there is a birational toroidal structure $(\pi, \widetilde{X}, D)$ with $U = \widetilde{X} \setminus D$ such that the weakly embedded rational polyhedral complex $\Pi_{\widetilde{X},U}$ is quasi-embedded, $\mathbf{D} \in \mathrm{bDiv}(\widetilde{X},U)$ and there is an ample divisor on $\widetilde{X}$ with support contained in D.*

As in the case of functions, one can define the top intersection product of a collection of toroidal b-divisors when there is at most one Weil b-divisor involved (all the other must be Cartier).

Definition 4.24 Let $\mathbf{D}_1, \ldots, \mathbf{D}_n$ be toroidal b-divisors on X and assume that at most one of the $\mathbf{D}_i$'s is not Cartier. Assume furthermore that there exists a birational toroidal structure $(\pi, \widetilde{X}, D)$ with $U = \widetilde{X} \setminus D$ such that for all i, $\mathbf{D}_i \in \mathrm{bDiv}(\widetilde{X},U)$ and that $\Pi = \Pi_{\widetilde{X},U}$ is quasi embedded. If the field has characteristic zero, by Lemma 4.23 this assumption is always satisfied. Without lost of generality, let $\mathbf{D}_1$ be the one that may be non Cartier. Let $\Pi' \in R_{\mathrm{sm}}(\Pi)$ such that all of the $\mathbf{D}_i$'s for $i = 2, \ldots, n$ are determined on $\widetilde{X}_{\Pi'}$. The top intersection product $\langle \mathbf{D}_1 \cdots \mathbf{D}_n \rangle$ is defined by

$$\langle \mathbf{D}_1 \cdots \mathbf{D}_n \rangle := \deg(D_{1,\Pi'} \cdot D_{2,\Pi'} \cdots D_{n,\Pi'}).$$

By the projection formula in algebraic geometry, this product is independent of the choice of birational toroidal structure satisfying the assumptions and of the common refinement Π'.

Remark 4.25 It follows from Corollary 4.7 that, in the case of characteristic zero, the top intersection product of a collection of Cartier b-divisors and a toroidal Weil b-divisor can be computed tropically. In the arbitrary characteristic case we have to assume furthermore that they are toroidal with respect to the same birational toroidal structure.

4.4 Top intersection product of nef toroidal Weil b-divisors

We start by recalling the definition of nef b-divisors.

Definition 4.26 Let X be a smooth variety over k. A Cartier b-divisor $\mathbf{E} \in \mathrm{CbDiv}(X)$ is said to be *nef* if $E_\pi \in \mathrm{Div}(X_\pi)$ is nef for some (hence any) determination E_π of $\mathbf{E}$. The set of nef toroidal Cartier b-divisors forms a cone in $\mathrm{CbDiv}(X)$, denoted by $\mathrm{CbDiv}^+(X)$. The cone of nef Weil b-divisors is the closure in $\mathrm{bDiv}(X)$ of $\mathrm{CbDiv}^+(X)$. It is denoted by $\mathrm{bDiv}^+(X)$.

We now fix temporarily a toroidal embedding $U \hookrightarrow X$ with $B = X \setminus U$ such that B is a snc divisor, $\Pi = \Pi_{(X,U)}$ is quasi-embedded and there is an ample divisor with support contained in B. Recall that this can always be achieved if $\mathrm{char}(k) = 0$. Choose a Euclidean metric on $N_{\mathbb{R}}^{|\Pi|}$ and denote by $\widehat{|\Pi|}$ the induced Euclidean conical space.

Nef toroidal Cartier b-divisors on (X, U) are defined as

$$\mathrm{CbDiv}^+(X, U) = \mathrm{CbDiv}(X, U) \cap \mathrm{CbDiv}^+(X)$$

and nef toroidal Weil b-divisors on (X, U), denoted by $\mathrm{bDiv}^+(X, U)$, are the closure of $\mathrm{CbDiv}^+(X, U)$ in $\mathrm{bDiv}(X, U)$.

Remark 4.27 By Proposition 4.17 and Remark 3.6 the inclusion between Cartier and Weil b-divisors $\mathrm{CbDiv}(X, U) \hookrightarrow \mathrm{bDiv}(X, U)$ can be factored as

$$\mathrm{CbDiv}\,(X, U) = \mathrm{PL}(|\Pi|) \to \mathrm{PL}\left(\widehat{|\Pi|}\right) \to \mathrm{Conic}\left(\widehat{|\Pi|}\right) \to \mathrm{Conic}(|\Pi|) = \mathrm{bDiv}\,(X, U)\,.$$

By Lemma 4.13 the image of $\mathrm{CbDiv}^+(X, U)$ in $\mathrm{PL}\left(\widehat{|\Pi|}\right)$ forms an admissible family of concave functions. By Theorem 2.13 the elements in the closure of $\mathrm{CbDiv}^+(X, U)$ in $\mathrm{Conic}\left(\widehat{|\Pi|}\right)$ are continuous functions. Since a continuous function is determined by its values on a dense subset and the topologies on $\mathrm{Conic}\left(\widehat{|\Pi|}\right)$ and $\mathrm{Conic}(|\Pi|)$ are both that of pointwise convergence, we deduce that the closure of $\mathrm{CbDiv}^+(X, U)$ in $\mathrm{Conic}\left(\widehat{|\Pi|}\right)$ is naturally homeomorphic to its closure in $\mathrm{Conic}(|\Pi|)$. The last one can be identified with the cone $\mathrm{bDiv}^+(X, U)$. Therefore, in order to work with nef toroidal Weil b-divisors, there is no difference between working on $|\Pi|$ or in $\widehat{|\Pi|}$.

Remark 4.28 A consequence of the preceding remark and Theorem 2.13 is that the closure of $\mathrm{CbDiv}^+(X, U)$ in $\mathrm{bDiv}(X, U)$ agrees with its sequential closure. It follows that if $\mathbf{D} \in \mathrm{bDiv}^+(X, U)$, then there is a sequence $(\mathbf{D}_i)_{i\in\mathbb{N}}$ of nef toroidal Cartier b-divisors converging to $\mathbf{D}$. Moreover, when we view $\mathbf{D}$ and the $\mathbf{D}_i$ as functions on $|\Pi|$, the convergence is uniform on compacts.

From now on we fix $\mathcal{C}$ = image of $\mathrm{CbDiv}^+(X, U)$ in $\mathrm{PL}\left(\widehat{|\Pi|}\right)$ as the admissible family of concave functions.

The following theorem describes nef toroidal b-divisors combinatorially. In view of Remark 4.27, it is a direct consequence of Theorem 2.13.

Theorem 4.29 *Let $\mathbf{D}$ be a nef toroidal b-divisor on X. Then the corresponding function $\phi_{\mathbf{D}}$ on $|\Pi|(\mathbb{Q})$ of Proposition 4.17 extends to a continuous function on $|\Pi|$, which we denote also by $\phi_{\mathbf{D}}$. The function $\phi_{\mathbf{D}}$ defines a $\mathcal{C}$-concave conical function on $\mathbb{S}^{\widehat{|\Pi|}}$ in the sense of Definition 2.12.*

Since we can view toroidal Cartier b-divisors as toroidal Weil b-divisor, there is a potential ambiguity when we say that a toroidal Cartier b-divisor is nef. Lemma 4.31 below shows that this potential ambiguity is not a real ambiguity in the toroidal case. It is also proved in more generality in [17, Corollary 4], the method of proof however is very different. Before stating it we make the following remark.

Remark 4.30 Let $\Pi'' \in \mathrm{R}_{\mathrm{sm}}(\Pi)$ and let D be a nef toroidal divisor on $X_{\Pi''}$. Then for any $\Pi' \leqslant \Pi''$ in $\mathrm{R}_{\mathrm{sm}}(\Pi)$ we have that $D \leqslant \pi^*\pi_* D$, where $\pi\colon X_{\Pi''} \to X_{\Pi'}$ denotes

the corresponding proper birational morphism. Indeed, the divisor $D - \pi^*\pi_* D$ is π-nef (i.e. has non-negative intersection with every curve contracted by π) and is π-exceptional. Hence, from the well-known Negativity Lemma (see e.g. [28, Lemma 3.39]), it follows that $D - \pi^*\pi_* D \leqslant 0$.

Lemma 4.31 *Let $\mathbf{D}$ be a nef Weil toroidal b-divisor and assume that $\mathbf{D}$ is Cartier. Then $\mathbf{D}$ is a nef toroidal Cartier b-divisor in the sense of Definition 4.26.*

Proof Let $\Pi' \in R_{sm}(\Pi)$ be such that $\mathbf{D}$ is determined on $X_{\Pi'}$. We have to show that $D_{\Pi'}$ is nef on $X_{\Pi'}$. For this, let $C \subseteq X_{\Pi'}$ be an irreducible curve. It suffices to show that the intersection product $D_{\Pi'} \cdot C$ is non-negative.

Let $\{B'_i \mid i \in I'\}$ be the irreducible components of the boundary divisor $B' = X_{\Pi'} \setminus U$ and for any subset $J' \subseteq I'$, denote by $B_{J'}$ the boundary intersection $\bigcap_{j \in J'} B_j$ (in particular, $B_\emptyset = X_{\Pi'}$). Let $K' \subseteq I'$ such that $B_{K'}$ is the minimal boundary intersection containing C.

If $\operatorname{codim}(B_{K'}) \geqslant 2$, we can find a subdivision $\Pi'' \geqslant \Pi'$ in $R_{sm}(\Pi)$ and a curve $\tilde{C} \subseteq X_{\Pi''}$ such that the following two conditions are satisfied:

(1) $\pi_* \tilde{C} = aC$ for some natural number $a > 0$.
(2) Denoting by $\{B''_i \mid i \in I''\}$ the irreducible components of the boundary divisor $B'' = X_{\Pi''} \setminus U$, the minimal boundary intersection $B_{K''}$ containing $\tilde{C}$ (for some subset $K'' \subseteq I''$) satisfies that $\operatorname{codim}(B_{K''}) = 1$.

If $\pi^* \tilde{C} D_{\Pi'} \cdot \tilde{C} \geqslant 0$, then using the projection formula, we get that $D_{\Pi'} \cdot \pi_* \tilde{C} = D_{\Pi'} \cdot C \geqslant 0$. Hence, replacing Π' by Π'', we may assume that $B_{K'}$ has codimension $\leqslant 1$.

Let $\{\mathbf{D}_i\}_{i \in \mathbb{N}}$ be a sequence of nef toroidal Cartier b-divisors converging to $\mathbf{D}$. We view them as toroidal Weil b-divisors. In particular, on $X_{\Pi'}$, we have that

$$D_{i,\Pi'} \xrightarrow[i \in \mathbb{N}]{} D_{\Pi'}$$

component-wise, and by continuity of the intersection product,

$$D_{i,\Pi'} \cdot C \xrightarrow[i \in \mathbb{N}]{} D_{\Pi'} \cdot C. \tag{4.2}$$

Now, for each $i \in \mathbb{N}$, let $\Pi_i \in R_{sm}(\Pi)$ be a determination of $\mathbf{D}_i$. We may assume that $\Pi_i \geqslant \Pi'$. Also, we let $\pi_i \colon X_{\Pi_i} \to X_{\Pi'}$ denote the corresponding proper birational morphism. Let C_i be the strict transform of the curve C under π_i. Note that this is well defined by the assumption that the minimal boundary intersection that contains C has codimension less or equal than one.

Using the projection formula, we compute

$$\begin{aligned} D_{i,\Pi'} \cdot C &= D_{i,\Pi'} \cdot \pi_{i*} C_i \\ &= \pi_i^* D_{i,\Pi'} \cdot C_i \\ &= \pi_i^* \pi_{i*} D_{i,\Pi_i} \cdot C_i \\ &= \left(\pi_i^* \pi_{i*} D_{i,\Pi_i} - D_{i,\Pi_i}\right) \cdot C_i + D_{i,\Pi_i} \cdot C_i \\ &\geqslant 0. \end{aligned}$$

Indeed, the first summand is non-negative since it follows from Remark 4.30 that both the terms $\pi_i^* \pi_{i*} D_{i,\Pi_i} - D_{i,\Pi_i}$ and C_i are effective and intersect properly. The second summand is non-negative since D_{i,Π_i} is nef and C_i is effective.

By (4.2), $D_{\Pi'} \cdot C$ is a limit of non-negative real numbers. Hence it is itself non-negative. This concludes the proof. □

The next is the main result of this paper.

Theorem 4.32 *The restriction of the top intersection product of toroidal Cartier b-divisors (Definition 4.24) to nef toroidal Cartier b-divisors*

$$(\mathrm{CbDiv}^+(X,U))^n \longrightarrow \mathbb{R}$$

can be extended continuously to a symmetric multilinear intersection product of nef toroidal Weil b-divisors

$$(\mathrm{bDiv}^+(X,U))^n \longrightarrow \mathbb{R}.$$

If $\mathbf{D}_1,\dots,\mathbf{D}_n$ is a collection of nef toroidal Weil b-divisors on X, then their top intersection product is given by

$$\langle \mathbf{D}_1 \cdots \mathbf{D}_n \rangle = \int_{\mathbb{S}^{|\widehat{\Pi}|}} \phi_{\mathbf{D}_1}(u)\, d\mu_{\mathbf{D}_2,\dots,\mathbf{D}_n},$$

where $\mu_{\mathbf{D}_2,\dots,\mathbf{D}_n}$ denotes the mixed Monge–Ampère measure induced by the collection of $\mathcal{C}$-concave conical functions $\phi_{\mathbf{D}_2},\dots,\phi_{\mathbf{D}_n}$ on $\mathbb{S}^{|\widehat{\Pi}|}$ from Definition 2.27.

Proof The proof is just putting together everything we have done up to now. Every nef toroidal Cartier b-divisor $\mathbf{D}$ on X defines a piecewise linear function $\phi_{\mathbf{D}}$ on $|\widehat{\Pi}|$ (Proposition 4.17). The family of piecewise linear functions on $|\widehat{\Pi}|$ obtained in this way forms an admissible family $\mathcal{C}$-concave functions on $|\widehat{\Pi}|$ (Lemma 4.13). The space of nef toroidal Weil b-divisors on (X,U) is homeomorphic to the space of $\mathcal{C}$-concave conical functions on $|\widehat{\Pi}|$ (Proposition 4.17 and Remark 4.27). Thus the result is a direct consequence of Corollary 2.28. □

Remark 4.33 Let $\mathbf{D}_1,\dots,\mathbf{D}_n$ be a collection of nef toroidal Weil b-divisors on (X,U). By Remark 4.28 there are sequences $(\mathbf{D}_{j,k})_{k\in\mathbb{N}}$, $j=1,\dots,n$ of nef toroidal Cartier b-divisors on (X,U) converging to them. Then, for each j, the sequence of functions $(\phi_{\mathbf{D}_{j,k}})_{k\in\mathbb{N}}$ converges uniformly on compacts to $\phi_{\mathbf{D}_j}$. Moreover

$$\langle \mathbf{D}_1 \cdots \mathbf{D}_n \rangle = \lim_{k\to\infty} \langle \mathbf{D}_{1,k} \cdots \mathbf{D}_{n,k} \rangle. \tag{4.3}$$

One has to be careful that the continuity condition (4.3) is only true when the sequences $(\mathbf{D}_{j,k})_{k\in\mathbb{N}}$ consist of nef toroidal Cartier b-divisors. Namely, one can construct sequences of toroidal Cartier b-divisors $(\mathbf{D}'_{j,k})_{k\in\mathbb{N}}$ such that, for each j, the sequence of functions $(\phi_{\mathbf{D}_{j,k}})_{k\in\mathbb{N}}$ converges uniformly on compacts to $\phi_{\mathbf{D}_j}$ and nevertheless the continuity condition (4.3) does not hold.

Remark 4.34 Since the intersection product is multilinear (for the semigroup law of $\mathrm{bDiv}^+(X,U)$), it can be extended by multilinearity to the space

$$\mathrm{bDiv}^+(X,U) - \mathrm{bDiv}^+(X,U)$$

of toroidal Weil b-divisors that are differences of nef ones.

Remark 4.35 The subspace $\mathbb{S}^{|\widehat{\Pi}|} \subseteq |\Pi|$ and the measure $\mu_{\mathbf{D}_2,\dots,\mathbf{D}_n}$ depend on the choice of the Euclidean metric, but the integral does not. A more canonical representation can be obtained using Corollary 3.17

$$\langle \mathbf{D}_1 \cdots \mathbf{D}_n \rangle = \int_{\mathbb{S}^{\Pi}} \phi_{\mathbf{D}_1}(u)\, d\eta_{\Pi,\phi_{\mathbf{D}_2},\dots,\phi_{\mathbf{D}_n}},$$

where $\mathbb{S}^{\Pi}$ is the lattice unit sphere of Definition 3.14.

For the remainder of the section we assume that k is of characteristic zero and work without the choice of a fixed toroidal structure.

Definition 4.36 The set of nef toroidal b-divisors is defined as the union over all toroidal structures of X

$$\mathrm{bDiv}^+(X)^{\mathrm{tor}} = \bigcup_{(\pi,\widetilde{X},D)} \mathrm{bDiv}^+(\widetilde{X}, \widetilde{X} \setminus D).$$

Theorem 4.37 *Let k be a field of characteristic zero, and X a variety over k of dimension n. Then the intersection product*

$$(\mathrm{CbDiv}^+(X))^n \to \mathbb{R}$$

can be extended to an intersection product

$$(\mathrm{bDiv}^+(X)^{tor})^n \to \mathbb{R}.$$

Moreover this intersection product can be computed tropically as the integral of a function with respect to a Monge-Ampère like measure. This extension is characterized uniquely by the following continuity property. Let $\mathbf{D}_1, \ldots, \mathbf{D}_n$ be nef toroidal b-divisors. and for each i, let $(\mathbf{D}_{i,j})_{j\geqslant 0}$ be a sequence of nef Cartier b-divisors converging to $\mathbf{D}_i$. Assume that the $\mathbf{D}_{i,j}$ are toroidal with respect to the same birational toroidal structure. Then

$$\langle \mathbf{D}_1, \ldots, \mathbf{D}_n \rangle = \lim_{j\to\infty} \langle \mathbf{D}_{1,j}, \ldots, \mathbf{D}_{n,j} \rangle \tag{4.4}$$

Proof Since the intersection product is invariant by change of ground field, we can assume that k is algebraically closed. Let $\mathbf{D}_1, \ldots, \mathbf{D}_n$ be nef toroidal b-divisors. Since k is of characteristic zero, using resolution of singularities, there is a birational toroidal structure $(\pi, \widetilde{X}, D)$ with $U = \widetilde{X} \setminus D$ such that

$$\mathbf{D}_i \in \mathrm{bDiv}^+(\widetilde{X}, U), \quad i = 1, \ldots, n.$$

By Proposition 3.39 we can assume furthermore that $\Pi = \Pi_{(\widetilde{X},U)}$ is quasi-embedded and that there is an ample divisor on $\widetilde{X}$ with support contained in D. Then by Theorem 4.32 and Remark 4.35 the intersection $\langle \mathbf{D}_1, \ldots, \mathbf{D}_n \rangle$ is well defined and there is a measure μ on $\mathbb{S}^\Pi$ such that

$$\langle \mathbf{D}_1, \ldots, \mathbf{D}_n \rangle = \int_{\mathbb{S}^\Pi} \phi_{\mathbf{D}_1} \, d\mu.$$

□

Remark 4.38 Nowadays the existence and the continuity of the intersection product in Theorem 4.37 can be deduced also from [17] where an intersection product of general b-divisors over a countable field is defined. Since the product defined in [17] is continuous, it agrees with the product defined here. In particular the continuity property (4.4) holds without the assumption that all Cartier b-divisors are toroidal with respect to the same toroidal structure.

5 Applications

Let $U \hookrightarrow X$ be a toroidal embedding as at the beginning of Section 4.1. That is, we assume that X is smooth, $B = X \setminus U$ is a snc divisor, that there is an effective ample divisor A with

support B and that the corresponding rational conical space $|\Pi|$ is quasi-embedded. Recall that, after Definition 4.1, by divisor we mean divisor with $\mathbb{R}$ coefficients.

A toroidal b-divisor on (X, U) is *big* if it has enough global sections (Definition 5.11). In this section, as an application of our results, we show a Hilbert–Samuel type formula for nef and big toroidal b-divisors on X relating the degree of a nef toroidal b-divisor both with its volume and with the volume of the associated convex Okounkov body (Definitions 5.3 and 5.6 and Theorem 5.14). As a corollary, we obtain the continuity of the volume function on the space of nef and big b-divisors (Corollary 5.16) and a Brunn–Minkowski type inequality (Corollary 5.17).

5.1 Volumes and convex Okounkov bodies of toroidal b-divisors

We start with the definition of the space of global sections of a toroidal b-divisor.

Definition 5.1 Let $F = k(X)$ be the field of rational functions of X. Then for any toroidal b-divisor $\mathbf{D} = (D_{\Pi'})_{\Pi' \in R_{sm}(\Pi)}$ on (X, U) one defines the space of global sections of $\mathbf{D}$ by

$$H^0(X, \mathbf{D}) = \{f \in F^\times \mid \text{b-div}(f) + \mathbf{D} \geqslant 0\} \cup \{0\} \subseteq F,$$

where $\text{b-div}(f)$ is the (Cartier) b-divisor on (X, U) induced by a rational function by setting

$$\text{b-div}(f) = \left(\text{div}_{X_{\Pi'}}(f)\right)_{\Pi' \in R_{sm}(\Pi)}, .$$

Remark 5.2 (1) We have that $H^0(X_\Sigma, \mathbf{D})$ is an intersection of finite-dimensional vector spaces

$$H^0(X, \mathbf{D}) = \bigcap_{\Pi' \in R_{sm}(\Pi)} H^0(X_{\Pi'}, D_{\Pi'}).$$

(2) We have a well defined map

$$H^0(X, \mathbf{D}) \times H^0(X, \mathbf{E}) \longrightarrow H^0(X, \mathbf{D} + \mathbf{E})$$

for any toroidal b-divisors $\mathbf{D}$ and $\mathbf{E}$ on (X, U).

Definition 5.3 Let $\mathbf{D}$ be a toroidal b-divisor. The volume of $\mathbf{D}$ is defined by

$$\text{vol}(\mathbf{D}) := \limsup_{\ell \to \infty} \frac{h^0(X, \ell\mathbf{D})}{\ell^n / n!},$$

where $h^0(X, \ell\mathbf{D})$ denotes the dimension of the space $H^0(X, \ell\mathbf{D})$.

We now associate a convex Okounkov body to $\mathbf{D}$.

Definition 5.4 Let $\mathbf{D}$ be a toroidal b-divisor on (X, U). We define the b-divisorial algebra $\mathcal{R}_{(X,U)}(\mathbf{D})$ associated to $\mathbf{D}$ by

$$\mathcal{R}_{(X,U)}(\mathbf{D}) := \bigoplus_{k \geqslant 0} H^0(X, k\mathbf{D})\, t^k.$$

This is a graded sub-k-algebra of $F[t]$.

One of the fundamental problems of algebraic geometry is the question about finite generation of divisorial algebras. It is clear that, in general, the b-divisorial algebra associated to a toroidal b-divisor is not finitely generated. However, the next proposition shows that it satisfies the weaker condition of being of almost integral type, which nevertheless, following [25], allows us to associate a convex Okounkov body to it.

Recall that a graded subalgebra $R \subseteq F[t]$ is of *integral type* if it is a finitely generated k-algebra and is a finite module over the algebra generated by R_1, while it is of *almost integral type* if it is contained in a graded subalgebra of integral type $R \subseteq A \subseteq F[t]$ (see [25, Section 2.3]).

Proposition 5.5 *Let* $\mathbf{D}$ *be a toroidal b-divisor on* (X, U). *Then the b-divisorial algebra* $\mathcal{R}_{(X,U)}(\mathbf{D}) \subseteq F[t]$ *is of almost integral type.*

Proof We clearly have

$$\mathcal{R}_{(X,U)}(\mathbf{D}) \subseteq \mathcal{R}_{X_{\Pi'}}(D_{\Pi'}),$$

for any $\Pi' \in R(|\Pi|)$, and the latter is an algebra of almost integral type by [25, Theorem 3.7]. □

We now briefly sketch the construction of the Okounkov body associated to $\mathcal{R}_{(X,U)}(\mathbf{D})$. For more details we refer to [25] and [30]. The choice of a (generic, infinitesimal) flag on X determines a valuation $\nu\colon F \setminus \{0\} \to \mathbb{Z}^n$ that can be extended to a valuation $\nu_t\colon F[t] \setminus \{0\} \to \mathbb{Z}^n \times \mathbb{Z}$.

The semigroup $S(\mathbf{D}) \subseteq \mathbb{Z}^n \times \mathbb{Z}$ is defined as the image of $\mathcal{R}_{(X,U)}(\mathbf{D})$ by the valuation ν_t. Then, for any integer $\ell \geqslant 0$, the equality $h^0(X, \ell\mathbf{D}) = \#(S(\mathbf{D}) \cap (\mathbb{Z}^n \times \{\ell\}))$ holds. Moreover, $S(\mathbf{D})$ satisfies the conditions (2.3–2.5) of [30]. One then defines the cone $C(\mathbf{D}) = \overline{\text{convhull}}(S(\mathbf{D}) \cup \{0\}) \subseteq \mathbb{R}^n \times \mathbb{R}$. This is a strictly convex cone.

Definition 5.6 Let $\mathbf{D}$ be a toroidal b-divisor on (X, U). The Okounkov body $\Delta_{\mathbf{D}} \subseteq \mathbb{R}^n$ is defined to be the *slice* of $C(\mathbf{D})$ at height 1, i.e.

$$\Delta_{\mathbf{D}} := C(\mathbf{D}) \cap (\mathbb{R}^n \times \{1\}).$$

It is a convex body (see [25, Theorem 2.30]).

Okounkov bodies have been useful to study geometric properties of divisors in terms of convex geometry. In particular, in the study of volumes of divisors. We will see that this extends to toroidal b-divisors. The following is a classical result.

Lemma 5.7 *Let* $\mathbf{D}$ *be a nef toroidal Cartier b-divisor on* (X, U). *Then*

$$\text{vol}(\mathbf{D}) = n!\text{vol}(\Delta_{\mathbf{D}}) = \mathbf{D}^n. \tag{5.1}$$

Proof Let $\Pi' \in R_{\text{sm}}(\Pi)$ be such that $\mathbf{D}$ is determined on $X_{\Pi'}$. Then the result follows from the well known case of nef Cartier divisors

$$\text{vol}(\mathbf{D}) = \text{vol}(D_{\Pi'}) = D_{\Pi'}^n = \mathbf{D}^n$$

combined with

$$\text{vol}(D_{\Pi'}) = n!\text{vol}(\Delta_{\mathbf{D}}).$$

□

In the next section we extend Equation 5.1 to nef and big Weil toroidal b-divisors on (X, U).

5.2 A Hilbert–Samuel formula

We start with two monotonicity lemmas for nef toroidal b-divisors. These play a key role.

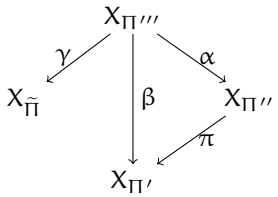
Lemma 5.8 *Let* $\mathbf{D} = (D_{\Pi'})_{\Pi' \in \mathsf{R}_{sm}(\Pi)}$ *be a nef toroidal b-divisor on* (X, U). *Let* $\Pi'' \geqslant \Pi'$ *be subdivisions in* $\mathsf{R}_{sm}(\Pi)$. *Then*

$$D_{\Pi''} \leqslant \pi^* D_{\Pi'},$$

where $\pi\colon X_{\Pi''} \to X_{\Pi'}$ *is the corresponding proper birational morphism. In particular, we get the following inclusion of spaces*

$$H^0(X_{\Pi''}, D_{\Pi''}) \subseteq H^0(X_{\Pi'}, D_{\Pi'}).$$

Proof Note that this is not a direct consequence of Remark 4.30 because the fact that $\mathbf{D}$ is nef does not imply that $D_{\pi''}$ is nef so a small argument is needed. Suppose first that $\mathbf{D}$ is a Cartier b-divisor and let $\widetilde{\Pi} \in \mathsf{R}_{\mathrm{sm}}(\Pi)$ be a determination of $\mathbf{D}$. In particular, we have that $D_{\widetilde{\Pi}}$ is nef. Let $\Pi'' \geqslant \Pi'$ be subdivisions in $\mathsf{R}_{\mathrm{sm}}(\Pi)$. Let Π''' be a common refinement of Π'' and $\widetilde{\Pi}$ and consider the following commutative diagram.



Since the pullback of a nef divisor is again nef, $D_{\Pi'''} = \gamma^* D_{\widetilde{\Pi}}$ is nef.

By Remark 4.30, $D_{\Pi'''} \leqslant \beta^*\beta_* D_{\Pi'''}$ and we conclude that

$$D_{\Pi''} = \alpha_* D_{\Pi'''} \leqslant \alpha_*\beta^*\beta_* D_{\Pi'''} = \alpha_*\beta^* D_{\Pi'} = \pi^* D_{\Pi'}$$

In the general case, choose a sequence $\{\mathbf{D}_i\}_{i\in\mathbb{N}}$ of nef Cartier b-divisors converging to $\mathbf{D}$. Then, by what was shown above, for each $i \in \mathbb{N}$ we have that

$$D_{i,\Pi''} \leqslant \pi^* D_{i,\Pi'}.$$

Hence, taking limits at both sides we deduce

$$D_{\Pi''} = \lim_{i\in\mathbb{N}} (D_{i,\Pi''}) \leqslant \lim_{i\in\mathbb{N}} \pi^*(D_{i,\Pi'}) = \pi^* D_{\Pi'}$$

as we wanted to show. □

The following is a monotone approximation lemma for nef toroidal b-divisors. Recall that we are assuming that there is an effective ample divisor A whose support is B. Since A is effective and the support of A is the whole B we deduce that ϕ_A is strictly negative in each ray of $\Pi \setminus \{0\}$. Since it is linear on each cone of Π, then the function $\phi_A|_{\mathbb{S}|\widehat{\Pi}|}$ is strictly negative.

Lemma 5.9 *Let* $\mathbf{D}$ *be a nef toroidal b-divisor on* (X, U). *Then there is a sequence of nef Cartier b-divisors* $(\mathbf{D}_i)_{i\in\mathbb{N}}$ *on* (X, U) *such that the following two properties are satisfied.*

(1) The sequence $(\mathbf{D}_i)_{i\in\mathbb{N}}$ *converges to* $\mathbf{D}$.

(2) if $i > j$, *then* $\mathbf{D}_j \geqslant \mathbf{D}_i$.

Proof Let $\left\{\mathbf{D}'_j\right\}_{j\in J}$ be a sequence of nef Cartier toroidal b-divisors converging to $\mathbf{D}$. By Theorem 2.13 the convergence

$$\phi_{\mathbf{D}'_i}|_{\mathbb{S}^{|\widehat{\Pi}|}} \xrightarrow[i\to\infty]{} \phi_{\mathbf{D}}|_{\mathbb{S}^{|\widehat{\Pi}|}}$$

is uniform. Let

$$\alpha := \inf_{x\in\mathbb{S}^{|\widehat{\Pi}|}} -\phi_A(x) > 0 \quad , \quad \beta := \sup_{x\in\mathbb{S}^{|\widehat{\Pi}|}} -\phi_A(x) \geqslant \alpha > 0,$$

and for each $i \in \mathbb{N}$, let

$$\delta_i := \sup_{x\in\mathbb{S}^{|\widehat{\Pi}|}} \left|\phi_{\mathbf{D}'_i}(x) - \phi_{\mathbf{D}}(x)\right|.$$

We know that

$$\delta_i \xrightarrow[i\to\infty]{} 0.$$

Now, choose a subsequence $\{i_k\}_{k\in\mathbb{N}}$ such that

$$\delta_{i_k} \leqslant \frac{1}{2k(k+1)}$$

and let

$$\mathbf{D}_k := \mathbf{D}'_{i_k} + \frac{1}{\alpha k}\mathbf{A}$$

for $k \in \mathbb{N}$, where $\mathbf{A}$ denotes the Cartier b-divisor induced by A. Then

$$\begin{aligned}\left(\phi_{\mathbf{D}_k} - \phi_{\mathbf{D}_{k+1}}\right)\Big|_{\mathbb{S}^{|\widehat{\Pi}|}} &= \left(\phi_{\mathbf{D}'_{i_k}} - \phi_{\mathbf{D}'_{i_{k+1}}} + \left(\frac{1}{\alpha k} - \frac{1}{\alpha(k+1)}\right)\phi_{\mathbf{A}}\right)\Big|_{\mathbb{S}^{|\widehat{\Pi}|}} \\ &\leqslant \frac{1}{k(k+1)} - \left(\frac{1}{\alpha k(k+1)}\right)\alpha \\ &= 0.\end{aligned}$$

Hence, $\mathbf{D}_k - \mathbf{D}_{k+1} \geqslant 0$ and thus $\mathbf{D}_k \geqslant \mathbf{D}_{k+1}$.

Moreover, we have

$$\begin{aligned}\left|\phi_{\mathbf{D}_k} - \phi_{\mathbf{D}}\right| &= \left|\phi_{\mathbf{D}'_{i_k}} + \frac{1}{\alpha k}\phi_{\mathbf{A}} - \phi_{\mathbf{D}}\right| \\ &\leqslant \left|\phi_{\mathbf{D}'_{i_k}} - \phi_{\mathbf{D}}\right| + \frac{1}{\alpha k}\left|\phi_{\mathbf{A}}\right| \\ &\leqslant \frac{1}{2k(k+1)} + \frac{1}{\alpha k}\beta \xrightarrow[k\to\infty]{} 0.\end{aligned}$$

Hence, we get that

$$\mathbf{D}_k \xrightarrow[k\to\infty]{} \mathbf{D}.$$

This concludes the proof. □

Remark 5.10 The above approximation lemma is also shown in a more general context in [17, Theorem 5].

Now, recall that a divisor on an algebraic variety is said to be big if it has strictly positive volume. We define big toroidal b-divisors analogously.

Definition 5.11 A toroidal b-divisor $\mathbf{D}$ on (X, U) is said to be *big* if it has positive volume, i.e. if $\mathrm{vol}(\mathbf{D}) > 0$.

Remark 5.12 If a toroidal b-divisor $\mathbf{D} = (D_{\Pi'})_{\Pi' \in R_{sm}(\Pi)}$ on (X, U) is big and nef, then by the monotonicity property of Lemma 5.8, it follows that $D_{\Pi'}$ is big for all $\Pi' \in R_{sm}(\Pi)$. Moreover, if $\{\mathbf{D}_i\}_{i\in\mathbb{N}}$ is a non-increasing sequence of nef Cartier b-divisors on (X, U) converging to $\mathbf{D}$, then, for all $i \in \mathbb{N}$, $\mathbf{D}_i$ has to be big as well.

We have the following lemma.

Lemma 5.13 *Let $\mathbf{D}$ be a nef and big b-divisor on (X, U) and let $\{\mathbf{D}_i\}_{i\in\mathbb{N}}$ be a non-increasing sequence of big and nef Cartier b-divisors on (X, U) converging to $\mathbf{D}$ (which exists by Lemma 5.9). Then*

$$\mathrm{vol}(\mathbf{D}) = \lim_{i\to\infty} \mathrm{vol}(\mathbf{D}_i).$$

Proof By Theorem 2.13 and by the monotonocity assumption, we have that the convergence

$$\phi_{\mathbf{D}_i} \xrightarrow[i\to\infty]{} \phi_{\mathbf{D}}$$

is non-decreasing and uniform on $\mathbb{S}^{|\widehat{\Pi}|}$. Let

$$\alpha := \inf_{x\in\mathbb{S}^{|\widehat{\Pi}|}} -\phi_A(x) > 0$$

as in the proof of Lemma 5.9. For each i let

$$a_i := \frac{2\sup_{x\in\mathbb{S}^{|\widehat{\Pi}|}} \left(\phi_{\mathbf{D}} - \phi_{\mathbf{D}_i}\right)}{\alpha}.$$

We have that

$$a_i \xrightarrow[i\to\infty]{} 0 \tag{5.2}$$

and for each i, the sequence of inequalities

$$\begin{aligned}\phi_{\mathbf{D}_i} - a_i\phi_A &\geqslant \phi_{\mathbf{D}_i} + a_i\alpha \\ &= \phi_{\mathbf{D}_i} + \frac{2\sup_{x\in\mathbb{S}^{|\widehat{\Pi}|}} \left(\phi_{\mathbf{D}} - \phi_{\mathbf{D}_i}\right)}{\alpha}\alpha \\ &= \phi_{\mathbf{D}_i} + 2\sup_{x\in\mathbb{S}^{|\widehat{\Pi}|}} \left(\phi_{\mathbf{D}} - \phi_{\mathbf{D}_i}\right) \geqslant \phi_{\mathbf{D}}\end{aligned}$$

is satisfied. Hence, we get that

$$\mathrm{vol}(\mathbf{D}_i) \geqslant \mathrm{vol}(\mathbf{D}) \geqslant \mathrm{vol}(\mathbf{D}_i - a_iA) \tag{5.3}$$

for each i. Now, set $\omega := \mathbf{D}_0 + A$. Then ω is a big and nef Cartier b-divisor. Moreover, by the monotonicity of the sequence, we have that $\omega \geqslant \mathbf{D}_i$ for all i. Thus, by [9, Corollary 3.4], we obtain

$$\mathrm{vol}(\mathbf{D}_i - a_iA) \geqslant \mathbf{D}_i^n - na_i(\mathbf{D}_i^{n-1}A) - Ca_i^2, \tag{5.4}$$

where C is a constant depending only on ω. Since the $\mathbf{D}_i$'s are nef Cartier b-divisors, we know that $\mathrm{vol}(\mathbf{D}_i) = \mathbf{D}_i^n$. Therefore, taking limits in (5.3) and (5.4), we get that

$$\liminf_i \mathrm{vol}(\mathbf{D}_i) \geqslant \mathrm{vol}(\mathbf{D}) \geqslant \limsup_i \left(\mathrm{vol}(\mathbf{D}_i) - na_i(\mathbf{D}_i^{n-1}A) - Ca_i^2\right).$$

Now, since $\left(\mathbf{D}_i^{n-1}A\right)$ is bounded, using (5.2), we have that $\lim_i \left(na_i(\mathbf{D}_i^{n-1}A) - Ca_i^2\right) = 0$. We conclude that

$$\liminf_i \mathrm{vol}(\mathbf{D}_i) \geqslant \mathrm{vol}(\mathbf{D}) \geqslant \limsup_i \mathrm{vol}(\mathbf{D}_i),$$

as we wanted to show. □

As a consequence, we get the following Hilbert–Samuel type formula.

Theorem 5.14 *Let $\mathbf{D}$ be a big and nef toroidal b-divisor on (X, U). Then*

$$\mathrm{vol}(\mathbf{D}) = n\mathcal{W}\mathrm{vol}(\Delta_{\mathbf{D}}) = \mathbf{D}^n.$$

Proof Write $\mathbf{D} = \lim_{i\to\infty} \mathbf{D}_i$ as a non-increasing limit of Cartier big and nef toroidal b-divisors on (X, U). Then, by Lemma 5.13, it follows that

$$\mathrm{vol}(\mathbf{D}) = \lim_{i\to\infty} \mathrm{vol}\,(\mathbf{D}_i) = \lim_{i\to\infty} \mathbf{D}_i^n = \mathbf{D}^n. \tag{5.5}$$

On the other hand, consider the semigroup $S(\mathbf{D})$ used in the construction of the Okounkov body $\Delta_{\mathbf{D}}$ (see discussion preceeding Definition 5.6) and set

$$S(\mathbf{D})_\ell = S(\mathbf{D}) \cap (\mathbb{Z}^n \times \{\ell\})$$

for any integer number $\ell > 0$. By [30, Proposition 2.1], we have that

$$\lim_{\ell\to\infty} \frac{\#S(\mathbf{D})_\ell}{\ell^n/n\mathcal{W}} = \mathrm{vol}_{\mathbb{R}^n}\left(\Delta_{\mathbf{D}}\right).$$

This implies that

$$\mathrm{vol}(\mathbf{D}) = \limsup_\ell \frac{h^0(X, \ell\mathbf{D})}{\ell/n\mathcal{W}} = \limsup_\ell \frac{\#S(\mathbf{D})_\ell}{\ell/n\mathcal{W}} = \lim_{\ell\to\infty} \frac{\#S(\mathbf{D})_\ell}{\ell/n\mathcal{W}} = \mathrm{vol}_{\mathbb{R}^n}\left(\Delta_{\mathbf{D}}\right),$$

as we wanted to show. □

Remark 5.15 As we will show in a forthcoming paper with D. Holmes and R. de Jong Theorem 5.14 is not true if we drop the condition of being toroidal.

We obtain the following two corollaries.

Corollary 5.16 *The function* vol *is continuous on the space of big and nef toroidal b-divisors on (X, U).*

Proof By Theorem 5.14 the volume function agrees with the degree function on the space of big and nef toroidal b-divisors on (X, U). By Theorem 4.32, it is continuous. □

The following is a Brunn–Minkowski type inequality.

Corollary 5.17 *Let $\mathbf{D}$ and $\mathbf{F}$ be two nef and big toroidal b-divisors on (X, U). Then the following Brunn–Minkowski type inequality holds true.*

$$(\mathbf{D}^n)^{1/n} + (\mathbf{F}^n)^{1/n} \geqslant ((\mathbf{D} + \mathbf{F})^n)^{1/n}.$$

Proof Consider the associated Okounkov bodies $\Delta_{\mathbf{D}}$ and $\Delta_{\mathbf{F}}$, respectively. Then, using Theorem 5.14, the inequality follows from a standard result in convex geometry about volumes of convex sets (see e.g. [33]). □

Remark 5.18 As in the last part of Section 4.4, we may assume that k is of characteristic zero and state all of the above results without the choice of a fixed toroidal structure.

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