Universal scaling and diagonal conductivity in the integral quantum Hall effect

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We perform a numerical finite-size study for the static homogeneous diagonal conductivity σ_{xx} at the critical filling factor 3/2 for different microscopic realizations of the random impurity potential. The variation of σ_{xx} with the system size defines a scaling function. It turns out to be independent of the particular realization of disorder and also of the Landau-level index. However, the diagonal conductivity in the second-lowest Landau level varies strongly with disorder. The universal critical conductivity is recovered only asymptotically when the correlation length of the potential is increased.
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The scaling scenario of the integer quantum Hall effect[1] developed by Pruisken and Khmel'nitzkiĭ[2] suggests that the delocalization transition, lying at the heart of the phenomenon, is similar in many respects to a second-order phase transition. In impurity potentials with particle-hole symmetry this transition occurs at the center of a Landau level at half-integer filling. Its properties, e.g., the critical exponents and the scaling functions, are thought to be universal with respect to the microscopic realization of the disorder and also the Landau-level index.[2–4] The idea of a universal delocalization transition is supported by experimental findings[5–7] and numerical calculations.[8–13] Moreover, the "law of corresponding states"[14] relates integer quantum Hall transitions to fractional ones and, also, these are believed to be in the same universality class.[4] However, it still is a matter of considerable debate, whether the critical conductivity is one of the universal properties or not.[15] Field-theoretical results suggest that the dissipative conductivity equals $e^2/2h$ independent of the character of the disorder and the Landau-level index.[3,4] The analysis of various network models[16] and also several numerical studies[11–13] corroborate this conclusion. The latter, however, have all been restricted to the lowest Landau level.

By contrast there is no convincing experimental evidence for a universal value of σ_{xx}^c.[16–18] In fact, the majority of experiments seems to be incompatible with the notion of a universal conductivity.[6,19,20] Macroscopic inhomogeneities of the electron density,[17] finite sample sizes,[12,13] screening, charging, and other correlation effects[21,22] might seriously impair the experimental observability. However in general electron interaction effects are expected not to change the universality class — at least, if the interaction potential is screened so that it drops faster than $1/r$.[23]

In this situation it clearly is desirable to put the theoretical ideas to a further test by calculating numerically the critical conductivity in the second-lowest Landau level (1LL). Numerical calculations for the 1LL are notoriously difficult as finite-size effects are known to be extremely important. In general, one has to consider at least one irrelevant scaling field when extrapolating the numerical data to the thermodynamic limit.[9] This extrapolation comes with uncertainties in the estimate of the critical conductivity, which make it difficult to compare the results for different realizations of the disorder. However, instead of first estimating σ_{xx}^c from scaling functions for different disorder realizations and then comparing it is simpler to compare the scaling functions directly. We show that all computed scaling functions for $\sigma_{xx}(L,N)$ collapse onto a single, universal curve after rescaling of the L *and* the σ_{xx} axes.

Our results on the diagonal Kubo conductivity calculated at the critical filling $n=3/2$ in the second-lowest Landau level can be summarized as follows: Within the range of system sizes considered the conductivity $\sigma_{xx}(\lambda,L,\varepsilon,N)$ as a function of the correlation length λ, the system width L, the imaginary frequency ε, and also the Landau level N obeys an universal scaling function:

$$\sigma_{xx}(\lambda,L,\varepsilon,N)=\mathcal{F}(a(\lambda,N)L\varepsilon^{1/2})/b(\lambda,N). \qquad (1)$$

Here $a(\lambda,N)$ and $b(\lambda,N)$ define the rescaling functions for the system size and the conductivity respectively. The critical conductivity is proportional to the inverse of $b(\lambda,N)$. For four different λ in the range from 0.0 to 1.5 (given in units of the magnetic length), it decays from $(1.07\pm0.03)e^2/h$ to $(0.44\pm0.04)e^2/h$.

Therefore our finite-size study does not lend support to the hypothesis that the critical conductivity is universal with respect to microscopic properties of the random potential in higher Landau levels. However, we cannot rule out universality as the conductivity turns out to be more sensitive to the system size than the localization length, so that our system sizes might be too small to see the universal behavior. In long-range potentials we find σ_{xx} to attain its semiclassical value $e^2/2h$ independent of the Landau-level index.[24]

We use static impurity potentials of varying correlation length λ and neglect electron-electron interactions completely. All states are assumed to lie in the second-lowest Landau level corresponding to a strong-field limit. For reasons of numerical efficiency the random Landau matrix model is used in calculating the matrix elements for varying

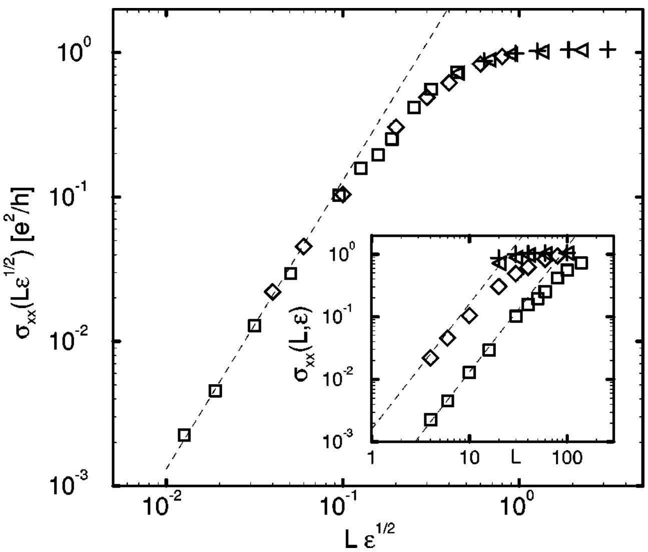

FIG. 1. $\sigma_{xx}(L\varepsilon^{1/2})$, the conductivity in the second-lowest Landau level for $\lambda=0$. In the inset the raw data are displayed. $\varepsilon=10^{-5}(\square)$, $\varepsilon=10^{-4}(\diamond)$, $\varepsilon=5\times10^{-4}(\triangleleft)$, $\varepsilon=10^{-3}(+)$. The dashed line indicates the power law εL^2.

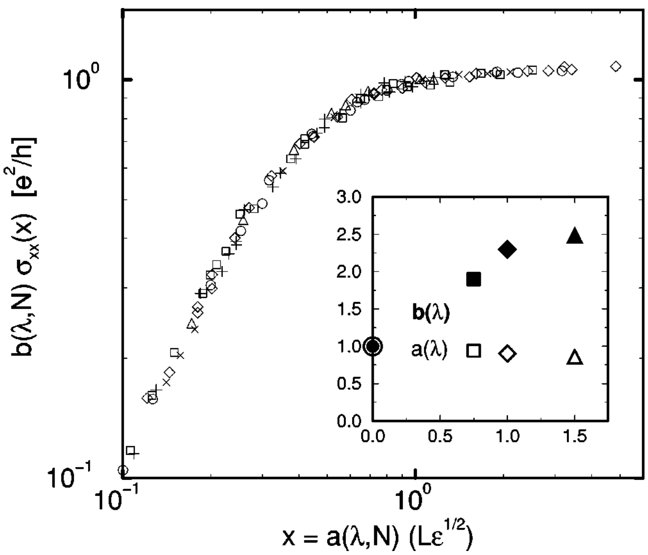

FIG. 3. Zoom into the critical regime of the universal scaling function for $\sigma_{xx}(N,\lambda,L\varepsilon^{1/2})$ for the second-lowest Landau level ($N=1$) and potential correlation lengths $\lambda=0(\bigcirc)$, $\lambda=0.75(\square)$, $\lambda=1.0(\diamond)$, and $\lambda=1.5(\triangle)$, and for $N=0$ with $\lambda=0(+)$, $\lambda=1$ $(\times)$. Inset: The scaling coefficients $a(\lambda)$ (empty symbols) and $b(\lambda)$ (filled symbols) for $N=1$.

λ.[25–27] So far this model has been successfully used in the accurate calculation of various critical properties in the integral quantum Hall effect.[9,12,13,25,28] As in previous work[12,13] we use a fast iterative Green-function method to calculate the static conductivity $\sigma_{xx}(L,\varepsilon,\lambda)$ as a function of λ and the sample width L. ε denotes the imaginary frequency. The thermodynamic limit is obtained in the limit $L\rightarrow\infty$ first and then $\varepsilon\rightarrow0^+$. To reduce statistical errors to a 2% level system lengths between $M=4\times10^5$ for the large systems and up to $M=10^7$ for the smaller systems have been necessary. To cover a wide parameter range the system width was varied within $L=8$ and $L=140$, ε within 10^{-5} and 10^{-3}, and λ within 0.0 to 1.5.

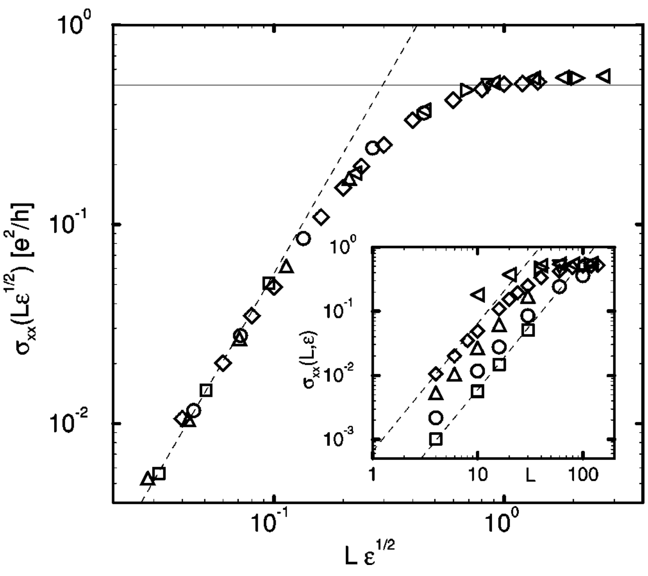

FIG. 2. $\sigma_{xx}(L\varepsilon^{1/2})$ in the second-lowest Landau level for $\lambda=0.75$. Inset: raw data, $\varepsilon=10^{-5}(\square)$, $\varepsilon=2\times10^{-5}(\bigcirc)$, $\varepsilon=5\times10^{-5}(\triangle)$, $\varepsilon=10^{-4}(\diamond)$, $\varepsilon=2\times10^{-4}(\triangledown)$, $\varepsilon=3\times10^{-4}(\triangleright)$, $\varepsilon=5\times10^{-4}(\triangleleft)$. The dashed line indicates the power law εL^2, and the solid line is $\sigma_{xx}=e^2/2h$.

In Figs. 1 and 2 the numerical results for correlation lengths $\lambda=0$ and $\lambda=0.75$ are depicted. (The complete data for all other λ will be published elsewhere.) Clearly the dynamical exponent z equals 2 in a diffusive system. Therefore all data collapse onto one curve when we plot the conductivities as function of $L\varepsilon^{1/2}$. The linear behavior of the scaling function at small arguments εL^2 is a consequence of the finite sample size. We have indicated it by dashed lines.

As shown in the scaling plot, Fig. 3, all data of $\sigma_{xx}(\lambda,L,\varepsilon,N)$ in the 1LL and also the data obtained for the 0LL with $\lambda=0,1$ can be mapped onto a universal scaling function when one rescales L and σ_{xx} axis.[29] This is our central result. Taken at face value our calculation supports the hypothesis of a universal transition with respect to disorder and the Landau-level index.

The rescaling factors $a(\lambda)$ for the system size and $b(\lambda)$ for the conductivity of the $N=1$ Landau level are displayed in the inset. Here it can clearly be seen that the amplitude of the scaling function $1/b(\lambda)$ that is proportional to the critical conductivity in the thermodynamic limit has a pronounced

TABLE I. Scaling factors and critical conductivities obtained from the data in Fig. 3. The value (*) has been calculated in Ref. 13 from the dynamic conductivity in the $\omega\rightarrow0$ limit.

N	λ $[l]$	a	b	$\sigma_{xx}^c(e^2/h)$
1	0.00	1.00	1.00	1.07 ± 0.03
1	0.75	0.94	1.90	0.56 ± 0.03
1	1.00	0.90	2.30	0.47 ± 0.04
1	1.50	0.86	2.48	0.44 ± 0.05
0	0.00	0.73	2.06	0.52 ± 0.04
0	1.00	0.78	2.31	0.46 ± 0.04
0	2.00			0.50 ± 0.02 (*)

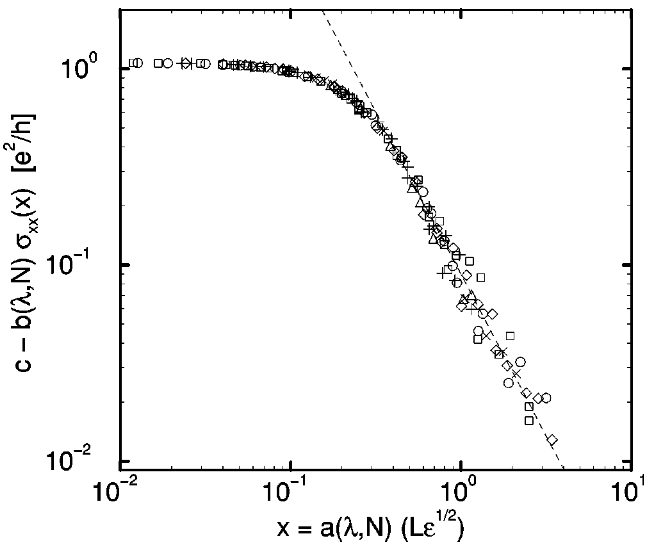

FIG. 4. Scaling towards the estimated fixed point value $c = (1.07 \pm 0.03)e^2/h$. (The symbols are the same as in Fig. 3.) The dashed line displays the power law with $y = 1.65 \pm 0.1$.

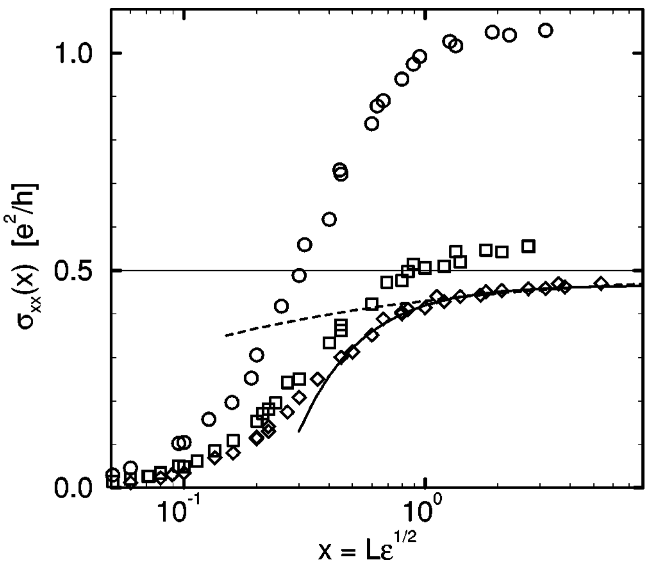

FIG. 5. The unscaled data for the second-lowest Landau level. (The symbols are the same as in Fig. 3.) The dashed line indicates the power law with $y = 0.4$ and the solid line is the fit with $y = 1.65$.

dependence on λ. The dependence of the factor $a(\lambda)$ rescaling the system size on the potential correlation length, on the other hand, is much weaker.

The data indicate a decrease of the critical conductivity between the short-range limit $\lambda = 0$ and $\lambda = 1.0$ by 50% as given in Table I. In the present case we use the data for $\lambda = 0$ in the 1LL for ''calibration'' that have the smallest statistical errors and the smallest variation of $\sigma_{xx}(L\varepsilon^{1/2})$ in the limit of a large argument.

In what follows we turn our attention to the scaling function $\mathcal{F}$ itself. Quite generally, near criticality one expects $\mathcal{F}$ to have an expansion

$$\mathcal{F}(x) = \mathcal{F}^c - a x^{-y} + \cdots \qquad (2)$$

that describes the finite size corrections to $\mathcal{F}^c$ in the thermodynamic limit $x \to \infty$ in terms of a power law.

As can be seen from Fig. 4 this is indeed the case. Our estimate for the exponent is $y = 1.65 \pm 0.1$. This result is not quite as accurate as the outcome for the conductivity: On the one hand, the power law proper can be observed only in a vicinity not larger than 30% of the fixed point value of σ_{xx} (see Fig. 4). On the other, statistical fluctuations and a systematical error in our data due to a finite epsilon keep us from approaching the maximum closer than approximately 2–3 %. Hence the accessible interval for the variation of σ_{xx} is not larger than an order of magnitude.

In what follows we discuss our results within the framework of the scaling theory of second-order phase transitions. According to the latter one would expect a scaling relation

$$\sigma_{xx}(g_1, g_2, .., L) = L^{x_\sigma} \mathcal{B}(g_1 L^{-y_{\mathrm{irr},1}}, g_2 L^{-y_{\mathrm{irr},2}}, ..), \qquad (3)$$

where x_σ denotes the scale dimension of the conductivity. Assuming that the scaling indices $y_{\mathrm{irr},1}, y_{\mathrm{irr},2}$ of the two most important irrelevant scaling fields are strictly positive one can expand the scaling function $\mathcal{B}$ in the coupling constants g_1, g_2 and obtains

$$\sigma_{xx}(g_1, g_2, .., L) = L^{x_\sigma}(\mathcal{B}^c + a_1 g_1 L^{-y_{\mathrm{irr},1}} + a_2 g_2 L^{-y_{\mathrm{irr},2}} + o(L^{-2y_{\mathrm{irr},1}}, L^{-2y_{\mathrm{irr},2}}, L^{-y_{\mathrm{irr},3}}). \qquad (4)$$

$\mathcal{B}^c$ is a number that coincides with the conductivity since $x_\sigma = 0$. The exponent $y_{\mathrm{irr},1}$ has been identified previously.[9] It describes the corrections due to the finite system size to the amplitude of the scaling function $\Lambda(L)$ of the localization length at criticality. In studies using the random Landau matrix model[9] and the Chalker network model[30] one finds similar results: $y_{\mathrm{irr},1} = 0.35 \pm 0.05$ in agreement with an analytical conjecture $y_{\mathrm{irr},1} = D_2/2 - 1/\nu$ where D_2 denotes the second of the generalized multifractal dimensions.[30] One expects $y_{\mathrm{irr},1}$ to describe the approach to the thermodynamic limit for samples large enough.

For $\lambda < 1$ we do not observe the exponent $y_{\mathrm{irr},1}$. One possible explanation is that the corresponding amplitude a_1 in the expansion 4 is small. Alternatively, the system sizes could be smaller than an irrelevant length scale so that we cannot see universal behavior including the universal conductivity. Instead we observe power-law behavior with an exponent $y_{\mathrm{irr},2} = 1.65 \pm 0.1$ While this scenario explains the discrepancy in the exponents and the nonuniversal conductivity, the fact that we see scaling of our data for all λ remains unexplained. Moreover, we cannot find any numerical clues that the conductivity approaches the universal value of 0.5 when the system size increases.

In the case of the long-range potential, $\lambda = 1$ the data are not incompatible with the expected exponent $y_{\mathrm{irr},1}$ (broken line) and $\sigma_{xx} = 0.5$ as can be seen in Fig. 5 where we show the unscaled data for $N = 1$ and $\lambda = 0, 0.75, 1$. [For comparison, the power law with $y = 1.65$, as obtained from the collapse of all data onto the universal scaling function (Fig. 4), is displayed.]

In conclusion we have found that the conductivity at the quantum phase transition between two Landau levels can be described by a universal scaling function that is independent of the Landau-level index and of the particular realization of the random impurity potential. This conjecture has been tested for $N=0,1$ and $\lambda=0$ to 1.5. In long-range potentials the universal conductivity is recovered.

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[1] K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).

[2] A. M. M. Pruisken, Phys. Rev. Lett. **61**, 1297 (1988); D. H. Khmel’nitskiĭ, Zh. Eksp. Teor. Fiz. Pis’ma Red. **38**, 454 (1983) [JETP Lett. **38**, 553 (1983)].

[3] M. P. A. Fisher, G. Grinstein, and S. M. Girvin, Phys. Rev. Lett. **64**, 587 (1990); M. P. A. Fisher, *ibid.* **65**, 923 (1990).

[4] S. Kivelson, D.-H. Lee, and S.-C. Zhang, Phys. Rev. B **46**, 2223 (1992).

[5] H. P. Wei, D. C. Tsui, M. A. Paalanen, and A. M. M. Pruisken, Phys. Rev. Lett. **61**, 1294 (1988); H. P. Wei, S. Y. Lin, D. C. Tsui, and A. M. M. Pruisken, Phys. Rev. B **45**, 3926 (1992).

[6] S. Koch, R. J. Haug, K. von Klitzing, and K. Ploog, Phys. Rev. Lett. **67**, 883 (1991); Phys. Rev. B **43**, 6828 (1991); *ibid.* **46**, 1596 (1992).

[7] L. W. Wong, H. W. Jiang, N. Trivedi, and E. Palm, Phys. Rev. B **51**, 18 033 (1995); R. J. F. Hughes *et al.*, J. Phys. C **6**, 4763 (1994).

[8] B. Huckestein, Rev. Mod. Phys. **67**, 357 (1995).

[9] B. Huckestein, Phys. Rev. Lett. **72**, 1080 (1994).

[10] B. Mieck, Z. Phys. B **90**, 427 (1993).

[11] Y. Huo and R. N. Bhatt, Phys. Rev. Lett. **68**, 1375 (1992); Y. Huo, E. Hetzel, and R. N. Bhatt, Phys. Rev. Lett. **70**, 481 (1993).

[12] B. M. Gammel and W. Brenig, Phys. Rev. Lett. **73**, 3286 (1994).

[13] B. M. Gammel and W. Brenig, Phys. Rev. B **53**, R13 279 (1996).

[14] J. K. Jain, S. A. Kivelson, and N. Trevedi, Phys. Rev. Lett. **64**, 1297 (1990).

[15] T. Ando, J. Phys. Soc. Jpn. **55**, 3199 (1986). Here the Thouless number has been calculated, which is assumed to be proportional to the conductivity.

[16] A. Szafer *et al.* in *Granular Nanoelectronics*, edited by D. K. Ferry (Plenum Press, New York, 1991); I. M. Ruzin and S. Feng, Phys. Rev. Lett. **74**, 154 (1995).

[17] I. M. Ruzin, N. R. Cooper, and B. I. Halperin, Phys. Rev. B **53**, 1558 (1996).

[18] P. T. Coleridge, Phys. Rev. Lett. **72**, 3917 (1994).

[19] M. Yamane, J. Wakabayashi, and S. Kawaji, J. Phys. Soc. Jpn. **58**, 1899 (1989).

[20] L. P. Rokhinson, B. Su, and V. J. Goldman, Solid State Commun. **96**, 309 (1995).

[21] W. Brenig, B. M. Gammel, and P. Kratzer, Z. Phys. B **103**, 417 (1997).

[22] I. V. Kukushkin, R. J. Haug, K. von Klitzing, and K. Eberl, Phys. Rev. B **51**, 18 045 (1995).

[23] A. M. M. Pruisken, Europhys. Lett. **31**, 543 (1995); D.-H. Lee and Z. Wang, Phys. Rev. Lett. **76**, 4014 (1996).

[24] F. Evers and W. Brenig, Z. Phys. B **94**, 155 (1994).

[25] B. Huckestein and B. Kramer, Phys. Rev. Lett. **64**, 1437 (1990).

[26] B. Mieck, Europhys. Lett. **13**, 453 (1990).

[27] In contrast to Refs. 8 and 25 we use normal deviates instead of box-distributed deviates, which is more physical and leads to slightly faster convergence in the Green-function iteration.

[28] Y. Ono, T. Ohtsuki, and B. Kramer, J. Phys. Soc. Jpn. **60**, 270 (1991); Y. Ono and S. Fukada, *ibid.* **61**, 1676 (1992).

[29] The data for $N=0$ with $\lambda=0$ and box-distributed deviates are taken from Ref. 12, the data for $\lambda=1$ have been newly calculated using normal deviates. Notably, even data points far from the fixed point, down to 10^{-3} on the vertical axis in Fig. 3 collapse onto the same universal curve.

[30] F. Evers and W. Brenig, Phys. Rev. B **57**, 1805 (1998).