Equivariant homotopy theory in the presence of bornologies

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Summery

This thesis has three branches. In the first part, we generalize – after recalling basic definitions, examples and properties – the notion of bornological coarse spaces to obtain a complete and co-complete category which has still very close connections to the original category of bornological coarse spaces. We proceed by defining equivariant generalized coarse homology theories, together with the stable co-complete ∞-category of equivariant generalized coarse motivic spectra and the universal equivariant coarse homology theory. We finish this part by proving that the fully faithful inclusion of equivariant bornological coarse spaces into its generalization constructed in this thesis induces an equivalence of generalized and non-generalized equivariant coarse motivic spectra and derive an equivalence of homotopy theories on both categories.

In the second part we introduce equivariant bornological topological spaces, as well as their generalization to a complete and co-complete category in a similar manner is in the first part. Building on the category of equivariant bornological topological spaces we define the notion of equivariant local homology theories, and we develop a universal such theory leading to the category of equivariant local motivic spaces and spectra.

The final part of this thesis is devoted to Elmendorf's theorem. This classical theorem provides an equivalence between the homotopy category of G-topological spaces (for a finite group G) and the category of presheaves on the orbit category $G\mathrm{Orb}$ of G. We apply similar constructions to the category of G-bornological topological spaces where we do not require G to be finite anymore. Then we introduce a special shape of equivariant local homology theories called Bredon-style theories (inspired by the classical notion of Bredon (co-)homology). We associate to every equivariant local homology theory a Bredon-style theory and construct a comparison morphism. By studying under which circumstances this comparison morphism is an equivalence we deduce special cases where the analogue of Elemendorf's theorem holds in our setup.

Contents

Introduction

Dissertatio mea est divisa in partes tres, quarum unam “Coarse geometry”, aliam “Local homology theory”, tertiam “Analogue of Elmendorf’s theorem” appellantur.

Coarse geometry

The first part of this thesis deals with coarse geometry. The basic idea and intuition of coarse geometry is to study spaces from a far-away viewpoint, i.e. the main issue of interest is the large-scale properties of a space. Intuitively speaking, two spaces that “look the same if you look at them sufficiently far away” are considered equivalent. For an illustrative example you may think of the integers and the real line. This idea was first formalized and introduced by John Roe in [Roe93] (see also [Roe96; Roe03]). His original motivation was to use coarse geometry to study index theory of complete Riemannian manifolds. To that end he introduced *coarse cohomology* in [Roe93]. Dually, *coarse homology* led to very fruitful development in this field of mathematics. Examples for coarse homology theories are coarse (ordinary) homology, coarse K-homology and coarsifications of locally finite homology theories. Especially the latter two examples are important as they lead to the theory of the *coarse assembly map* (c.f. [Roe96, Sec. 5]) and thus the *coarse Baum-Connes conjecture* concerning invertability of this assembly map (see [HR95; Yu95a; Yu95b]). This conjecture (though meanwhile known to be wrong in general: [HLS02]) has a huge impact: Injectivity yields results to the positive curvature question (e.g. [Roe16]) having consequences for the topology of a manifold. Another example: Coarse Baum-Connes conjecture would imply the Novikov-conjecture (e.g. [FWY20; RTY14]) concerning oriented homotopy invariance of higher signatures of a manifold.

In order to grasp the underlying basic structure of the proof of the coarse Baum-Connes conjecture under certain assumptions due to [Wri05], the authors Bunke and Engel developed a new axiomatic approach in coarse geometry in [BE20] by introducing the notion of bornological coarse spaces. These spaces are coarse spaces equiped with a “compatible” bornology (for a more detailed explanation of the benefits of that, we refer to the introduction of chapter 1). Every “classical” coarse space defined by Roe also constitutes a bornological coarse space in a canonical way. Using this new category of bornological coarse spaces, the authors in [BE20] axiomatise *coarse homology theories* and analyse the proof by [Wri05]. It turns out that this proof is not specific to the coarse assembly map for coarse K-homology, but in fact shows the claim for a general class of natural transformation between certain coarse homology theories in the language of [BE20] (see [BE20; BE20a]).

Using this as a starting point, it led to a new field of research. In [BEKW] they generalize everything to the equivariant world, in particular they introduced equivariant coarse ordinary homology and equivariant coarse algebraic K-homology. Also they develop equivariant coarse Waldhausen K-homology in [BKW18], equivariant coarse K-homology associated to left-exact ∞-categories in [BCKW] and topological equivariant coarse K-homology associated to C^*-categories in [BE20a], as well as the dual notion, namely coarse cohomology theory in [BE17]. The latter was used by the same authors to prove an open conjecture of Klein about dualizing spectra of groups. In his thesis [Cap19] develops equivariant coarse cyclic homology and coarse Hochschild homology which are related to equivariant coarse algebraic K-homology via trace maps.

In Chapter 1 of this thesis we recall basic definitions, examples and properties of bornological coarse spaces. We also elaborate on connections to the classical coarse spaces as well as to other generalizations of Roe's coarse category, e.g. by [Mit01; Har20] and many more. We also study categorical properties of the category of bornological coarse spaces and we will see that it is neither complete nor co-complete. Motivated by that we proceed by a generalization of these spaces, where we remove the bornological-axiom that any finite subset of a space is bounded. We may think of this generalization in analogy to “allowing spaces to be non-Hausdorff”. In fact, we give some nice examples that underlines this analogy. However, by this generalization we obtain the category of generalized bornological coarse spaces which proves to be bi-complete and has very close connections to the category of bornological coarse spaces due to [BE20]. In particular, we deduce a precise criterion for a diagram to admit a colimit in the latter category. Furthermore, in preparation for Chapter 2, we study coarse equivalence as the generalization of the classical notion of coarse equivalence introduced by Roe. We elaborate helpful properties and we prove that the ∞-category obtained by inverting these coarse equivalences is equivalent to the nerve of the 1-categorical homotopy category of bornological coarse spaces modulo "closeness". The latter category is known as the "coarse category" in the classical coarse geometry literature.

In Chapter 2 we develop in complete analogy to [BE20; BEKW] the notion of equivariant generalized coarse homology theory. It is a functor $F\colon \boldsymbol{G}\mathbf{B\widetilde{orn}Coarse} \to \mathcal{C}$ from the category of equivariant generalized bornological coarse spaces into a stable co-complete ∞-category $\mathcal{C}$ satisfying certain axioms (coarse excision, U-continuity, vanishing on flasques and coarse invariance). We also construct in a motivic way the *universal equivariant generalized coarse homology theory*

$$\mathrm{Y\tilde{o}}_G\colon \boldsymbol{G}\mathbf{B\widetilde{orn}Coarse} \longrightarrow \boldsymbol{G}\mathbf{S\tilde{p}}\mathcal{X}$$

with values in *equivariant generalized coarse motivic spectra*. This functor is universal in the sense that a colimit-preserving functor $\boldsymbol{G}\mathbf{S\tilde{p}}\mathcal{X} \to \mathcal{C}$ is the same as a equivariant coarse homology theory via precomposition with $\mathrm{Y\tilde{o}}_G$. The key theorem in that chapter is the equivalence between the ∞-categories of generalized and non-generalized equivariant coarse motivic spectra, which implies an equivalence between the category of equivariant coarse homology theories developed in [BEKW] and generalized equivariant coarse homology theories developed in this thesis.

Local homology theory

As mentioned above the coarse Baum-Connes conjecture concers invertibility of the coarse assembly map, a comparison morphism between the coarsification of locally finite K-homology and coarse K-homology. After setting up the new framework of bornological coarse spaces and coarse homology theories on them, the authors in [BE20] studied transfering the existing proof of coarse Baum-Connes ([Wri05]) to a more general situation, namely an assembly map between an arbitrary coarse homology theory and the coarsification of its associated locally finite homology theory.

The notion of locally finite homology theory – also known as Borel-Moore homology (originally due to [BM60] and in the form of locally finite homology theory in e.g. [Spa93; HR96]) – has been axiomatised in the context of coarse geometry by [BE20, Sec. 7] to be a functor

$$F\colon \mathbf{TopBorn} \longrightarrow \mathcal{C}$$

from the category of topological bornological spaces to a complete and co-complete stable ∞-category having certain properties. The domain category consists of topological spaces equiped with a bornology which is "compatible" with the topology. The functor is a locally finite homology theory on **TopBorn** if in addition to homotopy invariance and excision, it fulfills the local finiteness condition, i.e. if for all spaces X we have

$$\lim_{B \in \mathcal{B}_X} F(X \setminus B) \simeq 0$$

where the limit runs over the members of the bornology $\mathcal{B}_X$ of X, i.e. over all bounded subsets of X. This can be understood as a generalization of [WW95] (also related work is done in [CP98]). An example for such a theory is analytic locally finite K-homology [BE20, Def 7.66]. To every such theory Bunke and Engel construct an associated coarse homology theory, called the *coarsification* of the locally finite homology theory. These motivate the generalization of coarse assembly maps (see discussion above and [BE20a]).

However, the local finiteness condition is formulated as a limit hence mapping out of such a theory is quite difficult. Also there cannot be constructed locally finite homology theories in a motivic fashion like it was done for coarse homology theories. Both shortcomes can be fixed if we replace the locally finite condition with the condition "vanishing on flasques", which is weaker but still powerful enough. We refer to [BE20a] for a more precise elaboration of that assertion. In that paper they construct a natural transformation between a coarse homology theory and the coarsification of its associated *local homology theory* (obtained by replacing the locally finiteness condition by vanishing on flasques). The local homology theories there are defined on a category of uniform bornological coarse spaces, not on topological coarse spaces, however they construct close connections between both notions.

In this thesis we define the notion of local homology theory on the category of topological bornological spaces and its equivariant generalization $\boldsymbol{G}$**TopBorn**. To that end, we repeat all basic definitions, constructions, properties and examples of topological bornological spaces and then introduce the notion of equivariant bornological topological spaces in Chapter 3. We also study categorical properties and – like in the coarse part of this thesis – we see that this category is neither co-complete nor complete. However, the same generalization that bi-completed the category of bornological coarse spaces, also yields a bi-complete generalization of $\boldsymbol{G}$**TopBorn**. Unfortunately, in this situation we do not expect that local homology theories (to be defined) on the category $\boldsymbol{G}$**TopBorn** and its generalization $\boldsymbol{G}\widetilde{\mathbf{TopBorn}}$ coincide.

Chapter 4 is then devoted to defining local homology theory on $\boldsymbol{G}\mathbf{TopBorn}$. Similarly as in Chapter 2, we also construct motivically the stable ∞-category of local motivic spectra and the universal equivariant local homology theory

$$\mathrm{Yo}_G \colon \boldsymbol{G}\mathbf{TopBorn} \longrightarrow \boldsymbol{G}\mathbf{Sp}\mathcal{TB}\,.$$

For related work of this proper equivariant homology theory we refer to e.g. [DHLPS; BEUV; Lüc02; Phi88; San05; Dwy08; BHS10; Bár14].

An analogue of Elmendorf's theorem

On the category $\boldsymbol{G}\mathbf{Top}$ of topological spaces with continuous G-action Bredon introduced in [Bre67] the "fine" homotopical structure where a morphism $f \colon X \to Y$ is called a *weak G-homotopy equivalence* if for all (closed) subgroups H of G the induced map on H-fixed point subspaces $f^H \colon X^H \to Y^H$ is a non-equivariant weak homotopy equivalence. Every G-space X gives rise to a "system of Fixed Point Sets" (the title of [Elm83]) via

$$\begin{aligned} \Phi \colon \boldsymbol{G}\mathbf{Top} &\longrightarrow \mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Top}) \\ X &\longmapsto (G/H \mapsto X^H) \end{aligned}$$

In [Elm83] Elmendorf constructed (for G compact) a functor which is inverse to Φ up to homotopy hence proving that Φ induces an equivalence of the respective homotopy categories, a fact which is nowadays known as "Elmendorf's theorem". It has been generalized to a simplicial Quillen equivalence in [DK84] and it has been reproven in numerous similar setups, e.g. [Sey83; CP96; Pia91; Ste10; Ste16; May96]. It has later been formulated for G-categories and G-posets ([Rub20; Boh+14]).

In modern language of ∞-categories Elmendorf's theorem gives an equivalence between $\boldsymbol{G}\mathbf{Top}[W_G^{-1}]$ and $\mathbf{PSh}(\boldsymbol{G}\mathbf{Orb})$ (where W_G denotes the collection of weak G-homotopy equivalences, and by $\mathbf{PSh}$ we mean space-valued presheaves). A consequence of this is that for a co-complete stable ∞-category $\mathcal{C}$ an equivariant homology theory on $\boldsymbol{G}\mathbf{Top}$ with values in $\mathcal{C}$ is equivalently given by a functor $\boldsymbol{G}\mathbf{Orb} \to \mathcal{C}$ (we refer to the introduction of section 5.2 for details).

In Chapter 5 we construct an analogue of Elmendorf's theorem in the context of equivariant bornological topological spaces and equivariant local homology theories. While we do not know (nor do we expect to) whether Elmendorf's theorem holds in this new situation, we elaborate partial answers to that question. More precisely we link the analogue of Elmendorf's inverse functor to the construction of local homology theories of a certain shape, called Bredon-style homology theories (c.f. [Jos07; Bre68; Ill75]). To every equivariant local homology theory we can associate an Bredon-style theory and we ask the question whether (or under which circumstances) the comparison morphism between those two homology theories is an equivalence.

We derive that for a finite group G we have an Elmendorf-style adjunction between $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$ (being the analogue of $\boldsymbol{G}\mathbf{Top}[W_G^{-1}]$) and $\mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB})$ (being the counterpart to the presheaf-category $\mathbf{PSh}(\boldsymbol{G}\mathbf{Orb})$ in Elmendorf's theorem), and that the unit of this adjunction is an equivalence, while the co-unit can be proven to be an equivalence only after restriction to a certain full subcategory of $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$.

Conventions

Within this thesis we freely deploy the language of ∞-categories in the form developed by [Cis19; Joy08; Lur09]. Also, we overcome size issues by using Grothendiecks theory of nested universes.

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Yesterday is history.
Tomorrow is a mystery.
And today? Today is a gift
That's why we call it the present.

ELEANOR ROOSEVELT

Chapter 1
Equivariant bornological coarse spaces

In this chapter we deal with the category of (equivariant) bornological coarse spaces as building blocks for coarse homology theories. Historically, the first instances of coarse spaces can be found in works of John Roe in [Roe93] or more detailed in [Roe03]. A coarse space is a set equipped with a collection of "entourages" which are considered as "keeping things under controll" (on a large-scale perspective). In order to capture the large-scale behaviour of a space (the basic idea of coarse geometry), it is enough to study coarse spaces up to "coarse equivalence". A prominant example for this notion is the coarse equivalence between the integers $\mathbb{Z}$ and the reals $\mathbb{R}$.

Every coarse space naturally gives a notion of "boundedness" (c.f. coarsely bounded subsets, Definition 1.1.15). This notion is closed under the operation of "controlled thickening" (defined via entourages). The notion of bounded subsets plays a huge role in classical coarse geometry.

Independently of coarse structures there is a rich theory of bornologies (especially in functional analysis) which goes back to [Mac45]. Basically a bornology is a nice categorical setup to keep track of subsets which are considered/declared bounded.

Now, the authors Bunke and Engel in [BE16] (meanwhile [BE20]) came up with the idea to merge these two notions to define a bornological coarse space – a coarse space equipped with a bornology such that these two structures are compatible in a suitable way. By doing so they introduce a bit more flexibility to the study of coarse spaces, since now we have more freedom over which subsets we call bounded. It turns out that the coarsely bounded subsets from classical coarse theory generate the minimal bornology compatible with the coarse space. In particular, the theory of bornological coarse spaces contains the classical theory of coarse spaces as special cases.

Taking the category of bornological coarse spaces as a starting point there has been a fruitful development in various directions. Relevant for the content of this particular chapter, we especially mention the generalization of bornological coarse spaces to the equivariant world, where a group acts on a bornological coarse space by automorphisms. This notion together with lifting the theory developed on non-equivariant bornological coarse spaces to this generalized setting is due to [BEKW]. There is an even more generalized version of equivariant bornological coarse spaces by the author of this thesis in [Hei19], which yield an bi-complete category and is very well-behaved with respect to the theory of ordinary equivariant bornological coarse spaces.

This chapter is organized as follows: In the first section we recollect the theory of bornological coarse spaces developed by [BE16] with additional remarks and examples. It is followed by a brief recap and close study of equivariant bornological coarse spaces in section 1.2. In the end of that section we motivate the generalization of equivariant bornological coarse spaces, which will then be introduced and intensively studied in section 1.3. In particular, we elaborate the deep connection to the category of non-generalized bornological coarse spaces. We finish this chapter with a section about coarse equivalence. There we recall the definition of closeness of morphisms and coarse equivalences which was lifted from classical coarse theory introduced by Roe to the category of bornological coarse spaces by Bunke and Engel. We then give several examples and properties which were motivated by not only [BE20], but also by [Mit01; Har20; Roe03; Wul20] and many others. We then briefly study coarse invariants and coarse invariance and we finish by a theorem comparing the "homotopy category" of generalized equivariant bornological coarse spaces (i.e. the category up to coarse equivalences) and the ∞-category obtained by inverting coarse equivalences via Dwyer-Kan localization.

1.1 The category of bornological coarse spaces

In this section we provide definitions, basic properties and examples for bornological coarse spaces which were introduced in this manner by Bunke and Engel in [BE20] and later generalized by those authors together with Kasprowski and Winges in [BEKW] to the notion of equivariant bornological coarse spaces which we will treat in section 1.2. An important class of examples and the main source of intuition is provided by metric spaces where we already have a notion and intuition of "boundedness" and "thickenings". However, the category of metric spaces is not well-behaved (e.g. non-existence of coproducts or infinite products) while the category of bornological coarse spaces behaves better in this respect (although the category of generalized bornological coarse spaces (see section 1.3) is even better-behaved). Therefore we may view the category of bornological coarse spaces as a "nice generalization" of metric spaces.

We start this section with definitions regaring to coarse spaces and first examples. We will see that a coarse structure gives rise to a canonical notion of bounded subsets (Definition 1.1.15) which forms a compatible bornology on the coarse space. This anker point is used to transition to the more flexible notion of bornological coarse spaces, where we allow to choose more freely what subsets we call "bounded". We finish this section with the categorical definition of bornological coarse spaces and several examples.

If not specified otherwise, during this section X will denote a set and G will denote a group. The powerset of X is denoted by $\mathcal{P}(X)$. For subsets U and V of $X \times X$ and B a subset of X we define

$$U^{-1} := \left\{(x,y) \in X \times X \,\middle|\, (y,x) \in U\right\} \tag{Inverses}$$

$$U \circ V := \left\{(x,y) \in X \times X \,\middle|\, \exists z \in X : \ (x,z) \in U, (z,y) \in V\right\} \tag{Composition}$$

$$U[B] := \left\{x \in X \,\middle|\, \exists b \in B : \ (x,b) \in U\right\}. \tag{Thickening}$$

Example. For a map $f\colon X \to X$ we denote by $\Gamma_f := \left\{(x, f(x)) \,\middle|\, x \in X\right\}$ the graph of f. Then for another map $g\colon X \to X$ we have

$$\Gamma_f \circ \Gamma_g = \Gamma_{g \circ f}.$$

We start by recalling the classical definition of a coarse structure on X given e.g. in [Roe03] followed by some easy first examples.

Definition 1.1.1. A *coarse structure* on X is a subset $\mathcal{C}_X$ of $\mathcal{P}(X \times X)$ which is closed under taking subsets, finite unions, inverses and compositions and which furthermore contains the diagonal Δ_X of X.
The elements in $\mathcal{C}_X$ are called *entourages* or *controlled subsets* of X. A *coarse space* is a set X equipped with a coarse structure $\mathcal{C}_X$.

Remark 1.1.2. There are slightly different conventions about the axiom $\Delta_X \in \mathcal{C}_X$. For example in [Mit01] a coarse structure is not required to contain the diagonal; and if it does, the coarse structure is called *unital*. However, in this thesis we will

always consider unital coarse structures and omit that adjective.
For motivating examples for the axioms of a coarse structure see e.g. [Bun21, Example 2.7, 2.8]. ♦

Example 1.1.3. The *maximal coarse structure* on X is given by $\mathcal{P}(X \times X)$. If X is endowed with this coarse structure we often write $X_{\max}$. ✠

Example 1.1.4. The collection of all subsets of the diagonal Δ_X forms a coarse structure on X. It is called the *minimal coarse structure* and we often write $X_{\min}$ for a set X equipped with the minimal coarse structure. ✠

Example 1.1.5 ([Roe03, Example 2.7]). The collection of all subsets U of $X \times X$ such that $U \setminus \Delta_X$ is finite forms a coarse structure on X, which the author in [Roe03] calls the *discrete coarse structure*. It is the minimal coarse structure on X such that the resulting coarse space is coarsely connected (see Definition 1.1.12). ✠

Definition 1.1.6. For any subset $\mathcal{S}$ of $\mathcal{P}(X \times X)$ we denote by $\mathcal{C}\langle\mathcal{S}\rangle$ the smallest coarse structure on X with $\mathcal{S} \subseteq \mathcal{C}\langle\mathcal{S}\rangle$. We call $\mathcal{C}\langle\mathcal{S}\rangle$ the coarse structure *generated by* $\mathcal{S}$.

Example 1.1.7. The minimal coarse structure is generated by the empty set. The maximal coarse structure is generated by $\{X \times X\}$. ✠

Example 1.1.8. Consider a metric space (X, d). For any r in $(0, \infty)$ we denote by

$$U_r := \left\{(x, y) \in X \times X \;\middle|\; \mathrm{d}(x, y) < r\right\}$$

the r-thickend neighborhoods of the diagonal of X. These generate a coarse structure

$$\mathcal{C}_X := \mathcal{C}\left\langle\left\{U_r \;\middle|\; r \in (0, \infty)\right\}\right\rangle$$

on X, which we call the *metric coarse structure* on X. We often denote the resulting coarse space by X_{d}. ✠

In [Wri03] the author studies properly positive scalar curvature using the following refinement of metric coarse structure:

Example 1.1.9. Consider a metric space (X, d). There exists the so-called C_0 *coarse structure* on X: A subset U of $X \times X$ is an entourage if “it gets thin at infinity”, i.e. if for all $\varepsilon > 0$ there exists a compact subset K of X such that for all (x, y) in $U \setminus (K \times K)$ we have $\mathrm{d}(x, y) < \varepsilon$. ✠

Example 1.1.10 ([Roe03, Example 2.8]). Consider a topological space X. We denote by $\mathcal{R}$ the collection of all relatively compact subsets of X. We call a subset U of $X \times X$ *symmetric* if $U^{-1} = U$. The coarse structure defined by

$$\mathcal{C}_X := \mathcal{C}\left\langle \left\{ U \in \mathcal{P}(X \times X) \text{ symmetric} \;\middle|\; \forall R \in \mathcal{R}: \; U[R] \in \mathcal{R} \right\} \right\rangle$$

is called the *indiscrete coarse structure* on X (c.f. Example 1.1.5). If X is compact, this coarse structure is simply the maximal one. On the other hand, if X is a proper[1] metric space, $\mathcal{C}_X$ agrees with the metric coarse structure (c.f. Example 1.1.8). ✠

Lemma 1.1.11. *Let X be a coarse space. Then the union $\bigcup_{U \in \mathcal{C}_X} U$ is an equivalence relation on X, i.e. $x \sim y$ if and only if $(x, y) \in U$ for some U in $\mathcal{C}_X$.*

Proof. Reflexivity comes from $\Delta_X \in \mathcal{C}_X$, symmetry and transitivity follow because $\mathcal{C}_X$ is closed under taking inverses and composition respectively. □

Definition 1.1.12. The equivalence classes of the relation in Lemma 1.1.11 are called *coarse components* of X. We write $\pi_0^{\text{coarse}}(X)$ for the set of coarse components of X. We say that X is *coarsely connected* if $\pi_0^{\text{coarse}}(X) \cong *$.

Example 1.1.13. Consider a set X with an extended metric d (i.e. infinite distances are allowed). In Example 1.1.8 we saw that d defines a coarse structure on X. The resulting coarse space X_{d} is coarsely connected if d is a metric. Note that the coarse structure has a countable generating system. ✠

The converse to the example above is also true. It is known as the "metrization theorem":

Proposition 1.1.14 (Metrization theorem, [Roe03, Thm 2.55])**.** *Let X be a coarse space such that $\mathcal{C}_X$ has a countable generating system $\mathcal{S} = \{U_1, U_2, U_3, \dots\}$. Then there exists an extended metric d on X such that the induced coarse structure coincides with $\mathcal{C}_X$. This extended metric is a metric if and only if X is coarsely connected.*

Proof. Inductively we define a sequence of entourages as follows: We set $V_0 := \Delta_X$. Now if V_i is already defined, we set

$$V_{i+1} := (V_i \circ V_i) \cup U_{i+1} \cup U_{i+1}^{-1}.$$

[1] a metric space is proper if closed bounded subsets are compact

One easily shows inductively that for all i in $\mathbb{N}$ we have $V_i^{-1} = V_i$ and $\Delta_X \subseteq V_i$. The latter implies that $V_i \subseteq V_i \circ V_i$ for all i and therefore we derive a sequence of inclusions

$$\Delta_X \subseteq V_i \subseteq V_i \circ V_i \subseteq V_{i+1}.$$

These inclusion relations (and symmetry of the V_i) ensure that the set

$$\tilde{\mathcal{C}} := \left\{ U \subseteq X \times X \;\middle|\; \exists i \in \mathbb{N} : \; U \subseteq V_i \right\}$$

is a coarse structure on X. However, by construction we know $V_i \in \mathcal{C}_X$ for all i (since $\mathcal{C}_X$ is closed under the operations defining V_i), hence $\tilde{\mathcal{C}} \subseteq \mathcal{C}_X$. On the other hand, each generator of $\mathcal{C}_X$ is contained in an entourage V_i for some i, hence $\mathcal{C}_X \subseteq \tilde{\mathcal{C}}$.
We define an extended metric d by

$$\mathrm{d}(x, y) := \inf \left\{ i \in \mathbb{N} \;\middle|\; (x, y) \in V_i \right\}.$$

Since $V_0 = \Delta_X$ we have $\mathrm{d}(x, y) = 0$ if and only if $x = y$. Also, symmetry of d is clear because $V_i = V_i^{-1}$. It remains to check the triangle inequality: Consider three distinct elements x, y, z in X. Assume $(x, y) \in V_n$ and $(y, z) \in V_m$. W.l.o.g. we may assume $n \geq m$ and thus $V_m \subseteq V_n$, hence $(x, z) \in V_n \circ V_n \subseteq V_{n+1}$ which implies

$$\mathrm{d}(x, z) \leq n + 1 \leq \mathrm{d}(x, y) + \mathrm{d}(y, z).$$

It is clear from the construction that this extended metric d induces the coarse structure $\tilde{\mathcal{C}}$ on X (and we already saw that $\tilde{\mathcal{C}} = \mathcal{C}_X$).
Finally it is clear that d is a metric if and only if X is coarsely connected. □

Every coarse space gives rise to the notion of boundedness:

Definition 1.1.15. Let X be a coarse space and U in $\mathcal{C}_X$. Consider a subset B of X. We say that B is *U-bounded* if $B \times B \subseteq U$.
The subset B is called *bounded* if it is V-bounded for some V in $\mathcal{C}_X$.

Example 1.1.16. Let X_{d} be a metric coarse space. Then the bounded subsets in the sense of Definition 1.1.15 are the metrically bounded subsets of (X, d). ✠

We have seen that in coarse spaces there exists a notion of bounded subsets. Thickenings of bounded subsets by controlled subsets remain bounded:

Lemma 1.1.17. *Let X be a coarse space. Consider a U-bounded subset B of X for some U in $\mathcal{C}_X$. Then for all entourages V in $\mathcal{C}_X$ the V-thickening $V[B]$ of B is also bounded.*

Proof. Consider an element (x, y) in $V[B] \times V[B]$. By definition there exist b, b' in B such that $(x, b), (y, b') \in V$. Since by assumption $(b, b') \in U$ we have

$$(x, y) \in V \circ U \circ V^{-1}.$$

Note that the latter is an entourage by axioms. □

The notion of a bornology (often encountered in functional analysis) is a way to axiomize the notion of "boundedness". It goes back to Mackey in [Mac45] and has been defined categorically in several sources e.g. [Hog77]. We recall the definition of a bornology on a set X:

Definition 1.1.18. A *bornology* on X is a subset $\mathcal{B}_X$ of $\mathcal{P}(X)$ which is closed under taking subsets and finite unions and which contains all finite subsets of X.

Recall that a filter on a set X is a subset $\mathcal{F}$ of $\mathcal{P}(X)$ which is closed under taking finite intersections and supersets, and such that $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$.

Example 1.1.19 (c.f. [NB11, p. 168]). Assume X is endowed with a bornology $\mathcal{B}_X$ such that $X \notin \mathcal{B}_X$ (equivalently $\mathcal{B}_X \neq \mathcal{P}(X)$). Then the subset $\mathcal{F}$ of $\mathcal{P}(X)$ obtained by taking memberwise complements of $\mathcal{B}_X$

$$\mathcal{F} := \left\{ X \setminus B \;\middle|\; B \in \mathcal{B}_X \right\}$$

is a filter on X. Moreover, this filter is free, i.e. $\bigcap_{F \in \mathcal{F}} F = \emptyset$. It is called the *filter at infinity.*
Conversely, if $\mathcal{F}'$ is a free filter on X then taking memberwise complements defines a bornology on X. ✠

We have seen previously that a coarse space gives rise to a certain class of bounded subsets. However, to obtain more flexibility we can endow a coarse space with a bornology in order to redefine which subsets we call bounded. These two structures need to be compatible in the sense that thickenings of bornology-bounded subsets by controlled subsets of the coarse structure remain bornology-bounded (c.f. Lemma 1.1.17). This construction is due to Bunke and Engels in [BE20]. We recall their precise definition in the following:

Definition 1.1.20. A *bornological coarse space* is a set X together with a coarse structure $\mathcal{C}_X$ on X and a bornology $\mathcal{B}_X$ on X such that $\mathcal{C}_X$ and $\mathcal{B}_X$ are *compatible*, i.e. for all U in $\mathcal{C}_X$ and all B in $\mathcal{B}_X$ we have $U[B] \in \mathcal{B}_X$.

Remark 1.1.21. For an entourage U and a subset B of X the thickening $U[B]$ is also called the U-penumbra $\mathrm{Pen}_U(B)$ of B in the literature (e.g. [Wul20]). ♦

Example 1.1.22. The *maximal bornology* on X is given by $\mathcal{P}(X)$. This bornology clearly is compatible with every coarse structure on X. Moreover, it is the only bornology compatible with the maximal coarse structure. If we equipp X with the maximal coarse structure and the maximal bornology we denote the resulting bornological coarse space by $X_{\mathrm{max,max}}$ or – to keep notation short – by X_{MM}.
Here, the first index refers to the coarse structure and the second one to the bornology. For example, if we equipp X with the maximal bornology and the minimal coarse structure, we refer to this space as $X_{\mathrm{min,max}}$ or X_{mM}. ✠

Example 1.1.23. The collection of all finite subsets of X constitutes a bornology on X, called the *minimal bornology*. It is compatible with for example the minimal or the discrete coarse structure (however the latter will not be considered within this thesis). The bornological coarse space obtained by equipping X with the minimal coarse structure and the minimal bornology is denoted by $X_{\mathrm{min,min}}$ or a bit shorter by X_{mm}. ✠

Definition 1.1.24. Consider a subset $\mathcal{S}$ of $\mathcal{P}(X)$. The smallest bornology on X which contains $\mathcal{S}$ is called the *bornology generated by* $\mathcal{S}$ and denoted by $\mathcal{B}\langle\mathcal{S}\rangle$.

The following lemma is not surprising but very convenient for checking compatibility: It is enough to check it for generators:

Lemma 1.1.25. *Consider a subset $\mathcal{S}$ of $\mathcal{P}(X \times X)$ which is closed under taking inverses. Let furthermore $\mathcal{T}$ be a subset of $\mathcal{P}(X)$ such that $X = \bigcup_{T\in\mathcal{T}} T$. If for any member U of $\mathcal{S}$ and any B in $\mathcal{T}$ the thickening $U[B]$ is contained in a finite union of elements of $\mathcal{T}$ then the coarse structure $\mathcal{C}\langle\mathcal{S}\rangle$ and the bornology $\mathcal{B}\langle\mathcal{T}\rangle$ are compatible.*

Proof. This is straightforward to check. Note that for two entourages U, V in $\mathcal{C}_X$ and a subset B of X we have $(U \circ V)[B] = U[V[B]]$ and $(U \circ V)^{-1} = V^{-1} \circ U^{-1}$. □

Example 1.1.26. Consider a metric space (X, d). The metrically-bounded subsets of X constitute a bornology on X. This bornology is compatible with the coarse structure induced from the metric. Indeed, the bornology is the bornology generated by $\left\{B_r(x) \;\middle|\; x \in X,\ r > 0\right\}$ and for a generating entourage

$$U_s = \left\{(x, y) \in X \times X \;\middle|\; \mathrm{d}(x, y) < s\right\}$$

and a generating bounded subset $B_r(x)$ we have

$$U_s[B_r(x)] \subseteq B_{r+s}(x).$$

The resulting bornological coarse space is often denoted by X_{d}. ✠

As we mentioned above, each coarse space gives rise to the notion of bounded subsets which is subject to the study of coarse geometry done by a variety of mathematicians (e.g. [Roe03; Har20; Mit01; Wri03]). The authors in [BE20] introduced more flexibility by allowing arbitrary compatible bornologies. In fact, the "classical" notion of bounded subsets in a coarse space is somehow the minimal compatible bornology:

Example 1.1.27. For any coarse space X the collection of bounded subsets in the sense of Definition 1.1.15 generate the minimal bornology on X which is compatible with the given coarse structure. This can be seen as follows: First the bornology generated by coarsely-bounded subsets is compatible with the coarse structure by Lemma 1.1.17. And for any other compatible bornology $\mathcal{B}$ on X we consider a non-empty U-bounded subset B of X for some U in $\mathcal{C}_X$. For any b in B the subset $\{b\}$ is an element in $\mathcal{B}$, hence by compatibility also $U[\{b\}]$ is in $\mathcal{B}$. But clearly $B \subseteq U[\{b\}]$ since $B \times B \subseteq U$. ✠

From now on, whenever we talk about bornological coarse spaces, the notion "bounded subset" refers to elements in the given bornology, not to bounded subsets in the sense of coarse spaces (c.f. Definition 1.1.15). We proceed by introducing the notion of morphisms between bornological coarse spaces. We decided to use the naming conventions in [BE20].

Definition 1.1.28. A map $f\colon X \to Y$ between bornological coarse spaces is called

- *controlled* if it maps entourages to entourages, i.e. if for all U in $\mathcal{C}_X$ we have $(f \times f)(U) \in \mathcal{C}_Y$.
- *proper* if for all B in $\mathcal{B}_Y$ we have $f^{-1}(B) \in \mathcal{B}_X$.
- *bornological* if images of bounded subsets are bounded.

Remark 1.1.29. There are various different name conventions. E.g. [Roe03] uses the adjective "bornologous" for controlled maps in our sense above while [Har20] calls these maps "coarsely uniform". And what we call bornological map often is called a "bounded map" (e.g. in [Hog77]). Often a map that is controlled and proper (for the bounded subsets of a coarse space) is called a "coarse map" (e.g. [Roe03; Mit01; Har20]). We will not adept this notation as proper and controlled maps will just be refered to as morphisms. ♦

Definition 1.1.30. A *morphism* between bornological coarse spaces is a map that is proper and controlled. The category of bornological coarse spaces with these morphisms is denoted by **BornCoarse**.

Earlier we had a useful lemma which essentially states that we can check compatibility of coarse and bornological structure on the respective generators. The same is true for properness and controlledness of morphisms:

Lemma 1.1.31. *Let $f\colon X \to Y$ be a map between bornological coarse spaces. Assume $\mathcal{B}_Y$ is generated by a set of generators $\mathcal{T}$ which cover Y. Moreover assume that $\mathcal{C}_X$ is generated by a set $\mathcal{S}$ which contains Δ_X and is closed under taking inverses. If for all B in $\mathcal{T}$ the pre-image $f^{-1}(T)$ is in $\mathcal{B}_X$ then f is proper. Likewise, if for any U in $\mathcal{S}$ the image $(f \times f)(U)$ is in $\mathcal{C}_Y$ the map f is controlled.*

Proof. These are immediately verified. We have to check that taking (pre-)images behaves well w.r.t. operations in the definition of a coarse structure or bornology. □

Example 1.1.32. Consider a bornological coarse space X and a set Y together with a map of sets $f\colon Y \to X$. Then we define a coarse structure

$$f^*\mathcal{C}_X := \mathcal{C}\left\langle\left\{U \subseteq X \times X \;\middle|\; (f \times f)(U) \in \mathcal{C}_X\right\}\right\rangle$$

on Y as well as a bornology

$$f^*\mathcal{B}_X := \mathcal{B}\left\langle\left\{f^{-1}(B) \;\middle|\; B \in \mathcal{B}_X\right\}\right\rangle$$

on Y. Using Lemma 1.1.25 and the fact that

$$U\left[f^{-1}(B)\right] \subseteq f^{-1}\Big(\big((f \times f)(U)\big)[B]\Big)$$

we immeditelly see that $f^*\mathcal{C}_X$ and $f^*\mathcal{B}_X$ are compatible. Moreover, by definition of these structures and Lemma 1.1.31 it is clear that f becomes a morphism of

bornological coarse spaces.
If f is the inclusion of a subset $Y \hookrightarrow X$ then we call the bornological coarse structure constructed here the *subspace structure* inherited from X. ✠

Lemma 1.1.33. *For a set X the assignment $X \mapsto X_{\min,\max}$ extends to a functor*

$$(-)_{\min,\max} \colon \mathbf{Set} \longrightarrow \mathbf{BornCoarse}.$$

Moreover this functor is part of an adjunction (with right adjoint the forgetful functor):

$$(-)_{\min,\max} \colon \mathbf{Set} \rightleftarrows \mathbf{BornCoarse} : F.$$

Proof. Any map $f \colon X \to Y$ of sets maps subsets of the diagonal into subsets of the diagonal. Therefore, for any set X and any bornological coarse space Z every map $f \colon X_{mM} \to Z$ is controlled. Properness of this map is trivially fulfilled, hence f is a morphism of bornological coarse spaces. This shows both assertions. □

The following lengthly example introduces the notion of *continuously controlled bornological coarse structure* on a Hausdorff space X. Continuously controlled coarse structures are used for example in the study of algebraic K-theory in [ACFP] or A-theory in [Vog95].

Example 1.1.34 ([Bun21, Examples 2.18 & 3.9]). Consider a Hausdorff space $\overline{X}$ and a subset A of $\overline{X}$. We set $X := \overline{X} \setminus A$[†]. We define a coarse structure $\mathcal{C}_X$ on X as follows: A subset U of $X \times X$ is an entourage in $\mathcal{C}_X$ precisely if for any net $(x_i, x_i')_{i\in I}$ in U the net $(x_i)_{i\in I}$ converges to a point y in A if and only if $(x_i')_{i\in I}$ converges to this point y.
We show for example that $\mathcal{C}_X$ is closed under composition: Assume U and V are entourages in $\mathcal{C}_X$. Consider a net $(x_i, x_i')_{i\in I}$ in $U \circ V$. By definition there exists a net $(x_i'')_{i\in I}$ such that for all i in I we have $(x_i, x_i'') \in U$ and $(x_i'', x_i') \in V$. Now, if $(x_i)_i$ converges to a point y in A, then – since $U \in \mathcal{C}_X$ – also the net $(x_i'')_i$ converges to y. But this implies that $(x_i')_i$ converges to y because $V \in \mathcal{C}_X$. The remaining axioms are similar and can be found in [Bun21, Ex. 2.18]. Now, we define a bornology on X by

$$\mathcal{B}_X := \left\{ B \subseteq X \;\middle|\; \overline{B} \cap A = \emptyset \right\}.$$

We finish this example by showing that $\mathcal{C}_X$ and $\mathcal{B}_X$ are compatible: For this we consider U in $\mathcal{C}_X$ and B in $\mathcal{B}_X$ and assume $U[B] \notin \mathcal{B}_X$. Then by definition there is

[†]We may think of $\overline{X}$ as a compactification of X and A the corresponding boundary

a point y in $\overline{U[B]} \cap A$. Thus there exists a net $(x_i)_{i\in I}$ in $U[B]$ converging to y in A. By definition of the U-thickening this gives a net $(b_i)_{i\in I}$ in B such that $(x_i, b_i)_{i\in I}$ is a net in U. But $(x_i)_i$ converges to y in A, hence $(b_i)_i$ converges to y in A. This contradicts $\overline{B} \cap A = \emptyset$. ✠

Remark 1.1.35. The construction of the continuously controlled coarse structure as in Example 1.1.34 above can be found e.g. in [Roe03, Def. 2.28] for the special case where X is a Hausdorff space, $\overline{X}$ its compactification and A the boundary $\overline{X} \setminus X$. This structure is also called *topological coarse structure* for given compactification. For example ([Roe03, Example 2.31]) if we consider a set X as a discrete topological space the continuously controlled coarse structure on X given by the Stone-Čech compactification equals the discrete coarse structure (see Example 1.1.5). On the other hand, the one-point-compactification gives the indiscrete coarse structure (c.f. Example 1.1.10) by [Roe03, Ex. 2.30]. ♦

We finish this section with a nice example due to [Bun21]. See also [NB11, Ex 6.7.11].

Example 1.1.36. Consider a uniformly locally finite bornological coarse space, i.e. for all entourages U in $\mathcal{C}_X$ and all x in X the U-thickening $U[\{x\}]$ is finite. For a ring R the set

$$\mathcal{A}(X) := \left\{A\colon X \times X \to R \;\middle|\; \operatorname{supp}(A) \in \mathcal{C}_X\right\}$$

forms an R-module with the usual pointwise operations. Indeed, for A, B in $\mathcal{A}(X)$ and r in R we have

$$\operatorname{supp}(A + B) \subseteq \operatorname{supp}(A) \cup \operatorname{supp}(B) \qquad \text{and} \qquad \operatorname{supp}(rA) \subseteq \operatorname{supp}(A)$$

which shows that $\mathcal{A}(X)$ is closed under addition and scalar multiplication.
Moreover, this R-module carries an associative algebra structure defined by "matrix multiplication": For A and B in $\mathcal{A}(X)$ we define their product $A * B$ as

$$(A * B)(x, y) := \sum_{z\in X} A(x, z)B(z, y).$$

First, this sum is well-defined (i.e. finite) because for an element z in X whose corresponding summand is non-zero we have in particular $B(z, y) \neq 0$, hence $(z, y) \in \operatorname{supp}(B)$, hence $z \in \operatorname{supp}(B)[\{y\}]$. The latter set is finite because X is uniformly locally finite by assumption. Also, consider an element (x, y) in $\operatorname{supp}(A * B)$. Then there exists necessarily an element z such that $A(x, z) \neq 0$ and $B(z, y) \neq 0$, hence

$(x, z) \in \operatorname{supp}(A)$ and $(z, y) \in \operatorname{supp}(B)$, hence

$$(x, y) \in \operatorname{supp}(A) \circ \operatorname{supp}(B)$$

which shows $\operatorname{supp}(A * B) \subseteq \operatorname{supp}(A) \circ \operatorname{supp}(B)$ hence $\operatorname{supp}(A * B) \in \mathcal{C}_X$.

We define a second R-module $\mathscr{C}(X)$ as the set of functions $X \to R$ with locally compact support, i.e.

$$\mathscr{C}(X) := \left\{ f \colon X \to R \;\middle|\; \forall B \in \mathcal{B}_X : \ \left| B \cap \operatorname{supp}(f) \right| < \infty \right\}.$$

There is an action of $\mathcal{A}(X)$ on $\mathscr{C}(X)$ defined by

$$(A \cdot f)(x) := \sum_{y \in X} A(x, y) f(y)$$

for all A in $\mathcal{A}(X)$ and f in $\mathscr{C}(X)$. First we see that this sum is finite because if y is an element in X whose summand is non-zero then in particular $y \in \operatorname{supp}(f)$ and $(x, y) \in \operatorname{supp}(A)$. The latter can be reformulated as

$$y \in (\operatorname{supp}(A))^{-1} [\{x\}] =: \tilde{B}$$

which is a bounded subset in $\mathcal{B}_X$ (since $\operatorname{supp}(A) \in \mathcal{C}_X$). But $\operatorname{supp}(f)$ is locally finite hence the intersection $\operatorname{supp}(f) \cap \tilde{B}$ is finite.
It remains to check that $A \cdot f$ has locally finite support: For this we consider a bounded subset B in $\mathcal{B}_X$ and an element x in the intersection $B \cap \operatorname{supp}(A \cdot f)$. From $(A \cdot f)(x) \neq 0$ we derive the existence of some y with $f(y) \neq 0$ and $A(x, y) \neq 0$. The latter gives $(x, y) \in \operatorname{supp}(A)$, i.e. $y \in \operatorname{supp}(A)^{-1}[\{x\}]$. But $x \in B$, hence in total:

$$y \in \operatorname{supp}(f) \cap (\operatorname{supp}(A))^{-1}[B] =: F$$

Note that F is finite since $f \in \mathscr{C}(X)$. Furthermore, X is uniformly locally bounded, hence every controlled thickening of F is finite. In particular $\operatorname{supp}(A)[F]$ is finite. Since we already saw $(x, y) \in \operatorname{supp}(A)$ we immediately derive $x \in \operatorname{supp}(A)[F]$, hence $B \cap \operatorname{supp}(A \cdot f) \subseteq \operatorname{supp}(A)[F]$ and thus the claim. ✠

1.2 Equivariant bornological coarse spaces

Shortly after introducing the notion of bornological coarse spaces in [BE16] the authors together with Kasprowski and Winges generalized this notion in [BEKW] to the equivariant setting where a group G acts on the bornological coarse spaces by automorphisms. One motivation for this was to define equivariant coarse homology theories which are important in the study of (coarse) assembly maps appearing in conjectures of types Farrell-Jones ([BL12; BFJR]) or Baum-Connes ([Yu95a; Yu95b]). Coarse homology theory will be the content of the next chapter.

In this section we recall the basic definitions of equivariant bornological coarse spaces and provide first constructions and several examples. We finish this section by pointing out a (actually "*the*") reason why the category of G-equivariant bornological coarse spaces is neither complete nor co-complete.

Throughout this section we fix a group G. Consider a bornological coarse space X and assume G acts on X by automorphisms. This action induces an action of G on $\mathcal{C}_X$. We denote by $\mathcal{C}_X^G$ the poset of those entourages in $\mathcal{C}_X$ which are G-invariant.

Definition 1.2.1. A *G-equivariant bornological coarse space* is a bornological coarse space X on which G acts by automorphisms such that $\mathcal{C}_X^G$ is cofinal in $\mathcal{C}_X$. A morphism $f\colon X \to Y$ between G-equivariant bornological coarse spaces is a G-equivariant map which is proper and controlled. We denote the category of G-equivariant bornological coarse spaces by $\boldsymbol{G}$**BornCoarse**.

Remark 1.2.2. The set $\mathcal{C}_X^G$ is cofinal in $\mathcal{C}_X$ if and only if $\mathcal{C}_X$ is closed under taking G-saturations ([Roe03, p. 23]), i.e. if for all U in $\mathcal{C}_X$ the subset

$$\bigcup_{g\in G} g\cdot U = \Big\{(gx, gy) \;\Big|\; g \in G,\; (x,y)\in U\Big\}$$

is contained in $\mathcal{C}_X$. In particular, for G finite this is obviously fulfilled. ♦

Example 1.2.3. Let X be a G-set[3]. Consider a collection $\mathcal{S}$ of G-invariant subsets of $X \times X$ and set $\mathcal{C}_X := \mathcal{C}\langle\mathcal{S}\rangle$. Then $\mathcal{C}_X^G$ is cofinal in $\mathcal{C}_X$. In words: If the generating system of a coarse structure consists of G-invariant subsets the cofinality-condition is fulfilled. ✠

Example 1.2.4. Let X be a G-set. Then the following spaces are G-equivariant bornological coarse spaces: $X_{\mathrm{min,max}}$, $X_{\mathrm{max,max}}$ and $X_{\mathrm{min,min}}$. ✠

[3] A G-set is a set with a G-action

Lemma 1.2.5. *Let Y be a G-set and X a G-equivariant bornological coarse space. Consider a G-equivariant map $f\colon Y \to X$. Then the pullback structures $f^*\mathcal{C}_X$ and $f^*\mathcal{B}_X$ (see Example 1.1.32) turn Y into a G-equivariant bornological coarse space and the map f becomes a morphism in* $\boldsymbol{G}$**BornCoarse**.
In particular, if $f\colon Y \hookrightarrow X$ is the inclusion of a G-invariant subset then we call the pullback structure the induced subspace structure *of X on Y.*

Proof. The set $\left\{(f\times f)^{-1}(U) \,\middle|\, U \in \mathcal{C}_X\right\}$ is cofinal in $f^*\mathcal{C}_X$. Since $\mathcal{C}_X^G$ is cofinal in $\mathcal{C}_X$ also the set $\left\{(f\times f)^{-1}(U) \,\middle|\, U \in \mathcal{C}_X^G\right\}$ is cofinal in $f^*\mathcal{C}_X$. By G-equivariance of f the latter set consists of G-invariant entourages. The remaining assertions follow from the non-equivariant case (Example 1.1.32). □

Example 1.2.6. Consider a metric space (X, d). Recall the bornological coarse structure on X induced by the metric (Example 1.1.8). Assume G acts on X isometrically. Clearly the generating entourages are G-invariant by isometry of the action, hence X_d is an object in $\boldsymbol{G}$**BornCoarse**. ✠

Example 1.2.7 ([Roe03, Example 2.13],[BEKW, Example 2.4]). The group G acts on itself by left multiplication. We equipp G with the minimal bornology and we define the following coarse structure:

$$\mathcal{C}_G := \mathcal{C}\left\langle\left\{G\cdot(B\times B) \,\middle|\, B \in \mathcal{B}_G\right\}\right\rangle.$$

This coarse structure is called the *canonical coarse structure*. It is compatible with $\mathcal{B}_G$ because the thickening of a singleton $\{x\}$ by a generating entourage $G\cdot(B\times B)$ is given by

$$\Big(G\cdot(B\times B)\Big)\Big[\{x\}\Big] = \left\{xb^{-1}b' \,\middle|\, b, b' \in B\right\}$$

which is clearly finite. The resulting G-equivariant bornological coarse space is denoted by $X_{\mathrm{can,min}}$ or X_{cm}. Clearly this space is coarsely connected. ✠

Lemma 1.2.8. *Let G be a finitely generated group. Any word metric* d *on G (which depends on the choice of a generating set) induces a bornological coarse structure G_{d}. This structure agrees with $G_{\mathrm{can,min}}$ (independent of the coice of* d*).*
More generally: If G is countable and d *is a proper left invariant metric on G, then $G_{cm} \cong G_{\mathrm{d}}$.*

Proof. See [Roe03, Example 2.13] for the first assertion and [Cap19, Example 1.1.22] for the generalization. □

Example 1.2.9. Let X be a G-equivariant bornological coarse space and let U be in $\mathcal{C}_X^G$. The coarse structure $\mathcal{C}'_U := \mathcal{C}\langle\{U\}\rangle$ is compatible with $\mathcal{B}_X$. The G-equivariant bornological coarse space obtained by replacing $\mathcal{C}_X$ with $\mathcal{C}'_U$ is denoted by X_U. Clearly the identity map of the set X gives a morphism $X_U \to X$. Furthermore, for any other entourage U' in $\mathcal{C}_X^G$ with $U \subseteq U'$ the identity map also gives a morphism $X_U \to X_{U'}$. We will use this fact later (Example 1.3.30 and Definition 2.1.17). ✠

Lemma 1.2.10 (c.f. Lemma 1.1.33)**.** *If $F\colon$ **GBornCoarse** $\to$ **GSet** denotes the forgetful functor, then we have an adjunction*

$$(-)_{mM}\colon \boldsymbol{G}\mathbf{Set} \rightleftarrows \boldsymbol{G}\mathbf{BornCoarse} : F.$$

Definition 1.2.11. We say that a G-equivariant bornological coarse space X is *complete* if the orbit of any bounded subset is again bounded, i.e. for any B in $\mathcal{B}_X$ we have $GB \in \mathcal{B}_X$. We denote the full subcategory of complete G-equivariant bornological coarse spaces by **GBornCoarse**$^\wedge$.

Lemma 1.2.12. *Let X be in **GBornCoarse**. Then we define a bornology*

$$G\mathcal{B}_X := \mathcal{B}\left\langle\left\{GB \;\middle|\; B \in \mathcal{B}_X\right\}\right\rangle$$

*on X. This bornology is compatible with $\mathcal{C}_X$ and replacing $\mathcal{B}_X$ by $G\mathcal{B}_X$ gives a complete G-equivariant bornological coarse space $\hat{X}$ in **GBornCoarse**$^\wedge$. This construction is functorial in X and gives the G-completion functor*

$$\widehat{(-)}\colon \boldsymbol{G}\mathbf{BornCoarse} \to \boldsymbol{G}\mathbf{BornCoarse}^\wedge.$$

*which is right adjoint to the inclusion functor **GBornCoarse**$^\wedge \to$ **GBornCoarse**.*

Proof. Since $\mathcal{C}_X^G$ is cofinal in $\mathcal{C}_X$ it is enough to check compatibility for G-invariant entourages. For such an entourage U and a bounded subset B in $\mathcal{B}_X$ an easy calculation (using G-invariance of U) shows

$$U[GB] = G(U[B]).$$

Thus $U[GB]$ is a generator of $G\mathcal{B}_X$ hence the compatibility. Functoriality and the adjoint-asserion both follow from the fact that for any morphism $f\colon X \to Y$ in **GBornCoarse** and any bounded subset B in $\mathcal{B}_Y$ we have $f^{-1}(GB) = Gf^{-1}(B)$ implied from the G-equivariance of f. □

Example 1.2.13 ([BEKW, Example 2.13]). Let X be a G-set equipped with a bornology $\mathcal{B}_X$. Assume that the action of G is such that for g in G and all B in $\mathcal{B}_X$ we have $gB \in \mathcal{B}_X$ and such that for all bounded subsets B in $\mathcal{B}_X$ the set

$$\left\{g \in G \;\middle|\; gB \cap B \neq \emptyset\right\}$$

is finite. Then the coarse structure $\mathcal{C}_X$ on X defined by

$$\mathcal{C}_X := \mathcal{C}\left\langle\left\{G \cdot (B \times B) \;\middle|\; B \in \mathcal{B}_X\right\}\right\rangle$$

is compatible with $\mathcal{B}_X$. Since $\mathcal{C}_X$ is generated by G-invariant entourages, we obtain a space X in $\boldsymbol{G}$**BornCoarse**. Actually $\mathcal{C}_X$ is the smallest G-invariant coarse structure compatible with $\mathcal{B}_X$ such that the resulting space is coarsely connected.
(Almost) the same construction can be found in [Roe03, Example 2.13]. ✠

Lemma 1.2.14. *Consider a complete G-equivariant bornological coarse space X. Denote by $\overline{X}$ the set of cosets ${}_G\backslash^{X}$ and let $\pi\colon X \to \overline{X}$ be the canonical projection. Then the bornological coarse structure on $\overline{X}$ defined by*

$$\mathcal{B}_{\overline{X}} := \left\{B \subseteq \overline{X} \;\middle|\; \pi^{-1}(B) \in \mathcal{B}_X\right\}$$
$$\mathcal{C}_{\overline{X}} := \mathcal{C}\left\langle\left\{(\pi \times \pi)(U) \;\middle|\; U \in \mathcal{C}_X\right\}\right\rangle.$$

turn $\overline{X}$ into a space in **BornCoarse**. *The assignment $Q\colon X \mapsto \overline{X}$ is functorial in X and moreover this functor Q is part of an adjunction*

$$Q\colon \boldsymbol{G}\mathbf{BornCoarse}^{\wedge} \rightleftarrows \mathbf{BornCoarse} :\mathrm{incl}.$$

Proof. First, $\mathcal{B}_{\overline{X}}$ is a bornology because for $\overline{x}$ in $\overline{X}$ we have $\pi^{-1}(\{\overline{x}\}) = G\{x\}$ for any x in the coset $\overline{x}$. However, $G\{x\}$ is bounded because X is complete. For compatibility we consider a bounded subset B of $\overline{X}$ and a generating entourage $\overline{U} := (\pi \times \pi)(U)$. By definition $\pi^{-1}(B)$ is bounded in X, hence so is its U-thickening $B' := U[\pi^{-1}(B)]$. By completeness of X also the orbit GB' is bounded. However, an easy calculation shows that

$$\pi^{-1}(\overline{U}[B]) \subseteq GB'$$

hence $\overline{U}[B]$ is bounded in $\mathcal{B}_{\overline{X}}$. Next, to see functoriality we consider two G-complete spaces X, Y in $\boldsymbol{G}\mathbf{BornCoarse}^\wedge$ and a morphism $f\colon X \to Y$. By equivariance of f we obtain an induced map $\overline{f}\colon \overline{X} \to \overline{Y}$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_X"'] & Y \arrow[d, "\pi_Y"] \\ \overline{X} \arrow[r, "\overline{f}"'] & \overline{Y} \end{tikzcd}$$

commutes. Using this commutativity and the definition of the bornological coarse structures on $\overline{X}$ and $\overline{Y}$ we immediately get that $\overline{f}$ is proper and controlled.
Finally, it is straightforward to see that for all X in $\boldsymbol{G}\mathbf{BornCoarse}$ and all Y in $\mathbf{BornCoarse}$ precomposition with the proper and controlled map $\pi\colon X \to \overline{X}$ induces a bijection

$$\mathrm{Hom}_{\mathbf{BornCoarse}}(\overline{X}, Y) \xrightarrow{\pi^*} \mathrm{Hom}_{\boldsymbol{G}\mathbf{BornCoarse}}(X, Y^{\mathrm{triv}})$$

where the supscript “triv” indicates that G acts trivially on Y. $\square$

Consider two spaces X and Y in $\boldsymbol{G}\mathbf{BornCoarse}$. For entourages U in $\mathcal{C}_X$ and U' in $\mathcal{C}_Y$ we define (notation due to [Wul20])

$$U \bar{\times} U' := \left\{ \Big((u_1, u_1'), (u_2, u_2') \Big) \in (X \times Y) \times (X \times Y) \;\Big|\; (u_1, u_2) \in U,\ (u_1', u_2') \in U' \right\}.$$

Example 1.2.15. We define the tensor product $X \otimes Y$ in $\boldsymbol{G}\mathbf{BornCoarse}$ as follows: The underlying set of $X \otimes Y$ is given by $X \times Y$ and G acts on this set factorwise. The bornological coarse structure is defined by

$$\begin{aligned} \mathcal{B}_{X\otimes Y} &:= \mathcal{B}\Big\langle \Big\{ B \times B' \;\Big|\; B \in \mathcal{B}_X,\ B' \in \mathcal{B}_Y \Big\} \Big\rangle \\ \mathcal{C}_{X\otimes Y} &:= \mathcal{C}\Big\langle \Big\{ U \bar{\times} U' \;\Big|\; U \in \mathcal{C}_X,\ U' \in \mathcal{C}_Y \Big\} \Big\rangle . \end{aligned}$$

Compatibility can be checked on generators where it immediately follows from comptability in both factors separately. Moreover, since $\mathcal{C}_X^G$ and $\mathcal{C}_Y^G$ are cofinal in $\mathcal{C}_X$ and $\mathcal{C}_Y$ respectively, we get that $\mathcal{C}_{X\otimes Y}^G$ is cofinal in $\mathcal{C}_{X\otimes Y}$. ✠

Lemma 1.2.16. *The construction of the tensor product in Example 1.2.15 extends to a bifunctor*

$$- \otimes - \colon \boldsymbol{G}\mathbf{BornCoarse} \times \boldsymbol{G}\mathbf{BornCoarse} \longrightarrow \boldsymbol{G}\mathbf{BornCoarse}$$

which gives rise to a symmetric monoidal structure on $\boldsymbol{G}$**BornCoarse** *with tensor unit the singleton.*

Proof. See for example [Cap19, p. 8]. □

Remark 1.2.17. The category $\boldsymbol{G}$**BornCoarse** has all non-empty products (see Corollary 1.3.37). However, it is not symmetric monoidal w.r.t. this product because the tensor unit for the product is necessarily the terminal object. But in $\boldsymbol{G}$**BornCoarse** there exists no terminal object (see Lemma 1.2.19). ♦

Example 1.2.18. For natural numbers n and m we consider the bornological coarse spaces $\mathbb{R}^n$ and $\mathbb{R}^m$. Then we have an isomorphism $\mathbb{R}^n \otimes \mathbb{R}^m \cong \mathbb{R}^{n+m}$. ✠

Let us take a look at categorical properties of $\boldsymbol{G}$**BornCoarse**. First we see that this category is not complete:

Lemma 1.2.19. *The category* $\boldsymbol{G}$**BornCoarse** *has no final object.*

Proof. Assume there exists a final object T in $\boldsymbol{G}$**BornCoarse**. By Lemma 1.2.10 the forgetful functor $F\colon \boldsymbol{G}\mathbf{BornCoarse} \to \boldsymbol{G}\mathbf{Set}$ is a right adjoint, hence preserves limits. Thus the underlying G-set $F(T)$ of T is terminal in $\boldsymbol{G}$**Set**, hence a singleton. Therefore T would have to be a singleton equipped with maximal bornology. This contradicts for example the properness of the morphism $\mathbb{R} \to T$ where we equip $\mathbb{R}$ with the bornological coarse structure induced from the standard metric on $\mathbb{R}$. □

In fact, the terminal object (i.e. the empty limit) is the only limit in the category $\boldsymbol{G}$**BornCoarse** that does not exist (see Corollary 1.3.37). Note furthermore that the only reason why it does not exist is the axiom that every one-point-subset needs to be bounded.

While $\boldsymbol{G}$**BornCoarse** is almost complete, it is far from being co-complete. We give one example here:

Example 1.2.20. The following diagram of spaces in $\boldsymbol{G}$**BornCoarse** with trivial G-action has no pushout in $\boldsymbol{G}$**BornCoarse**:

$$\begin{tikzcd} \mathbb{N}_{mM} \arrow[r] \arrow[d] & \mathbb{N}_{MM} \\ \mathbb{N}_{mm} & \end{tikzcd}$$

where both morphisms are the identity on the underlying sets. In fact, there cannot even exist a space T in $\boldsymbol{G}$**BornCoarse** together with morphisms α, β such that the diagram

$$\begin{tikzcd} \mathbb{N}_{mM} \arrow[r] \arrow[d] & \mathbb{N}_{MM} \arrow[d, "\alpha"] \\ \mathbb{N}_{mm} \arrow[r, "\beta"'] & T \end{tikzcd} \tag{†}$$

commutes: Assume that there is a commutative square as in (†). Let $t := \alpha(0)$. Then the subset $B := \{t\}$ is bounded. Furthermore $\mathbb{N} \times \mathbb{N}$ is a controlled set in $\mathbb{N}_{MM}$, hence $U := (\alpha \times \alpha)(\mathbb{N} \times \mathbb{N}) \in \mathcal{C}_T$. So, by compatibility of the bornological and coarse structure, the U-thickening $U[B]$ of B is bounded in T. But then by properness of β, the pre-image $\beta^{-1}(U[B])$ is bounded in $\mathbb{N}_{mm}$, hence finite. However since (†) commutes, the maps α and β coincide, thus we get $\beta^{-1}(U[B]) = \mathbb{N}$, because for any n in $\mathbb{N}$ we have $\beta(n) = \alpha(n) \in U[B] = U[\{\alpha(0)\}]$ (since $(\alpha(n), \alpha(0)) \in U$). Thus the pre-image is not finite, a contradiction. ✠

In order to motivate the generalization of bornological coarse spaces (see section 1.3) we want to point out the following: The reason we gave for the non-existence of the pushout in Example 1.2.20 was again (c.f. the remark after Lemma 1.2.19) solely the fact that by axioms of a bornology every one-point-subset of a space needs to be bounded. In fact, it is not only the solely reason *we gave*, it is the solely reason *there is*. To make this precise we look at this pseudo-example:

Example 1.2.21. Consider the set of natural numbers $\mathbb{N}$ equipped with the maximal coarse structure and with a "bornology" whose only element is the empty set. Let us denote this "space" by $\mathbb{N}_{M,\emptyset}$. Then the following square (where every morphism is the identity of the underlying sets) is a pushout (if it would exist):

$$\begin{tikzcd} \mathbb{N}_{mM} \arrow[r] \arrow[d] & \mathbb{N}_{MM} \arrow[d] \\ \mathbb{N}_{mm} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathbb{N}_{M,\emptyset}. \end{tikzcd}$$

Indeed let T be a test object and let γ, δ be morphisms such that the solid diagram

$$\begin{tikzcd} \mathbb{N}_{mM} \arrow[r] \arrow[d] & \mathbb{N}_{MM} \arrow[d] \arrow[ddr, bend left, "\gamma"] & \\ \mathbb{N}_{mm} \arrow[r] \arrow[drr, bend right, "\delta"'] & \mathbb{N}_{M,\emptyset} \arrow[dr, dotted, "\phi"] & \\ & & T \end{tikzcd}$$

commutes. Viewed as diagram in **Set** there is a unique map ϕ as indicated in the diagram making everything commute. This map is controlled because γ is controlled (and $\gamma = \phi$ as map of sets). Furthermore ϕ is proper, since the very same calculation as in Example 1.2.20 shows, that ϕ cannot hit any bounded point. ✠

1.3 Equivariant generalized bornological coarse spaces

In Lemma 1.2.19 and Example 1.2.20 we saw that the category ***G*BornCoarse** is neither complete nor co-complete. Also we elaborated that there seems to be a common reason for the lack of existence of (co-)limits, namely the requirement that in a bornological space every one-point-subset is bounded by axioms. In this section we define the notion of *generalized bornology* where we just drop that requirement. This can be compared with e.g. transitioning into considering non-Hausdorff spaces in topology: We would not think of a generic point to be bounded. Also several examples in the previous sections required the restriction to Hausdorff spaces to ensure the bornology-axiom of bounded one-point-sets (see Examples 1.3.7 and 1.3.8 for an elaboration). Considering this generalized notion of bornology we will see that we obtain a similar theory as we did for "classical" bornologies, but the resulting category of G-equivariant generalized bornological coarse spaces is both complete and co-complete. We will also study the connections between this new category and ***G*BornCoarse**. In particular, we will derive that ***G*BornCoarse** has all non-empty limits and we will obtain a precise condition which decides if a diagram in ***G*BornCoarse** admits a colimit or not.
We copied large parts of this section from our previous paper [Hei19].

We fix a group G. If not specified otherwise, X denotes a set in the following.

Definition 1.3.1. A *generalized bornology* on X is a subset $\mathcal{B}_X$ of $\mathcal{P}(X)$ which is closed under taking subsets and finite unions. The elements of $\mathcal{B}_X$ are called *bounded subsets of* X. The set X equipped with a generalized bornology $\mathcal{B}_X$ is called a *generalized bornological space*.

Definition 1.3.2. A map $f\colon X \to Y$ between generalized bornological spaces is called *proper*, if for all B in $\mathcal{B}_Y$ we have $f^{-1}(B) \in \mathcal{B}_X$.

Example 1.3.3. Every (classical) bornology on X is a generalized bornology on X. Thus we can adapt the definitions of the bornology induced by a metric on X, as well as the notions of the maximal and the minimal bornology (although the latter is no longer the "true" minimal one). ✠

Example 1.3.4. The *trivial bornology* $\mathcal{B}_\emptyset := \{\emptyset\}$ is a generalized bornology on X. We write $X_\emptyset$ for the generalized bornological space X together with the trivial bornology. For trivial reasons any map into $X_\emptyset$ is proper. ✠

Definition 1.3.5. Let X be a generalized bornological space.

- The *set of bounded points* of X is defined as
$$X_b := \left\{ x \in X \;\middle|\; \{x\} \in \mathcal{B}_X \right\}.$$
Its complement $X_h := X \setminus X_b$ is called the *set of unbounded points* of X.
- The space X is called *locally bounded* if $X = X_b$.
- We call a subset D of X *small* if $D \subseteq X_b$ and *big* otherwise.

Remark 1.3.6. We have $X_b = \bigcup_{B \in \mathcal{B}_X} B$, hence $\mathcal{B}_X$ is an (ordinary) bornology on X iff X is locally bounded. ♦

In the introductory words of this section we compared transitioning from bornologies into generalized bornologies to considering also non-Hausdorff spaces. To elaborate a bit on that we give the following two examples:

Example 1.3.7. Let X be a topological space. The collection of all relatively quasi compact subsets (subsets whose closure is quasi compact) forms a generalized bornology on X. If X is Hausdorff then this generalized bornological space is locally bounded. ✠

Example 1.3.8 (c.f. continuously controlled structure in Example 1.1.34). Consider a topological space $\overline{X}$ and a subset A of $\overline{X}$. We set $X := \overline{X} \setminus A$ and we define a generalized bornology
$$\mathcal{B}_X := \left\{ B \subseteq X \;\middle|\; \overline{B} \cap A = \emptyset \right\}$$
on X. If X is Hausdorff then $\mathcal{B}_X$ is a classical bornology. ✠

Example 1.3.9. We have seen in Example 1.1.19 that taking memberwise complements gives a one-to-one correspondence between non-maximal bornologies on X and free filters on X. This correspondence extends to a correspondence between all filters on X and non-maximal generalized bornologies on X. ✠

Definition 1.3.10. Consider a subset $\mathcal{S}$ of $\mathcal{P}(X)$. The smallest generalized bornology on X containing $\mathcal{S}$ is denoted by $\tilde{\mathcal{B}}\langle\mathcal{S}\rangle$. It is called the generalized bornology *generated by* $\mathcal{S}$.

Remark 1.3.11. For a subset $\mathcal{S}$ of $\mathcal{P}(X)$ the generated generalized bornology $\tilde{\mathcal{B}}\langle\mathcal{S}\rangle$ consists precisely of those subsets B of X which are contained in a finite union of members of $\mathcal{S}$. ♦

Definition 1.3.12. Let X be a set equipped with a coarse structure $\mathcal{C}_X$ and a generalized bornology $\mathcal{B}_X$. We say, that $\mathcal{B}_X$ and $\mathcal{C}_X$ are *compatible*, if for all bounded sets B in $\mathcal{B}_X$ and all entourages U in $\mathcal{C}_X$, the U-thickening $U[B]$ is again a bounded subset of X.

Like in the setting of classical bornologies, compatibility can be checked on generators:

Lemma 1.3.13. *Consider a subset $\mathcal{S}$ of $\mathcal{P}(X \times X)$ which is closed under taking inverses and consider a subset $\mathcal{T}$ of $\mathcal{P}(X)$. If for any B in $\mathcal{T}$ and any U in $\mathcal{S}$ the U-thickening of B is contained in $\tilde{\mathcal{B}}\langle\mathcal{T}\rangle$ then the structures $\mathcal{C}\langle\mathcal{S}\rangle$ and $\tilde{\mathcal{B}}\langle\mathcal{T}\rangle$ are compatible.*

Proof. Analogous to the proof of Lemma 1.1.25. □

Definition 1.3.14. A *generalized bornological coarse space* is a set X together with a coarse structure $\mathcal{C}_X$ and a generalized bornology on $\mathcal{B}_X$, such that $\mathcal{B}_X$ and $\mathcal{C}_X$ are compatible. A morphism $f\colon X \to Y$ between generalized bornological coarse spaces is a map $f\colon X \to Y$ that is proper and controlled.
The category of generalized bornological coarse spaces is denoted by $\mathbf{\widetilde{Born}Coarse}$.

Example 1.3.15. By trivial reasons the trivial bornology is compatible with every coarse structure. Hence $\mathcal{C}_{\max}$ is not only compatible with the maximal but also with the trivial generalized bornology (cf. Example 1.1.22). Equipping X with the maximal coarse structure and the trivial bornology the resulting space is denoted by $X_{\max,\emptyset}$, or $X_{M,\emptyset}$. ✠

Definition 1.3.16. A *G-equivariant generalized bornological coarse space* is a generalized bornological coarse space X in $\mathbf{\widetilde{Born}Coarse}$ together with an action of G on X by automorphisms such that the set $\mathcal{C}_X^G$ of G-invariant entourages is cofinal in $\mathcal{C}_X$. A morphism of G-equivariant generalized bornological coarse spaces is a morphism of the underlying generalized bornological coarse spaces that is in addition G-equivariant. The category of G-equivariant generalized bornological coarse spaces is denoted by $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$.

Remark 1.3.17. In other words, a G-equivariant generalized bornological coarse space is an object in $\mathrm{Fun}(BG, \mathbf{\widetilde{Born}Coarse})$ such that for all U in $\mathcal{C}_X$ the set

$$\bigcup_{g \in G} g \cdot U$$

is again an entourage in $\mathcal{C}_X$. This condition is always fulfilled if G is finite. ♦

Lemma 1.3.18. *The construction in Example 1.3.15 above extends to a functor*

$$(-)_{\max,\emptyset}\colon \boldsymbol{G}\mathbf{Set} \longrightarrow \boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$$

which is part of an adjunction

$$F\colon \boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse} \rightleftarrows \boldsymbol{G}\mathbf{Set} : (-)_{\max,\emptyset},$$

where F denotes the forgetful functor.

Proof. Follows immediately from the definitions. □

Lemma 1.3.19. *The functor $(-)_{mM}\colon \boldsymbol{G}\mathbf{Set} \to \boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$ is part of an adjunction*

$$(-)_{mM}\colon \boldsymbol{G}\mathbf{Set} \rightleftarrows \boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse} : F$$

where $F\colon \boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse} \to \boldsymbol{G}\mathbf{Set}$ denotes the forgetful functor.

Proof. Like in the classical setting. See Lemma 1.2.10. □

Let X be a G-equivariant generalized bornological coarse space and let Y be a G-set. Consider a G-equivariant map $f\colon Y \to X$ of G-sets. We can define the *pullback structures* on Y (c.f. Lemma 1.2.5)

$$\begin{aligned} f^*\mathcal{C}_X &:= \mathcal{C}\left\langle\left\{U \subseteq Y \times Y \;\middle|\; (f \times f)(U) \in \mathcal{C}_X\right\}\right\rangle \\ f^*\mathcal{B}_X &:= \tilde{\mathcal{B}}\left\langle\left\{f^{-1}(B) \;\middle|\; B \in \mathcal{B}_X\right\}\right\rangle. \end{aligned}$$

Lemma 1.3.20. *The structures $f^*\mathcal{C}_X$ and $f^*\mathcal{B}_X$ are compatible and endow Y with the structure of a G-equivariant generalized bornological coarse space. Moreover the map f becomes a morphism $Y \to X$ in $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$.*
If $f\colon Y \hookrightarrow X$ is the inclusion of a G-invariant subset, we call this structure the induced subspace structure.

Proof. Same proof as in the classical case. See Lemma 1.2.5. □

The construction of the tensor product (c.f. Example 1.2.15) generalizes to $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$:

Example 1.3.21. Consider two spaces X and Y in $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$. We define their *tensor product* $X \otimes Y$ as follows: The underlying G-set is given by $X \times Y$ (where G

acts factorwise). The generalized bornology and the coarse structure are defined by

$$\mathcal{C}_{X\otimes Y} := \mathcal{C}\left\langle\left\{U\bar{\times}U' \;\middle|\; U\in\mathcal{C}_X,\ U'\in\mathcal{C}_Y\right\}\right\rangle$$
$$\mathcal{B}_{X\otimes Y} := \tilde{\mathcal{B}}\left\langle\left\{B\times B' \;\middle|\; B\in\mathcal{B}_X,\ B'\in\mathcal{B}_Y\right\}\right\rangle.$$

This construction extends to a bi-functor

$$(-)\otimes(-)\colon \boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}\times\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}\longrightarrow\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}$$

which is part of a symmetric monoidal structure on $\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}$ with tensor unit the one point space $\{*\}_{MM}$. ✠

However, in this new category $\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}$ we have also a symmetric monodial structure w.r.t to the product (c.f. Remark 1.2.17):

Example 1.3.22. For two spaces X and Y in $\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}$ we define their *product* $X\times Y$ as follows: We equipp the underlying G-equivariant coarse space of $X\otimes Y$ with the bornology

$$\mathcal{B}_{X\times Y} := \tilde{\mathcal{B}}\left\langle\left\{B\times Y \;\middle|\; B\in\mathcal{B}_X\right\}\cup\left\{X\times B' \;\middle|\; B'\in\mathcal{B}_Y\right\}\right\rangle.$$

We show that this is indeed the product of X and Y in $\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}$ later in greater generality (see Proposition 1.3.23). The tensor unit of this monoidal structure is given by the one point space $\{*\}_{m\emptyset}$ (which is terminal in $\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}$). ✠

We proceed with the proofs that $\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}$ is complete and co-complete. For this we show that it has all (co-)products and all (co-)equalizers. Since the forgetful functor $F\colon \boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}\to\boldsymbol{G}\mathbf{Set}$ is both left and right adjoint (see Lemmas 1.3.18 and 1.3.19), we already know what the underlying G-set of the respective (co-)limit will be and we only need to construct a suitable G-equivariant generalized bornological coarse structure.

Proposition 1.3.23. *The category* $\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}$ *has all products.*

Proof. Let $(X_i)_{i\in I}$ be a family of G-equivariant generalized bornological coarse spaces. The underlying G-set of the product will be the product $X := \prod_{i\in I} X_i$ of the underlying G-sets. We define the coarse structure by

$$\mathcal{C}_X := \mathcal{C}\left\langle\left\{\bar{\prod}_{i\in I}U_i \;\middle|\; \forall i\in I:\ U_i\in\mathcal{C}_{X_i}\right\}\right\rangle$$

and the generalized bornology by

$$\mathcal{B}_X := \tilde{\mathcal{B}}\Big\langle\Big\{B_j \times \prod\nolimits_{i\neq j} X_i \;\Big|\; j \in I,\ B_j \in \mathcal{B}_{X_j}\Big\}\Big\rangle\text{[4]}.$$

Compatibility of $\mathcal{C}_X$ and $\mathcal{B}_X$ can be checked on generators for which the assertion is clear. Also since G-invariant entourages are cofinal in $\mathcal{C}_{X_i}$ for all i we immediately get the same for $\mathcal{C}_X$. Therefore X together with these structures constitutes an object in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$.
The canonical projections $\pi_i\colon X \to X_i$ are morphisms in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ by construction. Now the space X represents the product $\prod_{i\in I} X_i$ in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$: For any test-object T in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ together with morphisms $f_i\colon T \to X_i$ there is a unique G-equivariant set-theoretical map $f\colon T \to X$ compatible with $(f_i)_{i\in I}$. Properness of that map can be checked on generators, hence we consider $B := B_j \times \prod_{i\neq j} X_j$ for some j in I. Then

$$f^{-1}(B) \subseteq f_j^{-1}(\pi_j(B)) = f_j^{-1}(B_j)$$

where the latter set is bounded in T since f_j is proper, hence $f^{-1}(B) \in \mathcal{B}_T$. For an entourage U in $\mathcal{C}_T$, we have $(f_i \times f_i)(U) \in \mathcal{C}_{X_i}$ for all i, hence

$$(f \times f)(U) = \bar{\prod_{i\in I}}(f_i \times f_i)(U) \in \mathcal{C}_X. \qquad \square$$

Proposition 1.3.24. *For any objects X and Y in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ and any two parallel morphisms $f, g\colon X \to Y$ the equalizer $\mathrm{Eq}(f,g)$ exists in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$.*

Proof. We define $E := \mathrm{Eq}(f,g)$ to be the set-theoretical equalizer of the maps f and g. In particular, E is a subset of X. By the G-equivariance of f and g we see that the subset E is G-invariant, hence we can equipp E with the subspace structure induced from X (see Lemma 1.3.20) and we obtain an inclusion morphism $\iota\colon E \hookrightarrow X$ which fulfills the universal property of the equalizer $\mathrm{Eq}(f,g)$ in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$: Let T be a test object in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ and let $h\colon T \to X$ be a morphism such that $f \circ h = g \circ h$. Then set-theoretically this map factors uniquely through E:

$$\begin{tikzcd} E \arrow[r, "\iota"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ T \arrow[u, dotted, "\ell"] \arrow[ur, "h"'] & & \end{tikzcd}$$

[4]We trust that the reader understands this slightly abused notation in the correct sense

Now ℓ is proper, because any generator A of $\iota^*\mathcal{B}_X$ is of the form $A = \iota^{-1}(B)$ for some B in $\mathcal{B}_X$, hence $\ell^{-1}(A) = h^{-1}(B)$, which is bounded because h is proper. Furthermore ℓ is controlled since for any U in $\mathcal{C}_T$ we have

$$(\iota \times \iota)\Big((\ell \times \ell)(U)\Big) = (h \times h)(U)$$

which is an entourage in X by controlledness of h, hence $(\ell \times \ell)(U) \in \iota^*\mathcal{C}_X$ which is the coarse structure on E.

Finally to see that ℓ is G-equivariant we consider elements γ in G and t in T. By equivariance of h and ι we have

$$\iota(\ell(\gamma \cdot t)) = h(\gamma \cdot t) = \gamma \cdot h(t) = \gamma \cdot \iota(\ell(t)) = \iota(\gamma \cdot \ell(t)).$$

By injectivity of ι we derive $\ell(\gamma \cdot t) = \gamma \cdot \ell(t)$. □

Theorem 1.3.25. *The category* $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ *is complete.*

Proof. This follows from [Mac71, Thm V.2.1] and Propositions 1.3.23 and 1.3.24. □

Proposition 1.3.26. *The category* $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ *has all coproducts.*

Proof. Let $(X_i)_{i \in I}$ be a family of G-equivariant generalized bornological coarse spaces. We denote by X the coproduct in $\boldsymbol{G}\mathbf{Set}$ of the underlying G-sets of X_i and we equip X with the following generalized bornology and coarse structure:

$$\begin{aligned} \mathcal{C}_X &:= \mathcal{C}\left\langle \bigcup\nolimits_{i \in I} \mathcal{C}_{X_i} \right\rangle \\ \mathcal{B}_X &:= \left\{ B \subseteq X \,\middle|\, \forall i \in I:\; B \cap X_i \in \mathcal{B}_{X_i} \right\}. \end{aligned}$$

Cofinality of G-invariant entourages in $\mathcal{C}_X$ follows immediately of the respective cofinality condition on $\mathcal{C}_{X_i}$. Next, we show that $\mathcal{C}_X$ and $\mathcal{B}_X$ are compatible. Since we can check this on generators of the coarse structure we consider an entourage U in $\mathcal{C}_{X_j}$ for some j in I as well as a subset B in $\mathcal{B}_X$. To see that $U[B]$ is bounded we observe that for all k in I we have

$$X_k \cap U[B] \subseteq \begin{cases} U[B \cap X_j] & \text{, if } k = j \\ \emptyset & \text{, else.} \end{cases}$$

Hence $X_k \cap U[B]$ is bounded in X_k for all k in I, hence $U[B] \in \mathcal{B}_X$.

We claim that X represents the coproduct $\coprod_{i \in I} X_i$ in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$: The obvious inclusions $X_i \hookrightarrow X$ are clearly morphisms, and for an object T in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ together with morphisms $f_i\colon X_i \to T$ we get a set-theoretical unique map $f\colon X \to T$ compatible with $(f_i)_{i\in I}$. Thus it suffices to show that this f is a morphism in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$. Since for all i in I the maps f_i are controlled, the map f maps generators of $\mathcal{C}_X$ to entourages, and hence is controlled. Now consider B in $\mathcal{B}_T$. For any i in I we have $f^{-1}(B) \cap X_i = f_i^{-1}(B)$, which is bounded by properness of f_i, hence by definition $f^{-1}(B)$ is bounded. $\square$

Proposition 1.3.27. *For any two objects X and Y in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ and any two morphisms $f, g\colon X \to Y$ the coequalizer* $\mathrm{CoEq}(f, g)$ *exists in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$.*

Proof. We define $E := \mathrm{CoEq}(f, g)$ to be the set-theoretical co-equalizer of the underlying maps f and g and we denote by π the canonical map $Y \to E$. Since f and g are G-equivariant we get an induced action of G on E such that π is G-equivariant. We define a coarse structure and a generalized bornology on E by

$$\begin{aligned}\mathcal{C}_E &:= \mathcal{C}\left\langle \left\{ (\pi\times\pi)(U) \;\middle|\; U \in \mathcal{C}_Y^G \right\} \right\rangle \\ \mathcal{B}_E &:= \tilde{\mathcal{B}}\left\langle \left\{ B \subseteq E \;\middle|\; \forall U \in \mathcal{C}_E :\; \pi^{-1}\Big(U[B]\Big) \in \mathcal{B}_Y \right\} \right\rangle .\end{aligned}$$

Equivariance of π ensures that all generators $(\pi \times \pi)(U)$ for all $U \in \mathcal{C}_Y^G$ are G-invariant, hence $\mathcal{C}_E^G$ is cofinal in $\mathcal{C}_E$. Furthermore $\mathcal{C}_E$ and $\mathcal{B}_E$ are compatible because for U in $\mathcal{C}_E$ and a generator B of $\mathcal{B}_E$ the U-thickening $B' := U[B]$ is a generator of $\mathcal{B}_E$: Indeed, for any entourage V in $\mathcal{C}_E$ we have

$$\pi^{-1}(V[B']) = \pi^{-1}\Big(V\Big[U[B]\Big]\Big) = \pi^{-1}\Big((V\circ U)[B]\Big) \in \mathcal{B}_Y$$

by assumption on B (note $V \circ U \in \mathcal{C}_E$). Therefore E is an object in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$. To see that E represents the coequalizer of f and g we first show that the G-equivariant map π is proper and controlled. For properness consider a generator B of $\mathcal{B}_E$. Since $\Delta_E \in \mathcal{C}_E$ we have by assumption on B that

$$\pi^{-1}(B) = \pi^{-1}(\Delta_E[B]) \in \mathcal{B}_Y.$$

Next, π is controlled because for any entourage U in $\mathcal{C}_Y$ there exists an entourage V in $\mathcal{C}_Y^G$ containing U. Thus $(\pi\times\pi)(U) \subseteq (\pi\times\pi)(V)$ and the latter is a generator of $\mathcal{C}_E$. Therefore π is a morphism in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$.

Finally, for a test object T in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ together with a morphism $p\colon X' \to T$

such that $p \circ f = p \circ g$, the set-theoretical map p factors through E:

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\pi"] \arrow[dr, "p"'] & E \arrow[d, dotted, "h"] \\ & & T \end{tikzcd}$$

It remains to verify that h is a morphism in $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$. First we show G-equivariance. For this we consider an element γ in G and e in E. Surjectivity of π provides y in Y with $\pi(y) = e$. Now by G-equivariance of π and p we see:

$$\begin{aligned} h(\gamma \cdot e) = h(\gamma \cdot \pi(y)) &= (h\pi)(\gamma \cdot y) = p(\gamma \cdot y) = \gamma \cdot p(y) = \gamma \cdot h(\pi(y)) \\ &= \gamma \cdot h(e). \end{aligned}$$

By assumption the map p is controlled, hence by commutativity of the diagram we immediately get that h maps generators of $\mathcal{C}_E$ to controlled sets. Therefore the map h is controlled. To check that it is also proper, consider a bounded subset B of T. We show, that $h^{-1}(B)$ is a generator of $\mathcal{B}_E$. For this let U be in $\mathcal{C}_E$. One immediatelly verifies, that

$$\pi^{-1}\Big(U[h^{-1}(B)]\Big) \subseteq p^{-1}\Big((h \times h)(U)[B]\Big).$$

Therefore it remains to show, that the supset is bounded in Y: Since h is controlled, $V := (h \times h)(U)$ is an entourage of T, hence the V-thickening of B is a bounded subset of T, so its pre-image under p is bounded in Y which concludes the proof. □

Theorem 1.3.28. *The category* $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$ *is co-complete.*

Proof. This follows from the dual argument of [Mac71, Theorem V.2.1] and Propositions 1.3.26 and 1.3.27. □

We have shown that $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$ is bi-complete. But we used that every (co-)limit can be expressed by (co-)products and (co-)equalizers which is not very explicit. For convenience we give concrete formulas for an arbitrary (co-)limit in $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$:

Remark 1.3.29. Let $\mathbf{I}$ be a small category and $D\colon \mathbf{I} \to \boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$ be a diagram. We describe the (co-)limit of D explicitly.
Let $F\colon \boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse} \to \boldsymbol{G}\mathbf{Set}$ be the forgetful functor and let $X := \lim_{\mathbf{I}} D$ be the limit of the diagram D in $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$. Since F is right adjoint (c.f.

Lemma 1.3.18) we know that $F(X)$ is the limit of the diagram $F \circ D$ in $\boldsymbol{G}\mathbf{Set}$ and can hence be viewed as subset of the product $\prod_{\mathbf{I}} FD(j)$. It comes with canonical G-equivariant maps $f_j \colon F(X) \to F(D(j))$ for all j in $\mathbf{I}$. The generalized bornological coarse structure on X is then given by

$$\mathcal{C}_X = \mathcal{C}\left\langle\left\{(X \times X) \cap \prod_{j \in \mathbf{I}} U_j \;\middle|\; \forall j \in \mathbf{I} \colon\; U_j \in \mathcal{C}_{D(j)}\right\}\right\rangle.$$
$$\mathcal{B}_X = \tilde{\mathcal{B}}\left\langle\left\{f_j^{-1}(B) \;\middle|\; j \in \mathbf{I},\; B \in \mathcal{B}_{D(j)}\right\}\right\rangle.$$

Similarly for the coproduct we let $Y := \operatorname{colim}_{\mathbf{I}} D$ be the colimit of the diagram D in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$. Since F is also left adjoint (c.f. Lemma 1.3.19) we know that $F(Y)$ is the colimit of $F \circ D$ in $\boldsymbol{G}\mathbf{Set}$. It comes with canonical G-equivariant morphisms $g_j \colon F(D(j)) \to F(Y)$ for all j in $\mathbf{I}$. The generalized bornological coarse structure on Y is then given by

$$\mathcal{C}_Y = \mathcal{C}\left\langle\left\{(g_j \times g_j)(U) \;\middle|\; j \in \mathbf{I},\; U \in \mathcal{C}_{D(j)}\right\}\right\rangle$$
$$\mathcal{B}_Y = \tilde{\mathcal{B}}\left\langle\left\{B \subseteq Y \;\middle|\; \forall U \in \mathcal{C}_Y,\; \forall j \in \mathbf{I} \colon\; g_j^{-1}\big(U[B]\big) \in \mathcal{B}_{D(j)}\right\}\right\rangle. \qquad \blacklozenge$$

Example 1.3.30. Consider a G-equivariant generalized bornological coarse space X. For any G-invariant entourage U in $\mathcal{C}_X^G$ we obtain a new G-equivariant generalized bornological coarse space X_U by replacing $\mathcal{C}_X$ with $\mathcal{C}\langle\{U\}\rangle$ (c.f. Example 1.2.9). We get an isomorphism

$$\operatorname*{colim}_{U \in \mathcal{C}_X^G} X_u \xrightarrow{\cong} X.$$

This will be used for definition of "U-continuity" of coarse homology theories (c.f. Definition 2.1.17). ✠

We proceed with the elaboration of some connections between the two categories $\boldsymbol{G}\mathbf{BornCoarse}$ and $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$, as well as consequences of those. Clearly we can view $\boldsymbol{G}\mathbf{BornCoarse}$ as a full subcategory of $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$, however there is no adjunction between these two categories. Nevertheless there are strong connections between limits and colimits in both these categories.

Proposition 1.3.31. *The functor $\iota \colon$ $\boldsymbol{G}\mathbf{BornCoarse} \hookrightarrow \boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ is a fully faithful embedding. But it has neither left nor right adjoint.*

Proof. The first statement is clear from definitions. Next, if ι had a right adjoint R, then R would preserve limits and in particular $R(\{*\}_{m\emptyset})$ would be a terminal object in $\boldsymbol{G}\mathbf{BornCoarse}$ which does not exist (c.f. Lemma 1.2.19). Finally, if we assume that ι has a left adjoint L then applying L to the square in Example 1.2.21 gives a pushout square

$$\begin{tikzcd} L(\mathbb{N}_{mM}) \arrow[r] \arrow[d] & L(\mathbb{N}_{MM}) \arrow[d] \\ L(\mathbb{N}_{mm}) \arrow[r] & L(\mathbb{N}_{M\emptyset}) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

in $\boldsymbol{G}\mathbf{BornCoarse}$. However, since ι is fully faithful, the co-unit of the adjunction $L \dashv \iota$ is a natural isomorphism. In particular we have a natural isomorphism $L(\mathbb{N}_{mM}) \cong \mathbb{N}_{mM}$ and similarly for $\mathbb{N}_{mm}$ and $\mathbb{N}_{MM}$. Hence we can identify the pushout diagram above with

$$\begin{tikzcd} \mathbb{N}_{mM} \arrow[r] \arrow[d] & \mathbb{N}_{MM} \arrow[d] \\ \mathbb{N}_{mm} \arrow[r] & L(\mathbb{N}_{M\emptyset}). \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

This directly contradicts Example 1.2.20. $\square$

Properness of morphisms provide the following very useful lemma:

Lemma 1.3.32. *Consider a morphism $f \colon X \to Y$ in $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$. For any bounded point y in Y the fiber $f^{-1}(\{y\})$ contains bounded points only. In particular, if Y is locally bounded, then so is X.*

Proof. This is precisely the properness-condition imposed on morphisms in the category $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$. $\square$

Recall the equivalence relation $\sim$ on a (generalized bornological) coarse space X defined by $x \sim y$ iff $(x, y) \in U$ for some U in $\mathcal{C}_X$ (c.f. Lemma 1.1.11).

Lemma 1.3.33. *Consider a space X in $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$. A coarse component of X is either locally bounded or consists of unbounded points only.*

Proof. Consider a coarse component M of X. Assume that it contains an unbounded point x and a bounded point y of X. Since M is coarsely connected there exists an entourage U in $\mathcal{C}_X$ such that $(x, y) \in U$. This means $x \in U[\{y\}]$ which is bounded by compatibility and $\{y\} \in \mathcal{B}_X$. But this would imply that x is bounded. $\square$

Consider a space X in $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$. The subsets X_b and X_h of bounded and unbounded points of X are G-invariant because G acts by automorphisms on X.

Hence we can equip X_b and X_h with the subspace structure induced from X and we obtain:

Corollary 1.3.34. *We have an isomorphism $X \cong X_b \amalg X_h$ in $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$.*

Proof. The inclusions $X_b \hookrightarrow X$ and $X_h \hookrightarrow X$ are morphisms by construction of the subspace structure. They induce a morphism $X_b \amalg X_h \to X$ whose underlying map of sets is the identity. Therefore it remains to see that the identity map of sets $X \to X_b \amalg X_h$ is proper and controlled, or in other words that every bounded subset of $X_b \amalg X_h$ is bounded in X and that every entourage of X is an entourage in the coproduct. For this we first notice that a bounded subset of $X_b \amalg X_h$ is a bounded subset of X_b, hence bounded in X. Now, for an entourage U of X we know that $U \subseteq X_b \times X_b$ or $U \subseteq X_h \times X_h$ by Lemma 1.3.33. Therefore it is an entourage in X_b or in X_h hence in $X_b \amalg X_h$. □

Although the fully faithful inclusion functor $\iota\colon \boldsymbol{G}\mathbf{BornCoarse} \to \boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$ is neither left nor right adjoint (Proposition 1.3.31) it still preserves all limits and colimits:

Proposition 1.3.35. *Let $\mathbf{I}$ be a small category and $D\colon \mathbf{I} \to \boldsymbol{G}\mathbf{BornCoarse}$ be a diagram. If the limit $\lim_{\mathbf{I}} D$ exists in $\boldsymbol{G}\mathbf{BornCoarse}$ then the canonical morphism*

$$\iota\Big(\lim_{\mathbf{I}} D\Big) \longrightarrow \lim_{\mathbf{I}}(\iota \circ D)$$

is an isomorphism in $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$. Similarly, if the colimit of D exists in $\boldsymbol{G}\mathbf{BornCoarse}$, then the canonical morphism

$$\operatorname*{colim}_{\mathbf{I}}(\iota \circ D) \longrightarrow \iota\Big(\operatorname*{colim}_{\mathbf{I}} D\Big)$$

is an isomorphism in $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$.

Proof. Assume the limit $L := \lim_{\mathbf{I}} D$ exists in $\boldsymbol{G}\mathbf{BornCoarse}$. Then we show that $\iota(L)$ fulfills the universal property of the limit of the diagram $\iota \circ D$: In fact for any object T in $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$ together with morphisms $f_i\colon T \to \iota(D(i))$ each[5] of the morphisms f_i implies that T is locally bounded (because $\iota(D(i))$ is) by Lemma 1.3.32. Hence we can view T as an object in $\boldsymbol{G}\mathbf{BornCoarse}$ and identify T with $\iota(T)$. Now

[5] note that the limit is not taken over the empty diagram, because this does not have a limit in $\boldsymbol{G}\mathbf{BornCoarse}$ as shown in Lemma 1.2.19

fully faithfulness of ι gives:

$$
\begin{aligned}
\mathrm{Hom}_{\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}}(T,\iota(L)) &\cong \mathrm{Hom}_{\boldsymbol{G}\mathbf{BornCoarse}}(T,L) \cong \lim_{i\in\mathbf{I}} \mathrm{Hom}_{\boldsymbol{G}\mathbf{BornCoarse}}(T,D(i)) \\
&\cong \lim_{i\in\mathbf{I}} \mathrm{Hom}_{\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}}(T,\iota(D(i)))
\end{aligned}
$$

which shows $\lim_{\mathbf{I}}(\iota\circ D)\cong\iota(L)$.
For the second assertion, we assume that the colimit $C := \mathrm{colim}_{\mathbf{I}} D$ exists. By cocompleteness of $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$ we know that also the colimit $\tilde{C} := \mathrm{colim}_{\mathbf{I}}(\iota\circ D)$ exists. Note that by fully faithfullness of ι we have

$$
\begin{aligned}
\mathrm{Hom}_{\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}}(\tilde{C},\iota(C)) &\cong \lim_{i\in\mathbf{I}} \mathrm{Hom}_{\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}}(\iota(D(i)),\iota(C)) \\
&\cong \lim_{i\in\mathbf{I}} \mathrm{Hom}_{\boldsymbol{G}\mathbf{BornCoarse}}(D(i),C) \\
&\cong \mathrm{Hom}_{\boldsymbol{G}\mathbf{BornCoarse}}(C,C)
\end{aligned}
$$

In particular there exists at least one morphism $\tilde{C}\to\iota(C)$ which implies (since $\iota(C)$ is locally bounded) that $\tilde{C}\in\boldsymbol{G}\mathbf{BornCoarse}$ by Lemma 1.3.32. We derive $C\cong\tilde{C}$ in $\boldsymbol{G}\mathbf{BornCoarse}$ by the usual “Universal-Property-Yoga”. □

Corollary 1.3.36. *Let $\mathbf{I}$ be a small category and $D\colon\mathbf{I}\to\boldsymbol{G}\mathbf{BornCoarse}$ be a diagram. The (co-)limit of D exists in $\boldsymbol{G}\mathbf{BornCoarse}$ if and only if the (co-)limit of $\iota\circ D$ in $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$ is locally bounded.*

Proof. Assume that the limit $L := \lim_{\mathbf{I}} D$ exists. Then $\iota(L)$ is the limit of $\iota\circ D$ by Proposition 1.3.35, hence $\lim_{\mathbf{I}}\iota\circ D$ is locally bounded. On the other hand, assume that $\tilde{L} := \lim_{\mathbf{I}}\iota\circ D$ is locally bounded. Then we can view it as an object in the full subcategory $\boldsymbol{G}\mathbf{BornCoarse}$ where it fulfills the universal property of $\lim_{\mathbf{I}} D$.
The same argument shows the assertion for colimits. □

Corollary 1.3.37. *The category $\boldsymbol{G}\mathbf{BornCoarse}$ has all non-empty limits.*

Proof. Let $\mathbf{I}$ be a small non-empty category and $D\colon\mathbf{I}\to\boldsymbol{G}\mathbf{BornCoarse}$ a diagram. Let $L := \lim_{\mathbf{I}}\iota\circ D$ be the limit in $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$. For any j in $\mathbf{I}$ the canonical projection $L\to\iota(D(j))$ implies that L is locally finite by Lemma 1.3.32. Thus Corollary 1.3.36 gives the claim. □

Corollary 1.3.36 gives a precise criterium for when a small diagram $D\colon\mathbf{I}\to\boldsymbol{G}\mathbf{BornCoarse}$ has a colimit or not. In [BEKW, Prop. 2.21] the authors define a condition on diagrams called "colimit-admissible" and show that this condition ensures the existence of the colimit of that diagram. In the following we show that their condition is equivalent to the condition in Corollary 1.3.36.

Let $F\colon \boldsymbol{G}\mathbf{BornCoarse} \to \boldsymbol{G}\mathbf{Set}$ be the forgetful functor.

Corollary 1.3.38. *Consider a small diagram $D\colon \mathbf{I} \to \boldsymbol{G}\mathbf{BornCoarse}$ and for all i in $\mathbf{I}$ we set $X_i := D(i)$ and we let $f_i\colon F(X_i) \to X := \operatorname{colim}_{\mathbf{I}}(F \circ D)$ be the structure morphism. Then the colimit of D exists in $\boldsymbol{G}\mathbf{BornCoarse}$ if and only if D is* colimit-admissible*, i.e. iff*

$$\forall b \in X\ \forall n \in \mathbb{N}\ \forall k, i_1, i_2, \ldots, i_n \in \mathbf{I},\ \forall (U_j)_{j \in \{i_1,\ldots,i_n\}} \in \prod_{j \in \{i_1,\ldots,i_n\}} \mathcal{C}_{X_j} :$$
$$f_k^{-1}\Big((f_{i_n} \times f_{i_n})(U_{i_n})\Big[\cdots (f_{i_1} \times f_{i_1})(U_{i_1})[\{b\}] \cdots\Big]\Big) \in \mathcal{B}_{X_k}.$$

Proof. First, assume that $\operatorname{colim}_{\mathbf{I}} D$ exists in $\boldsymbol{G}\mathbf{BornCoarse}$. Since the underlying set equals X (c.f. Lemma 1.3.19 and Proposition 1.3.35) we refer to the colimit also by X. Now for all b in X the set $\{b\}$ is bounded, and for all j in $\{i_1, \ldots, i_n\}$ the morphism f_j is controlled, hence $(f_j \times f_j)(U_j)$ is an entourage in X. Thus the successively thickening of $\{b\}$ by these entourages stays bounded and therefore the its pre-image under f_k is bounded in X_k which shows the "colimit-admissible"-condition.

On the other hand, assume that the diagram is colimit-admissible. By Corollary 1.3.36 it is enough to see that the colimit $C := \operatorname{colim}_{\mathbf{I}} \iota \circ D$ in $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$ is locally finite, hence that for all b in C we have $\{b\} \in \mathcal{B}_C$. Looking at the definition of the generalized bornology of the colimit (Remark 1.3.29) we see that this condition is just a shorter way to express the colimit-admissible-condition (Note that $\mathcal{C}_C$ is closed under taking composition and use that $U[V[B]] = (U \circ V)[B]$ for any two entourages U and V and any subset B of C). $\square$

1.4 Coarse equivalence

One of the most fundamental notions in coarse geometry is "coarse equivalence". The study of coarse spaces not up to isomorphisms, but up to coarse equivalences (it is often called the "coarse category") is one of the most important and basic things in coarse geometry and it is done in almost every paper working in the coarse world (e.g. [Roe93; Mit01; Wri03; HPR97; Wul20; DL98; MNS20; HR94] and many more).

The probably most famous example is the coarse equivalence between the integers $\mathbb{Z}$ and the reals $\mathbb{R}$ allowing us to "discretize" spaces. More generally, the universal cover of a compact manifold is (under a small condition) coarsely equivalent to the (discrete) fundamental group as a consequence of Švarc-Milnor's theorem (Theorem 1.4.22).

In [BE20] the authors adapt the notion of coarse equivalence to their category **BornCoarse** and later to $\boldsymbol{G}$**BornCoarse** in [BEKW]. Note that equivariance has a non-trivial impact in the theory of coarse equivalences. There are several examples (even for finite groups) for spaces in $\boldsymbol{G}$**BornCoarse** that are coarsely equivalent only after forgetting the group action (c.f. Example 1.4.26).

Coarse equivalences play the role of "homotopy equivalences" in the coarse setting and is thus an important notion for (equivariant) coarse homology theory, with which we will deal in the following chapter.

In this section we lift the notion of coarse equivalence to the category of G-equivariant generalized bornological coarse spaces. We then give basic properties and generalize classical facts to this new setting. Also we give some examples from classical coarse geometry as well as some examples developed in the setting of bornological coarse spaces. We will also provide an equivalent definition of coarse equivalence (Proposition 1.4.17). Furthermore, we examine some coarse invariants by (counter-)examples.

Viewing coarse equivalences as weak equivalences W of the category $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ we can look at the Dwyer-Kan localization $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}[W^{-1}]$ in the realm of ∞-categories ([DK80; Hin16]). We will finish this section by showing that in our situation this ∞-category is in fact (the nerve of) an ordinary 1-category, namely the homotopy category $\mathbf{ho}(\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse})$ (which will also be defined in this section and plays the role of "coarse category" mentioned above). Since the category $\boldsymbol{G}$**BornCoarse** is a full subcategory of $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ all the statements made in this section also hold for the non-generalized setting.

We fix a group G and a G-equivariant generalized bornological coarse space X.

Definition 1.4.1. Consider two morphisms $f, g \colon X \to Y$ in $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$. We say that f *and* g *are close to each other* if the subset $(f \times g)(\Delta_X)$ is an entourage in $\mathcal{C}_Y$.

Example 1.4.2. Consider $\mathbb{R}$ as a bornological coarse space endowed with the structure induced from the standard metric. The maps $f, g \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x + 1$ and $g(x) := -x$ are morphisms in **BornCoarse**. The morphism f is close to the identity $\mathrm{id}_{\mathbb{R}}$ because

$$(f \times \mathrm{id}_{\mathbb{R}})(\Delta_{\mathbb{R}}) = \Big\{(x+1, x) \;\Big|\; x \in \mathbb{R}\Big\} \subseteq U_2$$

where U_2 was the metric entourage of all (x, y) in $\mathbb{R} \times \mathbb{R}$ with $|x - y| < 2$.
However, g is not close to the identity because for all $r > 0$ we have

$$(g \times \mathrm{id}_{\mathbb{R}})(x) = (-x, x) \notin U_r$$

for all $x > \frac{r}{2}$. ✠

Lemma 1.4.3. *Being close to each other defines an equivalence relation on the collection of morphisms in* $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$.

Proof. Reflexivity follows because every subset of the diagonal is an entourage. Symmetry follows because the coarse structure is closed under taking inverses and finally since the composition of entourages is an entourage, we obtain transitivity. □

In the introductory words we already mentioned that coarse equivalence will play the role of homotopy invariance in the coarse world. Therefore closeness of maps plays the role of being homotopic, which in turn is often characterized by usage of "inverval objects". In fact this is also true for closeness of maps in our setting:

Definition 1.4.4. The G-equivariant bornological coarse space $\{0,1\}_{MM}$ with trivial G-action is called the *interval object* in $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$ and denoted by $\mathbb{I}$.

Lemma 1.4.5. *Two morphisms* $f, g \colon X \to Y$ *in* $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$ *are close if and only if the map*

$$h \colon \mathbb{I} \otimes X \longrightarrow Y$$

defined by

$$(i,x) \longmapsto \begin{cases} f(x) & , \ i = 0 \\ g(x) & , \ i = 1. \end{cases}$$

is a morphism in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$.

Proof. Properness and G-equivariance of h both follow from properness and equivariance of f and g. The only thing to show is that h is controlled if and only if f and g are close. First, assume h is controlled. The subset $\{(0,1)\}$ is an entourage on $\mathbb{I}$, hence $U := \{(0,1)\} \bar{\times} \Delta_X$ is an entourage on $\mathbb{I} \otimes X$, hence the image

$$(h \times h)(U) = \Big\{ (f(x), g(x)) \;\Big|\; x \in X \Big\} = (f \times g)(\Delta_X)$$

is an entourage on Y, hence f is close to g.
On the other hand, we assume that f and g are close and we show that h is controlled. Since every entourage U' on $\mathbb{I} \otimes X$ is contained in $U := (\{0,1\} \times \{0,1\}) \bar{\times} U_X$ for some U_X in $\mathcal{C}_X$, it is enough to see that $(h \times h)(U)$ is an entourage on Y. But we see

$$(h \times h)(U) = (f \times f)(U_X) \cup (f \times g)(U_X) \cup (g \times f)(U_X) \cup (g \times g)(U_X).$$

By controlledness of f and g the first and last subset in this union is an entourage on Y. Also the third is the inverse of the second, hence it is enough to see that $(f \times g)(U_X) \in \mathcal{C}_Y$, but an elementary calculation shows

$$(f \times g)(U_X) \subseteq (f \times f)(U_X) \circ (f \times g)(\Delta_X)$$

hence we are through (note that $(f \times g)(\Delta_X) \in \mathcal{C}_Y$ since f is close to g). □

Corollary 1.4.6. *For any space X in* $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ *the canonical inclusions $\iota_0, \iota_1 \colon X \to \mathbb{I} \otimes X$ are close.*

Proof. The corresponding morphism h in Lemma 1.4.5 is the identity. □

Remark 1.4.7. In contrast to being homotopic, being close does not involve a *choice* of a homotopy $h \colon \mathbb{I} \otimes X \to Y$. The map h is fixed and being a morphism is a property. This will be very important later (Theorem 1.4.36). ♦

Since being close is characterized by an interval object, we get usual properties like:

Corollary 1.4.8. *For any pair of close morphisms f, g and any other morphisms h and k such that the respective compositions are well-defined we get that kfh is close to kgh.*

Lemma 1.4.9. *Consider two morphisms $f, g\colon X \to Y$ in* $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ *and two subsets A, B of X such that $X = A \cup B$. If $f\big|_A$ is close to $g\big|_A$ and $f\big|_B$ is close to $g\big|_B$ then f is close to g.*

Proof. Just calculate $(f \times g)(\Delta_X) = (f \times g)(\Delta_A \cup \Delta_B) = (f \times g)(\Delta_A) \cup (f \times g)(\Delta_B)$ and use that $\mathcal{C}_Y$ is closed under finite unions. □

Remark 1.4.10. The statements in Corollary 1.4.8 and Lemma 1.4.9 follow from the axioms on a coarse structure. The converse is also true: If for all sets S we have an equivalence relation on the set of maps $S \to X$ satisfying Corollary 1.4.8 and Lemma 1.4.9 then there exists a unique coarse structure on X such that the equivalence relation is closeness w.r.t. that coarse structure. This is due to [Roe03, Prop 2.15]. ♦

The closeness-relation on the set of morphisms induces in a natural way the notion of coarse equivalence:

Definition 1.4.11. A morphism $f\colon X \to Y$ in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ is called a *coarse equivalence* if it is invertible up to closeness, i.e. if there exists a morphism $g\colon Y \to X$ in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ such that fg and gf are close to the respective identities. In this case we that X and Y are *coarsely equivalent* and we write $X \simeq_c Y$.

Example 1.4.12. Consider a non-empty space X in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$. Assume that X carries the maximal coarse structure $\mathcal{C}_X = \mathcal{P}(X \times X)$. Then either X carries the trivial bornology or the maximal one (follows immediately from compatibility). In the first case X is coarsely equivalent to $\{*\}_{MM}$, in the latter case it is coarsely equivalent to the terminal object $\{*\}_{M,\emptyset}$. Both assertions follow immediately, because every morphism $X \to X$ is close to id_X by maximality of $\mathcal{C}_X$. ✠

Example 1.4.13. The projection $p\colon X \otimes \mathbb{I} \to X$ is a coarse equivalence with inverse morphism for example given by $\iota_0\colon X \to X \otimes \mathbb{I}$, $x \mapsto (x, 0)$. Indeed, clearly $p\iota_0 = \mathrm{id}_X$ and on the other hand $\iota_0 p$ is close to the identity on $X \otimes \mathbb{I}$ because an easy calculation shows

$$(\iota_0 p \times \mathrm{id}_{X\otimes\mathbb{I}})(\Delta_{X\otimes\mathbb{I}}) \subseteq \Delta_X \bar{\times} (\{0\} \times \{0,1\}).$$

The supset above is an entourage on $X \otimes \mathbb{I}$ beause the coarse structure on $\mathbb{I}$ is the maximal one. ✠

The following notions are motivated by [Har19].

Definition 1.4.14. A morphism $f\colon X \to Y$ in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ is called

- *coarsely injective* if for all entourages U in $\mathcal{C}_Y$ we have $(f \times f)^{-1}(U) \in \mathcal{C}_X$.
- *coarsely surjective* if there exists an entourage U in $\mathcal{C}_Y$ such that $U[\mathrm{im}(f)] = Y$.
- *coarsely bijective* if it is coarsely injective and coarsely surjective.

Remark 1.4.15. In the literature a coarsely surjective map is also called “dense” and a coarsely injective map is referred to as “coarse embedding”. Note that the latter is an adequate notion in the category of coarse spaces, but not in the category of bornological coarse spaces where we would also want that an embedding is bornological (i.e. images of bounded subsets are bounded). ⧫

Definition 1.4.16. A morphism $f\colon X \to Y$ in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ is called a *coarse embedding* if $f\colon X \to \mathrm{im}(f)$ is a coarse equivalence (where $\mathrm{im}(f)$ is endowed[6] with the subspace structure induced by Y).

We gave the definition of coarse injectivity and coarse surjectivity because:

Proposition 1.4.17. *Let $f\colon X \to Y$ be a morphism in* $\mathbf{\widetilde{Born}Coarse}$. *Then f is a coarse equivalence if and only if it is bornological and coarsely bijective.*

Proof. First, assume that f is a coarse equivalence with coarse inverse $g\colon Y \to X$. By definition we obtain entourages

$$\begin{aligned} V &:= (fg \times \mathrm{id}_Y)(\Delta_Y) \quad \text{in } \mathcal{C}_Y \\ U &:= (gf \times \mathrm{id}_X)(\Delta_X) \quad \text{in } \mathcal{C}_X. \end{aligned}$$

Now for any bounded subset B in $\mathcal{B}_X$ a quick calculation shows

$$f(B) \subseteq g^{-1}\Big(U[B]\Big)$$

and this supset is bounded by properness of g, hence $f(B) \in \mathcal{B}_Y$, hence f is bornological. Furthermore, for any y in Y we have $\Big(y, f(g(y))\Big) \in V^{-1}$, hence $y \in V^{-1}[\mathrm{im}(f)]$,

[6] note that $\mathrm{im}(f)$ is G-invariant by G-equivariance of f

hence f is coarsely surjective. Finally, for any entourage W in $\mathcal{C}_Y$ an easy calculation shows

$$(f \times f)^{-1}(W) \subseteq U^{-1} \circ (g \times g)(W) \circ U.$$

Since this supset is an entourage in $\mathcal{C}_X$ (by controlledness of g), we see that f is coarsely injective.

To see the other implication, we assume that f is bornological and coarsely bijective. We have to define a proper controlled map $g\colon Y \to X$ such that fg and gf are close to the respective identities.

By assumption there exists an entourage V in $\mathcal{C}_Y$ such that

$$V[\mathrm{im}(f)] = Y. \tag{*}$$

For any y in Y there exists by (*) some x_y in X such that $(y, f(x_y)) \in V$. We define $g(y) := x_y$ and obtain a map $g\colon Y \to X$. For properness of g we consider a bounded subset B in $\mathcal{B}_X$. For any y in $g^{-1}(B)$ we know by construction of g that $\big(y, f(g(y))\big) \in V$, also $g(y) \in B$, therefore $y \in V[f(B)]$. Hence we have shown $g^{-1}(B) \subseteq V[f(B)]$. Note that the latter supset is bounded because f is bornological by assumption.

To see that g is controlled, we consider an entourage U in $\mathcal{C}_Y$. A straightforward calculation shows

$$(g \times g)(U) \subseteq (f \times f)^{-1}\big(V^{-1} \circ U \circ V\big).$$

Note that the supset is controlled because f is coarsely injective. Thus g is controlled, hence a morphism $Y \to X$ in $\mathbf{\widetilde{Born}Coarse}$.

It remains to see that the compositions are close to the respective identities: Both follows because by construction of g we immediately get:

$$\begin{aligned} (fg \times \mathrm{id}_Y)(\Delta_Y) &\subseteq V^{-1} \\ (gf \times \mathrm{id}_X)(\Delta_X) &\subseteq (f \times f)^{-1}(V^{-1}). \end{aligned}$$

Note that the latter supset is an entourage because f is coarsely injective. □

Remark 1.4.18. A coarse equivalence $f\colon X \to Y$ in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ is bornological and coarsely bijective with the very same proof as above in Proposition 1.4.17. However, the proof of the converse is not true in general because the potential inverse cannot be defined G-equivariant. Moreover, not only does the proof above not go through, but also the statement is wrong as shown in Example 1.4.25. ♦

Example 1.4.19. Let X be in $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$ and consider a G-invariant subset A of X endowed with the subspace structure. Then the inclusion $A \hookrightarrow X$ is a coarse embedding. ✠

Example 1.4.20. Consider a space X in $\widetilde{\mathbf{Born}}\mathbf{Coarse}$, a G-invariant subset A of X and an entourage U in $\mathcal{C}_X^G$ with $\Delta_X \subseteq U$. Then the inclusion $A \hookrightarrow U[A]$ is a coarse equivalence, where both subsets are endowed with the subspace structure induced from X. Indeed, $A \hookrightarrow U[A]$ is a coarse embedding (Example 1.4.19) and it is coarsely surjective by construction[7]. ✠

Example 1.4.21. Consider the reals $\mathbb{R}$ as space in **BornCoarse** equipped with the structure induced by the Euclidean metric and we equip the subset of integers $\mathbb{Z}$ with the subspace structure induced from $\mathbb{R}$. Then the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a coarse equivalence by Example 1.4.20 because clearly $U_1[\mathbb{Z}] = \mathbb{R}$, where U_1 in $\mathcal{C}_{\mathbb{R}}$ is the metric entourage defined in Example 1.1.8.
With the same line of argument we see that $\mathbb{N}$ is coarsely equivalent to $\mathbb{R}_{\geq 0}$. ✠

More generally:

Theorem 1.4.22 (Švarc-Milnor)**.** *Let G act properly and co-compactly by isometries on a path metric space X, then G (equipped with any word metric[8]) is coarsely equivalent to X.*
In particular, for any compact manifold M with universal cover $\tilde{M}$ and fundamental group G acting on $\tilde{M}$ properly and co-compactly we have $G \simeq_c \tilde{M}$.

Proof. See [Nor04, Thm 1.2.6] or [Roe06, p. 669]. □

Lemma 1.4.23. *Let (X, d) and (Y, d') be two metric spaces and let $f\colon X \to Y$ be a quasi-isometry. Then we can view f as a morphism in* **BornCoarse***. Moreover, f is a coarse embedding, hence if f is coarsely surjective then f is a coarse equivalence.*

Proof. By assumption there exist constants $c, d \in \mathbb{R}^+$ such that for all x, x' in X:

$$c^{-1}\mathrm{d}(x,x') - d \leq \mathrm{d}'(f(x), f(x')) \leq c\mathrm{d}(x,x') + d. \tag{*}$$

[7] note that $\Delta_X \subseteq U$
[8] In this case G is automatically finitely generated and thus any word metric induces the same coarse structure, c.f. Lemma 1.2.8

We check that f is a morphisms, i.e. controlled and proper. Both can be checked on generators. For controlledness the latter inequality in (*) immediately shows

$$(f \times f)(U_r) \subseteq U_{cr+d}.$$

For properness we first see that we can enlarge every generating bounded subset which is hit by f to a bounded subset $B_r(f(x))$ for some x in X. Then the lower inequality of (*) gives that

$$f^{-1}(B_r(f(x))) \subseteq B_{c(r+d)}(x).$$

Hence it is left to show that the morphism f is a coarse equivalence. By assumption f is coarsely surjective, and with analoguous arguments as above we see that (*) immediately gives that f is bornological and coarsely injective. □

Remark 1.4.24. Being a quasi-isometry is stronger than being a coarse embedding. An example for a coarse equivalence between two metric spaces which is not a quasi-isometry can be found in [Bun21, p. 21]. ♦

In contrast to the upper Example 1.4.20 we see that in the equivariant world thickenings are not in general coarse equivalences (the same example is also a counterexample for the equivariant version of Lemma 1.4.23):

Example 1.4.25 ([BEKW, Ex. 3.2]). We consider the space $X := \mathbb{C}$ with the bornological coarse structure induced from the Euclidean metric. Let the group $G := \mathbb{Z}$ act on X by

$$g \cdot x := \exp(2\pi \mathrm{i} \theta g)x,$$

for some θ in $(0,1)$[‡‡]. Since G acts by isometries we can view X as an object in $\boldsymbol{G}$**BornCoarse** (see Example 1.2.6). Consider the G-invariant subset $A := X \setminus \{0\}$ and let U be the metric entourage U_1 (c.f. Example 1.1.8), then clearly $U[A] = X$, hence the coarse embedding $A \hookrightarrow X$ is coarsely surjective, nevertheless it is *not* a coarse equivalence because there is no G-fixed point in A which could serve as an image of 0 under the inverse morphism $X \to A$. ✠

This example does not rest on the group being infinite. The same argument of this example also shows:

[‡‡]The authors in [BEKW] require θ to be non-rational, which is not necessary

Example 1.4.26. Let $G := \mathbb{Z}/2\mathbb{Z}$ operate on $X := \mathbb{R}$ (with the standard-metric structure) by multiplication with (-1). The subset $A := \mathbb{R} \setminus \{0\}$ is G-invariant and coarsely dense in X, however the inclusion is no coarse equivalence (as we do not have a G-fixed point in A that could serve as an image of 0 under the potential inverse). However, clearly the inclusion is a coarse equivalence after forgetting the G-action and considering both spaces as objects in **BornCoarse**. ✠

Definition 1.4.27. A subset A of a space X in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ is called *nice* if for any entourage U in $\mathcal{C}_X^G$ with $\Delta_X \subseteq U$ the inclusion $A \hookrightarrow U[A]$ of A into its U-thickening is a coarse equivalence.

Example 1.4.28 (c.f. [BEKW, Ex. 3.4]). For any X in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ and any Y in $\mathbf{\widetilde{Born}Coarse}$ and any subset A of Y the subset $A \otimes X$ is nice in $Y \otimes X$. This follows because the thickening takes place only in the first factor where we have trivial G-action and whenever G acts trival, any thickening is a coarse equivalence by Example 1.4.20. ✠

Example 1.4.29. For any G-fixed point x in X the subset $\{x\}$ is nice in X. This is because the proof of Proposition 1.4.17 goes through in this case (because G-equivariance of the inverse map is trivially fulfilled). ✠

Another interesting topic related to coarse equivalences are coarse invariants. We give some examples:

Example 1.4.30. Being locally bounded is a coarse invariant, i.e. for every coarse equivalence $f\colon X \to Y$, the space X is locally bounded if and only if Y is locally bounded. This follows immediately from Lemma 1.3.32. ✠

Example 1.4.31. Being countable is not a coarse invariant because we know that $\mathbb{Z}$ and $\mathbb{R}$ are coarsely equivalent (c.f. Example 1.4.21). ✠

Coarse invariants can be useful for proving that two given (G-equivariant generalized bornological) coarse spaces are not coarsely equivalent. In fact, up until now it is not easy to show that $\mathbb{N}$ is not coarsely equivalent to $\mathbb{R}$.

Example 1.4.32. For metric spaces the authors of [FLL11] define a coarse invariant denoted by σ which essentially counts "how many different ways there are to ∞". Indeed, we get $\sigma(\mathbb{N}) = 1$ and $\sigma(\mathbb{R}) = 2$, hence $\mathbb{N} \not\simeq_c \mathbb{R}$. ✠

Another way to show that, is using coarse ordinary homology which vanishes on $\mathbb{N}$ but not on $\mathbb{R}$. For details we refer to e.g. [BE20]. However, this is another type of invariant, namely a "coarsely invariant" functor on $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$.

Let $\mathbf{C}$ be any ∞-category and consider a functor $F\colon \boldsymbol{G}\widetilde{\mathbf{BornCoarse}} \to \mathbf{C}$.

Proposition 1.4.33. *The following are equivalent:*

1. *For all pairs of close morphisms $f, g\colon X \to Y$ in $\boldsymbol{G}\widetilde{\mathbf{BornCoarse}}$ the induced morphisms*
$$F(f), F(g)\colon F(X) \longrightarrow F(Y)$$
are equivalent in $\mathbf{C}$.

2. *For all spaces X in $\boldsymbol{G}\widetilde{\mathbf{BornCoarse}}$ the canonical projection induces an equivalence*
$$F(X \otimes \mathbb{I}) \xrightarrow{\simeq} F(X)$$
in $\mathbf{C}$.

3. *For any coarse equivalence $f\colon X \to Y$ in $\boldsymbol{G}\widetilde{\mathbf{BornCoarse}}$ the induced morphism $F(f)\colon F(X) \to F(Y)$ is an equivalence in $\mathbf{C}$.*

If F satisfies one of these equivalent conditions we say that F is coarsely invariant *(c.f. Definition 2.1.17).*

Proof. To see that (1) implies (3) we consider a coarse equivalence $f\colon X \to Y$. By definition there exists a morphism $g\colon Y \to X$ such that fg and gf are close to the respective identities. Applying the functor F gives by assumption

$$\mathrm{id} \simeq F(\mathrm{id}) \simeq F(fg) \simeq F(f)F(g)$$

and similarly $F(g)F(f) \simeq \mathrm{id}$, hence $F(f)$ is an equivalence in $\mathbf{C}$.
Since the projection $X \otimes \mathbb{I} \to X$ is a coarse equivalence by Example 1.4.13 clearly (3) implies (2).
To see that (2) implies (1) we consider a pair of close morphisms $f, g\colon X \to Y$ and we denote by $p\colon X \otimes \mathbb{I} \to X$ the projection and by $\iota_0, \iota_1\colon X \to X \otimes \mathbb{I}$ the respective inclusions. Since clearly $p\iota_0 = p\iota_1 = \mathrm{id}_X$ we get

$$F(p)F(\iota_0) \simeq F(\mathrm{id}_X) \simeq \mathrm{id}_{F(X)} \simeq F(\mathrm{id}_X) \simeq F(p)F(\iota_1)$$

and since $F(p)$ is invertible we get $F(\iota_0) \simeq F(\iota_1)$. Since f is close to g by assumption, there exists a morphism $h\colon X \otimes \mathbb{I} \to Y$ such that $h\iota_0 = f$ and $h\iota_1 = g$ by

Lemma 1.4.5. Applying F we get

$$F(f) \simeq F(h\iota_0) \simeq F(h)F(\iota_0) \simeq F(h)F(\iota_1) \simeq F(h\iota_1) \simeq F(g). \qquad \square$$

Lemma 1.4.34. *The functor* $\pi_0^{\text{coarse}} \colon \boldsymbol{G}\mathbf{B\widetilde{orn}Coarse} \to \mathbf{Set}$ *(c.f. Definition 1.1.12) is coarsely invariant. In particular, being coarsely connected is a coarse invariant.*

Proof. That π_0^{coarse} is a functor is immediate from controlledness of morphisms in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$. Now for all spaces X in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ the projection induces the following map of sets

$$\pi_0^{\text{coarse}}(X \otimes \mathbb{I}) \longrightarrow \pi_0^{\text{coarse}}(X), \qquad [(x,i)] \longmapsto [x].$$

Surjectivity of this map is clear and injectivity follows because the coarse structure on $\mathbb{I}$ is maximal, i.e. if x and y are in the same coarse component of X, then so are (x,i) and (y,j) in $X \otimes \mathbb{I}$. $\square$

All of this section suggested that we want to treat coarse equivalences as equivalences, i.e. invertible. There are two ways of doing this: The classical functor to the 1-categorical homotopy category (defined below) which inverts coarse equivalences by construction. The other way is to invert the coarse equivalences by performing a Dwyer-Kan localization (c.f. [Hin16]) at the collection of coarse equivalences yielding an ∞-category $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}[\{\text{coarse equivalences}\}^{-1}]$. In general both constructions give different results, however in this case of coarse equivalences both processes produce the same category. We want to emphazise again that the same results hold for the non-generalized situation of G-equivariant bornological coarse spaces $\boldsymbol{G}\mathbf{BornCoarse}$.

Definition 1.4.35. The category $\mathbf{ho}(\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse})$ is defined as follows: The objects are all spaces in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ and morphisms are closeness classes of morphisms in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$.

This is a well-defined category by Corollary 1.4.8. It is also clear from construction that coarse equivalences become isomorphisms in $\mathbf{ho}(\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse})$.

Let $\mathcal{W}$ be the collection of coarse equivalences in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$. The Dwyer-Kan localization

$$\ell \colon \boldsymbol{G}\mathbf{B\widetilde{orn}Coarse} \longrightarrow \boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}[\mathcal{W}^{-1}]$$

of $G\mathbf{B\widetilde{orn}Coarse}$ at $\mathcal{W}$ is the universal ∞-category inverting $\mathcal{W}$. More precisely, for any ∞-category $\mathbf{C}$ precomposing with ℓ induces an equivalence of ∞-categories between the functor category $\mathrm{Fun}(G\mathbf{B\widetilde{orn}Coarse}[\mathcal{W}^{-1}], \mathbf{C})$ and the full subcategory of $\mathrm{Fun}(G\mathbf{B\widetilde{orn}Coarse}, \mathbf{C})$ spanned by those functors F such that $F(f)$ is an equivalence in $\mathbf{C}$ for all f in $\mathcal{W}$.

Theorem 1.4.36. *We have an equivalence of ∞-categories between the Dwyer-Kan localization $G\mathbf{B\widetilde{orn}Coarse}[\mathcal{W}^{-1}]$ and (the nerve of) $\mathbf{ho}(G\mathbf{B\widetilde{orn}Coarse})$.*

The essential reason for this is the following: Usually (e.g. in the case of topological spaces and homotopies) a homotopy between two maps is a *choice*, and different choices of homotopies give rize to a homotopy between homotopies, which again is a choice, and so on and so forth. These "higher coherences" are captures by the modern language of ∞-categories and thus it is only natural that inverting homotopy equivalences yields an honest ∞-category. However, in the case of coarse equivalences, being close does *not* involve a choice. Two morphisms $f, g\colon X \to Y$ are close if the uniquely determined induced map $h\colon X \otimes \mathbb{I} \to Y$ is a morphism. This is a *property*, not a *choice*. This fact enters the proof in the statement which was outsourced to Lemma 1.4.37 below.

With that in mind, we proceed by proving Theorem 1.4.36. It essentially follows from [Lur17, Example 1.3.4.8] and we thank Thomas Nikolaus for pointing out this example and Sebastian Wolf for a helpful discussion.

Proof. We define a 2-category $\mathscr{C}$ as follows: The underlying 1-category is our category $G\mathbf{B\widetilde{orn}Coarse}$ and for two 1-morphisms f, g in $\mathrm{Hom}_{G\mathbf{B\widetilde{orn}Coarse}}(X, Y)$ we define

$$\mathrm{Hom}^2_{\mathscr{C}}(f,g) := \begin{cases} * & , \text{ if } f \text{ is close to } g \\ \emptyset & , \text{ else.} \end{cases}$$

This is a well-defined 2-category by Lemma 1.4.3. Since every 2-morphism is invertible (as being close is symmetric), we can view $\mathscr{C}$ as groupoid-enriched. Thus viewing $\mathscr{C}$ as simplicial category, it is Kan-enriched ([GJ09, Lemma 3.5]), hence a fibrant simplicial category (c.f. [Lur09, Remark 1.1.4.3]). The isomorphism of categories proven in Lemma 1.4.37 gives an *isomorphism* (not just an equivalence) of simplicial sets

$$\mathrm{Map}_{\mathscr{C}}(X \otimes \mathbb{I}, Y) \xrightarrow{\cong} \mathrm{Map}_{\mathbf{Set}_\Delta}(\Delta^1, \mathrm{Map}_{\mathscr{C}}(X, Y))$$

for all spaces X, Y in $\mathscr{C}$. Hence, by [Lur17, Ex. 1.3.4.8] we get an equivalence of ∞-categories

$$\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}[\mathcal{W}^{-1}] \simeq \mathrm{N}(\mathscr{C}).$$

Now observe that clearly $\mathrm{N}(\mathscr{C})$ is equivalent to the nerve of $\mathbf{ho}(\boldsymbol{G}\mathbf{\widetilde{Born}Coarse})$. □

We denote by $\mathbb{\Delta}$ the category with two objects $\{0, 1\}$ and the only non-identity morphism $0 \to 1$.

Lemma 1.4.37. *In the notation of the proof above: For all spaces X in $\mathscr{C}$ we have a functor*

$$\varepsilon_X \colon \mathbb{\Delta} \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X, X \otimes \mathbb{I})$$

such that for all Y in $\mathscr{C}$ postcomposition induces an isomorphism[10]

$$\mathrm{Hom}_{\mathscr{C}}(X \otimes \mathbb{I}, Y) \xrightarrow{\cong} \mathrm{Fun}(\mathbb{\Delta}, \mathrm{Hom}_{\mathscr{C}}(X, Y))$$

of categories.

Proof. We define for all spaces X in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ a functor

$$\varepsilon_X \colon \mathbb{\Delta} \to \mathrm{Hom}_{\mathscr{C}}(X, X \otimes \mathbb{I})$$

as follows: On objects we have $\varepsilon_X(0) := \iota_0$ and $\varepsilon_X(1) := \iota_1$ where $\iota_j \colon X \to X \otimes \mathbb{I}$ denotes the canonical inclusion morphism. By Corollary 1.4.6 we know ι_0 and ι_1 are close, hence $\mathrm{Hom}^2_{\mathscr{C}}(\varepsilon_X(0), \varepsilon_X(1)) = *$ which ensures that ε_X is a well-defined functor.

Furthermore for any additional space Y in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ we obtain a functor

$$\mathcal{F} \colon \mathrm{Hom}_{\mathscr{C}}(X \otimes \mathbb{I}, Y) \longrightarrow \mathrm{Fun}(\mathbb{\Delta}, \mathrm{Hom}_{\mathscr{C}}(X, Y))$$

defined on objects by

$$(h \colon X \otimes \mathbb{I} \to Y) \longmapsto h_* \circ \varepsilon_X.$$

Here we use that h_* defines a functor $\mathrm{Hom}_{\mathscr{C}}(X, X \otimes \mathbb{I}) \to \mathrm{Hom}_{\mathscr{C}}(X, Y)$ which is well-defined on the 2-morphisms by Corollary 1.4.8.

We have not yet defined $\mathcal{F}$ on the 2-morphisms in $\mathrm{Hom}_{\mathscr{C}}(X \otimes \mathbb{I}, Y)$. To do this we first unravel definitions: For two functors $F, G \in \mathrm{Fun}(\mathbb{\Delta}, \mathrm{Hom}_{\mathscr{C}}(X, Y))$ a natural

[10] not(!) just an equivalence!

transformation $F \to G$ is precisely the information that $F(0)$ is close to $G(0)$ and $F(1)$ is close to $G(1)$. Hence, for a 2-morphism $(*\colon h \to \tilde{h})$ in $\mathcal{C}$ we can define $\mathcal{F}(*)$ which is equivalent to the information that $h\iota_0$ is close to $\tilde{h}\iota_0$ and also $h\iota_1$ is close to $\tilde{h}\iota_1$, which both follow since h is close to $\tilde{h}$ (and again Corollary 1.4.8).

Now, by writing out every definition it is obviously true that the functor $\mathcal{F}$ is bijective on objects. The more crucial observation is that $\mathcal{F}$ is also bijective on morphisms: As we said above, a morphism $F \to G$ in $\mathrm{Fun}(\Delta, \mathrm{Hom}_{\mathscr{C}}(X, Y))$ is precisely the information that $F(0)$ is close to $G(0)$ as well as $F(1)$ is close to $G(1)$. Therefore either there exists no morphism $F \to G$ or there is a *unique* one. The same is of course true in $\mathrm{Hom}_{\mathscr{C}}(X \otimes \mathbb{I}, Y)$, since by definition the 2-morphism-sets are either empty or a singleton. The only thing left to show is that if there exists a transformation $\eta\colon F \to G$ in $\mathrm{Fun}(\Delta, \mathrm{Hom}_{\mathscr{C}}(X, Y))$ then the corresponding 2-Hom-set in $\mathrm{Hom}_{\mathscr{C}}(X \otimes \mathbb{I}, Y)$ is non-empty. Unraveling all the notions this boils down to the following statement: Consider two morphisms $h, \tilde{h}\colon X \otimes \mathbb{I} \to Y$ in $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$ and assume that $h\iota_0$ is close to $\tilde{h}\iota_0$ as well as $h\iota_1$ is close to $\tilde{h}\iota_1$, then h is close to $\tilde{h}$. This follows directly from Lemma 1.4.9. $\square$

Chapter 2
Equivariant coarse homology theory

In this chapter we introduce and study equivariant generalized coarse homology theories in the spirit of [BE20]. The first appearances of coarse homology is in [Roe93] (although in form of *co*homology). Important examples of (classical) coarse homology theories are coarse ordinary homology, coarse K-homology and coarsifications of locally finite homology theories. They are also due to Roe. Upon this notion of coarse homology, plenty of work has been build by several matematicians in various directions. From studying coarse Baum-Connes conjecture ([HR95; Wri05]), Farrell-Jones conjecture ([BLR08; UW19]) and applications towards the Novikov conjecture (e.g. [FWY20; GTY12; RTY14]) over (coarse) assembly maps (e.g. [EM06; Wul16; BE20a]) to Roe algebras ([BF21; BCL20]).

In chapter 1 of this thesis we learned about bornological coarse spaces as a generalization of coarse spaces with more flexibility in the notion of boundedness. The authors Bunke and Engel which invented these spaces in that language proceeded in [BE20] to lift the axioms of coarse homology theory to their category **BornCoarse**. Also the three important examples listed above can naturally be extended to the category of bornological coarse spaces. Soon after this, they introduced equivariant bornological coarse spaces in [BEKW] and lifted the notion of coarse homology to this new setting resulting in the notion of equivariant coarse homology theory. Building on this new language they also realized and elaborated that the already existing proofs for cases of Farrell-Jones, coarse Baum-Connes and Novikov conjectures are not specific to K-theory but extend to natural transformations of arbitrary coarse homology theories if axiomatised appropriately.

This chapter is organized as follows: In the first section we define all necessary notions to formulate the axioms for equivariant generalized coarse homology theory. We also give some nice examples and we elaborate some facts about these notions. We proceed in section 2.2 by constructing the "unstable version" of the universal equivariant generalized coarse homology, namely the category of equivariant generalized coarse motivic spaces $\boldsymbol{G}\mathbf{\widetilde{Spc}}\mathcal{X}$ together with the coarse homology theory $\widetilde{\mathrm{Y}\mathrm{o}}_G \colon \boldsymbol{G}\mathbf{\widetilde{Born}Coarse} \to \boldsymbol{G}\mathbf{\widetilde{Spc}}\mathcal{X}$. It is obtained by embedding $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ into its presheaf-category followed by a localization that enforces the "dual counterparts" of the axioms for coarse homology theory to become equivalences. What's left to obtain the universal equivariant generalized coarse homology theory, is stability of the target-category. This is obtained in section 2.3, where we postcompose with stabilization functor which inverts formally the suspension functor. We then briefly summerize properties of this universal coarse homology theory $\widetilde{\mathrm{Y}\mathrm{o}}^s_G \colon \boldsymbol{G}\mathbf{\widetilde{Born}Coarse} \to \boldsymbol{G}\mathbf{\widetilde{Sp}}\mathcal{X}$ and elaborate a nice excision property. Finally, in section 2.4 we will prove that the categories $\boldsymbol{G}\mathbf{\widetilde{Sp}}\mathcal{X}$ and $\boldsymbol{G}\mathbf{Sp}\mathcal{X}$ are equivalent, hence we have an equivalence between the ∞-category of equivariant coarse homology theories (developed in [BEKW] and basis for all the following works) and the ∞-category of equivariant generalized coarse homology theories developed in this chapter.

During this whole chapter we fix a group G and a G-equivariant generalized bornological coarse space X.

2.1 Equivariant coarse homology theories on generalized bornological coarse spaces

We use this section to develop the notion of equivariant (generalized) coarse homology theory. Such a theory is a functor on the category of G-equivariant generalized bornological coarse spaces with values in a stable ∞-category satisfying several axioms. This notion is a direct generalization of coarse homology theories in [BE20; BEKW]. A reader not familiar with the work of these papers can restrict the definitions given in this section to $\boldsymbol{G}\mathbf{BornCoarse}$ and obtain the non-generalized theory. The content of this section is vastly inspired by [BEKW], but also examples from different sources found its way into this section.

We will not give an explicit example for a equivariant coarse homology theory, although there exist plenty of interesting ones. The interested reader is refered to [BE20; BEKW; Cap19; BE20b; BC20] among others.

Definition 2.1.1. A *G-equivariant big family* on X is a filtered family $\mathcal{Y} := (Y_i)_{i \in I}$ of G-invariant subsets of X such that for all i in I and all entourages U in $\mathcal{C}_X$ there exists an j in I with $U[Y_i] \subseteq Y_j$.

Given a functor $F\colon G\widetilde{\mathbf{BornCoarse}} \to \mathbf{C}$ into some co-complete ($\infty-$)category $\mathbf{C}$ we often want to evaluate F on a big family $\mathcal{Y}$ on X. We define this by

$$F(\mathcal{Y}) := \operatorname*{colim}_{i\in I} F(Y_i).$$

Example 2.1.2. If G acts trivially on X, then the bornology $\mathcal{B}_X$ is a big family on X by compatibility of $\mathcal{B}_X$ and $\mathcal{C}_X$.
For non-trivial G-action on X we can consider the G-completion (c.f. Definition 1.2.11) $\hat{X}$ of X. The family of G-invariant bounded subsets is a G-equivariant big family on $\hat{X}$. ✠

Example 2.1.3. Let A be a G-invariant subset of X and by $\mathcal{C}_X^{G,\Delta}$ we denote the subset of G-invariant entourages U in $\mathcal{C}_X^G$ with $\Delta_X \subseteq U$. Then the family of thickenings $\{U[A]\}_{U\in\mathcal{C}_X^{G,\Delta}}$ is a G-equivariant big family on X.
If G acts trivially, each member of this big family is coarsely equivalent to A (Example 1.4.20). Otherwise this is not always the case (c.f. Example 1.4.25). ✠

Lemma 2.1.4. *Consider a morphism $f\colon Z \to X$ in $G\widetilde{\mathbf{BornCoarse}}$ and let $\mathcal{Y}$ be a G-equivariant big family on X. Then memberwise taking the pre-image defines a G-equivariant big family $f^{-1}\mathcal{Y}$ on Z.*
In particular, if Z is a G-invariant subset of X and f is the inclusion, memberwise taking intersection gives a big family $Z\cap\mathcal{Y}$ on Z (considered as space with the induced subspace structure).

Proof. By G-equivariance of f the members of $f^{-1}\mathcal{Y}$ are G-invariant. Consider a member $f^{-1}(Y_i)$ in $f^{-1}\mathcal{Y}$ and an entourage U in $\mathcal{C}_Z$. Then $V := (f\times f)(U)$ is an entourage in $\mathcal{C}_X$ hence there exists Y_j in $\mathcal{Y}$ with $V[Y_i] \subseteq Y_j$. Now

$$U\left[f^{-1}(Y_i)\right] \subseteq f^{-1}(Y_j)$$

shows that $f^{-1}\mathcal{Y}$ is big and we are done. □

Definition 2.1.5. A *G-equivariant complementary pair* on X is a pair $(Z,\mathcal{Y})$ consisting of a G-equivariant big family $\mathcal{Y}$ and a G-invariant subset Z of X such that $\mathcal{Y}$ and Z *cover* X, i.e. there exists k in I with $Y_k \cup Z = X$.

Example 2.1.6. We consider the space $X = \mathbb{R}$ in **BornCoarse** equiped with the structure induced from the standard metric. Furthermore we consider the subspace $Z := [0,\infty)$ and the family $\mathcal{Y} := \left\{Y_n := (-\infty, n] \;\middle|\; n \in \mathbb{N}\right\}$. Remember that $\mathcal{C}_X$ was generated by entourages $U_r = \Big\{(x,y) \in X \times X \;\Big|\; \mathrm{d}(x,y) \leq r\Big\}$ for $r > 0$. Clearly we have $U_r[Y_n] \subseteq Y_{n+r}$ hence the family $\mathcal{Y}$ is big. We obtain a complementary pair $(Z, \mathcal{Y})$ on X. ✠

Example 2.1.7 ([Bun21, Ex 4.23]). Let $\overline{X}$ be a Hausdorff space, A be a subset of $\overline{X}$ and $X := \overline{X} \setminus A$ equiped with the continuously controlled bornological coarse structure (c.f. Example 1.1.34). Let Z be an open neighborhood of A in $\overline{X}$ and let $\mathcal{Y} := \mathcal{B}_X$. Then $\mathcal{Y}$ is a big family (Example 2.1.2). Furthermore $(Z, \mathcal{Y})$ is a complementary pair because $X \setminus Z$ is a member in $\mathcal{Y}$ (meaning $\overline{X \setminus Z} \cap A = \emptyset$). ✠

We proceed to the notion of flasqueness. In the context of coarse spaces it was introduced by Higson, Pedersen and Roe in [HPR97, sec 10]. We will require that (equivariant) coarse homology theories vanish of "flasque" spaces. We first give the definition of classical flasqueness due to [HPR97] in this new setting of G-equivariant (generalized) bornological coarse spaces. Then we generalize the notion slightly to define "generalized flasque spaces" which has a more categorical description. Then we show however that vanishing on flasque spaces is equivalent to vanishing on generalized flasque spaces.

Definition 2.1.8. The space X in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ is called *flasque* if it admits an endomorphism $f \colon X \to X$ with the following properties:

1. The morphisms f and id_X are close to each other,
2. for any entourage U in $\mathcal{C}_X$ the union $\bigcup_{k\in\mathbb{N}} (f^k \times f^k)(U)$ is an entourage in $\mathcal{C}_X$,
3. for any bounded subset B in $\mathcal{B}_X$ there exists a non-negative integer k such that $f^k(X) \cap B = \emptyset$, or equivalently $(f^k)^{-1}(B) = \emptyset$.

In this situation we say that flasqueness is *implemented* by f.

Example 2.1.9. Let X in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ consist of unbounded points only, i.e. $X = X_h$. Then the identity morphism id_X implements flasqueness of X. Indeed, conditions (1) and (2) are trivially fulfilled and since the only bounded subset of X is the empty set, also (3) is clear. ✠

Example 2.1.10. Let $X := \mathbb{R}^+$ be the set of positive reals. Let G act trivially on X and equip this set with the structure of a bornological coarse space induced by the standard metric. Then X is flasque and flasqueness is implemented by the shift

$$f \colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+, \qquad x \longmapsto x + 1.$$

In fact, closeness of f and id_X is true because $(f \times \mathrm{id}_X)(\Delta_X) = U_1$. Furthermore, for any k in $\mathbb{N}$ and any generating entourage U_r in $\mathcal{C}_X$ we have $(f_k \times f_k)(U_r) = U_r$, hence also condition (2) is fulfilled. Finally, clearly for any bounded subset B of X there exists some $k > \sup B$ which gives $(f^k)(X) \cap B = (k+1, \infty) \cap B = \emptyset$. ✠

Since $\mathbb{N}$ is coarsely equivalent to $\mathbb{R}^+$ (Example 1.4.21), also $\mathbb{N}$ is flasque. Moreover, $\mathbb{N}$ is in a certain sense the "universal example" of a flasque space: For any space X in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ we can consider the tensor product $X \otimes \mathbb{N}$ where G acts trivially on the $\mathbb{N}$-part. This gives a flasque space:

Lemma 2.1.11. *For all spaces X in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ the space $X \otimes \mathbb{N}$ is flasque.*

Proof. Let $f\colon \mathbb{N} \to \mathbb{N}$ be the shift-endomorphism $n \mapsto n+1$ like in Example 2.1.10. Then flasqueness of $X \otimes \mathbb{N}$ is implemented by

$$\mathrm{id}_X \otimes f\colon X \otimes \mathbb{N} \longrightarrow X \otimes \mathbb{N}.$$

Since f is close to $\mathrm{id}_\mathbb{N}$ we get that $\mathrm{id}_X \otimes f$ is close to $\mathrm{id}_{X \otimes \mathbb{N}}$. Furthermore, any generating entourage of $X \otimes \mathbb{N}$ is of the form $U_1 \bar{\times} U_2$ with U_1 in $\mathcal{C}_X$ and U_2 in $\mathcal{C}_\mathbb{N}$. Since f implements flasqueness of $\mathbb{N}$, there exists an entourage V_2 in $\mathcal{C}_\mathbb{N}$ such that $(f^k \times f^k)(U_2) \subseteq V_2$ for all k in $\mathbb{N}$. Thus clearly

$$\Big((\mathrm{id}_X \otimes f)^k \times (\mathrm{id}_X \otimes f)^k\Big)(U_1 \bar{\times} U_2) \subseteq U_1 \bar{\times} V_2.$$

Finally, any bounded subset B in $\mathcal{B}_{X \otimes \mathbb{N}}$ is contained in $B_1 \times B_2$ for B_1 in $\mathcal{B}_X$ and B_2 in $\mathcal{B}_\mathbb{N}$. By properties of f there exists k in $\mathbb{N}$ such that $f^k(X) \cap B_2 = \emptyset$. Thus $(\mathrm{id}_X \otimes f)^k(X \otimes \mathbb{N}) \cap (B_1 \times B_2) = \emptyset$. □

As mentioned above this lemma, the spaces of the form $X \otimes \mathbb{N}$ are somehow the universal examples. More precisely: Every flasque space is a retract of $X \otimes \mathbb{N}$:

Lemma 2.1.12. *Let X be a flasque space in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$. Then there exists a retract morphism $r\colon X \otimes \mathbb{N} \to X$ such that $r\iota_0 = \mathrm{id}_X$, where ι_0 denotes the canonical inclusion morphism $x \mapsto (x, 0)$.*

Proof. Follows directly from Lemma 2.1.14 and Proposition 2.1.15. □

The converse is not true, because the "classical" notion of flasqueness is a bit too rigid. Therefore we make it slightly more flexible:

Definition 2.1.13. A space X in $G\widetilde{\mathbf{Born}}\mathbf{Coarse}$ is called *generalized flasque* if there exists a sequence $(f_i)_{i\in\mathbb{N}}$ of endomorphisms $f_i\colon X \to X$ such that:

1. $f_0 = \mathrm{id}_X$,
2. for all i in $\mathbb{N}$ the morpisms f_i and f_{i+1} are close *uniformly close*, i.e. there exists an entourage W in $\mathcal{C}_X$ such that $(f_i \times f_{i+1})(\Delta_X) \subseteq W$ for all i in $\mathbb{N}$.
3. for any entourage U in $\mathcal{C}_X$ the union $\bigcup_{i\in\mathbb{N}}(f_i \times f_i)(U)$ is an entourage in $\mathcal{C}_X$,
4. for all bounded subsets B in $\mathcal{B}_X$ there exists a positive integer N such that for all integers $i \geq N$ we have $f_i^{-1}(B) = \emptyset$.

Lemma 2.1.14. *Every flasque space X in $G\widetilde{\mathbf{Born}}\mathbf{Coarse}$ is generalized flasque.*

Proof. Let $f\colon X \to X$ implement flasqueness of X. We choose $f_i := f^i$ for all i in $\mathbb{N}$. Then conditions (1) and (3) are clear. For condition (2) we observe that f is close to id_X because f implements flasqueness on X. Thus $W := (\mathrm{id}_X \times f)(\Delta_X)$ is an entourage in $\mathcal{C}_X$ and for all i in $\mathbb{N}$ we clearly have

$$(f_i \times f_{i+1})(\Delta_X) = (f^i \times f^{i+1})(\Delta_X) \subseteq (\mathrm{id} \times f)(\Delta_X) = W.$$

Finally, for any bounded subset B of X there exists k in $\mathbb{N}$ with $(f^k)(X) \cap B = \emptyset$. Hence for all $i \geq k$ we have $f_i(X) \cap B \subseteq f_k(X) \cap B = \emptyset$. $\square$

This new notion of generalized flasqueness is actually *equivalent* to being a retract of the cylinder $X \otimes \mathbb{N}$ as we indicated above:

Proposition 2.1.15. *A space X in $G\widetilde{\mathbf{Born}}\mathbf{Coarse}$ is generalized flasque if and only if there exists a morphism $r\colon X \otimes \mathbb{N} \to X$ such that $r\iota_0 = \mathrm{id}_X$.*

Proof. First, assume X is generalized flasque. Let $(f_i)_{i\in\mathbb{N}}$ be the sequence of morphisms implementing generalized flasqueness. We define a map

$$r\colon X \otimes \mathbb{N} \longrightarrow X, \qquad (x, i) \longmapsto f_i(x).$$

We claim that this is our desired retract. It is clear that $r\iota_0 = \mathrm{id}_X$ and that r is G-equivariant because the morphisms f_i are and G acts trivially on $\mathbb{N}$. It remains to verify that r is proper and controlled. For properness we consider a bounded subset B in $\mathcal{B}_X$. Then there exists N in $\mathbb{N}$ such that for all integers $k \geq N$ we have $f_k^{-1}(B) = \emptyset$. Therefore we see

$$r^{-1}(B) = \bigcup_{k \in \mathbb{N}} f_k^{-1}(B) \times \{k\} = \bigcup_{k=0}^{N} f_k^{-1}(B) \times \{k\}$$

By properness of the morphisms f_k the subsets $f_k^{-1}(B)$ are bounded in X, hence this *finite* union is bounded in $X \otimes \mathbb{N}$.
To see that r is controlled one easily checks that $\mathcal{C}_{X\otimes\mathbb{N}}$ is generated by entourages of the form $U \bar{\times} \Delta_{\mathbb{N}}$ for U in $\mathcal{C}_X$ and $\Delta_X \bar{\times} V$ where $V = \Big\{(n,m) \in \mathbb{N} \times \mathbb{N} \;\Big|\; |n-m| \leq R\Big\}$ for some integer $R \geq 0$. More precisely: By definition of the tensor product, the coarse structure is generated by products of the form $U \bar{\times} V$ and it is easy to see that

$$U \bar{\times} V \subseteq \Big(U \bar{\times} \Delta_{\mathbb{N}}\Big) \circ \Big(\Delta_X \bar{\times} V\Big).$$

Thus it suffices to check controlledness of r on this two types of generators. First, we easily see that

$$(r \times r)(U \bar{\times} \Delta_{\mathbb{N}}) = \bigcup_{n \in \mathbb{N}} (r \times r)\Big(U \bar{\times} \{(n,n)\}\Big) = \bigcup_{n \in \mathbb{N}} (f_n \times f_n)(U)$$

which is an entourage in $\mathcal{C}_X$ by condition (3) of generalized flasqueness.
For the other type of generator, we notice that

$$V = \bigcup_{i=0}^{R} \left(\left(\bigcup_{n \in \mathbb{N}} \{(n, n+i)\} \right) \cup \left(\bigcup_{n \in \mathbb{N}} \{(n+i, n)\} \right) \right).$$

Since $\mathcal{C}_X$ is closed under finite unions and inverses (and taking images behaves well with these two operations) we see that it is enough to check that

$$S_i := (r \times r) \left(\Delta_X \bar{\times} \left(\bigcup_{n \in \mathbb{N}} \{(n, n+i)\} \right) \right)$$

is an entourage in $\mathcal{C}_X$ for all $0 \leq i \leq R$.
Plugging in the definitions we see

$$S_i = \bigcup_{n \in \mathbb{N}} (r \times r)(\Delta_X \bar{\times} \{(n, n+i)\}) = \bigcup_{n \in \mathbb{N}} (f_n \times f_{n+i})(\Delta_X).$$

By condition (2) of generalized flasqueness there exists W in $\mathcal{C}_X$ such that for all k in $\mathbb{N}$ we have $(f_k \times f_{k+1})(\Delta_X) \subseteq W$. Inductively we derive $(f_n \times f_{n+i})(\Delta_X) \subseteq W^i$ for all n and i in $\mathbb{N}$. Therefore:

$$S_i = \bigcup_{n\in\mathbb{N}} (f_n \times f_{n+i})(\Delta_X) \subseteq \bigcup_{n\in\mathbb{N}} W^i = W^i$$

which is an entourage in $\mathcal{C}_X$ and we are through.

To see the converse, we assume we have a retract $r\colon X \otimes \mathbb{N} \to X$. Then we define the sequence of endomorphisms by $f_n := r \circ \iota_n$ where $\iota_n\colon X \to X \otimes \mathbb{N}$ is the canonical inclusion morphism $x \mapsto (x, n)$. By retract-property of r we immediately get $f_0 = \mathrm{id}_X$, i.e. condition (1). Now we let $V_1 := \Big\{(n,m) \in \mathbb{N} \times \mathbb{N} \;\Big|\; |n-m| \leq 1\Big\}$. Then $T := \Delta_X \bar{\times} V_1$ is an entourage on $X \otimes \mathbb{N}$. we get for all k in $\mathbb{N}$ that

$$(f_k \times f_{k+1})(\Delta_X) = (r \times r)\Big((\iota_k \times \iota_{k+1})(\Delta_X)\Big) \subseteq (r \times r)(T)$$

is an entourage in $\mathcal{C}_X$ since r is controlled. Since T was independent of k this gives condition (2). Next, for any entourage U in $\mathcal{C}_X$ we have

$$\bigcup_{n\in\mathbb{N}} (f_n \times f_n)(U) = (r \times r)\left(\bigcup_{n\in\mathbb{N}} (\iota_n \times \iota_n)(U)\right) = (r \times r)(U \bar{\times} \Delta_{\mathbb{N}})$$

where the latter subset is an entourage on X because r is controlled. This gives condition (3). Finally, to see condition (4) we consider a bounded subset B in $\mathcal{B}_X$. Since r is proper the pre-image $r^{-1}(B)$ is bounded in $X \otimes \mathbb{N}$. In particular, there exists R in $\mathbb{N}$ such that $r^{-1}(B) \subseteq X \times [0, R]$. We get for all integers $k > R$ that

$$f_k^{-1}(B) = \iota_k^{-1}(r^{-1}(B)) \subseteq \iota_k^{-1}(X \times [0, R]) = \emptyset.$$

This finishes the proof. $\square$

Thus, X being generalized flasque can be expressed by being a retract of the flasque space $X \otimes \mathbb{N}$ instead of giving a sequence of endomorphisms satisfying certain properties. In this description, it is almost obvious that "generalized flasqueness" is a coarsly invariant property, while it is not clear if the classical notion of flasqueness is preserved under coarse equivalences.

However, the condition "vanishing on flasques" is equivalent to "vanishing on generalized flasques":

Let $\mathbf{C}$ be a co-complete stable ∞-category and let

$$F\colon \boldsymbol{G}\mathbf{B\widetilde{orn}Coarse} \to \mathbf{C}$$

be a functor.

Proposition 2.1.16. *The following are equivalent:*

1. *for all flasque spaces X in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ we have $F(X) \simeq 0$,*
2. *for all generalized flasque spaces X we have $F(X) \simeq 0$.*

Proof. For (2) $\Rightarrow$ (1) we use that every flasque space is generalized flasque (c.f. Lemma 2.1.14). Hence it suffices to show (1) $\Rightarrow$ (2): For this we consider a generalized flasque space X. Then by Proposition 2.1.15 there exists a retract $r\colon X \otimes \mathbb{N} \to X$ with $r\iota_0 = \mathrm{id}_X$. Note that $X \otimes \mathbb{N}$ is flasque by Lemma 2.1.11, hence applying the functor F to $\mathrm{id}_X\colon X \to X \otimes \mathbb{N} \to X$ we get

$$\mathrm{id}_{F(X)}\colon F(X) \longrightarrow 0 \longrightarrow F(X)$$

hence $F(X) \simeq 0$. □

Recall that for any G-invariant entourage U of X we constructed a new space X_U by replacing $\mathcal{C}_X$ with $\mathcal{C}\langle\{U\}\rangle$ and we obtained a compatible system of inclusion morphisms $X_U \to X$ (for U in $\mathcal{C}_X^G$) which induced an isomorphism (see Example 1.3.30)

$$\operatorname*{colim}_{U \in \mathcal{C}_X^G} X_U \xrightarrow{\cong} X.$$

We proceed to the definition of a G-equivariant coarse homology theory on the category $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$:

Definition 2.1.17. The functor F is a $\mathbf{C}$-*valued G-equivariant generalized coarse homology theory on* $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ if it satisfies:

1. (*Excision*) We have $F(\emptyset) \simeq 0$ and for all spaces X in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ and all G-equivariant complementary pairs $(Z, \mathcal{Y})$ on X the following square

$$\begin{tikzcd} F(Z \cap \mathcal{Y}) \arrow[r] \arrow[d] & F(Z) \arrow[d] \\ F(\mathcal{Y}) \arrow[r] & F(X) \end{tikzcd}$$

is a pushout square in $\mathbf{C}$.

2. (*Coarse invariance*) If $f\colon X \to Y$ is a coarse equivalence in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ then the induced morphism $F(f)\colon F(X) \to F(Y)$ is an equivalence in $\mathbf{C}$.

3. (*Vanishing on flasques*) If X in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ is flasque then $F(X) \simeq 0$.

4. (*U-continuity*) For every X in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ the canonical morphism

$$\operatorname*{colim}_{U \in \mathcal{C}_X^G} F(X_U) \longrightarrow F(X)$$

is an equivalence in $\mathbf{C}$.

Remark 2.1.18. The "vanishing on flasque"-condition on F is equivalent to require that $F(X \otimes \mathbb{N}) \simeq 0$ for all spaces X in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ (use Proposition 2.1.16). ⧫

Remark 2.1.19. The functor F fulfills the coarse invariance condition if and only if for all spaces X in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ the canonical projection induces an equivalence $F(X \otimes \mathbb{I}) \to F(X)$ in $\mathbf{C}$ (seen in Proposition 1.4.33). ⧫

For a space X in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ and a G-equivariant big family $\mathcal{Y}$ on X we define

$$F(X, \mathcal{Y}) := \operatorname{cofib}\Big(F(\mathcal{Y}) \to F(X)\Big).$$

Lemma 2.1.20. *The functor $F\colon \boldsymbol{G}\mathbf{B\widetilde{orn}Coarse} \to \mathbf{C}$ satisfies excision if and only if for all spaces X in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ and all G-equivariant complementary pairs $(Z, \mathcal{Y})$ on X the induced morphism of cofibers*

$$F(Z, Z \cap \mathcal{Y}) \longrightarrow F(X, \mathcal{Y})$$

is an equivalence in $\mathbf{C}$.

Proof. This is an easy consequence of stability of $\mathbf{C}$ (see [Lur17, Lem 1.2.4.14]). □

We finish this section with a nice example of a contravariant functor on the category $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ which satisfies the excision property (but has unstable target). The example was adapted from [Bun21].

Example 2.1.21. Let R be a ring. For all spaces X in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ we define the R-valued locally finite functions on X by

$$C^{\mathrm{lf}}(X; R) := \Big\{ f\colon X \to R \,\Big|\, \operatorname{supp}(f) \in \mathcal{B}_X \Big\}.$$

This set is an R-module because for $f, g\colon X \to Y$ we have

$$\operatorname{supp}(f+g) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g)$$

and for r in R we have $\operatorname{supp}(rf) \subseteq \operatorname{supp}(f)$.
The assignment $X \mapsto C^{\mathrm{lf}}(X; R)$ extends to a functor

$$C^{\mathrm{lf}}\colon \boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}^{op} \longrightarrow \mathbf{Mod}_R$$

because for any morphism $\phi\colon X \to Y$ in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ the induced pullback

$$\phi^*\colon C^{\mathrm{lf}}(Y; R) \longrightarrow C^{\mathrm{lf}}(X; R)$$

is well-defined since $\operatorname{supp}(\phi^*(f)) = \operatorname{supp}(f \circ \phi) \subseteq \phi^{-1}(\operatorname{supp}(f))$ and ϕ is proper.
We claim that this functor satisfies excision: For this we consider an G-equivariant complementary pair $(Z, \mathcal{Y})$ on X and we show that the induced map of cofibers

$$C^{\mathrm{lf}}(X, \mathcal{Y}) \longrightarrow C^{\mathrm{lf}}(Z, Z \cap \mathcal{Y})$$

is an isomorphism in $\mathbf{Mod}_R$. The surjectivity of

$$C^{\mathrm{lf}}(X)/C^{\mathrm{lf}}(\mathcal{Y}) \longrightarrow C^{\mathrm{lf}}(Z)/C^{\mathrm{lf}}(Z \cap \mathcal{Y})$$

is clear since we can extend any function $f\colon Z \to X$ by zero to whole X without changing the support. For injectivity we consider a class $[f]$ in $C^{\mathrm{lf}}(X)/C^{\mathrm{lf}}(\mathcal{Y})$ such that the class of the restriction $\left[f\big|_Z\right]$ vanishes. Hence there exists Y in $\mathcal{Y}$ such that $\operatorname{supp}\left(f\big|_Z\right) \subseteq Y$. Moreover, since $(Z, \mathcal{Y})$ are covering, the complement $X \setminus Z$ is contained in a member Y' of $\mathcal{Y}$. Since $\mathcal{Y}$ is filtered, there exists a member $\tilde{Y}$ containing both Y and Y', and we conclude $\operatorname{supp}(f) \subseteq \tilde{Y}$, hence $[f] = 0$. ✠

2.2 Equivariant generalized coarse motivic spaces

In this section we define the category of G-equivariant generalized coarse motivic spaces. It is in some sense the unstable prestage of equivariant generalized coarse motivic spectra which is the target of the universal equivariant generalized coarse homology theory.
The category of G-equivariant generalized coarse motivic spaces will be obtained by considering the ∞-category of space-valued presheaves on $\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}$ and then localize to obtain the desired properties that lead to coarse homology theories. The whole approach is along the lines of [BE20] which in turn was inspired by similar constructions in $\mathbb{A}^1$-homotopy theory (see [Hoy17]) or differential cohomology theory (see [BNV16; BG16]).
Localizations require a bit of care with set-theoretical size-issues. We will not focus on these in this thesis. It can be treated by introducing three Grothendieck universes such that the objects in categories like $\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}$ are declared "very small", then categories like $\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}$ are called "small" and localizations like the category of equivariant generalized coarse motivic spectra $\boldsymbol{G}\mathbf{S\widetilde{pc}}\mathcal{X}$ are "large". A very careful elaboration on this can be found in [BE20, Chapter 3].

We denote by $\mathbf{Spc}$ the ∞-category of spaces (or anima, or homotopy types, ...). In the following we denote by

$$\mathrm{yo}\colon \boldsymbol{G}\mathbf{B\widetilde{or}nCoarse} \longrightarrow \mathbf{PSh}(\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}) := \mathrm{Fun}(\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}^{op}, \mathbf{Spc})$$

the Yoneda-embedding into the ∞-category of space-valued presheaves.

Remark 2.2.1. By [Lur09, Thm 5.1.5.6] (or [BE20, p. 23] for this precise setting) precomposition with yo induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\lim}(\mathbf{PSh}(\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}), \mathbf{Spc}) \xrightarrow{\simeq} \mathbf{PSh}(\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse})$$

where the sup-script "lim" stands for "limit-preserving". In particular, for two presheaves E, F on $\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}$ we obtain

$$E(F) \simeq \lim_{\mathrm{yo}(X)\to F} E(X). \qquad \blacklozenge$$

Example 2.2.2. Consider a space X in $\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}$ and a G-equivariant big family $\mathcal{Y}$ on X. We have already defined the evaluation of yo on the big family $\mathcal{Y}$ by

$\mathrm{yo}(\mathcal{Y}) := \mathrm{colim}_{Y\in\mathcal{Y}}\,\mathrm{yo}(Y)$ (c.f. below Definition 2.1.1). Hence for a presheaf E we interpret E as limit-preserving functor on the presheaf-category and obtain by a cofinality-argument

$$E(\mathrm{yo}(\mathcal{Y})) \simeq E\Big(\operatorname*{colim}_{Y\in\mathcal{Y}} \mathrm{yo}(Y)\Big) \simeq \lim_{Y\in\mathcal{Y}} E(\mathrm{yo}(Y)) \simeq \lim_{Y\in\mathcal{Y}} E(Y).$$

To shorten notation we often write $E(\mathcal{Y})$ instead of $E(\mathrm{yo}(\mathcal{Y}))$. ✠

Definition 2.2.3. Consider a space X in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ together with a G-equivariant complementary pair $(Z,\mathcal{Y})$ on X and a presheaf E in $\mathbf{PSh}(\boldsymbol{G}\mathbf{\widetilde{Born}Coarse})$. We say that E *satisfies descent w.r.t.* $(X,Z,\mathcal{Y})$ if the following diagram is cartesian in $\mathbf{Spc}$:

$$\begin{tikzcd} E(X) \arrow[r] \arrow[d] & E(Z) \arrow[d] \\ E(\mathcal{Y}) \arrow[r] & E(Z\cap\mathcal{Y}). \end{tikzcd}$$

Proposition 2.2.4. *There exists a Grothendieck topology τ_χ on $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ such that τ_χ-sheaves are exactly those presheaves E on $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ which satisfy $E(\emptyset)\simeq *$ and descent w.r.t. $(X,Z,\mathcal{Y})$ for all spaces X in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ and all G-equivariant complementary pairs $(Z,\mathcal{Y})$ on X.*

Proof. Consider a G-equivariant generalized bornological coarse space X together with a G-equivariant completementary pair $(Z,\mathcal{Y})$. We define the following set-valued sieves on X:

$$\begin{aligned}
\mathscr{S}_Z(W) &:= \Big\{ W \xrightarrow{f} Z \hookrightarrow X \;\Big|\; f\in \mathrm{Hom}_{\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}}(W,Z)\Big\}\\
\mathscr{S}_{\mathcal{Y}}(W) &:= \Big\{ W \xrightarrow{f} Y \hookrightarrow X \;\Big|\; Y\in\mathcal{Y},\ f\in \mathrm{Hom}_{\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}}(W,Y)\Big\}\\
\mathscr{S}_{Z\cap\mathcal{Y}}(W) &:= \mathscr{S}_Z(W)\cap\mathscr{S}_{\mathcal{Y}}(W)
\end{aligned}$$

and furthermore we define $\mathscr{S}_{(Z,\mathcal{Y})}$ as the pushout in $\mathbf{PSh}_{\mathbf{Set}}(\boldsymbol{G}\mathbf{\widetilde{Born}Coarse})$:

$$\begin{tikzcd} \mathscr{S}_{Z\cap\mathcal{Y}} \arrow[r] \arrow[d] & \mathscr{S}_Z \arrow[d] \\ \mathscr{S}_{\mathcal{Y}} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathscr{S}_{(Z,\mathcal{Y})} \end{tikzcd} \tag{†}$$

Since pushouts in functor categories are calculated objectwise, we get that $\mathscr{S}_{(Z,\mathcal{Y})}$ consists of all morphisms $W\to X$ that factor through Z or a member of $\mathcal{Y}$.

Postcomposing with the functor $\mathbf{Set} \longrightarrow \mathbf{Spc}$ we can view the sieves defined above as sieves with values in $\mathbf{Spc}$, i.e. as presheaves in $\mathbf{PSh}(\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse})$. Since the pushout in (†) is taken along monomorphisms, it is still a pushout diagram when considered as diagram of space-values presheaves.

Claim 1: A presheaf E in $\mathbf{PSh}(\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse})$ fulfills descent w.r.t. $(X, Z, \mathcal{Y})$ if and only if the induced morphism

$$E(X) \longrightarrow \lim_{\mathrm{yo}(W) \to \mathscr{S}_{(Z,\mathcal{Y})}} E(W)$$

is an equivalence.

Proof of Claim 1: By Remark 2.2.1 we can view E as limit-preserving functor

$$\mathbf{PSh}(\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse})^{op} \longrightarrow \mathbf{Spc}.$$

Since (†) is a pushout, it is a pullback in $\mathbf{PSh}(\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse})^{op}$, hence the following square is also a pullback:

$$\begin{tikzcd} E(\mathscr{S}_{(Z,\mathcal{Y})}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E(\mathscr{S}_{\mathcal{Y}}) \arrow[d] \\ E(\mathscr{S}_Z) \arrow[r] & E(\mathscr{S}_{Z\cap\mathcal{Y}}). \end{tikzcd}$$

Since the collection $\left\{\mathrm{yo}(Y) \to \mathscr{S}_{\mathcal{Y}} \;\middle|\; Y \in \mathcal{Y}\right\}$ is cofinal in $\{\mathrm{yo}(W) \to \mathscr{S}_{\mathcal{Y}}\}$ we get by [Lur09, Prop. 4.1.1.8] the second equivalence in:

$$E(\mathscr{S}_{\mathcal{Y}}) \simeq \lim_{\mathrm{yo}(W) \to \mathscr{S}_{\mathcal{Y}}} E(W) \simeq \lim_{\substack{\mathrm{yo}(Y) \to \mathscr{S}_{\mathcal{Y}} \\ Y \in \mathcal{Y}}} E(Y) \simeq \lim_{Y \in \mathcal{Y}} E(Y) \simeq E(\mathcal{Y}).$$

The same argument shows $E(\mathscr{S}_Z) \simeq E(Z)$ and $E(\mathscr{S}_{Z\cap\mathcal{Y}}) \simeq E(Z \cap \mathcal{Y})$. Therefore we can identify the upper pullback diagram with the following one:

$$\begin{tikzcd} E(\mathscr{S}_{(Z,\mathcal{Y})}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E(\mathcal{Y}) \arrow[d] \\ E(Z) \arrow[r] & E(Z \cap \mathcal{Y}). \end{tikzcd}$$

Now, the inclusions induce morphisms $E(X) \to E(\mathcal{Y})$ and $E(X) \to E(Z)$ which agree after postcomposing with the morphism into $E(Z \cap \mathcal{Y})$, hence the universal property of pullbacks gives a unique morphism η as depicted in the diagram below:

$$\begin{tikzcd}
E(X) \arrow[dr, dotted, "\eta"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& E(\mathscr{S}_{(Z,\mathcal{Y})}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E(\mathcal{Y}) \arrow[d] \\
& E(Z) \arrow[r] & E(Z \cap \mathcal{Y}).
\end{tikzcd}$$

Obviously η is an equivalence if and only if the outer diagram is a pullback, i.e. iff E satisfies descent w.r.t. $(X, Z, \mathcal{Y})$. $\square_{\text{Claim 1}}$.

Now we define $\mathcal{M}$ as the collection of all presheaves in $\mathbf{PSh}(\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse})$ which satisfy descent w.r.t. $(X, Z, \mathcal{Y})$ for all spaces X in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ and all G-equivariant complementary pairs $(Z, \mathcal{Y})$ on X. Furthermore, we let $\tau_\square$ be the finest Grothendieck topology on $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ such that all presheaves in $\mathcal{M}$ are $\tau_\square$-sheaves (this exists by [Sta20, 00Z9]) and moreover, by $\tau_{\mathscr{S}}$ we denote the Grothendieck topology generated by all sieves $\mathscr{S}_{(Z,\mathcal{Y})}$ for all G-equivariant complementary pairs on all spaces in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$.
By *Claim 1* we have $\tau_{\mathscr{S}} \subseteq \tau_\square$ and therefore $\mathbf{Sh}^{\tau_\square} \subseteq \mathbf{Sh}^{\tau_{\mathscr{S}}}$.
Again, by *Claim 1* we have $\mathbf{Sh}^{\tau_{\mathscr{S}}} \subseteq \mathcal{M}$ and by definition we have $\mathcal{M} \subseteq \mathbf{Sh}^{\tau_\square}$. All together we get

$$\mathbf{Sh}^{\tau_\square} \subseteq \mathbf{Sh}^{\tau_{\mathscr{S}}} \subseteq \mathcal{M} \subseteq \mathbf{Sh}^{\tau_\square}.$$

Thus the Grothendieck topology $\tau_\square$ fulfills the claim. $\square$

To shorten notation we will refer to τ_χ-sheaves simply as sheaves.

Lemma 2.2.5. *The Grothendieck topology τ_χ defined in Proposition 2.2.4 is subcanonical.*

Proof. [BE20, Lem 3.12]. See also Lemma 4.2.4 for almost exactly that proof. $\square$

Remark 2.2.6. By Lemma 2.2.5 the presheaf $\mathrm{yo}(\emptyset)$ is a sheaf. Actually, it is the initial sheaf $\emptyset_{\mathbf{Sh}}$ in $\mathbf{Sh}(\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse})$ because for every sheaf E we have $E(\emptyset) \simeq *$ by sheaf-condition, hence the Yoneda-lemma lets us deduce

$$\mathrm{Hom}_{\mathbf{Sh}(\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse})}(\mathrm{yo}(\emptyset), E) \simeq E(\emptyset) \simeq *$$

hence $\mathrm{yo}(\emptyset)$ is inital in $\mathbf{Sh}(\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse})$. ♦

In the following we denote by E a sheaf in $\mathbf{Sh}(\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse})$.

Definition 2.2.7. We say that E is *coarsely invariant* if for all generalized bornological coarse spaces X in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ the canonical projection $X \otimes \mathbb{I} \to X$ is sent to an equivalence $E(X) \to E(X \otimes \mathbb{I})$ in $\mathbf{Spc}$.

Remark 2.2.8. Like in the covariant case (Proposition 1.4.33) we can show that E is coarsely invariant if and only if it sends coarse equivalences in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ to equivalences in $\mathbf{Spc}$. ♦

Recall that for a category $\mathbf{C}$ and a collection of morphisms $\mathcal{S}$ in $\mathbf{C}$ we call an object D in $\mathbf{C}$ *local w.r.t.* $\mathcal{S}$ (or $\mathcal{S}$*-local*) if for any morphism $f\colon C \to C'$ in $\mathcal{S}$ the induced map

$$\mathrm{Hom}_{\mathbf{C}}(C', D) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(C, D)$$

is an equivalence ([Lur09, Def 5.5.4.1]).

Lemma 2.2.9. *A sheaf E is coarsely invariant if and only if it is local w.r.t. the collection of morphisms* $\mathcal{M} := \left\{\mathrm{yo}(X \otimes \mathbb{I}) \to \mathrm{yo}(X) \;\middle|\; X \in \boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}\right\}$.

Proof. By Lemma 2.2.5 the morphisms in $\mathcal{M}$ are morphisms of sheaves. Now by definition a sheaf is local w.r.t. $\mathcal{M}$ if and only if for all X in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ the induced morphism

$$\mathrm{Hom}_{\mathbf{Sh}(\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse})}(\mathrm{yo}(X), E) \longrightarrow \mathrm{Hom}_{\mathbf{Sh}(\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse})}(\mathrm{yo}(X \otimes \mathbb{I}), E)$$

is an equivalence in $\mathbf{Spc}$. Using the Yoneda-lemma this morphism can be identified with

$$E(X) \longrightarrow E(X \otimes \mathbb{I})$$

which shows the lemma. □

Definition 2.2.10. We say that E is *vanishing on flasques* or simply E *vanishes on flasques* if for all flasque spaces X in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ the object $E(X)$ is final in $\mathbf{Spc}$, i.e. $E(X) \simeq *$.

Remark 2.2.11. We could also require that E vanishes on generalized flasque spaces. But vanishing on flasques and vanishing on generalized flasques turns out to be equivalent: In fact in Proposition 2.1.16 we have shown this for functors with valued in a co-complete stable ∞-category, but the same proof works here (just replace the 0-object by the terminal object $*$). ♦

Lemma 2.2.12. *A sheaf E is vanishing on flasques if and only if it is local w.r.t. the collection* $\mathcal{M} := \left\{\emptyset_{\mathbf{Sh}} \to \mathrm{yo}(X) \;\middle|\; X \in \boldsymbol{G}\mathbf{\widetilde{Born}Coarse} \text{ flasque}\right\}$.

Proof. Again by Lemma 2.2.5 the collection $\mathcal{M}$ consists of morphisms of sheaves. Now for a sheaf E being local w.r.t. $\mathcal{M}$ means that for all flasque spaces X in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ the induced morphism

$$\mathrm{Hom}_{\mathbf{Sh}(\boldsymbol{G}\mathbf{\widetilde{Born}Coarse})}(\mathrm{yo}(X), E) \longrightarrow \mathrm{Hom}_{\mathbf{Sh}(\boldsymbol{G}\mathbf{\widetilde{Born}Coarse})}(\emptyset_{\mathbf{Sh}}, E) \simeq *$$

is an equivalence in $\mathbf{Spc}$. However, the domain of this morphism is equivalent to $E(X)$ by the Yoneda-lemma. □

In a similar manner we deal with the last axiom "U-continuity":

Definition 2.2.13. The sheaf E is called *U-continuous* if for all spaces X in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ the collection of inclusions $X_U \to X$ induce an equivalence

$$E(X) \xrightarrow{\simeq} \lim_{U \in \mathcal{C}_X^G} E(X_U).$$

Remark 2.2.14. Recall that for any X in $\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}$ we had an isomorphism $X \cong \mathrm{colim}_{U \in \mathcal{C}_X^G} X_U$ (Example 1.3.30). Hence we call E "U-continuous" if it preserves these special colimits. ♦

Lemma 2.2.15. *A sheaf E is U-continuous if and only if it is local w.r.t. the collection of morphisms* $\mathcal{M} := \left\{\mathrm{colim}_{U \in \mathcal{C}_X^G} \mathrm{yo}(X_U) \to \mathrm{yo}(X) \;\middle|\; X \in \boldsymbol{G}\mathbf{\widetilde{Born}Coarse}\right\}$.

Proof. Follows like above using $\mathrm{Hom}_{\mathbf{Sh}(\boldsymbol{G}\mathbf{\widetilde{Born}Coarse})}(\mathrm{yo}(X), E) \simeq E(X)$ and

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Sh}(\boldsymbol{G}\mathbf{\widetilde{Born}Coarse})}\left(\mathop{\mathrm{colim}}_{U \in \mathcal{C}_X^G} \mathrm{yo}(X_U), E\right) &\simeq \lim_{U \in \mathcal{C}_X^G} \mathrm{Hom}_{\mathbf{Sh}(\boldsymbol{G}\mathbf{\widetilde{Born}Coarse})}(\mathrm{yo}(X_U), E) \\ &\simeq \lim E(X_U).\end{aligned}$$

□

Definition 2.2.16. The full subcategory of $\mathbf{Sh}(\boldsymbol{G}\mathbf{\widetilde{Born}Coarse})$ consisting of those sheaves that are U-continuous, coarsely invariant and vanishing on flasque is called the category of *G-equivariant generalized coarse motivic spaces* and denoted by $\boldsymbol{G}\mathbf{\widetilde{Spc}}\mathcal{X}$.

Proposition 2.2.17. *The full subcategory $\boldsymbol{G}\mathbf{\widetilde{Spc}}\mathcal{X}$ of $\mathbf{Sh}(\boldsymbol{G}\mathbf{\widetilde{Born}Coarse})$ is localizing, i.e. the inclusion fits into an adjunction*

$$\mathcal{L} \colon \mathbf{Sh}(\boldsymbol{G}\mathbf{\widetilde{Born}Coarse}) \rightleftarrows \boldsymbol{G}\mathbf{\widetilde{Spc}}\mathcal{X} : \mathrm{incl}.$$

Proof. The objects in $\boldsymbol{G}\mathbf{Spc}\widetilde{\mathcal{X}}$ are precisely those sheaves E on $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$ that are local w.r.t. the collection of morphisms listed in Lemmas 2.2.9, 2.2.12, and 2.2.15. Hence the proposition follows from [Lur09, Prop 5.5.4.15]. □

We denote by $\widetilde{\mathrm{Yo}}_G$ the composition

$$\widetilde{\mathrm{Yo}}_G\colon \boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse} \xrightarrow{\mathrm{yo}} \mathbf{Sh}(\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}) \xrightarrow{\mathcal{L}} \boldsymbol{G}\mathbf{Spc}\widetilde{\mathcal{X}}.$$

Remark 2.2.18. We can also describe the full subcategory $\mathbf{Sh}(\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse})$ of sheaves inside $\mathbf{PSh}(\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse})$ as local objects. In fact, a presheaf E' on $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$ fulfills the sheaf-condition if and only if it is local w.r.t. the morphisms $\emptyset_{\mathbf{PSh}} \to \mathrm{yo}(\emptyset)$ and

$$\mathrm{yo}(\mathcal{Y}) \coprod_{\mathrm{yo}(\mathcal{Y}\cap Z)} \mathrm{yo}(Z) \longrightarrow \mathrm{yo}(X)$$

for all spaces X in $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$ and all G-equivariant complementary pairs $(Z, \mathcal{Y})$ on X. ◆

Corollary 2.2.19. *The functor* $\widetilde{\mathrm{Yo}}_G\colon \boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse} \to \boldsymbol{G}\mathbf{Spc}\widetilde{\mathcal{X}}$ *has the following properties:*

1. *For all spaces X in $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$ and all G-equivariant complementary pairs $(\mathcal{Y}, Z)$ on X the following square is a pushout in $\boldsymbol{G}\mathbf{Spc}\widetilde{\mathcal{X}}$:*

$$\begin{tikzcd} \widetilde{\mathrm{Yo}}_G(\mathcal{Y}\cap Z) \arrow[r] \arrow[d] & \widetilde{\mathrm{Yo}}_G(\mathcal{Y}) \arrow[d] \\ \widetilde{\mathrm{Yo}}_G(Z) \arrow[r] & \widetilde{\mathrm{Yo}}_G(X). \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

2. *For all coarse equivalences $f\colon X \to Y$ in $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$ the induced morphisms $\widetilde{\mathrm{Yo}}_G(X) \to \widetilde{\mathrm{Yo}}_G(Y)$ is an equivalence in $\boldsymbol{G}\mathbf{Spc}\widetilde{\mathcal{X}}$.*

3. *For all spaces X in $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$ the canonical projection induces an equivalence $\widetilde{\mathrm{Yo}}_G(X \otimes \mathbb{I}) \to \widetilde{\mathrm{Yo}}_G(X)$.*

4. *For all generalized flasque spaces X in $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$ the object $\widetilde{\mathrm{Yo}}_G(X)$ is initial in $\boldsymbol{G}\mathbf{Spc}\widetilde{\mathcal{X}}$.*

5. *For any space X in $\boldsymbol{G}\mathbf{Bo\widetilde{rn}Coarse}$ the induced morphism*

$$\operatorname*{colim}_{U \in \mathcal{C}_X^G} \mathrm{Y\tilde{o}}_G(X_U) \longrightarrow \mathrm{Y\tilde{o}}_G(X)$$

is an equivalence in $\boldsymbol{G}\mathbf{S\widetilde{pc}}\mathcal{X}$.

Proof. By construction, the localization $\mathbf{PSh}(\boldsymbol{G}\mathbf{Bo\widetilde{rn}Coarse}) \to \boldsymbol{G}\mathbf{S\widetilde{pc}}\mathcal{X}$ enforces those morphisms which are local w.r.t. $\boldsymbol{G}\mathbf{S\widetilde{pc}}\mathcal{X}$ to become equivalences. By Remark 2.2.18 we therefore get that

$$\mathrm{Y\tilde{o}}_G(\mathcal{Y}) \coprod_{\mathrm{Y\tilde{o}}_G(\mathcal{Y} \cap Z)} \mathrm{Y\tilde{o}}_G(Z) \longrightarrow \mathrm{Y\tilde{o}}_G(X)$$

is an equivalence, i.e. property (1). Property (2) and (3) are equivalent for any functor (see Proposition 1.4.33), hence it suffices to show (3). However, properties (3) and (5) follow with the same argument as (1) using that applying $\mathcal{L}$ to the morphisms listed in Lemmas 2.2.9 and 2.2.15 makes them equivalences. Finally, for property (4) we use that for all flasque spaces X the morphism

$$\mathcal{L}(\emptyset_{\mathbf{PSh}}) \to \mathcal{L}(\mathrm{yo}(X)) \simeq \mathrm{Y\tilde{o}}_G(X)$$

is an equivalence and $\mathcal{L}$ is left adjoint, hence $\mathcal{L}(\emptyset_{\mathbf{PSh}}) \simeq \emptyset_{\boldsymbol{G}\mathbf{S\widetilde{pc}}\mathcal{X}}$. □

For any co-complete category $\mathbf{C}$ we denote by $\mathrm{Fun}^{\mathcal{X}}(\boldsymbol{G}\mathbf{Bo\widetilde{rn}Coarse}, \mathbf{C})$ the full subcategory of those functors $\boldsymbol{G}\mathbf{Bo\widetilde{rn}Coarse} \to \mathbf{C}$ that have the same properties as listed for $\mathrm{Y\tilde{o}}_G$ in Corollary 2.2.19.

Theorem 2.2.20. *Precomposition with $\mathrm{Y\tilde{o}}_G$ induces an equivalence of ∞-categories*

$$\mathrm{Y\tilde{o}}_G^*\colon \mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{S\widetilde{pc}}\mathcal{X}, \mathbf{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathcal{X}}(\boldsymbol{G}\mathbf{Bo\widetilde{rn}Coarse}, \mathbf{C})$$

where the sup-script "colim" stands for "colimit-preserving".

Proof. By [Lur09, Thm 5.5.4.20] precomposition with the localization functor gives an equivalence of ∞-categories between $\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{S\widetilde{pc}}\mathcal{X}, \mathbf{C})$ and the full subcategory of $\mathrm{Fun}^{\mathrm{colim}}(\mathbf{PSh}(\boldsymbol{G}\mathbf{S\widetilde{pc}}\mathcal{X}), \mathbf{C})$ consisting of those functors which send all morphisms listed in Lemmas 2.2.9, 2.2.12, and 2.2.15, and Remark 2.2.18 to equivalences in $\mathbf{C}$. Furthermore, precomposition with the Yoneda-embedding gives an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{colim}}(\mathbf{PSh}(\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}),\mathbf{C}) \xrightarrow{\simeq} \mathrm{Fun}(\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse},\mathbf{C})$$

by [Lur09, Thm 5.1.5.6]. Restricting this equivalence to the upmentioned full subcategory of $\mathrm{Fun}(\mathbf{PSh}(\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}),\mathbf{C})$ gives exactly the functors in the category $\mathrm{Fun}^{\chi}(\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse},\mathbf{C})$ as claimed. $\square$

2.3 Equivariant generalized coarse motivic spectra

In this section we stabilize the category $\boldsymbol{G}\mathbf{S}\widetilde{\mathbf{pc}}\mathcal{X}$ of G-equivariant generalized coarse motivic spaces by formally inverting the suspension endofunctor Σ (see below). The resulting category of G-equivariant generalized coarse motivic spectra $\boldsymbol{G}\mathbf{S}\tilde{\mathbf{p}}\mathcal{X}$ together with the stabil version $\widetilde{\mathrm{Yo}}^s_G$ of the Yoneda-embedding $\widetilde{\mathrm{Yo}}_G$ will turn out to be the universal G-equivariant coarse homology theory on $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$ in the sense that every other such theory factors through $\widetilde{\mathrm{Yo}}^s_G$. A reference for stability in the realms of ∞-category is [Lur17, sec. 1].

As in the previous section, the construction mimics the one in [BE20, sec. 4]. We also refer to this source for a motivating analogy to the stabilization of the ∞-category of spaces and for a careful examination of the set-theoretical size-issues.

Construction 2.3.1. We denote by $\boldsymbol{G}\mathbf{S}\widetilde{\mathbf{pc}}\mathcal{X}_{*/}$ the pointed ∞-category of pointed G-equivariant generalized coarse motivic spaces. Also we let

$$\Sigma\colon \boldsymbol{G}\mathbf{S}\widetilde{\mathbf{pc}}\mathcal{X}_{*/} \longrightarrow \boldsymbol{G}\mathbf{S}\widetilde{\mathbf{pc}}\mathcal{X}_{*/}$$

denote the suspension functor defined such that for all X in $\boldsymbol{G}\mathbf{S}\widetilde{\mathbf{pc}}\mathcal{X}_{*/}$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

is a pushout. For stability we want that Σ is invertible, i.e. we invert Σ by the following construction:

$$\boldsymbol{G}\mathbf{S}\tilde{\mathbf{p}}\mathcal{X} := \boldsymbol{G}\mathbf{S}\widetilde{\mathbf{pc}}\mathcal{X}_{*/}[\Sigma^{-1}] = \operatorname{colim}\left\{\boldsymbol{G}\mathbf{S}\widetilde{\mathbf{pc}}\mathcal{X}_{*/} \xrightarrow{\Sigma} \boldsymbol{G}\mathbf{S}\widetilde{\mathbf{pc}}\mathcal{X}_{*/} \xrightarrow{\Sigma} \boldsymbol{G}\mathbf{S}\widetilde{\mathbf{pc}}\mathcal{X}_{*/} \xrightarrow{\Sigma} \dots\right\},$$

where the colimit is takin in the ∞-category $\mathbf{Pr^L}$ of presentable ∞-categories with left-adjoint functors. We call the category $\boldsymbol{G}\mathbf{S}\tilde{\mathbf{p}}\mathcal{X}$ the category of *G-equivariant generalized coarse motivic spectra.* Furthermore we denote by

$$\Sigma^{\mathrm{mot}}_+\colon \boldsymbol{G}\mathbf{S}\widetilde{\mathbf{pc}}\mathcal{X} \longrightarrow \boldsymbol{G}\mathbf{S}\widetilde{\mathbf{pc}}\mathcal{X}_{*/} \longrightarrow \boldsymbol{G}\mathbf{S}\tilde{\mathbf{p}}\mathcal{X}$$

the composition of adjoining a basepoint and inverting Σ. This functor fits into an adjunction

$$\Sigma^{\mathrm{mot}}_+\colon \boldsymbol{G}\mathbf{S}\widetilde{\mathbf{pc}}\mathcal{X} \rightleftarrows \boldsymbol{G}\mathbf{S}\tilde{\mathbf{p}}\mathcal{X} :\Omega^{\mathrm{mot}}.$$

Finally, we define

$$\mathrm{Y}\widetilde{\mathrm{o}}_G^s := \Sigma_+^{\mathrm{mot}} \circ \mathrm{Y}\widetilde{\mathrm{o}}_G \colon \boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse} \longrightarrow \boldsymbol{G}\mathbf{S}\widetilde{\mathbf{p}}\mathcal{X}.$$

Lemma 2.3.2. *For every co-complete stable ∞-category $\mathbf{C}$ precomposition with Σ_+^{mot} induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{S}\widetilde{\mathbf{p}}\mathcal{X}, \mathbf{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{S}\widetilde{\mathbf{pc}}\mathcal{X}, \mathbf{C}).$$

Proof. For $\mathbf{C}$ presentable this is [Lur17, Cor 1.4.4.5]. For gernalization to all co-complete stable categories $\mathbf{C}$ is due to Denis-Charles Cisinski and Thomas Nikolaus and can be found in [BE20, Lemma 4.4]. □

As indicated in the introductory words of this section, the functor $\mathrm{Y}\widetilde{\mathrm{o}}_G^s$ is the universal G-equivariant coarse homology theory on $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$. We will elaborate on this in the following. First, this functor is indeed a G-equivariant coarse homology theory in the sense of Definition 2.1.17:

Theorem 2.3.3. *The functor $\mathrm{Y}\widetilde{\mathrm{o}}_G^s \colon \boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse} \to \boldsymbol{G}\mathbf{S}\widetilde{\mathbf{p}}\mathcal{X}$ has the following properties:*

1. *For any space X in $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$ and any G-equivariant complementary pair $(Z, \mathcal{Y})$ on X the following square is a pushout in $\boldsymbol{G}\mathbf{S}\widetilde{\mathbf{p}}\mathcal{X}$:*

$$\begin{tikzcd} \mathrm{Y}\widetilde{\mathrm{o}}_G^s(\mathcal{Y} \cap Z) \arrow[r] \arrow[d] & \mathrm{Y}\widetilde{\mathrm{o}}_G^s(Z) \arrow[d] \\ \mathrm{Y}\widetilde{\mathrm{o}}^s(\mathcal{Y}) \arrow[r] & \mathrm{Y}\widetilde{\mathrm{o}}_G^s(X). \end{tikzcd}$$

2. *For any space X in $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$ the projection induces an equivalence*

$$\mathrm{Y}\widetilde{\mathrm{o}}_G^s(X \otimes \mathbb{I}) \xrightarrow{\simeq} \mathrm{Y}\widetilde{\mathrm{o}}_G^s(X).$$

3. *For all (generalized) flasque spaces X in $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$ we have $\mathrm{Y}\widetilde{\mathrm{o}}_G^s(X) \simeq 0$.*

4. *For all spaces X in $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$ the induced morphism*

$$\operatorname*{colim}_{U \in \mathcal{C}_X^G} \mathrm{Y}\widetilde{\mathrm{o}}_G^s(X_U) \longrightarrow \mathrm{Y}\widetilde{\mathrm{o}}_G^s(X)$$

is an equivalence in $\boldsymbol{G}\mathbf{S}\widetilde{\mathbf{p}}\mathcal{X}$.

In particular, $\mathrm{Y\tilde{o}}^s_G$ *is a* G*-equivariant coarse homology theory on* $G\mathbf{Bo\widetilde{rn}Coarse}$.

Proof. This follows immediately from the corresponding properties of $\mathrm{Y\tilde{o}}_G$ in Corollary 2.2.19 and the fact that Σ^{mot}_+ preserves equivalences and colimits because it is a left adjoint. □

To formulate in which sense $\mathrm{Y\tilde{o}}^s_G$ is the "universal" G-equivariant coarse homology theory on $G\mathbf{Bo\widetilde{rn}Coarse}$, for all co-complete stable ∞-categories $\mathbf{C}$ we denote by $G\mathbf{\widetilde{Homol}Theo}\mathcal{X}_{\mathbf{C}}$ the full subcategory of $\mathrm{Fun}(G\mathbf{Bo\widetilde{rn}Coarse}, \mathbf{C})$ consisting of those functors that are G-equivariant coarse homology theories in the sense of Definition 2.1.17.

Theorem 2.3.4. *For any co-complete stable* ∞*-category* $\mathbf{C}$ *precomposition with* $\mathrm{Y\tilde{o}}^s_G$ *induces an equivalence of* ∞*-categories*

$$(\mathrm{Y\tilde{o}}^s_G)^* \colon \mathrm{Fun}^{\mathrm{colim}}(G\mathbf{S\tilde{p}}\mathcal{X}, \mathbf{C}) \xrightarrow{\simeq} G\mathbf{\widetilde{Homol}Theo}\mathcal{X}_{\mathbf{C}}.$$

Proof. This is an immediate consequence of the definitions and combining Theorem 2.2.20 and Lemma 2.3.2. □

We finish this section with a few additional properties of the functor $\mathrm{Y\tilde{o}}^s_G$ (which therefore also hold for any G-equivariant generalized coarse homology theory).

Recall that for a functor e.g. $\mathrm{Y\tilde{o}}^s_G$ and a G-equivariant big family $\mathcal{Y}$ on X we defined

$$\mathrm{Y\tilde{o}}^s_G(X, \mathcal{Y}) := \mathrm{cofib}\left(\mathrm{Y\tilde{o}}^s_G(\mathcal{Y}) \to \mathrm{Y\tilde{o}}^s_G(X)\right).$$

Lemma 2.3.5. *We have a fiber sequence*

$$\cdots \to \mathrm{Y\tilde{o}}^s_G(\mathcal{Y}) \to \mathrm{Y\tilde{o}}^s_G(X) \to \mathrm{Y\tilde{o}}^s_G(X, \mathcal{Y}) \to \Sigma\mathrm{Y\tilde{o}}^s_G(\mathcal{Y}) \to \dots$$

Proof. Clear from definition of $\mathrm{Y\tilde{o}}^s_G(X, \mathcal{Y})$ and from stability of $G\mathbf{S\tilde{p}}\mathcal{X}$. □

Lemma 2.3.6. *For any* G*-equivariant complementary pair* $(Z, \mathcal{Y})$ *on* X *we have an induced equivalence of cofibers*

$$\mathrm{Y\tilde{o}}^s_G(Z, Z \cap \mathcal{Y}) \longrightarrow \mathrm{Y\tilde{o}}^s_G(X, \mathcal{Y})$$

in $G\mathbf{S\tilde{p}}\mathcal{X}$.

Proof. See Lemma 2.1.20. □

A G-equivariant complementary pair $(Z, \mathcal{Y})$ consists of a subspace Z and a big family $\mathcal{Y}$. This is to ensure that the intersection of Z and $\mathcal{Y}$ is "not too small" (c.f. classical topology). We can also formulate an excision with two fixed subspaces as long as they intersect "enough":

Definition 2.3.7. A *(G-equivariant) coarsely excisive pair* on X is a pair (Y, Z) consisting of two G-invariant subsets Y and Z (equipped with the subspace structure induced from X) such that:

1. they cover X, i.e. $X = Y \cup Z$.
2. For every entourage U in $\mathcal{C}_X$ there exists an entourage V in $\mathcal{C}_X$ such that
$$U[Y] \cap U[Z] \subseteq W[Y \cap Z].$$
3. There exists a cofinal subset $\mathcal{S}$ of $\mathcal{C}_X^G$ such that for all entourages U in $\mathcal{S}$ the intersection $U[Y] \cap Z$ is nice (c.f. Definition 1.4.27) and so is Y.

Example 2.3.8. Consider two generalized Γ-bornological coarse spaces Y and Z. We identify them with their respective image in the coproduct $X := Y \amalg Z$. Then the pair (Y, Z) is coarsely excisive for X as we will show now: The coarse structure of X is generated by $\mathcal{C}_Y \cup \mathcal{C}_Z$. Since $Y \cap Z = \emptyset$, for any entourage U in $\mathcal{C}_Y$ we have $U[Z] = \emptyset$ and the same is true for thickenings of Y by entourages of Z. Therefore, for any entourage U in $\mathcal{C}_X$ we have $U[Y] \subseteq Y$ and $U[Z] \subseteq Z$. Therefore obtain $U[Y] \cap U[Z] = \emptyset$ and thus conditions (2) und (3) are fulfilled trivially. $\maltese$

Example 2.3.9. The pair (X_b, X_h) of bounded and unbounded points of X is coarsely excisive. Indeed, by Corollary 1.3.34 we have $X \cong X_b \amalg X_h$. Hence this is a special case of Example 2.3.8. $\maltese$

Proposition 2.3.10. *Consider a coarsely excisive pair (Y, Z) on X. Then the following square is a pushout in $\boldsymbol{G}\mathbf{S\tilde{p}}\mathcal{X}$:*

$$\begin{tikzcd}
\mathrm{Y\tilde{o}}^s_G(Y \cap Z) \arrow[r] \arrow[d] & \mathrm{Y\tilde{o}}^s_G(Y) \arrow[d] \\
\mathrm{Y\tilde{o}}^s_G(Z) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{Y\tilde{o}}^s_G(X).
\end{tikzcd}$$

Proof. By condition (3) on a G-equivariant coarsely excisive pair we can choose a cofinal subset $\mathcal{S}$ of $\mathcal{C}_X^G$ such that all entourages in $\mathcal{S}$ contain Δ_X and for all of them the intersection $U[Y] \cap Z$ is nice. We define $\mathcal{Y} := \{U[Y]\}_{U \in \mathcal{S}}$. This is a G-equivariant big family on X (c.f. Example 2.1.3) and since $Y \cup Z = X$ the pair $(Z, \mathcal{Y})$ is a G-equivariant complementary pair. By Theorem 2.3.3 the following square is a pushout:

$$\begin{tikzcd}
\mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}(\mathcal{Y} \cap Z) \arrow[r] \arrow[d] & \mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}(\mathcal{Y}) \arrow[d] \\
\mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}(Z) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}(X).
\end{tikzcd}$$

Since Y is nice, all morphisms $Y \to U[Y]$ (for U in $\mathcal{S}$) are coarse equivalences, hence $\mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}(Y) \to \mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}(U[Y])$ are equivalences in $\boldsymbol{G}\mathbf{S}\tilde{\mathbf{p}}\mathcal{X}$ and we get an equivalence

$$\mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}(Y) \simeq \mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}(\mathcal{Y}).$$

To see that $\mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}(Z \cap \mathcal{Y}) \simeq \mathrm{Y}\tilde{\mathrm{o}}^{S}_{G}(Z \cap Y)$ we first note that since $Y \cap Z$ is nice and $V[Y] \cap Z$ are nice for all V in $\mathcal{S}$, the same argument as above gives equivalences

$$\mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}(Y \cap Z) \simeq \mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}\left(\{U[Y \cap Z]\}_{U \in \mathcal{S}}\right) \tag{*}$$

and

$$\mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}(V[Y] \cap Z) \simeq \mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}\left(\Big\{U\Big[V[Y] \cap Z\Big]\Big\}_{U \in \mathcal{S}}\right). \tag{†}$$

However, for all entourages U, V in $\mathcal{S}$ we have inclusions

$$U[Y \cap Z] \hookrightarrow U[V[Y] \cap Z]$$

which give the first morphism in the chain

$$\begin{aligned}
\mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}\left(\{U[Y \cap Z]\}_{U \in \mathcal{S}}\right) \longrightarrow \mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}\left(\Big\{U\Big[V[Y] \cap Z\Big]\Big\}_{U \in \mathcal{S}}\right) &\underset{(†)}{\simeq} \mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}(V[Y] \cap Z) \\
\longrightarrow \mathrm{Y}\tilde{\mathrm{o}}^{s}_{G}(\mathcal{Y} \cap Z). &
\end{aligned} \tag{♣}$$

Note that $V[Y] \cap Z$ is a member of $\mathcal{Y} \cap Z$ which gives the latter morphism in the upper chain.

Now, using condition (2) of a G-equivariant coarsely excisive pair, we can find for any entourage U in $\mathcal{S}$ an entourage W in $\mathcal{S}$ such that $U[Y] \cap Z \subseteq W[Y \cap Z]$ which induces a morphism

$$\widetilde{\mathrm{Yo}}^s_G(\mathcal{Y} \cap Z) \longrightarrow \widetilde{\mathrm{Yo}}^{\mathcal{S}}_G(\{U[Y \cap Z]\}_{U \in \mathcal{S}})$$

which is inverse to (♣). Precomposing this equivalence with

$$\varphi\colon \mathrm{Yo}^s_G(Y \cap Z) \longrightarrow \mathrm{Yo}^s_G(\mathcal{Y} \cap Z)$$

gives the equivalence of (*), hence by the 2-out-of-3-property of equivalences we deduce that φ is an equivalence which was the claim. □

Corollary 2.3.11. *We have an equivalence of G-equivariant generalized coarse motivic spectra*

$$\widetilde{\mathrm{Yo}}^s_G(X) \simeq \widetilde{\mathrm{Yo}}^s_G(X_b) \oplus \widetilde{\mathrm{Yo}}^s_G(X_h) \simeq \widetilde{\mathrm{Yo}}^s_G(X_b).$$

Proof. The pair (X_b, X_h) is coarsely excisive by Example 2.3.9 and $X_b \cap X_h = \emptyset$. Thus

$$\begin{tikzcd} \widetilde{\mathrm{Yo}}^s_G(\emptyset) \arrow[r] \arrow[d] & \widetilde{\mathrm{Yo}}^s_G(X_b) \arrow[d] \\ \widetilde{\mathrm{Yo}}^s_G(X_h) \arrow[r] & \widetilde{\mathrm{Yo}}^s_G(X) \end{tikzcd}$$

is a pushout. But $\widetilde{\mathrm{Yo}}^s_G(\emptyset) \simeq 0$, hence we get the first equivalence of the claim. The latter equivalence follows because $\widetilde{\mathrm{Yo}}^s_G$ vanishes on flasques and X_h is flasque by Example 2.1.9. □

2.4 Equivalence of generalized and non-generalized coarse homology theories

Building on the universal coarse homology theory defined in [BE20] and its equivariant generalization in [BEKW] there has been a rich development in various directions. To name only a few of the resulting papers: [BE20b; BCKW; BKW18; BEKWb; BE20a; BC20; BE17; Bun18; BEKWa; Cap19]

Instead of verifying that analoguous work can be developed on this new genereralized setting of G-equivariant generalized coarse homology theories we show that both theories are equivalent. More precisely, the inclusion functor $\iota\colon \boldsymbol{G}\mathbf{BornCoarse} \to \boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ induces an equivalence of ∞-categories

$$\boldsymbol{G}\mathbf{S\tilde{p}}\mathcal{X} \xrightarrow{\simeq} \boldsymbol{G}\mathbf{Sp}\mathcal{X}$$

and in particular an equivalence between the ∞-category of G-equivariant coarse homology theories developed in [BEKW] and G-equivariant generalized coarse homology theories developed here in section 2.1.

Let $\mathbf{Sp}$ denote the stable ∞-category of spectra (i.e. spectrum objects in $\mathbf{Spc}_*$). For any category $\mathbf{C}$ we denote by $\mathbf{PSh}_{\mathbf{Sp}}(\mathbf{C}) := \mathbf{Fun}(\mathbf{C}^{op}, \mathbf{Sp})$ the ∞-category of spectrum-valued presheaves on $\mathbf{C}$. This category is equivalent to the subcategory of spectrum objects in $\mathbf{PSh}(\mathbf{C})$ by [Lur17, Rmk 1.4.2.9].

Definition 2.4.1. We denote by $\mathbf{Sh}_{\mathbf{Sp}}(\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse})$ the full subcategory of those presheaves E in $\mathbf{PSh}_{\mathbf{Sp}}(\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse})$ satisfying:

1. For all X, Y in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ the natural morphism

$$E(X \amalg Y) \longrightarrow E(X) \oplus E(Y)$$

   is an equivalence in $\mathbf{Sp}$.

2. For all spaces X in $\boldsymbol{G}\mathbf{B\widetilde{orn}Coarse}$ with $X_b = \emptyset$ we have $E(X) \simeq 0$.

Likewise we denote by $\mathbf{Sh}_{\mathbf{Sp}}(\boldsymbol{G}\mathbf{BornCoarse})$ the full subcategory of those presheaves E in $\mathbf{PSh}_{\mathbf{Sp}}(\boldsymbol{G}\mathbf{BornCoarse})$ satisfying the conditions (1) and (2) above[1].

[1]Note that condition (2) simply means $E(\emptyset) \simeq 0$ in this case

Lemma 2.4.2. *The full subcategories defined above in Definition 2.4.1 are localizing, i.e. the respective inclusions fit into adjunctions*

$$\tilde{\mathcal{L}}\colon \mathbf{PSh}_{\mathbf{Sp}}(\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}) \rightleftarrows \mathbf{Sh}_{\mathbf{Sp}}(\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}) :\mathrm{incl.}$$

and

$$\mathcal{L}\colon \mathbf{PSh}_{\mathbf{Sp}}(\boldsymbol{G}\mathbf{BornCoarse}) \rightleftarrows \mathbf{Sh}_{\mathbf{Sp}}(\boldsymbol{G}\mathbf{BornCoarse}) :\mathrm{incl.}$$

Proof. Analoguous to the proofs in section 2.2. □

Lemma 2.4.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $i\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful embedding. Then we obtain (basically by left Kan extension) a pair of adjoint functors*

$$i_!\colon \mathbf{PSh}(\mathcal{C}) \rightleftarrows \mathbf{PSh}(\mathcal{D}) :i^*.$$

Moreover $i_!$ is fully faithful, hence the unit $\mathrm{id} \to i^* i_!$ *is an equivalence.*

Proof. This classical result from [SGA4, Exp. 1, Prop. 5.6] is given in the language of ∞-categories in [Lur09, Prop 5.3.5.11]. □

With these two sheaf-categories we proceed to the main proof of this section:

Theorem 2.4.4. *The inclusion $\iota\colon \boldsymbol{G}\mathbf{BornCoarse} \to \boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$ induces an equivalence of ∞-categories*

$$\iota^*\colon \mathbf{Sh}_{\mathbf{Sp}}(\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}) \xrightarrow{\simeq} \mathbf{Sh}_{\mathbf{Sp}}(\boldsymbol{G}\mathbf{BornCoarse}).$$

Proof. Consider the following diagram

$$\begin{tikzcd}
\boldsymbol{G}\mathbf{BornCoarse} \arrow[r, "\mathrm{yo}"] \arrow[d, "\iota"'] & \mathbf{PSh}_{\mathbf{Sp}}(\boldsymbol{G}\mathbf{BornCoarse}) \arrow[r, shift left, "\mathcal{L}"] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "\iota_!"'] \arrow[d, phantom, "\dashv"] & \mathbf{Sh}_{\mathbf{Sp}}(\boldsymbol{G}\mathbf{BornCoarse}) \arrow[l, shift left, "j"] \arrow[d, dotted, shift right, "\mathcal{F}"'] \\
\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse} \arrow[r, "\mathrm{yo}"'] & \mathbf{PSh}_{\mathbf{Sp}}(\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}) \arrow[r, "\tilde{\mathcal{L}}"'] \arrow[u, shift right, "\iota^*"'] & \mathbf{Sh}_{\mathbf{Sp}}(\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}). \arrow[u, dotted, shift right, "\iota^*"']
\end{tikzcd}$$

It is clear, that ι^* restricts to a functor (which we also denote by ι^*)

$$\iota^*\colon \mathbf{Sh}_{\mathbf{Sp}}(\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}) \longrightarrow \mathbf{Sh}_{\mathbf{Sp}}(\boldsymbol{G}\mathbf{BornCoarse}).$$

Further we define $\mathcal{F}$ as the composition $\mathcal{F} := \tilde{\mathcal{L}} \circ \iota_! \circ j$, where j is the inclusion of the full subcategory $\mathbf{Sh}_{\mathbf{Sp}}(\boldsymbol{G}\mathbf{BornCoarse}) \hookrightarrow \mathbf{PSh}_{\mathbf{Sp}}(\boldsymbol{G}\mathbf{BornCoarse})$.

Claim 1: The functor ι^* detects equivalences.
In fact, let $E \to F$ be a morphism in $\mathbf{Sh}_{\mathbf{Sp}}(\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse})$ such that the induced morphism $\iota^* E \to \iota^* F$ is an equivalence. Now for all spaces X in $\boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$, Corollary 1.3.34 provides an isomorphism $X \cong X_b \amalg X_h$. Thus using the sheaf conditions we obtain:

$$\begin{aligned} E(X) &\simeq E(X_b \amalg X_h) \xrightarrow{\simeq} E(X_b) \oplus E(X_h) \simeq E(X_b) \simeq E(\iota(X_b)) \\ &\simeq \iota^* E(X_b) \xrightarrow{\simeq} \iota^* F(X_b) \simeq F(X_b) \simeq F(X_b) \oplus F(X_h) \simeq F(X_b \amalg X_h) \simeq F(X). \end{aligned}$$

Claim 2: We have an equivalence $\mathcal{L}\iota^* \simeq \iota^* \tilde{\mathcal{L}}$. We show this claim later.

Claim 3: We have an equivalence $\iota^* \mathcal{F} \simeq \mathrm{id}$.
Fully faithfulness of ι gives by Lemma 2.4.3 that the unit of $\iota_! \dashv \iota^*$ is an equivalence $\mathrm{id} \xrightarrow{\simeq} \iota^* \iota_!$. Using this and claim 2 we obtain:

$$\iota^* \mathcal{F} \simeq \iota^* \tilde{\mathcal{L}} \iota_! j \simeq \mathcal{L} \iota^* \iota_! j \simeq \mathcal{L} j \simeq \mathrm{id}.$$

Claim 4: $\mathcal{F}\iota^* \simeq \mathrm{id}$.
By claim 3, we have $\iota^* \mathcal{F} \simeq \mathrm{id}$. In particular we get $\iota^* \mathcal{F} \iota^* \simeq \iota^*$. But ι^* detects equivalences by claim 1, thus $\mathcal{F}\iota^* \simeq \mathrm{id}$.

It remains to verify claim 2. The strategy will be to define two concrete models for the localization functors $\mathcal{L}$ and $\tilde{\mathcal{L}}$ and verify claim 2 by an explicit calculation. We start with constructing a model for $\tilde{\mathcal{L}}$:
To shorten notation, let $\mathcal{C} := \boldsymbol{G}\mathbf{BornCoarse}$ and $\mathcal{D} := \boldsymbol{G}\widetilde{\mathbf{Born}}\mathbf{Coarse}$. We define a category $\mathcal{E}$ with objects $(X, \mathscr{S})$, where X is in $\mathcal{D}$ and $\mathscr{S}$ is a finite decomposition of X into unions of coarse components, which is fine enough that each component is contained in X_b or in X_h (cf. Lemma 1.3.33). A morphism $(X, \mathscr{S}) \to (X', \mathscr{S}')$ is a morphism f in $\mathrm{Mor}_{\mathcal{D}}(X, X')$ such that $f^* \mathscr{S}' \subseteq \mathscr{S}$. Moreover, we define another category $\mathcal{E}'$ with objects $(X, \mathscr{S}, S)$, where $(X, \mathscr{S})$ is an object in $\mathcal{E}$ and S is in $\mathscr{S}$ with $S \subseteq X_b$. We have forgetful functors

$$\begin{aligned} \alpha \colon \mathcal{E}' &\longrightarrow \mathcal{E}, \qquad (X, \mathscr{S}, S) \longmapsto (X, \mathscr{S}) \\ \beta \colon \mathcal{E} &\longrightarrow \mathcal{D}, \qquad\quad (X, \mathscr{S}) \longmapsto X. \end{aligned}$$

These forgetful functors induce restrictions

$$\alpha^* \colon \mathbf{PSh}_{\mathbf{Sp}}(\mathcal{E}) \to \mathbf{PSh}_{\mathbf{Sp}}(\mathcal{E}'), \quad \beta^* \colon \mathbf{PSh}_{\mathbf{Sp}}(\mathcal{D}) \to \mathbf{PSh}_{\mathbf{Sp}}(\mathcal{E}).$$

We denote by α_* the right Kan extension of α^* and by $\beta_!$ the left Kan extension of β^*. Finally we define a functor $F\colon \mathbf{PSh}_{\mathbf{Sp}}(\mathcal{D}) \to \mathbf{PSh}_{\mathbf{Sp}}(\mathcal{E}')$ by

$$F(E)\Big((X,\mathscr{S},S)\Big) := E(S).$$

For $(X,\mathscr{S},S)$ in $\mathcal{E}'$ the inclusion $S \hookrightarrow X$ gives a natural transformation $\alpha^*\beta^* \to F$, which corresponds via the adjunction $\alpha^* \dashv \alpha_*$ to a natural transformation $\beta^* \to \alpha_* F$. Applying $\beta_!$ to this natural transformation and using $\beta_!\beta^* \simeq \mathrm{id}$[2], we obtain a natural transformation

$$\mathrm{id} \to \beta_!\alpha_* F =: \tilde{\mathscr{L}}.$$

We claim, that $\tilde{\mathscr{L}}$ models the localization $\tilde{\mathcal{L}}$.

To show this, we must check that $\tilde{\mathscr{L}}$ has values in $\mathbf{Sh}_{\mathbf{Sp}}(\mathcal{D})$ and that it is idempotent or – equivalently – that $E \to \tilde{\mathscr{L}}E$ is an equivalence for all τ-sheaves E. First, consider a presheaf E in $\mathbf{PSh}_{\mathbf{Sp}}(\mathcal{D})$ and let X, Y in $\mathcal{D}$. Then for $Z := X \amalg Y$ we calculate

$$\tilde{\mathscr{L}}(E)(Z) \simeq \operatorname*{colim}_{\beta(Z,\mathscr{S})\to Z} \alpha_* F(E)((Z,\mathscr{S})) \simeq \operatorname*{colim}_{\beta(Z,\mathscr{S})\to Z} \lim_{(Z,\mathscr{S})\to\alpha(Z,\mathscr{S},S)} \underbrace{F(E)(Z,\mathscr{S},S)}_{\simeq E(S)}.$$

We can choose $\mathscr{S}$ in the colimit fine enough such that each S belongs to either X or Y. Also, the colimit is filtered, hence commutes with finite products, hence:

$$\begin{aligned}
\tilde{\mathscr{L}}(E)(Z) &\simeq \operatorname*{colim}_{\beta(Z,\mathscr{S})\to Z} \left(\lim_{\substack{(Z,\mathscr{S})\to\alpha(Z,\mathscr{S},S) \\ S\subseteq X}} E(S) \times \lim_{\substack{(Z,\mathscr{S})\to\alpha(Z,\mathscr{S},S) \\ S\subseteq Y}} E(S) \right) \\
&\simeq \operatorname*{colim}_{\beta(Z,\mathscr{S})\to Z} \lim_{\substack{(Z,\mathscr{S})\to\alpha(Z,\mathscr{S},S) \\ S\subseteq X}} E(S) \times \operatorname*{colim}_{\beta(Z,\mathscr{S})\to Z} \lim_{\substack{(Z,\mathscr{S})\to\alpha(Z,\mathscr{S},S) \\ S\subseteq Y}} E(S) \\
&\simeq \tilde{\mathscr{L}}(E)(X) \times \tilde{\mathscr{L}}(E)(Y).
\end{aligned}$$

For the last equivalence we used again a cofinality argument to identify both factors separately.

The second sheaf condition is immediate: By definition we require for any S in $\mathscr{S}$ to be a subset of X_b. Therefore, if $X_b = \emptyset$, we take the colimit over the empty limit and thus obtain 0. So we see, that $\tilde{\mathscr{L}}(E)$ is indeed a sheaf in $\mathbf{Sh}_{\mathbf{Sp}}(\mathcal{D})$.

[2] $\beta_!\beta^*(E)(X) \simeq \operatorname{colim}_{\beta(X,\mathscr{S})\to X} \beta^*(E)((X,\mathscr{S})) \simeq \operatorname{colim} E(X) \simeq E(X)$, because $(X,\mathscr{S})$ is cofinal in the collection $\beta(Y,\mathscr{S}) \to X$

Next consider a sheaf E in $\mathbf{Sh}_{\mathbf{Sp}}(\mathcal{D})$ and calculate

$$\tilde{\mathscr{L}}(E)(X) \simeq \operatorname*{colim}_{(X,\mathscr{S})} \lim_{(X,\mathscr{S},S)} E(S) \simeq \operatorname*{colim}_{(X,\mathscr{S})} \prod_{S\in\mathscr{S}} E(S) \simeq \operatorname*{colim}_{(X,\mathscr{S})} E(X) \simeq E(X),$$

where the third equivalence is the sheaf condition on E (note that $\mathscr{S}$ is *finite* by construction). This shows, that $\tilde{\mathscr{L}}$ is a model for $\tilde{\mathcal{L}}$.

As a second step we define $\mathscr{L} := \iota^* \tilde{\mathscr{L}} \iota_! \colon \mathbf{PSh}_{\mathbf{Sp}}(\mathcal{C}) \to \mathbf{Sh}_{\mathbf{Sp}}(\mathcal{C})$. We claim that $\mathscr{L}$ models the localization $\mathcal{L}$ and further we claim, that this localization commutes with restriction ι^*, more precisely: $\mathscr{L}\iota^* \simeq \iota^*\tilde{\mathscr{L}}$, which would complete the proof of claim 2.

The second assertion follows from explicit calculation: Let E be in $\mathbf{PSh}_{\mathbf{Sp}}(\mathcal{D})$ and let X in $\mathcal{C}$, then

$$\begin{aligned}
\mathscr{L}\iota^*(E)(X) &\simeq \tilde{\mathscr{L}}\iota_!\iota^*(E)(\iota(X)) \simeq \tilde{\mathscr{L}}\iota_!\iota^*(E)(X) \\
&\simeq \operatorname*{colim}_{\beta(X,\mathscr{S})\to X} \lim_{(X,\mathscr{S})\to\alpha(X,\mathscr{S},S)} \operatorname*{colim}_{\iota(Y)\to S} \iota^*(E)(Y) \\
&\simeq \operatorname*{colim}_{\beta(X,\mathscr{S})\to X} \lim_{(X,\mathscr{S})\to\alpha(X,\mathscr{S},S)} E(S) \\
&\simeq \tilde{\mathscr{L}}(E)(X) = \iota^*\tilde{\mathscr{L}}(E)(X).
\end{aligned}$$

For the equivalence in the third line we used that any S in $\mathscr{S}$ is contained in X_b, hence is cofinal in the collection $\iota(Y) \to S$.
It remains to show, that $\mathscr{L}$ models the localization $\mathcal{L}$, i.e. we have to check that it is idempotent. We already know that $\iota^*\iota_! \simeq \mathrm{id}$ and that $\tilde{\mathscr{L}}$ is idempotent, hence

$$\mathscr{L}\mathscr{L} \simeq \mathscr{L}\mathscr{L}\iota^*\iota_! \simeq \iota^*\tilde{\mathscr{L}}\tilde{\mathscr{L}}\iota_! \simeq \iota^*\tilde{\mathscr{L}}\iota_! \simeq \mathscr{L}\iota^*\iota_! \simeq \mathscr{L}. \qquad \square$$

To arrive at the main theorem of this section, we localize both sheaf categories in the theorem above at certain collections of morphisms and we will see that these localizations produce the categories $\boldsymbol{G}\mathbf{S}\tilde{\mathbf{p}}\mathcal{X}$ and $\boldsymbol{G}\mathbf{Sp}\mathcal{X}$ respectively.

Let $\mathscr{C}$ be one of the categories $\boldsymbol{G}\mathbf{BornCoarse}$ or $\boldsymbol{G}\mathbf{B}\widetilde{\mathbf{orn}}\mathbf{Coarse}$.

Definition 2.4.5. We denote by $\mathbf{Sh}^{\mathrm{mot}}_{\mathbf{Sp}}(\mathscr{C})$ the full subcategory of those sheaves E in $\mathbf{Sh}_{\mathbf{Sp}}(\mathscr{C})$ such that:

1. For all spaces X in $\mathscr{C}$ the canonical projection $X \otimes \mathbb{I} \to X$ is sent to an equivalence of spectra
$$E(X) \xrightarrow{\simeq} E(X \otimes \mathbb{I}).$$

2. For all spaces X in $\mathscr{C}$ and all G-equivariant complementary pairs $(Z,\mathcal{Y})$ on X the natural square
$$\begin{tikzcd} E(X) \arrow[r] \arrow[d] & E(Z) \arrow[d] \\ E(\mathcal{Y}) \arrow[r] & E(Z\cap\mathcal{Y}) \end{tikzcd}$$
is a pullback in $\mathbf{Sp}$.

3. For all flasque spaces X in $\mathscr{C}$ we have $E(X)\simeq 0$.

4. For all spaces X in $\mathscr{C}$ the canonical morphism
$$E(X)\longrightarrow \lim_{U\in\mathcal{C}_X} E(X_U)$$
is an equivalence in $\mathbf{Sp}$.

Remark 2.4.6. Assume that a presheaf E in $\mathbf{PSh}_{\mathbf{Sp}}(\mathscr{C})$ is local w.r.t. the following set of morphisms

- For all spaces X in $\mathscr{C}$ the morphism $\mathrm{yo}(X\otimes\mathbb{I})\to\mathrm{yo}(X)$ induced from the projection,
- for all spaces X in $\mathscr{C}$ and all G-equivariant complementary pairs $(Z,\mathcal{Y})$ on X the induced morphism of cofibers $\mathrm{yo}(Z,Z\cap\mathcal{Y})\to\mathrm{yo}(X,\mathcal{Y})$,
- for all flasque spaces X in $\mathscr{C}$ the morphism $\emptyset_{\mathbf{PSh}}\to\mathrm{yo}(X)$,
- for all spaces X in $\mathscr{C}$ the canonical morphism $\mathrm{colim}_{U\in\mathcal{C}_X^G}\mathrm{yo}(X_U)\to\mathrm{yo}(X)$.

Then E fulfills the sheaf conditions in Definition 2.4.1. Indeed, Condition (1) follows because X and Y are a G-equivariant coarsely excisive pair on $X\amalg Y$ by Example 2.3.8 and thus $E(X\amalg Y)\simeq E(X)\oplus E(Y)$ by combining the dual arguments in the proofs of Proposition 2.3.10 and Lemma 2.1.20. Condition (2) follows because E vanishes on flasque spaces and every space X with $X_b=\emptyset$ is flasque by Example 2.1.9.
In particular, $\mathbf{Sh}^{\mathrm{mot}}_{\mathbf{Sp}}(\mathscr{C})$ is equivalent to the full subcategory of $\mathbf{PSh}_{\mathbf{Sp}}(\mathscr{C})$ spanned by those presheaves that are local w.r.t. the list of morphisms above. ♦

Theorem 2.4.7. *We have equivalences of ∞-categories*
$$\mathbf{Sh}^{\mathrm{mot}}_{\mathbf{Sp}}(\boldsymbol{G}\mathbf{BornCoarse})\simeq \boldsymbol{G}\mathbf{Sp}\mathcal{X}$$

and

$$\mathbf{Sh}^{\mathrm{mot}}_{\mathbf{Sp}}(\boldsymbol{G}\mathbf{B\widetilde{or}n Coarse}) \simeq \boldsymbol{G}\mathbf{S\tilde{p}}\mathcal{X}.$$

In particular we obtain an equivalence of ∞*-categories*

$$\boldsymbol{G}\mathbf{S\tilde{p}}\mathcal{X} \xrightarrow{\simeq} \boldsymbol{G}\mathbf{Sp}\mathcal{X}.$$

Before we prove this, we give an immediate consequence:
Let $\mathbf{C}$ be a co-complete stable ∞-category. Recall that by $\boldsymbol{G}\mathbf{H\widetilde{omol}Theo}\mathcal{X}_{\mathbf{C}}$ we denoted the ∞-category of G-equivariant generalized coarse homology theories on $\boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}$ with values in $\mathbf{C}$. The non-generalized counterpart on $\boldsymbol{G}\mathbf{BornCoarse}$ will be denoted by $\boldsymbol{G}\mathbf{HomolTheo}\mathcal{X}_{\mathbf{C}}$.

Corollary 2.4.8. *The inclusion functor* $\iota\colon \boldsymbol{G}\mathbf{BornCoarse} \to \boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}$ *induces an equivalence of* ∞*-categories*

$$\iota^*\colon \boldsymbol{G}\mathbf{H\widetilde{omol}Theo}\mathcal{X}_{\mathbf{C}} \xrightarrow{\simeq} \boldsymbol{G}\mathbf{HomolTheo}\mathcal{X}_{\mathbf{C}}.$$

Proof. Combine Theorems 2.3.4 and 2.4.7. □

Now we proof Theorem 2.4.7:

Proof. We show the claim for the generalized categories, i.e. we show the equivalence

$$\mathbf{Sh}^{\mathrm{mot}}_{\mathbf{Sp}}(\boldsymbol{G}\mathbf{B\widetilde{or}n Coarse}) \simeq \boldsymbol{G}\mathbf{S\tilde{p}}\mathcal{X}.$$

To shorten notation a bit, we let $\mathscr{C} := \boldsymbol{G}\mathbf{B\widetilde{or}nCoarse}$ and we denote by $\mathcal{S}$ the collection of morphisms listed in Remark 2.4.6.
Let $\mathcal{D}$ be any stable presentable ∞-category. By [Lur09, Prop 5.5.4.20] and the last sentence in Remark 2.4.6 we have an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{colim}}(\mathbf{Sh}^{\mathrm{mot}}_{\mathbf{Sp}}(\mathscr{C}), \mathcal{D}) \simeq \mathrm{Fun}^{\chi}(\mathbf{PSh}_{\mathbf{Sp}}(\mathscr{C}), \mathcal{D})$$

where by the latter we mean the full subcategory of $\mathrm{Fun}^{\mathrm{colim}}(\mathbf{PSh}_{\mathbf{Sp}}(\mathscr{C}), \mathcal{D})$ spanned by those functors F which send the morphisms in $\mathcal{S}$ to equivalences in $\mathcal{D}$.
Now, by [Lur17, Remark 1.4.2.9] we have for any ∞-category $\mathcal{G}$ an equivalence between the category $\mathrm{Fun}(\mathcal{G}, \mathbf{Sp}(\mathbf{Spc}))$ of functors from $\mathcal{G}$ to spectrum objects in $\mathbf{Spc}$ (i.e. spectra) and the spectrum objects $\mathbf{Sp}(\mathrm{Fun}(\mathcal{G}, \mathbf{Spc}))$ of $\mathrm{Fun}(\mathcal{G}, \mathbf{Spc})$. In

particular, we have an equivalence $\mathbf{PSh}_{\mathbf{Sp}}(\mathscr{C}) \simeq \mathbf{Sp}(\mathbf{PSh}(\mathscr{C}))$ which induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\chi}(\mathbf{PSh}_{\mathbf{Sp}}(\mathscr{C}), \mathcal{D}) \simeq \mathrm{Fun}^{\chi}(\mathbf{Sp}(\mathbf{PSh}(\mathscr{C})), \mathcal{D}) \simeq \mathrm{Fun}^{\chi}(\mathbf{PSh}(\mathscr{C}), \mathcal{D}),$$

where the latter equivalence is due to [Lur17, Cor 1.4.4.5]
Now, [Lur09, Thm 5.1.5.6] gives an equivalence between colimit-preserving functors $\mathrm{Fun}^{\mathrm{colim}}(\mathbf{PSh}(\mathscr{C}), \mathcal{D})$ and the functor category $\mathrm{Fun}(\mathscr{C}, \mathcal{D})$. This equivalence restricts to an equivalence of ∞-categories

$$\mathrm{Fun}^{\chi}(\mathbf{PSh}(\mathscr{C}), \mathcal{D}) \simeq \boldsymbol{G}\widetilde{\mathbf{HomolTheo}}\mathcal{X}_{\mathcal{D}}.$$

Finally, by Theorem 2.2.20 we have an equivalence of ∞-categories

$$\boldsymbol{G}\widetilde{\mathbf{HomolTheo}}\mathcal{X}_{\mathcal{D}} \simeq \mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{S}\tilde{\mathbf{p}}\mathcal{X}, \mathcal{D}).$$

All together we have shown that we have an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\mathbf{Sh}^{\mathrm{mot}}_{\mathbf{Sp}}(\mathscr{C}), \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{S}\tilde{\mathbf{p}}\mathcal{X}, \mathcal{D})$$

proving the claim. The equivalence

$$\mathbf{Sh}^{\mathrm{mot}}_{\mathbf{Sp}}(\boldsymbol{G}\mathbf{BornCoarse}) \simeq \boldsymbol{G}\mathbf{Sp}\mathcal{X}$$

follows completely analoguous, just replace the reference to Theorem 2.2.20 by [BEKW, Cor 4.10]. $\square$

Chapter 3

Equivariant bornological topological spaces

This rather short chapter is kind of preliminary in nature. In chapter 4 we will talk about local homology theories which were inspired by locally finite homology (Borel-Moore homology). To axiomatize locally finite homology theories we need a notion of topology and a notion of boundedness, hence it is only natural to define socalled bornological topological spaces, which are topological spaces equipped with a compatible bornology (which plays the role of compact subsets in "classical" locally finite theory). The idea for this is due to [BE20], however in this chapter we are going to define a slightly more general notion: equivariant bornological topological spaces, where a group acts on the spaces by automorphisms. This raises slightly more technical problems than in the generalization of bornological coarse spaces to the equivariant notion we have seen in chapter 1.

The basic idea to add a topology to coarse spaces is quite a bit older, since there has been introduced coarse spaces with a "compatible" topology (or the other way round) e.g. in [Mit01]. Interestingly, in the functional analysis book [NB11] which deals with a quite different kind of mathematical language there is also a mention of bornology compatible with a given topology.

In the first section of this chapter we define the category of equivariant bornological topological spaces, we give several examples and show connection to the up mentioned notion of coarse spaces together with a topology. Moreover, we study some categorical properties. Not surprisingly the category $\boldsymbol{G}\mathbf{TopBorn}$ of equivariant bornological topological spaces is neither complete nor co-complete which motivated section 3.2, where we generalize this category in the same spirit as in chapter 1. We obtain a bi-complete category $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$ and we infer existence of certain (co-)limits in $\boldsymbol{G}\mathbf{TopBorn}$. We also elaborate an illustrating example why we do not think that – in contrast to the coarse situation in chapter 2 – equivariant local homology theories (defined in the subsequent chapter) on $\boldsymbol{G}\mathbf{TopBorn}$ and on $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$ are equivalent.

Like in the previous chapters G denoted a (not necessarily finite) group throughout.

3.1 Equivariant bornological topological spaces

In e.g. the works [Mit01; MNS20] the authors declare a given topology compatible with a coarse structure, if every entourage is contained in an open entourage and if the closure of a bounded set is compact, where "bounded" does not refer to members of a bornology but to U-bounded subsets (for some entourage U), see Definition 1.1.15.
In [BE20] the authors define a compatibility-condition of a topology and a bornology which results in the category **TopBorn** of bornological topological spaces, which will be generalized in this section to the equivariant world. However, every coarse structure gives rise to the minimal bornology which is compatible with given coarse structure. If we start from the classical notion of a coarse space together with a compatible topology as mentioned above and we view this space as an object in **BornCoarse** with the minimal compatible bornology, then the compatibility-condition introduced by [BE20] is implied by compatibility of coarse structure and topology (c.f. Lemma 3.1.3).
Independently, in the functional analysis book [NB11] the authors work with bornologies as a categorical way of treating the concept of boundedness. As an exercise they declare a bornology on a given topological space "proper" if it fulfills axioms which are obviously equivalent to those stated for compatibility in [BE20] (see Remark 3.1.6 for details).

In this section we adapt the notion of bornological topological spaces from [BE20] and we generalize these to the equivariant setting where a group G acts on bornological topological spaces by automorphisms. We obtain the category ***G*TopBorn** of G-equivariant bornological topological spaces. We finish this section with some basic examples and properties.

In this section we fix a set X.

Definition 3.1.1 ([BE20, Def. 7.1])**.** Consider a topology $\mathcal{O}_X$ and a bornology $\mathcal{B}_X$ on X. We say that these two structures are *compatible* if $\mathcal{B}_X$ is closed under taking topological closure and if $\mathcal{O}_X \cap \mathcal{B}_X$ is cofinal in $\mathcal{B}_X$.

Remark 3.1.2. The compatibility condition means in other words that the closure of every bounded subset is again bounded, and if every bounded subset has a bounded open neighborhood. ♦

Lemma 3.1.3. *Consider a topological space $(X, \mathcal{O}_X)$ equipped with a coarse structure $\mathcal{C}_X$ such that topology and coarse structure are compatible in the sense of [MNS20, Def 1.2], i.e. every entourage is contained in an open entourage and the closure of any U-bounded subset (for any U in $\mathcal{C}_X$) is compact. Let $\mathcal{B}_X$ denote the minimal $\mathcal{C}_X$-compatible bornology on X, i.e. the set of all U-bounded subsets for some U in $\mathcal{C}_X$ (c.f. Example 1.1.27). Then $\mathcal{O}_X$ and $\mathcal{B}_X$ are compatible.*

Proof. For any bounded subset B in $\mathcal{B}_X$ we have an entourage U in $\mathcal{C}_X$ such that $B \times B \subseteq U$. By compatibility, there exists an open entourage V in $\mathcal{C}_X$ containing U. Without loss of generality we can assume that U (and hence V) contains the diagonal Δ_X. This gives $B \subseteq V[B]$ and also we have $V[B] \in \mathcal{B}_X$ by Lemma 1.1.17. We claim that $V[B]$ is open in X, hence we have constructed a bounded open neighborhood of B which is one of the compatibility-axioms for $\mathcal{B}_X$ and $\mathcal{O}_X$. To see the openness of $V[B]$ we first note that

$$V[B] = \bigcup_{b \in B} V[\{b\}]$$

hence it suffices to see that $V[\{b\}]$ is open for any b in B. For any x in $V[B]$ we know $(x, b) \in V$ and since this set is open in $X \times X$ there exist open subsets S_1, S_2 of X such that $(x, b) \in S_1 \times S_2$ and $S_1 \times S_2 \subseteq V$. This implies that S_1 is an open neighborhood of x inside $V[\{b\}]$, hence $V[\{b\}]$ is open.
It is left to see that for all B in $\mathcal{B}_X$ the closure $\overline{B}$ is in $\mathcal{B}_X$. By compatibility of $\mathcal{O}_X$ and $\mathcal{C}_X$ we know that $\overline{B}$ is compact. Furthermore, for any b in $\overline{B}$ the subset $\{b\}$ is bounded, hence – as we just showed – there exists a bounded open neighborhood U_b of b. Since $\overline{B}$ is compact the open cover

$$\overline{B} = \bigcup_{b \in \overline{B}} U_b$$

has a finite subcover which is bounded as finite union of bounded subsets. □

Definition 3.1.4. A *bornological topological space* is a set X equipped with a topology and a compatible bornology. A map $f : X \to Y$ between bornological topological spaces is called a *morphism* if it is continuous and proper[1]. The category of bornological topological spaces is denoted by **TopBorn**.

[1]in the bornological sense

Example 3.1.5 ([BE20, Ex 7.5]). Let X be a locally compact Hausdorff space. The collection of relatively compact subsets forms a compatible bornology on X (proven in greater generality in Example 3.1.11), hence we can view a locally compact Hausdorff space as a bornological topological space. A continuous map between locally compact Hausdorff spaces becomes a morphism in **TopBorn** if it is proper in the topological sense. ✠

Remark 3.1.6. Consider a topological space X equipped with a bornology $\mathcal{B}_X$. The authors in [NB11, p. 174] call the bornology *closed* if $\mathcal{B}_X = \mathcal{B}\left\langle\left\{\overline{B} \;\middle|\; B \in \mathcal{B}_X\right\}\right\rangle$ and they call it *open* if $\mathcal{B}_X = \mathcal{B}\left\langle\left\{\mathrm{int}(B) \;\middle|\; B \in \mathcal{B}_X\right\}\right\rangle$. Finally, they call $\mathcal{B}_X$ *proper* if it is both open and closed. It is very easy to see that $\mathcal{B}_X$ is proper if and only if it is compatible with $\mathcal{O}_X$ in the sense of Definition 3.1.1. ♦

Proposition 3.1.7 ([NB11, (6.203) on page 174]). *Let X be a bornological topological space. Assume there exists a pseudometric*[2] *which induces the topology $\mathcal{O}_X$ on X and assume further that $\mathcal{B}_X$ has a countable base (i.e. a countable subset $\mathcal{S}$ of $\mathcal{B}_X$ such that every bounded subset B in $\mathcal{B}_X$ is contained in some member of $\mathcal{S}$). Then there exists a pseudometric on X (possibly different from the above) which induces the topology $\mathcal{O}_X$ on X and the bornology $\mathcal{B}_X$ (i.e. $\mathcal{B}_X$ consists of the pseudometrically bounded subsets of X).*

We proceed to the main content of this section: The generalization of bornological topological spaces to the G-equivariant setting.

Definition 3.1.8. A *G-equivariant bornological topological space* is a space X in **TopBorn** such that G acts on X by automorphisms. The category of G-equivariant bornological topological spaces together with G-equivariant continuous proper maps is denoted by ***G*TopBorn**.

Lemma 3.1.9. *Consider a topological space X. Let $\mathcal{M}$ be a subset of $\mathcal{P}(X)$ such that $X = \bigcup_{M \in \mathcal{M}} M$. Assume that for all B in $\mathcal{M}$ we have $\overline{B} \in \mathcal{B}\langle\mathcal{M}\rangle$ and we have an open neighborhood U of B that is an element in $\mathcal{B}\langle\mathcal{M}\rangle$ then $\mathcal{B}\langle\mathcal{M}\rangle$ and $\mathcal{O}_X$ are compatible.*

Proof. Both compatibility conditions are stable under taking subsets. Also, since both $\mathcal{B}\langle\mathcal{M}\rangle$ and $\mathcal{O}_X$ are closed under finite unions and taking the closure commutes with finite unions, the claim follows. □

[2]a pseudometric d on X does not imply $x = y$ if $d(x, y) = 0$

Example 3.1.10. Let X be in $\boldsymbol{G}\mathbf{TopBorn}$. Then $\mathcal{B}_X$ contains all relatively compact subsets of X. Indeed, consider a relatively compact subset B of X. By axioms, for all b in $\overline{B}$ we have $\{b\} \in \mathcal{B}_X$ hence there exists by compatibility an open bounded subset U_b in $\mathcal{B}_X \cap \mathcal{O}_X$ with $b \in U_b$. Since B is relatively compact, the open covering $\overline{B} \subseteq \bigcup_{b\in\overline{B}} U_b$ has a finite subcovering $\overline{B} \subseteq \bigcup_{b\in I} U_b$ for some finite subset I of $\overline{B}$. Now, since all U_b are bounded, also the *finite* union $U := \bigcup_{b\in I} U_b$ is bounded and clearly $B \subseteq \overline{B} \subseteq U$, hence $B \in \mathcal{B}_X$ as claimed. ✠

Not only are all relatively compact subsets contained in any compatible bornology, under mild conditions (locally compact) the set of relatively compact subsets forms a compatible bornology on X and is hence the minimal compatible bornology:

Example 3.1.11. Consider a locally compact topological space X with a G-action by automorphisms. Then the family of relatively compact subsets forms a bornology on X: Indeed, every finite subset of X is compact, hence bounded. Furthermore, subsets and finite unions of relatively compact are relatively compact, hence the family of relatively compact subsets form a bornology. This bornology is always compatible with $\mathcal{O}_X$: It is clear that it is closed under taking closures, so the only thing to verify is that each relatively compact subset has a relatively compact open neighborhood. To see this we argue similarly as in the previous example: Consider a relatively compact subset R of X. Then, since X is locally compact for every x in $\overline{R}$ there exist open subsets U_x and compact subsets K_x of X with $x \in U_x \subseteq K_x$. Since $\overline{R}$ is compact, there exists a finite subset $I \subseteq \overline{R}$ such that $R \subseteq \overline{R} \subseteq \bigcup_{x\in I} U_x =: U$. Now U is an open neighborhood of R and it is relatively compact since it is contained in the finite union $\bigcup_{x\in I} K_x$ of compact (and closed) subsets.
In particular, we can view each locally compact space with G-action by automorphisms as an object in $\boldsymbol{G}\mathbf{TopBorn}$. ✠

Example 3.1.12. Let (X, d) be a metric space and let G act on X by isometries. The metric induces a bornology $\mathcal{B}_\mathrm{d}$ as well as a topology $\mathcal{O}_\mathrm{d}$ on X. In this case $\mathcal{B}_\mathrm{d}$ and $\mathcal{O}_\mathrm{d}$ are compatible and the resulting G-equivariant bornological topological space is denoted by X_d. The same notation is used for the bornological coarse space obtained from the metric, but it will always be clear from the context which case applies. ✠

Remark 3.1.13. The topological space $\mathbb{R}^n$ (for any positive integer n) with the standard-topology is locally compact, hence carries a bornological topological structure by Example 3.1.11. But as a metric space it also inherits a bornological topological structure by Example 3.1.12. These two structures coincide in this case by the Heine-Borel-theorem. ♦

Example 3.1.14. For any G-set X the discrete topology and the maximal bornology are compatible. Equipping X with these structures, the resulting bornological topological space is denoted by X_{dM}. The assignment $X \mapsto X_{dM}$ is obviously functorial and in fact this functor is part of an adjunction

$$(-)_{dM} : \boldsymbol{G}\mathbf{Set} \rightleftarrows \boldsymbol{G}\mathbf{TopBorn} : V$$

where V denotes the forgetful functor. ✠

Example 3.1.15. The category $\boldsymbol{G}\mathbf{TopBorn}$ has no final object. Indeed, if there was a final object T in $\boldsymbol{G}\mathbf{TopBorn}$ then the forgetful $V : \boldsymbol{G}\mathbf{TopBorn} \to \boldsymbol{G}\mathbf{Set}$ would preserve it since it is a right adjoint (see Example 3.1.14). Therefore, the underlying set of T would be the singleton. The only bornological topological structure possible on a singleton is the maximal one. Thus, if T is final then every object in $\boldsymbol{G}\mathbf{TopBorn}$ is bounded (as pre-image of the bounded space T) which is a contradiction since for example the metric space $\mathbb{R}$ can be seen as a bornological topological space which is not bounded. ✠

The final object would be the empty product in $\boldsymbol{G}\mathbf{TopBorn}$. We will see later (Corollary 3.2.7) that in fact $\boldsymbol{G}\mathbf{TopBorn}$ has all other (i.e. non-empty) limits.

Example 3.1.16. Let X be a G-equivariant bornological topological space and consider a G-set Y together with a G-equivariant map $f : Y \to X$. We can endow Y with a bornology and topology as follows:

$$\mathcal{B}_Y := f^*\mathcal{B}_X = \mathcal{B}\left\langle \left\{ f^{-1}(B) \;\middle|\; B \in \mathcal{B}_X \right\} \right\rangle, \qquad \mathcal{O}_Y := f^*\mathcal{O}_X = \left\{ f^{-1}(U) \;\middle|\; U \in \mathcal{O}_X \right\}.$$

These two structures turn Y into a G-equivariant bornological topological space anf f into a morphism as we show in the following:
It is clear that $\mathcal{B}_Y$ and $\mathcal{O}_Y$ define a bornology, topology respectively in such a way that f becomes proper, continuous repectively. So the only thing to verify is that they are compatible. By Lemma 3.1.9 we can check that on generators, so consider a subset $f^{-1}(B)$ for some B in $\mathcal{B}_X$. By compatibility there exists an open bounded neighborhood U of B in $\mathcal{O}_X \cap \mathcal{B}_X$. By continuity of f the subset $f^{-1}(U)$ is an open neighborhood of $f^{-1}(B)$ and it is a generator of $\mathcal{B}_Y$ hence it is bounded. Furthermore the closure $\overline{B}$ of B in X is bounded, hence $f^{-1}\left(\overline{B}\right) \in \mathcal{B}_Y$. This subset is closed and

contains $f^{-1}(B)$, hence also

$$\overline{f^{-1}(B)} \subseteq f^{-1}\left(\overline{B}\right).$$

This shows that the closure of $f^{-1}(B)$ is bounded and we are through.
In particular we can consider every G-invariant subset of a G-equivariant bornological topological space as a subspace with these induced subspace structures. ✠

Definition 3.1.17. Let X be a topological space and $\mathcal{M}$ be a collection on subsets of X. Then by $\mathcal{B}\langle\langle\mathcal{M}\rangle\rangle$ we denote the smallest bornology on X which contains $\mathcal{M}$ and which is compatible with the topology $\mathcal{O}_X$.

Note that this is well-defined because the maximal bornology on a space is always compatible with any topology.

Definition 3.1.18. For a G-equivariant bornological topological space X we define its *G-completion* $B_G X$ as the following space in $\boldsymbol{G}$**TopBorn**: The underlying topological space equals X but the bornology is replaced by

$$\mathcal{B}_{B_G X} := \mathcal{B}\left\langle\left\langle\left\{GB \;\middle|\; B \in \mathcal{B}_X\right\}\right\rangle\right\rangle.$$

We say that X is *G-complete* if $X = B_G X$.

Remark 3.1.19. A space X in $\boldsymbol{G}$**TopBorn** is G-complete iff for any bounded subset B in $\mathcal{B}_X$ also its orbit GB is a bounded subset of X. ♦

Lemma 3.1.20. *If the group G is finite, then every space X in* $\boldsymbol{G}$**TopBorn** *is G-complete.*

Proof. Consider a bounded subset B in $\mathcal{B}_X$. Since G acts by automorphisms, also the subsets gB are bounded in X for all g in G. Now, since G is finite we derive that also

$$GB = \bigcup_{g \in G} gB$$

is bounded in X. In total we have seen $\left\{GB \;\middle|\; B \in \mathcal{B}_X\right\} \subseteq \mathcal{B}_X$ which implies $\mathcal{B}_{B_G X} \subseteq \mathcal{B}_X$. On the other hand, for a bounded subset B in $\mathcal{B}_X$ we have $B \subseteq GB$ and $GB \in \mathcal{B}_{B_G X}$, hence also $B \in \mathcal{B}_{B_G X}$. All together we derive $\mathcal{B}_X = \mathcal{B}_{B_G X}$. □

Lemma 3.1.21. *Consider a G-complete G-equivariant bornological topological space X and the canonical projection $\pi\colon X \twoheadrightarrow X/G$ to the set of cosets. We endow X/G with the quotient topology and the bornology*

$$\mathcal{B}_{X/G} = \left\{ B \subseteq X/G \;\middle|\; \pi^{-1}(B) \in \mathcal{B}_X \right\}.$$

The bornology $\mathcal{B}_{X/G}$ is compatible with the quotient topology and therefore we can reguard X/G as an object in **TopBorn**.
This construction extends to a functor

$$Q\colon \boldsymbol{G}\mathbf{TopBorn}^{\wedge} \longrightarrow \mathbf{TopBorn}$$

where $\boldsymbol{G}\mathbf{TopBorn}^{\wedge}$ denotes the full subcategory of $\boldsymbol{G}\mathbf{TopBorn}$ consisting of G-complete spaces.

Proof. For a subset S of X we see that $\pi^{-1}(\pi(S)) = GS$. Consider an open subset U of X. Since G acts by automorphisms, $GU = \bigcup_{g\in G} gU$ is also open in X. In particular the projection map π is open. From this we can derive (from Lemma 3.1.22) that for any subset T of X/G we have

$$\pi^{-1}\left(\overline{T}\right) = \overline{\pi^{-1}(T)}. \tag{*}$$

First we want to show that $\mathcal{B}_{X/G}$ is indeed a bornology. For this we notice that any coset $[x]$ in X/G is contained in $\mathcal{B}_{B_G X}$ because $\pi^{-1}([x]) = G\{x\}$ and the latter set is bounded in X because X was G-complete (and $\{x\} \in \mathcal{B}_X$). The rest of the axioms for a bornology are clearly fulfilled since the pre-image operation commutes with finite unions and the subset-relation.
Next, to see compatibility of $\mathcal{B}_{B_G X}$ and $\mathcal{O}_{B_G X}$ we consider a bounded subset B of X/G. Then by definition $\pi^{-1}(B)$ is bounded in X, hence by compatibility there exists an open bounded neighborhood U of $\pi^{-1}(B)$ in X. Since π is an open map the image $\pi(U)$ is open in X/G. We claim that this is a bounded open neighborhood of B in X/G. By surjectivity of π we have $B = \pi(\pi^{-1}(B)) \subseteq \pi(U)$ and by definition $\pi(U)$ is bounded iff $\pi^{-1}(\pi(U)) = GU$ is bounded in X, but X is G-complete and U is bounded in X. Hence B has a bounded open neighborhood in X/G. It remains to see that the closure $\overline{B}$ is bounded as well. For this we must see that $\pi^{-1}(\overline{B})$ is bounded in X, but using (*) we have

$$\pi^{-1}\left(\overline{B}\right) = \overline{\pi^{-1}(B)}.$$

The latter set is the closure of the (by assumption) bounded subset $\pi^{-1}(B)$ hence is bounded itself by compatibility.
Functoriality of the assignment $X \mapsto X/G$ follows immediately from the G-equivariance of the morphisms in ***G*TopBorn** and the commutativity of

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_X"'] & Y \arrow[d, "\pi_Y"] \\ X/G \arrow[r, "{[x]\mapsto f(x)}"'] & Y/G. \end{tikzcd}$$

□

In the proof above we needed this short technical lemma:

Lemma 3.1.22. *Let $f\colon X \to Y$ be a quotient map of topological spaces. If f is open, then for any subset S of Y we have*

$$\overline{f^{-1}(S)} = f^{-1}\left(\overline{S}\right).$$

Proof. For a subset S of Y we denote its complement by $S^{\complement}$. Now we calculate

$$\begin{aligned} f^{-1}\left(\overline{S}\right) &= f^{-1}\left(\bigcap_{\substack{S\subseteq A\\ A\text{ closed}}} A\right) = \bigcap_{\substack{S\subseteq A\\ A\text{ closed}}} f^{-1}(A) = \bigcap_{\substack{S\subseteq U^{\complement}\\ U\text{ open}}} f^{-1}\left(U^{\complement}\right) \\ &= \bigcap_{\substack{S\subseteq U^{\complement}\\ U\text{ open}}} f^{-1}(U)^{\complement} = \left(\bigcup_{\substack{S\subseteq U^{\complement}\\ U\text{ open}}} f^{-1}(U)\right)^{\complement} \overset{(*)}{=} \left(\bigcup_{\substack{f^{-1}(S)\subseteq V^{\complement}\\ V\text{ open}}} V\right)^{\complement} \\ &= \bigcap_{\substack{f^{-1}(S)\subseteq V^{\complement}\\ V\text{ open}}} V^{\complement} = \bigcap_{\substack{f^{-1}(S)\subseteq A\\ A\text{ closed}}} A = \overline{f^{-1}(S)}. \end{aligned}$$

The key-step is marked with (*) and we need to verify that step: First, consider an open subset U of Y such that $S \subseteq U^{\complement}$. Then $V := f^{-1}(U)$ is an open subset of X and of coarse $f^{-1}(S) \subseteq f^{-1}\left(U^{\complement}\right) = f^{-1}(U)^{\complement} = V^{\complement}$. Thus every member of the left-hand-side union of (*) is also a member of the right-hand-side union of (*). On the other hand, consider an open subset V of X such that $f^{-1}(S) \subseteq V^{\complement}$. Since f is an open map, the subset $U := f(V)$ is open in Y and $V \subseteq f^{-1}(U)$. If we check $S \subseteq U^{\complement}$ then every member of the right-hand-side union of (*) is a subset of a member of the left-hand-side union of (*) and we are done. So first notice that $f^{-1}(S) \subseteq V^{\complement}$ is equivalent to $V \subseteq f^{-1}(S)^{\complement}$. And now

$$U = f(V) \subseteq f\left(f^{-1}(S)^{\complement}\right) = f\left(f^{-1}\left(S^{\complement}\right)\right) \subseteq S^{\complement}$$

which again is equivalent to $S \subseteq U^{\complement}$. □

Lemma 3.1.23. *The functor Q in Lemma 3.1.21 is part of an adjunction*

$$Q\colon \boldsymbol{G}\mathbf{TopBorn}^{\wedge} \rightleftarrows \mathbf{TopBorn} : \mathrm{incl}.$$

where the right adjoint is the fully faithful inclusion which equipps a bornological topological space with the trivial G-action. The resulting space is then clearly G-complete.

Proof. Any morphism $f\colon X \to Y$ in $\boldsymbol{G}\mathbf{TopBorn}$ where G acts trivially on the target Y is necessarily constant when restricted to an orbit (this follows from G-equivariance). Hence we easily check that for all X in $\boldsymbol{G}\mathbf{TopBorn}$ and all Y in $\mathbf{TopBorn}$ the following maps are well-defined and mutually inverse bijections:

$$\begin{aligned} \mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(X, \mathrm{incl}(Y)) &\longrightarrow \mathrm{Hom}_{\mathbf{TopBorn}}(X/G, Y) \\ f &\longmapsto \big([x] \mapsto f(x)\big) \\ g \circ \pi &\longleftarrow\!\shortmid\ g \end{aligned}$$

where $\pi\colon X \to X/G$ is the canonical projection. □

Lemma 3.1.24. *Consider two G-equivariant bornological topological spaces X and Y. Then we can define their* tensor product *$X \otimes Y$ as follows: The underlying topological space is the product space $X \times Y$ and the bornology is generated by subsets of the form $B \times B'$ where $B \in \mathcal{B}_X$ and $B' \in \mathcal{B}_Y$. This construction gives rise to a symmetric monodial structure*

$$(-) \otimes (-)\colon \boldsymbol{G}\mathbf{TopBorn} \times \boldsymbol{G}\mathbf{TopBorn} \longrightarrow \boldsymbol{G}\mathbf{TopBorn}$$

with tensor unit the one-point-space.

We finish this section with another source for examples of bornological topological spaces. The authors in [BE20] introduced the notion of uniform bornological coarse spaces (later generalized to equivariant version in [BEKW]). These spaces give rise to equivariant bornological topological spaces:

Definition 3.1.25. A *uniform structure* $\mathcal{U}_X$ on a bornological coarse space is a subset of $\mathcal{P}(X \times X)$ which is closed under taking finite intersections, inverses, compositions and supersets such that every member U of $\mathcal{U}_X$ contains the diagonal Δ_X and admits another member V in $\mathcal{U}_X$ such that $V \circ V \subseteq U$. A morphism $f\colon X \to Y$ between bornological coarse spaces both endowed with a uniform structure is called *uniformly continuous* if $f^{-1}(\mathcal{U}_Y) \subseteq \mathcal{U}_X$. The uniform structure $\mathcal{U}_X$ is called *compatible* with the coarse structure $\mathcal{C}_X$ if the intersection $\mathcal{C}_X \cap \mathcal{U}_X$ is non-empty.

A G-equivariant uniform bornological coarse space is a space X in $\boldsymbol{G}\mathbf{BornCoarse}$ together with a compatible uniform structure $\mathcal{U}_X$ such that $\mathcal{U}_X^G$ is cofinal in $\mathcal{U}_X$ (in the sense that every member in $\mathcal{U}_X$ contains a member of $\mathcal{U}_X^G$) and such that the G-action on X is uniformly continuous, i.e. G acts via automorphisms on X. The category of G-equivariant uniform bornological coarse spaces is denoted by $\boldsymbol{G}\mathbf{UBC}$.

For X in $\boldsymbol{G}\mathbf{UBC}$ the uniform structure on X defines a topology $\mathcal{O}_X$ on X as follows: A subset U of X is open iff for all x in U there exists a member V in $\mathcal{U}_X$ such that $x \in V$ and $V \subseteq U$.

Remark 3.1.26. Consider a space X in $\boldsymbol{G}\mathbf{UBC}$. If $\bigcap_{U \in \mathcal{U}_X} U = \Delta_X$ then the induced topology on X is Hausdorff: Indeed, for two distinct points x, y in X there exists a symmetric U in $\mathcal{U}_X$ with $(x, y) \notin U$ (otherwise $(x, y) \in \bigcap_U U = \Delta_X$). By axioms there exists a V in $\mathcal{U}_X$ (w.l.o.g. symmetric) such that $V \circ V \subseteq U$. Now the open neighborhoods $V[\{x\}]$ of x and $V[\{y\}]$ of y are disjoint, because assume $U[\{x\}] \cap U[\{y\}]$ contains an element z, i.e. $(z, x) \in V$ and (z, y) in V then $(x, y) \in V \circ V \subseteq U$, a contradiction. ♦

Lemma 3.1.27. *We have a functor* $\boldsymbol{G}\mathbf{UBC} \to \boldsymbol{G}\mathbf{TopBorn}$ *which equipps a space* X *in* $\boldsymbol{G}\mathbf{UBC}$ *with the induced topology and forgets the coarse structure.*

Proof. Consider a morphism $f\colon X \to Y$ in $\boldsymbol{G}\mathbf{UBC}$ and let M be an open subset of Y. We show that $f^{-1}(M)$ is open in X: Consider an element x in $f^{-1}(M)$. Then – since M is open in Y – there exists a member U in $\mathcal{U}_Y$ such that $f(x) \in U$ and $U \subseteq M$. By uniform continuity of f, the pre-image $V := (f \times f)^{-1}(U)$ is in $\mathcal{U}_X$ and by construction we get $V[\{x\}] \subseteq f^{-1}(M)$ (and of course $x \in V[\{x\}]$ since $\Delta_X \subseteq V$). □

3.2 Equivariant generalized bornological topological spaces

As we saw in Example 3.1.15 the category $\boldsymbol{G}\mathbf{TopBorn}$ is not complete. Actually it is also not co-complete. Thus, in this section we proceed with an variation of equivariant bornological topological spaces in the same fashion as for equivariant bornological coarse spaces in section 1.3: We replace the bornology with a generalized bornology and obtain a complete and cocomplete category $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$ containing $\boldsymbol{G}\mathbf{TopBorn}$ as full subcategory.

Definition 3.2.1. A *G-equivariant generalized bornological topological space* is a G-set X equipped with a generalized bornology $\mathcal{B}_X$ and a topology $\mathcal{O}_X$ such that $\mathcal{B}_X$ and $\mathcal{O}_X$ are compatible in the same sense as in the non-generalized case (c.f. Definition 3.1.1). The morphisms between G-equivariant generalized bornological topological spaces are G-equivariant continuous and proper maps. This category will be denoted by $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$.

Example 3.2.2. For a locally compact topological space X the collection of relatively quasi compact subsets forms a generalized bornology, which is an ordinary bornology if X is Hausdorff (c.f. Example 3.1.5). ✠

Example 3.2.3. Consider a G-set X. The trivial generalized bornology is obviously compatible with every topology on X. In particular with the discrete topology $\mathcal{P}(X)$ and the indiscrete topology $\{\emptyset, X\}$. Both combinations define spaces in $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$ denoted by $X_{d,\emptyset}$ and $X_{triv,\emptyset}$ respectively. Likewise, the maximal bornology is compatible with every topology and we obtain spaces X_{dM} (c.f. Example 3.1.14) and $X_{triv,M}$ in $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$. ✠

Lemma 3.2.4. *The assignments $X \mapsto X_{dM}$ and $X \mapsto X_{triv,\emptyset}$ extend to functors*

$$(-)_{dM},\ (-)_{triv,\emptyset} \colon \boldsymbol{G}\mathbf{Set} \longrightarrow \boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$$

which are part of adjunctions

$$(-)_{dM} \colon \boldsymbol{G}\mathbf{Set} \underset{\perp}{\rightleftarrows} \boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}} \colon U$$

and

$$U \colon \boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}} \underset{\perp}{\rightleftarrows} \boldsymbol{G}\mathbf{Set} \colon (-)_{triv,\emptyset}$$

where $U \colon \boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}} \to \boldsymbol{G}\mathbf{Set}$ is the forgetful functor.

Proof. The assertions about $(-)_{dM}$ are shown in Example 3.1.14. For the assertions about $(-)_{triv,\emptyset}$ we only realize that every G-equivariant map $f\colon X \to Y_{triv,\emptyset}$ (for X in $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$) is proper and continuous by trivial reasons. $\square$

We proceed with the proof that the category $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$ is complete and cocomplete. In contrast to the coarse situation in section 1.3 we do not split up the proof and show that there exists (co-)products and (co-)equalizers but we construct directly every (co-)limit. For this we use that the category $\boldsymbol{G}\mathbf{Top}$ of G-equivariant topological spaces is bi-complete and that the forgetful functor $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}} \to \boldsymbol{G}\mathbf{Top}$ is left- and right-adjoint (c.f. Lemma 3.2.4).

Theorem 3.2.5. *The category* $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$ *is complete.*

Proof. Let $\mathbf{I}$ be a small category and consider a diagram $D\colon \mathbf{I} \to \boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$. By post-composing with the forgetful functor $V\colon \boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}} \to \boldsymbol{G}\mathbf{Top}$ we obtain a diagram $VD\colon \mathbf{I} \to \boldsymbol{G}\mathbf{Top}$ in the complete category $\boldsymbol{G}\mathbf{Top}$. We denote by

$$L := \lim_{\mathbf{I}} VD$$

the limit of this diagram in $\boldsymbol{G}\mathbf{Top}$. For all i in $\mathbf{I}$ we have the canonical G-equivariant continuous map $\pi_i\colon L \to VD(i)$. We define a generalized bornology on L by

$$\mathcal{B}_L := \mathcal{B}\left\langle \left\{ \pi_i^{-1}(B) \;\middle|\; i \in \mathbf{I},\ B \in \mathcal{B}_{D(i)} \right\} \right\rangle .$$

We want to see that this defines a generalized bornological topological space (i.e. that $\mathcal{O}_L$ and $\mathcal{B}_L$ are compatible) and that this resulting space represents the limit $\lim_{\mathbf{I}} D$ in $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$. The compatibility can be checked on generators of the generalized bornology (c.f. Lemma 3.1.9) hence we consider a subset $\pi_i^{-1}(B)$ for some i in $\mathbf{I}$ and some bounded subset B in $\mathcal{B}_{D(i)}$. By compatibility of $\mathcal{B}_{D(i)}$ and $\mathcal{O}_{D(i)}$ there exists a bounded open subset U of $D(i)$ with $B \subseteq U$ and we have $\overline{B} \in \mathcal{B}_{D(i)}$. But then $\pi_i^{-1}(U)$ is bounded in L (it is even a generator of $\mathcal{B}_L$) as well as an open neighborhood of $\pi_i^{-1}(B)$. Moreover, by continuity of π_i we have

$$\overline{\pi_i^{-1}(B)} \subseteq \pi_i^{-1}\left(\overline{B}\right)$$

where the latter supset is a generator of $\mathcal{B}_L$. Hence the closure of $\pi_i^{-1}(B)$ is in $\mathcal{B}_L$ as well which shows compatibility as claimed.
To see that L represents the limit of D in $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$ we first note that the canonical

maps $\pi_i\colon L \to D(i)$ are proper by construction of $\mathcal{B}_L$. Furthermore we consider a test object T in $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$ together with a compatible system of morphisms $t_i\colon T \to D(i)$. Since L is the limit of VD in $\boldsymbol{G}\mathbf{Top}$ there exists a unique G-equivariant continuous map $t\colon T \to L$ such that $\pi_i \circ t = t_i$. Hence we only need to verify that this map t is proper: Consider a generator $\pi_i^{-1}(B)$ of $\mathcal{B}_L$ for some i in $\mathbf{I}$ and B in $\mathcal{B}_{D(i)}$. By commutativity we have $t^{-1}\left(\pi_i^{-1}(B)\right) = t_i^{-1}(B)$ which is bounded by properness of t_i. □

The co-completeness is not so easy to show. The obvious candidate for the bornology on the colimit is not compatible in general. The author thanks Ulrich Bunke for the useful hint how to generate a compatible bornology.

Theorem 3.2.6. *The category* $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$ *is co-complete.*

Proof. Similarly as in the proof of Theorem 3.2.5 we consider a small index category $\mathbf{I}$ and a diagram $D\colon \mathbf{I} \to \boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$. We denote by

$$C := \operatorname*{colim}_{\mathbf{I}} VD$$

the colimit of the postcomposition of D with the forgetful functor

$$V\colon \boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}} \to \boldsymbol{G}\mathbf{Top}$$

in $\boldsymbol{G}\mathbf{Top}$. This colimit comes with canonical G-equivariant continuous maps

$$\iota_i\colon D(i) \to C.$$

Now we define a generalized bornology on C that is compatible with $\mathcal{O}_C$ and afterwards we verify that the resulting G-equivariant generalized bornological topological space represents the colimit of D.
Consider the following generalized bornology on C:

$$\mathcal{M} := \left\{ B \subseteq C \;\middle|\; \forall i \in \mathbf{I} :\ \iota_i^{-1}(B) \in \mathcal{B}_{D(i)} \right\}.$$

This is indeed a generalized bornology which can be easily seen. However, in general it is not compatible with $\mathcal{O}_C$. It is clear that $\mathcal{M}$ is the largest generalized bornology such that all structure maps ι_i become morphisms in $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$. Hence we define a suitable sub-bornology of $\mathcal{M}$ in the following way:

$$\mathcal{B}_C := \left\{ B \subseteq C \;\middle|\; \exists (B_n)_{n\in\mathbb{N}} \text{ s.t. } \begin{array}{ll} \overline{B_n} \subseteq B_{n+1} & \forall n \in \mathbb{N} \\ B_n \in \mathcal{O}_C \cap \mathcal{M} & \forall n \in \mathbb{N} \\ B \subseteq B_0 \subseteq B_1 \subseteq \cdots \subseteq C & \end{array} \right\}.$$

Claim 1: The set $\mathcal{B}_C$ is a generalized bornology. Assume $B \in \mathcal{B}_C$ witnessed by a sequence of spaces $(B_n)_{n\in\mathbb{N}}$. Then for every subset B' of B the very same sequence of spaces witnesses that $B' \in \mathcal{B}_C$. Hence $\mathcal{B}_C$ is closed under taking subsets. To see that it is also closed under taking finite unions, we consider elements $B^1, \ldots, B^k$ in $\mathcal{B}_C$ where the membership of B^j is witnessed by a sequence $(B^j_n)_{n\in\mathbb{N}}$ for all j in $\{1, \ldots, k\}$. Then for the union $B := \bigcup_{j=1}^k B^j$ we define a sequence $(B_n)_{n\in\mathbb{N}}$ by $B_n := \bigcup_{j=1}^k B^j_n$ for all n in $\mathbb{N}$. Since taking closures commutes with finite unions, it is clear that $\overline{B}_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$. Also, since both $\mathcal{O}_C$ and $\mathcal{M}$ are closed under taking finite unions, each B_n is contained in $\mathcal{O}_C \cap \mathcal{M}$. Finally, by construction one easily sees $B \subseteq B_0 \subseteq B_1 \subseteq \ldots$. All together this shows that the sequence $(B_n)_{n\in\mathbb{N}}$ witnesses that $B \in \mathcal{B}_C$. $\diamond_{\text{Claim 1}}$.

Claim 2: $\mathcal{B}_C$ and $\mathcal{O}_C$ are compatible. Consider an element B in $\mathcal{B}_C$ witnessed by the sequence $(B_n)_{n\in\mathbb{N}}$. We note that then the space B_1 is also in $\mathcal{B}_C$ witnessed by the shifted sequence $(B_{n+2})_{n\in\mathbb{N}}$. By construction B_1 is an open neighborhood of B and it is bounded as we just verified. Moreover, we have $\overline{B} \subseteq \overline{B_0} \subseteq B_1$ and since B_1 is bounded, also $\overline{B}$ is bounded. $\diamond_{\text{Claim 2}}$.

Claim 3: For all i in $\mathbf{I}$ the structure maps $\iota_i\colon D(i) \to C$ are morphisms in the category $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$. By the construction of $\mathcal{B}_C$ and the fact that the generalized bornology $\mathcal{M}$ is closed under taking subsets, it is clear that $\mathcal{B}_C \subseteq \mathcal{M}$. Hence the claim. $\diamond_{\text{Claim 3}}$.

Claim 4: The G-equivariant generalized bornological topological space C fulfills the universal property of the colimit of D. Consider a test object T together with a compatible system of morphisms $t_i\colon T \to D(i)$ for all i in $\mathbf{I}$. Since C is the colimit of VD in $\boldsymbol{G}\mathbf{Top}$ we get a unique morphism $t\colon C \to T$ such that $t\iota_i = t_i$ for all i in $\mathbf{I}$. The last thing left to show is that this map t is proper. For this we consider a bounded subset B in $\mathcal{B}_T$. Inductively we define a sequence of open bounded subspaces of T as follows:

By compatibility of $\mathcal{B}_T$ and $\mathcal{O}_T$ there exists a bounded open neighborhood U_0 of B. Now assume U_n is already defined. Again by compatibility also the closure $\overline{U_n}$ of U_n is bounded, hence there exists an open bounded neighborhood U_{n+1} of $\overline{U_n}$.

We wanted to see that $t^{-1}(B)$ is in $\mathcal{B}_C$ and we claim that this is witnessed by the sequence $(B_n)_{n\in\mathbb{N}}$ where we define $B_n := t^{-1}(U_n)$ for all n in $\mathbb{N}$. First, by construction we have $B \subseteq U_0 \subseteq U_1 \subseteq \dots$ which immediatelly gives $t^{-1}(B) \subseteq B_0 \subseteq B_1 \subseteq \dots$. Next, by continuity of t we have $B_n \in \mathcal{O}_T$ for all n in $\mathbb{N}$ and also we have $B_n \in \mathcal{M}$ because $\iota_i^{-1}(B_n) = t_i^{-1}(U_n)$ which is bounded in $D(i)$ since $U_i \in \mathcal{B}_T$ and t_i is proper for all i in $\mathbf{I}$. Finally, we see that by continuity of t:

$$\overline{B_n} = \overline{t^{-1}(U_n)} \subseteq t^{-1}\left(\overline{U_n}\right) \subseteq t^{-1}(U_{n+1}) = B_{n+1}. \qquad \square$$

Similarly as in the coarse case in chapter 1, we use the bi-completeness of this new category $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$ to classify all limits and colimits that exist in $\boldsymbol{G}\mathbf{TopBorn}$.

Corollary 3.2.7. *The category* $\boldsymbol{G}\mathbf{TopBorn}$ *has all non-empty limits.*

Proof. Consider a diagram $D\colon \mathbf{I} \to \boldsymbol{G}\mathbf{TopBorn}$ for some small non-empty index category $\mathbf{I}$. Post-composing with the inclusion functor $\iota\colon \boldsymbol{G}\mathbf{TopBorn} \to \boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$ and taking the limit we get an object $L := \lim_{\mathbf{I}} \iota D$ in $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$. For any[3] i in $\mathbf{I}$ we get a structure morphism $\pi_i\colon L \to \iota(D(i))$ which is proper. Since $\iota(D(i))$ is locally bounded, properness of π_i enforces L to be locally bounded, hence an object of $\boldsymbol{G}\mathbf{TopBorn}$. $\square$

Lemma 3.2.8. *The fully faithful inclusion functor*

$$\iota\colon \boldsymbol{G}\mathbf{TopBorn} \longrightarrow \boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$$

preserves all colimits and limits.

Proof. Consider a diagram $D\colon \mathbf{I} \to \boldsymbol{G}\mathbf{TopBorn}$ and assume that its limit exists. Denote the limit by $L := \lim_{\mathbf{I}} D$. We show that $\iota(L)$ fulfills the universal property of the limit $\lim_{\mathbf{I}} \iota D$: For any test object T in $\boldsymbol{G}\mathbf{Top}\widetilde{\mathbf{Born}}$ together with a compatible family of morphisms $t_i\colon T \to \iota(D(i))$ any of these morphisms[4] implies that T is locally bounded, hence can be seen as an object in $\boldsymbol{G}\mathbf{TopBorn}$. Hence the universal property of the limit L in $\boldsymbol{G}\mathbf{TopBorn}$ together with the fully faithfullness of the inclusion ι give the limit-property for $\iota(L)$. A more detailed argumentation can be found in the proof of the analoguous statement for the category of (generalized) bornological coarse spaces in Proposition 1.3.35.

[3] note that $\mathbf{I}$ is non-empty by assumption

[4] Note that there exists at least one such morphism, since for the empty index category the limit in $\boldsymbol{G}\mathbf{TopBorn}$ would not have existed (c.f. Example 3.1.15)

Now let us consider a diagram $D' \colon \mathbf{I} \to \boldsymbol{G}\mathbf{TopBorn}$ whose colimit $C := \operatorname{colim}_{\mathbf{I}} D'$ exists in $\boldsymbol{G}\mathbf{TopBorn}$. By co-completeness of $\boldsymbol{G}\widetilde{\mathbf{TopBorn}}$ also the colimit $\tilde{C} := \operatorname{colim}_{\mathbf{I}} \iota D'$ exists in $\boldsymbol{G}\widetilde{\mathbf{TopBorn}}$ and by its universal property there exists a unique morphism $\tilde{C} \to C$ which shows that $\tilde{C}$ is locally bounded since C is. We conclude $\iota(C) \cong \tilde{C}$ in $\widetilde{\mathbf{TopBorn}}$ (again, for more detail we refer to the proof of Proposition 1.3.35), hence the claim. $\square$

Corollary 3.2.9. *The colimit of a diagram $D \colon \mathbf{I} \to \boldsymbol{G}\mathbf{TopBorn}$ for some index category $\mathbf{I}$ exists in $\boldsymbol{G}\mathbf{TopBorn}$ if and only if the colimit of $\iota D \colon \mathbf{I} \to \boldsymbol{G}\widetilde{\mathbf{TopBorn}}$ in $\boldsymbol{G}\widetilde{\mathbf{TopBorn}}$ is locally bounded. In that case this locally bounded colimit represents the colimit of D.*
I.e. the colimit of D exists in $\boldsymbol{G}\mathbf{TopBorn}$ if every point of the colimit of the underlying spaces in $\boldsymbol{G}\mathbf{Top}$ has an increasing sequence of open neighborhoods $(B_n)_n$ such that the closure of B_n is contained in B_{n+1} and such that the pre-images of each B_n in each $D(i)$ is bounded.

Proof. If the colimit of D exists, then it is also the colimit of ιD by Lemma 3.2.8 and clearly it is locally bounded. On the other hand if the colimit of ιD is locally bounded, then by fully faithfullness of ι we see that it fulfills the universal property of the colimit of D in $\boldsymbol{G}\mathbf{TopBorn}$. $\square$

For example we immediatelly get:

Corollary 3.2.10. *The category $\boldsymbol{G}\mathbf{TopBorn}$ has all small coproducts.*

Remark 3.2.11. For convenience, we state explicitly the bornology of the coproduct $C := \coprod_{i \in I} X_i$ for a family of spaces $(X_i)_{i \in I}$ in $\boldsymbol{G}\mathbf{TopBorn}$: It is given by

$$\mathcal{B}_C = \left\{ B \subseteq C \,\middle|\, \forall i \in I : \; B \cap X_i \in \mathcal{B}_{X_i} \right\}. \qquad \blacklozenge$$

Up until here, the generalization of $\boldsymbol{G}\mathbf{TopBorn}$ was very similar to the one of the category $\boldsymbol{G}\mathbf{BornCoarse}$ in section 1.3. We proceeded in section 2.3 with the definition of equivariant generalized coarse motivic spectra as universal coarse homology theory and proved in section 2.4 that at this point the generalized and non-generalized story merge in the sence that the fully faithful inclusion functor defines an equivalence between generalized and non-generalized motivic spectra. We would like to have a similar result for bornological topological spaces (after defining appropriate notions of motivic spectra, section 4.3). But we do not expect that this holds in this situation. A crucial point in the coarse situation was that every generalized bornological coarse space X can be written as a coproduct $X_b \coprod X_u$ of its subspaces of bounded points and unbounded points (see Corollary 1.3.34). This is *not* true in the topological situation[5]:

Example 3.2.12. We consider the space $X := \{0\} \cup \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N}^+ \right\}$ with the subspace topology induced by $\mathbb{R}$ and endowed with the bornology consisting of all finite subsets of X that do not contain 0. Since every bounded subset of X is therefore both open and closed, it is clear that $\mathcal{B}_X$ and $\mathcal{O}_X$ are compatible. We have subspaces $X_b = X \setminus \{0\}$ and $X_u = \{0\}$ and the inclusions $X_b \hookrightarrow X$ and $X_u \hookrightarrow X$ induce a morphism $X_b \coprod X_u \to X$ but this is *not* an isomorphism because e.g. the subset $X_u = \{0\}$ is not open in X. ✠

[5] The author thanks for helpful feedback of the audiance at the SFB internal meeting in Windberg 2020

Chapter 4 Equivariant local homology theory

As mentioned in the previous chapter, the category of bornological topological spaces was originally introduced by [BE20] in order to axiomatise *locally finite homology theories*. These are functors

$$F\colon \mathbf{TopBorn} \longrightarrow \mathcal{C}$$

into a complete and co-complete stable ∞-category, which – in addition to the "usual" homotopy invariance and excision property – satisfies the *local finiteness condition*, i.e. for all spaces X in **TopBorn** we have

$$\lim_{B \in \mathcal{B}_X} F(X \setminus B) \simeq 0.$$

This notion is related to, but differs from [WW95]. To every locally finite homology theory there exists an associated coarse homology theory, its *coarsification*. Most notably there exists an assembly map between coarse K-homology and the coarsification of locally finite K-homology, which is subject to the coarse Baum-Connes conjecture. The initial motivation of [BE20] to axiomatise locally finite homology theories and coarsifications, was to generalize this situation to arbitrary natural transformations between coarse homology theories and coarsifications of their associated locally finite theory. However, constructing these natural transformations failed in general due to the fact that the local finiteness condition being a limit-condition. Most recently the same authors overcame this problem by replacing locally finite theories by local homology theories defined on the category of uniform bornological coarse spaces.

In this chapter we introduce an equivariant version of local homology theory on the category of equivariant bornological topological spaces. This notion is closely related to locally finite homology theories on **TopBorn** and local homology theories on uniform bornological coarse spaces. In section 4.1 we define this notion of local homology theory and elaborate on first properties and examples. In the subsequent sections we construct motivically the ∞-categories of equivariant local motivic spaces (important for the analogue of Elmendorf's Theorem in Chapter 5) and equivariant local motivic spectra, together with the universal equivariant local homology theory

$$\mathrm{Yo}_G \colon \boldsymbol{G}\mathbf{TopBorn} \longrightarrow \boldsymbol{G}\mathbf{Sp}\mathcal{TB}\,.$$

This whole approach is very analoguous to the constructions in Chapter 2.

There has been a recent development in proper equivariant stable homology theory. We refer to [DHLPS] as an entry point, but also refer to [BEUV; Lüc02; Phi88; San05; Dwy08; BHS10; Bár14] for further reading.

4.1 Equivariant local homology theory

In this section we define the notion of G-equivariant local homology theories in the context of G-equivariant bornological topological spaces and we construct the universal such theory in section 4.3. In principal we could define this notion also for equivariant generalized bornological topological spaces. Every single definition immediately generalizes without adjusting anything. However, in contrast to the coarse situation in chapter 2, we do not expect that we have an equivalence between equivariant local homology theories on $\boldsymbol{G}\mathbf{TopBorn}$ and on $\boldsymbol{G}\widetilde{\mathbf{TopBorn}}$ (see the comment before Example 3.2.12).

Within this section $\mathcal{C}$ always denotes a co-complete stable ∞-category and G denotes a group. Consider a G-equivariant bornological topological space X.

Definition 4.1.1. A *G-equivariant big family* on X is a filtered family $\mathcal{Y} := (Y_i)_{i\in I}$ of G-invariant subsets of X such that for each i in I there exists a j in I such that Y_j contains an open neighborhood of the closure $\overline{Y_i}$ of Y_i.

Example 4.1.2. If G acts trivially on X then the bornology $\mathcal{B}_X$ is a G-equivariant big family on X by compatibility. ✠

We use the notion of big families to define complementary pairs for excision properties:

Definition 4.1.3. A *G-equivariant complementary pair* on X is a tuple $(\mathcal{Y}, \mathcal{Z})$ of two G-equivariant big families $\mathcal{Y} = (Y_i)_{i\in I}$ and $\mathcal{Z} = (Z_j)_{j\in J}$ which *cover* X, i.e. there exist indices i in I and j in J such that $X = Y_i \cup Z_j$.

Let $F\colon \boldsymbol{G}\mathbf{TopBorn} \to \mathcal{C}$ be a functor. We define the evaluation of F on a big family $\mathcal{Y}$ on a G-equivariant bornological topological space X by

$$F(\mathcal{Y}) := \operatorname*{colim}_{i\in I} F(Y_i).$$

Here we consider every member of the big family $\mathcal{Y}$ as objects in $\boldsymbol{G}\mathbf{TopBorn}$ with the induced subspace structure inherited from X (c.f. Example 3.1.16).

Definition 4.1.4. We say that F is *weakly excisive* (or *satisfies weak excision*) if $F(\emptyset) \simeq 0$ and if for every G-equivariant bornological topological space X and every G-equivariant complementary pair $(\mathcal{Y}, \mathcal{Z})$ on X the following square is a pushout:

$$\begin{tikzcd} F(\mathcal{Y}\cap\mathcal{Z}) \arrow[r] \arrow[d] & F(\mathcal{Y}) \arrow[d] \\ F(\mathcal{Z}) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & F(X). \end{tikzcd}$$

Here by $\mathcal{Y}\cap\mathcal{Z}$ we mean the big family with members $(Y_i \cap Z_j)_{(i,j)\in I\times J}$.

This notion of weak excisiveness is a generalization of the two stronger notions of “open excision” and “closed excision”. Which we will show in the following two examples:

Example 4.1.5. We say that F satisfies *closed excision* if for every G-equivariant bornological topological space X and every pair of G-invariant closed subsets Y, Z of X the following square is a pushout:

$$\begin{tikzcd} F(Y\cap Z) \arrow[r] \arrow[d] & F(Y) \arrow[d] \\ F(Z) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & F(X). \end{tikzcd}$$

If F satisfies closed excision then it also satisfies weak excision as we show in the following:

Consider an arbitrary space X in $\boldsymbol{G}\mathbf{TopBorn}$ and a G-equivariant complementary pair $(\mathcal{Y} = (Y_i)_{i\in I}, \mathcal{Z} = (Z_j)_{j\in J})$ on X. By assumption there exist indizes i_0 in I and j_0 in J such that $Y_{i_0} \cup Z_{j_0} = X$. For any i in I there exists by definition an index k_i

in I such that $i \leq k_i$ and the inclusion $Y_i \hookrightarrow Y_{k_i}$ factors through $\overline{Y_i}$. Hence we can consider the collection of subspaces $\mathcal{F}_{\mathcal{Y}} := \left\{\overline{Y_i}\right\}_{\substack{i\in I\\ i_0\leq i}}$ as being cofinal in $\mathcal{Y}$ and likewise for the big family $\mathcal{Z}$. This gives $F(\mathcal{Y}) \simeq F(\mathcal{F}_{\mathcal{Y}})$ and likewise $F(\mathcal{Z}) \simeq F(\mathcal{F}_{\mathcal{Z}})$. Now, for any members $\overline{Y_i}$ in $\mathcal{F}_{\mathcal{Y}}$ and $\overline{Z_j}$ in $\mathcal{F}_{\mathcal{Z}}$ the following square

$$\begin{tikzcd}
F\left(\overline{Y_i} \cap \overline{Z_j}\right) \arrow[r] \arrow[d] & F\left(\overline{Y_i}\right) \arrow[d] \\
F\left(\overline{Z_j}\right) \arrow[r] & F(X)
\end{tikzcd}$$

is a pushout since F fulfills closed excision (note that $\overline{Y_i}$ and $\overline{Z_j}$ are G-invariant since G acts by automorphisms and Y_i and Z_j are G-invariant). Taking the colimit we obtain a pushout diagram

$$\begin{tikzcd}
F(\mathcal{F}_{\mathcal{Y}} \cap \mathcal{F}_{\mathcal{Z}}) \arrow[r] \arrow[d] & F(\mathcal{F}_{\mathcal{Y}}) \arrow[d] \\
F(\mathcal{F}_{\mathcal{Z}}) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & F(X).
\end{tikzcd}$$

Combining this with the identifications $F(\mathcal{Y}) \simeq F(\mathcal{F}_{\mathcal{Y}})$ and $F(\mathcal{Z}) \simeq F(\mathcal{F}_{\mathcal{Z}})$ as well as $F(\mathcal{F}_{\mathcal{Y}} \cap \mathcal{F}_{\mathcal{Z}}) \simeq F(\mathcal{Y} \cap \mathcal{Z})$ we obtain the claim. ✠

Example 4.1.6. We say that F satisfies *open excision* if for every G-equivariant bornological topological space X and every pair of G-invariant open subsets U, V of X the following square is a pushout:

$$\begin{tikzcd}
F(U \cap V) \arrow[r] \arrow[d] & F(U) \arrow[d] \\
F(V) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & F(X).
\end{tikzcd}$$

If F satisfies open excision then it also satisfies weak excision which follows with analoguous arguments as we used in Example 4.1.5 above. We only replace the closures in the construction of the cofinal families with open neighborhoods of that closure. ✠

Definition 4.1.7. We define the *interval object* $\mathcal{I}$ in ***G*TopBorn** as the set $[0, 1]$ equipped with the trivial G-action and the metric bornology and topology.

With this interval object we define homotopy between morphisms in ***G*TopBorn** in the usual way. Note that additionally to the "standard notion of topological homotopy" we have properness in this situation. Usually in the literature this is referred to as "proper homotopy" (e.g. [EHR88]) but we will not use this notation as properness is always automatically encoded in the notion of a morphism in $\boldsymbol{G}$**TopBorn**.

Definition 4.1.8. Consider two objects X and Y in $\boldsymbol{G}$**TopBorn** and two morphisms $f, g\colon X \to Y$. We say that f and g are *homotopic* if there exists a morphism $h\colon X \otimes \mathcal{I} \to Y$ such that $f = h\iota_0$ and $g = h\iota_1$ where $\iota_j\colon X \hookrightarrow X \otimes \mathcal{I}$ (for $j \in \{0,1\}$) denote the usual inclusions. In this situation we write $f \sim g$.

We have the usual properties of homotopy of maps:

Lemma 4.1.9. *Being homotopic is an equivalence relation on* $\mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(X, Y)$ *for all* X, Y *in* $\boldsymbol{G}$**TopBorn**. *Moreover, if* $f, g\colon X \to Y$ *are homotopic and if* $h\colon Z \to X$ *and* $k\colon Y \to W$ *are morphisms in* $\boldsymbol{G}$**TopBorn** *then we have* $kfh \sim kgh$ *as well.*

Proof. It is well-known that being homotopic is an equivalence relation. The only thing to check is that the proof of that behaves well with the properness of our homotopies. For a morphism f in $\boldsymbol{G}$**TopBorn** the constant homotopy $H_1(x,t) := f(x)$ from f to f is proper since $H_1^{-1}(B) = f^{-1}(B) \times \mathcal{I}$ and $\mathcal{I}$ carries the maximal bornology. To check symmetry note that if H_2 is a homotopy between a morphism f and a morphism g, then the homotopy from g to f is given by the composition $H_2 \circ tw$ where $tw(x,t) := (x, 1-t)$ is proper since it only operates on the $\mathcal{I}$-part and there we have the maximal bornology. Finally, for transitivity the usual composition of two homotopies H_3 and H_4 is proper as the X-part of the preimage of a bounded subset B is contained in the X-part of the union $H_3^{-1}(B) \cup H_4^{-1}(B)$ and in the $\mathcal{I}$-part everything is bounded.

To see the last claim we consider maps as in the assertion. Let $H\colon X \otimes \mathcal{I} \to Y$ be the homotopy between f and g. Then the following composition of morphisms witnesses the homotopy $kfh \sim kgh$:

$$Z \otimes \mathcal{I} \xrightarrow{h \otimes \mathcal{I}} X \otimes \mathcal{I} \xrightarrow{H} Y \xrightarrow{k} W.$$

Note that $(-) \otimes \mathcal{I}\colon \boldsymbol{G}\mathbf{TopBorn} \to \boldsymbol{G}\mathbf{TopBorn}$ is a functor, hence $h \otimes \mathcal{I}$ is a morphism. ☐

Definition 4.1.10. We say that the functor $F\colon \boldsymbol{G}\mathbf{TopBorn} \to \mathcal{C}$ is *homotopy invariant* if the projection $X \otimes \mathcal{I} \to X$ induces an equivalence $F(X \otimes \mathcal{I}) \xrightarrow{\simeq} F(X)$.

Finally, we want that local homology theories vanish on flasque spaces. Recall that in the coarse world we had three different notions of (generalized) flasqueness (Definitions 2.1.8 and 2.1.13 and Proposition 2.1.15) and for a functor vanishing on these types of flasque spaces was equivalent (see Proposition 2.1.16). It turns out that we can transfer these three flasqueness-definitions into the world of bornological topological spaces, however vanishing on the different types is no longer equivalent[1]. With an eye towards chapter 5, we want that flasqueness is stable under G-completion (Definition 3.1.18) hence we put this into the definition right away.

Let X be a G-equivariant bornological topological space.

Definition 4.1.11. The space X is called *flasque* if its G-completetion $B_G X$ is flasque in the following sense: A G-complete space X is called *flasque* if it admits an endomorphism $f\colon X \to X$ which is homotopic to the identity $f \sim \mathrm{id}_X$ and for all bounded subsets B in $\mathcal{B}_X$ there exists a positive integer n such that $f^n(X) \cap B = \emptyset$ (or equivalently $(f^n)^{-1}(B) = \emptyset$).

Remark 4.1.12. Recall that if G acts trivially or if G is finite, every space X in $\boldsymbol{G}$**TopBorn** is G-complete (see Lemma 3.1.20). ♦

We also give the topological analogues to the other two flasqueness-definitions from the coarse world.

Definition 4.1.13. The space X is called *r-flasque*[2] if its G-completion $B_G X$ admits a retract $r\colon B_G X \otimes \mathbb{R}_{\geq 0} \to B_G X$ in $\boldsymbol{G}$**TopBorn** (i.e. $r\iota_0 = \mathrm{id}_X$).

Definition 4.1.14. The space X is called *generalized flasque* if it admits a sequence of endomorphisms $(f_n)_{n\in\mathbb{N}}$ such that:

- We have $f_0 = \mathrm{id}_X$.
- For all positive integers n we have $f_n \sim f_{n+1}$.
- For all bounded subsets B in $\mathcal{B}_X$ there exists an integers N such that for all $n \geq N$ we have $f^n(X) \cap GB = \emptyset$.

The standard-metric on $\mathbb{R}_{\geq 0}$ defines a bornological topological structure on this set. We consider it as a space in $\boldsymbol{G}$**TopBorn** by letting G act trivially.

[1] at least the author sees no reason why this should be the case
[2] the "r" stands for retract

Example 4.1.15. The space $X \otimes \mathbb{R}_{\geq 0}$ is flasque. Indeed, since G acts trivially on the second factor we have $B_G(X \otimes \mathbb{R}_{\geq 0}) \cong B_G X \otimes \mathbb{R}_{\geq 0}$. Hence we replace X by $B_G X$ and thus assume that X is G-complete and we proof the axioms for flasqueness. A candidate for the endomorphism implementing flasqueness is given by

$$f\colon X \otimes \mathbb{R}_{\geq 0} \longrightarrow X \otimes \mathbb{R}_{\geq 0}, \qquad (x,t) \longmapsto (x, t+1).$$

G acts trivially on the $\mathbb{R}$-part, hence f is G-equivariant. Also f is clearly continuous and proper, hence a morphism in $\boldsymbol{G}$**TopBorn**. The following is clearly a well-defined homotopy from f to $\mathrm{id}_{X \otimes \mathbb{R}_{\geq 0}}$:

$$\begin{aligned} h\colon X \otimes \mathbb{R}_{\geq 0} \otimes \mathcal{I} &\longrightarrow X \otimes \mathbb{R}_{\geq 0} \\ (x,t,s) &\longmapsto (x, t+s). \end{aligned}$$

Finally, for any bounded subset B of $X \otimes \mathbb{R}_{\geq 0}$ there exists a bounded subset B' of X and real numbers a, b in $\mathbb{R}_{\geq 0}$ such that B is contained in $B' \times [a,b]$. We set $n := b+1$ and obtain clearly $f^n(X \otimes \mathbb{R}_{\geq 0}) \cap B = \emptyset$. ✠

Definition 4.1.16. We say that the functor $F\colon \boldsymbol{G}\mathbf{TopBorn} \to \mathcal{C}$ *vanishes on flasques, r-flasques or generalized flasques* if for all flasque, r-flasque or generalized flasque spaces X we have $F(X) \simeq 0$.

Building on the comment above Definition 4.1.11 we have some relations between flasqueness, r-flasqueness and generalized flasqueness, and vanishing on the respective spaces.

Lemma 4.1.17. *Every flasque space X in* $\boldsymbol{G}$**TopBorn** *is r-flasque.*

Proof. By definitons we may assume that X is G-complete. Assume that flasqueness of X is implemented by f. Then by definition we have a homotopy

$$h\colon X \otimes \mathcal{I} \longrightarrow X$$

such that $h\iota_0 = \mathrm{id}_X$ and $h\iota_1 = f$.
We define a retract $r\colon X \otimes \mathbb{R}_{\geq 0} \to X$ as follows: We can cover $X \otimes \mathbb{R}_{\geq 0}$ by

$$X \otimes \mathbb{R}_{\geq 0} = \bigcup_{n \in \mathbb{N}} X \otimes [n, n+1]$$

On each of these pieces we define

$$r\colon X\otimes[n,n+1]\longrightarrow X,\qquad (x,t)\longmapsto f^n(h(x,t-n)).$$

The overlaps are elements (x,n) for x in X and n in $\mathbb{N}$. If we consider (x,n) as an element in $X\otimes[n-1,n]$ then we have defined

$$r(x,n)=f^{n-1}(h(x,t-(n-1)))=f^{n-1}(h(x,1))=f^{n-1}(f(x))=f^n(x).$$

If on the other hand we consider (x,n) as an element in $X\otimes[n,n+1]$ we have

$$r(x,n)=f^n(h(x,n-n))=f^n(h(x,0))=f^n(\mathrm{id}_X(x))=f^n(x).$$

Hence we obtain a well-defined G-equivariant continuous map $r\colon X\otimes\mathbb{R}_{\geq 0}\to X$. Also the retract-property is clear because $r\iota_0=\mathrm{id}_X$ by construction. It remains to see that r is proper. For this we consider a bounded subset B in $\mathcal{B}_X$. By flasqueness of X there exists k in $\mathbb{N}$ such that $f^k(X)\cap B=\emptyset$, which we use in the following calculation at the step marked with "!":

$$\begin{aligned}
r^{-1}(B)&=\Big\{(x,t)\in X\otimes\mathbb{R}_{\geq 0}\ \Big|\ r(x,t)\in B\Big\}\\
&=\bigcup_{n\in\mathbb{N}}\Big\{(x,t+n)\in X\otimes\mathbb{R}_{\geq 0}\ \Big|\ t\in[0,1),\ r(x,t+n)\in B\Big\}\\
&=\bigcup_{n\in\mathbb{N}}\Big\{(x,t+n)\in X\otimes\mathbb{R}_{\geq 0}\ \Big|\ t\in[0,1),\ f^n(h(x,t))\in B\Big\}\\
&\stackrel{!}{=}\bigcup_{n=0}^{k}\Big\{(x,t+n)\in X\otimes\mathbb{R}_{\geq 0}\ \Big|\ t\in[0,1),\ f^n(h(x,t))\in B\Big\}
\end{aligned}$$

Now f and h are proper, hence $(f^n\circ h)^{-1}(B)$ is bounded and so is their translation in the $\mathbb{R}$-part, hence this finite union is bounded and we are through. □

Lemma 4.1.18. *Every r-flasque space X in* $\boldsymbol{G}$**TopBorn** *is generalized flasque.*

Proof. We may assume that X is G-complete. Let $\pi\colon X\otimes\mathbb{R}_{\geq 0}\to X$ implement flasqueness of X. We obtain the sequence by defining $f_n:=\pi\circ\iota_n$ where for all n in $\mathbb{N}$ the map $\iota_n\colon X\to X\otimes\mathbb{R}_{\geq 0}$ is defined by $x\mapsto(x,n)$. Properness of ι_n can be checked on generators of $\mathcal{B}_{X\otimes\mathbb{R}_{\geq 0}}$ and the pre-image of such a generator $B\times[s,t]$ is contained in B which is bounded in X. Additionally, ι_n are G-equivariant since G acts trivially on the $\mathbb{R}_{\geq 0}$-part. Hence all the maps f_n are indeed morphisms and we check that they implement generalized flasqueness on X. First, by the retract-property we have

$f_0 = \pi\iota_0 = \mathrm{id}_X$ as claimed. Next, consider a bounded subset B of X. By properness of π we know that $\pi^{-1}(B)$ is bounded in $X \otimes \mathbb{R}_{\geq 0}$. Hence there exists a bounded subset B' of X and a positive integer N such that $\pi^{-1}(B) \subseteq B' \times [0, N]$. Thus, for all integers $n > N$ we conclude

$$\begin{aligned} f_n^{-1}(GB) &= \iota_n^{-1}\left(\pi^{-1}(GB)\right) = \iota_n^{-1}\left(\pi^{-1}\left(\bigcup_{g\in G} gB\right)\right) = \iota_n^{-1}\left(\bigcup_{g\in G} g\cdot\pi^{-1}(B)\right) \\ &\subseteq \iota_n^{-1}\left(\bigcup_{g\in G} g\cdot(B'\times[0,N])\right) = \iota_n^{-1}(GB'\times[0,N]) = \emptyset. \end{aligned}$$

Finally, for an arbitrarily chosen natural number n we construct a homotopy between f_n and f_{n+1}: For this we let

$$\begin{aligned} h\colon X\otimes\mathcal{I} &\hookrightarrow X\otimes\mathbb{R}_{\geq 0} \xrightarrow{\pi} X. \\ (x,t) &\mapsto (x,t+n) \mapsto \pi(x,t+n) \end{aligned}$$

The first part of this composition is a morphism since on the first factor it is the identity and on the second factor continuity is clear, G-equivariance is true since G acts trivially on these factors and properness is fulfilled since the bornology on $\mathcal{I}$ is maximal. In total, h is a morphism $X \to X$. Next we observe

$$(h\iota_0)(x) = h(x,0) = \pi(x,n) = (\pi\iota_n)(x) = f_n(x)$$

and analoguously $(h\iota_1)(x) = f_{n+1}(x)$ which shows $f_n \sim f_{n+1}$. □

Remark 4.1.19. We cannot modify the proof of Lemma 4.1.17 to show that a generalized flasque space is r-flasque (this was true in the setting of ***G*****BornCoarse**), because we cannot control the homotopies between the morphisms f_n and f_{n+1} to ensure that a resulting retract is proper. ◆

Proposition 4.1.20. *If the functor F vanishes on generalized flasques, then so it does on flasques.*
It vanishes on r-flasques if and only if it vanishes on flasques.

Proof. Every flasque space is r-flasque (Lemma 4.1.17 and every r-flasque space is generalized flasque (Lemma 4.1.18) hence vanishing on generalized flasques implies vanishing on flasques.
With the same argument vanishing on r-flasques implies vanishing on flasques. So it

remains to see that vanishing on flasques implies vanishing on r-flasques. For this we assume F vanishes on flasques and we consider an r-flasque space X. By definition we have a retract r, hence

$$\mathrm{id}_X \colon X \xrightarrow{\iota_0} X \otimes \mathbb{R}_{\geq 0} \xrightarrow{r} X.$$

Applying the functor F gives

$$\mathrm{id}_{F(X)} \colon F(X) \longrightarrow F(X \otimes \mathbb{R}_{\geq 0}) \longrightarrow F(X).$$

But since F vanishes on flasques and $X \otimes \mathbb{R}_{\geq 0}$ is flasque (Example 4.1.15) we have $F(X \otimes \mathbb{R}_{\geq 0}) \simeq 0$, hence $F(X) \simeq 0$. □

Recall that by $\mathcal{C}$ we denoted a co-complete stable ∞-category.

Definition 4.1.21. A *G-equivariant local homology theory with values in $\mathcal{C}$* is a functor $F \colon \boldsymbol{G}\mathbf{TopBorn} \to \mathcal{C}$ which satisfies

1. weak excision,
2. homotopy invariance,
3. vanishing on flasques.

4.2 Equivariant local motivic spaces

In order to construct the universal G-equivariant local homology theory we proceed in a similar manner as in the coarse situation: As a first step we take the presheaf-category on $\boldsymbol{G}\mathbf{TopBorn}$ and take an appropriate localization giving the ∞-category of local motivic spaces. These construction will be done in this section. It is similar to the work done in section 2.2. In particular, set-theoretical issues will be handled as mentioned there.

Let

$$\mathrm{yo}\colon \boldsymbol{G}\mathbf{TopBorn} \to \mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn}) := \mathrm{Fun}(\boldsymbol{G}\mathbf{TopBorn}^{\mathrm{op}}, \mathbf{Spc})$$

denote the Yoneda-embedding into the ∞-category of space-valued presheaves. By [Lur09, Thm 5.1.5.6] (or [BE20, Rem 3.9] for this precise formulation), precomposing with yo induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\lim}\left(\mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn})^{\mathrm{op}}, \mathbf{Spc}\right) \xrightarrow{\simeq} \mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn}). \tag{4.2.1}$$

Recall that we use this identification to evaluate a presheaf on $\boldsymbol{G}\mathbf{TopBorn}$ also on presheaves. More precisely, for two presheaves E and F in $\mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn})$ we have

$$E(F) \simeq \lim_{\mathrm{yo}(X)\to F} E(X).$$

Example 4.2.1. Consider a space X in $\boldsymbol{G}\mathbf{TopBorn}$ and a G-equivariant big family $\mathcal{Y} = (Y_i)_{i\in I}$ on X. We have already defined the evaluation $\mathrm{yo}(\mathcal{Y}) = \mathrm{colim}_{i\in I}\,\mathrm{yo}(Y_i)$. Hence, for a presheaf E in $\mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn})$ we have

$$E(\mathrm{yo}(\mathcal{Y})) \simeq \lim_{i\in I} E(\mathrm{yo}(Y_i)) \simeq \lim_{i\in I} E(Y_i).$$ ✠

To keep notation a bit shorter we write $E(\mathcal{Y})$ instead of $E(\mathrm{yo}(\mathcal{Y}))$.

Definition 4.2.2. We say that a presheaf E in $\mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn})$ satisfies *weak excision* if $E(\emptyset) \simeq *$ and if for all X in $\boldsymbol{G}\mathbf{TopBorn}$ and all G-equivariant complementary pairs $(\mathcal{Y}, \mathcal{Z})$ on X the following square is a pullback:

$$\begin{tikzcd} E(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E(\mathcal{Z}) \arrow[d] \\ E(\mathcal{Y}) \arrow[r] & E(\mathcal{Y}\cap\mathcal{Z}). \end{tikzcd}$$

If the diagram above is a pullback, we say that E *satisfies descent w.r.t.* $(X, \mathcal{Y}, \mathcal{Z})$.

Proposition 4.2.3. *There exists a Grothendieck topology $\tau_{\mathcal{L}}$ on $\boldsymbol{G}$**TopBorn** such that a presheaf is a $\tau_{\mathcal{L}}$-sheaf if and only if it satisfies weak excision.*

Proof. The proof is almost verbatim the same as the proof of the corresponding assertion in the setting of equivariant bornological coarse spaces in Proposition 2.2.4. $\square$

Lemma 4.2.4. *The Grothendieck topology $\tau_{\mathcal{L}}$ is subcanonical.*

Proof. It is to show that for all spaces W in $\boldsymbol{G}$**TopBorn** the represented presheaf $\mathrm{yo}(W)$ is a sheaf. Let

$$y\colon \boldsymbol{G}\mathbf{TopBorn} \to \mathbf{PSh}_{\mathbf{Set}}(\boldsymbol{G}\mathbf{TopBorn}) := \mathrm{Fun}(\boldsymbol{G}\mathbf{TopBorn}^{\mathrm{op}}, \mathbf{Set})$$

be the usual 1-categorical Yoneda-embedding. Then we have $\mathrm{yo}(W) \simeq \iota(y(W))$ where $\iota\colon \mathbf{Set} \to \mathbf{Spc}$. Since ι preserves limits and both sheaf-conditions are limit-conditions, it suffices to show the assertions for y instead of yo.

First, $y(W)(\emptyset) = \mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(\emptyset, W) = *$ is clear, hence we only need to verify the pullback-condition. For this we consider a space X in $\boldsymbol{G}$**TopBorn** together with a G-equivariant complementary pair $(\mathcal{Y} = (Y_i)_{i\in I}, \mathcal{Z} = (Z_j)_{j\in J})$. By definition there exist indizes i_0 in I and j_0 in J such that $Y_{i_0} \cup Z_{j_0} = X$. We define a G-equivariant complementary pair $(\mathcal{Y}', \mathcal{Z}')$ by $\mathcal{Y}' := (Y_i)_{\substack{i\in I\\ i_0\le i}}$ and $\mathcal{Z}' := (Z_j)_{\substack{j\in J\\ j_0\le j}}$. Now for all Y_i in $\mathcal{Y}'$ and Z_j in $\mathcal{Z}'$ we claim that

$$\begin{tikzcd}
\mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(X, W) \arrow[r] \arrow[d] & \mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(Y_i, W) \arrow[d] \\
\mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(Z_j, W) \arrow[r] & \mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(Y_i \cap Z_j, W)
\end{tikzcd} \tag{*}$$

is a pullback in **Set** where the maps are given by the restriction to the respective subspaces. Since $Y_i \cup Z_j = X$ we see that the following map is well-defined and injective:

$$\begin{aligned}
\mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(X, W) &\longrightarrow \mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(Y_i, W) \times_{\mathrm{Hom}(Y_i\cap Z_j, W)} \mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(Z_j, W) \\
f &\longmapsto \left(f\big|_{Y_i}, f\big|_{Z_j}\right).
\end{aligned}$$

For surjectivity we consider morphisms $f\colon Y_i \to W$ and $g\colon Z_j \to W$ which agree on the intersection $Y_i \cap Z_j$. Then we can glue these two morphisms to a well-defined continuous G-equivariant map $h\colon X \to W$. This map is also proper because for a bounded subset B in $\mathcal{B}_W$ we have $h^{-1}(B) \subseteq f^{-1}(B) \cup g^{-1}(B)$.
In total, (*) is indeed a pullback. Taking the limit of (*) gives a pullback diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(X,W) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \lim\limits_{Y_i \in \mathcal{Y}'} \mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(Y_i,W) \arrow[d] \\
\lim\limits_{Z_j \in \mathcal{Z}'} \mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(Z_j,W) \arrow[r] & \lim\limits_{\mathcal{Y}' \cap \mathcal{Z}'} \mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(Y_i \cap Z_j, W).
\end{tikzcd}$$

Since $\mathcal{Y}'$ is cofinal in $\mathcal{Y}$ and $\mathcal{Z}'$ is cofinal in $\mathcal{Z}$ we get by (the dual statement of) [Mac71, Thm IX.3.1] that

$$\lim_{Y_i \in \mathcal{Y}'} \mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(Y_i,W) \cong \lim_{Y_i \in \mathcal{Y}'} y(W)(Y_i) \cong \lim_{Y_i \in \mathcal{Y}} y(W)(Y_i) \cong y(W)(\mathcal{Y})$$

and likewise for the other limits. Hence

$$\begin{tikzcd}
y(W)(X) \arrow[r] \arrow[d] & y(W)(\mathcal{Y}) \arrow[d] \\
y(W)(\mathcal{Z}) \arrow[r] & y(W)(\mathcal{Y} \cap \mathcal{Z})
\end{tikzcd}$$

is a pullback diagram and we are through. □

Remark 4.2.5. Like in Remark 2.2.6 the sheaf-condition together with the Yoneda-lemma gives immediately that $\mathrm{yo}(\emptyset)$ is the initial presheaf in $\mathbf{Sh}(\boldsymbol{G}\mathbf{TopBorn})$. ♦

The sheaf-condition imposed above corresponds to weak excisiveness of equivariant local homology theories. Next, we localize the category of $\tau_{\mathcal{L}}$-sheaves in order to enforce equivalences that correspond to vanishing on flasques and homotopy invariance.

Definition 4.2.6. Consider a presheaf E on $\boldsymbol{G}\mathbf{TopBorn}$. We say that

1. E is *homotopy invariant* if the projection induces an equivalence of spaces

$$E(X) \xrightarrow{\simeq} E(X \otimes \mathcal{I}).$$

2. *E vanishes on flasques* if $E(X)$ is terminal in **Spc** for all flasque (or equivalently r-flasque) spaces X in $\boldsymbol{G}$**TopBorn**.

The full subcategory of $\mathbf{Sh}(\boldsymbol{G}\mathbf{TopBorn})$ consisting of sheaves that vanish on flasques and which are homotopy invariant is denoted by $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$. We call this ∞-category the *category of G-equivariant local motivic spaces.*

Proposition 4.2.7. *The full subcategory $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$ of $\mathbf{Sh}(\boldsymbol{G}\mathbf{TopBorn})$ is localizing, i.e. it fits into a localization adjunction*

$$\mathcal{L}\colon \mathbf{Sh}(\boldsymbol{G}\mathbf{TopBorn}) \rightleftarrows \boldsymbol{G}\mathbf{Spc}\mathcal{TB} : \mathrm{incl.}$$

Proof. Consider the following collections of morphisms:

$$\mathcal{M}_1 := \left\{ \mathrm{yo}(X \otimes \mathcal{I}) \to \mathrm{yo}(X) \;\middle|\; X \in \boldsymbol{G}\mathbf{TopBorn} \right\}$$
$$\mathcal{M}_2 := \left\{ \mathrm{yo}(\emptyset) \to \mathrm{yo}(X) \;\middle|\; X \in \boldsymbol{G}\mathbf{TopBorn} \text{ flasque} \right\}$$

Both collections consist of morphisms of sheaves by Lemma 4.2.4. An object E in the category $\mathbf{Sh}(\boldsymbol{G}\mathbf{TopBorn})$ is local w.r.t. $\mathcal{M}_1$ if and only if

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Sh}(\boldsymbol{G}\mathbf{TopBorn})}(\mathrm{yo}(X), E) \arrow[r] \arrow[d, equal, "\wr"] & \mathrm{Hom}_{\mathbf{Sh}(\boldsymbol{G}\mathbf{TopBorn})}(\mathrm{yo}(X \otimes \mathcal{I}), E) \arrow[d, equal, "\wr"] \\
E(X) \arrow[r] & E(X \otimes \mathcal{I})
\end{tikzcd}$$

is an equivalence in **Spc**, i.e. if and only if it is homotopy invariant. Similarly the sheaf E vanishes on flasques if and only if it is local w.r.t. $\mathcal{M}_2$. Hence the full subcategory $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$ consist of precisely those sheafs that are local w.r.t. $\mathcal{M}_1 \cup \mathcal{M}_2$. Hence the claim follows from [Lur09, Prop 5.5.4.15]. □

We denote by Yo_G the composition

$$\mathrm{Yo}_G\colon \boldsymbol{G}\mathbf{TopBorn} \xrightarrow{\mathrm{yo}} \mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn}) \xrightarrow{\mathrm{sheafify}} \mathbf{Sh}(\boldsymbol{G}\mathbf{TopBorn}) \xrightarrow{\mathcal{L}} \boldsymbol{G}\mathbf{Spc}\mathcal{TB}.$$

Remark 4.2.8. Since every represented presheaf is already a sheaf (Lemma 4.2.4) we could omit the sheafification-step in the composition above. ⧫

Corollary 4.2.9. *By construction the functor* Yo_G *has the following properties:*

1. *For all spaces* X *in* $\boldsymbol{G}\mathbf{TopBorn}$ *and all* G*-equivariant complementary pairs* $(\mathcal{Y}, \mathcal{Z})$ *on* X *the following square is a pushout:*

$$\begin{tikzcd} \mathrm{Yo}_G(\mathcal{Y} \cap \mathcal{Z}) \arrow[r] \arrow[d] & \mathrm{Yo}_G(\mathcal{Y}) \arrow[d] \\ \mathrm{Yo}_G(\mathcal{Z}) \arrow[r] & \mathrm{Yo}_G(X) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}.$$

2. *For all spaces* X *in* $\boldsymbol{G}\mathbf{TopBorn}$ *to projection* $X \otimes \mathcal{I} \to X$ *induces an equivalence*

$$\mathrm{Yo}_G(X \otimes \mathcal{I}) \xrightarrow{\simeq} \mathrm{Yo}_G(X).$$

3. *For all flasque spaces* X *in* $\boldsymbol{G}\mathbf{TopBorn}$ *we have* $\mathrm{Yo}_G(X) \simeq \emptyset_{\mathbf{Sh}}$.

4. *For all r-flasque spaces* X *in* $\boldsymbol{G}\mathbf{TopBorn}$ *we have* $\mathrm{Yo}_G(X) \simeq \emptyset_{\mathbf{Sh}}$.

Proof. Property (1) follows from Lemma 4.2.4 and the fact that the localization functor $\mathcal{L}$ is a left adjoint, hence preserves the pushout. Properties (2) und (3) follow from the fact that the localization by construction enforces the respective morphisms to become equivalences. Equivalence of properties (3) and (4) can be shown exactly as in Proposition 4.1.20. □

A special example for the excision property of Yo_G is the case of coproducts:

Corollary 4.2.10. *Consider two* G*-equivariant bornological topological spaces* Y *and* Z *in* $\boldsymbol{G}\mathbf{TopBorn}$*. Then their coproduct* $X := Y \amalg Z$ *exists in* $\boldsymbol{G}\mathbf{TopBorn}$ *by Corollary* 3.2.10 *and we have an equivalence*

$$\mathrm{Yo}_G(Y \amalg Z) \simeq \mathrm{Yo}_G(Y) \amalg \mathrm{Yo}_G(Z)$$

of G*-equivariant local motivic spaces.*

Proof. The set $\mathcal{Y} := \{Y\}$ is a G-equivariant big family on X because Y is both open and closed in X. Likewise $\mathcal{Z} := \{Z\}$ is a G-equivariant big family on X. The intersection $\mathcal{Y} \cap \mathcal{Z}$ is clearly empty. By excisiveness of Yo_G (Corollary 4.2.9) the following square is a pushout in $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$:

$$\begin{tikzcd} \mathrm{Yo}_G(\mathcal{Y} \cap \mathcal{Z}) \arrow[r] \arrow[d] & \mathrm{Yo}_G(\mathcal{Y}) \arrow[d] \\ \mathrm{Yo}_G(\mathcal{Z}) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{Yo}_G(X). \end{tikzcd}$$

Now we have $\mathrm{Yo}_G(\mathcal{Y} \cap \mathcal{Z}) = \mathrm{Yo}_G(\emptyset) \simeq \emptyset_{\boldsymbol{G}\mathbf{Spc}\mathcal{TB}}$, hence we obtain

$$\mathrm{Yo}_G(X) \simeq \mathrm{Yo}_G(\mathcal{Y}) \amalg \mathrm{Yo}_G(\mathcal{Z}) \simeq \mathrm{Yo}_G(Y) \amalg \mathrm{Yo}_G(Z). \qquad \square$$

Remark 4.2.11. Sheafification can also be seen as a localization

$$\mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn}) \to \mathbf{Sh}(\boldsymbol{G}\mathbf{TopBorn})$$

in the fashion of Proposition 4.2.7. Sheaves are precisely those presheaves that are local w.r.t. to the following collection of morphisms:

1. For all spaces X in $\boldsymbol{G}\mathbf{TopBorn}$ and all G-equivariant complementary pairs $(\mathcal{Y}, \mathcal{Z})$ on X the induced morphism

$$\mathrm{yo}(\mathcal{Y}) \coprod_{\mathrm{yo}(\mathcal{Y} \cap \mathcal{Z})} \mathrm{yo}(\mathcal{Z}) \to \mathrm{yo}(X).$$

2. The morphism $\emptyset_{\mathbf{PSh}} \to \mathrm{yo}(\emptyset)$. ♦

Let $\mathcal{C}$ be a co-complete ∞-category. We denote by $\mathrm{Fun}^{\mathcal{TB}}(\boldsymbol{G}\mathbf{TopBorn}, \mathcal{C})$ the full subcategory of $\mathrm{Fun}(\boldsymbol{G}\mathbf{TopBorn}, \mathcal{C})$ spanned by those functors F which satisfy weak excision, are homotopy invariant and for which $F(X)$ is initial in $\mathcal{C}$ for all flasque spaces X in $\boldsymbol{G}\mathbf{BornCoarse}$.

Definition 4.2.12. The functors in $\mathrm{Fun}^{\mathcal{TB}}(\boldsymbol{G}\mathbf{TopBorn}, \mathcal{C})$ are called *unstable G-equivariant local homology theories.*

Proposition 4.2.13. *For any co-complete ∞-category $\mathcal{C}$ precomposition with* Yo_G *induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Spc}\mathcal{TB}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathcal{TB}}(\boldsymbol{G}\mathbf{TopBorn}, \mathcal{C}).$$

Proof. Let $\mathcal{S}$ denote the collection of morphism listed in Proposition 4.2.7 and Remark 4.2.11, hence $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$ are precisely the $\mathcal{S}$-local objects in $\mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn})$.

Then by [Lur09, Prop 5.5.4.20] we have an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Spc}\mathcal{TB},\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{colim},\mathcal{S}}(\mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn}),\mathcal{C})$$

where the latter category is the full subcategory of $\mathbf{PSh}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{TopBorn})$ spanned by those colimit-preserving functors which send the morphisms in $\mathcal{S}$ to equivalences in $\mathcal{C}$. Now the equivalence of ∞-categories (due to [Lur09, Thm. 5.1.5.6])

$$\mathrm{yo}^*\colon \mathrm{Fun}^{\mathrm{colim}}(\mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn}),\mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(\boldsymbol{G}\mathbf{TopBorn},\mathcal{C})$$

restricts to an equivalence of ∞-categories

$$\mathrm{yo}^*\colon \mathrm{Fun}^{\mathrm{colim},\mathcal{S}}(\mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn}),\mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathcal{TB}}(\boldsymbol{G}\mathbf{TopBorn},\mathcal{C})$$

as we indicate in the following: We show e.g. that for a functor F in the category $\mathrm{Fun}^{\mathrm{colim},\mathcal{S}}(\mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn}),\mathcal{C})$ the composition $F\circ\mathrm{yo}$ satisfies weak excision. For this we consider a space X in $\boldsymbol{G}\mathbf{TopBorn}$ together with a G-equivariant complementary pair $(\mathcal{Y},\mathcal{Z})$ on X. The following morphism in $\mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn})$ is a member of $\mathcal{S}$:

$$\mathrm{yo}(\mathcal{Y}) \coprod_{\mathrm{yo}(\mathcal{Y}\cap\mathcal{Z})} \mathrm{yo}(\mathcal{Z}) \longrightarrow \mathrm{yo}(X).$$

Hence F sends this morphism to an equivalence in $\mathcal{C}$. But F is also colimit-preserving hence

$$F(\mathrm{yo}(\mathcal{Y})) \coprod_{F(\mathrm{yo}(\mathcal{Y}\cap\mathcal{Z}))} F(\mathrm{yo}(\mathcal{Z})) \longrightarrow F(\mathrm{yo}(X))$$

is an equivalence in $\mathcal{C}$, hence we have a pushout square in $\mathcal{C}$:

$$\begin{tikzcd}
(F\circ\mathrm{yo})(\mathcal{Y}\cap\mathcal{Z}) \arrow[r] \arrow[d] & (F\circ\mathrm{yo})(\mathcal{Y}) \arrow[d] \\
(F\circ\mathrm{yo})(\mathcal{Z}) \arrow[r] & (F\circ\mathrm{yo})(X). \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

This shows that $F\circ\mathrm{yo}$ satisfies weak excision. The remaining axioms follow analoguous. $\square$

4.3 Equivariant local motivic spectra

Following the analogy of chapter 2 we stabilize the category of G-equivariant local motivic spaces to obtain the category of G-equivariant local motivic spectra. The induced functor $\mathrm{Yo}^s_G \colon \boldsymbol{G}\mathbf{TopBorn} \to \boldsymbol{G}\mathbf{Sp}\mathcal{TB}$ will be the universal G-equivariant local homology theory in the sense that every other such theory factors through Yo^s_G. The process of stabilizing will mimic the one in chapter 2, hence we refer to there for details.

Let $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}_*$ denote the category of pointed G-equivariant local motivic spaces[3]. We have an endofunctor

$$\begin{aligned}\Sigma \colon \boldsymbol{G}\mathbf{Spc}\mathcal{TB}_* &\longrightarrow \boldsymbol{G}\mathbf{Spc}\mathcal{TB}_* \\ X &\longmapsto \operatorname{colim}\Big(* \leftarrow X \rightarrow *\Big).\end{aligned}$$

A way to invert this functor Σ is given by taking the colimit

$$\boldsymbol{G}\mathbf{Spc}\mathcal{TB}_*[\Sigma^{-1}] := \operatorname{colim}\Big\{\boldsymbol{G}\mathbf{Spc}\mathcal{TB}_* \xrightarrow{\Sigma} \boldsymbol{G}\mathbf{Spc}\mathcal{TB}_* \xrightarrow{\Sigma} \boldsymbol{G}\mathbf{Spc}\mathcal{TB}_* \xrightarrow{\Sigma} \dots\Big\},$$

where the colimit is taken in the category $\mathbf{Pr^L}$ of presentable ∞-categories with left adjoint functors.

Definition 4.3.1. We define the category of *G-equivariant local motivic spectra* as

$$\boldsymbol{G}\mathbf{Sp}\mathcal{TB} := \boldsymbol{G}\mathbf{Spc}\mathcal{TB}_*[\Sigma^{-1}]$$

and we denote the composition

$$\Sigma^{mot}_+ \colon \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \to \boldsymbol{G}\mathbf{Spc}\mathcal{TB}_* \to \boldsymbol{G}\mathbf{Sp}\mathcal{TB}\,.$$

Finally this gives rise to the *universal G-equivariant local homology theory*

$$\mathrm{Yo}^s_G \colon \boldsymbol{G}\mathbf{TopBorn} \xrightarrow{\mathrm{Yo}_G} \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \xrightarrow{\Sigma^{mot}_+} \boldsymbol{G}\mathbf{Sp}\mathcal{TB}\,.$$

The name "universal G-equivariant local homology theory" is justified for Yo^s_G by the following theorem:

[3]It is the co-slice category $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}_{*/}$ under the terminal object

Theorem 4.3.2. *The functor* $\mathrm{Yo}^s_G\colon \boldsymbol{G}\mathbf{TopBorn} \to \boldsymbol{G}\mathbf{Sp}\mathcal{TB}$ *is a* G*-equivariant local homology theory. Furthermore for all stable co-complete* ∞*-categories* $\mathcal{C}$ *precomposition with* Yo^s_G *induces an equivalence of* ∞*-categories*

$$\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Sp}\mathcal{TB}, \mathcal{C}) \xrightarrow{\simeq} \boldsymbol{G}\mathbf{LocHomTheo}_{\mathcal{C}}$$

where the colim-supscript means "colimit-preserving" and the target category is the category of $\mathcal{C}$*-valued* G*-equivariant local homology theories.*

Proof. The functor Σ^{mot}_+ fits into an adjunction

$$\Sigma^{mot}_+\colon \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \rightleftarrows \boldsymbol{G}\mathbf{Sp}\mathcal{TB} :\Omega^{mot}.$$

In particular it preserves colimits. Therefore, all the properties listed in Corollary 4.2.9 for the functor Yo_G are preserved by the functor Σ^{mot}_+ and hold thusly also for the functor Yo^s_G which shows immediately that Yo^s_G is a G-equivariant local homology theory.
Furthermore, for a stable co-complete ∞-category $\mathcal{C}$ precomposition with Yo_G induces by Proposition 4.2.13 an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Spc}\mathcal{TB}, \mathcal{C}) \xrightarrow{\simeq} \boldsymbol{G}\mathbf{LocHomTheo}_{\mathcal{C}}.$$

So the only thing left to show is that precomposition with Σ^{mot}_+ gives an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Sp}\mathcal{TB}, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Spc}\mathcal{TB}, \mathcal{C}).$$

This follows like in the coarse situation (c.f. Lemma 2.3.2). □

For a big family $\mathcal{Y}$ on X we define

$$\mathrm{Yo}^s_G(X, \mathcal{Y}) := \mathrm{cofib}\left(\mathrm{Yo}^s_G(\mathcal{Y}) \to \mathrm{Yo}^s_G(X)\right).$$

Corollary 4.3.3. *We have a fiber sequence in* $\boldsymbol{G}\mathbf{Sp}\mathcal{TB}$

$$\cdots \to \mathrm{Yo}^s_G(\mathcal{Y}) \to \mathrm{Yo}^s_G(X) \to \mathrm{Yo}^s_G(X, \mathcal{Y}) \to \Sigma\mathrm{Yo}^s_G(\mathcal{Y}) \to \ldots$$

Lemma 4.3.4. *For any space X in $\boldsymbol{G}$**TopBorn** and any G-equivariant complementary pair $(\mathcal{Y}, \mathcal{Z})$ on X the induced morphism of cofibers*

$$\mathrm{Yo}^s_G(\mathcal{Z}, \mathcal{Z} \cap \mathcal{Y}) \longrightarrow \mathrm{Yo}^s_G(X, \mathcal{Y})$$

*is an equivalence in $\boldsymbol{G}$**Sp**$\mathcal{TB}$.*

Proof. Analoguous to Lemma 2.3.6. □

4.4 Examples

We finish this chapter with an outline to examples, inspired by [BE20, Sec 7]. The details are unfortunately work in progress (c.f. [Hei21])

We fix a complete and co-complete stable ∞-category $\mathcal{C}$.

A nice source of examples is provided by locally finite homology theories as they are in particular local homology theories:

Definition 4.4.1. A functor $F\colon \boldsymbol{G}\mathbf{TopBorn} \to \mathcal{C}$ is called a *locally finite homology theory*, if it satisfies weak excision, homotopy invariance, and if for all spaces X in $\boldsymbol{G}\mathbf{TopBorn}$ we have

$$\lim_{B \in \mathcal{B}_X^G} F(X \setminus B) \simeq 0$$

where the limit runs over all G-invariant bounded subsets of X.

Proposition 4.4.2. *Consider a locally finite homology theory $F\colon \boldsymbol{G}\mathbf{TopBorn} \to \mathcal{C}$ and a flasque space X in $\boldsymbol{G}\mathbf{TopBorn}$. Then $F(X) \simeq 0$ and thus every locally finite homology theory gives an example of a local homology theory on $\boldsymbol{G}\mathbf{TopBorn}$.*

Proof. See [BE20, Lem 7.21]. □

We proceed by constructing locally finite homology theories: We consider a functor

$$F\colon \boldsymbol{G}\mathbf{TopBorn} \longrightarrow \mathcal{C}.$$

For a morphism $f\colon Y \to X$ in $\boldsymbol{G}\mathbf{TopBorn}$ we write

$$F(X, Y) := \mathrm{Cof}(F(Y) \longrightarrow F(X)).$$

Construction 4.4.3. We define the *locally finite evaluation* of F by

$$F^{\mathrm{lf}}(X) := \lim_{B \in \mathcal{B}_X^G} F(X, X \setminus B).$$

Lemma 4.4.4. *The functor F^{lf} is locally finite.*

Proof. Analoguous to [BE20, Lem 7.17]. □

Now, a key obersation is that taking locally finite evaluation preserves homotopy invariance and weak excisiveness:

Proposition 4.4.5. *Assume that F is homotopy invariant and weakly excisive, then so is its locally finite evaluation F^{lf}. In particular F^{lf} is then a locally finite homology theory.*

Proof. See [BE20, Lem 7.35, 7.36]. □

Example 4.4.6. We denote by $\ell\colon \mathbf{Ch} \to \mathbf{Ch}_\infty := \mathbf{Ch}[W^{-1}]$ the Dwyer-Kan localization of the category of chain complex of Abelian groups by quasi-isomorphisms (c.f. [BE20, (6.4)]). Consider the singular chain complex functor C^{Sing}. We can restrict to the subcomplexes consisting of G-equivariant chains and obtain a functor $C^{Sing}_G\colon \boldsymbol{G}\mathbf{Top} \to \mathbf{Ch}$. Post-composing with ℓ and pre-composing with the forgetful functor $\mathcal{F}_{\mathcal{B}}\colon \boldsymbol{G}\mathbf{TopBorn} \to \boldsymbol{G}\mathbf{Top}$ we obtain a homotopy invariant, weakly excisive (c.f. [BE20, Ex 7.23]) functor

$$\ell \circ C^{Sing}_G \circ \mathcal{F}_{\mathcal{B}}\colon \boldsymbol{G}\mathbf{TopBorn} \longrightarrow \mathbf{Ch}_\infty.$$

In light of Proposition 4.4.5 we obtain a locally finite homology theory

$$(\ell \circ C^{Sing}_G \circ \mathcal{F}_{\mathcal{B}})^{\mathrm{lf}}\colon \boldsymbol{G}\mathbf{TopBorn} \longrightarrow \mathbf{Ch}_\infty.$$

✠

Chapter 5 Elmendorf's Theorem

We fix a group G and by $\boldsymbol{G}\mathbf{Orb}$ we denote the category of (non-empty) transitive G-sets with G-equivariant maps. Naturally, every G-topological space X in $\boldsymbol{G}\mathbf{Top}$ defines a "presheaf" E in $\mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Top})$ by $E(S) := \mathrm{Map}_{\boldsymbol{G}\mathbf{Set}}(S, X)$. Since this assignment is functorial in X we obtain a functor

$$\Phi\colon \boldsymbol{G}\mathbf{Top} \to \mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Top}).$$

This construction goes back to Elmendorf in [Elm83]. In that paper, Elmendorf also proves that there exists a functor $\Lambda\colon \mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Top}) \to \boldsymbol{G}\mathbf{Top}$ basically defined by evaluation at $G/\{e\}$ and that this functor is adjoint to Φ. Furthermore he constructs a less obvious functor that was inverse to Φ up to homotopy, showing that the homotopy categories of $\boldsymbol{G}\mathbf{Top}$ and $\mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Top})$ are equivalent. This classical result in known as "Elmendorf's Theorem". It has been generalized by several mathematicians e.g. [Sey83; DK84; CP96; MS93a].

Later, Piacenza reproved this theorem using the language of model categories in [Pia91], where he showed that Elmendorfs functor Φ and its inverse (up to homotopy) defines a Quillen equivalence (for suitable model structures on the respective categories). Elmendorf's original inverse functor can be considered the cofibrant replacement followed by Λ (c.f. [Boh+14]).

In more modern developments Elmendorf's theorem gets generalized to several other situations, e.g. [Ste10; Ste16], Elmendorf's theorem for G-categories and G-posets [Rub20; Boh+14]. We also want to refer to [Blu17, Sec. 1.3] for Elmendorf's theorem in the language of ∞-categories.

In [BC19] the authors define an analogue of Elmendorf's theorem for bornological coarse spaces with partial results towards the question if Elmendorf's theorem holds in that situation. In this chapter we proceed similarly for bornological topological spaces. In section 5.1 we define the local motivic orbit functor, which plays the role

of the functor Φ. For a more detailed analogy we refer to the introduction of section 5.1, below.
In section 5.2 we introduce the notion of Bredon-style local homology theory. These local homology theories are of a specific shape and it is closely related to Elmendorf's inverse functor. For each equivariant local homology theory there exists an associated Bredon-style homology theory together with a comparison morphism, raising the question under which conditions this comparison morphism is an equivalence.
Finally in section 5.3, we tackle this question and give some partial answers. These answers eventually prove our analogue of Elmendorf's theorem in some special cases.

5.1 The topological motivic orbit functor

To outline the aim of this section, we start by a reformulation of the classical Elmendorf theorem in greater detail than in the chapter introduction. We consider the category $\boldsymbol{G}\mathbf{Top}$ and we denote by W_G the appropriate collection of equivariant weak equivalences, i.e. the collection of morphisms $f\colon X \to Y$ in $\boldsymbol{G}\mathbf{Top}$ such that for all subgroups H of G the induced morphism on fixed points $f^H\colon X^H \to Y^H$ is a non-equivariant weak equivalence in $\mathbf{Top}$ (see c.f. [Bre67; MS93b]). We now construct a functor

$$\begin{aligned} \Psi\colon \boldsymbol{G}\mathbf{Top} &\longrightarrow \mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Top}) \\ X &\longmapsto (G/H \mapsto X^H). \end{aligned}$$

If $\ell\colon \mathbf{Top} \to \mathbf{Spc}$ denotes the functor which inverts weak equivalences, then by definition of W_G and construction of Ψ we get that $\ell_*(\Psi(f))$ becomes invertible in $\mathbf{Spc}$ for all f in W_G, hence $\ell_* \circ \Psi$ factors through the localization $\boldsymbol{G}\mathbf{Top}[W_G^{-1}]$ making the following diagram commute:

$$\begin{tikzcd} \boldsymbol{G}\mathbf{Top} \arrow[r, "\Psi"] \arrow[d] & \mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Top}) \arrow[r, "\ell_*"] & \mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}) \\ \boldsymbol{G}\mathbf{Top}[W_G^{-1}] \arrow[urr, dotted, "\hat{\Psi}"'] & & \end{tikzcd}$$

Elmendorfs theorem states (in this new language) that the induced functor $\hat{\Psi}$ is an equivalence between $\boldsymbol{G}\mathbf{Top}[W_G^{-1}]$ and $\mathbf{PSh}(\boldsymbol{G}\mathbf{Orb})$.

Our goal in this chapter is to formulate an analogue of this theorem in context of bornological topological spaces and equivariant local homology theories. To this end, we want to construct an analogue of the functor Ψ and also prove that it factorizes through $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$. To highlight the connection to "classical" Elmendorf above, we note that the functor Ψ can be reformulated as $\Psi(X)(S) := \mathrm{Map}_{\boldsymbol{G}\mathbf{Set}}(S, X)$ for all X in $\boldsymbol{G}\mathbf{Top}$ and S in $\boldsymbol{G}\mathbf{Orb}$. Now in the situation above we replace $\boldsymbol{G}\mathbf{Top}$ by $\boldsymbol{G}\mathbf{TopBorn}$ and $\mathbf{Spc}$ by $\mathbf{Spc}\mathcal{TB}$ and thus of course ℓ by Yo. Also the localization $\boldsymbol{G}\mathbf{Top} \to \boldsymbol{G}\mathbf{Top}[W_G^{-1}]$ is replaced by $\mathrm{Yo}_G\colon \boldsymbol{G}\mathbf{TopBorn} \to \boldsymbol{G}\mathbf{Spc}\mathcal{TB}$. Then we arrive at the following situation:

$$\begin{tikzcd}
\boldsymbol{G}\mathbf{TopBorn} \arrow[r, "Y"] \arrow[d, "\mathrm{Yo}_G"'] & \mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{TopBorn}) \arrow[r, "\mathrm{Yo}_*"] & \mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB}) \\
\boldsymbol{G}\mathbf{Spc}\mathcal{TB}\,. \arrow[urr, dotted, "\hat{Y}"'] & &
\end{tikzcd}$$

The goal of this section is to define a functor Y as depicted above such that $Y(X)(S) = \mathrm{Map}_{\boldsymbol{G}\mathbf{Set}}(S, X)$ (together with a suitable structure of a bornological topological space) and to prove that it has a factorization $\hat{Y}$ as shown in the diagram. The functor $\hat{Y}$ will be called the topological motivic orbit functor.

Construction 5.1.1. For a G-bornological topological space X and a G-orbit S we consider the set

$$X^{(S)} := \mathrm{Map}_{\boldsymbol{G}\mathbf{Set}}(S, X).$$

For any point s in S the evaluation map $\mathrm{ev}_s\colon X^{(S)} \to X^{G_s}$ gives a bijection between $X^{(S)}$ and the subspace X^{G_s} of X consisting of G_s-fixed points, where G_s denotes the stabilizer subgroup of s in G. We equipp X^{G_s} with the subspace structure induced by $X^{G_s} \subseteq X \hookrightarrow B_G X$ and use this bijection to endow $X^{(S)}$ with the structure of a bornological topological space. Hence we view $X^{(S)}$ as an object in $\mathbf{TopBorn}$. This construction is independent of the coice of the point s in S by Lemma 5.1.2.
For a G-equivariant map $\phi\colon S \to T$ of G-orbits we get an induced map $X^{(T)} \to X^{(S)}$ which is a morphism in $\mathbf{TopBorn}$ because the space X^{G_t} is simply a subspace of X^{G_s}.
Finally it is clear that this construction is functorial in X, hence we obtain a functor

$$y\colon \boldsymbol{G}\mathbf{TopBorn} \longrightarrow \mathrm{Fun}\left(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{TopBorn}\right), \qquad X \longmapsto \left(S \mapsto X^{(S)}\right).$$

By post-composing with the functor $\mathrm{Yo}\colon \mathbf{TopBorn} \to \mathbf{Spc}\mathcal{TB}$ we obtain a functor

$$Y\colon \boldsymbol{G}\mathbf{TopBorn} \xrightarrow{y} \mathrm{Fun}\left(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{TopBorn}\right) \xrightarrow{\mathrm{Yo}_*} \mathrm{Fun}\left(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB}\right).$$

Lemma 5.1.2. *The definition of the boronological topological structure on $X^{(S)}$ is independent of the choice of s in S.*

Proof. For another point t in S there exists an element g in G such that $gs = t$. Then we have $G_t = gG_sg^{-1}$ and one can easily show $X^{G_t} = gX^{G_s}$. Since G acts on B_GX by automorphisms the claim follows. □

Theorem 5.1.3. *The functor Y is an unstable G-equivariant local homology theory in the sense of Definition 4.2.12.*

Proof. First, for homotopy invariance we show that for all X in $\boldsymbol{G}$**TopBorn** the projection $X \otimes \mathcal{I} \to X$ induces an equivalence $Y(X \otimes \mathcal{I}) \xrightarrow{\simeq} Y(X)$ of functors in the ∞-category $\mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB})$. By [Cis19, Cor 3.5.12] this can be tested on objects, hence for an arbitrary G-orbit S we want to show that

$$\mathrm{Yo}\left(\mathrm{Map}_{\boldsymbol{G}\mathbf{Set}}(S, X \otimes \mathcal{I})\right) \longrightarrow \mathrm{Yo}\left(\mathrm{Map}_{\boldsymbol{G}\mathbf{Set}}(S, X)\right)$$

is an equivalence in $\mathbf{Spc}\mathcal{TB}$. For this, we first see that $(X \otimes \mathcal{I})^{G_s} \cong X^{G_s} \otimes \mathcal{I}$ because G acts trivially on $\mathcal{I}$. Note that this is not just a bijection of sets but an isomorphism in **TopBorn** since both spaces carry the subspace structure of $B_G(X \otimes \mathcal{I}) \cong B_G(X) \otimes \mathcal{I}$. We deduce

$$\mathrm{Map}_{\boldsymbol{G}\mathbf{Set}}(S, X \otimes \mathcal{I}) \cong \mathrm{Map}_{\boldsymbol{G}\mathbf{Set}}(S, X) \otimes \mathcal{I}$$

in **TopBorn** and by construction (c.f. Corollary 4.2.9) the functor Yo sends the morphism

$$\mathrm{Map}_{\boldsymbol{G}\mathbf{Set}}(S, X) \otimes \mathcal{I} \to \mathrm{Map}_{\boldsymbol{G}\mathbf{Set}}(S, X)$$

to an equivalence in $\mathbf{Spc}\mathcal{TB}$.

Next, we show vanishing on flasques. Consider a flasque space X in $\boldsymbol{G}$**TopBorn**. Then X is r-flasque (Lemma 4.1.17), hence we have a retract

$$r\colon B_GX \otimes \mathbb{R}_{\geq 0} \longrightarrow B_GX.$$

For any G-orbit S we choose s in S. The subspace X^{G_s} of G_s-fixed points in B_GX is r-flasque as well: Indeed, the restriction of r to $X^{G_s} \otimes \mathbb{R}_{\geq 0}$ takes values in X^{G_s} because r is G-equivariant and G acts trivially on the $\mathbb{R}_{\geq 0}$-part. Thus r restricts to a retract

$$X^{G_s} \otimes \mathbb{R}_{\geq 0} \longrightarrow X^{G_s}$$

hence $X^{(S)}$ is r-flasque. However, the functor Yo vanishes on r-flasque spaces (Corollary 4.2.9) and therefore $Y(X)(S) \simeq \mathrm{Yo}(X^{(S)})$ is initial in $\mathbf{Spc}\mathcal{TB}$. Since this is true for any G-orbit S we have shown that $Y(X)$ is initial in $\mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{op}, \mathbf{Spc}\mathcal{TB})$. Finally, to see weak excision we consider a space X in $\boldsymbol{G}\mathbf{TopBorn}$ together with two G-equivariant big families $\mathcal{Y}, \mathcal{Z}$ on X which cover X. Choose an arbitrary G-orbit S. Now memberwise taking the G_s-fixed points gives two big families $\mathcal{Y}^{G_s}$ and $\mathcal{Z}^{G_s}$ on X^{G_s} which are covering. In particular, by Corollary 4.2.9 the following square is a pushout:

$$\begin{tikzcd}
\mathrm{Yo}(\mathcal{Y}^{G_s} \cap \mathcal{Z}^{G_s}) \arrow[r] \arrow[d] & \mathrm{Yo}(\mathcal{Y}^{G_s}) \arrow[d] \\
\mathrm{Yo}(\mathcal{Z}^{G_s}) \arrow[r] & \mathrm{Yo}(X^{G_s}). \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Using that colimits in functor categories are computed objectwise we get

$$Y(\mathcal{Z})(S) \simeq (\operatorname*{colim}_{Z_i \in \mathcal{Z}} Y(Z_i))(S) \simeq \operatorname*{colim}_{Z_i \in \mathcal{Z}} (Y(Z_i)(S)) \simeq \operatorname*{colim}_{Z_i \in \mathcal{Z}} \mathrm{Yo}\left(Z_i^{G_s}\right) \simeq \mathrm{Yo}\left(\mathcal{Z}^{G_s}\right)$$

and similarly for $\mathrm{Yo}(\mathcal{Y}^{G_s})$ and $\mathrm{Yo}(\mathcal{Y}^{G_s} \cap \mathcal{Z}^{G_s})$. We obtain therefore the following pushout diagram:

$$\begin{tikzcd}
Y(\mathcal{Y} \cap \mathcal{Z})(S) \arrow[r] \arrow[d] & Y(\mathcal{Y})(S) \arrow[d] \\
Y(\mathcal{Z})(S) \arrow[r] & Y(X)(S). \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Since this is true for any G-orbit S, the diagram

$$\begin{tikzcd}
Y(\mathcal{Y} \cap \mathcal{Z}) \arrow[r] \arrow[d] & Y(\mathcal{Y}) \arrow[d] \\
Y(\mathcal{Z}) \arrow[r] & Y(X)
\end{tikzcd}$$

is a pushout in $\mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB})$ and thus the excisiveness of Y. $\square$

Corollary 5.1.4. *The functor Y factors essentually unique through $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$. More precisely, there exists an essentually unique colimit-preserving functor $\hat{Y}$ making the following diagram commute:*

$$\begin{tikzcd}
\boldsymbol{G}\mathbf{TopBorn} \arrow[r, "Y"] \arrow[d, "\mathrm{Yo}_G"'] & \mathrm{Fun}\,(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB}) \\
\boldsymbol{G}\mathbf{Spc}\mathcal{TB}\,. \arrow[ur, dotted, "\hat{Y}"'] &
\end{tikzcd}$$

Proof. See Proposition 4.2.13. □

Definition 5.1.5. We call the functor $\hat{Y}$ the *local motivic orbit functor*.

Remark 5.1.6. Since $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$ and $\mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB})$ are presentable and $\hat{Y}$ preserves all colimits it is part of an adjunction

$$\hat{Y}\colon \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \rightleftarrows \mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB}) : Z.$$

If the analogue of Elmendorf's theorem (c.f. introductory words of this section) holds in this situation, then Z is the inverse of $\hat{Y}$. ♦

5.2 Bredon-style local homology theories

To motivate the content of this section, we recall that by Elmendorf's theorem the functor $\Psi\colon \boldsymbol{G}\mathbf{Top} \to \mathbf{PSh}(\boldsymbol{G}\mathbf{Orb})$ defined by $\Psi(X)(S) := \ell(\mathrm{Map}_{\boldsymbol{G}\mathbf{Set}}(S, X))$ induces an equivalence between $\mathbf{PSh}(\boldsymbol{G}\mathbf{Orb})$ and $\boldsymbol{G}\mathbf{Top}[W_G^{-1}]$ (for details of notation we refer to the introduction of section 5.1). Let us consider a co-complete ∞-category $\mathcal{C}$ and a functor $\mathtt{F}\colon \boldsymbol{G}\mathbf{Orb} \to \mathcal{C}$. By the universal property of the Yoneda-embedding $\mathrm{yo}\colon \boldsymbol{G}\mathbf{Orb} \to \mathbf{PSh}(\boldsymbol{G}\mathbf{Orb})$[1] there exists an essentially unique colimit-preserving functor $\mathtt{F}^G\colon \mathbf{PSh}(\boldsymbol{G}\mathbf{Orb}) \to \mathcal{C}$ defined as left Kan extension of $\mathtt{F}$ along yo. To highlight the connection to the constructions during this section we rewrite this Kan extension $\mathtt{F}^G$ in form of a co-end (c.f. [Lor20, (2.27)]:

$$\mathtt{F}^G\colon \mathbf{PSh}(\boldsymbol{G}\mathbf{Orb}) \longrightarrow \mathcal{C}, \qquad E \longmapsto \int^{(S,T)\in \boldsymbol{G}\mathbf{Orb}\times\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}} \mathtt{F}(S) \otimes_{\mathbf{Spc}} E(T).$$

Precomposing $\mathtt{F}^G$ with Ψ gives a $\mathcal{C}$-valued equivariant homology theory $\boldsymbol{G}\mathbf{Top} \to \mathcal{C}$.

In summary, the Elmendorf theorem gives an equivalence between $\mathcal{C}$-valued equivariant homology theories and functors $\boldsymbol{G}\mathbf{Orb} \to \mathcal{C}$. To transport this to our setting of bornological topological spaces, we note that by [Lur09, Thm 5.1.5.6] we have an equivalence $\mathcal{C} \simeq \mathrm{Fun}^{\mathrm{colim}}(\mathbf{Spc}, \mathcal{C})$. Thus a $\mathcal{C}$-valued equivariant homology theory is essentially the same as a family of non-equivariant $\mathcal{C}$-valued homology theories index functorially over $\boldsymbol{G}\mathbf{Orb}$ (i.e. a functor $\boldsymbol{G}\mathbf{Orb} \to \mathrm{Fun}^{\mathrm{colim}}(\mathbf{Spc}, \mathcal{C})$).

Replacing again $\boldsymbol{G}\mathbf{Top}$ by $\boldsymbol{G}\mathbf{TopBorn}$ and $\mathbf{Spc}$ by $\mathbf{Spc}\mathcal{TB}$ like in the introductory text of section 5.1, we obtain the following: For a functor

$$\mathtt{F}\colon \boldsymbol{G}\mathbf{Orb} \to \mathrm{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{TB}, \mathcal{C})$$

(i.e. a family of non-equivariant local homology theories) we can define[2]

$$\mathtt{F}^G\colon \mathrm{Fun}\,(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB}\,) \longrightarrow \mathcal{C}, \qquad \int^{(S,T)\in \boldsymbol{G}\mathbf{Orb}\times\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}} \mathtt{F}(S)\big(E(T)\big).$$

Then precomposition with the topological motivic orbit functor $\hat{Y}$ gives a colimit-preserving functor

$$\mathtt{F}^G \circ \hat{Y}\colon \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \longrightarrow \mathcal{C}$$

i.e. an unstable equivariant local homology theory on $\boldsymbol{G}\mathbf{TopBorn}$. A local homol-

[1] see [Lur09, Thm 5.1.5.6]
[2] this will be done in detail during this section

ogy theory of this form is called an Bredon-style local homology theory. See e.g. [Bre67; Ill75; Jos07; Sán08; Put18] for a motivation of the nomenclature. Also we refer to [Blu17, sec. 1.4] for a nice consequence of Elmendorf's theorem to Bredon (co-)homology.

However, since we do not know if Elmendorf's theorem holds in this situation, we do not know (in contrast to the classical topological case) whether all equivariant local homology theories are of this form.

In this section we make the construction mentioned and motivated above concrete. Also, we construct for any equivariant local homology theory F an associated Bredon-style theory F^{Bredon} and we construct functorially a comparison morphism $F^{\mathrm{Bredon}} \to F$. Similar work in another setting was done in [DL98].

During this and the subsequent section we use co-ends quite repeatedly. We assume familiarity with this notion. [Lor20; MMSS; GHN20; Gla15] are good first references for that.

We fix a group G (not necessarily finite) and a co-complete ∞-category $\mathcal{C}$. Remember that pullback along the functor

$$\mathrm{Yo}_G\colon \boldsymbol{G}\mathbf{TopBorn} \longrightarrow \boldsymbol{G}\mathbf{Spc}\mathcal{TB}$$

induces an equivalence between the ∞-category of unstable G-equivariant $\mathcal{C}$-valued local homology theories and $\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Spc}\mathcal{TB}, \mathcal{C})$ (see Proposition 4.2.13).

Construction 5.2.1. We consider a family of unstable local homology theories indexed functorially over the orbit-category $\boldsymbol{G}\mathbf{Orb}$, i.e. a functor

$$\mathtt{F}\colon \boldsymbol{G}\mathbf{Orb} \to \mathrm{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{TB}, \mathcal{C}).$$

For any functor $E\colon \boldsymbol{G}\mathbf{Orb}^{\mathrm{op}} \to \mathbf{Spc}\mathcal{TB}$ we form a new functor

$$\mathtt{F} \star E\colon \boldsymbol{G}\mathbf{Orb} \times \boldsymbol{G}\mathbf{Orb}^{\mathrm{op}} \longrightarrow \mathcal{C}, \qquad (S,T) \longmapsto \mathtt{F}(S)\big(E(T)\big).$$

Moreover, we define

$$\mathtt{F}^G(E) := \int^{\boldsymbol{G}\mathbf{Orb}} \mathtt{F} \star E.$$

This expression is functorial in $\mathtt{F}$ and E. In particular we have constructed a functor

$$\mathtt{F}^G\colon \mathrm{Fun}\,(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB}) \to \mathcal{C}$$

and thus a also a functor

$$(-)^G\colon \mathrm{Fun}\Big(\boldsymbol{G}\mathbf{Orb}, \mathrm{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{TB}, \mathcal{C})\Big) \longrightarrow \mathrm{Fun}\Big(\mathrm{Fun}\left(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB}\right), \mathcal{C}\Big)$$
$$\mathtt{F} \longmapsto \left(E \mapsto \int^{(S,T)} \mathtt{F}(S)\big(E(T)\big)\right).$$

Remark 5.2.2. Consider a functor $\mathtt{F}$ as in Construction 5.2.1. For all G-orbits S the functor $\mathtt{F}(S)$ is colimit-preserving. Moreover, since the co-end is defined as a colimit (c.f. [Lor20, Def 7.3.3]), also forming the co-end preserves colimits. Hence $\mathtt{F}^G$ preserves colimits:

$$\mathtt{F}^G\left(\mathrm{colim}_i E_i\right) \simeq \int^{(S,T)} \mathtt{F}(S)\left(\left(\mathrm{colim}_i E_i\right)(T)\right) \simeq \int^{(S,T)} \mathtt{F}(S)\left(\mathrm{colim}_i E_i(T)\right)$$
$$\simeq \mathrm{colim}_i \int^{*(S,T)} \mathtt{F}(S)(E_i(T)) \simeq \mathrm{colim}_i \mathtt{F}^G(E_i).$$

In particular, the functor $(-)^G$ defined in Construction 5.2.1 refines to a functor

$$(-)^G\colon \mathrm{Fun}\Big(\boldsymbol{G}\mathbf{Orb}, \mathrm{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{TB}, \mathcal{C})\Big) \longrightarrow \mathrm{Fun}^{\mathrm{colim}}\Big(\mathrm{Fun}\left(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB}\right), \mathcal{C}\Big).$$

♦

Recall the local motivic orbit functor $\hat{Y}\colon \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \to \mathrm{Fun}\left(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB}\right)$ from Corollary 5.1.4.

Lemma 5.2.3. *For any functor* $\mathtt{F}\colon \boldsymbol{G}\mathbf{Orb} \to \mathrm{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{TB}, \mathcal{C})$ *as in Construction* 5.2.1 *the composition*

$$\mathtt{F}^G \circ \hat{Y}\colon \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \longrightarrow \mathcal{C}$$

preserves colimits and hence defines an unstable G-equivariant local homology theory

$$\mathtt{F}^G \circ Y\colon \boldsymbol{G}\mathbf{TopBorn} \longrightarrow \mathcal{C}.$$

Proof. By construction the functor $\hat{Y}$ preserves colimits (c.f. Corollary 5.1.4) and $\mathtt{F}^G$ preserves colimits by Remark 5.2.2. □

Definition 5.2.4. We call unstable G-equivariant local homology theories of the form $\mathtt{F}^G \circ Y$ *Bredon-style local homology theories.*

In light of Proposition 4.2.13 we also refer to functors of the form $\mathtt{F}^G \circ \hat{Y}$ as Bredon-style local homology theories.

As elaborated in the introduction of this section, every equivariant homology theory on $\boldsymbol{G}\mathbf{Top}$ is (equivalent to) an Bredon-style theory, however we do not expect the same result in the setting of $\boldsymbol{G}\mathbf{TopBorn}$ (see Question 5.2.13).

Our next goal is to construct for every unstable G-equivariant local homology theory F an associated Bredon-style one F^{Bredon} together with a comparison morphism $F^{\mathrm{Bredon}} \to F$. The rough idea for the family of non-equivariant local homology theories $\mathtt{F}$ will be to twist a local motivic space X in $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$ with an orbit S and then evaluate via F. To that end we first have to define this twist:

Lemma 5.2.5. *The functor*

$$\boldsymbol{G}\mathbf{Set} \times \boldsymbol{G}\mathbf{TopBorn} \xrightarrow{(-)_{dM} \otimes (-)} \boldsymbol{G}\mathbf{TopBorn} \xrightarrow{\mathrm{Yo}_G} \boldsymbol{G}\mathbf{Spc}\mathcal{TB}$$

has an essentially unique factorization

$$\begin{tikzcd} \boldsymbol{G}\mathbf{Set} \times \boldsymbol{G}\mathbf{TopBorn} \arrow[r] \arrow[d, "\mathrm{id} \times \mathrm{Yo}_G"'] & \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \\ \boldsymbol{G}\mathbf{Set} \times \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \arrow[ur, dotted, "Tw"'] & \end{tikzcd}$$

which preserves colimits in its second argument.

Proof. By Proposition 4.2.13 it is enough to check that the assignment

$$F\colon X \mapsto \mathrm{Yo}_G(S_{dM} \otimes X)$$

is an unstable G-equivariant local homology theory. For homotopy invariance we first notice that by associativity $(S_{dM} \otimes X) \otimes \mathcal{I} \cong S_{dM} \otimes (X \otimes \mathcal{I})$, hence the morphism

$$\begin{tikzcd} F(X \otimes \mathcal{I}) \arrow[rr] \arrow[d, equal] & & F(X) \arrow[d, equal] \\ \mathrm{Yo}_G(S_{dM} \otimes (X \otimes \mathcal{I})) & & \mathrm{Yo}_G(S_{dM} \otimes X) \end{tikzcd}$$

induced by the projection can be identified with the morphism

$$\mathrm{Yo}_G((S_{dM} \otimes X) \otimes \mathcal{I}) \longrightarrow \mathrm{Yo}_G(S_{dM} \otimes X)$$

which is an equivalence by homotopy invariance of Yo_G (c.f. Corollary 4.2.9).

Next, we consider a flasque G-equivariant bornological topological space X. Then

X is r-flasque (Lemma 4.1.17), hence we have a retract

$$r\colon B_G X \otimes \mathbb{R}_{\geq 0} \longrightarrow B_G X.$$

Now we tensor this morphism with S_{dM} and since the bornology on S_{dM} is maximal (i.e. S_{dM} is G-complete) we can identify $S_{dM} \otimes B_G X$ with $B_G(S_{dM} \otimes X)$ and we obtain a retract

$$\mathrm{id} \otimes r\colon B_G(S_{dM} \otimes X) \otimes \mathbb{R}_{\geq 0} \longrightarrow B_G(S_{dM} \otimes X)$$

which shows that $S_{dM} \otimes X$ is r-flasque, hence $F(X) \simeq \mathrm{Yo}(S_{dM} \otimes X)$ is initial in $\mathbf{Spc}\mathcal{TB}$ (by Corollary 4.2.9), hence F vanishes on flasques.
Finally, for excisiveness of F we consider a G-equivariant bornological topological space X and a G-equivariant complementary pair $(\mathcal{Y}, \mathcal{Z})$ on X. It is immediate that memberwise tensoring the big families with S_{dM} gives two G-invariant big families $S_{dM} \otimes \mathcal{Y}$ and $S_{dM} \otimes \mathcal{Z}$ which cover $S_{dM} \otimes X$. Hence excisiveness of Yo (Corollary 4.2.9) implies excisiveness of F and we are through. □

Definition 5.2.6. For a G-orbit S and a G-equivariant local motivic space X in $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$ we write $S_{dM} \otimes X := Tw(S, X)$.

Remark 5.2.7. Consider a G-equivariant local motivic space X in $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$. We can write X as colimit

$$X \simeq \operatorname*{colim}_{\mathrm{yo}(X_i) \to X} \mathrm{Yo}_G(X_i).$$

Therefore we derive a concrete formula for $S_{dM} \otimes X$ given by

$$S_{dM} \otimes X \simeq \operatorname*{colim}_{\mathrm{yo}(X_i) \to X} \mathrm{Yo}_G(S_{dM} \otimes X_i). \qquad \blacklozenge$$

We fix an unstable G-equivariant local homology theory corresponding (Proposition 4.2.13) to a colimit-preserving functor

$$F\colon \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \longrightarrow \mathcal{C}.$$

As mentioned above, we want to construct an associated Bredon-style theory.

Construction 5.2.8. We define a family of unstable $\mathcal{C}$-valued non-equivariant local homology theories (indexed functorially over $\boldsymbol{G}\mathbf{Orb}$) by

$$\mathtt{F}\colon \boldsymbol{G}\mathbf{Orb} \longrightarrow \mathrm{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{TB}, \mathcal{C}), \qquad S \longmapsto \Big(X \mapsto F\left(S_{dM} \otimes X\right)\Big).$$

Note that $\mathtt{F}(S)$ is colimit-preserving by Lemma 5.2.5 and because F preserves colimits, hence $\mathtt{F}$ is a well-defined. Furthermore, this construction in functorial in F, hence we obtain a functor

$$\begin{aligned}\mathscr{F}\colon \mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Spc}\mathcal{TB}, \mathcal{C}) &\longrightarrow \mathrm{Fun}\left(\boldsymbol{G}\mathbf{Orb}, \mathrm{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{TB}, \mathcal{C})\right)\\ F &\longmapsto \Big(S \mapsto \big(X \mapsto F(S_{dM} \otimes X)\big)\Big).\end{aligned}$$

Recall the functor $(-)^G$ from Remark 5.2.2. The composition[3]

$$\mathscr{B} := \left(\hat{Y}\right)^* \circ (-)^G \circ \mathscr{F}\colon \mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Spc}\mathcal{TB}, \mathcal{C}) \longrightarrow \mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Spc}\mathcal{TB}, \mathcal{C})$$

is called the *Bredon-approximation functor*.
For any functor F in $\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Spc}\mathcal{TB}, \mathcal{C})$ we denote by $F^{\mathrm{Bredon}} := \mathscr{B}(F)$ the *Bredon-style local homology theory associated to F*.

Remark 5.2.9. For convenience of the reader we give a more explicit formula for the Bredon-approximation functor $\mathscr{B}$ constructed in Construction 5.2.8. For a functor F in $\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Spc}\mathcal{TB}, \mathcal{C})$ and a G-equivariant local motivic space X in $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$ we have

$$\begin{aligned}F^{\mathrm{Bredon}}(X) &\simeq \mathscr{B}(F)(X) \simeq \mathscr{F}(F)^G\left(\hat{Y}(X)\right)\\ &\simeq \int^{(S,T)} \mathscr{F}(F)(S)\left(\hat{Y}(X)(T)\right) \simeq \int^{(S,T)} F\left(S_{dM} \otimes \hat{Y}(X)(T)\right).\end{aligned}$$

If we denote by $F' := F \circ \mathrm{Yo}_G$ the corresponding unstable G-equivariant local homology theory $\boldsymbol{G}\mathbf{TopBorn} \to \mathcal{C}$, then for all G-equivariant bornological topological spaces X we obtain

$$(F')^{\mathrm{Bredon}}(X) \simeq \int^{(S,T)} F'\left(S_{dM} \otimes X^{(T)}\right)$$

because all constructions commute with colimits. ♦

[3]Note that $\mathscr{B}$ takes values in colimit-preserving functors by Lemma 5.2.3

We finish this section by constructing a comparison morphism $F^{\mathrm{Bredon}} \to F$.

Lemma 5.2.10. *For a G-equivariant bornological topological space X and a G-orbit S the evaluation map*

$$S \times \mathrm{Map}_{\boldsymbol{G}\mathbf{Set}}(S, X) \longrightarrow X$$

induces a morphism

$$e_S\colon S_{dM} \otimes X^{(S)} \longrightarrow X$$

in $\boldsymbol{G}\mathbf{TopBorn}$. In particular, we obtain for any fixed G-orbit S a natural transformation

$$S_{dM} \otimes (-)^{(S)} \longrightarrow \mathrm{id}_{\boldsymbol{G}\mathbf{TopBorn}}$$

of functors $\boldsymbol{G}\mathbf{TopBorn} \to \boldsymbol{G}\mathbf{TopBorn}$. Moreover, for a morphism $\phi\colon S \to T$ in $\boldsymbol{G}\mathbf{Orb}$ the following square commutes for all X in $\boldsymbol{G}\mathbf{TopBorn}$:

$$\begin{tikzcd}
S_{dM} \otimes X^{(T)} \arrow[r, "\mathrm{id} \otimes \phi^*"] \arrow[d, "\phi_{dM} \otimes \mathrm{id}"'] & S_{dM} \otimes X^{(S)} \arrow[d, "e_S"] \\
T_{dM} \otimes X^{(T)} \arrow[r, "e_T"'] & X.
\end{tikzcd}$$

Proof. We only check that the evaluation map e_S is a morphism in $\boldsymbol{G}\mathbf{TopBorn}$. The rest of the lemma follows obviously. First, note that G acts trivially on $X^{(S)}$, hence

$$e_S(g \cdot (s, f)) = e_S(g \cdot s, f) = f(g \cdot s) = g \cdot f(s) = g \cdot e_S(s, f)$$

showing G-equivariance of e_S. To see continuity and properness we fix some s_0 in S. Every s in S is of the form $s = g \cdot s_0$ for some g in G. By G-equivariance of maps in $X^{(S)}$ we derive for every subset Y of X that

$$e_s^{-1}(Y) = e_{s_0}^{-1}(g^{-1} \cdot Y), \tag{*}$$

where $e_s\colon X^{(S)} \to X$ denotes the evaluation map $f \mapsto f(s)$ for every s in S. However, if U is an open subset of X then all translates $g \cdot U$ are open since G acts by automorphisms and since we used e_{s_0} to transfer the topology of X to $X^{(S)}$ the pre-images $e_{s_0}^{-1}(g^{-1} \cdot U)$ are open in $X^{(S)}$ for all g in G. Therefore, also the subset

$$e_S^{-1}(U) = \bigcup_{s \in S} \{s\} \times e_s^{-1}(U) \overset{(*)}{=} \bigcup_{s \in S} \{s\} \times e_{s_0}^{-1}(g^{-1} \cdot U)$$

is open (note that the topology on S_{dM} is discrete) and thus e_S is continuous. Similarly, if B is a bounded subset of X, then GB is bounded in $B_G X$ and since we transfered the bornology of $B_G X$ to $X^{(S)}$ via e_{s_0} the pre-image $e_{s_0}^{-1}(GB)$ is bounded in $X^{(S)}$. We derive that the pre-image

$$e_S^{-1}(B) = \bigcup_{s\in S} \{s\} \times e_s^{-1}(B) \overset{(*)}{\subseteq} S \times e_{s_0}^{-1}(GB)$$

is bounded since the latter supset is bounded (note that $\mathcal{B}_S$ is maximal). $\square$

The statement of the lemma above descents to the level of G-equivariant local motivic spaces. More precisely:

Corollary 5.2.11. *For a fixed G-orbit S and G-equivariant local motivic space X the evaluation morphism e_S in Lemma 5.2.10 induces an evaluation map*

$$\varepsilon_S \colon S_{dM} \otimes \hat{Y}(X)(S) \longrightarrow X$$

which is functorial in X. Moreover, for any morphism $\phi\colon S \to T$ in $\boldsymbol{G}$**Orb** *the following square commutes:*

$$\begin{CD} S_{dM} \otimes \hat{Y}(X)(T) @>{\mathrm{id} \otimes \hat{Y}(X)(\phi)}>> S_{dM} \otimes \hat{Y}(X)(S) \\ @V{\phi_{dM} \otimes \mathrm{id}}VV @VV{\varepsilon_S}V \\ T_{dM} \otimes \hat{Y}(X)(T) @>>{\varepsilon_T}> X. \end{CD} \tag{5.2.1}$$

Proof. We write X as a colimit $X \simeq \mathrm{colim}_i \operatorname{Yo}_G(X_i)$. Then for any i we have the evaluation morphism $e_S\colon S_{dM} \otimes X_i^{(S)} \to X_i$ functorially in X_i. Applying Yo_G and then taking the colimit therefore provides a morphism

$$\operatorname*{colim}_i \operatorname{Yo}_G\left(S_{dM} \otimes X_i^{(S)}\right) \longrightarrow \operatorname*{colim}_i \operatorname{Yo}_G(X_i) \simeq X \tag{*}$$

By construction and definition (Remark 5.2.7 and Corollary 5.1.4) we have

$$\operatorname*{colim}_i \operatorname{Yo}_G\left(S_{dM} \otimes X_i^{(S)}\right) \simeq S_{dM} \otimes \hat{Y}(X)(S).$$

Therefore the morphism in (*) gives the morphism ε_S in the claim. The remaining statements follow construction of ε_S and the corresponding statements on the level of bornological topological spaces. $\square$

Recall that F denoted a colimit-preserving functor $\boldsymbol{G}\mathbf{Spc}\mathcal{TB} \to \mathcal{C}$ (associated to the unstable G-equivariant local homology theory $F \circ \mathrm{Yo}_G$). For this functor we wanted to construct a comparison map between its associated Bredon-style homology theory F^{Bredon} and F.

Construction 5.2.12. Consider a G-equivariant local motivic space X. Since for all G orbits S and T and all morphisms $\phi\colon S \to T$ in $\boldsymbol{G}\mathbf{Orb}$ the square (5.2.1) in Corollary 5.2.11 commutes, the universal property of the coend provides a (unique) morphism

$$\varepsilon^G\colon \int^{(S,T)} S_{dM} \otimes \hat{Y}(X)(T) \longrightarrow X.$$

Now we apply the functor F to ε^G and obtain:

$$\beta_{F,X}\colon F\left(\int^{(S,T)} S_{dM} \otimes \hat{Y}(X)(T)\right) \longrightarrow F(X).$$

Since F preserves colimits and the coend is constructed as a colimit (c.f. [Lor20, Def 7.3.3]) we can commute F with the coend and see that the domain of $\beta_{F,X}$ is equivalent to

$$F\left(\int^{(S,T)} S_{dM} \otimes \hat{Y}(X)(T)\right) \simeq \int^{(S,T)} F\left(S_{dM} \otimes \hat{Y}(X)(T)\right) \simeq F^{\mathrm{Bredon}}(X).$$

Since the comparison map $\beta_{F,X}$ is natural in F and X we obtain the *comparison morphism*

$$\beta\colon \mathscr{B} \longrightarrow \mathrm{id}_{\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Spc}\mathcal{TB},\mathcal{C})}\,.$$

Question 5.2.13. This raises the following questions (in decreasing strength):

1. Is β an equivalence, i.e. do we have an equivalence

$$\beta_F\colon F^{\mathrm{Bredon}} \longrightarrow F$$

   for all F in $\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Spc}\mathcal{TB},\mathcal{C})$?

2. If not: For which functors F is $F^{\mathrm{Bredon}} \to F$ an equivalence?

3. For a functor F in $\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Spc}\mathcal{TB},\mathcal{C})$, is there a full subcategory $\mathcal{S}$ of $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$ such that $F^{\mathrm{Bredon}}(X) \to F(X)$ is an equivalence for all X in $\mathcal{S}$?

5.3 An analogue of Elmendorf's theorem

In the introduction to this chapter we explained Elmendorf's theorem. In section 5.1 we started to construct an analogue statement of this theorem in the realm of G-equivariant bornological topological spaces and G-equivariant local homology theories by defining the local motivic orbit functor. In classical Elmendorf situation this functor is an equivalence. In section 5.2 we have then elaborated on the relation between Elmendorf's theorem and Bredon-style homology theories. Building on that, we start this section by studying the comparison morphisms constructed in section 5.2. We will see that it is – under some conditions – an equivalence when restricted to a certain full subcategory of $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$ (Proposition 5.3.9 and subsequent corollaries).

If we restrict to finite groups – an assumption which was also stated in classical Elmendorf theorem[4] – then we deduce further that the Bredon-approximation functor is idempotent, i.e. a Bousfield localization (Proposition 5.3.14).

Finally, in (5.3.1) we define a candidate for the inverse $\mathcal{A}$ of $\hat{Y}$. The construction of this functor is inspired by Elmendorf's construction as we motivated in the introduction of section 5.2. Again assuming that G is finite we are able to prove that $\mathcal{A}$ is left adjoint to $\hat{Y}$ (Corollary 5.3.20). We finish the section (and this thesis) by calculating that the unit of the adjunction is (if G is finite) an equivalence (Proposition 5.3.21) and the co-unit becomes an equivalence after restriction to a full subcategory (follows from Corollary 5.3.10).

We start this section by recalling the following two statements which are known under several names like the *(Co-)Yoneda formula*, the *Density theorem* or *Ninja Yoneda lemma.*

Lemma 5.3.1. *For a functor* $F\colon \boldsymbol{G}\mathbf{Orb} \to \mathcal{C}$ *and a functor* $G\colon \boldsymbol{G}\mathbf{Orb}^{\mathrm{op}} \to \mathcal{C}$ *we have*

$$\int^{(S,T)\in \boldsymbol{G}\mathbf{Orb}\times \boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}} \mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(T,-) \otimes_{\mathbf{Set}} F(S) \simeq F$$

and

$$\int^{(S,T)\in \boldsymbol{G}\mathbf{Orb}\times \boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}} \mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(-,S) \otimes_{\mathbf{Set}} G(T) \simeq G.$$

Proof. See [MMSS, Lem 1.6] or [Lor20, Prop 2.2.1] and also [BC19]. □

[4] some of the generalizations however allowed e.g. infinite compact Lie groups

For any set M and a G-equivariant bornological topological space X we denote by $M \otimes_{\mathbf{Set}} X$ the G-equivariant bornological topological space given by the "M-fold coproduct" $\coprod_M X$ in $\boldsymbol{G}\mathbf{TopBorn}$[5]. Likewise, for a G-equivariant local motivic space X by $M \otimes_{\mathbf{Set}} X$ we mean the coproduct $\coprod_M X$ in $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$.

Lemma 5.3.2. *Let M be a finite set and X be a space in $\boldsymbol{G}\mathbf{TopBorn}$. Then we have an equivalence*

$$\mathrm{Yo}_G\left(M_{dM} \otimes X\right) \simeq M \otimes_{\mathbf{Set}} \mathrm{Yo}_G(X)$$

of G-equivariant local motivic spaces.

Proof. We first show that the obvious bijection of sets $M \times X \cong \coprod_M X$ gives an isomorphism of spaces $M_{dM} \otimes X \cong M \otimes_{\mathbf{Set}} X$ in $\boldsymbol{G}\mathbf{TopBorn}$. For the bornology we note that a generating bounded subset of $M_{dM} \otimes X$ is of the form $M \times B$ for some B in $\mathcal{B}_X$ (c.f. Lemma 3.1.24). The corresponding subset in $M \otimes_{\mathbf{Set}} X$ is given by $\coprod_M B$ which is bounded in $M \otimes_{\mathbf{Set}} X$ by Remark 3.2.11. On the other hand, a generating bounded subset of $M \otimes_{\mathbf{Set}} X$ is of the form $\coprod_{m \in M} B_m$ for B_m in $\mathcal{B}_X$. It corresponds to the following subset of $M_{dM} \otimes X$:

$$\Big\{(m, x) \;\Big|\; m \in M,\ x \in B_m\Big\}.$$

This subset is contained in $M \times B$ for $B := \bigcup_{m \in M} B_m$. Note that B is bounded in X since M was finite.
For the topologies we observe that subsets of the form $\{m\} \times U$ (for U in $\mathcal{O}_X$) forms a basis for the topology on $M_{dM} \otimes X$. With this we easily derive the claim.

To complete the proof of this lemma we note that by Corollary 4.2.10 we have

$$\mathrm{Yo}_G(X \amalg X) \simeq \mathrm{Yo}_G(X) \amalg \mathrm{Yo}_G(X).$$

Using induction (note that M is finite!) we derive

$$\mathrm{Yo}_G(M \otimes_{\mathbf{Set}} X) \simeq M \otimes_{\mathbf{Set}} \mathrm{Yo}_G(X). \qquad \square$$

[5]Note that arbitrary coproducts exist in $\boldsymbol{G}\mathbf{TopBorn}$ by Corollary 3.2.10

Corollary 5.3.3. *For all finite sets M and all (unstable) equivariant local homology theories $F\colon$ **GTopBorn** $\to \mathcal{C}$ we have*

$$F(M_{dM} \otimes X) \simeq M \otimes_{\mathbf{Set}} F(X).$$

Proof. By Proposition 4.2.13 the functor F factors as $F' \circ \mathrm{Yo}_G$ for some colimit-preserving functor $F'\colon \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \to \mathcal{C}$. Now

$$\begin{aligned} F(M_{dM} \otimes X) &\simeq F'(\mathrm{Yo}_G(M_{dM} \otimes X)) \overset{\text{Lemma 5.3.2}}{\simeq} F'(M \otimes_{\mathbf{Set}} \mathrm{Yo}_G(X)) \\ &\overset{!}{\simeq} W \otimes_{\mathbf{Set}} F'(\mathrm{Yo}_G(X)) \simeq W \otimes_{\mathbf{Set}} F(X). \end{aligned}$$

Here the equivalence marked with (!) is true since F' preserves colimits. □

Corollary 5.3.4. *For all finite sets M and all functors F' in* $\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Spc}\mathcal{TB}, \mathcal{C})$ *we have an equivalence*

$$F'(M_{dM} \otimes X) \simeq M \otimes_{\mathbf{Set}} F'(X).$$

Proof. We set $F := F' \circ \mathrm{Yo}_G$ and obtain an unstable equivariant local homology theory $F\colon$ **GTopBorn** $\to \mathcal{C}$. Now we write $X \simeq \mathrm{colim}_i \mathrm{Yo}_G(X_i)$, then we have $M_{dM} \otimes X \simeq \mathrm{colim}_i \mathrm{Yo}_G(M_{dM} \otimes X_i)$ by Remark 5.2.7. We calculate using that F' preserves colimits:

$$\begin{aligned} F'(M_{dM} \otimes X) &\simeq \operatorname*{colim}_i F(M_{dM} \otimes X_i) \overset{!}{\simeq} \operatorname*{colim}_i M \otimes_{\mathbf{Set}} F(X_i) \\ &\simeq M \otimes_{\mathbf{Set}} \operatorname*{colim}_i F(X_i) \simeq M \otimes_{\mathbf{Set}} F'(X). \end{aligned}$$

Here the equivalence marked with (!) is Corollary 5.3.3. □

The proof of Lemma 5.3.2 used finiteness at two places: To ensure that a certain union of bounded subsets of X is bounded and to inductively use excision for Yo_G. If the space X is bounded itself, then for the first assertion here, we no longer need M to be finite. But excision w.r.t. infinite coproducts in not true in general. This motivates the notion of hyperexcision which we define in the following.

For a set M and a space X in **GTopBorn** we have a family of morphisms

$$\iota_m\colon X \longrightarrow M_{dM} \otimes X, \qquad x \longmapsto (m, x)$$

for all m in M. These are indeed morphisms (i.e. proper and continuous) with similar arguments as in the proof of Lemma 5.3.2. For a functor $F\colon \boldsymbol{G}\mathbf{TopBorn} \to \mathcal{C}$ we get a family of induced morphisms

$$F(\iota_m)\colon F(X) \longrightarrow F(M_{dM} \otimes X)$$

which induces a morphism

$$M \otimes_{\mathbf{Set}} F(X) \longrightarrow F(M_{dM} \otimes X).$$

In the following we fix a colimit-preserving functor $F'\colon \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \to \mathcal{C}$ with corresponding unstable local homology theory $F := F' \circ \mathrm{Yo}_G$ (c.f. Proposition 4.2.13).

We denote by $\boldsymbol{G}\mathbf{TopBorn}_{\mathbf{bd}}$ the full subcategory of $\boldsymbol{G}\mathbf{TopBorn}$ consisting of the bounded spaces, i.e. all spaces X with $X \in \mathcal{B}_X$.

Definition 5.3.5. We say that the local homology theory F is *hyperexcisive* if for all sets M and all spaces X in $\boldsymbol{G}\mathbf{TopBorn}_{\mathbf{bd}}$ the induced morphism

$$M \otimes_{\mathbf{Set}} F(X) \longrightarrow F(M_{dM} \otimes X)$$

is an equivalence in $\mathcal{C}$.
The functor F' is called hyperexcisive, if the corresponding $F \simeq F' \circ \mathrm{Yo}_G$ is hyperexcisive.

Let us denote by $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}_{\mathbf{bd}}$ the full subcategory of $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$ generated under colimits by objects $\mathrm{Yo}_G(X)$ for X in $\boldsymbol{G}\mathbf{TopBorn}_{\mathbf{bd}}$.

Corollary 5.3.6. *The functor F' is hyperexcisive if and only if for all sets M and all spaces X in $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}_{\mathbf{bd}}$ we have an equivalence*

$$M \otimes_{\mathbf{Set}} F'(X) \xrightarrow{\simeq} F'(M_{dM} \otimes X).$$

Proof. If we write $X \simeq \operatorname{colim}_i \mathrm{Yo}_G(X_i)$, then by Remark 5.2.7 we have

$$M_{dM} \otimes X \simeq \operatorname*{colim}_i \mathrm{Yo}_G(M_{dM} \otimes X).$$

Using that F' preserves colimits, we see

$$F'(M_{dM} \otimes X) \simeq \operatorname*{colim}_i F'(\mathrm{Yo}_G(M_{dM} \otimes X_i)) \simeq \operatorname*{colim}_i F(M_{dM} \otimes X_i).$$

On the other hand since $M \otimes_{\mathbf{Set}} (-)$ commutes with colimits we see

$$M \otimes_{\mathbf{Set}} F'(X) \simeq M \otimes_{\mathbf{Set}} \operatorname*{colim}_i F(X_i) \simeq \operatorname*{colim}_i (M \otimes_{\mathbf{Set}} F(X_i)) .$$

With that, the assertions become clear. □

Recall that the functor

$$\begin{aligned} Y \colon \boldsymbol{G}\mathbf{TopBorn} &\longrightarrow \operatorname{Fun}(\boldsymbol{G}\mathbf{Orb}^{op}, \mathbf{Spc}\mathcal{TB}) \\ X &\longmapsto \left(S \mapsto X^{(S)}\right) \end{aligned}$$

has a unique colimit-preserving factorization

$$\begin{tikzcd} \boldsymbol{G}\mathbf{TopBorn} \arrow[r, "Y"] \arrow[d, "\mathrm{Yo}_G"'] & \operatorname{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB}) \\ \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \, . \arrow[ur, dotted, "\hat{Y}"'] & \end{tikzcd}$$

The following lemma or more precisely its subsequent corollary will be useful at several places within this section.

Lemma 5.3.7. *Consider two G-orbits R and S and a bornological topological space X in* **TopBorn**. *Then we have an isomorphism*

$$(S_{dM} \otimes X)^{(R)} \cong \mathrm{Map}_{\boldsymbol{G}\mathbf{Set}}(R, S)_{dM} \otimes X$$

of bornological topological spaces, where the structure on the left hand side is defined as in Construction 5.1.1.

Proof. We fix some r in R and we denote by G_r the stabilizer of r. By construction the space $(S_{dM} \otimes X)^{(R)}$ carries the structure transported via the bijection to the subspace $(S_{dM} \otimes X)^{G_r}$ of G_r-fixed points in the G-completion $B_G(S_{dM} \otimes X)$. However, since $S_{dM} \otimes X$ is G-complete (G acts trivially on X), the set $(S_{dM} \otimes X)^{G_r}$ carries the subspace structure of $S_{dM} \otimes X$. Now, since G acts trivially on X we obtain

$$(S_{dM} \otimes X)^{G_r} = (S^{G_r})_{dM} \otimes X$$

as bornological topological spaces which gives the claim. □

Corollary 5.3.8. *For G-orbits R and S and a non-equivariant local motivic space X in $\mathbf{Spc}\mathcal{TB}$ we have an equivalence*

$$\hat{Y}(S_{dM} \otimes X)(R) \simeq \mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(R,S)_{dM} \otimes X$$

in $\mathbf{Spc}\mathcal{TB}$.

Proof. We write $X \simeq \mathrm{colim}_i \,\mathrm{Yo}(X_i)$. Then we have $S_{dM} \otimes X \simeq \mathrm{colim}_i \,\mathrm{Yo}_G(S_{dM} \otimes X_i)$ (c.f. Remark 5.2.7). We calculate using that $\hat{Y}$ preserves colimits

$$\begin{aligned}
\hat{Y}(S_{dM} \otimes X)(R) &\simeq \operatorname*{colim}_i \hat{Y}(\mathrm{Yo}_G(S_{dM} \otimes X_i))(R) \simeq \operatorname*{colim}_i Y(S_{dM} \otimes X_i)(R)\\
&\simeq \operatorname*{colim}_i \mathrm{Yo}\left(\mathrm{Map}_{\boldsymbol{G}\mathbf{Set}}(R, S_{dM} \otimes X_i)\right)\\
&\overset{!}{\simeq} \operatorname*{colim}_i \mathrm{Yo}\left(\mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(R,S)_{dM} \otimes X_i\right)\\
&\overset{!!}{\simeq} \mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(R,S)_{dM} \otimes \operatorname*{colim}_i \mathrm{Yo}(X_i) \simeq \mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(R,S)_{dM} \otimes X.
\end{aligned}$$

Here the equivalence marked with (!) is Lemma 5.3.7 and the equivalence marked with (!!) is again Remark 5.2.7. □

Recall the questions raised in Question 5.2.13. We get a first answer to the weakest question there. We fix a functor F in $\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Spc}\mathcal{TB}, \mathcal{C})$, an orbit T in $\boldsymbol{G}\mathbf{Orb}$ and a non-equivariant local motivic space X in $\mathbf{Spc}\mathcal{TB}$.

Proposition 5.3.9. *Assume* one *of the following conditions:*

1. *The group G is finite,*
2. *the functor F is hyperexcisive and X is bounded, i.e. X is contained in $\mathbf{Spc}\mathcal{TB}_{\mathbf{bd}}$.*

Then the Bredon-approximation

$$\beta_{F,T_{dM}\otimes X} \colon F^{\mathrm{Bredon}}(T_{dM} \otimes X) \longrightarrow F(T_{dM} \otimes X)$$

is an equivalence in $\mathcal{C}$.

Proof. Consider two G-orbits R and S. We calculate

$$\begin{aligned}
F\left(S_{dM} \otimes \hat{Y}(T_{dM} \otimes X)(R)\right) &\overset{!}{\simeq} F\left(S_{dM} \otimes \mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(R,T)_{dM} \otimes X\right)\\
&\simeq F\left(\mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(R,T)_{dM} \otimes (S_{dM} \otimes X)\right)\\
&\overset{!!}{\simeq} \mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(R,T) \otimes_{\mathbf{Set}} F(S_{dM} \otimes X) \qquad (*)
\end{aligned}$$

where the equivalence marked with (!) is by Corollary 5.3.8 and the equivalence marked with (!!) is either due to Corollary 5.3.3 if G is finite (and therefore the set $\mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(R,T)$ is finite) or due to Corollary 5.3.6 if F is hyperexcisive (note that if $X \in \mathbf{Spc}\mathcal{TB}_{\mathbf{bd}}$ then $S_{dM} \otimes X \in \boldsymbol{G}\mathbf{Spc}\mathcal{TB}_{\mathbf{bd}}$).

Now we deduce the claim by

$$\begin{aligned} F^{\mathrm{Bredon}}(T_{dM} \otimes X) &\overset{\text{Remark 5.2.9}}{\simeq} \int^{(S,R)} F\left(S_{dM} \otimes \hat{Y}(T_{dM} \otimes X)(R)\right) \\ &\overset{(*)}{\simeq} \int^{(S,R)} \mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(R,T) \otimes_{\mathbf{Set}} F(S_{dM} \otimes X) \\ &\overset{\text{Lemma 5.3.1}}{\simeq} F(T_{dM} \otimes X). \end{aligned}$$

□

Two immediate consequences also provide partial answers to Question 5.2.13. For a nice formulation of the consequences we introduce some notation: By

$$\boldsymbol{G}\mathbf{Spc}\mathcal{TB}\ \langle \boldsymbol{G}\mathbf{Orb} \otimes \mathbf{Spc}\mathcal{TB}\,]\rangle$$

we denote the full subcategory of $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$ generated under colimits by objects of the form $S_{dM} \otimes X$ for G-orbits S and non-equivariant local motivic spaces X in $\mathbf{Spc}\mathcal{TB}$. Similarly by

$$\boldsymbol{G}\mathbf{Spc}\mathcal{TB}\ \langle \boldsymbol{G}\mathbf{Orb} \otimes \mathbf{Spc}\mathcal{TB}_{\mathbf{bd}}\rangle$$

we denote the full subcategory generated under colimits by objects of the form $S_{dM} \otimes X$ for S in $\boldsymbol{G}\mathbf{Orb}$ and X in $\mathbf{Spc}\mathcal{TB}_{\mathbf{bd}}$.

Recall that F denoted a functor in $\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{G}\mathbf{Spc}\mathcal{TB}, \mathcal{C})$. Let X be in $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$.

Corollary 5.3.10. *If the group G is finite and X belongs to the full subcategory $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}\ \langle \boldsymbol{G}\mathbf{Orb} \otimes \mathbf{Spc}\mathcal{TB}\rangle$, then the comparison morphism*

$$F^{\mathrm{Bredon}}(X) \longrightarrow F(X)$$

is an equivalence in $\mathcal{C}$.

Proof. By Proposition 5.3.9 the comparison map is an equivalence for all spaces of the form $S_{dM} \otimes X_i$ for X_i in $\mathbf{Spc}\mathcal{TB}$. Now since our X from the assertion can be written as a colimit of spaces of this form and both functors F^{Bredon} and F preserve colimits, we derive the result. □

Corollary 5.3.11. *If the functor F is hyperexcisive and the space X belongs to the full subcategory $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}\,\langle \boldsymbol{G}\mathbf{Orb} \otimes \mathbf{Spc}\mathcal{TB}_{\mathbf{bd}}\rangle$ then the comparison morphism*

$$F^{\mathrm{Bredon}}(X) \longrightarrow F(X)$$

is an equivalence in $\mathcal{C}$.

Proof. Literally the same as for Corollary 5.3.10. □

Recall the functors $(-)^G$ from Remark 5.2.2 and the functors $\mathscr{F}, \mathscr{B}$ from Construction 5.2.8.

Lemma 5.3.12. *If the group G is finite, then we have an equivalence*

$$\mathscr{F} \circ (\hat{Y})^* \circ (-)^G \simeq \mathrm{id}\,.$$

Proof. For any functor $\mathtt{F}\colon \boldsymbol{G}\mathbf{Orb} \to \mathrm{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{TB}, \mathcal{C})$, any G-orbit T and any non-equivariant local motivic space X in $\mathbf{Spc}\mathcal{TB}$ we calculate

$$\begin{aligned}
(\mathscr{F} \circ (\hat{Y})^* \circ (-)^G)(\mathtt{F})(T)(X) &\overset{\mathrm{Def}}{\simeq} \mathtt{F}^G\left(\hat{Y}\left(T_{dM} \otimes X\right)\right)\\
&\simeq \int^{(S,R)} \mathtt{F}(S)\left(\hat{Y}(T_{dM} \otimes X)(R)\right)\\
&\overset{!}{\simeq} \int^{(S,R)} \mathtt{F}(S)\left(\mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(R,T)_{dM} \otimes X\right)\\
&\overset{!!}{\simeq} \int^{(S,R)} \mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(R,T) \otimes_{\mathbf{Set}} \mathtt{F}(S)(X)\\
&\overset{!!!}{\simeq} \mathtt{F}(T)(X).
\end{aligned}$$

Here the equivalence marked with (!) is due to Corollary 5.3.8, equivalence (!!) is Corollary 5.3.4 and the equivalence marked with (!!!) is the co-Yoneda formula Lemma 5.3.1. □

Remark 5.3.13. The same proof as for Lemma 5.3.12 also shows that for a non-finite group G and all functors $\mathtt{F}\colon \boldsymbol{G}\mathbf{Orb} \to \mathrm{Fun}^{\mathrm{colim}}(\mathbf{Spc}\mathcal{TB}, \mathcal{C})$ which take values in hyperexcisive functors and all G-orbits T and all spaces X in $\mathbf{Spc}\mathcal{TB}_{\mathbf{bd}}$ we have an equivalence

$$(\mathscr{F} \circ (\hat{Y})^* \circ (-)^G)(\mathtt{F})(T)(X) \simeq \mathtt{F}(T)(X).$$

We only have to replace the reference for the equivalence marked with (!!) by Corollary 5.3.6. ♦

Proposition 5.3.14. *If G is finite, then the Bredon-approximation functor is an idempotent endofunctor, hence a Bousfield-localization.*

Proof. By definition we have $\mathscr{B} = (\hat{Y})^* \circ (-)^G \circ \mathscr{F}$, hence using Lemma 5.3.12 we get:

$$\mathscr{B} \circ \mathscr{B} \simeq (\hat{Y})^* \circ (-)^G \circ \underbrace{\mathscr{F} \circ (\hat{Y})^* \circ (-)^G}_{\simeq \mathrm{id}} \circ \mathscr{F} \simeq (\hat{Y})^* \circ (-)^G \circ \mathscr{F} \simeq \mathscr{B}. \qquad \square$$

We now return to our analogue of Elmendorf's Theorem, i.e. the question if (or when) the functor $\hat{Y}$ is an equivalence.

The colimit-preserving (c.f. Remark 5.2.2) functor

$$\mathcal{A} := \left((-)^G \circ \mathscr{F}\right)(\mathrm{id}_{\boldsymbol{G}\mathbf{Spc}\mathcal{TB}})\colon \mathrm{Fun}\,(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB}\,) \longrightarrow \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \tag{5.3.1}$$

is our candidate for the inverse.

Remark 5.3.15. Plugging in the definitions we get a pointwise formula for $\mathcal{A}$: For E in $\mathrm{Fun}\,(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB}\,)$ we have

$$\mathcal{A}(E) \simeq \mathscr{F}(\mathrm{id})^G(E) \simeq \int^{(S,T)} S_{dM} \otimes E(T). \qquad \blacklozenge$$

First we show that the functor $\mathcal{A}$ is left[6] adjoint to $\hat{Y}$ e.g. if G is finite, which requires some preliminary work.

In the following G will always be a finite group.

Lemma 5.3.16. *For all G-orbits S we have an adjunction*

$$S_{dM} \otimes (-)\colon \mathbf{TopBorn} \rightleftarrows \boldsymbol{G}\mathbf{TopBorn} : (-)^{(S)}.$$

Proof. For all X in $\boldsymbol{G}\mathbf{TopBorn}$ the evaluation morphism $e_S\colon S_{dM} \otimes X^{(S)} \to X$ from Lemma 5.2.10 will be the counit of the claimed adjunction. Since we already saw that it is a morphism, we only show that for all X in $\boldsymbol{G}\mathbf{TopBorn}$ and all

[6] not to be confused with Remark 5.1.6

non-equivariant Z in **TopBorn** the following composition of maps is a bijection:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{TopBorn}}\left(X, Z^{(S)}\right) &\xrightarrow{S_{dM}\otimes(-)} \mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}\left(S_{dM}\otimes X, S_{dM}\otimes Z^{(S)}\right)\\ &\xrightarrow{(e_S)_*} \mathrm{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(S_{dM}\otimes X, Z).\end{aligned}$$

A morphism f in $\mathrm{Hom}_{\mathbf{TopBorn}}(X, Z^{(S)})$ is sent to the map $\hat{f}(x,s) := f(x)(s)$ therefore the asignment $f \mapsto \hat{f}$ is clearly injective. To see surjectivity consider a morphism $h\colon S_{dM}\otimes X \to Z$. We claim that the obvious candidate f for a pre-image defined by $f(x)(s) := h(s,x)$ is an element in $\mathrm{Hom}_{\mathbf{TopBorn}}(X, Z^{(S)})$ which finishes the claim. For this we identify $Z^{(S)}$ with the subspace Z^H of Z consising of H-fixed points where H denotes the stabilizer subgroup of a fixed s_0 in S. By G-equivariance of h we have that $f(x)(s_0)$ is indeed an H-fixed point (i.e. the map $f\colon X \to Z^H$ is well-defined). Hence it suffices to check that f is continuous and proper.
Since G is finite, we have $Z \cong B_G Z$ hence Z^H carries the subspace structure of Z. We consider a bounded subset B of Z^H, then B is also bounded in Z hence the pre-image $h^{-1}(B)$ is bounded in $S_{dM}\otimes X$. Thus there exist bounded subsets B_1 of S_{dM} and B_2 of X such that $h^{-1}(B) \subseteq B_1 \times B_2$. One immediatelly sees that $f^{-1}(B) \subseteq B_2$ hence f is proper. For continuity we consider an open subset U of Z_H. Hence there is an open subset $\tilde{U}$ of Z such that $U = \tilde{U} \cap Z^H$. By continuity of h we have that $h^{-1}(\tilde{U})$ is open in $S_{dM}\otimes X$. Furthermore the topology on S_{dM} is discrete, hence $\{s_0\}\times X$ is also open in $S_{dM}\otimes X$. We obtain an open subset $V := h^{-1}(\tilde{U}) \cap \left(\{s_0\}\times X\right)$ of $S_{dM}\otimes X$. Since the topology on $S_{dM}\otimes X$ is the product topology, the canoncial projection $\pi\colon S_{dM}\otimes X \to X$ is open, thus $\pi(V)$ is an open subset of X and an easy calculation shows $f^{-1}(U) = \pi(V)$. □

To see that this adjunction descents to motives we have to take a closer look at the construction of the functor $\hat{Y}$. Recall the construction of the functors y and Y in Construction 5.1.1. Also let

$$\mathrm{yo}\colon \mathbf{TopBorn} \to \mathbf{PSh}(\mathbf{TopBorn})$$

denote the Yoneda embedding. Finally we denote by $\mathcal{L}_G$ the localization functor $\mathcal{L}_G\colon \mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn}) \to \boldsymbol{G}\mathbf{Spc}\mathcal{TB}$ (see Remark 4.2.8) and its non-equivariant counterpart by $\mathcal{L}$.

Lemma 5.3.17. *The diagram in Corollary 5.1.4 extends to a commuting diagram* *as depicted below. The functor Λ is colimit-preserving.*

$$\begin{tikzcd}
\boldsymbol{G}\mathbf{TopBorn} \arrow[r, "\mathrm{yo}_*\circ y"] \arrow[rr, bend left, "Y"] \arrow[d, "\mathrm{yo}"'] \arrow[dd, bend right, "\mathrm{Yo}_G"'] & \mathrm{Fun}\left(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{PSh}(\mathbf{TopBorn})\right) \arrow[r, "\mathcal{L}_*"] & \mathrm{Fun}\left(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB}\right) \\
\mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn}) \arrow[ur, dotted, "\Lambda"'] \arrow[d, "\mathcal{L}_G"'] & & \\
\boldsymbol{G}\mathbf{Spc}\mathcal{TB} \arrow[uurr, "\hat{Y}"'] & &
\end{tikzcd}$$

Proof. The existence of Λ is [Lur09, Thm. 5.1.5.6], which also gives that Λ is colimit-preserving. To see compatibility with $\hat{Y}$ we want to check that the composition $\mathcal{L}_* \circ \Lambda$ sends the collection $\mathcal{S}$ of morphisms which are inverted by $\mathcal{L}_G$ to equivalences in $\mathrm{Fun}\left(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB}\right)$. We recall (Proposition 4.2.7 and Remark 4.2.11) that $\mathcal{S}$ consisted of the following morphisms:

- $\mathrm{yo}(X \otimes \mathcal{I}) \to \mathrm{yo}(X)$ induced form the projection,
- $\emptyset_{\mathbf{PSh}} \to \mathrm{yo}(X)$ for all flasque spaces X in $\boldsymbol{G}\mathbf{TopBorn}$,
- $\mathrm{yo}(\mathcal{Y}) \amalg_{\mathrm{yo}(\mathcal{Y}\cap\mathcal{Z})} \mathrm{yo}(\mathcal{Z}) \to \mathrm{yo}(X)$ for complementary pairs $(\mathcal{Y}, \mathcal{Z})$ on X.

We show that for example the morphisms $\mathrm{yo}(X \otimes \mathcal{I}) \to \mathrm{yo}(X)$ are sent by $\mathcal{L}_* \circ \Lambda$ to equivalences, for the remaining morphisms the argument is analogous[7]:

$$\begin{aligned}(\mathcal{L}_* \circ \Lambda)(\mathrm{yo}(X \otimes \mathcal{I})) &\simeq \mathcal{L}_*\Big((\mathrm{yo}_* \circ y)(X \otimes \mathcal{I})\Big) \simeq Y(X \otimes \mathcal{I}) \\ &\xrightarrow[\text{Theorem } 5.1.3]{\simeq} Y(X) \simeq \cdots \simeq (\mathcal{L}_* \circ \Lambda)(\mathrm{yo}(X)).\end{aligned}$$ □

Corollary 5.3.18. *If Z is an equivariant local motivic space in $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$ then for all G-orbits S the presheaf $\Lambda(Z)(S)$ in $\mathbf{PSh}(\mathbf{TopBorn})$ is a non-equivariant local motivic space (i.e. an object in $\mathbf{Spc}\mathcal{TB}$). In particular we derive*

$$\Lambda(Z)(S) \simeq \mathcal{L}(\Lambda(Z)(S)) \simeq \hat{Y}(\mathcal{L}_G(Z))(S) \simeq \hat{Y}(Z)(S).$$

Proof. We only have to see that $\Lambda(Z)(S)$ is local w.r.t. to the collection of morphisms defining the localization $\mathcal{L}$. However, plugging in the definitions and using that Z is local w.r.t. the collection corresponding to the localization $\mathcal{L}_G$ we immediately derive this. □

[7] note that both Λ and $\mathcal{L}$ are colimit-preserving.

Proposition 5.3.19. *For all G-orbits S we have an adjunction*

$$S_{dM} \otimes (-) \colon \mathbf{Spc}\mathcal{TB} \rightleftarrows \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \colon \hat{Y}(-)(S).$$

Proof. Consider a local motivic space $X \simeq \operatorname{colim}_i \operatorname{Yo}(X_i)$ in $\mathbf{Spc}\mathcal{TB}$ and an equivariant local motivic space $Z \simeq \operatorname{colim}_j \operatorname{Yo}_G(Z_j)$ in $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$. We calculate

$$\begin{aligned}
\operatorname{Hom}_{\boldsymbol{G}\mathbf{Spc}\mathcal{TB}}(S_{dM} \otimes X, Z) &\simeq \operatorname{Hom}_{\boldsymbol{G}\mathbf{Spc}\mathcal{TB}}\left(\operatorname{colim}_i \operatorname{Yo}_G(S_{dM} \otimes X_i), Z\right) \\
&\simeq \lim_i \operatorname{Hom}_{\boldsymbol{G}\mathbf{Spc}\mathcal{TB}}(\operatorname{Yo}_G(S_{dM} \otimes X_i), Z) \\
&\simeq \lim_i \operatorname{Hom}_{\mathbf{PSh}(\boldsymbol{G}\mathbf{TopBorn})}(\operatorname{yo}(S_{dM} \otimes X_i), Z) \\
&\simeq \lim_i Z(S_{dM} \otimes X_i) \simeq \lim_i \operatorname*{colim}_j \operatorname{yo}(Z_j)(S_{dM} \otimes X_i) \\
&\simeq \lim_i \operatorname*{colim}_j \operatorname{Hom}_{\boldsymbol{G}\mathbf{TopBorn}}(S_{dM} \otimes X_i, Z_j) \\
&\simeq \lim_i \operatorname*{colim}_j \operatorname{Hom}_{\mathbf{TopBorn}}(X_i, y(Z_j)(S)) \\
&\simeq \lim_i \operatorname*{colim}_j \operatorname{yo}\Big(y(Z_j)(S)\Big)(X_i) \\
&\simeq \lim_i \operatorname{Hom}_{\mathbf{PSh}(\mathbf{TopBorn})}(\operatorname{yo}(X_i), \operatorname{colim}_j \operatorname{yo}(y(Z_j)(S))) \\
&\simeq \operatorname{Hom}_{\mathbf{PSh}(\mathbf{TopBorn})}(\operatorname*{colim}_i \operatorname{yo}(X_i), \Lambda(Z)(S)) \\
&\simeq \operatorname{Hom}_{\mathbf{PSh}(\mathbf{TopBorn})}(X, \hat{Y}(Z)(S)) \\
&\simeq \operatorname{Hom}_{\mathbf{Spc}\mathcal{TB}}(X, \hat{Y}(Z)(S)).
\end{aligned}$$

$\square$

Corollary 5.3.20. *We have an adjunction*

$$\mathcal{A} \colon \operatorname{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB}) \rightleftarrows \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \colon \hat{Y}.$$

Proof. Consider an equivariant local motivic space X in $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}$ and a functor E in $\operatorname{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB})$. Then we calculate

$$\begin{aligned}
\operatorname{Hom}_{\boldsymbol{G}\mathbf{Spc}\mathcal{TB}}(\mathcal{A}(E), X) &\simeq \operatorname{Hom}_{\boldsymbol{G}\mathbf{Spc}\mathcal{TB}}\left(\int^{(S,T)} S_{dM} \otimes E(T), X\right) \\
&\simeq \int_{(S,T)} \operatorname{Hom}_{\boldsymbol{G}\mathbf{Spc}\mathcal{TB}}(S_{dM} \otimes E(T), X) \\
&\simeq \int_{(S,T)} \operatorname{Hom}_{\mathbf{Spc}\mathcal{TB}}(E(T), \hat{Y}(X)(S)) \\
&\simeq \operatorname{Hom}_{\operatorname{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB})}(E, \hat{Y}(X)).
\end{aligned}$$

The last equivalence can be found in e.g. [GHN20, Prop.5.1] or [Gla15, Prop 2.3]. $\square$

Still assuming that G is finite, we also get:

Proposition 5.3.21. *For all functors E in* $\mathrm{Fun}(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB})$ *we have an equivalence*

$$E \xrightarrow{\simeq} \hat{Y}(\mathcal{A}(E)).$$

Proof. We can test this equivalence objectwise by [Cis19, Cor 3.5.12], hence for any G-orbit T we want to show an equivalence

$$E(T) \simeq \hat{Y}(\mathcal{L}(E))(T)$$

in $\mathbf{Spc}\mathcal{TB}$. For this we write $E(T)$ as $E(T) \simeq \mathrm{colim}_i \,\mathrm{Yo}(X_i)$ for non-equivariant bornological topological spaces X_i in **TopBorn**.
For another G-orbit S we get $S_{dM} \otimes E(T) \simeq \mathrm{colim}_i \,\mathrm{Yo}_G(S_{dM} \otimes X_i)$ by Remark 5.2.7.
Now we calculate using that $\hat{Y}$ preserves colimits:

$$\hat{Y}(S_{dM} \otimes E(T)) \simeq \operatorname*{colim}_i \hat{Y}(\mathrm{Yo}_G(S_{dM} \otimes X_i)) \simeq \operatorname*{colim}_i Y(S_{dM} \otimes X_i).$$

Thus we obtain for any G-orbit R that

$$\begin{aligned}
\hat{Y}(S_{dM} \otimes E(T))(R) &\simeq \operatorname*{colim}_i Y(S_{dM} \otimes X_i)(R) \simeq \operatorname*{colim}_i \mathrm{Yo}\left((S_{dM} \otimes X_i)^{(R)}\right) \\
&\overset{\text{Lemma 5.3.7}}{\simeq} \operatorname*{colim}_i \mathrm{Yo}\left(\mathrm{Map}_{\boldsymbol{G}\mathbf{Set}}(R,S)_{dM} \otimes X_i\right) \\
&\overset{!}{\simeq} \operatorname*{colim}_i \left(\mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(R,S) \otimes_{\mathbf{Set}} \mathrm{Yo}(X_i)\right) \\
&\simeq \mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(R,S) \otimes_{\mathbf{Set}} \operatorname*{colim}_i \mathrm{Yo}(X_i) \\
&\simeq \mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(R,S) \otimes_{\mathbf{Set}} E(T).
\end{aligned}$$

Here the equivalence marked with (!) is due to Lemma 5.3.2 which can be applied since G is finite by assumption, hence $\mathrm{Map}_{\boldsymbol{G}\mathbf{Set}}(R,S)$ is finite.
We derive an equivalence

$$\hat{Y}(S_{dM} \otimes E(T)) \simeq \mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(-,S) \otimes_{\mathbf{Set}} E(T). \tag{†}$$

Now we finish this proof with (note that $\hat{Y}$ commutes with colimits, in particular with the co-end):

$$
\begin{aligned}
\hat{Y}(\mathcal{L}(E)) &\simeq \hat{Y}\left(\mathscr{F}(\mathrm{id}_{\boldsymbol{G}\mathbf{Spc}\mathcal{TB}})^G(E)\right) \simeq \hat{Y}\left(\int^{(S,T)} S_{dM} \otimes E(T)\right) \\
&\simeq \int^{(S,T)} \hat{Y}\left(S_{dM} \otimes E(T)\right) \overset{\dagger}{\simeq} \int^{(S,T)} \mathrm{Map}_{\boldsymbol{G}\mathbf{Orb}}(-,S) \otimes_{\mathbf{Set}} E(T) \\
&\simeq E,
\end{aligned}
$$

where the last equivalence is the Yoneda formula (Lemma 5.3.1). □

Now we consider the other composition:

$$\mathcal{L} \circ \hat{Y} \simeq (\hat{Y})^*(\mathcal{L}) \simeq (\hat{Y})^*\left((-)^G \circ \mathscr{F}\right)(\mathrm{id}_{\boldsymbol{G}\mathbf{Spc}\mathcal{TB}}) \simeq \mathscr{B}(\mathrm{id}_{\boldsymbol{G}\mathbf{Spc}\mathcal{TB}}) \simeq (\mathrm{id}_{\boldsymbol{G}\mathbf{Spc}\mathcal{TB}})^{\mathrm{Bredon}}.$$

However, we do not know if the comparison morphism

$$\beta_{\mathrm{id}} \colon \mathcal{L} \circ \hat{Y} \simeq (\mathrm{id}_{\boldsymbol{G}\mathbf{Spc}\mathcal{TB}})^{\mathrm{Bredon}} \longrightarrow \mathrm{id}_{\boldsymbol{G}\mathbf{Spc}\mathcal{TB}}$$

is an equivalence in general, not even for G finite. But for G finite we have elaborated (Corollary 5.3.10) that

$$\beta_{\mathrm{id},X} \colon (\mathcal{L} \circ \hat{Y})(X) \longrightarrow X$$

is an equivalence for all X in the full subcategory $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}\, \langle \boldsymbol{G}\mathbf{Orb} \otimes \mathbf{Spc}\mathcal{TB} \rangle$ giving a partial answer.

To summarize these results:

Theorem 5.3.22 (analogue version of Elmendorf)**.** *If the group G is finite then the functors $\mathcal{A}$ and $\hat{Y}$ are an adjoint pair*

$$\mathcal{A} \colon \mathrm{Fun}\,(\boldsymbol{G}\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}\mathcal{TB}) \rightleftarrows \boldsymbol{G}\mathbf{Spc}\mathcal{TB} \colon \hat{Y}$$

and we have $\mathrm{id} \xrightarrow{\simeq} \hat{Y} \circ \mathcal{A}$*. Furthermore we have* $\mathcal{A} \circ \hat{Y} \xrightarrow{\simeq} \mathrm{id}$ *after restriction to the full subcategory* $\boldsymbol{G}\mathbf{Spc}\mathcal{TB}\, \langle \boldsymbol{G}\mathbf{Orb} \otimes \mathbf{Spc}\mathcal{TB} \rangle$.

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