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The Blow-Up of Manifolds with Corners

Submanifold Properties of Iterated Blow-Ups
and
Lifting Properties of Vector Fields

Master Thesis
(corrected version)

Faculty of Mathematics

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Preface

About this document

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From classical mechanics to blow-ups

In physics, in particular in mechanics, one is interested in N-particle systems, for example a system of N electrons around a nucleus. These electrons feel an attractive force towards the nucleus, and repulsive forces among each other, which can be modelled by an antiderivative to said forces, called a *potential*.

Such a potential is a function $V\colon \mathbb{R}^{3N} \to \mathbb{R} \cup \{\infty\}$, where the $3N$ components of the domain stand for the positions of all of the electrons in three-dimensional space. The Schrödinger equation which describes eigenstates of this N-particle system is of the form

$$\mathcal{H}\psi + V\psi = E\psi, \tag{0.1}$$

where $\mathcal{H}$ is a differential operator and $E \in \mathbb{R}$.

Now the problem is that V is not finite everywhere: Indeed, it becomes singular when some electrons are at the same point in space, i. e. they collide. For example, the first two electrons colliding corresponds to the satisfaction of all of the three collision equations $x_1 = x_4$, $x_2 = x_5$ and $x_3 = x_6$. For large enough N, more equations need to be taken into account, which then correspond not only to two, but also to multiple electrons colliding. Additionally, one needs to consider the case of one or multiple electrons falling into the nucleus leading to even more collision equations.

When trying to numerically solve (0.1), it is important to optimise the behaviour of the algorithm around these singularities to obtain results of high precision. This can be done by "blowing up" $\mathbb{R}^{3N}$ along the singularity submanifolds which replaces the potential landscape by a regular one whilst retaining many of the core properties of the original potential.

In this thesis, we will introduce the abstract concept of the blow-up of manifolds with corners and we will prove some properties of particular kinds of blow-ups, which show up in the application to the N-body problem.

Therefore, we present the concept of a manifold with corners, which is a direct generalisation of the ordinary concept of smooth manifolds, together with different kinds of submanifold notions. These will make it possible to talk about the blow-up construction.

Having this abstract foundation, we will then iterate these blow-ups and ask for different kinds of submanifold properties in this context. More precisely, some results from [2] regarding images of product maps in the context of these iterated blow-ups are discussed and new insights and stronger versions of these theorems are presented, giving a complete classification of the types of submanifolds to expect depending on the type of blow-up.

Lastly, we will consider vector fields on manifolds with corners and we will answer the question whether or not a given vector field lifts into a blow-up along a p-submanifold. It turns out that there is a nice necessary and sufficient condition for this to happen, which can then be used to investigate special kinds of lifts:

Translation vector fields and affine maps on $\mathbb{R}^n$ can be extended to the spherical compactification $\overline{\mathbb{R}^n}$, and for a k-dimensional linear subspace V of $\mathbb{R}^n$, we will characterise the vector fields which lift into $\left[\overline{\mathbb{R}^n} \colon \partial\overline{V}\right]$.

This is particularly interesting because this type of blow-up shows up in the treatment of the N-body problem using iterated blow-ups, which was discussed in [2, p. 32f, Example 5.13].

1 Manifolds with corners and their submanifolds

For the application purposes mentioned in the preface, one can not only consider ordinary smooth manifolds, but one needs a particular generalisation of those: manifolds with corners. This is because after the blow-up procedure, even a smooth manifold without corners will in general become a manifold *with* corners.

For a start, we introduce the following model spaces (see [2, p. 4]):

Definition 1.1 (model spaces)**.** For $n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$ we define

$$\mathbb{R}^n_k \coloneqq [0, \infty)^k \times \mathbb{R}^{n-k} \subset \mathbb{R}^n_k.$$

Since we do not only want to consider topological manifolds with corners, but smooth manifolds, we have to introduce a suitable notion of smoothness, which can be found in [2, p. 4, Def. 1.1]:

Definition 1.2. For $U \subset \mathbb{R}^n_k$ and $V \subset \mathbb{R}^m_l$ open and $f = (f_1, \dots, f_m) \colon U \to V$, we say f is *smooth* if there exists an open neighbourhood W of U in $\mathbb{R}^n$ such that f extends to a smooth function $\tilde{f} \colon W \to \mathbb{R}^m$.

f is a *diffeomorphism* if f is bijective and both f and f^{-1} are smooth.

In this sense, using the notation of [2, p. 4f., Def. 1.2], we may now define corner charts which will locally model manifolds with corners.

Definition 1.3. Let M be a Hausdorff space. A *(corner) chart* on M is a tuple (U, ϕ), where $U \subset M$ is open and $\phi \colon U \to \Omega$ is a homeomorphism onto the open subset $\Omega \subset \mathbb{R}^n_k$.

We say, two corner charts (U, ϕ) and (U', ϕ') with values in $\mathbb{R}^n_k$ and $\mathbb{R}^{n'}_{k'}$ are *compatible*, if $V \coloneqq U \cap U' = \emptyset$ or if the map

$$\phi' \circ \phi^{-1} \colon \phi(V) \to \phi'(V)$$

is a diffeomorphism.

As in [2, p. 5, Def. 1.3, 1.4], we now proceed in the same way as with ordinary manifolds to define atlases and smooth maps between manifolds:

Definition 1.4 (Corner atlas, manifold with corners)**.** Given a Hausdorff space M, a *(corner) atlas* $\mathcal{A} = \{(U_a, \phi_a) \mid a \in A\}$ on M is a family of compatible corner charts such that $M \subset \bigcup_{a \in A} U_a$.

We say, two corner atlases are *equivalent* if their union again forms a corner atlas. A *manifold with corners* is a paracompact Hausdorff space together with an equivalence class of corner atlases. For the rest of this chapter, if not defined otherwise, n will always be the dimension of M.

Definition 1.5 (Smooth maps between manifolds)**.** A map $f\colon M \to M'$ between two manifolds with corners is *smooth* if for any two charts (U, ϕ) of M and (U', ϕ') of M', the map $\phi' \circ f \circ \phi$ is smooth. Analogously, *diffeomorphisms* are defined in the obvious way.

As manifolds with corners seem to be very similar to manifolds without corners, it is rather tempting to think that the notion of a submanifold of a manifold with corners is as straightforward and canonical as usual. And indeed, one can define submanifolds of manifolds with corners in quite an ordinary fashion and get a sensible object ([2, p. 6f., Def. 1.8]):

Definition 1.6 (Submanifold)**.** We call a subset S of a manifold with corners M a *submanifold* if for each $p \in S$ there exists some $k \in \{0, \ldots, n\}$ and a corner chart $\phi\colon U \to \Omega \subset \mathbb{R}^n_k$ together with natural numbers $n' \leq n$ and $k' \leq n'$ and a matrix $G \in \mathrm{GL}(n, \mathbb{R})$ such that $p \in U$ and

1. $G \cdot \left(\mathbb{R}^{n'}_{k'} \times \{0\}^{n-n'}\right) \subset \mathbb{R}^n_k$,
2. $\phi(S \cap U) = G \cdot \left(\mathbb{R}^{n'}_{k'} \times \{0\}^{n-n'}\right) \cap \Omega$.

The first property assures that the matrix G suitably embeds the local model of S into the one of M, whereas the second statement says that, locally, S sits inside M as a model subspace up to the action of G.

This definition is surely helpful, but there are some downsides to this approach:

For example, in general, the image of a submanifold under a diffeomorphism onto its image is not a submanifold, as we will see later in 1.20. Also, certain constructions like the blow-up of a manifold with corners along a submanifold only work if this submanifold has some additional properties. So it is necessary to introduce different kinds of submanifolds, each of them with their own advantages and disadvantages.

For example, a slightly less restrictive version introduced in [2, p. 8, Def. 1.112] is the following:

Definition 1.7 (Weak submanifold)**.** Let M be a manifold with corners and $S \subset M$. We say, S is a *weak submanifold* if, for each $p \in S$, there exists a $k \in \{0, \ldots, n\}$ and a chart $\phi\colon U \to \Omega \subset \mathbb{R}^n_k$ such that $p \in U$ and

$$\phi(S \cap U) \text{ is a submanifold with corners of } \mathbb{R}^n.$$

The main difference to an ordinary submanifold is that in local coordinates, we do not obtain a submanifold of the local model $\mathbb{R}^n_k$ of M, but one of the whole Euclidean space $\mathbb{R}^n$. This type of submanifold will be of great importance since images of submanifolds under a diffeomorphism onto its image are always weak submanifolds.

In contrast to that, a more restrictive version of a submanifold are so-called p-submanifolds. These will be modelled by the following submanifolds of $\mathbb{R}^n_k$ (see [2, p. 8, Def. 1.13]):

Definition 1.8. Let $I \subset \{1, \dots, n\}$. We define

$$L_I \coloneqq \{x = (x_1, \dots, x_n) \in \mathbb{R}^n_k \mid x_i = 0 \text{ if } i \in I\}.$$

We say, $b \coloneqq \#(I \cap \{1, \dots, k\})$ is the *boundary depth* of L_I, $c \coloneqq \#I$ is the *codimension* of L_I and $d \coloneqq n - c$ its *dimension.*

These sets are special submanifolds of $\mathbb{R}^n_k$, in the sense that they are factors of products of the form $\mathbb{R}^n_k \cong \mathbb{R}^{n_1}_{k_1} \times \mathbb{R}^{n_2}_{k_2}$. As in [2, p. 9, Def. 1.14], the idea for p-submanifolds now is to model them locally as such a factor L_I of the *product* $\mathbb{R}^n_k = L_I \times L_I^{\perp}$ which explains where the "*p*" in p-submanifold comes from.

Definition 1.9 (p-submanifold)**.** A subset P of a manifold with corners M is a *p-submanifold* if, for each $x \in P$, there exists a chart (U, ϕ) with $x \in U$ and $I \subset \{1, \dots, n\}$ such that

$$\phi(P \cap U) = L_I \cap \phi(U).$$

Dimension, codimension and boundary depth are defined analogously to Definition 1.8.

Then there is a type of submanifold which lies in between p-submanifolds and submanifolds: those without an interior boundary. Although the name suggests that the definition uses a correspondence between the boundaries of the submanifold and the ambient manifold, we actually think of them to locally be the intersection of a linear subspace of $\mathbb{R}^n$ with $\mathbb{R}^n_k$ (see [2, p. 11, Def. 1.21]):

Definition 1.10 (wib-submanifold)**.** Let M be a manifold with corners and let $S \subset M$ be a submanifold. Then S is called a *wib-submanifold* or a *submanifold without interior boundary* if, for every $p \in S$, there exists a chart $\phi \colon U \to \phi(U) \subset \mathbb{R}^n_k$, and some linear subspace L of $\mathbb{R}^n$ such that $p \in S$ and

$$\phi(S \cap U) = L \cap \phi(U).$$

Using the notation G for the linear map in 1.6, we have $L = G \cdot (\mathbb{R}^{n'} \times \{0\})$. So in other words, a submanifold is a wib-submanifold if and only if one can choose $k' = 0$.

We see that any boundary of S in the definition above can only occur as the intersection of S with the boundary of $\mathbb{R}^n_k$ because of the local description. Hence, the boundary of a wib-submanifold is completely contained in the boundary of the ambient manifold which explains the choice of the name "submanifold without interior boundary".

Remark 1.11. One can immediately observe that by choosing $L = \operatorname{span}_{\mathbb{R}^n}(L_I)$ in the definition above, we obtain that every p-submanifold is a wib-submanifold.

As already mentioned in the previous definitions, the four notions of submanifolds introduced so far can be arranged from being stricter to being less strict. Indeed, we may state the following remark:

Remark 1.12. Let M be a manifold with corners and let $S \subset M$ be a subset; then

$$\begin{aligned} &S \text{ is a } p\text{-submanifold} \Rightarrow S \text{ is a } wib\text{-submanifold} \\ &\qquad \Rightarrow S \text{ is a submanifold} \Rightarrow S \text{ is a weak submanifold.} \end{aligned} \tag{1.1}$$

It is worth mentioning that we have not yet discussed whether or not these implications are proper, i. e. if we really have four distinct types of submanifolds. But before we discuss this topic, we will introduce even more submanifold notions. To be precise, we will introduce two more kinds of submanifolds in between p-submanifolds and ordinary submanifolds, the so-called d- and b-submanifolds which were first defined by Melrose in [4, p. I.12, Def. 1.7.4]:

Definition 1.13 (d-submanifold)**.** A submanifold S of a manifold M with corners is called a *d-submanifold* if at each point $p \in S$ there exists a coordinate chart (ϕ, U) with $U \subset \mathbb{R}^n_k$ such that

$$\phi(U \cap S) = L \cap \phi(U),$$

where L is of the form

$$L = \{x \in \mathbb{R}^n_k \mid x_{l+1} = \ldots x_k = 0, x_{k+1} \geq 0, \ldots, x_{k+j} \geq 0, x_{k+j+1} = \ldots = x_{k+j+r} = 0\}$$

with $l, r, j \geq 0$, $l \leq k$ and $r + j + k \leq n$.

The name d-submanifold stands for "decomposable", since its local description is a decomposition in the sense above. Looking at this decomposition, we can also see that it is actually a generalisation of a p-submanifold:

Remark 1.14. Using the notation of the above definition, we can reformulate the definition of a p-submanifold: A d-submanifold is a p-submanifold if L as above can always be chosen with $j = 0$.

This is indeed a restriction: For example, consider the model case $M = \mathbb{R}^2_1$, $S = \mathbb{R}^2_2$, where $j = 1$, $r = l = 0$. Here, $S \subset M$ is a d-submanifold, but not a p-submanifold.

The last type of submanifolds we would like to discuss are b-submanifolds; therefore we need to define a certain kind of smooth maps between manifolds with corners, so-called b-maps, which Melrose introduced in [4, p. I.21, Def. 1.12.8]:

Definition 1.15 (b-maps)**.** Let M and N be manifolds with corners and assume that there exist complete families $(\rho_H)_{H \in \mathcal{M}_1(M)}$ and $(\rho'_G)_{G \in \mathcal{M}_1(N)}$ of boundary defining functions of M and N, respectively.

Then we call a C^∞-map $F\colon M \to N$ a *b-map* if for each $G \in \mathcal{M}_1(N)$

$$\text{either } F^*\rho'_G \equiv 0 \text{ or}$$
$$F^*\rho'_G = a_G \prod_{H \in \mathcal{M}_1(M)} \rho_H^{e(H,G)} \text{ with } 0 < a_G \in C^\infty(M),\ e(H,G) \in \mathbb{N}_0.$$

We call F an *interior b-map* if for no $G \in \mathcal{M}_1(N)$ the first case occurs, otherwise it is called a *boundary b-map.*

Although this definition is rather technical, it allows us to very easily define b-submanifolds. This is because for d-submanifolds the canonical inclusion is always a b-map (see [4, p. I.22, Prop. 1.12.1]). So it is only natural to use this property as a condition for this new type of submanifold which it conveniently also inherits its name from.

Definition 1.16 (b-submanifold)**.** Let $S \subset M$ be a submanifold of a manifold with corners. We call S a *b-submanifold* if the inclusion $\iota_S \colon S \to M$ is a b-map ([4, p. I.22, Def. 1.12.9]).

As before, we see that these new types of submanifolds fit nicely in between p-submanifolds and ordinary submanifolds ([4, p. I.22]):

Remark 1.17. By definition, any d-submanifold is a b-submanifold, and every b-submanifold is a submanifold. This leads to the following chain of statements:

$$\begin{aligned} &S \text{ is a } p\text{-submanifold} \Rightarrow S \text{ is a } d\text{-submanifold} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow S \text{ is a } b\text{-submanifold} \Rightarrow S \text{ is a submanifold.} \end{aligned}$$

As mentioned before, one might wonder if the implications in 1.12 and 1.17 are proper, and indeed they are. As a proof for this statement, we will consider a range of examples and counterexamples.

Example 1.18 (A wib-submanifold, which is not p)**.** Given a manifold N with corners, the Hausdorff space $M := N \times N$ is also a manifold with corners. In [2, p. 10, Example 1.17], it was stated that for $\partial N \neq \emptyset$ the *diagonal* $\Delta_N := \{(p,p) \in M \mid p \in N\}$ is a wib-submanifold of M (choose $L = \{x \in \mathbb{R}^{2n} \mid \forall i \in \{1, \dots, n\} : x_i = x_{n+i}\}$), but not a p-submanifold.

Example 1.19 (A submanifold, which is not wib)**.** Let $\mathbb{E} := \{x \in \mathbb{R}^n \mid |x| < 1\}$. Then $\overline{\mathbb{E}} \subset \mathbb{R}^n$ is a submanifold, but not a wib-submanifold. In an illustrative sense, the boundary of $\overline{\mathbb{E}}$, which is $\mathbb{S}^{n-1}$, lies in the interior of $\mathbb{R}^n$, so it is not "without an interior boundary".

The precise argument, however, is as follows: There really is no choice for L in this context, since the dimensions of $\overline{\mathbb{E}}$ and $\mathbb{R}^n$ match. Thus, $L = \mathbb{R}^n$, but at any boundary point, one can never find a chart ϕ and a coordinate neighbourhood U around $x \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$ such that

$$\phi(\overline{\mathbb{E}} \cap U) = \mathbb{R}^n \cap \phi(U) = \phi(U).$$

This is because U is an open subset of $\mathbb{R}^n$ whereas $\overline{\mathbb{E}} \cap U$ is not.

Previously, we stated that in general, the image of a diffeomorphism onto its image is not a submanifold. As an illustration, consider the following example from [2, p. 7, Ex. 1.11]:

Example 1.20 (A weak submanifold, which is not a submanifold)**.** Let $M := \mathbb{R}^2_1$ and $S := f(\mathbb{R}^2_1)$ where $f \colon \mathbb{R}^2_1 \to \mathbb{R}^2_1$, $(x,y) \mapsto (x + y^2, y)$. Then S is a submanifold of $\mathbb{R}^2$, i. e. a weak submanifold of $\mathbb{R}^2_1$, but it is not a submanifold of $\mathbb{R}^2_1$.

Proof. Suppose, $S \subset M$ were a submanifold. Because S contains a non-empty open subset of M in every neighbourhood of zero, the local model of S around zero needs to be two-dimensional. Indeed, since the local model of M around zero is $\mathbb{R}^2_1$ and S has no corner of depth two, we are only left with the model $\mathbb{R}^2_1$ of S.

So, if we choose a corner charts $\phi\colon U \to \phi(U) \subset \mathbb{R}^2_1$ with $0 \in U$ and a matrix $G \in \mathrm{GL}(2, \mathbb{R})$ with

$$G \cdot \left(\mathbb{R}^2_1\right) \subset \mathbb{R}^2_1,$$

we get:

$$\phi(S \cap U) = G \cdot \mathbb{R}^2_1 \cap \phi(U). \tag{1.2}$$

Now, as a diffeomorphism, ϕ needs to preserve boundaries, hence for G we obtain

$$G \cdot (\{0\} \times \mathbb{R}) = \{0\} \times \mathbb{R},$$

which restricts this matrix to the form

$$G = \begin{pmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{pmatrix}$$

with $g_{11} > 0$ and $g_{22} \neq 0$. In summary, we have

$$G \cdot \mathbb{R}^2_1 = \mathbb{R}^2_1$$

and thus, equation (1.2) transforms to

$$\phi(S \cap U) = \phi(U).$$

But since ϕ is a diffeomorphism, this means that

$$S \cap U = U.$$

This is a contradiction because in any given neighbourhood around zero, there are points of $\mathbb{R}^2_1$ that do not lie in S. Therefore, this proves the claim. $\square$

Remark 1.21. So far, we have only discussed that the implications in 1.12 are proper. But as already said, this also holds for the implications in 1.17:

First, we have seen in 1.14 that not every d-submanifold is a p-submanifold.

Furthermore, the diagonal of $\mathbb{R}^2_2$ shows that there are b-submanifolds which are no d-submanifolds (the proof works the same way as in the case of showing that the diagonal is not a p-submanifold).

And lastly, there exist submanifolds which are no b-submanifolds, as will be shown in the following example:

Example 1.22. Consider $M := \mathbb{R}^2_1$ and the subset $S := \left\{(x, y) \in \mathbb{R}^2_1 \mid |y| \leq x\right\}$. Then S is a submanifold of M, but it is not a b-submanifold. Indeed, we can easily verify that the inclusion map can not be a b-map.

Also, instead of just considering the two implication chains of submanifolds separately, we might wonder if there is any connection between them. But this is not the case as well: Indeed, all notions of submanifolds introduced are distinct, as we can see from the following statements:

Example 1.23 (A wib-submanifold, which is not a d-submanifold)**.** The diagonal of $\mathbb{R}^2_2$ is a wib-submanifold, but it is not a d-submanifold because of the very same reasons it is not a p-submanifold.

Example 1.24 (A d-submanifold, which is not a wib-submanifold)**.** The submanifold $\mathbb{R}^2_2 \subset \mathbb{R}^2_1$ discussed in 1.14 is a d-submanifold, but it is not a wib-submanifold, because part of the boundary lies in the interior of the containing manifold with corners.

Remark 1.25. Since every d-submanifold is a b-submanifold, but d-submanfolds can have an interior boundary, we also know that being a b-submanifold does not imply being a wib-submanifold.

This shall be enough about the differences between different submanifold types. But there is one type of submanifolds which will become highly important when introducing the blow-up of manifolds later on: the p-submanifolds. Hence it is worth remarking a few basic properties (see [2, p. 9, Lemma 1.16]):

Lemma 1.26. *Let $P \subset Q \subset M$ be manifolds with corners.*

1. *If P is a p-submanifold of M, then P is locally closed, i. e. it is the intersection of a closed subset with an open subset.*

2. *If both P and Q are p-submanifolds of M, then P is a p-submanifold of Q.*

3. *If P is a p-submanifold of Q and Q is a p-submanifold of M, then P is a p-submanifold of M.*

Proof. We fix an atlas $\mathcal{A} = \{(U, \phi)\}$.

1. Let (U, ϕ) be a coordinate chart as in Definition 1.13. By the definition of a p-submanifold, we have
$$\phi(P \cap U) = L_I \cap \phi(U),$$
where L_I is closed and $\phi(U)$ is open.

2. Locally, a p-submanifold can be expressed by defining functions: Let $x^1, \ldots, x^l$ be such defining functions for the p-submanifold P of codimension l in a neighbourhood of $x \in P$. I.e., $x^j \in C^\infty(\phi(U))$, d$x$ has full rank on $\phi(P \cap U)$ and $\phi(P \cap U) = \bigcap_{j=1}^{l} x^{-1}(0)$.

   Now choose $I \subset \{1, \ldots, l\}$ such that $(dx^i|_p)_{i \in I}$ is a basis of $T^*_x Q$. Then, the functions $(x^i)_{i \in I}$ define P as a p-submanifold of Q in a (possibly smaller) neighbourhood of x.

3. Let $x^1, \dots, x^k$ be functions locally defining P as a p-submanifold of Q and let $x^{k+1}, \dots, x^l$ be functions locally defining Q as a p-submanifold of M. Then $x^1, \dots, x^l$ locally define P as a p-submanifold of M. $\square$

At this point we have gathered all of the equipment needed to proceed with the blow-up of manifolds with corners which will be introduced in the following chapter.

2 The blow-up of a manifold with corners

Now that we have a good notion of manifolds with corners, we may introduce the concept of a blow-up. Illustratively, up to diffeomorphism, blowing up a manifold M along a p-submanifold P can be described as removing a small open neighbourhood around P from M, leaving behind a manifold with additional boundary or corners. But the precise definition of a blow-up is a lot better since we will be able to achieve a blow-up diffeomorphic to said illustrative one, which preserves the manifold structure outside of P. In other words, the blow-up $[M \colon P]$ of M along P is identical to M in $M \setminus P$.

In order to get this property, we will define the blow-up as a set by replacing every point x in P by its inward pointing spherical normal bundle, which is the set of inward-pointing unit length vectors in the tangent space T_xM orthogonal to P. Therefore, we need the following definition ([2, p. 10, Def. 1.18]):

Definition 2.1. Let $P \subset M$ be a p-submanifold of a manifold M with corners. Then

$$N^M(P) \coloneqq TM|_P/TP$$

is called the *normal bundle of P in M*.

We call the image $N_+^M P$ of $T^+M|_P$ in $N^M P$ the *inward pointing normal fibre bundle of P in M*. The set $\mathbb{S}(N_+^M P)$ of unit vectors in $N_+^M P$ is called the *inward pointing spherical normal bundle of P in M* which is equipped with a fibre bundle projection

$$\mathbb{S}(N_+^M P) \to P.$$

As said before, we can now define the blow-up by replacing P with its inward pointing spherical normal bundle (see [2, p. 12, Def. 2.1]):

Definition 2.2. Let M be a manifold with corners and let P be a closed p-submanifold of M. Let $\mathbb{S}(N_+^M P)$ be the inward pointing spherical normal bundle of P in M. We define the *blow-up of M along P* as a set to be the following disjoint union:

$$[M \colon P] \coloneqq (M \setminus P) \sqcup \mathbb{S}(N_+^M P).$$

Naturally, this set comes with a map $\beta = \beta_{M,P}$, the so-called blow-down map

$$\begin{aligned} \beta_{M,P} \colon [M \colon P] &\to M \\ x &\mapsto \begin{cases} x & \text{if } x \in M \setminus P, \\ \pi(x) & \text{else,} \end{cases} \end{aligned}$$

where $\pi \colon \mathbb{S}(N_+^M P) \to P$ is the fibre bundle projection.

Of course, this definition of a blow-up only makes sense as a set, since the smooth structure of the disjoint union above does not match our imagination of a blow-up as removing an open neighbourhood around P. So, following [2, p. 12f.], we try to find a different smooth structure we can equip the blow-up with. As in many other cases, it is useful to first look at simple model cases (i. e. the local descriptions of p-submanifolds) and see whether we can construct a suitable smooth structure.

Recall that every p-submanifold sits inside the ambient manifold as a factor of a product. So every blow-up has a local model of the form

$$\begin{aligned}[\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} : \mathbb{R}^n_l \times \{0\}] &:= \left(\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} \setminus \mathbb{R}^n_l \times \{0\}\right) \sqcup \mathbb{R}^n_l \times \mathbb{S}^{n'-1}_{l'} \\ &= \mathbb{R}^n_l \times \left(\mathbb{S}^{n'-1}_{l'} \sqcup \left(\mathbb{R}^{n'}_{l'} \setminus \{0\}\right)\right).\end{aligned} \tag{2.1}$$

In these local models, we may write down a map inspired by polar coordinates:

$$\begin{aligned}\kappa\colon \mathbb{R}^n_l \times \mathbb{S}^{n'-1}_{l'} \times [0,\infty) &\to \mathbb{R}^n_l \times \left(\mathbb{S}^{n'-1}_{l'} \sqcup \left(\mathbb{R}^{n'}_{l'} \setminus \{0\}\right)\right) \\ (x,\xi,r) &\mapsto \begin{cases}(x,\xi) \in \mathbb{R}^n_l \times \mathbb{S}^{n'-1}_{l'} & \text{if } r = 0 \\ (x, r\xi) \in \mathbb{R}^n_l \times \left(\mathbb{R}^{n'}_{l'} \setminus \{0\}\right) & \text{if } r > 0.\end{cases}\end{aligned} \tag{2.2}$$

It is an easy calculation that κ is bijective. Under this map, the blow-down map transforms to

$$\begin{aligned}\beta\colon \mathbb{R}^n_l \times \mathbb{S}^{n'-1}_{l'} \times [0,\infty) &\to \mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} \\ (x,\xi,r) &\mapsto (x, r\xi).\end{aligned} \tag{2.3}$$

The map κ^{-1} has the advantage that it maps the blow-up of the model space bijectively to a space with a natural smooth structure, namely

$$\mathbb{R}^n_l \times \mathbb{S}^{n'-1}_{l'} \times [0,\infty)$$

together with the smooth structure of the product of manifolds with corners. So for any open subset $U \subset \mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'}$, we can endow

$$[U\colon U \cap (\mathbb{R}^n_l \times \{0\})] = \beta^{-1}(U) \subset [\mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'} : \mathbb{R}^n_l \times \{0\}]$$

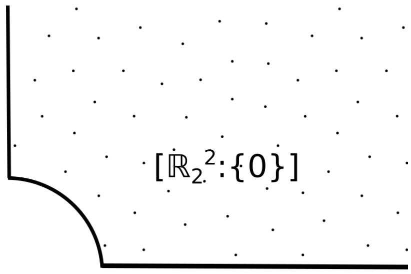
with the induced structure of a manifold with corners. Also, by construction, this turns κ into a diffeomorphism of manifolds with corners.

In order to get an illustration of blow-ups of model cases, we give some basic examples.

Example 2.3 (Simple model case)**.** One of the easiest examples is blowing up the first quadrant of the plane along the origin: Let $M := \mathbb{R}^2_2$ and $P := \{(0,0)\}$. Then, using equation (2.1), we obtain

$$\begin{aligned}[M\colon P] &= \left[\mathbb{R}^2_2 \colon \{0\}\right] \\ &= \mathbb{S}^{2-1}_2 \sqcup \left(\mathbb{R}^2_2 \setminus \{0\}\right) \\ &= \mathbb{S}^1_2 \sqcup \left(\mathbb{R}^2_2 \setminus \{0\}\right).\end{aligned}$$

If we now use the map κ from (2.2), this can be transformed to $\mathbb{S}^1_2 \times [0,\infty)$, which can be pictured as follows:



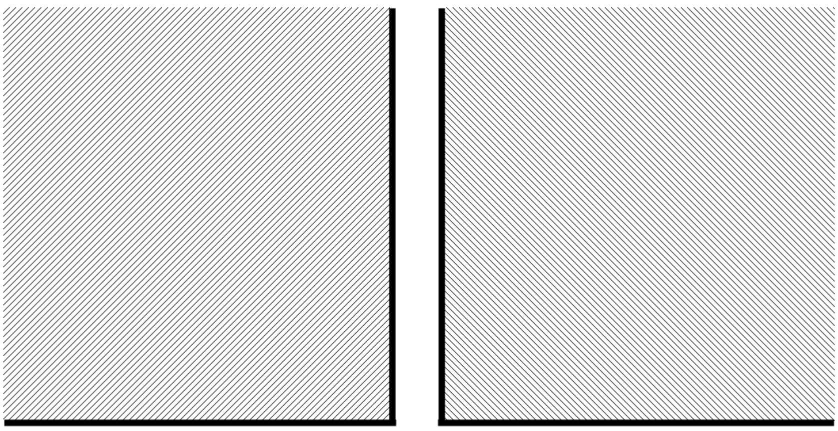
Example 2.4 (Another simple model case)**.** Consider the manifold with corners $M \coloneqq \mathbb{R}^2_1$ together with its p-submanifold $P \coloneqq [0,\infty) \times \{0\}$. First, we again write M and P in the notation of equation (2.1):

$$M = \mathbb{R}^2_1 = \mathbb{R}^1_1 \times \mathbb{R}^1_0, \quad P = [0,\infty) \times \{0\} = \mathbb{R}^1_1 \times \{0\}.$$

Then, using equation (2.1) and the map κ from (2.2), we can write the blow-up $[M \colon P]$ as follows:

$$\begin{aligned}
[M \colon P] &= \left[\left(\mathbb{R}^1_1 \times \mathbb{R}^1_0\right) : \left(\mathbb{R}^1_1 \times \{0\}\right)\right] \\
&= \left(\mathbb{R}^2_1 \setminus ([0,\infty) \times \{0\})\right) \sqcup \mathbb{S}\left(N_+^{\mathbb{R}^2_1}([0,\infty) \times \{0\})\right) \\
&= \mathbb{R}^1_1 \times \left(\mathbb{S}^{1-1}_0 \sqcup \left(\mathbb{R}^1_0 \setminus \{0\}\right)\right) \\
&\cong \mathbb{R}^1_1 \times \mathbb{S}^0 \times [0,\infty) \\
&= [0,\infty) \times \mathbb{S}^0 \times [0,\infty)\,.
\end{aligned}$$

Since $\mathbb{S}^0$ is just two points, this is diffeomorphic to $\bigsqcup_{i=1}^2 [0,\infty)^2$, which might be depicted like this:



At this point, we have endowed blow-ups of model cases with a suitable smooth structure, but of course we would like to turn any blow-up into a (smooth) manifold with corners. Therefore we need the following lemma from [2, p. 13, Lemma 2.2]:

Lemma 2.5. *Let $P_i \subset M_i$ $(i \in \{1,2\})$ be closed p-submanifolds and let $\phi\colon M_1 \to M_2$ be a diffeomorphism such that $\phi(P_1) = P_2$. Then there exists a unique map $\phi^\beta\colon [M_1\colon P_1] \to [M_2\colon P_2]$ which is bijective and makes the following diagram commute:*

$$\begin{tikzcd} {[M_1\colon P_1]} \arrow[r, "\phi^\beta"] \arrow[d, "\beta_{M_1,P_1}"] & {[M_2\colon P_2]} \arrow[d, "\beta_{M_2,P_2}"] \\ M_1 \arrow[r, "\phi"] & M_2 \end{tikzcd}$$

It is functorial in the following sense:

$$(\phi \circ \psi)^\beta = \phi^\beta \circ \phi^\beta.$$

In the case that the M_i are open subsets of $\mathbb{R}^n_k$, ϕ^β is a diffeomorphism.

Applying this lemma to corner charts allows us to derive an atlas of the blow-up from an atlas of the original manifold (see [2, p. 13, Lemma 2.3]).

Lemma 2.6. *Let $\mathcal{A} = \{(U_a, \phi_a) \mid a \in A\}$ be an atlas of a manifold with corners M and let P be a closed p-submanifold of M together with the blow-down map $\beta = \beta_{M,P}\colon [M\colon P] \to M$.*

We endow the blow-up $[M\colon P]$ with the smallest topology which makes all the maps $(\phi_a^\beta)_{a\in A}$ continuous. Then

$$\beta^*(\mathcal{A}) \coloneqq \{(\beta^{-1}(U_a), \phi_a^\beta) \mid a \in A\}$$

is an atlas of $[M\colon P]$.

If $\mathcal{A}$ and $\mathcal{A}'$ are compatible atlases of M, then $\beta^*(\mathcal{A})$ and $\beta^*(\mathcal{A}')$ are compatible atlases of $[M\colon P]$.

Because of this compatibility property of atlases of the blow-up, we can use the smooth structure on $[M\colon P]$ induced by any lifted atlas of M ([2, p. 14, Def. 2.4]):

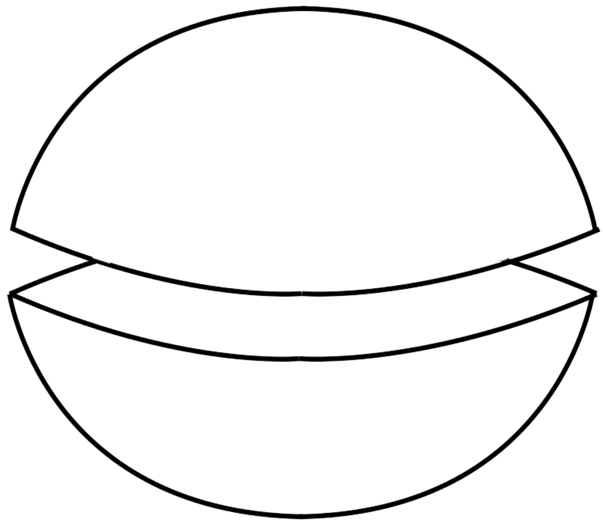
Definition 2.7. Given a manifold M with corners and a closed p-submanifold $P \subset M$, we endow $[M\colon P]$ with the smooth structure defined by the atlas $\beta^*(\mathcal{A})$ for any atlas $\mathcal{A}$ of M.

In summary, we see that the blow-up of a manifold with corners is again a manifold with corners. This allows us to create blow-ups along any p-submanifold of any manifold, for example the blow-up of the 2-sphere along its equator.

Example 2.8. Let $M \coloneqq \mathbb{S}^2$ be the 2-dimensional unit sphere and $P \coloneqq \mathbb{S}^1$ be its equator. Then, in local coordinates near P, $[M\colon P]$ is modelled by the blow-up $\left[\mathbb{R}^2\colon \mathbb{R}^1 \times \{0\}\right]$, which can be computed in the very same way as in the previous examples. This yields the following local structure:

$$\mathbb{R} \times \mathbb{S}^0 \times [0,\infty) = \bigsqcup_{i=1}^{2} (\mathbb{R} \times [0,\infty))\,. \tag{2.4}$$

This is diffeomorphic to the process of removing a small open neighbourhood around the equator.



Having seen these examples and knowing that the blow-up is again a manifold, it is quite natural to ask whether one can repeat this process. This leads to the concept of iterated blow-ups which will be discussed below.

3 Submanifold properties of iterated blow-ups

The goal of this chapter is to introduce iterated blow-ups together with some important statements and to partly disprove three parts of a conjecture in [2, p. 28, Rem. 4.14] related to these statements. Therefore we need to discuss iterated blow-ups of different generality and complexity:

3.1 Iterated blow-ups of disjoint p-submanifolds

There are several possibilities to iterate the procedure of a blow-up, and depending on the relation of the p-submanifolds, delicate situations might appear. However, the easiest case is the iterated blow-up of two *disjoint* p-submanifolds: In this case, P lifts to a p-submanifold of $[M\colon Q]$ because $[M\colon Q] = M$ in $M \setminus Q$ and vice versa. So both iterated blow-ups are well-defined and since both single blow-ups are in some sense independent of each other, it does not matter in which order we do the blow-up. Indeed, there is a lemma stated in [2, Lemma 3.7] which turns this into a precise statement:

Lemma 3.1. *Let M be a manifold with corners and let P and Q be two closed, non-trivial, disjoint p-submanifolds of M. Then there is a unique, smooth, natural map*

$$\zeta_{M,Q,P}\colon\ [[M\colon Q] : P] \to [M\colon P]$$

which restricts to the identity outside of $P \cup Q$. Furthermore, the product map

$$\mathcal{B}_{M,Q,P} \coloneqq \left(\zeta_{M,Q,P}, \beta_{[M\colon Q],P}\right)\colon\ [[M\colon Q] : P] \to [M\colon P] \times [M\colon Q]$$

is proper in each component.

Its image is a weak submanifold and $\mathcal{B}_{M,Q,P}$ is a diffeomorphism onto its image.

Proof. The proof can be found in [2, p. 19]. □

Reading through this statement, the question arises if the image of the product map is really just a weak submanifold. In the paper, they conjectured that this statement could be refined in such a way that said image is actually a wib-submanifold and a b-submanifold ([2, p. 28, Rem. 4.14]). Considering some basic examples, this is quite tempting to think, but it turns out that, in general, it is not even an ordinary submanifold:

Proposition 3.2. *The image of the map $\mathcal{B}_{M,Q,P}$ of Lemma 3.1, in general, is not a submanifold. More specifically, it is a submanifold if and only if everywhere the local model for any of the blow-ups $[M\colon P]$ and $[M\colon Q]$ is of the form*

$$\left[\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \colon \mathbb{R}^n_k \times \{0\}\right],$$

where either $n' = 1$ or $n' \geq 2$ and $k' = 0$, and in this case, it is actually always a wib-submanifold.

In order to prove this proposition, we will apply some simplifications without any loss of generality:

Remark 3.3. Since being a submanifold is a local property, it is sufficient to only consider points in a neighbourhood around P, and we may assume

$$[M\colon Q, P] = \left[\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \colon \mathbb{R}^n_k\right], \tag{3.1}$$

$$[M\colon P] = \left[\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \colon \mathbb{R}^n_k\right], \tag{3.2}$$

$$[M\colon Q] = \mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'}.$$

In this setting, the product map locally becomes

$$\mathcal{B} := \mathcal{B}_{M,Q,P} = \mathrm{id} \times \beta_{\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'}, \mathbb{R}^n_k} \colon \left[\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \colon \mathbb{R}^n_k\right] \to \left[\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \colon \mathbb{R}^n_k\right] \times \left(\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'}\right)$$

Now we can simplify the blow-up in (3.1) and (3.2), since both the ambient manifold and the p-submanifold contain common factors of lines and half-lines. Thus, we may take the diffeomorphism

$$\left[\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \colon \mathbb{R}^n_k\right] \cong \mathbb{R}^n_k \times \left[\mathbb{R}^{n'}_{k'} \colon \{0\}\right]$$

in order to obtain the following description of the product map:

$$\mathcal{B} = \delta_{\mathbb{R}^n_k} \times \mathrm{id} \times \beta_{\mathbb{R}^{n'}_{k'}, \{0\}} \colon \mathbb{R}^n_k \times \left[\mathbb{R}^{n'}_{k'} \colon \{0\}\right] \to (\mathbb{R}^n_k)^2 \times \left[\mathbb{R}^{n'}_{k'} \colon \{0\}\right] \times \mathbb{R}^{n'}_{k'},$$

where $\delta_{\mathbb{R}^n_k} \colon \mathbb{R}^n_k \to (\mathbb{R}^n_k)^2$ is the diagonal map. Since images of such diagonal maps are always wib-submanifolds, this part of the map is well-behaved, i. e. if this factor embeds as a wib-submanifold, so does the whole map. Thus, in order to prove Proposition 3.2, we may only consider the rest of the product map

$$\tilde{\mathcal{B}} = \mathrm{id} \times \beta_{\mathbb{R}^{n'}_{k'}, \{0\}} \colon \left[\mathbb{R}^{n'}_{k'} \colon \{0\}\right] \to \left[\mathbb{R}^{n'}_{k'} \colon \{0\}\right] \times \mathbb{R}^{n'}_{k'}.$$

At this point, it becomes clear that the submanifold property can only depend on the values of n' and k'. But before we can proceed with the full proof, we will prove the case $n' = 1$. This is because the one-dimensional models behave rather differently than the higher-dimensional ones which already became clear in the statement of the proposition in the case $n' = 1$.

Example 3.4 ($n' = 1$)**.** If $n' = 1$, the image of the product map is always a submanifold. To see that, we first remark that there are only two possible local models corresponding to $k' = 0$ and $k' = 1$ respectively: $\mathbb{R}^1_0 = \mathbb{R}$ and $\mathbb{R}^1_1 = [0, \infty)$.

For $k' = 0$, since $\mathbb{S}^0$ just consists of two points, the product map for $k' = 0$ is given by

$$\begin{aligned} \tilde{\mathcal{B}} \colon \{-1, 1\} \times [0, \infty) &\to \{-1, 1\} \times [0, \infty) \times \mathbb{R}, \\ (\xi, r) &\mapsto (\xi, r, r\xi). \end{aligned}$$

Clearly, the image consists of two connected components, each one belonging to a connected component of the ambient space:

$$\begin{aligned} \{1\} \times \{(x, x) \mid x \in [0, \infty)\} &\subset \{-1, 1\} \times [0, \infty) \times \mathbb{R}, \\ \{-1\} \times \{(x, -x) \mid x \in [0, \infty)\} &\subset \{-1, 1\} \times [0, \infty) \times \mathbb{R}. \end{aligned}$$

So the connected components of the image boil down to the weak submanifolds

$$\begin{aligned} \{(x, x) \mid x \in [0, \infty)\} &\subset [0, \infty) \times \mathbb{R}, \\ \{(x, -x) \mid x \in [0, \infty)\} &\subset [0, \infty) \times \mathbb{R}, \end{aligned}$$

which can both be bent to $\mathbb{R}^1_1 \subset \mathbb{R}^2_1$ and therefore are actually p-submanifolds. In particular, the whole image of the product map is a wib-submanifold.

In the other case, $k' = 1$, the product map takes the form

$$\begin{aligned} \tilde{\mathcal{B}} \colon \{1\} \times [0, \infty) &\to \{1\} \times [0, \infty) \times [0, \infty)\,, \\ (\xi, r) &\mapsto (\xi, r, r\xi), \end{aligned}$$

so the image image of this map (after omitting the single point $\{1\}$) is the diagonal in $\mathbb{R}^2_1$, which is a wib-submanifold as well.

Having treated the special case separately, we are now capable of proving 3.2 for $n' \geq 2$:

Proof of Proposition 3.2. Recall the following local description of the product map:

$$\tilde{\mathcal{B}} = \left(\mathrm{id}, \beta_{\mathbb{R}^{n'}_{k'}, \{0\}}\right) : \left[\mathbb{R}^{n'}_{k'} : \{0\}\right] \to \left[\mathbb{R}^{n'}_{k'} : \{0\}\right] \times \mathbb{R}^{n'}_{k'}.$$

First, we want to express this blow-up in concrete terms using the diffeomorphism κ from (2.2):

$$\left[\mathbb{R}^{n'}_{k'} : \{0\}\right] \cong \mathbb{S}^{n'-1}_{k'} \times [0, \infty)\,.$$

This yields the following description of $\tilde{\mathcal{B}}$ (which, in an abuse of notation, we will also call $\tilde{\mathcal{B}}$):

$$\begin{aligned} \tilde{\mathcal{B}} \colon \mathbb{S}^{n'-1}_{k'} \times [0, \infty) &\to \mathbb{S}^{n'-1}_{k'} \times [0, \infty) \times \mathbb{R}^{n'}_{k'}, \\ (\xi, r) &\mapsto (\xi, r, r\xi), \end{aligned} \tag{3.3}$$

Now there are two distinct cases in which different statements hold: In the no-corner case $k' = 0$, we will use $(n'-1)$-dimensional spherical coordinates to show that the image of $\tilde{\mathcal{B}}$ is a wib-submanifold. However, if there is a proper corner, i. e. $k' > 0$, we extract the first component using the Mercator projection. This component is a half-line because $k' \geq 1$, and at the boundary the Jacobian of $\tilde{\mathcal{B}}$ will show that the image of $\tilde{\mathcal{B}}$ is not even a submanifold.

Case $k' = 0$: Since $n' \geq 2$ and $k' = 0$, we can locally parametrise $\mathbb{S}^{n'-1}_{k'} = \mathbb{S}^{n'-1}$ smoothly around any given point ξ, i. e. we can choose $(n'-1)$-dimensional spherical coordinates $\phi\colon \mathbb{R}^{n'-1} \supset U \xrightarrow{\sim} \mathbb{S}^{n'-1}_{k'}$ where U is an open neighbourhood of 0 and $\operatorname{im}(\phi)$ is an open neighbourhood of ξ. This yields another local description of $\tilde{\mathcal{B}}$:

$$\begin{aligned} \tilde{\mathcal{B}}\colon \mathbb{R}^{n'}_1 \supset [0,\infty) \times U &\to [0,\infty) \times U \times \mathbb{R}^{n'}_{k'}, \\ (r, x_1, \ldots, x_{n'-1}) &\mapsto (r, x_1, \ldots, x_{n-1}, r\phi(x_1, \ldots, x_{n'-1})), \end{aligned} \tag{3.4}$$

At this point, it is crucial that $k' = 0$ because in this case, there exists the smooth map

$$\begin{aligned} F\colon ([0,\infty) \times U) \times V &\to ([0,\infty) \times U) \times \mathbb{R}^{n'} \\ (x, y) &\mapsto (x, y - x_1 \cdot \phi(x_2, \ldots, x_{n'})), \end{aligned}$$

where $V \subset \mathbb{R}^{n'}$ is a sufficiently small open neighbourhood of 0. Indeed, this is a diffeomorphism onto its image, since the inverse map is just given by

$$(x, y) \mapsto (x, y + x_1 \cdot \phi(x_2, \ldots, x_{n'})).$$

If we combine the parametrisation and the local diffeomorphism, we locally obtain the commutative diagram

$$\begin{tikzcd} {[0,\infty) \times U} \arrow[r, "\tilde{\mathcal{B}}"] \arrow[dr, "\hat{\mathcal{B}}"'] & {[0,\infty) \times U \times V} \arrow[d, "F"] \\ & {[0,\infty) \times U \times \mathbb{R}^{n'}.} \end{tikzcd}$$

We can now calculate for any $(r, x) \in [0,\infty) \times U$:

$$\hat{\mathcal{B}}(r, x) = F(\tilde{\mathcal{B}}(r, x)) = ((r, x), r\phi(x) - r\phi(x)) = ((r, x), 0)$$

and see that $\hat{\mathcal{B}}$ is a projection onto the first components. More precisely, locally around zero we have

$$\operatorname{im}(\hat{\mathcal{B}}) \cong \Delta_{\mathbb{R}^n_k} \times \mathbb{R}^{n'}_1 \times \{0\} \subset (\mathbb{R}^n_k)^2 \times \mathbb{R}^{n'}_1 \times \mathbb{R}^{n'}.$$

This subset is a combination of a diagonal and the image of a projection, hence it is indeed a wib-submanifold. Note that this construction only works since $\mathbb{R}^{n'} = \mathbb{R}^{n'}_{k'}$ is a vector space for $k' = 0$ and therefore subtraction is defined; for $k' > 0$ the corner $\mathbb{R}^{n'}_{k'}$ is not a vector space and hence does not admit any such diffeomorphism F.

Case $k' > 0$: The problem with this case is that the image of $\tilde{\mathcal{B}}$ is tangent to a boundary face at zero whilst not being locally contained in the boundary, similar to example 1.20.

So our goal is to exploit the Jacobian of $\tilde{\mathcal{B}}$ to show that its image is tangential to the boundary at zero, but any points in any sufficiently small neighbourhood already lie in the interior of $\mathbb{R}^{n'}_{k'}$. Such a tangential behaviour cannot be flattened by any diffeomorphism, so it cannot be a submanifold.

We start by considering the local description of the product map from (3.3), but this time we use the *Mercator* diffeomorphism

$$\begin{aligned}\Phi\colon \mathbb{S}^{n'-1}_{k'} \setminus \{N\} &\xrightarrow{\sim} [0,\infty) \times \mathbb{S}^{n'-2}_{k'-1}\\ \xi = (\xi_1, \dots, \xi_{n'}) &\mapsto \frac{1}{\sqrt{\xi_2^2 + \ldots + \xi_{n'}^2}} \cdot \xi,\end{aligned}$$

where $N := (1, 0, \dots, 0) \in \mathbb{S}^{n'-1}_{k'}$ is the north pole.

Now we choose a point $\tilde{\xi}$ such that $\tilde{\xi}_1 = 0$ and any sufficiently small neighbourhood looks like an open subset of $[0,\infty) \times \mathbb{S}^{n'-2}$, i. e. $\tilde{\xi}$ is on the edge $\xi_1 = 0$ of the sphere orthant and the boundary depth is precisely 1.

Here we can use rather obvious coordinates to parametrise this lower-dimensional sphere in an open neighbourhood U' of $\tilde{\xi}$. Without loss of generality, this parametrisation is assumed to be the identity $\phi = \mathrm{id}$ for $n' = 2$ and the ordinary $(n'-2)$-dimensional spherical coordinates for $n' > 2$:

$$\begin{aligned}\phi\colon \mathbb{R}^{n'-2} \supset U &\xrightarrow{\sim} U' \subset \mathbb{S}^{n'-2}\\ \phi_1(x_1, \dots, x_{n'-2}) &= \cos(x_1),\\ \phi_2(x_1, \dots, x_{n'-2}) &= \sin(x_1)\cos(x_2),\\ &\vdots\\ \phi_{n'-2}(x_1, \dots, x_{n'-2}) &= \sin(x_1) \cdot \ldots \cdot \sin(x_{n'-3})\cos(x_{n'-2}),\\ \phi_{n'-1}(x_1, \dots, x_{n'-2}) &= \sin(x_1) \cdot \ldots \cdot \sin(x_{n'-3})\sin(x_{n'-2}).\end{aligned}$$

Combining these two diffeomorphisms, we get a diffeomorphism

$$\begin{aligned}\Psi := \Phi^{-1} \circ (\mathrm{id}, \phi)\colon\ [0,\infty) \times U &\xrightarrow{\sim} U'' \subset \mathbb{S}^{n'-1}_{k'} \setminus \{N\}\\ x = (x_1, \tilde{x}) &\mapsto \frac{(x_1, \phi(\tilde{x}))}{\|(x_1, \phi(\tilde{x}))\|},\end{aligned}$$

where we used the shorthand notation $\tilde{x} := (x_2, \dots, x_{n'-1})$. Using Ψ, we obtain a different description of the product map:

$$\begin{aligned}\hat{\mathcal{B}}\colon\ [0,\infty) \times U \times [0,\infty) &\to [0,\infty) \times U \times [0,\infty) \times \mathbb{R}^{n'}_{k'}\\ (x, r) = (x_1, \tilde{x}, r) &\mapsto \left(x, r, r\frac{(x_1, \phi(\tilde{x}))}{\|(x_1, \phi(\tilde{x}))\|}\right)\end{aligned} \tag{3.5}$$

We observe that $\hat{\mathcal{B}}$ is an injective immersion because its differential takes the form

$$\mathrm{D}\hat{\mathcal{B}}_{(x,r)} = \begin{pmatrix} \mathbb{1}_{n'} \\ * \end{pmatrix},$$

where $*$ stands for entries which are not calculated any further, since the columns are linearly independent anyway. Hence, the differential has full rank everywhere and therefore is injective.

If we now consider any point $(x,r) \in [0,\infty) \times U \times [0,\infty)$ with $x_1 = r = 0$, we claim that the differential $\mathrm{D}\hat{\mathcal{B}}$ at (x,r) is a map

$$\mathrm{D}\hat{\mathcal{B}}_{(x,r)} \colon \mathbb{R} \times \mathbb{R}^{n'-2} \times \mathbb{R} \longrightarrow \Big(\mathbb{R} \times \mathbb{R}^{n'-2} \times \mathbb{R}\Big) \times \{0\} \times \mathbb{R}^{n'-1},$$

so $\mathrm{D}\hat{\mathcal{B}}^{n+1} = 0$. In order to obtain this, we just compute all the partial derivatives of

$$\hat{\mathcal{B}}^{n'+1}(x,r) = \frac{rx_1}{\|(x_1, \phi(\tilde{x}))\|}$$

at the point (x,r) with $x_1 = r = 0$:

$$\begin{aligned}
\frac{\partial \hat{\mathcal{B}}^{n'+1}}{\partial x_1}(x,r) &= \frac{r}{\|(x_1,\phi(\tilde{x}))\|}\bigg|_{r=0} + rx_1 \frac{\partial}{\partial x_1}\frac{1}{\|(x_1,\phi(\tilde{x}))\|}\bigg|_{x_1=0} = 0,\\
\frac{\partial \hat{\mathcal{B}}^{n'+1}}{\partial x_i}(x,r) &= rx_1 \frac{\partial}{\partial x_i}\frac{1}{\|(x_1,\phi(\tilde{x}))\|}\bigg|_{x_1=r=0} = 0,\\
\frac{\partial \hat{\mathcal{B}}^{n'+1}}{\partial r}(x,r) &= \frac{x_1}{\|(x_1,\phi(\tilde{x}))\|}\bigg|_{x_1=0} = 0,
\end{aligned}$$

where $i \in \{2, \dots, n'-1\}$.

If we now denote the coordinates in the target space of $\hat{\mathcal{B}}$ by z^i, we obtain

$$\mathrm{im}(\mathrm{D}\hat{\mathcal{B}}) \subset \ker \mathrm{d}z^{n'+1},$$

where $z^{n'+1}$ is the boundary defining function of the first component of $\mathbb{R}^{n'}_{k'}$.

Now suppose $\mathrm{im}(\hat{\mathcal{B}})$ were a submanifold. Then we would find some diffeomorphism $F \colon V \xrightarrow{\sim} W$ between open subsets $V, W \subset M := [0,\infty) \times U \times [0,\infty) \times \mathbb{R}^{n'}_{k'}$ such that

$$F(\hat{\mathcal{B}}) = L \cap M,$$

where $L \subset \mathbb{R}^{2n'}$ is a linear subspace. As a diffeomorphism, F must map boundary faces of codimension l onto one another:

$$F((\partial M)_l \cap V) = (\partial M)_l \cap W.$$

In particular, without loss of generality, we can assume that F maps $\{z^{n'+1} = 0\}$ to $\{z^1 = 0\}$.

Using this diffeomorphism, we get another product map $\Phi = F \circ \hat{\mathcal{B}}$ which now maps onto the linear subspace L introduced above. Looking at its differential at zero, we obtain

$$\mathrm{D}\Phi_0 = \mathrm{D}F_0 \mathrm{D}\hat{\mathcal{B}}_0.$$

But since $\mathrm{D}\hat{\mathcal{B}}^{n'+1}(0) = 0$ and $\mathrm{D}F$ maps the $(n'+1)$-st component to the first component, we also have $\mathrm{D}\Phi^1(0) = 0$, hence

$$\mathrm{D}\Phi(0) \in \{z^1 = 0\}.$$

Now the image of Φ is a subset of some linear subspace, so this implies that the whole image of Φ lies in the boundary face $\{z^1 = 0\}$. Because F, as a diffeomorphism, maps boundary faces to boundary faces, this would also imply that, locally, we have that the image $\operatorname{im}\hat{\mathcal{B}} \subset \{z^{n'+1} = 0\}$.

But this is obviously not true because there exist arbitrarily small non-zero points $(x, r) \in [0, \infty) \times U \times [0, \infty)$ for which

$$\hat{\mathcal{B}}^{n'+1}(x, r) = \frac{rx}{\|(x, \phi(\tilde{x}))\|} \neq 0,$$

i. e. their images doe not lie in the boundary face $\{z^{n'+1} = 0\}$. This is a contradiction, and therefore $\operatorname{im}(\hat{\mathcal{B}})$ cannot be a submanifold. □

To get a better intuition of the different situations considered in the general proof, we want to give another basic example where the image is a wib-submanifold:

Example 3.5. Let $M := \mathbb{R}^3_1 = \mathbb{R}^1_1 \times \mathbb{R}^2$, let $Q := (0, -1, 0) + (\mathbb{R}^2_1 \times \{0\})$, $P := \mathbb{R}^1_1 \times \{0\}$ and let β be the double blow-down map $[M : Q, P] \to M$. In an abuse of notation, we will write $\mathbb{R}^1_1$ instead of $\mathbb{R}^1_1 \times \{0\}$ and $\mathcal{B}$ instead of $\mathcal{B}_{M,Q,P}$. In the double blow-up $[M : P, Q]$, there are essentially 4 types of points $x \in [M : Q, P]$ one can find in a neighbourhood around blown-up P in $[M : Q, P]$ (as locally depicted in figure 3.1):

- Type 1: $(\beta(x) \notin P \cup Q, \beta(x) \notin \partial M)$
  At such points, every small enough neighbourhood will look like $\mathbb{R}^3$ and will not intersect with P or Q. Thus, each of the blow-ups is basically just given by the interior of M, which is modelled by $\mathbb{R}^3$. So, locally, the product map is given by the diagonal

$$\begin{aligned} \mathcal{B} \colon \mathbb{R}^3 &\to \mathbb{R}^3 \times \mathbb{R}^3 \\ x &\mapsto (x, x), \end{aligned}$$

  whose image is known to be a wib-submanifold.

- Type 2: $(\beta(x) \in P, \beta(x) \notin \partial M)$
  These points lie on the boundary of a cylinder, so their local model in $[M : P, Q]$ and in $[M : P]$ is $\mathbb{R}^3_1$; $[M : Q]$, however, is locally modelled by $\mathbb{R}^3$ outside of Q. Hence, the product map takes the form

$$\begin{aligned} \mathcal{B} \colon \mathbb{R}^3_1 &\to \mathbb{R}^3_1 \times \mathbb{R}^3 \\ x &\mapsto (x, x_1\phi(\tilde{x})), \end{aligned}$$

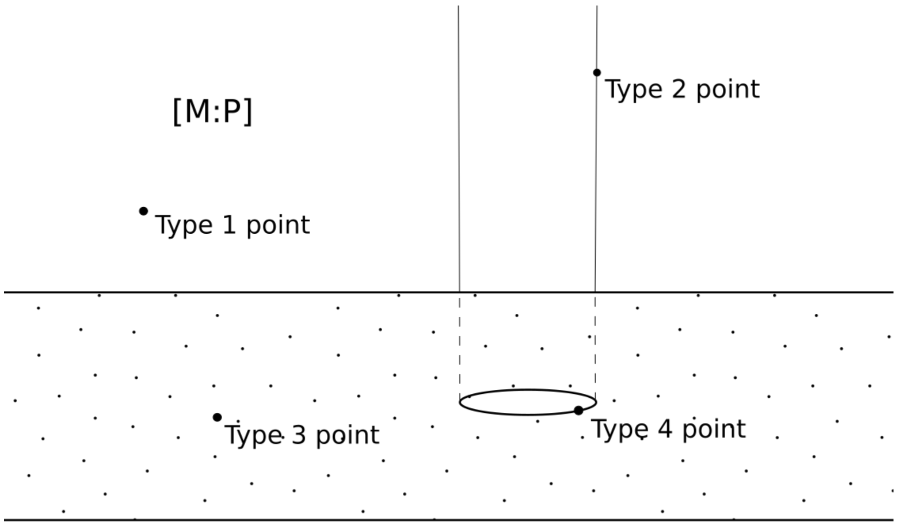


Figure 3.1: The model-case blow-up $\left[\mathbb{R}^3_1 \colon \mathbb{R}^1_1\right]$

where, as before, ϕ is the polar coordinate function and $\tilde{x} := (x_2, x_3)$. This is because, in a neighbourhood of x , the two blow-ups $[M \colon P, Q]$ and $[M \colon P]$ are the same, and the induced map component is just the identity. In the second component, we essentially have a blow-down map.

We can now compose with the local diffeomorphism $\mathbb{R}^3_1 \times \mathbb{R}^3 \to \mathbb{R}^3_1 \times \mathbb{R}^3, (x, y) \mapsto (x, y - x_1\phi(\tilde{x}))$ from (3.1) in order to obtain the easier description of $\mathcal{B}$ as a projection:

$$\begin{aligned} \mathcal{B} \colon \mathbb{R}^3_1 &\to \mathbb{R}^3_1 \times \mathbb{R}^3 \\ x &\mapsto (x, 0). \end{aligned}$$

Here we see that the image of $\mathcal{B}$ is actually a p-submanifold, in particular a wib-submanifold.

- Type 3: $(\beta(x) \notin P, \beta(x) \in \partial M)$
The local model around such a point is $\mathbb{R}^3_1$ and all three blow-ups involved are trivial. So we obtain the diagonal map

$$\begin{aligned} \mathcal{B} \colon \mathbb{R}^3_1 &\to \mathbb{R}^3_1 \times \mathbb{R}^3_1 \\ x &\mapsto (x, x), \end{aligned}$$

whose image is a wib-submanifold.

- Type 4: $(\beta(x) \in P, \beta(x) \in \partial M)$
These points lie on the one-dimensional intersection sphere of the cylinder of

$[M\colon P]$ and the boundary face of M. So their local model in $[M\colon P]$ is $\mathbb{R}^3_2$ whereas in M it is just $\mathbb{R}^3_1$. The coordinates we need to choose in the blow-up are cylindrical coordinates, so up to diffeomorphism, the product map is given by

$$\begin{aligned} \mathcal{B}\colon \mathbb{R}^3_2 &\to \mathbb{R}^3_2 \times \mathbb{R}^3_1 \\ (x_1, x_2, x_3) &\mapsto (x_1, x_2, x_3, x_1, x_2 \cos x_3, x_2 \sin x_3). \end{aligned}$$

After permuting coordinates

$$\begin{aligned} \mathcal{B}\colon \mathbb{R}^3_2 &\to [0,\infty)^2 \times [0,\infty) \times \mathbb{R} \times \mathbb{R}^2 \\ (x_1, x_2, x_3) &\mapsto (x_1, x_1, x_2, x_3, x_2 \cos x_3, x_2 \sin x_3), \end{aligned}$$

we see that the image of $\mathcal{B}$ is a combination of a diagonal and a map of the form $x \mapsto (x, \phi(x))$ as seen before (here, ϕ is again the polar coordinate map). Hence, this is also a wib-submanfold.

Of course, there are also points which lie in a neighbourhood of Q in $[M\colon Q, P]$, but the arguments at these points are very similar, since everything is local.

3.2 Iterated blow-up of nested p-submanifolds

Since we now have a good understanding of when the image of the product map is a wib-submanifold in the disjoint case, we may consider more complicated situations, e. g. two nested p-submanifolds. Therefore we recall Lemma 4.10 from [2, p. 23]:

Lemma 3.6 (nested blow-up)**.** *Let M be a manifold with corners, let Q be a p-submanifold of P and P a p-submanifold of M. Then there exists a unique, smooth, natural map*

$$\zeta_{M,Q,P}\colon\ [M\colon Q, P] := [[M\colon Q] :\ [P\colon Q]] \to [M\colon P]$$

which restricts to the identity on $M \setminus P$. Furthermore, the product map

$$\mathcal{B}_{M,Q,P} := \left(\zeta_{M,Q,P}, \beta_{[M\,:\,Q],[P\,:\,Q]}\right)\colon\ [M\colon Q, P] \to [M\colon P] \times [M\colon Q]$$

is proper in each component, its image is a weak submanifold and $\mathcal{B}_{M,Q,P}$ is a diffeomorphism onto its image.

As in the disjoint case, it was believed that a stronger version of this lemma holds, namely that the image is actually a wib-submanifold. However, this turns out to be wrong:

Proposition 3.7. *Let M, P and Q be as in 3.6 and for $Z \in \{M, P, Q\}$ and $q \in Q$ let $\dim(Z, q)$ denote the dimension of Z at q. Then, if there exists a $q \in Q$ such that $\dim(Q, q) < \dim(P, q) < \dim(M, q)$, the image of the map $\mathcal{B}_{M,Q,P}$ is not a submanifold.*

In order to prove this, we need to define special kinds of sphere orthants (see [2, p. 22]):

Definition 3.8. Let $k, k', n, n' \in \mathbb{N}$. Then we define

$$\mathbb{S}^{n,n'}_{k,k'} := \mathbb{S}^{n+n'} \cap \left(\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}\right).$$

It is important to remark that

$$\mathbb{S}^{n,n'}_{k,k'} = \mathbb{S}^{n+n'} \cap \left(\mathbb{R}^n_k \times \mathbb{R}^{n'+1}_{k'}\right) \cong \mathbb{S}^{n+n'}_{k+k'} = \mathbb{S}^{n+n'} \cap \mathbb{R}^{n+n'+1}_{k+k'},$$

so this new kind of sphere orthant is obtained by the classical one using a diffeomorphism permuting the coordinates.

Proof of Proposition 3.7. We will use results obtained in the proof of the original lemma in [2]. First, using 3.3 and the dimension inequality in 3.7 itself, we can assume that we are in the following model case:

$$\begin{cases} M := \mathbb{R}^m_{k_m} \times \mathbb{R}^p_{k_p} \\ P := \{0\} \times \mathbb{R}^p_{k_p} \\ Q := \{0\}. \end{cases}$$

Using the local description of the blow-up, we obtain:

$$\begin{aligned} [M\colon P] &\simeq \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1}, \\ [M\colon Q] &\simeq \mathbb{S}^{m,p-1}_{k_m,k_p} \times [0,\infty)\,, \\ [[M\colon Q]\colon\ [P\colon Q]] &\simeq \mathbb{S}^{m-1}_{k_m} \times \mathbb{S}^p_{k_p+1} \times [0,\infty)\,. \end{aligned}$$

Under these diffeomorphisms, the blow-down map $\beta_{[M\colon Q],[P\colon Q]}$ becomes

$$\begin{aligned} \beta_{[M\colon Q],[P\colon Q]}\colon \mathbb{S}^{m-1}_{k_m} \times \mathbb{S}^p_{k_p+1} \times [0,\infty) &\to \mathbb{S}^{m,p-1}_{k_m,k_p} \times [0,\infty) \\ (\phi,\psi,t) &\mapsto (\psi_1\phi, \tilde{\psi}, t). \end{aligned} \tag{3.6}$$

The other factor of the product map is given by

$$\begin{aligned} \zeta_{M,Q,P}\colon \mathbb{S}^{m-1}_{k_m} \times \mathbb{S}^p_{k_p+1} \times [0,\infty) &\to \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1} \\ (\phi,\psi,t) &\mapsto (\phi, t\psi) \end{aligned}$$

If we put all this together, we obtain the following local description of the product map:

$$\begin{aligned} \mathcal{B}_{M,Q,P}\colon \mathbb{S}^{m-1}_{k_m} \times \mathbb{S}^p_{k_p+1} \times [0,\infty) &\to \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1} \times \mathbb{S}^{m,p-1}_{k_m,k_p} \times [0,\infty) \\ (\phi,\psi,t) &\mapsto (\phi, t\psi, \psi_1\phi, \tilde{\psi}, t). \end{aligned}$$

Now let $N := \mathbb{S}^{m-1}_{k_m} \times \mathbb{S}^p_{k_p+1} \times [0,\infty)$ and $M := \mathbb{S}^{m-1}_{k_m} \times \mathbb{R}^{p+1}_{k_p+1} \times \mathbb{S}^{m,p-1}_{k_m,k_p} \times [0,\infty)$, i. e. $\mathcal{B}_{M,Q,P}\colon N \to M$. Furthermore let $S := \operatorname{im}(\mathcal{B}_{M,Q,P})$. If S were a submanifold of M, it would satisfy the following:

For every $p \in \partial S \cap \partial M$ and every inward-pointing vector $X \in T_pS \subset T_pM$, which is tangential to the boundary of T_pM, there exists an integral curve in ∂M, i. e. a smooth map $\gamma : [0,\epsilon) \to M$ with $\dot{\gamma}(0) = X$ and $\gamma(0) = p$.

We show that this is not the case by constructing an inward-pointing vector $X \in T_pS$ at a particular point $p \in \partial S \cap \partial M$ satisfying two conditions:

1. X is tangential to the boundary of T_pM.
2. Every integral curve γ of X fulfils $\gamma(s) \notin \partial M$ for every $s > 0$. In other words, the image of any integral curve lies in the interior of M except for the starting point p.

It is clear that both statements together contradict the above mentioned statement for submanifolds. The question now is which point p and tangent vector X one should take. The construction works as follows:

Let $\Phi_q \in \partial\mathbb{S}^{m-1}_{k_m}$, $\Psi_q \in \partial\mathbb{S}^{p}_{k_p+1}$ and set $q := (\Phi_q, \Psi_q, 0) \in \partial N$. The desired point p is then given by $p := \mathcal{B}_{M,Q,P}(q)$ and the tangent vector X is defined as $\mathrm{d}B_q(\Phi_0, \Psi_0, t_0)$, where Φ_0 and Ψ_0 are inward-pointing tangent vectors at Φ_q and Ψ_q, respectively, and $t \in \mathbb{R}_{>0}$.

Now we can prove the first statement by showing that X is tangential to the boundary in the $\mathbb{R}^{p+1}_{k_p+1}$-component. Therefore we denote the corresponding component of $\mathcal{B}_{M,Q,P}$ by $\mathcal{B}_2$. Then

$$\mathrm{d}_q\mathcal{B}_2(\Phi_0, \Psi_0, t_0) = 0 \cdot \Phi_0 + t_0 \cdot \Psi_q,$$

which is tangential to the boundary of $\mathbb{R}^{p+1}_{k_p+1}$ because Ψ_q itself is.

For the second statement we consider any integral curve γ of X and use that $\mathcal{B}_{M,Q,P}$ is an immersion and a homeomorphism onto its image. This implies that $\tau := \mathcal{B}^{-1}_{M,Q,P} \circ \gamma$ is a smooth curve in N with $\mathrm{d}B_q(\dot\tau(0)) = X$. By the choice of Φ_0, Ψ_0 and t_0, we have that $\tau(s) \in \mathring{N}$ for every $s > 0$, hence:

$$\forall s > 0 \quad \mathcal{B}_{M,Q,P}(\tau(s)) = \gamma(s) \in \mathring{M}.$$

Since γ was chosen arbitrarily, this shows the second statement and therefore the overall claim. $\square$

3.3 Iterated blow-up along clean semilattices

The last generalisation of iterated blow-ups considered in this thesis is the one along so-called clean semilattices which we will now introduce by recalling definitions and results of [2, Section 2.3]. In the following, let M always be a manifold with corners.

Definition 3.9 (clean intersection)**.**

1. We say, p-submanifolds $X_1, \ldots, X_k$ of M *intersect cleanly* or *have a clean intersection* if
   a) $Y := X_1 \cap \ldots \cap X_k$ is a p-submanifold of M (possibly empty),
   b) for all $x \in Y$, $T_xY = T_xX_1 \cap \ldots \cap T_xX_k$.
2. For a locally finite (unordered) set $\mathcal{F}$ of p-submanifolds of M, we call $\mathcal{F}$ a *cleanly intersecting family* if any $X_1, \ldots, X_j \in \mathcal{F}$ have a clean intersection.

Definition 3.10 ((meet) semilattice)**.** A *(meet) semilattice* is a partially ordered set $\mathcal{L}$ such that, for every two $x, y \in \mathcal{L}$, there is a greatest common lower bound $x \cap y \in \mathcal{L}$ of x and y. For our purposes, the partial order will always be the ordinary inclusion of sets and the lower bound of any two elements is obtained by the usual intersection.

Definition 3.11 (clean semilattice)**.** A subset $\emptyset \in \mathcal{S} \subset \mathcal{P}(M)$ of closed p-submanifolds of M is called a *clean semilattice* if it is a cleanly intersecting family and a semilattice with respect to the set-theoretic inclusion.

The important thing about clean semilattices of p-submanifolds is that we can iteratively blow up minimal elements. In this blow-up, the original semilattice lifts to a family of p-submanifolds which form again a clean semilattice. Precisely, we have

Proposition 3.12 (blowing up minimal elements)**.** *Let $\mathcal{S}$ be a clean semilattice of p-submanifolds of M and let P be a minimal element of $S \setminus \{\emptyset\}$. Setting $Q' := [Q\colon P \cap Q]$, the set*

$$\mathcal{S}' := \left\{Q' \mid Q \in \mathcal{S}\right\}$$

is a clean semilattice of $[M\colon P]$ with $\emptyset = \emptyset' = P' \in \mathcal{S}'$.

With this result in mind, it is only natural to order a clean semilattice such that it is compatible with the inclusion. Because then we can blow up iteratively while being sure to have a clean semilattice at each step.

Definition 3.13. An ordering $(P_i)_{i=1}^k = (P_1, \ldots, P_k)$ of $\mathcal{S} \setminus \emptyset$ is called *compatible with the inclusion* if

$$P_i \subset P_j \Rightarrow i \leq j.$$

Proposition 3.14. *Let $\mathcal{S}$ be a clean semilattice and $(P_i)_{i=1}^k$ be an ordering of $\mathcal{S} \setminus \emptyset$ compatible with the inclusion. Then $\left[M\colon (P_i)_{i=1}^k\right]$ is defined.*

The last object we need to introduce for the main theorem of this section is the graph blow-up as in [2, p. 17, Def. 3.1]:

Definition 3.15. Let $\mathcal{F}$ be a locally finite set of closed p-submanifolds of the manifold with corners M. Then, the *graph blow-up* $\{M\colon \mathcal{F}\}$ of M along $\mathcal{F}$ is defined by

$$\{M\colon \mathcal{F}\} := \overline{\left\{(x, \ldots, x) \mid x \in M \setminus \bigcup \mathcal{F}\right\}} \subset \prod_{Y \in \mathcal{F}} [M\colon Y]\,.$$

Now we have introduced everything to give a similar statement about the image of a suitable product map for the iterative blow-up of a clean semilattice, as stated in [2, p. 26, Theorem 4.12]:

Theorem 3.16. *Let $\mathcal{S} = (P_j)_{j=1,\ldots,k}$ be a clean semilattice of closed p-submanifolds of M. Then, for each $P \in \mathcal{S}$, there exists a unique smooth map $\phi_{\mathcal{S},P}\colon [M\colon \mathcal{S}] \to [M\colon P]$ which restricts to the identity on $M \setminus \bigcup_{Q \in \mathcal{S}}$, and the induced map*

$$\mathcal{B}_{\mathcal{S}} := (\phi_{\mathcal{S},P_0}, \ldots, \phi_{\mathcal{S},P_k})\colon [M\colon \mathcal{S}] \to \prod_{j=0}^{k} [M\colon P_j]$$

is proper in each component. Furthermore, the image of $\mathcal{B}_{\mathcal{S}}$ is a weak submanifold of $\prod_{j=0}^{k} [M\colon P_j]$ and $\mathcal{B}_{\mathcal{S}}$ maps $[M\colon \mathcal{S}]$ diffeomorphically onto $\{M\colon \mathcal{S}\}$, i. e.

$$[M\colon \mathcal{S}] \xrightarrow{\sim} \{M\colon \mathcal{S}\}\,.$$

And, as before, we can actually give a stronger statement about the image of the product map:

Theorem 3.17. *In the situation of 3.16, the image of the product map $\mathcal{B}_\mathcal{S}$ is a submanifold if and only if the following holds: There do not exist elements $Q \subset P \subset M$ in $\mathcal{S}$ and a point $q \in Q$ such that $\dim(Q,q) < \dim(P,q) < \dim(M,q)$. Additionally, at every point in M the local model of the iterated blow-up (which is just a single blow-up because of the above dimension restriction) is of the form*

$$\left[\mathbb{R}^n_k \times \mathbb{R}^{n'}_{k'} \colon \mathbb{R}^n_k \times \{0\}\right],$$

where either $n' = 1$ or $n' \geq 2$ and $k' = 0$. In these cases, it is actually also a wib-submanifold.

Proof. This proof is rather easy, since we can use the results from 3.2 and 3.7. The procedure will be an induction on the number $k+1$ of elements in $\mathcal{S} = (P_j)_{j=0,\dots,k}$, as in the proof of 3.16, which can be found in [2, p. 26f.]. First of all, the case $k = 0$ is trivial, so consider $k > 0$. Then we have three different cases:
Case $k = 1$: If $\mathcal{S}$ contains $1 + 1 = 2$ elements, we have $\mathcal{S} = (\emptyset, P)$ and the product map is the same as in 3.2, therefore obtaining the same result.
Case $k = 2$: If $\mathcal{S}$ contains $2 + 1 = 3$ elements, we have $\mathcal{S} = (\emptyset, Q, P)$ with either $Q \subset P$ or $Q \cap P = \emptyset$. If they do not intersect, we are again in the same case as in 3.2, whereas otherwise we have the nested situation from 3.7. So, in the disjoint case, the fact whether or not the image of the product map is a submanifold, depends on the same parameters as in said Proposition. But in the nested case, the image will never be a submanifold. This is why we demand the p-submanifolds in the semilattice $\mathcal{S}$ to be pairwise disjoint.
Case $k \geq 3$: Suppose, we know the statement for $k-1$ and that the elements of $\mathcal{S}$ are ordered compatible with the inclusion. Then, the first element P_1 is minimal in $\mathcal{S} \setminus \emptyset$ with respect to $\subset$. Therefore, using the definition

$$\mathcal{S}' := \left\{P'_j := [P_j \colon P_1] \mid j = 2, \dots, k\right\}$$

of the blown-up lattice, the blow-up $[M \colon \mathcal{S}]$ is defined and the product map is given by

$$\begin{aligned}
[M \colon \mathcal{S}] := \left[[M \colon P_1] : \mathcal{S}'\right] &\xrightarrow{\mathcal{B}_{\mathcal{S}'}} \prod_{j=1}^{k} [[M \colon P_1] : \ [P_j \colon P_1]] \\
&\xrightarrow{\Phi} [M \colon P_1] \times \prod_{j=2}^{k} ([M \colon P_1] \times [M \colon P_j]),
\end{aligned}$$

where $\Phi := \mathrm{id} \times \prod_{j=2}^{k} \mathcal{B}_{M,P_1,P_j}$. Now either P_j and P_1 are disjoint for any j or there exists some $j \in \{2, \dots, k\}$ such that $P_1 \subset P_j$. So, again, in the disjoint case, all of the components of this map arise from product maps as in 3.2.

Similarly, in the nested case, there is at least one pair of factors which are the same as in 3.7, which means that the results of said propositions generalise to the current setting. Indeed, the proof presented there can be applied to the i- and j-components in the very same way, leading to the same result. This shows the claim. ☐

4 Lifting vector fields to blow-ups

As mentioned in [2, p. 32f, Example 5.13], in the N-body problem, one considers the manifold $X := \mathbb{R}^n$ (where $n := 3N$) and the subspaces

$$Y_j := \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^{3N} \mid x_j = 0\} \text{ and}$$
$$Y_{ij} := \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^{3N} \mid x_i = x_j\}.$$

So in this notation, every $x_i \in \mathbb{R}^3$ stands for the position of a single particle in three-dimensional space and the subspaces Y_i and Y_{ij} are these configurations of the N particles where one particle flies into the nucleus or two particles collide, respectively.

The spaces $\overline{Y}_i$ and $\overline{Y_{ij}}$ form a finite clean semilattice $\mathcal{S}$ of p-submanifolds of the so-called spherical compactification $\overline{X}$ of X, which will be introduced later (see [2, p. 31, equation (29)]).

We are now interested in the corresponding iterated blow-up $\left[\overline{X} : \mathcal{S}\right]$ or more precisely, in the first step of this iteration: the blow-up $\left[\overline{\mathbb{R}^n} : \partial\overline{V}\right]$, where V is a minimal element of $\mathcal{S}$ with respect to the inclusion.

On $\mathbb{R}^n$, there a two obvious types of vector fields: translation vector fields and – more generally – affine maps. The question is whether or not we can extend these vector fields to the spherical compactification $\overline{\mathbb{R}^n}$ and if these extensions then lift into the blow-up $\left[\overline{\mathbb{R}^n} : \partial\overline{V}\right]$.

But before we can investigate this, we first of all try to classify when a general vector field lifts into a given blow-up. Therefore, it is crucial to introduce a suitable notion of vector fields on manifolds with corners:

Definition 4.1 (Vector fields on manifolds with corners)**.** Let M be a manifold with corners, let $\mathcal{B}_M := \{H_1, \dots, H_k\}$ be the set of its boundary hyperfaces and for every $H \in \mathcal{B}_M$ let $i_H \colon H \to M$ be the immersion of H into M. Then we define

$$\mathcal{V}_M := \{X \in \Gamma(TM) \mid \forall H \in \mathcal{B}_M : i^*(X) \in \Gamma(TH)\},$$

the vector space of all vector fields on M which are tangent to all boundary hyperfaces (see [1, p. 12]).

As blow-ups come naturally with a blow-down map, a lift of a vector field should be a vector field in the blow-up, whose push-forward along the blow-down map coincides with the original vector field. More generally, we can consider lifts of vector fields along any surjection between manifolds with corners (similar to [1, p. 12f]):

Definition 4.2 (Lift of a vector field)**.** Let M, N be manifolds with corners, let $\beta \colon M \to N$ be a smooth surjective map and let $X \in \mathcal{V}_N$. Furthermore, let $\mathrm{d}\beta$ denote the differential of β.

Then, we say X *lifts to* $W \in \mathcal{V}_M$ if and only if

$$\mathrm{d}\beta(W) = X \circ \beta$$

As said before, we wonder whether or not a given vector field on a manifold with corners lifts to a blow-up along a p-submanifold. An answer was given in [1, p. 14, Prop. 3.2], and it nicely characterises exactly those vector fields admitting a lift:

Proposition 4.3 (Lifting vector fields)**.** *Let M be a manifold with corners, let $P \subset M$ be a p-submanifold and let $X \in \mathcal{V}_M$. Then, X admits a lift $W \in \mathcal{V}_{[M:\,P]}$ if and only if*

$$X|_P \in \Gamma(TP), \tag{4.1}$$

i. e. X is tangential to P.

Proof. One direction is rather obvious: Suppose, X lifts to a vector field $W \in \mathcal{V}_{[M:\,P]}$. Then, by definition, W is tangent to the boundary hyperface $N_+^M(P)$. Hence, the pushed-forward vector field $X \circ \beta = \mathrm{d}\beta_{M,P} W$ has to be tangent to the image of $N_+^M(P)$ under $\beta_{M,P}$, which is precisely P. For the converse, we use that lifting vector fields is a local property, so we may restrict to the model case

$$V := \mathbb{R}^n_l \times \mathbb{R}^{n'}_{l'},$$
$$P := \mathbb{R}^n_l \times \{0\}.$$

with the blow-down map $\beta\colon\ [V:\,P] \to V$.

Now suppose $X|_P \in \Gamma(TP)$, that means we have $X_{n+1\ldots n+n'}(x,0) = 0$ for all $x \in \mathbb{R}^n_l$. We want to construct a lift of X by setting values on the spherical normal bundle of the blow-up. In the notation of above, let Q be the orthogonal complement of P in V, i. e. $Q = \{0\} \times \mathbb{R}^{n'}_{l'}$. Then the blow-up of this model case in its original definition is given by

$$[V:\,P] := V \setminus P \sqcup (P \times \mathbb{S}Q),$$

and therefore we need to define the lift on $P \times \mathbb{S}Q$.

First of all, we realize that X can be decomposed with respect to P and Q:

$$X = X_P + X_Q,$$

where X_P and X_Q are the projections of the vector field X onto TP and TQ respectively.

For any point $(x,\xi) \in V$, the differential of X_Q at (x,ξ) is a homomorphism from V to Q, i. e.

$$\mathrm{d}_{(x,\xi)} X_Q \in \mathrm{Hom}(V,Q),$$

and its restriction to P vanishes:

$$\mathrm{d}_{(x,\xi)} X_Q|_P = 0.$$

Therefore, we can view this differential as an endomorphism of $Q \cong V/P$:

$$\mathrm{d}_{(x,\xi)} X_Q \in \mathrm{Hom}(V/P,Q) \cong \mathrm{Hom}(Q,Q).$$

Hence, since $\mathbb{S}Q \subset Q \subset V$, one can consider this at $\xi \in \mathbb{S}Q$ and decompose it into a radial part and a part tangential to $\mathbb{S}Q$ at ξ:

$$\mathrm{d}_{(x,\xi)} X_Q(\xi) = \lambda(\xi) \cdot \xi + \hat{X}(\xi),$$

where $\lambda \in C^\infty(\mathbb{S}Q)$ and $\hat{X} \in \Gamma(\mathbb{S}Q)$. Since $P \times \mathbb{S}Q$ is going to be a boundary hyperface in the blow-up, the lift of X needs to be tangential to it. In other words, for the lift we set

$$\begin{aligned} \tilde{X} \colon\ [V \colon P] = (V \setminus P) \sqcup (P \times \mathbb{S}Q) &\to T\,[V \colon P] = T(V \setminus P) \sqcup T(P \times \mathbb{S}Q) \\ V \setminus P \ni (x, y) &\mapsto X(x, y) \\ P \times \mathbb{S}Q \ni (x, \xi) &\mapsto \Big(X_P(x, \xi), \hat{X}(\xi) \Big), \end{aligned}$$

where we replaced X_Q at the boundary by the non-radial part of its differential.

Surely, this map is well-defined and smooth in $V \setminus P$ as well as on $P \times \mathbb{S}Q$ in all directions which are tangent to this boundary hyperface. Hence, it only remains to check that $\tilde{X}$ is smooth on $P \times \mathbb{S}Q$ in radial direction.

Therefore, we recall the map κ from (2.2), which endowed the blow-up with its smooth structure:

$$\begin{aligned} (V \setminus P) \sqcup (P \times \mathbb{S}Q) &\xleftarrow[\kappa]{\sim} P \times \mathbb{S}Q \times [0, \infty) \\ (x, r\xi) &\hookleftarrow (x, \xi, r) (\text{if } r > 0), \\ (r, \xi) &\hookleftarrow (x, \xi, 0). \end{aligned}$$

Now, on $V \setminus P$, the pulled-back vector field of X along κ is given by

$$\begin{aligned} \mathrm{d}\kappa^{-1}(X(x, r\xi)) &= \mathrm{D}\kappa^{-1}(x, r\xi) \cdot X(x, r\xi) \\ &= \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & \frac{1}{r}(\mathbb{1}_{n'} - \xi \otimes \xi) \\ 0 & \xi^T \end{pmatrix} \cdot X(x, r\xi) \\ &= \begin{pmatrix} X_P(x, r\xi) \\ \frac{1}{r}\pi_\xi(X_Q(x, r\xi)) \\ \xi \cdot X_Q(x, r\xi), \end{pmatrix} \end{aligned}$$

where $\pi_\xi := \mathbb{1}_{n'} - \xi \otimes \xi$.

Since we are given $X_Q(x, 0) = X_{n+1 \ldots n+n'}(x, 0) = 0$, we may transform the expression in the middle of the upper vector to

$$\begin{aligned} &\frac{1}{r} \cdot \pi_\xi(X_Q(x, r\xi) - X_Q(x, 0)) \\ &= \pi_\xi \left(\frac{1}{r} \cdot (X_Q(x, r\xi) - X_Q(x, 0)) \right) \\ &\xrightarrow{r \to 0} \pi_\xi \big(\partial_{(0,\xi)} X_Q(x, 0) \big) \\ &= \pi_\xi(\mathrm{D}X_Q(x, 0) \cdot (0, \xi)) \\ &= \pi_\xi \big(\mathrm{d}_{(x,0)} X_Q(\xi) \big) \end{aligned}$$

In total, we obtain

$$\begin{aligned}
\mathrm{d}\kappa^{-1}(X(x, r\xi)) &= \begin{pmatrix} X_P(x, r\xi) \\ \frac{1}{r}\pi_\xi(X_Q(x, r\xi)) \\ \xi \cdot (X_Q(x, r\xi)) \end{pmatrix} \\
&\xrightarrow{r\to 0} \begin{pmatrix} X_P(x, 0) \\ \pi_\xi\big(\mathrm{d}_{(x,0)} X_Q(\xi)\big) \\ 0 \end{pmatrix} \\
&= \mathrm{d}\kappa^{-1} \begin{pmatrix} X_P(x, \xi) \\ \pi_\xi\big(\mathrm{d}_{(x,\xi)} X_Q(\xi)\big) \end{pmatrix} \\
&= \mathrm{d}\kappa^{-1}\Big(\tilde{X}(\xi)\Big),
\end{aligned}$$

so it exactly fits with our definition of $\tilde{X}$. This proves the claim. □

We now want to apply this to the case, where M is a particular kind of compactification of $\mathbb{R}^n$, the so-called spherical compactification (see also [2, p. 28f]):

Definition 4.4. Let $n \in \mathbb{N}$. Then, as a set, the *spherical compactification* $\overline{\mathbb{R}^n}$ of $\mathbb{R}^n$ is defined as

$$\overline{\mathbb{R}^n} := \mathbb{R}^n \sqcup \mathbb{S}^{n-1}.$$

We endow it with a smooth structure by pulling back the natural smooth structure of $\mathbb{S}^n_1$ along the bijection

$$\begin{aligned}
\Theta_n \colon \overline{\mathbb{R}^n} &\to \mathbb{S}^n_1 \\
\mathbb{R}^n \ni x &\mapsto \frac{1}{\|(1, x)\|}(1, x), \\
\mathbb{S}^{n-1} \ni v &\mapsto (0, v)
\end{aligned}$$

with the inverse

$$\begin{aligned}
\Theta_n^{-1} \colon \mathbb{S}^n_1 &\to \overline{\mathbb{R}^n} \\
(y_0, \ldots, y_n) &\mapsto \frac{1}{y_0}(y_1, \ldots, y_n) \text{ if } y_0 > 0, \\
(0, v) &\mapsto v.
\end{aligned}$$

In the following, we will fix $M = \overline{\mathbb{R}^n}$ for some $n \in \mathbb{N}$ and $P = \partial\overline{V}$, where $V \subset \mathbb{R}^n$ is a k-dimensional linear subspace.

Definition 4.5 (Translation vector field)**.** For any $X \in \mathbb{R}^n$, the *translation vector field of* X, also denoted by X, is defined as

$$\begin{aligned}
X \colon \mathbb{R}^n &\to T\mathbb{R}^n \\
x &\mapsto X,
\end{aligned}$$

where X is viewed as an element in $T_x\mathbb{R}^n$ using the canonical vector space isomorphism $\mathbb{R}^n \cong T_x\mathbb{R}^n$. It is clear that $X \in \mathcal{V}_{\mathbb{R}^n}$, since it is smooth as a constant map and $\mathbb{R}^n$ has no boundary hyperfaces.

The first thing to ask now is whether or not this vector field extends to $\overline{\mathbb{R}^n}$. Therefore, we consider the global chart Θ_n of $\overline{\mathbb{R}^n}$ which endowed it with the structure of a smooth manifold:

$$\begin{aligned}\Theta_n\colon \overline{\mathbb{R}^n} &\longrightarrow \mathbb{S}^n_1\\ \mathbb{R}^n \ni x &\mapsto \frac{1}{\|(1,x)\|}(1,x),\\ \mathbb{S}^{n-1} \ni v &\mapsto (0,v).\end{aligned}$$

Using this chart, we can push forward X to a vector field on $\mathbb{S}^n_1 \setminus \partial\mathbb{S}^n_1$ and look for a possible extension there. For all $y = (y_0, \dots y_n) \in \mathbb{S}^n_1$ with $y_0 > 0$, we calculate this pushed-forward vector field using the shorthand notation $\overline{y} \coloneqq (y_1, \dots, y_n)$:

$$\begin{aligned}((\Theta_n)_*X)(y_0,\dots,y_n) &= \mathrm{d}_{\frac{1}{y_0}\overline{y}}\Theta_n\left(X\left(\frac{1}{y_0}\overline{y}\right)\right)\\ &= \mathrm{d}_{\frac{\overline{y}}{y_0}}\Theta_n(X)\\ &= \mathrm{D}\Theta_n\left(\frac{\overline{y}}{y_0}\right)\cdot X.\end{aligned}$$

If we want to extend $((\Theta_n)_*X)$ to all of $\mathbb{S}^n_1$, we need to look at the limit $y_0 \to 0$. Therefore, we compute the differential above:

$$\begin{aligned}\mathrm{D}\Theta_n(x) &= \begin{pmatrix} -\frac{x}{\|(1,x)\|^3} \\ \left(\frac{\delta_{ij}}{\|(1,x)\|} - \frac{x_i x_j}{\|(1,x)\|^3}\right)_{i,j}\end{pmatrix}\\ &= \frac{1}{\|(1,x)\|}\begin{pmatrix} -\frac{x}{\|(1,x)\|^2} \\ \mathbb{1}_n - \frac{x\otimes x}{\|(1,x)\|^2}\end{pmatrix}.\end{aligned}$$

If $x = \frac{\overline{y}}{y_0}$, we have $\|(1,x)\| = \left\|\left(1, \frac{\overline{y}}{y_0}\right)\right\| = \frac{\|y\|}{y_0}$ and

$$\begin{aligned}\mathrm{D}\Theta_n\left(\frac{\overline{y}}{y_0}\right) &= \frac{1}{\left\|\left(1,\frac{\overline{y}}{y_0}\right)\right\|}\begin{pmatrix} -\frac{\overline{y}}{y_0\left\|\left(1,\frac{\overline{y}}{y_0}\right)\right\|^2} \\ \mathbb{1}_n - \frac{\overline{y}\otimes\overline{y}}{y_0^2\left\|\left(1,\frac{\overline{y}}{y_0}\right)\right\|^2}\end{pmatrix}\\ &= \frac{1}{\left\|\left(1,\frac{\overline{y}}{y_0}\right)\right\|}\begin{pmatrix} -\frac{\overline{y}}{\|y\|\left\|\left(1,\frac{\overline{y}}{y_0}\right)\right\|} \\ \mathbb{1}_n - \frac{\overline{y}\otimes\overline{y}}{\|y\|^2}\end{pmatrix}\end{aligned}$$

We are now interested in the limit of this differential for $y_0 \to 0$ whilst maintaining the constraint $y \in \mathbb{S}^n_1$. Therefore, for any $(0,\overline{y}) \in \partial\mathbb{S}^n_1$ we consider the meridian

$$\begin{aligned}\gamma_{\overline{y}}\colon [0,1) &\to \mathbb{S}^n_1\\ t &\mapsto \left(t, \sqrt{1-t^2}\,\overline{y}\right).\end{aligned}$$

Let us now fix an arbitrary bounded sequence $(y_k)_{k\in\mathbb{N}}$ with $y_k \in \mathbb{S}^n_1$ and a sequence $(t_k)_{k\in\mathbb{N}}$ with $t_k > 0$ and $\lim_{k\to\infty} t_k = 0$.

Then, for any $k \in \mathbb{N}$, the differential $\mathrm{D}\Theta_n$ along $\Theta_n^{-1} \circ \gamma_{y_k}$ at t_k is given by

$$\begin{aligned}
\mathrm{D}\Theta_n\big(\Theta_n^{-1}(\gamma_{y_k}(t_k))\big) &= \mathrm{D}\Theta_n\left(\frac{\overline{\gamma_{y_k}(t_k)}}{\gamma_{y_k}(t_k)_0}\right) \\
&= \mathrm{D}\Theta_n\left(\frac{\sqrt{1-t_k^2}}{t_k} y_k\right) \\
&= \frac{1}{\left\| \left(1, \frac{\sqrt{1-t_k^2}}{t_k} y_k\right) \right\|} \begin{pmatrix} -\frac{\sqrt{1-t_k^2}\, y_k}{\left\| \left(1, \frac{\sqrt{1-t_k^2}}{t_k} y_k\right)\right\|} \\ \mathbb{1}_n - (1-t_k^2) y_k \otimes y_k \end{pmatrix}
\end{aligned}$$

A short and straightforward investigation shows that this expression converges to zero for $k \to \infty$. Hence, also $\mathrm{D}\Theta_n\left(\frac{\overline{y}}{y_0}\right) \cdot X$ decays to zero for $y_0 \to 0$ if $\overline{y} \neq 0$.

Note that we could assume $y_0 \neq 0$ without any loss of generality since y is a point on a sphere and this inequality is true in any neighbourhood of $\partial \mathbb{S}^n_1$ which does not contain the north pole.

To sum it up, the only possible extension of X to $\overline{\mathbb{R}^n}$ is the extension by zero on the sphere, i. e. $X(v) := 0$ for $v \in \mathbb{S}^{n-1}$. Since the extension of $\mathrm{D}\Theta_n$ to $y_0 = 0$ is smooth, this also holds for the vector field X; hence, $X \in \mathcal{V}_{\overline{\mathbb{R}^n}}$.

As said previously, we would like to know whether or not X lifts to the blow-up $\left[\overline{\mathbb{R}^n} \colon \partial\overline{V}\right]$, where $V \subset \mathbb{R}^n$ is a linear subspace. But this is now trivial: Since $\partial\overline{V} \subset \partial\overline{\mathbb{R}^n} = \mathbb{S}^{n-1}$ and $X|_{\mathbb{S}^{n-1}} = 0$, we can apply 4.3, which yields the desired lift of X to $\left[\overline{\mathbb{R}^n} \colon \partial\overline{V}\right]$.

So, translations seem to be rather easy - they always lift to a blow-up of the given kind. A more general type of vector fields on $\mathbb{R}^n$, however, are affine maps:

Definition 4.6 (affine maps, motions)**.** An *affine map on* $\mathbb{R}^n$ is a vector field in $\mathcal{V}_{\mathbb{R}^n}$ of the form $M_{A,X} \colon x \mapsto A \cdot x + X$, where $A \in M_n(\mathbb{R})$ and $X \in \mathbb{R}^n$.

$M_{A,X}$ is called a *motion* if $A \in \mathcal{O}(n)$.

This is motivated by the fact that, more generally, a motion is an isometry of a metric space, and the isometries of Euclidean space are precisely the orthogonal affine maps (see also [3, p. 21, Def. 4.1.1]).

Similar to translations, we may now wonder whether or not affine linear maps extend to $\overline{\mathbb{R}^n}$, and, as a next step, if they also lift to blow-ups at infinity. We recall the definition of $\overline{\mathbb{R}^n}$:

$$\overline{\mathbb{R}^n} := \mathbb{R}^n \sqcup \mathbb{S}^{n-1}.$$

Our goal now is to give a reasonable extension of $M_{A,X}$ to $\overline{\mathbb{R}^n}$ by defining values on $\mathbb{S}^{n-1}$.

First of all, the translation part X is going to vanish at infinity, so let us just consider $M_{A,0}$: As $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, we can restrict the vector field $M_{A,0}$ to $\mathbb{S}^{n-1}$ and decompose it into a radial component and one tangential to the sphere:

$$\begin{aligned}\mathcal{M}_{A,0} \colon \mathbb{S}^{n-1} &\to T\mathbb{R}^n \\ x &\mapsto A \cdot x = \lambda(x) \cdot x + B(x),\end{aligned}$$

where $\lambda \in C^\infty(\mathbb{S}^{n-1})$ and $B \in \Gamma(\mathbb{S}^{n-1})$.

Since the extension of $M_{A,0}$ and hence of $M_{A,X}$ should be an element of $\mathcal{V}_{\overline{\mathbb{R}^n}}$, it has to be tangent to the boundary hyperface at infinity, i. e. tangent to $\mathbb{S}^{n-1}$. Thus, the contribution of λ has to vanish, and therefore we set

$$\begin{aligned}\overline{M}_{A,X} \colon \overline{\mathbb{R}^n} = \mathbb{R}^n \sqcup \mathbb{S}^{n-1} &\to T\overline{\mathbb{R}^n} = T\mathbb{R}^n \sqcup T\mathbb{S}^{n-1} \\ \mathbb{R}^n \ni x &\mapsto A \cdot x, \\ \mathbb{S}^{n-1} \ni v &\mapsto B(v),\end{aligned}$$

where $B(v)$ is defined as before.

This vector field now turns out to be smooth: Indeed, it is clearly smooth in the interior of $\overline{\mathbb{R}^n}$, as $x \mapsto A \cdot x$ is smooth; and the extension is smooth on $\partial\overline{\mathbb{R}^n}$ in every direction tangential to the boundary hyperface, because B is smooth.

Hence, it remains to check if the extension $\overline{M}_{A,X}$ is smooth in the radial direction at the boundary. Therefore, we consider the chart Θ_n from before and compute the pushed-forward vector field along Θ_n:

$$\begin{aligned}(\Theta_n)_* M_{A,X}(y_0, \ldots, y_n) &= \mathrm{d}_{\frac{\overline{y}}{y_0}} \Theta_n \left(A \cdot \frac{\overline{y}}{y_0} + X \right) \\ &= \mathrm{d}_{\frac{\overline{y}}{y_0}} \Theta_n \left(A \cdot \frac{\overline{y}}{y_0} \right) + \mathrm{d}_{\frac{\overline{y}}{y_0}} \Theta_n(X).\end{aligned}$$

As said before, in the limit $y_0 \to 0$ the second summand vanishes as it is a translation. The first summand computes:

$$\begin{aligned}\mathrm{d}_{\frac{\overline{y}}{y_0}} \Theta_n \left(A \cdot \frac{\overline{y}}{y_0} \right) &= \mathrm{D}\Theta_n \left(\frac{\overline{y}}{y_0} \right) \cdot A \cdot \frac{\overline{y}}{y_0} \\ &= \frac{1}{y_0 \left\| \left(1, \frac{\overline{y}}{y_0}\right) \right\|} \begin{pmatrix} -\frac{\overline{y}}{\|y\| \left\| \left(1, \frac{\overline{y}}{y_0}\right) \right\|} \\ \mathbb{1}_n - \frac{\overline{y} \otimes \overline{y}}{\|y\|^2} \end{pmatrix} \cdot A \cdot \overline{y} \\ &= \frac{1}{\|y\|} \begin{pmatrix} -\frac{\overline{y}}{\|y\| \left\| \left(1, \frac{\overline{y}}{y_0}\right) \right\|} \\ \mathbb{1}_n - \frac{\overline{y} \otimes \overline{y}}{\|y\|^2} \end{pmatrix} \cdot A \cdot \overline{y}.\end{aligned}$$

Similar to before, in order to show that the extension is smooth in radial direction at infinity, we again consider the meridian $\gamma_{\overline{y}}$ for $\overline{y} \in \partial\mathbb{S}^n_1$. Also, let $(y_k)_{k\in\mathbb{N}}$ with $y_k \in \mathbb{S}^n_1$ be again a convergent sequence, whose limit $\overline{y}$ lies in $\partial\mathbb{S}^n_1$, and let $(t_k)_{k\in\mathbb{N}}$ be a sequence with $t_k > 0$ and $\lim_{k\to\infty} t_k = 0$.

Then, for any $k \in \mathbb{N}$, we compute

$$\begin{aligned}
\frac{1}{\gamma_{y_k}(t)_0} \mathrm{D}\Theta_n \left(\frac{\overline{\gamma_{y_k}(t_k)}}{\gamma_{y_k}(t_k)_0} \right) &= \frac{1}{t_k} \mathrm{D}\Theta_n \left(\frac{\sqrt{1-t_k^2}}{t_k} \right) \\
&= \frac{1}{t_k \left\| \left(1, \frac{\sqrt{1-t_k^2}}{t_k} y_k \right) \right\|} \begin{pmatrix} -\frac{\sqrt{1-t_k^2} y_k}{\left\| \left(1, \frac{1-t_k^2}{t_k} y_k \right) \right\|} \\ \mathbb{1}_n - (1-t_k^2) y_k \otimes y_k \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\sqrt{1-t_k^2} y_k}{\left\| \left(1, \frac{1-t_k^2}{t_k} y_k \right) \right\|} \\ \mathbb{1}_n - (1-t_k^2) y_k \otimes y_k \end{pmatrix} \\
&\xrightarrow{k \to \infty} \begin{pmatrix} 0 \\ \mathbb{1}_n - \overline{y} \otimes \overline{y} \end{pmatrix}.
\end{aligned}$$

So, the smooth limit of $M_{A,X}$ at points $v \in \partial \overline{\mathbb{R}^n} = \mathbb{S}^{n-1}$ is precisely the image of $A \cdot v$ under the projection $\pi_{\overline{y}} \colon \mathbb{R}^n|_{\mathbb{S}^{n-1}} \to T\mathbb{S}^{n-1}$, which is B:

$$\begin{aligned}
M_{A,X} \colon \mathbb{S}^{n-1} &\to T\mathbb{S}^{n-1} \\
v &\mapsto (\mathbb{1}_n - v \otimes v) \cdot A \cdot v = B(v).
\end{aligned}$$

This shows that the extension $\overline{M}_{A,X}$ of $M_{A,X}$ is smooth. Since it is unique, we will just denote it by $M_{A,X}$ in the following.

As said previously, we would like to know when $M_{A,X}$ lifts to $\left[\overline{\mathbb{R}^n} \colon \partial\overline{V}\right]$, where $V \subset \mathbb{R}^n$ is a k-dimensional linear subspace. Proposition 4.3 tells us that this is the case if and only if $M_{A,X}$ is tangential to $\partial\overline{V}$. But by definition of the extension, this is equivalent to $A \cdot v \in V \cap \mathbb{S}^{n-1}$ for every $v \in V \cap \mathbb{S}^{n-1}$. Since A is linear, we obtain:

$$M_{A,X} \text{ lifts to } \left[\overline{\mathbb{R}^n} \colon \partial\overline{V}\right] \Leftrightarrow A(V) \subset V.$$

Hence, we can now summarise the results of this chapter in the following proposition:

Proposition 4.7. *Let $n \in \mathbb{N}$, let $V \subset \mathbb{R}^n$ be a linear subspace, let $A \in M_n(\mathbb{R})$ and let $X \in \mathbb{R}^n$. Furthermore, let $\overline{\mathbb{R}^n}$ and $\overline{V}$ be the spherical compactifications of $\mathbb{R}^n$ and V, respectively.*

Then, the following hold:

1. *The translation vector field X smoothly extends to $\overline{\mathbb{R}^n}$ by zero at infinity, and this extension always lifts to the blow-up $\left[\overline{\mathbb{R}^n} \colon \partial\overline{V}\right]$.*
2. *The affine map $M_{A,X} \colon x \mapsto A \cdot x + X$ smoothly extends to $\overline{\mathbb{R}^n}$ by setting $\mathbb{S}^{n-1} \ni v \mapsto (\mathbb{1}_n - v \otimes v) \cdot A \cdot v$, and this extension lifts to the blow-up $\left[\overline{\mathbb{R}^n} \colon \partial\overline{V}\right]$ if and only if $A(V) \subset V$, i. e. A restricts to an endomorphism of V.*

Remark 4.8. Since the second part of the previous proposition is an extension to the first one, it is important to note that their statements coincide if $A = 0$. In this case, $A(V) = \{0\} \subset V$ is true, so both parts yield that translations lift into the blow-up $\left[\overline{\mathbb{R}^n} \colon \partial\overline{V}\right]$.

5 Summary

We have seen that manifolds with corners inherit many of the properties of ordinary manifolds and plenty of the definitions and theorems are fairly straightforward. But on the other hand, there is not only one good notion of a submanifold, but rather several different ones with different properties.

This came particularly clear when considering blow-ups, since a sensible definiton of this construction can only be given for p-submanifolds, but their images under injective immersions such as the product map $\mathcal{B}$ are, in general, not even submanifolds.

This was discussed further in the next section where it turned out that no nesting of the p-submanifolds is allowed for the image of $\mathcal{B}$ to be a submanifold. It led to the theorem that the statement only holds for "trivial" clean semilattices, i. e. those whose elements were all pairwise disjoint. And even in this case, many restrictions to the local models of the p-submanifolds had to be given.

So, the conjecture in [2, p. 28, Rem. 4.14] turns out to be wrong in general, since, if the image of $\mathcal{B}$ is not a submanifold, in particular, in can neither be a wib- nor a b-submanifold.

In the last section, we investigated lifts of vector fields on manifolds with corners, leading to two main results:

First, a vector field on M lifts to $[M : P]$ if and only if it is tangential to P at P. And second, translations and affine maps on $\mathbb{R}^n$ extend to $\overline{\mathbb{R}^n}$. But while translations always lift to $\left[\overline{\mathbb{R}^n} : \partial\overline{V}\right]$, where V is a linear subspace of $\mathbb{R}^n$, general affine maps only do if they restrict to an endomorphism of V.

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Declaration of Academic Integrity

Hereby, I declare that I have composed the presented thesis independently on my own and without any other resources than the ones indicated. All thoughts taken directly or indirectly from external sources are properly denoted as such.

This paper has neither been previously submitted to another authority nor has it been published yet.

Furthermore, the presented print versions and the presented electronic version of this thesis are identical, and I am aware of the legal consequences in § 26 Sec. 6.

Regensburg, 14 May 2021

Alexander König
Name and Signature