Chaotic and localization properties of a realistic generalization of the complex SYK model

I. INTRODUCTION

SYK blablabla

II. THE MODEL

As the goal of this notes is to derive a realistic, potentially experimentally accessible Hamiltonian, we will derive an effective Hamiltonian from the well known quartic complex SYK model.

Its Hamiltonian is in general given as

$$\mathcal{H}_{\mathrm{cSYK}} = \sum_{i,j,k,l}^{N} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l. \tag{1}$$

The kernel J_{ijkl} is a complex valued matrix with randomly distributed elements sampled from a Gaussian distribution with zero mean and variance $\langle J_{ijkl}^2 \rangle = J^2/(3N^3)$. Antisymmetry is enforced to ensure hermiticity of the Hamiltonian.

This Hamiltonian is known to exhibit quantum chaotic dynamics and to follow predictions from random matrix theory, whose details depend on the specific parameters of the Hamiltonian, i.e. the number of flavors N and the particle content N_f.

This model is exactly solvable and has very interesting properties, such as being an exact model for a non-Fermi liquid. However, there is no known experimental realization, which primarily is due to the fact that Coulomb interactions are known to be restricted to density interactions. Here, though, we observe the presence of off-diagonal elements, in the sense that the interaction changes the occupation configuration of a state in Fock space.

However, here we try to derive a model, which overcomes these obvious shortcomings while still preserving important signatures of the original SYK model.

First, we state to consider finite size systems, meaning that - in contrary to the conformal and large flavor limit of the analytical solution carried out in literature we focus on systems where N is finite.

Analysing the structure of the Hamiltonian we can make a rather obvious observation: There are three cases which are distinct with respect to their overall structure:
(i) All four indices i, j, k, l are pairwise distinct. This means the corresponding terms in the Hamiltonian feature a "pairwise hopping" between flavor sites.
(ii) Only three indices are distinct. A density of a single flavor site mediates one-particle hopping. At this stage this is not what is known as a kinetic hopping term, but features a similar physical effect.
(iii) Only two indices are distinct. The corresponding terms only include density interactions. These terms can be found in nature, e.g. in the Coulomb interaction.

Case (i) cannot be included in a regular Coulomb type interaction. For the proposed effective model we discard all those terms. This of course is a mayor simplification, since these terms dominate in the large N limit.

Case (iii) can be realized in a realistic model. However, we know that the Hamiltonian in Fock space is diagonal if we only include these terms. Hence it is impossible to exhibit spectral properties as those found in the SYK model or random matrices, where spectral correlations, e.g. in the form of Wigner-Dyson Level statistics occur. Hence we need to include case (ii) in some way.

We do this as follows: Introducing additional correlations to the matrix elements, enforcing elements of the type J_{ijjk} to be independent of the density operator index j, can lead to an intresting identity. We label these additionally correlated matrix elemnts $J_{ijjk} = t_{ik}$. Then we can write

$$\begin{aligned}\mathcal{H} &= \sum_{i,j,k} J_{ijjk} c_i^\dagger c_k n_j = \sum_{i,j,k} t_{ik} c_i^\dagger c_k n_j \\ &= \sum_{ij} t_{ij} c_i^\dagger c_k \sum_{j\neq i,k} n_j = \sum_{ij} t_{ik} c_i^\dagger c_k (\hat{N} - 1).\end{aligned} \tag{2}$$

Here n_j is the number operator at site j and $\hat{N}$ is the total number operator. Since the total number of particles is conserved by the Hamiltonian, it can be replaced with its eigenvalue, assuming we only consider one symmetry block, i.e. a specific filling of the system.

In this case we transformed case (ii) of the SYK Hamiltonian to a quadratic – and hence in principle realizable – kinetic term. Defining $U_{ij} = J_{ijji}$ we write for the model Hamiltonian which is in the following tested with respect to its SYK properties

$$\mathcal{H} = a(N_f - 1) \sum_{ij} t_{ij} c_i^\dagger c_j + \sum_{ij} U_{ij} n_i n_j. \tag{3}$$

In order to investigate this model with respect to the importance of the different terms, i.e. to the interaction strength we introduce the parameter a. When $a = 1$ we recover the effective model derived from the complex SYK, when $a \to 0$ we have a purely density interaction Hamiltonian (which in some sense is trivial) and if $a \to \infty$ it converges towards a non-interacting but all-to-all hopping model. Note that there are no quadratic onsite terms, which distinguishes the limit $a \to \infty$ from a quadratic complex SYK model.

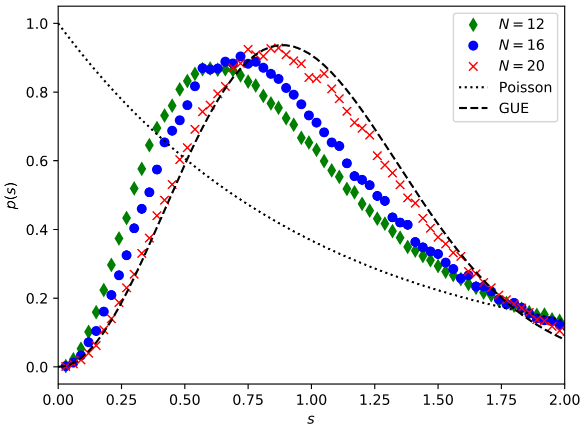

FIG. 1. Convergence of the LSS towards GUE prediction. We calculate the bulk LSS of the effective SYK-like model ($a = 1$) for a quater filling and different system sizes. The disorder average is carried out for $N = 12$ over $N_{avg} = 5000$, for $N = 16$ over $N_{avg} = 200$ and for $N = 20$ over $N_{avg} = 20$ realizations. The dashed lines represent the Poisson distribution $p_P(s)$ which predicts LSS of integrable systems, and the GUE Wigner-Dyson curve $p_{GUE}(s)$ predicting the LSS of Gaussian unitary ensembles. As the system size increases, the LSS converges to the SYK and GUE prediction.

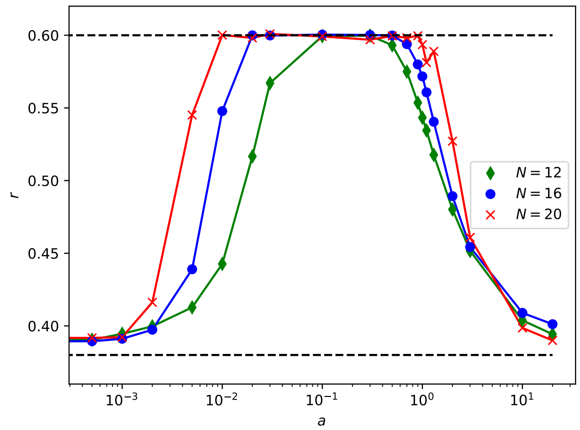

FIG. 2. Phase transition between Poisson- and WD-LSS. We vary a and measure $\langle r \rangle$ for different system sizes at a quarter filling. The dashed baselines are obtained using RMT Poissonian and GUE ensembles. We observe a sharpening phase transition from a Poissonian phase at $a = 0$ to a chaotic phase at $a = \mathcal{O}(1)$ plus a transition to another Poissonian phase at $a \to \infty$.

III. LEVEL SPACING STATISTICS

We turn to calculating the level spacing statistics of this model. This is a first important test, if we are at all able to reproduce SYK physics with this simplified effective model. For the bare SYK Hamiltonian (1) we use the very well-known prediction from random matrix theory which states that the level statistics should follow random Gaussian ensembles. In practice we calculate the mean level spacing s_i of the bulk spectrum, for the finite size Hamiltonian, by exact diagonalization. A first study in Fig. 1 shows, that for $a = 1$ the spectrum converges to the RMT prediction for a Gaussian unitary ensemble (GUE) when the system size is increased. This suggests that the effective model described above is at least capable of reproducing the SYK behavior of the level correlations.

Subsequently we calculate the level spacing ratio, defined as

$$r_i = \frac{\min(s_i, s_{i+1})}{\max(s_i, s_{i+1})}. \tag{4}$$

The average value $\langle r \rangle$ is a measure for the "distance" of a LSS distribution from the integrable (Poissonian) or chaotic (RMT) phase.

In order to investigate the importance of the various terms of the Hamiltonian (2) we vary a and measure $\langle r \rangle$. By this we should see if there are phase transitions from a chaotic (in the SYK like parameter regime) phase to an integrable (in the pure interaction, so $a \to 0$) phase. The results of such study are shown in Fig. 2.

Here we see two jumps in the parameter $\langle r \rangle$ which correspond to phase transitions from a SYK like phase to Poissonian phases. The nature of the latter has to be investigated further. It is clear that the limit $a = 0$ is rather trivial since the Hamiltonian is then diagonal in the Fock space and does not obey Level repulsion. It is certainly integrable. The SYK like phase is supposed to be delocalized and chaotic, indicating a resemblence of the original SYK Hamiltonian. This is already a major result, since it shows the possibility to obtain SYK like physics with a much more realistic model as derived in Eq. (2).

IV. WAVE FUNCTION STATISTICS

The phase at $a \to \infty$ is not clear so far. Since we want to investigate all the different states and phases in the context of many-body localization we will calculate the wave function statistics.

In particular, we want to see if the wave functions are localized in Fock space, or delocalized. This can be seen in the context of an insulator-conductor transition.

We will look at the wave function moments:

$$I_q = \frac{1}{\nu} \sum_i \langle | \langle i \mid \psi \rangle |^{2q} \, \delta(E_\psi) \rangle, \tag{5}$$

where the average is taken over all eigenvectors $|\psi\rangle$ and disorder realizations, $\nu = \sum_\psi \delta(E_\psi)$ is the density of states at zero energy and $|i\rangle$ labels the Fock basis states.

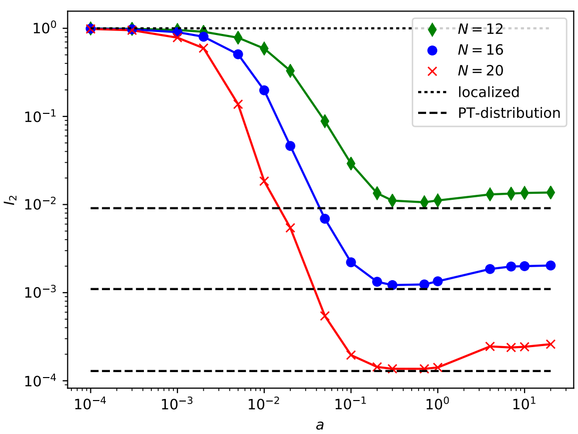

FIG. 3. Localization of the wave function. Shown is the avaerage value of I_2 versus the interaction strength paramter a. The lower baselines are obtained by diagonalizing random unitary hermitian matrices of the appropriate size. We see a strong localization transition when the interaction becomes stronger, which is consistent with the transition we found in the LSS. At strong hopping there is no such clear transition. Instead we find an intermediate state, where the wave functions are rather delocalized, but do not saturate the RMT prediction. A weak localization can be observed. The disorder average for $N = 12$ ($N = 16$) is performed over $N_{avg} = 5000$ ($N_{avg} = 200$) realizations. We fix the the particle number to fulfill a quarter filling.

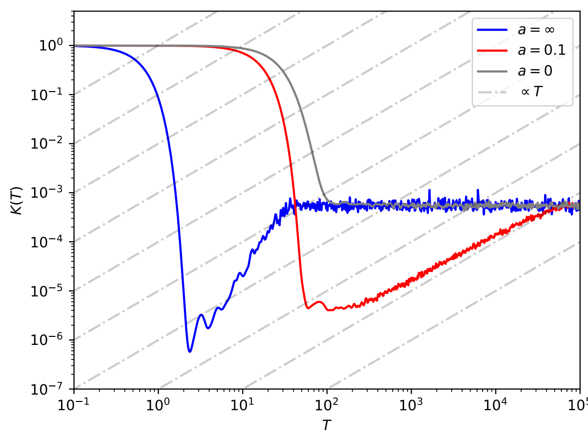

FIG. 4. Spectral form factor of different phases. We present the spectral form factor for the three different parameter regimes described above. We find for the "trivial" limit – pure density interactions – no ramp, which points at chaotic behavior. The plateau is assumed immediately. For intermediate a – well within the WD-LSS and delocalized phase, the spectral form factor follows the prediction from RMT and SYK: It has a clear linear ramp. In the quadratic limit, we see a superlinear, also super-power law behavior resembling the exponential ramp described for integrable many-body systems which emerged from a single-particle chaotic model. $N = 16$, $N_f = 4$.

We take only the central part of the spectrum and assume a constant density of states. For the second moment we obtain an approximate formula

$$I_2 = \sum_i \langle | \langle i \mid \psi \rangle |^4 \rangle. \tag{6}$$

It can be easily verified that for completely localized wave functions, it holds $I_q = 1$ while for delocalized, ergodically distributed ones we will obtain the so called Porter-Thomas (PT) distribution

$$I_q = q! D^{1-q}, \tag{7}$$

where $D = 2^N$ is the Fock space dimension. This states that the components of the wave functions are independtly distrubted Gaussian variables.

In Fig. 3 we observe a localization transition when increasing the relative strength of the interaction. The phase transition is consistent with what we found in the previous section analyzing the spectrum.

However, for larger a we do not observe such a transition. Where we expect the phase transtition to occur, according to the previous section, an interesting intermediate phase is assumed, where the localization is present but very weak. This essentially non-interacting phase has to be analyzed more carefully in the following section.

V. SPECTRAL FORM FACTOR

Since we obtained a very strange but intriguing result in the last section, namely, the quadratic limit $a \to \infty$ has delocalized states but Poissonian statistics, we calculate the spectral form factor, giving access to the exact chaotic dynamics in a regime called "ramp" regime.

The SFF at infinity temperature is given by

$$K(T) = \frac{\langle | Z(iT) |^2 \rangle}{\langle | Z(0) |^2 \rangle}, \tag{8}$$

where $Z(iT)$ is the partition function given at iT and the average is performed over disorder realizations. The quantity $K(T)$ can be interpreted as a Fourier transformed correlator between energy levels.

Here, for chaotic dynamics we expect a linear ramp, while for integrable or other systems there is different behavior. I.e. in the case of the quadratic Majorana SYK model it was shown to exhibit an exponential ramp, indicating a large symmetry class and hence rather integrable, and Poissonian dynamics.

Particularly interesting will be the regime $a \to \infty$ which exhibits delocalized or only very weakly localized states but Poissonian statistics, indicating a non-trivial phase between chaos and integrability. In Ref. [?] the authors look at a model like the one we obtain in the latter limit. (Note however the absence of onsite energies.) They populate a chaotic single particle level, which is certainly true also in our case, since the single particle Hamiltonian with a random hopping is identical to a Gaussian unitary ensemble, with many particles. Thereby, they obtain a phase, where the statistics of level spacings is Poissonian, indicating integrability. This affects the spectral form factor in such a way that the chaotic linear ramp between the Heisenberg time and the plateau time for the chaotic single particle Hamiltonian transforms to a exponential ramp, indicating integrability.

This is a very interesting behavior in the context of random matrix theory. It quantifies, additionally to the wave function statistics discussed above, the difference between the two Poissonian phases of the model (2) when performing the limit $a \to \infty$ and $a \to 0$, since the latter phase does not have a ramp in the sense of single- or many-body chaos.

In Fig. 4 we report the exact spectral form factor $K(T)$ for different parameter regimes of a. We see a clear distinction between the previously seen different phases. The pure interaction model is trivial, it exhibits no ramp to indicate quantum chaos. The non-interacting limit has an interesting behavior consistent with Ref. [**?**], which reports an exponential ramp for a similar model, indicating integrability in a many-particle system, where the single particle Hamiltonian – identival to a GUE ensemble – is chaotic. This is also in perfect agreement with what we found in the level spacing statistics.

In the intermediate regime, where we obtained previously a WD-phase with delocalized states, we expect the SYK prediction to hold. This means we should observe a linear ramp, indicating quantum chaos. This prediction holds very nicely.

VI. CONCLUSION

We derived a potentially realizable candidate for a non-Fermi liquid in the spirit of the complex SYK model. This only features a hopping term, known from simple condensed matter physics, as well as a density interaction, known from the Coulomb interaction. In principle it should be possible to realize this Hamiltonian (2) in nature.

Subsequently we checked if this model is capable of reproducing some key features of the original SYK model, particularly its quantum chaotic properties. This is indeed possible, as the system size increases.

Additionally, we investigated how changing the relative strength of the interaction term varies the physical properties of the model when analyzing quantum chaos.

We saw that this model features three distinct interesting phases:
(i) In the pure interaction model, the Hamiltonian becomes trivial. It is diagonal in occupation basis, hence the energy levels are uncorrelated, and the states localized in Fock space.
(ii) Increasing the hopping strength we observe a phase transition into a thermalizing, delocalized phase, where SYK physics happens.
(iii) In the non-interacting limit, we observe again integrabiliy, however it is still a complex phase: Even though Poissonian eigen value statistics are assumed, states are rather delocalized and a ramp in the spectral form factor is assumed which is distinct from phase (i) and (ii), so exponential and in agreement with the literature.

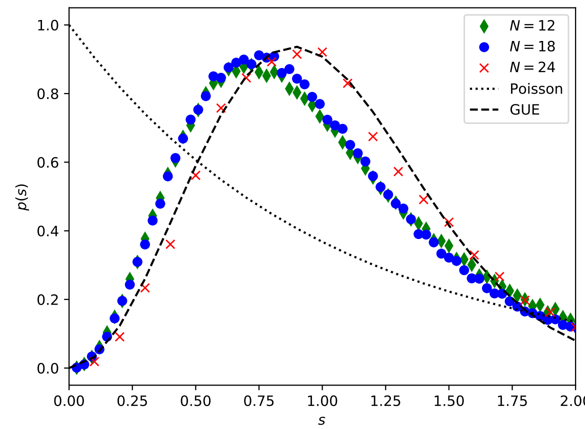

FIG. 5. Convergence of the LSS in the derived model. We fix the filling to a sixth of the site number N and calculate the bulk LSS for system sizes up to $N = 24$. We see an agreement of the convergence described in the main text, since for larger N the statistics still resemble the Wigner-Dyson distribution. The disorder average is performed over $N_{avg} = 10000, 500, 1$ realizations, respectively.

Appendix A: Thermodynamic limit for approximate SYK model

We take $a = 1$, so we have the originally derived SYK model. In order to confirm the LSS convergence to a GUE Wigner-Dyson distribution, we perform a study where we fix the filling to be a sixth of the sites number N. Hence we are able to go to higher system sizes than before, since the Fock space dimension is smaller. In Fig. 5 we observe – similarly to Fig. 1 – a convergence for large N when keeping the density of particles constant. Since we can perform this up to $N = 24$ in the case of a small filling, this is additional evidence for the coonvergence of the model 2 to the GUE statistic, and hence the eigenvalue behavior of the original complex SYK model.