Spin Hamiltonian in quantum gates

I. HAMILTONIAN AND JORDAN-WIGNER TRANSFORMATION

We are interested in the following Hamiltonian

$$\mathcal{H} = \sum_{ij} t_{ij} c_i^\dagger c_j + \sum_{ij} U_{ij} n_i n_j. \tag{1}$$

where t_{ij}, U_{ij} are zero-mean Gaussian random numbers with a fixed variance. They are chosen such that the Hamiltonian is hermitesch, i.e. $t_{ij} = t_{ji}$ and $U_{ij} = U_{ji}$.

We want to write this Hamiltonian in spin matrices, therefore, we apply a Jordan Wigner transformation $c_i = \left[\prod_{j=0}^{i-1} Z_j\right] \sigma_i^-$, where $\sigma_i^- = (X_i - iY_i)/2$.

Inserting this we obtain

$$n_i n_j = \frac{1}{4}(1 + Z_i + Z_j + Z_i Z_j) \tag{2}$$

and

$$c_i^\dagger c_j = -\left[\prod_{\zeta=\zeta_1+1}^{\zeta_2-1} Z_\zeta\right] \sigma_i^+ \sigma_j^- \tag{3}$$

where $\{\zeta_1 < \zeta_2\} = \{i, j\}$ as a set. In the second term of the Hamiltonian only the term featuring $Z_i Z_j$ remains since the other terms have a free sum over the random variable and therefore average out. Hence, we can write

$$H_2 = \frac{1}{4} \sum_{ij} U_{ij} Z_i Z_j. \tag{4}$$

The first term of the Hamiltonian is a little more involved We make a simplification: We leave out the Z string in order to avoid coupling to a high number of qubits.

We have to check if this implification changes the observations.

Let us look at the indiviudal transfer matrices, which do not commute.

$$T_2^{ij} = \exp\left(\frac{it}{4N_{trot}} U_{ij} Z_i Z_j\right) \tag{5}$$

since $(Z_i Z_j)^2 = 1$ we can write

$$\begin{aligned} T_2^{ij} &= \cos\left(\frac{t}{4N_{trot}} U_{ij}\right) + i \sin\left(\frac{t}{4N_{trot}} U_{ij}\right) Z_i Z_j \\ &= \mathrm{CNOT}(j,i) X_i R_i\left(\frac{t}{4N_{trot}} U_{ij}\right) X_i \\ &\times R_i\left(-\frac{t}{4N_{trot}} U_{ij}\right) \mathrm{CNOT}(j,i), \end{aligned} \tag{6}$$

where R is the rotation gate, and $\mathrm{CNOT}(c,t)$ has the control qubit c and the target quibit t. N_{trot} is the timestep number used in the Trotter approximation. For the first part of the Hamiltonian we rewrite it in such a way that the summands are individually hermitian (which is not the case in the form it is written above).

$$H_1 = -\sum_{i<j} t_{ij} (\sigma_i^+ \sigma_j^- + \sigma_j^+ \sigma_i^-). \tag{7}$$

Then we write the definition of the lowering and raising operators such that

$$H_1 = -\frac{1}{4} \sum_{ij} t_{ij} (X_i X_j + Y_i Y_j). \tag{8}$$

Interesting note: The simplified spin Hamiltonian hence is just a sum of all-to-all coupled Pauli matrices.

$$H_{spin} = -\frac{1}{4} \sum_{ij} t_{ij} \vec{S}_i \vec{S}_j, \tag{9}$$

which is a random quantum Heisenberg magnet. There are two individual transfer matrices of H_1, which read

$$T_X^{ij} = \cos\left(\frac{t}{4N_{trot}} t_{ij}\right) + i \sin\left(\frac{t}{4N_{trot}} t_{ij}\right) X_i X_j \tag{10}$$

$$T_Y^{ij} = \cos\left(\frac{t}{4N_{trot}} t_{ij}\right) + i \sin\left(\frac{t}{4N_{trot}} t_{ij}\right) Y_i Y_j. \tag{11}$$

Gate representation of these two matrices.

Subsequently we need to set up a trotter formula for the 3 transfer matrix blocks, where $T_A = \prod_{ij} T_A^{ij}$ for $A = X, Y, Z$.