Interaction of Mean Curvature Flow and a Diffusion Equation

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To ***?***, who is the best reason to finally finish this work.
To $\mathcal{S}$amuel, who is the best reason for its delay.
And to $\mathcal{S}$imon, who is the reason for more than he knows.

Abstract

We consider a geometric problem consisting of an evolution equation for a closed hypersurface coupled to a parabolic equation on this evolving surface. More precisely, the evolution of the hypersurface is determined by a scaled mean curvature flow that depends on a quantity defined on the surface via a diffusion equation. This system arises as a gradient flow of a simple energy functional.
Assuming suitable parabolicity conditions, we derive short-time existence for the system. The proof is based on linearization and a contraction argument. For this, we parameterize the hypersurface via a height function and thus the system, originally defined on an evolving surface, can be transformed onto a fixed reference surface. The result is formulated in a classical sense, holds for the case of embedded and immersed hypersurfaces alike and provides an existence time independent of small changes in the initial surface.
Afterwards, several properties of the solution are analyzed. Emphasis is placed on to what extent the surface in our setting evolves the same as for the usual mean curvature flow. To this end, we show that the surface area is stricly decreasing but give an example of a surface that exists for infinite times nevertheless. Moreover, mean convexity is conserved whereas convexity is not. Finally, we construct an embedded hypersurface that develops a self-intersection in the course of time.

Zusammenfassung

Wir betrachten ein geometrisches Problem, bestehend aus einer Evolutionsgleichung für eine geschlossene Hyperfläche und einer an diese gekoppelte parabolische Gleichung auf der evolvierenden Fläche. Dabei unterliegt die Entwicklung der Hyperfläche einem skalierten mittleren Krümmungsfluss, der von einer Größe abhängt die wiederum durch eine Diffusionsgleichung auf der Fläche bestimmt ist. Dieses System geht als Gradientenfluss aus einem einfachen Energiefunktional hervor.
Unter geeigneten Annahmen zur Sicherstellung der Parabolizität zeigen wir die Existenz von Lösungen für kurze Zeiten. Der Beweis beruht auf Linearisierung und einem Kontraktionsargument. Dafür parameterisieren wir die Hyperfläche mittels einer Höhenfunktion, wodurch das ursprünglich auf einer evolvierenden Fläche definierte System auf eine feste Referenzfläche überführt werden kann. Das Resultat ist im Sinne von klassischen Lösungen formuliert, gilt gleichermaßen für eingebettete sowie immersierte Hyperflächen und liefert eine von kleinen Änderungen der Anfangsfläche unabhängige Existenzzeit.
Anschließend werden verschiedene Eigenschaften der Lösung untersucht, unter besonderer Berücksichtigung der Frage inwiefern sich unsere Flächen wie beim normalen mittleren Krümmungsfluss verhalten. So nimmt etwa der Flächeninhalt streng monoton ab, aber wir können dennoch ein Beispiel für eine Fläche angeben, die für unendliche Zeiten existiert. Außerdem wird gezeigt, dass für eine Lösung die mittlere Konvexität erhalten bleibt, die Konvexität jedoch nicht. Schließlich konstruieren wir eine eingebettete Hyperfläche, die im Laufe der Zeit eine Selbstdurchdringung entwickelt.

Contents

Chapter 1

Introduction

Many aspects of our daily life can be modeled by partial differential equations: Medical processes in our body, the weather forecast and traffic problems are just some famous examples. We can describe both stationary situations, as the shape of a droplet imposed by certain forces, and the temporal development of systems, such as the conduction of heat in a solid. In this thesis, we focus on the time-dependent case and thus discuss equations of evolution. Particularly, we deal with geometric evolution equations that describe the evolution of geometric objects or geometric quantities. Typical examples for such geometric objects underlying geometric evolution laws are evolving hypersurfaces, whose evolution often is determined by curvature terms.

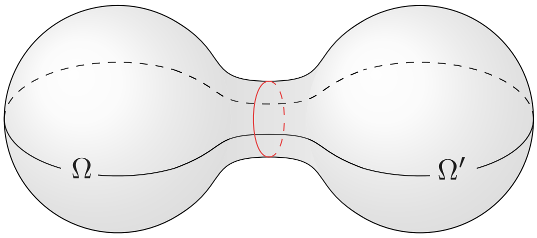
The most fundamental curvature driven evolution law is the mean curvature flow, which is also known as curve shortening flow in one dimension. It evolves a surface such that its velocity in normal direction equals the mean curvature of the surface and it decreases the surface area most efficiently. This flow has been studied in detail both from an analytical (see e.g. [Hui84], [GH86], [Gig06] and [Man11]) and a numerical (e.g. [DDE05]) point of view. For an overview of its applications in natural sciences, we refer to [Gar13]. In Section 3.1, we give a short introduction to its properties. Other important geometric flows include the surface diffusion flow, where the normal velocity equals the Laplacian of the mean curvature and which decreases the surface area but conserves the enclosed volume (see [EMS98] for an overview), as well as the Willmore flow (or elastic flow in one dimension) which is a gradient flow of the Willmore functional describing the total bending energy of a surface (see [KS01], [KS02] and [KS04] for an introduction).

We can also prescribe the evolution of the geometric object and instead search to solve a PDE defined on this moving object. Problems like this arise naturally, as several physical or biological phenomena are not formulated on fixed domains, but the domain can move or change its shape during the evolution. This is the case, e.g., in epitaxial growth or for growing biological systems. Furthermore, this setting leads to interesting mathematical effects and insights.

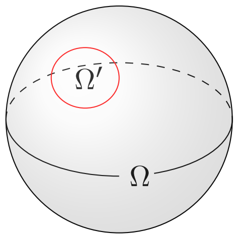
Even for a fixed, i.e., non-evolving, (curved) surface, the simple diffusion equation is worth an investigation as the behavior of its solution depends heavily on the geometry of the surface. It is a well-known fact that solutions to diffusion equations (without sources) spread out in the course of time. In [Eck08], this result is derived rigorously in an integral sense for a solution u to the diffusion equation $\partial_t u - \Delta_M u = 0$ on a closed surface M, where Δ_M is the Laplace-Beltrami operator on M. With $\overline{u}(t)$ denoting the average value of $u(t,\cdot)$ on M and $\mathrm{d}\mathcal{A}$ being the volume element of M, we obtain

$$\int_M \left|u(t,\cdot) - \overline{u}(0)\right|^2 \mathrm{d}\mathcal{A} \leq e^{-2C_p t} \int_M \left|u(0,\cdot) - \overline{u}(0)\right|^2 \mathrm{d}\mathcal{A} \to 0 \tag{1.1}$$

for $t \to \infty$. The constant $C_p > 0$ in the exponential decay rate is the optimal constant from the Poincaré inequality and therefore depends on the surface M. Ecker gives the following geometric interpretation of this constant (see [Eck08]): If a small loop on M can separate two large regions, the constant C_p has to be small, but if any small loop can only separate a small region from a large one, the constant C_p typically is larger. As the size of the constant C_p is related directly to the speed of the diffusion by (1.1), therefore, in general, a quantity is distributed much faster, e.g., on a sphere than on a surface that consists of two spheres connected by a thin tube (see Figure 1.1).



(a) A small loop can separate two large regions Ω and Ω'. Therefore, the speed of diffusion is smaller.



(b) A small loop can only separate a small region Ω' from a large region Ω. Hence, diffusion proceeds faster.

Figure 1.1: A red loop separating two regions Ω and Ω' for a dumbbell (a) and a sphere (b).

Obviously, the problem becomes more interesting and challenging if we proceed from a fixed to an evolving surface and solve PDEs thereon. Nevertheless, we still assume the motion of the hypersurface to be precribed. This setting is a prominent area of research, and there exist many contributions to the topic. For example, Dziuk and Elliott consider a scalar conservation law on a closed evolving hypersurface in [DE13]. Therein, they prove the existence and uniqueness of weak solutions and then focus on numerical approximation by finite element methods. A more general approach to solve parabolic PDEs on evolving spaces can be found in the publications of Alphonse, Elliott and Stinner (see [AES15a], [AES15b] and [Alp16]). Their aim is to formulate a framework for abstract evolving spaces, placing emphasis on suitable (a-priori not at all clear) definitions of time derivatives. In this abstract setting, they prove existence and uniqueness of solutions and use a Galerkin ansatz for numerical approximation. The results can be applied, e.g., to moving domains and hypersurfaces; examples can be found in [AES15b].
In this work, we combine both of the tasks explained above: We discuss a parabolic PDE on an evolving hypersurface, where the evolution of the geometry is not given but part of the problem. To our knowledge, there is not yet much literature on this interesting coupling. For the one-dimensional curve case there exist some first results: Pozzi and

Stinner investigate the numerical approximation of such a coupling problem and develop (semi-discrete) finite element schemes for the curve shortening flow (in [PS17]) and the elastic flow (in [PS19]) coupled with a diffusion equation on the curve. Barrett, Deckelnick and Styles consider a slightly more general version of the problem from [PS17], enhance the numerical analysis and end up with a fully discrete scheme (see [BDS17]). For higher dimensions, studying finite element methods for coupled problems is a difficult task. A first error analysis for the case of two-dimensional, closed surfaces has been achieved by Kovács, Li, Lubich and Power Guerra in [KLLPG17], leading to a FEM semi-discretization for regularized versions of geometric evolution equations. Kovács and Lubich extend these ideas and receive a full-discretization, again for regularized versions of geometric evolution equations (see [KL18]). Both results apply to the coupling of a regularized mean curvature flow and a diffusion equation. In the later work [KLL20], Kovács, Li and Lubich finally obtain a result without regularization and present a fully discrete FE algorithm for the coupled problem of mean curvature flow and a diffusion equation for two-dimensional closed surfaces. For the case of two-dimensional surfaces that can be represented as the graph of some function, Deckelnick and Styles investigate the problem from [BDS17] and derive a fully discrete finite element scheme (see [DS21]).
The considerations in [PS17], [BDS17], [DS21] and [KLL20] are of special interest to us because the problem statements therein are very similar to ours. We will get back to this later, after having explicitely formulated the problem discussed in this work, to precisely compare the settings. However, all the contributions listed above are exclusively concerned with numerical analysis and do not discuss well-posedness of the continuous problem. Contrarily, our work only contains analytic results and does not address any numerical approximation.
The problem dealt with in this work arises from a physical setting that will be explained in the following. We consider an evolving, closed hypersurface $\Gamma = (\Gamma(t))_{t\in[0,T]}$ in $\mathbb{R}^{d+1}$; imagine, e.g., the membrane of a cell in $\mathbb{R}^3$. On this hypersurface, there is some quantity, for example cholesterol molecules on the cell membrane, and the concentration $c : \Gamma \to \mathbb{R}_{\geq 0}$ of this quantity can vary in space and time. We assume a closed setting, which means that the quantity can neither vanish from nor be deposited onto the surface. This implies that the total mass m of the quantity is conserved and thus

$$\int_{\Gamma(t)} c(t)\,\mathrm{d}\mathcal{A} = m$$

is constant in time, where $\mathrm{d}\mathcal{A}$ denotes again the volume element of the surface. So, the system we wish to examine consists of the evolving hypersurface Γ and the concentration c and we express its (Gibbs) energy at a time $t \in [0,T]$ by

$$\mathrm{E}\big(\Gamma(t), c(t)\big) := \int_{\Gamma(t)} G\big(c(t)\big)\,\mathrm{d}\mathcal{A} \tag{1.2}$$

with the help of a (Gibbs) energy density $G : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Due to entropic effects, physically relevant energy densities penalize both very small and very big concentration values, so that the extreme cases of $c \approx 0$ and $c \approx \infty$ are avoided. Later on, parabolicity conditions impose further conditions on the shape of G, see Section 3.4. In order to analyze how the system evolves to decrease this energy most efficiently, we consider a gradient flow of the energy functional E which leads to the equations

$$V = \big(G(c) - G'(c)c\big)H, \tag{1.3a}$$

$$\partial^{\square} c = \Delta_{\Gamma}\big(G'(c)\big) + cHV \tag{1.3b}$$

defined on the evolving hypersurface Γ. Here, V denotes the normal velocity and H the mean curvature of the hypersurface, and the differential operators $\partial^{\square}$ and Δ_{Γ} symbolize the normal time derivative and the Laplace-Beltrami operator on Γ, respectively. A solution to these equations consists of an evolving closed hypersurface Γ and a concentration function c defined on Γ. In particular, the evolution of the geometry is not given but part of the problem.

The first equation (1.3a) is a variation of the usual mean curvature flow $V = H$ and the second equation (1.3b) is a diffusion equation for the concentration c on the surface Γ. Hence, in this work, we discuss the coupling of a mean curvature flow-type equation and a diffusion equation, similar as in [PS17], [BDS17], [DS21] and [KLL20]. In contrast to all these previous contributions that concern modifications $V = H + f(c)$ of the mean curvature flow resulting from an additive term $f(c)$ (with $f(c) = c$ in the case of [KLL20]), we deal with a multiplicatively scaled version $V = g(c)H$, $g(c) \coloneqq G(c) - G'(c)c$ of the mean curvature flow. This seams more natural to us, as it arises from the physical situation explained above. Also, while the diffusion equations in the previous literature all are semilinear, i.e., $\partial^{\square} c = \alpha\Delta_{\Gamma} c + \text{l.o.t.}$ with a constant $\alpha > 0$, our second equation $\partial^{\square} c = G''(c)\Delta_{\Gamma} c + \text{l.o.t.}$ is quasilinear. Be reminded that [PS17] and [BDS17] only consider the one-dimensional case of closed curves and [DS21] and [KLL20] restrict to the case of two-dimensional surfaces, represented as graph of a function or being closed, respectively. Our results however apply to closed hypersurfaces of arbitrary dimension. Finally, all four of the mentioned contributions address numerical analysis exclusively whereas this work is purely analytic: We present a short-time existence result and investigate several properties of the solution to our system of equations, placing emphasis on whether or not the hypersurface in our setting evolves as for the usual mean curvature flow. More precisely, we show that for our hypersurfaces, mean convexity is conserved but convexity is not. Furthermore, embedded hypersurfaces can develop self-intersections in the course of time and even though the surface area is decreasing permanently, closed hypersurfaces may exist for infinitely long times.

Our system of equations (1.3) is defined on an evolving hypersurface so that usual analytic methods can not be applied directly. But as we only consider the case of codimension 1, the evolving hypersurface can be parameterized over a fixed reference surface via a real valued parameterization called height function ρ. This procedure is explained in Section 2.1.6. Then, transforming the system (1.3) onto the fixed reference surface (see Section 3.6) yields a system consisting of an equation for the height function ρ and another one for the transformed concentration $\tilde{c}$.

Both equations are of second order, as the mean curvature and the Laplace-Beltrami operator are (quasilinear and linear, respectively) differential operators of second order. Due to $HV \sim H^2$, second order derivatives of the height function occur quadratically such that the system is fully non-linear. But as these derivatives of the height function appear in the equation for the concentration only, both equations remain quasilinear when considered separately. Suitable assumptions on the energy density function G ensure parabolicity of

the system (see Section 3.4). Hence, we consider a system of two parabolic, quasilinear differential equations of second order that are, of course, coupled.
From a mathematical point of view, this coupling makes the problem interesting and challenging. For the proof of short-time existence, it is resolved by a spitting ansatz: As a first step, we solve the first equation for ρ with an arbitrary concentration $\tilde{c}$ and then, we solve the second equation for $\tilde{c}$ where we insert the solution function $\rho_{\tilde{c}}$ from the first equation. The approach for solving both equations has the same structure, relying, as usual for parabolic, not fully linear equations, on a linearization and a contraction argument. Nevertheless, the second order derivatives of the height function occuring in the equation for the concentration necessitate handling the second equation more carefully than the first, where the concentration only appears in lower order terms. Also, the quadratic occurence of these derivatives makes it clear that we have to use solution spaces that form an algebra with pointwise multiplication. Sobolev spaces do not have this property. Instead, we will work with little Hölder spaces, which in particular implies that we solve the transformation of the system (1.3) in a classical sense. Finally, since it shall be applied in Section 5.3 to prove the formation of self-intersections, we need to formulate the short-time existence result for the case of immersed hypersurfaces.
We close the introduction by summarizing the content of this thesis. Chapter 2 provides the mathematical background our work is based on. We explain the differential geometry needed to describe evolving hypersurfaces and quantities thereon and discuss (little) Hölder spaces that will be used as solution spaces for the short-time existence result in Chapter 4. Moreover, we introduce the theory of semigroups and their generators which play an important role for the linearization of our equations and we end with formulating parabolic maximum principles on evolving hypersurfaces that enable us to prove conservation properties in Chapter 5. None of these sections gives a full introduction to the corresponding topic, but we restrict to the definitions and statements that will be used in the further chapters.
A deeper discussion of the geometric problem, the physical setting and the resulting system of equations can be found in Chapter 3. First, both of the equations are addressed independently and particularly, an overview of the properties of the mean curvature flow is contained in Section 3.1. Then, we derive formally how the system of equations arises as a gradient flow from the energy (1.2). The energy density G and especially how the parabolicity conditions govern its shape is the topic of the subsequent section. There, we also explain that the parabolicity conditions ensure the decrease of surface area for our hypersurfaces, which is a behavior known from the usual mean curvature flow. A first difference between the evolution of the hypersurface in our setting compared to the usual mean curvature flow is derived in Section 3.5: For the simple example of a radial symmetric setting it is shown that, even under the parabolicity conditions, the initially sphere-shaped hypersurface does not necessarily collapse in finite time. We finish the third chapter by transforming the system (1.3) to a fixed reference surface which enables the application of typical analytic methods.
The proof of the short-time existence result is the content of Chapter 4. It contains two equally structured sections, dealing separately with the first and the second equation. The combined result is given in Section 4.3. It is formulated for the case of immersed hypersurfaces and yields a uniform lower bound on the existence time that allows for small changes

in the initial value of the height function.
The last Chapter 5 deals with properties of the solution to our system of equations. As a start, we investigate the conservation of convexity and mean convexity of the evolving hypersurface: Whereas the latter is conserved by our scaled mean curvature flow, the former is not - in contrast to the usual mean curvature flow. Afterwards, we discuss the formation of self-intersections which is impossible for the usual mean curvature flow but can occur for our scaled mean curvature flow. Finally, several properties of the concentration function are analyzed.

Chapter 2

Mathematical Foundations and Notations

2.1 Differential Geometry

This section provides an overview of the geometrical background we need to precisely state the problem that was presented in the introduction. It does not aim to give a full summary on differential geometry but it recaps basic definitions and properties of hypersurfaces and clarifies the notation we use to describe them. In particular, we omit most of the proofs and refer to appropriate literature.
The first part deals with the static case: We consider embedded as well as immersed submanifolds of $\mathbb{R}^n$ and explain how the term embedded / immersed hypersurface is used in this work. Then, we discuss the relevant differential operators on such hypersurfaces and introduce some geometric curvature quantities. Afterwards, the dynamic case of evolving hypersurfaces is addressed, i.e., hypersurfaces that move in time. Once again, we are concerned with differentiating on such objects; this time focussing on time derivatives. Moreover, we describe the velocity with which a hypersurface evolves. Finally, we present the special case of evolving hypersurfaces that are parameterized over a fixed reference surface via a height function. This concept is used in Chapter 4 to prove the existence of short-time solutions to our geometric problem.
For a general introduction to differential geometry of submanifolds of $\mathbb{R}^n$, we suggest [Bär10] for basic aspects, [dC16] for a broader overview and [Wal15] for a very descriptive presentation. In [KH15], both submanifolds of $\mathbb{R}^n$ and manifolds without taking account of an ambient space are adressed and we refer to [Lee03] for an in-depth discussion of smooth manifolds. As an orientation for the section at hand, we used [BGN20, Section 2] because it provides a good overview of the geometry of (evolving) hypersurfaces. A detailed introduction to this topic is given in [PS16, Chapter 2].
In the following, let $d, n, m \in \mathbb{N}_{>0}$ be dimensions with $d \leq n$ and we use $s \in \mathbb{R}_{\geq 0}$ to describe regularities. For $s \notin \mathbb{N}_{\geq 0}$, the integer part

$$\lfloor s \rfloor := \max\{k \in \mathbb{N}_{\geq 0} \,|\, k \leq s\} \in \mathbb{N}_{\geq 0}$$

is the order of differentiability and $s - \lfloor s \rfloor \in (0,1)$ the real exponent of the Hölder condition. In the interest of readability, we use the abbreviation $\mathcal{C}^s \in \{C^s, h^s\}$ such that $\mathcal{C}^s$ is a space

of differentiable functions if $s \in \mathbb{N}_{\geq 0}$ and a Hölder or little Hölder space, else. We assume submanifolds to be of regularity $1+s$ because many concepts require differentiability of submanifolds.
We presume that the reader has basic knowledge on (little) Hölder spaces. If not, we suggest to assume all the regularity parameters occuring in this section to be integers and refer to the next Section 2.2 for an introduction to (little) Hölder spaces. In particular, as in Definition 2.72, we use the notation

$$C_b^k(\overline{W}, \mathbb{R}^m)$$

for the set of all k-times continuously differentiable functions on the open subset $W \subset \mathbb{R}^d$, whose derivatives up to the order k are bounded and continuously extendable onto $\overline{W}$. If the domain W is bounded, we omit the index b.
If not stated otherwise, subsets of $\mathbb{R}^d$ are equipped with the subspace topology induced by the standard topology on $\mathbb{R}^d$.

2.1.1 Embedded Hypersurfaces

The goal of this section is to give an exact definition of the term embedded hypersurface, as it is used in this work. For this, we firstly recap the definition of an embedded submanifold of $\mathbb{R}^n$. In this work, a submanifold never contains a boundary. As we will need Hölder-continuous functions defined on submanifolds later on, we state in particular the meaning of Hölder-regularity for submanifolds and use special charts and local parameterizations that are compatible with this Hölder regularity.

Definition 2.1 (Immersion, Embedding and Diffeomorphism). *Let $W \subset \mathbb{R}^m$ be an open subset. A map $\gamma \in \mathcal{C}_b^{1+s}(\overline{W}, \mathbb{R}^n)$ with $m \leq n$ is called a*

(i) $\mathcal{C}^{1+s}$-immersion if its differential $D\gamma(x) : \mathbb{R}^m \to \mathbb{R}^n$ is injective for every $x \in \overline{W}$,

(ii) $\mathcal{C}^{1+s}$-embedding if γ is a $\mathcal{C}^{1+s}$-immersion and additionally a homeomorphism onto its image $\gamma(\overline{W})$,

(iii) $\mathcal{C}^{1+s}$-diffeomorphism if $m = n$ holds, $\gamma(W) \subset \mathbb{R}^n$ is open and γ is bijective onto its image $\gamma(\overline{W})$ with $\gamma^{-1} \in \mathcal{C}_b^{1+s}\big(\gamma(\overline{W}), \mathbb{R}^m\big)$.

Definition 2.2 (Hölder-continuous Chart). *Let $M \subset \mathbb{R}^n$. A pair (ϕ, U) is called a (d-dimensional) $\mathcal{C}^{1+s}$-chart of M if $U \subset \mathbb{R}^n$ is an open, bounded and convex subset and $\phi : \overline{U} \to \phi(\overline{U}) \subset \mathbb{R}^n$ is a $\mathcal{C}^{1+s}$-diffeomorphism with $\phi(\overline{U} \cap M) = \phi(\overline{U}) \cap \big(\mathbb{R}^d \times \{0\}\big)$.*

Definition 2.3 (Hölder-continuous embedded Submanifold). *$M \subset \mathbb{R}^n$ is called a d-dimensional $\mathcal{C}^{1+s}$-embedded submanifold if for every point $p \in M$ there exists a d-dimensional $\mathcal{C}^{1+s}$-chart (ϕ_p, U_p) of M with $p \in U_p$.*

In particular, any open subset $U \subset M$ of an embedded submanifold M is again an embedded submanifold. Besides the definition via charts, it will be useful to describe submanifolds with the help of local parameterizations. Again, we will carefully state the meaning of Hölder-regularity. The equivalence of both definitions for embedded submanifolds is given in Lemma 2.5.

Definition 2.4 (Hölder-continuous local Parameterization)**.** *Let $M \subset \mathbb{R}^n$. A pair (γ, W) is called a (d-dimensional) $\mathcal{C}^{1+s}$-local parameterization of M if $W \subset \mathbb{R}^d$ is an open, bounded and convex subset and $\gamma : \overline{W} \to \mathbb{R}^n$ is a $\mathcal{C}^{1+s}$-embedding such that $\gamma(W) \subset M$ is an open subset with $\gamma(\overline{W}) \subset M$. Choosing the local parameterization (γ, W) sufficiently small means that $\gamma(W) \subset M$ is sufficiently small.*

Lemma 2.5. *$M \subset \mathbb{R}^n$ is a d-dimensional $\mathcal{C}^{1+s}$-embedded submanifold if and only if for every point $p \in M$ there exists a d-dimensional $\mathcal{C}^{1+s}$-local parameterization (γ_p, W_p) of M with $p \in \gamma_p(W_p)$.*

Proof. Fix $p \in M$. Let (ϕ, U) be a d-dimensional $\mathcal{C}^{1+s}$-chart of M with $p \in U$. Define

$$\widetilde{W} := \mathrm{pr}_{\mathbb{R}^d}\big(\phi(U \cap M)\big) \quad \text{and} \quad \tilde{\gamma} := \big(\phi^{-1}\big)_{|\phi(U \cap M)} \circ (\mathrm{Id}, 0).$$

Then, $\widetilde{W} \subset \mathbb{R}^d$ is an open subset and $\tilde{\gamma} : \overline{\widetilde{W}} \to \mathbb{R}^n$ is a $\mathcal{C}^{1+s}$-embedding with $\tilde{\gamma}(\overline{\widetilde{W}}) \subset M$ and $p \in \tilde{\gamma}(\widetilde{W})$. By choosing a suitable subset $W \subset \widetilde{W}$, hence $(\gamma := \tilde{\gamma}_{|W}, W)$ is a d-dimensional $\mathcal{C}^{1+s}$-local parameterization of M with $p \in \gamma(W)$.
Conversely, given a d-dimensional $\mathcal{C}^{1+s}$-local parameterization (γ, W) of M with $p \in \gamma(W)$, we choose a basis extension $A \in \mathbb{R}^{n \times (n-d)}$ such that $\big(D\gamma_{|\gamma^{-1}(p)}, A\big) \in \mathbb{R}^{n \times n}$ is invertible. From the inverse mapping theorem (see e.g. [Lan12, Theorem XIV.1.2]) we get that

$$(\gamma, A) : W \times \mathbb{R}^{n-d} \to \mathbb{R}^n,\ (x, y) \mapsto \gamma(x) + Ay$$

is locally invertible. The local inverse $\tilde{\phi} : \tilde{U} \to \mathbb{R}^n$ provides the desired d-dimensional chart (ϕ, U) of M with $p \in U$ by choosing a suitable subset $U \subset \tilde{U}$ and $\phi := \tilde{\phi}_{|U}$. The $\mathcal{C}^{1+s}$-regularity of ϕ follows from

$$D\phi = \big(D\gamma, A\big)^{-1}_{|\circ\phi}$$

because the inverse of a Hölder-continuous, invertible matrix-valued function is Hölder-continuous as well (see Remark 2.106). □

In the following remark, we give a summary of the notation used to describe an embedded submanifold and gather some properties of the characterizing functions.

Remark 2.6. *(i) In contrast to the usual literature on submanifolds, we restrict to bounded and convex sets in Definitions 2.2 and 2.4 and assume the corresponding charts and local parameterizations to be well-defined on the closure of these sets. This is possible w.l.o.g., because we can always achieve these properties by choosing the sets smaller. It does result though in the fact, that a submanifold $M \subset \mathbb{R}^n$ can be covered by finitely many charts or local parameterizations if and only if it is closed, i.e., if it is compact as a subset of $\mathbb{R}^n$. For example, the simple submanifold of an open subset of $\mathbb{R}^n$ needs infinitely many charts or local parameterizations.*
We assume convexity, because this guarantees the same embedding properties known for continuously differentiable functions also for Hölder regular functions (see Lemma 2.88). Also, on a convex set, a continuously differentiable function always is Lipschitz continuous. Due to $1 + s \geq 1$, thus $\phi : \overline{U} \to \mathbb{R}^n$ and $\gamma : \overline{W} \to \mathbb{R}^n$ are Lipschitz

continuous for any chart (ϕ, U) and any local parameterization (γ, W).
Furthermore, as U and W are convex, they are connected and we can assume them to have a regular boundary as defined in Section 2.2.3, again by choosing the sets smaller if necessary.

(ii) Let $M \subset \mathbb{R}^n$ be a d-dimensional $\mathcal{C}^{1+s}$-embedded submanifold. By Definition 2.3 and Lemma 2.5, for every $p \in M$ there exists a $\mathcal{C}^{1+s}$-chart (ϕ_p, U_p) with $p \in U_p$ and a $\mathcal{C}^{1+s}$-local parameterization (γ_p, W_p) with $p \in \gamma_p(W_p)$. We define $V_p \coloneqq \gamma_p(W_p)$ and then $V_p \subset M$ is an open subset and we can assume $V_p \subset U_p$. Moreover, we define

$$\varphi_p \coloneqq pr_{\mathbb{R}^d} \circ \phi_{p|\overline{V_p}} : \overline{V_p} \to \mathbb{R}^d.$$

On account of the properties of ϕ_p, also $\varphi_p : \overline{V_p} \to \mathbb{R}^d$ is a homeomorphism onto its image and Lipschitz continuous. By the proof of Lemma 2.5, we can assume $\varphi_p = \gamma_p^{-1}$ and thus $\varphi_p(V_p) = W_p$.
If M is a closed submanifold, i.e., compact as a subset of $\mathbb{R}^n$, it suffices to use finitely many charts $(\phi_l, U_l)_{l=1,\ldots,L}$ and local parameterizations $(\gamma_l, W_l)_{l=1,\ldots,L}$ to cover the embedded submanifold. W.l.o.g., we can assume the existence of further open subsets $A_l \subset M$ with $\overline{A_l} \subset V_l$ and

$$M \subset \bigcup_{l=1}^{L} A_l.$$

We state a well-known characterization for submanifolds that is proven e.g. in [dC16, Section 2.2, Proposition 2].

Lemma 2.7. *Let $V \subset \mathbb{R}^n$ be an open subset and let $\phi \in C^k(V, \mathbb{R})$. For every regular value $a \in \phi(V)$, i.e., $D\phi(p) \neq 0$ for every $p \in \phi^{-1}(\{a\})$, the set $\phi^{-1}(\{a\}) \subset \mathbb{R}^n$ is a $(n-1)$-dimensional C^k-embedded submanifold.*

An important concept when discussing submanifolds is the so-called tangent space, a linear approximation of the submanifold.

Definition 2.8 (Tangent Space)**.** *Let $M \subset \mathbb{R}^n$ be a C^1-embedded submanifold and $p \in M$. The tangent space T_pM of M at p is then defined as*

$$T_pM \coloneqq \left\{ v \in \mathbb{R}^n \,\middle|\, \text{there exists a } C^1\text{-map } c : I_0 \to M \text{ with } c(0) = p, c'(0) = v \right\},$$

where $I_0 \in \left\{ (-\varepsilon, \varepsilon), [0, \varepsilon), (-\varepsilon, 0] \right\}$ is a sufficiently small interval around 0.

The tangent space is local, i.e., $T_pM = T_pU$ holds for every open subset $U \subset M$ and $p \in U$, which can be seen directly from the following characterization of the tangent space with the help of coordinate vectors.

Remark 2.9 (Basis of the Tangent Space)**.** *Let $M \subset \mathbb{R}^n$ be a d-dimensional C^1-embedded submanifold. For any local parameterization (γ, W) of M we have*

$$T_{\gamma(x)}M = \operatorname{span}\left\{ \partial_i \gamma_{|x} \,\middle|\, i = 1, \ldots, d \right\}$$

for all $x \in \overline{W}$. The $\partial_i \gamma$ are called coordinate vectors with respect to the local parameterization (γ, W). In particular, $T_{\gamma(x)}M \subset \mathbb{R}^n$ is a d-dimensional linear subspace of $\mathbb{R}^n$. A proof of these simple statements is given in [Bär10, Proposition 3.2.2].

The normal space is the orthogonal complement to the tangent space. For submanifolds with codimension 1 it is characterized by a single vector called normal.

Definition 2.10 (Orientability and Normal)**.** *A d-dimensional C^1-embedded submanifold $M \subset \mathbb{R}^{d+1}$ is called orientable if there exists a continuous unit normal ν_M, i.e., a continuous vector field $\nu_M : M \to \mathbb{R}^{d+1}$ with $|\nu_M(p)| = 1$ and $\nu_M(p) \perp T_pM$ for all $p \in M$.*

In the literature (see e.g. [Lee03, Chapter 15, Orientations of Manifolds]), orientability of a submanifold usually is defined via compatible choices of orientation on the tangent spaces. We state a relation between the two definitions in the next lemma. For the convenience of the reader, the orientation of a vector space as well as the generalized cross product which is used in the proof below are defined in the Appendix (see Definitions A.1 and A.4).

Lemma 2.11. *Let $M \subset \mathbb{R}^{d+1}$ be an orientable d-dimensional C^1-embedded submanifold. There exists a choice of orientation for each tangent space such that for all $p \in M$ there exists a local parameterization (γ, W) around p with $\{\partial_1\gamma_{|x}, ..., \partial_d\gamma_{|x}\}$ positively oriented for all $x \in \overline{W}$. Such a local parameterization is called a positively oriented local parameterization. Moreover, due to connectedness of W, for every local parameterization (γ, W), either $\{\partial_1\gamma, ..., \partial_d\gamma\}$ or $\{-\partial_1\gamma, \partial_2\gamma, ..., \partial_d\gamma\}$ is positively oriented on the whole domain $\overline{W}$.*

Proof. Let $\nu : M \to \mathbb{R}^{d+1}$ be the continuous unit normal to the orientable submanifold M. Let $p \in M$ be fixed and let (γ, W) be any local parameterization around $p =: \gamma(x_0)$. Then, $\{\partial_1\gamma_{|x}, ..., \partial_d\gamma_{|x}\}$ is a basis of $T_{\gamma(x)}M$ for all $x \in \overline{W}$ by Remark 2.9. As $T_{\gamma(x)}M$ is a d-dimensional subspace of $\mathbb{R}^{d+1}$, there exist exactly two choices for a vector $\tilde{\nu}\big(\gamma(x)\big)$ (which differ by sign) such that $\big|\tilde{\nu}\big(\gamma(x)\big)\big| = 1$ and $\tilde{\nu}\big(\gamma(x)\big) \perp T_{\gamma(x)}M$ hold. With the help of the generalized cross product $\mathcal{K} : (\mathbb{R}^{d+1})^d \to \mathbb{R}^{d+1}$ as in Definition A.4, we have

$$\tilde{\nu}\big(\gamma(x)\big) = \pm\bar{\mathcal{K}}(x) := \pm\frac{\mathcal{K}\big(\partial_1\gamma_{|x}, ..., \partial_d\gamma_{|x}\big)}{\big|\mathcal{K}\big(\partial_1\gamma_{|x}, ..., \partial_d\gamma_{|x}\big)\big|}.$$

By reparameterization of γ, we can assume w.l.o.g. that $\nu(p) = +\bar{\mathcal{K}}(x_0)$ holds. We have to show $\nu \circ \gamma(x) = +\bar{\mathcal{K}}(x)$ for all $x \in \overline{W}$.
Because ν is continuous on M and the generalized cross product conserves regularity, both $\nu \circ \gamma$ and $\bar{\mathcal{K}}$ are continuous on $\overline{W}$. So, for a sufficiently small $\delta > 0$,

$$\big|\bar{\mathcal{K}}(x_0) - \bar{\mathcal{K}}(x)\big| < \frac{1}{2} \qquad \text{and} \qquad \big|\nu\big(\gamma(x_0)\big) - \nu\big(\gamma(x)\big)\big| < \frac{1}{2}$$

hold for all $x \in \overline{W} \cap B_\delta(x_0)$. Then, we have $\nu\big(\gamma(x)\big) = +\bar{\mathcal{K}}(x)$ for all $x \in \overline{W} \cap B_\delta(x_0)$ because otherwise

$$\big|\nu\big(\gamma(x_0)\big) - \nu\big(\gamma(x)\big)\big| = \big|\bar{\mathcal{K}}(x_0) + \bar{\mathcal{K}}(x)\big| \geq 2\,\big|\bar{\mathcal{K}}(x_0)\big| - \big|\bar{\mathcal{K}}(x_0) - \bar{\mathcal{K}}(x)\big| > 2 - \frac{1}{2} = \frac{3}{2}$$

and thus a contradiction follows. As $\overline{W}$ is compact, $\nu \circ \gamma$ and $\bar{\mathcal{K}}$ are even uniformly continuous on $\overline{W}$ and then inductively, $\nu\big(\gamma(x)\big) = +\bar{\mathcal{K}}(x)$ follows for all $x \in \overline{W}$. □

The representation of the normal via the generalized cross product as in the proof above yields the following statement concerning the regularity of the normal.

Remark 2.12 (Regularity of the Normal). *Let $M \subset \mathbb{R}^{d+1}$ be an orientable d-dimensional $\mathcal{C}^{1+s}$-embedded submanifold. Its unit normal ν_M fulfills $\nu_M \in \mathcal{C}^s(M, \mathbb{R}^{d+1})$, which means that for any point $p \in M$, there exists a local parameterization (γ_p, W_p) of M around p with $\nu_M \circ \gamma_p \in \mathcal{C}^s(\overline{W_p}, \mathbb{R}^{d+1})$. If additionally M is closed or $s \in \mathbb{N}_{\geq 0}$, this implies $\nu_M \circ \gamma \in \mathcal{C}^s(\overline{W}, \mathbb{R}^{d+1})$ for every local parameterization (γ, W) of M (see Remark 2.76(iii)).*

Now, we can formulate exactly the use of the term embedded hypersurface in this work. Besides of being an embedded submanifold of codimension 1, it includes some further properties. In particular, our hypersurfaces just as our submanifolds never contain a boundary.

Definition 2.13 (Embedded Hypersurface). *A $\mathcal{C}^{1+s}$-embedded submanifold $M \subset \mathbb{R}^{d+1}$ is called a $\mathcal{C}^{1+s}$-embedded (closed) hypersurface if it is d-dimensional and orientable as well as connected (and compact) as a subset of $\mathbb{R}^{d+1}$. A 1-dimensional embedded hypersurface is also called an embedded curve.*

The reader should note that in this work, a curve always has codimension 1, i.e., it is surrounded by $\mathbb{R}^2$ and not any other $\mathbb{R}^n$ with $n > 2$.

Remark 2.14. *As a generalization of the Jordan curve theorem, the Jordan–Brouwer separation theorem tells us that any d-dimensional embedded submanifold $M \subset \mathbb{R}^{d+1}$ (as always in this work without boundary) that is compact and connected as a subset of $\mathbb{R}^{d+1}$ already is orientable (see [Sam69] or [Lim88]). Therefore, requiring orientability for our closed hypersurfaces is actually redundant.*

As a next step, we discuss mappings with domain on embedded submanifolds. We introduce the differential of a mapping and then deal with immersions, embeddings and diffeomorphisms defined on embedded hypersurfaces as well as some relations between them.
For mappings with domain on an embedded submanifold, regularity properties are defined on the euclidean space via pullback by local parameterizations. For this, the pullback-function and hence the embedded submanifold obviously have to be sufficiently smooth.

Definition/Lemma 2.15 (Hölder-regularity on embedded Submanifolds). *Let $M \subset \mathbb{R}^n$ be a $\mathcal{C}^{1+s}$-embedded submanifold and let $r \in \mathbb{R}_{\geq 0}$ with $r \leq 1 + s$. A mapping $f : M \to \mathbb{R}^m$ is of regularity $f \in \mathcal{C}^r(M, \mathbb{R}^m)$ if for any point $p \in M$, there exists a local parameterization (γ_p, W_p) of M around p such that the mapping fulfills $f \circ \gamma_p \in \mathcal{C}^r(\overline{W_p}, \mathbb{R}^m)$.*
If additionally M is closed or $r \in \mathbb{N}_{\geq 0}$, this implies $f \circ \gamma \in \mathcal{C}^r(\overline{W}, \mathbb{R}^m)$ for every local parameterization (γ, W) of M (cf. Definition 2.75 and Remark 2.76(iii)).

As we have defined differentiability of a mapping, we can now introduce the differential of a mapping.

Definition 2.16 (Differential). *Let M_1, M_2 be C^1-embedded submanifolds and let the map $f : M_1 \to M_2$ be differentiable. The differential of f in $p \in M_1$ is defined as the linear mapping*

$$\mathrm{d}_p f : T_p M_1 \to T_{f(p)} M_2, \; v \mapsto \mathrm{d}_p f[v]$$

such that for any C^1-map $c : (-\varepsilon, \varepsilon) \to M_1$ with $c(0) = p$ we have

$$\mathrm{d}_p f\big[c'(0)\big] \coloneqq \big(f \circ c\big)'(0).$$

Remark 2.17. *The differential is well-defined, in particular it is independent of the chosen map c, and linear. For a local parameterization (γ, W) of the d-dimensional submanifold M, the differential of $f : M \to \mathbb{R}^m$ is characterized through*

$$\mathrm{d}_{\gamma(x)} f\big[\partial_i \gamma_{|x}\big] = \partial_i (f \circ \gamma)_{|x}$$

with $i = 1, ..., d$ and $x \in \overline{W}$. Moreover, the differential fulfills a chain rule, meaning that for mappings $f : M_1 \to M_2$ and $g : M_2 \to M_3$ between embedded submanifolds M_1, M_2 and M_3 we have $\mathrm{d}_p(g \circ f) = \mathrm{d}_{f(p)} g \circ \mathrm{d}_p f$. For the proof of these properties of the differential, we refer to [Bär10, Proposition 3.2.7] and [Lee03, Proposition 3.6].

Having explained the differential of a mapping defined on an embedded submanifold, we can now transfer the definition of immersions, embeddings and diffeomorphisms on open subsets of $\mathbb{R}^d$ from Definition 2.1 to the more general case of mappings that have an embedded submanifold as domain.

Definition 2.18 (Immersion, Embedding and Diffeomorphism on Submanifolds)**.** *Let $M \subset \mathbb{R}^n$ be a d-dimensional $\mathcal{C}^{1+s}$-embedded submanifold. A mapping $\theta \in \mathcal{C}^{1+s}(M, \mathbb{R}^m)$ with $d \leq m$ is called a*

(i) $\mathcal{C}^{1+s}$-immersion if its differential $\mathrm{d}_p\theta : T_pM \to \mathbb{R}^m$ is injective for all $p \in M$,

(ii) $\mathcal{C}^{1+s}$-embedding if θ is a $\mathcal{C}^{1+s}$-immersion and additionally a homeomorphism onto its image $\theta(M)$,

(iii) $\mathcal{C}^{1+s}$-diffeomorphism if $M' \coloneqq \theta(M) \subset \mathbb{R}^m$ also is a d-dimensional $\mathcal{C}^{1+s}$-embedded submanifold and $\theta : M \to M'$ is bijective with $\theta^{-1} \in \mathcal{C}^{1+s}(M', \mathbb{R}^n)$.

In the following lemmas, we gather some relations between the three terms defined above.

Lemma 2.19. *Let $M \subset \mathbb{R}^n$ be a $\mathcal{C}^{1+s}$-embedded submanifold and let $\theta : M \to \mathbb{R}^m$ be a $\mathcal{C}^{1+s}$-immersion. Then, locally θ is a $\mathcal{C}^{1+s}$-embedding, i.e., for all $p \in M$ there exists an open neighborhood $U \subset M$ around p such that $\theta_{|U} : U \to \mathbb{R}^m$ is a $\mathcal{C}^{1+s}$-embedding.*

Proof. For every open subset $U \subset M$, $\theta_{|U} : U \to \mathbb{R}^m$ is a $\mathcal{C}^{1+s}$-immersion on the $\mathcal{C}^{1+s}$-embedded submanifold U by restriction. So, we only have to prove that for any $p \in M$, there exists an open neighborhood $U \subset M$ around p such that $\theta_{|U} : U \to \mathbb{R}^m$ is a homeomorphism onto its image.
Let $M \subset \mathbb{R}^n$ be of dimension d. Fix $p \in M$ and choose a local parameterization (γ, W) of M around p. Thus, $W \subset \mathbb{R}^d$ is an open, bounded and convex subset and $\gamma : \overline{W} \to \mathbb{R}^n$ is an embedding, in particular an immersion, with $\gamma(\overline{W}) \subset M$ and $p \in \gamma(W)$. As combination of immersions, also $\theta \circ \gamma : \overline{W} \to \mathbb{R}^m$ is an immersion. By Lemma A.7, $\theta \circ \gamma$ thus locally is an embedding. So, there exists an open subset $\widetilde{W} \subset \mathbb{R}^d$ with $\gamma^{-1}(p) \in \widetilde{W} \subset W$ such that $\theta \circ \gamma_{|\overline{\widetilde{W}}}$ is an embedding, in particular a homeomorphism onto its image. Then, $U \coloneqq \gamma(\widetilde{W}) \subset M$ is an open neighborhood around p and as combination and restriction of homeomorphisms, $\theta_{|U} = (\theta \circ \gamma) \circ \gamma^{-1}_{|\gamma(\widetilde{W})} : U \to \mathbb{R}^m$ is also a homeomorphism onto its image. □

Lemma 2.20. *Let $M \subset \mathbb{R}^{d+1}$ be an orientable, d-dimensional $\mathcal{C}^{1+s}$-embedded submanifold and let $\theta : M \to \mathbb{R}^{d+1}$ be a $\mathcal{C}^{1+s}$-immersion. There exists a vector field $\nu : M \to \mathbb{R}^{d+1}$ of regularity $\mathcal{C}^s(M, \mathbb{R}^{d+1})$ such that $|\nu(p)| = 1$ and $\nu(p) \perp \mathrm{d}_p\theta(T_pM)$ hold for all $p \in M$.*

Proof. Fix $p \in M$. Because $\theta : M \to \mathbb{R}^{d+1}$ is an immersion, its differential $\mathrm{d}_p\theta : T_pM \to \mathbb{R}^{d+1}$ is injective and thus the image $\mathrm{d}_p\theta(T_pM) \subset \mathbb{R}^{d+1}$ is also a d-dimensional linear subspace. Hence, there exist exactly two choices of $\nu(p)$ (which differ by sign) such that $|\nu(p)| = 1$ and $\nu(p) \perp \mathrm{d}_p\theta(T_pM)$ hold. Let $\{v_1, ..., v_d\}$ be any positively oriented basis of T_pM and choose the sign of $\nu(p)$ such that $\{\mathrm{d}_p\theta[v_1], ..., \mathrm{d}_p\theta[v_d], \nu(p)\}$ is positively oriented in $\mathbb{R}^{d+1}$. This choice of sign for $\nu(p)$ is independent of the choice of the positively oriented basis $\{v_1, ..., v_d\}$ of T_pM: Indeed, let $\{w_1, ..., w_d\}$ be another positively oriented basis of T_pM and let $B \in \mathrm{Gl}(d, \mathbb{R})$ be the transformation matrix with $B(v_1, ..., v_d) = (w_1, ..., w_d)$. Because $\mathrm{d}_p\theta$ is linear, we have

$$\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \big(\mathrm{d}_p\theta[v_1], ..., \mathrm{d}_p\theta[v_d], \nu(p)\big) = \big(\mathrm{d}_p\theta[w_1], ..., \mathrm{d}_p\theta[w_d], \nu(p)\big).$$

As $\det \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} = \det B > 0$ holds, $\{\mathrm{d}_p\theta[v_1], ..., \mathrm{d}_p\theta[v_d], \nu(p)\}$ is positively oriented in $\mathbb{R}^{d+1}$ if and only if $\{\mathrm{d}_p\theta[w_1], ..., \mathrm{d}_p\theta[w_d], \nu(p)\}$ is. Thus, $\nu(p)$ is well-defined for all $p \in M$. In particular, for a negatively oriented basis $\{v_1, ..., v_d\}$ of T_pM, using the negative of ν, $\{\mathrm{d}_p\theta[v_1], ..., \mathrm{d}_p\theta[v_d], -\nu(p)\}$ is positively oriented in $\mathbb{R}^{d+1}$ (see Remark A.2).
It remains to show that $\nu : M \to \mathbb{R}^{d+1}$ is of the claimed regularity. Let $p \in M$ and let (γ, W) be a local parameterization of M around p with $\theta \circ \gamma \in \mathcal{C}^{1+s}(\overline{W}, \mathbb{R}^{d+1})$. Set $\sigma \coloneqq +1$ if (γ, W) is positively oriented and $\sigma \coloneqq -1$ else. Let $\mathcal{K} : (\mathbb{R}^{d+1})^d \to \mathbb{R}^{d+1}$ be the generalized cross product as in Definition A.4. Due to its properties, the vector

$$\tilde{\nu}_{|\gamma(x)} \coloneqq \sigma \frac{\mathcal{K}\big(\mathrm{d}_{\gamma(x)}\theta[\partial_1\gamma_{|x}], ..., \mathrm{d}_{\gamma(x)}\theta[\partial_d\gamma_{|x}]\big)}{\big|\mathcal{K}\big(\mathrm{d}_{\gamma(x)}\theta[\partial_1\gamma_{|x}], ..., \mathrm{d}_{\gamma(x)}\theta[\partial_d\gamma_{|x}]\big)\big|} = \sigma \frac{\mathcal{K}\big(\partial_1(\theta \circ \gamma)_{|x}, ..., \partial_d(\theta \circ \gamma_{|x})\big)}{\big|\mathcal{K}\big(\partial_1(\theta \circ \gamma)_{|x}, ..., \partial_d(\theta \circ \gamma)_{|x}\big)\big|}$$

fulfills $|\tilde{\nu}(\gamma(x))| = 1$ and $\tilde{\nu}(\gamma(x)) \perp \mathrm{d}_{\gamma(x)}\theta(T_{\gamma(x)}M) = \mathrm{span}\ \mathrm{d}_{\gamma(x)}\theta\big(\{\partial_1\gamma_{|x}, ..., \partial_d\gamma_{|x}\}\big)$ for all $x \in \overline{W}$ and in addition, $\{\mathrm{d}_{\gamma(x)}\theta[\partial_1\gamma_{|x}], ..., \mathrm{d}_{\gamma(x)}\theta[\partial_d\gamma_{|x}], \sigma\tilde{\nu}(\gamma(x))\}$ is positively oriented in $\mathbb{R}^{d+1}$ for all $x \in \overline{W}$. By uniqueness, we thus have $\nu(\gamma(x)) = \tilde{\nu}(\gamma(x))$ for all $x \in \overline{W}$. Because the generalized cross product $\mathcal{K}$ conserves regularity and $\theta \circ \gamma \in \mathcal{C}^{1+s}(\overline{W}, \mathbb{R}^{d+1})$ holds, we have $\nu \circ \gamma = \tilde{\nu} \circ \gamma \in \mathcal{C}^s(\overline{W}, \mathbb{R}^{d+1})$ and therefore $\nu \in \mathcal{C}^s(M, \mathbb{R}^{d+1})$ follows. $\square$

Lemma 2.21. *Let $M \subset \mathbb{R}^n$ be a d-dimensional $\mathcal{C}^{1+s}$-embedded submanifold and let the map $\theta : M \to \mathbb{R}^m$ be a $\mathcal{C}^{1+s}$-embedding. Then the following hold:*

(i) *$\theta(M) \subset \mathbb{R}^m$ is also a d-dimensional $\mathcal{C}^{1+s}$-embedded submanifold. More precisely, for every $p \in M$, there exists a local parameterization (γ_p, W_p) of M around p such that $(\gamma_{\theta,p} \coloneqq \theta \circ \gamma_p, W_p)$ is a local parameterization of $\theta(M)$ around $\theta(p)$. If additionally M is closed or $s \in \mathbb{N}_{\geq 0}$, every local parameterization (γ, W) of M yields a local parameterization $(\gamma_\theta \coloneqq \theta \circ \gamma, W)$ of $\theta(M)$.*
If M is connected, compact or orientable (for $n = m = d + 1$), then also $\theta(M)$ is. In particular, if M is an embedded (closed) hypersurface, then also $\theta(M)$ is an embedded (closed) hypersurface.

(ii) *$\mathrm{d}_p\theta : T_pM \to T_{\theta(p)}\theta(M)$ is a linear isomorphism for all $p \in M$.*

(iii) *We have $\theta^{-1} \in \mathcal{C}^{1+s}\big(\theta(M), \mathbb{R}^n\big)$.*

In particular, $\theta : M \to \theta(M)$ is a $\mathcal{C}^{1+s}$-diffeomorphism.

Proof.

Ad (i) Let $p \in M$ and choose a local parameterization (γ, W) of M around p with $\theta \circ \gamma \in \mathcal{C}^{1+s}(\overline{W}, \mathbb{R}^m)$, i.e, $W \subset \mathbb{R}^d$ is an open, bounded and convex subset and $\gamma : \overline{W} \to \mathbb{R}^n$ is a $\mathcal{C}^{1+s}$-embedding with image $\gamma(\overline{W}) \subset M$. As a combination of embeddings, $\gamma_\theta = \theta \circ \gamma : \overline{W} \to \mathbb{R}^m$ is also an embedding and $\gamma_\theta(\overline{W}) \subset \theta\big(\gamma(\overline{W})\big) \subset \theta(M)$ holds. So, (γ_θ, W) is a $\mathcal{C}^{1+s}$-local parameterization of $\theta(M)$ around $\theta(p)$. In the case that additionally M is closed or $s \in \mathbb{N}_{\geq 0}$, we have $\theta \circ \gamma \in \mathcal{C}^{1+s}(\overline{W}, \mathbb{R}^m)$ for every local parameterization (γ, W) and the considerations above then yield the additional claim.
According to Lemma 2.5, $\theta(M)$ hence is a d-dimensional $\mathcal{C}^{1+s}$-submanifold. Because $\theta : M \to \theta(M)$ is a homeomorphism, connectedness and compactness transfer from M to $\theta(M)$. If M and thus $\theta(M)$ are compact, orientability follows directly with Remark 2.14. But also if M is not compact, orientability transfers from M to $\theta(M)$: Let $\tilde{\nu} \in \mathcal{C}^s(M, \mathbb{R}^{d+1})$ be the vector field from Lemma 2.20. As $\theta : M \to \theta(M)$ is a $\mathcal{C}^{1+s}$-embedding, $\nu := \tilde{\nu} \circ \theta^{-1} \in C^0\big(\theta(M), \mathbb{R}^{d+1}\big)$ is well-defined and we have $|\nu(z)| = 1$ and $\nu(z) \perp \mathrm{d}_p\theta(T_pM)$ for all $z = \theta(p) \in \theta(M)$. By (ii), $\mathrm{d}_p\theta(T_pM) = T_z\theta(M)$ holds and thus $\nu : \theta(M) \to \mathbb{R}^{d+1}$ is a continuous unit normal to $\theta(M)$. Hence, $\theta(M)$ is orientable.

Ad (ii) As $M \subset \mathbb{R}^n$ and $\theta(M) \subset \mathbb{R}^m$ are both embedded submanifolds of the same dimension, also their tangent spaces have the same dimension. Hence the linear, injective differential $\mathrm{d}_p\theta : T_pM \to T_{\theta(p)}\theta(M)$ is bijective for all $p \in M$.

Ad (iii) Let $z \in \theta(M)$, define $p := \theta^{-1}(z) \in M$ and choose a local parameterization (γ, W) of M around p such that $(\gamma_\theta = \theta \circ \gamma, W)$ is a local parameterization of $\theta(M)$ around z, which exists by (i). Then, $\theta^{-1} \circ \gamma_\theta = \gamma \in \mathcal{C}^{1+s}(\overline{W}, \mathbb{R}^n)$ holds and thus $\theta^{-1} \in \mathcal{C}^{1+s}\big(\theta(M), \mathbb{R}^n\big)$ follows. □

2.1.2 Immersed Hypersurfaces

Embedded hypersurfaces can never touch themselves. This is a problem when we want to describe self-intersections of surfaces as in Section 5.3. Now, we introduce so-called immersed hypersurfaces, which allow exactly for those self-intersections, but apart from that differ as little as possible from the embedded hypersurfaces discussed so far.

Definition 2.22 (Hölder-continuous immersed Submanifold)**.** *Let $M \subset \mathbb{R}^n$ be a d-dimensional $\mathcal{C}^{1+s}$-embedded submanifold and let $\theta : M \to \mathbb{R}^m$ be a $\mathcal{C}^{1+s}$-immersion. Then, $\Sigma := \theta(M) \subset \mathbb{R}^m$ is called a d-dimensional $\mathcal{C}^{1+s}$-immersed submanifold with reference submanifold M and global parameterization θ.*

The reader should note that in this work, just as for embedded submanifolds, an immersed submanifold never contains a boundary. Moreover, we remark that we do not use any topological structure on the immersed submanifold Σ itself but only consider the topology on the (embedded) reference manifold M.
In general, an immersed submanifold is not an embedded submanifold because it can have

self-intersections. If $\theta : M \to \Sigma$ is not only an immersion but also an embedding, then Σ does not have any self-intersections but is an embedded submanifold (see Lemma 2.21). In particular, as locally any immersion $\theta : M \to \Sigma$ is an embedding (see Lemma 2.19), for every sufficiently small open subset $U \subset M$, $\theta_{|U} : U \to \theta(U)$ is an embedding and then $\theta(U) \subset \Sigma$ is an embedded submanifold. We call $\theta(U) \subset \Sigma$ an *(embedded) patch of* Σ. As a consequence of Lemmas 2.19 and 2.21, the following proposition holds.

Proposition 2.23 (Hölder-cont. local Parameterization for immersed Submanifolds)**.** *Let $\Sigma = \theta(M)$ be a $\mathcal{C}^{1+s}$-immersed submanifold. For every point $p \in M$ there exists an open neighborhood $U \subset M$ such that $\theta(U) \subset \Sigma$ is an embedded patch, i.e., a $\mathcal{C}^{1+s}$-embedded submanifold of the same dimension as Σ. We thus say that the immersed submanifold Σ locally is embedded. If M is closed, Σ can be covered by finitely many embedded patches. Furthermore, for any $p \in U$, there exists a local parameterization (γ_p, W_p) of M around p with $\gamma_p(\overline{W_p}) \subset U$ such that*

$$\left(\gamma_{\theta,p} := \theta \circ \gamma_p, W_p\right)$$

is a local parameterization of $\theta(U)$. If M is closed or $s \in \mathbb{N}_{\geq 0}$, every local parameterization (γ, W) of M with $\gamma(\overline{W}) \subset U$ yields a local parameterization $(\gamma_\theta = \theta \circ \gamma, W)$ of $\theta(U)$.

The reader should note that Σ being locally an embedded submanifold does not imply that for every point $z \in \Sigma$ there exists an open neighborhood $V \subset \Sigma$ such that V is an embedded submanifold!

In points of self-intersection, it does not make sense to speak of "the" tangent space of Σ. Instead, we have to distinguish different tangent spaces for each preimage. This can of course be done by working locally - as Σ locally is an embedded submanifold, locally the tangent space is well-defined: Let $p \in M$ and let $\theta(U) \subset \Sigma$ be an embedded patch with $p \in U$. Then, $T_{\theta(p)}\theta(U)$ is well-defined. Because $\theta_{|U}$ is an embedding, $\theta_{|U} : U \to \theta(U)$ is a diffeomorphism and thus

$$\mathrm{d}_p\theta : T_pU = T_pM \to T_{\theta(p)}\theta(U)$$

is a linear isomorphism. So, $T_{\theta(p)}\theta(U) = \mathrm{d}_p\theta(T_pM)$ holds. In particular, $T_{\theta(p)}\theta(U)$ is a linear subspace of the same dimension as T_pM.
The relation between the two tangent spaces T_pM and $T_{\theta(p)}\theta(U)$ can also be seen from the definition of the tangent space via C^1-maps: Because $\theta_{|U} : U \to \theta(U)$ is a diffeomorphism, there exists a bijection between the C^1-maps $c : I_0 \to U$ with $c(0) = p$ and the C^1-maps $\eta : I_0 \to \theta(U)$ with $\eta(0) = \theta(p)$ given through $\eta = \theta \circ c$. Due to $\eta'(0) = (\theta \circ c)'(0) = \mathrm{d}_p\theta\big[c'(0)\big]$, we thus have

$$v \in T_pU = T_pM \Leftrightarrow \mathrm{d}_p\theta[v] \in T_{\theta(p)}\theta(U).$$

As the tangent space is local, $T_{\theta(p)}\theta(U)$ is independent of the concrete choice of U. But it does depend of course on the embedded patch of Σ in which $\theta(p)$ is considered, which means as mentioned above that regarding the whole immersed submanifold Σ, we have to define a tangent space not for every $\theta(p) \in \Sigma$ but for every $p \in M$.

Definition 2.24 (Tangent Space for immersed Submanifolds). *For every $p \in M$, the tangent space of a C^1-immersed submanifold $\Sigma = \theta(M)$ at $\theta(p)$ is defined as*

$$T_p\Sigma \coloneqq \mathrm{d}_p\theta(T_pM).$$

The representation of the tangent space $T_p\Sigma$ via the differential $\mathrm{d}_p\theta$ directly yields a basis for the tangent space:

Remark 2.25 (Basis of the Tangent Space for immersed Submanifolds). *Let $\Sigma = \theta(M)$ be a d-dimensional C^1-immersed submanifold. For any local parameterization (γ, W) of M, we have*

$$T_{\gamma(x)}\Sigma = \operatorname{span}\left\{\partial_i\gamma_{\theta\,|x} \,\middle|\, i = 1, ..., d\right\}$$

for all $x \in \overline{W}$.

This follows from Remarks 2.9 and 2.17, because $\left\{\partial_i\gamma_{|x} \,\middle|\, i = 1, ..., d\right\}$ is a basis of $T_{\gamma(x)}M$ and $\mathrm{d}_{\gamma(x)}\theta : T_{\gamma(x)}M \to T_{\gamma(x)}\Sigma$ is a linear isomorphism with $\mathrm{d}_{\gamma(x)}\theta[\partial_i\gamma_{|x}] = \partial_i(\theta \circ \gamma)_{|x} = \partial_i\gamma_{\theta\,|x}$.

Like in the embedded case, we will now focus on submanifolds with codimension 1 and introduce the orthogonal complement to the tangent space, which then is characterized by a single vector called normal. Just as we needed to distinguish different tangent spaces for different preimages of the immersed submanifold, we will also define the normal on the embedded submanifold instead of on its immersed image.

Definition 2.26 (Orientability and Normal for immersed Submanifolds). *A d-dimensional C^1-immersed submanifold $\Sigma = \theta(M) \subset \mathbb{R}^{d+1}$ is called orientable if there exists a continuous unit normal ν_Σ, i.e., a continuous vector field $\nu_\Sigma : M \to \mathbb{R}^{d+1}$ that fulfills $|\nu_\Sigma(p)| = 1$ and $\nu_\Sigma(p) \perp T_p\Sigma$ for all $p \in M$.*

On account of Lemma 2.20, orientability transfers directly from the embedded submanifold M to the immersed submanifold $\Sigma = \theta(M)$. In particular, the following proposition holds.

Proposition 2.27. *Let $\Sigma = \theta(M) \subset \mathbb{R}^{d+1}$ be a d-dimensional $\mathcal{C}^{1+s}$-immersed submanifold such that $M \subset \mathbb{R}^{d+1}$ is orientable. Then Σ is also orientable with $\nu_\Sigma \in \mathcal{C}^s(M, \mathbb{R}^{d+1})$.*

We use the term hypersurface in the immersed case entirely analogously to the one in the embedded situation.

Definition 2.28 (Immersed Hypersurface). *A $\mathcal{C}^{1+s}$-immersed submanifold $\Sigma = \theta(M) \subset \mathbb{R}^{d+1}$ is called a $\mathcal{C}^{1+s}$-immersed (closed) hypersurface, if $M \subset \mathbb{R}^{d+1}$ is an embedded (closed) hypersurface. A 1-dimensional immersed hypersurface is also called an immersed curve.*

The embedded submanifold M is an embedded (closed) hypersurface, if it is d-dimensional and orientable as well as connected (and compact) as a subset of $\mathbb{R}^{d+1}$. By Proposition 2.27 and because $\theta : M \to \mathbb{R}^{d+1}$ is continuous, these properties transfer directly to the immersed submanifold Σ: If it is an immersed (closed) hypersurface, it is d-dimensional and orientable as well as connected (and compact) as a subset of $\mathbb{R}^{d+1}$.

Choosing $\theta = \mathrm{Id}$, every embedded submanifold M can be described as an immersed submanifold $\Sigma = \mathrm{Id}(M)$ with $\Sigma = M$. Therefore, all the definitions and statements gathered in the following for the immersed case particularly hold in the embedded case. Even more, as every immersed submanifold locally is embedded (see Proposition 2.23), every term introduced for immersed submanifolds can also be defined on the corresponding embedded patches. Then, for every locally defined term, these definitions are perfectly conform such that the term on the immersed submanifold equals the pullback of the term defined on the embedded patch.

Difficulties arise for statements concerning Hölder regularity because defining Hölder-regularity on non-closed submanifolds is cumbersome (see Remark 2.76(iii)). All embedded or immersed hypersurfaces occuring in the later parts of this work are closed. So, at a first thought, it seems as if it suffices to restrict to the easier closed case. But the embedded patches of an (even closed) immersed submanifold are of course not closed in general. As we want to transfer statements from the embedded setting to the immersed setting via these embedded patches, we indeed need to discuss non-closed submanifolds. Nevertheless, only the necessary definitions and statements are formulated for the general non-closed case. In particular, results on Hölder regularity for certain quantities will only be given for closed submanifolds.

2.1.3 Space Derivatives and Curvature Terms

This section deals with differentiating on submanifolds. We have already introduced the differential of a mapping with domain on an embedded submanifold in Definition 2.16. From this, we derive several differential operators. Moreover, the differential of the unit normal leads to important curvature terms.

Definition/Lemma 2.29 (First Fundamental Form). *The first fundamental form of a d-dimensional C^1-immersed submanifold $\Sigma = \theta(M) \subset \mathbb{R}^n$ in $p \in M$ is the restriction*

$$g_p : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$$

of the euclidean inner product on $\mathbb{R}^n$ to the linear subspace $T_p\Sigma \subset \mathbb{R}^n$, which defines a Riemannian metric $\tilde{g} = \big(g_p(\mathrm{d}_p\theta[\cdot], \mathrm{d}_p\theta[\cdot])\big)_{p\in M}$ on M. Its representation with respect to a local parameterization (γ, W) of M is given by

$$g^{\theta}_{ij\,|x} = \partial_i \gamma_{\theta\,|x} \cdot \partial_j \gamma_{\theta\,|x}$$

for $i, j = 1, \ldots, d$ and $x \in \overline{W}$ with $\gamma_\theta := \theta \circ \gamma$ as in Proposition 2.23. We denote its inverse by $[g^{ij}_\theta]_{i,j} := \big([g^\theta_{ij}]_{i,j}\big)^{-1}$.

If $\theta = \mathrm{Id}$, we omit the sub- and superscript θ and simply write γ as well as g_{ij} and g^{ij}.

Remark 2.30. *(i) Let $\Sigma = \theta(M)$ be a d-dimensional $\mathcal{C}^{1+s}$-immersed submanifold with M closed or $s \in \mathbb{N}_{\geq 0}$. Then, $g^\theta_{ij}, g^{ij}_\theta \in \mathcal{C}^s(\overline{W}, \mathbb{R})$ holds for every local parameterization (γ, W) of M and $i, j = 1, \ldots, d$ due to $\gamma_\theta \in \mathcal{C}^{1+s}(\overline{W}, \mathbb{R}^n)$ and Remark 2.106.*

(ii) The first fundamental form $[g^\theta_{ij}]_{i,j}$ and its inverse $[g^{ij}_\theta]_{i,j}$ are symmetric and uniformly elliptic on $\overline{W}$, i.e., there exists a constant $c > 0$ such that

$$\xi^\top [g^\theta_{ij}(x)]_{i,j}\xi \geq c|\xi|^2 \text{ and } \xi^\top [g^{ij}_\theta(x)]_{i,j}\xi \geq c|\xi|^2$$

holds for all $x \in \overline{W}$ and $\xi \in \mathbb{R}^d$. As $\overline{W}$ is compact, this is clear for the first fundamental form, and the statement for the inverse follows from the symmetry of the form.
We explain this more detailed in the following, where we omit the sub- and superscripts θ in the interest of readability. Define $F : \overline{W} \times \mathbb{R}^d \to \mathbb{R}$, $F(x,\xi) := \left|\sum_{i=1}^d \xi_i \partial_i \gamma(x)\right|^2$. Because F is continuous, it attains its minimum on the compact set $\overline{W} \times \mathcal{S}$, with $\mathcal{S} := \{\xi \in \mathbb{R}^d, |\xi| = 1\}$. So, $F(x,\xi) \geq c$ holds for some $c > 0$ and for all $(x,\xi) \in \overline{W} \times \mathcal{S}$ and then

$$\xi^\top [g_{ij}(x)]_{i,j}\xi = \sum_{i,j=}^{d} \xi_i \partial_i \gamma(x) \cdot \partial_j \gamma(x) \xi_j = F(x,\xi) = F\left(x, \frac{\xi}{|\xi|}\right)|\xi|^2 \geq c|\xi|^2$$

follows for all $x \in \overline{W}$ and $\xi \in \mathbb{R}^d \backslash \{0\}$. Analogously, because $\tilde{F} : \overline{W} \times \mathbb{R}^d \to \mathbb{R}$, $\tilde{F}(x,\eta) := \left|[g_{ij}(x)]_{i,j} \cdot \eta\right|^2$ is continuous, we have

$$\left|[g_{ij}(x)]_{i,j} \cdot \eta\right|^2 = \tilde{F}\left(x, \frac{\eta}{|\eta|}\right)|\eta|^2 \leq \tilde{c}|\eta|^2$$

for all $x \in \overline{W}$ and $\eta \in \mathbb{R}^d \setminus \{0\}$. Thus, due to the symmetry of $[g_{ij}]_{i,j}$, defining $\eta_x := [g^{ij}(x)]_{i,j} \cdot \xi$ leads to

$$\xi^\top [g^{ij}]_{i,j}\xi = ([g_{ij}]_{i,j}\eta_x)^\top \cdot \eta_x = \eta_x^\top [g_{ij}]_{i,j}\eta_x \geq c|\eta_x|^2 \geq \frac{c}{\tilde{c}}|\xi|^2$$

for all $x \in \overline{W}$ and $\xi \in \mathbb{R}^d \setminus \{0\}$.

Now, we introduce some differential operators on hypersurfaces. The Riemannian metric our hypersurface is endowed with allows to generalize concepts known from the Euclidean geometry: We define the surface gradient and the surface divergence. Combining them, we obtain the surface Hessian and the surface Laplacian, which is called Laplace-Beltrami operator.

Definition/Lemma 2.31 (Surface Gradient)**.** *Let $\Sigma = \theta(M) \subset \mathbb{R}^{d+1}$ be a C^1-immersed hypersurface. For $f \in C^1(M, \mathbb{R})$, the surface gradient $\nabla_\Sigma f(p)$ in $p \in M$ is the unique vector $v(p) \in T_p\Sigma$ such that*

$$\left(\mathrm{d}_p f \circ (\mathrm{d}_p\theta)^{-1}\right)[w] = g_p\big(v(p), w\big)$$

holds for all $w \in T_p\Sigma$. One easily checks that its representation with respect to a local parameterization (γ, W) of M is given by

$$\nabla_\Sigma f \circ \gamma = \sum_{i,j=1}^{d} g_\theta^{ij}\, \partial_i(f \circ \gamma)\, \partial_j \gamma_\theta.$$

Definition/Lemma 2.32 (Surface Divergence)**.** *Let $\Sigma = \theta(M) \subset \mathbb{R}^{d+1}$ be a C^1-immersed hypersurface. For $F \in C^1(M, \mathbb{R}^{d+1})$ and an orthonormal basis $\{v_1, ..., v_d\}$ of $T_p\Sigma$, the surface divergence $\mathrm{div}_\Sigma F(p)$ in $p \in M$ is defined as*

$$\mathrm{div}_\Sigma F(p) = \sum_{i=1}^{d} \left(\mathrm{d}_p F \circ (\mathrm{d}_p\theta)^{-1}\right)[v_i] \cdot v_i.$$

Its representation with respect to a local parameterization (γ, W) of M is given by

$$\mathrm{div}_\Sigma F \circ \gamma = \sum_{i,j=1}^{d} g_\theta^{ij}\, \partial_i (F \circ \gamma) \cdot \partial_j \gamma_\theta.$$

This formula shows in particular, that the definition of the surface divergence is independent of the choice of the orthonormal basis $\{v_1, ..., v_d\}$.

Proof. Let (γ, W) be a local parameterization of M around $p = \gamma(x)$, in particular $\{\partial_1 \gamma_{\theta\,|x}, ..., \partial_d \gamma_{\theta\,|x}\}$ is a basis of $T_p\Sigma$. We use

$$v_i = \sum_j \alpha_i^j \partial_j \gamma_{\theta\,|x} \quad \text{and} \quad \partial_r \gamma_{\theta\,|x} = \sum_l \beta_r^l v_l$$

as changes of basis and obtain

$$\begin{aligned}
g_p(v_i, \partial_r\gamma_\theta) &= \sum_j \alpha_i^j g_{jr}^\theta &&\Rightarrow \sum_r g_p(v_i, \partial_r\gamma_\theta) g_\theta^{rk} = \sum_{r,j} \alpha_i^j g_{jr}^\theta g_\theta^{rk} = \alpha_i^k,\\
g_p(v_i, \partial_r\gamma_\theta) &= \sum_l \beta_r^l \delta_{il} = \beta_r^i &&\Rightarrow \alpha_i^k = \sum_r g_p(v_i, \partial_r\gamma_\theta) g_\theta^{rk} = \sum_r \beta_r^i g_\theta^{rk},\\
g_{rs}^\theta = g_p(\partial_r\gamma_\theta, \partial_s\gamma_\theta) &= \sum_{i,j} \beta_r^i \beta_s^j \delta_{ij} = \sum_i \beta_r^i \beta_s^i &&\Rightarrow \sum_i \alpha_i^k \alpha_i^l = \sum_{i,r,s} \beta_r^i \beta_s^i g_\theta^{rk} g_\theta^{sl} = \sum_{r,s} g_{rs}^\theta g_\theta^{rk} g_\theta^{sl} = g_\theta^{kl},
\end{aligned}$$

where we omitted the evaluation in x in the interest of readability. So, the representation of the surface divergence with respect to the local parameterization (γ, W) of M is given by

$$\begin{aligned}
\mathrm{div}_\Sigma F(p) &= \sum_i \big(\mathrm{d}_p F \circ (\mathrm{d}_p\theta)^{-1}\big)[v_i] \cdot v_i = \sum_{i,k,l} \alpha_i^k \alpha_i^l\, \mathrm{d}_p F[\partial_k \gamma_{|x}] \cdot \partial_l \gamma_{\theta\,|x}\\
&= \sum_{k,l} \left(\sum_i \alpha_i^k \alpha_i^l \right) \partial_k (F \circ \gamma)_{|x} \cdot \partial_l \gamma_{\theta\,|x} = \sum_{k,l} g_\theta^{kl}\, \partial_k (F\circ\gamma)_{|x} \cdot \partial_l \gamma_{\theta\,|x}.
\end{aligned}$$

□

Definition/Lemma 2.33 (Laplace-Beltrami Operator)**.** *Let $\Sigma = \theta(M) \subset \mathbb{R}^{d+1}$ be a C^2-immersed hypersurface. For $f \in C^2(M, \mathbb{R})$,*

$$\Delta_\Sigma f \coloneqq \mathrm{div}_\Sigma\big(\nabla_\Sigma f\big)$$

is called Laplace-Beltrami operator of f. One easily checks that its representation with respect to a local parameterization (γ, W) of M is given by

$$\begin{aligned}
\big(\Delta_\Sigma f\big) \circ \gamma &= \sum_{i,j=1}^{d} g_\theta^{ij}\, \partial_i (\nabla_\Sigma f \circ \gamma) \cdot \partial_j \gamma_\theta\\
&= \sum_{i,j=1}^{d} g_\theta^{ij}\, \partial_i \partial_j (f \circ \gamma) \;+ \sum_{i,j,k,l=1}^{d} g_\theta^{ij}\, \partial_k (f\circ\gamma) \partial_i (g_\theta^{kl} \partial_l \gamma_\theta) \cdot \partial_j \gamma_\theta.
\end{aligned}$$

Definition/Lemma 2.34 (Surface Hessian)**.** *Let $\Sigma = \theta(M) \subset \mathbb{R}^{d+1}$ be a C^2-immersed hypersurface. For $f \in C^2(M, \mathbb{R})$,*

$$D_\Sigma^2 f \quad \text{with} \quad [D_\Sigma^2 f]_{rs} \coloneqq \big[\nabla_\Sigma\big([\nabla_\Sigma f]_r\big)\big]_s \text{ for } r, s = 1, ..., d+1$$

is called surface Hessian of f. To clarify the structure of rows and columns, we state the following formula for its representation with respect to a local parameterization (γ, W) of M that can be checked easily:

$$\left[D_\Sigma^2 f \circ \gamma\right]_{rs} = e_r^\top \cdot \left(D_\Sigma^2 f \circ \gamma\right) \cdot e_s = \sum_{i,j=1}^{d} g_\theta^{ij} \left(\partial_i (\nabla_\Sigma f \circ \gamma) \cdot e_r\right) \left(\partial_j \gamma_\theta \cdot e_s\right).$$

Remark 2.35. *Let $\Sigma = \theta(M) \subset \mathbb{R}^{d+1}$ be a $\mathcal{C}^{1+s}$-immersed hypersurface with M closed or $s \in \mathbb{N}_{\geq 0}$. For $f \in \mathcal{C}^{1+s}(M, \mathbb{R})$ and $F \in \mathcal{C}^{1+s}(M, \mathbb{R}^{d+1})$ with $s \geq 1$ if necessary, we have*

$$\begin{aligned} &\nabla_\Sigma f \in \mathcal{C}^s\left(M, \mathbb{R}^{d+1}\right), \quad \mathrm{div}_\Sigma F \in \mathcal{C}^s(M, \mathbb{R}), \\ &\Delta_\Sigma f \in \mathcal{C}^{s-1}(M, \mathbb{R}), \quad D_\Sigma^2 f \in \mathcal{C}^{s-1}\left(M, \mathbb{R}^{(d+1)\times(d+1)}\right). \end{aligned}$$

Moreover,

$$\Delta_\Sigma f = \sum_{r=1}^{d+1} \left[D_\Sigma^2 f\right]_{rr}$$

holds. Both statements follow directly from the representations of the differential operators with respect to a local parameterization.

Lemma 2.36 (Surface Derivatives in Extreme Points)**.** *Let $\Sigma = \theta(M)$ be a C^2-immersed hypersurface and let $f \in C^2(M, \mathbb{R})$ have a maximum in $p \in M$. Then, we have*

$$\nabla_\Sigma f(p) = 0 \qquad \textit{and} \qquad D_\Sigma^2 f(p) \leq 0.$$

Proof. Let (γ, W) be a local parameterization of M around $p = \gamma(x)$. In particular, $f \circ \gamma \in C^2(\overline{W})$ has a maximum in $x \in W$ and thus $D(f \circ \gamma)(x) = 0$ as well as $D^2(f \circ \gamma)(x) \leq 0$ hold. Hence,

$$\nabla_\Sigma f(p) = \sum_{i,j=1}^{d} g_\theta^{ij}(x)\, \partial_i (f \circ \gamma)(x)\, \partial_j \gamma_\theta(x) = 0$$

follows. For any $\xi \in \mathbb{R}^{d+1}$, define $\eta = (\eta_1, \ldots, \eta_d) \in \mathbb{R}^d$ by

$$\eta_i \coloneqq \sum_{j=1}^{d} g_\theta^{ij}(x) \left(\partial_j \gamma_\theta(x) \cdot \xi\right).$$

Then, we also have

$$\begin{aligned}
\xi^\top D^2_\Sigma f(p)\xi &= \sum_{i,j=1}^d g_\theta^{ij}(x)\big(\partial_i\big(\nabla_\Sigma f\circ\gamma\big)(x)\cdot\xi\big)\big(\partial_j\gamma_\theta(x)\cdot\xi\big)\\
&= \sum_{i,j,k,l=1}^d g_\theta^{ij}(x)\big(\partial_i\big(g_\theta^{kl}\partial_k(f\circ\gamma)\partial_l\gamma_\theta\big)(x)\cdot\xi\big)\big(\partial_j\gamma_\theta(x)\cdot\xi\big)\\
&= \sum_{i,j,k,l=1}^d g_\theta^{ij}(x)g_\theta^{kl}(x)\partial_i\partial_k(f\circ\gamma)(x)\big(\partial_l\gamma_\theta(x)\cdot\xi\big)\big(\partial_j\gamma_\theta(x)\cdot\xi\big)\\
&\quad+\sum_{i,j,k,l=1}^d g_\theta^{ij}(x)\big(\partial_i\big(g_\theta^{kl}\partial_l\gamma_\theta\big)(x)\cdot\xi\big)\partial_k(f\circ\gamma)(x)\big(\partial_j\gamma_\theta(x)\cdot\xi\big)\\
&= \sum_{i,k=1}^d\left(\sum_{j=1}^d g_\theta^{ij}(x)\big(\partial_j\gamma_\theta(x)\cdot\xi\big)\right)\partial_i\partial_k(f\circ\gamma)(x)\left(\sum_{l=1}^d g_\theta^{kl}(x)\big(\partial_l\gamma_\theta(x)\cdot\xi\big)\right)\\
&= \sum_{i,k=1}^d \eta_i\partial_i\partial_k(f\circ\gamma)(x)\eta_k\\
&= \eta^\top D^2(f\circ\gamma)(x)\eta \le 0.
\end{aligned}$$

In particular, for a normal vector $\xi\perp T_p\Sigma$, the calculation above implies

$$\xi^\top D^2_\Sigma f(p)\xi = 0. \qquad\square$$

The differential describes how a mapping changes along the hypersurface. Curvature of a hypersurface can be characterized by the changing of the unit normal. Being interested in curvature terms, it therefore is natural to examine the differential of the unit normal which leads to the so-called shape operator.

Definition 2.37 (Shape Operator). *Let $\nu\in C^1(M,\mathbb{R}^{d+1})$ be the unit normal of a C^2-immersed hypersurface $\Sigma=\theta(M)\subset\mathbb{R}^{d+1}$. The shape operator*

$$\mathcal{S}_p := -\mathrm{d}_p\nu\circ(\mathrm{d}_p\theta)^{-1} : T_p\Sigma\to T_p\Sigma$$

of Σ in $p\in M$ is defined as the pushforward of the negative differential of the unit normal.

Remark 2.38. *The shape operator is well-defined because for a local parameterization (γ,W) of M,*

$$\mathcal{S}_{\gamma(x)}\big[\partial_i\gamma_{\theta\,|x}\big] = -\mathrm{d}_{\gamma(x)}\nu\big[\partial_i\gamma_{|x}\big] = -\partial_i(\nu\circ\gamma)_{|x}$$

is perpendicular to $\nu\circ\gamma_{|x}$ for $i=1,\dots,d$ and $x\in\overline{W}$, which follows directly from differentiating $|\nu\circ\gamma_{|x}|^2=1$ with respect to x_i. So, $\mathcal{S}_{\gamma(x)}\big[\partial_i\gamma_{\theta\,|x}\big]\in T_{\gamma(x)}\Sigma$ holds. As $\{\partial_i\gamma_{\theta\,|x}\}$ is a basis of $T_{\gamma(x)}\Sigma$, we thus have $\mathcal{S}_p[v]\in T_p\Sigma$ for all $v\in T_p\Sigma$.

With the help of the shape operator, we can define the second fundamental form.

Definition/Lemma 2.39 (Second Fundamental Form). *The second fundamental form of a C^2-immersed hypersurface $\Sigma = \theta(M) \subset \mathbb{R}^{d+1}$ with unit normal ν is defined as*

$$h_p : T_p\Sigma \times T_p\Sigma \to \mathbb{R}, \quad h_p(v, w) \coloneqq g_p\big(\mathcal{S}_p[v], w\big) = \mathcal{S}_p[v] \cdot w$$

in $p \in M$. Its representation with respect to a local parameterization (γ, W) of M is given by

$$h^\theta_{ij\,|x} = \mathcal{S}_{\gamma(x)}\big[\partial_i \gamma_{\theta\,|x}\big] \cdot \partial_j \gamma_{\theta\,|x} = -\partial_i(\nu \circ \gamma)_{|x} \cdot \partial_j \gamma_{\theta\,|x} = \nu \circ \gamma_{|x} \cdot \partial_i \partial_j \gamma_{\theta\,|x}$$

for $i = 1, ..., d$ and $x \in \overline{W}$, where the last identity follows from differentiating $(\nu \circ \gamma) \cdot \partial_j \gamma_\theta = 0$ with respect to x_i.

Remark 2.40. *The second fundamental form is symmetric, which follows from the representation $h^\theta_{ij} = (\nu \circ \gamma) \cdot \partial_i \partial_j \gamma_\theta$ with Schwarz' theorem. Thus, the shape operator is self-adjoint, i.e.,*

$$g_p\big(\mathcal{S}_p[v], w\big) = h_p(v, w) = h_p(w, v) = g_p\big(v, \mathcal{S}_p[w]\big)$$

holds for all $v, w \in T_p\Sigma$. Hence, there exists an orthonormal basis of $T_p\Sigma$ consisting of eigenvectors of the shape operator $\mathcal{S}_p$.

Using the eigenvectors and eigenvalues of the shape operator, we introduce some important curvature terms.

Definition 2.41 (Principal Curvatures and Mean Curvature). *Let $\Sigma = \theta(M) \subset \mathbb{R}^{d+1}$ be a C^2-immersed hypersurface with unit normal ν. Let $p \in M$ and let $\{v_1, ..., v_d\}$ be an orthonormal basis of $T_p\Sigma$ consisting of eigenvectors of the shape operator $\mathcal{S}_p$. The principal curvatures $\kappa_i(p)$ of Σ in p are defined as the eigenvalues of the shape operator, so*

$$h_p(v_i, v_i) = g_p\big(\mathcal{S}_p[v_i], v_i\big) = g_p\big(\kappa_i(p) v_i, v_i\big) = \kappa_i(p)$$

holds. The mean curvature

$$H(p) = \mathrm{trace}(\mathcal{S}_p) = \kappa_1(p) + ... + \kappa_d(p)$$

of Σ in p is the trace of the shape operator. To clarify the corresponding hypersurface, we also use the notation H_Σ.

Note that despite the term "mean", the mean curvature is defined as the sum of the principal curvatures and not their mean value.

Remark 2.42. *Let $\Sigma = \theta(M)$ be a C^2-immersed hypersurface with unit normal ν. Then, by definition of the mean curvature and the surface divergence, we have*

$$H = -\mathrm{div}_\Sigma \nu \quad \text{on } M.$$

Remark 2.43. *There is no standard sign convention for the mean curvature. In this work, we assign a negative mean curvature to convex surfaces. In other words, we always use the outer unit normal. We consider an (embedded) sphere $\partial B_R(0) = \{p \in \mathbb{R}^{d+1} \,|\, |p| = R\}$*

with $\theta = \mathrm{Id}$ as an example. The outer unit normal $\nu(p) = \frac{p}{R}$ equals a scaled identity on the sphere and hence its differential $\mathrm{d}_p\nu = \frac{\mathrm{Id}}{R}$ is also a scaled identity. Therefore, all the eigenvalues of the shape operator $\mathcal{S}_p = -\mathrm{d}_p\nu$ are $-\frac{1}{R}$. Thus, the mean curvature is given by $H(p) = -\frac{d}{R}$ and in particular, it is negative.
However, the mean curvature vector $H\nu$ is independent on the choice of the unit normal.

A closed, embedded hypersurface can not have vanishing mean curvature. With an analogous argumentation, the same statement holds in the immersed case.

Proposition 2.44. *Let $\Sigma = \theta(M) \subset \mathbb{R}^{d+1}$ be a C^2-immersed closed hypersurface. The mean curvature on Σ is not the zero function.*

Proof. Because Σ is compact, there exists $R > 0$ such that Σ is contained in the ball $B_R(0) \subset \mathbb{R}^{d+1}$, i.e., $\Sigma \subset B_R(0)$ holds. Decreasing R yields the existence of $R_0 > 0$ with $\Sigma \subset B_R(0)$ for every $R > R_0$ and Σ touching the sphere $\partial B_{R_0}(0)$, i.e., there exists a point $z_0 \in \Sigma \cap \partial B_{R_0}(0)$. As Σ is closed, z_0 is not a boundary point. Let $\theta(U) \subset \Sigma$ be an embedded patch of Σ around z. Due to $|z_0| = R_0$ and $|z| \leq R_0$ for every $z \in \theta(U)$, in the point z_0, the surface $\theta(U)$ must bend inwards at least as much as the sphere $\partial B_{R_0}(0)$. With the characterization of the mean curvature from Remark 2.42,

$$\left|H_\Sigma(z_0)\right| = \left|H_{\theta(U)}(z_0)\right| \geq \left|H_{\partial B_{R_0}(0)}(z_0)\right| = \frac{d}{R_0} > 0$$

follows. In particular, H_Σ is not the zero function. □

We now introduce partitions of unity for submanifolds. They are particularly useful to construct a global function from several locally defined ones. As an example, this procedure is used in Definition 2.47 for the integration on submanifolds.

Lemma 2.45 (Partition of Unity)**.** *Let $M \subset \mathbb{R}^n$ be a $\mathcal{C}^{1+s}$-embedded submanifold and let $\mathcal{V} = (V_l)_{l\in\mathscr{L}}$ be an arbitrary open cover of M. Then there exists a partition of unity subordinate to $\mathcal{V}$, i.e., a family $(\psi_l)_{l\in\mathscr{L}}$ of functions $\psi_l \in \mathcal{C}^{1+s}(M,\mathbb{R})$ such that*

(i) $\operatorname{supp}\psi_l \coloneqq \overline{\{p \in M \mid \psi_l(p) \neq 0\}} \subset V_l$ *for all $l \in \mathscr{L}$,*

(ii) every point in M has a neighborhood that intersects $\operatorname{supp}\psi_l$ *only for finitely many l,*

(iii) $0 \leq \psi_l(p) \leq 1$ *for all $l \in \mathscr{L}$ and $p \in M$ as well as $\sum_{l\in\mathscr{L}} \psi_l(p) = 1$ for all $p \in M$.*

Proof. see [AE09, Proposition XI.1.20 and Remark XI.1.21c] □

Remark 2.46. *Patching together locally defined functions to obtain a global one is easy if the locally defined functions agree in points of overlapping domains. But even if the local definitions do not agree, in the literature it is still often assumed without further arguments that they can be patched together to a well-defined global function. Implicitly, then a strategy based on a partition of unity is used which we will explain in the following. Let X be a Banach space, let $M \subset \mathbb{R}^n$ be a d-dimensional $\mathcal{C}^{1+s}$-embedded submanifold and let $(\gamma_l, W_l)_{l\in\mathscr{L}}$ be an arbitrary set of local parameterizations such that M is covered by* *the open sets $\mathcal{V} = (V_l)_{l\in\mathscr{L}}$ with $V_l \coloneqq \gamma_l(W_l)$. Assume $(u_l)_{l\in\mathscr{L}}$ to be a set of locally defined functions $u_l \circ \gamma_l : \overline{W_l} \to X$. With a partition of unity $(\psi_l)_{l\in\mathscr{L}}$ subordinate to $\mathcal{V}$,*

$$u \coloneqq \sum_{l\in\mathscr{L}} \psi_l u_l$$

yields a well-defined global function $u : M \to X$. (Note that due to Lemma 2.45(ii), $\sum_{l\in\mathscr{L}}$ reduces to a well-defined finite sum in every point of M.) The idea is to prove that u inherits all desired properties, e.g. regularity, from the functions u_l.
In particular, if M is a closed submanifold and $r \in \mathbb{R}_{\geq 0}$ with $r \leq 1 + s$, then $u_l \circ \gamma_l \in \mathcal{C}^r(\overline{W_l}, X)$ for all $l \in \mathscr{L}$ implies $u \in \mathcal{C}^r(M, X)$ with $\|u\|_{C^r(M,X)} \lesssim \sum_l \|u_l \circ \gamma_l\|_{C^r(\overline{W_l},X)}$. This is clear for $r \in \mathbb{N}_{\geq 0}$ as differentiability on two separated sets implies differentiability on the union of these sets. Because Hölder regularity does not have this property, the proof for Hölder regular functions is more involved and is explained in the Appendix (see Lemma A.15). Additionally, if Y and Z are further Banach spaces, $U \subset Y$ an open subset, $m \in \mathbb{N}_{\geq 0}$ and $F_l \circ \gamma_l \in C^m_{(b)}\big(U, \mathcal{C}^r(\overline{W_l}, X)\big)$ or rather $F_l \circ \gamma_l \in C^m_{(b)}\big(U, \mathcal{L}\big(Z, \mathcal{C}^r(\overline{W_l}, X)\big)\big)$ is a set of locally defined functionals, then we have $F \coloneqq \sum_l \psi_l F_l \in C^m_{(b)}\big(U, \mathcal{C}^r(M, X)\big)$ or $F \in C^m_{(b)}\big(U, \mathcal{L}\big(Z, \mathcal{C}^r(M, X)\big)\big)$, respectively (see Corollary A.16).

Definition 2.47 (Integration on Submanifolds). *Let $\Sigma = \theta(M) \subset \mathbb{R}^n$ be a d-dimensional C^1-immersed submanifold. For a local parameterization (γ, W) of M and a function $f : M \to \mathbb{R}$ such that $f \circ \gamma$ is Lesbesgue-integrable on $\overline{W}$, we define*

$$\int_{\gamma_\theta(\overline{W})} f \,\mathrm{d}\mathcal{H}^d \coloneqq \int_{\overline{W}} f \circ \gamma \sqrt{\det g^\theta} \,\mathrm{d}\mathcal{L}^d,$$

where $g^\theta \coloneqq [g^\theta_{ij}]_{i,j=1,\dots,d}$ and $\mathcal{L}^d$ is the d-dimensional Lebesgue measure.
The integral $\int_\Sigma f \,\mathrm{d}\mathcal{H}^d$ is defined using a partition of unity and the patching strategy from Remark 2.46: Let $(\gamma_l, W_l)_{l\in\mathscr{L}}$ be a set of local parameterizations such that M is covered by the open sets $\mathcal{V} \coloneqq \big(\gamma_l(W_l)\big)_{l\in\mathscr{L}}$ and let $(\psi_l)_{l\in\mathscr{L}}$ be a partition of unity subordinate to $\mathcal{V}$. For suitable $f : M \to \mathbb{R}$, define

$$\int_\Sigma f \,\mathrm{d}\mathcal{H}^d \coloneqq \sum_{l\in\mathscr{L}} \int_{\gamma_{\theta,l}(\overline{W_l})} \psi_l f \,\mathrm{d}\mathcal{H}^d. \tag{2.1}$$

Here, "suitable" means that $f \circ \gamma_l$ is integrable on $\overline{W_l}$ for all $l \in \mathscr{L}$, which is the case for example if $f \in C^0(M, \mathbb{R})$.

The summation in (2.1) is well-defined if we can reduce to a finite sum, e.g., if Σ is closed. In particular, (2.1) then is independent of the concrete choice of the set of local parameterizations and of the partition of unity (see [AE09, Sections XII.1 and XII.2, particularly Remark 2.1(d)] for details). Thus, for any $f : M \to \mathbb{R}$ with $\operatorname{supp} f = \overline{\{p \in M \mid f(p) \neq 0\}} \subset \gamma(W)$ and $f \circ \gamma$ Lebesgue-integrable on $\overline{W}$ for a local parameterization (γ, W) of M, we have

$$\int_\Sigma f \,\mathrm{d}\mathcal{H}^d = \int_{\gamma_\theta(\overline{W})} f \,\mathrm{d}\mathcal{H}^d,$$

which we justify in the following: Let $(\gamma_l, W_l)_{l=1,\dots,L}$ be a finite set of local parameterizations of M with $(\gamma, W) = (\gamma_{l_0}, W_{l_0})$ for $l_0 \in \{1, \dots, L\}$ such that M is covered by the open sets $\mathcal{V} := \big(\gamma_l(W_l)\big)_{l=1,\dots,L}$ and let $(\psi_l)_{l=1,\dots,L}$ be a partition of unity subordinate to $\mathcal{V}$. With the help of a cut-off function $\xi : M \to [0,1]$ such that $\xi \equiv 1$ on $\operatorname{supp} f$ and $\xi \equiv 0$ on $M \setminus \gamma(W)$, we define a new partition of unity subordinate to $\mathcal{V}$ by

$$\widetilde{\psi}_{l_0} := \psi_{l_0} + \xi \sum_{l \neq l_0} \psi_l \qquad \text{and} \qquad \widetilde{\psi}_l := (1-\xi)\psi_l \text{ for } l \neq l_0.$$

We obtain $\widetilde{\psi}_{l_0} \equiv 1$ as well as $\widetilde{\psi}_l \equiv 0$ for $l \neq l_0$ on $\operatorname{supp} f$, which finally implies

$$\int_{\gamma_{\theta,l_0}(\overline{W_{l_0}})} \widetilde{\psi_{l_0}} f \,\mathrm{d}\mathcal{H}^d = \int_{\gamma_\theta(\overline{W})} f \,\mathrm{d}\mathcal{H}^d \qquad \text{and} \qquad \int_{\gamma_{\theta,l}(\overline{W_l})} \widetilde{\psi}_l f \,\mathrm{d}\mathcal{H}^d = 0 \text{ for } l \neq l_0.$$

Since a closed submanifold has no boundary, the following generalization of Gauß' theorem to closed submanifolds holds.

Proposition 2.48 (Gauß' Theorem on closed Submanifolds)**.** *Let $\Sigma = \theta(M) \subset \mathbb{R}^n$ be a d-dimensional C^2-immersed closed submanifold and let $F \in C^1(M,\mathbb{R}^n)$ with $F(p) \in T_p\Sigma$ for every $p \in M$ as well as $f \in C^2(M,\mathbb{R})$ and $g \in C^1(M,\mathbb{R})$. Then we have*

$$\int_\Sigma \operatorname{div}_\Sigma F \,\mathrm{d}\mathcal{H}^d = 0 \qquad \textit{and} \qquad \int_\Sigma g \Delta_\Sigma f \,\mathrm{d}\mathcal{H}^d = -\int_\Sigma \nabla_\Sigma g \cdot \nabla_\Sigma f \,\mathrm{d}\mathcal{H}^d.$$

Proof. In the embedded case with $\theta = \mathrm{Id}$, these formulas are well-known, see for example [BGN20, Theorem 21 and Remark 22i]. For general immersed hypersurfaces, $(g_p)_{p\in M}$ still yields a Riemannian metric and therefore the same arguments as for embedded hypersurfaces can be used to prove the statement (cf. [Bär10, Theorem 5.1.7, which relies on Lemmas 5.1.5 and 5.1.6 therein]). ☐

2.1.4 Evolving Hypersurfaces

As we are familiar with the basic properties of static hypersurfaces now, we can pass on to the concept of hypersurfaces that move in time. First, the notation used to characterize such evolving hypersurfaces is explained carefully. Then, we introduce a way of describing the velocity with which a hypersurface moves. The discussion of differentiating in time in this setting leads to the so-called material and normal time derivative and we give a formula for the normal time derivative of the mean curvature. The last Section 2.1.6 is dedicated to hypersurfaces parameterized via height functions, a special type of evolving hypersurface that we use to prove the existence of short-time solutions to our geometric PDE in Chapter 4.
As before, $\mathcal{C}^s \in \{C^s, h^s\}$ for $s \in \mathbb{R}_{\geq 0}$ continues to describe the regularity in space, whereas $\mathcal{C}^r \in \{C^r, h^r\}$ for $r \in \mathbb{R}_{\geq 0}$ is used for the newly emerging regularity in time. As we will work with parabolic equations later on, we already define the evolving hypersurfaces in a way that suits the parabolic setting and therefore always assume them to be of regularity $1+r$ in time and $2+s$ in space. In particular, for every fixed time, differentiability of the occuring static hypersurface is guaranteed.

We use the same approach to define evolving hypersurfaces in the embedded as well as in the immersed setting, except that we choose global parameterizations that are embeddings or immersions at every time t, respectively.

Definition 2.49 (Evolving Hypersurface)**.** *Let $M \subset \mathbb{R}^{d+1}$ be a $\mathcal{C}^{2+s}$-embedded (closed) hypersurface and let $T \in (0,\infty)$. Furthermore, let $\theta : [0,T] \times M \to \mathbb{R}^{d+1}$ with*

$$\theta \in \mathcal{C}^{1+r}\big([0,T], \mathcal{C}^s(M,\mathbb{R}^{d+1})\big) \cap \mathcal{C}^r\big([0,T], \mathcal{C}^{2+s}(M,\mathbb{R}^{d+1})\big)$$

such that $\theta_t \coloneqq \theta(t,\cdot) : M \to \mathbb{R}^{d+1}$ is an embedding / immersion for all $t \in [0,T]$. With $\Gamma(t) \coloneqq \Gamma_t \coloneqq \theta_t(M)$, we call

$$\Gamma \coloneqq \big\{\{t\} \times \Gamma(t) \,\big|\, t \in [0,T]\big\}$$

a $\mathcal{C}^{1+r}$- $\mathcal{C}^{2+s}$-evolving embedded / immersed (closed) hypersurface with reference surface M and global parameterization θ.

For a non-closed hypersurface M, the space $\mathcal{C}^s(M,\mathbb{R}^m)$ is not normed and thus the set $\mathcal{C}^r\big([0,T], \mathcal{C}^s(M,\mathbb{R}^m)\big)$ is not well-defined in the usual sense as in Definition 2.75. Instead, we define

$$\mathcal{C}^r\big([0,T], \mathcal{C}^s(M,\mathbb{R}^m)\big) \coloneqq \Big\{f : [0,T] \times M \to \mathbb{R}^m \,\Big|\, \forall p \in M : \exists \text{ local param. } (\gamma_p, W_p) \\ \text{with } f \circ \gamma_p \in \mathcal{C}^r\big([0,T], \mathcal{C}^s(\overline{W_p},\mathbb{R}^m)\big)\Big\}.$$

We state that an evolving hypersurface Γ is well-defined which means in particular that for any time t, Γ_t is an embedded or immersed hypersurface again, respectively. In the immersed case, Proposition 2.23 thus yields that Γ_t locally is an embedded hypersurface. But we obtain even more by the next proposition: The whole evolving immersed hypersurface Γ locally is an evolving embedded hypersurface. This means that the locality can be chosen independently of the time t: For $U \subset M$ sufficiently small, $\Gamma_{|U} \coloneqq \big\{\{t\} \times \theta_t(U) \,\big|\, t \in [0,T]\big\}$ is an evolving embedded hypersurface. We call $\Gamma_{|U} \subset \Gamma$ an *(embedded) patch of* Γ.

Proposition 2.50. *Let $\Gamma = \big\{\{t\} \times \theta_t(M) \,\big|\, t \in [0,T]\big\}$ be a $\mathcal{C}^{1+r}$- $\mathcal{C}^{2+s}$-evolving embedded / immersed (closed) hypersurface with reference surface M and global parameterization θ. Then, $\Gamma_t = \theta_t(M)$ is a $\mathcal{C}^{2+s}$-embedded / immersed (closed) hypersurface for all $t \in [0,T]$. Additionally, for every point $p \in M$ there exists an open neighborhood $U \subset M$ such that*

$$\Gamma_{|U} \coloneqq \big\{\{t\} \times \theta_t(U) \,\big|\, t \in [0,T]\big\}$$

is an embedded patch of Γ, i.e., it is a $\mathcal{C}^{1+r}$- $\mathcal{C}^{2+s}$-evolving embedded hypersurface. We thus say that the evolving immersed hypersurface Γ locally is an evolving embedded hypersurface. Furthermore, for any $p \in U$, there exists a local parameterization (γ_p, W_p) of M around p with $\gamma_p(\overline{W_p}) \subset U$ such that

$$\big(\gamma_{\theta(t),p} \coloneqq \theta_t \circ \gamma_p, W\big)$$

with $\gamma_{\theta,p} \in \mathcal{C}^r\big([0,T], \mathcal{C}^{2+s}(\overline{W},\mathbb{R}^{d+1})\big)$ is a local parameterization of $\theta_t(U) \subset \Gamma_t$ for all $t \in [0,T]$. If M is closed or $s \in \mathbb{N}_{\geq 0}$, every local parameterization (γ, W) of M with $\gamma(\overline{W}) \subset U$ yields a local parameterization $(\gamma_{\theta(t)} = \theta_t \circ \gamma, W)$ of $\theta_t(U)$ with

$$\gamma_\theta \in \mathcal{C}^{1+r}\big([0,T], \mathcal{C}^s(\overline{W},\mathbb{R}^{d+1})\big) \cap \mathcal{C}^r\big([0,T], \mathcal{C}^{2+s}(\overline{W},\mathbb{R}^{d+1})\big).$$

Proof. In the immersed case, $\Gamma_t = \theta_t(M)$ is a $\mathcal{C}^{2+s}$-immersed (closed) hypersurface by definition. Moreover, for fixed $t \in [0,T]$, any sufficiently small open subset $U \subset M$ yields an embedded submanifold $\theta_t(U) \subset \Gamma_t$ by Proposition 2.23. We have to prove that we can choose this open subset $U \subset M$ independently of t.
For this, we first show that locally, $\theta_t : M \to \mathbb{R}^{d+1}$ is an embedding independently of $t \in [0,T]$, i.e., that for every point $p \in M$ there exists an open neighborhood $U \subset M$ such that $\theta_{t|U}$ is an embedding for every $t \in [0,T]$. For any open subset $U \subset M$, $\theta_{t|U}$ is a $\mathcal{C}^{2+s}$-immersion on the $\mathcal{C}^{2+s}$-embedded submanifold U by restriction for every $t \in [0,T]$. So, we only have to prove that for every $p \in M$, there exists an open neighborhood $U \subset M$ around p such that $\theta_{t|U}$ is a homeomorphism onto its image for all $t \in [0,T]$.
We fix $p \in M$ and choose a local parameterization (γ, W) of M around p. Thus, $W \subset \mathbb{R}^d$ is an open, bounded and convex subset and $\gamma : \overline{W} \to \mathbb{R}^{d+1}$ is an embedding, in particular an immersion, with $\gamma(\overline{W}) \subset M$ and $p \in \gamma(W)$. We define $\gamma_\theta \coloneqq \theta\big(\cdot, \gamma(\cdot)\big) : [0,T] \times \overline{W} \to \mathbb{R}^{d+1}$ with $\gamma_\theta \in C^1([0,T] \times \overline{W}, \mathbb{R}^{d+1})$. As combination of immersions, also $\gamma_{\theta(t)} = \theta_t \circ \gamma : \overline{W} \to \mathbb{R}^{d+1}$ is an immersion for all $t \in [0,T]$. By Lemma A.10, $\gamma_{\theta(t)}$ thus locally is an embedding independently of $t \in [0,T]$, i.e., there exists an open subset $\widetilde{W} \subset \mathbb{R}^d$ with $\gamma^{-1}(p) \in \widetilde{W} \subset W$ such that $\gamma_{\theta(t)|\overline{\widetilde{W}}}$ is an embedding, hence a homeomorphism onto its image, for all $t \in [0,T]$.
Then, $U \coloneqq \gamma(\widetilde{W}) \subset M$ is an open neighborhood around p and as a combination and restriction of homeomorphisms, $\theta_{t|U} = \gamma_{\theta(t)|\widetilde{W}} \circ \gamma^{-1}_{|\gamma(\widetilde{W})} : U \to \mathbb{R}^{d+1}$ is also a homeomorphism onto its image for all $t \in [0,T]$.
So, locally, $\theta_t : M \to \mathbb{R}^{d+1}$ is an embedding independently of $t \in [0,T]$. In particular, for a sufficiently small open subset $U \subset M$, we have $\theta_{|[0,T]\times U} \in \mathcal{C}^{1+r}\big([0,T], \mathcal{C}^s(U, \mathbb{R}^{d+1})\big)$ and $\theta_{|[0,T]\times U} \in \mathcal{C}^r\big([0,T], \mathcal{C}^{2+s}(U, \mathbb{R}^{d+1})\big)$ by restriction and $\theta_{|[0,T]\times U}(t, \cdot) : U \to \mathbb{R}^{d+1}$ is an embedding for all $t \in [0,T]$. Thus,

$$\Gamma_{|U} \coloneqq \big\{ \{t\} \times \theta_t(U) \,\big|\, t \in [0,T] \big\}$$

is a $\mathcal{C}^{1+r}$- $\mathcal{C}^{2+s}$-evolving embedded hypersurface.
For every $p \in U$, choose a local parameterization (γ, W) of M around p with $\gamma(\overline{W}) \subset U$ and $\theta \circ \gamma \in \mathcal{C}^r\big([0,T], \mathcal{C}^{2+s}(\overline{W}, \mathbb{R}^{d+1})\big)$. As in Proposition 2.23 (see the proof of Lemma 2.21), $(\gamma_{\theta(t)}, W)$ then is a local parameterization of $\theta_t(U) = \Gamma_{|U}(t) \subset \Gamma(t)$ and by construction, $\gamma_\theta \in \mathcal{C}^r\big([0,T], \mathcal{C}^{2+s}(\overline{W}, \mathbb{R}^{d+1})\big)$ holds.
If M is closed or $s \in \mathbb{N}_{\geq 0}$ and (γ, W) is any local parameterization of M with $\gamma(\overline{W}) \subset U$, Proposition 2.23 yields that $(\gamma_{\theta(t)}, W)$ is a local parameterization of $\theta_t(U) = \Gamma_{|U}(t) \subset \Gamma(t)$ and then $\gamma_\theta \in \mathcal{C}^{1+r}\big([0,T], \mathcal{C}^s(\overline{W}, \mathbb{R}^{d+1})\big) \cap \mathcal{C}^r\big([0,T], \mathcal{C}^{2+s}(\overline{W}, \mathbb{R}^{d+1})\big)$ holds by construction.
In the embedded case, it follows from Lemma 2.21(i) that Γ_t is a $\mathcal{C}^{2+s}$-embedded (closed) hypersurface for every $t \in [0,T]$. The remaining claims clearly hold with $U \coloneqq M$. □

For an evolving hypersurface Γ with reference surface $M \subset \mathbb{R}^{d+1}$, we use the notation $\nu : [0,T] \times M \to \mathbb{R}^{d+1}$ to describe the continuous unit normals $\nu(t, \cdot)$ of Γ_t.

Proposition 2.51. *Let Γ be a $\mathcal{C}^{1+r}$- $\mathcal{C}^{2+s}$-evolving immersed hypersurface with reference surface $M \subset \mathbb{R}^{d+1}$. There exists a vector field $\nu_\Gamma \in \mathcal{C}^r\big([0,T], \mathcal{C}^{1+s}(M, \mathbb{R}^{d+1})\big)$ such that $\nu_{\Gamma(t)} \coloneqq \nu_\Gamma(t, \cdot)$ is a continuous unit normal to $\Gamma(t)$ for all $t \in [0,T]$.*

Proof. For fixed $t \in [0,T]$, there exists a continuous unit normal to the $\mathcal{C}^{2+s}$-immersed hypersurface $\Gamma_t = \theta_t(M)$ due to Proposition 2.27, i.e., there exists $\nu_{\Gamma(t)} \in C^0(M,\mathbb{R}^{d+1})$ with $|\nu_{\Gamma(t)}(p)| = 1$ and $\nu_{\Gamma(t)}(p) \perp T_p\Gamma_t = \mathrm{d}_p\theta_t(T_pM)$ for all $p \in M$. For every local parameterization (γ, W) of M with $\theta \circ \gamma \in \mathcal{C}^r\big([0,T], \mathcal{C}^{2+s}(\overline{W},\mathbb{R}^{d+1})\big)$, we can express this normal

$$\nu_{\Gamma(t)} \circ \gamma = \pm\frac{\mathcal{K}\big(\mathrm{d}_{\gamma(\cdot)}\theta[\partial_1\gamma],\dots,\mathrm{d}_{\gamma(\cdot)}\theta[\partial_d\gamma]\big)}{\big|\mathcal{K}\big(\mathrm{d}_{\gamma(\cdot)}\theta[\partial_1\gamma],\dots,\mathrm{d}_{\gamma(\cdot)}\theta[\partial_d\gamma]\big)\big|} = \pm\frac{\mathcal{K}\big(\partial_1(\theta_t\circ\gamma),\dots,\partial_d(\theta_t\circ\gamma)\big)}{\big|\mathcal{K}\big(\partial_1(\theta_t\circ\gamma),\dots,\partial_d(\theta_t\circ\gamma)\big)\big|} \text{ on } \overline{W}$$

with the help of the generalized cross product $\mathcal{K} : (\mathbb{R}^{d+1})^d \to \mathbb{R}^{d+1}$ (as in Definition A.4). Because the cross product conserves regularity, $\nu_\Gamma\circ\gamma \in \mathcal{C}^r\big([0,T], \mathcal{C}^{1+s}(\overline{W},\mathbb{R}^{d+1})\big)$ and hence $\nu_\Gamma \in \mathcal{C}^r\big([0,T], \mathcal{C}^{1+s}(M,\mathbb{R}^{d+1})\big)$ follows. □

Similar as in the static case, every evolving embedded hypersurface is an immersed one. Therefore, even if the proposition above as well as the following definitions and statements are formulated only for the immersed case, they obviously hold analogously for the embedded case.

2.1.5 Time Derivatives and Velocity Terms

In this section, we introduce some further quantities on evolving hypersurfaces. First, the velocity with which a hypersurface evolves is described. Afterwards, we deal with time derivatives on evolving hypersurfaces and introduce the concept of the material and the normal time derivative. Finally, a formula for the normal time derivative of the mean curvature is stated. The crucial issue through the whole section is to characterize everything independently of the global parameterization.

Definition 2.52 (Total and Normal Velocity)**.** *Let Γ be a C^1-C^2-evolving immersed hypersurface with global parameterization θ and normal ν as in Proposition 2.51. We define its total velocity*

$$V^{tot} \coloneqq \partial_t\theta$$

and its normal velocity

$$V \coloneqq V^{tot}\cdot\nu.$$

To clarify the corresponding hypersurface, we also use the notation V_Γ^{tot} and V_Γ.

Note that the normal velocity V is a real number. The velocity of the hypersurface Γ in normal direction is given by the normal velocity vector $V\nu$. Together with the tangential part of the velocity, it adds up to the total velocity

$$V^{\mathrm{tot}} = V\nu + \big(V^{\mathrm{tot}}\big)_{\mathrm{tan}}.$$

As the tangential part can be eliminated by a suitable changing of the global parameterization, the only interesting part of the total velocity is the normal velocity.

In contrast to the total velocity, the normal velocity is independent of the global parameterization. In fact, for any C^1-map $\eta : (t-\varepsilon, t+\varepsilon) \to \mathbb{R}^{d+1}$ with $\eta(\tau) \in \Gamma(\tau)$ for all $\tau \in (t-\varepsilon, t+\varepsilon)$ and $\eta(t) = p$, the normal velocity of Γ in (t,p) is given by

$$V_{|(t,p)} = \eta'(t) \cdot \nu_{|(t,p)}.$$

A proof of this statement can be found in [BGN20, Remark 24(ii)].

Remark 2.53. *Due to construction, if Γ is a $\mathcal{C}^{1+r}$- $\mathcal{C}^{2+s}$-evolving immersed hypersurface with reference surface $M \subset \mathbb{R}^{d+1}$, then we have*

$$V^{tot} \in \mathcal{C}^r\big([0,T], \mathcal{C}^s(M, \mathbb{R}^{d+1})\big) \qquad \textit{and} \qquad V \in \mathcal{C}^r\big([0,T], \mathcal{C}^s(M, \mathbb{R})\big).$$

Next, we will discuss differentiating mappings $f : [0,T] \times M \to \mathbb{R}$ describing a function on an evolving hypersurface Γ with reference surface M. We have already introduced space derivatives for the case of static hypersurfaces in Section 2.1.3 and we can transfer the definitions to the time-dependent case of evolving hypersurfaces by setting, e.g. for the surface gradient,

$$(\nabla_\Gamma f)(t,p) := \big(\nabla_{\Gamma(t)} f(t,\cdot)\big)(p)$$

for any $(t,p) \in [0,T] \times M$. Now, we focus on time derivatives.

Definition 2.54 (Material and Normal Time Derivative)**.** *Let Γ be a C^1- C^2-evolving immersed hypersurface with reference surface M and total velocity V^{tot}. Then, for a mapping $f \in C^1([0,T] \times M)$, we define the material time derivative*

$$\partial^\circ f := \partial_t f$$

and the normal time derivative

$$\partial^\square f := \partial^\circ f - V^{tot} \cdot \nabla_\Gamma f.$$

The importance of the material time derivative is understandable only when considering evolving embedded hypersurfaces $\Gamma = \big\{\{t\} \times \theta_t(M) \,\big|\, t \in [0,T]\big\}$. Then, we can not only define functions $f : [0,T] \times M \to \mathbb{R}$ but also $\check{f} : \Gamma \to \mathbb{R}$ is well-defined.
For fixed t, the function $\check{f}(t,\cdot) : \Gamma(t) \to \mathbb{R}$ is defined on the fixed embedded hypersurface $\Gamma(t)$ and therefore we can differentiate in space as we did in Section 2.1.3.
But differentiating in time then is more involved: It is not possible to fix z and vary t because the function $\check{f}$ is only defined in points (t,z) with $z \in \Gamma(t)$. This means that z depends on t and therefore variations in t always imply variations in z. We can solve this problem by differentiating $\check{f} \circ \theta$ instead of $\check{f}$ as the two variables t and p of $\check{f} \circ \theta$ can be varied independently. Then,

$$\partial^\circ \check{f} := \big(\partial_t(\check{f} \circ \theta)\big) \circ \theta^{-1} \tag{2.2}$$

is called the material time derivative of $\check{f}$. The name "material" time derivative arises, because it describes how a mapping changes along a trajectory of a material point. We can picture the trajectory as the motion of a bit of matter fixed on the hypersurface during

the evolution of the hypersurface. In particular, and also obvious from (2.2), the material time derivative depends heavily on the global parameterization.
The normal time derivative is a way to define a time derivative that is independent of the global parameterization: Similar to the procedure we applied to extract the normal velocity from the total velocity of an evolving hypersurface, we use the material time derivative but subtract the changing of the mapping that is due to the tangential movement of the evolving hypersurface. Because the independency of the normal time derivative from tangential shifts is not obvious from the definition, we state the following identification: For any C^1-map $\eta : (t-\varepsilon, t+\varepsilon) \to \mathbb{R}^{d+1}$ with $\eta(\tau) \in \Gamma(\tau)$ that points in normal direction, i.e. $\eta'(\tau) = |\eta'(\tau)|\nu\big(\tau, \eta(\tau)\big)$, we have

$$\partial^{\square} f_{|(t,\eta(t))} = \frac{\mathrm{d}}{\mathrm{d}\tau}_{|\tau=t} f\big(\tau, \eta(\tau)\big).$$

This statement is proven in [BGN20, Remark 29(iii) and (iv)] and shows clearly that the normal time derivative is independent of the global parameterization.

Remark 2.55. *For a $\mathcal{C}^{1+r}$- $\mathcal{C}^{2+s}$-evolving immersed hypersurface Γ with reference surface M and $f \in \mathcal{C}^{1+\tau}\big([0,T], \mathcal{C}^{\sigma}(M)\big)$ with $\tau \le r$ and $\sigma \le s$, we have*

$$\partial^{\circ} f \in \mathcal{C}^{\tau}\big([0,T], \mathcal{C}^{\sigma}(M)\big).$$

If additionally M is closed or $\sigma \in \mathbb{N}_{\ge 0}$, for $f \in \mathcal{C}^{1+\tau}\big([0,T], \mathcal{C}^{\sigma}(M)\big) \cap \mathcal{C}^{\tau}\big([0,T], \mathcal{C}^{1+\sigma}(M)\big)$,

$$\partial^{\square} f \in \mathcal{C}^{\tau}\big([0,T], \mathcal{C}^{\sigma}(M)\big)$$

follows on account of Remarks 2.35 and 2.53.

As we use the vector field $\nu_\Gamma : [0,T] \times M \to \mathbb{R}^{d+1}$ for an evolving hypersurface Γ to describe the normal $\nu_\Gamma(t,p)$ of the hypersurface $\Gamma(t)$ in a point $\theta(t,p) \in \Gamma(t)$, we denote by $H_\Gamma : [0,T] \times M \to \mathbb{R}$ the function for which $H_\Gamma(t,p)$ is the mean curvature of the hypersurface $\Gamma(t)$ in a point $\theta(t,p) \in \Gamma(t)$. If the corresponding evolving hypersurface is clear, we also omit the index Γ and simply write ν and H.

Remark 2.56. *On account of Remark 2.42 and Proposition 2.51, for a $\mathcal{C}^{1+r}$- $\mathcal{C}^{2+s}$-evolving immersed hypersurface Γ with reference surface M and unit normal ν, we have*

$$H = -\mathrm{div}_\Gamma \nu \in \mathcal{C}^{r}\big([0,T], \mathcal{C}^{s}(M)\big).$$

Having clarified the regularity of the mean curvature, we give a formula for its normal time derivative.

Proposition 2.57 (Normal Time Derivative of the Mean Curvature)**.** *We assume Γ to be a $(C^2$- $C^2) \cap (C^1$- $C^4)$-evolving immersed hypersurface with unit normal ν, mean curvature H and normal velocity V. Then,*

$$\partial^{\square} H = \Delta_\Gamma V + V\big|\nabla_\Gamma \nu\big|^2$$

holds on $\Gamma(t)$ for every $t \in [0,T]$.

Proof. Because the evolving immersed hypersurface Γ locally is embedded, the claim follows directly from the analogous statement for evolving embedded hypersurfaces which is proven for example in [BGN20, Lemma 39(ii)]. $\square$

Now, we state the so-called transport theorem for evolving hypersurfaces, which enables us to differentiate integrals over moving surfaces in time. It will be an important argument in Chapter 3.

Proposition 2.58 (Transport Theorem)**.** *Let Γ be a C^1-C^2-evolving immersed closed hypersurface with reference surface $M \subset \mathbb{R}^{d+1}$, mean curvature H and normal velocity V. For $f \in C^1([0,T] \times M)$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} f \,\mathrm{d}\mathcal{H}^d = \int_{\Gamma(t)} \partial^{\square} f - fHV \,\mathrm{d}\mathcal{H}^d.$$

Proof. As, obviously, integration is not defined locally, the statement does not transfer directly from the embedded to the immersed case and we have to argue more subtly. The evolving immersed hypersurface Γ locally is embedded but its embedded patches are not closed in general. The assumed compactness of the evolving surface in [BGN20, Theorem 32] is not necessary for the first statement therein, such that it also holds for the non-closed embedded patches. Together with a partition of unity, the statement thus transfers to the immersed case and yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} f \,\mathrm{d}\mathcal{H}^d = \int_{\Gamma(t)} \partial^{\circ} f + f \mathrm{div}_{\Gamma} V^{\mathrm{tot}} \,\mathrm{d}\mathcal{H}^d = \int_{\Gamma(t)} \partial^{\circ} f - \nabla_{\Gamma} f \cdot V^{\mathrm{tot}} + \mathrm{div}_{\Gamma}(f V^{\mathrm{tot}}) \,\mathrm{d}\mathcal{H}^d.$$

For the decomposition $V^{\mathrm{tot}} = V\nu + (V^{\mathrm{tot}})_{\mathrm{tan}}$ of the total velocity V^{tot} into the normal part $V\nu$ and the tangential part $(V^{\mathrm{tot}})_{\mathrm{tan}}$, Proposition 2.48 implies

$$\int_{\Gamma(t)} \mathrm{div}_{\Gamma}(f V^{\mathrm{tot}}) \,\mathrm{d}\mathcal{H}^d = \int_{\Gamma(t)} \mathrm{div}_{\Gamma}(f V \nu) \,\mathrm{d}\mathcal{H}^d = \int_{\Gamma(t)} f V \mathrm{div}_{\Gamma} \nu \,\mathrm{d}\mathcal{H}^d = -\int_{\Gamma(t)} f V H \,\mathrm{d}\mathcal{H}^d,$$

where we used $H = -\mathrm{div}_{\Gamma}\nu$ from Remark 2.56 in the last step. Thus, we finally have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} f \,\mathrm{d}\mathcal{H}^d = \int_{\Gamma(t)} \partial^{\circ} f - \nabla_{\Gamma} f \cdot V^{\mathrm{tot}} + \mathrm{div}_{\Gamma}(f V^{\mathrm{tot}}) \,\mathrm{d}\mathcal{H}^d = \int_{\Gamma(t)} \partial^{\square} f - f V H \,\mathrm{d}\mathcal{H}^d. \quad \square$$

2.1.6 Parameterization via Height Function

In this section, we discuss evolving closed hypersurfaces Γ defined by special global parameterizations of the form

$$\theta_\rho(t,p) = \bar{\theta}(p) + \rho(t,p)\nu_\Sigma(p),$$

where $\Sigma = \bar{\theta}(M)$ is a fixed, i.e. time-independent, immersed closed hypersurface, named the *immersed reference surface*, in distinction from the (embedded) reference surface M. The function $\rho : [0,T] \times M \to \mathbb{R}$ is a so-called height function. This name arises from the fact that $\bar{\theta}(p) \in \Sigma$ and $\theta_\rho(t,p) \in \Gamma(t)$ differ only in normal direction; so if we flatten Σ locally, $\bar{\theta}(p)$ and $\theta_\rho(t,p)$ lie exactly above one another and the difference in height is $\rho(t,p)$.

We start with the embedded case, where we choose $\bar{\theta} = \mathrm{Id}$ and thus $\Sigma = M$ is an embedded closed hypersurface. For such hypersurfaces, we can define a tubular neighborhood which can be interpreted as the extension of the hypersurface to an open set in $\mathbb{R}^{d+1}$. In particular, extending a mapping defined on an embedded closed hypersurface constantly in normal direction to the tubular neighborhood allows us to use the calculus in $\mathbb{R}^{d+1}$ to compute quantities for this mapping, which is frequently used in our source [BGN20].

Definition 2.59 (Tubular Neighborhood)**.** *Let $M \subset \mathbb{R}^{d+1}$ be a C^2-embedded closed hypersurface with unit normal ν_M. For $R > 0$, the tubular neighborhood $\mathcal{T}_R(M)$ of M is defined as*

$$\mathcal{T}_R(M) \coloneqq \left\{ z \in \mathbb{R}^{d+1} \,\middle|\, z = p + r\nu_M(p) \text{ for } p \in M \text{ and } |r| < R \right\}.$$

For $R > 0$ sufficiently small,

$$M \times (-R, R) \to \mathcal{T}_R(M),\ (p, r) \mapsto p + r\nu_M(p)$$

is a bijection, see for example [PS16, Section 2.3]. For this reason, there exist functions $\Pi_M : \mathcal{T}_R(M) \to M$ and $\mathrm{d}_M : \mathcal{T}_R(M) \to (-R, R)$ with

$$z = p + r\nu_M(p) \iff \Pi_M(z) = p,\ \mathrm{d}_M(z) = r.$$

It turns out that d_M is the signed distance function to M, with $\mathrm{d}_M(z)$ being positive if z lies in the direction in which ν_M points and negative otherwise, and Π_M is the projection onto M, so we have

$$\Pi_M(z) = \operatorname*{arg\,min}_{p \in M} \|z - p\|_{\mathbb{R}^{d+1}}.$$

For R sufficiently small, $\Pi_M \in C^1\big(\mathcal{T}_R(M), \mathbb{R}^{d+1}\big)$ and $\mathrm{d}_M \in C^2\big(\mathcal{T}_R(M), \mathbb{R}\big)$ hold, see once again [PS16, Section 2.3]. For a further discussion of tubular neighborhoods of embedded hypersurfaces, besides the detailed but maybe hard to follow [PS16, Section 2.3], we recommend [BF12, Section III.3.2], where especially the dependence of the size of the tubular neighborhood on the curvature of the surface is adressed.

Proposition 2.60. *Let $M \subset \mathbb{R}^{d+1}$ be a C^2-embedded closed hypersurface with unit normal ν_M. Furthermore, let $\rho \in C^1(M, \mathbb{R})$ with $\|\rho\|_{C^0(M,\mathbb{R})}$ sufficiently small. Then the following hold:*

(i) $\theta_\rho : M \to \mathbb{R}^{d+1}$, $\theta_\rho(p) \coloneqq p + \rho(p)\nu_M(p)$ *is a homeomorphism onto its image.*

(ii) $\mathrm{d}_p\theta_\rho : T_pM \to \mathbb{R}^{d+1}$ *is injective with* $\nu_M(p) \notin \mathrm{d}_p\theta_\rho(T_pM)$ *for all* $p \in M$.

In particular, θ_ρ is an embedding.

Proof.

Ad (i) Due to the bijection $M \times (-R, R) \to \mathcal{T}_R(M)$ mentioned above, θ_ρ is injective for $\|\rho\|_{C^0(M,\mathbb{R})}$ sufficiently small. Therefore θ_ρ is bijective onto its image. As ρ and ν_M are continuous on M, also θ_ρ is. And because we have $\theta_\rho^{-1} = \Pi_{M\,|\theta_\rho(M)}$, continuity of θ_ρ^{-1} follows from the continuity of Π_M.

Ad (ii) Let $p \in M$ and let (γ, W) be a local parameterization of M with $p = \gamma(x)$ for $x \in \overline{W}$. Then $T_pM = \operatorname{span}\{\partial_i\gamma_{|x}\}_{i=1,\dots,d}$ holds and $(\partial_i\gamma_{|x})_{i=1,\dots,d}$ are linearly independent with $\nu_{M\,|\gamma(x)} \perp \operatorname{span}\{\partial_i\gamma_{|x}\}_{i=1,\dots,d}$. Because of

$$\mathrm{d}_p\theta_\rho[\partial_i\gamma_{|x}] = \partial_i\gamma_{|x} + \partial_i(\rho\circ\gamma)_{|x}\nu_{M\,|\gamma(x)} + \rho_{|\gamma(x)}\partial_i(\nu_M\circ\gamma)_{|x},$$

for $\|\rho\|_{C^0(M,\mathbb{R})}$ sufficiently small, $\big(\mathrm{d}_p\theta_\rho[\partial_1\gamma_{|x}],\dots,\mathrm{d}_p\theta_\rho[\partial_d\gamma_{|x}],\nu_{M\,|\gamma(x)}\big)$ are also linearly independent, which we explain more detailed in the following.
Let $\alpha_1,\dots,\alpha_{d+1} \in \mathbb{R}$ with

$$\begin{aligned}
0 &= \sum_{i=1}^d \alpha_i \mathrm{d}_p\theta_\rho[\partial_i\gamma_{|x}] + \alpha_{d+1}\nu_{M\,|\gamma(x)} \\
&= \sum_{i=1}^d \alpha_i\Big(\partial_i\gamma_{|x} + \partial_i(\rho\circ\gamma)_{|x}\nu_{M\,|\gamma(x)} + \rho_{|\gamma(x)}\partial_i(\nu_M\circ\gamma)_{|x}\Big) + \alpha_{d+1}\nu_{M\,|\gamma(x)} \\
&= \sum_{i=1}^d \alpha_i\partial_i\gamma_{|x} + \rho_{|\gamma(x)}\sum_{i=1}^d \alpha_i\partial_i(\nu_M\circ\gamma)_{|x} + \left(\sum_{i=1}^d \alpha_i\partial_i(\rho\circ\gamma)_{|x} + \alpha_{d+1}\right)\nu_{M\,|\gamma(x)}.
\end{aligned}$$

We have $\partial_i\gamma_{|x}, \partial_i(\nu_M\circ\gamma)_{|x} \in T_{\gamma(x)}M$ due to Remark 2.38 and $\nu_{M\,|\gamma(x)} \perp T_{\gamma(x)}M$. So,

$$0 = \sum_{i=1}^d \alpha_i\partial_i\gamma_{|x} + \rho_{|\gamma(x)}\sum_{i=1}^d \alpha_i\partial_i(\nu_M\circ\gamma)_{|x} \tag{2.3}$$

and

$$0 = \sum_{i=1}^d \alpha_i\partial_i(\rho\circ\gamma)_{|x} + \alpha_{d+1} \tag{2.4}$$

hold independently. For $\|\rho\|_{C^0(M,\mathbb{R})}$ sufficiently small, (2.3) yields $\alpha_1,\dots,\alpha_d = 0$ and then $\alpha_{d+1} = 0$ follows with (2.4). Thus, the claimed linear independency does indeed hold. In particular, we have $\nu_M(p) \notin \mathrm{d}_p\theta_\rho(T_pM)$ for all $p \in M$. As the differential $\mathrm{d}_p\theta_\rho : T_pM \to \mathbb{R}^{d+1}$ is linear, its injectivity follows directly from $T_pM = \operatorname{span}\{\partial_i\gamma_{|x}\}_{i=1,\dots,d}$ and the linear independency of $(\mathrm{d}_p\theta_\rho[\partial_i\gamma_{|x}])_{i=1,\dots,d}$. □

We now turn to the immersed case where $\bar\theta$ is no longer an embedding but only an immersion. As an immersed closed hypersurface, $\Sigma = \bar\theta(M)$ may have self-intersections and thus does not possess a tubular neighborhood which we used to prove that θ_ρ is a well-defined global parameterization in the embedded case (see Proposition 2.60). But, on the other hand, in the immersed situation we do not need $\theta_\rho(t,\cdot)$ to be an embedding but only an immersion. This relaxation enables us to do without the tubular neighborhood in the immersed case and it turns out that the same conditions as in the embedded case ($\rho \in C^1$ and $\|\rho\|_{C^0}$ sufficiently small) are sufficient for θ_ρ to be a well-defined global parameterization. In the proof of Proposition 2.62, we even state an explicit bound for the required smallness of the height function.

Lemma 2.61. *Let $\Sigma = \theta(M)$ be a C^1-immersed closed hypersurface. Then, we have*

$$\inf_{p\in M}\inf_{\substack{v\in T_pM,\\ |v|=1}} \big|\mathrm{d}_p\theta[v]\big| > 0.$$

Proof. Let $d \coloneqq \dim M$ and choose a local parameterization (γ, W) of M. In particular, $\gamma \in C^1(\overline{W}, \mathbb{R}^{d+1})$ is an embedding with $\gamma(\overline{W}) \subset M$. Set

$$v(\alpha, x) \coloneqq \frac{\sum_i \alpha^i \partial_i \gamma(x)}{|\sum_i \alpha^i \partial_i \gamma(x)|}$$

for $\alpha \in \mathbb{R}^d \setminus \{0\}$ and $x \in \overline{W}$. Then $v(\alpha, x) \in T_{\gamma(x)} M$ holds with $|v(\alpha, x)| = 1$ for every $\alpha \in \mathbb{R}^d \setminus \{0\}$ and $x \in \overline{W}$. Moreover, for $\beta \coloneqq \frac{\alpha}{|\alpha|}$, we have

$$v(\beta, x) = \frac{\sum_i \beta^i \partial_i \gamma(x)}{|\sum_i \beta^i \partial_i \gamma(x)|} = \frac{\frac{1}{|\alpha|} \sum_i \alpha^i \partial_i \gamma(x)}{\left|\frac{1}{|\alpha|} \sum_i \alpha^i \partial_i \gamma(x)\right|} = \frac{\sum_i \alpha^i \partial_i \gamma(x)}{|\sum_i \alpha^i \partial_i \gamma(x)|} = v(\alpha, x)$$

for every $x \in \overline{W}$. So, with $\mathcal{S} \coloneqq \{\alpha \in \mathbb{R}^d, |\alpha| = 1\}$,

$$\left\{v \in T_{\gamma(x)} M,\ |v| = 1\right\} = \left\{v(a, x) \,\middle|\, \alpha \in \mathcal{S}\right\}$$

and in particular

$$\inf_{\substack{v \in T_{\gamma(x)} M, \\ |v| = 1}} \left|\mathrm{d}_{\gamma(x)} \theta[v]\right| = \inf_{\alpha \in \mathcal{S}} \left|\mathrm{d}_{\gamma(x)} \theta\big[v(\alpha, x)\big]\right|$$

follows for every $x \in \overline{W}$. We have

$$\left|\mathrm{d}_{\gamma(x)} \theta\big[v(\alpha, x)\big]\right| = \left|\mathrm{d}_{\gamma(x)} \theta\left[\frac{\sum_i \alpha^i \partial_i \gamma(x)}{|\sum_i \alpha^i \partial_i \gamma(x)|}\right]\right| = \left|\frac{\sum_i \alpha^i \mathrm{d}_{\gamma(x)} \theta[\partial_i \gamma(x)]}{|\sum_i \alpha^i \partial_i \gamma(x)|}\right| = \frac{\left|\sum_i \alpha^i \partial_i (\theta \circ \gamma)(x)\right|}{|\sum_i \alpha^i \partial_i \gamma(x)|}$$

for all $\alpha \in \mathcal{S}$ and $x \in \overline{W}$. Due to $\theta \in C^1(M, \mathbb{R}^{d+1})$, $\gamma \in C^1(\overline{W}, \mathbb{R}^{d+1})$ with $\gamma(\overline{W}) \subset M$ and $\partial_i \gamma \neq 0$ on $\overline{W}$ for all $i = 1, ..., d$ by the immersion property of γ, thus

$$(\alpha, x) \mapsto \left|\mathrm{d}_{\gamma(x)} \theta\big[v(\alpha, x)\big]\right| \in C^0(\mathcal{S} \times \overline{W})$$

follows. Because θ is an immersion, $\left|\mathrm{d}_{\gamma(x)} \theta\big[v(\alpha, x)\big]\right| > 0$ holds for all $(\alpha, x) \in \mathcal{S} \times \overline{W}$ and then compactness of $\mathcal{S} \times \overline{W}$ implies

$$\inf_{x \in \overline{W}} \inf_{\substack{v \in T_{\gamma(x)} M, \\ |v| = 1}} \left|\mathrm{d}_{\gamma(x)} \theta[v]\right| = \inf_{x \in \overline{W}} \inf_{\alpha \in \mathcal{S}} \left|\mathrm{d}_{\gamma(x)} \theta\big[v(\alpha, x)\big]\right| > 0.$$

Finally, as M is compact, it can be covered by finitely many local parameterizations (γ, W) and therefore the claim follows. ☐

Proposition 2.62. *Let $\Sigma = \bar{\theta}(M) \subset \mathbb{R}^{d+1}$ be a C^2-immersed closed hypersurface with unit normal ν_Σ. Furthermore, let $\rho \in C^1(M, \mathbb{R})$ with $\|\rho\|_{C^0(M, \mathbb{R})}$ sufficiently small. Then,*

$$\theta_\rho : M \to \mathbb{R}^{d+1}, \quad \theta_\rho(p) \coloneqq \bar{\theta}(p) + \rho(p) \nu_\Sigma(p)$$

is an immersion. In particular, $\Sigma_\rho \coloneqq \theta_\rho(M)$ is an immersed closed hypersurface and for any sufficiently small local parameterization (γ, W) of M, $(\gamma_\rho \coloneqq \theta_\rho \circ \gamma, W)$ is a local parameterization of an embedded patch of Σ_ρ and

$$\left(\partial_1 \gamma_{\rho|x}, ..., \partial_d \gamma_{\rho|x}, \nu_{\Sigma|\gamma(x)}\right) \subset \mathbb{R}^{d+1}$$

are linearly independent for every $x \in \overline{W}$.

Proof. We have $\nu_\Sigma \in C^1(M, \mathbb{R}^{d+1})$ by Proposition 2.27 and thus $\theta_\rho = \bar{\theta} + \rho\nu_\Sigma \in C^1(M, \mathbb{R}^{d+1})$. For any local parameterization (γ, W) of M, the domain $\overline{W} \subset \mathbb{R}^d$ is compact and hence

$$\begin{aligned} S_{(\gamma,W)} &:= \sup_{x\in\overline{W}} \left\| \mathrm{d}_{\gamma(x)}\nu_\Sigma \right\|_{\mathcal{L}(T_{\gamma(x)}M, \mathbb{R}^{d+1})} \\ &\lesssim \sup_{x\in\overline{W}} \max_{i=1,\dots,d} \frac{\left|\mathrm{d}_{\gamma(x)}\nu_\Sigma\big(\partial_i\gamma(x)\big)\right|}{|\partial_i\gamma(x)|} = \sup_{x\in\overline{W}} \max_{i=1,\dots,d} \frac{|\partial_i(\nu_\Sigma \circ \gamma)(x)|}{|\partial_i\gamma(x)|} < \infty \end{aligned}$$

holds. As M is compact, it can be covered by finitely many local parameterizations $(\gamma_l, W_l)_{l=1,\dots,L}$ and therefore

$$S := \sup_{p\in M} \left\| \mathrm{d}_p\nu_\Sigma \right\|_{\mathcal{L}(T_pM, \mathbb{R}^{d+1})} \leq \max_{l=1,\dots,L} S_{(\gamma_l, W_l)} < \infty$$

follows. Remark 2.42 and Proposition 2.44 imply $S \neq 0$ and by Lemma 2.61, we have

$$I := \inf_{p\in M} \inf_{\substack{v\in T_pM, \\ |v|=1}} \left|\mathrm{d}_p\bar{\theta}[v]\right| > 0.$$

Hence,

$$R := \frac{I}{2S} > 0$$

is well-defined. Assume $\|\rho\|_{C^0(M)} \leq R$. For all $p \in M$ and $v \in T_pM$

$$\mathrm{d}_p\bar{\theta}[v], \mathrm{d}_p\nu_\Sigma[v] \in T_p\Sigma \qquad \text{and} \qquad \nu_\Sigma(p) \perp T_p\Sigma \tag{2.5}$$

hold. Due to $\rho(p), \mathrm{d}_p\rho[v] \in \mathbb{R}$, we thus have

$$\begin{aligned} \left|\mathrm{d}_p\theta_\rho[v]\right|^2 &= \left|\mathrm{d}_p\bar{\theta}[v] + \mathrm{d}_p\rho[v]\nu_\Sigma(p) + \rho(p)\mathrm{d}_p\nu_\Sigma[v]\right|^2 \\ &= \left|\mathrm{d}_p\bar{\theta}[v] + \rho(p)\mathrm{d}_p\nu_\Sigma[v]\right|^2 + \left|\mathrm{d}_p\rho[v]\right|^2 \\ &\geq \left|\mathrm{d}_p\bar{\theta}[v] + \rho(p)\mathrm{d}_p\nu_\Sigma[v]\right|^2 \end{aligned}$$

and then

$$\left|\mathrm{d}_p\theta_\rho[v]\right| \geq \left|\mathrm{d}_p\bar{\theta}[v] + \rho(p)\mathrm{d}_p\nu_\Sigma[v]\right| \geq \left|\mathrm{d}_p\bar{\theta}[v]\right| - R\left|\mathrm{d}_p\nu_\Sigma[v]\right| \geq I - RS = \frac{I}{2} > 0$$

follows for $|v| = 1$. In particular, $\mathrm{d}_p\theta_\rho : T_pM \to \mathbb{R}^{d+1}$ is injective and therefore $\theta_\rho : M \to \mathbb{R}^{d+1}$ is an immersion.
The fact that Σ_ρ is an immersed closed hypersurface follows by definition and then Proposition 2.23 yields that (γ_ρ, W) is a local parameterization of an embedded patch of Σ_ρ. It remains to show that

$$\big(\partial_1\gamma_{\rho|x}, \dots, \partial_d\gamma_{\rho|x}, \nu_{\Sigma|\gamma(x)}\big) \subset \mathbb{R}^{d+1}$$

are linearly independent for every $x \in \overline{W}$. For this, fix $x \in \overline{W}$ and let $\alpha_1, ..., \alpha_{d+1} \in \mathbb{R}$ with

$$\begin{aligned}
0 &= \sum_{i=1}^{d} \alpha_i \partial_i \gamma_{\rho\,|x} + \alpha_{d+1} \nu_{\Sigma\,|\gamma(x)} \\
&= \sum_{i=1}^{d} \alpha_i \Big(\partial_i (\bar{\theta} \circ \gamma)_{|x} + \partial_i (\rho \circ \gamma)_{|x} \nu_{\Sigma\,|\gamma(x)} + \rho_{|\gamma(x)} \partial_i (\nu_\Sigma \circ \gamma)_{|x} \Big) + \alpha_{d+1} \nu_{\Sigma\,|\gamma(x)} \\
&= \sum_{i=1}^{d} \alpha_i \Big(\mathrm{d}_{\gamma(x)} \bar{\theta} [\partial_i \gamma_{|x}] + \partial_i (\rho \circ \gamma)_{|x} \nu_{\Sigma\,|\gamma(x)} + \rho_{|\gamma(x)} \mathrm{d}_{\gamma(x)} \nu_\Sigma [\partial_i \gamma_{|x}] \Big) + \alpha_{d+1} \nu_{\Sigma\,|\gamma(x)} \\
&= \sum_{i=1}^{d} \alpha_i \mathrm{d}_{\gamma(x)} \bar{\theta} [\partial_i \gamma_{|x}] + \rho_{|\gamma(x)} \sum_{i=1}^{d} \alpha_i \mathrm{d}_{\gamma(x)} \nu_\Sigma [\partial_i \gamma_{|x}] + \left(\sum_{i=1}^{d} \alpha_i \partial_i (\rho \circ \gamma)_{|x} + \alpha_{d+1} \right) \nu_{\Sigma\,|\gamma(x)}.
\end{aligned}$$

With the statement in (2.5),

$$0 = \sum_{i=1}^{d} \alpha_i \mathrm{d}_{\gamma(x)} \bar{\theta} [\partial_i \gamma_{|x}] + \rho_{|\gamma(x)} \sum_{i=1}^{d} \alpha_i \mathrm{d}_{\gamma(x)} \nu_\Sigma [\partial_i \gamma_{|x}] \tag{2.6}$$

and

$$0 = \sum_{i=1}^{d} \alpha_i \partial_i (\rho \circ \gamma)_{|x} + \alpha_{d+1} \tag{2.7}$$

hold independently. For $\|\rho\|_{C^0(M)}$ sufficiently small, Equation (2.6) yields $\alpha_1, ..., \alpha_d = 0$ and then $\alpha_{d+1} = 0$ follows with Equation (2.7). So, the claimed linear independency does indeed hold. □

With this preparatory work, we can show that for suitable height functions, the special global parameterizations introduced at the beginning of Section 2.1.6 really yield well-defined evolving hypersurfaces.

Corollary 2.63. *Let $\Sigma = \bar{\theta}(M) \subset \mathbb{R}^{d+1}$ be a $\mathcal{C}^{3+s}$-embedded / immersed closed hypersurface with unit normal ν_Σ and let $T \in (0, \infty)$. Furthermore, let $\rho : [0,T] \times M \to \mathbb{R}$ be a function with $\rho \in \mathcal{C}^{1+r}\big([0,T], \mathcal{C}^s(M)\big) \cap \mathcal{C}^r\big([0,T], \mathcal{C}^{2+s}(M)\big)$ and $\|\rho\|_{C^0([0,T]\times M)}$ sufficiently small. We define*

$$\theta_\rho : [0,T] \times M \to \mathbb{R}^{d+1}, \;\; \theta_\rho(t,p) \coloneqq \bar{\theta}(p) + \rho(t,p) \nu_\Sigma(p).$$

Then, with $\Gamma_\rho(t) \coloneqq \theta_\rho(t, M)$,

$$\Gamma_\rho \coloneqq \Big\{ \{t\} \times \Gamma_\rho(t) \,\Big|\, t \in [0,T] \Big\}$$

is a $\mathcal{C}^{1+r}$- $\mathcal{C}^{2+s}$-evolving embedded / immersed closed hypersurface with reference surface M and global parameterization θ_ρ, called the evolving embedded / immersed hypersurface parameterized via the height function ρ.

Proof. We have $\theta_\rho \in \mathcal{C}^{1+r}\big([0,T], \mathcal{C}^s(M, \mathbb{R}^{d+1})\big) \cap \mathcal{C}^r\big([0,T], \mathcal{C}^{2+s}(M, \mathbb{R}^{d+1})\big)$ by construction and according to Propositions 2.60 and 2.62, $\theta_\rho(t, \cdot) : M \to \mathbb{R}^{d+1}$ is an embedding / immersion for all $t \in [0,T]$. □

In the following final remark, we introduce some basic notation for evolving closed hypersurfaces parameterized via height functions and recap some important regularity properties for later look-up.

Remark 2.64. *Let Γ_ρ be a $\mathcal{C}^{1+r}$- $\mathcal{C}^{2+s}$-evolving immersed closed hypersurface parameterized via a height function ρ as in Corollary 2.63 with reference surface $M \subset \mathbb{R}^{d+1}$ and global parameterization $\theta_\rho : [0,T] \times M \to \mathbb{R}^{d+1}$. Moreover, let $\Sigma = \bar{\theta}(M)$ be the corresponding immersed reference surface with unit normal ν_Σ. We use the notation $\theta_{\rho(t)} \coloneqq \theta_\rho(t,\cdot)$ and $\Gamma_\rho(t) \coloneqq \Gamma_{\rho(t)} \coloneqq \theta_{\rho(t)}(M)$ for all $t \in [0,T]$. Given a sufficiently small local parameterization $(\gamma = \varphi^{-1}, W)$ of M, we define*

$$\varphi_{\rho(t)} \coloneqq \varphi \circ \theta_{\rho(t)}^{-1} \qquad \text{and} \qquad \gamma_{\rho(t)} \coloneqq \theta_{\rho(t)} \circ \gamma$$

as well as $[g_{ij}^{\rho(t)}]_{i,j} \coloneqq [g_{ij}^{\theta_{\rho(t)}}]_{i,j}$ and $[g^{ij}_{\rho(t)}]_{i,j} \coloneqq \big([g_{ij}^{\rho(t)}]_{i,j}\big)^{-1} = [g^{ij}_{\theta_{\rho(t)}}]_{i,j}$ as in Definition 2.29. Due to Proposition 2.50, $\big(\gamma_{\rho(t)} = \varphi_{\rho(t)}^{-1}, W\big)$ is a local parameterization of $\Gamma_{\rho(t)}$ for all $t \in [0,T]$ and we have

$$\gamma_\rho \in \mathcal{C}^{1+r}\big([0,T], \mathcal{C}^s(\overline{W}, \mathbb{R}^{d+1})\big) \cap \mathcal{C}^r\big([0,T], \mathcal{C}^{2+s}(\overline{W}, \mathbb{R}^{d+1})\big).$$

We gather some regularity statements deduced in the sections above and introduce an abbreviatory notation to express the dependence on the height function: We use

$$\nu_\rho \coloneqq \nu_{\Gamma_\rho} \in \mathcal{C}^r\big([0,T], \mathcal{C}^{1+s}(M, \mathbb{R}^{d+1})\big)$$

for the unit normal as in Proposition 2.51,

$$H_\rho \coloneqq H_{\Gamma_\rho} \in \mathcal{C}^r\big([0,T], \mathcal{C}^s(M)\big)$$

for the mean curvature as in Remark 2.56 and

$$V_\rho^{tot} \coloneqq V_{\Gamma_\rho}^{tot} \in \mathcal{C}^r\big([0,T], \mathcal{C}^s(M, \mathbb{R}^{d+1})\big)$$
$$V_\rho \coloneqq V_{\Gamma_\rho} \in \mathcal{C}^r\big([0,T], \mathcal{C}^s(M)\big)$$

for the total and normal velocity as in Remark 2.53. Also, we recap some important surface derivatives from Section 2.1.3 and introduce a similar short notation for the dependence on the height function. For $f \in C^0\big([0,T], C^1(M,\mathbb{R})\big)$ and $F \in C^0\big([0,T], C^1(M, \mathbb{R}^{d+1})\big)$, we define

$$\nabla_\rho f \coloneqq \nabla_{\Gamma_\rho} f \qquad \text{and} \qquad \operatorname{div}_\rho F \coloneqq \operatorname{div}_{\Gamma_\rho} F$$

as well as for $f \in C^0\big([0,T], C^2(M,\mathbb{R})\big)$

$$\Delta_\rho f \coloneqq \Delta_{\Gamma_\rho} f.$$

Their representations with respect to a sufficiently small local parameterization (γ, W) of M are given by

$$\begin{aligned}
\nabla_\rho f \circ \gamma &= \sum_{i,j=1}^{d} g_\rho^{ij}\, \partial_i(f\circ\gamma)\, \partial_j\gamma_\rho, \\
\mathrm{div}_\rho F \circ \gamma &= \sum_{i,j=1}^{d} g_\rho^{ij}\, \partial_i(F\circ\gamma)\cdot \partial_j\gamma_\rho, \\
\Delta_\rho f \circ \gamma &= \sum_{i,j=1}^{d} g_\rho^{ij}\, \partial_i\partial_j(f\circ\gamma) + \sum_{k,l=1}^{d} g_\rho^{ij}\, \partial_i\big(g_\rho^{kl}\partial_l\gamma_\rho\big)\cdot\partial_j\gamma_\rho\,\partial_k(f\circ\gamma).
\end{aligned}$$

From these formulas it is obvious that functions of regularity $f \in \mathcal{C}^\tau\big([0,T], \mathcal{C}^\sigma(M,\mathbb{R})\big)$ and $F \in \mathcal{C}^\tau\big([0,T], \mathcal{C}^\sigma(M,\mathbb{R}^{d+1})\big)$ for $\tau, \sigma \in \mathbb{R}_{\geq 0}$ with $\tau \leq r$, $\sigma \leq 2+s$ and $\sigma \geq 1$ (or even $\sigma \geq 2$ if necessary), fulfill

$$\begin{aligned}
\nabla_\rho f &\in \mathcal{C}^\tau\big([0,T], \mathcal{C}^{\sigma-1}(M,\mathbb{R}^{d+1})\big), \\
\mathrm{div}_\rho F &\in \mathcal{C}^\tau\big([0,T], \mathcal{C}^{\sigma-1}(M,\mathbb{R})\big) \text{ and} \\
\Delta_\rho f &\in \mathcal{C}^\tau\big([0,T], \mathcal{C}^{\sigma-2}(M,\mathbb{R})\big).
\end{aligned}$$

Finally, by Definitions 2.52 and 2.54, the normal time derivative of a function of regularity $f \in \mathcal{C}^{1+\tau}\big([0,T], \mathcal{C}^\sigma(M)\big) \cap \mathcal{C}^\tau\big([0,T], \mathcal{C}^{1+\sigma}(M)\big)$ for $\tau, \sigma \in \mathbb{R}_{\geq 0}$ with $\tau \leq r$ and $\sigma \leq s$ is given by

$$\partial^\square f = \partial_t f - \partial_t\theta_\rho \cdot \nabla_\rho f$$

and thus $\partial^\square f \in \mathcal{C}^\tau\big([0,T], \mathcal{C}^\sigma(M,\mathbb{R})\big)$ holds.

2.2 Hölder Spaces

This section is dedicated to Hölder and little Hölder spaces. In short, they are intermediate spaces between spaces of continuously differentiable functions, where any little Hölder space is that subspace of the corresponding Hölder space in which the smooth functions are dense. We do not give a full introduction to these spaces but only gather the definitions and properties used in this work.

There are two possibilities to define (little) Hölder spaces: The first one is straight-forward using appropriate seminorms (see Sections 2.2.1 and 2.2.2) and the second one is based on interpolation theory (see Section 2.2.3). Both of these approaches will be useful in Section 2.2.4 to deduce properties of those spaces. The last Section 2.2.5 addresses composition operators and analyzes in particular the regularity of operators acting on (little) Hölder spaces by composition with a sufficiently smooth function. As further literature for interpolation theory and its application to (little) Hölder spaces, [Lun12, Chapter 1] is recommended.

In the following, let $(X, \|\cdot\|_X)$ be a Banach space and let $d, n \in \mathbb{N}_{>0}$ be dimensions. We use $k \in \mathbb{N}_{\geq 0}$ to describe the order of differentiability and $\alpha \in (0,1)$ for the so-called Hölder exponent. As before, we sometimes use the short notation $s \in \{k, k+\alpha\} \subset \mathbb{R}_{\geq 0}$ with

$$\lfloor s \rfloor := \max\{l \in \mathbb{N}_{\geq 0} \,|\, l \leq s\} = k.$$

2.2.1 Hölder Seminorms

At first, we introduce two Hölder seminorms. We define them for functions with domain in an arbitrary subset $\Omega \subset \mathbb{R}^d$, such that they can be applied later for functions defined on the closure of open sets as well as on embedded submanifolds.

Definition 2.65 (Hölder Seminorms). *Let $\Omega \subset \mathbb{R}^d$ be an arbitrary subset, let X be a Banach space and let $\alpha \in (0,1)$. For $f : \Omega \to X$, we define the Hölder seminorm*

$$[f]_{C^\alpha(\Omega,X)} \coloneqq \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|f(x)-f(y)\|_X}{|x-y|^\alpha}.$$

Furthermore, for $R \in (0,\infty]$, we set

$$[f]^R_{C^\alpha(\Omega,X)} \coloneqq \sup_{\substack{x,y\in\Omega \\ 0<|x-y|<R}} \frac{\|f(x)-f(y)\|_X}{|x-y|^\alpha}.$$

In the following, we will also need the uniform norm, which we denote by

$$\|f\|_{C^0(\Omega,X)} \coloneqq \sup_{x\in\Omega} \|f(x)\|_X.$$

Now, we gather some basic properties of these seminorms: First, we state how the Hölder seminorms behave for finite products of Banach spaces X_i, for different values of α and for Lipschitz continuous functions. Afterwards, we discuss the composition of functions and the union of sets.

Remark 2.66. *Let $\Omega \subset \mathbb{R}^d$ be an arbitrary set, let $\alpha \in (0,1)$ and let $R \in (0,\infty]$. Furthermore, let $X_1, ..., X_n$ be Banach spaces and define $\widetilde{X} \coloneqq \prod_{i=1}^n X_i$, such that $\|\tilde{x}\|_{\widetilde{X}} \sim \sum_{i=1}^n \|x_i\|_{X_i}$ holds for all $\tilde{x} = (x_i)_{i=1,...,n} \in \widetilde{X}$. Then, for all $f : \Omega \to \widetilde{X}$, we have*

$$[f]^R_{C^\alpha(\Omega,\widetilde{X})} \sim \sum_{i=1}^n [f_i]^R_{C^\alpha(\Omega,X_i)}.$$

Remark 2.67. *Let $\Omega \subset \mathbb{R}^d$ be a bounded subset, let X be a Banach space, let $\alpha, \alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 \leq \alpha_2$ and let $R \in (0,\infty]$. Then, for all $f : \Omega \to X$, we have*

$$[f]^R_{C^{\alpha_1}(\Omega,X)} \leq [f]^R_{C^{\alpha_2}(\Omega,X)} C(R)^{\alpha_2-\alpha_1},$$

where $C(R) = \min\{R, 2|\Omega|\} \in (0,\infty)$ fulfills $\lim_{R\to 0} C(R) = 0$, as well as

$$[f]^R_{C^{\alpha_1}(\Omega,X)} \leq \left([f]^R_{C^{\alpha_2}(\Omega,X)}\right)^{\frac{\alpha_1}{\alpha_2}} \left(2\|f\|_{C^0(\Omega,X)}\right)^{1-\frac{\alpha_1}{\alpha_2}}.$$

Moreover, for all Lipschitz continuous $f : \Omega \to X$ with Lipschitz constant L,

$$[f]^R_{C^\alpha(\Omega,X)} \leq L\, C(R)^{1-\alpha}$$

holds where we choose again $C(R) = \min\{R, 2|\Omega|\}$.

Lemma 2.68 (Composition with a Lipschitz continuous Function)**.** *Let $\Omega \subset \mathbb{R}^d$ be an arbitrary subset, let X be a Banach space and let $\alpha \in (0,1)$. Furthermore, let $d' \in \mathbb{N}_{>0}$ be another dimension and let $\varphi : \Omega \to \varphi(\Omega) \subset \mathbb{R}^{d'}$ be Lipschitz continuous, i.e., there exists a constant $L \geq 0$ with*

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|\varphi(x) - \varphi(y)|}{|x-y|} \leq L.$$

Then, there exists a constant $C = C(L) \geq 0$ such that for all $f : \varphi(\Omega) \to X$ and $R \in (0,\infty]$

$$[f \circ \varphi]^R_{C^\alpha(\Omega,X)} \leq C[f]^{LR}_{C^\alpha(\varphi(\Omega),X)}$$

holds.

Proof. If φ is constant, the claim is clear. So, assume that φ is not constant in the following. Then, by definition of the supremum,

$$[f \circ \varphi]^R_{C^\alpha(\Omega,X)} = \sup_{\substack{x,y\in\Omega \\ 0<|x-y|<R}} \frac{\left\| f\big(\varphi(x)\big) - f\big(\varphi(y)\big) \right\|_X}{|x-y|^\alpha} = \sup_{\substack{x,y\in\Omega \\ 0<|x-y|<R \\ \varphi(x)\neq\varphi(y)}} \frac{\left\| f\big(\varphi(x)\big) - f\big(\varphi(y)\big) \right\|_X}{|x-y|^\alpha}$$

holds. For any $x, y \in \Omega$ with $|x-y| < R$, we have $|\varphi(x) - \varphi(y)| < LR$. Therefore,

$$\begin{aligned}
[f \circ \varphi]^R_{C^\alpha(\Omega,X)} &= \sup_{\substack{x,y\in\Omega \\ 0<|x-y|<R \\ \varphi(x)\neq\varphi(y)}} \frac{\left\| f\big(\varphi(x)\big) - f\big(\varphi(y)\big) \right\|_X}{|x-y|^\alpha} \\
&\leq \sup_{\substack{x,y\in\Omega \\ 0<|\varphi(x)-\varphi(y)|<LR}} \frac{\left\| f\big(\varphi(x)\big) - f\big(\varphi(y)\big) \right\|_X}{|x-y|^\alpha} \\
&= \sup_{\substack{x,y\in\Omega \\ 0<|\varphi(x)-\varphi(y)|<LR}} \frac{\left\| f\big(\varphi(x)\big) - f\big(\varphi(y)\big) \right\|_X}{|\varphi(x)-\varphi(y)|^\alpha} \left(\frac{|\varphi(x)-\varphi(y)|}{|x-y|} \right)^\alpha \\
&\leq L^\alpha \sup_{\substack{u,v\in\varphi(\Omega) \\ 0<|u-v|<LR}} \frac{\|f(u) - f(v)\|_X}{|u-v|^\alpha} = L^\alpha [f]^{LR}_{C^\alpha(\varphi(\Omega),X)}
\end{aligned}$$

follows. □

Later on, Lemma 2.68 will be used to prove the fact that the composition $f \circ \varphi$ of a Hölder regular function f and a Hölder regular and globally Lipschitz continuous function φ is Hölder regular again.

We will call a function Hölder regular, if its Hölder seminorm is bounded. In particular, we thus impose a uniform condition for Hölder regularity and therefore a function being Hölder regular on two different sets not necessarily has to be Hölder regular on the union of these sets. But the following Lemma 2.69 states that this problem can be solved if the sets overlap in a suitable way: Then, the Hölder seminorm on their union can be controlled by the Hölder seminorms on the separated sets. As we will see later, this implies in particular that defining Hölder regularity on embedded submanifolds via local coordinates works smoothly in spite of the uniform condition for Hölder regularity.

Lemma 2.69 (Union of Sets)**.** *Let X be a Banach space, let $\alpha \in (0,1)$ and let $L \in \mathbb{N}$. Furthermore, let $M \subset \mathbb{R}^d$ be a compact set, let $A_l \subset M$ be compact and $U_l \subset M$ be open subsets with $A_l \subset U_l$ for all $l \in \{1, ..., L\}$. There exists a constant $C = C(A_l, U_l) > 0$, such that for all $f : \bigcup_{l=1}^{L} U_l \to X$*

$$[f]_{C^\alpha(\bigcup_{l=1}^{L} A_l, X)} \leq C \sum_{l=1}^{L} \Big(\|f\|_{C^0(U_l,X)} + [f]_{C^\alpha(U_l,X)} \Big)$$

holds. In addition, for sufficiently small $R \in (0, \infty]$, we have

$$[f]^R_{C^\alpha(\bigcup_{l=1}^{L} A_l, X)} \leq \sum_{l=1}^{L} [f]^R_{C^\alpha(U_l,X)}.$$

Proof. We only show the claim for $L = 2$; the general statement then follows by mathematical induction: Because we can always choose an open subset $\widetilde{U_l} \subset M$ and a compact subset $\widetilde{A_l} \subset M$ with $A_l \subset \widetilde{U_l} \subset \widetilde{A_l} \subset U_l$,

$$\begin{aligned}
[f]_{C^\alpha(\bigcup_{l=1}^{L+1} A_l, X)} &= [f]_{C^\alpha(\bigcup_{l=1}^{L-1} A_l \cup (A_L \cup A_{L+1}), X)} \\
&\leq C \left(\sum_{l=1}^{L+1} \|f\|_{C^0(U_l,X)} + \sum_{l=1}^{L-1} [f]_{C^\alpha(U_l,X)} + [f]_{C^\alpha(\widetilde{U_L} \cup \widetilde{U_{L+1}}, X)} \right)
\end{aligned}$$

with

$$[f]_{C^\alpha(\widetilde{U_L} \cup \widetilde{U_{L+1}}, X)} \leq [f]_{C^\alpha(\widetilde{A_L} \cup \widetilde{A_{L+1}}, X)} \leq C \left(\sum_{l=1}^{L+1} \|f\|_{C^0(U_l,X)} + [f]_{C^\alpha(U_L,X)} + [f]_{C^\alpha(U_{L+1},X)} \right)$$

holds. So, we restrict to $L = 2$ in the following. For $x, y \in A_1 \cup A_2$, we only need to differ three different cases:

(i) $x, y \in U_1$,

(ii) $x, y \in U_2$,

(iii) $x \in A_1 \setminus U_2$ and $y \in A_2 \setminus U_1$ or vice versa.

This is due to the following considerations: W.l.o.g. we can assume $x \in A_1$. If $x \in A_1 \cap U_2$ and $y \in A_1$, we have case (i) and if $y \in A_2$ we have case (ii). If $x \in A_1 \setminus U_2$ and $y \in A_2 \setminus U_1$, we have case (iii). Otherwise, if $y \notin A_2 \setminus U_1$, then $y \in U_1$ or $y \notin A_2$ and thus $y \in A_1 \subset U_1$ holds and hence we have case (i).
Due to

$$\big(A_1 \setminus U_2\big) \cap \big(A_2 \setminus U_1\big) \subset U_1 \cap U_1^C = \varnothing,$$

we have

$$|x-y| \neq 0 \quad \text{for all } x \in A_1 \setminus U_2 \text{ and } y \in A_2 \setminus U_1.$$

As $A_i \setminus U_j \subset M$ and $M \subset \mathbb{R}^d$ are compact sets, also $A_i \setminus U_j \subset \mathbb{R}^d$ is a compact set and thus

$$I \coloneqq \inf\left\{|x-y| \,\middle|\, x \in A_1 \setminus U_2 \text{ and } y \in A_2 \setminus U_1\right\} \in (0,\infty)$$

follows. With

$$C(R) \coloneqq \begin{cases} I^{-\alpha} & \text{if } R > I, \\ -\infty & \text{if } R \leq I, \end{cases}$$

any $f : U_1 \cup U_2 \to X$ fulfills

$$\begin{aligned} \sup_{\substack{x \in A_1 \setminus U_2,\, y \in A_2 \setminus U_1 \\ 0<|x-y|<R}} \frac{\|f(x)-f(y)\|_X}{|x-y|^\alpha} &\leq \sup_{\substack{x \in A_1 \setminus U_2,\, y \in A_2 \setminus U_1 \\ 0<|x-y|<R}} \frac{1}{|x-y|^\alpha}\Big(\|f\|_{C^0(U_1,X)} + \|f\|_{C^0(U_2,X)}\Big) \\ &\leq C(R)\Big(\|f\|_{C^0(U_1,X)} + \|f\|_{C^0(U_2,X)}\Big). \end{aligned}$$

These preliminary considerations lead to

$$\begin{aligned} [f]^R_{C^\alpha(A_1 \cup A_2,X)} &= \sup_{\substack{x,y \in A_1 \cup A_2 \\ 0<|x-y|<R}} \frac{\|f(x)-f(y)\|_X}{|x-y|^\alpha} \\ &= \max\left\{ \sup_{\substack{x,y \in U_1 \\ 0<|x-y|<R}} \frac{\|f(x)-f(y)\|_X}{|x-y|^\alpha}, \sup_{\substack{x,y \in U_2 \\ 0<|x-y|<R}} \frac{\|f(x)-f(y)\|_X}{|x-y|^\alpha}, \right. \\ &\qquad\qquad \left. \sup_{\substack{x \in A_1 \setminus U_2,\, y \in A_2 \setminus U_1 \\ 0<|x-y|<R}} \frac{\|f(x)-f(y)\|_X}{|x-y|^\alpha} \right\} \\ &= \max\left\{[f]^R_{C^\alpha(U_1,X)}, [f]^R_{C^\alpha(U_2,X)}, C(R)\Big(\|f\|_{C^0(U_1,X)} + \|f\|_{C^0(U_2,X)}\Big)\right\}. \end{aligned}$$

Therefore, with a constant $C \coloneqq \max\{1, I^{-\alpha}\} > 0$ depending only on A_1, A_2, U_1 and U_2, we have

$$[f]_{C^\alpha(A_1 \cup A_2,X)} \leq C\Big(\Big(\|f\|_{C^0(U_1,X)} + [f]_{C^\alpha(U_1,X)}\Big) + \Big(\|f\|_{C^0(U_2,X)} + [f]_{C^\alpha(U_2,X)}\Big)\Big)$$

and for sufficiently small $R > 0$,

$$[f]^R_{C^\alpha(A_1 \cup A_2,X)} \leq [f]^R_{C^\alpha(U_1,X)} + [f]^R_{C^\alpha(U_2,X)}$$

holds. □

Lemma 2.69 also holds for $M = \mathbb{R}^d$, but as we will need the statement later on to define the Hölder spaces on embedded closed manifolds M, we formulated it for compact sets M.

2.2.2 Definition of the Hölder Spaces

In the section above, we introduced the Hölder seminorms and collected some of its basic properties. With this preparatory work, we can now move on to the concrete defintion of Hölder and little Hölder spaces. We start by defining them on the closure $\overline{W}$ of an open subset $W \subset \mathbb{R}^d$. In a second step, we transfer the definition to embedded submanifolds.

Definition on the Closure of Open Sets

At first, we introduce our notation for continuous and continuously differentiable functions.

Definition 2.70 (Continuous and Differentiable Functions)**.** *Let X, Y be Banach spaces, let $U \subset Y$ be an open subset and let $k \in \mathbb{N}_{>0}$. With*

$$C^0(U, X) \quad \text{and} \quad C^k(U, X),$$

we denote, respectively, the continuous and k-times continuously Fréchet-differentiable functions $f : U \to X$. Moreover, we use

$$C_b^0(U, X) \quad \text{and} \quad C_b^k(U, X)$$

to describe functions in $C^0(U, X)$ or $C^k(U, X)$, respectively, that are bounded and whose Fréchet-derivatives up to order k are bounded as functions on U.

Now, we extend this definition to the closure $\overline{W}$ of an open subset $W \subset \mathbb{R}^d$.

Definition/Lemma 2.71 (Cont. and Diff. Functions on the Closure of Open Sets)**.** *Let $W \subset \mathbb{R}^d$ be an open subset and let X be a Banach space. We define the Banach space*

$$C_b^0(\overline{W}, X) \coloneqq \left\{ f : \overline{W} \to X \text{ continuous} \,\middle|\, \|f\|_{C^0(\overline{W}, X)} < \infty \right\}$$

of all continuous and bounded functions on $\overline{W}$, which is endowed with the uniform norm

$$\|f\|_{C^0(\overline{W}, X)} \coloneqq \sup_{x \in \overline{W}} \|f(x)\|_X.$$

For $k \in \mathbb{N}_{>0}$,

$$C_b^k(\overline{W}, X) \coloneqq \left\{ f \in C^k(W, X) \,\middle|\, \forall |\beta| \le k : D^\beta f \in C_b^0(\overline{W}, X) \right\}$$

denotes the set of all k times continuously Fréchet-differentiable functions on W, whose derivatives up to the order k are bounded and continuously extendable onto $\overline{W}$. Endowed with the norm

$$\|f\|_{C^k(\overline{W}, X)} \coloneqq \sum_{|\beta| \le k} \left\| D^\beta f \right\|_{C^0(\overline{W}, X)},$$

it forms a Banach space. If W is bounded, we omit the index b.

With this, we can formulate the definition of Hölder and little Hölder spaces, which are used throughout this whole thesis.

Definition/Lemma 2.72 (Hölder Spaces on the Closure of Open Sets)**.** *Let $W \subset \mathbb{R}^d$ be an open subset and let X be a Banach space. For $\alpha \in (0,1)$, we define the Hölder space*

$$C^\alpha(\overline{W}, X) := \left\{ f \in C_b^0(\overline{W}, X) \,\middle|\, [f]_{C^\alpha(\overline{W},X)} < \infty \right\}$$

and the little Hölder space

$$h^\alpha(\overline{W}, X) := \left\{ f \in C^\alpha(\overline{W}, X) \,\middle|\, \lim_{R\to 0} [f]^R_{C^\alpha(\overline{W},X)} = 0 \right\}.$$

Together with the norm

$$\|f\|_{C^\alpha(\overline{W},X)} := \|f\|_{C^0(\overline{W},X)} + [f]_{C^\alpha(\overline{W},X)}$$

both the space $C^\alpha(\overline{W}, X)$ and its subspace $h^\alpha(\overline{W}, X)$ are Banach spaces. For $k \in \mathbb{N}_{>0}$, we define (little) Hölder spaces of higher order as

$$C^{k+\alpha}(\overline{W}, X) := \left\{ f \in C_b^k(\overline{W}, X) \,\middle|\, \forall |\beta| = k : D^\beta f \in C^\alpha(\overline{W}, X) \right\},$$
$$h^{k+\alpha}(\overline{W}, X) := \left\{ f \in C^{k+\alpha}(\overline{W}, X) \,\middle|\, \forall |\beta| = k : D^\beta f \in h^\alpha(\overline{W}, X) \right\}$$

both endowed with the norm

$$\|f\|_{C^{k+\alpha}(\overline{W},X)} := \|f\|_{C^k(\overline{W},X)} + \sum_{|\beta|=k} \left[D^\beta f\right]_{C^\alpha(\overline{W},X)}.$$

Note, that we assume a Hölder regular function to fulfill not only a local, but a uniform Hölder condition!
Because $\left(C_b^k(\overline{W}, X), \|\cdot\|_{C^k(\overline{W},X)}\right)$ is a Banach space, proving that the spaces $C^{k+\alpha}(\overline{W}, X)$ and $h^{k+\alpha}(\overline{W}, X)$ are complete reduces to showing that they are closed as subsets of $C_b^k(\overline{W}, X)$ with respect to the norm $\|\cdot\|_{C^{k+\alpha}(\overline{W},X)}$. This proof is straight-forward (see [Alt16, Section 3.7]).

Remark 2.73. *(i) For $s \in \mathbb{R}_{\geq 0}$, we use the short notation*

$$\mathcal{C}_b^s(\overline{W}, X)$$

as in Section 2.1, meaning $C_b^s(\overline{W}, X)$ if $s \in \mathbb{N}_{\geq 0}$ and $C^s(\overline{W}, X)$ or $h^s(\overline{W}, X)$ else. If $X = \mathbb{R}$, we use the abbreviation

$$\mathcal{C}_b^s(\overline{W}) := \mathcal{C}_b^s(\overline{W}, \mathbb{R}).$$

If W is bounded, we omit the index b.

(ii) On account of Remark 2.66, we have

$$f \in \mathcal{C}_b^s(\overline{W}, \mathbb{R}^n) \Leftrightarrow f_i \in \mathcal{C}_b^s(\overline{W}, \mathbb{R}) \text{ for all } i = 1, ..., n$$

with $\|f\|_{C^s(\overline{W},\mathbb{R}^n)} \sim \sum_{i=1}^n \|f_i\|_{C^s(\overline{W},\mathbb{R})}$.

Definition on embedded Submanifolds

For any $k \in \mathbb{N}_{\geq 0}$ and $\alpha \in (0,1)$ used in the following, let $M \subset \mathbb{R}^n$ be a d-dimensional $C^1 \cap C^k$- or $C^1 \cap \mathcal{C}^{k+\alpha}$-embedded submanifold, respectively, as defined in Section 2.1.1. In particular, if $d \neq n$, M is not the closure of an open subset of $\mathbb{R}^n$. We will define Hölder functions on M with the help of local parameterizations. If M is closed, we choose a finite set $(\gamma_l, W_l)_{l=1,\dots,L}$ of local parameterizations of M such that there exists a set of charts $(\phi_l, U_l)_{l=1,\dots,L}$ with $\gamma_l(W_l) \subset U_l$ and $\mathrm{pr}_{\mathbb{R}^d} \circ \phi_{l|\gamma_l(\overline{W_l})} = \gamma_l^{-1}$ and a set of compact subsets $A_l \subset \gamma_l(W_l)$ with $M \subset \bigcup_l A_l$, which is possible due to Remark 2.6(ii).

Definition/Lemma 2.74 (Continuous Functions on Submanifolds). *Let X be a Banach space. We define the set*

$$C^0(M,X) \coloneqq \{f : M \to X \ \textit{continuous}\}$$

of all continuous functions on M. If M is closed, any continuous function $f : M \to X$ is bounded and thus $C^0(M,X)$ can be endowed with the uniform norm

$$\|f\|_{C^0(M,X)} \coloneqq \sup_{p \in M} \|f(p)\|_X$$

to form a Banach space.

Because continuity of a function on two different sets implies continuity on the union of these sets,

$$\begin{aligned} C^0(M,X) &= \Big\{ f : M \to X \,\Big|\, \text{for every } p \in M \text{ there exists a local parameterization } (\gamma_p, W_p) \\ &\qquad\qquad \text{with } p \in \gamma_p(W_p) \text{ and } f \circ \gamma_p \in C^0\big(\overline{W_p}, X\big) \Big\} \\ &= \Big\{ f : M \to X \,\Big|\, f \circ \gamma \in C^0\big(\overline{W}, X\big) \text{ for all local parameterizations } (\gamma, W) \Big\} \end{aligned}$$

holds. If M is closed, we can reduce to the finite set $(\gamma_l, W_l)_{l=1,\dots,L}$ of local parameterizations and obtain

$$C^0(M,X) = \Big\{ f : M \to X \,\Big|\, f \circ \gamma_l \in C^0\big(\overline{W_l}, X\big) \text{ for all } l = 1,\dots,L \Big\}$$

endowed with the equivalent norms

$$\|\cdot\|_{C^0(M,X)} \sim \sum_{l=1}^{L} \|\cdot \circ \gamma_l\|_{C^0(\overline{W_l},X)}.$$

Following this equivalent formulation in local coordinates, we introduce the definition for differentiability and Hölder regularity on embedded submanifolds (cf. Definition 2.15).

Definition/Lemma 2.75 (Hölder Spaces on Submanifolds). *Let X be a Banach space, let $k \in \mathbb{N}_{\geq 0}$ and let $\alpha \in (0,1)$. We define the sets*

$$\begin{aligned} C^k(M,X) &\coloneqq \Big\{ f : M \to X \,\Big|\, \forall p \in M : \exists (\gamma_p, W_p) \ \textit{with } f \circ \gamma_p \in C^k\big(\overline{W_p}, X\big) \Big\} \ \textit{for } k \neq 0, \\ C^{k+\alpha}(M,X) &\coloneqq \Big\{ f : M \to X \,\Big|\, \forall p \in M : \exists (\gamma_p, W_p) \ \textit{with } f \circ \gamma_p \in C^{k+\alpha}\big(\overline{W_p}, X\big) \Big\} \ \textit{and} \\ h^{k+\alpha}(M,X) &\coloneqq \Big\{ f : M \to X \,\Big|\, \forall p \in M : \exists (\gamma_p, W_p) \ \textit{with } f \circ \gamma_p \in h^{k+\alpha}\big(\overline{W_p}, X\big) \Big\}. \end{aligned}$$

If M is closed, they can be endowed with the norms

$$\|f\|_{C^k(M,X)} := \sum_{l=1}^{L} \|f \circ \gamma_l\|_{C^k(\overline{W_l},X)} \text{ for } k \neq 0 \text{ or}$$
$$\|f\|_{C^{k+\alpha}(M,X)} := \sum_{l=1}^{L} \|f \circ \gamma_l\|_{C^{k+\alpha}(\overline{W_l},X)},$$

respectively, to form the Banach spaces of differentiable functions and (little) Hölder functions on the embedded closed submanifold M.

If M is closed, the completeness of $C^k(\overline{W_l},X)$, $C^{k+\alpha}(\overline{W_l},X)$ and $h^{k+\alpha}(\overline{W_l},X)$ by Lemma 2.72 transfers directly to the functions spaces with domain M, such that $C^k(M,X)$, $C^{k+\alpha}(M,X)$ and $h^{k+\alpha}(M,X)$ indeed are Banach spaces with the indicated norms.

Remark 2.76. *(i) As usual, for $s \in \mathbb{R}_{\geq 0}$, we use the short notation*

$$\mathcal{C}^s(M,X)$$

for either $C^s(M,X)$ or $h^s(M,X)$. If $X = \mathbb{R}$, we use again the abbreviation

$$\mathcal{C}^s(M) := \mathcal{C}^s(M,\mathbb{R}).$$

(ii) On account of Remark 2.73(ii), we have

$$f \in \mathcal{C}^s(M,\mathbb{R}^n) \Leftrightarrow f_i \in \mathcal{C}^s(M,\mathbb{R}) \text{ for all } i = 1,...,n$$

with $\|f\|_{\mathcal{C}^s(M,\mathbb{R}^n)} \sim \sum_{i=1}^n \|f_i\|_{\mathcal{C}^s(M,\mathbb{R})}$.

(iii) The definition of Hölder regularity on non-closed submanifolds is cumbersome. But if M is closed or $s \in \mathbb{N}_{\geq 0}$ holds, we obtain the simple characterization

$$\mathcal{C}^s(M,X) := \Big\{f : M \to X \,\Big|\, f \circ \gamma \in \mathcal{C}^s\big(\overline{W},X\big) \text{ for all } (\gamma,W)\Big\}$$

as in the case of continuous functions. We give a short sketch of the proof here and refer to Lemma A.14 in the Appendix for a detailed explanation.
Let $f \in \mathcal{C}^s(M,X)$ and let (γ,W) be an arbitrary local parameterization of M. In particular, for every $p \in M$, there exists a local parameterization (γ_p,W_p) of M around p with $f \circ \gamma_p \in \mathcal{C}^s(\overline{W_p},X)$. We define

$$V_p := \gamma^{-1}\big(\gamma(W) \cap \gamma_p(W_p)\big) \text{ for all } p \in M$$

and can show

$$f \circ \gamma = (f \circ \gamma_p) \circ (\gamma_p^{-1} \circ \gamma) \in \mathcal{C}^s(\overline{V_p},X) \text{ for all } p \in M,$$

because $\gamma_p^{-1} \circ \gamma \in \mathcal{C}^s(\overline{V_p},\mathbb{R}^d)$ follows from representing $\gamma_p^{-1} = pr_{\mathbb{R}^d} \circ \phi_{p|\gamma_p(\overline{W_p})}$ with a suitable chart $\phi_p \in \mathcal{C}^s(\overline{U_p},\mathbb{R}^n)$ as in Remark 2.6(ii). The justification for this is clear if $s \in \mathbb{N}_{\geq 0}$ and for general s and closed M, Lemma 2.68 can be used. As

$M = \bigcup_p \gamma_p(W_p)$ implies $\overline{W} = \bigcup_p \overline{V_p}$, the claim $f \circ \gamma \in C^s(\overline{W}, X)$ follows obviously for $s \in \mathbb{N}_{\geq 0}$. In contrast to differentiability, Hölder regularity does not transfer directly to the union $\overline{W}$. But as M is closed, we can use Lemma 2.69 to circumvent this problem and finally receive

$$f \circ \gamma \in \mathcal{C}^s(\overline{W}, X).$$

In the following lemma, we state some intuitive representations for Hölder functions on embedded submanifolds.

Lemma 2.77. *Let M be closed, let X be a Banach space, let $k \in \mathbb{N}_{\geq 0}$ and let $\alpha \in (0,1)$.*

(i) We have

$$C^{k+\alpha}(M, X) = \left\{ f \in C^k(M, X) \,\middle|\, \left[D^\beta(f \circ \gamma_l)\right]_{C^\alpha(\overline{W_l}, X)} < \infty \text{ for all } |\beta| = k \text{ and } l \right\},$$

$$h^{k+\alpha}(M, X) = \left\{ f \in C^{k+\alpha}(M, X) \,\middle|\, \lim_{R \to 0} \left[D^\beta(f \circ \gamma_l)\right]^R_{C^\alpha(\overline{W_l}, X)} = 0 \text{ for all } |\beta| = k \text{ and } l \right\}$$

with

$$\|\cdot\|_{C^{k,\alpha}(M,X)} = \|\cdot\|_{C^k(M,X)} + \sum_{l=1}^{L} \sum_{|\beta|=k} \left[D^\beta(\cdot \circ \gamma_l)\right]_{C^\alpha(\overline{W_l}, X)}.$$

(ii) We have

$$C^\alpha(M, X) = \left\{ f \in C^0(M, X) \,\middle|\, [f]_{C^\alpha(M,X)} < \infty \right\} \text{ and}$$
$$h^\alpha(M, X) = \left\{ f \in C^\alpha(M, X) \,\middle|\, \lim_{R \to 0} [f]^R_{C^\alpha(M,X)} = 0 \right\}.$$

Furthermore,

$$\|\cdot\|_{C^\alpha(M,X)} \sim \|\cdot\|_{C^0(M,X)} + [\cdot]_{C^\alpha(M,X)},$$

are equivalent norms on $\mathcal{C}^\alpha(M, X)$ and for sufficiently small $R \in (0, \infty]$

$$\sum_l [\cdot \circ \gamma_l]^{\sim R}_{C^\alpha(\overline{W_l}, X)} \sim [\cdot]^R_{C^\alpha(M,X)}$$

holds.

(iii) We have

$$C^1(M, \mathbb{R}) = \left\{ f \in C^0(M, \mathbb{R}) \,\middle|\, \nabla_M f \in C^0\big(M, \mathbb{R}^n\big) \right\},$$
$$C^{1+\alpha}(M, \mathbb{R}) = \left\{ f \in C^1(M, \mathbb{R}) \,\middle|\, [\nabla_M f]_{C^\alpha(M,\mathbb{R}^n)} < \infty \right\} \text{ as well as}$$
$$h^{1+\alpha}(M, \mathbb{R}) = \left\{ f \in C^{1+\alpha}(M, \mathbb{R}) \,\middle|\, \lim_{R \to 0} [\nabla_M f]^R_{C^\alpha(M,\mathbb{R}^n)} = 0 \right\}$$

and

$$\|\cdot\|_{C^1(M,\mathbb{R})} \sim \|\cdot\|_{C^0(M,\mathbb{R})} + \|\nabla_M \cdot\|_{C^0(M,\mathbb{R}^n)} \text{ as well as}$$
$$\|\cdot\|_{C^{1+\alpha}(M,\mathbb{R})} \sim \|\cdot\|_{C^1(M,\mathbb{R})} + [\nabla_M \cdot]_{C^\alpha(M,\mathbb{R}^n)}$$

are equivalent norms on $C^1(M, \mathbb{R})$ or $\mathcal{C}^{1+\alpha}(M, \mathbb{R})$, respectively.

(iv) For $k \geq 2$, we have

$$C^2(M,\mathbb{R}) = \left\{ f \in C^1(M,\mathbb{R}) \,\middle|\, D_M^2 f \in C^0\big(M,\mathbb{R}^{n\times n}\big) \right\},$$
$$C^{2+\alpha}(M,\mathbb{R}) = \left\{ f \in C^2(M,\mathbb{R}) \,\middle|\, \big[D_M^2 f\big]_{C^\alpha(M,\mathbb{R}^{n\times n})} < \infty \right\} \text{ as well as}$$
$$h^{2+\alpha}(M,\mathbb{R}) = \left\{ f \in C^{2+\alpha}(M,\mathbb{R}) \,\middle|\, \lim_{R\to 0} \big[D_M^2 f\big]^R_{C^\alpha(M,\mathbb{R}^{n\times n})} = 0 \right\}$$

and

$$\|\cdot\|_{C^2(M,\mathbb{R})} \sim \|\cdot\|_{C^1(M,\mathbb{R})} + \|D_M^2\cdot\|_{C^0(M,\mathbb{R}^{n\times n})} \text{ as well as}$$
$$\|\cdot\|_{C^{2+\alpha}(M,\mathbb{R})} \sim \|\cdot\|_{C^2(M,\mathbb{R})} + \big[D_M^2\cdot\big]_{C^\alpha(M,\mathbb{R}^{n\times n})}$$

are equivalent norms on $C^2(M,\mathbb{R})$ or $\mathcal{C}^{2+\alpha}(M,\mathbb{R})$, respectively.

Proof.

Ad (i) Definition 2.75 and Remark 2.76(iii) directly imply the statement.

Ad (ii) The claim follows with Lemmas 2.68 and 2.69 and is performed in detail in the Appendix (see Lemma A.17).

Ad (iii) As M is a $\mathcal{C}^{1+\alpha}$-embedded submanifold, $\partial_i\gamma_l \in \mathcal{C}^\alpha(\overline{W_l},\mathbb{R}^n)$ and $g_l^{ij} \in \mathcal{C}^\alpha(\overline{W_l},\mathbb{R})$ hold. Due to

$$\nabla_M f\circ\gamma_l = \sum_{i,j} g_l^{ij}\partial_i(f\circ\gamma_l)\partial_j\gamma_l \qquad \text{and} \qquad \partial_i(f\circ\gamma_l) = (\nabla_M f\circ\gamma_l)\cdot\partial_i\gamma_l,$$

the claim follows with the fact that Hölder spaces are algebras (see Proposition 2.94) and the statement in (ii).

Ad (iv) As M is a $\mathcal{C}^{2+\alpha}$-embedded submanifold, $\partial_i\gamma_l \in \mathcal{C}^{1+\alpha}(\overline{W_l},\mathbb{R}^n)$ and $g_l^{ij} \in \mathcal{C}^{1+\alpha}(\overline{W_l},\mathbb{R})$ hold. Due to

$$[D_M^2 f\circ\gamma_l]_{rs} = \sum_{i,j} g_l^{ij}\partial_i(\nabla_M f\circ\gamma_l)\cdot e_r\,\partial_j\gamma_l\cdot e_s \text{ and } \partial_i(\nabla_M f\circ\gamma_l) = [D_M^2 f\circ\gamma_l]\cdot\partial_i\gamma_l,$$

the claim follows again with the fact that Hölder spaces are algebras (see Proposition 2.94) and the statement in (ii). □

2.2.3 Hölder spaces as Interpolation Spaces

If $W \subset \mathbb{R}^d$ is an open, bounded and convex subset, every $f \in C^1(\overline{W},X)$ is Lipschitz continuous on $\overline{W}$ with Lipschitz constant $L \leq \|f\|_{C^1(\overline{W},X)}$. Thus, by Remark 2.67, the Hölder space $C^\alpha(\overline{W},X)$ and the little Hölder space $h^\alpha(\overline{W},X)$ are *intermediate spaces* between the spaces $C^0(\overline{W},X)$ and $C^1(\overline{W},X)$ of continuous and continuously differentiable funtions, meaning

$$C^1(\overline{W},X) \hookrightarrow \mathcal{C}^\alpha(\overline{W},X) \hookrightarrow C^0(\overline{W},X),$$

where the symbol $\hookrightarrow$ denotes a continuous embedding. To analyze this more precisely, we make use of interpolation theory. It turns out that Hölder and little Hölder spaces of all orders are real interpolation spaces between the spaces of continuous and m-times continuously differentiable functions for m sufficiently large (see Lemma 2.86).

Short Introduction to Interpolation Theory

First, we list a few basic statements on interpolation theory. In particular, we define the so-called K-method for introducing real interpolation spaces and formulate the reiteration theorem which is used later on to analyze interpolation spaces of (little) Hölder spaces. For a full introduction to interpolation theory, we refer to [Lun12, Chapter 1] and the listed literature therein.
In the following, let X and Y be Banach spaces with $X \hookrightarrow Y$, let $\theta \in [0,1]$ and let $p \in [1,\infty]$. We are interested in *intermediate spaces* E with $X \hookrightarrow E \hookrightarrow Y$.

Definition 2.78. *Let E be a Banach space with $X \hookrightarrow E \hookrightarrow Y$. If there exists a constant $C > 0$ with*

$$\|x\|_E \le C\|x\|_Y^{1-\theta}\|x\|_X^\theta$$

for all $x \in X$, then we write $E \in J_\theta(Y,X)$.

Lemma 2.79. *Let $E \in J_\theta(Y,X)$, let $\alpha, \alpha_Y, \alpha_X \in [0,1]$ with $\alpha = (1-\theta)\alpha_Y + \theta\alpha_X$ and let $I \subset \mathbb{R}$ be a closed interval. There exists a constant $C > 0$ with*

$$\begin{aligned} &\|u\|_{C^\alpha(I,E)} \le C\|u\|_{C^{\alpha_Y}(I,Y)}^{1-\theta}\|u\|_{C^{\alpha_X}(I,X)}^\theta \quad \textit{and} \\ &[u]_{C^\alpha(I,E)}^R \le C\big([u]_{C^{\alpha_Y}(I,Y)}^R\big)^{1-\theta}\big([u]_{C^{\alpha_X}(I,X)}^R\big)^\theta \end{aligned}$$

for all $u \in C^{\alpha_Y}(I,Y) \cap C^{\alpha_X}(I,X)$ and $R \in (0,\infty]$, where we set $[u]_{C^0(I,Z)}^R \coloneqq 2\|u\|_{C^0(I,Z)}$ and $[u]_{C^1(I,Z)}^R \coloneqq \|u\|_{C^1(I,Z)}$ for any Banach space Z.

Proof. For $u \in C^{\alpha_Y}(I,Y) \cap C^{\alpha_X}(I,X)$,

$$\|u\|_{C^0(I,E)} = \sup_{t\in I}\|u(t)\|_E \lesssim \sup_{t\in I}\|u(t)\|_Y^{1-\theta}\|u(t)\|_X^\theta \le \|u\|_{C^0(I,Y)}^{1-\theta}\|u\|_{C^0(I,X)}^\theta$$

and

$$\begin{aligned} [u]_{C^\alpha(I,E)}^R &= \sup_{\substack{t,s\in I\\ 0<|t-s|<R}} \frac{\|u(t)-u(s)\|_E}{|t-s|^\alpha} \\ &\lesssim \sup_{\substack{t,s\in I\\ 0<|t-s|<R}} \frac{\|u(t)-u(s)\|_Y^{1-\theta}\|u(t)-u(s)\|_X^\theta}{|t-s|^{\alpha_Y(1-\theta)+\alpha_X\theta}} \le \big([u]_{C^{\alpha_Y}(I,Y)}^R\big)^{1-\theta}\big([u]_{C^{\alpha_X}(I,X)}^R\big)^\theta \end{aligned}$$

hold, which proves the claim. □

Definition 2.80. *For $y \in Y$ and $t > 0$, we define*

$$K(y,t,Y,X) \coloneqq \inf_{\substack{y=a+b,\\ a\in Y, b\in X}} \big\{\|a\|_Y + t\|b\|_X\big\}.$$

Let E be a Banach space with $X \hookrightarrow E \hookrightarrow Y$. If there exists a constant $C > 0$ with

$$K(y,t,Y,X) \le Ct^\theta\|y\|_E$$

for all $y \in E$ and $t > 0$, then we write $E \in K_\theta(Y,X)$.

Definition/Lemma 2.81 (Real Interpolation Spaces)**.** *Let $\theta \in (0,1]$. We define the sets*

$$
\begin{aligned}
(Y,X)_{\theta,p} &\coloneqq \left\{ y \in Y \,\middle|\, t \mapsto t^{-\theta-1/p} K(y,t,Y,X) \in L^p(0,\infty) \right\} \quad \textit{and} \\
(Y,X)_{\theta} &\coloneqq \left\{ y \in Y \,\middle|\, \lim_{t\to 0} t^{-\theta} K(y,t,Y,X) = 0 \right\}.
\end{aligned}
$$

Together with the norms

$$
\begin{aligned}
\|y\|_{\theta,p} &\coloneqq \left\| t^{-\theta-1/p} K(y,t,Y,X) \right\|_{L^p(0,\infty)} \quad \textit{and} \\
\|y\|_{\theta} &\coloneqq \|y\|_{\theta,\infty} = \left\| t^{-\theta} K(y,t,Y,X) \right\|_{L^\infty(0,\infty)},
\end{aligned}
$$

they form Banach spaces by [Lun12, Proposition 1.2.4] and [Lun12, Corollar 1.2.5], which are called real interpolation spaces between Y and X.

Remark 2.82. *Let $X_1, ..., X_n$ and $Y_1, ..., Y_n$ be finitely many Banach spaces and define $X \coloneqq \prod_{i=1}^n X_i$ as well as $Y \coloneqq \prod_{i=1}^n Y_i$. Then*

$$
(Y,X)_{\theta,p} = \prod_{i=1}^n (Y_i, X_i)_{\theta,p} \qquad \textit{and} \qquad (Y,X)_{\theta} = \prod_{i=1}^n (Y_i, X_i)_{\theta}
$$

hold for any $\theta \in (0,1]$.

Lemma 2.83. *For $\theta \in (0,1)$, we have*

$$
(Y,X)_{\theta,p} \in K_\theta(Y,X) \cap J_\theta(Y,X) \qquad \textit{and} \qquad (Y,X)_{\theta} \in K_\theta(Y,X) \cap J_\theta(Y,X).
$$

Proof. For any Banach space E, [Lun12, Definition 1.2.14] and [Lun12, Proposition 1.2.13] yield

$$
E \in K_\theta(Y,X) \cap J_\theta(Y,X) \Leftrightarrow (Y,X)_{\theta,1} \subset E \subset (Y,X)_{\theta,\infty}.
$$

Moreover, with [Lun12, Proposition 1.2.3], we have

$$
X \subset (Y,X)_{\theta,p_1} \subset (Y,X)_{\theta,p_2} \subset (Y,X)_{\theta} \subset (Y,X)_{\theta,\infty} \subset \overline{X}^{\|\cdot\|_Y}
$$

for $1 \le p_1 \le p_2 < \infty$. □

Proposition 2.84 (Reiteration Theorem)**.** *Let $\theta \in (0,1)$ and $0 \le \theta_1 < \theta_2 \le 1$ as well as $w = (1-\theta)\theta_1 + \theta\theta_2$. For Banach spaces E_1, E_2 with $E_i \in K_{\theta_i}(Y,X) \cap J_{\theta_i}(Y,X)$*

$$
(E_1, E_2)_{\theta,p} = (Y,X)_{w,p} \qquad \textit{and} \qquad (E_1, E_2)_{\theta} = (Y,X)_{w}
$$

hold with equivalent norms.

Proof. The reiteration theorem is proven in [Lun12, Theorem 1.2.15]. □

Application to Hölder Spaces

In the introduction to this Section 2.2.3, we assumed convexity for the subset $W \subset \mathbb{R}^d$ to obtain results on embedding properties of Hölder spaces $\mathcal{C}^\alpha(\overline{W})$. Lunardi does not need convexity but also uses an additional condition. She assumes that W has a regular boundary, which she defines in the following way (see [Lun12, Section 0.1, pages 2 and 3]): *There exists a (at most countable) collection of open balls $\big(B_r(x_j)\big)_{j\in\mathbb{N}}$ covering ∂W such that there exists an integer $k \in \mathbb{N}_{\geq 0}$ with the property that $\bigcap_{j\in J} B_r(x_j) = \varnothing$ holds for all $J \subset \mathbb{N}$ with more than k elements. Moreover, we assume that there is an $\varepsilon > 0$ such that $B_{r/2}(x_j)$ still covers an ε-neighborhood of ∂W for all $j \in \mathbb{N}$, and that there exist coordinate transformations φ_j such that $\varphi_j : \overline{B_r(x_j)} \to \overline{B_1(0)} \subset \mathbb{R}^d$ is a C^∞-diffeomorphism, mapping $\overline{B_r(x_j)} \cap W$ onto the upper half ball $\{y \in \overline{B_1(0)} \,|\, y_d > 0\}$ and mapping $\overline{B_r(x_j)} \cap \partial W$ onto the basis $\{y \in \overline{B_1(0)} \,|\, y_d = 0\}$.* In fact, Lunardi even distinguishes uniformly C^k-, $C^{k+\alpha}$- and $h^{k+\alpha}$- boundaries which are defined in an analogous way but, of course, result in a weaker condition on W. For simplicity, we will always assume that the open subset $W \subset \mathbb{R}^d$ has a regular, i.e., a uniformly C^∞-boundary and refer the reader to [Lun12] for the more detailed discussion with less regular boundaries. Note, that Lunardi omits the index b in her notation of $C_b^0(\overline{W}, X)$ and $C_b^m(\overline{W}, X)$ even if W is unbounded.

First, we prove a lemma ensuring that we can transfer the representations deduced in [Lun12] for (little) Hölder spaces defined on the closure of open sets with regular boundary to (little) Hölder spaces defined on embedded closed submanifolds. Then, we state these representations for both kinds of (little) Hölder spaces.

In the following, let $M \subset \mathbb{R}^n$ be a d-dimensional C^m-embedded closed submanifold for $m \in \mathbb{N}_{\geq 1}$. By Remark 2.6(ii), there exists a finite set $(\gamma_l, W_l)_{l=1,\dots,L}$ of local parameterizations of M such that W_l are bounded and have regular boundaries and there exist compact sets $A_l \subset M$ with $A_l \subset \gamma_l(W_l)$ such that

$$M \subset \bigcup_{l=1}^{L} A_l$$

holds. As in Section 2.2.2, Lemma 2.69 yields that using local coordinates works smoothly even though (uniform) Hölder regularity does not transfer from two seperated sets to the union of these sets.

Lemma 2.85. *Let $m \in \mathbb{N}_{\geq 1}$, $\theta \in (0,1]$ and $p \in [1,\infty]$. Then, for any function $u : M \to \mathbb{R}$, we have*

$$u \in \big(C^0(M), C^m(M)\big)_{\theta,p} \Leftrightarrow u \circ \gamma_l \in \big(C^0(\overline{W_l}), C^m(\overline{W_l})\big)_{\theta,p} \text{ for all } l = 1,\dots,L \text{ and}$$
$$u \in \big(C^0(M), C^m(M)\big)_{\theta} \Leftrightarrow u \circ \gamma_l \in \big(C^0(\overline{W_l}), C^m(\overline{W_l})\big)_{\theta} \text{ for all } l = 1,\dots,L.$$

Proof. Let $(\psi_l)_{l=1,\dots,L} \subset C^m(M)$ be a partition of unity subordinate to $\big(\gamma_l(W_l)\big)_{l=1,\dots,L}$. We define

$$S : u \mapsto [u \circ \gamma_l]_{l=1,\dots,L}$$

for any $u : M \to \mathbb{R}$ and

$$R : [v_l]_{l=1,\dots,L} \mapsto \sum_{l=1}^{L} \psi_l (v_l \circ \gamma_l^{-1})$$

for any $v_l : \overline{W_l} \to \mathbb{R}$ for $l = 1,...,L$. Clearly, $S \in \mathcal{L}\big(C^k(M), \prod_{l=1}^L C^k(\overline{W_l})\big)$ holds for $k \in \{0,m\}$. With Remark 2.46, we also have $R \in \mathcal{L}\big(\prod_{l=1}^L C^k(\overline{W_l}), C^k(M)\big)$ for $k \in \{0,m\}$. Because $RSu = u$ holds for any $u : M \to \mathbb{R}$, the operator R is a retraction in the sense of [Tri78, Section 1.2.4], and the operator S is its coretraction. Hence, [Tri78, Section 1.2.4] implies that S yields isomorphisms

$$\big(C^0(M), C^m(M)\big)_{\theta,p} \to \left(\prod_{l=1}^L C^0(\overline{W_l}), \prod_{l=1}^L C^m(\overline{W_l})\right)_{\theta,p} \qquad \text{and}$$
$$\big(C^0(M), C^m(M)\big)_{\theta} \to \left(\prod_{l=1}^L C^0(\overline{W_l}), \prod_{l=1}^L C^m(\overline{W_l})\right)_{\theta}.$$

On account of Remark 2.82, thus the claim follows. □

With the help of Lemma 2.85, we can deduce the following statement for the representation of Hölder and little Hölder spaces.

Lemma 2.86 (Hölder Spaces as Interpolation Spaces)**.** *Let $m \in \mathbb{N}_{\geq 1}$ and $\theta \in (0,1)$ with $\theta m \notin \mathbb{N}$ and let $W \subset \mathbb{R}^d$ be an open subset with regular boundary. Then,*

$$\big(C_b^0(\overline{W}), C_b^m(\overline{W})\big)_{\theta,\infty} = C^{\theta m}(\overline{W}) \quad \textit{and} \quad \big(C_b^0(\overline{W}), C_b^m(\overline{W})\big)_{\theta} = h^{\theta m}(\overline{W}) \ \textit{as well as}$$
$$\big(C^0(M), C^m(M)\big)_{\theta,\infty} = C^{\theta m}(M) \quad \textit{and} \quad \big(C^0(M), C^m(M)\big)_{\theta} = h^{\theta m}(M)$$

hold with equivalent norms.

Proof. For domains $\overline{W}$, the proof is given in [Lun12, Corollary 1.2.19] and the statement for submanifolds M then follows directly with Lemma 2.85. □

Lemma 2.87 (Hölder Spaces as Interpolation Spaces II)**.** *Let $0 < \sigma_1 < \sigma_2$ with $\sigma_1, \sigma_2 \notin \mathbb{N}$ and let $\theta \in (0,1)$ as well as $w = (1-\theta)\sigma_1 + \theta\sigma_2$ with $w \notin \mathbb{N}$. Furthermore, let $m \in \mathbb{N}_{\geq 1}$ with $m - 1 < \sigma_2 < m$. Finally, let $W \subset \mathbb{R}^d$ be an open subset with regular boundary. Then,*

$$\big(C^{\sigma_1}(\overline{W}), C^{\sigma_2}(\overline{W})\big)_{\theta,\infty} = C^w(\overline{W}),$$
$$\big(h^{\sigma_1}(\overline{W}), h^{\sigma_2}(\overline{W})\big)_{\theta,\infty} = C^w(\overline{W}) \ \textit{and}$$
$$\big(h^{\sigma_1}(\overline{W}), h^{\sigma_2}(\overline{W})\big)_{\theta} = h^w(\overline{W})$$

as well as

$$\big(C^{\sigma_1}(M), C^{\sigma_2}(M)\big)_{\theta,\infty} = C^w(M),$$
$$\big(h^{\sigma_1}(M), h^{\sigma_2}(M)\big)_{\theta,\infty} = C^w(M) \ \textit{and}$$
$$\big(h^{\sigma_1}(M), h^{\sigma_2}(M)\big)_{\theta} = h^w(M)$$

hold with equivalent norms.

Proof. Because the proof is identical for both kinds of domains, we omit the domain $\overline{W}$ or M in the following. Due to $0 < \sigma_i < m$, there exist $\theta_i \in (0,1)$ with $\sigma_i = \theta_i m$. Because of $\sigma_1 < \sigma_2$, also $\theta_1 < \theta_2$ holds. Lemmas 2.86 and 2.83 yield

$$C^{\sigma_i} = C^{\theta_i m} = (C_b^0, C_b^m)_{\theta_i,\infty} \in K_{\theta_i}(C_b^0, C_b^m) \cap J_{\theta_i}(C_b^0, C_b^m).$$

With the reiteration theorem (Proposition 2.84) and Lemma 2.86,

$$(C^{\sigma_1}, C^{\sigma_2})_{\theta,\infty} = (C_b^0, C_b^m)_{(1-\theta)\theta_1+\theta\theta_2,\infty} = C^{m((1-\theta)\theta_1+\theta\theta_2)} = C^w$$

follows. Analogously, Lemmas 2.86 and 2.83 yield

$$h^{\sigma_i} = h^{\theta_i m} = (C_b^0, C_b^m)_{\theta_i} \in K_{\theta_i}(C_b^0, C_b^m) \cap J_{\theta_i}(C_b^0, C_b^m).$$

Again, with the reiteration theorem (Proposition 2.84) and Lemma 2.86,

$$(h^{\sigma_1}, h^{\sigma_2})_{\theta,\infty} = (C_b^0, C_b^m)_{(1-\theta)\theta_1+\theta\theta_2,\infty} = C^{m((1-\theta)\theta_1+\theta\theta_2)} = C^w$$

as well as

$$(h^{\sigma_1}, h^{\sigma_2})_{\theta} = (C_b^0, C_b^m)_{(1-\theta)\theta_1+\theta\theta_2} = h^{m((1-\theta)\theta_1+\theta\theta_2)} = h^w$$

follow. □

2.2.4 Basic Properties of Hölder Spaces

Obviously, for two open subsets $W_1 \subset W_2 \subset \mathbb{R}^d$, we can restrict any function $f : \overline{W_2} \to X$ to $\overline{W_1}$ and thus we have, by a slight abuse of notation, $\mathcal{C}_b^s(\overline{W_2}, X) \subset \mathcal{C}_b^s(\overline{W_1}, X)$. In the following, we will discuss embeddings $\mathcal{C}_b^{s_2}(\overline{W}, X) \hookrightarrow \mathcal{C}_b^{s_1}(\overline{W}, X)$ of Hölder spaces for different Hölder exponents $s_1 \le s_2$. Because clearly $h^s(\overline{W}, X) \hookrightarrow C^s(\overline{W}, X) \hookrightarrow C_b^{\lfloor s \rfloor}(\overline{W}, X)$ holds for $s \notin \mathbb{N}_{\ge 0}$, we limit our considerations to the case $\mathcal{C}_b^{s_2}(\overline{W}, X) \hookrightarrow h^{s_1}(\overline{W}, X)$. For this, we use both the ansatz from Section 2.2.2 via the Hölder seminorms as well as the one from Section 2.2.3 identifying Hölder spaces as interpolation spaces. Thus, the same additional conditions on the open subset W as above are necessary: We assume W to be bounded and convex, or, if $X = \mathbb{R}^n$, to have a regular boundary. After analyzing embeddings of Hölder spaces, we show that the product as well as the composition of two Hölder regular functions is Hölder regular again under weak additional requirements.

Lemma 2.88 (Embeddings of Hölder Spaces)**.** *Let $W \subset \mathbb{R}^d$ be an open, bounded and convex subset and let X be a Banach space. For any $s_1, s_2 \in \mathbb{R}_{>0}$ with $s_1 < s_2$ and $s_1 \notin \mathbb{N}$,*

$$\mathcal{C}_b^{s_2}(\overline{W}, X) \hookrightarrow h^{s_1}(\overline{W}, X)$$

holds. For $X = \mathbb{R}^m$, the statement also holds if $W \subset \mathbb{R}^d$ is an open subset with regular boundary. If $M \subset \mathbb{R}^n$ is a d-dimensional $C^1 \cap \mathcal{C}^{s_2}$-embedded closed submanifold, we have analogously

$$\mathcal{C}^{s_2}(M, X) \hookrightarrow h^{s_1}(M, X).$$

Proof. If W is convex, every $f \in C^1(\overline{W}, X)$ is Lipschitz continuous with Lipschitz constant $L \le \|f\|_{C^1(\overline{W},X)}$. Hence, Remark 2.67 yields the claim in the case of bounded and convex W. If $X = \mathbb{R}$ and W has regular boundary, the statement follows from Lemma 2.86 using Lemma 2.83 and [Lun12, Proposition 1.2.3]. On account of Remark 2.73(ii), this implies the statement for $X = \mathbb{R}^m$. Then, the statement for submanifolds follows directly with Definition 2.75 and Remark 2.76(iii). □

In particular, for $d = 1$, the following remark holds.

Remark 2.89. *Let $T \in (0,1]$, let X be a Banach space and let $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 < \alpha_2$. We have $\mathcal{C}^{\alpha_2}([0,T],X) \hookrightarrow \mathcal{C}^{\alpha_1}([0,T],X)$ with*

$$\|f\|_{C^{\alpha_1}([0,T],X)} \le 2T^{\alpha_2-\alpha_1}\|f\|_{C^{\alpha_2}([0,T],X)} + \|f(0)\|_X$$

for all $f \in \mathcal{C}^{\alpha_2}\big([0,T],X\big)$. This is due to

$$\begin{aligned}
[f]_{C^{\alpha_1}([0,T],X)} &= \sup_{\substack{t,s\in[0,T]\\ t\neq s}} |t-s|^{-\alpha_1}\|f(t)-f(s)\|_X \\
&\le T^{\alpha_2-\alpha_1} \sup_{\substack{t,s\in[0,T]\\ t\neq s}} |t-s|^{-\alpha_2}\|f(t)-f(s)\|_X \\
&= T^{\alpha_2-\alpha_1}[f]_{C^{\alpha_2}([0,T],X)}
\end{aligned}$$

and

$$\begin{aligned}
\|f\|_{C^0([0,T],X)} &\le \sup_{\substack{t\in[0,T]\\ t\neq 0}} \|f(t)-f(0)\|_X + \|f(0)\|_X \\
&\le T^{\alpha_2} \sup_{\substack{t\in[0,T]\\ t\neq 0}} |t|^{-\alpha_2}\|f(t)-f(0)\|_X + \|f(0)\|_X \\
&\le T^{\alpha_2} \sup_{\substack{t,s\in[0,T]\\ t\neq s}} |t-s|^{-\alpha_2}\|f(t)-f(s)\|_X + \|f(0)\|_X \\
&= T^{\alpha_2}[f]_{C^{\alpha_2}([0,T],X)} + \|f(0)\|_X \\
&\le T^{\alpha_2-\alpha_1}[f]_{C^{\alpha_2}([0,T],X)} + \|f(0)\|_X.
\end{aligned}$$

Lemma 2.90 (Embeddings of Hölder Spaces in Time and Space)**.** *Let $T \in (0,\infty)$ and let $\alpha, \beta \in (0,1)$. Furthermore, let $M \subset \mathbb{R}^n$ be a d-dimensional $h^{2+\alpha}$-embedded closed submanifold. Define $X := h^\alpha(M)$, $Y := h^{1+\alpha}(M)$ and $Z := h^{2+\alpha}(M)$. Then, there exists $\gamma \in (0,1)$ with $\gamma > \beta$ such that*

$$h^{1+\beta}([0,T],X) \cap h^\beta([0,T],Z) \hookrightarrow h^\gamma([0,T],Y)$$

is a continuous embedding.

Proof. With $\theta := \frac{1+\alpha}{2} \in (0,1)$, Lemmas 2.86 and 2.83 yield

$$Y = h^{2\theta}(M) = \big(C^0(M), C^2(M)\big)_\theta \in J_\theta\big(C^0(M), C^2(M)\big).$$

Due to $Z \hookrightarrow C^2(M) \hookrightarrow Y \hookrightarrow X \hookrightarrow C^0(M)$,

$$Y \in J_\theta(X,Z)$$

follows. Define $\gamma := (1-\theta) + \theta\beta$ such that $\gamma \in (0,1)$ with $\gamma > \beta$ holds. For any function $f \in h^{1+\beta}([0,T],X) \cap h^\beta([0,T],Z)$, Lemma 2.79 then yields

$$\|f\|_{C^\gamma([0,T],Y)} \lesssim \|f\|^{1-\theta}_{C^1([0,T],X)}\|f\|^\theta_{C^\beta([0,T],Z)} \le \|f\|_{C^{1+\beta}([0,T],X)} + \|f\|_{C^\beta([0,T],Z)}$$

and

$$[f]^R_{C^\gamma([0,T],Y)} \lesssim \|f\|^{1-\theta}_{C^1([0,T],X)} \big([f]^R_{C^\beta([0,T],Z)}\big)^\theta$$

for all $R \in (0,\infty]$ such that in particular $f \in h^\gamma([0,T],Y)$ follows. $\square$

Lemma 2.91 (Compact Embeddings of Hölder Spaces)**.** *Let $W \subset \mathbb{R}^d$ be an open and bounded subset. Additionally, let W be convex or have regular boundary. For every $s_1, s_2 \in \mathbb{R}_{>0} \setminus \mathbb{N}$ with $s_1 < s_2$,*

$$\mathcal{C}^{s_2}(\overline{W}) \hookrightarrow h^{s_1}(\overline{W})$$

is a compact embedding. If $M \subset \mathbb{R}^n$ is a d-dimensional $C^1 \cap \mathcal{C}^{s_2}$-embedded closed submanifold, analogously

$$\mathcal{C}^{s_2}(M) \hookrightarrow h^{s_1}(M)$$

is a compact embedding.

Proof. By Lemma 2.88, $\mathcal{C}^{s_2}(\overline{W}) \hookrightarrow h^{s_1}(\overline{W})$ is a continuous embedding. It remains to show that every bounded sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{C}^{s_2}(\overline{W})$ contains a in $h^{s_1}(\overline{W})$ convergent subsequence. In the interest of readability, let $s_2 = k + \alpha_2$ with $k \in \mathbb{N}_{\geq 0}$ and $\alpha_2 \in (0,1)$.
By Lemma 2.88, $(D^\beta u_n)_{n\in\mathbb{N}} \subset \mathcal{C}^{\alpha_2}(\overline{W})$ is bounded for all $|\beta| \leq k$. Because $\overline{W}$ is compact, $(D^\beta u_n)_{n\in\mathbb{N}}$ thus is relatively compact in $C^0(\overline{W})$ by the theorem of Arzelà-Ascoli. Hence, there exists a convergent subsequence $D^\beta u_n \to u^\beta$ in $C^0(\overline{W})$ for all $|\beta| \leq k$. With $u \coloneqq u^0$, due to the uniform convergence, we have $u \in C^k(\overline{W})$ with $D^\beta u = u^\beta$ and $u_n \to u$ in $C^k(\overline{W})$. Moreover, since $|D^\beta u(x) - D^\beta u(y)| = \lim_{n\to\infty} |D^\beta u_n(x) - D^\beta u_n(y)| \leq C|x-y|^{\alpha_2}$ holds for all $x, y \in \overline{W}$ and $|\beta| = k$, also $u \in \mathcal{C}^{\alpha_2}(\overline{W})$ holds.
If $s_1 < k$, another application of Lemma 2.88 yields $u \in h^{s_1}(\overline{W})$ with $u_n \to u$ in $h^{s_1}(\overline{W})$. If $s_1 \geq k$, then we have $s_1 = k + \alpha_1$ with $\alpha_1 \in [0, \alpha_2)$ and hence

$$[D^\beta(u_n - u)]^R_{C^{\alpha_1}(\overline{W})} \leq \big([D^\beta(u_n-u)]^R_{C^{\alpha_2}(\overline{W})}\big)^{\frac{\alpha_1}{\alpha_2}} \big(2\|D^\beta(u_n-u)\|_{C^0(\overline{W})}\big)^{1-\frac{\alpha_1}{\alpha_2}}$$

for every $|\beta| = k$ with Remark 2.67. In particular, $u \in \mathcal{C}^{k+\alpha_1}(\overline{W})$ and $u_n \to u$ in $\mathcal{C}^{k+\alpha_1}(\overline{W})$ follow.
The second statement for submanifolds follows directly from the first, because for a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{C}^{s_2}(M)$, the existence of $u_l \in h^{s_1}(\overline{W_l})$ with $u_n \circ \gamma_l \to u_l$ in $h^{s_1}(\overline{W_l})$ implies the existence of a function $u \in h^{s_1}(M)$ with $u \circ \gamma_l = u_l$ on $\overline{W_l}$. $\square$

Lemma 2.92 (Dense Embeddings of little Hölder Spaces)**.** *Let $W \subset \mathbb{R}^d$ be an open subset with regular boundary. For any $s_1, s_2 \in \mathbb{R}_{>0} \setminus \mathbb{N}$ with $s_1 < s_2$,*

$$h^{s_2}(\overline{W}, \mathbb{R}^m) \hookrightarrow h^{s_1}(\overline{W}, \mathbb{R}^m)$$

is a dense embedding. If $M \subset \mathbb{R}^n$ is a d-dimensional $C^1 \cap h^{s_2}$-embedded closed submanifold, analogously

$$h^{s_2}(M, \mathbb{R}^m) \hookrightarrow h^{s_1}(M, \mathbb{R}^m)$$

is a dense embedding.

Proof. The proof is identical for both statements; so we omit the domain $\overline{W}$ or M in the following. By Lemma 2.88, $h^{s_2} \hookrightarrow h^{s_1}$ is a continuous embedding. Let $k \in \mathbb{N}$ with $s_2 < k$. There exist $\theta_1, \theta_2 \in (0,1)$ with $\theta_1 < \theta_2$ such that $\theta_i k = s_i$ holds for $i = 1,2$. With [Lun12, Proposition 1.2.12] and Lemma 2.86,

$$C_b^k \hookrightarrow (C_b^0, C_b^k)_{\theta_i} = h^{\theta_i k} = h^{s_i}$$

is a dense embedding for $i = 1,2$ and $m = 1$. Thus the claim follows for $m = 1$ and then Remark 2.73(ii) yields the claim for $m > 1$. □

Lemma 2.93 (Dense Embedding of C^∞ to little Hölder Spaces)**.** *Let $W \subset \mathbb{R}^d$ be an open and bounded subset with regular boundary. For any $s \in \mathbb{R}_{>0} \setminus \mathbb{N}$,*

$$C^\infty(\overline{W}) \hookrightarrow h^s(\overline{W})$$

is a dense embedding. If $M \subset \mathbb{R}^n$ is a d-dimensional C^∞-embedded closed submanifold, analogously

$$C^\infty(M) \hookrightarrow h^s(M)$$

is a dense embedding.

Proof. Partially, the proof is identical for both statements; and then we omit the domain $\overline{W}$ or M. Let $m \in \mathbb{N}$ with $s < m$. There exists $\theta \in (0,1)$ with $\theta m = s$. With [Lun12, Proposition 1.2.12] and Lemma 2.86,

$$C^m \hookrightarrow (C^0, C^m)_\theta = h^{\theta m} = h^s$$

is a dense embedding. So, what is left to show is that $C^\infty \hookrightarrow C^m$ is also a dense embedding. This is true for domains $\overline{W}$, because by [RR06, Theorems 7.58 and 7.60], any $u \in C^m(\overline{W})$ can be extended to $\tilde{u} \in C^m(\mathbb{R}^d)$ with compact support, and then on account of [Wer09, Proposition IV.9.3, Proposition IV.9.6 and Corollary IV.9.7], $\tilde{u}$ can be approximated by smooth functions using convolution with mollifiers.
Let $M \subset \mathbb{R}^n$ be a C^∞-embedded closed submanifold with a finite set of local parameterizations $(\gamma_l, W_l)_{l=1,\dots,L}$ such that $M \subset \bigcup_l \gamma_l(W_l)$ holds. Moreover, let $(\psi_l)_{l=1,\dots,L}$ be a partition of unity subordinate to $\big(\gamma_l(W_l)\big)_{l=1,\dots,L}$. Then, on account of the considerations for domains $\overline{W}$, any $u \in C^m(M)$ locally can be approximated by smooth $(u_l^n)_{n\in\mathbb{N}} \subset C^\infty(\mathbb{R}^d)$ in $\|\cdot\|_{C^m(\overline{W_l})}$. By Remark 2.46, $(u^n)_{n\in\mathbb{N}}$ with $u^n := \sum_l \psi_l(u_l^n \circ \gamma_l^{-1}) \in C^\infty(M)$ approximates u in $\|\cdot\|_{C^m(M)}$. □

Proposition 2.94 (Pointwise Product in Hölder Spaces)**.** *Let $W \subset \mathbb{R}^d$ be an open, bounded and convex subset and let $s \in \mathbb{R}_{\geq 0}$. Furthermore, let X_1, X_2, X be Banach spaces with a $\mathbb{R}$-bilinear operation $\cdot : X_1 \times X_2 \to X$ such that $\|u_1 \cdot u_2\|_X \lesssim \|u_1\|_{X_1}\|u_2\|_{X_2}$ holds for all $u_1 \in X_1$, $u_2 \in X_2$.*
Then, with pointwise multiplication, $f \cdot g \in \mathcal{C}_b^s(\overline{W}, X)$ with

$$\|f \cdot g\|_{C^s(\overline{W},X)} \leq C\|f\|_{C^s(\overline{W},X_1)}\|g\|_{C^s(\overline{W},X_2)}$$

holds for all $f \in \mathcal{C}_b^s(\overline{W}, X_1)$, $g \in \mathcal{C}_b^s(\overline{W}, X_2)$. *In particular, for* $s = \alpha \in (0,1)$, *we have*

$$\begin{aligned}
&[f \cdot g]^R_{C^\alpha(\overline{W},X)} \le \|f\|_{C^0(\overline{W},X_1)} [g]^R_{C^\alpha(\overline{W},X_2)} + [f]^R_{C^\alpha(\overline{W},X_1)} \|g\|_{C^0(\overline{W},X_2)} \quad \textit{and} \\
&\|f \cdot g\|_{C^\alpha(\overline{W},X)} \le \|f\|_{C^\alpha(\overline{W},X_1)} \|g\|_{C^\alpha(\overline{W},X_2)}
\end{aligned}$$

for all $f \in \mathcal{C}^\alpha(\overline{W}, X_1)$, $g \in \mathcal{C}^\alpha(\overline{W}, X_2)$ *and* $R \in (0, \infty]$. *For* $X_1 = X_2 = \mathbb{R}^n$ *and* $X = \mathbb{R}$, *the statements also hold if* $W \subset \mathbb{R}^d$ *is an open subset with regular boundary.*
If $M \subset \mathbb{R}^n$ *is a d-dimensional* $C^1 \cap \mathcal{C}^s$*-embedded closed submanifold, with pointwise multiplication, analogously* $f \cdot g \in \mathcal{C}^s(M, X)$ *with*

$$\|f \cdot g\|_{C^s(M,X)} \le C \|f\|_{C^s(M,X_1)} \|g\|_{C^s(M,X_2)}$$

holds for all $f \in \mathcal{C}^s(M, X_1)$, $g \in \mathcal{C}^s(M, X_2)$.

Proof. Let $s = k + \alpha$ with $k \in \mathbb{N}_{\ge 0}$ and $\alpha \in [0,1)$. Assume $k = 0$ at first and let $f : \overline{W} \to X_1$ and $g : \overline{W} \to X_2$. We have

$$\begin{aligned}
\|(f \cdot g)(x) - (f \cdot g)(y)\|_X &\le \big\|f(x)\big(g(x) - g(y)\big)\big\|_X + \big\|\big(f(x) - f(y)\big)g(y)\big\|_X \\
&\le \|f(x)\|_{X_1} \|g(x) - g(y)\|_{X_2} + \|f(x) - f(y)\|_{X_1} \|g(y)\|_{X_2} \qquad (2.8) \\
&\le \|f\|_{C^0(\overline{W},X_1)} \|g(x) - g(y)\|_{X_2} + \|f(x) - f(y)\|_{X_1} \|g\|_{C^0(\overline{W},X_2)}
\end{aligned}$$

for all $x, y \in \overline{W}$. So, $f \in \mathcal{C}_b^0(\overline{W}, X_1)$ and $g \in \mathcal{C}_b^0(\overline{W}, X_2)$ imply $f \cdot g \in \mathcal{C}_b^0(\overline{W}, X)$ with

$$\|f \cdot g\|_{C^0(\overline{W},X)} = \sup_{x \in \overline{W}} \|f(x) \cdot g(x)\|_X \le \sup_{x \in \overline{W}} \|f(x)\|_{X_1} \|g(x)\|_{X_2} \le \|f\|_{C^0(\overline{W},X_1)} \|g\|_{C^0(\overline{W},X_2)}.$$

For $\alpha > 0$ and $R \in (0, \infty]$, Equation (2.8) yields

$$\begin{aligned}
[f \cdot g]^R_{C^\alpha(\overline{W},X)} &= \sup_{\substack{x,y \in \overline{W} \\ 0 < |x-y| < R}} \frac{\|(f \cdot g)(x) - (f \cdot g)(y)\|_X}{|x - y|^\alpha} \\
&\le \|f\|_{C^0(\overline{W},X_1)} [g]^R_{C^\alpha(\overline{W},X_2)} + [f]^R_{C^\alpha(\overline{W},X_1)} \|g\|_{C^0(\overline{W},X_2)},
\end{aligned}$$

and thus $f \in \mathcal{C}^\alpha(\overline{W}, X_1)$ and $g \in \mathcal{C}^\alpha(\overline{W}, X_2)$ imply $f \cdot g \in \mathcal{C}^\alpha(\overline{W}, X)$ with

$$\begin{aligned}
[f \cdot g]^R_{C^\alpha(\overline{W},X)} &\le \|f\|_{C^0(\overline{W},X_1)} [g]^R_{C^\alpha(\overline{W},X_2)} + [f]^R_{C^\alpha(\overline{W},X_1)} \|g\|_{C^0(\overline{W},X_2)} \quad \text{and} \\
\|f \cdot g\|_{C^\alpha(\overline{W},X)} &= \|f \cdot g\|_{C^0(\overline{W},X)} + [f \cdot g]_{C^\alpha(\overline{W},X)} \\
&\le \|f\|_{C^0(\overline{W},X_1)} \|g\|_{C^0(\overline{W},X_2)} + \|f\|_{C^0(\overline{W},X_1)} [g]_{C^\alpha(\overline{W},X_2)} \\
&\qquad + [f]_{C^\alpha(\overline{W},X_1)} \|g\|_{C^0(\overline{W},X_2)} \\
&\le \|f\|_{C^\alpha(\overline{W},X_1)} \|g\|_{C^\alpha(\overline{W},X_2)}.
\end{aligned}$$

Now, let $k > 0$ and let $f \in \mathcal{C}_b^s(\overline{W}, X_1)$ and $g \in \mathcal{C}_b^s(\overline{W}, X_2)$. By Lemma 2.88, $D^\gamma f \in \mathcal{C}_b^\alpha(\overline{W}, X_1)$ and $D^\gamma g \in \mathcal{C}_b^\alpha(\overline{W}, X_2)$ hold for all $|\gamma| \le k$ with $\|D^\gamma f\|_{C^\alpha(\overline{W},X_1)} \le \|f\|_{C^s(\overline{W},X_1)}$ as well as $\|D^\gamma g\|_{C^\alpha(\overline{W},X_2)} \le \|g\|_{C^s(\overline{W},X_2)}$. Because we have

$$D^\beta(f \cdot g) \sim \sum_{\gamma^1 + \gamma^2 = \beta} D^{\gamma^1} f \cdot D^{\gamma^2} g$$

for all $|\beta| \le k$, the first part implies $D^\beta(f \cdot g) \in \mathcal{C}_b^\alpha(\overline{W}, X)$ with

$$\|D^\beta(f \cdot g)\|_{C^\alpha(\overline{W},X)} \le C\|f\|_{C^s(\overline{W},X_1)}\|g\|_{C^s(\overline{W},X_2)}$$

for all $|\beta| \le k$. In particular, $f \cdot g \in \mathcal{C}_b^s(\overline{W}, X)$ follows with

$$\|f \cdot g\|_{C^s(\overline{W},X)} \le C\|f\|_{C^s(\overline{W},X_1)}\|g\|_{C^s(\overline{W},X_2)}.$$

The second statement for submanifolds follows directly from the first. □

Remark 2.95. *We can choose an open subset $W \subset \mathbb{R}^d$ without additional assumptions in Proposition 2.94, if*

$$D^\beta f \in \mathcal{C}_b^{s-\lfloor s \rfloor}(\overline{W}, X_1) \quad \text{and} \quad D^\beta g \in \mathcal{C}_b^{s-\lfloor s \rfloor}(\overline{W}, X_2) \quad \text{with}$$
$$\|D^\beta f\|_{C^{s-\lfloor s \rfloor}(\overline{W},X_1)} \lesssim \|f\|_{C^s(\overline{W},X_1)} \quad \text{and} \quad \|D^\beta g\|_{C^{s-\lfloor s \rfloor}(\overline{W},X_2)} \lesssim \|g\|_{C^s(\overline{W},X_2)}$$

still hold for all $|\beta| \le \lfloor s \rfloor$. Then, also

$$D^\beta(f \cdot g) \in \mathcal{C}_b^{s-\lfloor s \rfloor}(\overline{W}, X) \quad \text{with} \quad \|D^\beta(f \cdot g)\|_{C^{s-\lfloor s \rfloor}(\overline{W},X)} \lesssim \|f\|_{C^s(\overline{W},X_1)}\|g\|_{C^s(\overline{W},X_2)}$$

holds for all $|\beta| \le \lfloor s \rfloor$. In particular, this is the case for $s < 1$.

Remark 2.96. *The most important examples for Proposition 2.94 are the following choices of X_1, X_2 and X:*

(i) *$X_1 = X_2 = X := \mathbb{R}$ and $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the multiplication of real numbers; this implies in particular that with pointwise multiplication, $\mathcal{C}_b^s(\overline{W})$ and $\mathcal{C}^s(M)$ are $\mathbb{R}$-algebras;*

(ii) *$X_1 = X_2 := \mathbb{R}^n$, $X := \mathbb{R}$ and $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the inner product on $\mathbb{R}^n$;*

(iii) *$X_1 := \mathcal{L}(Y, X)$, $X_2 := Y$ for arbitrary Banach spaces X, Y and the bilinear operation $\cdot : \mathcal{L}(Y, X) \times Y \to X, (F, u) \mapsto F[u]$ is the evaluation of the linear operator.*
This implies in particular that for any open subset $W \subset \mathbb{R}^d$ and any $\alpha \in [0, 1)$, the conditions $F \in \mathcal{C}_b^\alpha(\overline{W}, \mathcal{L}(Y, X))$ and $u \in \mathcal{C}_b^\alpha(\overline{W}, Y)$ yield $F[u] \in \mathcal{C}_b^\alpha(\overline{W}, X)$ with

$$\|F[u]\|_{C^\alpha(\overline{W},X)} \le \|F\|_{C^\alpha(\overline{W},\mathcal{L}(Y,X))}\|u\|_{C^\alpha(\overline{W},Y)}.$$

Proposition 2.97 (Composition of Hölder Functions)**.** *Let $W_1 \subset \mathbb{R}^{d_1}$, $W_2 \subset \mathbb{R}^{d_2}$ be open, bounded and convex subsets, let X, Y, Z be Banach spaces, $U \subset Y$ an open subset and let $s \in \mathbb{R}_{\ge 0}$. Furthermore, let $\varphi \in \mathcal{C}_b^s(\overline{W_1}, \mathbb{R}^{d_2})$ such that $\varphi(\overline{W_1}) \subset \overline{W_2}$ holds and $\varphi : \overline{W_1} \to \mathbb{R}^{d_2}$ is Lipschitz continuous, i.e., there exists a constant $L \ge 0$ with*

$$\sup_{\substack{x,y \in \overline{W_1} \\ x \neq y}} \frac{|\varphi(x) - \varphi(y)|}{|x - y|} \le L.$$

Then, for any $m \in \mathbb{N}_{\ge 0}$, we have

(i) $F \in \mathcal{C}_b^s(\overline{W_2}, X) \quad \Rightarrow \quad F \circ \varphi \in \mathcal{C}_b^s(\overline{W_1}, X)$,

(ii) $F \in C^m\big(U, \mathcal{C}_b^s(\overline{W_2}, X)\big) \;\Rightarrow\; F \circ \varphi \in C^m\big(U, \mathcal{C}_b^s(\overline{W_1}, X)\big)$ *and*

(iii) $F \in C^m\big(U, \mathcal{L}\big(Z, \mathcal{C}_b^s(\overline{W_2}, X)\big)\big) \;\Rightarrow\; F \circ \varphi \in C^m\big(U, \mathcal{L}\big(Z, \mathcal{C}_b^s(\overline{W_1}, X)\big)\big)$

and there exists a constant $C_\varphi \geq 0$ *such that*

$$\|F \circ \varphi\|_{C^s(\overline{W_1}, X)} \leq C_\varphi \|F\|_{C^s(\overline{W_2}, X)}$$

holds for all $F \in \mathcal{C}_b^s(\overline{W_2}, X)$.
For $X = \mathbb{R}^n$*, the statements also hold if* $W_1 \subset \mathbb{R}^{d_1}$*,* $W_2 \subset \mathbb{R}^{d_2}$ *are open subsets with regular boundaries.*

Proof. Let $s = k + \alpha$ with $k \in \mathbb{N}_{\geq 0}$ and $\alpha \in [0, 1)$.

Ad (i) for $k = 0$: Clearly, $F \circ \varphi \in C_b^0(\overline{W_1}, X)$ holds with $\|F \circ \varphi\|_{C^0(\overline{W_1}, X)} \leq \|F\|_{C^0(\overline{W_2}, X)}$. For $\alpha > 0$, due to Lemma 2.68 there exists a constant $C_\varphi \geq 0$ with

$$[F \circ \varphi]^R_{C^\alpha(\overline{W_1}, X)} \leq C_\varphi [F]^{LR}_{C^\alpha(\overline{W_2}, X)}.$$

In particular, we thus have $F \circ \varphi \in \mathcal{C}^\alpha(\overline{W_1}, X)$ with

$$\begin{aligned}\|F \circ \varphi\|_{C^\alpha(\overline{W_1}, X)} &= \|F \circ \varphi\|_{C^0(\overline{W_1}, X)} + [F \circ \varphi]_{C^\alpha(\overline{W_1}, X)} \\ &\leq C_\varphi \left(\|F\|_{C^0(\overline{W_2}, X)} + [F]_{C^\alpha(\overline{W_2}, X)}\right) = C_\varphi \|F\|_{C^\alpha(\overline{W_2}, X)}.\end{aligned}$$

Ad (i) for $k > 0$: Clearly, $F \circ \varphi \in C_b^k(\overline{W_1}, X)$ holds with $\|F \circ \varphi\|_{C^k(\overline{W_1}, X)} \leq C_\varphi \|F\|_{C^k(\overline{W_2}, X)}$. So, assume $\alpha > 0$ in the following. On account of the chain rule, the derivatives of $F \circ \varphi$ consist of compositions of derivatives of F and derivatives of φ; more precisely, we have

$$D^\beta (F \circ \varphi)(x) \sim \sum_{l=1}^{|\beta|} D^l F\big(\varphi(x)\big) \left(\sum_{\substack{(\gamma^1, \dots, \gamma^l) \\ \sum_i \gamma^i = \beta}} \big[D^{\gamma^1} \varphi(x)\big] \dots \big[D^{\gamma^l} \varphi(x)\big] \right) \tag{2.9}$$

for all $|\beta| \leq k$ and $x \in \overline{W_1}$. Lemma 2.88 yields $D^\gamma \varphi \in \mathcal{C}^\alpha(\overline{W_1}, \mathbb{R}^{d_2})$ for all $|\gamma| \leq k$ and $D^l F \in \mathcal{C}^\alpha\big(\overline{W_2}, \mathcal{L}(\mathbb{R}^{d_2}, \dots, X)\big)$ for all $l \leq k$ with $\|D^l F\|_{C^\alpha(\overline{W_2}, \mathcal{L}(\mathbb{R}^{d_2}, \dots, X))} \leq \|F\|_{C^s(\overline{W_2}, X)}$. Due to the first part, $D^l F \circ \varphi \in \mathcal{C}^\alpha\big(\overline{W_1}, \mathcal{L}(\mathbb{R}^{d_2}, \dots, X)\big)$ follows for all $l \leq k$ with

$$\|D^l F \circ \varphi\|_{C^\alpha(\overline{W_1}, \mathcal{L}(\mathbb{R}^{d_2}, \dots, X))} \leq C_\varphi \|F\|_{C^s(\overline{W_2}, X)}.$$

On account of Proposition 2.94, where we choose $X_1 \coloneqq \mathcal{L}\big(\mathbb{R}^{d_2}, \mathcal{L}(\mathbb{R}^{d_2}, \dots, X)\big)$ and $X_2 \coloneqq \mathbb{R}^{d_2}$ as in Remark 2.96(iii), we thus obtain recursively

$$\big(D^l F \circ \varphi\big)[D^{\gamma^1} \varphi] \dots [D^{\gamma^l} \varphi] \in \mathcal{C}^\alpha(\overline{W_1}, X)$$

for all $l \leq k$ and $|\gamma^l| \leq k$ with

$$\big\|\big(D^l F \circ \varphi\big)[D^{\gamma^1} \varphi] \dots [D^{\gamma^l} \varphi]\big\|_{\mathcal{C}^\alpha(\overline{W_1}, X)} \leq C_\varphi \|F\|_{C^s(\overline{W_2}, X)}.$$

Finally, Equation (2.9) yields $D^\beta(F\circ\varphi)\in\mathcal{C}^\alpha(\overline{W_1},X)$ for all $|\beta|=k$ and therefore $F\circ\varphi\in\mathcal{C}^s(\overline{W_1},X)$ with

$$\|F\circ\varphi\|_{C^s(\overline{W_1},X)}\le\|F\circ\varphi\|_{C^k(\overline{W_1},X)}+\sum_{|\beta|=k}\|D^\beta(F\circ\varphi)\|_{C^\alpha(\overline{W_1},X)}\le C_\varphi\|F\|_{C^s(\overline{W_2},X)}.$$

Ad (ii), (iii): Let $F\in C^m\big(U,\mathcal{C}^s_b(\overline{W_2},X)\big)$ and $G\in C^m\big(U,\mathcal{L}\big(Z,\mathcal{C}^s_b(\overline{W_2},X)\big)\big)$. Due to part (i), we have $(F\circ\varphi)(u)=F(u)\circ\varphi\in\mathcal{C}^s_b(\overline{W_1},X)$ and $(G\circ\varphi)(u)[z]=G(u)[z]\circ\varphi\in\mathcal{C}^s_b(\overline{W_1},X)$ for all $u\in U$ and $z\in Z$. The estimate

$$\|\cdot\circ\varphi\|_{C^s(\overline{W_1},X)}\le C_\varphi\|\cdot\|_{C^s(\overline{W_2},X)}$$

proven in (i) yields $(G\circ\varphi)(u)\in\mathcal{L}\big(Z,\mathcal{C}^s_b(\overline{W_1},X)\big)$ for all $u\in U$ as well as

$$u\mapsto(F\circ\varphi)(u)\in C^0\big(U,\mathcal{C}^s_b(\overline{W_1},X)\big)\text{ and}$$
$$u\mapsto(G\circ\varphi)(u)\in C^0\big(U,\mathcal{L}\big(Z,\mathcal{C}^s_b(\overline{W_1},X)\big)\big).$$

We have $D^m(F\circ\varphi)(u)=D^mF(u)\circ\varphi$ and $D^m(G\circ\varphi)(u)[z]=D^mG(u)[z]\circ\varphi$ for all $u\in U$ and $z\in Z$. So, the claim for $m>0$ follows recursively. □

Remark 2.98. *We can choose open subsets $W_1\subset\mathbb{R}^{d_1}$, $W_2\subset\mathbb{R}^{d_2}$ without additional assumptions in Proposition 2.97, if*

$$D^\beta\varphi\in\mathcal{C}_b^{s-\lfloor s\rfloor}(\overline{W_1},\mathbb{R}^{d_2})\quad\textit{and}\quad D^\beta F\in\mathcal{C}_b^{s-\lfloor s\rfloor}(\overline{W_2},X)$$
$$\textit{with}\quad\|D^\beta F\|_{C^{s-\lfloor s\rfloor}(\overline{W_2},X)}\lesssim\|F\|_{C^s(\overline{W_2},X)}$$

still hold for all $|\beta|\le\lfloor s\rfloor$. Then, also

$$D^\beta(F\circ\varphi)\in\mathcal{C}_b^{s-\lfloor s\rfloor}(\overline{W_1},X)\quad\textit{with}\quad\|D^\beta(F\circ\varphi)\|_{C^{s-\lfloor s\rfloor}(\overline{W_1},X)}\le C_\varphi\|F\|_{C^s(\overline{W_2},X)}$$

holds for all $|\beta|\le\lfloor s\rfloor$. In particular, this is the case for $s<1$.

Lemma 2.99. *Let $M\subset\mathbb{R}^n$ be a d-dimensional $C^1\cap\mathcal{C}^s$-embedded closed submanifold for $s\in\mathbb{R}_{\ge0}$ and let $(\gamma_l,W_l)_{l=1,\dots,L}$ be a finite set of local parameterizations with $M\subset\bigcup_l\gamma_l(W_l)$ as in Remark 2.6(ii). Furthermore, let X,Y,Z be Banach spaces and let $U\subset Y$ an open subset. Then, for any $m\in\mathbb{N}_{\ge0}$, we have*

(i) $F\in C^m\big(U,\mathcal{C}^s(M,X)\big)$
$\Leftrightarrow F:U\times M\to X$ *and* $F\circ\gamma_l\in C^m\big(U,\mathcal{C}^s(\overline{W_l},X)\big)$ *for all $l=1,\dots,L$ as well as*

(ii) $F\in C^m\big(U,\mathcal{L}\big(Z,\mathcal{C}^s(M,X)\big)\big)$
$\Leftrightarrow F:U\times Z\times M\to X$ *and* $F\circ\gamma_l\in C^m\big(U,\mathcal{L}\big(Z,\mathcal{C}^s(\overline{W_l},X)\big)\big)$ *for all $l=1,\dots,L$.*

Proof. By definition, we have

$$F(u)\in\mathcal{C}^s(M,X)\Leftrightarrow F(u):M\to X\text{ and }(F\circ\gamma_l)(u)\in\mathcal{C}^s(\overline{W_l},X)\;\forall l,$$
$$G(u)[z]\in\mathcal{C}^s(M,X)\Leftrightarrow G(u)[z]:M\to X\text{ and }(G\circ\gamma_l)(u)[z]\in\mathcal{C}^s(\overline{W_l},X)\;\forall l$$

for any $u \in U$ and $z \in Z$. The equivalence $\|\cdot\|_{\mathcal{C}^s(M,X)} \sim \sum_l \|\cdot \circ \gamma_l\|_{\mathcal{C}^s(\overline{W_l},X)}$ implies

$$G(u) \in \mathcal{L}\big(Z, \mathcal{C}^s(M,X)\big) \Leftrightarrow G(u) : Z \times M \to X \text{ and } (G \circ \gamma_l)(u) \in \mathcal{L}\big(Z, \mathcal{C}^s(\overline{W_l}, X)\big)\ \forall l$$

for any $u \in U$ as well as

$$\begin{aligned} F \in C^0\big(U, \mathcal{C}^s(M,X)\big) &\Leftrightarrow F : U \times M \to X \text{ and } F \circ \gamma_l \in C^0\big(U, \mathcal{C}^s(\overline{W_l}, X)\big)\ \forall l, \\ G \in C^0\big(U, \mathcal{L}\big(Z, \mathcal{C}^s(M,X)\big)\big) &\Leftrightarrow G : U \times Z \times M \to X \text{ and} \\ &\qquad G \circ \gamma_l \in C^0\big(U, \mathcal{L}\big(Z, \mathcal{C}^s(\overline{W_l}, X)\big)\big)\ \forall l. \end{aligned}$$

Due to $D^m(F \circ \gamma_l)(u) = D^m F(u) \circ \gamma_l$ and $D^m(G \circ \gamma_l)(u)[z] = D^m G(u)[z] \circ \gamma_l$ for any $u \in U$ and $z \in Z$, the claim for $m > 0$ follows recursively. □

2.2.5 Regularity of Composition Operators

This section deals with the composition of a sufficiently smooth function with a (little) Hölder function. We formulate conditions on which the composition is a (little) Hölder function again and then discuss regularity properties for operators acting by composition with a sufficiently smooth function. In particular, we obtain the following result on matrix inversion: For a Hölder regular, matrix-valued mapping whose image is contained in the invertible matrices also the mapping onto the inverse of each image is Hölder regular.

In the following, let $W \subset \mathbb{R}^d$ be an open, bounded and convex subset and let $s \in \mathbb{R}_{\geq 0}$. Moreover, let X, Y, Z be Banach spaces and for any subset $U \subset Y$ define

$$\mathcal{C}^s(\overline{W}, U) := \big\{ u \in \mathcal{C}^s(\overline{W}, Y) \,\big|\, u(x) \in U \text{ for all } x \in \overline{W} \big\}.$$

As a start, we consider the composition of a linear operator with a (little) Hölder function.

Lemma 2.100. *Let $g : \overline{W} \to \mathcal{L}(Y,Z)$ and define $G(v) : \overline{W} \to Z$, $\big(G[v]\big)(x) := g(x)\big[v(x)\big]$ for any function $v : \overline{W} \to Y$. If $g \in \mathcal{C}^s(\overline{W}, \mathcal{L}(Y,Z))$, then $G \in \mathcal{L}\big(\mathcal{C}^s(\overline{W}, Y), \mathcal{C}^s(\overline{W}, Z)\big)$ holds with*

$$\|G\|_{\mathcal{L}(\mathcal{C}^s(\overline{W},Y), \mathcal{C}^s(\overline{W},Z))} \lesssim \|g\|_{C^s(\overline{W}, \mathcal{L}(Y,Z))}.$$

Proof. The well-definedness of $G : \mathcal{C}^s(\overline{W}, Y) \to \mathcal{C}^s(\overline{W}, Z)$ with

$$\|G(v)\|_{C^s(\overline{W},Z)} \lesssim \|g\|_{C^s(\overline{W},\mathcal{L}(Y,Z))} \|v\|_{C^s(\overline{W},Y)}$$

for every $v \in \mathcal{C}^s(\overline{W}, Y)$ is fulfilled due to Proposition 2.94 and Remark 2.96(iii). The linearity of G follows directly from the linearity of g. □

Now, we can formulate the main result of this section.

Proposition 2.101. *Let $U \subset Y$ be an open subset and $K \subset U$ a convex subset. Furthermore, let $f : U \to Z$ and define $F(u) : \overline{W} \to Z$, $\big(F(u)\big)(x) := f\big(u(x)\big)$ for any function $u : \overline{W} \to U$. Then the following hold:*

(i) *If* $f \in C^{\lfloor s\rfloor+1}(U,Z)$ *with* $f \in C_b^{\lfloor s\rfloor+1}(K,Z)$*, then we have* $F(u) \in \mathcal{C}^s(\overline{W},Z)$ *for all* $u \in \mathcal{C}^s(\overline{W},K)$*. In addition, for any* $R>0$ *there exists a* $C(R)>0$ *such that*

$$\|F(u)\|_{C^s(\overline{W},Z)} \le C(R)$$

holds for all $u \in \mathcal{C}^s(\overline{W},K)$ *with* $\|u\|_{C^s(\overline{W},Y)} \le R$*.*

(ii) *If* $f \in C^{\lfloor s\rfloor+2}(U,Z)$ *with* $f \in C_b^{\lfloor s\rfloor+2}(K,Z)$*, then* $F \in C^0\big(\mathcal{C}^s(\overline{W},K),\mathcal{C}^s(\overline{W},Z)\big)$*. In particular, for any* $R>0$ *there exists a* $C(R)>0$ *such that we have*

$$\|F(u_1)-F(u_2)\|_{C^s(\overline{W},Z)} \le C(R)\|u_1-u_2\|_{C^s(\overline{W},Y)}$$

for all $u_1,u_2 \in \mathcal{C}^s(\overline{W},K)$ *with* $\|u_j\|_{C^s(\overline{W},Y)} \le R$*. Moreover,* $F \in C_b^0\big(\mathcal{B},\mathcal{C}^s(\overline{W},Z)\big)$ *holds for all subsets* $\mathcal{B} \subset \mathcal{C}^s(\overline{W},K)$ *that are bounded in* $\mathcal{C}^s(\overline{W},Y)$*.*

(iii) *If* $f \in C^{k+\lfloor s\rfloor+2}(U,Z)$ *with* $f \in C_b^{k+\lfloor s\rfloor+2}(K,Z)$*, then* $F \in C^k\big(\mathcal{C}^s(\overline{W},V),\mathcal{C}^s(\overline{W},Z)\big)$ *and* $F \in C_b^k\big(\mathcal{B},\mathcal{C}^s(\overline{W},Z)\big)$ *hold for any* $k \in \mathbb{N}_{\ge 0}$*, any open subset* $V \subset K$ *and any bounded subset* $\mathcal{B} \subset \mathcal{C}^s(\overline{W},V)$*.*

Proof. First, we prove the statements (i) and (ii) for $s \in [0,1)$, i.e. $\lfloor s\rfloor = 0$.

Ad (i) Let $u \in \mathcal{C}^s(\overline{W},K)$. We have

$$\|F(u)\|_{C^0(\overline{W},Z)} = \sup_{x\in\overline{W}} \big\|f\big(u(x)\big)\big\|_Z \le \|f\|_{C^0(K,Z)} < \infty$$

and due to the mean value theorem

$$[F(u)]^r_{C^s(\overline{W},Z)} = \sup_{\substack{x,y\in\overline{W}\\ 0<\|x-y\|<r}} \frac{\big\|f\big(u(x)\big)-f\big(u(y)\big)\big\|_Z}{|x-y|^s} \le \|Df\|_{C^0(K,\mathcal{L}(Y,Z))}[u]^r_{C^s(\overline{W},Y)}$$

for all $r \in (0,\infty]$. In particular, $F(u) \in \mathcal{C}^s(\overline{W},Z)$ holds.
Let $R>0$ be fixed. For any $u \in \mathcal{C}^s(\overline{W},K)$ with $\|u\|_{C^s(\overline{W},Y)} \le R$ we have as before

$$\begin{aligned}\|F(u)\|_{C^s(\overline{W},Z)} &\le \|f\|_{C^0(K,Z)} + \|Df\|_{C^0(K,\mathcal{L}(Y,Z))}\|u\|_{C^s(\overline{W},Y)}\\ &\le \|f\|_{C^1(K,Z)}(1+R) =: C(R).\end{aligned}$$

Ad (ii) Let $u_1,u_2 \in \mathcal{C}^s(\overline{W},K)$ with $\|u_j\|_{C^s(\overline{W},Y)} \le R$. Due to the mean value theorem,

$$\begin{aligned}\|F(u_1)-F(u_2)\|_{C^0(\overline{W},Z)} &= \sup_{x\in\overline{W}} \big\|f\big(u_1(x)\big)-f\big(u_2(x)\big)\big\|_Z\\ &\le \sup_{x\in\overline{W}} \|Df\|_{C^0(K,\mathcal{L}(Y,Z))}\|u_1(x)-u_2(x)\|_Y\\ &\le \|f\|_{C^1(K,Z)}\|u_1-u_2\|_{C^0(\overline{W},Y)}\end{aligned}$$

holds and applying the mean value theorem twice yields

$$
\begin{aligned}
[F(u_1)-F(u_2)]_{C^s(\overline{W},Z)} &= \sup_{x\neq y\in\overline{W}} \frac{\big\|f\big(u_1(x)\big)-f\big(u_2(x)\big)-f\big(u_1(y)\big)+f\big(u_2(y)\big)\big\|_Z}{|x-y|^s} \\
&\leq \sup_{x\neq y\in\overline{W}} \left\{ \frac{\|D^2 f\|_{C^0(K,\mathcal{L}(Y,\mathcal{L}(Y,Z)))}\frac{\|(u_1+u_2)(x)-(u_1+u_2)(y)\|_Y}{2}\|u_1(x)-u_2(x)\|_Y}{|x-y|^s} \right. \\
&\qquad \left. +\frac{\|Df\|_{C^0(K,\mathcal{L}(Y,Z))}\|u_1(x)-u_2(x)-u_1(y)+u_2(y)\|_Y}{|x-y|^s} \right\} \\
&\leq \|f\|_{C^2(K,Z)}\frac{[u_1+u_2]_{C^s(\overline{W},Y)}}{2}\|u_1-u_2\|_{C^0(\overline{W},Y)}+\|f\|_{C^1(K,Z)}[u_1-u_2]_{C^s(\overline{W},Y)} \\
&\leq \|f\|_{C^2(K,Z)}\big(R\|u_1-u_2\|_{C^0(\overline{W},Y)}+[u_1-u_2]_{C^s(\overline{W},Y)}\big).
\end{aligned}
$$

Together, we obtain

$$
\begin{aligned}
\|F(u_1)-F(u_2)\|_{C^s(\overline{W},Z)} &\leq \|f\|_{C^2(K,Z)}(1+R)\|u_1-u_2\|_{C^s(\overline{W},Y)} \\
&=: C(R)\|u_1-u_2\|_{C^s(\overline{W},Y)}.
\end{aligned}
$$

Boundedness of the function $F:\mathcal{B}\to\mathcal{C}^s(\overline{W},Z)$ for a bounded subset $\mathcal{B}\subset\mathcal{C}^s(\overline{W},K)$ follows directly from the estimate in (i).

The general statements (i) and (ii) for arbitrary $s\in\mathbb{R}_{\geq 0}$ follow by mathematical induction on $\lfloor s\rfloor$, using Lemma 2.88 and the fact that differentiability of f and u implies differentiability of $F(u)$ and we have $\partial_{x_i}\big(F(u)\big)=A(u)\big(\partial_{x_i}u\big)$ with $A(v):\overline{W}\to\mathcal{L}(Y,Z)$, $\big(A(v)\big)(x):=Df\big(v(x)\big)$ for any function $v:\overline{W}\to U$. Applying the inductive hypothesis and Lemma 2.100 on A conclude the inductive step.

Ad (iii) We show the claim using mathematical induction: If $k=0$, it reduces to the statement of (ii). Assume that the claim is satisfied for a fixed $k\in\mathbb{N}_0$ and choose a function $f\in C^{k+\lfloor s\rfloor+3}(U,Z)\cap C_b^{k+\lfloor s\rfloor+3}(K,Z)$ as well as an open subset $V\subset K$ and a bounded subset $\mathcal{B}\subset C^s(\overline{W},V)$. In particular, we have $Df\in C^{k+\lfloor s\rfloor+2}(U,\mathcal{L}(Y,Z))\cap C_b^{k+\lfloor s\rfloor+2}(K,\mathcal{L}(Y,Z))$ and, for any $u:\overline{W}\to U$, we define $A(u):\overline{W}\to\mathcal{L}(Y,Z)$, $\big(A(u)\big)(x):=Df\big(u(x)\big)$. With this, the inductive hypothesis yields

$$
F\in C^k\big(\mathcal{C}^s(\overline{W},V),\mathcal{C}^s(\overline{W},Z)\big)\cap C_b^k\big(\mathcal{B},\mathcal{C}^s(\overline{W},Z)\big)
$$

and $A\in C^k\big(\mathcal{C}^s(\overline{W},V),\mathcal{C}^s(\overline{W},\mathcal{L}(Y,Z))\big)\cap C_b^k\big(\mathcal{B},\mathcal{C}^s(\overline{W},\mathcal{L}(Y,Z))\big)$. Applying Lemma 2.100,

$$
A\in C^k\big(\mathcal{C}^s(\overline{W},V),\mathcal{L}\big(\mathcal{C}^s(\overline{W},Y),\mathcal{C}^s(\overline{W},Z)\big)\big)\cap C_b^k\big(\mathcal{B},\mathcal{L}\big(\mathcal{C}^s(\overline{W},Y),\mathcal{C}^s(\overline{W},Z)\big)\big)
$$

follows. It remains to show that F is Fréchet-differentiable and its differential is given by $DF=A$. Then, $F\in C^{k+1}\big(\mathcal{C}^s(\overline{W},V),\mathcal{C}^s(\overline{W},Z)\big)\cap C_b^{k+1}\big(\mathcal{B},\mathcal{C}^s(\overline{W},Z)\big)$ holds and so the claim is also satisfied for $k+1$.
Fix $u_0\in\mathcal{C}^s(\overline{W},V)$. Due to $f\in C_b^{\lfloor s\rfloor+3}(V,Z)$, we have $D^2f\in C_b^{\lfloor s\rfloor+1}(V,\mathcal{L}(Y,\mathcal{L}(Y,Z)))$.

Therefore, the statement of (i) yields $D^2 f(u_0 + \theta h) \in \mathcal{C}^s\big(\overline{W}, \mathcal{L}(Y, \mathcal{L}(Y, Z))\big)$ with $\|D^2 f(u_0 + \theta h)\|_{C^s(\overline{W}, \mathcal{L}(Y, \mathcal{L}(Y,Z)))} \le C(\|u_0\|_{C^s(\overline{W},Y)} + 1) =: C(u_0)$ for all $\theta \in [0,1]$ and $h \in \mathcal{C}^s(\overline{W}, Y)$ with $\|h\|_{C^s(\overline{W},Y)}$ sufficiently small. With a taylor expansion, we receive for such h

$$
\begin{aligned}
&\big(F(u_0 + h) - F(u_0) - A(u_0)h\big)(x) \\
&= f\big(u_0(x) + h(x)\big) - f\big(u_0(x)\big) - Df\big(u_0(x)\big)h(x) \\
&= \int_0^1 (1-\theta) D^2 f\big(u_0(x) + \theta h(x)\big) h(x)^2 \, \mathrm{d}\theta.
\end{aligned}
$$

On account of the triangle inequality for integrals and Lemma 2.100, this implies

$$
\begin{aligned}
&\big\|F(u_0 + h) - F(u_0) - A(u_0)h\big\|_{C^s(\overline{W},Z)} \\
&\le \int_0^1 (1-\theta) \|D^2 f(u_0 + \theta h)\|_{C^s(\overline{W}, \mathcal{L}(Y, \mathcal{L}(Y,Z)))} \|h\|^2_{C^s(\overline{W},Y)} \, \mathrm{d}\theta \\
&\le C(u_0) \|h\|^2_{C^s(\overline{W},Y)}.
\end{aligned}
$$

So, F is Fréchet-differentiable in u_0 with $DF(u_0) = A(u_0)$. □

We derive two corollaries from this main result: The first one reduces to the case of a compact subset $K \subset U$ and the second one deals with a finite dimensional setting.

Corollary 2.102. *Let $U \subset Y$ be an open subset and $K \subset U$ a compact and convex subset. Furthermore, let $f : U \to Z$ and define $F(u) : \overline{W} \to Z$, $\big(F(u)\big)(x) := f\big(u(x)\big)$ for any function $u : \overline{W} \to U$. Then the following hold:*

(i) If $f \in C^{\lfloor s \rfloor + 1}(U, Z)$, then $F(u) \in \mathcal{C}^s(\overline{W}, Z)$ for all $u \in \mathcal{C}^s(\overline{W}, K)$. Moreover, for any $R > 0$ there exists a constant $C(K,R) > 0$ such that

$$\|F(u)\|_{C^s(\overline{W},Z)} \le C(K,R)$$

holds for all $u \in \mathcal{C}^s(\overline{W}, K)$ with $\|u\|_{C^s(\overline{W},Y)} \le R$.

(ii) If $f \in C^{\lfloor s \rfloor + 2}(U, Z)$, then $F \in C^0\big(\mathcal{C}^s(\overline{W}, K), \mathcal{C}^s(\overline{W}, Z)\big) \cap C^0_b\big(\mathcal{B}, \mathcal{C}^s(\overline{W}, Z)\big)$ holds for all bounded subsets $\mathcal{B} \subset \mathcal{C}^s(\overline{W}, K)$. Moreover, for any $R > 0$ there exists a constant $C(K,R) > 0$ such that

$$\|F(u_1) - F(u_2)\|_{C^s(\overline{W},Z)} \le C(K,R) \|u_1 - u_2\|_{C^s(\overline{W},Y)}$$

holds for all $u_1, u_2 \in \mathcal{C}^s(\overline{W}, K)$ with $\|u_j\|_{C^s(\overline{W},Y)} \le R$.

(iii) If $f \in C^{k + \lfloor s \rfloor + 2}(U, Z)$, then $F \in C^k\big(\mathcal{C}^s(\overline{W}, V), \mathcal{C}^s(\overline{W}, Z)\big) \cap C^k_b\big(\mathcal{B}, \mathcal{C}^s(\overline{W}, Z)\big)$ holds for any $k \in \mathbb{N}_{\ge 0}$, any open subset $V \subset K$ and any bounded subset $\mathcal{B} \subset \mathcal{C}^s(\overline{W}, V)$.

Proof.

Ad (i) We have $f \in C^{\lfloor s \rfloor + 1}(U, Z)$ and $K \subset U$ compact. Thus, $f \in C_b^{\lfloor s \rfloor + 1}(K, Z)$ holds with the convex subset $K \subset Y$. Proposition 2.101(i) implies $F(u) \in \mathcal{C}^s(\overline{W}, Z)$ for all $u \in \mathcal{C}^s(\overline{W}, K)$ and

$$\|F(u)\|_{\mathcal{C}^s(\overline{W}, Z)} \leq C(K, R)$$

for all $u \in \mathcal{C}^s(\overline{W}, K)$ with $\|u\|_{\mathcal{C}^s(\overline{W}, Y)} \leq R$.

Ad (ii) We have $f \in C^{\lfloor s \rfloor + 2}(U, Z)$ and $K \subset U$ compact. Thus, $f \in C_b^{\lfloor s \rfloor + 2}(K, Z)$ holds with the convex subset $K \subset Y$. Proposition 2.101(ii) yields $F \in C^0\big(\mathcal{C}^s(\overline{W}, K), \mathcal{C}^s(\overline{W}, Z)\big)$ and $F \in C_b^0\big(\mathcal{B}, \mathcal{C}^s(\overline{W}, Z)\big)$ for all bounded subsets $\mathcal{B} \subset \mathcal{C}^s(\overline{W}, K)$ as well as

$$\|F(u_1) - F(u_2)\|_{\mathcal{C}^s(\overline{W}, Z)} \leq C(K, R) \|u_1 - u_2\|_{\mathcal{C}^s(\overline{W}, Y)}$$

for all $u_1, u_2 \in \mathcal{C}^s(\overline{W}, K)$ with $\|u_j\|_{\mathcal{C}^s(\overline{W}, Y)} \leq R$.

Ad (iii) As K is convex with $V \subset K$, also the convex hull $\operatorname{conv} V \subset K$ of V is a subset of K. Its interior $\tilde{V} := (\operatorname{conv} V)^\circ$ therefore is an open and convex set with $\tilde{V} \subset K$. We then have $f \in C^{k + \lfloor s \rfloor + 2}(U, Z)$ and $K \subset U$ compact, $\tilde{V} \subset K$. Thus, $f \in C_b^{k + \lfloor s \rfloor + 2}(\tilde{V}, Z)$ holds with the open and convex subset $\tilde{V} \subset Y$. Proposition 2.101(iii) yields $F \in C^k\big(\mathcal{C}^s(\overline{W}, \tilde{V}), \mathcal{C}^s(\overline{W}, Z)\big)$ and $F \in C_b^k\big(\mathcal{B}, \mathcal{C}^s(\overline{W}, Z)\big)$ for all bounded subsets $\mathcal{B} \subset \mathcal{C}^s(\overline{W}, \tilde{V})$. As $V \subset Y$ is open, $V \subset \tilde{V}$ holds and therefore $F \in C^k\big(\mathcal{C}^s(\overline{W}, V), \mathcal{C}^s(\overline{W}, Z)\big)$ and $F \in C_b^k\big(\mathcal{B}, \mathcal{C}^s(\overline{W}, Z)\big)$ for all bounded subsets $\mathcal{B} \subset \mathcal{C}^s(\overline{W}, V)$ follows. □

Corollary 2.103. *Let $f : U \to \mathbb{R}^N$ for an open subset $U \subset \mathbb{R}^M$ and define $F(u) : \overline{W} \to \mathbb{R}^N$, $\big(F(u)\big)(x) := f\big(u(x)\big)$ for any function $u : \overline{W} \to U$. Then the following hold:*

(i) If $f \in C^{\lfloor s \rfloor + 1}(U, \mathbb{R}^N)$, then $F(u) \in \mathcal{C}^s(\overline{W}, \mathbb{R}^N)$ for all $u \in \mathcal{C}^s(\overline{W}, U)$.

(ii) If $f \in C^{k + \lfloor s \rfloor + 2}(U, \mathbb{R}^N)$, then $F \in C^k\big(\mathcal{C}^s(\overline{W}, U), \mathcal{C}^s(\overline{W}, \mathbb{R}^N)\big) \cap C_b^k\big(\mathcal{B}, \mathcal{C}^s(\overline{W}, \mathbb{R}^N)\big)$ holds for any $k \in \mathbb{N}_{\geq 0}$ and any bounded subset $\mathcal{B} \subset \mathcal{C}^s(\overline{W}, \mathcal{A})$ with $\mathcal{A} \subset \mathbb{R}^M$ closed and $\mathcal{A} \subset U$.

Proof. Assume $U = \mathbb{R}^M$ at first.

Ad (i) Let $f \in C^{\lfloor s \rfloor + 1}(\mathbb{R}^M, \mathbb{R}^N)$ and $u \in \mathcal{C}^s(\overline{W}, \mathbb{R}^M)$. With $R := \max_{x \in \overline{W}} |u(x)| < \infty$ and the compact and convex subset $K := \overline{B_R(0)} \subset \mathbb{R}^M$ we have $u \in \mathcal{C}^s(\overline{W}, K)$. Thus, Corollary 2.102(i) yields $F(u) \in \mathcal{C}^s(\overline{W}, \mathbb{R}^N)$.

Ad (ii) Let $f \in C^{k + \lfloor s \rfloor + 2}(\mathbb{R}^M, \mathbb{R}^N)$. Corollary 2.102(iii) yields $F \in C^k\big(\mathcal{C}^s(\overline{W}, V), \mathcal{C}^s(\overline{W}, \mathbb{R}^N)\big)$ and $F \in C_b^k\big(\mathcal{B}, \mathcal{C}^s(\overline{W}, \mathbb{R}^N)\big)$ for any open and bounded subset $V \subset \mathbb{R}^M$ and any bounded subset $\mathcal{B} \subset \mathcal{C}^s(\overline{W}, V)$. Therefore, $F \in C^k\big(\mathcal{C}^s(\overline{W}, \mathbb{R}^M), \mathcal{C}^s(\overline{W}, \mathbb{R}^N)\big)$ and $F \in C_b^k\big(\mathcal{B}, \mathcal{C}^s(\overline{W}, \mathbb{R}^N)\big)$ hold for all bounded subsets $\mathcal{B} \subset \mathcal{C}^s(\overline{W}, \mathbb{R}^M)$.

Now, assume $U \subsetneq \mathbb{R}^M$ and $f \in C^l(U,\mathbb{R}^N)$. For any compact subset $K \subset U$, choose a cut-off function $\xi \in C^\infty(\mathbb{R}^m,\mathbb{R})$ with $\xi \equiv 1$ on K, $\xi \equiv 0$ on $\mathbb{R}^M \setminus U$ and $0 \le \xi \le 1$. Then, define $\tilde{f} := \xi f \in C^l(\mathbb{R}^M,\mathbb{R}^N)$ so that $\tilde{f}_{|K} = f_{|K}$ and $\tilde{f}_{|\mathbb{R}^M\setminus U} \equiv 0$ hold. Furthermore, define $\tilde{F}(u) : \overline{W} \to \mathbb{R}^N$, $\big(\tilde{F}(u)\big)(x) := \tilde{f}\big(u(x)\big)$ for any function $u : \overline{W} \to \mathbb{R}^M$ so that $\tilde{F} = F$ follows on $\mathcal{C}^s(\overline{W},K)$.

Ad (i) For $u \in \mathcal{C}^s(\overline{W},U)$, the image $K := u(\overline{W}) \subset U$ is compact and $u \in \mathcal{C}^s(\overline{W},K)$ holds. For $l = \lfloor s \rfloor + 1$, we thus have $F(u) = \tilde{F}(u) \in \mathcal{C}^s(\overline{W},\mathbb{R}^N)$.

Ad (ii) If $l = k + \lfloor s \rfloor + 2$ holds, we have $\tilde{F} \in C^k\big(\mathcal{C}^s(\overline{W},\mathbb{R}^M),\mathcal{C}^s(\overline{W},\mathbb{R}^N)\big)$, which implies $\tilde{F} \in C^k\big(\mathcal{C}^s(\overline{W},U),\mathcal{C}^s(\overline{W},\mathbb{R}^N)\big)$. Because the compact set $K \subset U$ is arbitrary, this directly yields $F \in C^k\big(\mathcal{C}^s(\overline{W},U),\mathcal{C}^s(\overline{W},\mathbb{R}^N)\big)$.
Moreover, for any bounded subset $\mathcal{B} \subset \mathcal{C}^s(\overline{W},\mathcal{A})$, there exists a constant $R > 0$ with $|u(x)| \le \|u\|_{C^s(\overline{W},\mathbb{R}^M)} \le R$ for every $u \in \mathcal{B}$ and $x \in \overline{W}$ and thus $\mathcal{B} \subset \mathcal{C}^s(\overline{W},K)$ holds with the compact set $K := \mathcal{A} \cap \overline{B_R(0)} \subset U$. So, $F = \tilde{F} \in C_b^k\big(\mathcal{B},\mathcal{C}^s(\overline{W},\mathbb{R}^N)\big)$ follows. □

The following regularity properties of derivatives of composition operators are intuitive, but we still prove them rigorously.

Lemma 2.104 (Derivatives of Composition Operators)**.** *Let V, Y, Z be Banach spaces and let $U \subset Y$ be an open subset. Then we have*

(i) $F \in C^k\big(U,\mathcal{C}^{1+s}(\overline{W},Z)\big) \quad \Rightarrow \quad \partial_i\big(F(\cdot)\big) \in C^k\big(U,\mathcal{C}^s(\overline{W},Z)\big)$ *and*

(ii) $F \in C^k\big(U,\mathcal{L}\big(V,\mathcal{C}^{1+s}(\overline{W},Z)\big)\big) \quad \Rightarrow \quad \partial_i\big(F(\cdot)[\cdot]\big) \in C^k\big(U,\mathcal{L}\big(V,\mathcal{C}^s(\overline{W},Z)\big)\big)$

for all $i = 1,\dots,d$. Analogous statements hold for bounded functionals when replacing every C^k by C_b^k.

Proof. We fix $i \in \{1,\dots,d\}$ and let $k = 0$ at first, i.e., we choose $F \in C^0\big(U,\mathcal{C}^{1+s}(\overline{W},Z)\big)$ and $G \in C^0\big(U,\mathcal{L}\big(V,\mathcal{C}^{1+s}(\overline{W},Z)\big)\big)$. This implies $\partial_i\big(F(u)\big), \partial_i\big(G(u)[v]\big) \in \mathcal{C}^s(\overline{W},Z)$ for every $u \in U$ and $v \in V$. Due to the linearity of the partial derivative ∂_i and the functional G, also $\partial_i\big(G(u)[\cdot]\big)$ is linear for all $u \in U$ with

$$\big\|\partial_i\big(G(u)[v]\big)\big\|_{C^s(\overline{W},Z)} \le \|G(u)[v]\|_{C^{1+s}(\overline{W},Z)} \le \|G(u)\|_{\mathcal{L}(V,\mathcal{C}^{1+s}(\overline{W},Z))} \tag{2.10}$$

for all $v \in V$ with $\|v\|_V \le 1$. In particular, $\partial_i\big(G(u)[\cdot]\big) \in \mathcal{L}\big(V,\mathcal{C}^s(\overline{W},Z)\big)$ holds for all $u \in U$. The linearity of the partial derivative ∂_i also implies

$$\|\partial_i\big(F(u_1)\big) - \partial_i\big(F(u_2)\big)\|_{C^s(\overline{W},Z)} \le \|F(u_1) - F(u_2)\|_{C^{1+s}(\overline{W},Z)} \text{ and}$$
$$\|\partial_i\big(G(u_1)[\cdot]\big) - \partial_i\big(G(u_2)[\cdot]\big)\|_{\mathcal{L}(V,\mathcal{C}^s(\overline{W},Z))} \le \|G(u_1) - G(u_2)\|_{\mathcal{L}(V,\mathcal{C}^{1+s}(\overline{W},Z))}$$

for $u_1, u_2 \in U$. On account of the continuity of F and G, thus $\partial_i\big(F(\cdot)\big) \in C^0\big(U,\mathcal{C}^s(\overline{W},Z)\big)$ as well as $\partial_i\big(G(\cdot)[\cdot]\big) \in C^0\big(U,\mathcal{L}\big(V,\mathcal{C}^s(\overline{W},Z)\big)\big)$ follow. In the case of boundedness of F and G, due to

$$\big\|\partial_i\big(F(u)\big)\big\|_{C^s(\overline{W},Z)} \le \|F(u)\|_{C^{1+s}(\overline{W},Z)}$$

for all $u \in U$ and Estimate (2.10), also $\partial_i\big(F(\cdot)\big)$ and $\partial_i\big(G(\cdot)[\cdot]\big)$ are bounded mappings. Now, let $k = 1$ and choose $F \in C^1_{(b)}\big(U, \mathcal{C}^{1+s}(\overline{W}, Z)\big)$. For all $u \in U$, we have

$$\begin{aligned}
&\lim_{\|y\|_Y \to 0} \frac{\big\|\partial_i\big(F(u+y)\big) - \partial_i\big(F(u)\big) - \partial_i\big(DF(u)[y]\big)\big\|_{C^s(\overline{W},Z)}}{\|y\|_Y} \\
&\le \lim_{\|y\|_Y \to 0} \frac{\|F(u+y) - F(u) - DF(u)[y]\|_{C^{1+s}(\overline{W},Z)}}{\|y\|_Y} = 0
\end{aligned}$$

and therefore $\partial_i\big(F(\cdot)\big) : U \to \mathcal{C}^s(\overline{W}, Z)$ is Fréchet-differentiable with

$$D\big(\partial_i\big(F(\cdot)\big)\big) = \partial_i\big(DF(\cdot)[\cdot]\big) \in C^0_{(b)}\big(U, \mathcal{L}\big(Y, \mathcal{C}^s(\overline{W}, Z)\big)\big)$$

on account of the first part of the proof. Hence, finally $\partial_i\big(F(\cdot)\big) \in C^1_{(b)}\big(U, \mathcal{C}^s(\overline{W}, Z)\big)$ follows. That $G \in C^1_{(b)}\big(U, \mathcal{L}\big(V, \mathcal{C}^{1+s}(\overline{W}, Z)\big)\big)$ implies $\partial_i\big(G(\cdot)[\cdot]\big) \in C^1_{(b)}\big(U, \mathcal{L}\big(V, \mathcal{C}^s(\overline{W}, Z)\big)\big)$ and the claim for $k > 1$ can be proven recursively. ☐

The next rather basic remark will be applied in Chapter 4 on the combination of regular composition operators.

Remark 2.105 (Products of Banach-valued Functions). *Let X, Y, Z and V be Banach spaces and let $U \subset Y$ an open subset. Additionally, let $(X, \cdot)$ be an $\mathbb{R}$-algebra such that $\|x_1 \cdot x_2\|_X \lesssim \|x_1\|_X \|x_2\|_X$ holds for all $x_1, x_2 \in X$. With this pointwise multiplication in X,*

$$\begin{aligned}
C^k(U,X) \times C^k(U,X) &\to C^k(U,X), \\
C^k(U,X) \times C^k\big(U,\mathcal{L}(Z,X)\big) &\to C^k\big(U,\mathcal{L}(Z,X)\big) \quad \textit{and} \\
C^k\big(U,\mathcal{L}(V,X)\big) \times C^k\big(U,\mathcal{L}(Z,X)\big) &\to C^k\big(U,\mathcal{L}\big(V,\mathcal{L}(Z,X)\big)\big)
\end{aligned}$$

are well-defined by [Ruz06, §2 Satz 2.7(ii)]. Analogous statements hold for bounded functionals when replacing every C^k by C^k_b.

The statements on regularity of composition operators deduced above imply in particular this useful property for the inverse of a matrix.

Remark 2.106 (Hölder Regularity for the Inverse of a Matrix). *Let $W \subset \mathbb{R}^d$ be an open, bounded and convex subset and let $s \in \mathbb{R}_{\ge 0}$. The set of invertible matrices*

$$U := \{A \in \mathbb{R}^{n\times n} \mid \det A \ne 0\}$$

is an open subset of $\mathbb{R}^{n\times n}$. For the matrix inversion mapping $f : U \to \mathbb{R}^{n\times n}$, $f(A) := A^{-1}$, we have $f \in C^1(U, \mathbb{R}^{n\times n})$ with

$$Df(A)[H] = -f(A) \cdot H \cdot f(A)$$

for all $A \in U$ and $H \in \mathbb{R}^{n\times n}$. Remark 2.105 yields $Df \in C^1(U, \mathcal{L}(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}))$ and then recursively, $f \in C^\infty(U, \mathbb{R}^{n\times n})$ follows. Corollary 2.103(ii) thus implies

$$(\cdot)^{-1} \in C^\infty\big(\mathcal{C}^s(\overline{W}, U), \mathcal{C}^s(\overline{W}, \mathbb{R}^{n\times n})\big) \cap C^\infty_b\big(\mathcal{B}, \mathcal{C}^s(\overline{W}, \mathbb{R}^{n\times n})\big)$$

for the inversion $(\cdot)^{-1}$ of matrices with $\mathcal{B} \subset \mathcal{C}^s(\overline{W}, \mathcal{A})$ an arbitrary bounded subset and $\mathcal{A} \subset \mathbb{R}^{n\times n}$ closed with $\mathcal{A} \subset U$. In particular, for any $A \in \mathcal{C}^s(\overline{W}, \mathbb{R}^{n\times n})$ with $\det A \ne 0$ on $\overline{W}$, also $A^{-1} \in \mathcal{C}^s(\overline{W}, \mathbb{R}^{n\times n})$ holds.

2.3 Generators of Semigroups

The theory of semigroups provides an abstract approach for solving linear, time-dependent problems

$$\partial_t u = Au \text{ in } (0,T), \tag{2.11a}$$

$$u(0) = u_0 \tag{2.11b}$$

for $u : [0,T) \to X$ with a Banach space X. In Chapter 4, the application of this theory to the linearization of our PDE will play an important role for proving the existence of short time solutions.

If X is finite dimensional, the solution to the problem (2.11) above clearly is given by $u(t) = \exp(At)u_0$. If X is infinite dimensional but $A : \mathcal{D}(A) \subset X \to X$ is a bounded linear operator, the exponential of the operator A still is well-defined via the power series

$$\exp(At) = \sum_{n\in\mathbb{N}} \frac{t^n}{n!} A^n$$

and the solution to problem (2.11) still can be expressed as $u(t) = \exp(At)u_0$ if $u_0 \in \mathcal{D}(A)$. For unbounded operators A, the power series does not converge in general so that a different approach is necessary. The theory of semigroups yields such a generalization of the concept of $\exp(At)$ for unbounded operators $A : \mathcal{D}(A) \subset X \to X$. It is based on the characterization

$$\exp(At) = \lim_{n\to\infty} \left(\mathrm{Id} + \tfrac{t}{n}A\right)^n$$

that, in general, also diverges for unbounded operators but allows for the modification

$$\exp(At) = \big(\exp(-At)\big)^{-1} = \lim_{n\to\infty} \left(\mathrm{Id} - \tfrac{t}{n}A\right)^{-n} = \lim_{n\to\infty} \left(\tfrac{n}{t}\right)^n \left(\tfrac{n}{t}\mathrm{Id} - A\right)^{-n}. \tag{2.12}$$

Instead of taking powers of A, we now take powers of the resolvent $\left(\frac{n}{t}\mathrm{Id} - A\right)^{-1}$ which, as long as it exists, is a bounded operator - even if A is unbounded.

As before, we do not wish to give a full introduction to semigroups, but only gather the definitions and statements used in this work. We refer to [RR06, Chapter 12] for a self-contained but simple and to [Paz92] for a very detailed discussion of so-called strongly continuous semigroups. Besides, [Lun12, Chapter 2] offers an approach that does not assume continuity of the semigroup from the very beginning.

2.3.1 Strongly Continuous Semigroups

If $A \in \mathbb{R}^{n\times n}$ is a matrix, $\exp(A\cdot 0) = \mathrm{Id}$ and $\exp\big(A(t+s)\big) = \exp(At)\exp(As)$ hold for all $t,s \in \mathbb{R}_{\geq 0}$. This motivates the following definition of a family of linear operators with the same properties.

Definition 2.107 (Semigroup and C^0-Semigroup)**.** *Let X be a Banach space. A family $\big(T(t)\big)_{t\geq 0} \subset \mathcal{L}(X,X)$ of linear operators is called a semigroup in X if*

$$T(0) = \mathrm{Id} \qquad \text{and} \qquad T(t+s) = T(t)T(s)$$

hold for all $t,s \geq 0$. Additionally, it is called a strongly continuous or C^0-semigroup if the mapping $[0,\infty) \to X$, $t \mapsto T(t)x$ is continuous for every $x \in X$.

It can be shown that it is sufficient to assume continuity of $t \mapsto T(t)x$ at $t = 0$ for every $x \in X$ for a semigroup to be strongly continuous (see [RR06, Remark 12.2]). Also, every strongly continuous semigroup fulfills a growth condition in the following sense (cf. [RR06, Theorem 12.7]).

Lemma 2.108. *Let X be a Banach space and let $\big(T(t)\big)_{t\geq 0}$ be a C^0-semigroup in X. There exist $\omega \in \mathbb{R}$ and $M > 0$ with*

$$\|T(t)\|_{\mathcal{L}(X,X)} \leq M \exp(\omega t) \text{ for every } t \geq 0.$$

Our aim is to generalize $\exp(At)$ for unbounded operators A by characterizing it via a semigroup $\big(T(t)\big)_{t\geq 0}$. So, we need to establish a connection between the unbounded operator and the semigroup. Because we have $\frac{\mathrm{d}}{\mathrm{d}t} \exp(At) = A \exp(At)$ and in particular $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \exp(At) = A$ for matrices $A \in \mathbb{R}^{n\times n}$, the following definition arises naturally.

Definition 2.109 (Infinitesimal Generator)**.** *Let X be a Banach space and let $\big(T(t)\big)_{t\geq 0}$ be a semigroup in X. Define*

$$Ax \coloneqq \lim_{t\searrow 0} \frac{T(t)x - x}{t}$$

and $\mathcal{D}(A)$ as the set of all $x \in X$ such that the limit above exists. Then, $A : \mathcal{D}(A) \to X$ is called the infinitesimal generator of the semigroup. We also say that A generates the semigroup.

From the definition, it is not clear that $\mathcal{D}(A) \neq \{0\}$. But the statement in Lemma 2.111 (cf. [RR06, Theorem 12.12]) guarantees that, indeed, the notion of infinitesimal generators is meaningful. To be able to formulate this lemma, we first have to define closed operators and introduce the so-called graph norm.

Definition 2.110. *Let $(X, \|\cdot\|_X)$ be a Banach space, let $\mathcal{D}(A) \subset X$ be a linear subset and let $A : \mathcal{D}(A) \to X$ be a linear operator. We call A a closed operator, if*

$$\big\{(x, Ax) \,\big|\, x \in \mathcal{D}(A)\big\} \subset X \times X$$

is closed and define the graph norm

$$\|x\|_{\mathcal{D}(A)} \coloneqq \|x\|_X + \|Ax\|_X$$

for $x \in \mathcal{D}(A)$.

Note that if A is a closed operator, $\big(\mathcal{D}(A), \|\cdot\|_{\mathcal{D}(A)}\big)$ is a Banach space and we have $A \in \mathcal{L}\big(\mathcal{D}(A), X\big)$.

Lemma 2.111. *Let X be a Banach space and let $A : \mathcal{D}(A) \subset X \to X$ generate a C^0-semigroup in X. Then, $\mathcal{D}(A) \subset X$ is dense and A is a closed operator.*

In particular, every strongly continuous semigroup has an infinitesimal generator. The main task when applying the theory of semigroups to PDEs often is to recognize such generators of semigroups. Many characterizations of generators set conditions on the resolvent set, whose definition we recall here.

Definition 2.112 (Resolvent Set). *Let X be a $\mathbb{K}$-Banach space, let $\mathcal{D}(A) \subset X$ be a linear subset and let $A : \mathcal{D}(A) \to X$ be a linear operator. The resolvent set of A is defined as all the values $\lambda \in \mathbb{K}$ such that $\lambda \mathrm{Id} - A : \mathcal{D}(A) \to X$ is bijective with $(\lambda \mathrm{Id} - A)^{-1} \in \mathcal{L}(X, X)$.*

Lemma 2.113. *Let X be a Banach space, let $\mathcal{D}(A) \subset X$ be a linear subset and let the linear operator $A : \mathcal{D}(A) \to X$ be such that the resolvent set is non-empty. Then, A is a closed operator.*

Proof. Let $\lambda \in \mathbb{K}$ be an element of the resolvent set of A. We have $(\lambda \mathrm{Id} - A)^{-1} \in \mathcal{L}(X, X)$ and thus $(\lambda \mathrm{Id} - A)^{-1}$ is a closed operator, i.e. $\{(z, (\lambda \mathrm{Id} - A)^{-1} z) \mid z \in X\} \subset X \times X$ is closed. Define

$$f : X \times X \to X \times X, \quad f(x, y) := (\lambda x - y, x).$$

Because f is continuous and

$$f\big(\{(x, Ax) \mid x \in \mathcal{D}(A)\}\big) = \{((\lambda \mathrm{Id} - A)x, x) \mid x \in \mathcal{D}(A)\} = \{(z, (\lambda \mathrm{Id} - A)^{-1} z) \mid z \in X\}$$

holds, also $\{(x, Ax) \mid x \in \mathcal{D}(A)\} \subset X \times X$ is closed and therefore A is a closed operator. □

Now, we can formulate a first characterization for generators: The theorem of Hille and Yosida (see [RR06, Theorem 12.17]) yields a sufficient and necessary condition for an operator to generate a strongly continuous semigroup.

Proposition 2.114 (Hille-Yosida). *Let X be a Banach space, let $A : \mathcal{D}(A) \to X$ be a closed operator with $\mathcal{D}(A) \subset X$ dense and let $\omega \in \mathbb{R}$, $M \geq 1$. Then, the following statements are equivalent:*

(a) The operator A generates a C^0-semigroup $(T(t))_{t \geq 0}$ with $\|T(t)\|_{\mathcal{L}(X,X)} \leq M \exp(\omega t)$.

(b) Every real number $\lambda > \omega$ belongs to the resolvent set of A and furthermore fulfills $\|(\lambda \mathrm{Id} - A)^{-n}\|_{\mathcal{L}(X,X)} \leq \frac{M}{(\lambda - \omega)^n}$ for every $n \in \mathbb{N}$.

The proof of [RR06, Theorem 12.17] yields

$$T(t) = \lim_{n \to \infty} \left(\mathrm{Id} - \tfrac{t}{n} A\right)^{-n}$$

which corresponds to Formula (2.12) for the matrix exponential that we predicted to be extendable to the case of unbounded operators. Additionally, if an operator A generates a C^0-semigroup, this C^0-semigroup is unique (see [Paz92, Theorem 1.2.6]). Therefore, using the notation $(\exp(At))_{t \geq 0}$ for the C^0-semigroup generated by an operator A is totally consistent.

2.3.2 Analytic Semigroups in Complex Banach Spaces

In Chapter 4, we will make use of a class of semigroups that yield better regularity properties for solutions to the initial value problem (2.11) than strongly continuous semigroups do in general. We do not discuss these regularity properties for different kinds of semigroups but refer the interested reader to [Paz92, Chapter 4]. In Proposition 2.131, we only

state the result for the special class of semigroups that is applied in Chapter 4.
For this, we first have to define this class of semigroups, which are called analytic semigroups. As we need a complex setting to describe analytic mappings, we restrict to complex Banach spaces for the time being. In the next section, we explain how we can return to a complex setting in the case of real Banach spaces.
So far, we have considered semigroups with domain $[0,\infty)$. Now, we want to extend this domain to a region in the complex plane. In order to preserve the semigroup structure, the new domain in the complex plane has to be of a suitable form. We will use sectors around $[0,\infty)$ and for this, we define

$$S_{\omega,\theta} := \left\{ z \in \mathbb{C} \setminus \{\omega\} \,\middle|\, |\arg(z-\omega)| < \theta \right\}$$

for $\omega \in \mathbb{R}$ and $\theta \in (0,\pi)$. Then,

$$S_\theta := S_{0,\theta}$$

is an extension of the open subset $(0,\infty) \subset \mathbb{R}$ to an open sector of $\mathbb{C}$.

Definition 2.115 (Analytic Semigroup)**.** *Let X be a complex Banach space and let $\theta \in \left(0, \frac{\pi}{2}\right)$. A family $\left(T(z)\right)_{z \in S_\theta \cup \{0\}} \subset \mathcal{L}(X,X)$ of linear operators is called a semigroup in X if*

$$T(0) = \mathrm{Id} \qquad \textit{and} \qquad T(z+w) = T(z)T(w)$$

hold for all $z, w \in S_\theta$. Additionally, it is called a strongly continuous or C^0-semigroup if the mapping $S_\theta \cup \{0\} \to X$, $z \mapsto T(z)x$ is continuous for every $x \in X$. Furthermore, it is called an analytic semigroup if the mapping $S_\theta \to \mathcal{L}(X,X)$, $z \mapsto T(z)$ is analytic.

Definition 2.116 (Analytic Semigroup II)**.** *Let X be a complex Banach space. A (C^0-) semigroup $\left(T(t)\right)_{t \geq 0}$ in X is called an analytic (C^0-)semigroup if there exists a $\theta \in \left(0, \frac{\pi}{2}\right)$ and an extension $\left(\widetilde{T}(z)\right)_{z \in S_\theta \cup \{0\}}$ such that the extension is an analytic (C^0-)semigroup.*

Again, we are interested in identifying generators of semigroups. Proposition 2.114 provided a characterization for strongly continuous semigroups. We refer to [Lun12, Chapter 2] for sufficient or necessary conditions for generators of analytic semigroups that are not strongly continuous. As only the results for both analytic and strongly continuous semigroups will be applied in this work, we restrict to this setting here. In [RR06, Theorem 12.31], Renardy and Rogers prove the following possibility to characterize analytic C^0-semigroups (note that they only call them analytic semigroups):

Proposition 2.117. *Let X be a complex Banach space, let $\mathcal{D}(A) \subset X$ be a dense, linear subset and let $A : \mathcal{D}(A) \to X$ be a linear operator. Then, the following statements are equivalent:*

(a) The operator A generates an analytic C^0-semigroup.

(b) There exist $\omega \in \mathbb{R}$ and $M > 0$ such that $S_{\omega,\frac{\pi}{2}}$ is contained in the resolvent set of A and $\|(\lambda \mathrm{Id} - A)^{-1}\|_{\mathcal{L}(X,X)} \leq \frac{M}{|\lambda - \omega|}$ holds for every $\lambda \in S_{\omega,\frac{\pi}{2}}$.

Proof. The statement is proven in [RR06, Theorem 12.31], provided that A is a closed operator. But if (a) holds, A is a closed operator due to Lemma 2.111 and if (b) holds, the resolvent set of A is non-empty so that Lemma 2.113 also implies that A is a closed operator. Hence, the additional condition of closedness is needless. □

Similar to [Lun12, Proposition 2.1.11], we show the following proposition.

Proposition 2.118. *Let X be a complex Banach space, let $\mathcal{D}(A) \subset X$ be a linear subset and let $A : \mathcal{D}(A) \to X$ be a linear operator. Then, the following statements are equivalent:*

(a) There exist $\omega \in \mathbb{R}$ and $M > 0$ such that $S_{\omega,\frac{\pi}{2}}$ is contained in the resolvent set of A and $\|(\lambda \mathrm{Id} - A)^{-1}\|_{\mathcal{L}(X,X)} \leq \frac{M}{|\lambda-\omega|}$ holds for every $\lambda \in S_{\omega,\frac{\pi}{2}}$.

(b) There exist $\omega \in \mathbb{R}$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$ and $M > 0$ such that $S_{\omega,\theta}$ is contained in the resolvent set of A and $\|(\lambda \mathrm{Id} - A)^{-1}\|_{\mathcal{L}(X,X)} \leq \frac{M}{|\lambda-\omega|}$ holds for every $\lambda \in S_{\omega,\theta}$.

Proof. Due to $S_{\omega,\frac{\pi}{2}} \subset S_{\omega,\theta}$ for every $\theta > \frac{\pi}{2}$, it is clear that (b) implies (a). So, we only have to show the other implication. For this, let $\theta := \pi - \arctan(2M) \in \left(\frac{\pi}{2}, \pi\right)$ and fix $\lambda \in S_{\omega,\theta} \setminus S_{\omega,\frac{\pi}{2}}$ such that $\lambda \in \mathbb{C} \setminus \{\omega\}$ holds with $\frac{\pi}{2} \leq |\arg(\lambda - \omega)| < \theta < \pi$. In particular, we have $\mathrm{Re}(\lambda - \omega) \leq 0$ and $\mathrm{Im}\ \lambda \neq 0$. We claim, that $\mathrm{Re}\ \lambda > \omega - \frac{|\mathrm{Im}\ \lambda|}{2M}$ holds. For $\mathrm{Re}(\lambda - \omega) = 0$, the statement is clear due to $\mathrm{Im}\ \lambda \neq 0$. So, let $\mathrm{Re}(\lambda - \omega) < 0$ and $\mathrm{Im}\ \lambda > 0$. We have

$$\pi + \arctan\left(\frac{|\mathrm{Im}\ \lambda|}{\mathrm{Re}\ \lambda - \omega}\right) = |\arg\ (\lambda - \omega)| < \theta = \pi + \arctan(-2M)$$

and because the arcus tangens is monotonically increasing, this implies

$$\frac{|\mathrm{Im}\ \lambda|}{\mathrm{Re}\ \lambda - \omega} < -2M \quad \Leftrightarrow \quad \mathrm{Re}\ \lambda >= \omega - \frac{|\mathrm{Im}\ \lambda|}{2M}$$

as claimed. With

$$\tilde{\omega} := \mathrm{Re}\ \lambda + \frac{|\mathrm{Im}\ \lambda|}{2M} > \omega \qquad \text{and} \qquad r := |\mathrm{Im}\ \lambda| > 0,$$

we have

$$\tilde{\omega} - \frac{r}{2M} \pm ir = \mathrm{Re}\ \lambda + \frac{|\mathrm{Im}\ \lambda|}{2M} - \frac{|\mathrm{Im}\ \lambda|}{2M} + i\mathrm{Im}\ \lambda = \mathrm{Re}\ \lambda + i\mathrm{Im}\ \lambda = \lambda.$$

For every $\tilde{w} > \omega$ and every $r > 0$, it holds that $\mathrm{Re}(\tilde{w} \pm ir) = \tilde{\omega} > \omega \Leftrightarrow \tilde{\omega} \pm ir \in S_{\omega,\frac{\pi}{2}}$ and thus $\tilde{w} \pm ir$ belongs to the resolvent set of A. By [Lun12, Proposition A.0.3], then also $B_R(\tilde{w} \pm ir)$ is contained in the resolvent set of A for

$$R := \left\|\left((\tilde{\omega} \pm ir)\mathrm{Id} - A\right)^{-1}\right\|_{\mathcal{L}(X,X)}^{-1} \geq \frac{|\tilde{w} \pm ir - \omega|}{M} \geq \frac{r}{M}.$$

In particular, $B_{\frac{r}{M}}(\tilde{\omega} \pm ir)$ is contained in and thus $\lambda = \tilde{\omega} - \frac{r}{2M} \pm ir$ belongs to the resolvent set of A. Furthermore, with $\lambda_0 := \tilde{w} \pm ir$, we have by [Lun12, Formula A.0.4]

$$\|(\lambda \mathrm{Id} - A)^{-1}\|_{\mathcal{L}(X,X)} \leq \sum_{n \in \mathbb{N}_{\geq 0}} |\lambda - \lambda_0|^n \|(\lambda_0 \mathrm{Id} - A)^{-(n+1)}\|_{\mathcal{L}(X,X)} \leq \sum_{n \in \mathbb{N}_{\geq 0}} \frac{r^n}{2^n M^n} \frac{M^{n+1}}{|\lambda_0 - \omega|^{n+1}}$$

with

$$|\lambda_0 - \omega|^2 = |\tilde{\omega} - \omega|^2 + |r|^2 \geq r^2 \quad \Leftrightarrow \quad |\lambda_0 - \omega| \geq r.$$

Moreover, we claim that there exists $C > 0$ with $r \geq C^{-1}|\lambda - \omega|$. For this, note at first that $\lambda \notin S_{\omega,\frac{\pi}{2}} \Leftrightarrow \mathrm{Re}(\lambda - \omega) \leq 0$ implies

$$\tilde{\omega} - \omega = \mathrm{Re}\ \lambda + \frac{|\mathrm{Im}\ \lambda|}{2M} - \omega \leq \frac{|\mathrm{Im}\ \lambda|}{2M} = \frac{r}{2M} \leq \frac{r}{M}.$$

With this,

$$|\lambda - \omega|^2 = \left|\tilde{\omega} - \omega - \frac{r}{2M} \pm ir\right|^2 = |\tilde{\omega} - \omega|^2 + \frac{r^2}{4M^2} - \frac{r}{M}(\tilde{\omega} - \omega) + r^2 \leq r^2\left(1 + \frac{1}{4M^2}\right)$$

follows and thus $r \geq C^{-1}|\lambda - \omega|$ holds with $C = \left(1 + \frac{1}{4M^2}\right)^{\frac{1}{2}} \geq 1$. Overall, we therefore have

$$\|(\lambda \mathrm{Id} - A)^{-1}\|_{\mathcal{L}(X,X)} \leq \sum_{n \in \mathbb{N}_{\geq 0}} \frac{1}{2^n}\frac{M}{r} = \frac{2M}{r} \leq \frac{2MC}{|\lambda - \omega|}.$$

As $\lambda \in S_{\omega,\theta} \setminus S_{\omega,\frac{\pi}{2}}$ was arbitrary, together with the assumptions in (a), we conclude that $S_{\omega,\theta}$ is contained in the resolvent set of A and

$$\|(\lambda \mathrm{Id} - A)^{-1}\|_{\mathcal{L}(X,X)} = \frac{2MC}{|\lambda - \omega|}$$

holds for every $\lambda \in S_{\omega,\theta}$. □

Remark 2.119. *On account of Propositions 2.117 and 2.118, in the case of a dense, linear subset $\mathcal{D}(A) \subset X$, the operator A generates an analytic C^0-semigroup if and only if it is sectorial in the sense of [Lun12, Definition 2.0.1]. We thus can apply all the results from [Lun12, Chapter 2] proven for sectorial operators also for our generators of analytic C^0-semigroups.*

2.3.3 Analytic Semigroups in Real Banach Spaces

As mentioned in the introduction to this Section 2.3, we want to apply the theory of semigroups in Chapter 4 to prove the existence of short-time solutions. There, we use little Hölder spaces which are not complex but real Banach spaces. Thus, we have to extend the concept of analytic semigroups in complex Banach spaces to the setting of real Banach spaces. This is achieved by complexification of the real Banach space and all occurring operators, which we present in the following remark.

Remark 2.120 (Complexification)**.** *Let $(X, +, \cdot, \|\cdot\|_X)$ be a real Banach space. We set $\widetilde{X} := X \times X$ and define*

$$\oplus : \quad \widetilde{X} \times \widetilde{X} \to \widetilde{X}, \quad \tilde{x} \oplus \tilde{y} := (x_1 + y_1, x_2 + y_2),$$

$$\odot : \quad \mathbb{C} \times \widetilde{X} \to \widetilde{X}, \quad c \odot \tilde{x} := (ax_1 - bx_2, ax_2 + bx_1) \text{ as well as}$$

$$\|\cdot\|_{\widetilde{X}} : \quad \widetilde{X} \to \mathbb{R}, \quad \|\tilde{x}\|_{\widetilde{X}} := \sup_{\varphi \in [0,2\pi]} \left(\|x_1 \cos\varphi - x_2 \sin\varphi\|_X^2 + \|x_1 \sin\varphi + x_2 \cos\varphi\|_X^2\right)^{1/2}$$

for all $\tilde{x} = (x_1, x_2), \tilde{y} = (y_1, y_2) \in \widetilde{X}$ *and* $c = a + ib \in \mathbb{C}$*. Then,* $(\widetilde{X}, \oplus, \odot, \|\cdot\|_{\widetilde{X}})$ *is a complex Banach space, called the complexification of* X*. In particular, for every* $x, y \in X$*,*

$$\begin{aligned}\|(x,0)\|_{\widetilde{X}}^2 = \|x\|_X^2 \le \frac{1}{2}\big(\|x-y\|_X^2 + \|x+y\|_X^2\big) \\ = \big\|x\cos\tfrac{\pi}{4} - y\sin\tfrac{\pi}{4}\big\|_X^2 + \big\|x\sin\tfrac{\pi}{4} + y\cos\tfrac{\pi}{4}\big\|_X^2 \le \|(x,y)\|_{\widetilde{X}}^2\end{aligned}$$

holds and for $(\tilde{x}_n)_{n\in\mathbb{N}} = \big((x_n^1, x_n^2)\big)_{n\in\mathbb{N}} \subset \widetilde{X}$ *and* $\tilde{x} = (x^1, x^2) \in \widetilde{X}$*, we hence have*

$$\tilde{x}_n \to \tilde{x} \text{ in } \widetilde{X} \quad \Leftrightarrow \quad x_n^i \to x^i \text{ in } X \text{ for } i = 1, 2.$$

For a linear operator $A : \mathcal{D}(A) \subset X \to X$*, define*

$$\mathcal{D}(\widetilde{A}) := \mathcal{D}(A) \times \mathcal{D}(A) \subset \widetilde{X} \qquad \text{and} \qquad \widetilde{A}\tilde{x} := (Ax_1, Ax_2)$$

for all $\tilde{x} = (x_1, x_2) \in \mathcal{D}(\widetilde{A})$*. Then,* $\widetilde{A} : \mathcal{D}(\widetilde{A}) \to \widetilde{X}$ *is a linear operator called the complexification of* A*. If* $A \in \mathcal{L}(X, X)$*, then also* $\widetilde{A} \in \mathcal{L}(\widetilde{X}, \widetilde{X})$ *holds. Furthermore,* $(T(t))_{t\ge0} \subset \mathcal{L}(X, X)$ *is a (*C^0*-)semigroup if and only if* $(\widetilde{T}(t))_{t\ge0} \subset \mathcal{L}(\widetilde{X}, \widetilde{X})$ *is a (*C^0*-) semigroup.*

Because a complex setting is necessary to describe analytic mappings, we have to extend to the complexification of a semigroup to clarify if it is an analytic semigroup. Matching the properties of (C^0-)semigroups and their complexifications from the remark above, we introduce the following definition.

Definition 2.121 (Analytic Semigroup for Real Banach Spaces)**.** *Let* X *be a real Banach space. A (*C^0*-)semigroup* $\big(T(t)\big)_{t\ge0}$ *in* X *is called an analytic (*C^0*-)semigroup if its complexification* $\big(\widetilde{T}(t)\big)_{t\ge0}$ *is an analytic (*C^0*-) semigroup in* $\widetilde{X}$*.*

As before, we are interested in identifying generators of semigroups. To transfer the characterization from Proposition 2.117 to the setting of real Banach spaces, we prove that an operator $A : \mathcal{D}(A) \subset X \to X$ generates an analytic C^0-semigroup in a real Banach space X if and only if its complexification $\widetilde{A} : \mathcal{D}(\widetilde{A}) \to \widetilde{X}$ generates an analytic C^0-semigroup in the complexification $\widetilde{X}$.

Lemma 2.122. *Let* $A : \mathcal{D}(A) \subset X \to X$ *generate an analytic* C^0*-semigroup* $\big(T(t)\big)_{t\ge0}$*. Its complexification* $\widetilde{A} : \mathcal{D}(\widetilde{A}) \to \widetilde{X}$ *generates the complexification* $\big(\widetilde{T}(t)\big)_{t\ge0}$*.*

Proof. By construction, the complexification $\widetilde{T}(t)(x_1, x_2) = \big(T(t)x_1, T(t)x_2\big)$ for $t \ge 0$ and $x_1, x_2 \in X$ defines an analytic C^0-semigroup $\big(\widetilde{T}(t)\big)_{t\ge0}$. On account of the convergence properties in $\widetilde{X}$ from Remark 2.120, we have

$$\begin{aligned}&\left\{\tilde{x} \in \widetilde{X} \,\middle|\, \lim_{t\searrow0} \frac{\widetilde{T}(t)\tilde{x} - \tilde{x}}{t} \text{ exists in } \widetilde{X}\right\} \\ &= \left\{(x_1, x_2) \in \widetilde{X} \,\middle|\, \lim_{t\searrow0} \frac{T(t)x_1 - x_1}{t} \text{ and } \lim_{t\searrow0} \frac{T(t)x_2 - x_2}{t} \text{ exist in } X\right\} \\ &= \big\{(x_1, x_2) \in \widetilde{X} \,\big|\, x_1, x_2 \in \mathcal{D}(A)\big\} = \mathcal{D}(\widetilde{A})\end{aligned}$$

as well as

$$\lim_{t\searrow 0}\frac{\widetilde{T}(t)\tilde{x}-\tilde{x}}{t}=\left(\lim_{t\searrow 0}\frac{T(t)x_1-x_1}{t},\lim_{t\searrow 0}\frac{T(t)x_2-x_2}{t}\right)=\big(Ax_1,Ax_2\big)=\widetilde{A}\tilde{x}$$

for all $\tilde{x}=(x_1,x_2)\in\mathcal{D}(\widetilde{A})$. Therefore, $\widetilde{A}$ generates $\big(\widetilde{T}(t)\big)_{t\geq 0}$. $\square$

Lemma 2.123. *Let $\mathcal{D}(A)\subset X$ be a linear subspace and let $A:\mathcal{D}(A)\to X$ be a linear operator. If its complexification $\widetilde{A}:\mathcal{D}(\widetilde{A})\to\widetilde{X}$ generates an analytic C^0-semigroup $\big(\widetilde{T}(t)\big)_{t\geq 0}$, then $\widetilde{T}(t)(X\times\{0\})\subset X\times\{0\}$ holds for every $t\geq 0$. In particular, A generates an analytic C^0-semigroup $\big(T(t)\big)_{t\geq 0}$ whose complexification is $\big(\widetilde{T}(t)\big)_{t\geq 0}$.*

Proof. By Remark 2.119, $\widetilde{A}:\mathcal{D}(\widetilde{A})\to\widetilde{X}$ is sectorial in the sense of [Lun12, Definition 2.0.1]. On account of [Lun12, Corollary 2.1.3], $\widetilde{T}(t)(X\times\{0\})\subset X\times\{0\}$ follows for all $t\geq 0$. With similar arguments as Lunardi uses in her proof of [Lun12, Corollary 2.1.3], $\big[\widetilde{T}(t)(x_1,x_2)\big]_1=\big[\widetilde{T}(t)(x_2,x_1)\big]_2$ holds for all $x_1,x_2\in X$ and $t\geq 0$:
By construction, $\big[\widetilde{A}(x_1,x_2)\big]_1=Ax_1=\big[\widetilde{A}(x_2,x_1)\big]_2$ is valid. For sufficiently large $n\in\mathbb{N}$, Proposition 2.117 yields well-definedness of the operator $\widetilde{A}_n:=n\widetilde{A}(n\mathrm{Id}_{\widetilde{X}}-\widetilde{A})^{-1}$ and we have $\big[\widetilde{A}_n(x_1,x_2)\big]_1=\big[\widetilde{A}_n(x_2,x_1)\big]_2$. Due to [Lun12, Proposition 2.1.2], $\widetilde{A}_n$ generates an analytic C^0-semigroup $\big(\widetilde{T}_n(t)\big)_{t\geq 0}$ and because $\widetilde{A}_n$ is a bounded operator, $\widetilde{T}_n(t)=\sum_{k\in\mathbb{N}}\frac{t^k}{k!}\widetilde{A}_n^k$ holds as in the proof of [Lun12, Corollary 2.1.3]. Thus, $\big[\widetilde{T}_n(t)(x_1,x_2)\big]_1=\big[\widetilde{T}_n(t)(x_2,x_1)\big]_2$ follows. Due to $\widetilde{T}_n(t)\tilde{x}\to\widetilde{T}(t)\tilde{x}$ for all $\tilde{x}\in\widetilde{X}$ by [Lun12, Proposition 2.1.2], we hence have $\big[\widetilde{T}(t)(x_1,x_2)\big]_1=\big[\widetilde{T}(t)(x_2,x_1)\big]_2$ as claimed.
Define

$$T(t):X\to X,\ \ T(t)x:=\big[\widetilde{T}(t)(x,0)\big]_1$$

for all $x\in X$ and $t\geq 0$. Then, $\big(T(t)\big)_{t\geq 0}\subset\mathcal{L}(X,X)$ is well-defined due to the properties of $\|\cdot\|_{\widetilde{X}}$ in Remark 2.120. With the considerations above, we have

$$\begin{aligned}\widetilde{T}(t)(x_1,x_2)&=\widetilde{T}(t)(x_1,0)+\widetilde{T}(t)(0,x_2)\\&=\Big(\big[\widetilde{T}(t)(x_1,0)\big]_1,0\Big)+\Big(\big[\widetilde{T}(t)(x_2,0)\big]_2,\big[\widetilde{T}(t)(x_2,0)\big]_1\Big)\\&=\big(T(t)x_1,0\big)+\big(0,T(t)x_2\big)=\big(T(t)x_1,T(t)x_2\big)\end{aligned}$$

for all $(x_1,x_2)\in\widetilde{X}$, such that $\widetilde{T}(t)$ is the complexification of $T(t)$. Finally, $\big(T(t)\big)_{t\geq 0}$ is an analytic C^0-semigroup because $\big(\widetilde{T}(t)\big)_{t\geq 0}$ is one. $\square$

Remark 2.124. *On account of Lemmas 2.122 and 2.123, in the case of a real Banach space X, an operator A generates an analytic C^0-semigroup $\big(T(t)\big)_{t\geq 0}$ in X if and only if its complexification $\widetilde{A}$ generates the complexified analytic C^0-semigroup $\big(\widetilde{T}(t)\big)_{t\geq 0}$ in $\widetilde{X}$. We thus can transfer all the results for generators of analytic C^0-semigroups in complex Banach spaces to generators of analytic C^0-semigroups in real Banach spaces.*

In particular, the transfer of the characterization for generators of analytic C^0-semigroups from Proposition 2.117 to the setting of real Banach spaces yields the following statement.

Proposition 2.125. *Let X be a real Banach space, let $\mathcal{D}(A) \subset X$ be a dense, linear subset and let $A : \mathcal{D}(A) \to X$ be a linear operator. Then, the following statements are equivalent:*

(a) The operator A generates an analytic C^0-semigroup.

(b) There exist $\omega \in \mathbb{R}$ and $M > 0$ such that $S_{\omega,\frac{\pi}{2}}$ is contained in the resolvent set of the complexification $\widetilde{A}$ and $\|(\lambda \mathrm{Id} - \widetilde{A})^{-1}\|_{\mathcal{L}(\widetilde{X},\widetilde{X})} \le \frac{M}{|\lambda-\omega|}$ holds for every $\lambda \in S_{\omega,\frac{\pi}{2}}$.

We also receive the following result for pertubations of generators in real Banach spaces.

Lemma 2.126 (Pertubation of Infinitesimal Generators)**.** *Let X be a real Banach space, let $A : \mathcal{D}(A) \subset X \to X$ generate an analytic C^0-semigroup and let $B \in \mathcal{L}\big(\mathcal{D}(B), X\big)$ be a bounded, linear operator such that $\mathcal{D}(B) \in J_\theta\big(X, \mathcal{D}(A)\big)$ holds for some $\theta \in [0,1)$. Then, also $A + B : \mathcal{D}(A) \to X$ generates an analytic C^0-semigroup.*

Here, J_θ is a certain class of intermediate spaces introduced in Definition 2.78.

Proof. By Lemma 2.111, $\mathcal{D}(A) \subset X$ is a dense subset. Thus, on account of Remarks 2.119, 2.120 and 2.124, the claim follows with [Lun12, Proposition 2.4.1(i)]. □

In the following, we discuss certain intermediate spaces between $\mathcal{D}(A)$ and X that will play an important role for the application of the theory of semigroups to differential equations. We define them for arbitrary Banach spaces and later state a relation between those intermediate spaces in a real setting and in the corresponding complexificated setting.

Definition 2.127. *Let $(X, \|\cdot\|_X)$ be a Banach space and let $A : \mathcal{D}(A) \subset X \to X$ generate an analytic C^0-semigroup $\big(T(t)\big)_{t\ge 0}$ in X. For $\beta \in (0,1)$, we define*

$$\mathcal{D}_A(\beta,\infty) := \left\{ x \in X \,\middle|\, \sup_{0<s\le 1} s^{1-\beta} \big\|AT(s)x\big\|_X < \infty \right\} \quad \textit{and}$$
$$\mathcal{D}_A(\beta) := \left\{ x \in \mathcal{D}_A(\beta,\infty) \,\middle|\, \lim_{s \searrow 0} s^{1-\beta} AT(s)x = 0 \right\}$$

with

$$\|x\|_{\mathcal{D}_A(\beta)} := \|x\|_{\mathcal{D}_A(\beta,\infty)} := \|x\|_X + \sup_{0<s\le 1} s^{1-\beta} \big\|AT(s)x\big\|_X.$$

The spaces $\mathcal{D}_A(\beta,\infty)$ and $\mathcal{D}_A(\beta)$ are intermediate spaces between the Banach spaces $\big(\mathcal{D}(A), \|\cdot\|_{\mathcal{D}(A)}\big)$ and $(X, \|\cdot\|_X)$ with $\mathcal{D}(A) \hookrightarrow \mathcal{D}_A(\beta) \hookrightarrow \mathcal{D}_A(\beta,\infty) \hookrightarrow X$. Even more, they are given as interpolation spaces:

Lemma 2.128. *Let X be a Banach space, let $A : \mathcal{D}(A) \subset X \to X$ generate an analytic C^0-semigroup and let $\beta \in (0,1)$. Then,*

$$\mathcal{D}_A(\beta,\infty) = \big(X, \mathcal{D}(A)\big)_{\beta,\infty} \quad \textit{and} \quad \mathcal{D}_A(\beta) = \big(X, \mathcal{D}(A)\big)_\beta$$

hold with equivalent norms.

Proof. By Lemma 2.111, $\mathcal{D}(A) \subset X$ is a dense subset. Thus, on account of Remark 2.119, the proof is given in [Lun12, Proposition 2.2.2]. □

Lemma 2.129. *Let X be a real Banach space, let $A : \mathcal{D}(A) \subset X \to X$ generate an analytic C^0-semigroup, let $\widetilde{A} : \mathcal{D}(\widetilde{A}) \to \widetilde{X}$ be its complexification and let $\beta \in (0,1)$. Then, we have*

$$\mathcal{D}_{\widetilde{A}}(\beta,\infty) = \mathcal{D}_A(\beta,\infty) \times \mathcal{D}_A(\beta,\infty) \qquad \text{and} \qquad \mathcal{D}_{\widetilde{A}}(\beta) = \mathcal{D}_A(\beta) \times \mathcal{D}_A(\beta).$$

Proof. Let Y be another real Banach space with $Y \hookrightarrow X$ and let $\widetilde{Y}$ be its complexification. With K as in Definition 2.80 and the properties of $\|\cdot\|_{\widetilde{X}}, \|\cdot\|_{\widetilde{Y}}$ from Remark 2.120, for every $\tilde{x} = (x_1, x_2) \in \widetilde{X}$ and $t > 0$, we have

$$\begin{aligned} K(\tilde{x}, t, \widetilde{X}, \widetilde{Y}) &= \inf_{\substack{\tilde{x}=\tilde{a}+\tilde{b}, \\ \tilde{a}\in\widetilde{X},\tilde{b}\in\widetilde{Y}}} \left\{ \|\tilde{a}\|_{\widetilde{X}} + t\|\tilde{b}\|_{\widetilde{Y}} \right\} \\ &\leq \inf_{\substack{x_i=a_i+b_i, \\ a_i\in X,b_i\in Y}} \left\{ \|a_1\|_X + \|a_2\|_X + t\|b_1\|_Y + t\|b_2\|_Y \right\} \\ &= K(x_1, t, X, Y) + K(x_2, t, X, Y) \end{aligned}$$

as well as

$$\begin{aligned} K(x_j, t, X, Y) &= \inf_{\substack{x_j=a_j+b_j, \\ a_j\in X,b_j\in Y}} \left\{ \|a_j\|_X + t\|b_j\|_Y \right\} \\ &\leq \inf_{\substack{x_i=a_i+b_i, \\ a_i\in X,b_i\in Y}} \left\{ \|(a_1,a_2)\|_{\widetilde{X}} + t\|(b_1,b_2)\|_{\widetilde{Y}} \right\} \\ &= K(\tilde{x}, t, \widetilde{X}, \widetilde{Y}) \end{aligned}$$

for $j = 1,2$. This implies in particular $(\widetilde{X},\widetilde{Y})_{\theta,p} = (X,Y)_{\theta,p} \times (X,Y)_{\theta,p}$ as well as $(\widetilde{X},\widetilde{Y})_{\theta} = (X,Y)_{\theta} \times (X,Y)_{\theta}$ for every $\theta \in (0,1]$ and $p \in [1,\infty]$. With Lemma 2.128 thus the claim follows. □

The following Definition 2.130 will be useful for investigating maximal regularity for the initial value problem (2.13), see below.

Definition 2.130. *Let X be a Banach space and let $A : \mathcal{D}(A) \subset X \to X$ generate an analytic C^0-semigroup. For $\beta \in (0,1)$ and $T > 0$, we define*

$$\left(h^\beta([0,T],X) \times \mathcal{D}(A)\right)_+ := \left\{ (f,x) \in h^\beta([0,T],X) \times \mathcal{D}(A) \,\middle|\, Ax + f(0) \in \mathcal{D}_A(\beta) \right\}$$

with

$$\|(f,x)\|_{(h^\beta([0,T],X)\times\mathcal{D}(A))_+} := \|f\|_{h^\beta([0,T],X)} + \|x\|_{\mathcal{D}(A)} + \|Ax + f(0)\|_{\mathcal{D}_A(\beta)}$$

for $(f,x) \in \left(h^\beta([0,T],X) \times \mathcal{D}(A)\right)_+$.

Again, for a real Banach space X and its complexification $\widetilde{X}$ as well as a generator A of an analytic C^0-semigroup in X and its complexification $\widetilde{A}$, we have

$$\left(h^\beta([0,T],\widetilde{X}) \times \mathcal{D}(\widetilde{A})\right)_+ = \left(h^\beta([0,T],X) \times \mathcal{D}(A)\right)_+ \times \left(h^\beta([0,T],X) \times \mathcal{D}(A)\right)_+.$$

This follows directly from Remark 2.66 and Lemma 2.129.

2.3.4 Application to PDEs

We return to our linear, time-dependent problem (2.11) that we extend to an inhomogeneous problem by a non-vanishing right-hand side f, such that we now observe the initial value problem

$$\partial_t u - Au = f \text{ in } (0,T), \tag{2.13a}$$
$$u(0) = u_0 \tag{2.13b}$$

for $u : [0,T) \to X$ with a Banach space X. We will show for arbitrary Banach spaces X, that if $A : \mathcal{D}(A) \subset X \to X$ generates an analytic C^0-semigroup, $Au_0 + f(0) \in \mathcal{D}_A(\beta)$ is the suitable compability condition such that $h^\beta([0,T],X)$ is of maximal regularity for the initial value problem (2.13) (see Proposition 2.131). Afterwards, we choose X and a subspace $X_0 \subset X$ as a little Hölder spaces and derive an improved regularity statement for preimages x with $Ax \in X_0$ (see Lemma 2.132). Finally, we discuss differential operators A acting on little Hölder spaces and formulate a condition that guarantees them to generate analytic C^0-semigroups (see Propositions 2.135 and 2.139).

Proposition 2.131 (Maximal Regularity)**.** *Let X be a Banach space and let the operator $A : \mathcal{D}(A) \subset X \to X$ generate an analytic C^0-semigroup. Furthermore, let $\beta \in (0,1)$ and $T \in (0,1]$. We have*

(i) $h^{1+\beta}([0,T],X) \cap h^\beta\big([0,T],\mathcal{D}(A)\big) \hookrightarrow C^1\big([0,T],\mathcal{D}_A(\beta)\big)$ *and*

(ii) $L_T : h^{1+\beta}([0,T],X) \cap h^\beta\big([0,T],\mathcal{D}(A)\big) \to \big(h^\beta([0,T],X) \times \mathcal{D}(A)\big)_+$, $L_T[\rho] := \binom{\partial_t\rho - A\rho}{\rho(0)}$ *is bijective with* $\sup_{0<T\le 1} \|L_T^{-1}\|_{\mathcal{L}} < \infty$.

Proof. If X is a real Banach space, let $\widetilde{A} : \mathcal{D}(\widetilde{A}) \to \widetilde{X}$ be the complexification of A. If X already is a complex Banach space, set $\widetilde{X} := X$ and $\widetilde{A} := A$. On account of Remarks 2.124 and 2.119, then $\widetilde{A} : \mathcal{D}(\widetilde{A}) \to \widetilde{X}$ is sectorial in the sense of [Lun12, Definition 2.0.1].

Ad (i) [Lun12, Proposition 2.2.12(ii)] yields the statement for the complex Banach space $\widetilde{X}$. Due to Remark 2.66 and Lemma 2.129, this also implies the statement for real Banach spaces X.

Ad (ii) Clearly, any $\rho \in h^{1+\beta}([0,T],X) \cap h^\beta\big([0,T],\mathcal{D}(A)\big)$ fulfills

$$L_T\rho = \big(\partial_t\rho - A\rho, \rho(0)\big) \in h^\beta([0,T],X) \times \mathcal{D}(A).$$

With (i), we also have $A\rho(0) + (\partial_t\rho - A\rho)(0) = \partial_t\rho(0) \in \mathcal{D}_A(\beta)$ and thus

$$L_T\rho \in \big(h^\beta([0,T],X) \times \mathcal{D}(A)\big)_+$$

holds. Hence, L_T is well-defined.
Let $(f,x) \in \big(h^\beta([0,T],X) \times \mathcal{D}(A)\big)_+$ and define

$$(\tilde{f},\tilde{x}) := \big((f,0),(x,0)\big) \in \big(h^\beta([0,T],\widetilde{X}) \times \mathcal{D}(\widetilde{A})\big)_+.$$

Moreover, set $\widetilde{L_T} := (L_T, L_T)$ such that we have $\widetilde{L_T}[\rho] = \binom{\partial_t\rho - \widetilde{A}\rho}{\rho(0)}$. By [Lun12, Definition 4.1.4], there exists a solution $\tilde{\rho} = (\rho_1, \rho_2)$ of $\widetilde{L_T}\tilde{\rho} = (\tilde{f}, \tilde{x})$ and due to [Lun12, Corollary 4.3.2],

$$\tilde{\rho} \in h^{1+\beta}([0,T], \widetilde{X}) \cap h^{\beta}\big([0,T], \mathcal{D}(\widetilde{A})\big)$$

holds. With Remark 2.66, we have in particular

$$\rho := \rho_1 \in h^{1+\beta}([0,T], X) \cap h^{\beta}\big([0,T], \mathcal{D}(A)\big)$$

with $L_T\rho = (f, x)$ in $[0,T] \times X$. Uniqueness of the solution follows with [Lun12, Proposition 4.1.2].

Finally, [Lun12, Theorem 4.3.1(iii)] yields boundedness of $\sup_{0<T\le 1} \|\widetilde{L_T}^{-1}\|_{\mathcal{L}}$, which also implies boundedness of $\sup_{0<T\le 1} \|L_T^{-1}\|_{\mathcal{L}}$. □

Lemma 2.132 (Improved Regularity for Preimages)**.** *Let $s_1, s_2 \in (0,2) \setminus \{1\}$ with $s_1 < s_2$. Let $M \subset \mathbb{R}^{d+1}$ be a h^{2+s_2}-embedded closed hypersurface and let $A : h^{2+s_i}(M) \to h^{s_i}(M)$ generate an analytic C^0-semigroup for both $i \in \{1,2\}$. Then, any $v \in h^{2+s_1}(M)$ with $Av \in h^{s_2}(M)$ already fulfills $v \in h^{2+s_2}(M)$.*

Proof. Because

$$A : h^{2+s_i}(M) \to h^{s_i}(M)$$

generates an analytic C^0-semigroup, the theorem of Hille and Yosida (Proposition 2.114) yields that

$$\lambda \mathrm{Id} - A : h^{2+s_i}(M) \to h^{s_i}(M)$$

is bijective for sufficiently large $\lambda > 0$. For any $v \in h^{2+s_1}(M) \subset h^{s_2}(M)$ with $Av \in h^{s_2}(M)$, we have $\lambda v - Av \in h^{s_2}(M)$. Due to the bijectivity of $\lambda \mathrm{Id} - A : h^{2+s_2}(M) \to h^{s_2}(M)$, there exists a unique $\tilde{v} \in h^{2+s_2}(M)$ with

$$\lambda\tilde{v} - A\tilde{v} = \lambda v - Av \text{ in } h^{s_2}(M) \hookrightarrow h^{s_1}(M).$$

Due to the bijectivity of $\lambda \mathrm{Id} - A : h^{2+s_1}(M) \to h^{s_1}(M)$, we have $v = \tilde{v}$ in $h^{2+s_1}(M)$ and therefore $v = \tilde{v} \in h^{s+s_2}(M)$ holds. □

In the following, we will prove that every symmetric, elliptic differential operator of second order generates an analytic C^0-semigroup in the setting of little Hölder spaces. For the readers convenience, we recall the definition of ellipticity.

Definition/Lemma 2.133 ((Uniform) Ellipticity)**.** *Let $\Omega \subset \mathbb{R}^d$ be an arbitrary subset. A matrix valued function $a : \Omega \to \mathbb{R}^{n\times n}$ is called*

(i) elliptic (or positive definite on Ω), if

$$\xi^\top a(x)\xi > 0$$

holds for every $x \in \Omega$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ and *(ii) uniformly elliptic, if there exists $C > 0$ so that*

$$\xi^\top a(x)\xi \geq C|\xi|^2$$

holds for every $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

If Ω is compact, the two properties coincide.

For simplicity, the following short notation is used.

Definition 2.134. *For two matrices $N, \widetilde{N} \in \mathbb{R}^{n\times n}$, we define*

$$N : \widetilde{N} := \sum_{i,j=1}^{n} N_{ij}\widetilde{N}_{ij}.$$

To begin with, we prove the claimed statement for little Hölder spaces defined on the whole space $\mathbb{R}^d$.

Proposition 2.135. *Let $s \in (0,2) \setminus \{1\}$. Furthermore, let $A \in \mathcal{L}\big(h^{2+s}(\mathbb{R}^d), h^s(\mathbb{R}^d)\big)$ be a symmetric, uniformly elliptic differential operator of second order, i.e.*

$$Av = a : D^2 v + b \cdot \nabla v + cv$$

holds for every $v \in h^{2+s}(\mathbb{R}^d)$, where $a \in h^s(\mathbb{R}^d, \mathbb{R}^{d\times d})$, $b \in h^s(\mathbb{R}^d, \mathbb{R}^d)$ and $c \in h^s(\mathbb{R}^d, \mathbb{R})$ hold and the matrix a is symmetric and uniformly elliptic on $\mathbb{R}^d$. Then,

$$A : \mathcal{D}(A) \to h^s(\mathbb{R}^d)$$

generates an analytic C^0-semigroup with $\mathcal{D}(A) = h^{2+s}(\mathbb{R}^d)$ and equivalent norms.

Proof. As a, b, c are Hölder-continuous on $\mathbb{R}^d$, they are in particular bounded and uniformly continuous such that the conditions of [Lun12, beginning of Section 3.1] are satisfied. On account of [Lun12, Theorem 3.1.14], the complexification of

$$A : \mathcal{D}(A) \subset h^s(\mathbb{R}^d) \to h^s(\mathbb{R}^d)$$

is sectorial in the sense of [Lun12, Definition 2.0.1]. Due to [Lun12, Corollar 3.1.16], we have $\mathcal{D}(A) = h^{2+s}(\mathbb{R}^d)$ with equivalence of the norms. As $h^{2+s}(\mathbb{R}^d) \subset h^s(\mathbb{R}^d)$ is dense by Lemma 2.92, finally Remarks 2.119 and 2.124 yield that

$$A : h^{2+s}(\mathbb{R}^d) \to h^s(\mathbb{R}^s)$$

generates an analytic C^0-semigroup. $\square$

We want to transfer this result for functions defined on the whole space $\mathbb{R}^d$ to functions defined on hypersurfaces. As a first step, we discuss extensions of functions that are only defined on an open subset of $\mathbb{R}^d$ to the whole space $\mathbb{R}^d$.

Remark 2.136. *Let $s \in \mathbb{R}_{\geq 0}$, let $M \subset \mathbb{R}^{d+1}$ be a $C^1 \cap \mathcal{C}^s$-embedded closed hypersurface and let (γ, W) be a local parameterization of M. Furthermore, let $w \in \mathcal{C}^s(\overline{W}, \mathbb{R})$ with* $\operatorname{supp} w \subset W$*. Then, by extension with zero, i.e.*

$$\widetilde{w} : \mathbb{R}^d \to \mathbb{R},\ \widetilde{w}(x) \coloneqq \begin{cases} w(x) & \text{if } x \in \overline{W}, \\ 0 & \text{else} \end{cases} \qquad \text{and}$$

$$\widetilde{w} \circ \gamma^{-1} : M \to \mathbb{R},\ \widetilde{w} \circ \gamma^{-1}(p) \coloneqq \begin{cases} w \circ \gamma^{-1}(p) & \text{if } p \in \gamma(\overline{W}), \\ 0 & \text{else,} \end{cases}$$

we can assume $w \in \mathcal{C}_b^s(\mathbb{R}^d, \mathbb{R})$ and $w \circ \gamma^{-1} \in \mathcal{C}^s(M, \mathbb{R})$ with

$$\|w \circ \gamma^{-1}\|_{C^s(M,\mathbb{R})} \sim \|w\|_{C^s(\overline{W},\mathbb{R})} \sim \|w\|_{C^s(\mathbb{R}^d,\mathbb{R})}$$

by Remark A.12 and Lemma A.13.

Lemma 2.137. *Let $s \in \mathbb{R}_{>0} \setminus \mathbb{N}$, let $W \subset \mathbb{R}^d$ be open, bounded and convex subset and let $K \subset \mathbb{R}^d$ be a compact subset with $K \subset W$. Furthermore, let $a \in h^s(\overline{W}, \mathbb{R}^n)$. There exists $\tilde{a} \in h^s(\mathbb{R}^d, \mathbb{R}^n)$ with $\tilde{a} = a$ on K. If $n = d \times d$ and the matrix a is symmetric and positive definite on $\overline{W}$, we can choose $\tilde{a}$ such that it is symmetric and uniformly elliptic on $\mathbb{R}^d$.*

Proof. Choose a cut-off function $\xi \in C^\infty(\mathbb{R}^d, \mathbb{R})$ with $0 \leq \xi \leq 1$, $\xi \equiv 1$ on K and $\xi \equiv 0$ on $\mathbb{R}^d \setminus \overline{W}$. Then, for an arbitrary $x_0 \in \overline{W}$, set $\varphi \coloneqq \xi \cdot \mathrm{Id} + (1 - \xi) \cdot x_0$. This definition implies $\varphi \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ with $\varphi_{|K} \equiv \mathrm{Id}$ and $\varphi_{|\mathbb{R}^d \setminus \overline{W}} \equiv x_0 \in \overline{W}$. Due to the convexity of W, also $\varphi(\overline{W}) \subset \overline{W}$ holds, so that $\varphi(\mathbb{R}^d) \subset \overline{W}$ follows. In particular, $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ is bounded and due to $\operatorname{supp}(\varphi - x_0) \subset \overline{W}$, also all derivatives of φ are bounded and hence $\varphi \in C_b^\infty(\mathbb{R}^d, \mathbb{R}^d)$ holds. We define

$$\tilde{a} \coloneqq a \circ \varphi.$$

On account of Lemma 2.88, the function φ satisfies the conditions of Proposition 2.97, which yields $\tilde{a} \in h^s(\mathbb{R}^d, \mathbb{R}^n)$. Obviously, $\tilde{a} = a$ holds on K. For $n = d \times d$, due to compactness of $\overline{W}$, any positive definite matrix valued function $a : \overline{W} \to \mathbb{R}^{d \times d}$ is uniformly elliptic on $\overline{W}$ (see Lemma 2.133). Because for every $x \in \mathbb{R}^d$ there exists $y \coloneqq \varphi(x) \in \overline{W}$ with $\tilde{a}(x) = a(y)$, for $n = d \times d$, symmetry and uniform ellipticity of the matrix a on $\overline{W}$ implies the same properties for the matrix $\tilde{a}$ on $\mathbb{R}^d$. □

Next, we prove a technical auxiliary lemma, that transfers the property of generating analytic C^0-semigroups from locally defined operators to the corresponding operator defined on a hypersurface.

Lemma 2.138. *Let $s \in \mathbb{R}_{>0} \setminus \mathbb{N}$, let $M \subset \mathbb{R}^{d+1}$ be a h^{2+s}-embedded closed hypersurface and let $(\gamma_l, W_l)_{l=1,\dots,L}$ be a finite set of local parameterizations of M with $M \subset \bigcup_l \gamma_l(W_l)$ as in Remark 2.6(ii). Moreover, let $A : h^{2+s}(M) \to h^s(M)$ and $A_l : h^{2+s}(\mathbb{R}^d) \to h^s(\mathbb{R}^d)$ for $l = 1, \dots, L$ be linear operators with*

$$\operatorname{supp} Au \subset \operatorname{supp} u \quad \text{and} \quad \operatorname{supp} A_l w \subset \operatorname{supp} w$$

for all $u \in h^{2+s}(M)$ and $w \in h^{2+s}(\mathbb{R}^d)$. Additionly, for every $\bar{u} \in h^{2+s}(M)$ and $\bar{w} \in h^{2+s}(\mathbb{R}^d)$, let $B_{\bar{u}} \in \mathcal{L}\big(h^{1+s}(M), h^s(M)\big)$ and $B_{l,\bar{w}} \in \mathcal{L}\big(h^{1+s}(\mathbb{R}^d), h^s(\mathbb{R}^d)\big)$, $l = 1, \dots, L$, be linear and bounded operators with

$$\operatorname{supp} B_{\bar{u}} u \subset \operatorname{supp} \bar{u} \cap \operatorname{supp} u \quad \textit{and} \quad \operatorname{supp} B_{l,\bar{w}} w \subset \operatorname{supp} \bar{w} \cap \operatorname{supp} w$$

for all $u \in h^{1+s}(M)$ and $w \in h^{1+s}(\mathbb{R}^d)$. We assume

$$A(\bar{u} \cdot u) = \bar{u} \cdot Au + B_{\bar{u}} u \quad \textit{and} \quad A_l(\bar{w} \cdot w) = \bar{w} \cdot A_l w + B_{l,\bar{w}} w$$

for all $\bar{u}, u \in h^{2+s}(M)$ and $\bar{w}, w \in h^{2+s}(\mathbb{R}^d)$. Finally, we assume A and A_l to be related by

$$(Au) \circ \gamma_l = A_l(u \circ \gamma_l) \ \textit{in} \ h^s(\mathbb{R}^d)$$

for every $l = 1, \dots, L$ and $u \in h^{2+s}(M)$ with $\operatorname{supp} u \subset \gamma_l(W_l)$, where $u \circ \gamma_l$ and $(Au) \circ \gamma_l$ are extensions onto $\mathbb{R}^d$ as in Remark 2.136. If all A_l generate analytic C^0-semigroups, then also A generates an analytic C^0-semigroup.

Proof. The proof is based on [PS16, Section 6.4].
Let $(\pi_l)_{l=1,\dots,L} \subset h^{2+s}(M)$ be a family of functions such that $(\pi_l^2)_{l=1,\dots,L}$ is a partition of unity subordinate to $\big(\gamma_l(W_l)\big)_{l=1,\dots,L}$. Such a family of functions can be constructed in the following way: Let $(K_l)_{l=1,\dots,L}$ be compact subsets of M with $K_l \subset \gamma_l(W_l)$ and $M \subset \bigcup_l K_l$ as in Remark 2.6(ii). Furthermore, let $(\psi_l)_{l=1,\dots,L} \subset h^{2+s}(M)$ be a family of functions with $\psi_l \geq 0$, $\psi_l > 0$ on K_l and $\operatorname{supp} \psi_l \subset \gamma_l(W_l)$. Define

$$\psi \coloneqq \sqrt{\sum_l \psi_l^2}.$$

Due to $\psi > 0$ on M, we have $\frac{1}{\psi} \in h^{2+s}(M)$ on account of Corollary 2.103(i). In particular,

$$\pi_l \coloneqq \frac{\psi_l}{\psi} \in h^{2+s}(M)$$

follows and due to $\operatorname{supp} \pi_l = \operatorname{supp} \psi_l \subset \gamma_l(W_l)$, $0 \leq \pi_l \leq 1$ and

$$\sum_l \pi_l^2 = \frac{\sum_l \psi_l^2}{\psi^2} \equiv 1 \text{ on } M,$$

the family $(\pi_l)_{l=1,\dots,L}$ fulfills the desired properties.
Moreover, we define

$$\begin{aligned} \vec{R}u &\coloneqq \big[(\pi_l u) \circ \gamma_l\big]_{l=1,\dots,L} && \text{for } u \in h^s(M, \mathbb{R}), \\ R\big([v_l]_{l=1,\dots,L}\big) &\coloneqq \sum_{l=1}^{L} \pi_l (v_l \circ \gamma_l^{-1}) && \text{for } [v_l]_{l=1,\dots,L} \in h^s(\mathbb{R}^d, \mathbb{R}^L). \end{aligned}$$

For $k \in \{0, 1, 2\}$, $\vec{R} \in \mathcal{L}\big(h^{k+s}(M), h^{k+s}(\mathbb{R}^d, \mathbb{R}^L)\big)$ and $R \in \mathcal{L}\big(h^{k+s}(\mathbb{R}^d, \mathbb{R}^L), h^{k+s}(M)\big)$ follow with Remark 2.136 and for every $u \in h^{k+s}(M)$,

$$R\vec{R}u = R\Big(\big[(\pi_l u) \circ \gamma_l\big]_{l=1,\dots,L}\Big) = \sum_{l=1}^{L} \pi_l^2 u = u$$

holds in $h^{k+s}(M)$. Another application of Remark 2.136 yields that

$$C_l : h^{1+s}(M) \to h^s(\mathbb{R}^d), \ \ C_l u \coloneqq -(B_{\pi_l} u) \circ \gamma_l \text{ and}$$
$$D_l : h^{1+s}(\mathbb{R}^d) \to h^s(M), \ \ D_l w \coloneqq (B_{l,(\pi_l \circ \gamma_l)} w) \circ \gamma_l^{-1}$$

are well-defined with $C_l \in \mathcal{L}\big(h^{1+s}(M), h^s(\mathbb{R}^d)\big)$ and $D_l \in \mathcal{L}\big(h^{1+s}(\mathbb{R}^d), h^s(M)\big)$. We set

$$\mathcal{A} \coloneqq [\text{diag } A_l]_{l=1,\dots,L}, \qquad \mathcal{C} \coloneqq [C_l R]_{l=1,\dots,L} \qquad \text{and} \qquad \mathcal{D} \coloneqq \vec{R}\big([D_l]_{l=1,\dots,L} \cdot [\cdot]\big)$$

such that $\mathcal{A} : h^{2+s}(\mathbb{R}^d, \mathbb{R}^L) \to h^s(\mathbb{R}^d, \mathbb{R}^L)$ is a well-defined linear operator and we have $\mathcal{C}, \mathcal{D} \in \mathcal{L}\big(h^{1+s}(\mathbb{R}^d, \mathbb{R}^L), h^s(\mathbb{R}^d, \mathbb{R}^L)\big)$. In the subsequent section of the proof, we show that $\mathcal{A} + \mathcal{C}, \mathcal{A} + \mathcal{D} : h^{2+s}(\mathbb{R}^d, \mathbb{R}^L) \to h^s(\mathbb{R}^d, \mathbb{R}^L)$ generate analytic C^0-semigroups.

With Lemma 2.92, $h^{2+s}(\mathbb{R}^d) \subset h^s(\mathbb{R}^d)$ and $h^{2+s}(\mathbb{R}^d, \mathbb{R}^L) \subset h^s(\mathbb{R}^d, \mathbb{R}^L)$ are dense subsets. By assumption, $A_l : h^{2+s}(\mathbb{R}^d) \to h^s(\mathbb{R}^d)$ generate analytic C^0-semigroups, so due to Proposition 2.125, there exist $\omega_l \in \mathbb{R}$ and $M_l > 0$ such that $S_{\omega_l, \frac{\pi}{2}}$ is contained in the resolvent set of the complexification $\widetilde{A}_l$ and

$$\big\|(\lambda \text{Id} - \widetilde{A}_l)^{-1}\big\|_{\mathcal{L}} \le \frac{M_l}{|\lambda - \omega_l|}$$

holds for all $\lambda \in S_{\omega_l, \frac{\pi}{2}}$. Define $\omega \coloneqq \max_l \omega_l$ and $M \coloneqq \max_l M_l$. Then, for every l,

$$S_{\omega, \frac{\pi}{2}} = \big\{\lambda \in \mathbb{C} \,\big|\, \text{Re}\lambda > w\big\} \subset \big\{\lambda \in \mathbb{C} \,\big|\, \text{Re}\lambda > w_l\big\} = S_{\omega_l, \frac{\pi}{2}}$$

is contained in the resolvent set of $\widetilde{A}_l$ and for every $\lambda \in S_{\omega, \frac{\pi}{2}}$, we have $|\lambda - w| \le |\lambda - w_l|$ and thus

$$\big\|(\lambda \text{Id} - \widetilde{A}_l)^{-1}\big\|_{\mathcal{L}} \le \frac{M_l}{|\lambda - w_l|} \le \frac{M}{|\lambda - w|}.$$

Because $\mathcal{A} = [\text{diag} A_l]_{l=1,\dots,L}$ implies $(\lambda \text{Id} - \widetilde{\mathcal{A}})^{-1} = \big[\text{diag } (\lambda \text{Id} - \widetilde{A}_l)^{-1}\big]_{l=1,\dots,L}$ for any $\lambda \in \mathbb{C}$, $S_{\omega, \frac{\pi}{2}}$ is also contained in the resolvent set of $\widetilde{\mathcal{A}}$ and

$$\big\|(\lambda \text{Id} - \widetilde{\mathcal{A}})^{-1}\big\|_{\mathcal{L}} \lesssim \frac{LM}{|\lambda - w|}$$

holds for all $\lambda \in S_{\omega, \frac{\pi}{2}}$. By Proposition 2.125, $\mathcal{A}$ thus generates an analytic C^0-semigroup. On account of Remarks 2.73(ii) and 2.82 as well as Lemmas 2.83 and 2.87, we have $h^{1+s}(\mathbb{R}^d, \mathbb{R}^L) \in J_{1/2}\big(h^s(\mathbb{R}^d, \mathbb{R}^L), h^{2+s}(\mathbb{R}^d, \mathbb{R}^L)\big)$. Hence, with Lemma 2.126, also $\mathcal{A} + \mathcal{C}$ and $\mathcal{A} + \mathcal{D}$ generate analytic C^0-semigroups.

For the corresponding complexifications and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\widetilde{\vec{R}}(\lambda \text{Id} - \widetilde{A})\tilde{u} &= \Big[\big(\pi_l \cdot (\lambda \text{Id} - \widetilde{A})\tilde{u}\big) \circ \gamma_l\Big]_{l=1,\dots,L} = \Big[\lambda \text{Id}\big((\pi_l \tilde{u}) \circ \gamma_l\big) - (\pi_l \cdot \widetilde{A}\tilde{u}) \circ \gamma_l\Big]_{l=1,\dots,L} \\ &= \Big[\lambda \text{Id}\big((\pi_l \tilde{u}) \circ \gamma_l\big) - \widetilde{A}_l\big((\pi_l \tilde{u}) \circ \gamma_l\big) - \widetilde{C}_l \tilde{u}\Big]_{l=1,\dots,L} = \big(\lambda \text{Id} - (\widetilde{\mathcal{A}} + \widetilde{\mathcal{C}})\big)\widetilde{\vec{R}}\tilde{u} \qquad (2.14)\end{aligned}$$

in $h^s\widetilde{(\mathbb{R}^d,\mathbb{R}^L)}$ for all $\tilde{u} \in h^{2+s}\widetilde{(M)}$ and

$$\begin{aligned}
&\widetilde{R}\Big(\lambda\mathrm{Id} - \big(\widetilde{\mathcal{A}} + \widetilde{\mathcal{D}}\big)\Big)[\widetilde{v_l}]_{l=1,\dots,L} = \widetilde{R}\Big(\lambda[\widetilde{v_l}]_{l=1,\dots,L} - \big[\widetilde{A_l}\widetilde{v_l}\big]_{l=1,\dots,L} - \widetilde{\vec{R}}\Big(\sum_l \widetilde{D_l}\widetilde{v_l}\Big)\Big) \\
&= \lambda\sum_l \pi_l(\widetilde{v_l}\circ\gamma_l^{-1}) - \sum_l \pi_l\big(\widetilde{A_l}\widetilde{v_l}\circ\gamma_l^{-1}\big) - \sum_l \widetilde{D_l}\widetilde{v_l} \\
&= \lambda\sum_l \pi_l(\widetilde{v_l}\circ\gamma_l^{-1}) - \sum_l \widetilde{A}\big(\pi_l(\widetilde{v_l}\circ\gamma_l^{-1})\big) = \big(\lambda\mathrm{Id} - \widetilde{A}\big)\widetilde{R}[\widetilde{v_l}]_{l=1,\dots,L}
\end{aligned} \tag{2.15}$$

in $\widetilde{h^s(M)}$ for all $[\widetilde{v_l}]_{l=1,\dots,L} \in h^{2+s}\widetilde{(\mathbb{R}^d,\mathbb{R}^L)}$. As $\mathcal{A}+\mathcal{C}$ and $\mathcal{A}+\mathcal{D}$ generate analytic C^0-semigroups, by Proposition 2.125, there exists $\omega \in \mathbb{R}$ such that for all $\lambda \in S_{\omega,\frac{\pi}{2}}$,

$$L_\lambda := \widetilde{R}\big(\lambda\mathrm{Id} - (\widetilde{\mathcal{A}} + \widetilde{\mathcal{C}})\big)^{-1}\widetilde{\vec{R}}, \quad R_\lambda := \widetilde{R}\big(\lambda\mathrm{Id} - (\widetilde{\mathcal{A}} + \widetilde{\mathcal{D}})\big)^{-1}\widetilde{\vec{R}} \quad \in \mathcal{L}\big(\widetilde{h^s(M)}, \widetilde{h^s(M)}\big)$$

holds with $L_\lambda\big(\widetilde{h^s(M)}\big), R_\lambda\big(\widetilde{h^s(M)}\big) \subset h^{2+s}\widetilde{(M)}$. The operators L_λ and R_λ are left and right inverse, respectively, of $\lambda\mathrm{Id} - \widetilde{A}$, as we have

$$\begin{aligned}
L_\lambda(\lambda\mathrm{Id} - \widetilde{A})\tilde{u} &= \widetilde{R}\big(\lambda\mathrm{Id} - (\widetilde{\mathcal{A}} + \widetilde{\mathcal{C}})\big)^{-1}\widetilde{\vec{R}}(\lambda\mathrm{Id} - \widetilde{A})\tilde{u} \overset{(2.14)}{=} \widetilde{R}\widetilde{\vec{R}}\tilde{u} = \tilde{u} \text{ for all } \tilde{u} \in h^{2+s}\widetilde{(M)}, \\
(\lambda\mathrm{Id} - \widetilde{A})R_\lambda\tilde{u} &= (\lambda\mathrm{Id} - \widetilde{A})\widetilde{R}\big(\lambda\mathrm{Id} - (\widetilde{\mathcal{A}} + \widetilde{\mathcal{D}})\big)^{-1}\widetilde{\vec{R}}\tilde{u} \overset{(2.15)}{=} \widetilde{R}\widetilde{\vec{R}}\tilde{u} = \tilde{u} \text{ for all } \tilde{u} \in \widetilde{h^s(M)}.
\end{aligned}$$

Hence, $R_\lambda = L_\lambda = (\lambda\mathrm{Id} - \widetilde{A})^{-1} \in \mathcal{L}\big(\widetilde{h^s(M)}, \widetilde{h^s(M)}\big)$ holds for all $\lambda \in S_{\omega,\frac{\pi}{2}}$. In particular, $S_{\omega,\frac{\pi}{2}}$ is contained in the resolvent set of $\widetilde{A}$ with

$$\big\|(\lambda\mathrm{Id} - \widetilde{A})^{-1}\big\|_{\mathcal{L}} = \|L_\lambda\|_{\mathcal{L}} \le C(R,\vec{R})\big\|\big(\lambda\mathrm{Id} - (\widetilde{\mathcal{A}} + \widetilde{\mathcal{C}})\big)^{-1}\big\|_{\mathcal{L}} \le C(R,\vec{R})\frac{M_{\widetilde{\mathcal{A}}+\widetilde{\mathcal{C}}}}{|\lambda - w|}$$

for all $\lambda \in S_{\omega,\frac{\pi}{2}}$. By Proposition 2.125, the operator A therefore generates an analytic C^0-semigroup. □

With this preparatory work, we can finally transfer Proposition 2.135 to the setting of hypersurfaces.

Proposition 2.139 (Differential Operators as Generators)**.** *Let $s \in (0,2)\setminus\{1\}$ and let $M \subset \mathbb{R}^{d+1}$ be a h^{2+s}-embedded closed hypersurface. Moreover, let $A \in \mathcal{L}\big(h^{2+s}(M), h^s(M)\big)$ be a symmetric, elliptic differential operator of second order, i.e. given a local parameterization (γ, W) of M,*

$$Au\circ\gamma = a : D^2(u\circ\gamma) + b\cdot\nabla(u\circ\gamma) + c(u\circ\gamma) \tag{2.16}$$

holds for every $u \in h^{2+s}(M)$, with $a \in h^s(\overline{W},\mathbb{R}^{d\times d})$, $b \in h^s(\overline{W},\mathbb{R}^d)$ and $c \in h^s(\overline{W},\mathbb{R})$ such that the matrix a is symmetric and positive definite on $\overline{W}$. Then,

$$A : h^{2+s}(M) \to h^s(M)$$

generates an analytic C^0-semigroup.

Proof. Let $(\gamma_l, W_l)_{l=1,\dots,L}$ be a finite set of local parameterizations of M and let $(U_l)_{l=1,\dots,L}$ be open subsets with $\overline{U_l} \subset \gamma_l(W_l)$ and $M \subset \bigcup_l U_l$ as in Remark 2.6(ii). Define the compact sets $K_l \coloneqq \gamma_l^{-1}(\overline{U_l}) \subset W_l$. For every $l = 1, \dots, L$, let $a_l \in h^s(\overline{W_l}, \mathbb{R}^{d\times d})$, $b_l \in h^s(\overline{W_l}, \mathbb{R}^d)$ and $c_l \in h^s(\overline{W_l}, \mathbb{R})$ be as in (2.16) with respect to the local parameterization (γ_l, W_l). We extend them to $\widetilde{a_l} \in h^s(\mathbb{R}^d, \mathbb{R}^{d\times d})$, $\widetilde{b_l} \in h^s(\mathbb{R}^d, \mathbb{R}^d)$ and $\widetilde{c_l} \in h^s(\mathbb{R}^d, \mathbb{R})$ with $\widetilde{a_l} = a_l$, $\widetilde{b_l} = b_l$ and $\widetilde{c_l} = c_l$ on K_l and $\widetilde{a_l}$ symmetric and uniformly elliptic on $\mathbb{R}^d$ as in Lemma 2.137. On account of Proposition 2.135,

$$A_l : \mathcal{D}(A_l) = h^{2+s}(\mathbb{R}^d) \to h^s(\mathbb{R}^d), \quad A_l v \coloneqq \widetilde{a_l} : D^2 v + \widetilde{b_l} \cdot \nabla v + \widetilde{c_l} v$$

generates an analytic C^0-semigroup for every $l = 1, \dots, L$.
With $\big(\widetilde{\gamma_l}, \widetilde{W_l}\big)_{l=1,\dots,L} \coloneqq \big(\gamma_{l|\gamma_l^{-1}(U_l)}, \gamma_l^{-1}(U_l)\big)_{l=1,\dots,L}$, we have a set of local parameterizations as in Lemma 2.138 such that

$$\begin{aligned}(Au) \circ \widetilde{\gamma_l} &= (Au) \circ \gamma_l = a_l : D^2(u \circ \gamma_l) + b_l \cdot \nabla(u \circ \gamma_l) + c_l(u \circ \gamma_l) \\ &= \widetilde{a_l} : D^2(u \circ \gamma_l) + \widetilde{b_l} \cdot \nabla(u \circ \gamma_l) + \widetilde{c_l}(u \circ \gamma_l) = A_l(u \circ \gamma_l) = A_l(u \circ \widetilde{\gamma_l}) \qquad (2.17)\end{aligned}$$

holds in $h^s(\mathbb{R}^d)$ for every $u \in h^{2+s}(M)$ with $\operatorname{supp} u \subset \widetilde{\gamma_l}(\widetilde{W_l}) = U_l \subset \gamma_l(K_l)$. Furthermore, define

$$B_{l,\bar{w}} w \coloneqq \big(2\widetilde{a_l} \cdot \nabla \bar{w}\big) \cdot \nabla w + \big(\widetilde{a_l} : D^2 \bar{w} + \widetilde{b_l} \cdot \nabla \bar{w}\big) w$$

for $\bar{w} \in h^{2+s}(\mathbb{R}^d)$ and $w \in h^{1+s}(\mathbb{R}^d)$. Then, we have $B_{l,\bar{w}} \in \mathcal{L}\big(h^{1+s}(\mathbb{R}^d), h^s(\mathbb{R}^d)\big)$ for all $\bar{w} \in h^{2+s}(\mathbb{R}^d)$ and, due to symmetry of $\widetilde{a_l}$,

$$A_l(\bar{w} \cdot w) = \bar{w} \cdot Aw + B_{l,\bar{w}} w \qquad (2.18)$$

holds for all $\bar{w}, w \in h^{2+s}(\mathbb{R}^d)$. Let $(\pi_l)_{l=1,\dots,L} \subset h^{2+s}(M)$ be a family of functions such that $(\pi_l^2)_{l=1,\dots,L}$ is a partition of unity subordinate to $(U_l)_{l=1,\dots,L}$ as in the proof of Lemma 2.138. Then, define

$$B_{\bar{u}} u \coloneqq \sum_{l=1}^{L} \Big(B_{l,(\pi_l \bar{u})\circ\gamma_l}\big((\pi_l u) \circ \gamma_l\big) - (\bar{u} \circ \gamma_l) B_{l,\pi_l\circ\gamma_l}\big((\pi_l u) \circ \gamma_l\big)\Big) \circ \gamma_l^{-1}$$

for $\bar{u} \in h^{2+s}(M)$ and $u \in h^{1+s}(M)$. On account of Remark 2.136, $B_{\bar{u}} \in \mathcal{L}\big(h^{1+s}(M), h^s(M)\big)$ is well-defined for all $\bar{u} \in h^{2+s}(M)$. With the notation $\bar{w}_l \coloneqq (\pi_l \bar{u}) \circ \gamma_l$ and $w_l \coloneqq (\pi_l u) \circ \gamma_l$, Equations (2.17) and (2.18) yield

$$\begin{aligned}A(\bar{u} \cdot u) - \bar{u} \cdot Au &= \sum_l \Big(A\big(\pi_l^2(\bar{u} \cdot u)\big) - \bar{u} \cdot A(\pi_l^2 u)\Big) \\ &= \sum_l \Big(A_l(\bar{w}_l \cdot w_l) - (\bar{u} \circ \gamma_l) \cdot A_l\big((\pi_l^2 u) \circ \gamma_l\big)\Big) \circ \gamma_l^{-1} \\ &= \sum_l \Big((\bar{w}_l \cdot A_l w_l) + B_{l,\bar{w}_l} w_l - (\bar{u} \circ \gamma_l)\Big((\pi_l \circ \gamma_l) \cdot A_l w_l + B_{l,(\pi_l\circ\gamma_l)} w_l\Big)\Big) \circ \gamma_l^{-1} \\ &= \sum_l \Big(B_{l,\bar{w}_l} w_l - (\bar{u} \circ \gamma_l) B_{l,(\pi_l\circ\gamma_l)} w_l\Big) \circ \gamma_l^{-1} = B_{\bar{u}} u\end{aligned}$$

for all $\bar{u}, u \in h^{2+s}(M)$. In particular, the assumptions of Lemma 2.138 are satisfied and therefore A generates an analytic C^0-semigroup. $\square$

2.4 Maximum Principles on Evolving Hypersurfaces

To prove conservation properties in Chapter 5, we want to apply maximum principles. In the literature, there are plenty of results on maximum principles for functions with domain in Euclidean space, see for example [Eva10, Section 7.1.4] or [RR06, Sections 4.1 and 4.4]. We want to formulate analogous maximum principles for functions defined on evolving closed hypersurfaces in the following setting:

Assumptions 2.140. *Let Γ be a C^1- C^2-evolving immersed closed hypersurface with reference surface $M \subset \mathbb{R}^{d+1}$ and global parameterization θ. Furthermore, let $A : [0,T] \times M \to \mathbb{R}^{(d+1)\times(d+1)}$, $B : [0,T] \times M \to \mathbb{R}^{d+1}$ and $C : [0,T] \times M \to \mathbb{R}$ be continuous with A symmetric and positive definite on $[0,T] \times M$. Let $w \in C^1\big([0,T], C^0(M)\big) \cap C^0\big([0,T], C^2(M)\big)$ and define*

$$\mathcal{L}w := -\partial^\square w + A{:}\, D_\Gamma^2 w + B \cdot \nabla_\Gamma w + Cw,$$

where the surface gradient ∇_Γ, the surface Hessian D_Γ^2 and the normal time derivative $\partial^\square$ were introduced in Definitions 2.31, 2.34 and 2.54, respectively.

In the assumptions above, we used the same notation for the element-wise matrix multiplication as in Section 2.3 that we recall here for the reader's convenience.

Definition 2.141. *For two matrices $A, \widetilde{A} \in \mathbb{R}^{n\times n}$, we define*

$$A{:}\, \widetilde{A} := \sum_{i,j=1}^{n} A_{ij} \widetilde{A}_{ij}.$$

We also recall that compactness of $[0,T] \times M$ implies that the matrix valued function A is uniformly elliptic on $[0,T] \times M$, i.e., there exists $C > 0$ so that

$$\xi^\top A(t,p)\xi \geq C|\xi|^2$$

holds for all $(t,p) \in [0,T] \times M$ and $\xi \in \mathbb{R}^{d+1}$ (see Lemma 2.133).

In order to apply our maximum principles to the mean curvature function, they have to hold without assumptions on the (sign of the) zeroth-order term C (cf. Remark 5.3). Therefore, we will be particularly concerned in proving maximum principles without condition on the zeroth-order term. This is possible, as, in contrast to the elliptic case, sign conditions usually can be evaded for parabolic equations.

2.4.1 Weak Maximum Principle

The typical weak maximum principles for parabolic equations as in [Eck12, Proposition 3.1] and [Eva10, Section 7.1: Theorem 8 and Theorem 9] impose sign conditions on the zeroth-order term, so they do not fulfill our requirements. Instead, we derive a suitable weak maximum principle by hand.

Lemma 2.142 (Hamilton's Trick)**.** *Let $M \subset \mathbb{R}^n$ be a C^1-embedded closed hypersurface and let $T \in (0,\infty)$. Furthermore, let $w \in C^1\big([0,T], C^0(M)\big)$. As M is closed, the value $w_{\max}(t) \coloneqq \max_M w(t,\cdot)$ is well-defined for every $t \in [0,T]$. Then, $w_{\max} : [0,T] \to \mathbb{R}$ is Lipschitz continuous, in particular differentiable almost everywhere, and in every point $t \in (0,T)$ in which $w_{\max}$ is differentiable,*

$$\partial_t w_{\max}(t) = \partial_t w(t,p)$$

holds, where $p \in M$ is an arbitrary point with $w(t,p) = w_{\max}(t)$.

Proof. The proof of this lemma can be found in [Man11, Lemma 2.1.3], but for the reader's convenience we explicate it here again.

Step 1: *$w_{\max}$ is Lipschitz continuous*
For fixed $p \in M$, we have $w(\cdot,p) \in C^1([0,T])$. So, by the mean value theorem, $w(\cdot,p) : [0,T] \to \mathbb{R}$ is Lipschitz continuous and $L \coloneqq \max_{[0,T]\times M} |\partial_t w|$ is a Lipschitz constant independent of $p \in M$. For $t, s \in [0,T]$ and suitable choices of $p, q \in M$ we thus have

$$\begin{aligned}
&w_{\max}(t) = w(t,p) \le w(s,p) + L|t-s| \le w_{\max}(s) + L|t-s|,\\
&w_{\max}(s) = w(s,q) \le w(t,q) + L|t-s| \le w_{\max}(t) + L|t-s|\\
&\Rightarrow \big|w_{\max}(t) - w_{\max}(s)\big| \le L|t-s|.
\end{aligned}$$

Hence, $w_{\max} : [0,T] \to \mathbb{R}$ is also Lipschitz continuous and therefore differentiable almost everywhere.

Step 2: *Formula for time derivative*
Let $t \in (0,T)$ be a point in which $w_{\max}$ is differentiable and let $p \in M$ such that $w(t,p) = w_{\max}(t)$ holds. We have $w(\cdot,p) \in C^1([0,T])$ and for $\varepsilon > 0$ sufficiently small, $[t-\varepsilon, t+\varepsilon] \subset [0,T]$ holds. By the mean value theorem, there exists $\tau_\varepsilon \in [t, t+\varepsilon]$ with

$$\frac{w_{\max}(t+\varepsilon) - w_{\max}(t)}{\varepsilon} \ge \frac{w(t+\varepsilon,p) - w(t,p)}{\varepsilon} = \partial_t w(\tau_\varepsilon, p).$$

Analogously, there exists $\bar{\tau}_\varepsilon \in [t-\varepsilon, t]$ with

$$\frac{w_{\max}(t) - w_{\max}(t-\varepsilon)}{\varepsilon} \le \frac{w(t,p) - w(t-\varepsilon,p)}{\varepsilon} = \partial_t w(\bar{\tau}_\varepsilon, p).$$

Overall, $\partial_t w_{\max}(t) = \partial_t w(t,p)$ follows in the limit $\varepsilon \to 0$. □

With the help of this relation between a function $w : [0,T] \times M \to \mathbb{R}$ and its corresponding maximum value function $w_{\max} : [0,T] \to \mathbb{R}$, we can now prove a weak maximum principle that suits our conditions.

Proposition 2.143 (Weak Maximum Principle)**.** *Let Assumptions 2.140 hold true with $\mathcal{L}w \ge 0$ on $[0,T] \times M$ and $\max_M w(0,\cdot) = 0$. Then, we have*

$$\max_{[0,T]\times M} w = 0.$$

Proof. We prove the statement by contradiction and assume

$$\max_{[0,T]\times M} w > 0,$$

such that there exists $t_1 \in (0,T]$ with $w_{\max}(t_1) > 0$.
By Hamilton's trick (Lemma 2.142), $w_{\max} : [0,T] \to \mathbb{R}$ is continuous. Due to $w_{\max}(0) = 0$ and $w_{\max}(t_1) > 0$, there exists $t_0 \in [0,t_1)$ with

$$w_{\max}(t_0) = 0 \quad \text{and} \quad w_{\max}(t) > 0 \text{ for all } t \in (t_0,t_1].$$

By Hamilton's trick (Lemma 2.142), $w_{\max} : [0,T] \to \mathbb{R}$ also is differentiable almost everywhere. Let $t \in (t_0,t_1)$ be such that $w_{\max}$ is differentiable in t. For any $p \in M$ with $w(t,p) = w_{\max}(t)$, Lemma 2.36 implies

$$\nabla_\Gamma w_{|(t,p)} = 0 \quad \text{and} \quad D^2_\Gamma w_{|(t,p)} \le 0. \tag{2.19}$$

With $K \coloneqq \max_{[0,T]\times M} |C|$, we have $C_{|(t,p)} w_{\max}(t) \le K w_{\max}(t)$ on account of $w_{\max}(t) > 0$. Hence,

$$\begin{aligned} 0 \le \mathcal{L}w_{|(t,p)} &= -\partial^\square w + A \colon D^2_\Gamma w + B \cdot \nabla_\Gamma w + Cw_{\,|(t,p)} \\ \Leftrightarrow \partial^\square w_{|(t,p)} &\le Cw_{|(t,p)} \le K w_{\max}(t) \end{aligned}$$

follows. (A proof why $A > 0$ and $D^2_\Gamma w \le 0$ imply $A \colon D^2_\Gamma w \le 0$ is given in [Eva10, Section 6.4: proof of Theorem 1].) Now, we compute $\partial^\square w_{|(t,p)}$. By Definition 2.54 and (2.19), we have

$$\partial^\square w_{|(t,p)} = \partial^\circ w_{|(t,p)} - V^{\text{tot}}_{\Gamma\,|(t,p)} \cdot \nabla_\Gamma w_{|(t,p)} = \partial_t w_{|(t,p)} = \partial_t w_{\max}(t).$$

We also used that by Hamilton's trick (Lemma 2.142), $\partial_t w_{|(t,p)} = \partial_t w_{\max}(t)$ holds as $w_{\max}$ is differentiable in t. Overall,

$$\partial_t w_{\max}(t) \le K w_{\max}(t)$$

follows for almost every $t \in (t_0,t_1)$. By Hamilton's trick (Lemma 2.142), $w_{\max} : [0,T] \to \mathbb{R}$ is Lipschitz continuous and thus absolute continuous. Therefore, Gronwall's inequality for absolute continuous functions ([Eva10, Appendix Inequalities]) yields

$$w_{\max}(t) \le w_{\max}(t_0) e^{K(t-t_0)} = 0 \text{ for all } t \in [t_0,t_1].$$

This is a contradiction to $w_{\max}(t) > 0$ for all $t \in (t_0,t_1]$. □

2.4.2 Strong Maximum Principle

A strong maximum principle without assumptions on the zeroth-order term can be found in the literature in the following setting for domains in Euclidean space:

Assumptions 2.144. *Let $W \subset \mathbb{R}^d$ be an open, bounded and connected subset and let $T \in (0,\infty)$. Moreover, let $a : [0,T] \times \overline{W} \to \mathbb{R}^{d\times d}$, $b : [0,T] \times \overline{W} \to \mathbb{R}^d$ and $c : [0,T] \times \overline{W} \to \mathbb{R}$ be continuous such that the matrix a is symmetric and positive definite on $[0,T] \times \overline{W}$. Let $v \in C^1\big([0,T], C^0(\overline{W})\big) \cap C^0\big([0,T], C^2(\overline{W})\big)$ and define*

$$\widetilde{\mathcal{L}} v \coloneqq -\partial_t v + a \colon D^2 v + b \cdot \nabla v + cv.$$

Again, compactness of $[0,T]\times\overline{W}$ implies that a is uniformly elliptic on $[0,T]\times\overline{W}$ (see Lemma 2.133). In this setting, [RR06, Theorem 4.26] yields the strong maximum principle stated below.

Lemma 2.145 (Strong Maximum Principle in Euclidean Space)**.** *Let Assumptions 2.144 hold true with $\widetilde{\mathcal{L}}v \geq 0$ on $[0,T]\times\overline{W}$. Define $\widetilde{m} \coloneqq \sup_{(0,T]\times W} v$ and let $(t_0,x_0)\in(0,T]\times W$ with $v(t_0,x_0)=\widetilde{m}$. Finally, assume one of the following conditions is fulfilled:*

(a) $c=0$ on $[0,T]\times\overline{W}$ and $\widetilde{m}$ arbitrary,

(b) $c\leq 0$ on $[0,T]\times\overline{W}$ and $\widetilde{m}\geq 0$,

(c) c arbitrary and $\widetilde{m}=0$.

Then, we have $v=\widetilde{m}$ on $[0,t_0]\times\overline{W}$.

We want to transfer this result to the setting of evolving closed hypersurfaces. For this, firstly, we construct a suitable differential operator in Euclidean space that corresponds to the differential operator on the hypersurface in a local neighborhood. Afterwards, we prove the analogous maximum principle to Lemma 2.145 in the setting of evolving closed hypersurfaces.

Lemma 2.146. *Let Assumptions 2.140 hold true and let (γ,W) be a sufficiently small local parameterization of M such that $(\gamma_\theta\coloneqq\theta\circ\gamma,W)$ is a local parameterization of an embedded patch of Γ. Set $g^\theta_{kl}\coloneqq\partial_k\gamma_\theta\cdot\partial_l\gamma_\theta$ and $[g^{kl}_\theta]_{k,l}\coloneqq\big([g^\theta_{kl}]_{k,l}\big)^{-1}$. Define $u_k\coloneqq\sum_{r=1}^d g^{kr}_\theta\partial_r\gamma_\theta$ as well as*

$$\begin{aligned}
a^{kl} &\coloneqq u_k^\top(A\circ\gamma)u_l,\\
b^k_1 &\coloneqq \sum_{r=1}^d u_r^\top(A\circ\gamma)\partial_r u_k,\\
b^k_2 &\coloneqq (B\circ\gamma+\partial_t\gamma_\theta)\cdot u_k,\\
b^k &\coloneqq b^k_1+b^k_2,\\
c &\coloneqq C\circ\gamma
\end{aligned}$$

for $k,l=1,\dots,d$. Furthermore, for any $v\in C^1\big([0,T],C^0(\overline{W})\big)\cap C^0\big([0,T],C^2(\overline{W})\big)$, define

$$\widetilde{\mathcal{L}}v\coloneqq-\partial_t v+a\colon D^2v+b\cdot\nabla v+cv.$$

Then, Assumptions 2.144 are satisfied and

$$(\mathcal{L}w)\circ\gamma=\widetilde{\mathcal{L}}(w\circ\gamma) \tag{2.20}$$

holds on $[0,T]\times\overline{W}$ for any $w\in C^1\big([0,T],C^0(M)\big)\cap C^0\big([0,T],C^2(M)\big)$.

Proof. We first prove that Assumptions 2.144 are satisfied. The regularity of Γ and A, B, C implies continuity of a, b, c on $[0,T] \times \overline{W}$. Symmetry of the matrix a follows directly from the symmetry of A. For any $\xi \in \mathbb{R}^d \setminus \{0\}$, we have

$$\eta \coloneqq \sum_{k=1}^{d} \xi_k u_k = \sum_{k,r=1}^{d} \xi_k \, g_\theta^{kr} \, \partial_r \gamma_\theta : [0,T] \times \overline{W} \to \mathbb{R}^{d+1} \setminus \{0\}$$

as well as

$$\xi^\top a \xi = \sum_{k,l=1}^{d} \xi_k a^{kl} \xi_l = \sum_{k,l=1}^{d} \xi_k u_k^\top (A \circ \gamma) u_l \xi_l = \eta^\top (A \circ \gamma) \eta.$$

Therefore, positive definiteness of A implies the same for the matrix a.
It remains to show (2.20).
Ad $\partial^\square w$: With Definition 2.54, we have

$$(\partial^\square w) \circ \gamma = \big(\partial^\circ w - V_\Gamma^{\mathrm{tot}} \cdot \nabla_\Gamma w\big) \circ \gamma = (\partial_t w - \partial_t \theta \cdot \nabla_\Gamma w) \circ \gamma = \partial_t v - (\partial_t \theta \cdot \nabla_\Gamma w) \circ \gamma.$$

Ad Cw: Clearly, we have

$$(Cw) \circ \gamma = cv.$$

Ad $B \cdot \nabla_\Gamma w$: With Lemma 2.31, we have

$$\big((B + \partial_t \theta) \cdot \nabla_\Gamma w\big) \circ \gamma = (B \circ \gamma + \partial_t \gamma_\theta) \sum_{k=1}^{d} u_k \partial_k v = b_2 \cdot \nabla v.$$

Ad $A : D_\Gamma^2 w$: With Lemmas 2.34 and 2.31, we have

$$\begin{aligned}
(A : D_\Gamma^2 w) \circ \gamma &= \sum_{i,j=1}^{d+1} \big(A^{ij} [D_\Gamma^2 w]_{ij}\big) \circ \gamma \\
&= \sum_{i,j=1}^{d+1} (A^{ij} \circ \gamma) \sum_{k=1}^{d} (u_k \cdot e_j) \, \partial_k \big([\nabla_\Gamma w]_i \circ \gamma\big) \\
&= \sum_{i,j=1}^{d+1} (A^{ij} \circ \gamma) \sum_{k,l=1}^{d} (u_k \cdot e_j) \, \partial_k \big((u_l \cdot e_i) \partial_l v\big) \\
&= \sum_{k,l=1}^{d} \big(u_k^\top (A \circ \gamma) u_l\big) \partial_k \partial_l v + \big(u_k^\top (A \circ \gamma) \partial_k u_l\big) \partial_l v \\
&= \sum_{k,l=1}^{d} a^{kl} \partial_k \partial_l v + \sum_{l=1}^{d} b_1^l \partial_l v = a : D^2 v + b_1 \cdot \nabla v.
\end{aligned}$$

Altogether,

$$\begin{aligned}
(\mathcal{L} w) \circ \gamma &= -\big(\partial^\square w\big) \circ \gamma + \big(A : D_\Gamma^2 w\big) \circ \gamma + \big(B \cdot \nabla_\Gamma w\big) \circ \gamma + (Cw) \circ \gamma \\
&= -\partial_t v + a : D^2 v + b_1 \cdot \nabla v + b_2 \cdot \nabla v + cv = \widetilde{\mathcal{L}} v
\end{aligned}$$

follows. $\square$

Proposition 2.147 (Strong Maximum Principle)**.** *Let Assumptions 2.140 hold true with* $\mathcal{L}w \geq 0$ *on* $[0,T] \times M$*. Define* $m := \sup_{(0,T]\times M} w$ *and let* $(t_0, p_0) \in (0,T] \times M$ *with* $w(t_0, p_0) = m$*. Finally, assume one of the following conditions is fulfilled:*

(a) $C = 0$ *on* $[0,T] \times M$ *and* m *arbitrary,*

(b) $C \leq 0$ *on* $[0,T] \times M$ *and* $m \geq 0$*,*

(c) C *arbitrary and* $m = 0$*.*

Then, we have $w = m$ *on* $[0, t_0] \times M$*.*

Proof. Choose a sufficiently small local parameterization (γ, W) of M around p_0 and let $x_0 \in W$ with $\gamma(x_0) = p_0$. By Remark 2.6(i), $W \subset \mathbb{R}^d$ is connected and by Proposition 2.50, $(\theta \circ \gamma, W)$ is a local parameterization of an embedded patch of Γ.

Step 1: $w = m$ *on* $[0, t_0] \times \gamma(\overline{W})$

Define

$$v := w \circ \gamma : [0,T] \times \overline{W} \to \mathbb{R}.$$

The regularity of w implies $v \in C^1\big([0,T], C^0(\overline{W})\big) \cap C^0\big([0,T], C^2(\overline{W})\big)$. With a, b, c and $\widetilde{\mathcal{L}}$ as in Lemma 2.146, Assumptions 2.144 are satisfied and we have

$$\widetilde{\mathcal{L}}v = (\mathcal{L}w) \circ \gamma \geq 0 \text{ on } [0,T] \times \overline{W}.$$

Define

$$\widetilde{m} := \sup_{(0,T]\times W} v.$$

Then, we have $m \geq \widetilde{m} \geq v(t_0, x_0) = w(t_0, p_0) = m$ and thus $\widetilde{m} = m$ holds. In particular, we have $(t_0, x_0) \in (0,T] \times W$ with $v(t_0, x_0) = \widetilde{m}$. Due to $c = C \circ \gamma$, also one of the following conditions is fulfilled:

(a) $c = 0$ on $[0,T] \times \overline{W}$ and $\widetilde{m}$ arbitrary,

(b) $c \leq 0$ on $[0,T] \times \overline{W}$ and $\widetilde{m} \geq 0$,

(c) c arbitrary and $\widetilde{m} = 0$.

So, the strong maximum principle in Euclidean space (Lemma 2.145) yields

$$v = \widetilde{m} \text{ on } [0, t_0] \times \overline{W} \quad \Leftrightarrow \quad w = m \text{ on } [0, t_0] \times \gamma(\overline{W}).$$

Step 2: $w = m$ *on* $[0, t_0] \times M$

Let (γ^l, W^l) be another local parameterization of M with $\gamma(W) \cap \gamma^l(W^l) \neq \varnothing$. Choose $t_0^l := t_0$ and $p_0^l \in \gamma(W) \cap \gamma^l(W^l)$. Then, $(t_0^l, p_0^l) \in (0,T] \times M$ holds with $w(t_0^l, p_0^l) = m$, because the first step yields $w = m$ on $[0, t_0] \times \gamma(\overline{W}) \ni (t_0^l, p_0^l)$. As in the first step,

$$w = m \text{ on } [0, t_0] \times \gamma^l(\overline{W^l})$$

follows. Because M is compact and connected, a finite cover $M \subset \bigcup_{l=1}^{L} \gamma^l(W^l)$ implies

$$w = m \text{ on } [0, t_0] \times M. \qquad \square$$

2.4.3 Combination of the Maximum Principles

We summarize this section by formulating a combination of the weak and the strong maximum principle on evolving closed hypersurfaces that will be used for several results in Chapter 5.

Corollary 2.148 (Combination of the Maximum Principles)**.** *Let Assumptions 2.140 hold true with $\mathcal{L}w \geq 0$ on $[0,T] \times M$ and $w(0,\cdot) \leq 0$ on M. Then, we have $w(t,\cdot) \leq 0$ for all $t \in [0,T]$ and there exists $t_0 \in [0,T]$ with*

$$w(t,\cdot) \equiv 0 \text{ for all } t \in (0,t_0] \quad \text{and} \quad w(t,\cdot) < 0 \text{ for all } t \in (t_0,T].$$

Proof. Define $w_{\max} : [0,T] \to \mathbb{R}$ with $w_{\max}(t) := \max_M w(t,\cdot)$, which is well-defined because M is compact.

Step 1: We first show $w(t,\cdot) \leq 0$ for all $t \in [0,T]$. If $w_{\max}(t) \leq 0$ for all $t \in [0,T]$, nothing is to show. If not, then on account of $w_{\max}(0) \leq 0$ and the continuity of $w_{\max} : [0,T] \to \mathbb{R}$ by Hamilton's trick (Lemma 2.142), there exists $t_0 \in [0,T)$ with

$$w_{\max}(t_0) = 0 \quad \text{and} \quad w_{\max}(t) \leq 0 \text{ for all } t \in [0,t_0).$$

In particular, $\max_M w(t_0,\cdot) = w_{\max}(t_0) = 0$ holds and thus the weak maximum principle (Proposition 2.143) implies

$$\max_{[t_0,T]\times M} w = 0 \quad \Leftrightarrow \quad w(t,\cdot) \leq 0 \text{ for all } t \in [t_0,T].$$

Together, $w(t,\cdot) \leq 0$ follows for all $t \in [0,T]$.

Step 2: It remains to show that there exists $t_0 \in [0,T]$ with $w(t,\cdot) \equiv 0$ for all $t \in (0,t_0]$ and $w(t,\cdot) < 0$ for all $t \in (t_0,T]$. If $w_{\max}(t) < 0$ holds for all $t \in (0,T]$, then the claim is satisfied with $t_0 = 0$. So, assume the existence of $t_1 \in (0,T]$ with $w_{\max}(t_1) = 0$. Case (c) of the strong maximum principle (Proposition 2.147) implies

$$w(t) \equiv 0 \text{ for all } t \in [0,t_1].$$

Hence, if we have $w_{\max}(T) = 0$, the claim is satisfied with $t_0 = T$. So, assume $w_{\max}(T) < 0$. Due to the continuity of $w_{\max} : [0,T] \to \mathbb{R}$ by Hamilton's trick (Lemma 2.142), there exists $t_0 \in [t_1,T)$ with

$$w_{\max}(t_0) = 0 \quad \text{and} \quad w_{\max}(t) < 0 \text{ for all } t \in (t_0,T].$$

With the considerations above, $w(t) \equiv 0$ follows for all $t \in [0,t_0]$. ∎

Chapter 3

The System of Equations

This chapter aims to give an introduction to the geometric problem this work is based on. More precisely, we are interested in solutions to the equations

$$V = \big(G(c) - G'(c)c\big)H, \tag{3.1a}$$

$$\partial^\square c = \Delta_\Gamma\big(G'(c)\big) + cHV. \tag{3.1b}$$

These equations are defined on an evolving closed hypersurface Γ. For simplicity, we assume in the following that Γ is an embedded hypersurface, if not stated otherwise. The mean curvature and normal velocity of the hypersurface are denoted by H and V, respectively (see Section 2.1). Furthermore, the function $c : \Gamma \to \mathbb{R}$ describes the concentration of a quantity on the surface Γ. The differential operators $\partial^\square$ and Δ_Γ are the normal time derivative and the Laplace-Beltrami operator as defined in Section 2.1. The function $G : \mathbb{R} \to \mathbb{R}$ is a (Gibbs) energy density and we will often use the notation

$$g(c) := G(c) - G'(c)c$$

which appears in the right hand side of (3.1a). A solution to the system (3.1) consists of an evolving closed hypersurface Γ and the concentration function $c : \Gamma \to \mathbb{R}$. In particular, we do not prescribe the evolution of the geometry but it is part of the problem.

In this chapter, we give an introduction to both of the equations. The first one dictates the evolution of the geometry. Due to its structure similar to the mean curvature flow, it is referred to as scaled mean curvature flow equation. The second one is a diffusion equation for the concentration c on the evolving surface Γ. The coupling of both equations makes the problem interesting and challenging.

Then, we explain why the equations can be seen as a suitable characterization of the physical situation presented in the introduction: We are interested in how a pair (Γ, c) evolves to decrease the (Gibbs) energy

$$\mathrm{E}(\Gamma, c) := \int_\Gamma G(c)\,\mathrm{d}\mathcal{H}^d \tag{3.2}$$

most efficiently while conserving the total mass of the quantity whose distribution is given by c. Here, $\mathrm{d}\mathcal{H}^d$ denotes the volume element of the d-dimensional surface $\Gamma(t)$. From this formulation (3.2) it is clear why the function G is called a (Gibbs) energy density. It turns

out that solutions of the system of equations (3.1) decrease the energy (3.2) and hence tend to approach (local) minima of (3.2).
Afterwards, we discuss physical and mathematical aspects of the shape of the energy density function G. Due to the conservation of mass, reducing the energy means finding a balance between the two opposing objectives of decreasing the surface area of Γ and decreasing the values of $G(c)$. The shape of G therefore has a strong impact on the evolution of the geometry.
This geometric evolution is then studied for the example of a radial symmetric situation. Emphasis is placed on whether properties are identical or similar when compared to the usual mean curvature flow.
Finally, we reformulate the equations (3.1) to a system defined on a fixed domain. This reformulation is used in Chapter 4 to prove the existence of short-time solutions to the system.

3.1 The Scaled Mean Curvature Flow Equation

This section is devoted to the scaled mean curvature flow equation (3.1a). If $G \equiv \alpha$ is a constant function then we also have $g(c) = G(c) - G'(c)c = \alpha$. Thus, for $\alpha = 1$, (3.1a) is just the usual mean curvature flow equation

$$V = H$$

and for $\alpha \neq 1$ we obtain a constantly scaled equation that behaves equally.
Ever since the study of curved shapes was no longer limited to the static case but also started to include time-dependent evolution in the middle of the last century, the mean curvature flow has been of great interest. First coming up in [Mul56] to describe a physical model, it has been analyzed extensively from a mathematical point of view afterwards. Three different approaches have been effectively applied for its investigation: To our knowledge, [Bra78] was the first (mathematical) publication on mean curvature flow in 1978, relying on geometric measure theory. Shortly after, [Hui84] and [GH86] discussed the evolution of convex surfaces under the mean curvature flow using classical PDEs. Finally, [ES95] led the way to analyzing mean curvature flow by motion of level sets and [CGG91] introduced the method of viscosity solutions.
We gather some properties of the mean curvature flow, which is also called curvature flow or curve-shortening flow in the case of (plane) curves. As a wonderful survey article on this topic, we recommend [Whi02]. First of all, surfaces that evolve under mean curvature flow turn smooth instantly. This is due to the parabolicity of the partial differential equation for the motion. But because the equation is non-linear, the general theory for parabolic equations only yields smoothing for a short time and does not forbid later singularities (see [Hui84] for a further discussion). The second property is the decrease of the surface area. In fact, the mean curvature flow turns out to be a gradient flow for the area functional which is explicated e.g. in [Gar13]. With the help of a parabolic maximum principle, one can show that embedded, closed surfaces remain embedded, i.e., do not develop self-intersections, and disjoint, closed surfaces remain disjoint (see [Man11, Sections 2.1 and 2.2] or [Eck12, Proposition 2.4]). This implies particularly that closed surfaces have finite lifespans: As explained e.g. in [Eck08], the statement reduces to the simple case of spheres,

because any closed surface can be surrounded by a sphere and letting them both evolve under mean curvature flow does not produce any collisions. A further consequence of the parabolic maximum principle is that mean convexity is conserved for closed surfaces (see [Man11, Proposition 2.4.1]).

Besides these relatively basic properties of the mean curvature flow, one of its most remarkable features is that convex, closed surfaces shrink to round points. This catchy formulation means firstly that any convex, closed surface stays convex and secondly that it becomes asymptotically spherical, i.e., the rescaled surface converges to a sphere. The result was proven in [GH86] for the curve case $d = 1$ and in [Hui84] for higher dimensions $d \geq 2$. For the curve case, there is an even stronger result (see [Gra87]): Under the curvature flow, any embedded, closed curve becomes convex and thus shrinks to a round point. In particular, it does not develop any singularities besides this collapse. This is not true for the higher dimensional case: For $d \geq 2$, embedded surfaces can form different kinds of singularities that are classified into singularities of type I and type II, depending on the blow up rate of the maximal curvature. In some sense, the singularities of type I behave more nicely as they only allow the rescaled surface in blow up points to have the shape of hyperplanes, spheres or cylinders (see [Man11, Chapter 3]). A typical example for a singularity of type II is a degenerated neckpinch, arising e.g. from the evolution of a dumbbell with precisely chosen thickness of the central part. For a discussion of type II singularities, we refer to [Man11, Chapter 4] and a detailed analysis for the formation of degenerate neckpinches can be found in [AV97]. To follow the subsequent chapters in this thesis, knowledge on the different types of singularities is not necessary. We merely need to remember that under the usual mean curvature flow, the surface area is decreasing, closed surfaces only have finite lifespans, convexity as well as mean convexity is conserved and embedded surfaces remain embedded.

In this work however, we deal with more general, non-constant functions G and therefore end up with a non-constantly scaled version

$$V = g(c)H$$

of the mean curvature flow. Our aim is to analyze which of the properties mentioned above transfer from the usual mean curvature flow to the scaled version. This depends of course heavily on the shape of the function G.

A futher discussion of the energy density G is the content of Section 3.4. There, we state two parabolicity conditions which G has to fulfill for mathematical reasons and briefly explain their effects. In particular, a parabolically scaled mean curvature flow leads to a decreasing surface area, just as the usual mean curvature flow. Nevertheless, in Section 3.5, we demonstrate for the simple example of a radial symmetric setting, that - in contrast to the usual mean curvature flow - a suitable choice of G for the parabolically scaled mean curvature flow prevents a closed surface from collapsing in finite time. Further properties of the parabolically scaled mean curvature flow are addressed in Chapter 5: As for the usual mean curvature flow, it conserves mean convexity (see Section 5.1), but for a non-constant scaling function g, a convex surface can turn non-convex in the course of time (see Section 5.2). Moreover, in Section 5.3, we show that a non-constant scaling function g can force initially embedded surfaces to develop self-intersections during the evolution.

3.2 The Diffusion Equation

Now, we discuss the second equation (3.1b). If there is no geometric evolution, i.e., if Γ is a non-moving hypersurface, the energy functional (3.2) is fully determined through $G(c)$ and therefore $G(c)$ can be understood as the free energy of the system. Then, $\mu = G'(c)$ is the chemical potential. Fick's first law $j = -\nabla\mu$ for the flux j and a constant diffusion coefficient, arbitrarily set to one, together with the continuity equation $\partial_t c = -\mathrm{div} j$ finally results in the diffusion equation $\partial_t c = \Delta\mu$. And, indeed, without geometric evolution, i.e. $V = 0$, the second equation (3.1b) reduces to the simple diffusion equation valid for a concentration independent diffusion coefficient

$$\partial_t c = \Delta_\Gamma\big(G'(c)\big)$$

as $\partial^\square c = \partial_t c$ holds for a non-moving hypersurface.

In the general case of a non-constant evolving hypersurface, the second equation (3.1b) is given by

$$\partial^\square c = \Delta_\Gamma\big(G'(c)\big) + cHV.$$

Again, we have a diffusion equation for the concentration c on the surface Γ and the changing of the geometry results in an additional source term that depends linearly on the concentration c.

We want the concentration c to describe the distribution of a quantity that can neither vanish from nor be added to the surface. The second equation guarantees this conservation of mass: For a solution (Γ, c) of (3.1b), the transport theorem (Proposition 2.58) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} c \,\mathrm{d}\mathcal{H}^d = \int_{\Gamma(t)} \partial^\square c - cHV \,\mathrm{d}\mathcal{H}^d = \int_{\Gamma(t)} \Delta_\Gamma\big(G'(c)\big)\,\mathrm{d}\mathcal{H}^d = 0, \tag{3.3}$$

where the last identity holds due to Gauß' theorem on hypersurfaces (Proposition 2.48), as $\Gamma(t)$ is closed.

3.3 Gradient Flow of the Energy Functional

In this section, we establish a connection between the equations (3.1) and the energy functional (3.2): The evolution of a pair (Γ, c) that decreases the energy (3.2) most efficiently is characterized by our equations (3.1). In other words, the system of equations is a gradient flow of the energy functional.

First of all, using the transport theorem (Proposition 2.58) and integration by parts (Proposition 2.48) for the closed surface $\Gamma(t)$, we get that a solution (Γ, c) of (3.1) fulfills

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(\Gamma, c) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma G(c)\,\mathrm{d}\mathcal{H}^d \\
&= \int_{\Gamma(t)} G'(c)\partial^\square c - G(c)HV \,\mathrm{d}\mathcal{H}^d \\
&= \int_{\Gamma(t)} G'(c)\Delta_\Gamma\big(G'(c)\big) + \big(G'(c)c - G(c)\big)HV \,\mathrm{d}\mathcal{H}^d \\
&= -\int_{\Gamma(t)} \big|\nabla_\Gamma G'(c)\big|^2 + V^2 \,\mathrm{d}\mathcal{H}^d \le 0.
\end{aligned}$$

Hence, a solution (Γ, c) of (3.1) can never increase the energy functional E. As long as the geometry of the system changes, i.e. $V \neq 0$, the energy will actually decrease. Also, assuming G' not to be constant, a non-uniform distribution of the quantity described by c in general also leads to an actual decrease of the energy. Note that due to $V = g(c)H$ by (3.1a) and because a closed hypersurface $\Gamma(t)$ can not have vanishing mean curvature H everywhere (Proposition 2.44), the condition $g > 0$ assumed for parabolicity reasons later on (see Section 3.4) implies an actual decrease of the energy.
In the following, we explain formally how the equations (3.1) can be seen as a gradient flow of the energy functional (3.2). This means that a solution (Γ, c) of (3.1) even decreases the energy functional in an optimal way. The techniques used in this section are based on corresponding arguments in [Gar13] for the usual mean curvature flow of evolving hypersurfaces. We extend these considerations to our setting with the additional concentration that describes a distribution on the surface. As we have seen above, a solution of (3.1) satisfies mass conservation. So, for a constant mass $m \in \mathbb{R}_{>0}$, we consider the set

$$M^m \coloneqq \Big\{ (\Sigma, c) \,\Big|\, \Sigma \subset \mathbb{R}^{d+1} \text{ smooth, closed, embedded hypersurface,} \\ c : \Sigma \to \mathbb{R} \text{ smooth concentration with } \int_\Sigma c \, \mathrm{d}\mathcal{H}^d = m \Big\}$$

of all surfaces Σ in $\mathbb{R}^{d+1}$ and concentrations $c : \Sigma \to \mathbb{R}$ such that the total mass of the quantity, whose distribution on Σ is described by c, equals m. In a formal way, we endow M^m with the tangent space

$$T_{(\Sigma,c)} M^m = \Big\{ \big(V, w\big) \,\Big|\, V, w : \Sigma \to \mathbb{R} \text{ smooth with } \int_\Sigma w - cHV \, \mathrm{d}\mathcal{H}^d = 0 \Big\}.$$

Here, V is a possible normal velocity of Σ, w is a variation of the concentration and the additional condition $\int_\Sigma w - cHV = 0$ with mean curvature H of Σ ensures that the change is such that mass conservation holds. We now elaborate how such a pair (V, w) of smooth functions $V, w : \Sigma \to \mathbb{R}$ arises as the differential of a map in M^m, hence as a "tangent vector" of M^m in a point (Σ, c).
Let $\theta_t : \Sigma \to \mathbb{R}^{d+1}$, $\theta_t(p) \coloneqq p + tV(p)\nu(p)$ with the smooth unit normal ν of Σ. According to Proposition 2.60, for $|t|$ sufficiently small, θ_t is a smooth diffeomorphism onto its image $\Gamma_t \coloneqq \theta_t(\Sigma)$ which again is a smooth, closed, embedded hypersurface in $\mathbb{R}^{d+1}$. In particular, $\big\{ \{t\} \times \Gamma_t \,\big|\, t \in (-\varepsilon, \varepsilon) \big\}$ is an evolving hypersurface as in Definition 2.49 and its normal velocity in $t = 0$ equals V. So, V really is the normal velocity of $\Sigma = \Gamma_0$ as already stated above.
Moreover, with

$$m_t \coloneqq \left(\int_{\Gamma_t} 1 \, \mathrm{d}\mathcal{H}^d \right)^{-1} \left(m - \int_{\Gamma_t} (c + tw) \circ \theta_t^{-1} \, \mathrm{d}\mathcal{H}^d \right) \in \mathbb{R},$$

$c_t \coloneqq (c + tw) \circ \theta_t^{-1} + m_t : \Gamma_t \to \mathbb{R}$ defines a smooth function on Γ_t. We consider $\eta(t) \coloneqq (\Gamma_t, c_t)$ for $t \in (-\varepsilon, \varepsilon)$ and claim that η is the map in M^m through (Σ, c) with differential (V, w). By construction, we have

$$\begin{aligned} \int_{\Gamma_t} c_t \, \mathrm{d}\mathcal{H}^d &= \int_{\Gamma_t} (c + tw) \circ \theta_t^{-1} + m_t \, \mathrm{d}\mathcal{H}^d \\ &= \int_{\Gamma_t} (c + tw) \circ \theta_t^{-1} \, \mathrm{d}\mathcal{H}^d + \left(m - \int_{\Gamma_t} (c + tw) \circ \theta_t^{-1} \mathrm{d}\mathcal{H}^d \right) = m \end{aligned}$$

for every $t \in (-\varepsilon, \varepsilon)$, such that $\eta(t) \in M^m$ holds for every $t \in (-\varepsilon, \varepsilon)$. Furthermore, we have $\eta(0) = (\Sigma, c)$, since

$$\int_{\Gamma_t} (c + tw) \circ \theta_t^{-1} \, \mathrm{d}\mathcal{H}^d{}_{|t=0} = \int_\Sigma c \, \mathrm{d}\mathcal{H}^d = m \tag{3.4}$$

implies $m_0 = 0$. In addition,

$$\partial^\square \big((c + tw) \circ \theta_t^{-1} \big)_{|t=0} = \frac{\mathrm{d}}{\mathrm{d}t} (c + tw)_{|t=0} - \partial_t \theta_t \cdot \nabla_{\Gamma_t} \big((c + tw) \circ \theta_t^{-1} \big)_{|t=0} = w - V\nu \cdot \nabla_\Sigma c = w$$

holds because as an element of the tangent space of Σ, the surface gradient $\nabla_\Sigma c$ is perpendicular to the normal ν. With the transport theorem (Proposition 2.58) and the additional condition for (V, w),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \int_{\Gamma_t} (c + tw) \circ \theta_t^{-1} \, \mathrm{d}\mathcal{H}^d &= \int_\Sigma \partial^\square \big((c + tw) \circ \theta_t^{-1} \big) - \big((c + tw) \circ \theta_t^{-1} \big)_{|t=0} HV \, \mathrm{d}\mathcal{H}^d \\ &= \int_\Sigma w - cHV \, \mathrm{d}\mathcal{H}^d = 0 \end{aligned} \tag{3.5}$$

follows. Equations (3.4) and (3.5) yield $\partial^\square m_{t\,|t=0} = \frac{\mathrm{d}}{\mathrm{d}t} m_{t\,|t=0} = 0$, such that we finally have

$$\partial^\square c_{t\,|t=0} = \partial^\square \big((c + tw) \circ \theta_t^{-1} \big)_{|t=0} + \partial^\square m_{t\,|t=0} = w.$$

As the normal velocity of the evolving hypersurface $\big\{ \{t\} \times \Gamma_t \,\big|\, t \in (-\varepsilon, \varepsilon) \big\}$ in $t = 0$ is V and the normal time derivative of c_t in $t = 0$ is w, the pair (V, w) fully determines the differential $\eta'(0)$. In particular, (V, w) can be interpreted as "tangential vector" to M^m in (Σ, c).

On the tangent space $T_{(\Sigma,c)} M^m$ we define an L^2-H^{-1}-inner product

$$\begin{aligned} \big\langle (V_1, w_1), (V_2, w_2) \big\rangle &\coloneqq \int_\Sigma V_1 V_2 \, \mathrm{d}\mathcal{H}^d + \int_\Sigma \nabla_\Sigma u_1 \cdot \nabla_\Sigma u_2 \, \mathrm{d}\mathcal{H}^d \\ &= \int_\Sigma V_1 V_2 \, \mathrm{d}\mathcal{H}^d + \int_\Sigma u_1 \big(w_2 - cHV_2 \big) \, \mathrm{d}\mathcal{H}^d \\ &= \int_\Sigma \big(V_1 - u_1 cH \big) V_2 \, \mathrm{d}\mathcal{H}^d + \int_\Sigma u_1 w_2 \, \mathrm{d}\mathcal{H}^d, \end{aligned}$$

where $-\Delta_\Sigma u_i = w_i - cHV_i$ holds on Σ. These functions u_i are well-defined, as due to Gauß' theorem on closed hypersurfaces (Proposition 2.48), $\int_\Sigma w_i - cHV_i \, \mathrm{d}\mathcal{H}^d = \int_\Sigma -\Delta_\Sigma u_i \, \mathrm{d}\mathcal{H}^d = 0$ is fulfilled for $(V_i, w_i) \in T_{(\Sigma,c)} M^m$. Using a Lax-Milgram type argument, one can show that this is exactly the solvability condition for $-\Delta_\Sigma u_i = w_i - cHV_i$ on Σ. The choice of an H^{-1}-inner product for the concentration part ensures the conservation of mass and the L^2-inner product for the surface part results in decreasing surface area just as for the usual mean curvature flow (see Section 3.4).

Now, we want to identify the gradient flow of the energy functional E (see (3.2)) with respect to this inner product. With the help of the transport theorem (Proposition 2.58), the total differential of E in (Σ, c) in direction of $(V, w) \in T_{(\Sigma,c)} M^m$ is given by

$$\begin{aligned} \delta \mathrm{E}(\Sigma, c)(V, w) &= \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{E}\big(\Gamma_t, c_t \big)_{|t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma_t} G(c_t) \, \mathrm{d}\mathcal{H}^d_{|t=0} \\ &= \int_\Sigma \partial^\square \big(G(c_t) \big)_{|t=0} - G(c) HV \, \mathrm{d}\mathcal{H}^d \\ &= \int_\Sigma G'(c) w - G(c) HV \, \mathrm{d}\mathcal{H}^d \end{aligned}$$

with Γ_t and c_t defined as before. If we choose $(V_g, w_g) \coloneqq \operatorname{grad} \mathrm{E}(\Sigma, c) \in T_{(\Sigma,c)}M^m$ as a notation for the gradient of E in (Σ, c), then for any direction $(V, w) \in T_{(\Sigma,c)}M^m$,

$$\big\langle \operatorname{grad} \mathrm{E}(\Sigma, c), (V, w) \big\rangle = \big\langle (V_g, w_g), (V, w) \big\rangle = \int_\Sigma \big(V_g - u_g cH\big) V \,\mathrm{d}\mathcal{H}^d + \int_\Sigma u_g w \,\mathrm{d}\mathcal{H}^d$$

holds with $-\Delta_\Sigma u_g = w_g - cHV_g$ as before. Since the gradient is defined through

$$\delta \mathrm{E}(\Sigma, c)(V, w) = \big\langle \operatorname{grad} \mathrm{E}(\Sigma, c), (V, w) \big\rangle$$

for all $(V, w) \in T_{(\Sigma,c)}M^m$, we obtain $u_g = G'(c)$ and

$$\begin{aligned} V_g &= u_g cH - G(c)H = \big(G'(c)c - G(c)\big)H, \\ w_g &= -\Delta_\Sigma u_g + cHV_g = -\Delta_\Sigma\big(G'(c)\big) + cHV_g. \end{aligned}$$

As explained above, the differential $\frac{\mathrm{d}}{\mathrm{d}t}(\Gamma_t, c_t)_{|t=0}$ is determined by the normal velocity V of the evolving hypersurface $\big\{\{t\} \times \Gamma_t \,\big|\, t \in (-\varepsilon, \varepsilon)\big\}$ in $t = 0$ and the normal time derivative $\partial^\square c$ of c_t in $t = 0$. The family $(\Gamma_t, c_t)_{t\in(-\varepsilon,\varepsilon)}$ hence is a solution to the desired gradient flow in $t = 0$ if and only if

$$(V, \partial^\square c) = -\operatorname{grad} \mathrm{E}(\Sigma, c) = -(V_g, w_g)$$

is valid. So, the gradient flow of the energy functional E with respect to our $L^2 - H^{-1}$-inner product is the system (3.1)

$$\begin{aligned} V &= \big(G(c) - G'(c)c\big)H, \\ \partial^\square c &= \Delta_\Sigma\big(G'(c)\big) + cHV \end{aligned}$$

as claimed.

3.4 The Energy Density Function G

Due to its role in the energy functional

$$\mathrm{E}(\Gamma, c) \coloneqq \int_\Gamma G(c) \,\mathrm{d}\mathcal{H}^d,$$

the function $G : \mathbb{R} \to \mathbb{R}$ is called an energy density function. This section is restricted to the physical relevant case of $c \geq 0$, so it suffices to consider $G : \mathbb{R}_{\geq 0} \to \mathbb{R}$. From a physical point of view, it seems natural to assign a high energy to the extreme cases of $c \approx 0$ and $c \approx \infty$ so that their occurrence is avoided.

So, seeking for a minimal energy, the concentration tends to assume values in the moderate area of G. On the other hand, the hypersurface Γ tends to minimize its surface area and, on account of mass conservation, thus will force the concentration to grow. The system hence needs to find a balance between these two trends. It is not clear which of them will prevail; whether the hypersurface will shrink to a single point as for the usual mean curvature flow or if the role of the additional concentration is significant enough to stop or at least slow down the shrinking of the surface. Obviously, the sought balance depends on the shape of G: A constant energy density function G implies independence of the concentration and results in the usual mean curvature flow, as explained in Section 3.1. Moderate growth of $G(c)$ for $c \to \infty$ leads to a weak effect of the concentration and therefore to a behavior still similar to the usual mean curvature flow whereas strong growth of $G(c)$ for $c \to \infty$ increases the impact of the additional concentration and thus allows for a different geometric behavior. In Section 3.5, this will be discussed for the example of a radial symmetric setting.

To show the existence of short-time solutions to (3.1) in Chapter 4, we need a more specific shape of G: The equations (3.1a) and (3.1b) have to be parabolic, so $g > 0$ and $G'' > 0$ have to hold. For readers not so familiar with geometric quantities it might be easier to deduce these parabolicity conditions from the transformed system (3.8), see Section 3.6. The second condition $G'' > 0$ implies that G is convex. This fits nicely to the physical idea of high energy for the extreme cases of $c \approx 0$ and $c \approx \infty$, but it prohibits interesting effects like phase transitions. Hence, we have to restrict to considering only one phase or to describing physical situations that generally allow for convex (Gibbs) energy densities. This is the case, e.g., if we neglect effects of internal energy and simply consider a (convex) mixing entropy density.

The first condition $g > 0$ is more involved and will be discussed now. As $G'' > 0$, we have

$$g'(c) = \big(G(c) - G'(c)c\big)' = G'(c) - G''(c)c - G'(c) = -G''(c)c < 0$$

for $c \geq 0$. Together, $g > 0$ and $g' < 0$ imply that $g'(c)$ is bounded for $c \to \infty$. So,

$$G''(c) = -\frac{g'(c)}{c}$$

tends to zero for $c \to \infty$ and therefore $G(c)$ can grow at most linearly for $c \to \infty$. It is even possible to specify the slope of this linear growth: As G is convex, we have

$$G(c) - G'(c_0)c \geq G(c_0) - G'(c_0)c_0 = g(c_0) > 0 \quad \Leftrightarrow \quad \frac{G(c)}{c} \geq G'(c_0)$$

for all $c, c_0 > 0$ and hence

$$\lim_{c_0 \to \infty} G'(c_0) \leq \inf_{c>0} \frac{G(c)}{c}.$$

Linear growth definitely is only moderate growth. On that account, we expect the geometric evolution in the parabolically scaled case to vary less from the one of the usual mean curvature flow than it would for arbitrary scaling.

A simple example of this effect is the decrease of surface area for the parabolically scaled setting: The transport theorem (Proposition 2.58) together with the scaled mean curvature flow equation yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \mathrm{d}\mathcal{H}^d = -\int_\Gamma HV \,\mathrm{d}\mathcal{H}^d = -\int_\Gamma g(c) H^2 \,\mathrm{d}\mathcal{H}^d.$$

Due to $g > 0$ and because a closed surface can not have vanishing mean curvature H everywhere (Proposition 2.44), the surface area of Γ is strictly decreasing, just as for the usual mean curvature flow. In particular, stationary solutions are not possible. Without the parabolicity condition, a different behavior of the surface would be possible of course. This is illustrated in Section 3.5 for the example of a radial symmetric situation. There we also observe that, even under the parabolicity conditions and despite the decreasing surface area, the evolution of the surface in the scaled setting may still differ considerably from the one of the usual mean curvature flow.

3.5 Example of a Radial Symmetric Setting

To get a first impression of the evolution prescribed by our system of equations (3.1), we study the radial symmetric case as in [GM21]. In particular, we are interested in whether the evolution of the geometry is the same as for the usual mean curvature flow or if the impact of the additional concentration is strong enough to change the usual behavior. The usual mean curvature flow forces a convex, closed surface to shrink to a round point in finite time. Obviously, if the surface in our example does shrink to a single point, it also shrinks to a round point just as for the usual mean curvature flow; but roundness is true anyway due to the construction of the surface as radial symmetric sphere. So, we focus on the collapsing of the surface and whether this happens in finite time.
A radial symmetric situation means that the hypersurface $\Gamma(t) = \partial B_{R(t)}(0) \subset \mathbb{R}^{d+1}$ is a sphere and the concentration $c(t) : \Gamma(t) \to \mathbb{R}$ is constant in space. As mass conservation is fulfilled,

$$m = \int_{\Gamma(t)} c(t)\,\mathrm{d}\mathcal{H}^d = c(t)\mathcal{H}^d\big(\Gamma(t)\big) = \alpha_d c(t) R(t)^d \quad \Leftrightarrow \quad c(t) = \frac{m}{\alpha_d} R(t)^{-d}$$

holds with a constant α_d only depending on the dimension d. So, the whole evolution of the pair $\big(\Gamma(t), c(t)\big)$ is characterized by the function $R : [0,T) \to (0,\infty)$. As in Remark 2.43, we have

$$\nu(t,p) = \frac{p}{R(t)} \qquad \text{and} \qquad H(t,p) = -\frac{d}{R(t)}$$

for the unit normal and the mean curvature of $\Gamma(t)$ in $p \in \Gamma(t)$. As

$$\theta : [0,T] \times \Gamma_0 \to \mathbb{R}^{d+1}, \quad \theta(t,z) \coloneqq \frac{R(t)}{R_0} z$$

defines a global parameterization for the evolving hypersurface, the normal velocity of $\Gamma(t)$ in $p = \theta(t,z) \in \Gamma(t)$ is given by

$$V(t,p) = \partial_t \theta(t,z) \cdot \nu(t,p) = \frac{R'(t)}{R_0} z \cdot \frac{p}{R(t)} = \frac{R'(t)}{R_0} z \cdot \frac{z}{R_0} = R'(t).$$

The normal time derivative of the constant-in-space function $c(t)$ in $p = \theta(t,z) \in \Gamma(t)$ is

$$\partial^\square c(t,p) = \partial^\circ c(t,p) = \frac{\mathrm{d}}{\mathrm{d}t} c(t) = -\frac{md}{\alpha_d} R(t)^{-d-1} R'(t)$$

and thus the concrete form of the second equation (3.1b)

$$\partial^{\square} c = -\frac{md}{\alpha_d} R^{-d-1} R' = 0 + \frac{m}{\alpha_d} R^{-d} \cdot \frac{-d}{R} \cdot R' = \Delta_\Gamma\big(G'(c)\big) + cHV$$

is automatically fulfilled. This was to be expected because the diffusion equation for spatially constant functions c reduces to $\partial^{\square} c = cHV$. As also the geometric quantities H and V are spatially constant, this reduced equation is equivalent to mass conservation (cf. Equation (3.3)) and we chose the time dependence of c such that mass conservation is fulfilled. The impact of the additional concentration hence is limited to the non-constant scaling factor $g(c)$ in the mean curvature flow equation. Thus, the question we seek to answer is whether there exists a shape of g such that the evolution of the geometry differs from the one for a constant function g, i.e., the usual mean curvature flow.
We now turn to the first equation which, in the radial symmetric case, transforms to

$$R' = V = g(c)H = -g\left(\frac{m}{\alpha_d} R^{-d}\right) \frac{d}{R} =: f(R).$$

Assuming the parabolicity condition $g > 0$, we have $f < 0$ on $(0, \infty)$ and therefore we can apply separation of variables which yields

$$t = \int_0^t \frac{R'(s)}{f(R(s))} \,\mathrm{d}s = \int_{R_0}^{R(t)} \frac{1}{f(z)} \,\mathrm{d}z =: F\big(R(t)\big).$$

Due to $f < 0$ on $(0, \infty)$, $F : (0, \infty) \to \mathbb{R}$ is strictly monotonic decreasing and thus bijective onto its image. Therefore we can define $R(t) := F^{-1}(t)$ for all $t \in F\big((0, \infty)\big)$; but as only positive times $t > 0$ are considered, we reduce to $t \in (0, T) := (0, \infty) \cap F\big((0, \infty)\big)$. We have

$$\begin{aligned} \lim_{R \searrow 0} F(R) &= \int_{R_0}^{0} \frac{1}{f(z)} \,\mathrm{d}z = -\int_0^{R_0} \frac{1}{f(z)} \,\mathrm{d}z > 0, \\ \lim_{R \nearrow \infty} F(R) &= \int_{R_0}^{\infty} \frac{1}{f(z)} \,\mathrm{d}z < 0 \end{aligned} \tag{3.6}$$

and thus $(0, T) = \big(0, F(0)\big)$. In particular,

$$R(t) = F^{-1}(t) \to F^{-1}(T) = F^{-1}\big(F(0)\big) = 0$$

holds for $t \to T$. So, just as for the usual mean curvature flow, the hypersurface shrinks to a single point under the scaled mean curvature flow with $g > 0$. Due to the assumed parabolicity condition $g > 0$, the strict decrease of the surface area was already known from Section 3.4.
Obviously, a negative scaling factor g changes the sign of the mean curvature flow equation and forces a convex surface to grow: With the same arguments as above, $g < 0$ and thus $f > 0$ lead to $(0, T) = \big(0, F(\infty)\big)$ with $R(t) \to F^{-1}\big(F(\infty)\big) = \infty$ for $t \to T$.
A scaling function

$$g(c) \begin{cases} > 0, & c < c_\star \\ = 0, & c = c_\star \\ < 0, & c > c_\star \end{cases}$$

whose sign changes at a fixed concentration $c_\star = \frac{m}{\alpha_d} R_\star^{-d} \in (0,\infty)$ leads to a stationary solution for $c = c_\star \Leftrightarrow R = R_\star$ because $g(c_\star) = 0$ prohibits any geometrical evolution. As above, $g > 0 \Leftrightarrow c < c_\star \Leftrightarrow R > R_\star$ implies a shrinking of the convex surface, whereas $g < 0 \Leftrightarrow c > c_\star \Leftrightarrow R < R_\star$ forces the surface to grow. Using the notation

$$f(z) = -g\left(\frac{m}{\alpha_d} z^{-d}\right)\frac{d}{z} \qquad \text{and} \qquad F(R) = \int_{R_0}^{R} \frac{1}{f(z)}\,\mathrm{d}z$$

as above, we can argue more precisely: If $R_0 \in (0,R_\star)$, then $F : (0,R_\star) \to \mathbb{R}$ is well-defined and strictly increasing because we have $f > 0$ on $(0,R_\star)$ and $f(z) = 0 \Leftrightarrow z = R_\star$. Thus $R(t) \coloneqq F^{-1}(t)$ is well-defined for all $t \in (0,T) = (0,\infty) \cap F((0,R_\star)) = (0,F(R_\star))$, where the last identity holds due to $0 = F(R_0) \in F((0,R_\star))$. Otherwise, if $R_0 \in (R_\star,\infty)$, then $F : (R_\star,\infty) \to \mathbb{R}$ is well-defined and strictly decreasing as $f < 0$ holds on $(R_\star,\infty)$, and hence $R(t) \coloneqq F^{-1}(t)$ is well-defined for all $t \in (0,T) = (0,\infty) \cap F((R_\star,\infty)) = (0,F(R_\star))$. In both cases, we have $R(t) \to F^{-1}(T) = R_\star$ for $t \to T$, which means that the solution tends to the stationary solution, either expanding a smaller surface or shrinking a larger surface. If the zero of the function g at $c_\star$ is such that $\frac{1}{f(z)}$ is integrable in a neighborhood of $z = R_\star$, the stationary solution is attained at the finite time $T = F(R_\star) < \infty$, otherwise we have $T = \infty$ and the geometry converges to the stationary solution, but does not reach it in any finite time.

So we see that scaling the mean curvature flow equation results in very different geometric evolutions. Assuming the parabolicity condition $g > 0$ however leads to a very similar behavior as for the usual mean curvature flow, meaning that convex surfaces shrink to a single point. But, in contrast to the usual mean curvature flow, even for $g > 0$, the final time T at which the surface collapses does not need to be finite for the scaled mean curvature flow. If $g > 0$, we have (see (3.6))

$$T = F(0) = -\int_0^{R_0} \frac{1}{f(z)}\,\mathrm{d}z = \frac{1}{d}\int_0^{R_0} \frac{z}{g\left(\frac{m}{\alpha_d} z^{-d}\right)}\,\mathrm{d}z$$

and for a suitable choice of g, this turns out to be $T = \infty$. Choose for example the power function

$$G(c) \coloneqq \frac{1}{1-s} c^s$$

with $s < -\frac{2}{d}$. It follows that

$$G''(c) = -s c^{s-2} \qquad \text{and} \qquad g(c) = G(c) - G'(c)c = c^s$$

and thus both parabolicity conditions $G'' > 0$ and $g > 0$ are fulfilled for the physical relevant case $c > 0$. Calculation of the final time T yields

$$T = \frac{1}{d}\int_0^{R_0} \frac{z}{g\left(\frac{m}{\alpha_d} z^{-d}\right)}\,\mathrm{d}z = C(m,d,s)\int_0^{R_0} z^{sd+1}\,\mathrm{d}z = \frac{C(m,d,s)}{sd+2}\, z^{sd+2}\Big|_0^{R_0}$$

with a positive constant $C(m,d,s)$ depending on the mass m, the dimension d and the power variable s. Due to the choice $s < -\frac{2}{d} \Leftrightarrow sd + 2 < 0$, this adds up to $T = \infty$ and therefore the surface does not shrink to a point in finite time, but exists as a sphere for all times.

We have seen using the simple example of radial symmetry that scaling the mean curvature flow with different factors g can produce considerably diverse geometric evolutions. Assuming though the parabolicity condition $g > 0$, a surface will always shrink and thus evolve very similar as for the usual mean curvature flow. But even a positive scaling factor $g > 0$ can slow down this behavior enforced by the mean curvature flow and prevent the surface from collapsing, at least in finite time. The additional concentration thus does have an effect on the geometric evolution of the surface in our system (3.1).

3.6 Reformulation onto a Fixed Domain

We wish to reformulate the system (3.1) in a way that enables us to prove the existence of short-time solutions in Chapter 4. For this, we assume that Γ_ρ is an evolving immersed closed hypersurface parameterized via a height function ρ as in Section 2.1.6. Then, the system

$$\left.\begin{aligned} V_{\Gamma_{\rho|U}} &= g(c)H_{\Gamma_{\rho|U}}, \\ \partial^\square c &= \Delta_{\Gamma_{\rho|U}}\big(G'(c)\big) + cH_{\Gamma_{\rho|U}}V_{\Gamma_{\rho|U}} \end{aligned}\right\} \quad \text{on } \Gamma_{\rho|U} \tag{3.7}$$

defined on every embedded patch $\Gamma_{\rho|U}$, $U \subset M$, of the evolving immersed hypersurface Γ_ρ can be transformed onto the fixed domain $[0,T] \times M$ with the help of the global parameterization $\theta_\rho : [0,T] \times M \to \mathbb{R}^{d+1}$, $\theta_\rho(t,z) \coloneqq \bar\theta(z) + \rho(t,z)\nu_\Sigma(z)$, where $\Sigma = \bar\theta(M)$ is a fixed immersed closed hypersurface.

We introduce the function $u : [0,T]\times M \to \mathbb{R}$ to describe the concentration on the immersed surface such that for every embedded patch $\Gamma_{\rho|U}$ of Γ_ρ

$$\begin{aligned} c \circ \theta_\rho &= u, \\ g(c) \circ \theta_\rho &= g(u), \\ \big(\nabla_{\Gamma_{\rho|U}} c\big) \circ \theta_\rho &= \nabla_\rho u, \\ \big(\Delta_{\Gamma_{\rho|U}} G'(c)\big) \circ \theta_\rho &= \Delta_\rho G'(u) \text{ and} \\ \big(\partial^\square c\big) \circ \theta_\rho &= \partial^\square u \end{aligned}$$

hold on $[0,T]\times U$. Here, we used the notation introduced in Remark 2.64 for the pullbacks of the surface gradient, the Laplace-Beltrami operator and the normal time derivative. Furthermore, we use the notation

$$\begin{aligned} \nu_\rho &\coloneqq \nu_{\Gamma_{\rho|U}} \circ \theta_\rho, \\ H(\rho) &\coloneqq H_{\Gamma_{\rho|U}} \circ \theta_\rho, \\ V_\rho &\coloneqq V_{\Gamma_{\rho|U}} \circ \theta_\rho \text{ and} \\ V_\rho^{\text{tot}} &\coloneqq V^{\text{tot}}_{\Gamma_{\rho|U}} \circ \theta_\rho \end{aligned}$$

for the pullbacks of the normal, the mean curvature and the normal and total velocity again as in Remark 2.64. Assuming ρ to be sufficiently small in an appropriate sense yields that

$$a(\rho) \coloneqq \frac{1}{\nu_\Sigma \cdot \nu_\rho}$$

is well-defined with $\frac{1}{2} \leq a(\rho) \leq C$ (see Remark 4.6).
Using all this notation and the formulas developed in Section 2.1, we can now transform the system (3.7). With the pullback of the total velocity $V_\rho^{\text{tot}} = \partial_t \theta_\rho = \partial_t \rho \, \nu_\Sigma$ of the evolving surface, we obtain

$$\begin{aligned} V_{\Gamma_{\rho|U}} \circ \theta_\rho &= V_\rho = V_\rho^{\text{tot}} \cdot \nu_\rho = \partial_t \rho \nu_\Sigma \cdot \nu_\rho = \frac{\partial_t \rho}{a(\rho)} \text{ and} \\ \left(\partial^\square c\right) \circ \theta_\rho &= \partial^\square u = \partial^\circ u - V_\rho^{\text{tot}} \cdot \nabla_\rho u = \partial_t u - \partial_t \rho \nu_\Sigma \cdot \nabla_\rho u \end{aligned}$$

for the normal velocity of the surface and the normal time derivative of the concentration.
So, finally, the formulation of the system (3.7) on the fixed domain $[0, T] \times M$ is given by

$$\partial_t \rho = g(u) a(\rho) H(\rho), \tag{3.8a}$$

$$\begin{aligned} \partial_t u &= \Delta_\rho G'(u) + \partial_t \rho \, \nu_\Sigma \cdot \nabla_\rho u + u H(\rho) V_\rho \\ &= \Delta_\rho G'(u) + g(u) a(\rho) H(\rho) \, \nu_\Sigma \cdot \nabla_\rho u + g(u) H(\rho)^2 u. \end{aligned} \tag{3.8b}$$

Chapter 4

Short-Time Existence

The topic of this chapter is the existence of short-time solutions to our system of equations

$$\partial_t \rho = g(u)a(\rho)H(\rho), \tag{4.1a}$$

$$\partial_t u = \Delta_\rho G'(u) + g(u)a(\rho)H(\rho)\,\nu_\Sigma \cdot \nabla_\rho u + g(u)H(\rho)^2 u \tag{4.1b}$$

transformed onto $[0,T] \times M$ with a fixed, embedded reference surface M, cf. Section 3.6. As therein, we define an evolving hypersurface Γ_ρ, parameterized via the height function $\rho : [0,T] \times M \to \mathbb{R}$ (see Section 2.1.6) and let $u : [0,T] \times M \to \mathbb{R}$ describe the concentration of a quantity on Γ_ρ. With the notation from Remark 2.64, $H(\rho), \nabla_\rho, \Delta_\rho$ and ν_ρ denote the pullbacks of the mean curvature, the surface gradient, the Laplace-Beltrami operator and the unit normal of Γ_ρ to $[0,T]\times M$, respectively, and ν_Σ is the unit normal of the immersed reference surface $\Sigma = \bar{\theta}(M)$. The functional $a(\rho) \coloneqq \frac{1}{\nu_\rho \cdot \nu_\Sigma}$ is well-defined by Lemma 4.5(ii), see below. Finally, $G, g : \mathbb{R} \to \mathbb{R}$ are real valued functions whose relation and properties will be specified in Assumptions 4.9.

As a start, several regularity properties of functionals are stated which will be useful throughout the whole chapter. Then, we list the conditions under which our short-time existence result holds (see Assumptions 4.9) and introduce the notations that will be used (see Notations 4.10). With this preparatory work, we can move on to the actual proof of short-time existence. As explained in Chapter 1, a splitting ansatz is applied: In Section 4.1, the first equation (4.1a) for the height function ρ is discussed. For an arbitrary concentration u, we obtain a short-time solution ρ_u of this equation which is then inserted into the second equation (4.1b) for the concentration u. Section 4.2 deals with the existence of short-time solutions to this reduced system, i.e., the second equation with inserted ρ_u. The combined result on short-time existence can be found in Section 4.3.

Notations 4.1. *Let $s \in \mathbb{R}_{>0} \setminus \mathbb{N}$ and let $\Sigma = \bar{\theta}(M) \subset \mathbb{R}^{d+1}$ be an h^{2+s}-immersed closed hypersurface. We define $X_s \coloneqq h^s(M)$, $Y_s \coloneqq h^{1+s}(M)$, $Z_s \coloneqq h^{2+s}(M)$ and for constants $R^\Sigma > 0$ and $R^c > 0$*

$$U^h_{s,1} \coloneqq \left\{\rho \in Y_s \,\middle|\, \|\rho\|_{C^1(M)} < 2R^\Sigma\right\}, \qquad U^c_s \coloneqq \left\{u \in Y_s \,\middle|\, \|u\|_{Y_s} < 2R^c\right\}.$$

We recall the notation and some properties for surfaces parameterized via height functions in the following lemma.

Lemma 4.2. *Let $s \in \mathbb{R}_{>0} \setminus \mathbb{N}$ and let $\Sigma = \bar{\theta}(M) \subset \mathbb{R}^{d+1}$ be an h^{2+s}-immersed closed hypersurface with unit normal ν_Σ. We use Notations 4.1. There exists a sufficiently small $R^\Sigma > 0$ such that for all $\rho \in U^h_{s,1}$*

$$\theta_\rho : M \to \mathbb{R}^{d+1},\ \theta_\rho(z) \coloneqq \bar{\theta}(z) + \rho(z)\nu_\Sigma(z)$$

is an h^{1+s}-immersion and $\Gamma_\rho \coloneqq \theta_\rho(M)$ is an h^{1+s}-immersed closed hypersurface. In particular, for any sufficiently small local parameterization (γ, W) of M and

$$\gamma_\rho \coloneqq \theta_\rho \circ \gamma,$$

(γ_ρ, W) is a local parameterization of an embedded patch of Γ_ρ.
Moreover, $\big(\partial_1\gamma_{\rho|x}, \ldots, \partial_d\gamma_{\rho|x}, \nu_\Sigma \circ \gamma_{|x}\big) \subset \mathbb{R}^{d+1}$ are linearly independent for every $x \in \overline{W}$, where

$$\partial_i\gamma_\rho = \partial_i\gamma + \partial_i(\rho \circ \gamma)(\nu_\Sigma \circ \gamma) + (\rho \circ \gamma)\partial_i(\nu_\Sigma \circ \gamma)$$

holds.

Proof. This is exactly the statement of Proposition 2.62. □

Now, we turn to the promised regularity statements.

Lemma 4.3. *Let $s \in \mathbb{R}_{>0} \setminus \mathbb{N}$ and let $\Sigma = \bar{\theta}(M) \subset \mathbb{R}^{d+1}$ be an h^{3+s}-immersed closed hypersurface. We use Notations 4.1. For $R^\Sigma > 0$ sufficiently small, there exist functions*

$$P \in C^\infty\Big(U^h_{s,1}, \mathcal{L}\big(Z_s, X_s\big)\Big) \text{ and } Q \in C^\infty\big(U^h_{s,1}, X_s\big)$$

such that the mean curvature $H(\rho)$ of the h^{2+s}-immersed closed hypersurface $\Gamma_\rho = \theta_\rho(M)$ from Lemma 4.2 is given by

$$H(\rho) = P(\rho)[\rho] + Q(\rho) \text{ in } X_s$$

for all $\rho \in U^h_{s,1} \cap Z_s$.

Proof. By [ES98, Lemma 3.1], for any sufficiently small local parameterization (γ, W) of M, we have

$$H(\rho)\circ\gamma = P(\rho)[\rho]\circ\gamma + Q(\rho)\circ\gamma \text{ with}$$
$$P(\rho)[u]\circ\gamma = \frac{1}{d}\left(\sum_{i,j=1}^{d} p_{ij}(\rho)\partial_i\partial_j(u\circ\gamma) + \sum_{k=1}^{d} p_k(\rho)\partial_k(u\circ\gamma)\right) \text{ and } Q(\rho)\circ\gamma = \frac{1}{d}q(\rho),$$

where $p_{ij}, p_i, q \in C^\infty\big(U^h_{s,1}, h^s(\overline{W})\big)$ hold for $R^\Sigma > 0$ sufficiently small. Hence,

$$P\circ\gamma \in C^\infty\Big(U^h_{s,1}, \mathcal{L}\big(Z_s, h^s(\overline{W})\big)\Big) \quad \text{and} \quad Q\circ\gamma \in C^\infty\big(U^h_{s,1}, h^s(\overline{W})\big)$$

follow and thus Remark 2.46 yields

$$P \in C^\infty\big(U^h_{s,1}, \mathcal{L}(Z_s, X_s)\big) \quad \text{and} \quad Q \in C^\infty(U^h_{s,1}, X_s).$$

Note that [ES98] assumes Σ to be a sphere. In [PS16, Section 2.2.5], the same statement is shown for an arbitrary embedded closed hypersurface Σ but as the proof therein is less clearly arranged, we chose to cite [ES98]. Both proofs reduce the statement to local coordinates and therefore neither the shape of a sphere nor the embeddedness property are necessary. Instead, the proofs can be transferred w.l.o.g. to our setting of an immersed closed hypersurface Σ, when choosing the local parameterization (γ, W) so small that $\theta_\rho\big(\gamma(\overline{W})\big)$ is a subset of an embedded patch of Σ and thus (γ_ρ, W) is a local parameterization of an embedded patch of Σ. $\square$

The fact that the mean curvature H has a quasilinear structure is the key argument to ensure that the PDE for the height function (4.1a) is also quasilinear. Even more, its main part $P(\rho)$ is elliptic, as we will see in the upcoming lemma.

Lemma 4.4. *Let $s \in \mathbb{R}_{>0} \setminus \mathbb{N}$ and let $\Sigma = \bar{\theta}(M) \subset \mathbb{R}^{d+1}$ be an h^{3+s}-immersed closed hypersurface. We use Notations 4.1 and choose P as in Lemma 4.3. For $R^\Sigma > 0$ sufficiently small and $\rho \in U^h_{s,1}$, $P(\rho) \in \mathcal{L}(Z_s, X_s)$ is a symmetric and elliptic differential operator of second order, i.e., given a sufficiently small local parameterization (γ, W) of M,*

$$P(\rho)[\cdot] \circ \gamma = \sum_{i,j} a^{ij} \partial_i \partial_j (\cdot \circ \gamma) + \textit{ lower order terms}$$

holds with a symmetric and positive definite coefficient matrix $[a^{ij}]_{i,j} \in h^s(\overline{W}, \mathbb{R}^{d\times d})$.

Proof. Let $\rho \in U^h_{s,1}$. With our sign convention, [ES98, Lemma 3.1] yields

$$P(\rho) \circ \gamma = \frac{1}{d} \sum_{i,j} p_{ij}(\rho) \partial_i \partial_j (\cdot \circ \gamma) + \text{ lower order terms}$$

with

$$p_{ij}(\rho) = \frac{w^{ij}(\rho)\left(1 + \sum_{k,l} w^{kl}(\rho)\partial_k(\rho\circ\gamma)\partial_l(\rho\circ\gamma)\right) - \sum_{k,l} w^{ik}(\rho) w^{jl}(\rho) \partial_k(\rho\circ\gamma)\partial_l(\rho\circ\gamma)}{\left(1 + \sum_{k,l} w^{kl}(\rho)\partial_k(\rho\circ\gamma)\partial_l(\rho\circ\gamma)\right)^{3/2}}$$

and $w_{kl}(\rho) = g^{\bar\theta}_{kl} + (\rho\circ\gamma)\big(\partial_k(\nu_\Sigma\circ\gamma)\cdot\partial_l\gamma_{\bar\theta} + \partial_l(\nu_\Sigma\circ\gamma)\cdot\partial_k\gamma_{\bar\theta}\big) + (\rho\circ\gamma)^2\big(\partial_k(\nu_\Sigma\circ\gamma)\cdot\partial_l(\nu_\Sigma\circ\gamma)\big)$ as well as $[w^{kl}(\rho)]_{k,l} = \big([w_{kl}(\rho)]_{k,l}\big)^{-1}$. In particular, $a^{ij}_\rho := \frac{1}{d} p_{ij}(\rho) \in h^s(\overline{W})$ holds for all $i,j = 1, \ldots, d$. On account of $\rho \in U^h_{s,1}$, we have $\|\rho\|_{C^1(\Sigma)} < 2R^\Sigma$. Thus, choosing $R^\Sigma > 0$ sufficiently small, symmetry and positive definiteness of $[g^{\bar\theta}_{ij}]_{i,j}$ and $[g^{ij}_{\bar\theta}]_{i,j}$ (see Remark 2.30(ii)) ensures the same for $[a^{ij}_\rho]_{i,j}$. $\square$

We gather some further regularity statements in the following lemma.

Lemma 4.5. *Let $s \in \mathbb{R}_{>0} \setminus \mathbb{N}$ and let $\Sigma = \bar{\theta}(M) \subset \mathbb{R}^{d+1}$ be an h^{3+s}-immersed closed hypersurface with unit normal ν_Σ. We use the notation $\nabla_\rho, \mathrm{div}_\rho, \Delta_\rho$ and ν_ρ as in Remark 2.64 as well as Notations 4.1. For $R^\Sigma > 0$ sufficiently small,* *(i)* $\rho \mapsto \big(\nabla_\rho : f \mapsto \nabla_\rho f\big) \in C^\infty\big(U^h_{s,1}, \mathcal{L}(Y_s, X_s^{d+1})\big)$ *and* $\rho \mapsto \big(\mathrm{div}_\rho : F \mapsto \mathrm{div}_\rho F\big) \in C^\infty\big(U^h_{s,1}, \mathcal{L}(Y_s^{d+1}, X_s)\big)$ *hold,*

(ii) $\rho \mapsto a(\rho) := \frac{1}{\nu_\rho \cdot \nu_\Sigma} \in C^\infty(U^h_{s,1}, X_s)$ *holds and*

(iii) there exist functions $D \in C^\infty\big(U^h_{s,1}, \mathcal{L}(Z_s, X_s)\big)$ *and* $J \in C^\infty\big(U^h_{1+s,1}, \mathcal{L}(Y_s, X_s)\big)$ *with* $J \in C^\infty_b\big(U^h_{1+s,1} \cap \mathcal{B}, \mathcal{L}(Y_s, X_s)\big)$ *for any bounded subset* $\mathcal{B} \subset Z_s$ *such that we have* $\Delta_\rho u = D(\rho)[u] + J(\rho)[u]$ *for all* $\rho \in U^h_{1+s,1}$ *and* $u \in Z_s$.
In particular, $\rho \mapsto \big(\Delta_\rho : f \mapsto \Delta_\rho f\big) \in C^\infty\big(U^h_{1+s,1}, \mathcal{L}(Z_s, X_s)\big)$ *follows.*

Proof.

Ad (i) Let $f : \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{d+1}$, $f(v_1, v_2, v_3, u_1, u_2) := v_1 + u_2 v_2 + u_1 v_3$. Because $\Sigma = \bar\theta(M)$ is an h^{2+s}-immersed hypersurface, we have $\partial_j\gamma, \nu_\Sigma \circ \gamma, \partial_j(\nu_\Sigma \circ \gamma) \in h^s(\overline{W}, \mathbb{R}^{d+1})$ for any sufficiently small local parameterization (γ, W) of M. Thus, smoothness of f and Corollary 2.103(ii) yield

$$F \in C^\infty\big(h^s(\overline{W}) \times h^s(\overline{W}), h^s(\overline{W}, \mathbb{R}^{d+1})\big) \cap C^\infty_b\big(B, h^s(\overline{W}, \mathbb{R}^{d+1})\big)$$

for $F : (u_1, u_2) \mapsto \partial_j\gamma + u_2(\nu_\Sigma \circ \gamma) + u_1\partial_j(\nu_\Sigma \circ \gamma)$ and arbitrary bounded subsets $B \subset h^s(\overline{W}) \times h^s(\overline{W})$. Additionally, $G : u \mapsto \big(u \circ \gamma, \partial_j(u \circ \gamma)\big) \in \mathcal{L}\big(Y_s, h^s(\overline{W}) \times h^s(\overline{W})\big)$ holds and therefore we have

$$\rho \mapsto \partial_j\gamma_\rho = F \circ G(\rho) \in C^\infty\big(Y_s, h^s(\overline{W}, \mathbb{R}^{d+1})\big) \cap C^\infty_b\big(\mathcal{B}, h^s(\overline{W}, \mathbb{R}^{d+1})\big)$$

for bounded subsets $\mathcal{B} \subset Y_s$. In particular, $\rho \mapsto g^\rho_{ij} = \partial_i\gamma_\rho \cdot \partial_j\gamma_\rho \in C^\infty\big(Y_s, h^s(\overline{W})\big)$ and $\rho \mapsto g^\rho_{ij} \in C^\infty_b\big(\mathcal{B}, h^s(\overline{W})\big)$ follow with Remark 2.105. According to Lemma 4.2, for $\rho \in U^h_{s,1}$ with $R^\Sigma > 0$ sufficiently small, $[g^\rho_{ij}]_{1 \le i,j \le d}$ is invertible on $\overline{W}$ and thus $\min_{\overline{W}} |\det[g^\rho_{ij}]| > 0$ holds. So, with the open subset $U := \{A \in \mathbb{R}^{d\times d} \,|\, \det A \neq 0\}$, we have $[g^\rho_{ij}] \in h^s(\overline{W}, U)$ for all $\rho \in U^h_{s,1}$. Even more, as $\rho \mapsto \min_{\overline{W}} |\det[g^\rho_{ij}]|$ is continuous as mapping on $C^1(M)$, there exists $\varepsilon > 0$ with $\min_{\overline{W}} |\det[g^\rho_{ij}]| \ge \varepsilon$ for all $\rho \in U^h_{s,1} \subset \{\rho \in C^1(M) \,|\, \|\rho\|_{C^1(M)} < 2R^\Sigma\}$ with $R^\Sigma > 0$ sufficiently small. For the closed subset $\mathcal{A} := \{A \in \mathbb{R}^{d\times d} \,|\, |\det A| \ge \varepsilon\} \subset U$, we thus have $[g^\rho_{ij}] \in h^s(\overline{W}, \mathcal{A})$ for all $\rho \in U^h_{s,1}$. In particular, $\rho \mapsto [g^\rho_{ij}] \in C^\infty\big(U^h_{s,1}, h^s(\overline{W}, U)\big) \cap C^\infty_b\big(U^h_{s,1} \cap \mathcal{B}, h^s(\overline{W}, \mathcal{A})\big)$ follows. By Remark 2.106, $(\cdot)^{-1} \in C^\infty\big(h^s(\overline{W}, U), h^s(\overline{W}, \mathbb{R}^{d\times d})\big) \cap C^\infty_b\big(\mathscr{B}, h^s(\overline{W}, \mathbb{R}^{d\times d})\big)$ holds for the inversion $(\cdot)^{-1}$ of matrices and any bounded subset $\mathscr{B} \subset h^s(\overline{W}, \mathcal{A})$. Hence, combination implies

$$\rho \mapsto g^{ij}_\rho \in C^\infty\big(U^h_{s,1}, h^s(\overline{W})\big) \cap C^\infty_b\big(U^h_{s,1} \cap \mathcal{B}, h^s(\overline{W})\big).$$

Due to $f \mapsto \partial_i(f \circ \gamma) \in \mathcal{L}\big(Y_s, h^s(\overline{W})\big)$ and $F \mapsto \partial_i(F \circ \gamma) \in \mathcal{L}\big(Y_s^{d+1}, h^s(\overline{W}, \mathbb{R}^{d+1})\big)$, we finally have

$$(\rho, f) \mapsto \nabla_\rho f \circ \gamma = \sum_{i,j} g^{ij}_\rho \partial_i(f \circ \gamma)\partial_j\gamma_\rho \in C^\infty\big(U^h_{s,1}, \mathcal{L}(Y_s, h^s(\overline{W}, \mathbb{R}^{d+1}))\big) \text{ and}$$

$$(\rho, F) \mapsto \mathrm{div}_\rho F \circ \gamma = \sum_{i,j} g^{ij}_\rho \partial_i(F \circ \gamma) \cdot \partial_j\gamma_\rho \in C^\infty\big(U^h_{s,1}, \mathcal{L}(Y_s^{d+1}, h^s(\overline{W}))\big)$$

with Remark 2.105. Now, the claim follows with Lemma 2.99.

Ad (iii) For any sufficiently small local parameterization (γ, W) of M, we have

$$\Delta_\rho f \circ \gamma = \sum_{i,j} g_\rho^{ij}\, \partial_i \partial_j (f \circ \gamma) + \sum_{i,j,k,l} g_\rho^{ij} \partial_i \big(g_\rho^{kl} \partial_l \gamma_\rho \big) \cdot \partial_j \gamma_\rho\, \partial_k (f \circ \gamma)$$

by Remark 2.64. We choose D as the principal part of Δ and define $J \coloneqq \Delta - D$ such that

$$D(\rho)[f] \circ \gamma = \sum_{i,j} g_\rho^{ij}\, \partial_i \partial_j (f \circ \gamma) \qquad \text{and}$$
$$J(\rho)[f] \circ \gamma = \sum_{i,j,k,l} g_\rho^{ij} \partial_i \big(g_\rho^{kl} \partial_l \gamma_\rho \big) \cdot \partial_j \gamma_\rho\, \partial_k (f \circ \gamma)$$

hold on $\overline{W}$. With Remark 2.46, $D(\rho)[f]$ and $J(\rho)[f]$ are well-defined on the whole hypersurface M. As in (i), we have $\rho \mapsto g_\rho^{ij} \in C^\infty\big(U_{s,1}^h, h^s(\overline{W})\big)$ and on account of $f \mapsto \partial_i \partial_j (f \circ \gamma) \in \mathcal{L}\big(Z_s, h^s(\overline{W})\big)$

$$(\rho, f) \mapsto D(\rho)[f] \circ \gamma \in C^\infty\big(U_{s,1}^h, \mathcal{L}(Z_s, h^s(\overline{W}))\big)$$

follows with Remark 2.105. But we only needed Σ to be an h^{2+s}-immersed hypersurface for the proof of (i). Thus, also

$$\rho \mapsto \partial_j \gamma_\rho \in C^\infty\big(Z_s, h^{1+s}(\overline{W}, \mathbb{R}^{d+1})\big) \cap C_b^\infty\big(\mathcal{B}, h^{1+s}(\overline{W}, \mathbb{R}^{d+1})\big) \text{ and}$$
$$\rho \mapsto g_\rho^{ij} \in C^\infty\big(U_{1+s,1}^h, h^{1+s}(\overline{W})\big) \cap C_b^\infty\big(U_{1+s,1}^h \cap \mathcal{B}, h^{1+s}(\overline{W})\big)$$

hold for bounded subsets $\mathcal{B} \subset Z_s$ with $h^{1+s}(\overline{W}, \mathbb{R}^m) \hookrightarrow h^s(\overline{W}, \mathbb{R}^m)$ for $m \in \{1, d+1\}$ due to Lemma 2.88. Furthermore, Lemma 2.104 yields

$$\rho \mapsto \partial_i \partial_l \gamma_\rho \in C^\infty\big(Z_s, h^s(\overline{W}, \mathbb{R}^{d+1})\big) \cap C_b^\infty\big(\mathcal{B}, h^s(\overline{W}, \mathbb{R}^{d+1})\big) \text{ and}$$
$$\rho \mapsto \partial_i g_\rho^{kl} \in C^\infty\big(U_{1+s,1}^h, h^s(\overline{W})\big) \cap C_b^\infty\big(U_{1+s,1}^h \cap \mathcal{B}, h^s(\overline{W})\big).$$

Due to $f \mapsto \partial_k (f \circ \gamma) \in \mathcal{L}\big(Y_s, h^s(\overline{W})\big)$, we hence have

$$(\rho, f) \mapsto J(\rho)[f] \circ \gamma \in C^\infty\big(U_{1+s,1}^h, \mathcal{L}(Y_s, h^s(\overline{W}))\big) \cap C_b^\infty\big(U_{1+s,1}^h \cap \mathcal{B}, \mathcal{L}(Y_s, h^s(\overline{W}))\big)$$

with Remark 2.105. Now, the claim follows with Remark 2.46.

Ad (ii) Let $\mathcal{K} : (\mathbb{R}^{d+1})^d \to \mathbb{R}^{d+1}$ be the generalized cross product as in Definition A.4; in particular $\mathcal{K}$ is smooth. For the open subset

$$U \coloneqq \big\{ (v_1, ..., v_{d+1}) \in (\mathbb{R}^{d+1})^{d+1} \,\big|\, (v_1, ..., v_{d+1}) \subset \mathbb{R}^{d+1} \text{ linearly independent} \big\},$$

the map $f : U \to \mathbb{R}$ with

$$f(v_1, ..., v_{d+1}) \coloneqq \frac{|\mathcal{K}(v_1, ..., v_d)|}{\mathcal{K}(v_1, ..., v_d) \cdot v_{d+1}}$$

is well-defined and also smooth. On that account, by Corollary 2.103(ii), we have $F \in C^\infty\big(h^s(\overline{W}, U), h^s(\overline{W})\big)$ with $\big(F(u)\big)(x) \coloneqq f\big(u(x)\big)$ for $u : \overline{W} \to U$. As in the proof of (i), $\rho \mapsto \partial_j \gamma_\rho \in C^\infty\big(Y_s, h^s(\overline{W}, \mathbb{R}^{d+1})\big)$ holds for any sufficiently small local parameterization (γ, W) of M and thus $G : \rho \mapsto (\partial_1\gamma_\rho, ..., \partial_d\gamma_\rho, \nu_\Sigma \circ \gamma) \in C^\infty\big(Y_s, h^s(\overline{W}, (\mathbb{R}^{d+1})^{d+1})\big)$. Due to Lemma 4.2, $\big(\partial_1\gamma_{\rho|x}, ..., \partial_d\gamma_{\rho|x}, \nu_\Sigma \circ \gamma_{|x}\big) \subset \mathbb{R}^{d+1}$ are linearly independent for every $x \in \overline{W}$ if $\rho \in U^h_{s,1}$ with $R^\Sigma > 0$ sufficiently small. Therefore, $G \in C^\infty\big(U^h_{s,1}, h^s(\overline{W}, U)\big)$ follows. Composition yields

$$\big(F \circ G\big)(\rho) = \frac{|K(\partial_1\gamma_\rho, ..., \partial_d\gamma_\rho)|}{K(\partial_1\gamma_\rho, ..., \partial_d\gamma_\rho) \cdot (\nu_\Sigma \circ \gamma)} = \frac{1}{(\nu_\rho \circ \gamma) \cdot (\nu_\Sigma \circ \gamma)} = a(\rho) \circ \gamma$$

and hence $\rho \mapsto a(\rho) \circ \gamma \in C^\infty\big(U^h_{s,1}, h^s(\overline{W})\big)$. The claim follows again with Lemma 2.99. □

Remark 4.6. *Let $s \in \mathbb{R}_{>0} \setminus \mathbb{N}$, let $\Sigma = \bar{\theta}(M)$ be an h^{3+s}-immersed closed hypersurface and let $\alpha \in (0,1)$ with $\alpha \leq s$. We use Notations 4.1. Due to the smoothness of $a : U^h_{\alpha,1} \to X_\alpha$, $a(\rho) \coloneqq \frac{1}{\nu_\rho \cdot \nu_\Sigma}$ by Lemma 4.5(ii) and $a(0) = \frac{1}{|\nu_\Sigma|^2} = 1$ for $0 \in U^h_{\alpha,1}$, we can choose $R^\Sigma > 0$ sufficiently small such that $a \geq \frac{1}{2}$ holds on $\big\{\rho \in Y_\alpha \,\big|\, \|\rho\|_{Y_\alpha} < 2R^\Sigma\big\}$. In particular, we thus have $a \geq \frac{1}{2}$ on the set U^h_s defined in Notations 4.10. Analogously, there exists a constant $C > 0$ such that $\|a(\rho)\|_{X_\alpha} \leq C$ holds for all $\rho \in U^h_s$.*

The pullback Δ_ρ of the Laplace-Betrami operator obviously is a linear operator, so that the PDE for the concentration (4.1b) is quasilinear. Its parabolicity relies mainly on the fact that Δ_ρ is an elliptic operator, as we state in the next lemma.

Lemma 4.7. *Let $s \in \mathbb{R}_{>0} \setminus \mathbb{N}$ and let $\Sigma = \bar{\theta}(M) \subset \mathbb{R}^{d+1}$ be an h^{3+s}-immersed closed hypersurface. We use the notation Δ_ρ as in Remark 2.64 as well as Notations 4.1. For $R^\Sigma > 0$ sufficiently small and $\rho \in U^h_{1+s,1}$, the pullback $\Delta_\rho \in \mathcal{L}(Z_s, X_s)$ of the Laplace-Beltrami operator is a symmetric and elliptic differential operator of second order, i.e., given a sufficiently small local parameterization (γ, W) of M,*

$$\Delta_\rho[\cdot] \circ \gamma = \sum_{i,j} a^{ij} \partial_i \partial_j (\cdot \circ \gamma) + \textit{ lower order terms}$$

holds with a symmetric and positive definite coefficient matrix $[a^{ij}]_{i,j} \in h^s(\overline{W}, \mathbb{R}^{d\times d})$.

Proof. Let $\rho \in U^h_{1+s,1}$. Remark 2.64 yields

$$\Delta_\rho[\cdot] \circ \gamma = \sum_{i,j} g^{ij}_\rho \partial_i \partial_j (\cdot \circ \gamma) + \text{ lower order terms.}$$

We have $\gamma_\rho \in h^{1+s}(\overline{W}, \mathbb{R}^{d+1})$ and thus on account of Remark 2.106, $g^{ij}_\rho \in h^s(\overline{W})$ follows for all $i, j = 1, ..., d$. By Remark 2.30(ii), $[g^{ij}_\rho]_{i,j} : \overline{W} \to \mathbb{R}^{d\times d}$ is symmetric and positive definite. □

We end the collection of regularity statements by a simple consequence of Section 2.2.5 on the regularity of composition operators that will be applied to the functions G and g later on.

Lemma 4.8. *Let $s \in \mathbb{R}_{>0} \setminus \mathbb{N}$ and let $\Sigma = \bar{\theta}(M) \subset \mathbb{R}^{d+1}$ be an h^{1+s}-immersed closed hypersurface. We use Notations 4.1. If $F \in C^{k+\lfloor s \rfloor+2}(\mathbb{R})$, we have*

$$u \mapsto F(u) \in C^k(X_s, X_s)$$

and in particular, $u \mapsto F(u) \in C^k(U_s^c, X_s)$.

Proof. Let (γ, W) be any sufficiently small local parameterization of M and let $R > 0$. Due to $F \in C_b^{k+\lfloor s \rfloor+2}\big((-R,R)\big)$, Proposition 2.101(iii) yields $F \in C^k\big(h^s(\overline{W},(-R,R)), h^s(\overline{W})\big)$. As $R > 0$ was arbitrary, $F \in C^k\big(h^s(\overline{W}), h^s(\overline{W})\big)$ holds. With $u \mapsto u \circ \gamma \in \mathcal{L}\big(X_s, h^s(\overline{W})\big)$ and Lemma 2.99 the claim follows. □

Having gathered these general regularity statements, we proceed to the more specific setting in which we will prove the existence of short-time solutions. First, we list the assumptions needed for our proof.

Assumptions 4.9.

(i) Let $\alpha \in (0,1)$ and $\beta \in (0, \frac{1}{2})$ with $2\beta + \alpha \notin \mathbb{N}$. Furthermore, let $G \in C^7(\mathbb{R})$ with $G'' > 0$ and $g := G - G' \cdot \mathrm{Id} > 0$.

(ii) Let $\Sigma = \bar{\theta}(M) \subset \mathbb{R}^{d+1}$ be an $h^{4+\alpha}$-immersed closed hypersurface with unit normal ν_Σ and let $R^\Sigma > 0$ be sufficiently small.

(iii) Let $u_0 \in h^{2+2\beta+\alpha}(M)$ and let $\delta_1 > R^\Sigma$ be arbitrary.

(iv) Let $R^c, R^h > 0$ be sufficiently large such that $2\|u_0\|_{C^{2+\alpha}(M)} \le R^c$ and $2\delta_1 \le R^h$ holds. Let $\delta_0 \in (0, R^\Sigma)$. Then, let $T \in (0,1)$ be sufficiently small such that

$$R^h T^\beta + \delta_0 < R^\Sigma \tag{4.2}$$

is valid.

We give a few comments on these assumptions and explain why they are postulated by refering to later statements. So, these comments will not be understandable for the reader yet, but serve as a later look-up. Choosing $\beta < \frac{1}{2}$ ensures that the embedding $Z_\alpha \hookrightarrow Y_{2\beta+\alpha}$ is compact and thus $K^c_{2\beta+\alpha}$, $K^h_{2\beta+\alpha}$ as in Definition 4.10(i) are compact sets in $Y_{2\beta+\alpha}$. Assuming the immersed hypersurface Σ to be of $h^{4+\alpha}$-regularity guarantees that we can apply Lemmas 4.3 and 4.5 for $s := 2\beta+\alpha$. Together with the C^7-regularity of G, this is used in Corollary 4.11 to gain regularity properties for our operators. As mentioned in Section 3.4, $G'' > 0$ and $g > 0$ ensure that our PDEs are parabolic. The $h^{2+2\beta+\alpha}$-regularity, which we assume for the initial value u_0 of the concentration, as well as for the initial value ρ_0 of the height function later on, makes sure that by applying our second order operators, we still end up with a $h^{2\beta+\alpha}$-regularity. This turns out to be the necessary compability condition and is used in Lemmas 4.16 and 4.24.
We will obtain a short-time existence result for any initial height function $\rho_0 \in h^{2+2\beta+\alpha}(M)$ with $\|\rho_0\|_{C^{2+2\beta+\alpha}(M)} < \delta_1$ and $\|\rho_0\|_{C^{1+\alpha}(M)} < \delta_0$. Particularly, $2\|\rho_0\|_{C^{2+\alpha}(M)} < 2\delta_1 \le R^h$ follows. As $\delta_1 > 0$ can be chosen arbitrarily large, $\|\rho_0\|_{C^{2+2\beta+\alpha}(M)} < \delta_1$ is not an actual restriction on ρ_0. To yield a suitable height function as in Lemma 4.2, the initial value ρ_0

only needs to be small in the C^1-norm. But to achieve $a(\rho_0) > 0$ with Remark 4.6, and also later on in the proofs of Theorem 4.18 and Proposition 4.25, smallness of ρ_0 in the $C^{1+\alpha}$-norm is necessary. This is why we set the condition $\|\rho_0\|_{C^{1+\alpha}(M)} < \delta_0$.
$R^\Sigma > 0$ being sufficiently small means such that Lemmas 4.2, 4.3, 4.4, 4.5 and 4.7 as well as Remark 4.6 hold. In particular, this implies that any function $\|\rho_t\| < R^\Sigma$ is a well-defined height function as in Lemma 4.2 and the regularity statements in terms of ρ_t hold for all the geometric quantities from Lemmas 4.3 and 4.5.
In the following, we will choose R^c and R^h even larger and $\delta_0 > 0$ and $T > 0$ even smaller, where T always has to be so small that Estimate (4.2) holds. Enlarging R^c and R^h increases the set of possible solutions to our system of PDEs, which we seek in balls with radii R^c and R^h. Then, Estimate (4.2) together with the Hölder-regularity of the solution guarantees that every $\|\rho\| \le R^h$ with initial value $\|\rho(0)\| < \delta_0$ fulfills $\|\rho(t)\| < R^\Sigma$. Particularly, $\rho(t)$ remains small in the $C^{1+\alpha}$-norm for all times $t \in [0,T]$ such that all the properties mentioned above hold; most importantly, $\rho(t)$ is a well-defined height function as in Lemma 4.2 for every $t \in [0,T]$.

Now, we give a summary of the notation used in the following sections. It relies on Notations 4.1, but is reduced to our more specific setting.

Notations 4.10. *Suppose Assumptions 4.9 are valid and let $s \in \{\alpha, 2\beta+\alpha\}$. This guarantees that $\Sigma = \bar\theta(M)$ is an h^{3+s}-immersed closed hypersurface and thus permits to use Notations 4.1: $X_s \coloneqq h^s(M)$, $Y_s \coloneqq h^{1+s}(M)$ and $Z_s \coloneqq h^{2+s}(M)$. We also recall*

$$\begin{aligned} U^h_{s,1} &\coloneqq \big\{\rho \in Y_s \,\big|\, \|\rho\|_{C^1(M)} < 2R^\Sigma\big\}, \qquad U^c_s \coloneqq \big\{u \in Y_s \,\big|\, \|u\|_{Y_s} < 2R^c\big\},\\ U^h_{1+s,1} &\coloneqq \big\{\rho \in Z_s \,\big|\, \|\rho\|_{C^1(M)} < 2R^\Sigma\big\} \end{aligned}$$

and define

$$U^h_s \coloneqq \big\{\rho \in Y_s \,\big|\, \|\rho\|_{Y_s} < 2R^h,\, \|\rho\|_{Y_\alpha} < 2R^\Sigma\big\}.$$

(i) Furthermore, we define

$$K^h_s \coloneqq \overline{\big\{\rho \in Z_\alpha \,\big|\, \|\rho\|_{Z_\alpha} \le R^h,\, \|\rho\|_{Y_\alpha} \le R^\Sigma\big\}}^{\|\cdot\|_{Y_s}}, \qquad K^c_s \coloneqq \overline{\big\{u \in Z_\alpha \,\big|\, \|u\|_{Z_\alpha} \le R^c\big\}}^{\|\cdot\|_{Y_s}}.$$

(ii) We use the following notation for spaces and sets with time-dependence

$$\begin{aligned} \mathbb{E}_{0,T} &\coloneqq h^\beta([0,T], X_\alpha),\\ \mathbb{E}_{1,T} &\coloneqq h^{1,\beta}([0,T], X_\alpha) \cap h^\beta([0,T], Z_\alpha),\\ M^c_T &\coloneqq \{u \in \mathbb{E}_{1,T} \,|\, \|u\|_{\mathbb{E}_{1,T}} \le R^c \text{ and } u(0) = u_0 \text{ in } Z_\alpha\},\\ M^h_T &\coloneqq \big\{\rho \in \mathbb{E}_{1,T} \,\big|\, \|\rho\|_{\mathbb{E}_{1,T}} \le R^h \text{ and } \|\rho(t)\|_{Y_\alpha} \le R^\Sigma \text{ for all } t \in [0,T]\big\} \text{ and}\\ M^h_{T,\rho_0} &\coloneqq \{\rho \in \mathbb{E}_{1,T} \,|\, \|\rho\|_{\mathbb{E}_{1,T}} \le R^h \text{ and } \rho(0) = \rho_0 \text{ in } Z_\alpha\}\\ &\qquad \text{for any } \rho_0 \in Z_{2\beta+\alpha} \text{ with } \|\rho_0\|_{Z_\alpha} \le R^h \text{ and } \|\rho_0\|_{Y_\alpha} < \delta_0. \end{aligned}$$

(iii) For the sake of completeness, we also define the operators used in the following sections. For $u, \rho \in \mathbb{E}_{1,T}$ and $u_1, \rho_1 \in Z_\alpha$, we set

$$\begin{aligned}
A^h_{u_1,\rho_1}[\rho] &:= g(u_1)a(\rho_1)P(\rho_1)[\rho],\\
A^h[\rho] &:= A^h_{u_0,0}[\rho] = g(u_0)a(0)P(0)[\rho],\\
G^h_u(\rho) &:= g(u)a(\rho)H(\rho) - A^h[\rho],\\
L^h[\rho] &:= \begin{pmatrix} \partial_t\rho - A^h[\rho] \\ \rho(0) \end{pmatrix},\\
A^c_{u_1,\rho_1}[u] &:= G''(u_1)\Delta_{\rho_1}u + g(u_1)a(\rho_1)H(\rho_1)\nu_\Sigma\cdot\nabla_{\rho_1}u + g(u_1)H(\rho_1)^2u,\\
A^c[u] &:= A^c_{u_0,0}[u] = G''(u_0)\Delta_\Sigma u + g(u_0)H_\Sigma^2 u,\\
G^c_{\rho_0}(u) &:= \Delta_{\rho_{u,\rho_0}}G'(u) + g(u)a(\rho_{u,\rho_0})H(\rho_{u,\rho_0})\nu_\Sigma\cdot\nabla_{\rho_{u,\rho_0}}u + g(u)H(\rho_{u,\rho_0})^2u - A^c[u],\\
L^c[u] &:= \begin{pmatrix} \partial_t u - A^c[u] \\ u(0) \end{pmatrix}.
\end{aligned}$$

Here, H, P, Q are the functionals from Lemma 4.3. Moreover, we have $a(\rho) := \frac{1}{\nu_\rho\cdot\nu_\Sigma}$ as in Lemma 4.5, where ν_ρ as well as the differential operators ∇_ρ, Δ_ρ were introduced in Remark 2.64; in particular, $\nu_0 = \nu_\Sigma$, $\nabla_0 = \nabla_\Sigma$, $\Delta_0 = \Delta_\Sigma$ and $H(0) = H_\Sigma$ hold in the case of $\rho = 0$. Finally, $\rho_{u,\rho_0} \in M^h_{T,\rho_0}$ is the solution from Theorem 4.18 associated with the concentration $u \in M^c_T$ and the initial value $\rho_0 \in Z_{2\beta+\alpha}$ with $\|\rho_0\|_{Z_{2\beta+\alpha}} < \delta_1$ and $\|\rho_0\|_{Y_\alpha} < \delta_0$.

Both our PDEs (4.1) are parabolic, quasilinear equations of second order (see Lemmas 4.3, 4.4, 4.5 and 4.7 as well as Remark 4.6) and will be solved by similar approaches. To underline this parallel structure, we use the same notation for all corresponding sets and operators and mark the association to the respective equation with a superscript, using the letter h for the first equation (4.1a) concerning the height function and the letter c for the second equation (4.1b) concerning the concentration function. Dependences of sets or operators will never be denoted by superscripts, but only by indices. To clarify this even more, we use the letters h and c only to denote the association to the equation; while height functions and concentrations will always be called ρ and u, respectively.
Whereas the initial value u_0 for the concentration is chosen fixed in Assumptions 4.9, our short-time existence result allows for small variations in the initial value ρ_0 of the height function. More precisely, for any $\rho_0 \in h^{2+2\beta+\alpha}(M)$ with $\|\rho_0\|_{C^{2+2\beta+\alpha}(M)} < \delta_1$ and $\|\rho_0\|_{C^{1+\alpha}(M)} < \delta_0$, we will obtain a solution to (4.1) on a time interval $[0,T]$ with T independent of ρ_0. This is crucial for the application in Section 5.3 to prove the formation of self-intersections. However, we thus can not linearize the system (4.1) around the initial value for the height function, as we do for the concentration. Instead, we linearize around the fixed value 0. This is possible, as due to $\|\rho_0\|_{C^{1+\alpha}(M)} < \delta_0$ all eligible initial values ρ_0 are close to the zero-function in a suitable sense.
We will solve our PDEs in the space $\mathbb{E}_{0,T}$ and therefore the solution functions lie in $\mathbb{E}_{1,T}$. To be precise, we seek the solution functions in M^c_T and M^h_T, which are the balls with radii R^c and R^h mentioned earlier. As forecasted, Estimate (4.2) guarantees that any $\rho \in M^h_{T,\rho_0}$

fulfills

$$\begin{aligned}\|\rho(t)\|_{Y_\alpha} &\le \|\rho(t)-\rho(0)\|_{Y_\alpha} + \|\rho(0)\|_{Y_\alpha}\\ &\le \|\rho\|_{h^\beta([0,T],Y_\alpha)} T^\beta + \|\rho_0\|_{Y_\alpha}\\ &< R^h T^\beta + \delta_0\\ &< R^\Sigma,\end{aligned}$$

i.e., $M^h_{T,\rho_0} \subset M^h_T$ holds. In particular, $\rho(t)$ is a well-defined height function as in Lemma 4.2 for every $\rho \in M^h_T$ and $t \in [0,T]$.

Now, we give a few crucial comments on embeddings of the sets defined above. Here, the superscripts c and h are omitted whenever the corresponding statement holds for both of them. As always, we set $s \in \{\alpha, 2\beta+\alpha\}$.

(i) We have $K_s \subset U_s$ due to $Z_\alpha \hookrightarrow Y_s$. Moreover, $K_s \subset Y_s$ is compact and convex, because $Z_\alpha \hookrightarrow Y_s$ is a compact embedding due to Lemma 2.91 with $\beta < \frac{1}{2}$. Obviously, $U_s \subset Y_s$ is open.

(ii) As $u \in M^c_T$, $\rho \in M^h_T$ fulfill $\|u(t)\|_{Z_\alpha} \le R^c$, $\|\rho(t)\|_{Z_\alpha} \le R^h$ and $\|\rho(t)\|_{Y_\alpha} \le R^\Sigma$ for every $t \in [0,T]$, the inclusions

$$\begin{aligned}M^h_T &\subset h^\beta\big([0,T], K^h_s\big) \cap h^\beta\big([0,T], U^h_{1+\alpha,1}\big) \subset h^\beta\big([0,T], U^h_s\big),\\ M^c_T &\subset h^\beta\big([0,T], K^c_s\big) \subset h^\beta\big([0,T], U^c_s\big)\end{aligned}$$

follow. Furthermore, for any $u \in M^c_T$ and any $\rho \in M^h_T$,

$$\begin{aligned}\|\rho\|_{h^\beta([0,T],Y_s)} &\le \|\rho\|_{h^\beta([0,T],Z_\alpha)} \le \|\rho\|_{\mathbb{E}_{1,T}} \le R^h,\\ \|u\|_{h^\beta([0,T],Y_s)} &\le \|u\|_{h^\beta([0,T],Z_\alpha)} \le \|u\|_{\mathbb{E}_{1,T}} \le R^c\end{aligned}$$

hold on account of $Z_\alpha \hookrightarrow Y_s$.

In the next corollary, we state some regularity properties for the components of the operators from Notations 4.10(iii). This corollary, and even more so the subsequent remark, are crucial for the proof of short-time existence as many of the following statements are based on this regularity.

Corollary 4.11. *We suppose Assumptions 4.9 are valid and use Notations 4.10. For $s \in \{\alpha, 2\beta+\alpha\}$,*

$$\begin{aligned}gaP &\in C^2\big(U^c_s \times U^h_s, \mathcal{L}(Z_s, X_s)\big),\\ gaQ &\in C^2(U^c_s \times U^h_s, X_s),\\ G''D &\in C^2\big(U^c_s \times U^h_s, \mathcal{L}(Z_s, X_s)\big),\\ G''J &\in C^2\big(U^c_s \times U^h_{1+s,1}, \mathcal{L}(Y_s, X_s)\big),\\ gaP\nu_\Sigma\cdot\nabla &\in C^2\big(U^c_s \times U^h_s, \mathcal{L}(Z_s, \mathcal{L}(Y_s, X_s))\big),\\ gaQ\nu_\Sigma\cdot\nabla &\in C^2\big(U^c_s \times U^h_s, \mathcal{L}(Y_s, X_s)\big),\\ gP^2\mathrm{Id}^c &\in C^2\big(U^c_s \times U^h_s, \mathcal{L}(Z_s, \mathcal{L}(Z_s, \mathcal{L}(X_s, X_s)))\big),\\ gPQ\mathrm{Id}^c &\in C^2\big(U^c_s \times U^h_s, \mathcal{L}\big(Z_s, \mathcal{L}(X_s, X_s)\big)\big) \text{ and}\\ gQ^2\mathrm{Id}^c &\in C^2\big(U^c_s \times U^h_s, \mathcal{L}(X_s, X_s)\big),\end{aligned}$$

hold. Furthermore, for $G'''\nabla_{(\cdot)}\cdot\nabla_{(\cdot)} : (u,\rho)\mapsto G'''(u)\nabla_\rho\cdot\nabla_\rho$*, we have*

$$G'''\,\nabla_{(\cdot)}\cdot\nabla_{(\cdot)} \in C^2\big(U_\alpha^c\times U_\alpha^h, \mathcal{L}(Y_\alpha,\mathcal{L}(Y_\alpha,X_\alpha))\big),$$
$$G'''\,\nabla_{(\cdot)}\cdot\nabla_{(\cdot)} \in C^1\big(U_{2\beta+\alpha}^c\times U_{2\beta+\alpha}^h, \mathcal{L}(Y_{2\beta+\alpha},\mathcal{L}(Y_{2\beta+\alpha},X_{2\beta+\alpha}))\big).$$

Proof. We have $2\beta+\alpha\in(0,2)\setminus\{1\}$ with $2\beta<1$. Because $\Sigma=\bar\theta(M)$ is an $h^{4+\alpha}$-immersed closed hypersurface and $G\in C^7(\mathbb{R})$, we can choose $s\in\{\alpha,2\beta+\alpha\}$ and mostly $k:=2$ in Lemmas 4.3, 4.5 and 4.8. (If $s>1$, we have to restrict to $k:=1$ in Lemma 4.8 for G''' and thus can only conclude C^1-differentiability for G'''.) Moreover, the inclusion $U_s^h\subset U_{s,1}^h$ holds. By considering functions independent of c or h as constant in these variables, multiple application of Remark 2.105 and $Z_s\hookrightarrow Y_s\hookrightarrow X_s$ proves the claims. ☐

Remark 4.12. *As $K_s^c\times K_s^h\subset U_s^c\times U_s^h$ is compact and convex, we can apply Corollary 2.102 in the following way: For any Banach space W_s and any functional $F\in C^2\big(U_s^c\times U_s^h,W_s\big)$, there exists a constant $C=C(R^\Sigma,R^c,R^h)$ such that $F(u_1,\rho_1)\in h^\beta\big([0,T],W_s\big)$ holds with*

$$\big\|F(u_1,\rho_1)\big\|_{h^\beta([0,T],W_s)}\le C \qquad \textit{and}$$
$$\big\|F(u_1,\rho_2)-F(u_2,\rho_2)\big\|_{h^\beta([0,T],W_s)}\le C\big(\|u_1-u_2\|_{h^\beta([0,T],Y_s)}+\|\rho_1-\rho_2\|_{h^\beta([0,T],Y_s)}\big)$$

for $u_i\in h^\beta\big([0,T],K_s^c\big)$, $\rho_i\in h^\beta\big([0,T],K_s^h\big)$ with $\|u_i\|_{h^\beta([0,T],Y_s)}\le R^c$, $\|\rho_i\|_{h^\beta([0,T],Y_s)}\le R^h$. In particular, these conditions are fulfilled for $u_i\in M_T^c$ and $\rho_i\in M_T^h$.
Except for $G''J$, all of the functionals listed in Corollary 4.11 can be estimated in this way. Because $G'''\nabla_{(\cdot)}\cdot\nabla_{(\cdot)}$ is a C^1-function only for $s=2\beta+\alpha$, Corollary 2.102 only yields the first of the two estimates stated above in that case. But if we restrict to $u_1=u_2$, the second estimate also holds: As in Corollary 4.11, we have $G'''\in C^1(U_s^c,X_s)$ and $\nabla_{(\cdot)}\cdot\nabla_{(\cdot)} : \rho\mapsto\nabla_\rho\cdot\nabla_\rho\in C^2(U_s^h,W_s)$ with $W_s:=\mathcal{L}(Y_s,\mathcal{L}(Y_s,X_s))$. Thus, Corollary 2.102 yields the existence of a constant $C=C(R^\Sigma,R^c,R^h)$ with

$$\begin{aligned}&\big\|G'''(u)\nabla_{\rho_1}\cdot\nabla_{\rho_1}-G'''(u)\nabla_{\rho_2}\cdot\nabla_{\rho_2}\big\|_{h^\beta([0,T],W_s)}\\ &\le\|G'''(u)\|_{h^\beta([0,T],X_s)}\big\|\nabla_{\rho_1}\cdot\nabla_{\rho_1}-\nabla_{\rho_2}\cdot\nabla_{\rho_2}\big\|_{h^\beta([0,T],W_s)}\\ &\le C\|\rho_1-\rho_2\|_{h^\beta([0,T],Y_s)}\end{aligned}$$

for all $u\in M_T^c$ and $\rho_i\in M_T^h$.
Due to $U_{1+s,1}^h\subset Z_s$ and $M_T^h\subset\mathbb{E}_{1,T}=h^{1+\beta}([0,T],X_\alpha)\cap h^\beta([0,T],Z_\alpha)$, we can not find a compact set $K\subset U_{1+s,1}^h$ with $M_T^h\subset h^\beta\big([0,T],K\big)$. Therefore, the functional $G''J$, which is defined on $U_s^c\times U_{1+s,1}^h$, has to be handled differently. But as $J:\rho\mapsto J(\rho)$ is bounded on bounded sets by Lemma 4.5(iii), we have $J:\rho\mapsto J(\rho)\in C_b^2\big(U_{1+s,1}^h\cap\overline{B_R^{Z_s}(0)},\mathcal{L}(Y_s,X_s)\big)$ for any $R>0$. As $U_{1+s,1}^h\cap\overline{B_R^{Z_s}(0)}\subset Z_s$ is convex, we can apply Proposition 2.101 instead of Corollary 2.102 to $G''J$. With $W_s:=\mathcal{L}(Y_s,X_s)$, this means that there exists a constant $C=C(R^\Sigma,R^c,R)$ such that $G''J(u_1,\rho_1)\in h^\beta\big([0,T],W_s\big)$ holds with

$$\big\|G''J(u_1,\rho_1)\big\|_{h^\beta([0,T],W_s)}\le C \qquad \textit{and}$$
$$\big\|G''J(u_1,\rho_2)-G''J(u_2,\rho_2)\big\|_{h^\beta([0,T],W_s)}\le C\big(\|u_1-u_2\|_{h^\beta([0,T],Y_s)}+\|\rho_1-\rho_2\|_{h^\beta([0,T],Z_s)}\big)$$

for $u_i \in h^\beta\big([0,T], K_s^c\big)$, $\rho_i \in h^\beta\big([0,T], U_{1+s,1}^h\big)$ with $\|u_i\|_{h^\beta([0,T],Y_s)} \le R^c$, $\|\rho_i\|_{h^\beta([0,T],Z_s)} \le R$. For $s = \alpha$, this is again fulfilled for $u_i \in M_T^c$ and $\rho_i \in M_T^h$ with $R = R^h$.

As preparation for the following two sections, we deduce a technical auxiliary corollary from Remark 4.12.

Corollary 4.13. *We suppose Assumptions 4.9 are valid and use Notations 4.10. For $u \in M_T^c$ and $\rho \in M_T^h$, we have $A^h[\rho] \in \mathbb{E}_{0,T}$ and $G_u^h(\rho) \in \mathbb{E}_{0,T}$ and*

$$\big(g(u)a(\rho)Q(\rho)\big)(t) \in X_{2\beta+\alpha} \qquad \text{as well as} \qquad \Big(G'''(u)\big|\nabla_\rho u\big|^2\Big)(t) \in X_{2\beta+\alpha}$$

holds for all $t \in [0,T]$.

Proof. We have $u, u_0 \in M_T^c$ and $\rho, 0 \in M_T^h \subset h^\beta([0,T], Z_\alpha)$. Thus, Remark 4.12 together with Lemma 2.100 yields $A^h[\rho], G_u^h(\rho) \in \mathbb{E}_{0,T}$. Furthermore, for every $t \in [0,T]$, we have $u(t) \in U_{2\beta+\alpha}^c$ and $\rho(t) \in U_{2\beta+\alpha}^h \subset Y_{2\beta+\alpha}$. Thus, Corollary 4.11 with $s \coloneqq 2\beta+\alpha$ yields the remaining claims. □

4.1 Short-Time Existence for ρ

This section deals with the first equation (4.1a)

$$\partial_t \rho = g(u)a(\rho)H(\rho)$$

for height functions ρ with initial value $\rho(0) = \rho_0$. We use the standard approach for parabolic, quasilinear partial differential equations of second order relying on linearization and a contraction argument, as explicated e.g. in [Lun12, Chapter 7]. For this, we first show that the linearization of the (elliptic) operator on the right hand side of the equation generates an analytic C^0-semigroup (see Proposition 4.14). In particular, the linearization of the initial value problem then yields an invertible operator (see Proposition 4.15).

Proposition 4.14. *We suppose Assumptions 4.9 are valid and use Notations 4.10. Then,*

$$A^h = A^h_{u_0,0} : Z_\alpha \to X_\alpha$$

generates an analytic C^0-semigroup with $\mathcal{D}_{A^h}(\beta) = X_{2\beta+\alpha}$. If $u \in M_T^c$ and $\rho \in M_T^h$, also

$$A^h_{u(t),\rho(t)} : Z_s \to X_s$$

generates an analytic C^0-semigroup for $s \in \{\alpha, 2\beta+\alpha\}$ and $t \in [0,T]$.

Proof. First, we prove the generation of the claimed semigroups. As $u_0 \in M_T^c$, $0 \in M_T^h$ hold, we treat $A^h_{u_0,0}$ as a special case of $A^h_{u(t),\rho(t)}$. Let $u \in M_T^c$, $\rho \in M_T^h$, choose $s \in \{\alpha, 2\beta+\alpha\}$ and fix $t \in [0,T]$. Then, $u(t) \in U_s^c$ and $\rho(t) \in U_s^h$ hold and thus $g\big(u(t)\big), a\big(\rho(t)\big) \in X_s$ follows with Lemmas 4.8 and 4.5(ii). Also, Lemma 4.4 yields that $P(\rho(t)) \in \mathcal{L}(Z_s, X_s)$ is a symmetric and elliptic differential operator of second order. Because we have

$$A^h_{u(t),\rho(t)} = g\big(u(t)\big)a\big(\rho(t)\big)P\big(\rho(t)\big)$$

with $g > 0$ and $a > 0$ by Assumption 4.9(i) and Remark 4.6, $A^h_{u(t),\rho(t)} \in \mathcal{L}(Z_s, X_s)$ is a symmetric and elliptic differential operator of second order, too. Due to Proposition 2.139, $A^h_{u(t),\rho(t)} : \mathcal{D}\big(A^h_{u(t),\rho(t)}\big) \subset X_s \to X_s$ therefore generates an analytic C^0-semigroup with $\mathcal{D}\big(A^h_{u(t),\rho(t)}\big) = Z_s$ and equivalent norms.
By Lemmas 2.128 and 2.87, finally

$$\mathcal{D}_{A^h_{u_0,0}}(\beta) = \big(X_\alpha, \mathcal{D}\big(A^h_{u_0,0}\big)\big)_\beta = (X_\alpha, Z_\alpha)_\beta = \big(h^\alpha(\Sigma), h^{2+\alpha}(\Sigma)\big)_\beta = h^{2\beta+\alpha}(\Sigma) = X_{2\beta+\alpha}$$

holds. □

Proposition 4.15. *We suppose Assumptions 4.9 are valid and use Notations 4.10. Then,*

$$L^h : \mathbb{E}_{1,T} \to (\mathbb{E}_{0,T} \times Z_\alpha)^h_+$$

is bijective with

$$\Lambda^h \coloneqq \sup_{0<T\le 1} \|\big(L^h\big)^{-1}\|_{\mathcal{L}((\mathbb{E}_{0,T}\times Z_\alpha)^h_+, \mathbb{E}_{1,T})} < \infty,$$

where

$$(\mathbb{E}_{0,T} \times Z_\alpha)^h_+ \coloneqq \big\{(f, f_0) \in (\mathbb{E}_{0,T} \times Z_\alpha) \,\big|\, f(0) + A^h[f_0] \in \mathcal{D}_{A^h}(\beta) = X_{2\beta+\alpha}\big\} \text{ with}$$
$$\|(f, f_0)\|_{(\mathbb{E}_{0,T}\times Z_\alpha)^h_+} \coloneqq \|f\|_{\mathbb{E}_{0,T}} + \|f_0\|_{Z_\alpha} + \|f(0) + A^h[f_0]\|_{X_{2\beta+\alpha}} \text{ for } (f, f_0) \in (\mathbb{E}_{0,T} \times Z_\alpha)^h_+.$$

In particular, $\Lambda^h = \Lambda^h(u_0)$ only depends on the initial value u_0.

Proof. By Proposition 4.14, A^h satisfies the conditions of Proposition 2.131, which yields bijectivity of

$$\widetilde{L^h} : \widetilde{\mathbb{E}_{1,T}} \to \big(\mathbb{E}_{0,T} \times \mathcal{D}(A^h)\big)_+,\ \rho \mapsto \begin{pmatrix} \partial_t \rho - A^h[\rho] \\ \rho(0) \end{pmatrix}$$

and the estimate

$$\sup_{0<T\le 1} \|\big(\widetilde{L^h}\big)^{-1}\|_{\mathcal{L}((\mathbb{E}_{0,T}\times\mathcal{D}(A^h))_+, \widetilde{\mathbb{E}_{1,T}})} < \infty,$$

where $\widetilde{\mathbb{E}_{1,T}} \coloneqq h^{1,\beta}([0,T], X_\alpha) \cap h^\beta\big([0,T], \mathcal{D}(A^h)\big)$. The claim follows with $\mathcal{D}(A^h) = Z_\alpha$ by Proposition 4.14. Because $A^h = A^h_{u_0,0}$ only depends on u_0, also $\mathcal{L}^h$ and then Λ^h only depend on u_0. □

As a next step, we prove a technical auxiliary lemma.

Lemma 4.16. *We suppose Assumptions 4.9 are valid and use Notations 4.10.*

(i) If $u \in M^c_T$ and $\rho \in M^h_T$ with $\rho(0) \in Z_{2\beta+\alpha}$, then $\big(G^h_u(\rho)\big)(0) = G^h_{u_0}\big(\rho(0)\big)$ holds in X_α and we have

$$\big(G^h_u(\rho), \rho(0)\big) \in \big(\mathbb{E}_{0,T} \times Z_\alpha\big)^h_+.$$

(ii) *There exists a constant* $N^h = N^h\big(R^c, \delta_1\big)$ *independent of* T, R^h *and* $u \in M_T^c$ *such that*

$$\big\|\big(G_u^h(\rho_0), \rho_0\big)\big\|_{(\mathbb{E}_{0,T}\times Z_\alpha)_+^h} \leq N^h$$

holds for all $\rho_0 \in Z_{2\beta+\alpha}$ *with* $\|\rho_0\|_{Z_{2\beta+\alpha}} < \delta_1$, $\|\rho_0\|_{Y_\alpha} < \delta_0$.

Proof.

Ad (i) We have $G_u^h(\rho) \in \mathbb{E}_{0,T}$ by Corollary 4.13, hence $\big(G_u^h(\rho), \rho(0)\big) \in \mathbb{E}_{0,T} \times Z_\alpha$ holds. Moreover, we have $u(0) = u_0 \in U_{2\beta+\alpha}^c$ and $\rho(0), 0 \in U_{2\beta+\alpha}^h \cap Z_{2\beta+\alpha}$. So, Corollary 4.11 yields

$$\begin{aligned}\big(G_u^h(\rho)\big)(0) &= \Big(\big(gaP\big)(u,\rho)[\rho] + \big(gaQ\big)(u,\rho) - \big(gaP\big)(u_0,0)[\rho]\Big)(0)\\ &= \big(gaP\big)\big(u_0,\rho(0)\big)\big[\rho(0)\big] + \big(gaQ\big)\big(u_0,\rho(0)\big) - \big(gaP\big)(u_0,0)\big[\rho(0)\big]\\ &= G_{u_0}^h\big(\rho(0)\big) \text{ in } X_{2\beta+\alpha} \hookrightarrow X_\alpha\end{aligned}$$

and therefore

$$A^h\big[\rho(0)\big] + \big(G_u^h(\rho)\big)(0) = g(u_0)a\big(\rho(0)\big)H\big(\rho(0)\big) \in X_{2\beta+\alpha}$$

follows with $X_{2\beta+\alpha} = \mathcal{D}_{A^h}(\beta)$ by Proposition 4.14.
Overall, we thus have $\big(G_u^h(\rho), \rho(0)\big) \in \big(\mathbb{E}_{0,T} \times Z_\alpha\big)_+^h$.

Ad (ii) We have $u, u_0 \in M_T^c$ and $\rho_0, 0 \in \widetilde{M_T^h}$ with $\widetilde{M_T^h}$ defined as M_T^h but with $\widetilde{R^h} := 2\delta_1$ instead of R^h. So, Remark 4.12 together with Lemma 2.100 yields

$$\begin{aligned}\big\|G_u^h(\rho_0)\big\|_{\mathbb{E}_{0,T}} &= \big\|\big(gaH\big)(u,\rho_0) - \big(gaP\big)(u_0,0)[\rho_0]\big\|_{\mathbb{E}_{0,T}}\\ &\leq \Big\|\Big(\big(gaP\big)(u,\rho_0) - \big(gaP\big)(u_0,0)\Big)[\rho_0]\Big\|_{\mathbb{E}_{0,T}} + \big\|\big(gaQ\big)(u,\rho_0)\big\|_{\mathbb{E}_{0,T}}\\ &\leq C(R^c,\delta_1)\big(\|u-u_0\|_{h^\beta([0,T],Y_\alpha)} + \|\rho_0\|_{Y_\alpha}\big)\|\rho_0\|_{Z_\alpha} + C(R^c,\delta_1)\\ &\leq C(R^c,\delta_1).\end{aligned}$$

We have

$$\big\|A^h[\rho_0] + \big(G_u^h(\rho_0)\big)(0)\big\|_{\mathcal{D}_{A^h}(\beta)} = \big\|\big(gaH\big)(u_0,\rho_0)\big\|_{X_{2\beta+\alpha}}$$

due to part (i). So, again, Remark 4.12 together with Lemma 2.100 yields

$$\begin{aligned}\big\|A^h[\rho_0] + \big(G_u^h(\rho_0)\big)(0)\big\|_{\mathcal{D}_{A^h}(\beta)} &\leq \big\|\big(gaP\big)(u_0,\rho_0)[\rho_0]\big\|_{X_{2\beta+\alpha}} + \big\|\big(gaQ\big)(u_0,\rho_0)\big\|_{X_{2\beta+\alpha}}\\ &\leq C(R^c,\delta_1)\big(\|\rho_0\|_{Z_{2\beta+\alpha}} + 1\big)\\ &\leq C\big(R^c,\delta_1\big).\end{aligned}$$

(As u_0 and ρ_0 are independent of t, there is also no time dependence in the application of Remark 4.12 in the estimate above.) Together,

$$\begin{aligned}&\big\|\big(G_u^h(\rho_0), \rho_0\big)\big\|_{(\mathbb{E}_{0,T}\times Z_\alpha)_+^h}\\ &= \big\|G_u^h(\rho_0)\big\|_{\mathbb{E}_{0,T}} + \|\rho_0\|_{Z_\alpha} + \big\|A^h[\rho_0] + \big(G_u^h(\rho_0)\big)(0)\big\|_{\mathcal{D}_{A^h}(\beta)}\\ &\leq C\big(R^c,\delta_1\big) =: N^h\end{aligned}$$

holds. □

The following proposition is the key point for the contraction argument.

Proposition 4.17. *We suppose Assumptions 4.9 are valid and use Notations 4.10. There exists* $\varepsilon > 0$ *with*

$$\begin{aligned}\|G^h_{u_1}(\rho_1) - G^h_{u_2}(\rho_2)\|_{\mathbb{E}_{0,T}} \le{}& C(R^\Sigma, R^c, R^h) T^\varepsilon \big(\|u_1 - u_2\|_{\mathbb{E}_{1,T}} + \|\rho_1 - \rho_2\|_{\mathbb{E}_{1,T}}\big)\\ &+ C(R^\Sigma, R^c, R^h)\|\rho_1(0) - \rho_2(0)\|_{Y_\alpha}\\ &+ C(R^\Sigma, \|u_0\|_{Z_\alpha}, \|\rho_1(0)\|_{Z_\alpha})\|\rho_1(0)\|_{Y_\alpha}\|\rho_1 - \rho_2\|_{\mathbb{E}_{1,T}}\end{aligned}$$

for any $u_1, u_2 \in M^c_T$ *and* $\rho_1, \rho_2 \in M^h_T$.

Proof. Remark 4.12 yields

$$\begin{aligned}&\big\|\big(gaQ\big)(u_1,\rho_1) - \big(gaQ\big)(u_2,\rho_2)\big\|_{\mathbb{E}_{0,T}}\\ &\le C(R^\Sigma, R^c, R^h)\big(\|u_1 - u_2\|_{h^\beta([0,T],Y_\alpha)} + \|\rho_1 - \rho_2\|_{h^\beta([0,T],Y_\alpha)}\big)\\ &\le C(R^\Sigma, R^c, R^h)\Big(T^{\gamma-\beta}\big(\|u_1 - u_2\|_{h^\gamma([0,T],Y_\alpha)} + \|\rho_1 - \rho_2\|_{h^\gamma([0,T],Y_\alpha)}\big) + \|\rho_1(0) - \rho_2(0)\|_{Y_\alpha}\Big)\\ &\le C(R^\Sigma, R^c, R^h)\Big(T^{\gamma-\beta}\big(\|u_1 - u_2\|_{\mathbb{E}_{1,T}} + \|\rho_1 - \rho_2\|_{\mathbb{E}_{1,T}}\big) + \|\rho_1(0) - \rho_2(0)\|_{Y_\alpha}\Big),\end{aligned}$$

where we used Remark 2.89 und Lemma 2.90 for the further estimate and $\gamma \in (0,1)$ with $\gamma > \beta$ is the exponent from Lemma 2.90. For $w \in \mathbb{E}_{1,T} \subset h^\beta\big([0,T], Z_\alpha\big)$ and using Lemma 2.100, we have analogously

$$\begin{aligned}&\Big\|\Big(\big(gaP\big)(u_1,\rho_1) - \big(gaP\big)(u_2,\rho_2)\Big)[w]\Big\|_{\mathbb{E}_{0,T}}\\ &\le C(R^\Sigma, R^c, R^h)\left(\|u_1 - u_2\|_{h^\beta([0,T],Y_\alpha)} + \|\rho_1 - \rho_2\|_{h^\beta([0,T],Y_\alpha)}\right)\|w\|_{h^\beta([0,T],Z_\alpha)}\\ &\le C(R^\Sigma, R^c, R^h)\Big(T^{\gamma-\beta}\big(\|u_1 - u_2\|_{\mathbb{E}_{1,T}} + \|\rho_1 - \rho_2\|_{\mathbb{E}_{1,T}}\big) + \|\rho_1(0) - \rho_2(0)\|_{Y_\alpha}\Big)\|w\|_{\mathbb{E}_{1,T}}.\end{aligned}$$

Finally, using $\widetilde{R^c} \coloneqq \|u_0\|_{Z_\alpha}$ and $\widetilde{R^h} \coloneqq \|\rho_1(0)\|_{Z_\alpha}$ instead of R^c and R^h, Remark 4.12 with Lemma 2.100 implies

$$\Big\|\Big(\big(gaP\big)(u_0,\rho_1(0)) - \big(gaP\big)(u_0,0)\Big)[w]\Big\|_{\mathbb{E}_{0,T}} \le C(R^\Sigma, \|u_0\|_{Z_\alpha}, \|\rho_1(0)\|_{Z_\alpha})\|\rho_1(0)\|_{Y_\alpha}\|w\|_{\mathbb{E}_{1,T}}$$

for $w \in \mathbb{E}_{1,T}$. Overall,

$$
\begin{aligned}
&\|G^h_{u_1}(\rho_1) - G^h_{u_2}(\rho_2)\|_{\mathbb{E}_{0,T}} \\
&= \Big\| \Big((gaH)(u_1,\rho_1) - (gaP)(u_0,0)[\rho_1] \Big) - \Big((gaH)(u_2,\rho_2) - (gaP)(u_0,0)[\rho_2] \Big) \Big\|_{\mathbb{E}_{0,T}} \\
&\le \Big\| \Big((gaP)(u_1,\rho_1) - (gaP)(u_0,\rho_1(0)) \Big)[\rho_1 - \rho_2] \Big\|_{\mathbb{E}_{0,T}} \\
&\quad + \Big\| \Big((gaP)(u_0,\rho_1(0)) - (gaP)(u_0,0) \Big)[\rho_1 - \rho_2] \Big\|_{\mathbb{E}_{0,T}} \\
&\quad + \Big\| \Big((gaP)(u_1,\rho_1) - (gaP)(u_2,\rho_2) \Big)[\rho_2] \Big\|_{\mathbb{E}_{0,T}} + \big\| (gaQ)(u_1,\rho_1) - (gaQ)(u_2,\rho_2) \big\|_{\mathbb{E}_{0,T}} \\
&\le C(R^\Sigma, R^c, R^h) T^{\gamma-\beta} \big(\|u_1 - u_0\|_{\mathbb{E}_{1,T}} + \|\rho_1 - \rho_1(0)\|_{\mathbb{E}_{1,T}} \big) \|\rho_1 - \rho_2\|_{\mathbb{E}_{1,T}} \\
&\quad + C(R^\Sigma, \|u_0\|_{Z_\alpha}, \|\rho_1(0)\|_{Z_\alpha}) \|\rho_1(0)\|_{Y_\alpha} \|\rho_1 - \rho_2\|_{\mathbb{E}_{1,T}} \\
&\quad + C(R^\Sigma, R^c, R^h) \Big(T^{\gamma-\beta} \big(\|u_1 - u_2\|_{\mathbb{E}_{1,T}} + \|\rho_1 - \rho_2\|_{\mathbb{E}_{1,T}} \big) + \|\rho_1(0) - \rho_2(0)\|_{Y_\alpha} \Big) \big(\|\rho_2\|_{\mathbb{E}_{1,T}} + 1 \big) \\
&\le C(R^\Sigma, R^c, R^h) \Big(T^{\gamma-\beta} \big(\|u_1 - u_2\|_{\mathbb{E}_T} + \|\rho_1 - \rho_2\|_{\mathbb{E}_{1,T}} \big) + \|\rho_1(0) - \rho_2(0)\|_{Y_\alpha} \Big) \\
&\quad + C(R^\Sigma, \|u_0\|_{Z_\alpha}, \|\rho_1(0)\|_{Z_\alpha}) \|\rho_1(0)\|_{Y_\alpha} \|\rho_1 - \rho_2\|_{\mathbb{E}_{1,T}}
\end{aligned}
$$

follows. □

With this preparatory work, we can now prove short-time existence for the first equation (4.1a).

Theorem 4.18. *We suppose Assumptions 4.9 are valid and use Notations 4.10. Therein, choose $R^h = R^h(R^c, u_0, \delta_1) > 0$ sufficiently large, choose $\delta_0 = \delta_0(R^\Sigma, u_0, \delta_1) \in (0, R^\Sigma)$ sufficiently small and choose $T = T(R^\Sigma, R^c, R^h, u_0, \delta_0) \in (0,1)$ sufficiently small. For any initial value $\rho_0 \in Z_{2\beta+\alpha}$ with $\|\rho_0\|_{Z_{2\beta+\alpha}} < \delta_1$ and $\|\rho_0\|_{Y_\alpha} < \delta_0$ and any concentration $u \in M^c_T$, there exists a unique solution $\rho := \rho_{u,\rho_0} \in M^h_{T,\rho_0}$ of*

$$
\begin{cases}
\partial_t \rho &= g(u)a(\rho)H(\rho) \quad \text{in } \mathbb{E}_{0,T}, \\
\rho(0) &= \rho_0 \quad \text{in } Z_\alpha.
\end{cases}
$$

Proof. We show the existence of a unique solution $\rho \in M^h_{T,\rho_0}$ of

$$
\begin{cases}
\partial_t \rho &= g(u)a(\rho)H(\rho) \quad \text{in } \mathbb{E}_{0,T} \\
\rho(0) &= \rho_0 \quad \text{in } Z_\alpha
\end{cases}
\quad \Leftrightarrow \quad L^h[\rho] = \begin{pmatrix} G^h_u(\rho) \\ \rho_0 \end{pmatrix} \text{ in } \mathbb{E}_{0,T} \times Z_\alpha. \tag{4.3}
$$

Equation (4.3) is well-defined because $A^h[\rho], G^h_u(\rho) \in \mathbb{E}_{0,T}$ holds for $\rho \in M^h_T$ and $u \in M^c_T$ by Corollary 4.13. Due to Lemma 4.16(i) and Proposition 4.15 it is equivalent to prove the existence of a unique $\rho \in M^h_{T,\rho_0}$ with

$$
L^h[\rho] = \begin{pmatrix} G^h_u(\rho) \\ \rho_0 \end{pmatrix} \text{ in } (\mathbb{E}_{0,T} \times Z_\alpha)^h_+ \quad \Leftrightarrow \quad \rho = \big(L^h\big)^{-1} \begin{pmatrix} G^h_u(\rho) \\ \rho_0 \end{pmatrix} =: K^h_{u,\rho_0}(\rho) \text{ in } \mathbb{E}_{1,T}.
$$

So, we show that $K^h_{u,\rho_0} : M^h_{T,\rho_0} \subset \mathbb{E}_{1,T} \to \mathbb{E}_{1,T}$ has a unique fixed point $\rho \in M^h_{T,\rho_0}$ using the Banach fixed-point theorem.

Step 1: Due to Lemma 4.16(i) and Proposition 4.15, $K^h_{u,\rho_0}(\rho) \in \mathbb{E}_{1,T}$ is well-defined for $\rho \in M^h_{T,\rho_0}$.

Step 2: We have to verify that K^h_{u,ρ_0} is a contraction on M^h_{T,ρ_0}. For any $\rho_1, \rho_2 \in M^h_{T,\rho_0}$

$$\begin{aligned}
&\|K^h_{u,\rho_0}(\rho_1) - K^h_{u,\rho_0}(\rho_2)\|_{\mathbb{E}_{1,T}} \\
&\leq \Lambda^h \big\| \big(G^h_u(\rho_1), \rho_0\big) - \big(G^h_u(\rho_2), \rho_0\big)\big\|_{(\mathbb{E}_{0,T}\times Z_\alpha)^h_+} \\
&= \Lambda^h \|G^h_u(\rho_1) - G^h_u(\rho_2)\|_{\mathbb{E}_{0,T}} \\
&\leq \Big(C(R^\Sigma, R^c, R^h, \Lambda^h)T^\varepsilon + C(R^\Sigma, \|u_0\|_{Z_\alpha}, \|\rho_0\|_{Z_\alpha}, \Lambda^h)\|\rho_0\|_{Y_\alpha}\Big)\|\rho_1 - \rho_2\|_{\mathbb{E}_{1,T}} \\
&\leq \Big(C(R^\Sigma, R^c, R^h, \Lambda^h)T^\varepsilon + C(R^\Sigma, u_0, \delta_1, \Lambda^h)\delta_0\Big)\|\rho_1 - \rho_2\|_{\mathbb{E}_{1,T}}
\end{aligned}$$

holds by Proposition 4.15, Lemma 4.16(i) as well as Proposition 4.17. For sufficiently small $\delta_0 > 0$ and sufficiently small $T > 0$,

$$\|K^h_{u,\rho_0}(\rho_1) - K^h_{u,\rho_0}(\rho_2)\|_{\mathbb{E}_{1,T}} \leq \frac{1}{4}\|\rho_1 - \rho_2\|_{\mathbb{E}_{1,T}}$$

follows. Because Λ^h only depends on u_0, δ_0 only depends on R^Σ, u_0 and δ_1 whereas T only depends on R^Σ, R^c, R^h and u_0.

Step 3: We have to show that $K^h_{u,\rho_0} : M^h_{T,\rho_0} \to M^h_{T,\rho_0}$ is a self-mapping. Any $\rho \in M^h_{T,\rho_0}$ fulfills $\big(K^h_{u,\rho_0}(\rho)\big)(0) = \rho_0$ in Z_α because $w := K^h_{u,\rho_0}(\rho)$ is a solution to

$$L^h w = \begin{pmatrix} [L^h w]_1 \\ w(0) \end{pmatrix} = \begin{pmatrix} G^h_u(\rho) \\ \rho_0 \end{pmatrix} \text{ in } \mathbb{E}_{0,T} \times Z_\alpha.$$

Furthermore, we have

$$\begin{aligned}
\|K^h_{u,\rho_0}(\rho)\|_{\mathbb{E}_{1,T}} &\leq \|K^h_{u,\rho_0}(\rho_0)\|_{\mathbb{E}_{1,T}} + \|K^h_{u,\rho_0}(\rho) - K^h_{u,\rho_0}(\rho_0)\|_{\mathbb{E}_{1,T}} \\
&\leq \Lambda^h \big\|\big(G^h_u(\rho_0), \rho_0\big)\big\|_{(\mathbb{E}_{0,T}\times Z_\alpha)^h_+} + \frac{1}{4}\|\rho - \rho_0\|_{\mathbb{E}_{1,T}} \\
&\leq \Lambda^h N^h + \frac{1}{4}\left(\|\rho\|_{\mathbb{E}_{1,T}} + 2\|\rho_0\|_{Z_\alpha}\right) \\
&\leq \frac{R^h}{2} + \frac{R^h}{2} = R^h,
\end{aligned}$$

where the first summand is bounded by Proposition 4.15 and Lemma 4.16(ii) and the second summand by the contraction-property (see step 2). R^h being sufficiently large thus means $R^h \geq 2\Lambda^h N^h$ and because Λ^h only depends on u_0 and N^h only depends on R^c and δ_1, we have $R^h = R^h\big(R^c, u_0, \delta_1\big)$. The two properties just deduced imply $K^h_{u,\rho_0}(\rho) \in M^h_{T,\rho_0}$ for all $\rho \in M^h_{T,\rho_0}$.

As M^h_{T,ρ_0} is a closed and non-empty subset of the Banach space $\mathbb{E}_{1,T}$, the Banach fixed-point theorem yields the existence of a unique $\rho \in M^h_{T,\rho_0}$ with

$$K^h_{u,\rho_0}(\rho) = \rho \text{ in } \mathbb{E}_{1,T} \quad \Leftrightarrow \quad \begin{cases} \partial_t \rho &= g(u)a(\rho)H(\rho) \text{ in } \mathbb{E}_{0,T}, \\ \rho(0) &= \rho_0 \quad\quad\quad\quad\;\, \text{in } Z_\alpha. \end{cases} \qquad \square$$

Now that we know that there exists a solution ρ_{u,ρ_0} to the first equation (4.1a), we analyze some of its properties. First, we discuss its dependence on the concentration u and the initial value ρ_0. The result in Proposition 4.19 will be necessary for the contraction argument for the second equation (4.1b). Afterwards, we state an improved regularity in space for the solution in Proposition 4.20.

Proposition 4.19. *We suppose that Assumptions 4.9 are valid and use Notations 4.10. Therein, choose $R^h > 0$ as large and choose $\delta_0 > 0$, $T > 0$ as small as in Theorem 4.18. There exists $\varepsilon > 0$ with*

$$\|\rho_1 - \rho_2\|_{\mathbb{E}_{1,T}} \le C(R^\Sigma, R^c, R^h, \Lambda^h, \delta_1)\big(T^\varepsilon \|u_1 - u_2\|_{\mathbb{E}_{1,T}} + \|\rho_{0,1} - \rho_{0,2}\|_{Z_{2\beta+\alpha}}\big)$$

for any $u_1, u_2 \in M_T^c$ and $\rho_{0,1}, \rho_{0,2} \in Z_{2\beta+\alpha}$ with $\|\rho_{0,i}\|_{Z_{2\beta+\alpha}} < \delta_1$ and $\|\rho_{0,i}\|_{Y_\alpha} < \delta_0$, where $\rho_i := \rho_{u_i,\rho_{0,i}} \in M_T^h$ is the solution from Theorem 4.18 associated with the concentration u_i and the initial value $\rho_{0,i}$, respectively.

Proof. As $\rho_i \in M_T^h$ is the solution from Theorem 4.18 associated with the concentration u_i and the initial value $\rho_{0,i}$, it is a fixed point of $(L^h)^{-1}\big(G_{u_i}^h(\cdot), \rho_{0,i}\big)$ as in the proof of Theorem 4.18. Therefore, we have

$$\begin{aligned}\|\rho_1 - \rho_2\|_{\mathbb{E}_{1,T}} &\le \Lambda^h \big\|\big(G_{u_1}^h(\rho_1), \rho_{0,1}\big) - \big(G_{u_2}^h(\rho_2), \rho_{0,2}\big)\big\|_{(\mathbb{E}_{0,T}\times Z_\alpha)_+^h} \\ &= \Lambda^h \big\|G_{u_1}^h(\rho_1) - G_{u_2}^h(\rho_2)\big\|_{\mathbb{E}_{0,T}} + \Lambda^h \|\rho_{0,1} - \rho_{0,2}\|_{Z_\alpha} \\ &\quad + \Lambda^h \big\|G_{u_0}(\rho_{0,1}) - G_{u_0}(\rho_{0,2}) + A^h[\rho_{0,1} - \rho_{0,2}]\big\|_{X_{2\beta+\alpha}}\end{aligned}$$

by Proposition 4.15 and Lemma 4.16(i). With $\delta_0 > 0$ and $T > 0$ as small as in Theorem 4.18, Proposition 4.17 yields

$$\begin{aligned}&\Lambda^h \big\|G_{u_1}^h(\rho_1) - G_{u_2}^h(\rho_2)\big\|_{\mathbb{E}_{0,T}} \\ &\le \frac{1}{4}\|\rho_1 - \rho_2\|_{\mathbb{E}_{1,T}} + C(R^\Sigma, R^c, R^h, \Lambda^h)\big(T^\varepsilon \|u_1 - u_2\|_{\mathbb{E}_{1,T}} + \|\rho_{0,1} - \rho_{0,2}\|_{Y_\alpha}\big).\end{aligned}$$

Due to $u_0 \in M_T^c$ and $\rho_{0,i} \in M_T^h$, Remark 4.12 together with Lemma 2.100 implies

$$\begin{aligned}&\big\|G_{u_0}(\rho_{0,1}) - G_{u_0}(\rho_{0,2}) + A^h[\rho_{0,1} - \rho_{0,2}]\big\|_{X_{2\beta+\alpha}} \\ &= \big\|\big(gaH\big)(u_0, \rho_{0,1}) - \big(gaH\big)(u_0, \rho_{0,2})\big\|_{X_{2\beta+\alpha}} \\ &\le \Big\|\Big(\big(gaP\big)(u_0, \rho_{0,1}) - \big(gaP\big)(u_0, \rho_{0,2})\Big)[\rho_{0,1}]\Big\|_{X_{2\beta+\alpha}} + \big\|\big(gaP\big)(u_0, \rho_{0,2})[\rho_{0,1} - \rho_{0,2}]\big\|_{X_{2\beta+\alpha}} \\ &\qquad + \big\|\big(gaQ\big)(u_0, \rho_{0,1}) - \big(gaQ\big)(u_0, \rho_{0,2})\big\|_{X_{2\beta+\alpha}} \\ &\le C(R^\Sigma, R^c, R^h)\big(\|\rho_{0,1} - \rho_{0,2}\|_{Y_{2\beta+\alpha}} \|\rho_{0,1}\|_{Z_{2\beta+\alpha}} + \|\rho_{0,1} - \rho_{0,2}\|_{Z_{2\beta+\alpha}} + \|\rho_{0,1} - \rho_{0,2}\|_{Y_{2\beta+\alpha}}\big) \\ &\le C(R^\Sigma, R^c, R^h, \delta_1)\|\rho_{0,1} - \rho_{0,2}\|_{Z_{2\beta+\alpha}}.\end{aligned}$$

(As u_0 and $\rho_{0,i}$ are all independent of t, there is also no time dependence in the appliciation of Lemma 2.100 in the estimate above.) Altogether, we thus have

$$\|\rho_1 - \rho_2\|_{\mathbb{E}_{1,T}} \le C(R^\Sigma, R^c, R^h, \Lambda^h, \delta_1)\big(T^\varepsilon \|u_1 - u_2\|_{\mathbb{E}_{1,T}} + \|\rho_{0,1} - \rho_{0,2}\|_{Z_{2\beta+\alpha}}\big). \qquad \square$$

Proposition 4.20. *We suppose that Assumptions 4.9 are valid and use Notations 4.10. Therein, choose $R^h > 0$ as large and choose $\delta_0 > 0$, $T > 0$ as small as in Theorem 4.18. Let $u \in M_T^c$ and $\rho_0 \in Z_{2\beta+\alpha}$ with $\|\rho_0\|_{Z_{2\beta+\alpha}} < \delta_1$ and $\|\rho_0\|_{Y_\alpha} < \delta_0$ be arbitrary and let $\rho := \rho_{u,\rho_0} \in M^h_{T,\rho_0}$ be the associated solution from Theorem 4.18. Then, $\rho(t) \in Z_{2\beta+\alpha}$ holds for all $t \in [0,T]$.*

Proof. By Proposition 4.14, A^h satisfies the conditions of Proposition 2.131 and thus $\rho \in M^h_{T,\rho_0} \subset \mathbb{E}_{1,T}$ fulfills $\partial_t \rho(t) \in \mathcal{D}_{A^h}(\beta) = X_{2\beta+\alpha}$ for all $t \in [0,T]$. In addition, by Corollary 4.13, we have $(gaQ)(u,\rho)(t) \in X_{2\beta+\alpha}$ for all $t \in [0,T]$. Hence,

$$\begin{aligned} A^h_{u(t),\rho(t)}\big[\rho(t)\big] &= (gaH)(u,\rho)(t) - (gaQ)(u,\rho)(t) \\ &= \partial_t \rho(t) - (gaQ)(u,\rho)(t) \in X_{2\beta+\alpha} \end{aligned}$$

holds for all $t \in [0,T]$. Because $A^h_{u(t),\rho(t)} : Z_{s_i} \to X_{s_i}$ generates an analytic C^0-semigroup for both $s_1 = \alpha$, $s_2 = 2\beta+\alpha$ (see Proposition 4.14), Lemma 2.132 finally yields $\rho(t) \in Z_{2\beta+\alpha}$ for all $t \in [0,T]$. □

4.2 Short-Time Existence for u

In this section, we discuss the second equation (4.1b)

$$\partial_t u = \Delta_\rho G'(u) + g(u)a(\rho)H(\rho)\,\nu_\Sigma \cdot \nabla_\rho u + g(u)H(\rho)^2 u$$

for concentrations u with initial value $u(0) = u_0$. As height function ρ, we insert the solution function ρ_{u,ρ_0} from Theorem 4.18 with initial value ρ_0. Both equations (4.1a) and (4.1b) are parabolic, quasilinear partial differential equations of second order. Due to this parallel structure, we apply the same approach as in Section 4.1 to solve this second equation, using linearization and a contraction argument.
First, we deduce a corollary from Remark 4.12, which contains the analogous statement to Corollary 4.13 but for A^c and G^c instead of A^h and G^h.

Corollary 4.21. *We suppose Assumptions 4.9 are valid and use Notations 4.10. Therein, choose $R^h > 0$ as large and choose $\delta_0 > 0$, $T > 0$ as small as in Theorem 4.18. Let $\rho_0 \in Z_{2\beta+\alpha}$ with $\|\rho_0\|_{Z_{2\beta+\alpha}} < \delta_1$ and $\|\rho_0\|_{Y_\alpha} < \delta_0$. For $u \in M_T^c$, we have $A^c[u] \in \mathbb{E}_{0,T}$ and $G^c_{\rho_0}(u) \in \mathbb{E}_{0,T}$.*

Proof. Let $\rho_{u,\rho_0} \in M^h_{T,\rho_0}$ be the solution from Theorem 4.18 associated with the concentration u and the initial value ρ_0. Then, we have $u, u_0 \in M_T^c \subset h^\beta([0,T], Z_\alpha)$ and $\rho_{u,\rho_0}, 0 \in M_T^h \subset h^\beta([0,T], Z_\alpha)$. Thus, Remark 4.12 together with Lemma 2.100 yields the statement. □

As in Section 4.1, we show that the linearization of the (elliptic) operator on the right hand side of the equation generates an analytic C^0-semigroup, which implies that the linearization of the initial value problem defines an invertible operator.

Proposition 4.22. *We suppose Assumptions 4.9 are valid and use Notations 4.10. Then,*

$$A^c = A^c_{u_0,0} : Z_\alpha \to X_\alpha$$

generates an analytic C^0-semigroup with $\mathcal{D}_{A^c}(\beta) = X_{2\beta+\alpha}$.
Let R^h be as large and let $\delta_0 > 0$, $T > 0$ be as small as in Theorem 4.18. If $\rho \coloneqq \rho_{u,\rho_0} \in M^h_{T,\rho_0}$ is the solution from Theorem 4.18 associated with the concentration $u \in M^c_T$ and the initial value $\rho_0 \in Z_{2\beta+\alpha}$ with $\|\rho_0\|_{Z_{2\beta+\alpha}} < \delta_1$ and $\|\rho_0\|_{Y_\alpha} < \delta_0$, also

$$A^c_{u(t),\rho(t)} : Z_s \to X_s$$

generates an analytic C^0-semigroup for $s \in \{\alpha, 2\beta + \alpha\}$ and $t \in [0, T]$.

Proof. First, we prove the generation of the claimed semigroups. Choose $s \in \{\alpha, 2\beta + \alpha\}$ and fix $t \in [0, T]$. If $\rho \coloneqq \rho_{u,\rho_0} \in M^h_{T,\rho_0}$ is the solution from Theorem 4.18 associated with the concentration u and the initial value ρ_0, $\rho(t) \in Z_s$ follows with Proposition 4.20. As also $\|\rho(t)\|_{C^1(M)} \le \|\rho(t)\|_{Y_\alpha} < R^\Sigma$ holds (see remark after Notations 4.10), we have $\rho(t) \in U^h_{1+s,1}$. For the following arguments, we do not need that ρ is a solution from Theorem 4.18, but only use $\rho(t) \in U^h_{1+s,1}$. Because $u_0 \in M^c_T$ and $0 \in U^h_{1+s,1}$ hold, we thus treat $A^c_{u_0,0}$ as a special case of $A^c_{u(t),\rho(t)}$. By Lemma 4.7, $\Delta_{\rho(t)} \in \mathcal{L}(Z_s, X_s)$ is a symmetric and elliptic differential operator of second order. Because we have

$$A^c_{u(t),\rho(t)} = G''\big(u(t)\big)\Delta_{\rho(t)} + \text{ lower order terms}$$

with $G'' > 0$ by Assumption 4.9(i), $A^c_{u(t),\rho(t)} \in \mathcal{L}(Z_s, X_s)$ is a symmetric and elliptic differential operator of second order, too. On account of Proposition 2.139, the operator $A^c_{u(t),\rho(t)} : \mathcal{D}\big(A^c_{u(t),\rho(t)}\big) \subset X_s \to X_s$ therefore generates an analytic C^0-semigroup with $\mathcal{D}\big(A^c_{u(t),\rho(t)}\big) = Z_s$ and equivalent norms.
By Lemmas 2.128 and 2.87, finally

$$\mathcal{D}_{A^c_{u_0,0}}(\beta) = \big(X_\alpha, \mathcal{D}\big(A^c_{u_0,0}\big)\big)_\beta = (X_\alpha, Z_\alpha)_\beta = \big(h^\alpha(\Sigma), h^{2+\alpha}(\Sigma)\big)_\beta = h^{2\beta+\alpha}(\Sigma) = X_{2\beta+\alpha}$$

holds. □

Proposition 4.23. *We suppose Assumptions 4.9 are valid and use Notations 4.10. Then,*

$$L^c : \mathbb{E}_{1,T} \to (\mathbb{E}_{0,T} \times Z_\alpha)^c_+$$

is bijective with

$$\Lambda^c \coloneqq \sup_{0<T\le 1} \|\big(L^c\big)^{-1}\|_{\mathcal{L}((\mathbb{E}_{0,T}\times Z_\alpha)^c_+, \mathbb{E}_{1,T})} < \infty,$$

where

$$(\mathbb{E}_{0,T} \times Z_\alpha)^c_+ \coloneqq \big\{(f, f_0) \in (\mathbb{E}_{0,T} \times Z_\alpha) \,\big|\, f(0) + A^c[f_0] \in \mathcal{D}_{A^c}(\beta) = X_{2\beta+\alpha}\big\} \text{ with}$$
$$\|(f, f_0)\|_{(\mathbb{E}_{0,T}\times Z_\alpha)^c_+} \coloneqq \|f\|_{\mathbb{E}_{0,T}} + \|f_0\|_{Z_\alpha} + \|f(0) + A^c[f_0]\|_{X_{2\beta+\alpha}} \text{ for } (f, f_0) \in (\mathbb{E}_{0,T} \times Z_\alpha)^c_+.$$

In particular, $\Lambda^c = \Lambda^c(u_0)$ only depends on the initial value u_0.

Proof. By Proposition 4.22, A^c satisfies the conditions of Proposition 2.131, which yields bijectivity of

$$\widetilde{L^c} : \widetilde{\mathbb{E}_{1,T}} \to \left(\mathbb{E}_{0,T} \times \mathcal{D}(A^c)\right)_+, \; u \mapsto \begin{pmatrix} \partial_t u - A^c[u] \\ u(0) \end{pmatrix}$$

and the estimate

$$\sup_{0<T\le 1} \left\| \left(\widetilde{L^c}\right)^{-1} \right\|_{\mathcal{L}((\mathbb{E}_{0,T}\times\mathcal{D}(A^c))_+, \widetilde{\mathbb{E}_{1,T}})} < \infty,$$

where $\widetilde{\mathbb{E}_{1,T}} := h^{1,\beta}([0,T], X_\alpha) \cap h^\beta\big([0,T], \mathcal{D}(A^c)\big)$. The claim follows with $\mathcal{D}(A^c) = Z_\alpha$ by Proposition 4.22. Because $A^c = A^c_{u_0,0}$ only depends on u_0, also $\mathcal{L}^c$ and then Λ^c only depend on u_0. □

We show a technical auxiliary lemma analogous to Lemma 4.16.

Lemma 4.24. *We suppose Assumptions 4.9 are valid and use Notations 4.10. Therein, choose $R^h > 0$ as large and choose $\delta_0 > 0$, $T > 0$ as small as in Theorem 4.18.*

(i) Let $\rho_0 \in Z_{2\beta+\alpha}$ with $\|\rho_0\|_{Z_{2\beta+\alpha}} < \delta_1$ and $\|\rho_0\|_{Y_\alpha} < \delta_0$ and let $u \in M^c_T$. Then

$$\big(G^c_{\rho_0}(u)\big)(0) = \Delta_{\rho_0} G'(u_0) + g(u_0)a(\rho_0)H(\rho_0)\nu_\Sigma \cdot \nabla_{\rho_0} u_0 + g(u_0)H(\rho_0)^2 u_0 - A^c[u_0]$$

holds in X_α. In particular, $\big(G^c_{\rho_0}(u)\big)(0)$ is independent of u. Furthermore, we have

$$\big(G^c_{\rho_0}(u), u_0\big) \in \big(\mathbb{E}_{0,T} \times Z_\alpha\big)^c_+.$$

(ii) There exists a constant $N^c = N^c\big(R^\Sigma, u_0, \delta_1\big)$ independent of T, R^c and R^h such that

$$\left\| \big(G^c_{\rho_0}(u_0), u_0\big) \right\|_{(\mathbb{E}_{0,T}\times Z_\alpha)^c_+} \le N^c$$

holds for all $\rho_0 \in Z_{2\beta+\alpha}$ with $\|\rho_0\|_{Z_{2\beta+\alpha}} < \delta_1$ and $\|\rho_0\|_{Y_\alpha} < \delta_0$.

Proof.

Ad (i) We have $G^c_{\rho_0}(u) \in \mathbb{E}_{0,T}$ by Corollary 4.21, hence $\big(G^c(u), u_0\big) \in \mathbb{E}_{0,T} \times Z_\alpha$ holds. Let $\rho := \rho_{u,\rho_0} \in M^h_{T,\rho_0}$ be the solution from Theorem 4.18 associated with the concentration $u \in M^c_T$ and the initial value ρ_0. Then, $u(0) = u_0 \in U^c_{2\beta+\alpha} \cap Z_{2\beta+\alpha}$ and $\rho(0) = \rho_0, 0 \in U^h_{2\beta+\alpha} \cap U^h_{1+2\beta+\alpha,1} \subset Z_{2\beta+\alpha}$ hold. So, Corollary 4.11 yields

$$\begin{aligned}
&\big(G^c_{\rho_0}(u)\big)(0) \\
&= \Big(\Delta_\rho G'(u) + g(u)a(\rho)H(\rho)\nu_\Sigma \cdot \nabla_\rho u + g(u)H(\rho)^2 u - G''(u_0)\Delta_\Sigma u - g(u_0)H^2_\Sigma u\Big)(0) \\
&= \Delta_{\rho_0} G'(u_0) + g(u_0)a(\rho_0)H(\rho_0)\nu_\Sigma \cdot \nabla_{\rho_0} u_0 + g(u_0)H(\rho_0)^2 u_0 \\
&\quad - G''(u_0)\Delta_\Sigma u_0 - g(u_0)H^2_\Sigma u_0 \text{ in } X_{2\beta+\alpha} \hookrightarrow X_\alpha
\end{aligned}$$

and therefore

$$\begin{aligned}
A^c[u_0] + \big(G^c_{\rho_0}(u)\big)(0) &= \Delta_{\rho_0} G'(u_0) + g(u_0)a(\rho_0)H(\rho_0)\nu_\Sigma \cdot \nabla_{\rho_0} u_0 + g(u_0)H(\rho_0)^2 u_0 \\
&\in X_{2\beta+\alpha}
\end{aligned}$$

follows with $X_{2\beta+\alpha} = \mathcal{D}_{A^c}(\beta)$ by Proposition 4.22.

Overall, we thus have $\big(G^c_{\rho_0}(u), u_0\big) \in \Big(\mathbb{E}_{0,T} \times Z_\alpha\Big)^c_+$.

Ad (ii) We have

$$\left\|\left(G^c_{\rho_0}(u_0), u_0\right)\right\|_{(\mathbb{E}_{0,T}\times Z_\alpha)^c_+} = \left\|G^c_{\rho_0}(u_0)\right\|_{\mathbb{E}_{0,T}} + \|u_0\|_{Z_\alpha} + \left\|A^c[u_0]+\left(G^c_{\rho_0}(u_0)\right)(0)\right\|_{\mathcal{D}_{A^c}(\beta)}.$$

Let $\rho := \rho_{u_0,\rho_0} \in M^h_{T,\rho_0} \subset h^\beta\big([0,T], Z_\alpha\big)$ be the solution from Theorem 4.18 associated with the concentration u_0 and the initial value ρ_0. We have $u_0 \in \widetilde{M^c_T}$ and $\rho, \rho_0, 0 \in \widetilde{M^h_T}$ with $\widetilde{M^c_T}$, $\widetilde{M^h_T}$ defined as M^c_T, M^h_T but with $\widetilde{R^c} := 2\|u_0\|_{Z_\alpha}$, $\widetilde{R^h} := \|\rho\|_{\mathbb{E}_{1,T}} \geq 2\|\rho_0\|_{Z_\alpha}$ instead of R^c, R^h. We thus can use Remark 4.12 and Lemma 2.100 to bound

$$\begin{aligned}\left\|G^c_{\rho_0}(u_0)\right\|_{\mathbb{E}_{0,T}} &= \big\|\Delta_\rho G'(u_0) + g(u_0)a(\rho)H(\rho)\nu_\Sigma\cdot\nabla_\rho u_0 + g(u_0)H(\rho)^2 u_0 \\ &\qquad - G''(u_0)\Delta_\Sigma u_0 - g(u_0)H^2_\Sigma u_0\big\|_{\mathbb{E}_{0,T}} \\ &\leq C\big(R^\Sigma, \|u_0\|_{Z_\alpha}, \|\rho\|_{\mathbb{E}_{1,T}}\big).\end{aligned}$$

We have

$$\begin{aligned}&\left\|A^c[u_0] + \left(G^c_{\rho_0}(u_0)\right)(0) - G''(u_0)J(\rho_0)[u_0]\right\|_{\mathcal{D}_{A^c}(\beta)} \\ &= \big\|G''(u_0)D(\rho_0)[u_0] + G'''(u_0)\left|\nabla_{\rho_0}u_0\right|^2 \\ &\qquad + g(u_0)a(\rho_0)H(\rho_0)\nu_\Sigma\cdot\nabla_{\rho_0}u_0 + g(u_0)H(\rho_0)^2u_0\big\|_{X_{2\beta+\alpha}}\end{aligned}$$

according to part (i). So, again, Remark 4.12 and Lemma 2.100 yield

$$\begin{aligned}&\left\|A^c[u_0] + \left(G^c_{\rho_0}(u_0)\right)(0) - G''(u_0)J(\rho_0)[u_0]\right\|_{\mathcal{D}_{A^c}(\beta)} \\ &\leq C\big(R^\Sigma, \|u_0\|_{Z_\alpha}, \|\rho\|_{\mathbb{E}_{1,T}}\big)\, C\big(\|u_0\|_{Z_{2\beta+\alpha}}, \|\rho_0\|_{Z_{2\beta+\alpha}}\big) \\ &\leq C\big(R^\Sigma, \|u_0\|_{Z_{2\beta+\alpha}}, \delta_1, \|\rho\|_{\mathbb{E}_{1,T}}\big).\end{aligned}$$

(As u_0 and ρ_0 are independent of t, there is also no time dependence in the application of Remark 4.12 in the estimate above.) Moreover, we have $\rho_0 \in U^h_{1+2\beta+\alpha,1} \cap B^{Z_{2\beta+\alpha}}_{\delta_1}(0)$ and therefore a last application of Remark 4.12 and Lemma 2.100 yields

$$\left\|G''(u_0)J(\rho_0)[u_0]\right\|_{\mathcal{D}_{A^c}(\beta)} \leq C\big(R^\Sigma, \|u_0\|_{Z_\alpha}, \delta_1\big)\|u_0\|_{Y_{2\beta+\alpha}} \leq C\big(R^\Sigma, \|u_0\|_{Z_{2\beta+\alpha}}, \delta_1\big).$$

So,

$$\left\|A^c[u_0] + \left(G^c_{\rho_0}(u_0)\right)(0)\right\|_{\mathcal{D}_{A^c}(\beta)} \leq C\big(R^\Sigma, \|u_0\|_{Z_{2\beta+\alpha}}, \delta_1, \|\rho\|_{\mathbb{E}_{1,T}}\big)$$

follows. Overall, we thus have

$$\left\|\left(G^c_{\rho_0}(u_0), u_0\right)\right\|_{(\mathbb{E}_{0,T}\times Z_\alpha)^c_+} \leq C\big(R^\Sigma, \|u_0\|_{Z_{2\beta+\alpha}}, \delta_1, \|\rho\|_{\mathbb{E}_{1,T}}\big).$$

Now, we have to explain why $\|\rho\|_{\mathbb{E}_{1,T}}$ can be bounded by a constant depending only on R^Σ, u_0 and δ_1. As ρ is the solution from Theorem 4.18, $\|\rho\|_{\mathbb{E}_{1,T}} \leq R^h$ holds with $R^h = R^h\big(R^c, u_0, \delta_1\big)$. Because ρ is associated to the concentration u_0, it suffices to use $R^h = R^h\big(2\|u_0\|_{Z_\alpha}, u_0, \delta_1\big)$ for the statement of Theorem 4.18. Thus, we have $\|\rho\|_{\mathbb{E}_{1,T}} \leq R^h = C\big(u_0, \delta_1\big)$ and therefore finally

$$\left\|\left(G^c_{\rho_0}(u_0), u_0\right)\right\|_{(\mathbb{E}_{0,T}\times Z_\alpha)^c_+} \leq C\big(R^\Sigma, u_0, \delta_1\big) =: N^c$$

follows. □

With the help of Proposition 4.19, an analogous statement to Proposition 4.17 holds which again will be the key point to the contraction argument.

Proposition 4.25. *We suppose that Assumptions 4.9 are valid and use Notations 4.10. Therein, choose $R^h > 0$ as large and choose $\delta_0 > 0$, $T > 0$ as small as in Theorem 4.18. There exists $\varepsilon > 0$ with*

$$\begin{aligned}\|G^c_{\rho_{0,1}}(u_1) - G^c_{\rho_{0,2}}(u_2)\|_{\mathbb{E}_{0,T}} \le{}& C(R^\Sigma, R^c, R^h, \Lambda^h, \delta_1)\big(T^\varepsilon \|u_1 - u_2\|_{\mathbb{E}_{1,T}} + \|\rho_{0,1} - \rho_{0,2}\|_{Z_{2\beta+\alpha}}\big)\\ &+ C(R^\Sigma, \|u_0\|_{Z_\alpha}, \delta_1)\delta_0 \|u_1 - u_2\|_{\mathbb{E}_{1,T}}\end{aligned}$$

for $u_1, u_2 \in M^c_T$ and initial values $\rho_{0,1}, \rho_{0,2} \in Z_{2\beta+\alpha}$ with $\|\rho_{0,i}\|_{Z_{2\beta+\alpha}} < \delta_1$ and $\|\rho_{0,i}\|_{Y_\alpha} < \delta_0$.

Proof. Let $\rho_i := \rho_{u_i,\rho_{0,i}} \in M^h_T$ be the solution from Theorem 4.18 associated with the concentration u_i and the initial value $\rho_{0,i}$. Using appropriate triangle inequalities (as in the proof of Proposition 4.17), Remark 4.12 together with Lemma 2.100 yields

$$\begin{aligned}&\Big\|\big(G^c_{\rho_{0,1}}(u_1) - G^c_{\rho_{0,2}}(u_2)\big) - \big(G''(u_1)D(\rho_1)[u_1 - u_2] - G''(u_0)D(0)[u_1 - u_2]\big)\Big\|_{\mathbb{E}_{0,T}}\\
&= \Big\|G''(u_1)D(\rho_1)[u_2] - G''(u_2)D(\rho_2)[u_2]\\
&\qquad + G''(u_1)J(\rho_1)[u_1] - G''(u_2)J(\rho_2)[u_2] - G''(u_0)J(0)[u_1 - u_2]\\
&\qquad + G'''(u_1)\big|\nabla_{\rho_1} u_1\big|^2 - G'''(u_2)\big|\nabla_{\rho_2} u_2\big|^2\\
&\qquad + g(u_1)a(\rho_1)H(\rho_1)\nu_\Sigma \cdot \nabla_{\rho_1} u_1 - g(u_2)a(\rho_2)H(\rho_2)\nu_\Sigma \cdot \nabla_{\rho_2} u_2\\
&\qquad + g(u_1)H(\rho_1)^2 u_1 - g(u_2)H(\rho_2)^2 u_2 - g(u_0)H^2_\Sigma[u_1 - u_2]\Big\|_{\mathbb{E}_{0,T}}\\
&\le C(R^\Sigma, R^c, R^h)\big(\|u_1 - u_2\|_{h^\beta([0,T],Y_\alpha)} + \|\rho_1 - \rho_2\|_{h^\beta([0,T],Y_\alpha)}\big)\\
&\qquad + C(R^\Sigma, R^c, R^h)\big(\|u_1 - u_0\|_{h^\beta([0,T],Y_\alpha)} + \|\rho_1\|_{h^\beta([0,T],Z_\alpha)}\big)\|u_1 - u_2\|_{h^\beta([0,T],Y_\alpha)}\\
&\qquad + C(R^\Sigma, R^c, R^h)\big(\|u_1 - u_2\|_{h^\beta([0,T],Y_\alpha)} + \|\rho_1 - \rho_2\|_{h^\beta([0,T],Z_\alpha)}\big)\\
&\le C(R^\Sigma, R^c, R^h)\big(\|u_1 - u_2\|_{h^\beta([0,T],Y_\alpha)} + \|\rho_1 - \rho_2\|_{h^\beta([0,T],Z_\alpha)}\big).\end{aligned}$$

Analogously, using $\widetilde{R^c} := \|u_0\|_{Z_\alpha}$ and $\widetilde{R^h} := \|\rho_{1,0}\|_{Z_\alpha}$ instead of R^c and R^h for the second summand, Remark 4.12 with Lemma 2.100 implies

$$\begin{aligned}&\big\|G''(u_1)D(\rho_1)[u_1 - u_2] - G''(u_0)D(0)[u_1 - u_2]\big\|_{\mathbb{E}_{0,T}}\\
&\le \big\|\big(G''(u_1)D(\rho_1) - G''(u_0)D(\rho_{0,1})\big)[u_1 - u_2]\big\|_{\mathbb{E}_{0,T}}\\
&\qquad + \big\|\big(G''(u_0)D(\rho_{0,1}) - G''(u_0)D(0)\big)[u_1 - u_2]\big\|_{\mathbb{E}_{0,T}}\\
&\le C(R^\Sigma, R^c, R^h)\big(\|u_1 - u_0\|_{h^\beta([0,T],Y_\alpha)} + \|\rho_1 - \rho_{0,1}\|_{h^\beta([0,T],Y_\alpha)}\big)\|u_1 - u_2\|_{h^\beta([0,T],Z_\alpha)}\\
&\qquad + C(R^\Sigma, \|u_0\|_{Z_\alpha}, \|\rho_{0,1}\|_{Z_\alpha})\|\rho_{0,1}\|_{Y_\alpha}\|u_1 - u_2\|_{h^\beta([0,T],Z_\alpha)}.\end{aligned}$$

So, together, we have

$$
\begin{aligned}
&\big\|G^c_{\rho_{0,1}}(u_1)-G^c_{\rho_{0,2}}(u_2)\big\|_{\mathbb{E}_{0,T}}\\
&\le C(R^\Sigma,R^c,R^h)\big(\|u_1-u_2\|_{h^\beta([0,T],Y_\alpha)}+\|\rho_1-\rho_2\|_{h^\beta([0,T],Z_\alpha)}\big)\\
&\quad+C(R^\Sigma,R^c,R^h)\big(\|u_1-u_0\|_{h^\beta([0,T],Y_\alpha)}+\|\rho_1-\rho_{0,1}\|_{h^\beta([0,T],Y_\alpha)}\big)\|u_1-u_2\|_{h^\beta([0,T],Z_\alpha)}\\
&\quad+C(R^\Sigma,\|u_0\|_{Z_\alpha},\delta_1)\delta_0\|u_1-u_2\|_{h^\beta([0,T],Z_\alpha)}.
\end{aligned}
$$

For the further estimate, we use Remark 2.89 und Lemma 2.90 and choose $\gamma\in(0,1)$ with $\gamma>\beta$ as the exponent from Lemma 2.90. We obtain

$$
\begin{aligned}
&\big\|G^c_{\rho_{0,1}}(u_1)-G^c_{\rho_{0,2}}(u_2)\big\|_{\mathbb{E}_{0,T}}\\
&\le C(R^\Sigma,R^c,R^h)\left(T^{\gamma-\beta}\|u_1-u_2\|_{\mathbb{E}_{1,T}}+\|\rho_1-\rho_2\|_{h^\beta([0,T],Z_\alpha)}\right)\\
&\quad+C(R^\Sigma,R^c,R^h)T^{\gamma-\beta}\big(\|u_1-u_0\|_{\mathbb{E}_{1,T}}+\|\rho_1-\rho_{0,1}\|_{\mathbb{E}_{1,T}}\big)\|u_1-u_2\|_{h^\beta([0,T],Z_\alpha)}\\
&\quad+C(R^\Sigma,\|u_0\|_{Z_\alpha},\delta_1)\delta_0\|u_1-u_2\|_{h^\beta([0,T],Z_\alpha)}\\
&\le C(R^\Sigma,R^c,R^h)\big(T^{\gamma-\beta}\|u_1-u_2\|_{\mathbb{E}_{1,T}}+\|\rho_1-\rho_2\|_{\mathbb{E}_{1,T}}\big)\\
&\quad+C(R^\Sigma,\|u_0\|_{Z_\alpha},\delta_1)\delta_0\|u_1-u_2\|_{\mathbb{E}_{1,T}}.
\end{aligned}
$$

Finally, due to Proposition 4.19,

$$
\begin{aligned}
\big\|G^c_{\rho_{0,1}}(u_1)-G^c_{\rho_{0,2}}(u_2)\big\|_{\mathbb{E}_{0,T}}\le{}& C(R^\Sigma,R^c,R^h,\Lambda^h,\delta_1)\big(T^\varepsilon\|u_1-u_2\|_{\mathbb{E}_{1,T}}+\|\rho_{0,1}-\rho_{0,2}\|_{Z_{2\beta+\alpha}}\big)\\
&+C(R^\Sigma,\|u_0\|_{Z_\alpha},\delta_1)\delta_0\|u_1-u_2\|_{\mathbb{E}_{1,T}}
\end{aligned}
$$

holds with an $\varepsilon>0$. □

The preparatory work above enables us to prove the short-time existence result for the second equation (4.1b).

Theorem 4.26. *We suppose Assumptions 4.9 are valid and use Notations 4.10. Therein, choose $R^c=R^c\big(R^\Sigma,u_0,\delta_1\big)>0$ sufficiently large and then, depending on this R^c, choose $R^h=R^h\big(R^c,u_0,\delta_1\big)>0$ as large as in Theorem 4.18. Also, choose $\delta_0=\delta_0\big(R^\Sigma,u_0,\delta_1\big)>0$ and $T=T\big(R^\Sigma,R^c,R^h,u_0,\delta_0,\delta_1\big)>0$ sufficiently small, but at least as small as in Theorem 4.18. For any initial value $\rho_0\in Z_{2\beta+\alpha}$ with $\|\rho_0\|_{Z_{2\beta+\alpha}}<\delta_1$ and $\|\rho_0\|_{Y_\alpha}<\delta_0$, there exists a unique solution $u:=u_{\rho_0}\in M^c_T$ of*

$$
\begin{cases}
\partial_t u &= \Delta_{\rho_u}G'(u)+g(u)a(\rho_u)H(\rho_u)\nu_\Sigma\cdot\nabla_{\rho_u}u+g(u)H(\rho_u)^2u \quad \text{in } \mathbb{E}_{0,T},\\
u(0) &= u_0 \quad \text{in } Z_\alpha,
\end{cases}
$$

where $\rho_u:=\rho_{u,\rho_0}\in M^h_{T,\rho_0}$ is the solution from Theorem 4.18 associated with the concentration u and the initial value ρ_0.

Proof. We show the existence of a unique solution $u\in M^c_T$ of

$$
\begin{aligned}
&\begin{cases}
\partial_t u &= \Delta_{\rho_u}G'(u)+g(u)a(\rho_u)H(\rho_u)\nu_\Sigma\cdot\nabla_{\rho_u}u+g(u)H(\rho_u)^2u \ \text{ in } \mathbb{E}_{0,T}\\
u(0) &= u_0 \quad \text{in } Z_\alpha
\end{cases}\\
&\Leftrightarrow L^c[u]=\begin{pmatrix}G^c_{\rho_0}(u)\\ u_0\end{pmatrix} \text{ in } \mathbb{E}_{0,T}\times Z_\alpha. \qquad (4.4)
\end{aligned}
$$

Equation (4.4) is well-defined because $A^c[u], G^c_{\rho_0}(u) \in \mathbb{E}_{0,T}$ holds for $u \in M^c_T$ by Corollary 4.21. Due to Lemma 4.24(i) and Proposition 4.23 it is equivalent to prove the existence of a unique $u \in M^c_T$ with

$$L^c[u] = \begin{pmatrix} G^c_{\rho_0}(u) \\ u_0 \end{pmatrix} \text{ in } (\mathbb{E}_{0,T} \times Z_\alpha)^c_+ \qquad \Leftrightarrow \qquad u = \big(L^c\big)^{-1} \begin{pmatrix} G^c_{\rho_0}(u) \\ u_0 \end{pmatrix} =: K^c_{\rho_0}(u) \text{ in } \mathbb{E}_{1,T}.$$

So, we show that $K^c_{\rho_0} : M^c_T \subset \mathbb{E}_{1,T} \to \mathbb{E}_{1,T}$ has a unique fixed point $u \in M^c_T$ using the Banach fixed-point theorem.

Step 1: Due to Lemma 4.24(i) and Proposition 4.23, $K^c_{\rho_0}(u) \in \mathbb{E}_{1,T}$ is well-defined for $u \in M^c_T$.

Step 2: We have to verify that $K^c_{\rho_0}$ is a contraction on M^c_T. For any $u_1, u_2 \in M^c_T$

$$\begin{aligned}
&\|K^c_{\rho_0}(u_1) - K^c_{\rho_0}(u_2)\|_{\mathbb{E}_{1,T}} \\
&\le \Lambda^c \big\| \big(G^c_{\rho_0}(u_1), u_0\big) - \big(G^c_{\rho_0}(u_2), u_0\big) \big\|_{(\mathbb{E}_{0,T} \times Z_\alpha)^c_+} \\
&= \Lambda^c \|G^c_{\rho_0}(u_1) - G^c_{\rho_0}(u_2)\|_{\mathbb{E}_{0,T}} \\
&\le \Big(C(R^\Sigma, R^c, R^h, \Lambda^c, \Lambda^h, \delta_1) T^\varepsilon + C(R^\Sigma, u_0, \delta_1, \Lambda^c)\delta_0 \Big) \|u_1 - u_2\|_{\mathbb{E}_{1,T}}
\end{aligned}$$

holds by Proposition 4.23, Lemma 4.24(i) as well as Proposition 4.25. For sufficiently small $\delta_0 > 0$ and sufficiently small $T > 0$,

$$\|K^c_{\rho_0}(u_1) - K^c_{\rho_0}(u_2)\|_{\mathbb{E}_{1,T}} \le \frac{1}{4} \|u_1 - u_2\|_{\mathbb{E}_{1,T}}$$

follows. Because Λ^c and Λ^h only depend on u_0, δ_0 only depends on R^Σ, u_0 and δ_1 whereas T only depends on R^Σ, R^c, R^h, u_0 and δ_1.

Step 3: We have to show that $K^c_{\rho_0} : M^c_T \to M^c_T$ is a self-mapping. Any $u \in M^c_T$ fulfills $\big(K^c_{\rho_0}(u)\big)(0) = u_0$ in Z_α because $w := K^c_{\rho_0}(u)$ is a solution to

$$L^c w = \begin{pmatrix} [L^c w]_1 \\ w(0) \end{pmatrix} = \begin{pmatrix} G^c_{\rho_0}(u) \\ u_0 \end{pmatrix} \text{ in } \mathbb{E}_{0,T} \times Z_\alpha.$$

Furthermore, we have

$$\begin{aligned}
\|K^c_{\rho_0}(u)\|_{\mathbb{E}_{1,T}} &\le \|K^c_{\rho_0}(u_0)\|_{\mathbb{E}_{1,T}} + \|K^c_{\rho_0}(u) - K^c_{\rho_0}(u_0)\|_{\mathbb{E}_{1,T}} \\
&\le \Lambda^c \big\| \big(G^c_{\rho_0}(u_0), u_0\big) \big\|_{(\mathbb{E}_{0,T} \times Z_\alpha)^c_+} + \frac{1}{4} \|u - u_0\|_{\mathbb{E}_{1,T}} \\
&\le \Lambda^c N^c + \frac{1}{4} \left(\|u\|_{\mathbb{E}_{1,T}} + 2\|u_0\|_{Z_\alpha} \right) \\
&\le \frac{R^c}{2} + \frac{R^c}{2} = R^c,
\end{aligned}$$

where the first summand is bounded by Proposition 4.23 and Lemma 4.24(ii) and the second summand by the contraction-property (see step 2). R^c being sufficiently large thus means $R^c \ge 2\Lambda^c N^c$ and because Λ^c only depends on u_0 and N^c only depends on R^Σ, u_0 and δ_1, we have $R^c = R^c\big(R^\Sigma, u_0, \delta_1\big)$. The two properties just deduced imply $K^c_{\rho_0}(u) \in M^c_T$ for all $u \in M^c_T$.

As M_T^c is a closed and non-empty subset of the Banach space $\mathbb{E}_{1,T}$, the Banach fixed-point theorem yields the existence of a unique $u \in M_T^c$ with

$$K^c_{\rho_0}(u) = u \text{ in } \mathbb{E}_{1,T} \quad \Leftrightarrow \quad \begin{cases} \partial_t u &= \Delta_{\rho_u} G'(u) + g(u)a(\rho_u)H(\rho_u)\nu_\Sigma \cdot \nabla_{\rho_u} u + g(u)H(\rho_u)^2 u \text{ in } \mathbb{E}_{0,T}, \\ u(0) &= u_0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } Z_\alpha. \end{cases} \qquad \square$$

As in Section 4.1, we now focus on properties of the solution u_{ρ_0} to the second equation (4.1b). To be precise, analogous statements to Propositions 4.19 and 4.20 are shown, analyzing the dependence of u_{ρ_0} on the initial value ρ_0 for the height function and providing an improved regularity in space for the solution.

Proposition 4.27. *We suppose that Assumptions 4.9 are valid and use Notations 4.10. Therein, choose $R^c > 0$, $R^h > 0$ as large and choose $\delta_0 > 0$, $T > 0$ as small as in Theorem 4.26. For initial values $\rho_{0,1}, \rho_{0,2} \in Z_{2\beta+\alpha}$ with $\|\rho_{0,i}\|_{Z_{2\beta+\alpha}} < \delta_1$ and $\|\rho_{0,i}\|_{Y_\alpha} < \delta_0$,*

$$\|u_1 - u_2\|_{\mathbb{E}_{1,T}} \le C(R^\Sigma, R^c, R^h, \Lambda^c, \Lambda^h, \|u_0\|_{Z_{2\beta+\alpha}}, \delta_1)\|\rho_{0,1} - \rho_{0,2}\|_{Z_{2\beta+\alpha}}$$

holds, where $u_i := u_{\rho_{0,i}} \in M_T^c$ is the solution from Theorem 4.26 associated with $\rho_{0,i}$, respectively.

Proof. As $u_i \in M_T^c$ is the solution from Theorem 4.26 associated with $\rho_{0,i}$, it is a fixed point of $(L^c)^{-1}\big(G^c_{\rho_{0,i}}(\cdot), u_0\big)$ as in the proof of Theorem 4.26. Therefore, we have

$$\begin{aligned} \|u_1 - u_2\|_{\mathbb{E}_{1,T}} &\le \Lambda^c \Big\| \big(G^c_{\rho_{0,1}}(u_1), u_0\big) - \big(G^c_{\rho_{0,2}}(u_2), u_0\big)\Big\|_{(\mathbb{E}_{0,T}\times Z_\alpha)^c_+} \\ &= \Lambda^c \big\|G^c_{\rho_{0,1}}(u_1) - G^c_{\rho_{0,2}}(u_2)\big\|_{\mathbb{E}_{0,T}} + \Lambda^c \Big\|\big(G^c_{\rho_{0,1}}(u_1)\big)(0) - \big(G^c_{\rho_{0,2}}(u_2)\big)(0)\Big\|_{X_{2\beta+\alpha}} \end{aligned}$$

by Proposition 4.23. With $\delta_0 > 0$ and $T > 0$ as small as in Theorem 4.26, Proposition 4.25 yields

$$\Lambda^c \big\|G^c_{\rho_{0,1}}(u_1) - G^c_{\rho_{0,2}}(u_2)\big\|_{\mathbb{E}_{0,T}} \le \frac{1}{4}\|u_1 - u_2\|_{\mathbb{E}_{1,T}} + C(R^\Sigma, R^c, R^h, \Lambda^c, \Lambda^h, \delta_1)\|\rho_{0,1} - \rho_{0,2}\|_{Z_{2\beta+\alpha}}.$$

We have

$$\begin{aligned} &\Big\|\Big(\big(G^c_{\rho_{0,1}}(u_1)\big)(0) - \big(G^c_{\rho_{0,2}}(u_2)\big)(0)\Big) - \Big(G''(u_0)J(\rho_{0,1})[u_0] - G''(u_0)J(\rho_{0,2})[u_0]\Big)\Big\|_{X_{2\beta+\alpha}} \\ &= \Big\|G''(u_0)D(\rho_{0,1})[u_0] - G''(u_0)D(\rho_{0,2})[u_0] + G'''(u_0)\big|\nabla_{\rho_{0,1}} u_0\big|^2 - G'''(u_0)\big|\nabla_{\rho_{0,2}} u_0\big|^2 \\ &\qquad + g(u_0)a(\rho_{0,1})H(\rho_{0,1})\nu_\Sigma \cdot \nabla_{\rho_{0,1}} u_0 - g(u_0)a(\rho_{0,2})H(\rho_{0,2})\nu_\Sigma \cdot \nabla_{\rho_{0,2}} u_0 \\ &\qquad + g(u_0)H(\rho_{0,1})^2 u_0 - g(u_0)H(\rho_{0,2})^2 u_0\Big\|_{X_{2\beta+\alpha}} \end{aligned}$$

according to Lemma 4.24(i). Due to $u_0 \in M_T^c$ and $\rho_{0,i} \in M_T^h$, we thus can use appropriate triangle inequalities (as in the proof of Proposition 4.19) and Remark 4.12 together with Lemma 2.100 to bound

$$\begin{aligned} &\Big\|\Big(\big(G^c_{\rho_{0,1}}(u_1)\big)(0) - \big(G^c_{\rho_{0,2}}(u_2)\big)(0)\Big) - \Big(G''(u_0)J(\rho_{0,1})[u_0] - G''(u_0)J(\rho_{0,2})[u_0]\Big)\Big\|_{X_{2\beta+\alpha}} \\ &\le C(R^\Sigma, R^c, R^h)\|\rho_{0,1} - \rho_{0,2}\|_{Z_{2\beta+\alpha}} C\big(\|u_0\|_{Z_{2\beta+\alpha}}, \|\rho_{0,1}\|_{Z_{2\beta+\alpha}}, \|\rho_{0,2}\|_{Z_{2\beta+\alpha}}\big) \\ &\le C(R^\Sigma, R^c, R^h, \|u_0\|_{Z_{2\beta+\alpha}}, \delta_1)\|\rho_{0,1} - \rho_{0,2}\|_{Z_{2\beta+\alpha}}. \end{aligned}$$

(As u_0 and $\rho_{0,i}$ are independent of t, there is also no time dependence in the application of Remark 4.12 in the estimate above.) Furthermore, we have $\rho_{0,i} \in U^h_{1+2\beta+\alpha,1} \cap B^{Z_{2\beta+\alpha}}_{\delta_1}(0)$ and therefore a last application of Remark 4.12 and Lemma 2.100 yields

$$\left\| G''(u_0)J(\rho_{0,1})[u_0] - G''(u_0)J(\rho_{0,2})[u_0] \right\|_{X_{2\beta+\alpha}} \le C(R^\Sigma, R^c, \delta_1)\|\rho_{0,1} - \rho_{0,2}\|_{Z_{2\beta+\alpha}}\|u_0\|_{Y_{2\beta+\alpha}}.$$

Altogether, we thus have

$$\|u_1 - u_2\|_{\mathbb{E}_{1,T}} \le C(R^\Sigma, R^c, R^h, \Lambda^c, \Lambda^h, \|u_0\|_{Z_{2\beta+\alpha}}, \delta_1)\|\rho_{0,1} - \rho_{0,2}\|_{Z_{2\beta+\alpha}}. \qquad \square$$

Proposition 4.28. *We suppose that Assumptions 4.9 are valid and use Notations 4.10. Therein, choose $R^c > 0$, $R^h > 0$ as large and choose $\delta_0 > 0$, $T > 0$ as small as in Theorem 4.26. Let $\rho_0 \in Z_{2\beta+\alpha}$ with $\|\rho_0\|_{Z_{2\beta+\alpha}} < \delta_1$ and $\|\rho_0\|_{Y_\alpha} < \delta_0$ and let $u := u_{\rho_0} \in M^c_T$ be the solution from Theorem 4.26 associated with ρ_0. Then, $u(t) \in Z_{2\beta+\alpha}$ holds for all $t \in [0,T]$.*

Proof. By Proposition 4.22, A^c satisfies the conditions of Proposition 2.131 and therefore $u \in M^c_T \subset \mathbb{E}_{1,T}$ fulfills $\partial_t u(t) \in \mathcal{D}_{A^c}(\beta) = X_{2\beta+\alpha}$ for all $t \in [0,T]$. In addition, by Corollary 4.13, we have $\left(G'''(u)\left|\nabla_\rho u\right|^2\right)(t) \in X_{2\beta+\alpha}$ for all $t \in [0,T]$, where $\rho := \rho_{u,\rho_0} \in M^h_{T,\rho_0}$ is the solution from Theorem 4.18 associated with the concentration u and the initial value ρ_0. Hence,

$$\begin{aligned}
&A^c_{u(t),\rho(t)}\left[u(t)\right] \\
&= \left(\Delta_\rho G'(u) + g(u)a(\rho)H(\rho)\nu_\Sigma \cdot \nabla_\rho u + g(u)H(\rho)^2 u\right)(t) - \left(G'''(u)\left|\nabla_\rho u\right|^2\right)(t) \\
&= \partial_t u(t) - \left(G'''(u)\left|\nabla_\rho u\right|^2\right)(t) \in X_{2\beta+\alpha}
\end{aligned}$$

holds for all $t \in [0,T]$. Because $A^c_{u(t),\rho(t)} : Z_{s_i} \to X_{s_i}$ generates an analytic C^0-semigroup for both $s_1 = \alpha$, $s_2 = 2\beta + \alpha$ (see Proposition 4.22), Lemma 2.132 finally yields $u(t) \in Z_{2\beta+\alpha}$ for all $t \in [0,T]$. $\square$

4.3 Analytic Short-Time Existence

Combining the results from Sections 4.1 and 4.2 yields our full statement on short-time existence. We formulate it in a self-contained way, such that the reader does not have to look up Assumptions 4.9 or Notations 4.10 that were continually used above.

Theorem 4.29. *Let $\alpha \in (0,1)$ and $\beta \in (0,\frac{1}{2})$ with $2\beta + \alpha \notin \mathbb{N}$ and let $G \in C^7(\mathbb{R})$ with $G'' > 0$ and $g := G - G' \cdot \mathrm{Id} > 0$. Moreover, let $\Sigma = \bar{\theta}(M)$ be an $h^{4+\alpha}$-immersed closed hypersurface with unit normal ν_Σ. Let $u_0 \in h^{2+2\beta+\alpha}(M)$ and $\delta_1 > 0$ be arbitrary. Then, choose $\delta_0 = \delta_0(\Sigma, u_0, \delta_1) > 0$ and $T = T(\Sigma, u_0, \delta_1) > 0$ sufficiently small. For every function $\rho_0 \in h^{2+2\beta+\alpha}(M)$ with $\|\rho_0\|_{C^{2+2\beta+\alpha}(M)} < \delta_1$ and $\|\rho_0\|_{C^{1+\alpha}(M)} < \delta_0$, there exists a solution (ρ, u) with $\rho, u \in \mathbb{E}_{1,T} := h^{1+\beta}\big([0,T], h^\alpha(M)\big) \cap h^\beta\big([0,T], h^{2+\alpha}(M)\big)$ to*

$$\begin{cases}
\partial_t \rho &= g(u)a(\rho)H(\rho) & \text{in } h^\beta\big([0,T], h^\alpha(M)\big), \\
\partial_t u &= \Delta_\rho G'(u) + g(u)a(\rho)H(\rho)\nu_\Sigma \cdot \nabla_\rho u + g(u)H(\rho)^2 u & \text{in } h^\beta\big([0,T], h^\alpha(M)\big), \\
\rho(0) &= \rho_0 & \text{in } h^{2+\alpha}(M), \\
u(0) &= u_0 & \text{in } h^{2+\alpha}(M).
\end{cases}$$

Furthermore, $\rho(t), u(t) \in h^{2+2\beta+\alpha}(M)$ as well as $\|\rho(t)\|_{C^{1+\alpha}(M)} < R^{\Sigma}$ hold for all $t \in [0,T]$ and there exists a constant $R = R(\Sigma, u_0, \delta_1) > 0$ independent of ρ_0 with $\|\rho\|_{\mathbb{E}_{1,T}}, \|u\|_{\mathbb{E}_{1,T}} \leq R$. For any two solutions, there exists $\overline{T} \in (0,T]$ such that the solutions coincide on $[0,\overline{T}]$.

Proof. For sufficiently small $R^{\Sigma} > 0$ and sufficiently large $R^c, R^h > 0$, choosing $\delta_0 > 0$ and $T > 0$ sufficiently small satisfies Assumptions 4.9 and the conditions of Theorems 4.18 and 4.26. The existence of a solution (ρ, u) with $\rho, u \in \mathbb{E}_{1,T}$ then follows directly from Theorems 4.18 and 4.26. With $R \coloneqq \max\{R^c, R^h\}$, we have $\|\rho\|_{\mathbb{E}_{1,T}}, \|u\|_{\mathbb{E}_{1,T}} \leq R$, where R^c, R^h and thus also R only depend on Σ, u_0 and δ_1 (see Theorems 4.18 and 4.26). The property $\rho(t), u(t) \in h^{2+2\beta+\alpha}(M)$ for all $t \in [0,T]$ is due to Propositions 4.20 and 4.28 and

$$\|\rho(t)\|_{C^{1+\alpha}(M)} \leq \frac{\|\rho(t) - \rho(0)\|_{C^{1+\alpha}(M)}}{|t-0|^{\beta}} T^{\beta} + \|\rho_0\|_{C^{1+\alpha}(M)} \leq \|\rho\|_{\mathbb{E}_{1,T}} T^{\beta} + \delta_0 < R^{\Sigma}$$

follows with Estimate (4.2).

To prove the stateted uniqueness property of the solution, assume that there exists a second solution $(\tilde{\rho}, \tilde{u})$ with $\tilde{\rho}, \tilde{u} \in \mathbb{E}_{1,T}$. Choose R^c and R^h as large as in Theorem 4.26, but at least as large such that $\|u\|_{\mathbb{E}_{1,T}}, \|\tilde{u}\|_{\mathbb{E}_{1,T}} \leq R^c$ and $\|\rho\|_{\mathbb{E}_{1,T}}, \|\tilde{\rho}\|_{\mathbb{E}_{1,T}} \leq R^h$ hold. Then, choose $\overline{T} > 0$ as small as in Theorem 4.26 but at least as small such that $\overline{T} \leq T$ holds. As δ_0 is independent of R^c and R^h, the conditions of Theorems 4.18 and 4.26 are satisfied. We hence receive a unique solution in

$$M_{\overline{T}} \coloneqq \left\{ (\bar{\rho}, \bar{u}) \in \mathbb{E}_{1,\overline{T}} \times \mathbb{E}_{1,\overline{T}} \,\middle|\, \|\bar{\rho}\|_{\mathbb{E}_{1,\overline{T}}} \leq R^h \text{ and } \|\bar{u}\|_{\mathbb{E}_{1,\overline{T}}} \leq R^c \right\}.$$

As we have $(\rho, u), (\tilde{\rho}, \tilde{u}) \in M_{\overline{T}}$, the two solutions coincide on $[0, \overline{T}]$. □

If we could apply a continuation argument to the two solutions (ρ, u) and $(\tilde{\rho}, \tilde{u})$ from the proof above, we could show that they coincide on the full time interval $[0,T]$ and thus obtain uniqueness of the solution. For this, we would need to ensure that for a solution (ρ, u) at any time t, the pair $\big(\rho(t), u(t)\big)$ fulfills the conditions for the initial values in Theorem 4.29. In particular, $\rho(t)$ needs to be bounded by $\delta_0\big(u(t)\big)$ in the appropriate norm. To achieve this, the dependence of δ_0 on u_0 should be controlled in a uniform way. We plan to analyze this in future work.

Chapter 5

Properties of Solutions

We consider a system consisting of an evolving closed hypersurface Γ and a concentration $c : \Gamma \to \mathbb{R}$ defined on this evolving hypersurface that satisfy the equations

$$\begin{cases} V &= g(c)H, \\ \partial^{\square} c &= \Delta_{\Gamma} G'(c) + cVH. \end{cases} \tag{5.1}$$

Again, $V = V_{\Gamma}$ is the normal velocity of the evolving hypersurface (see Definition 2.52) and $H = H_{\Gamma}$ is that function on the evolving hypersurface for which $H(t)$ is the mean curvature of the hypersurface $\Gamma(t)$ (see Definition 2.41). We denote the normal time derivative with $\partial^{\square}$ and the Laplace-Beltrami operator with Δ_{Γ} (see Definitions 2.54 and 2.33, respectively).
In Chapter 4, we proved that for sufficiently smooth initial data, i.e., a closed hypersurface $\Gamma_0 \subset \mathbb{R}^{d+1}$ parameterized suitably over a reference surface and a concentration $c_0 : \Gamma_0 \to \mathbb{R}$, as well as sufficiently smooth $G : \mathbb{R} \to \mathbb{R}$ with $G'' > 0$ and $g \coloneqq G - G' \cdot \mathrm{Id} > 0$, there exists a short time-solution to (5.1) or, in other words, an evolution starts. Now, we discuss some properties of such solutions.

5.1 Conservation of Mean Convexity

A solution of (5.1) conserves its mean convexity: If the initial hypersurface Γ_0 is mean convex, then the evolving hypersurface remains mean convex for all further times. To show this, we want to apply maximum principles to $w = -H$. On account of Remark 5.3, maximum principles without (sign) conditions on the zeroth-order term are necessary. Hence, the typical weak maximum principles as in [Eck12, Proposition 3.1] and [Eva10, §7.1: Theorem 8 and Theorem 9] (or rather their transfer to hypersurfaces) can not be used. Instead, we derived a suitable weak maximum principle in Section 2.4 (see Proposition 2.143). A strong maximum principle without assumptions on the zeroth-order term can be found in the literature for domains in $\mathbb{R}^d$ (see [RR06, Theorem 4.26]) and is transferred to hypersurfaces in Proposition 2.147.

We prove the conservation of mean convexity not only for hypersurfaces that evolve by the scaled mean curvature flow but also for more general evolving hypersurfaces that satisfy the following assumptions:

Assumptions 5.1. *Let Γ be a $(C^2\text{-}\,C^2)\cap(C^1\text{-}\,C^4)$-evolving immersed closed hypersurface with reference surface $M\subset\mathbb{R}^{d+1}$, normal ν and mean curvature H that evolves with normal velocity $V=V(H)$. Furthermore, we assume*

$$\Delta_\Gamma V + V\left|\nabla_\Gamma \nu\right|^2 = A\colon D_\Gamma^2 H + B\cdot\nabla_\Gamma H + CH \ \text{ on } [0,T]\times M$$

with continuous $A:[0,T]\times M\to\mathbb{R}^{(d+1)\times(d+1)}$, $B:[0,T]\times M\to\mathbb{R}^{d+1}$ and $C:[0,T]\times M\to\mathbb{R}$ such that A is symmetric and positive definite on $[0,T]\times M$.

Here, ∇_Γ, Δ_Γ and D_Γ^2 denote the surface gradient, the Laplace-Beltrami operator and the surface Hessian, respectively, introduced in Definitions 2.31, 2.33 and 2.34. As in Definition 2.141, $A\colon\widetilde{A}:=\sum_{i,j}A_{ij}\widetilde{A}_{ij}$ is the inner product of two matrices A and $\widetilde{A}$. With Proposition 2.57, we have

$$\partial^\square H = \Delta_\Gamma V + V\left|\nabla_\Gamma\nu\right|^2.$$

The regularity of the evolving hypersurface guarantees in particular that all occuring derivatives of the mean curvature are well-defined as Remark 2.56 implies

$$H\in C^1\big([0,T],C^0(M)\big)\cap C^0\big([0,T],C^2(M)\big).$$

Under these conditions, mean convexity of the hypersurface is conserved. Even more, the mean curvature instantly turns strictly positive.

Theorem 5.2 (Conservation of Mean Convexity)**.** *Suppose Assumptions 5.1 are valid with $H(0)\geq 0$ on M. Then, $H(t)>0$ holds on M for all $t\in(0,T]$.*

Proof. We define $w:=-H$ as well as

$$\mathcal{L}w := -\partial^\square w + A\colon D_\Gamma^2 w + B\cdot\nabla_\Gamma w + Cw$$

so that Assumptions 2.140 are satisfied. We have

$$\mathcal{L}w = -\mathcal{L}H = \partial^\square H - \left(\Delta_\Gamma V + V\left|\nabla_\Gamma\nu\right|^2\right) = 0 \text{ on } [0,T]\times M$$

and $w(0)=-H(0)\leq 0$ on M. With Corollary 2.148, $H(t)=-w(t)\geq 0$ follows on M for all $t\in[0,T]$ and there exists $t_0\in[0,T]$ with

$$\begin{aligned} &H(t)=-w(t)=0 \text{ on } M \text{ for all } t\in(0,t_0] \text{ and}\\ &H(t)=-w(t)>0 \text{ on } M \text{ for all } t\in(t_0,T]. \end{aligned}$$

As $\Gamma(t)=\theta_t(M)$ is a closed hypersurface, $H(t)=0$ can not hold on the whole surface M (see Proposition 2.44). Hence, we have $t_0=0$ and then $H(t)>0$ follows on M for all $t\in(0,T]$. □

In the following remark, we state that the scaled mean curvature flow $V=g(c)H$ satisfies Assumptions 5.1. In particular, it thus conserves mean convexity and any initially mean convex hypersurface turns strictly mean convex instantly.

Remark 5.3. *Let Γ be a $(C^2\text{-}\,C^2)\cap(C^1\text{-}\,C^4)$-evolving immersed closed hypersurface with reference surface $M\subset\mathbb{R}^{d+1}$, normal ν and mean curvature H that evolves with normal velocity*

$$V=g(c)H,$$

where $g\in C^2(\mathbb{R})$ with $g>0$ and $c\in C^0\big([0,T],C^2(M)\big)$. (Notably, this is fulfilled for the usual mean curvature flow with $g\equiv 1$.) Then, Assumptions 5.1 are satisfied. In particular, we have

$$\begin{aligned}
&\Delta_\Gamma V+V\left|\nabla_\Gamma\nu\right|^2\\
&=\Delta_\Gamma\big(g(c)H\big)+g(c)\left|\nabla_\Gamma\nu\right|^2H\\
&=\operatorname{div}_\Gamma\Big(\nabla_\Gamma\big(g(c)H\big)\Big)+g(c)\left|\nabla_\Gamma\nu\right|^2H\\
&=\operatorname{div}_\Gamma\Big(g'(c)\nabla_\Gamma c\,H+g(c)\nabla_\Gamma H\Big)+g(c)\left|\nabla_\Gamma\nu\right|^2H\\
&=g''(c)|\nabla_\Gamma c|^2H+g'(c)\Delta_\Gamma c\,H+2g'(c)\nabla_\Gamma c\cdot\nabla_\Gamma H+g(c)\Delta_\Gamma H+g(c)\left|\nabla_\Gamma\nu\right|^2H\\
&=g(c)\Delta_\Gamma H+2g'(c)\nabla_\Gamma c\cdot\nabla_\Gamma H+\Big(g(c)\left|\nabla_\Gamma\nu\right|^2+g'(c)\Delta_\Gamma c+g''(c)|\nabla_\Gamma c|^2\Big)H\\
&=A\colon D_\Gamma^2H+B\cdot\nabla_\Gamma H+CH
\end{aligned}$$

with

$$\begin{aligned}
A&\coloneqq g(c)\mathrm{Id},\\
B&\coloneqq 2g'(c)\nabla_\Gamma c,\\
C&\coloneqq g(c)\left|\nabla_\Gamma\nu\right|^2+g'(c)\Delta_\Gamma c+g''(c)|\nabla_\Gamma c|^2.
\end{aligned}$$

The assumptions and $\nu\in C^0\big([0,T],C^1(M,\mathbb{R}^{d+1})\big)$ by Proposition 2.51 imply continuity of A,B and C on $[0,T]\times M$. Moreover, A is cleary symmetric and as $g>0$ holds also positive definite on $[0,T]\times M$.

5.2 Non-Conservation of Convexity

Huisken showed in [Hui84, Theorem 4.3] that, in addition to mean convexity, the usual mean curvature flow also conserves convexity. This result can not be transferred to the scaled mean curvature flow: In contrast to the usual mean curvature flow where we have $\partial^\square H=\Delta_\Gamma H+\left|\nabla_\Gamma\nu\right|^2H$, first order derivatives of H occur in

$$\partial^\square H=g(c)\Delta_\Gamma H+2\nabla_\Gamma g(c)\cdot\nabla_\Gamma H+\big(\Delta_\Gamma g(c)+\left|\nabla_\Gamma\nu\right|^2g(c)\big)H$$

for the scaled mean curvature flow. In the evolution equation

$$\partial_t h_{ij}=g\Delta h_{ij}+\sum_{k,l}g^{kl}\nabla_i g\nabla_j h_{kl}+\text{ lower order terms}$$

for the second fundamental form $[h_{ij}]$ (cf. [Hui84, Theorem 3.4] for the usual mean curvature flow), they produce additional first order terms in various directions. Therefore,

Hamiltons maximum principle ([Hui84, Theorem 4.1]) can not be applied to the second fundamental form and thus we can not conclude the conservation of convexity as in [Hui84, Theorem 4.3].
Actually, it is possible to construct examples of closed hypersurfaces evolving with the scaled mean curvature flow that loose their convexity in the course of time and we will do so in the following. Obviously, the dimension of the surface has to fulfill $d > 1$, as otherwise convexity and mean convexity coincide and the latter is conserved by Section 5.1. We choose $d = 2$ and consider a rotationally symmetric structure for the hypersurface and the concentration defined thereon. However, this rotationally symmetric setting can be constructed in dimensions $d > 2$ analogously. The idea of the construction is a hypersurface shaped as a long cylinder, whose first principal curvature is 0 but the second is positive. Together, the surface thus has positive mean curvature and will shrink under the scaled mean curvature flow. A clever choice of the concentration forces the cylinder to shrink faster in the middle than at the ends, which turns the first principal curvature negative and therefore makes the surface non-convex.
We now construct this example explicitly. Fix an energy density $G \in C^7(\mathbb{R})$ with $G'' > 0$ and $g \coloneqq G - G' \cdot \mathrm{Id} > 0$. Let $1 \le x_0 < x_1 < x_2 < x_3 < x_4 < x_5 \in \mathbb{R}$. We define the function $w_0 : [x_0 - 1, x_5 + 1] \to \mathbb{R}_{\ge 0}$ with

$$w_0(x) \coloneqq \begin{cases} \big(1 - (x_0 - x)^6\big)^{1/6}, & \text{if } x \in [x_0 - 1, x_0], \\ 1, & \text{if } x \in [x_0, x_5], \\ \big(1 - (x - x_5)^6\big)^{1/6}, & \text{if } x \in [x_5, x_5 + 1]. \end{cases}$$

In particular, $w_0 \in C^5\big((x_0 - 1, x_5 + 1)\big)$ holds. Moreover, we define the surface of revolution

$$\Sigma \coloneqq \left\{ \gamma(x, \varphi) \coloneqq \begin{bmatrix} x \\ w_0(x)\cos\varphi \\ w_0(x)\sin\varphi \end{bmatrix} \,\middle|\, x \in [x_0 - 1, x_5 + 1], \varphi \in [0, 2\pi] \right\}.$$

This Σ will be the reference surface as well as the inital surface. Furthermore, we choose $c_0 \in C^\infty\big([x_0 - 1, x_5 + 1]\big)$ with $c_{0,I} \le c_0(x) \le c_{0,O}$ for all $x \in [x_0 - 1, x_5 + 1]$ and

$$c_0(x) = \begin{cases} c_{0,O}, & \text{if } x \in [x_0 - 1, x_1], \\ c_{0,I}, & \text{if } x \in [x_2, x_3], \\ c_{0,O}, & \text{if } x \in [x_4, x_5 + 1] \end{cases}$$

for constant values $0 < c_{0,I} < c_{0,O}$. As $c_0 > 0$ and thus $g'(c_0) = -G''(c_0)c_0 < 0$, also $g(c_0) : [x_0 - 1, x_5 + 1] \to \mathbb{R}_{>0}$ holds with $g_I \coloneqq g(c_{0,I}) \ge g(c_0)(x) \ge g(c_{0,O}) \eqqcolon g_O$ for all $x \in [x_0 - 1, x_5 + 1]$ and

$$g(c_0)(x) = \begin{cases} g_O, & \text{if } x \in [x_0 - 1, x_1], \\ g_I, & \text{if } x \in [x_2, x_3], \\ g_O, & \text{if } x \in [x_4, x_5 + 1], \end{cases}$$

with $g_I > g_O > 0$. Finally, we define the rotationally symmetric function $u_0 : \Sigma \to \mathbb{R}$ with

$$u_0\big(\gamma(x, \varphi)\big) \coloneqq c_0(x)$$

for all $x \in [x_0 - 1, x_5 + 1]$ and $\varphi \in [0, 2\pi]$. An illustration of the initial data w_0 and $g(c_0)$ can be found in Figure 5.1.

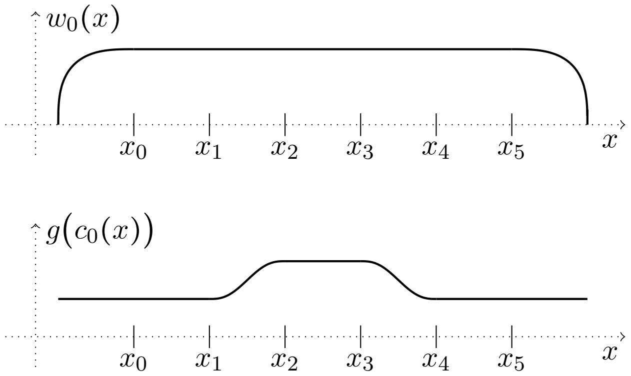


Figure 5.1: Plot of the initial data. The function w_0 generates the initial surface Γ_0 via revolution, which is shaped like a long cylinder with smoothly closed endings. The initial concentration c_0 is chosen such that scaling the mean curvature flow with $g(c_0)$ increases the velocity of the flow in the middle of the cylinder compared to the sides.

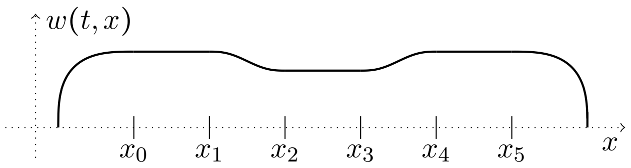


Figure 5.2: A possible shape of the function w at a time $t > 0$. The generated surface of revolution obviously is not convex.

Lemma 5.4. *Let the surface Σ and the functions w_0 and u_0 be as above. Then, Σ is a C^5-embedded closed and convex hypersurface of $\mathbb{R}^3$ and we have $u_0 \in C^5(\Sigma)$.*

Proof. Define the auxiliary function

$$\Phi(x, y, z) := (x - x_5)^6 + (y^2 + z^2)^{6/2}$$

for $[x, y, z] \in (x_5, \infty) \times \mathbb{R}^2$. Then, $\Phi \in C^\infty\big((x_5, \infty) \times \mathbb{R}^2\big)$ holds and 1 is a regular value of this function Φ. Hence,

$$\begin{aligned}
\Phi^{-1}(1) &= \Big\{[x, y, z] \in (x_5, \infty) \times \mathbb{R}^2 \,\Big|\, (y^2 - z^2)^{6/2} = 1 - (x - x_5)^6\Big\} \\
&= \Big\{[x, y, z] \in (x_5, x_5 + 1] \times \mathbb{R}^2 \,\Big|\, y^2 + z^2 = \big(1 - (x - x_5)^6\big)^{2/6} = w_0(x)^2\Big\} \\
&= \left\{ \begin{bmatrix} x \\ w_0(x)\cos\varphi \\ w_0(x)\sin\varphi \end{bmatrix} \,\middle|\, x \in (x_5, x_5 + 1], \varphi \in [0, 2\pi] \right\}
\end{aligned}$$

is a 2-dimensional C^∞-embedded submanifold of $\mathbb{R}^3$ (see Lemma 2.7). In particular, the "spherical shells" at the ends of Σ are C^∞-embedded submanifolds. As w_0 and thus also Σ are only of regularity C^5 in the "glueing points" x_0 and x_5, overall, Σ is a 2-dimensional C^5-embedded submanifold of $\mathbb{R}^3$. Clearly, Σ also is convex, compact, connected and orientable, such that Σ is a C^5-embedded closed and convex hypersurface of $\mathbb{R}^3$, and we have $u_0 \in C^5(\Sigma)$. □

Before we show that the evolution of the convex initial surface $\Gamma_0 \coloneqq \Sigma$ turns non-convex over time, as an auxiliary step, we derive formulas for the mean curvature and the normal velocity of surfaces of revolution.

Lemma 5.5 (Mean Curvature for Surfaces of Revolution)**.** *Let $a, b \in \mathbb{R}$ with $a < b$. Furthermore, let $w \in C^2([a,b])$ with $w(a) = w(b) = 0$ and $w > 0$ on (a,b) such that the surface of revolution*

$$\Gamma \coloneqq \left\{ \gamma(x,\varphi) \coloneqq \begin{bmatrix} x \\ w(x)\cos\varphi \\ w(x)\sin\varphi \end{bmatrix} \,\middle|\, x \in [a,b], \varphi \in [0,2\pi] \right\}$$

is a C^2-embedded closed hypersurface in $\mathbb{R}^3$. Its mean curvature in every point $\gamma(x,\varphi)$ with $x \in (a,b)$ and $\varphi \in [0,2\pi]$ is given by

$$H_{|(x,\varphi)} = \frac{1}{\sqrt{1+|w'(x)|^2}} \left(\frac{w''(x)}{1+|w'(x)|^2} - \frac{1}{w(x)} \right).$$

Proof. The tangential space of the hypersurface Γ in a point $\gamma(x,\varphi)$ with $x \in (a,b)$ and $\varphi \in [0,2\pi]$ is spanned by the vectors

$$\partial_x \gamma_{|(x,\varphi)} = \begin{bmatrix} 1 \\ w'(x)\cos\varphi \\ w'(x)\sin\varphi \end{bmatrix} \quad \text{and} \quad \partial_\varphi \gamma_{|(x,\varphi)} = \begin{bmatrix} 0 \\ -w(x)\sin\varphi \\ w(x)\cos\varphi \end{bmatrix}.$$

The (outer) unit normal in a point $\gamma(x,\varphi)$ with $x \in (a,b)$ and $\varphi \in [0,2\pi]$ thus is given by

$$\nu_{|(x,\varphi)} \coloneqq \frac{\partial_\varphi \gamma_{|(x,\varphi)} \times \partial_x \gamma_{|(x,\varphi)}}{\left\| \partial_\varphi \gamma_{|(x,\varphi)} \times \partial_x \gamma_{|(x,\varphi)} \right\|} = \frac{1}{\sqrt{1+|w'(x)|^2}} \begin{bmatrix} -w'(x) \\ \cos\varphi \\ \sin\varphi \end{bmatrix}.$$

For its differential $\mathrm{d}_{\gamma(x,\varphi)}\nu$ in the point $\gamma(x,\varphi)$, we have

$$-\mathrm{d}_{\gamma(x,\varphi)}\nu\big[\partial_x \gamma_{|(x,\varphi)}\big] = -\frac{\mathrm{d}}{\mathrm{d}\tau}_{|\tau=0} \nu(x+\tau,\varphi) = \frac{w''(x)}{\big(1+|w'(x)|^2\big)^{3/2}} \partial_x \gamma_{|(x,\varphi)}$$

and

$$-\mathrm{d}_{\gamma(x,\varphi)}\nu\big[\partial_\varphi \gamma_{|(x,\varphi)}\big] = -\frac{\mathrm{d}}{\mathrm{d}\tau}_{|\tau=0} \nu(x,\varphi+\tau) = \frac{-1}{w(x)} \frac{1}{\sqrt{1+|w'(x)|^2}} \partial_\varphi \gamma_{|(x,\varphi)}.$$

In particular, $\partial_x\gamma$ and $\partial_\varphi\gamma$ are eigenvectors of $-\mathrm{d}_{(\cdot)}\nu$ and the corresponding eigenvalues

$$\kappa_{1\,|(x,\varphi)} \coloneqq \frac{w''(x)}{\big(1+|w'(x)|^2\big)^{3/2}} \qquad \text{and} \qquad \kappa_{2\,|(x,\varphi)} \coloneqq \frac{-1}{w(x)} \frac{1}{\sqrt{1+|w'(x)|^2}}$$

are the principal curvatures of the surface Γ in $\gamma(x,\varphi)$. Hence, the mean curvature of Γ in $\gamma(x,\varphi)$ is given by

$$H_{|(x,\varphi)} = \kappa_{1\,|(x,\varphi)} + \kappa_{2\,|(x,\varphi)} = \frac{1}{\sqrt{1+|w'(x)|^2}} \left(\frac{w''(x)}{1+|w'(x)|^2} - \frac{1}{w(x)} \right). \qquad \square$$

Lemma 5.6 (Normal Velocity for Surfaces of Revolution)**.** *Let $T > 0$ and $a, b \in C^0([0,T])$ with $a(t) < b(t)$ for all $t \in [0,T]$. Moreover, for every $t \in [0,T]$, let $w(t) \in C^2\big([a(t), b(t)]\big)$ with $\partial_t w(t) \in C^0\big((a(t), b(t))\big)$ if $t \in (0,T)$ and $w(t, a(t)) = w\big(t, b(t)\big) = 0$ as well as $w(t) > 0$ on $\big(a(t), b(t)\big)$ such that the surface of revolution*

$$\Gamma \coloneqq \left\{ \{t\} \times \gamma(t,x,\varphi) \coloneqq \{t\} \times \begin{bmatrix} x \\ w(t,x)\cos\varphi \\ w(t,x)\sin\varphi \end{bmatrix} \,\middle|\, t \in [0,T], x \in \big[a(t), b(t)\big], \varphi \in [0, 2\pi] \right\}$$

is a C^1- C^2-evolving embedded closed hypersurface with $\Gamma(t) \subset \mathbb{R}^3$. Its normal velocity in every point $\big(t, \gamma(t,x,\varphi)\big)$ with $t \in (0,T)$, $x \in \big(a(t), b(t)\big)$ and $\varphi \in [0,2\pi]$ is given by

$$V_{|(t,x,\varphi)} = \frac{1}{\sqrt{1 + |\partial_x w(t,x)|^2}} \partial_t w(t,x).$$

Proof. The proof of Lemma 5.5 yields

$$\nu_{|(t,x,\varphi)} = \frac{1}{\sqrt{1 + |\partial_x w(t,x)|^2}} \begin{bmatrix} -\partial_x w(t,x) \\ \cos\varphi \\ \sin\varphi \end{bmatrix}$$

for the unit normal of Γ in a point $\big(t, \gamma(t,x,\varphi)\big)$ with $t \in (0,T)$, $x \in \big(a(t), b(t)\big)$ and $\varphi \in [0,2\pi]$. Hence, the normal velocity of Γ in $\big(t, \gamma(t,x,\varphi)\big)$ is given by

$$V_{|(t,x,\varphi)} = \nu_{|(t,x,\varphi)} \cdot \partial_t \gamma_{|(t,x,\varphi)} = \frac{1}{\sqrt{1 + |\partial_x w(t,x)|^2}} \partial_t w(t,x). \qquad \square$$

Theorem 5.7 (Non-Conservation of Convexity)**.** *Let the energy density G, the surface Σ and the function u_0 be as above. The initial hypersurface $\Gamma_0 \coloneqq \Sigma$ is a convex surface whose evolution under* (5.1) *with initial concentration u_0 does not stay convex.*

Proof. We know with Lemma 5.4 that Γ_0 is convex. So, we only have to show that its evolution under (5.1) with inital concentration u_0 turns non-convex.

Step 1: *Application of the short time existence result*
By assumption, we have $G \in C^7(\mathbb{R})$ with $G'' > 0$ and $g > 0$. Let $\alpha \in (0,1)$ and $\beta \in (0, \frac{1}{2})$ with $2\beta + \alpha \notin \mathbb{N}$. Then, with Lemma 5.4, $\Sigma \subset \mathbb{R}^3$ is an $h^{4+\alpha}$-embedded closed hypersurface and we denote its normal by ν_Σ. Moreover, we have $\rho_0 \coloneqq 0, u_0 \in h^{2+2\beta+\alpha}(\Sigma)$ with $\|\rho_0\|_{C^{2+2\beta+\alpha}(\Sigma)} < \delta_1$ and $\|\rho_0\|_{C^{2+\alpha}(\Sigma)} < \delta_0$ for any $\delta_0, \delta_1 > 0$. For $T > 0$ sufficiently small, Theorem 4.29 thus yields the existence of $\rho, u \in h^{1+\beta}\big([0,T], h^\alpha(\Sigma)\big) \cap h^\beta\big([0,T], h^{2+\alpha}(\Sigma)\big)$ with

$$\begin{cases} \partial_t \rho &=& g(u)a(\rho)H(\rho) & \text{on } [0,T] \times \Sigma, \\ \partial_t u &=& \Delta_\rho G'(u) + g(u)a(\rho)H(\rho)\nu_\Sigma \cdot \nabla_\rho u + g(u)H(\rho)^2 u & \text{on } [0,T] \times \Sigma, \\ \rho(0) &=& \rho_0 & \text{on } \Sigma, \\ u(0) &=& u_0 & \text{on } \Sigma. \end{cases}$$

With $\theta_\rho(t,z) := z+\rho(t,z)\nu_\Sigma(z)$ and $\Gamma_\rho(t) := \theta_\rho(t,\Sigma)$, hence the evolving hypersurface $\Gamma_\rho := \big\{\{t\}\times\Gamma_\rho(t)\,\big|\, t\in[0,T]\big\}$ and the function $u\circ\theta_\rho^{-1}$ are a solution of (5.1) with $\Gamma_\rho(0) = \Sigma = \Gamma_0$ and $u\circ\theta_\rho^{-1}(0) = u_0$. (For the notation concerning evolving hypersurfaces Γ_ρ parameterized via height functions ρ, we refer to Section 2.1.6, and in particular to Remark 2.64 for the definition of the pullback operators $\nabla_\rho, \Delta_\rho, H(\rho)$ and ν_ρ. As in Lemma 4.5(ii), we define $a(\rho) := \frac{1}{\nu_\Sigma\cdot\nu_\rho}$.) Due to the uniqueness property of the solution (see Theorem 4.29), the rotational symmetry of $\Sigma = \Gamma_0$ and u_0 implies that also $\Gamma_\rho(t)$ and $u(t)$ are rotationally symmetric. In particular, for every $t\in[0,T]$, there exists $w(t) : [\tilde{x_0}(t),\tilde{x_5}(t)]\to\mathbb{R}$ with

$$\Gamma_\rho(t) = \left\{\begin{bmatrix} x \\ w(t,x)\cos\varphi \\ w(t,x)\sin\varphi\end{bmatrix}\,\middle|\, x\in[\tilde{x_0}(t),\tilde{x_5}(t)], \varphi\in[0,2\pi]\right\}$$

and $c : [0,T]\times[x_0-1,x_5+1]\to\mathbb{R}$ with $u\big(t,\gamma(x,\varphi)\big) = c(t,x)$ for all $x\in[x_0-1,x_5+1]$ and $\varphi\in[0,2\pi]$. For $T>0$ sufficiently small, we can assume $[x_0,x_5]\subset[\tilde{x_0}(t),\tilde{x_5}(t)]$ for all $t\in[0,T]$ as well as $w>0$ on $[0,T]\times[x_0,x_5]$. Then we have $c\in\mathbb{E}_{1,T} := h^{1+\beta}\big([0,T],h^\alpha([x_0,x_5])\big)\cap h^\beta\big([0,T],h^{2+\alpha}([x_0,x_5])\big)$. On account of $\nu_\Sigma\circ\gamma(x,\varphi) = (0,\cos\varphi,\sin\varphi)$ on $[x_0,x_5]$,

$$w(t,x) = 1+\rho\big(t,\gamma(x,\varphi)\big)$$

follows for all $x\in[x_0,x_5]$ and $\varphi\in[0,2\pi]$ and hence we also have $w\in\mathbb{E}_{1,T}$.

Step 2: *Estimate of the Hölder-functions*

As $w,c\in\mathbb{E}_{1,T}$ holds, we have

$$\begin{aligned}&\|w(t)-w_0\|_{C^0([x_0,x_5])} + \|\partial_x w(t)\|_{C^0([x_0,x_5])} + \|\partial_{xx} w(t)\|_{C^0([x_0,x_5])}\\ &= \|w(t)-w(0)\|_{C^2([x_0,x_5])} \le \|w\|_{\mathbb{E}_{1,T}}\cdot T^\beta\end{aligned}$$

as well as

$$\|c(t)-c_0\|_{C^0([x_0,x_5])} \le \|c\|_{\mathbb{E}_{1,T}}\cdot T^\beta$$

for all $t\in[0,T]$. For $T>0$ sufficiently small, we can assume $0\le c\le 2c_{0,O}$ and then $g\in C^1(\mathbb{R})$ implies

$$\big\|g\big(c(t)\big)-g(c_0)\big\|_{C^0([x_0,x_5])} \le \|g\|_{C^1([0,2c_{0,O}])}\|c(t)-c_0\|_{C^0([x_0,x_5])} \lesssim \|c\|_{\mathbb{E}_{1,T}}\cdot T^\beta$$

for all $t\in[0,T]$. Hence, for arbitrary $\tilde{\varepsilon}>0$, we can choose $T>0$ sufficiently small such that for any $(t,x)\in[0,T]\times[x_0,x_5]$,

$$\frac{\partial_{xx}w(t,x)}{1+|\partial_x w(t,x)|^2} - \frac{1}{w(t,x)}\begin{cases}\le \frac{\|w\|\cdot T^\beta}{1+0} - \frac{1}{w_0(x)+\|w\|\cdot T^\beta} & \le -\frac{1}{w_0(x)}+\tilde{\varepsilon},\\ \ge \frac{-\|w\|\cdot T^\beta}{1+\|w\|^2T^{2\beta}} - \frac{1}{w_0(x)-\|w\|\cdot T^\beta} & \ge -\frac{1}{w_0(x)}-\tilde{\varepsilon}\end{cases}$$

as well as

$$g\big(c(t,x)\big)\begin{cases}\le g\big(c_0(x)\big)+\tilde{\varepsilon},\\ \ge g\big(c_0(x)\big)-\tilde{\varepsilon}\end{cases}$$

hold. Overall,

$$g\big(c(t,x)\big)\left(\frac{\partial_{xx}w(t,x)}{1+|\partial_x w(t,x)|^2}-\frac{1}{w(t,x)}\right)\begin{cases}\leq -g\big(c_0(x)\big)\frac{1}{w_0(x)}+\varepsilon,\\ \geq -g\big(c_0(x)\big)\frac{1}{w_0(x)}-\varepsilon\end{cases}$$

follows, where we can choose $\varepsilon > 0$ sufficiently small such that $g_O + 2\varepsilon < g_I$ is valid. Lemmas 5.5 and 5.6 yield that rotationally symmetric solutions of (5.1) fulfill

$$\partial_t w = g(c)\left(\frac{\partial_{xx}w}{1+|\partial_x w|^2}-\frac{1}{w}\right).$$

So, for every $(t,x) \in [0,T] \times [x_0, x_5]$,

$$w(t,x) = w_0(x) + \int_0^t \partial_t w(s,x)\mathrm{d}s \begin{cases}\leq w_0(x) + \left(-g\big(c_0(x)\big)\frac{1}{w_0(x)}+\varepsilon\right)\cdot t,\\ \geq w_0(x) + \left(-g\big(c_0(x)\big)\frac{1}{w_0(x)}-\varepsilon\right)\cdot t\end{cases}$$

holds. For all $x \in [x_2, x_3]$ and all $y \in [x_0, x_1] \cup [x_4, x_5]$, we have $w_0(x) = w_0(y) = 1$ and $g\big(c_0(x)\big) = g_I$, $g\big(c_0(y)\big) = g_O$; and hence for every $t \in (0,T]$

$$\begin{aligned} w(t,y) &\geq w_0(y) + \left(-g\big(c_0(y)\big)\frac{1}{w_0(y)} - \varepsilon\right)\cdot t = 1 + (-g_O - \varepsilon)\cdot t \\ &> 1 + (-g_I + \varepsilon)\cdot t = w_0(x) + \left(-g\big(c_0(x)\big)\frac{1}{w_0(x)} + \varepsilon\right)\cdot t \geq w(t,x) \end{aligned}$$

follows. Thus, $\Gamma_\rho(t)$ is not convex for $t \in (0,T]$. A possible shape of the function $w(t,\cdot)$ is illustrated in Figure 5.2. □

5.3 Formation of Self-Intersections

The usual mean curvature flow does not allow for self-intersections: As a consequence of the maximum principle for parabolic differential equations, an initially embedded surface will remain embedded as long as it exists. This property does not transfer to our scaled mean curvature flow and we develop a concrete example for the occuring of self-intersections in this section.

An embedded hypersurface can obviously never have a self-intersection. As an initially embedded surface will stay embedded at least for a small time, a short-time existence result only for the case of embedded surfaces would provide the evolution of this surface only as long as there does not occur a self-intersection (yet). To describe self-intersections it is thus necessary to use the theory of immersed hypersurfaces and it is crucial that we proved the short-time existence result also for the case of immersed surfaces.

The idea is to start with a very thin, curved tube as in Figure 5.4. The curvature forces both sides of the tube to move to the right. Choosing an initial concentration that increases the evolution for the left side in comparison with the right side of the tube will produce a self-intersection. To rigorously prove this, we have to ensure that firstly the evolution provided by the short-time existence result lasts long enough such that the self-intersection

will occur during that time and secondly that the concentration does not distribute too fast but that the difference in concentration from the left to the right side of the tube persists long enough such that the self-intersection will form. Both of this will be achieved by choosing the initial tube sufficiently thin. Therefore, we need to ensure that both the final time (up to which the existence result guarantees the existence of an evolution) as well as the evolution driving force (i.e. the concentration) can be controlled independently of the initial thickness of the tube. For this reason, we formulated the short-time existence result in a version that allows for small changes in the initial hypersurface. In particular, it yields a final time and a bound on the solution which are both independent of the initial thickness of the tube. The latter can then be used to control the driving force.
Now, we construct this example concretely. We set $d = 1$, so we will be dealing with curves instead of hypersurfaces, but the example can easily be extended to more dimensions using rotation arguments. First, we fix an energy density function $G \in C^7(\mathbb{R})$ with $G'' > 0$ and $g := G - G' \cdot \mathrm{Id} > 0$. Additionally, we assume that g is not a constant function, because otherwise we would obtain a constantly scaled mean curvature flow which of course never develops self-intersections. As reference surface, we choose $\Sigma = F(M)$ as illustrated in Figure 5.3: Σ is an immersed curve, consisting of a circular arc with radius $R > 0$ that is smoothly connected at the endings such that it forms a closed curve. A possibility of splitting the arc smoothly is to add appropriate exponential terms to the parameterization of the arc, as illustrated in Figure 5.5. Then, the free ends can be connected easily to form a curve of arbitrary smoothness. We choose Σ to be of differentiability C^5 at least.
By construction, there exist two preimages of $(0,0) \in \Sigma$ in M, which we call z_l and z_r. Choosing the sign of the unit normal field ν_Σ of Σ as in Figure 5.3, we get

$$\begin{aligned} F(z_l) &= (0,0), & F(z_r) &= (0,0), \\ \nu_\Sigma(z_l) &= (-1,0), & \nu_\Sigma(z_r) &= (+1,0). \end{aligned}$$

We fix a constant height function $\rho_0 > 0$ which will be scaled with a small $\varepsilon \in (0,1)$ and we define $\rho_0^\varepsilon := \varepsilon\rho_0$. Finally, we choose an initial concentration $u_0 \in C^4(M)$ with

$$g\big(u_0(z_l)\big) > g\big(u_0(z_r)\big),$$

which is possible as g is not constant.

Theorem 5.8 (Formation of Self-Intersections)**.** *Let the energy density G, the reference curve $\Sigma = F(M)$ and the functions ρ_0 and u_0 be as above. For sufficiently small $\varepsilon > 0$, the initial curve*

$$\Gamma_0^\varepsilon = \left\{ F(z) + \varepsilon\rho_0\nu_\Sigma(z) \,\middle|\, z \in M \right\}$$

is an embedded curve whose evolution under (5.1) *with initial concentration u_0 leads to a self-intersection.*

Proof. The short-time existence result (Theorem 4.29) yields the existence of a $T > 0$ and an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ there exists a solution $(\rho^\varepsilon, u^\varepsilon) : [0,T] \times M \to \mathbb{R}^2$ of

$$\begin{cases} \partial_t \rho^\varepsilon &= g(u^\varepsilon)a(\rho^\varepsilon)H(\rho^\varepsilon) & \text{on } [0,T] \times M, \\ \partial_t u^\varepsilon &= \Delta_{\rho^\varepsilon} G'(u^\varepsilon) + g(u^\varepsilon)a(\rho^\varepsilon)H(\rho^\varepsilon)\nu_\Sigma \cdot \nabla_{\rho^\varepsilon} u^\varepsilon + g(u^\varepsilon)H(\rho^\varepsilon)^2 u^\varepsilon & \text{on } [0,T] \times M, \\ \rho^\varepsilon(0) &= \rho_0^\varepsilon = \varepsilon\rho_0 & \text{on } M, \\ u^\varepsilon(0) &= u_0 & \text{on } M. \end{cases}$$

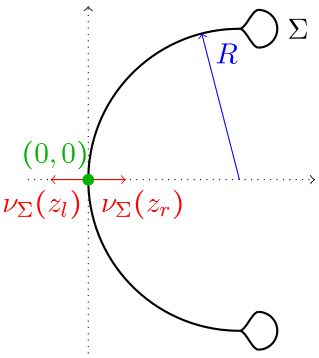


Figure 5.3: The immersed reference curve Σ, consisting of a circular arc with radius R that is smoothly connected at the endings. The points $z_l, z_r \in M$ are both preimages of $(0,0) \in \Sigma$. We choose the sign of the unit normal ν_Σ such that it points outwards in a neighborhood of z_l and inwards in a neighborhood of z_r.

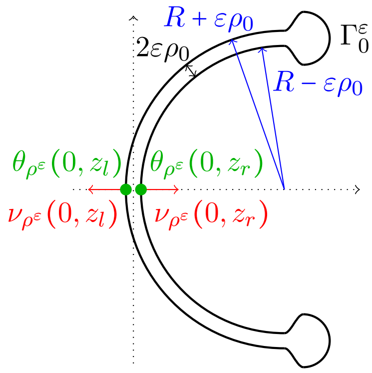


Figure 5.4: The initial curve Γ_0^ε, consisting of two smoothly connected circular arcs with radii $R \pm \varepsilon\rho_0$. Its unit normal $\nu_{\rho^\varepsilon} \circ \left(\theta_{\rho^\varepsilon}(0,\cdot)\right)^{-1}$ points outwards on the left arc and inwards on the right arc. As Γ_0^ε is embedded, the images of $z_l, z_r \in M$ on Γ_0^ε do not coincide.

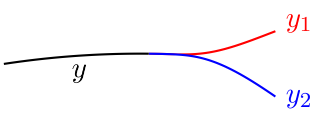


Figure 5.5: After reparameterization, $y(x) = \sqrt{R^2 - x^2}$ parameterizes the circular arc. Adding appropriate exponential terms leads to smooth continuations $y_1(x) = \sqrt{R^2 - x^2} + \exp\left(\frac{-1}{x}\right)$ and $y_2(x) = \sqrt{R^2 - x^2} - \exp\left(\frac{-1}{x}\right)$, $x > 0$, of y that split the arc into two parts.

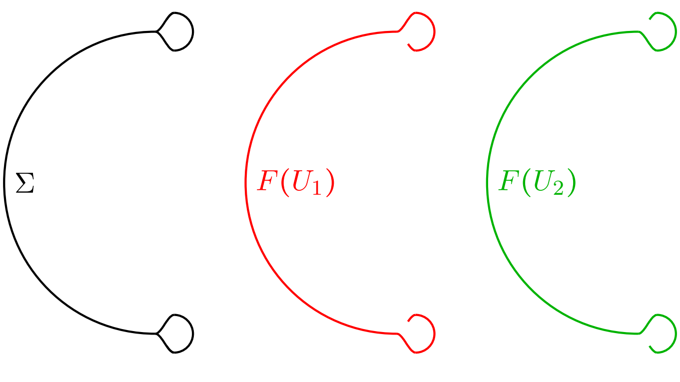


Figure 5.6: The immersed surface Σ can be covered by two embedded patches, called $F(U_1)$ and $F(U_2)$ with $U_1, U_2 \subset M$.

(Again, we refer to Section 2.1.6 for an introduction to evolving hypersurfaces parameterized via height functions, and in particular to Remark 2.64 for the definition of the pullback operators $\nabla_\rho, \Delta_\rho, H(\rho)$ and ν_ρ. As in Lemma 4.5(ii), we define $a(\rho) \coloneqq \frac{1}{\nu_\Sigma \cdot \nu_\rho}$.) Because we chose Σ, u_0 and ρ_0^ε sufficiently smooth, $\rho^\varepsilon, u^\varepsilon \in \mathbb{E}_{1,T} = h^{1+\beta}\big([0,T], h^\alpha(M)\big) \cap h^\beta\big([0,T], h^{2+\alpha}(M)\big)$ holds for any $\alpha \in (0,1)$ and $\beta \in (0, \frac{1}{2})$ with $2\beta + \alpha \notin \mathbb{N}$, and there exist $R^h, R^c > 0$ such that $\|\rho^\varepsilon\|_{\mathbb{E}_{1,T}} \le R^h$ and $\|u^\varepsilon\|_{\mathbb{E}_{1,T}} \le R^c$ hold independently of $\varepsilon \in (0, \varepsilon_0]$. With the notation from Section 2.1.6, the evolving curve is described by the global parameterization

$$\theta_{\rho^\varepsilon} : [0,T] \times M \to \mathbb{R}^2, \quad \theta_{\rho^\varepsilon}(t,z) \coloneqq F(z) + \rho^\varepsilon(t,z)\nu_\Sigma(z).$$

Step 1: *Embeddedness of the inital hypersurface*

As Σ is closed, it can be covered by finitely many embedded patches. As shown in Figure 5.6, two embedded patches are sufficient to cover Σ. Let $U_1, U_2 \subset M$ be the preimages of these embedded patches $F(U_1), F(U_2)$. We can choose $\varepsilon > 0$ sufficiently small, such that for both embedded patches $F(U_i) \subset \Sigma$,

$$\theta_{\rho^\varepsilon}(0,\cdot) \circ F^{-1} : F(U_i) \to \mathbb{R}^2, \quad p \mapsto p + \rho_0^\varepsilon \nu_\Sigma\big(F^{-1}(p)\big) = p + \rho_0^\varepsilon \nu_{F(U_i)}(p)$$

is an embedding (see Proposition 2.60, where we assume w.l.o.g. that $F(U_i)$ is expanded to an embedded closed hypersurface.). In particular, for every $z_1, z_2 \in U_i$ with $z_1 \neq z_2$ we have $F(z_1) \neq F(z_2)$ and thus $\theta_{\rho^\varepsilon}(0, z_1) \neq \theta_{\rho^\varepsilon}(0, z_2)$. For every $z_1 \in U_1 \setminus U_2$ and $z_2 \in U_2 \setminus U_1$, we clearly have

$$\theta_{\rho^\varepsilon}(0,z_1) = F(z_1) + \rho_0^\varepsilon \nu_\Sigma(z_1) \neq F(z_2) + \rho_0^\varepsilon \nu_\Sigma(z_2) = \theta_{\rho^\varepsilon}(0,z_2)$$

because the initial height function $\rho_0^\varepsilon = \varepsilon\rho_0$ is positiv everywhere and $\nu_{\Sigma|U_1\setminus U_2}$ points outwards whereas $\nu_{\Sigma|U_2\setminus U_1}$ points inwards, and so z_1 and z_2 are driven apart by $\theta_{\rho^\varepsilon}(0,\cdot)$. Altogether, if $\varepsilon > 0$ is sufficiently small, for any $z_1, z_2 \in M$ with $z_1 \neq z_2$ also $\theta_{\rho^\varepsilon}(0,z_1) \neq \theta_{\rho^\varepsilon}(0,z_2)$ holds. This implies that for $\varepsilon > 0$ sufficiently small,

$$\theta_{\rho^\varepsilon}(0,\cdot) : M \to \mathbb{R}^2, \quad \theta_{\rho^\varepsilon}(0,z) \coloneqq F(z) + \rho_0^\varepsilon \nu_\Sigma(z)$$

is injective and thus an embedding. Therefore, the initial curve

$$\Gamma_0^\varepsilon = \big\{\theta_{\rho^\varepsilon}(0,z) \,\big|\, z \in M\big\}$$

is an embedded curve. The curve Γ_0^ε is illustrated in Figure 5.4 and we collect some of its geometric quantities now. Because ρ_0^ε is constant, the curve Γ_0^ε consists of two circular arcs, the inner one with radius $R - \varepsilon\rho_0$ and the outer one with radius $R + \varepsilon\rho_0$. Due to the chosen sign of the normal, the (mean) curvature of the inner arc is positiv and that of the outer arc is negative. Especially in our fixed points z_l and z_r, we thus have

$$H(\rho^\varepsilon)(0,z_l) = \frac{-1}{R + \varepsilon\rho_0} \qquad \text{and} \qquad H(\rho^\varepsilon)(0,z_r) = \frac{1}{R - \varepsilon\rho_0}$$

as well as $\nu_{\rho^\varepsilon}(0,z_l) = (-1,0) = \nu_\Sigma(z_l)$ and $\nu_{\rho^\varepsilon}(0,z_r) = (+1,0) = \nu_\Sigma(z_r)$ (cf. Remark 2.43). So,

$$a(\rho^\varepsilon)(0,z_l) = 1 \qquad \text{and} \qquad a(\rho^\varepsilon)(0,z_r) = 1$$

follow.

Step 2: *Formation of self-intersection*

We want to show that the evolution of Γ_0^ε forms a self-intersection. For any $t \in [0,T]$,

$$\begin{aligned}\theta_{\rho^\varepsilon}(t,z_l) &= F(z_l) + \rho^\varepsilon(t,z_l)\nu_\Sigma(z_l) = (0,0) + \rho^\varepsilon(t,z_l)(-1,0) = \big(-\rho^\varepsilon(t,z_l),0\big),\\ \theta_{\rho^\varepsilon}(t,z_r) &= F(z_r) + \rho^\varepsilon(t,z_r)\nu_\Sigma(z_r) = (0,0) + \rho^\varepsilon(t,z_r)(+1,0) = \big(\rho^\varepsilon(t,z_r),0\big)\end{aligned}$$

holds. In particular,

$$\big[\theta_{\rho^\varepsilon}(0,z_l)\big]_1 = -\varepsilon\rho_0 < \varepsilon\rho_0 = \big[\theta_{\rho^\varepsilon}(0,z_r)\big]_1$$

holds and if we have

$$\big[\theta_{\rho^\varepsilon}(T_0,z_l)\big]_1 > \big[\theta_{\rho^\varepsilon}(T_0,z_r)\big]_1 \Leftrightarrow -\rho^\varepsilon(T_0,z_l) > \rho^\varepsilon(T_0,z_r) \tag{5.2}$$

for a $T_0 \in (0,T]$, then a self-intersection with $\theta_{\rho^\varepsilon}(T_1,z_l) = \theta_{\rho^\varepsilon}(T_1,z_r)$ occured at a time $T_1 \in (0,T_0)$. All that is left to prove is thus the existence of a $T_0 \in (0,T]$ with (5.2) for sufficiently small $\varepsilon > 0$.

As we have $g\big(u_0(z_l)\big) > g\big(u_0(z_r)\big)$, there exists a $K \in \mathbb{R}_{>0}$ with

$$g\big(u_0(z_l)\big) = g\big(u_0(z_r)\big) + K. \tag{5.3}$$

Choose $T_0 \in (0,T]$ so small that $R^h T_0^\beta \le \frac{K}{8R}$. Then,

$$\begin{aligned}\big\|\partial_t\rho^\varepsilon(t,\cdot) - \partial_t\rho^\varepsilon(0,\cdot)\big\|_{C^0(M)} &\le \|\partial_t\rho^\varepsilon\|_{h^\beta([0,T],h^\alpha(M))}T_0^\beta\\ &\le \|\rho^\varepsilon\|_{\mathbb{E}_{1,T}}T_0^\beta \le R^h T_0^\beta \le \frac{K}{8R}\end{aligned} \tag{5.4}$$

holds for all $t \in [0,T_0]$ and $\varepsilon \in (0,\varepsilon_0]$. Now, choose $\varepsilon = \varepsilon(T_0) \in (0,\varepsilon_0]$ so small that $\varepsilon \le \frac{R}{\sqrt{2}\rho_0}$ and that

$$\varepsilon < \frac{T_0 K R}{8\rho_0\big(R^2 + 2g(u_0(z_r))\big)}.$$

Then, with $H(\rho_0^\varepsilon)(z_l) = \frac{-1}{R+\varepsilon\rho_0}$ and $H(\rho_0^\varepsilon)(z_r) = \frac{1}{R-\varepsilon\rho_0}$,

$$-H(\rho_0^\varepsilon)(z_l) \ge \frac{1}{2R} \qquad \text{and} \qquad H(\rho_0^\varepsilon)(z_l) + H(\rho_0^\varepsilon)(z_r) = \frac{2\varepsilon\rho_0}{R^2 - \varepsilon^2\rho_0^2} \le \frac{4\varepsilon\rho_0}{R^2} \tag{5.5}$$

hold and we have

$$2\varepsilon\rho_0\left(1 + \frac{2}{R^2}g\big(u_0(z_r)\big)\right) < \frac{T_0 K}{4R}. \tag{5.6}$$

With these preliminary considerations and $a(\rho_0^\varepsilon)(z_l) = a(\rho_0^\varepsilon)(z_r) = 1$, we can compute

$$
\begin{aligned}
-\rho^\varepsilon(T_0, z_l) &= -\Big(\rho_0^\varepsilon + \int_0^{T_0} \partial_t \rho^\varepsilon(t, z_l)\, \mathrm{d}t\Big) \\
&\overset{(5.4)}{\geq} -\rho_0^\varepsilon - \frac{T_0 K}{8R} - T_0 \partial_t \rho^\varepsilon(0, z_l) \\
&= -\varepsilon\rho_0 - \frac{T_0 K}{8R} + T_0 g\big(u_0(z_l)\big) a(\rho_0^\varepsilon)(z_l)\big(- H(\rho_0^\varepsilon)(z_l)\big) \\
&\overset{(5.3)}{=} -\varepsilon\rho_0 - \frac{T_0 K}{8R} + T_0 K\big(- H(\rho_0^\varepsilon)(z_l)\big) + T_0 g\big(u_0(z_r)\big)\big(- H(\rho_0^\varepsilon)(z_l)\big) \\
&\overset{(5.5)}{\geq} -\varepsilon\rho_0 - \frac{T_0 K}{8R} + \frac{T_0 K}{2R} - \frac{4\varepsilon\rho_0 T_0}{R^2} g\big(u_0(z_r)\big) + T_0 g\big(u_0(z_r)\big) H(\rho_0^\varepsilon)(z_r) \\
&\geq -\varepsilon\rho_0 + \frac{3T_0 K}{8R} - \frac{4\varepsilon\rho_0}{R^2} g\big(u_0(z_r)\big) + T_0 \partial_t \rho^\varepsilon(0, z_r) \\
&\overset{(5.4)}{\geq} -\varepsilon\rho_0 + \frac{3T_0 K}{8R} - \frac{4\varepsilon\rho_0}{R^2} g\big(u_0(z_r)\big) - \frac{T_0 K}{8R} + \int_0^{T_0} \partial_t \rho^\varepsilon(t, z_r) \mathrm{d}t \\
&= -2\varepsilon\rho_0 + \frac{T_0 K}{4R} - \frac{4\varepsilon\rho_0}{R^2} g\big(u_0(z_r)\big) + \Big(\rho_0^\varepsilon + \int_0^{T_0} \partial_t \rho^\varepsilon(t, z_r) \mathrm{d}t\Big) \\
&= -2\varepsilon\rho_0 \Big(1 + \frac{2}{R^2} g\big(u_0(z_r)\big)\Big) + \frac{T_0 K}{4R} + \rho^\varepsilon(T_0, z_r) \\
&\overset{(5.6)}{>} \rho^\varepsilon(T_0, z_r).
\end{aligned}
$$

By (5.2), the evolution of Γ_0^ε hence developped a self-intersection. $\square$

5.4 Properties of the Concentration

In this section we discuss that the so called "concentration" $c : \Gamma \to \mathbb{R}$ really satisfies the most important properties of a physical concentration. First, the concentration should describe the distribution of a quantity whose mass is conserved. Second, the concentration should always be non-negative. We will analyze these features in the following setting.

Assumptions 5.9. *Let $G \in C^3(\mathbb{R})$ and $g \coloneqq G - G' \cdot \mathrm{Id}$. Moreover, let Γ be a C^1- C^2-evolving immersed closed hypersurface with reference surface $M \subset \mathbb{R}^{d+1}$ and mean curvature H that evolves with normal velocity*

$$V = g(c)H,$$

where $c \in C^1\big([0,T], C^0(M)\big) \cap C^0\big([0,T], C^2(M)\big)$ is a function with

$$\partial^\square c = \Delta_\Gamma G'(c) + cVH.$$

As a start, we show that the quantity whose concentration is described by the function c fulfills mass conservation.

Theorem 5.10 (Conservation of Mass)**.** *Suppose Assumptions 5.9 are valid. There exists a constant $m \in \mathbb{R}$ (specifying the mass of the quantity whose concentration is described by* *the function c) such that*

$$\int_M c(t,p)\,\mathrm{d}\mathcal{H}^d(p) = m$$

holds for all $t \in [0,T]$.

Proof. With the help of the transport theorem (Proposition 2.58) and Gauß' theorem on closed hypersurfaces (Proposition 2.48), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_M c(t,p)\,\mathrm{d}\mathcal{H}^d(p) &= \int_M \partial^\square c(t,p) - c(t,p)V(t,p)H(t,p)\,\mathrm{d}\mathcal{H}^d(p)\\ &= \int_M \Delta_{\Gamma(t)} G'\big(c(t,p)\big)\,\mathrm{d}\mathcal{H}^d(p)\\ &= 0.\end{aligned}$$

□

A concentration always is non-negative. Therefore, we show in the following theorem that non-negativity of the concentration is conserved. Even more, if an initially non-negative concentration is not the zero-function, then it instantly turns strictly positive.

Theorem 5.11 (Positivity of the Concentration)**.** *Suppose that Assumptions 5.9 are valid with* $G'' > 0$.

(i) Let $c(0) \geq 0$ *on* M*. Then* $c(t) \geq 0$ *holds on* M *for all* $t \in [0,T]$.

(ii) Let $c(0) \geq 0$ *on* M *with* $c(0) \not\equiv 0$*. Then* $c(t) > 0$ *holds on* M *for all* $t \in (0,T]$.

Proof. On account of $V = g(c)H$, we have

$$\begin{aligned}\Delta_\Gamma G'(c) + cVH &= G''(c)\Delta_\Gamma c + G'''(c)\left|\nabla_\Gamma c\right|^2 + g(c)H^2 c\\ &= A : D_\Gamma^2 c + B \cdot \nabla_\Gamma c + Cc\end{aligned}$$

with

$$\begin{aligned}A &\coloneqq G''(c)\mathrm{Id},\\ B &\coloneqq G'''(c)\nabla_\Gamma c,\\ C &\coloneqq g(c)H^2,\end{aligned}$$

where $A : \widetilde{A} \coloneqq \sum_{i,j} A_{ij}\widetilde{A}_{ij}$ is the inner product of two matrices A and $\widetilde{A}$ as in Definition 2.141. The assumptions and $H \in C^0\big([0,T]\times M\big)$ by Remark 2.56 imply continuity of $A : [0,T]\times M \to \mathbb{R}^{(d+1)\times(d+1)}$, $B : [0,T]\times M \to \mathbb{R}^{d+1}$ and $C : [0,T]\times M \to \mathbb{R}$. Moreover, the matrix A clearly is symmetric and, due to $G'' > 0$, also positive definite on $[0,T]\times M$. With this, we define $w \coloneqq -c$ as well as

$$\mathcal{L}w \coloneqq -\partial^\square w + A : D_\Gamma^2 w + B \cdot \nabla_\Gamma w + Cw.$$

In particular, Assumptions 2.140 are satisfied. We have

$$\mathcal{L}w = -\mathcal{L}c = \partial^\square c - \big(\Delta_\Gamma G'(c) + cVH\big) = 0 \text{ on } [0,T]\times M$$

and $w(0) = -c(0) \le 0$ on M. With Corollary 2.148, $c(t) = -w(t) \ge 0$ follows on M for all $t \in [0,T]$ and there exists $t_0 \in [0,T]$ with

$$c(t) = -w(t) = 0 \text{ on } M \text{ for all } t \in (0,t_0] \quad \text{and} \quad c(t) = -w(t) > 0 \text{ on } M \text{ for all } t \in (t_0,T].$$

Under the assumptions of (ii), we have $c(0) \not\equiv 0$. By Theorem 5.10, thus $c(t) \equiv 0$ can not hold for any $t > 0$. Consequently, we then have $t_0 = 0$ and $c(t) > 0$ follows on M for all $t \in (0,T]$. □

The concentration does not only stay non-negative, but the minimal concentration even increases monotonically. Additionally, if the hypersurface is mean convex, the increase of the minimal concentration even is strictly monotonic.

Theorem 5.12 (Growth of the Minimal Concentration)**.** *Suppose Assumptions 5.9 are valid with $G'' > 0$ and $g > 0$. Let $c_{\min} : [0,T] \to \mathbb{R}$, $c_{\min}(t) \coloneqq \min_M c(t,\cdot)$ be the minimum function of c on M.*

(i) Let $c(0) \ge 0$. Then, $c_{\min}$ is monotonically increasing.

(ii) Let Γ be of regularity $(C^2\text{-}C^2) \cap (C^1\text{-}C^4)$. Furthermore, let $H(0) \ge 0$, $c(0) \ge 0$ and $c(0) \not\equiv 0$. Then, $c_{\min}$ is strictly increasing.

Proof. With Hamilton's trick (Lemma 2.142), $c_{\min} : (0,T) \to \mathbb{R}$ is well-defined and Lipschitz continuous. In particular, it is differentiable almost everywhere, and in every time $t \in (0,T)$ in which $c_{\min}$ is differentiable, we have

$$\partial_t c_{\min\,|t} = \partial_t c_{|(t,p)}$$

where $p \in M$ is an arbitrary point with $c(t,p) = c_{\min}(t)$. For such a point p, we have $\nabla_\Gamma c(t,p) = 0$ and $\Delta_\Gamma c(t,p) \ge 0$ by Lemma 2.36 and then

$$\partial^\square c_{|(t,p)} = \partial^\circ c_{|(t,p)} - V^{\text{tot}}_{\Gamma\,|(t,p)} \cdot \nabla_\Gamma c_{|(t,p)} = \partial^\circ c_{|(t,p)} = \partial_t c_{|(t,p)}$$

follows with Definition 2.54. For every point $p \in M$ with $c(t,p) = c_{\min}(t)$ we thus have

$$\begin{aligned}
\partial_t c_{\min\,|t} &= \partial^\square c_{|(t,p)} \\
&= \Delta_\Gamma G'(c) + cVH_{|(t,p)} \\
&= G''(c)\Delta_\Gamma c + G'''(c)\left|\nabla_\Gamma c\right|^2 + g(c)H^2 c_{|(t,p)} \\
&\ge g(c)H^2 c_{|(t,p)}.
\end{aligned}$$

The assumptions in (i) and Theorem 5.11(i) imply $c(t) \ge 0$ for all $t \in [0,T]$ such that $\partial_t c_{\min} \ge 0$ follows almost everywhere. Hence, $c_{\min}$ is monotonically increasing.
The assumptions in (ii), Section 5.1 and Theorem 5.11(ii) yield $H(t) > 0$ and $c(t) > 0$ for all $t \in (0,T]$ such that $\partial_t c_{\min} > 0$ follows almost everywhere. Hence, $c_{\min}$ is strictly increasing. □

Appendix

In this Appendix, we recall a few basic definitions from linear algebra and gather some technical proofs. We assume $d, n, m \in \mathbb{N}_{>0}$ to be integers and fix $s \in \mathbb{R}_{\geq 0}$.

A.1 Orientation of Vector Spaces and Generalized Cross Product

Definition A.1 (Orientation of a Vector Space)**.** *Let V be a d-dimensional real vector space. We define an equivalence relation on the set of all ordered bases of V by calling two bases $(v_1, ..., v_d)$ and $(w_1, ..., w_d)$ equally oriented if the transition matrix $B \in Gl(d, \mathbb{R})$ determined by the relations $v_i = \sum_j B_{ij} w_j$ has positive determinant. An orientation of V is a choice of one of these two equivalence classes. Any basis in the given orientation is called positively oriented.*

Remark A.2. *Because the determinant is linear in every column,*

$$\det \tilde{B} = -\det B$$

holds for $B, \tilde{B} \in Gl(d, \mathbb{R})$ with $\tilde{B}_{i,j_0} = -B_{i,j_0}$ for a fixed j_0 and $\tilde{B}_{ij} = B_{ij}$ for all $j \neq j_0$. This implies that by changing the sign of one vector, every not positively oriented basis $(w_1, ..., w_d)$ of a d-dimensional vector space V turns into a positively oriented basis: With $\tilde{w}_{j_0} := -w_{j_0}$ and $\tilde{w}_j := w_j$ for all $j \neq j_0$, $(\tilde{w}_1, ..., \tilde{w}_d)$ is a positively oriented basis of V.

Remark A.3. *We fix an orientation of $\mathbb{R}^d$ by calling the standard basis $(e_1, ..., e_d)$ with $[e_j]_i = \delta_{ij}$ positively oriented.*

Definition A.4 (Generalized Cross Product)**.** *We call*

$$\mathcal{K} : (\mathbb{R}^{d+1})^d \to \mathbb{R}^{d+1},\, \mathcal{K}(v_1, ..., v_d) := \big(\det[v_1, ..., v_d, e_j]\big)_{j=1,...,d+1}$$

the generalized cross product of $\mathbb{R}^{d+1}$. Here, $e_j \in \mathbb{R}^{d+1}$ is the standard basis vector given by $[e_j]_i = \delta_{ij}$.

Remark A.5. *The generalized cross product fulfills the same properties as the usual cross product in $\mathbb{R}^3$. In particular, we have*

$$\mathcal{K}(v_1, ..., v_d) \cdot v_j = 0 \text{ for all } j = 1, ..., d$$

and if $(v_1, ..., v_d) \subset \mathbb{R}^{d+1}$ *are linearly independent, then* $\mathcal{K}(v_1, ..., v_d) \neq 0$ *holds and*

$$\big(v_1, ..., v_d, \mathcal{K}(v_1, ..., v_d)\big)$$

is a positively oriented basis of $\mathbb{R}^{d+1}$*. Furthermore, the generalized cross product conserves regularity, i.e., if* $v_j \in \mathcal{C}^{k+\alpha}(\overline{W}, \mathbb{R}^{d+1})$ *holds for all* $j = 1, ..., d$ *and any* $k \in \mathbb{N}_0$*,* $\alpha \in (0,1)$ *and an open subset* $W \subset \mathbb{R}^m$*, then we also have* $\mathcal{K}(v_1, ..., v_d) \in \mathcal{C}^{k+\alpha}(\overline{W}, \mathbb{R}^{d+1})$*.*

A.2 Immersions and Embeddings on domains in $\mathbb{R}^d$

The aim of this section is to show that, for domains in $\mathbb{R}^d$, any immersion locally is an embedding. Furthermore, we prove that for a time dependent function which is an immersion at every time t, the local neighborhood on which it is an embedding can be chosen independently of the time t. This will be used in Section 2.1 for the corresponding statements on hypersurfaces.

Definition A.6. *Let* $W \subset \mathbb{R}^d$ *be an open subset and let* $\gamma \in C^1(W, \mathbb{R}^m)$*. The function* γ *is called an immersion if* $D\gamma(x) : \mathbb{R}^d \to \mathbb{R}^m$ *is injective for all* $x \in W$*. It is called an embedding if additionally it is a homeomorphism onto its image.*
For any subset $A \subset W$*, we say that* $\gamma_{|A}$ *is an immersion if* $D\gamma(x) : \mathbb{R}^d \to \mathbb{R}^m$ *is injective for all* $x \in A$*, and we say that* $\gamma_{|A}$ *is an embedding if additionally* $\gamma_{|A}$ *is a homeomorphism onto its image. In particular, we say that locally* $\gamma_{|A}$ *is an embedding if for all* $x \in A$ *there exists an open neighborhood* $W_x \subset W$ *such that* $\gamma_{|W_x \cap A}$ *is an embedding.*

Lemma A.7 (An Immersion locally is an Embedding)**.** *Let* $W \subset \mathbb{R}^d$ *be an open subset and* $A \subset \mathbb{R}^d$ *any subset with* $A \subset W$*. Let* $\gamma \in C^1(W, \mathbb{R}^m)$ *such that* $\gamma_{|A}$ *is an immersion. Then, locally* $\gamma_{|A}$ *is an embedding.*

Proof. Due to the immersion property of γ, we have $d \leq m$. So, we can define

$$F : W \times \mathbb{R}^{m-d} \to \mathbb{R}^m,\ F(z_1, ..., z_m) \coloneqq \gamma(z_1, ..., z_d) + (0, ..., 0, z_{d+1}, ..., z_m).$$

Then, $\gamma = F \circ i_{\mathbb{R}^d}$ holds on W, where $i_{\mathbb{R}^d} : \mathbb{R}^d \to \mathbb{R}^m$, $x \mapsto (x, 0)$ is the trivial inclusion of $\mathbb{R}^d$ into $\mathbb{R}^m$. The function F is continuously differentiable on $W \times \mathbb{R}^{m-d}$ with

$$DF(z) = \begin{pmatrix} D\gamma(z_1, ..., z_d) & \begin{matrix} 0 \\ \\ \mathrm{Id}_{\mathbb{R}^{m-d}} \end{matrix} \end{pmatrix}.$$

For fixed $x \in A$,

$$D\gamma(x) = \begin{pmatrix} [\partial_i \gamma(x)]_{i=1,...,d} \\ [\partial_i \gamma(x)]_{i=d+1,...,m} \end{pmatrix} : \mathbb{R}^d \to \mathbb{R}^m$$

is injective. Thus, by rearranging the components of γ, we can assume w.l.o.g. that $[\partial_i \gamma(x)]_{i=1,...,d} \in \mathbb{R}^{d \times d}$ is invertible. Then, also $DF(x, 0)$ is invertible. The inverse function theorem yields the existence of an open subset $V \subset \mathbb{R}^m$ with $(x, 0) \in V \subset W \times \mathbb{R}^{m-d}$ such that $F_{|V} : V \to F(V)$ is a diffeomorphism. Define $W_x \coloneqq \mathrm{pr}_{\mathbb{R}^d}\big(V \cap (\mathbb{R}^d \times \{0\}^{m-d})\big)$, where

$\mathrm{pr}_{\mathbb{R}^d} : \mathbb{R}^m \to \mathbb{R}^d$, $z = (z_1, ..., z_m) \mapsto (z_1, ..., z_d)$ is the projection from $\mathbb{R}^m$ onto $\mathbb{R}^d$. Then, $W_x \subset \mathbb{R}^d$ is an open subset with $x \in W_x \subset W$ and $i_{\mathbb{R}^d|W_x} : W_x \to V \cap (\mathbb{R}^d \times \{0\}^{m-d})$ is a homeomorphism. As combination of homeomorphisms, also

$$\gamma_{|W_x} = F_{|V \cap (\mathbb{R}^d \times \{0\}^{m-d})} \circ i_{\mathbb{R}^d|W_x} : W_x \to \gamma(W_x)$$

is a homeomorphism and thus $\gamma_{|W_x \cap A} : W_x \cap A \to \mathbb{R}^m$ is an embedding. Since $x \in A$ was arbitrary, $\gamma_{|A}$ locally is thus an embedding. □

Remark A.8. *Lemma A.7 implies in particular that any immersion $\gamma : W \to \mathbb{R}^m$ with an open subset $W \subset \mathbb{R}^d$ locally is an embedding.*

Definition A.9. *Let $W \subset \mathbb{R}^d$ be an open subset and let $\gamma \in C^1\big([0,T] \times W, \mathbb{R}^{d+1}\big)$. We say that locally $\gamma(t,\cdot) : W \to \mathbb{R}^{d+1}$ is an embedding for fixed $t \in [0,T]$ if*

$$\forall t \in [0,T], \forall x \in W : \exists \text{ open neighborhood } W_{tx} \subset W \text{ of } x \text{ s.t. } \gamma(t,\cdot)_{|W_{tx}} \text{ is an embedding.}$$

On the other hand, we say that locally $\gamma(t,\cdot) : W \to \mathbb{R}^{d+1}$ is an embedding independently of $t \in [0,T]$ if

$$\forall x \in W : \exists \text{ open neighborhood } W_x \subset W \text{ of } x \text{ s.t. } \gamma(t,\cdot)_{|W_x} \text{ is an embedding } \forall t \in [0,T].$$

Lemma A.10 (Time-Independence of Locality)**.** *Let $W \subset \mathbb{R}^d$ be an open subset and let $\gamma \in C^1\big([0,T] \times W, \mathbb{R}^{d+1}\big)$ such that $\gamma(t,\cdot) : W \to \mathbb{R}^{d+1}$ is an immersion for all $t \in [0,T]$. Then locally $\gamma(t,\cdot) : W \to \mathbb{R}^{d+1}$ is an embedding independently of $t \in [0,T]$.*

It is clear by Lemma A.7 that $\gamma(t,\cdot) : W \to \mathbb{R}^{d+1}$ is an embedding for fixed $t \in [0,T]$. The claim of this lemma is therefore that W_x can be chosen independently of $t \in [0,T]$.

Proof. We extend γ onto an open subset $I \subset \mathbb{R}$ with $[0,T] \subset I$ such that $\gamma \in C^1(I \times W, \mathbb{R}^{d+1})$ is valid. This is possible due to [RR06, Theorem 7.58]. Then, we define

$$F : I \times W \to \mathbb{R}^{d+2},\ F(t,x) \coloneqq \big(t, \gamma(t,x)\big).$$

The function F is continuously differentiable on $I \times W$ with

$$DF(t,x) = \begin{pmatrix} \partial_t F_1 & D_x F_1 \\ \partial_t F_2 & D_x F_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \partial_t \gamma(t,x) & D\gamma_t(x) \end{pmatrix}$$

for all $(t,x) \in I \times W$. As $\gamma_t \coloneqq \gamma(t,\cdot) : W \to \mathbb{R}^{d+1}$ is an immersion for all $t \in [0,T]$, $D\gamma_t(x) : \mathbb{R}^d \to \mathbb{R}^{d+1}$ is injective for all $x \in W$ and $t \in [0,T]$. Due to the structure of DF, also $DF(t,x) : \mathbb{R}^{d+1} \to \mathbb{R}^{d+2}$ is injective for all $(t,x) \in [0,T] \times W$ and so $F : [0,T] \times W \to \mathbb{R}^{d+2}$ is an immersion.
Hence, locally, $F : [0,T] \times W \to \mathbb{R}^{d+2}$ is an embedding, i.e., for all $t \in [0,T]$ and $x \in W$ there exists $\varepsilon_t > 0$ and an open subset $\widetilde{W}_t \subset \mathbb{R}^d$ with $(t - \varepsilon_t, t + \varepsilon_t) \subset I$ and $x \in \widetilde{W}_t \subset W$ such that

$$F_{|((t-\varepsilon_t, t+\varepsilon_t) \cap [0,T]) \times \widetilde{W}_t}$$

is an embedding. Because the compact set $[0,T]$ is covered by the open sets $(t-\varepsilon_t, t+\varepsilon_t)$ with $t \in [0,T]$, there exists a finite subcover $(t_i - \varepsilon_{t_i}, t_i + \varepsilon_{t_i})_{i=1,\dots,m}$ with

$$[0,T] \subset \bigcup_{i=1}^{m} (t_i - \varepsilon_{t_i}, t_i + \varepsilon_{t_i}).$$

We define

$$I_i := (t_i - \varepsilon_{t_i}, t_i + \varepsilon_{t_i}) \cap [0,T] \text{ for all } i = 1,\dots,m \qquad \text{and} \qquad \widetilde{W} := \bigcap_{i=1}^{m} \widetilde{W}_{t_i}.$$

Then, $\widetilde{W} \subset \mathbb{R}^d$ is an open subset with $x \in \widetilde{W} \subset W$ and for all $t \in [0,T]$ there exists $i \in \{1,\dots,m\}$ with $t \in I_i$. Furthermore, $F_{|I_i \times \widetilde{W}}$ is an embedding for all $i = 1,\dots,m$.
Now, we have to prove that this implies that $\gamma(t,\cdot)_{|\widetilde{W}}$ is an embedding for all $t \in [0,T]$. Fix $\bar{t} \in [0,T]$ and $i \in \{1,\dots,m\}$ with $\bar{t} \in I_i$. Because

$$F_{|I_i \times \widetilde{W}} : (t,x) \mapsto \big(t, \gamma(t,x)\big)$$

is an embedding, $F_{|I_i \times \widetilde{W}}$ is injective and $(F_{|I_i \times \widetilde{W}})^{-1} : F(I_i \times \widetilde{W}) \to I_i \times \widetilde{W}$ is continuous. Injectivity of $\gamma(\bar{t},\cdot)_{|\widetilde{W}}$ follows directly from the injectivity of $F_{|I_i \times \widetilde{W}}$. Thus, $\big(\gamma(\bar{t},\cdot)_{|\widetilde{W}}\big)^{-1} : \gamma(\bar{t},\widetilde{W}) \to \widetilde{W}$ is well-defined. Due to

$$\big(F_{|I_i \times \widetilde{W}}\big)^{-1}(\bar{t},p) = \big(\bar{t}, \big(\gamma(\bar{t},\cdot)_{|\widetilde{W}}\big)^{-1}(p)\big)$$

and

$$\begin{aligned}
\gamma(\bar{t},\widetilde{W}) &= \big\{p \in \mathbb{R}^{d+1} \,\big|\, \exists x \in \widetilde{W} \text{ with } \gamma(\bar{t},x) = p\big\} \\
&= \big\{p \in \mathbb{R}^{d+1} \,\big|\, \exists x \in \widetilde{W} \text{ with } F(\bar{t},x) = \big(\bar{t},\gamma(\bar{t},x)\big) = (\bar{t},p)\big\} \\
&= \big\{p \in \mathbb{R}^{d+1} \,\big|\, (\bar{t},p) \in F(I_i \times \widetilde{W})\big\},
\end{aligned}$$

continuity of $\big(\gamma(\bar{t},\cdot)_{|\widetilde{W}}\big)^{-1} : \gamma(\bar{t},\widetilde{W}) \to \widetilde{W}$ follows directly from the continuity of $(F_{|I_i \times \widetilde{W}})^{-1} : F(I_i \times \widetilde{W}) \to I_i \times \widetilde{W}$. Therefore, $\gamma(\bar{t},\cdot)_{|\widetilde{W}}$ is an embedding. $\square$

A.3 Technical and Auxiliary Statements for Hölder Spaces

In this section, we gather some statements that are used in Section 2.2 on Hölder spaces and whose proofs are quite technical. First, we analyze how Hölder regularity of functions with compact support transfers to larger domains. As a second step, the cumbersome definition of Hölder regularity on submanifolds from Definition 2.75 is simplified for the case of closed submanifolds (see Lemma A.14). Afterwards, we explain the patching strategy from Remark 2.46 in detail. Finally, the proof of Lemma 2.77(ii) is performed, which yields a representation for Hölder functions on embedded submanifolds.

Lemma A.11. *Let $W \subset \mathbb{R}^d$ be open and bounded and let $W' \subset \mathbb{R}^d$ be open with $W' \subset W$. Moreover, let X be a Banach space and let $u : W \to X$ be such that we have $\operatorname{supp} u \subset W'$ and $u \in \mathcal{C}^s(\overline{W'}, X)$. This implies $u \in \mathcal{C}^s(\overline{W}, X)$ with $\|u\|_{C^s(\overline{W},X)} \lesssim \|u\|_{C^s(\overline{W'},X)}$.*

Proof. Because $\operatorname{supp} u \subset \mathbb{R}^d$ is compact and $W' \subset \mathbb{R}^d$ is open with $\operatorname{supp} u \subset W' \subset W$, obviously, $u \in C^{\lfloor s \rfloor}(\overline{W}, X)$ holds with $\|u\|_{C^{\lfloor s \rfloor}(\overline{W},X)} = \|u\|_{C^{\lfloor s \rfloor}(\overline{W'},X)}$. But to transfer (uniform) Hölder regularity from $\overline{W'}$ to the larger set $\overline{W}$, we need a more subtle argument. In the following, assume $\alpha := s - \lfloor s \rfloor \in (0,1)$ and fix $|\beta| = \lfloor s \rfloor$. We will show $D^\beta u \in \mathcal{C}^\alpha(\overline{W}, X)$ with $[D^\beta u]_{C^\alpha(\overline{W},X)} \lesssim \|D^\beta u\|_{C^\alpha(\overline{W'},X)}$.
For this, we introduce open subsets $\mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{R}^d$ with $\operatorname{supp} u \subset \mathcal{O}_1 \subset \overline{\mathcal{O}_1} \subset \mathcal{O}_2 \subset \overline{\mathcal{O}_2} \subset W'$, which is possible due to the fact that $\operatorname{supp} u$ is compact and W' is open. Now, define $A_1 := \overline{\mathcal{O}_2}$, $A_2 := \overline{W} \setminus \mathcal{O}_2$, $U_1 := W'$ and $U_2 := \overline{W} \setminus \overline{\mathcal{O}_1}$. Then, $A_1, A_2 \subset \overline{W}$ are compact, $U_1, U_2 \subset \overline{W}$ are open with $A_1 \subset U_1$, $A_2 \subset U_2$ and $\overline{W} \subset A_1 \cup A_2$. Therefore, Lemma 2.69 yields

$$[D^\beta u]_{C^\alpha(\overline{W},X)} = [D^\beta u]_{C^\alpha(A_1 \cup A_2,X)} \lesssim \|D^\beta u\|_{C^0(\overline{W},X)} + [D^\beta u]_{C^\alpha(\overline{U_1},X)} + [D^\beta u]_{C^\alpha(\overline{U_2},X)}$$

as well as

$$[D^\beta u]^R_{C^\alpha(\overline{W},X)} = [D^\beta u]^R_{C^\alpha(A_1 \cup A_2,X)} \le [D^\beta u]^R_{C^\alpha(\overline{U_1},X)} + [D^\beta u]^R_{C^\alpha(\overline{U_2},X)}$$

for $R \in (0, \infty]$ sufficiently small. By assumption, $D^\beta u \in \mathcal{C}^\alpha(\overline{U_1}, X)$ holds. On account of

$$\overline{U_2} \cap \operatorname{supp} u = (\overline{W} \setminus \mathcal{O}_1) \cap \operatorname{supp} u = \varnothing,$$

we have $u \equiv 0$ on $\overline{U_2}$ and thus also $D^\beta u \in \mathcal{C}^\alpha(\overline{U_2}, X)$ holds with $[D^\beta u]_{C^\alpha(\overline{U_2},X)} = 0$. Hence, $D^\beta u \in \mathcal{C}^\alpha(\overline{W}, X)$ follows with $[D^\beta u]_{C^\alpha(\overline{W},X)} \lesssim \|D^\beta u\|_{C^\alpha(\overline{W'},X)}$. □

Remark A.12. *On account of the remark after Lemma 2.69, the statement from Lemma A.11 also holds for $W = \mathbb{R}^d$ if $W' \subset \mathbb{R}^d$ remains bounded and we restrict to bounded functions $\mathcal{C}^s_b(\mathbb{R}^d, X)$.*

Lemma A.13. *Let $M \subset \mathbb{R}^n$ be a $C^1 \cap \mathcal{C}^s$-embedded closed submanifold. Futher, let X be a Banach space and let $f : M \to X$ be such that there exists a local parameterization (γ, W) of M with $\operatorname{supp} f \subset \gamma(W)$ and $f \circ \gamma \in \mathcal{C}^s(\overline{W}, X)$. This implies $f \in \mathcal{C}^s(M, X)$ with $\|f\|_{C^s(M,X)} = \|f \circ \gamma\|_{C^s(\overline{W},X)}$.*

Proof. We fix a set of local parameterizations $(\gamma_p, W_p)_{p \in M}$ of M with $M \subset \bigcup_{p \in M} \gamma_p(W_p)$ and $(\gamma_{p_0}, W_{p_0}) = (\gamma, W)$ for some $p_0 \in M$. As $\operatorname{supp} f \subset \gamma(W)$ is compact, we can assume w.l.o.g. that $\operatorname{supp} f \cap \gamma_p(W_p) = \varnothing$ for all $p \neq p_0$. In particular, we have $f \circ \gamma_p \equiv 0$ on $\overline{W_p}$ for all $p \neq p_0$ and thus $f \circ \gamma_p \in \mathcal{C}^s(\overline{W_p}, X)$ for all $p \in M$. This implies $f \in \mathcal{C}^s(M, X)$. Due to the closedness of M, we can reduce to a finite set $(\gamma_l, W_l)_{l=1,\dots,L} \subset (\gamma_p, W_p)_{p \in M}$ of local parameterizations with $M \subset \bigcup_{l=1}^L \gamma_l(W_l)$ and $(\gamma_1, W_1) = (\gamma, W)$ and hence obtain

$$\|f\|_{C^s(M,X)} = \sum_{l=1}^L \|f \circ \gamma_l\|_{C^s(\overline{W_l},X)} = \|f \circ \gamma\|_{C^s(\overline{W},X)}.$$ □

Lemma A.14. *Let $M \subset \mathbb{R}^n$ be a $C^1 \cap \mathcal{C}^s$-embedded submanifold such that M is closed or $s \in \mathbb{N}_{\ge 0}$ holds. Furthermore, let X be a Banach space and let $f \in \mathcal{C}^s(M, X)$, i.e., for every $p \in M$ there exists a local parameterization (γ_p, W_p) of M with $p \in W_p$ and $f \circ \gamma_p \in \mathcal{C}^s(\overline{W_p}, X)$. For any further local parameterization (γ, W) of M, this implies $f \circ \gamma \in \mathcal{C}^s(\overline{W}, X)$.*

Proof. Let $d \in \mathbb{N}_{>0}$ be the dimension of M. By choosing the sets W_p smaller if necessary, we can assume that the local parameterizations (γ_p, W_p) from the assumption allow for suitable charts (ϕ_p, U_p) with $\gamma_p(W_p) \subset U_p$ and $\mathrm{pr}_{\mathbb{R}^d} \circ \phi_{p|\gamma_p(\overline{W_p})} = \gamma_p^{-1}$ as in Remark 2.6(ii). We use $\alpha \coloneqq s - \lfloor s \rfloor$ as short notation and fix $p \in M$.
By definition, the sets $U_p \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^d$ are open, bounded and convex. Moreover, $\phi_p \in \mathcal{C}^s(\overline{U_p}, \mathbb{R}^n) \cap C^1(\overline{U_p}, \mathbb{R}^n)$ and $\gamma \in \mathcal{C}^s(\overline{W}, \mathbb{R}^n) \cap C^1(\overline{W}, \mathbb{R}^n)$ hold and $\phi_p : \overline{U_p} \to \mathbb{R}^n$ and $\gamma : \overline{W} \to \mathbb{R}^n$ are Lipschitz continuous by Remark 2.6(i). We define $V \coloneqq \gamma(W)$, $V_p \coloneqq \gamma_p(W_p)$ and

$$W_{\cap p} \coloneqq \gamma^{-1}(V \cap V_p) = \gamma^{-1}\big(\gamma(W) \cap \gamma_p(W_p)\big).$$

Then, $W_{\cap p} \subset \mathbb{R}^d$ is an open and bounded subset with $\overline{W_{\cap p}} \subset \overline{W}$. Because of the inclusions $\gamma(\overline{W_{\cap p}}) \subset \gamma_p(\overline{W_p})$ and $(\mathrm{pr}_{\mathbb{R}^d} \circ \phi_p)\big(\gamma_p(\overline{W_p})\big) \subset \overline{W_p}$, the differentiability statement

$$f \circ \gamma = (f \circ \gamma_p) \circ (\mathrm{pr}_{\mathbb{R}^d} \circ \phi_p) \circ \gamma \in C^{\lfloor s \rfloor}(\overline{W_{\cap p}}, \mathbb{R})$$

follows directly by composition of differentiable operators. Showing Hölder regularity for the composition $f \circ \gamma$ on $\overline{W_{\cap p}}$ is more involved and we will use Proposition 2.97 (which relies on Lemma 2.68) for this. As we have $\gamma(\overline{W_{\cap p}}) \subset \overline{U_p}$ and γ is Lipschitz continuous, Proposition 2.97 and Remark 2.98 yield

$$\phi_p \circ \gamma \in \mathcal{C}^s\big(\overline{W_{\cap p}}, \mathbb{R}^n\big)$$

with $D^\beta(\phi_p \circ \gamma) \in \mathcal{C}^\alpha(\overline{W_{\cap p}}, \mathbb{R}^n)$ for all $|\beta| \le \lfloor s \rfloor$. On account of $\gamma(\overline{W_{\cap p}}) \subset \gamma_p(\overline{W_p})$, this implies

$$\gamma_p^{-1} \circ \gamma = \mathrm{pr}_{\mathbb{R}^d} \circ \phi_p \circ \gamma \in \mathcal{C}^s\big(\overline{W_{\cap p}}, \mathbb{R}^d\big)$$

with $D^\beta(\gamma_p^{-1} \circ \gamma) \in \mathcal{C}^\alpha(\overline{W_{\cap p}}, \mathbb{R}^d)$ for all $|\beta| \le \lfloor s \rfloor$. Due to the Lipschitz continuity of ϕ_p and γ, also $\gamma_p^{-1} \circ \gamma : \overline{W_{\cap p}} \to \mathbb{R}^d$ is Lipschitz continuous and we have $(\gamma_p^{-1} \circ \gamma)(\overline{W_{\cap p}}) \subset \overline{W_p}$. With $f \circ \gamma_p \in \mathcal{C}^s(\overline{W_p}, X)$, another application of Proposition 2.97 and Remark 2.98 yields

$$f \circ \gamma = (f \circ \gamma_p) \circ (\gamma_p^{-1} \circ \gamma) \in \mathcal{C}^s\big(\overline{W_{\cap p}}, X\big)$$

with $\|f \circ \gamma\|_{C^s(\overline{W_{\cap p}}, X)} \lesssim \|f \circ \gamma_p\|_{C^s(\overline{W_p}, X)}$.
We have to show $f \circ \gamma \in \mathcal{C}^s(\overline{W}, X)$ on the whole set $\overline{W}$. Because $M = \bigcup_{p \in M} \gamma_p(W_p)$ implies $\overline{W} = \bigcup_{p \in M} \overline{W_{\cap p}}$, the differentiability statement $f \circ \gamma \in C^{\lfloor s \rfloor}(\overline{W}, X)$ follows directly. But to transfer (uniform) Hölder regularity to a union of sets, we have to argue more subtly: As in Remark 2.6(ii), we can assume the existence of open subsets $A_p \subset M$ with $K_p \coloneqq \overline{A_p} \subset \gamma_p(W_p)$ and $M \subset \bigcup_{p \in M} A_p$. Then, as M is closed, we can reduce to a finite set $(A_l)_{l=1,\dots,L}$ of open subsets with $M \subset \bigcup_{l=1}^L A_l$ and correspondingly to finite sets $(K_l)_{l=1,\dots,L}$ of compact sets and $(\gamma_l, W_l)_{l=1,\dots,L}$ of local parameterizations. We use the notation $V_l \coloneqq \gamma_l(W_l)$ and $W_{\cap l} \coloneqq \gamma^{-1}(V \cap V_l)$ as above. The set $\overline{W} \subset \mathbb{R}^d$ is compact and for every $l \in \{1, \dots, L\}$,

$$\gamma^{-1}\big(\overline{V \cap K_l}\big) \quad \text{and} \quad \gamma^{-1}\big(\overline{V} \cap V_l\big)$$

are a compact and an open subset of $\overline{W}$, respectively, with

$$\gamma^{-1}\big(\overline{V\cap K_l}\big)\subset\gamma^{-1}\big(\overline{V}\cap K_l\big)\subset\gamma^{-1}\big(\overline{V}\cap V_l\big).$$

Furthermore, we have

$$\begin{aligned}\bigcup_{l=1}^{L}\gamma^{-1}\big(\overline{V\cap K_l}\big)=\gamma^{-1}\left(\overline{V\cap\bigcup_l K_l}\right)=\gamma^{-1}\big(\overline{V\cap M}\big)=\gamma^{-1}\big(\overline{V}\big)=\overline{W}\\ \text{and}\quad\gamma^{-1}\big(\overline{V}\cap V_l\big)\subset\gamma^{-1}\left(\overline{\overline{V}\cap V_l}\right)=\gamma^{-1}\big(\overline{V\cap V_l}\big)=\overline{W_{\cap l}}\end{aligned}$$

for every $l\in\{1,...,L\}$. Therefore, Lemma 2.69 yields

$$\begin{aligned}&[D^\beta(f\circ\gamma)]_{C^\alpha(\overline{W},X)}=[D^\beta(f\circ\gamma)]_{C^\alpha(\bigcup_l\gamma^{-1}(\overline{V\cap K_l}),X)}\\&\lesssim\sum_{l=1}^{L}\|D^\beta(f\circ\gamma)\|_{C^\alpha(\gamma^{-1}(\overline{V}\cap V_l),X)}\le\sum_{l=1}^{L}\|D^\beta(f\circ\gamma)\|_{C^\alpha(\overline{W_{\cap l}},X)}\end{aligned}$$

and

$$\begin{aligned}&[D^\beta(f\circ\gamma)]^R_{C^\alpha(\overline{W},X)}=[D^\beta(f\circ\gamma)]^R_{C^\alpha(\bigcup_l\gamma^{-1}(\overline{V\cap K_l}),X)}\\&\le\sum_{l=1}^{L}[D^\beta(f\circ\gamma)]^R_{C^\alpha(\gamma^{-1}(\overline{V}\cap V_l),X)}\le\sum_{l=1}^{L}[D^\beta(f\circ\gamma)]^R_{C^\alpha(\overline{W_{\cap l}},X)}\end{aligned}$$

for every $|\beta|=\lfloor s\rfloor$ and for sufficiently small $R>0$. Hence, $D^\beta(f\circ\gamma)\in\mathcal{C}^\alpha(\overline{W},X)$ follows for all $|\beta|=\lfloor s\rfloor$ and this proves the claim. □

Lemma A.15. *Let $M\subset\mathbb{R}^n$ be a $C^1\cap\mathcal{C}^s$-embedded closed submanifold and let $(\gamma_l,W_l)_{l=1,...,L}$ be a finite set of local parameterizations of M with $M\subset\bigcup_l\gamma_l(W_l)$. Moreover, let X be a Banach space and for every $l\in\{1,...,L\}$, let $u_l:\gamma_l(\overline{W_l})\to X$ be with $u_l\circ\gamma_l\in\mathcal{C}^s(\overline{W_l},X)$. Choose $(\psi_l)_{l=1,...,L}$ to be a partition of unity subordinate to $\big(\gamma_l(W_l)\big)_{l=1,...,L}$. Then,*

$$u:=\sum_{l=1}^{L}\psi_l u_l\in\mathcal{C}^s(M,X)$$

holds with $\|u\|_{C^s(M,X)}\lesssim\sum_l\|u_l\circ\gamma_l\|_{C^s(\overline{W_l},X)}$.

Proof. For $l=1,...,L$, we define

$$\tilde{u}_l:=\psi_l u_l:M\to X.$$

Then, supp $\tilde{u}_l\subset\gamma_l(W_l)$ holds and we have $\tilde{u}_l\circ\gamma_l=(\psi_l\circ\gamma_l)\cdot(u_l\circ\gamma_l)\in\mathcal{C}^s(\overline{W_l},X)$ with $\|\tilde{u}_l\circ\gamma_l\|_{C^s(\overline{W_l},X)}\lesssim\|u_l\circ\gamma_l\|_{C^s(\overline{W_l},X)}$ by Proposition 2.94. Therefore, Lemma A.13 yields $\tilde{u}_l\in\mathcal{C}^s(M,X)$ with $\|\tilde{u}_l\|_{C^s(M,X)}=\|\tilde{u}_l\circ\gamma_l\|_{C^s(\overline{W_l},X)}\lesssim\|u_l\circ\gamma_l\|_{C^s(\overline{W_l},X)}$. This implies

$$u=\sum_{l=1}^{L}\psi_l u_l=\sum_{l=1}^{L}\tilde{u}_l\in\mathcal{C}^s(M,X)$$

with $\|u\|_{C^s(M,X)}\lesssim\sum_l\|u_l\circ\gamma_l\|_{C^s(\overline{W_l},X)}$. □

Corollary A.16. *Let $M \subset \mathbb{R}^n$ be a $C^1 \cap \mathcal{C}^s$-embedded closed submanifold and we fix a finite set of local parameterizations $(\gamma_l, W_l)_{l=1,\dots,L}$ of M with $M \subset \bigcup_l \gamma_l(W_l)$. Moreover, let X, Y, Z be Banach spaces, let $U \subset Y$ be an open subset and let $k \in \mathbb{N}_{\geq 0}$. For every $l \in \{1, \dots, L\}$, let F_l, G_l be with*

$$u \mapsto F_l(u) \circ \gamma_l \in C^k_{(b)}\big(U, \mathcal{C}^s(\overline{W_l}, X)\big) \text{ and}$$
$$(u,z) \mapsto G_l(u)[z] \circ \gamma_l \in C^k_{(b)}\big(U, \mathcal{L}\big(Z, \mathcal{C}^s(\overline{W_l}, X)\big)\big).$$

Finally, let $(\psi_l)_{l=1,\dots,L}$ be a partition of unity subordinate to $\big(\gamma_l(W_l)\big)_{l=1,\dots,L}$. Then, we have

$$u \mapsto F(u) := \sum_l \psi_l F_l(u) \in C^k_{(b)}\big(U, \mathcal{C}^s(M, X)\big) \text{ and}$$
$$(u,z) \mapsto G(u)[z] := \sum_l \psi_l G_l(u)[z] \in C^k_{(b)}\big(U, \mathcal{L}\big(Z, \mathcal{C}^s(M, X)\big)\big).$$

Proof. By Lemma A.15, $F(u), G(u)[z] \in \mathcal{C}^s(M, X)$ holds for all $u \in U$ and $z \in Z$. Due to the estimate for the norms in Lemma A.15, also $G(u) \in \mathcal{L}\big(Z, \mathcal{C}^s(M,X)\big)$ holds for all $u \in U$ and we have

$$u \mapsto F(u) := \sum_l \psi_l F_l(u) \in C^0_{(b)}\big(U, \mathcal{C}^s(M, X)\big) \text{ and}$$
$$(u,z) \mapsto G(u)[z] := \sum_l \psi_l G_l(u)[z] \in C^0_{(b)}\big(U, \mathcal{L}\big(Z, \mathcal{C}^s(M, X)\big)\big).$$

It also implies

$$\begin{aligned}
&\lim_{\|y\|_Y \to 0} \frac{\|F(u+y) - F(u) - \sum_l \psi_l DF_l(u)[y]\|_{C^s(M,X)}}{\|y\|_Y} \\
&= \lim_{\|y\|_Y \to 0} \frac{\big\|\sum_l \psi_l\big(F_l(u+y) - F_l(u) - DF_l(u)[y]\big)\big\|_{C^s(M,X)}}{\|y\|_Y} \\
&\lesssim \lim_{\|y\|_Y \to 0} \sum_{l=1}^{L} \frac{\big\|\big(F_l(u+y) - F_l(u) - DF_l(u)[y]\big) \circ \gamma_l\big\|_{C^s(\overline{W_l},X)}}{\|y\|_Y} = 0
\end{aligned}$$

for all $u \in U$. Thus, F is Fréchet-differentiable with $DF(u)[y] = \sum_l \psi_l DF_l(u)[y]$. Hence, $DF_l \circ \gamma_l \in C^0_{(b)}\big(U, \mathcal{L}\big(Y, \mathcal{C}^s(\overline{W_l}, X)\big)\big)$ implies $DF \in C^0_{(b)}\big(U, \mathcal{L}\big(Y, \mathcal{C}^s(M,X)\big)\big)$ and therefore $F \in C^1_{(b)}\big(U, \mathcal{C}^s(M, X)\big)$.
Finally, the statement $G \in C^1_{(b)}\big(U, \mathcal{L}\big(Z, \mathcal{C}^s(M,X)\big)\big)$ as well as the claim for $k > 1$ follow recursively. □

Lemma A.17. *Let $M \subset \mathbb{R}^n$ be a C^1-embedded closed submanifold, let X be a Banach space and let $\alpha \in (0,1)$. We have*

$$C^\alpha(M, X) = \Big\{f \in C^0(M, X) \,\Big|\, [f]_{C^\alpha(M,X)} < \infty\Big\} \text{ and}$$
$$h^\alpha(M, X) = \Big\{f \in C^\alpha(M, X) \,\Big|\, \lim_{R \to 0} [f]^R_{C^\alpha(M,X)} = 0\Big\}.$$

Furthermore,

$$\|\cdot\|_{C^\alpha(M,X)} \sim \|\cdot\|_{C^0(M,X)} + [\cdot]_{C^\alpha(M,X)}$$

are equivalent norms on $\mathcal{C}^\alpha(M,X)$ *and for sufficiently small* $R \in (0,\infty]$

$$\sum_l [\cdot \circ \gamma_l]^{\sim R}_{C^\alpha(\overline{W_l},X)} \sim [\cdot]^R_{C^\alpha(M,X)}$$

holds for any finite set of local parameterizations $(\gamma_l, W_l)_{l=1,\ldots,L}$ *of* M *for which there exists a set of charts* $(\phi_l, U_l)_{l=1,\ldots,L}$ *with* $\gamma_l(W_l) \subset U_l$ *and* $\mathrm{pr}_{\mathbb{R}^d} \circ \phi_{l\,|\gamma_l(\overline{W_l})} = \gamma_l^{-1}$ *and a set of compact subsets* $A_l \subset \gamma_l(W_l)$ *with* $M \subset \bigcup_l A_l$*.*

Proof. Let $f : M \to X$ and define $V_l \coloneqq \gamma_l(W_l)$. Because $\gamma_l : \overline{W_l} \to \gamma_l(\overline{W_l})$ is Lipschitz continuous by Remark 2.6(i), Lemma 2.68 yields

$$[f \circ \gamma_l]^{\sim R}_{C^\alpha(\overline{W_l},X)} \lesssim [f]^R_{C^\alpha(\gamma_l(\overline{W_l}),X)} \leq [f]^R_{C^\alpha(M,X)}$$

and thus

$$\sum_l [f \circ \gamma_l]^{\sim R}_{C^\alpha(\overline{W_l},X)} \lesssim [f]^R_{C^\alpha(M,X)}$$

for any $R \in (0,\infty]$. Analogously, because $\varphi_l \coloneqq \gamma_l^{-1} = \mathrm{pr}_{\mathbb{R}^d} \circ \phi_{l\,|\overline{V_l}} : \overline{V_l} \to \overline{W_l}$ is Lipschitz continuous by Remark 2.6(i), Lemma 2.68 yields

$$[f]^R_{C^\alpha(\overline{V_l},X)} = [(f \circ \gamma_l) \circ \varphi_l]^R_{C^\alpha(\gamma_l(\overline{W_l}),X)} \lesssim [f \circ \gamma_l]^{\sim R}_{C^\alpha(\overline{W_l},X)}$$

for all $R \in (0,\infty]$. With Lemma 2.69,

$$[f]_{C^\alpha(M,X)} = [f]_{C^\alpha(\bigcup_l A_l,X)} \lesssim \sum_l \|f\|_{C^0(V_l,X)} + [f]_{C^\alpha(V_l,X)} \lesssim \sum_l \|f \circ \gamma_l\|_{C^\alpha(\overline{W_l},X)}$$

and

$$[f]^R_{C^\alpha(M,X)} = [f]^R_{C^\alpha(\bigcup_l A_l,X)} \leq \sum_l [f]^R_{C^\alpha(V_l,X)} \lesssim \sum_l [f \circ \gamma_l]^{\sim R}_{C^\alpha(\overline{W_l},X)}$$

follow for $R > 0$ sufficiently small. Together, we hence have

$$\|f\|_{C^0(M,X)} + [f]_{C^\alpha(M,X)} \sim \sum_l \|f \circ \gamma_l\|_{C^\alpha(\overline{W_l},X)} = \|f\|_{C^\alpha(M,X)}$$

$$\text{and } [f]^R_{C^\alpha(M,X)} \sim \sum_l [f \circ \gamma_l]^{\sim R}_{C^\alpha(\overline{W_l},X)} \text{ for } R > 0 \text{ sufficiently small.}$$

In particular, this implies

$$\begin{aligned} C^\alpha(M,X) &= \{f : M \to X \,|\, f \circ \gamma_l \in C^\alpha(\overline{W_l},X) \text{ for all } l = 1,\ldots,L\} \\ &= \{f \in C^0(M,X) \,|\, \|f \circ \gamma_l\|_{C^\alpha(\overline{W_l},X)} < \infty \text{ for all } l = 1,\ldots,L\} \\ &= \{f \in C^0(M,X) \,|\, \|f\|_{C^0(M,X)} + [f]_{C^\alpha(M,X)} < \infty\} \\ &= \{f \in C^0(M,X) \,|\, [f]_{C^\alpha(M,X)} < \infty\} \end{aligned}$$

as well as

$$\begin{aligned} h^\alpha(M,X) &= \{f : M \to X \,|\, f \circ \gamma_l \in h^\alpha(\overline{W_l},X) \text{ for all } l = 1,\ldots,L\} \\ &= \{f \in C^\alpha(M,X) \,|\, \lim_{R\to 0} [f \circ \gamma_l]^R_{C^\alpha(\overline{W_l},X)} = 0 \text{ for all } l = 1,\ldots,L\} \\ &= \{f \in C^\alpha(M,X) \,|\, \lim_{R\to 0} [f]^R_{C^\alpha(M,X)} = 0\}. \end{aligned}$$

□

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