Supplement to Long Memory and the Term Strukture of Risk: Some Monte Carlo Results on Semiparametric Long Memory Estimation

Peter C. Schotman[a,b,c] Rolf Tschernig[d] Jan Budek[e]

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Affiliations: [a] Maastricht University, [b] CEPR, [c] NETSPAR, [d] University of Regensburg, [e] KPMG

Correspondence address: Rolf Tschernig
Institute of Economics and Econometrics
University of Regensburg, Universittsstr. 31
93040 Regensburg
E-Mail: rolf.tschernig@wiwi.uni-regensburg.de
Phone: (+49) 941 943 2737
Fax: (+49) 941 943 4917

1 Introduction

In this supplement we document the results of two small Monte Carlo simulations covering the estimation of the long memory properties of univariate and bivariate time series data. These Monte Carlo results provide the basis for the choice of the semiparametric estimators in the main paper *Long Memory and the Term Strukture of Risk*.

The following estimators are considered: the Gaussian semiparametric estimator (GSP) introduced in Künsch (1987) and discussed by Robinson and Henry (1999), the exact Whittle likelihood estimator (EW) proposed by Shimotsu and Phillips (2005), and the bivariate Whittle (BEW) likelihood estimator proposed by Sun and Phillips (2004).

The last estimator is specifically designed to estimating the degree of long memory if the underlying long memory process is polluted by additive noise. An example that was recently studied by Sun and Phillips (2004) is the nominal interest rate. They observe that the observable real interest rate $r_{tb,t+1}$ is actually an unknown quantity at time t when the nominal interest rate $r_{nom,t}$ is fixed for period $t+1$:

$$r_{nom,t} = r_{tb,t+1} + inflation_{t+1} \tag{1}$$

$$= E_t[r_{tb,t+1}] - \eta_{t+1} + E_t[inflation_{t+1}] + \eta_{t+1} \tag{2}$$

The persistence or more specific the degree of long memory characterizing the nominal rate is therefore determined by the *expected* real interest rate expected at the end of period t. When using ex-post real interest rates $r_{tb,t+1}$, the object of interest $E_t[r_{tb,t+1}]$ is polluted by the noise component η_{t+1}.

Since interest rates are also characterized by conditional heteroskedasticity, it is worth noting that among all estimators mentioned only the GSP estimator was shown to have an asymptotic distribution that is robust to conditional heteroskedasticity (Robinson and Henry (1999)).

The choice of bandwidth, that is, the number J of Fourier frequencies that is used for estimation is an important issue. We compare the various methods for automatic bandwidth choice proposed by Henry and Robinson (1996) and compare them to simple rule of thumb bandwidths. Note that the automatic bandwidth estimators are designed to adapt to potentially underlying short memory. Therefore we report the results of a small Monte Carlo study that compares their bandwidth estimators with simple rule-of-thumb methods.

All Monte Carlo simulations are based on variants of the ARFIMA(1,d,0) process

$$(1 - \alpha L)(1 - L)^d x_t = \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_t^2)$$

The data generating mechanisms including sample sizes that were selected include also specifications of ARFIMA models similar to those estimated for the state variables in main paper. Each realization is obtained by applying the Choleski decomposition to the $T \times T$ covariance matrix of the process where the exact autocovariance function is computed using the method suggested by Sowell (1992). In all simulations sample size is $T = 218$.

Section 2 will report the MC results on bandwidth selection. The MC results on the various semiparametric estimators are contained in section 3.

2 Bandwidth selection for semiparametric long memory estimation

The considered ARFIMA$(1, d, 0)$ specifications are:

- white noise, $d = \alpha = 0$,
- pure long memory processes with $d = 0.4, 0.7, 0.9$, $\alpha = 0$,
- AR(1) processes with $d = 0$ and $\alpha = 0.5, 0.9$,
- ARFIMA(1,d,0) processes with $(d, \alpha) = (0.4, 0.5), (0.7, 0.5), (0.9, 0.5)$.

The error variance is set to $\sigma_t^2 = 1$. The number of replications is 1000.

The semiparametric estimation is based on the GSP estimator which underlies the theoretical derivation of the plug-in bandwidth selectors of Henry and Robinson (1996). Since the asymptotic estimation properties of the GSP estimator depend on the level of the long memory parameter, the estimators are applied to the original series and its first differences.

The bandwidth selectors of Henry and Robinson (HR) (1996) are called 'first feasible' and 'direct'. Thus, the following bandwidth estimation methods are considered:

- first feasible and direct method
- rule-of-thumb: $m = floor(T^a)$ with $a = 0.50, 0.55$

Table 2 displays the mean squared error for each bandwidth selector and Table 1 displays the number of low Fourier frequencies underlying the estimation. The conclusions of this small Monte Carlo study are:

- the bandwidth selectors of Henry and Robinson always select a larger number of frequencies than the rule-of-thumb estimators;
- **pure short memory processes (AR(1))**:
  - if $\alpha = 0.5$, taking first differences produces a large bias and the HR methods perform worse than the rule-of-thumb estimators due to the inclusion of too many frequencies;
  - if $\alpha = 0.9$, all methods exhibit a large mean squared error between 0.3 and 0.7; for this case and sample size it is hard to distinguish long memory and 'strong' short memory;
  - overall, the rule-of-thumb with $\alpha = 0.5$ performs best; Both automatic rules fail dramatically!
- **pure long memory**:
  the direct HR method is by far the best if one correctly differences or not. The reason for this result is that they use many more frequencies than the naive rule-of-thumb bandwidths.
- **short and long memory**:
  now the rule-of-thumb bandwidth estimators perform much better, independently of prior differencing or not. The automatic bandwidth selectors do not recognize the short memory component adequately so that too many frequencies are taken into account.

Overall result: a **robust choice** is to take the level series with the **rule-of-thumb bandwidth** $J = floor\,(218^{0.5}) = 14$;[1] it exhibits the smallest mean squared error for $d = 0$ and $\alpha = 0.9$ while performing quite well for all other situations; in contrast, taking first differences leads to a worse performance for short memory processes. For $d > 0.5$, taking first differences may be worthwhile.

[1] In the main paper $J = ceil(218^{0.5}) = 15$ is used.

3 Semiparametric long memory estimation of potentially noise polluted long memory processes

In this section we document the results of a small Monte Carlo simulation covering the long memory estimation of univariate and bivariate time series possibly exhibiting conditional heteroskedasticity and long memory perturbated by noise.

The data generating mechanisms that were selected in particular include specifications of ARFIMA models similar to those estimated for the state variables in the main paper. All bivariate specifications are either directly or indirectly based on

$$(1-\alpha L)(1-L)^d x_{it} = \varepsilon_{it}, \quad \varepsilon_{it} \sim N(0, \sigma_{it}^2), \quad i = 1, 2,$$

where the data generating mechanisms w.r.t. to the conditional mean are:

- white noise, $d = \alpha = 0$,
- pure long memory with $d = 0.4, 0.8$, $\alpha = 0$,
- short memory: AR(1) with $d = 0$ and $\alpha = 0.5, 0.8$,
- short and long memory: ARFIMA(1,d,0) processes with $(d, \alpha) = (0.4, 0.5), (0.8, -0.2), (0.8, 0.2)$.

W.r.t. the (conditional) variances two variants will be considered:

- homoskedastic errors
$$\sigma_{it}^2 = 1, \quad \varepsilon_t \text{ is i.i.d.}$$
- heteroskedastic GARCH(1,1) errors:
$$\begin{aligned} \sigma_{it}^2 &= (1 - 0.2 - 0.7) + 0.2\varepsilon_{it-1}^2 + 0.7\sigma_{it}^2, \quad \sigma_{i0}^2 = 1 \\ \varepsilon_{it} &= \sigma_{it}\xi_{it}, \quad \xi_{it} \sim i.i.d.N(0,1) \end{aligned}$$

Finally, each of the above specification is also polluted by adding noise

$$y_{1t} = x_{2t} - u_t, \quad u_t \sim i.i.d.N(0, 12), \quad u_t, \xi_{it} \text{ are independent} \tag{3}$$
$$y_{2t} = x_{2t} + u_t, \tag{4}$$

in order to represent the case given by (2).

Sun and Phillips (2004) derive the asymptotic bias for the univariate EW estimator for that case and provide empirical evidence for this bias in GSP and EW long memory estimates of the U.S. real interest rate. This bias can be reduced by a variant of a bivariate exact Whittle (BEW) estimator that explicitly takes into account the specific noise structure in (4). Sun and Phillips (2004) show in a small Monte Carlo study that their estimator reduces the bias, however, at the cost of possibly increasing the variance.

Tables 3 to 6 contain the Monte Carlo results for all four setups. All simulations are done for sample size $T = 218$ and with $m = ceil\left(T^{0.5}\right) = 15$ frequencies. In order to guarantee convergence of the BEW estimator the range of long memory estimates is restricted to the interval $[-1, 2]$. The number of replications is 500.

For each DGP the tables show the mean squared error, the bias and the standard deviation for each estimator. Since the asymptotic estimation properties of the GSP estimator depend on d, it is applied to the level series as well as to the differenced series.

Homoskedastic errors

1. **No noise added**:

   (a) **all processes**:

   - the GSP estimator based on levels and the EW estimator perform equally well while the GSP estimator based on first differences only works well if d is nonstationary;
   - the BEW is almost always outperformed;
   - the best estimators can handle DGPs with exclusive short memory if it is not too strong, say an AR component of 0.5. If, however, one chooses 0.8 instead, all semiparametric estimators considered exhibit a large bias of about 0.3. This result is well known from other Monte Carlo studies, including the one of section 2, and is caused by a rather similar shape of the spectral densities at low frequencies.

   Note that the results w.r.t. differencing or not are in line with the Monte Carlo study for univariate time series presented in the previous section

2. **Noise added**:

    (a) **short memory**: the GSP estimator for level data performs best for stationary AR processes, followed by the EW estimator and the BEW estimator.

    (b) **stationary long and short memory**:
    the GSP estimator for first differences does slightly better than the BEW estimator;

    (c) **nonstationary long and short memory**: the BEW estimator is clearly the best. Its bias is negligible compared to the large bias of about 0.3 for the GSP and EW estimators. The latter bias corresponds to the asymptotic bias derived by Sun and Phillips (2004, equation (9)).

Heteroskedastic errors

The results for the conditionally heteroskedastic errors are not much different from the homoskedastic case.

Recommendations

If one explicitly expects the DGP to be

1. **without an additive noise component**,

    - take the **EW estimator**. Alternatively, one may take the **GSP estimator** in first differences but reestimate the series in levels in case a large d close to the stationarity boundary was estimated. If the level estimate is smaller, take it. In this way one can protect oneself against a potentially large positive bias of the GSP estimator in first differences in case of a pure short memory process. These results are independent of the volatility of errors.

2. **polluted by additive noise** and

    - **nonstationary long memory (and possibly short memory)**, take the **BEW estimator**; if BEW estimator is not available all other competitors are more or less alike.

- **stationary long memory (and possibly short memory)**, take the **GSP estimator based on first differences**;

Overall results: except for the nonstationary long memory case with additive noise where the BEW estimator is clearly advantageous, the **GPS estimator** after first differencing seems to be more robust against additive noise than the EW estimator. In order to avoid a potential bias to due erroneously first differencing one reestimates the series in levels and takes the level estimate if the latter is smaller.

References

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Table 1: Estimation of bandwidth: Included frequencies

d	AR	HR: first feasible		HR: direct		$\alpha = 0.50$		$\alpha = 0.55$	
		level	1st diff	level	1st diff	level	1st diff	level	1st diff
0	0	50.1	44.3	94.3	83.6	14	14	19	19
	0.5	36.6	31.0	58.1	58.0	14	14	19	19
	0.9	52.5	46.5	87.2	77.4	14	14	19	19
0.4	0	51.1	46.6	92.3	91.1	14	14	19	19
	0.5	41.2	32.0	64.9	53.9	14	14	19	19
0.7	0	51.1	48.1	88.5	92.8	14	14	19	19
	0.5	47.2	34.1	75.1	55.5	14	14	19	19
0.9	0	50.6	49.2	50.6	93.8	14	14	19	19
	0.5	53.5	35.7	85.9	57.2	14	14	19	19

Sample size: 218, error variance: 1, number of replications: 1000

Table 2: Estimation of bandwidth: Mean squared error

d	AR	HR: first feasible		HR: direct		$\alpha = 0.50$		$\alpha = 0.55$	
		level	1st diff	level	1st diff	level	1st diff	level	1st diff
0	0	.0089	.0647	.0047	.0621	.0370	.1628	.0246	.1180
	0.5	.0501	.1009	.0703	.1351	.0380	.1038	.0285	.0864
	0.9	.6156	.6577	.6025	.7071	.3318	.3903	.4072	.4591
0.4	0	.0090	.0138	.0056	.0098	.0368	.0417	.0242	.0296
	0.5	.0532	.0577	.0713	.0823	.0387	.0440	.0292	.0364
0.7	0	.0098	.0099	.0070	.0057	.0383	.0369	.0248	.0251
	0.5	.0543	.0520	.0627	.0749	.0416	.0387	.0308	.0248
0.9	0	.0087	.0091	.0076	.0049	.0357	.0370	.0225	.0247
	0.5	.0353	.0505	.0321	.0501	.0373	.0381	.0257	.0286

Sample size: 218, error variance: 1, number of replications: 1000

Table 3: Semiparametric long memory estimators: Monte Carlo results: Homoskedastic case

d	AR			GSP		EW	BEW
				level	1st diff	level	level
					$\sigma_e^2 = 0$		
0	0	MSE	$\hat{d}_1$	**0.033356**	0.14201	0.034342	0.071710
			$\hat{d}_2$	0.035240	0.15861	**0.034989**	0.062450
		Bias	$\hat{d}_1$	**-0.028643**	0.27303	-0.031693	-0.046999
			$\hat{d}_2$	-0.0088115	0.29890	**-0.0080991**	-0.014750
		Std	$\hat{d}_1$	**0.18038**	0.25975	0.18258	0.26363
			$\hat{d}_2$	0.18752	0.26319	**0.18688**	0.24946
0	0.5	MSE	$\hat{d}_1$	**0.033433**	0.088358	0.034425	0.062441
			$\hat{d}_2$	**0.037665**	0.10529	0.037802	0.059281
		Bias	$\hat{d}_1$	0.029042	0.20825	0.028549	**0.017636**
			$\hat{d}_2$	**0.049010**	0.23228	0.052655	0.052089
		Std	$\hat{d}_1$	**0.18053**	0.21211	0.18333	0.24926
			$\hat{d}_2$	0.18778	0.22658	**0.18716**	0.23784
0	0.8	MSE	$\hat{d}_1$	**0.11212**	0.16994	0.11418	0.13082
			$\hat{d}_2$	**0.12753**	0.19568	0.13089	0.14935
		Bias	$\hat{d}_1$	**0.28070**	0.36613	0.28246	0.28499
			$\hat{d}_2$	**0.30252**	0.39683	0.31040	0.31847
		Std	$\hat{d}_1$	**0.18256**	0.18944	0.18547	0.22272
			$\hat{d}_2$	0.18976	0.19547	**0.18584**	0.21893
0.4	0	MSE	$\hat{d}_1$	0.033546	0.037178	**0.032304**	0.053976
			$\hat{d}_2$	0.034957	0.041313	**0.032725**	0.052681
		Bias	$\hat{d}_1$	**-0.021637**	0.036185	-0.022897	0.025269
			$\hat{d}_2$	**0.0028487**	0.061849	0.0055298	0.041589
		Std	$\hat{d}_1$	0.18187	0.18939	**0.17827**	0.23095
			$\hat{d}_2$	0.18695	0.19362	**0.18082**	0.22572
0.4	0.5	MSE	$\hat{d}_1$	0.034247	0.039650	**0.033271**	0.057379
			$\hat{d}_2$	0.038992	0.044710	**0.036416**	0.055395
		Bias	$\hat{d}_1$	**0.034384**	0.068843	0.035058	0.072597
			$\hat{d}_2$	**0.060403**	0.095321	0.064581	0.093401
		Std	$\hat{d}_1$	0.18184	0.18684	**0.17900**	0.22827
			$\hat{d}_2$	0.18800	0.18874	**0.17957**	0.21604
0.8	-0.2	MSE	$\hat{d}_1$	0.035876	0.033337	**0.032306**	0.052575
			$\hat{d}_2$	0.031373	0.034095	**0.030805**	0.055867
		Bias	$\hat{d}_1$	**-0.012283**	-0.026535	-0.014597	0.037183
			$\hat{d}_2$	0.018791	**-0.0057433**	0.016361	0.067926
		Std	$\hat{d}_1$	0.18901	0.18064	**0.17915**	0.22626
			$\hat{d}_2$	0.17612	0.18456	**0.17475**	0.22639
0.8	0	MSE	$\hat{d}_1$	0.035789	0.033263	**0.032027**	0.055645
			$\hat{d}_2$	0.031559	0.034442	**0.031170**	0.055027
		Bias	$\hat{d}_1$	**-0.0084308**	-0.022561	-0.0099874	0.043497
			$\hat{d}_2$	0.022842	**-0.0020305**	0.021739	0.072099
		Std	$\hat{d}_1$	0.18899	0.18098	**0.17868**	0.23185
			$\hat{d}_2$	0.17617	0.18557	**0.17521**	0.22322
0.8	0.2	MSE	$\hat{d}_1$	0.035678	0.032989	**0.031953**	0.055037
			$\hat{d}_2$	0.032029	0.034854	**0.031749**	0.053574
		Bias	$\hat{d}_1$	**-1.1187e-005**	-0.012936	-0.00048381	0.050693
			$\hat{d}_2$	0.031537	**0.0071747**	0.031789	0.078216
		Std	$\hat{d}_1$	0.18889	0.18117	**0.17875**	0.22906
			$\hat{d}_2$	0.17617	0.18655	**0.17532**	0.21784

Table 4: Semiparametric long memory estimators: Monte Carlo results: Homoskedastic case, noise added

d	AR			GSP		EW	BEW
				level	1st diff	level	level
				$\sigma_e^2 = 12 * \sigma_u^2$			
0	0	MSE	$\hat{d}_1$	**0.038322**	0.15380	0.038792	0.041453
			$\hat{d}_2$	**0.039414**	0.15089	0.040208	0.048955
		Bias	$\hat{d}_1$	-0.030718	0.29753	-0.030504	**-0.030297**
			$\hat{d}_2$	**-0.031519**	0.29405	-0.032666	-0.032284
		Std	$\hat{d}_1$	**0.19333**	0.25549	0.19458	0.20133
			$\hat{d}_2$	**0.19601**	0.25383	0.19784	0.21889
0	0.5	MSE	$\hat{d}_1$	**0.036093**	0.14285	0.036673	0.056457
			$\hat{d}_2$	**0.036016**	0.14107	0.036538	0.056690
		Bias	$\hat{d}_1$	-0.016065	0.28459	-0.015387	**0.0045110**
			$\hat{d}_2$	**-0.0096082**	0.28142	-0.010998	0.018247
		Std	$\hat{d}_1$	**0.18930**	0.24870	0.19088	0.23756
			$\hat{d}_2$	**0.18953**	0.24876	0.19083	0.23740
0	0.8	MSE	$\hat{d}_1$	**0.048914**	0.15643	0.049854	0.12078
			$\hat{d}_2$	0.058190	0.16546	**0.057265**	0.12737
		Bias	$\hat{d}_1$	**0.12651**	0.33037	0.12754	0.21541
			$\hat{d}_2$	0.14725	0.34033	**0.14699**	0.24683
		Std	$\hat{d}_1$	**0.18141**	0.21747	0.18327	0.27272
			$\hat{d}_2$	0.19107	0.22279	**0.18884**	0.25776
0.4	0	MSE	$\hat{d}_1$	0.12417	**0.061777**	0.12345	0.099315
			$\hat{d}_2$	0.11758	**0.059831**	0.11602	0.10547
		Bias	$\hat{d}_1$	-0.30044	**-0.073303**	-0.29882	-0.080777
			$\hat{d}_2$	-0.28477	-0.066566	-0.28355	**-0.059814**
		Std	$\hat{d}_1$	**0.18412**	0.23750	0.18481	0.30462
			$\hat{d}_2$	0.19103	0.23537	**0.18874**	0.31920
0.4	0.5	MSE	$\hat{d}_1$	0.056079	**0.043859**	0.054637	0.076616
			$\hat{d}_2$	0.054321	**0.044296**	0.051788	0.085130
		Bias	$\hat{d}_1$	-0.15311	-0.020702	-0.15041	**-0.0072117**
			$\hat{d}_2$	-0.12758	**-0.0035278**	-0.12668	0.034146
		Std	$\hat{d}_1$	0.18066	0.20840	**0.17892**	0.27670
			$\hat{d}_2$	0.19505	0.21044	**0.18905**	0.28977
0.8	-0.2	MSE	$\hat{d}_1$	0.13869	0.12192	0.13836	**0.082740**
			$\hat{d}_2$	0.12144	0.11112	0.12322	**0.081053**
		Bias	$\hat{d}_1$	-0.31907	-0.29098	-0.32376	**-0.057383**
			$\hat{d}_2$	-0.28748	-0.26991	-0.29669	**-0.017172**
		Std	$\hat{d}_1$	0.19206	0.19301	**0.18313**	0.28186
			$\hat{d}_2$	0.19697	0.19563	**0.18760**	0.28418
0.8	0	MSE	$\hat{d}_1$	0.10631	0.099463	0.10593	**0.076903**
			$\hat{d}_2$	0.091623	0.088999	0.093400	**0.077800**
		Bias	$\hat{d}_1$	-0.26438	-0.25207	-0.27005	**-0.043223**
			$\hat{d}_2$	-0.23120	-0.22904	-0.24209	**4.9126e-005**
		Std	$\hat{d}_1$	0.19083	0.18954	**0.18166**	0.27393
			$\hat{d}_2$	0.19537	0.19116	**0.18652**	0.27893
0.8	0.2	MSE	$\hat{d}_1$	0.076363	0.076327	0.075401	**0.072448**
			$\hat{d}_2$	**0.065068**	0.066990	0.066176	0.066882
		Bias	$\hat{d}_1$	-0.20139	-0.20296	-0.20733	**-0.018244**
			$\hat{d}_2$	-0.16666	-0.17784	-0.17858	**0.013331**
		Std	$\hat{d}_1$	0.18923	0.18744	**0.18004**	0.26854
			$\hat{d}_2$	0.19312	0.18869	**0.18516**	0.25827

Table 5: Semiparametric long memory estimators: Monte Carlo results: GARCH(1,1) case

d	AR			GSP		EW	BEW
				level	1st diff	level	level
					$\sigma_e^2 = 0$		
0	0	MSE	$\hat{d}_1$	**0.039869**	0.15230	0.041714	0.079580
			$\hat{d}_2$	**0.040271**	0.16069	0.040506	0.071247
		Bias	$\hat{d}_1$	**-0.032521**	0.27567	-0.034870	-0.051609
			$\hat{d}_2$	-0.013208	0.30504	**-0.012151**	-0.026351
		Std	$\hat{d}_1$	**0.19701**	0.27624	0.20124	0.27734
			$\hat{d}_2$	**0.20024**	0.26007	0.20089	0.26562
0	0.5	MSE	$\hat{d}_1$	**0.039337**	0.096534	0.041711	0.072827
			$\hat{d}_2$	**0.042236**	0.10769	0.042656	0.068293
		Bias	$\hat{d}_1$	0.024887	0.21007	0.025457	**0.013413**
			$\hat{d}_2$	**0.044591**	0.23477	0.048574	0.045650
		Std	$\hat{d}_1$	**0.19677**	0.22892	0.20264	0.26953
			$\hat{d}_2$	**0.20062**	0.22929	0.20074	0.25731
0	0.8	MSE	$\hat{d}_1$	**0.11551**	0.17604	0.12034	0.13836
			$\hat{d}_2$	**0.13038**	0.20013	0.13462	0.15722
		Bias	$\hat{d}_1$	**0.27505**	0.36532	0.27846	0.27971
			$\hat{d}_2$	**0.29801**	0.39766	0.30609	0.31143
		Std	$\hat{d}_1$	**0.19965**	0.20635	0.20688	0.24521
			$\hat{d}_2$	0.20387	0.20493	**0.20230**	0.24542
0.4	0	MSE	$\hat{d}_1$	0.039589	0.043172	**0.038913**	0.063995
			$\hat{d}_2$	0.040661	0.044302	**0.038319**	0.060900
		Bias	$\hat{d}_1$	-0.025809	0.036941	-0.025051	**0.023186**
			$\hat{d}_2$	-0.0019904	0.061315	**0.0013194**	0.038849
		Std	$\hat{d}_1$	0.19729	0.20447	**0.19567**	0.25191
			$\hat{d}_2$	0.20164	0.20135	**0.19575**	0.24370
0.4	0.5	MSE	$\hat{d}_1$	0.039680	0.046785	**0.039325**	0.071111
			$\hat{d}_2$	0.043934	0.048600	**0.041473**	0.067000
		Bias	$\hat{d}_1$	**0.030042**	0.066681	0.033024	0.074272
			$\hat{d}_2$	**0.055468**	0.093484	0.060241	0.091958
		Std	$\hat{d}_1$	0.19692	0.20576	**0.19554**	0.25611
			$\hat{d}_2$	0.20213	0.19965	**0.19454**	0.24196
0.8	-0.2	MSE	$\hat{d}_1$	0.040391	0.041526	**0.038043**	0.060782
			$\hat{d}_2$	0.037511	0.039592	**0.036495**	0.057298
		Bias	$\hat{d}_1$	-0.018640	-0.030254	**-0.017830**	0.042780
			$\hat{d}_2$	0.014789	**-0.0096823**	0.013994	0.059785
		Std	$\hat{d}_1$	0.20011	0.20152	**0.19423**	0.24280
			$\hat{d}_2$	0.19311	0.19874	**0.19052**	0.23178
0.8	0	MSE	$\hat{d}_1$	0.040251	0.041510	**0.037883**	0.061995
			$\hat{d}_2$	**0.037630**	0.039692	0.043413	0.064873
		Bias	$\hat{d}_1$	-0.014844	-0.026505	**-0.012950**	0.048375
			$\hat{d}_2$	0.018778	**-0.0059345**	0.022483	0.068353
		Std	$\hat{d}_1$	0.20008	0.20201	**0.19420**	0.24424
			$\hat{d}_2$	**0.19307**	0.19914	0.20714	0.24536
0.8	0.2	MSE	$\hat{d}_1$	0.039992	0.040654	**0.037468**	0.063221
			$\hat{d}_2$	0.037982	0.039737	**0.037115**	0.058555
		Bias	$\hat{d}_1$	-0.0065270	-0.016745	**-0.0036641**	0.057455
			$\hat{d}_2$	0.027398	**0.0033916**	0.029006	0.073557
		Std	$\hat{d}_1$	0.19987	0.20093	**0.19353**	0.24479
			$\hat{d}_2$	0.19295	0.19931	**0.19046**	0.23053

Table 6: Semiparametric long memory estimators: Monte Carlo results: GARCH(1,1) case, noise added

d	AR			GSP		EW	BEW
				level	1st diff	level	level
				$\sigma_e^2 = 12 * \sigma_u^2$			
0	0	MSE	$\hat{d}_1$	**0.038305**	0.15391	0.038606	0.044995
			$\hat{d}_2$	**0.039376**	0.15165	0.040022	0.045874
		Bias	$\hat{d}_1$	-0.030741	0.29819	**-0.029954**	-0.041290
			$\hat{d}_2$	**-0.032624**	0.29234	-0.033803	-0.035282
		Std	$\hat{d}_1$	**0.19329**	0.25494	0.19419	0.20806
			$\hat{d}_2$	**0.19573**	0.25727	0.19718	0.21126
0	0.5	MSE	$\hat{d}_1$	**0.036720**	0.14345	0.037541	0.053139
			$\hat{d}_2$	**0.036376**	0.14114	0.036935	0.057135
		Bias	$\hat{d}_1$	-0.016491	0.28554	-0.015417	**0.0028129**
			$\hat{d}_2$	**-0.012325**	0.27876	-0.014081	0.019069
		Std	$\hat{d}_1$	**0.19091**	0.24884	0.19314	0.230500
			$\hat{d}_2$	**0.19033**	0.25186	0.19167	0.238277
0	0.8	MSE	$\hat{d}_1$	**0.049003**	0.15771	0.050642	0.13145
			$\hat{d}_2$	0.057810	0.16432	**0.057019**	0.13071
		Bias	$\hat{d}_1$	**0.12135**	0.33142	0.12292	0.21229
			$\hat{d}_2$	0.14066	0.33753	**0.13967**	0.23983
		Std	$\hat{d}_1$	**0.18514**	0.21880	0.18850	0.29391
			$\hat{d}_2$	0.19500	0.22449	**0.19368**	0.27055
0.4	0	MSE	$\hat{d}_1$	0.12440	**0.060304**	0.12181	0.10676
			$\hat{d}_2$	0.12193	**0.062593**	0.12162	0.10724
		Bias	$\hat{d}_1$	-0.30111	**-0.071461**	-0.29842	-0.080210
			$\hat{d}_2$	-0.29048	-0.070323	-0.29003	**-0.058408**
		Std	$\hat{d}_1$	0.18368	0.23494	**0.18098**	0.31674
			$\hat{d}_2$	0.19378	0.24010	**0.19366**	0.32222
0.4	0.5	MSE	$\hat{d}_1$	0.060323	**0.045502**	0.058687	0.083493
			$\hat{d}_2$	0.057352	**0.046311**	0.054741	0.092026
		Bias	$\hat{d}_1$	-0.15990	-0.021931	-0.15609	**-0.0067312**
			$\hat{d}_2$	-0.13504	**-0.0083887**	-0.13434	0.021892
		Std	$\hat{d}_1$	0.18642	0.21218	**0.18527**	0.28887
			$\hat{d}_2$	0.19778	0.21504	**0.19156**	0.30257
0.8	-0.2	MSE	$\hat{d}_1$	0.14901	0.12763	0.14798	**0.086770**
			$\hat{d}_2$	0.13073	0.11706	0.13213	**0.084403**
		Bias	$\hat{d}_1$	-0.33056	-0.29675	-0.33394	**-0.062983**
			$\hat{d}_2$	-0.29622	-0.27626	-0.30591	**-0.027691**
		Std	$\hat{d}_1$	0.19934	0.19891	**0.19096**	0.28776
			$\hat{d}_2$	0.20734	0.20184	**0.19634**	0.28920
0.8	0	MSE	$\hat{d}_1$	0.11628	0.10557	0.11528	**0.083334**
			$\hat{d}_2$	0.099922	0.095289	0.10141	**0.080376**
		Bias	$\hat{d}_1$	-0.27690	-0.25897	-0.28092	**-0.043120**
			$\hat{d}_2$	-0.24012	-0.23629	-0.25155	**-0.016400**
		Std	$\hat{d}_1$	0.19902	0.19623	**0.19068**	0.28544
			$\hat{d}_2$	0.20558	0.19863	**0.19528**	0.28303
0.8	0.2	MSE	$\hat{d}_1$	0.085223	0.082894	0.083820	**0.078577**
			$\hat{d}_2$	**0.072167**	0.073188	0.073066	0.076477
		Bias	$\hat{d}_1$	-0.21455	-0.21125	-0.21857	**-0.024434**
			$\hat{d}_2$	-0.17558	-0.18576	-0.18813	**0.0058863**
		Std	$\hat{d}_1$	0.19797	0.19563	**0.18986**	0.27925
			$\hat{d}_2$	0.20332	0.19667	**0.19410**	0.27648

Notes: The sample size is $T = 218$ that corresponds to the data sample. The error variance of the long memory processes $Var(u_{x_e}) = 1$. All estimates are based on $m = ceil(T^{0.5})$ frequencies. 500 replications are used. Further details are found in the text.