# Three slices of 3-manifold topology:

## spatial graphs, surface complexes and simplicial volumes

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## Summary


This thesis presents three research projects, with one chapter devoted to each.

The first chapter establishes a canonical decomposition of piecewise-linear spatial graphs (that is, graphs embedded in 3-spheres), into pieces that are non-separable by 2-spheres, and have no topologically cut vertices – such spatial graphs are called "blocks". This was motivated by the goal of producing an algorithm for testing the isomorphism type of spatial graphs, by encoding them as 3-manifolds with boundary pattern and applying a theorem of Matveev. Operations of disjoint union and vertex sum of spatial graphs are defined, and a combinatorial gadget for specifying iterated vertex sums is introduced, called a "tree of spatial graphs". The main results of this chapter are that each spatial graph is the disjoint union of non-separable pieces in a unique way, and each piece is the realization of a unique tree of blocks.

Chapter 2 studies a simplicial complex defined for compact oriented smooth manifolds $M$ (of any dimension) with a chosen codimension-1 homology class $\phi$. This complex $\mathcal{S}^\dagger(M, \phi)$ relates the properly embedded hypersurfaces in $M$ representing $\phi$, and its definition is similar to that of other classical complexes, such as the curve complex of a surface or the Kakimizu complex of a knot, with the difference that hypersurfaces are not taken up to isotopy. We show that $\mathcal{S}^\dagger(M, \phi)$ is connected and simply connected. We also show connectedness of a similar complex adapted to the 3-dimensional case, where only Thurston norm-realizing surfaces are considered. The connectedness results are transported to the complexes where hypersurfaces are taken up to isotopy, and in dimension 2 also the simple connectedness result is extended.

The last chapter presents the proof that oriented compact connected 3-manifolds with toroidal boundary that are prime and not covered by $\mathbb{S}^3$ satisfy integral approximation of simplicial volume. The computation of stable integral simplicial volume for such a manifold $M$ is carried out in terms of its integral foliated simplicial volume, which in turn is obtained by assembling the values for each of the JSJ pieces of $M$. The assembly requires a variant of integral foliated simplicial volume that takes into account the norm of the boundary of fundamental cycles. The aim of this chapter is to highlight the overall strategy of the argument, breaking down the main ingredients of the proof, so some of the more technically involved arguments are omitted.

# Contents

# Chapter 0

# Introduction

A pervasive principle in 3-manifold topology is the idea of understanding manifolds by finding interesting surfaces along which to cut them. The resulting pieces might decrease some relevant notion of complexity, allowing one to apply inductive arguments, or they might even be canonical, breaking the objects of study into elementary building blocks of the theory.

A remarkable example are Haken hierarchies, which are sequences of 3-manifolds obtained by starting with a compact 3-manifold and successively cutting it along properly embedded two-sided surfaces, until a disjoint union of 3-balls is reached. They are, for instance, a cornerstone of Waldhausen's work on deforming homotopy equivalences between 3-manifolds into homeomorphisms [Wal68], and of the solution to the homeomorphism problem for (piecewise-linear) Haken manifolds [Mat07, Chapter 6].

Another foundational result with the same "divide and conquer" flavour is the Prime Decomposition Theorem, which is ultimately about cutting 3-manifolds along separating 2-spheres [Mil62]. More concretely, it states that every compact oriented 3-manifold is the connected sum of 3-manifolds that cannot be non-trivially decomposed (called "prime"), and the summands are unique up to homeomorphism. This reduces the problem of classifying closed oriented 3-manifolds, to the prime ones. Once restricted to prime manifolds, one can use the JSJ Decomposition Theorem, which asserts the existence of a collection of incompressible embedded tori that further cut the manifold into canonical pieces, with the tori also unique up to isotopy [JS79][Joh79, Chapter III].

Broadening our view whilst confined to the 3-manifold setting, one can also consider the Prime Decomposition Theorem for knots, whose statement has a similar form to that of the homonymous result for 3-manifolds [BZ03, Theorem 7.12]. Here, instead of compact oriented 3-manifolds, one considers oriented knots in $\mathbb{S}^3$, and the connected sum operation of knots, also known as "band sum". Decomposing a knot $K$ as a connected sum amounts to finding a smoothly embedded 2-sphere in $\mathbb{S}^3$ that intersects $K$ transversely at two points. Cutting at this 2-sphere yields two knotted arcs in 3-balls, and capping each of them with an unknotted arc in a 3-ball recovers the summands.

These examples, far from being exhaustive, set the tone for the dissertation, which is a compilation of three research projects. Whilst they are concerned with different objects and have distinct aims, the ideas in the above paragraphs are a common underlying theme. The first of these projects [Fri+21] deals with piecewse-linear spatial graphs in 3-spheres, and one of its major goals is to establish the existence of canonical decompositions for spatial graphs. These decompositions are obtained by cutting the ambient 3-sphere along 2-spheres, and are thus reminiscent of prime decompositions of knots. In the second project [HQ20], we study a simplicial complex encoding relationships between different hypersurfaces representing a fixed homology class in a smooth manifold (of any dimension). The 3-dimensional case has an adaptation that allows one to define an invariant of the homology class, and this invariant is computed by cutting along the relevant surfaces. Finally, the third project [Fau+21] is about stable integral simplicial volume, a real-valued invariant of oriented compact connected manifolds. There, we exploit the properties of the JSJ decomposition in order to assemble previously known results into a computation for all compact oriented connected 3-manifolds with toroidal boundary that are not covered by $\mathbb{S}^3$.

Each chapter of this thesis is devoted to one of the aforementioned projects, and can be read independently of the others. We now give a brief summary of each one.

## 0.1 Canonical decompositions of spatial graphs

Chapter 1 is based on joint work with Stefan Friedl, Lars Munser and Yuri Santos-Rego [Fri+21], and it follows very closely the exposition in Sections 1 to 5 of that article.

The idea is to use a theorem of Matveev, building on work of Haken, in order to produce an algorithm that tests whether two spatial graphs are isomorphic. By a spatial graph we mean a 1-dimensional subcomplex of a piecewise-linear 3-sphere with a decomposition into vertices and edges, and an isomorphism is an orientation-preserving piecewise-linear homeomorphism of the ambient 3-spheres mapping vertices to vertices and edges to edges bijectively.

Matveev's Theorem asserts the existence of an algorithm for testing whether two compact piecewise-linear 3-manifolds are (piecewise-linearly) homeomorphic. These manifolds are even allowed to come equipped with 1-dimensional subcomplexes of their boundary (which are required to be preserved by homeomorphisms). But the theorem also comes with an additional assumption on these "3-manifolds with boundary pattern" – that they should be "Haken". If one ignores this Haken condition, then a basic idea for an algorithm to compare the isomorphism type of two given spatial graphs would be to consider complements of regular neighborhoods of their vertices and edges. One would then mark the boundaries with curves encoding the information of how to reconstruct the spatial graphs, and compare these "marked graph exteriors" using Matveev's Theorem.

It turns out that one cannot in general expect the marked graph exterior of a spatial graph to be Haken. However, we can reduce the problem of comparing spatial graphs to that case by decomposing them into spatial graphs whose marked exteriors are Haken, in a canonical way. The goal of Chapter 1 is to explain this decomposition.

A class of spatial graphs whose marked exteriors are excluded by the Haken requirement are those that admit non-trivial splittings along 2-spheres – in the language of Chapter 1, they are a "disjoint union" of non-empty pieces. But at least intuitively, this situation seems easily circumvented: one should be able to decompose the spatial graphs into non-separable pieces, and then compare these pieces. Implicit in this idea is the statement that such a decomposition into non-separable pieces is unique. The first part of Chapter 1 is devoted to studying the algebraic properties of the disjoint union operation on spatial graphs, and to giving a proof that the decomposition of a spatial graph as the disjoint union of non-separable pieces is indeed unique (Proposition 1.3.5).

Although it is hardly a surprising statement, understanding the proof that spatial graphs have canonical decompositions as disjoint unions of non-separable pieces illuminates the approach needed to overcome the next obstacle. Among non-separable spatial graphs, the ones whose marked exteriors are not Haken, and thus resist our approach using Matveev's Theorem, are those that have a cut vertex. This means that they can be obtained by joining two spatial graphs (other than one-point graphs) at a vertex. We call this procedure a "vertex sum", and proving that it is a well-defined operation (on spatial graphs with a marked vertex) is one of the most technically demanding tasks in Chapter 1.

Inspired by the program carried out with the disjoint union operation, we proceed to show that non-separable spatial graphs admit unique decompositions as iterated vertex sums of non-separable spatial graphs without cut vertices (which we call "blocks"). Making this statement precise requires setting up terminology that packages the combinatorics of which vertices from which spatial graphs are glued, and to this end we introduce the notion of a "tree of blocks". Chapter 1 culminates with a proof of the following result (Propositions 1.4.24 and 1.4.25):

**Theorem A** (Canonical decomposition as a tree of blocks)**.** *Every non-separable spatial graph other than a one-point graph admits a unique decomposition as a tree of blocks.*

This result completes the reduction of comparing the isomorphism type of spatial graphs to those with Haken marked exteriors. The details on how these marked exteriors are constructed and the verification that they indeed satisfy the hypotheses of Matveev's Theorem are left out of this thesis; the reader is pointed to the article [Fri+21, Sections 6 and 7].

## 0.2 The complex of hypersurfaces in a homology class

The work presented in Chapter 2 is part of a joint project with Gerrit Herrmann [HQ20], and it follows the exposition in Sections 1 to 5 of that preprint.

This work was motivated by a question that arose in the context of Gerrit Herrmann's doctoral dissertation: he was studying a real-valued invariant of an irreducible and boundary-irreducible oriented compact connected 3-manifold $M$ with toroidal boundary, and an oriented Thurston norm-realizing surface $S$ properly embedded in $M$. More concretely, he had shown that if $M \setminus\!\!\setminus S$ is the 3-manifold obtained by cutting $S$ along $M$, and $S_-$ is the part of $\partial(M \setminus\!\!\setminus S)$ corresponding to the negative side of $S$, there is a well-defined $\ell^2$-torsion $\tau^{(2)}(M \setminus\!\!\setminus S, S_-)$ of the pair $(M \setminus\!\!\setminus S, S_-)$. He had also observed that when choosing a different surface $S' \subset M$ in the same homology class, this quantity remains unchanged:

$$\tau^{(2)}(M \setminus\!\!\setminus S', S'_-) = \tau^{(2)}(M \setminus\!\!\setminus S, S_-),$$

provided that $S$ and $S'$ are disjoint.

This prompted the question of whether the disjointness hypothesis on $S, S'$ could be suppressed, and so the above $\ell^2$-torsion would in fact be an invariant of the homology class represented by the surfaces. A natural strategy is to show that given any two Thurston norm-realizing surfaces $S, S'$ representing the same homology class $\phi \in \mathrm{H}_2(M, \partial M)$, there is a sequence $S = S_0, S_1, \ldots, S_m = S'$ of homologous Thurston norm-realizing surfaces, with every two consecutive ones $S_{i-1}, S_i$ disjoint. The project that arose from pursuing this line of inquiry is the subject of Chapter 2.

The above idea can be phrased in terms of the simplicial complex $\mathcal{T}^\dagger(M, \phi)$, whose vertices are properly embedded surfaces in $M$ representing $\phi$ that realize its Thurston norm and have no null-homologous parts, and whose simplices are sets of pairwise disjoint surfaces. The statement then becomes:

**Theorem B** ($\mathcal{T}^\dagger(M, \phi)$ is connected)**.** *Let $M$ be an irreducible and boundary-irreducible oriented compact smooth 3-manifold, and let $\phi \in \mathrm{H}_2(M, \partial M)$. Then $\mathcal{T}^\dagger(M, \phi)$ is connected.*

This theorem is re-stated and proved as Theorem 2.4.6 below.

The core ideas of the proof do not rely on the surfaces considered being Thurston norm-realizing, and not even in the dimension of $M$ – we need only consider codimension 1 submanifolds representing a fixed homology class. We thus applied our techniques to a different simplicial complex $\mathcal{S}^\dagger(M, \phi)$, defined for an oriented compact connected manifold $M$ (of any dimension) and a codimension-1 homology class $\phi$. The vertices are properly embedded hypersurfaces representing $\phi$, and the simplices are sets of pairwise-disjoint hypersurfaces. The two main results we obtained, which are the main focus of Chapter 2, are summarized in the following theorem.

**Theorem C** ($\mathcal{S}^\dagger(M, \phi)$ is connected and simply connected)**.** *Let $M$ be an oriented compact connected $n$-manifold, and let $\phi \in \mathrm{H}_{n-1}(M, \partial M)$. Then:*

- *The complex $\mathcal{S}^\dagger(M, \phi)$ is connected (Theorem 2.3.3).*
- *The complex $\mathcal{S}^\dagger(M, \phi)$ is simply connected (Theorem 2.5.1).*

We also transported these results to the complexes $\mathcal{S}(M, \phi)$ and $\mathcal{T}(M, \phi)$ whose definition is similar, except that all hypersurfaces are taken only *up to isotopy.* These complexes are thus more similar to other classically studied complexes of hypersurfaces, such as the curve complex of a surface, or the Kakimizu complex of a knot. The connectedness results are straightforward consequences, but we were only able to show that $\mathcal{S}(M, \phi)$ is simply connected in the case where $M$ has dimension 2.

Theorem B was employed to the question about $\ell^2$-torsion mentioned above, and also for re-proving the classical fact that all Seifert surfaces for a knot in a rational homology 3-sphere are tube-equivalent. These applications are omitted from this thesis and can be found in the preprint [HQ20, Sections 6 and 7].

## 0.3 Stable integral simplicial volume of 3-manifolds

Chapter 3 is about a joint project with Daniel Fauser, Clara Löh and Marco Moraschini [Fau+21].

This work is primarily concerned with two versions of simplicial volume, an invariant of oriented compact manifolds whose original definition has seen a wide array of adaptations. The classical simplicial volume $\|M\|$ of an oriented compact manifold $M$ is the semi-norm of its real fundamental class, as induced by the $\ell^1$-norm on the real singular chain complex (with respect to the basis given by singular simplices). One way of producing real fundamental cycles for $M$, and thus bound $\|M\|$ from above, is to push forward an *integral* fundamental cycle for a finite covering of $M$, and then divide it by the covering degree. The infimum of all estimates obtained in this manner is called the "stable integral simplicial volume" of $M$, and denoted by $\|M\|_{\mathbb{Z}}^\infty$. Manifolds $M$ for which $\|M\| = \|M\|_{\mathbb{Z}}^\infty$ are said to satisfy "integral approximation", and whereas it is well-known that 2-manifolds satisfy integral approximation, this was known in dimension 3 only for some classes of manifolds, such as graph manifolds not covered by $\mathbb{S}^3$ and hyperbolic manifolds.

The main contribution of our article was to adapt and assemble the previously known results on Seifert-fibered and hyperbolic 3-manifolds into the following result (re-stated as Theorem 3.2.4 below).

**Theorem D** (Integral approximation for non-elliptic prime 3-manifolds)**.** *Let $M$ be an oriented compact connected 3-manifold with toroidal boundary. If $M$ is prime and not covered by $\mathbb{S}^3$, then $M$ satisfies integral approximation of simplicial volume.*

Chapter 3 is a new account of our proof, whose purpose is to break down the main ingredients and tools used in the proof, making the overall argument more transparent, in particular to non-experts in simplicial volume. In order to keep the exposition concise, some of the more technically involved proofs have thus been omitted, pointing the interested reader to the relevant sections of the article.

The proof computes $\|M\|_{\mathbb{Z}}^{\infty}$ by relating it to the "integral foliated simplicial volume" $|M|$, a variant of simplicial volume suited to the application of methods from ergodic theory. Using a version of integral foliated simplicial volume that takes into account also the norm of the boundaries of fundamental cycles, one is able to assemble previously known computations for the JSJ pieces of $M$ into a global upper bound. The additivity of boundary-controlled integral foliated simplicial volume with respect to gluing along the JSJ tori relies on the fact that profinite completions of fundamental groups of 3-manifolds behave well with respect to the graph of groups decomposition induced from the JSJ splitting.

The boundary-controlled integral foliated simplicial volume $|N|_{\partial}$ of each hyperbolic JSJ piece $N$ of $M$ is obtained by comparing it to $|\operatorname{int}(N)|$. In turn, this requires computing the stable integral simplicial volume of (not necessarily compact) oriented complete hyperbolic 3-manifolds of finite volume. For the sake of keeping the exposition concise we include only a sketch of that argument.

We also briefly point out an application of stable integral simplicial volume in bounding the asymptotic growth of homology along towers of finite covers of a manifold, and discuss implications of Theorem D in this context.

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# Chapter 1

# Canonical decompositions of spatial graphs

## 1.1 Introduction: algorithmic recognition of spatial graphs

### 1.1.1 Motivation

Spatial graphs are finite graphs embedded in oriented 3-spheres – not merely as subspaces, but with an explicit decomposition into vertices and edges. A precise definition will be given using piecewise-linear (henceforth abbreviated as "PL") topology (Definition 1.2.1). An isomorphism of spatial graphs is then an orientation-preserving PL homeomorphism of the ambient 3-spheres mapping vertices to vertices and edges to edges bijectively (Definition 1.2.3). This chapter presents one part of the machinery used in joint work with Stefan Friedl, Lars Munser and Yuri Santos-Rego [Fri+21], in order to establish the following result, and the exposition will very closely follow the text in Sections 1 to 5 of that article.

**Theorem 1.1.1** (Algorithmic detection of spatial graphs [Fri+21, Theorem 1.1])**.** *There exists an algorithm for determining whether two spatial graphs are isomorphic.*

In fact, Theorem 1.1.1 was proved in a broader context, where the spatial graphs are allowed to come equipped with additional decorations, such as colorings of vertices and/or edges, and orientations of the edges [Fri+21, Theorem 7.14]. This additional data must of course be respected by isomorphisms.

The proof uses as a main ingredient a theorem of Matveev that extends work of Haken [Mat07, Theorem 6.1.6], concerning PL 3-manifolds equipped with a 1-dimensional subcomplex of their boundary (called a "boundary pattern" [Mat07, Definition 3.3.9]). Matveev's Theorem states that it is possible to algorithmically detect whether two such 3-manifolds with boundary pattern are PL-homeomorphic (via a homeomorphism respecting the boundary patterns), provided that they are "Haken" [Mat07, Definition 6.1.5]).

The basic idea for proving Theorem 1.1.1 is to construct a PL 3-manifold with boundary pattern out of a spatial graph, its "marked exterior" [Fri+21, Definition 6.5], such that two spatial graphs are isomorphic precisely if their marked exteriors are PL-homeomorphic. This marked exterior is built by removing a suitably chosen "small" open neighborhood of the spatial graph, and then marking the boundary with a pattern that allows for easily reconstructing the spatial graph. One then uses Matveev's Theorem to compare the marked exteriors.

The main difficulty in using Matveev's Recognition Theorem to deduce Theorem 1.1.1 lies in the fact that Matveev's Theorem applies only to 3-manifolds with boundary pattern that are Haken. This property encompasses three conditions: one about existence of non-trivial embedded surfaces (being "sufficiently large"), one about triviality of embedded 2-spheres ("irreducibility"), and one about triviality of properly embedded discs ("boundary-irreducibility"). Whilst the "sufficiently large" requirement is easy to guarantee, the marked exterior of a spatial graph may very well fail to be irreducible and boundary-irreducible. In this chapter we lay out the theory of decompositions for spatial graphs designed to circumvent this issue.

### 1.1.2 Decomposition results

For an outline of the strategy, consider first the irreducibility requirement. The starting point is the observation [Fri+21, Proposition 6.9] that for a spatial graph $\Gamma$, irreducibility of its marked exterior is equivalent to $\Gamma$ not being the "disjoint union" $\Gamma_1 \sqcup \Gamma_2$ of non-empty spatial graphs $\Gamma_1, \Gamma_2$, where this disjoint union is the operation of placing $\Gamma_1, \Gamma_2$ "next to one another" in the same ambient 3-sphere (see Definition 1.3.3 for the precise notion). A non-empty spatial graph that is not a non-trivial disjoint union is called a "piece" (Definition 1.3.10). In Proposition 1.3.5, we show that the decomposition of a spatial graph into pieces is canonical in a suitable sense. This reduces the task of determining whether two spatial graphs are isomorphic, to testing whether the pieces in their decompositions are pairwise isomorphic. As these pieces have irreducible marked exteriors, we are one step closer to being able to apply Matveev's theorem.

The next step is to find a decomposition of non-separable graphs into spatial graphs whose marked exteriors are moreover boundary-irreducible. The strategy is similar to the one in the previous paragraph, except that the role of the disjoint union operation is played by the operation of "vertex sum" (Definition 1.4.3). Roughly, the vertex sum of two spatial graphs, each with a distinguished vertex, is obtained by "gluing them" along those vertices. For non-separable spatial graphs $\Gamma$, there is a very close correspondence between $\Gamma$ having boundary-irreducible marked exterior, and $\Gamma$ being indecomposable as a non-trivial vertex sum [Fri+21, Propositions 6.11 and 6.15]. We will see that non-separable spatial graphs admit a canonical decomposition as an iterated vertex sum (Propositions 1.4.24 and 1.4.25) of non-separable spatial graphs without cut vertices (which we call "blocks", see Definition 1.4.21). This reduces the comparison of the isomorphism type of two non-separable spatial graphs, to comparing the blocks in their decomposition. Except for one easy special case, these blocks have marked exteriors amenable to Matveev's algorithm.

The iterated vertex sums from the previous paragraph are allowed to be performed along different vertices, so the canonical decomposition must come bundled with the combinatorial data of which vertices from different blocks are glued to which. To package this information, the notion of a "tree of spatial graphs" is introduced (Definition 1.4.14), and in case the spatial graphs being glued are blocks, we call it a "tree of blocks" (Definition 1.4.21). The main results on decompositions as iterated vertex sums are summarized in the following theorem (see Propositions 1.4.24 and 1.4.25 for precise statements). Proving it is the main goal of the current chapter.

**Theorem 1.1.2** (Canonical decomposition as a tree of blocks)**.** *Every non-separable spatial graph other than a one-point graph admits a unique decomposition as a tree of blocks.*

This theory of decompositions has analogues in the setting of abstract graphs [Jun05, Excercise 8.3.3]. In the topological setting, Suzuki [Suz70] has established a unique factorization result with respect to a "composition" operation similar in spirit to our vertex sum, but only for connected 1-subcomplexes of the 3-sphere, and only up to a "neighborhood congruence" relation. Theorem 1.1.2 differs from Suzuki's in the following: our spatial graphs come with vertex/edge decompositions and possibly decorations, we broaden the connectedness assumption to non-separability, and we have no identification "up to neighborhood congruence", instead keeping track of the vertices along which to glue.

### 1.1.3 The piecewise-linear setting

We work in the PL setting because it is the natural home for results in computational topology such as Theorem 1.1.1, and it is the framework of the machinery developed by Matveev on which its proof relies. This is a standard framework in the field [Suz70; Kau89; Yam89; Bar15], although a theory of smooth, rather than PL, spatial graphs has also been introduced by Friedl and Herrmann [FH20; FH21].

The reader is referred to the textbook of Rourke and Sanderson [RS72] for the standard notions in PL topology. We will often give precise references for the results we import, but knowledge of basic concepts such as that of a polyhedron, or a PL manifold (possibly oriented, or with boundary) will be assumed. In particular, PL spaces (also called polyhedra) are by definition subspaces of some $\mathbb{R}^n$ whose points admit a star neighborhood. The ambient space $\mathbb{R}^n$ is equipped with the metric induced from the $\ell^1$-norm, so by "balls" and "spheres" we always mean polyhedra that are PL-homeomorphic to cubes and their boundaries, respectively.

We will also make heavy usage of regular neighborhoods. If $X \subseteq P$ are polyhedra, with $X$ compact, then one may think of a regular neighborhood of $X$ in $P$ as a "small, well-behaved neighborhood" of $X$ that deformation-retracts onto $X$ [RS72, Chapter 3]. If $P_0$ is a closed sub-polyhedron of $P$, there is also the notion of a regular neighborhood $N$ of $X$ in the pair $(P, P_0)$ [RS72, p. 52]. In this case we use a lighter notation than the one in Rourke-Sanderson, who would instead have written that *the pair* $(N, N \cap P_0)$ is a regular neighborhood of *the pair* $(X, X \cap P_0)$ in $(P, P_0)$.

### 1.1.4 Outline of this chapter

After laying out the elementary terms in the theory of spatial graphs (Section 1.2), we describe the operation of disjoint union of spatial graphs (Section 1.3), proving that the decomposition as a disjoint union of pieces is unique.

This program is mirrored in Section 1.4, where we define the vertex sum operation on pointed spatial graphs, establish a framework for describing iterated vertex sums (as trees of spatial graphs), and show uniqueness of decompositions as trees of blocks, thus completing the proof of Theorem 1.1.2. Showing the vertex sum operation is well-defined is one of the most technically demanding points of this program, with most of the hard work contained in the proof of Proposition 1.4.4.

Section 1.5 is an addendum explaining how to adapt the theory developed so far to spatial graphs decorated with additional structure, namely edge orientations, vertex colorings, and edge colorings.

## 1.2 Basic definitions

In this short section, we define spatial graphs and introduce other basic terminology. We take a moment to remind the reader that all spaces considered are polyhedra: subspaces of some euclidean space $\mathbb{R}^n$ having local cone neighborhoods at every point, and PL maps are defined as preserving this local cone structure [RS72, Chapter 1]. Standard models of balls and spheres are defined using the $\ell^1$-norm, so they are effectively cubes and their boundaries. Orientations of PL manifolds are defined as PL isotopy classes of embeddings of balls [RS72, pp. 43–46].

**Definition 1.2.1.** A **spatial graph** $\Gamma$ is a triple $(\mathcal{S}, V, E)$, where:

- $\mathcal{S}$ is an oriented PL 3-sphere, called the **ambient sphere** of $\Gamma$. We will occasionally say that "$\Gamma$ is a spatial graph in $\mathcal{S}$".
- $V$ is a finite subset of $\mathcal{S}$, whose elements are called **vertices** of $\Gamma$, and
- $E$ is a finite set of subpolyhedra of $\mathcal{S}$, called **edges** of $\Gamma$, such that:
  - each edge is PL-homeomorphic to an interval or to a PL 1-sphere,
  - each edge that is PL-homeomorphic to an arc intersects $V$ precisely its end-points,
  - each edge that is PL-homeomorphic to a circle contains precisely one element of $V$ (such edges are called **loops**),
  - for every two distinct edges $e, e'$, we have $e \cap e' \subseteq V$.

The **support** of $\Gamma$ is the union

$$|\Gamma| := V \cup \bigcup_{e \in E} e \subset \mathcal{S}.$$

The **underlying graph** $\langle\Gamma\rangle$ of $\Gamma$ is the (undirected) abstract graph with vertex set $V$ and edge set $E$, where each edge is incident to the one or two elements of $V$ that it contains. We will say that an edge of $\Gamma$ is **incident** to a vertex if this is true in $\langle\Gamma\rangle$. The **degree** of a vertex $v$ is the same as its degree in $\langle\Gamma\rangle$, that is, the number of edges incident to $v$, with loops counting twice. A vertex of degree 0 is called an **isolated vertex**, and a vertex of degree 1 is called a **leaf**.

Observe that the two subsets $|\Gamma|$ and $V$ of $\mathcal{S}$ determine the edge set, since there is a canonical bijection between $E$ and $\pi_0(|\Gamma| \setminus V)$.

**Definition 1.2.2.** A **sub-graph** of a spatial graph $\Gamma = (\mathcal{S}, V, E)$ is a spatial graph $\Gamma' = (\mathcal{S}, V', E')$, where $V' \subseteq V$ and $E' \subseteq E$.

**Definition 1.2.3.** Let $\Gamma_1 = (\mathcal{S}_1, V_1, E_1)$ and $\Gamma_2 = (\mathcal{S}_2, V_2, E_2)$ be spatial graphs. An **isomorphism** $\Phi\colon \Gamma_1 \to \Gamma_2$ is a PL homeomorphism of triples $\Phi\colon (\mathcal{S}_1, |\Gamma_1|, V_1) \to (\mathcal{S}_2, |\Gamma_2|, V_2)$ respecting the orientation of the ambient spheres. If such $\Phi$ exists, we say $\Gamma_1, \Gamma_2$ are **isomorphic** and write $\Gamma_1 \cong \Gamma_2$.

By the characterization of the elements of $E_1$ in terms of $|\Gamma_1| \setminus V_1$ (and similarly for $E_2$), such $\Phi$ also induces a bijection $E_1 \to E_2$, and we get an induced isomorphism of abstract graphs $\langle\Phi\rangle\colon \langle\Gamma_1\rangle \to \langle\Gamma_2\rangle$.

We emphasize that there is stark loss of information in the passage from a spatial graph to its underlying graph. In fact, one could claim that much of the field of knot theory is the study of the isomorphism classes of spatial graphs whose underlying graph is comprised of one vertex and one loop.

It will be convenient to loosen the notation by allowing ourselves to write an equality of spatial graphs "$\Gamma_1 = \Gamma_2$" whenever $\langle\Gamma_1\rangle = \langle\Gamma_2\rangle$ and there is an isomorphism $\Phi\colon \Gamma_1 \to \Gamma_2$ such that $\langle\Phi\rangle$ is the identity morphism.

Up to isomorphism, there is a unique spatial graph with no vertices (and hence also no edges), which we call the **empty spatial graph**, and denote by $\mathbf{0}$. Similarly, since the group of PL self-homeomorphisms of a 3-sphere acts transitively on its points [RS72, Lemma 3.33], there is a unique spatial graph (up to isomorphism) with a single vertex and no edges. We call it a **one-point** spatial graph and denote it by $\mathbf{1}$.

## 1.3 The disjoint union of spatial graphs

We want to define and establish properties of two operations on spatial graphs. For now, we focus on the disjoint union operation, and this will double as a warm-up for studying the more delicate vertex sum operation (Section 1.4). These operations implement constructions that are straightforward to define for abstract graphs, but in the setting of spatial graphs, a rigorous treatment requires some care.

### 1.3.1 Assembling spatial graphs through disjoint unions

In order to define the disjoint union of spatial graphs, we will need the following theorem:

**Theorem 1.3.1** (Disc Theorem [RS72, Theorem 3.34]). *Every two orientation-preserving PL embeddings of an $n$-ball into the interior of a connected, oriented $n$-manifold $M$ are PL-ambient-isotopic relative $\partial M$.*

The above reference does not state that the ambient isotopy fixes $\partial M$, but a closer inspection reveals that the stronger conclusion does follow from the proof. Later, we also present a stronger version of the Disc Theorem, which does include the boundary condition (Theorem 1.4.6).

**Definition 1.3.2.** An **enclosing ball** for a spatial graph $\Gamma$ in $\mathcal{S}$, is a PL-embedded 3-ball $B \subset \mathcal{S}$ such that $|\Gamma| \subset \operatorname{int}(B)$.

**Definition 1.3.3.** For each $i \in \{1, 2\}$, let $\Gamma_i = (\mathcal{S}_i, V_i, E_i)$ be a spatial graph, and let $B_i$ be an enclosing ball for $\Gamma_i$. Moreover, let $f\colon \partial B_1 \to \partial B_2$ be an orientation-reversing PL homeomorphism. Then the spatial graph

$$\Gamma_1 \sqcup_f \Gamma_2 := (B_1 \cup_f B_2, V_1 \sqcup V_2, E_1 \sqcup E_2),$$

where $B_1 \cup_f B_2$ denotes the 3-sphere obtained by attaching $B_1$ to $B_2$ using $f$, is said to be a **disjoint union** of $\Gamma_1$ and $\Gamma_2$.

We remark that gluing polyhedra along sub-polyhedra is a valid operation in the PL setting [RS72, Exercise 2.27 (2)].

As one would expect, the underlying graph $\langle \Gamma_1 \sqcup_f \Gamma_2 \rangle$ is the disjoint union $\langle \Gamma_1 \rangle \sqcup \langle \Gamma_2 \rangle$, as usually defined for abstract graphs.

The next lemma and proposition show that the isomorphism type of a disjoint union of two spatial graphs does not depend on the choice of enclosing balls $B_1, B_2$, nor on the attaching map $f$.

**Lemma 1.3.4** (Uniqueness of enclosing balls). *Let $\Gamma$ be a spatial graph in $\mathcal{S}$, and let $B, B'$ be enclosing balls for $\Gamma$. Then every orientation-preserving PL homeomorphism $\Phi_\partial\colon \partial B \to \partial B'$ extends to an orientation-preserving PL homeomorphism $\Phi_B\colon B \to B'$ that restricts to the identity on $|\Gamma|$.*

*Proof.* Fix a regular neighborhood $N_\Gamma$ of $|\Gamma|$ in $\mathcal{S}$ that is disjoint from $\partial B \cup \partial B'$. Since $N_\Gamma$ and $\mathcal{S}$ are 3-manifolds, the subspace $\overline{N_\Gamma} := \mathcal{S} \setminus \operatorname{int}(N_\Gamma)$ is also a 3-manifold [RS72, Corollary 3.14]. Moreover, since the closure of the complement of a PL-embedded $n$-ball in an $n$-sphere is always an $n$-ball [RS72, Corollary 3.13], we see that $\overline{B} := \mathcal{S} \setminus \operatorname{int}(B)$ is a 3-ball contained in $\operatorname{int}(\overline{N_\Gamma})$ (and similarly for $\overline{B'} := \mathcal{S} \setminus \operatorname{int}(B')$).

Since a PL homeomorphism between the boundaries of two balls always extends to a PL homeomorphism of their interior [RS72, Lemma 1.10], we may extend $\Phi_\partial$ to an orientation-preserving PL homeomorphism $\Phi_{\overline{B}}\colon \overline{B} \to \overline{B'}$. Then we apply the Disc

Theorem (Theorem 1.3.1) to produce a PL ambient isotopy of $\overline{N_\Gamma}$ taking the inclusion $\overline{B} \hookrightarrow \overline{N_\Gamma}$ to the composition

$$\overline{B} \xrightarrow{\Phi_{\overline{B}}} \overline{B'} \hookrightarrow \overline{N_\Gamma}.$$

Since this ambient isotopy keeps $\partial N_\Gamma$ fixed, its final homeomorphism $\Phi_{\overline{N_\Gamma}} \colon \overline{N_\Gamma} \to \overline{N_\Gamma}$ can be extended to all of $\mathcal{S}$ by setting it to the identity on $N_\Gamma$. This extension $\Phi_{\mathcal{S}} \colon \mathcal{S} \to \mathcal{S}$, when restricted to $B$, is a PL homeomorphism $\Phi_B \colon B \to B'$ satisfying the conclusion of the lemma. $\square$

**Proposition 1.3.5** (Well-definedness of the disjoint union)**.** *For each $i \in \{1,2\}$, let $\Gamma_i$ be a spatial graph in $\mathcal{S}_i$, and let $B_i, B_i'$ be two enclosing balls for $\Gamma_i$. Moreover, let $f \colon \partial B_1 \to \partial B_2$ and $f' \colon \partial B_1' \to \partial B_2'$ be orientation-reversing PL homeomorphisms. Then there is an isomorphism $\Phi \colon \Gamma_1 \sqcup_f \Gamma_2 \cong \Gamma_1 \sqcup_{f'} \Gamma_2$ such that $\langle \Phi \rangle$ is the identity on $\langle \Gamma_1 \rangle \sqcup \langle \Gamma_2 \rangle$.*

Note that in the abbreviated notation introduced in Section 1.2, the conclusion of this proposition can be rephrased as "$\Gamma_1 \sqcup_f \Gamma_2 = \Gamma_1 \sqcup_{f'} \Gamma_2$".

*Proof.* Lemma 1.3.4 provides an orientation-preserving PL homeomorphism $\Phi_1 \colon B_1 \to B_1'$ restricting to the identity on $|\Gamma_1|$. Using the same lemma, let $\Phi_2 \colon B_2 \to B_2'$ be an orientation-preserving PL homeomorphism fixing $|\Gamma_2|$ and whose restriction to $\partial B_2$ is $f' \circ \Phi_1|_{\partial B_1} \circ f^{-1}$. Then the maps $\Phi_i$ assemble to a PL homeomorphism $\Phi \colon B_1 \sqcup_f B_2 \to B_1' \sqcup_{f'} B_2'$ giving the desired isomorphism between $\Gamma_1 \sqcup_f \Gamma_2$ and $\Gamma_1 \sqcup_{f'} \Gamma_2$. The fact that each $\Phi_i$ restricts to the identity on $|\Gamma_i|$ implies that $\langle \Phi \rangle$ is the identity on $\langle \Gamma_1 \rangle \sqcup \langle \Gamma_2 \rangle$. $\square$

The disjoint union $\Gamma_1 \sqcup_f \Gamma_2$ is thus well-defined without specifying enclosing balls $B_1$, $B_2$ nor the attaching map $f$, up to isomorphism of spatial graphs respecting the underlying combinatorial structure. Hence we will from now on most of the time suppress the $f$-subscript from the notation.

We now collect two elementary observations:

**Lemma 1.3.6** (Disjoint union summands as sub-graphs)**.** *Let $\Gamma = \Gamma_1 \sqcup \Gamma_2$ be a disjoint union of spatial graphs, and denote by $\Gamma_1'$ the sub-graph of $\Gamma$ obtained by discarding all vertices and edges of $\Gamma_2$. Then $\Gamma_1' = \Gamma_1$.*

*Proof.* For each $i \in \{1,2\}$, let $\mathcal{S}_i$ be the ambient sphere for $\Gamma_i$, let $B_i \subset \mathcal{S}_i$ be the enclosing ball from which the disjoint union was formed, and let $f \colon \partial B_1 \to \partial B_2$ be the attaching map. Our task is to find a PL homeomorphism $\Phi \colon \mathcal{S}_1 \to B_1 \cup_f B_2$ that restricts to the identity on $|\Gamma_1|$. Setting $\Phi$ as the identity map on $B_1$, we are left to find a PL homeomorphism $\overline{B_1} \to B_2$ extending $f$, where $\overline{B_1} := \mathcal{S}_1 \setminus \operatorname{int}(B_1)$. Such an extension always exits [RS72, Lemma 1.10]. $\square$

When working with such a disjoint union, we will often refer to the summand $\Gamma_1$ as a sub-graph of $\Gamma$ without explicit mention of this lemma.

**Lemma 1.3.7** (Disjoint union of isomorphisms)**.** *Let $\Phi_1 \colon \Gamma_1 \to \Gamma_1'$ and $\Phi_2 \colon \Gamma_2 \to \Gamma_2'$ be isomorphisms of spatial graphs. Then there exists an isomorphism*

$$\Phi_1 \sqcup \Phi_2 \colon \Gamma_1 \sqcup \Gamma_2 \to \Gamma_1' \sqcup \Gamma_2'$$

*such that for each $i \in \{1,2\}$ the underlying isomorphism of abstract graphs $\langle \Phi_1 \sqcup \Phi_2 \rangle$ restricts to $\langle \Phi_i \rangle$ on $\langle \Gamma_i \rangle$.*

*Proof.* Form the disjoint union $\Gamma_1 \sqcup_f \Gamma_2$ using a suitable PL homeomorphism $f \colon \partial B_1 \to \partial B_2$ between the boundaries of enclosing balls for $\Gamma_1, \Gamma_2$. Writing $B_1' := \Phi_1(B_1)$, $B_2' := \Phi_2(B_2)$, and defining $f' \colon \partial B_1' \to \partial B_2'$ as $f' := \Phi_2|_{\partial B_2} \circ f \circ \Phi_1^{-1}|_{\partial B_1'}$, we can form the disjoint union $\Gamma_1' \sqcup_{f'} \Gamma_2'$. The restrictions $\Phi_i|_{B_i}$ then assemble to the desired isomorphism $\Phi_1 \sqcup \Phi_2$. $\square$

Note that this lemma strongly depends on the fact that the ambient 3-spheres of spatial graphs carry an orientation, which is preserved by isomorphisms. If we were to drop this requirement, then a spatial graph $\Gamma$ comprised of one vertex and one edge in the shape of a trefoil would be isomorphic to its mirror-image $\widetilde{\Gamma}$. The spatial graphs $\Gamma \sqcup \Gamma$ and $\Gamma \sqcup \widetilde{\Gamma}$ would however not be isomorphic.

We finish this subsection by recording basic algebraic properties of the disjoint union.

**Proposition 1.3.8** (Properties of the disjoint union)**.** *Let $\Gamma_1, \Gamma_2, \Gamma_3$ be spatial graphs. Then:*

- identity element: $\Gamma_1 \sqcup \mathbf{0} = \Gamma_1$,
- commutativity: $\Gamma_1 \sqcup \Gamma_2 = \Gamma_2 \sqcup \Gamma_1$,
- associativity: $(\Gamma_1 \sqcup \Gamma_2) \sqcup \Gamma_3 = \Gamma_1 \sqcup (\Gamma_2 \sqcup \Gamma_3)$.

*Proof.* The first statement follows from Lemma 1.3.6.

Commutativity is straightforward if one uses the same enclosing balls for both disjoint unions, and mutually inverse attaching maps.

For associativity, we need only a bit of care when choosing enclosing balls along which to perform the disjoint unions. The idea should be clear from the schematic in Figure 1.3.1, but we now supply a bit more detail. Denote the ambient 3-sphere of each $\Gamma_i$ by $\mathcal{S}_i$. Let $B_1$, $B_3$ be enclosing balls for $\Gamma_1, \Gamma_3$, respectively, and let $B_{21}$, $B_{23}$ be enclosing balls for $\Gamma_2$ such that $\operatorname{int}(B_{21}) \cup \operatorname{int}(B_{23}) = \mathcal{S}_2$. Equivalently, the 3-balls $\overline{B_{21}} := \mathcal{S} \setminus \operatorname{int}(B_{21})$ and $\overline{B_{23}} := \mathcal{S} \setminus \operatorname{int}(B_{23})$ should be disjoint. After fixing attaching maps $f_1 \colon \partial B_1 \to \partial B_{21}$, $f_3 \colon \partial B_{23} \to \partial B_3$, it follows that $B_1 \cup_{f_1} (B_{21} \cap B_{23})$ is an enclosing ball for $\Gamma_1 \sqcup_{f_1} \Gamma_2$, and $(B_{21} \cap B_{23}) \cup_{f_3} B_3$ is an enclosing ball for $\Gamma_2 \sqcup_{f_3} \Gamma_3$. The spatial graphs $(\Gamma_1 \sqcup_{f_1} \Gamma_2) \sqcup_{f_3} \Gamma_3$ and $\Gamma_1 \sqcup_{f_1} (\Gamma_2 \sqcup_{f_3} \Gamma_3)$ are then the same on the nose. ☐

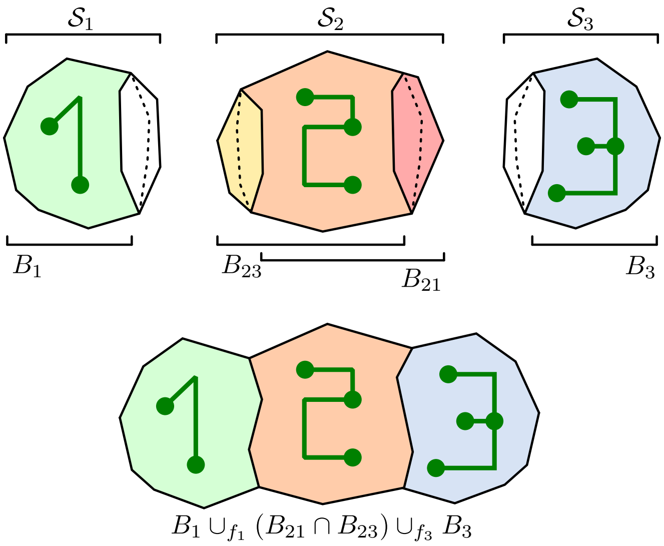



Figure 1.3.1: The proof of associativity of the disjoint union, with ambient spheres and enclosing balls depicted one dimension below. Top: the ambient spheres and enclosing balls for the spatial graphs $\Gamma_i$. Bottom: the spatial graph $\Gamma_1 \sqcup_{f_1} \Gamma_2 \sqcup_{f_3} \Gamma_3$ in its ambient sphere $B_1 \cup_{f_1} (B_{21} \cap B_{23}) \cup_{f_3} B_3$.

Commutativity and associativity allow us to unambiguously write down iterated disjoint unions. More precisely, if $\{\Gamma_i\}_{i \in I}$ is a collection of spatial graphs indexed by a finite set $I$, then $\bigsqcup_{i \in I} \Gamma_i$ is well-defined up to isomorphism inducing the identity on $\bigsqcup_{i \in I} \langle \Gamma_i \rangle$.

### 1.3.2 Decomposing spatial graphs as disjoint unions

We now start working towards proving that every spatial graph can be expressed as an iterated disjoint union in a canonical way. We will (often implicitly) use the fact that every PL-embedded 2-sphere in a PL 3-sphere decomposes it into two 3-balls. Although the topological version of this statement is known to be false by the famous counter-example of the "Alexander horned sphere", it holds in the PL setting, also due to work of Alexander [Ale24].

We begin with a simple observation.

**Lemma 1.3.9** (If it looks like a disjoint union, it is a disjoint union)**.** *Let $\Gamma$ be a spatial graph in $\mathcal{S}$ and $S \subset \mathcal{S} \setminus |\Gamma|$ a PL-embedded 2-sphere. Denote the closures of the two components of $\mathcal{S} \setminus S$ by $B_1$ and $B_2$. For each $i \in \{1, 2\}$, let $\Gamma_i$ be the sub-graph of $\Gamma$ comprised of the vertices and edges that are contained in $B_i$. Then $\Gamma = \Gamma_1 \sqcup \Gamma_2$.*

*Proof.* We use Lemma 1.3.6 to regard each $\Gamma_i$ as a sub-graph of $\Gamma_1 \sqcup \Gamma_2$, and take $B_i$ as an enclosing ball for $\Gamma_i$. If $f \colon S \to S$ is the identity map, then $\Gamma = \Gamma_1 \sqcup_f \Gamma_2$. □

Of course by definition of the disjoint union, if $\Gamma = \Gamma_1 \sqcup \Gamma_2$, then there exists such a sphere $S$.

**Definition 1.3.10.** Let $\Gamma$ be a spatial graph in $\mathcal{S}$.

- If $S \subset \mathcal{S} \setminus |\Gamma|$ is a 2-sphere and $\Gamma_1, \Gamma_2$ are as in Lemma 1.3.9, we say that "$S$ decomposes $\Gamma$ as $\Gamma_1 \sqcup \Gamma_2$".
- $\Gamma$ is said to be **separable** if it is the disjoint union of two non-empty spatial graphs; otherwise it is **non-separable**.
- If $S$ is a 2-sphere in $\mathcal{S}$ decomposing $\Gamma$ as $\Gamma_1 \sqcup \Gamma_2$ with $\Gamma_1, \Gamma_2$ non-empty, then $S$ is called a **separating sphere** for $\Gamma$.
- We will call a spatial graph a **piece** if it is non-empty and non-separable. We will also say that a spatial graph $\Lambda$ is a piece of $\Gamma$ if $\Lambda$ is a piece and $\Gamma = \Gamma' \sqcup \Lambda$ for some $\Gamma'$.

We use the word "piece", rather than "component", to avoid suggesting that for such $\Lambda$, the support $|\Lambda|$ (or equivalently the underlying graph $\langle \Lambda \rangle$) would have to be connected. Indeed, a spatial graph with non-connected support may very well be non-separable. Take, for example, a spatial graph isotopic to a Hopf link, such as the one in Figure 1.3.2.

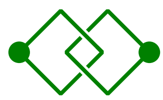


Figure 1.3.2: The notion of non-separability of spatial graphs does not coincide with the notion of connectedness. The depicted spatial graph $\Gamma$ is non-separable, but its support $|\Gamma|$ and underlying graph $\langle \Gamma \rangle$ are disconnected.

Every spatial graph $\Gamma$ can be decomposed as a disjoint union of finitely many pieces: if $\Gamma = \mathbf{0}$ we take the empty union, and if $\Gamma$ is itself a piece, we take a disjoint union indexed over a one-element set. If $\Gamma$ is non-empty and not a piece, then it can be expressed as a disjoint union of two non-empty graphs $\Gamma = \Gamma_1 \sqcup \Gamma_2$, each $\Gamma_i$ thus having strictly fewer vertices than $\Gamma$. By induction on the number of vertices, the $\Gamma_i$ can be decomposed into pieces, and hence so can $\Gamma$.

We now work towards proving that such a decomposition is unique.

**Lemma 1.3.11** (Spheres sort pieces)**.** *Let $\Lambda$ be a piece in $\mathcal{S}$, and let $S \subset \mathcal{S} \setminus |\Lambda|$ be a PL-embedded 2-sphere. Denote the closures of the two components of $\mathcal{S} \setminus S$ by $B_1, B_2$. Then $|\Lambda|$ is contained in exactly one of the $B_i$.*

*Proof.* Since $\Lambda \neq \mathbf{0}$, certainly $|\Lambda|$ cannot be contained in both $B_i$. For each $i \in \{1, 2\}$, denote by $\Lambda_i$ the sub-graph of $\Lambda$ whose vertices and edges are contained in $B_i$. Then by Lemma 1.3.9 we have that $S$ decomposes $\Lambda$ as $\Lambda_1 \sqcup \Lambda_2$. Since $\Lambda$ is non-separable, one of the summands, say $\Lambda_1$, is empty. By the first part of Proposition 1.3.8, it follows that $\Lambda_2 = \Lambda$. □

**Proposition 1.3.12** (Uniqueness of decomposition into pieces)**.** *Let $(\Lambda_i)_{i\in I_1}$ and $(\Lambda_i)_{i\in I_2}$ be collections of pieces indexed by finite sets $I_1, I_2$. Then for every isomorphism of spatial graphs $\Phi\colon \bigsqcup_{i\in I_1} \Lambda_i \to \bigsqcup_{i\in I_2} \Lambda_i$, there is a bijection $f\colon I_1 \to I_2$ such that for each $i \in I_1$, the PL homeomorphism $\Phi$ is an isomorphism of the sub-graphs $\Phi\colon \Lambda_i \to \Lambda_{f(i)}$.*

*Proof.* Write $\Gamma_1 := \bigsqcup_{i\in I_1} \Lambda_i$ and $\Gamma_2 := \bigsqcup_{i\in I_2} \Lambda_i$. We induct on the cardinality of $I_1$.

If $I_1 = \emptyset$ then $\Gamma_1 = \mathbf{0} = \Gamma_2$, so since for all $i \in I_2$ we know $\Lambda_i$ is non-empty, we conclude $I_2 = \emptyset$ and there is nothing left to show. If $I_1$ contains only one element $i_1$, then $\Gamma_1 = \Lambda_{i_1}$ is a piece. Hence $\Gamma_2$ is also a piece and therefore, again since the $\Lambda_i$ are non-empty, we conclude $I_2 = \{i_2\}$. We thus set $f(i_1) := i_2$.

If $I_1$ has more than one element, we partition it into two non-empty subsets $I_1 = I_1^+ \sqcup I_1^-$. Let $S_1$ be a 2-sphere decomposing $\Gamma_1$ as $\big(\bigsqcup_{i\in I_1^+} \Lambda_i\big) \sqcup \big(\bigsqcup_{i\in I_1^-} \Lambda_i\big)$, and write $\Gamma_1^+ := \bigsqcup_{i\in I_1^+} \Lambda_i$ and $\Gamma_1^- := \bigsqcup_{i\in I_1^-} \Lambda_i$.

Now $S_2 := \Phi(S_1)$ is a 2-sphere in the ambient sphere of $\Gamma_2$ disjoint from $|\Gamma_2|$. One side of $S_2$ corresponds to the "+"-summand of $\Gamma_1$, and the other to the "−"-summand. By Lemma 1.3.11, for each $i \in I_2$, we have $|\Lambda_i|$ contained in either the "+"-side or the "−"-side of $S_2$. Partition $I_2$ accordingly as $I_2 = I_2^+ \sqcup I_2^-$, and write $\Gamma_2^\pm := \bigsqcup_{i\in I_2^\pm} \Lambda_i$.

Since $\Phi$ maps the support $|\Gamma_1^+|$ into $|\Gamma_2^+|$, and similarly for "−", we conclude $\Phi$ doubles as a pair of isomorphisms of sub-graphs $\Phi^\pm\colon \Gamma_1^\pm \to \Gamma_2^\pm$. Both $I_1^\pm$ have fewer elements than $I_1$, so by induction we obtain bijections $f^\pm\colon I_1^\pm \to I_2^\pm$, which assemble to the required $f\colon I_1 \to I_2$. □

## 1.4 The vertex sum of spatial graphs

We now define another operation, the "vertex sum", whose input data is a pair of spatial graphs with distinguished vertices. The relevant case of the construction is when they are non-separable, but we will nevertheless formulate our definitions and statements without this hypothesis until it becomes indispensable. The overall structure of this section will be rather similar to that of the previous one, with many definitions and results having obvious counterparts.

### 1.4.1 Defining the vertex sum

**Definition 1.4.1.** A **pointed spatial graph** is a pair $(\Gamma, v)$ of a spatial graph $\Gamma$ and a vertex $v$ of $\Gamma$. The underlying graph of a pointed spatial graph is pointed with the same distinguished vertex: $\langle(\Gamma, v)\rangle := (\langle\Gamma\rangle, v)$. An isomorphism of pointed spatial graphs is an isomorphism of the spatial graphs that preserves the distinguished vertices.

**Definition 1.4.2.** An **enclosing ball** for a pointed spatial graph $(\Gamma, v)$ in $\mathcal{S}$, is a PL-embedded 3-ball $B \subset \mathcal{S}$ such that $|\Gamma| \subset B$ and $|\Gamma| \cap \partial B = \{v\}$.

**Definition 1.4.3.** For each $i \in \{1, 2\}$, let $\Gamma_i = (\mathcal{S}_i, V_i, E_i)$ be a non-empty spatial graph, let $v_i \in V_i$, and let $B_i$ be an enclosing ball for $(\Gamma_i, v_i)$. Moreover, let $f\colon \partial B_1 \to \partial B_2$ be an orientation-reversing PL homeomorphism mapping $v_1$ to $v_2$. We consider the spatial graph

$$\Gamma_1 \,{}_{v_1}\!\bullet_{v_2}\, \Gamma_2 := (B_1 \cup_f B_2, (V_1 \sqcup V_2)/v_1 \sim v_2, E_1 \sqcup E_2),$$

where $B_1 \cup_f B_2$ denotes the PL 3-sphere obtained by attaching $B_1$ to $B_2$ using $f$, and define the pointed spatial graph $(\Gamma_1 \,{}_{v_1}\!\bullet_{v_2}\, \Gamma_2, v_1 = v_2)$ to be a **vertex sum** of $(\Gamma_1, v_1)$ and $(\Gamma_2, v_2)$.

We will use the same notation to denote the analogous operation on pointed abstract graphs. For every such vertex sum of pointed spatial graphs we thus have $\langle \Gamma_1 \,{}_{v_1}\!\bullet_{v_2}\, \Gamma_2 \rangle = \langle \Gamma_1 \rangle \,{}_{v_1}\!\bullet_{v_2}\, \langle \Gamma_2 \rangle$.

We retrace the steps taken when discussing the disjoint union of spatial graphs, our next goal being to show that the vertex sum of pointed spatial graphs is independent of the choice of enclosing spheres and the attaching map $f$. The key is the following proposition.

**Proposition 1.4.4** (Uniqueness of enclosing balls for pointed spatial graphs)**.** *Let $(\Gamma, v)$ be a pointed spatial graph in $\mathcal{S}$, let $B, B'$ be enclosing balls for $(\Gamma, v)$. Then every orientation-preserving PL homeomorphism $\Phi_\partial \colon (\partial B, v) \to (\partial B', v)$ extends to an orientation-preserving PL homeomorphism $\Phi_B \colon B \to B'$ restricting to the identity on $|\Gamma|$.*

Proving this proposition will require substantially more work than its non-pointed counterpart, Lemma 1.3.4, as one should expect from the very particular behavior demanded of $\Phi$ near $v$. One of the necessary ingredients will be a generalization of the Disc Theorem (Theorem 1.3.1).

**Definition 1.4.5** ([RS72, p. 50])**.** An **unknotted ball pair** $(B, B_0)$ is a pair of polyhedra PL-homeomorphic to a standard ball pair $([-1,1]^n, [-1,1]^m \times \{0\}^{n-m})$ (for some $n \geq m \geq 0$). A **PL manifold pair** $(M, M_0)$ is a pair of polyhedra that are manifolds, such that $\partial M \cap M_0 = \partial M_0$ ("properness"), and such that each point of $M_0$ has a neighborhood in $(M, M_0)$ PL-homeomorphic to an unknotted ball pair ("local flatness")[1].

**Theorem 1.4.6** (Disc Theorem for pairs [RS72, Theorem 4.20])**.** *Let $(M, M_0)$ be a pair of connected, oriented PL manifolds, let $(B, B_0)$ be an unknotted ball pair with the same dimensions, and let $\iota_1, \iota_2 \colon (B, B_0) \to (\mathrm{int}(M), \mathrm{int}(M_0))$ be PL embeddings that preserve the orientation on both components. Then there is a PL ambient isotopy of $(M, M_0)$ relative $\partial M$ that carries $\iota_1$ to $\iota_2$.*

The reason we need the Disc Theorem for pairs is because it has the following corollary:

**Corollary 1.4.7** (Disc Theorem at the boundary)**.** *Let $M$ be a connected, oriented PL $n$-manifold, let $N \subseteq \partial M$ be a connected PL-embedded $(n-1)$-manifold that is closed in $\partial M$, let $B$ be a PL $n$-ball, and $D \subset \partial B$ a PL $(n-1)$-ball. For every two orientation-preserving PL embeddings $\iota_1, \iota_2 \colon (B, D) \to (\mathrm{int}(M) \cup \mathrm{int}(N), \mathrm{int}(N))$, there is a PL ambient isotopy of $(M, N)$ relative $\partial M \setminus \mathrm{int}(N)$ carrying $\iota_1$ to $\iota_2$.*

*Proof.* We consider the double $\mathrm{D}_N(M)$ of $M$ along $N$, which is a union of two copies of $M$ glued along the identity map on $N$, one of the copies with its orientation reversed. Using the fact that $N$ is closed in $\partial M$ one sees that $(\mathrm{D}_N(M), N)$ is a PL manifold pair, and its boundary is $(\mathrm{D}_{\partial N}(\partial M \setminus \mathrm{int}(N)), \partial N)$. Doubling also $B$ along $D$ yields an unknotted ball pair $(\mathrm{D}_D(B), D)$.

[1]The definition given by Rourke-Sanderson on p. 50 requires only that $M, M_0$ both be manifolds, but the remark on p. 51 adds the local flatness and properness conditions.

Now, the maps $\iota_1, \iota_2$ extend to orientation-preserving PL embeddings

$$\mathrm{D}(\iota_1), \mathrm{D}(\iota_2) \colon (\mathrm{D}_D(B), D) \to (\mathrm{int}(\mathrm{D}_N(M)), \mathrm{int}(N)).$$

Theorem 1.4.6 yields a PL ambient isotopy of $(\mathrm{D}_N(M), N)$ relative $\mathrm{D}_{\partial N}(\partial M \setminus \mathrm{int}(N))$ that carries $\mathrm{D}(\iota_1)$ to $\mathrm{D}(\iota_2)$. A connectivity argument shows that it restricts to an isotopy from $\iota_1$ to $\iota_2$ relative $\partial M \setminus \mathrm{int}(N)$. $\square$

We will also need the observation in Lemma 1.4.9 below, but before stating it we remind the reader of some standard terminology.

**Definition 1.4.8.** Let $P$ be a polyhedron in some $\mathbb{R}^n$, and let $v \in \mathbb{R}^n$. We write $vP$ to denote the polyhedron comprised of all points of the form $tp + (1-t)v$, with $p \in P$ and $t \in [0,1]$. If each point of $vP$ admits a unique such expression, we say $vP$ is a **cone** with base $P$ and vertex $v$.

Given two cones $vP, wQ$ with bases $P, Q$ and vertices $v, w$, respectively, and a PL map $f \colon P \to Q$, the **cone** of $f$ (with respect to $v, w$) [RS72, Exercise 1.6(3)] is the PL map $vP \to wQ$ given by

$$tp + (1-t)v \mapsto tf(p) + (1-t)w.$$

**Lemma 1.4.9** (Interpolating annulus)**.** *Let $A_0$ be a PL annulus in some $\mathbb{R}^n$, and let $v \in \mathbb{R}^n$ be such that $vA_0$ is a cone with base $A_0$ and vertex $v$. Denote the two boundary circles of $A_0$ by $\gamma_0, \delta_0$, and let $\gamma \subset v\gamma_0$ and $\delta \subset v\delta_0$ be PL circles such that $v\gamma, v\delta$ are cones with bases $\gamma, \delta$ respectively, and vertex $v$. Then there exists a PL annulus $A \subset vA_0$ with $\partial A = \gamma \cup \delta$, such that $vA$ is a cone with base $A$ and vertex $v$.*

This lemma is illustrated in Figure 1.4.1.

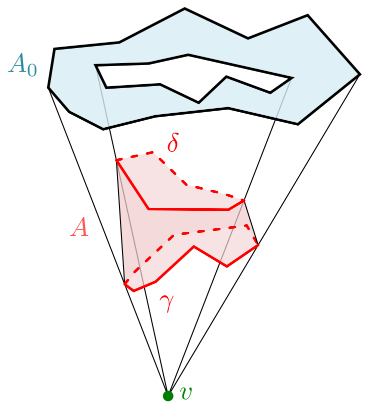



Figure 1.4.1: The setup of Lemma 1.4.9. The PL annulus $A$ “interpolates” between the PL circles $\gamma, \delta$.

*Proof.* We may assume without loss of generality that $A_0 = C \times [0,1] \subset \mathbb{R}^n$ for some PL circle $C \subset \mathbb{R}^{n-1}$, with $\gamma_0 = C \times \{0\}$ and $\delta_0 = C \times \{1\}$, because for every PL homeomorphism $C \times [0,1] \to A_0$, the cone $v(C \times [0,1]) \to vA_0$ preserves cones at $v$.

Choose a finite set of points in $\gamma \subset v(C \times \{0\})$ subdividing $\gamma$ into straight line segments (see Rourke-Sanderson for details [RS72, Theorem 2.2]). Pushing these points radially onto $\gamma_0 = C \times \{0\}$ and projecting onto $C$ yields a finite set of points in $C$ (note that since $v\gamma$ is a cone by assumption, no two points from $\gamma$ get pushed onto the same point of $\gamma_0$). Doing the same with $\delta$ yields a second finite subset of $C$. Finally, choose a third finite subset of $C$ subdividing $C$ itself into straight line segments. We denote by $p_1, \ldots, p_k$ the points in the union of these three subsets, ordered cyclically around $C$. The indices $1, \ldots, k$ should thus be interpreted as lying in $\mathbb{Z}/k$. We now push the points $(p_1, 0), \ldots, (p_k, 0) \in \gamma_0$ radially onto $\gamma$ to obtain points $p_1^\gamma, \ldots, p_k^\gamma$. Similarly, pushing $(p_1, 1), \ldots, (p_k, 1)$ radially yields $p_1^\delta, \ldots, p_k^\delta$.

Since the points $p_i$ subdivide $C$ into straight line segments, we see that for each $i \in \mathbb{Z}/k$, the points $(p_i, 0), (p_{i+1}, 0), (p_i, 1), (p_{i+1}, 1)$ are the vertices of a rectangle $R_i$ contained in $A_0$. In particular, the cone $vR_i \subset vA_0$ is convex. This is the crucial observation that will allow us to find the desired annulus $A$.

For each $i \in \mathbb{Z}/k$, denote by $T_i^\gamma$ the triangle spanned by the points $p_i^\gamma, p_{i+1}^\gamma, p_i^\delta$, and by $T_i^\delta$ the one spanned by $p_i^\delta, p_{i+1}^\delta, p_{i+1}^\gamma$. By the previous observation, both of these triangles are contained in $vR_i$. The union $A := \bigcup_{i \in \mathbb{Z}/k} (T_i^\gamma \cup T_i^\delta)$ is then a PL annulus embedded in $vA_0$, with $\partial A = \gamma \cup \delta$. It is also easy to see that each point of $A$ lies in a unique ray from $v$ through a point in $A$, whence the cone condition on $vA$ follows. $\square$

Finally we are equipped to prove Proposition 1.4.4.

*Proof of Proposition 1.4.4.* As in the proof of Lemma 1.3.4, write $\overline{B} := \mathcal{S} \setminus \operatorname{int}(B)$ and $\overline{B'} := \mathcal{S} \setminus \operatorname{int}(B')$, and choose any extension of $\Phi_\partial$ to a PL homeomorphism $\Phi_{\overline{B}} \colon \overline{B} \to \overline{B'}$. We will find an extension $\Phi_{\mathcal{S}} \colon \mathcal{S} \to \mathcal{S}$ of $\Phi_{\overline{B}}$ that fixes $|\Gamma|$, and whose restriction $\Phi_B$ to $B$ will therefore satisfy the conclusion of the lemma. The construction of this extension $\Phi_{\mathcal{S}}$ is somewhat intricate, so we need to introduce some notation, which we illustrate in Figure 1.4.2.

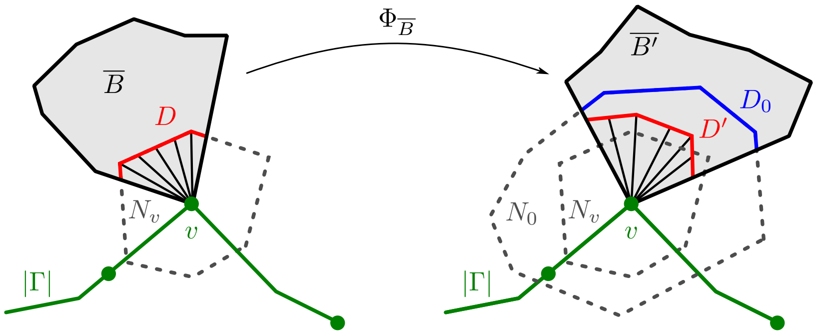



Figure 1.4.2: The 3-balls $\overline{B}$ and $\overline{B'}$ in the setup of the proof of Lemma 1.4.4. We emphasize the action of $\Phi_{\overline{B}}$ on $vD$ as the cone of a PL homeomorphism $D \to D'$.

First, choose a star neighborhood $N_0$ for $v$ in the pair $(\mathcal{S}, \overline{B'} \cup |\Gamma|)$. More explicitly, $N_0$ is a 3-ball such that the polyhedron $(\overline{B'} \cup |\Gamma|) \cap N_0$ is a cone with base its intersection with $\partial N_0$, and with vertex $v$ [RS72, p. 50]. In particular, $D_0 := \overline{B'} \cap \partial N_0$ is a disc and $\overline{B'} \cap N_0$ is the cone $vD_0$ with base $D_0$ and vertex $v$.

We then pick a smaller star neighborhood $N_v \subset \operatorname{int}(N_0)$ of $v$ in $(\mathcal{S}, \overline{B'} \cup |\Gamma|)$, such that $N_v$ is also a star neighborhood of $v$ in $(\mathcal{S}, \overline{B} \cup |\Gamma|)$, and $\overline{B} \cap N_v$ is mapped conically by $\Phi_{\overline{B}}$ into $\operatorname{int}(N_0)$. Denoting by $D$ the disc $\overline{B} \cap \partial N_v$, so $\overline{B} \cap N_v$ is a cone $vD$ with base $D$ and vertex $v$, this means that $\Phi_{\overline{B}}(vD)$ is a cone $vD'$ with base the disc $D' := \Phi_{\overline{B}}(D)$ and vertex $v$, and that $\Phi_{\overline{B}}|_{vD} \colon vD \to vD'$ is the cone of $\Phi_{\overline{B}}|_D \colon D \to D'$. The existence of such $N_v$ follows from the definitions of PL map and polyhedron, say, by taking $N_v$ to be a sufficiently small $\epsilon$-neighborhood of $v$. We will denote by $\overline{N_v}$ the 3-ball $\mathcal{S} \setminus \operatorname{int}(N_v)$.

In order to apply the disc theorem at the boundary, we will first need to move $\overline{B'}$ into a nicer configuration. More precisely, we will use the following fact, illustrated in Figure 1.4.3.

**Claim.** *There exists an orientation-preserving PL homeomorphism $\Psi\colon \mathcal{S} \to \mathcal{S}$ such that:*

- *$\Psi$ maps the pair $(\Phi_{\overline{B}}(\overline{B} \cap \overline{N_v}), D')$ into the pair $(\overline{N_v}, \partial N_v)$,*
- *writing $\widetilde{D} := \Psi(D')$, the map $\Psi$ is given on $vD'$ as the cone $vD' \to v\widetilde{D}$ of the PL homeomorphism $D' \to \widetilde{D}$, and*
- *$\Psi$ fixes $|\Gamma|$.*

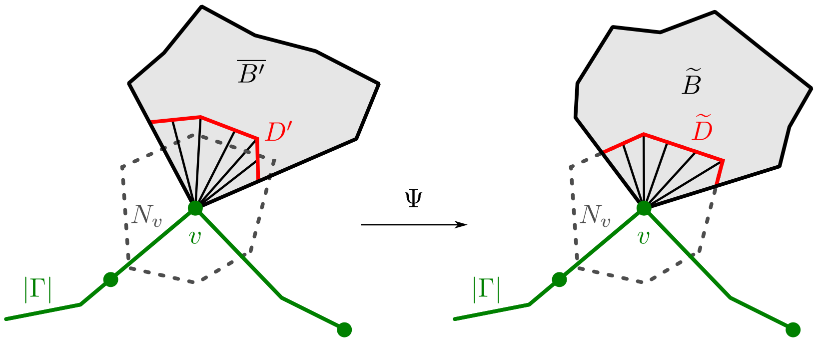



Figure 1.4.3: The 3-ball $\overline{B'}$ and its $\Psi$-image $\widetilde{B}$. The disc $D'$ is mapped by $\Psi$ to a disc $\widetilde{D}$ in $\partial N_v$, and $\Psi$ acts on $vD'$ as the cone of this map.

Assume for the moment that this claim holds, and let us see how to use the resulting $\Psi$ to construct the desired extension $\Phi_{\mathcal{S}}$ of $\Phi_{\overline{B}}$.

Let $\widetilde{B}$ be the 3-ball $\Psi(\overline{B'})$, and choose a regular neighborhood $N_\Gamma$ of $|\Gamma| \cap \overline{N_v}$ in $\overline{N_v}$, small enough to be disjoint from $\overline{B}$ and $\widetilde{B}$. Moreover, denote by $M$ the closure in $\overline{N_v}$ of $\overline{N_v} \setminus N_\Gamma$, and consider the closed codimension-0 submanifold $N := \partial N_v \cap M$ of $\partial M$.

By construction of $\Psi$, its restriction to $\Phi_{\overline{B}}(\overline{B} \cap \overline{N_v})$ is a PL homeomorphism of pairs $(\Phi_{\overline{B}}(\overline{B} \cap \overline{N_v}), D') \to (\widetilde{B} \cap \overline{N_v}, \widetilde{D})$. We may thus apply the Disc Theorem at the boundary (Corollary 1.4.7) to the inclusion $(\overline{B} \cap \overline{N_v}, D) \hookrightarrow (M, N)$ and the composition

$$(\overline{B} \cap \overline{N_v}, D) \xrightarrow{\Phi_{\overline{B}}} (\Phi_{\overline{B}}(\overline{B} \cap \overline{N_v}), D') \xrightarrow{\Psi} (\widetilde{B} \cap \overline{N_v}, \widetilde{D}) \hookrightarrow (M, N),$$

with the maps labeling the arrows appropriately restricted. This is illustrated in Figure 1.4.4.

The final PL homeomorphism $\widetilde{\Phi}_M\colon M \to M$ of the resulting PL isotopy of $M$ extends the composition $\Psi|_{\Phi_{\overline{B}}(\overline{B} \cap \overline{N_v})} \circ \Phi_{\overline{B}}|_{\overline{B} \cap \overline{N_v}}$ and fixes $\partial M \setminus \mathrm{int}(N) = \partial M \cap N_\Gamma$. We may thus extend $\widetilde{\Phi}_M$ to a PL homeomorphism $\widetilde{\Phi}_{\overline{N_v}}\colon \overline{N_v} \to \overline{N_v}$ by setting it to be the identity on $N_\Gamma$. In particular, $\widetilde{\Phi}_{\overline{N_v}}$ fixes $|\Gamma| \cap \overline{N_v}$. Finally, extend $\widetilde{\Phi}_{\overline{N_v}}$ to a PL homeomorphism $\widetilde{\Phi}_{\mathcal{S}}\colon \mathcal{S} \to \mathcal{S}$ by defining it on $N_v = v(\partial N_v)$ as the cone of the already prescribed PL homeomorphism $\partial N_v \to \partial N_v$.

The restriction $\widetilde{\Phi}_{\mathcal{S}}|_{\overline{B}}$ is now the composition $\Psi|_{\overline{B}} \circ \Phi_{\overline{B}}$: indeed, we have already seen that the two maps agree on $\overline{B} \cap \overline{N_v}$, and on $\overline{B} \cap N_v = vD$ both are defined as the cone of $D \xrightarrow{\Phi_{\overline{B}}} D' \xrightarrow{\Psi} \widetilde{D}$. Moreover, $\widetilde{\Phi}_{\mathcal{S}}$ clearly fixes $|\Gamma|$. Hence, the map $\Phi_{\mathcal{S}} := \Psi^{-1} \circ \widetilde{\Phi}_{\mathcal{S}}$ extends $\Phi_{\overline{B}}$ and fixes $|\Gamma|$, as desired. We are only left to prove the above claim.

*Proof of the claim.* Most of the work consists of adding enough detail to our pictures that the map can be made explicit. The construction is illustrated in Figure 1.4.5.

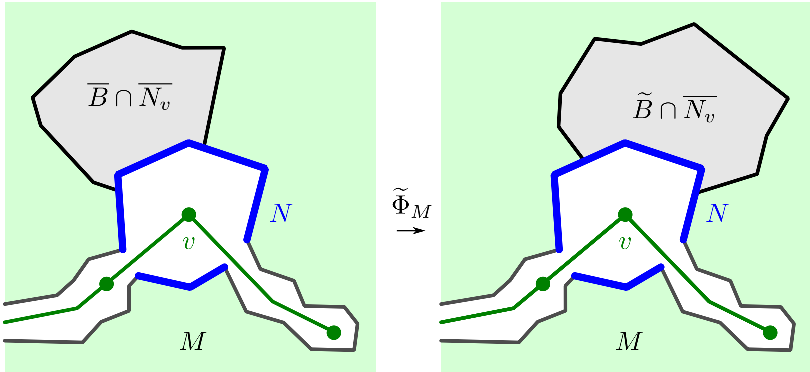



Figure 1.4.4: Applying the Disc Theorem at the boundary in order to ambiently isotope the pair $(\overline{B} \cap \overline{N_v}, D)$ onto $(\widetilde{B} \cap \overline{N_v}, \widetilde{D})$ within $(M, N)$.

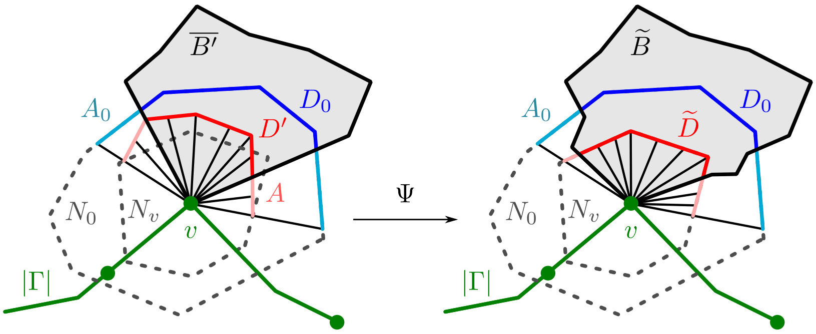



Figure 1.4.5: The construction of $\Psi$. Outside the 3-ball $vD_0^+$, the map $\Psi$ is defined as the identity. On $D'^+$, it is an extension of the identity $\partial D'^+ \to \partial \widetilde{D}^+$ to the interior of the discs, and this in turn is coned to a PL homeomorphism $vD'^+ \to v\widetilde{D}^+$. On the remaining 3-ball $C$, we take any extension of the already prescribed map on $\partial C$.

Choose a collar for $\partial D_0$ in $B' \cap \partial N_0$, that is, a PL embedding $c\colon \partial D_0 \times [0,1] \to B' \cap \partial N_0$ such that $c(-,0)$ is the identity on $\partial D_0$, and $c(\partial D_0 \times [0,1[)$ is an open neighborhood of $\partial D_0$ in $B' \cap \partial N_0$. We may also assume that the image $A_0$ of $c$ is disjoint from $|\Gamma|$. See Rourke-Sanderson for a discussion on collars [RS72, p. 24].

Let $D_0^+$ be the "enlarged disc" $D_0 \cup A_0$, and consider the 3-ball $vD_0^+$, which is a cone with base $D_0^+$ and vertex $v$. We will define $\Psi$ as the identity on $\mathcal{S} \setminus \mathrm{int}(vD_0^+)$, and then find a suitable extension of the identity on $\partial(vD_0^+)$ to all of $vD_0^+$.

Denote by $\widetilde{D}^+$ the disc $vD_0^+ \cap \partial N_v$ and consider the PL circles $\partial D'$ and $\partial \widetilde{D}^+$, each lying in the cone of a distinct component of $\partial A_0$. Each of these circles is the base of a cone with vertex $v$, so we can use Lemma 1.4.9 to find an annulus $A$ with $\partial A = \partial D' \cup \partial \widetilde{D}^+$, such that $vA$ is a cone with base $A$ and vertex $v$. We will denote by $D'^+$ the disc $D' \cup A$. Notice that by construction, we have $\partial D'^+ = \partial \widetilde{D}^+$.

To define $\Psi$ inside $vD_0^+$, we first choose any extension of the identity map $\partial D'^+ \to \partial \widetilde{D}^+$ to a PL homeomorphism $D'^+ \to \widetilde{D}^+$. Since both $vD'^+$ and $v\widetilde{D}^+$ are cones at $v$, we can define $\Psi$ on $vD'^+$ as the cone of the above PL homeomorphism $D'^+ \to \widetilde{D}^+$. Note that this is consistent with the definition of $\Psi$ as the identity on $\partial(vD_0^+)$.

It remains only to define $\Psi$ on $vD_0^+ \setminus vD'^+$, whose closure in $\mathcal{S}$ is a 3-ball $C$ (because it is the complement in $vD_0^+$ of an open regular neighborhood of a boundary point).

Writing $\widetilde{C}$ to denote the closure in $\mathcal{S}$ of $vD_0^+ \setminus v\widetilde{D}^+$, which is another 3-ball, this amounts to choosing a PL homeomorphism $C \to \widetilde{C}$ that agrees with the already prescribed map $\partial C \to \partial\widetilde{C}$. We choose any extension, and this completes the definition of $\Psi$. It is straightforward to verify that all required conditions on $\Psi$ are satisfied.

With the claim established, the proof is complete. □

We now collect the dividends from our work proving Proposition 1.4.4.

**Proposition 1.4.10** (Well-definedness of the vertex sum)**.** *Any two vertex sums of pointed spatial graphs* $(\Gamma_1, v_1), (\Gamma_2, v_2)$ *are isomorphic via an isomorphism that induces the identity on* $(\langle\Gamma_1\rangle \,{}_{v_1}\!\bullet_{v_2} \langle\Gamma_2\rangle, v_1 = v_2)$*.*

*Proof.* The argument can be copied almost word-by-word from the proof of Proposition 1.3.5, with the role of Lemma 1.3.4 now of course being played by Proposition 1.4.4. □

We can now rest at ease knowing that the ambiguity about enclosing balls and attaching maps in the notation "$\Gamma_1 \,{}_{v_1}\!\bullet_{v_2} \Gamma_2$" is immaterial.

We remark that in the purely combinatorial setting of abstract graphs, we can just as well define the vertex sum along an ordered $k$-tuple of distinct vertices. For spatial graphs, however, we would need a generalization of the notion of an enclosing ball: a 3-ball containing the support of the spatial graph, and whose boundary intersects it precisely at the $k$ distinguished vertices. But such balls could very well be non-unique in the sense of Proposition 1.4.4, as we illustrate in Figure 1.4.6. Hence our efforts to define a vertex sum of spatial graphs along multiple vertices would necessarily fall short, unless we were willing to also encode the data for the enclosing balls into the operation.

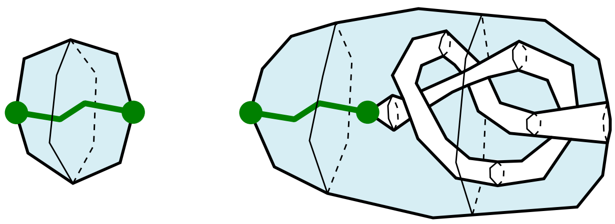


Figure 1.4.6: An example of non-uniqueness of enclosing balls for spatial graphs with more than one distinguished vertex. We depict a spatial graph comprised of exactly two vertices and one edge connecting them. If both vertices are distinguished, one can find not only the obvious enclosing ball on the left, but also more complicated ones, such as the one on the right.

There are analogues of Lemmas 1.3.6 and 1.3.7 for vertex sums, whose proofs are the same:

**Lemma 1.4.11** (Vertex summands as sub-graphs)**.** *Let* $\Gamma = \Gamma_1 \,{}_{v_1}\!\bullet_{v_2} \Gamma_2$ *be a vertex sum of pointed spatial graphs, and denote by* $\Gamma_1'$ *the sub-graph of* $\Gamma$ *obtained by discarding all vertices and edges that are not in* $\Gamma_1$*. Then* $\Gamma_1' = \Gamma_1$*.*

We take a brief moment to note how we have slightly extended our ongoing abuse of notation when writing "$\Gamma_1' = \Gamma_1$". Implicit in this statement is an equality between the vertex $v_1$ of $\Gamma_1$ and the vertex of $\Gamma_1 \,{}_{v_1}\!\bullet_{v_2} \Gamma_2$ obtained from the identification $v_1 \sim v_2$. We will allow ourselves to make such abuses in several harmless situations.

**Lemma 1.4.12** (Vertex sum of isomorphisms)**.** *Consider two isomorphisms of pointed spatial graphs* $\Phi_1\colon (\Gamma_1, v_1) \to (\Gamma_1', v_1')$ *and* $\Phi_2\colon (\Gamma_2, v_2) \to (\Gamma_2', v_2')$*. Then there exists an isomorphism*

$$\Phi_1 {}_{v_1}\!\bullet_{v_2} \Phi_2\colon (\Gamma_1 {}_{v_1}\!\bullet_{v_2} \Gamma_2, v_1 = v_2) \to (\Gamma_1' {}_{v_1'}\!\bullet_{v_2'} \Gamma_2', v_1' = v_2')$$

*such that for each* $i \in \{1, 2\}$ *the underlying isomorphism* $\langle \Phi_1 {}_{v_1}\!\bullet_{v_2} \Phi_2 \rangle$ *restricts to* $\langle \Phi_i \rangle$ *on* $\langle \Gamma_i \rangle$*.*

We also collect the following properties of the vertex sum, analogous to the ones given in Proposition 1.3.8 for the disjoint union.

**Proposition 1.4.13** (Properties of the vertex sum)**.** *Let* $(\Gamma_1, v_1), (\Gamma_2, v_2), (\Gamma_3, v_3)$ *be pointed spatial graphs. Then:*

- *identity element:* $(\Gamma_1 {}_{v_1}\!\bullet \mathbf{1}, v_1) = (\Gamma_1, v_1)$,
- *commutativity:* $(\Gamma_1 {}_{v_1}\!\bullet_{v_2} \Gamma_2, v_1 = v_2) = (\Gamma_2 {}_{v_2}\!\bullet_{v_1} \Gamma_1, v_1 = v_2)$,
- *associativity: for* $v_{21}$ *and* $v_{23}$ *(not necessarily distinct) vertices of* $\Gamma_2$*, we have*

$$(\Gamma_1 {}_{v_1}\!\bullet_{v_{21}} \Gamma_2) {}_{v_{23}}\!\bullet_{v_3} \Gamma_3 = \Gamma_1 {}_{v_1}\!\bullet_{v_{21}} (\Gamma_2 {}_{v_{23}}\!\bullet_{v_3} \Gamma_3).$$

*Proof.* The proofs are identical to their counterparts in Proposition 1.3.8, save for the following obvious modifications:

- The first item relies on vertex summands being sub-graphs (Lemma 1.4.11), rather than disjoint union summands being sub-graphs (Lemma 1.3.6).
- For proving associativity in the case $v_{21} = v_{23}$, the requirement on the enclosing balls is that $\mathrm{int}(B_1) \cup \mathrm{int}(B_2) = \mathcal{S} \setminus \{v_{21}\}$ (equivalently, $\overline{B_{21}} \cap \overline{B_{23}} = \{v_{21}\}$). □

### 1.4.2 Iterated vertex sums and trees of spatial graphs

At this juncture, we would like to make a claim about how the basic algebraic properties in Proposition 1.4.13 allow us to write down iterated vertex sums without keeping track of the order in which they are performed (as in the last paragraph of Subsection 1.3.1). We will indeed establish such a statement, but since we wish to allow for several vertices in each summand to be used (as in the associative property), we need to introduce terminology that allows us to package the more involved combinatorics.

Before doing that, however, let us spend a moment on the comparatively easy case where only one vertex of each summand is used. We denote by $(\bigstar_{i\in I}(\Gamma_i, v_i), v)$ the vertex sum of a collection of pointed spatial graphs $(\Gamma_i, v_i)$ indexed by a finite set $I$ (with $\mathbf{1}$ being the vertex sum over the empty set). If $V_i$ is the vertex set of $\Gamma_i$, then the vertex set of $\bigstar_{i\in I}(\Gamma_i, v_i)$ is $\left(\bigsqcup_{i\in I} V_i\right)/\!\sim$, where $v_i \sim v_{i'}$ for all $i, i' \in I$. The distinguished vertex $v$ of this vertex sum is the one obtained from identifying all the $v_i$. Here it is clear from commutativity and the "$v_{21} = v_{23}$" case of associativity in Proposition 1.4.13 that the omission of parentheses or an ordering of $I$ is immaterial – all choices yield pointed spatial graphs that are isomorphic via maps that induce the identity on the underlying pointed graph $(\bigstar_{i\in I}(\langle \Gamma_i \rangle, v_i), v)$.

To formalize iterated vertex sums where the vertices along which to sum are allowed to vary, we use the following notion:

**Definition 1.4.14.** A **tree of spatial graphs** is a tuple $(T, I, J, L, (\Gamma_i)_{i\in I}, (v(l))_{l\in L})$, where:

- $T$ is an abstract finite tree with vertex set $I \sqcup J$ and edge set $L$.
- The partition of the vertex set of $T$ into $I$ and $J$ is a bipartition of $T$, that is, each edge $l \in L$ has one of its endpoints in $I$, and the other in $J$. We will write $i(l), j(l)$, respectively, to denote the endpoints of $l$ in $I$ and $J$.
- Each vertex in $J$ is adjacent to at least two edges of $T$. Equivalently, all degree-one vertices of $T$ are in $I$, and $T$ is not comprised of only one vertex in $J$.
- The $\Gamma_i$ are spatial graphs indexed by elements of $I$.
- For each $l \in L$, $v(l)$ is a vertex of the spatial graph $\Gamma_{i(l)}$.
- If two different edges $l, l' \in L$ satisfy $i(l) = i(l')$, then $v(l) \neq v(l')$.

One should think of such $\mathcal{T} = (T, I, J, L, (\Gamma_i)_{i \in I}, (v(l))_{l \in L})$ as a blueprint for assembling a spatial graph $[\mathcal{T}]$, called its **realization**, out of the $\Gamma_i$ through iterated vertex sums. Roughly, when two distinct edges $l, l' \in L$ satisfy $j(l) = j(l')$ (and hence $i(l) \neq i(l')$), we understand this as an instruction to perform the vertex sum of $\Gamma_i, \Gamma_{i'}$ along $v_l, v_{l'}$. Before making this more precise, we invite the reader to study the example in Figure 1.4.7.

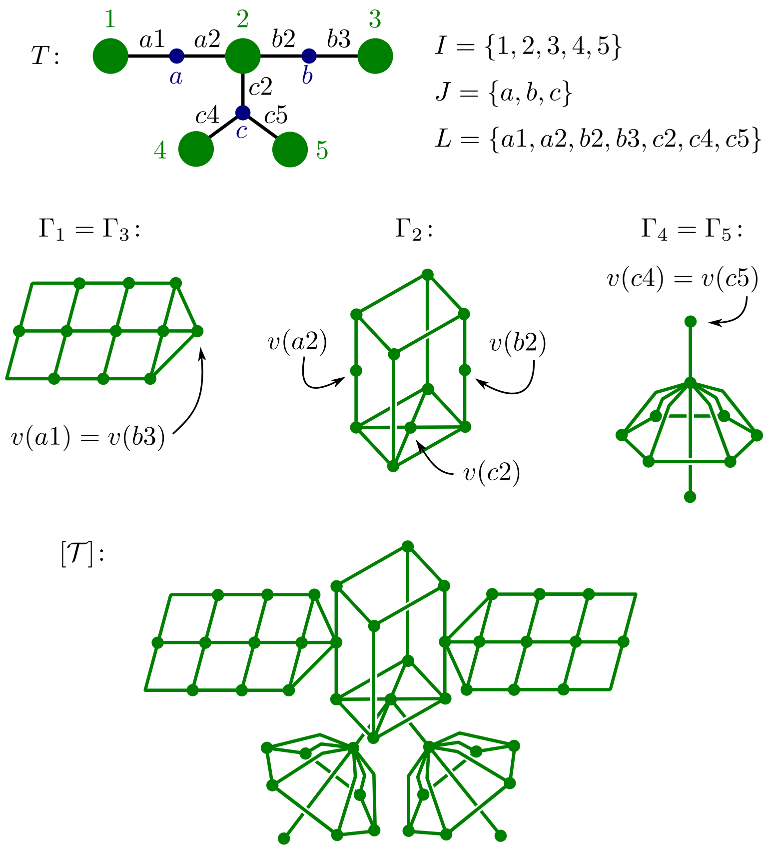



Figure 1.4.7: From a tree of spatial graphs $\mathcal{T} = (T, I, J, L, (\Gamma_i)_{i \in I}, (v(l))_{l \in L})$, we assemble its realization, the *spatial* graph $[\mathcal{T}]$.

We will use an inductive argument to define, given a tree of spatial graphs $\mathcal{T}$ as above, its realization $[\mathcal{T}]$, and show that the underlying graph $\langle [\mathcal{T}] \rangle$ is what one expects:

- the vertex set of $\langle[\mathcal{T}]\rangle$ is $\left(\bigsqcup_{i\in I} V_i\right)/\!\sim$, where $V_i$ is the vertex set of $\Gamma_i$ and $v(l) \sim v(l')$ whenever $j(l) = j(l')$,
- the edge set of $\langle[\mathcal{T}]\rangle$ is $\bigsqcup_{i\in I} E_i$, where $E_i$ is the edge set of $\Gamma_i$, each edge being incident to the one or two vertices it contains.

We induct on the cardinality of $J$. If $J = \emptyset$, then either $T$ is the empty tree, in which case we set $[\mathcal{T}] := \mathbf{0}$, or $T$ has a single vertex $i_0 \in I$, in which case $[\mathcal{T}] := \Gamma_{i_0}$. Either way, $\langle[\mathcal{T}]\rangle$ is as claimed.

For the inductive step, we first introduce the following notation: for each edge $l \in L$, the sub-graph of $T$ obtained by removing $l$ has precisely two connected components, each containing one endpoint of $l$. We will designate by $T_l$ the component that contains $i(l)$. Moreover, we denote by $I_l, J_l, L_l$, respectively, the subsets of $I, J, L$ comprised of vertices/edges in $T_l$. This allows us to define a new tree of spatial graphs $\mathcal{T}_l := (T_l, I_l, J_l, L_l, (\Gamma_i)_{i\in I_l}, (v(l'))_{l'\in L_l})$.

Now, if $J$ contains at least one vertex $j_0$ (whose choice we will soon show to be immaterial) let $L_0 \subseteq L$ be the set of edges incident to $j_0$. For each $l \in L_0$, note that $J_l$ has strictly fewer elements than $J$. Hence we have by induction constructed realizations $[\mathcal{T}_l]$, whose underlying graphs $\langle[\mathcal{T}_l]\rangle$ are as described above. In particular, $\langle[\mathcal{T}_l]\rangle$ has $v(l)$ as a vertex, and hence so does $[\mathcal{T}_l]$. We define

$$[\mathcal{T}] := \mathop{\bigstar}_{l\in L_0} ([\mathcal{T}_l], v(l)),$$

and call it a realization of $\mathcal{T}$.

Showing that $\langle[\mathcal{T}]\rangle$ is as claimed is now a matter of bookkeeping. One way of seeing it is to compare the claimed description of $\langle[\mathcal{T}]\rangle$ above with $\bigsqcup_{l\in L_0}\langle[\mathcal{T}_l]\rangle$: these graphs differ only in that the vertices $v(l)$ with $l \in L_0$ are identified in the former, but not in the latter. This identification is exactly what one obtains from the vertex sum $\bigstar_{l\in L_0}(\langle[\mathcal{T}_l]\rangle, v(l))$.

This finishes *an* inductive construction of the realization $[\mathcal{T}]$ with $\langle[\mathcal{T}]\rangle$ independent of choices. Next we show that $[\mathcal{T}]$ itself is independent of the choice of vertex $j_0$.

**Lemma 1.4.15** (Well-definedness of the realization)**.** *For every tree of spatial graphs $\mathcal{T} = (T, I, J, L, (\Gamma_i)_{i\in I}, (v(l))_{l\in L})$, any two realizations of $\mathcal{T}$ are isomorphic via an isomorphism that induces the identity on $\langle[\mathcal{T}]\rangle$.*

*Proof.* We again proceed by induction on the cardinality of $J$. When $J$ has at most one element, no choices are made in defining $[\mathcal{T}]$, so there is nothing to show.

Suppose then that $J$ contains two elements $j_1 \neq j_2$. For each $k \in \{1, 2\}$, denote by $[\mathcal{T}]_k$ the realization of $\mathcal{T}$ constructed by splitting $T$ at $j_k$. Moreover, let $L_k \subset L$ be the set of edges incident with $j_k$, and consider, for each $l \in L_k$, the tree of spatial graphs $\mathcal{T}_l := (T_l, I_l, J_l, L_l, (\Gamma_i)_{i\in I_l}, (v(l'))_{l'\in L_l})$ defined as before.

Now, there is exactly one edge $l_1 \in L_1$ such that the tree $T_{l_1}$ contains the vertex $j_2$, and one edge $l_2 \in L_2$ such that $T_{l_2}$ contains $j_1$. Since the intersection of two sub-trees of a tree is always itself a tree, we see that $\dot{T} := T_{l_1} \cap T_{l_2}$ is a tree, and indeed we have a tree of spatial graphs

$$\dot{\mathcal{T}} := (\dot{T}, \dot{I}, \dot{J}, \dot{L}, (\Gamma_i)_{i\in\dot{I}}, (v(l'))_{l'\in\dot{L}}),$$

where $\dot{I} := I_{l_1} \cap I_{l_2}$, $\dot{J} := J_{l_1} \cap J_{l_2}$, and $\dot{L} := L_{l_1} \cap L_{l_2}$. The tree $\dot{T}$ is illustrated in Figure 1.4.8.

By inductive hypothesis, for each $k \in \{1, 2\}$ the realization $[\mathcal{T}_{l_k}]$ is well-defined. One then easily checks that

$$[\mathcal{T}_{l_1}] = [\dot{\mathcal{T}}]\ {}_{v(l_2)}{\bullet}_{v_2} \mathop{\bigstar}_{l\in L_2\setminus\{l_2\}} ([\mathcal{T}_l], v(l)),$$

$$[\mathcal{T}_{l_2}] = [\dot{\mathcal{T}}]\ {}_{v(l_1)}{\bullet}_{v_1} \mathop{\bigstar}_{l\in L_1\setminus\{l_1\}} ([\mathcal{T}_l], v(l)),$$

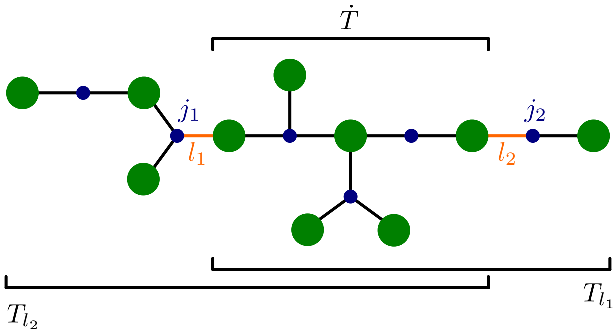



Figure 1.4.8: The setup in the proof of Lemma 1.4.15. We illustrate an example tree $T$ together with the relevant sub-trees $T_{l_1}, T_{l_2}$, and their intersection $\dot{T}$. Larger vertices represent elements of $I$, and smaller ones elements of $J$.

where $v_k$ denotes the result of identifying the vertices $v(l)$ with $l \in L_k \setminus \{l_k\}$.

Having established all the notation, we are ready to wrap up the proof and see that it boils down to an application of the "$v_{21} \neq v_{23}$" case of the associative property in Proposition 1.4.13:

$$\begin{aligned}
[\mathcal{T}]_1 &= [\mathcal{T}_{l_1}] \; {}_{v(l_1)}\!\bullet_{v_1} \left( \bigstar_{l \in L_1 \setminus \{l_1\}} ([\mathcal{T}_l], v(l)) \right) \\
&= \left( \left( \bigstar_{l \in L_2 \setminus \{l_2\}} ([\mathcal{T}_l], v(l)) \right) {}_{v_2}\!\bullet_{v(l_2)} [\dot{\mathcal{T}}] \right) {}_{v(l_1)}\!\bullet_{v_1} \left( \bigstar_{l \in L_1 \setminus \{l_1\}} ([\mathcal{T}_l], v(l)) \right) \\
&= \left( \bigstar_{l \in L_2 \setminus \{l_2\}} ([\mathcal{T}_l], v(l)) \right) {}_{v_2}\!\bullet_{v(l_2)} \left( [\dot{\mathcal{T}}] \; {}_{v(l_1)}\!\bullet_{v_1} \left( \bigstar_{l \in L_1 \setminus \{l_1\}} ([\mathcal{T}_l], v(l)) \right) \right) \\
&= \left( \bigstar_{l \in L_2 \setminus \{l_2\}} ([\mathcal{T}_l], v(l)) \right) {}_{v_2}\!\bullet_{v(l_2)} [\mathcal{T}_{l_2}] \\
&= [\mathcal{T}]_2. \qquad \square
\end{aligned}$$

Since realizations of trees are constructed by iterated vertex sums, the observations in Lemmas 1.4.11 and 1.4.12 have the following straightforward generalizations.

**Lemma 1.4.16** (Sub-graphs of the realization of a tree of spatial graphs)**.** *Let $\mathcal{T} = (T, I, J, L, (\Gamma_i)_{i \in I}, (v(l))_{l \in L})$ be a tree of spatial graphs, and for each $i \in I$, let $\Gamma'_i$ be the sub-graph of $[\mathcal{T}]$ comprised of the vertices and edges of $\Gamma_i$. Then $\Gamma'_i = \Gamma_i$.*

**Lemma 1.4.17** (Trees of isomorphisms)**.** *For each $k \in \{1, 2\}$, consider a tree of spatial graphs $\mathcal{T}_k = (T_k, I_k, J_k, L_k, (\Gamma_i)_{i \in I_k}, (v(l))_{l \in L_k})$. Fix also the data of:*

- *an isomorphism of trees $f \colon T_1 \to T_2$ such that $f(I_1) = I_2$ (hence also $f(J_1) = J_2$),*
- *for each $i \in I_1$, an isomorphism of spatial graphs $\Phi_i \colon \Gamma_i \to \Gamma_{f(i)}$, such that the collection $(\Phi_i)_{i \in I_1}$ respects the assignments $l \mapsto v(l)$ on $L_1$ and $L_2$, that is, for every $l \in L_1$, we have $\Phi_{i(l)}(v(l)) = v(f(l))$.*

*Then there is an isomorphism $\Phi \colon [\mathcal{T}_1] \to [\mathcal{T}_2]$ such that for every $i \in I_1$, the underlying isomorphism $\langle \Phi \rangle$ restricts to $\langle \Phi_i \rangle$ on the sub-graph $\langle \Gamma_i \rangle$ of $\langle [\mathcal{T}_1] \rangle$.*

### 1.4.3 Decomposing pieces as trees of blocks

As a first step towards establishing the existence of a canonical expression of a piece as the realization of a tree of spatial graphs, we give an analogue of Lemma 1.3.9, whose proof is essentially the same:

**Lemma 1.4.18** (If it looks like a vertex sum, it is a vertex sum)**.** *Let $\Gamma$ be a spatial graph in $\mathcal{S}$ and $S \subset \mathcal{S}$ a PL-embedded 2-sphere that intersects $|\Gamma|$ precisely at one vertex $v$ of $\Gamma$. Denote the closures of the two components of $\mathcal{S} \setminus S$ by $B_1$ and $B_2$. For each $i \in \{1, 2\}$, let $\Gamma_i$ be the sub-graph of $\Gamma$ comprised of the vertices and edges that are contained in $B_i$. Then $\Gamma = \Gamma_1 \,{}_v\!\bullet_v \Gamma_2$.*

We draw the reader's attention to the fact that from now on several statements will include a non-separability assumption on spatial graphs. Incidentally, we collect the following observation:

**Lemma 1.4.19** (Vertex sum preserves non-separability)**.** *Let $(\Gamma_1, v_1), (\Gamma_2, v_2)$ be pointed spatial graphs. Then $\Gamma_1 \,{}_{v_1}\!\bullet_{v_2} \Gamma_2$ is non-separable if and only if both $\Gamma_1, \Gamma_2$ are non-separable.*

*Proof.* If one of the vertex summands, say $\Gamma_1$, is separable, denote by $\mathcal{S}_1$ its ambient sphere and let $S \subset \mathcal{S}_1$ be a separating sphere. Choose an enclosing ball $B_1$ for $(\Gamma_1, v_1)$ that contains $S$ in its interior. Then if we use $B_1$ to form the vertex sum, $S$ will be contained in the ambient sphere $\mathcal{S}$ of $\Gamma_1 \,{}_{v_1}\!\bullet_{v_2} \Gamma_2$, with both sides of $S$ intersecting $|\Gamma_1 \,{}_{v_1}\!\bullet_{v_2} \Gamma_2|$. Hence $S$ will be a separating sphere for $\Gamma_1 \,{}_{v_1}\!\bullet_{v_2} \Gamma_2$.

Conversely, suppose $S$ is a separating sphere for $\Gamma_1 \,{}_{v_1}\!\bullet_{v_2} \Gamma_2$. The component of $\mathcal{S} \setminus S$ that does not contain the vertex $v_1 = v_2$ has non-empty intersection with the support of one of the summands, say $|\Gamma_1|$. Then both components of $\mathcal{S} \setminus S$ intersect $|\Gamma_1|$ and so, regarding $\Gamma_1$ as a sub-graph of $\Gamma_1 \,{}_{v_1}\!\bullet_{v_2} \Gamma_2$, we see $S$ is a separating sphere for $\Gamma_1$. $\square$

Since realizations of trees of spatial graphs are constructed by iterated vertex sums, we have, more generally:

**Corollary 1.4.20** (Realizations of trees of spatial graphs preserve non-separability.)**.** *Let $\mathcal{T} = (T, I, J, L, (\Gamma_i)_{i \in I}, (v(l))_{l \in L})$ be a tree of spatial graphs. Then $[\mathcal{T}]$ is non-separable if and only if all the $\Gamma_i$ are non-separable.*

**Definition 1.4.21.** Let $\Gamma$ be a spatial graph in $\mathcal{S}$.

- If $S \subset \mathcal{S}$ is a PL-embedded 2-sphere as in Lemma 1.4.18, we say that "$S$ decomposes $\Gamma$ as $\Gamma_1 \,{}_v\!\bullet_v \Gamma_2$".
- Suppose $\Gamma$ is non-separable. If $S$ is a 2-sphere decomposing $\Gamma$ as $\Gamma_1 \,{}_v\!\bullet_v \Gamma_2$, with $\Gamma_1, \Gamma_2$ both non-isomorphic to $\mathbf{1}$, then $v$ is called a **cut vertex** of $\Gamma$ and $S$ is a **cut sphere** of $\Gamma$.
- $\Gamma$ is called a **block** if it is a piece that has no cut vertices and is not isomorphic to $\mathbf{1}$.
- A tree of spatial graphs $\mathcal{T} = (T, I, J, L, (\Lambda_i)_{i \in I}, (v(l))_{l \in L})$ where each $\Lambda_i$ is a block is called a **tree of blocks**. In that case we also say that $\mathcal{T}$ is a tree of blocks for $[\mathcal{T}]$.

There is also a standard notion of cut vertex for a connected abstract graph $G$ that is similar in spirit: a vertex $v$ in an abstract graph $G$ is **cut** if $G$ is the union of two sub-graphs $G_1, G_2$ intersecting precisely at $v$, with neither $G_i$ comprised only of a single vertex. We should however note that it is possible for a vertex of a spatial graph $\Gamma$ to be cut in $\langle\Gamma\rangle$ but not in $\Gamma$, as exemplified in Figure 1.4.9.

One of the reasons one should care about having a spatial graph expressed as the realization of a tree of blocks is because cut vertices are easily read-off. In order to make this connection precise, we need an analogue of Lemma 1.3.11:

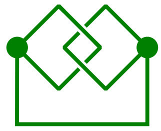


Figure 1.4.9: The notion of cut vertex for spatial graphs does not coincide with the one for abstract graphs. Both vertices of the depicted graph $\Gamma$ are cut in $\langle\Gamma\rangle$, but not in $\Gamma$.

**Lemma 1.4.22** (Spheres sort blocks)**.** *Let $\Lambda$ be a block in $\mathcal{S}$, and let $S \subset \mathcal{S}$ be a PL-embedded 2-sphere that intersects $|\Lambda|$ either at a single vertex of $\Lambda$, or not at all. Denote the closures of the two components of $\mathcal{S} \setminus S$ by $B_1, B_2$. Then $|\Lambda|$ is contained in exactly one of the $B_i$.*

*Proof.* Since $\Lambda$ is a piece, the case where $S \cap |\Lambda| = \emptyset$ follows from Lemma 1.3.11.

If $S \cap |\Lambda|$ is comprised precisely of one vertex $v$ of $\Lambda$, then since $\Lambda \ncong \mathbf{1}$, certainly $|\Lambda|$ cannot be contained in both $B_i$. By Lemma 1.4.18, $S$ decomposes $\Lambda$ as $\Lambda_1 \,{}_v{\bullet}_v\, \Lambda_2$. Since $\Lambda$ has no cut vertices, one of the summands, say $\Gamma_1$, is isomorphic to $\mathbf{1}$. This means $\Lambda = \Lambda_2$. □

**Proposition 1.4.23** (Cut vertices in the realization of a tree of blocks)**.** *Let $\mathcal{T} = (T, I, J, L, (\Lambda_i)_{i\in I}, (v(l))_{l\in L})$ be a tree of blocks. For each $j \in J$, denote by $v(j)$ the vertex of $[\mathcal{T}]$ that results from identifying all $v(l)$ with $l$ incident to $j$. Then the correspondence $j \mapsto v(j)$ is a bijection between $J$ and the set of cut vertices of $[\mathcal{T}]$.*

*Proof.* To see that each $v(j)$ is cut: by definition of a tree of spatial graphs, $j$ has degree at least 2, so if $L_0 \subseteq L$ is the set of edges incident to $j$, one may write some non-trivial partition $L_0 = L_1 \sqcup L_2$. For each $k \in \{1, 2\}$, choose some $l_k \in L_k$. The spatial graph $\Lambda_{i(l_k)}$, being a block, has an edge, and hence also $[\mathcal{T}_{l_k}]$ has an edge. Thus each vertex summand in

$$[\mathcal{T}] = \left(\mathop{\bigstar}_{l\in L_1} ([\mathcal{T}_l], v(l))\right) {}_{v(l_1)}{\bullet}_{v(l_2)} \left(\mathop{\bigstar}_{l\in L_2} ([\mathcal{T}_l], v(l))\right),$$

has an edge and so is not isomorphic to $\mathbf{1}$. Thus $v(j) = v(l_1) = v(l_2)$ is cut.

It is clear from the vertex set of $[\mathcal{T}]$, as given by the description of $\langle[\mathcal{T}]\rangle$, that the assignment $j \mapsto v(j)$ is injective.

Conversely, suppose $v$ is a vertex of $[\mathcal{T}]$ that does not result from such an identification, and consider a PL-embedded 2-sphere $S$ in the ambient sphere $\mathcal{S}$ of $[\mathcal{T}]$ intersecting $|[\mathcal{T}]|$ precisely at $v$. Say $S$ decomposes $[\mathcal{T}]$ as $\Gamma_1 \,{}_v{\bullet}_v\, \Gamma_2$ – we aim to show that one of the $\Gamma_i$ is isomorphic to $\mathbf{1}$. Our assumption on $v$ implies that the edges of $[\mathcal{T}]$ incident to $v$ all come from the same block $\Lambda_i$. Using Lemma 1.4.16 to regard $\Lambda_i$ as a sub-graph of $[\mathcal{T}]$, we see from Lemma 1.4.22 that all edges of $[\mathcal{T}]$ incident to $v$ are in one of the $\Gamma_i$, say in $\Gamma_1$. Hence $v$ is a vertex of $\Gamma_2$ without incident edges. All we need is to show that $\Gamma_2$ is non-separable, and it will follow that $\Gamma_2 \cong \mathbf{1}$. But since $[\mathcal{T}]$ is non-separable by Corollary 1.4.20, non-separability of $\Gamma_2$ follows from Lemma 1.4.19. □

Having at least somewhat motivated the usefulness of trees of blocks, we now establish their existence for non-separable graphs (with the exception of $\mathbf{1}$).

**Proposition 1.4.24** (Existence of trees of blocks)**.** *Every non-separable spatial graph $\Gamma \ncong \mathbf{1}$ is the realization of some tree of blocks.*

*Proof.* We use induction on the number of edges in $\Gamma$ to produce a tree of blocks $\mathcal{T} = (T, I, J, L, (\Lambda_i)_{i\in I}, (v(l))_{l\in L})$ realizing $\Gamma$. If $\Gamma$ has no edges, then since $\Gamma$ is non-separable we either have $\Gamma = \mathbf{0}$ or $\Gamma \cong \mathbf{1}$. The second case is ruled out by assumption, and in the first one we take $T$ to be the empty tree.

Now suppose $\Gamma$ has at least one edge. If $\Gamma$ is a block, then we are done by taking $T$ to be a tree with a single vertex $i_0 \in I$ and $\Lambda_{i_0} := \Gamma$. If $\Gamma$ is not a block, then it can be expressed as $\Gamma_1\, {}_v{\bullet}_v\, \Gamma_2$ with each $\Gamma_i$ not isomorphic to $\mathbf{1}$, and also non-separable by Lemma 1.4.19. Hence each of $\Gamma_1, \Gamma_2$ has at least one edge, and thus both have fewer edges than $\Gamma$ and our induction hypothesis applies to them.

For each $k \in \{1, 2\}$, let $\mathcal{T}_k = (T_k, I_k, J_k, L_k, (\Lambda_i)_{i \in I_k}, (v(l))_{l \in L_k})$ be a tree of blocks for $\Gamma_k$. We will construct $T$ as illustrated in Figure 1.4.10 from modified versions $T'_k$ of the $T_k$, according to the following two cases:

- If $v$ is not a cut vertex of $\Gamma_k$, so by Proposition 1.4.23 there is no edge $l_k \in L_k$ with $v(l_k) = v$, construct a new tree $T'_k$ from $T_k$ by adding a new vertex $j_k$ and a new edge $l_k$ connecting $j_k$ to the vertex $i_k \in I$ such that $\Lambda_{i_k}$ contains $v$. We also write

  $$J'_k := J_k \sqcup \{j_k\}, \quad L'_k := L_k \sqcup \{l_k\}, \quad L^0_k := \{l_k\},$$

  and set $v(l_k) := v$. Note that in this case, $T'_k$ with its vertex set partitioned as $I_k \sqcup J'_k$ is no longer admissible as the tree in a tree of spatial graphs, since $j_k$ is a leaf.

- If $v$ is a cut vertex of $\Gamma_k$, then there is a corresponding $j_k \in J_k$, whose set of incident edges we denote by $L^0_k$. In $\Gamma_k$, the vertices $v(l)$ with $l \in L^0_k$ are identified into the vertex $v$. For convenience, we write $T'_k := T_k, J'_k := J_k, L'_k := L_k$.

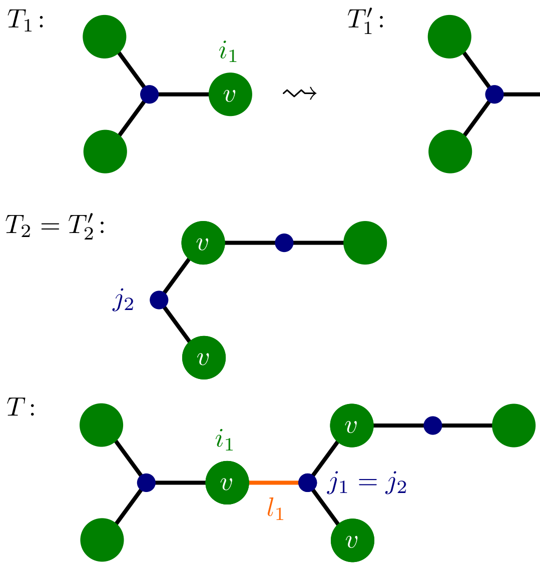



Figure 1.4.10: Constructing $T$ from $T_1$ and $T_2$. In the depicted trees, large vertices represent elements of $I_1, I_2$, and small vertices represent elements of $J_1, J_2$. We also indicate the elements of $I_1, I_2$ whose corresponding blocks contain the vertex $v$. In this example, $v$ is not a cut vertex in $\Gamma_1$, but it is in $\Gamma_2$ (where it corresponds to $j_2$). Accordingly, $T'_1$ is obtained from $T_1$ by adding a vertex $j_1$ and an edge $l_1$, whereas $T'_2 = T_2$. $T$ is then obtained from $T'_1$ and $T'_2$ by identifying $j_1$ with $j_2$.

We define

$$T := T'_1 \, {}_{j_1}\!\bullet_{j_2} \, T'_2, \qquad I := I_1 \sqcup I_2,$$
$$J := (J'_1 \sqcup J'_2)/j_1 \sim j_2, \qquad L := L'_1 \sqcup L'_2,$$

and this turns $\mathcal{T}$ into a tree of spatial graphs whose realization is $\Gamma$:

$$\begin{aligned}
[\mathcal{T}] &= \mathop{\bigstar}_{\substack{l \in L \\ j(l)=j_1=j_2}} ([\mathcal{T}_l], v(l)) \\
&= \left( \mathop{\bigstar}_{l \in L_1^0} ([\mathcal{T}_l], v(l)) \right) {}_v\!\bullet_v \left( \mathop{\bigstar}_{l \in L_2^0} ([\mathcal{T}_l], v(l)) \right) \\
&= [\mathcal{T}_1] \, {}_v\!\bullet_v \, [\mathcal{T}_2] \\
&= \Gamma_1 \, {}_v\!\bullet_v \, \Gamma_2.
\end{aligned}$$

□

We now give a uniqueness statement for the expression of non-separable spatial graphs as the realization of a tree of blocks.

**Proposition 1.4.25** (Uniqueness of decomposition into a tree of blocks)**.** *For each $k \in \{1, 2\}$, let $\mathcal{T}_k = (T_k, I_k, J_k, L_k, (\Lambda_i)_{i \in I_k}, (v(l))_{l \in L_k})$ be a tree of blocks, and let $\Phi \colon [\mathcal{T}_1] \to [\mathcal{T}_2]$ be an isomorphism. Then there is an isomorphism of trees $f \colon T_1 \to T_2$ satisfying $f(I_1) = I_2$, such that:*

- *for each $i \in I_1$, the map $\Phi$ is an isomorphism of sub-graphs $\Lambda_i \to \Lambda_{f(i)}$, and*
- *for each $l \in L_1$, we have $\Phi(v(l)) = v(f(l))$.*

Note that the second item implies that $f$ respects the bijective correspondence given by Proposition 1.4.23 between the $J_k$ and the set of cut vertices of $[\mathcal{T}_k]$. In other words, for each $j \in J_1$ we have $\Phi(v(j)) = v(f(j))$.

*Proof.* We will proceed by induction on the cardinality of $I_1$. If $I_1 = \emptyset$, then $[\mathcal{T}_1] = \mathbf{0} = [\mathcal{T}_2]$, so $T_2$ is the empty tree and there is nothing to show. If $I_1$ is comprised of a single element $i_1$, and hence $J_1 = \emptyset$, then $[\mathcal{T}_1] = \Lambda_{i_1}$ is a block, so $[\mathcal{T}_2]$ is a block. In particular, $[\mathcal{T}_2]$ has no cut vertices and so by Proposition 1.4.23 we conclude $J_2 = \emptyset$. Hence $I_2$ contains exactly one element $i_2$, with $[\mathcal{T}_2] = \Lambda_{i_2}$. We are thus done with this case by setting $f(i_1) := i_2$.

Assume now that $I_1$ contains at least two elements, and so $J_1 \neq \emptyset$. Choose $j_1 \in J_1$, write $v_1 := v(j_1)$, and let $S_1$ be a cut sphere for $[\mathcal{T}_1]$ decomposing it as $[\mathcal{T}_1] = \Gamma_1^+ \, {}_{v_1}\!\bullet_{v_1} \, \Gamma_1^-$, so $\Gamma_1^+, \Gamma_1^-$ are pieces non-isomorphic to $\mathbf{1}$. Similarly, we have that $v_2 := \Phi(v_1)$ is a cut vertex for $[\mathcal{T}_2]$, so let $j_2 \in J_2$ be the corresponding element. The sphere $S_2 := \Phi(S_1)$ is now a cut sphere for $[\mathcal{T}_2]$ decomposing it as $[\mathcal{T}_2] = \Gamma_2^+ \, {}_{v_2}\!\bullet_{v_2} \, \Gamma_2^-$, with the map $\Phi$ giving a pair of isomorphisms of sub-graphs $\Phi^\epsilon \colon \Gamma_1^\epsilon \to \Gamma_2^\epsilon$, for each $\epsilon \in \{+, -\}$.

Let $k \in \{1, 2\}$. Our goal is to extract from each $\mathcal{T}_k$ a description of the spatial graphs $\Gamma_k^+, \Gamma_k^-$ as realizations of trees of blocks, to which we will then apply the induction hypothesis. The procedure is entirely analogous for all four spatial graphs, so let us also fix a sign $\epsilon \in \{+, -\}$.

Denote by $L_k^0 \subseteq L_k$ the set of edges incident to $j_k$, and recall that Lemma 1.4.22 tells us that for each $i \in I_k$, the block $\Lambda_i$ is a sub-graph of exactly one among $\Gamma_k^+, \Gamma_k^-$. We consider the partition $L_k^0 = L_k^{0+} \sqcup L_k^{0-}$, where an edge $l \in L_k^0$ is in $L_k^{0\epsilon}$ if and only if $\Lambda_{i(l)}$ is a sub-graph of $\Gamma_k^\epsilon$.

Consider the decomposition of $[\mathcal{T}_k]$ as

$$[\mathcal{T}_k] = \left( \mathop{\bigstar}_{l \in L_k^{0+}} ([(\mathcal{T}_k)_l], v(l)) \right) {}_{v_k}\!\bullet_{v_k} \left( \mathop{\bigstar}_{l \in L_k^{0-}} ([(\mathcal{T}_k)_l], v(l)) \right).$$

We claim that this is the same as the decomposition given by $S_k$, that is,

$$\Gamma_k^\epsilon = \bigstar_{l \in L_k^{0\epsilon}}([(\mathcal{T}_k)_l], v(l)).$$

To see this, first notice that since all $\Lambda_i$ are non-separable, Corollary 1.4.20 tells us that each $[(\mathcal{T}_k)_l]$ is non-separable. Now, for each $l \in L_k^0$, it follows from Proposition 1.4.23 that $v(l)$ is not a cut vertex of $[(\mathcal{T}_k)_l]$. Therefore, $S_k$ decomposes $[(\mathcal{T}_k)_l]$ as a trivial vertex sum $[(\mathcal{T}_k)_l]\ {}_{v(l)}\bullet\ \mathbf{1}$. In other words, $|[(\mathcal{T}_k)_l]|$ is entirely contained in one side of $S_k$, which must of course be the same as $|\Lambda_{i(l)}|$, since $\Lambda_{i(l)}$ is a sub-graph of $|[(\mathcal{T}_k)_l]|$. From here, we get that each $\bigstar_{l \in L_k^{0\epsilon}}([(\mathcal{T}_k)_l], v(l))$ is a sub-graph of $\Gamma_k^\epsilon$, whence the above description of the $\Gamma_k^\epsilon$ holds.

Next, we write down an explicit tree of blocks $\mathcal{T}_k^\epsilon$ for $\bigstar_{l \in L_k^{0\epsilon}}([(\mathcal{T}_k)_l], v(l))$. If $L_k^{0\epsilon}$ has only one element $l_k^\epsilon$, put $\mathcal{T}_k^\epsilon := (\mathcal{T}_k)_{l_k^\epsilon}$. Otherwise, recover the notation introduced when defining the realization of a tree of spatial graphs

$$(\mathcal{T}_k)_l = ((T_k)_l, (I_k)_l, (J_k)_l, (L_k)_l, (\Lambda_i)_{i \in (I_k)_l}, (v(l'))_{l' \in (L_k)_l}),$$

and set $\mathcal{T}_k^\epsilon := (T_k^\epsilon, I_k^\epsilon, J_k^\epsilon, L_k^\epsilon, (\Lambda_i)_{i \in I_k^\epsilon}, (v(l))_{l \in L_k^\epsilon})$ to be the tree of blocks comprised of the branches of $\mathcal{T}_k$ at $j_k$ that stem from edges in $L_k^{0\epsilon}$. Explicitly, $T_k^\epsilon$ is the sub-tree of $T_k$ with vertex and edge sets given by

$$I_k^\epsilon := \bigsqcup_{l \in L_k^{0\epsilon}} (I_k)_l, \quad J_k^\epsilon := \{j_k\} \sqcup \bigsqcup_{l \in L_k^{0\epsilon}} (J_k)_l, \quad L_k^\epsilon := L_k^{0\epsilon} \sqcup \bigsqcup_{l \in L_k^{0\epsilon}} (J_k)_l.$$

Observe that in the first case $[\mathcal{T}_k^\epsilon]$ does not have $v_k$ as a cut vertex, and in the second case it does, with $j_k$ being the corresponding element of $J_k^\epsilon$.

It is now clear that, in either case, $[\mathcal{T}_k^\epsilon] = \bigstar_{l \in L_k^{0\epsilon}}([(\mathcal{T}_k)_l], v(l)) = \Gamma_i^\epsilon$. By induction hypothesis, the spatial graph isomorphisms $\Phi^\epsilon \colon [\mathcal{T}_1^\epsilon] \to [\mathcal{T}_2^\epsilon]$ yield tree isomorphisms $f^\epsilon \colon T_1^\epsilon \to T_2^\epsilon$, which we now want to assemble to the desired $f \colon T_1 \to T_2$. On each sub-tree $T_1^\epsilon$ of $T_1$, we want to set $f = f^\epsilon$, but we have to ensure that $f^+$ and $f^-$ agree where they overlap, and we must also define $f$ on the vertices and edges of $T_1$ that are not in one of the $T_1^\epsilon$.

Fix $\epsilon \in \{+, -\}$ for this paragraph. The isomorphism $\Phi^\epsilon$ ensures that $v_1$ is a cut vertex of $[\mathcal{T}_1^\epsilon]$ if and only if $v_2$ is a cut vertex of $[\mathcal{T}_2^\epsilon]$. If this is the case, then for both $k \in \{1, 2\}$, the vertex $j_k$ of $T_k$ is in $T_k^\epsilon$, along with the edges in $L_k^{0\epsilon}$. Moreover, in this situation we have in $\mathcal{T}_2^\epsilon$ that $v(j_2) = v_2 = \Phi^\epsilon(v_1) = \Phi^\epsilon(v(j_1)) = v(f^\epsilon(j_1))$, whence by injectivity of $j \mapsto v(j)$ it follows that $j_2 = f^\epsilon(j_1)$. On the other hand, if one (hence both) $v_k$ is not cut in $[\mathcal{T}_k^\epsilon]$, then the corresponding $j_k$ and the unique edge $l_k^\epsilon$ in $L_k^{0\epsilon}$ are not in $T_k^\epsilon$. In this situation, $i(l_k^\epsilon)$ is the only element of $I_k^\epsilon$ whose corresponding block $\Lambda_{i(l_k^\epsilon)}$ contains $v_k$ as a vertex.

We now consider the following three cases:

- If for one (hence both) $k \in \{1, 2\}$ the vertex $v_k$ is cut in $[\mathcal{T}_k^+]$ and $[\mathcal{T}_k^-]$, then the two sub-trees $T_k^+, T_k^-$ jointly cover all of $T_k$, and they overlap precisely at the vertex $j_k$. As we have seen that $f^+(j_1) = j_2 = f^-(j_1)$, we are allowed to glue together the $f^\epsilon$ into the desired $f \colon T_1 \to T_2$.

- Suppose for both $k \in \{1, 2\}$, the vertex $v_k$ is cut in $[\mathcal{T}_k^+]$ but not in $[\mathcal{T}_k^-]$ (the reverse situation being analogous). Then $T_k^+$ and $T_k^-$ do not overlap, and they jointly they cover all of $T_k$ except for the edge $l_k^-$ described above. In this case, we extend the definition of $f^+, f^-$ to all of $T_1$ by setting $f(l_1^-) := l_2^-$. This respects the endpoints of the edge: we have seen that $f^-(j_1) = j_2$, and the characterization of $i(l_k^\epsilon)$ given above, together with the fact that $\Phi^-(v_1) = v_2$, shows that $f^-(i(l_1^-)) = i(l_2^-)$.

- If for both $k \in \{1, 2\}$ the vertex $v_k$ is not cut in $[\mathcal{T}_k^+]$ nor in $[\mathcal{T}_k^-]$, then the trees $T_k^+, T_k^-$ are disjoint and cover all of $T_k$ except for the vertex $j_k$ and its only two incident edges $j_k^+, j_k^-$. We extend $f^+, f^-$ by putting $f(j_1) := j_2$ and, for each $\epsilon \in \{+, -\}$, setting $f(l_1^\epsilon) := l_2^\epsilon$. This clearly respects the incidence of each $l_1^\epsilon$ at the endpoint $j_1$, and for the other endpoint we argue exactly as in the previous item.

Having defined the isomorphism $f : T_1 \to T_2$, almost all stated properties are directly inherited from the $f^\epsilon$. We are only left to check that, in the second and third cases above, the definition of $f$ on the new edge(s) $l_1^\epsilon$ satisfies $\Phi(v(l_1^\epsilon)) = v(f(l_1^\epsilon))$. And indeed it does: $\Phi(v(l_1^\epsilon)) = \Phi(v_1) = v_2 = v(l_2^\epsilon) = v(f(l_1^\epsilon))$. □

For the purpose of proving Theorem 1.1.1, this proposition is meant to work in tandem with the result about uniqueness of decomposition into pieces (Proposition 1.3.12): That proposition reduces the isomorphism problem for spatial graphs to decomposing them as disjoint unions of pieces, and comparing the pieces. Now given the task of testing whether two pieces are isomorphic, we further decompose each of them as the realization of a tree of blocks. Then Proposition 1.4.25 guarantees that two pieces are isomorphic if and only if the blocks in the decompositions are pairwise isomorphic, via isomorphisms respecting the overall combinatorial structure of the tree. Though this might seem like little gain, comparing the isomorphism type of blocks is then within reach using Matveev's Theorem.

## 1.5 Extension to decorated spatial graphs

The theory developed so far can be easily generalized to spatial graphs equipped with additional structure. Three natural extensions are directed spatial graphs, and spatial graphs with colorings of the edges and/or vertices. In this section, we formalize these concepts and comment on how the operations and decomposition results given so far are adapted to these other settings. All main proofs will however carry over with no need for additional insight, so we will omit them.

**Definition 1.5.1.** A **directed** spatial graph is spatial graph $\Gamma$ together with a choice of orientation of each edge. If $e$ is a non-loop edge of $\Gamma$ and $h\colon [-1, 1] \to e$ is a PL homeomorphism orienting $e$, we say the vertex $h(-1)$ is the **source** of $e$, and $h(1)$ is its **target**. When $e$ is a loop, the only vertex of $\Gamma$ contained in $e$ is simultaneously the source and the target. We denote the source and target of an edge $e$ by $s(e)$ and $t(e)$, respectively.

An isomorphism of directed spatial graphs is an isomorphism of the underlying spatial graphs such that the induced PL homeomorphisms between the edges are all orientation-preserving.

**Definition 1.5.2.** Let $\Gamma = (\mathcal{S}, V, E)$ be a spatial graph. A **vertex coloring** of $\Gamma$ is a function $f\colon V \to \mathbb{N}$ from the vertex set to the non-negative integers. For each vertex $v \in V$, we refer to $f(v)$ as "the color of $v$". Given two spatial graphs $\Gamma_1 = (\mathcal{S}_1, V_1, E_1), \Gamma_2 = (\mathcal{S}_2, V_2, E_2)$ with vertex colorings $f_1, f_2$, an isomorphism $\Phi\colon \Gamma_1 \to \Gamma_2$ is said to preserve the vertex coloring if the induced bijection of vertices $\Phi|_{V_1} : V_1 \to V_2$ satisfies $f_1 = f_2 \circ \Phi|_{V_1}$.

In an entirely similar fashion, we define an **edge coloring** $g\colon E \to \mathbb{N}$, and what it means for an isomorphism of spatial graphs to preserve the edge coloring.

One may consider spatial graphs with any (possibly empty) combination of these three types of structure, and we will broadly refer to such spatial graphs as **decorated**.

By two spatial graphs carrying a decoration "of the same type", we mean that the combination of additional structures is the same.

Most of the notions we have discussed until now transfer without difficulty to the setting of decorated graphs. For example, an isomorphism of spatial graphs with decorations of the same type is an isomorphism of the corresponding undecorated spatial graphs that respects all additional pieces of structure. Sub-graphs of decorated spatial graphs inherit a decoration of the same type in an obvious way, and also the underlying abstract graph of a decorated spatial graph inherits a decoration:

- If $\Gamma$ is a directed spatial graph, then the source and target functions on the edge set make $\langle\Gamma\rangle$ a directed abstract graph.
- Vertex and edge colorings of abstract graphs are defined in exactly the same way as for spatial graphs, and if $\Gamma$ is decorated with a vertex and/or edge coloring, then so is $\langle\Gamma\rangle$, in an obvious manner.

The induced decoration on $\langle\Gamma\rangle$ determines the decoration of $\Gamma$, except for one ambiguity: if $\Gamma$ is directed, the orientation of a loop $e$ cannot be inferred from $s(e)$ and $t(e)$. We record the following straightforward consequence:

**Lemma 1.5.3** (Probing compatibility with decorations through underlying graphs)**.** *Let $\Gamma_1, \Gamma_2$ be spatial graphs with decorations of the same type, and let $\Phi\colon \Gamma_1^- \to \Gamma_2^-$ be an isomorphism between the corresponding undecorated spatial graphs. Assume moreover that the $\Gamma_i$ have no loops, or are not directed. Then $\Phi$ respects the decorations on the $\Gamma_i$ if and only if $\langle\Phi\rangle\colon \langle\Gamma_1^-\rangle \to \langle\Gamma_2^-\rangle$ respects the decorations on the $\langle\Gamma_i\rangle$.*

We now summarize the adaptations of the main definitions and statements regarding the operations of disjoint union and vertex sum:

- The disjoint union of spatial graphs with decorations of the same type is defined as the disjoint union of the underlying spatial graphs, and it carries a decoration of the same type in the obvious way.
- The vertex sum of two pointed spatial graphs with decorations of the same type is similarly defined provided that, in case a vertex coloring is part of the decoration, the basepoints are of the same color.
- Isomorphisms between decorated spatial graphs can be assembled along disjoint unions and vertex sums, in the sense of Lemmas 1.3.7 and 1.4.12.
- In the setting of decorated spatial graphs, there is still a well-defined identity element $\mathbf{0}$ for the disjoint union, but if a vertex coloring is part of the decoration, there is one isomorphism type $\mathbf{1}_c$ of one-point spatial graph for each color $c \in \mathbb{N}$.
- The properties of disjoint union and vertex sum listed in Propositions 1.3.8 and 1.4.13 hold for spatial graphs with decorations of the same type. If a vertex coloring is part of the decoration, the occurrence of $\mathbf{1}$ in the first item of Proposition 1.4.13 should be read as $\mathbf{1}_c$, where $c$ is the color of $v_1$.
- The definitions of separating sphere, separable spatial graph, and piece remain unchanged in the decorated setting (Definition 1.3.10). Every decorated spatial graph can be expressed as an iterated disjoint union of (decorated) pieces, in a way that is unique in the sense of Proposition 1.3.12.
- In the definition of a tree of spatial graphs (Definition 1.4.14), all $\Gamma_i$ should have a decoration of the same type. Moreover, if a vertex coloring is part of the decoration,

we require that for each $j \in J$, the vertices $v(l)$ with $l$ adjacent to $j$ all be of the same color. Realizations of trees of decorated spatial graphs are then well-defined in the sense of Lemma 1.4.15, carrying a canonical decoration of the same type.

- In the definitions of cut vertex, cut sphere and block (Definition 1.4.21), if we are in the colored vertex setting, occurrences of the expression "not isomorphic to $\mathbf{1}$" should be read as "not isomorphic to any $\mathbf{1}_c$". The definition of a tree of blocks remains unchanged.
- Propositions 1.4.24 and 1.4.25 apply to decorated graphs: every non-separable decorated spatial graph that is not isomorphic to a one-point graph is the realization of a tree of decorated blocks in a unique way.

We finish by pointing out that vertex colorings are used in an essential way for proving our main algorithmic recognition result for pieces [Fri+21, Proposition 7.11], even when the pieces do not come with vertex colorings. More concretely, vertex colorings are used in encoding the requirement that the isomorphisms between blocks respect the combinatorial structure of the trees of blocks (the second item of Proposition 1.4.25).

# Chapter 2

# The complex of hypersurfaces in a homology class

This chapter is an exposition of the main results in a joint project with Gerrit Herrmann [HQ20]. There, we establish connectedness and simple connectedness of a simplicial complex associated to a oriented compact smooth manifold and a codimension-1 homology class. The definition of this complex is akin to classical complexes of hypersurfaces, such as the curve complex of a surface, or the Kakimuzu complex of a knot, but we do not take hypersurfaces *up to isotopy*, and we do not place restrictions on the dimension of the manifold. In the article, we also apply our results to re-prove a classical result in knot theory, and to define an invariant for 2-dimensional homology classes in 3-manifolds, but these applications are not presented here; the interested reader is pointed to Sections 6 and 7 of the preprint. This chapter is a revised version of Sections 1 to 5 of the same preprint.

## 2.1 Introduction: Simplicial complexes of hypersurfaces

### 2.1.1 Connecting homologous hypersurfaces

Given a smooth manifold $M$, by a **submanifold** of $M$ we mean a smooth manifold $S$ contained in $M$, whose inclusion $S \hookrightarrow M$ is a smooth embedding transverse to the boundary $\partial M$. We will say that $S$ is **properly embedded** if $S \cap \partial M = \partial S$. When its codimension $\dim M - \dim S$ equals 1, we call $S$ a **hypersurface**.

For an oriented compact smooth manifold $M$ of dimension $n$ and a codimension-1 (integral) homology class $\phi \in \mathrm{H}_{n-1}(M, \partial M)$, it is well-known that there is a properly embedded oriented compact hypersurface $S \subset M$ **representing** $\phi$, that is, for which $\phi$ is the image of the fundamental class $[S]$ under the inclusion-induced map $\mathrm{H}_{n-1}(S, \partial S) \to \mathrm{H}_{n-1}(M, \partial M)$. (One can find such an $S$ as the pre-image of a regular value of a smooth map $M \to \mathbb{S}^1$ classifying the Poincaré dual of $\phi$.) In this chapter, we relate the various such embedded hypersurfaces, the first main theorem being the following:

**Theorem 2.1.1** (Sequentially disjoint hypersurfaces)**.** *Let $M$ be an oriented compact smooth manifold of dimension $n$, let $\phi \in \mathrm{H}_{n-1}(M, \partial M)$ be a codimension-1 homology class, and let $S, S'$ be properly embedded oriented hypersurfaces in $M$ representing $\phi$. Then there is a sequence $S = S_0, S_1, \ldots, S_m = S'$ of properly embedded oriented hypersurfaces in $M$, all representing $\phi$, such that each two consecutive $S_i$ are disjoint.*

Our proof produces the intermediate hypersurfaces $S_i$ rather explicitly. Namely, it will be clear that they can all be chosen to lie in an arbitrarily small neighborhood of the union $S \cup S'$, and if $S, S'$ have disjoint boundaries, then for every $S_i$, each connected component of $\partial S_i$ is isotopic in $\partial M$ to a component of $\partial S$ or $\partial S'$.

Theorem 2.1.1 will be re-stated and proved as Theorem 2.3.3. It is rephrased in the language of a simplicial complex $\mathcal{S}^\dagger(M, \phi)$, whose vertices are the hypersurfaces representing $\phi$, and whose $k$-simplices are sets of $k + 1$ pairwise-disjoint hypersurfaces (Definition 2.3.2). With that terminology in place, Theorem 2.1.1 is the statement that $\mathcal{S}^\dagger(M, \phi)$ is connected.

### 2.1.2 Decomposing oriented surgeries

The overall idea of the proof of Theorem 2.1.1 is to first replace $S$ and $S'$ with hypersurfaces intersecting transversely, and then to perform a surgery procedure that yields a hypersurface $\Sigma$ representing the class $2\phi$. The surface $\Sigma$ is then the disjoint union of two hypersurfaces $T_0, T_1$, each representing $\phi$, and we can observe that at least one of the $T_i$ must have strictly fewer intersections with both $S$ and $S'$ than $S$ and $S'$ have with one another. The theorem then follows by an inductive argument.

In fact, our proof provides a finer control on the number of components in the intersection of the $T_i$ with $S$ and $S'$. This yields a linear upper bound on the distance between two transverse hypersurfaces $S, S'$ in $\mathcal{S}^\dagger(M, \phi)$, in terms of the number of components in $S \cap S'$ (Proposition 2.3.4).

These ideas are further developed into a more technically involved argument showing the second main result of this chapter, which is re-stated and proved in the text as Theorem 2.5.1:

**Theorem 2.1.2** (Simple connectedness of $\mathcal{S}^\dagger(M, \phi)$)**.** *Let $M$ be an oriented compact smooth manifold of dimension $n$ and let $\phi \in \mathrm{H}_{n-1}(M, \partial M)$ be a codimension-1 homology class. Then $\mathcal{S}^\dagger(M, \phi)$ is simply connected.*

We emphasize that both Theorems 2.1.1 and 2.1.2 are provided with dimension-agnostic proofs. We are not aware of similar earlier results in dimensions above 3.

### 2.1.3 Other complexes of hypersurfaces

Similar complexes of hypersurfaces have been studied before, but traditionally their vertices are embedded submanifolds *up to isotopy*. A classical example is the curve complex $\mathcal{C}(M)$ of a surface $M$. The vertices of $\mathcal{C}(M)$ are the isotopy classes of (unoriented) simple closed curves in $M$, and a finite set of isotopy classes is a simplex whenever those isotopy classes can be represented by pairwise-disjoint curves. The curve complex was introduced by Harvey [Har81] and has proved a fruitful tool in the study of surface mapping class groups; see for example the exposition in the book of Farb and Margalit [FM12, Section 4.1]. By fixing a primitive homology class $x \in \mathrm{H}_1(S)$ and taking the subcomplex $\mathcal{C}_x(M)$ of $\mathcal{C}(M)$ spanned by the isotopy classes of curves that can be oriented to represent $x$, Hatcher and Margalit have given a new proof of the fact that the Torelli group of $M$ is generated by certain easy to understand mapping classes [HM12, Theorem 1]. A key step is showing that if $M$ has genus at least 3, the complex $\mathcal{C}_x(M)$ is connected [HM12, Theorem 2]. Irmer established various geometric properties of a variant of this complex where the vertices are homotopy classes of multicurves in a fixed homology class [Irm12].

Moving to dimension 3, Kakimizu has studied complexes of isotopy classes of Seifert surfaces in a link exterior that are incompressible, or that have minimal genus. For non-split links, Kakimizu proved that such complexes are connected [Kak92, Theorem A].

Przytycki and Schultens have adapted his complex of minimal Seifert surfaces in a knot exterior to a wider class of 3-manifolds and proved contractibility of the complex in that setting [PS12, Theorem 1.1].

More recently, Bowden, Hensel and Webb considered a variant $\mathcal{C}^\dagger(M)$ of the curve complex where the vertices are actual curves, rather than isotopy classes. This allowed them to show that the space of unbounded quasi-morphisms on $\mathrm{Diff}_0(M)$ (for an orientable closed surface $M$ of positive genus) is infinite-dimensional [BHW20, Theorem 1.2]. We follow their usage of the "dagger" superscript in the notation for our complex $\mathcal{S}^\dagger(M,\phi)$, to emphasize that hypersurfaces are not taken up to isotopy.

### 2.1.4 Additional results

When $M$ is assumed to have dimension 3 and to be irreducible and boundary-irreducible (Definition 2.4.1), it is particularly interesting to focus on surfaces that are "efficient representatives" of $\phi$. Concretely, they are Thurston norm-realizing (Definition 2.4.3). With that in mind, we will adjust the proof of Theorem 2.1.1 to the subcomplex $\mathcal{T}^\dagger(M,\phi)$ of $\mathcal{S}^\dagger(M,\phi)$ spanned by the vertices that realize the Thurston norm and have no homologically trivial parts (Definition 2.4.5):

**Theorem 2.1.3** ($\mathcal{T}^\dagger(M,\phi)$ is connected)**.** *Let $M$ be an irreducible and boundary-irreducible oriented compact smooth 3-manifold, and let $\phi \in \mathrm{H}_2(M,\partial M)$. Then the complex $\mathcal{T}^\dagger(M,\phi)$ is connected.*

This result is re-stated and proved as Theorem 2.4.6.

We will also consider the simplicial complexes $\mathcal{S}(M,\phi)$ and $\mathcal{T}(M,\phi)$ defined similarly to their "dagger" counterparts, but with all hypersurfaces taken up to smooth proper isotopy, and with a finite set of isotopy classes spanning a simplex if they can all be simultaneously disjointly realized (Definitions 2.6.1 and 2.6.3).

**Theorem 2.1.4** (Dropping the daggers)**.** *Let $M$ be an oriented compact smooth manifold of dimension $n$, let $\phi \in \mathrm{H}_{n-1}(M,\partial M)$ be a codimension-1 homology class. Then:*

- *the complex $\mathcal{S}(M,\phi)$ is connected (Corollary 2.6.2),*
- *if $M$ is a reducible and boundary-irreducible 3-manifold, then $\mathcal{T}(M,\phi)$ is connected (Corollary 2.6.4),*
- *if $n = 2$, then $\mathcal{S}(M,\phi)$ is simply connected (Corollary 2.6.5).*

The first two items will follow directly from previous results, with upper bounds on distance also inherited. The third one, however, requires an additional input available only in dimension 2 – the Bigon Criterion (Theorem 2.6.8). It is unclear to us whether it is possible to circumvent it and show simple connectedness of $\mathcal{S}(M,\phi)$ in all dimensions. A sufficient condition would be an affirmative answer to Question 2.6.10 below.

Applications of some of these theorems can be found in the preprint [HQ20]. In Section 6 thereof, we use Theorem 2.1.1 to give an alternative proof of the classical theorem that all Seifert surfaces for a knot in a rational homology 3-sphere are tube-equivalent. In Section 7, we explain how Theorem 2.1.3 has been used to construct an $\ell^2$-invariant for 2-dimensional homology classes in 3-manifolds (satisfying standard hypotheses). See also Herrmann's doctoral dissertation [Her19, Section 4.2] for a more detailed exposition of this application.

### 2.1.5 Outline of this chapter

In Section 2.2, we set up useful terminology for discussing collections of embedded submanifolds, and prove a lemma regarding general position.

Section 2.3 introduces the main character of this chapter, the simplicial complex $\mathcal{S}^\dagger(M,\phi)$ of a manifold $M$ and a codimension-1 homology class $\phi$ of $M$. There, we prove the first main result, Theorem 2.3.3. The core of the argument is Proposition 2.3.4, which gives a distance bound between vertices of $\mathcal{S}^\dagger(M,\phi)$ corresponding to transverse hypersurfaces. In Section 2.4, we adjust these arguments to prove connectedness of the complex $\mathcal{T}^\dagger(M,\phi)$ of Thurston norm-realizing surfaces in a reducible and boundary-irreducible oriented compact smooth 3-manifold $M$ (Theorem 2.4.6).

The second main result, that $\mathcal{S}^\dagger(M,\phi)$ is simply connected, is presented in Section 2.5, as Theorem 2.5.1. The proof is similar in spirit to that of connectedness, but more technically involved.

In Section 2.6, we consider the complexes $\mathcal{S}(M,\phi)$ and $\mathcal{T}(M,\phi)$ where all hypersurfaces are taken up to smooth proper isotopy. We explain how connectedness of $\mathcal{S}^\dagger(M,\phi)$ and $\mathcal{T}^\dagger(M,\phi)$ implies connectedness of these smaller complexes, with similar upper bounds on distance (Corollaries 2.6.2 and 2.6.4). We also transfer the simple connectedness result for $\mathcal{S}^\dagger(M,\phi)$ to $\mathcal{S}(M,\phi)$, in the case where $M$ has dimension 2 (Corollary 2.6.5).

In Section 2.7, we briefly comment on a generalization of the results in the 2-dimensional case to the setting of graphs embedded in 2-dimensional CW-complexes (under certain regularity assumptions). This connects to work of Turaev [Tur02], who has used such graphs to represent 1-dimensional cohomology classes, and defined an analogue of the Thurston norm.

## 2.2 A note on general position

The proofs of all main results in this chapter involve performing geometric constructions on families of submanifolds of a fixed smooth ambient manifold $M$. These procedures can only be carried out if the submanifolds involved satisfy a "general position" assumption, which we explain in the current section. The reader who is uninterested in the technical details involved in perturbing manifolds into general position is invited to read only until the statement of Proposition 2.2.2, and then skip to the next section.

Wall's book [Wal16, Section 1.5] is the main reference for all definitions in differential topology that are not explicitly stated here. We deviate from his terminology only in the use of the word "**proper**": for us, it always refers to the condition $S \cap \partial M = \partial S$ on a submanifold $S \subset M$, or, more generally, to the condition $f^{-1}(\partial M) = \partial N$ on a smooth map of manifolds $f\colon N \to M$. Since we will deal exclusively with compact spaces, the alternative notion of properness [Wal16, Section A.2] will not be needed.

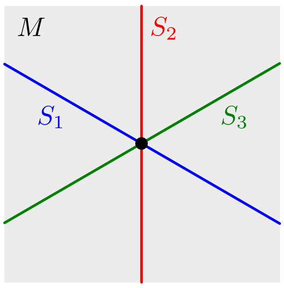
We now introduce a notion of transversality for finite sets of properly embedded submanifolds in an ambient manifold.

**Definition 2.2.1.** Let $M$ be a smooth manifold. A finite set $\mathcal{U} = \{S_1, \dots, S_k\}$ of proper submanifolds of $M$ is **transverse** if for every subset $I \subset \{1,\dots,k\}$, the intersection $\bigcap_{i\in I} S_i$ is a submanifold of $M$, and for every pair of disjoint subsets $I, J \subseteq \{1,\dots,k\}$, the ubmanifolds $\bigcap_{i\in I} S_i$ and $\bigcap_{j\in J} S_j$ are transverse.

Note that since submanifolds are by definition transverse to the boundary, Definition 2.2.1 requires in particular that all intersections $\bigcap_{i\in I} S_i$ be transverse to $\partial M$. It then follows also that $\bigcap_{i\in I} S_i$ is properly embedded: indeed, one can see in general that for any submanifold $T$ of $M$, the fact that $T$ is transverse to $\partial M$ implies that $T \cap \partial M \subseteq \partial T$, so in particular $\left(\bigcap_{i\in I} S_i\right) \cap \partial M \subseteq \partial\left(\bigcap_{i\in I} S_i\right)$. Conversely, each point

$p \in \partial\big(\bigcap_{i\in I} S_i\big)$ is in the boundary of some $S_i$, and hence, since $S_i$ is properly embedded, we have $p \in \partial M$, and so $p \in \big(\bigcap_{i\in I} S_i\big) \cap \partial M$.

Figure 2.2.1 exemplifies two phenomena that Definition 2.2.1 excludes.



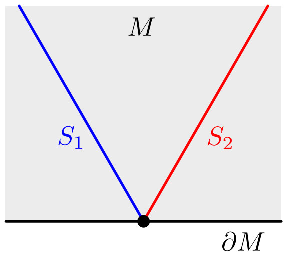




Figure 2.2.1: Submanifolds of a surface $M$ that do not form a transverse set. Left: the curves $S_1, S_2, S_3$ meet pairwise-transversely at a point, but the intersection $S_1 \cap S_2$ is not transverse to $S_3$. Right: for two curves $S_1, S_2$ meeting transversely at $\partial M$, the intersection $S_1 \cap S_2$ is not transverse to $\partial M$, so it is not a submanifold of $M$.

The goal of the current section is to establish the following statement, which justifies thinking of transverse sets as being in general position.

**Proposition 2.2.2** (Transverse approximation)**.** *Suppose $\mathcal{U}$ is a transverse set of properly embedded submanifolds of a compact smooth manifold $M$, and let $f\colon T \hookrightarrow M$ be a proper embedding of a compact manifold $T$. Then $f$ can be perturbed by an arbitrarily small proper isotopy to a proper embedding $g\colon T \hookrightarrow M$, such that for the modified manifold $T' := g(T)$, the set $\mathcal{U} \cup \{T'\}$ is transverse.*

Here the phrase "arbitrarily small isotopy" warrants some explanation. Given two smooth manifolds $T, M$, the set $\mathrm{C}^\infty(T, M)$ of smooth maps $T \to M$ is typically endowed with either the $C^\infty$ topology or the $W^\infty$ topology, which are the same if $T$ is compact [Wal16, Appendix A.4]; we will thus no longer care to distinguish them. If we consider the subspace $\mathrm{C}^\infty_\partial(T, M)$ of proper maps, we can make the statement of Proposition 2.2.2 precise by expressing it in terms of this topology. This translation relies on the following result.

**Proposition 2.2.3** (Stability of proper embeddings)**.** *If $T, M$ are smooth manifolds, with $T$ compact, and $f\colon T \hookrightarrow M$ is a proper embedding, then there is a neighborhood $U$ of $f$ in $\mathrm{C}^\infty_\partial(T, M)$ such that every $g \in U$ is a proper embedding that is properly isotopic to $f$.*

*Proof sketch.* If we do not insist that the isotopy connecting $f$ and $g$ be proper, then this statement is proved in Wall's book [Wal16, Proposition 4.4.4]. But the stronger result actually follows from the same argument, with almost no modification. Indeed, that proof uses a map $H\colon W \times [0,1] \to M$, where $W$ is an appropriate neighborhood of the diagonal in $M \times M$. This map $H$ is constructed by putting a Riemannian metric on $M$ and using the existence of unique geodesics between pairs of points that are close enough.

But if one starts with a Riemannian metric for which $\partial M$ is totally geodesic (which we can do [Wal16, Proposition 2.3.7 (i)]), then geodesics connecting boundary points are contained in the boundary, and this fact translates into properness of the isotopy that is ultimately produced between $f$ and $g$. $\square$

One can therefore refine Proposition 2.2.2 as follows.

**Proposition 2.2.4** (Denseness of transverse proper embeddings). *Let $\mathcal{U}$ be a transverse set of properly embedded submanifolds of a compact smooth manifold $M$, and let $T$ be a compact smooth manifold. Then the set of proper embeddings $f\colon T \hookrightarrow M$ making $\mathcal{U} \cup \{f(T)\}$ transverse is dense in the (open) subset of $\mathrm{C}^\infty_\partial(T, M)$ consisting of proper embeddings.*

The main tool one uses in order to establish statements of this type is Thom's Transversality Theorem [Wal16, Theorem 4.5.6]. We will not need its full power, only the following corollary.

**Theorem 2.2.5** (Elementary Transversality Theorem). *Let $T, M$ be smooth manifolds, with $T$ compact, and let $S$ be a closed submanifold of $M$. Then the set of maps $f\colon T \to M$ transverse to $S$ is open and dense in $\mathrm{C}^\infty(T, M)$.*

*Suppose further that $f_0\colon T \to M$ is a smooth map such that the restriction $f_0|_{\partial T}$ is transverse to $S$, and consider the subspace $\mathrm{C}^\infty(T, M; f_0, \partial T) \subseteq \mathrm{C}^\infty(T, M)$ of maps whose restriction to $\partial T$ agrees with $f_0$. Then the set of maps $f \in \mathrm{C}^\infty(T, M; f_0, \partial T)$ transverse to $S$ is open and dense in $\mathrm{C}^\infty(T, M; f_0, \partial T)$.*

The proof of the first part of the Elementary Transversality Theorem can be found in the book by Golubitsky and Guillemin [GG73, Corollary 4.12], and the second statement follows from a stronger version of Thom's Transversality Theorem [Wal16, Proposition 4.5.7], using the same argument.

Before proving Proposition 2.2.4, we state and prove two lemmas, the first of which is a mere linear-algebraic observation.

**Lemma 2.2.6** (3-fold transversality). *Let $V$ be a finite-dimensional vector space (over any field), and let $T, S, R$ be pairwise transverse subspaces of $V$, that is, $T+S = T+R = S+R = V$. Then the following conditions are equivalent:*

- $T + (S \cap R) = V$,
- $S + (T \cap R) = V$,
- $R + (T \cap S) = V$.

*Proof.* A straightforward dimension count shows that each condition is equivalent to

$$\dim T + \dim S + \dim R - \dim(T \cap S \cap R) = 2 \dim V. \qquad \square$$

**Lemma 2.2.7** (Transversality criterion). *Let $\mathcal{U} := \{S_1, \ldots, S_k\}$ be a transverse set of properly embedded submanifolds of a compact smooth manifold $M$, and let $T$ be a properly embedded submanifold of $M$ such that for every non-empty subset $I \subseteq \{1, \ldots, k\}$ the following conditions hold:*

- *$T$ is transverse to $\bigcap_{i\in I} S_i$, and*
- *$\partial T$ is transverse to $\bigcap_{i\in I} \partial S_i$ in $\partial M$.*

*Then $\mathcal{U} \cup \{T\}$ is a transverse set.*

*Proof.* Two conditions need to be verified, for all disjoint subsets $I, J \subseteq \{1, \ldots, k\}$:

1. the intersection $T \cap \bigcap_{i\in I} S_i$ is a submanifold of $M$,
2. the submanifold $T \cap \bigcap_{i\in I} S_i$ is transverse to $\bigcap_{j\in J} S_j$.

For proving (1), the fact that $T$ is transverse to $\bigcap_{i\in I} S_i$ tells us that $T \cap \bigcap_{i\in I} S_i$ is a manifold embedded in $T$ [Wal16, Lemma 4.5.1], and hence in $M$. We are thus left to show that $T \cap \bigcap_{i\in I} S_i$ is transverse to $\partial M$.

Since $T$ is transverse to $\partial M$, the tangent space $\mathrm{T}_p(T)$ at each boundary point $p \in \partial T$ has a 1-dimensional subspace $R$ such that

$$\mathrm{T}_p(T) = \mathrm{T}_p(\partial T) \oplus R, \qquad \mathrm{T}_p(M) = \mathrm{T}_p(\partial M) \oplus R.$$

Assuming now that $p$ is in $\partial M \cap T \cap \bigcap_{i\in I} S_i$, we obtain

$$\begin{aligned}
\mathrm{T}_p(M) &= \mathrm{T}_p(\partial M) + R && \text{(second equality)}\\
&= \mathrm{T}_p\Big(\bigcap_{i\in I} \partial S_i\Big) + \mathrm{T}_p(\partial T) + R && \textstyle(\partial T \text{ transverse to } \bigcap_{i\in I} \partial S_i \text{ in } \partial M)\\
&= \mathrm{T}_p\Big(\bigcap_{i\in I} \partial S_i\Big) + \mathrm{T}_p(T) && \text{(first equality)}\\
&= \Big(\mathrm{T}_p\Big(\bigcap_{i\in I} S_i\Big) \cap \mathrm{T}_p(\partial M)\Big) + \mathrm{T}_p(T) && \text{(all } S_i \text{ properly embedded)}\\
&= \Big(\mathrm{T}_p\Big(\bigcap_{i\in I} S_i\Big) \cap \mathrm{T}_p(T)\Big) + \mathrm{T}_p(\partial M) && \textstyle(T \text{ transverse to } \bigcap_{i\in I} S_i\text{, Lemma 2.2.6})\\
&= \Big(\mathrm{T}_p\Big(T \cap \bigcap_{i\in I} S_i\Big)\Big) + \mathrm{T}_p(\partial M).
\end{aligned}$$

Therefore, $T \cap \bigcap_{i\in I} S_i$ is transverse to $\partial M$.

Condition (2) follows from a straightforward application of Lemma 2.2.6 to the tangent spaces of $T$, $\bigcap_{i\in I} S_i$ and $\bigcap_{j\in J} S_j$, at points where all these submanifolds meet. □

Finally, we tackle the main result of this section.

*Proof of Proposition 2.2.4.* We will show that every proper submanifold $T \subseteq M$ (or, to be more precise, its inclusion $\iota\colon T \hookrightarrow M$) can be approximated arbitrarily well by a proper embedding $f\colon T \hookrightarrow M$ for which $f(T)$ satisfies the conditions in Lemma 2.2.7.

Applying the first part of the Elementary Transversality Theorem, we see that for each $I \subseteq \{1, \ldots, k\}$, the set of embeddings $\partial T \hookrightarrow \partial M$ transverse to $\partial\big(\bigcap_{i\in I} S_i\big)$ in $\partial M$ is open and dense in $\mathrm{C}^\infty(\partial T, \partial M)$. Hence, so is the set of embeddings simultaneously satisfying this transversality condition for all (finitely many) subsets $I$. We can thus approximate the restriction $\iota|_{\partial T}$ arbitrarily well by a map $f_\partial\colon \partial T \hookrightarrow \partial M$ transverse to all $\partial\big(\bigcap_{i\in I} S_i\big)$. By Proposition 2.2.3, we may take $f_\partial$ to be an embedding.

One can now use a small isotopy from $\iota|_{\partial T}$ to $f_\partial$ in order to approximate $\iota$ by a proper embedding $f_0\colon T \to M$ that differs from $\iota$ by a small proper isotopy supported in a collar neighborhood of $\partial M$, and such that $f_0|_{\partial T} = f_\partial$ and $f_0$ is transverse to $\partial M$. This construction relies on the existence of tubular neighborhoods for submanifolds with boundary [Wal16, Theorem 2.3.8].

Finally, we note that for each $I \subseteq \{1, \ldots, k\}$, the fact that $f_\partial$ is transverse to $\partial\big(\bigcap_{i\in I} S_i\big)$ in $\partial M$ implies that $f_\partial$ is transverse to $\bigcap_{i\in I} S_i$ in $M$, and so we can apply the second part of the Elementary Transversality Theorem to conclude that the set of maps in $\mathrm{C}^\infty(T, M; f_0, \partial T)$ that are transverse to $\bigcap_{i\in I} S_i$ is open and dense. Thus, as before, the set of maps satisfying this transversality condition for all subsets $I$ is also dense, and so we can approximate $f_0$ arbitrarily well by such a map $f$. Again by Proposition 2.2.3 we can take $f$ to be a proper embedding. The submanifold $f(T)$ satisfies the conditions in Lemma 2.2.7, so we are done. □

## 2.3 The complex $\mathcal{S}^\dagger(M, \phi)$ is connected.

Throughout this chapter, we will make heavy usage of the (combinatorial) notion of a simplicial complex, so we briefly recall the basic definitions.

**Definition 2.3.1.** A **simplicial complex** $S$ is the data of a set $\mathrm{V}(S)$, called the **vertex set** of $S$, and a subset of the power set of $\mathrm{V}(S)$, whose elements are called **simplices**, such that:

- Each simplex $\sigma$ is non-empty and finite. If $\sigma$ has $k+1$ elements, we say $\sigma$ has **dimension** $k$, or that it is a $k$**-simplex**. Simplices of dimension 1 and 2 will sometimes be called **edges** and **triangles**, respectively.
- Every non-empty subset of a simplex $\sigma$ is also a simplex, and is said to be a **face** of $\sigma$.

The (possibly infinite) supremum among the dimensions of all simplices in $S$ is the **dimension** of $S$.

A **map of simplicial complexes** $T \to S$ is a function $\mathrm{V}(T) \to \mathrm{V}(S)$ taking simplices of $T$ to simplices of $S$ (although not necessarily preserving their dimensions). A **subcomplex** $T$ of $S$ is a simplicial complex with $\mathrm{V}(T) \subseteq \mathrm{V}(S)$ and whose simplices are also simplices of $S$. In particular, for each $k \in \mathbb{N}$, the $k$**-skeleton** of $S$ is the subcomplex of $S$ with the same vertex set as $S$, but only the simplices of dimension at most $k$.

Every simplicial complex $S$ gives rise to a topological space $|S|$, its **geometric realization**, constructed as follows: for each simplex $\sigma$ of $S$, let $\Delta_\sigma$ be a copy of the standard simplex of dimension $\dim(\sigma)$ in $\mathbb{R}^{\dim(\sigma)+1}$, with the vertices of $\Delta_\sigma$ labeled by the vertices of $\sigma$. Then take

$$|S| := \Big( \bigsqcup_{\sigma \text{ simplex of } S} \Delta_\sigma \Big) / \sim,$$

where $\sim$ is generated by the affine maps $\Delta_\tau \hookrightarrow \Delta_\sigma$ given on vertices by the inclusions $\tau \hookrightarrow \sigma$ whenever $\tau$ is a face of $\sigma$. This is a functorial construction: every map of simplicial complexes $S \to T$ induces a continuous map $|S| \to |T|$ by extending the assignment on vertices $\mathrm{V}(S) \to \mathrm{V}(T)$ to affine maps on the $\Delta_\sigma$ for all simplices $\sigma$ of $S$.

We will often blur the distinction between a simplicial complex $S$ and its geometric realization, writing statements like "$S$ is connected" when referring to topological features of $|S|$. Also, when showing connectedness of a simplicial complex, it suffices to prove it for the 1-skeleton, which is a graph. In that case, we will always employ the equivalent combinatorial notion of connectedness for graphs, as the existence of a sequence of adjacent edges between any two vertices.

The current chapter is devoted to the study of the following simplicial complex.

**Definition 2.3.2.** Let $M$ be an oriented smooth manifold of dimension $n$, and $\phi \in \mathrm{H}_{n-1}(M, \partial M)$ a codimension-1 homology class. We denote by $\mathcal{S}^\dagger(M, \phi)$ the simplicial complex defined as follows:

- The vertices are the (possibly disconnected) properly embedded oriented smooth hypersurfaces in $M$ representing $\phi$.
- A set of $k+1$ hypersurfaces as above is a $k$-simplex if those hypersurfaces are pairwise-disjoint.

The first main result of this chapter, which we shall prove in this section, is the following.

**Theorem 2.3.3** (Connectedness of $\mathcal{S}^\dagger(M,\phi)$)**.** *Let $M$ be an oriented compact smooth $n$-manifold, and $\phi \in \mathrm{H}_{n-1}(M,\partial M)$ a codimension-1 homology class. Then $\mathcal{S}^\dagger(M,\phi)$ is connected.*

The codimension hypothesis is essential. For example, every non-trivial element $\phi$ of $\mathrm{H}_2(\mathbb{C}\mathrm{P}^2)$ has non-zero algebraic self-intersection, so the analogously defined complex $\mathcal{S}^\dagger(\mathbb{C}\mathrm{P}^2,\phi)$ has no edges, although there are clearly infinitely many vertices. Similarly, the assumption that $M$ is orientable cannot be dropped, as one sees by taking $M = \mathbb{R}\mathrm{P}^2$. In this case, the generator of $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2) \cong \mathbb{Z}/2$, when reduced to $\mathbb{Z}/2$-coefficients, has non-trivial algebraic self-intersection, and this again obstructs the existence of edges in the simplicial complex.

Throughout this section, $M$ will always be an oriented compact smooth manifold of dimension $n$, and $\phi \in \mathrm{H}_{n-1}(M,\partial M)$ a codimension-1 homology class. We will also denote by $|X|$ the number of connected components of a topological space $X$ and by $d^\dagger_{\mathcal{S}}(S_0,S_1)$ the path length distance between hypersurfaces $S_0$ and $S_1$ in the 1-skeleton of $\mathcal{S}^\dagger(M,\phi)$.

Theorem 2.3.3 will follow from the following statement, where we focus only on connecting *transverse* vertices (see Definition 2.2.1).

**Proposition 2.3.4** (Distance bound on transverse hypersurfaces)**.** *Let $S_0, S_1 \subset M$ be a transverse pair of hypersurfaces representing $\phi$. If $k$ is a non-negative integer such that $|S_0 \cap S_1| < 2^k$, then $d^\dagger_{\mathcal{S}}(S_0,S_1) \leq 2^k$.*

In particular, if $S_0 \cap S_1 \neq \emptyset$, then by choosing $k$ to satisfy $2^{k-1} \leq |S_0 \cap S_1| < 2^k$ we obtain the coarser (but easier to remember) estimate

$$d^\dagger_{\mathcal{S}}(S_0,S_1) \leq 2^k = 2 \cdot 2^{k-1} \leq 2\,|S_0 \cap S_1|.$$

*Proof.* We proceed by induction over $k$, and we will abbreviate $C := S_0 \cap S_1$. If $k = 0$, then we have $|C| = 0$, which means the $S_i$ are disjoint. Hence they are connected by an edge and $d^\dagger_{\mathcal{S}}(S_0,S_1) = 1 \leq 2^0$.

For positive $k$, we will find a hypersurface $T$ such that

- $T$ represents $\phi$,
- $T$ is transverse to both $S_i$, and
- $T$ has controlled intersection with each $S_i$, in the following sense:

$$|T \cap S_i| \leq \frac{|C|}{2} < 2^{k-1}.$$

By induction, it will follow that $d^\dagger_{\mathcal{S}}(T,S_i) \leq 2^{k-1}$ for each $i$, and hence from the triangle inequality $d^\dagger_{\mathcal{S}}(S_0,S_1) \leq 2^k$.

The overall strategy for finding $T$ is to perform a certain surgery procedure on $S_0, S_1$ in order to produce a third hypersurface $\Sigma \subset M$ representing the homology class $2\phi$. We then show that $\Sigma$ is the disjoint union of two hypersurfaces $T_0, T_1$, each representing $\phi$. Moreover, the set $\{S_0, S_1, T_0, T_1\}$ is transverse, and we will observe that at least one of the $T_m$ satisfies

$$|T_m \cap S_i| \leq \frac{|C|}{2} \quad \text{for each } i \in \{0,1\}$$

(all these intersections are compact submanifolds of $M$, and hence have finitely many components). This $T_m$ will be our desired $T$.

To construct $\Sigma$, we begin by observing that the normal bundle of the codimension-2 submanifold $C$ of $M$ is trivial. Indeed, since $S_0, S_1$ are both oriented, the orientation of $M$ induces framings of $S_0, S_1$, which jointly provide a framing of $C$. Hence, there is an open neighborhood $U$ of $C$ in $M$ that is diffeomorphic to $C \times \mathbb{R}^2$ via a diffeomorphism that identifies $S_0 \cap U$ with $C \times \mathbb{R} \times 0$, and $S_1 \cap U$ with $C \times 0 \times \mathbb{R}$, all respecting orientations.

We construct $\Sigma$ as follows (see Figure 2.3.1):

1. Start with the union $S_0 \cup S_1$.

2. Replace a small neighborhood of $C$ in $S_0 \cup S_1$ by a pair of smooth ramps connecting each side of $C$ in $S_0$ to a side of $C$ in $S_1$, in such a way that the resulting hypersurface inherits a consistent orientation from the $S_i$. We make this construction more precise in the following paragraph, but the idea should be clear from the top right of Figure 2.3.1.

   Consider the bump function $\mathrm{Bp}\colon \mathbb{R} \to \mathbb{R}$ defined in Wall's book [Wal16, Section 1.1], which satisfies

   $$\begin{cases} \mathrm{Bp}(t) = 0 & \text{for } t \leq 0, \\ \mathrm{Bp}(t) \in \,]0,1[ & \text{for } 0 < t < 1, \\ \mathrm{Bp}(t) = 1 & \text{for } t \geq 1. \end{cases} \quad \text{and}$$

   We replace $(S_0 \cup S_1) \cap U$ by the hypersurface corresponding, in $C \times \mathbb{R}^2$, to $C \times R$, where $R \subset \mathbb{R}^2$ is the union of the two curves parameterized by

   $$t \mapsto \mathrm{Bp}(t)(t,0) + (1 - \mathrm{Bp}(t))(0,t),$$
   $$t \mapsto \mathrm{Bp}(t)(0,t) + (1 - \mathrm{Bp}(t))(t,0),$$

   with $t \in \mathbb{R}$.

   Note that the resulting hypersurface represents the homology class $2\phi$, since the region $C \times K$ of $M$, where

   $$K := \{tX \in \mathbb{R}^2 \mid t \in [0,1], X \in R \cap \mathbb{D}^2\} \quad \text{(suitably oriented)},$$

   exhibits the new hypersurface as homologous to $[S_0] + [S_1]$.

3. Push this hypersurface slightly along its framing, so it intersects $S_0$ and $S_1$ transversely, along a pair of copies of $C$.

We will say that any hypersurface $\Sigma$ constructed in this manner is an **oriented surgery** of $S_0$ and $S_1$.

Our next goal is to show that $\Sigma$ is the disjoint union of two (possibly disconnected) hypersurfaces $T_0, T_1$, each representing $\phi$. We will say that the hypersurfaces $T_0, T_1$ are obtained by **decomposing the oriented surgery** $\Sigma$ of $S_0, S_1$. Let $f\colon M \to \mathbb{S}^1$ be a continuous map with $f^{-1}(1) = \Sigma$ (such a map can be constructed by collapsing to a point the complement of an open tubular neighborhood of $\Sigma$ in $M$). Regarding $\phi$ as an element of $\mathrm{H}^1(M)$ and $f$ as a classifying map for $2\phi$, we see that (for any basepoint) the induced map $f_*\colon \pi_1(M) \to \mathbb{Z}$ factors through $2\mathbb{Z} \hookrightarrow \mathbb{Z}$, and hence $f$ lifts to the double covering $\mathbb{S}^1 \to \mathbb{S}^1$, $z \mapsto z^2$. If $g\colon M \to \mathbb{S}^1$ is the lifted map, then taking $T_0 := g^{-1}(1)$ and $T_1 := g^{-1}(-1)$ yields $\Sigma = T_0 \sqcup T_1$. Moreover, since $g$ is a classifying map for $\phi$, we conclude $T_0$ and $T_1$ both represent $\phi$.

All that is left is to see that at least one among $T_0, T_1$ satisfies the claimed control on intersection with both $S_i$. In fact, we will show more: there are non-negative integers $N_0, N_1$ with $|C| = N_0 + N_1$, such that, for each $m \in \{0,1\}$,

$$|T_m \cap S_0| = |T_m \cap S_1| = N_m.$$

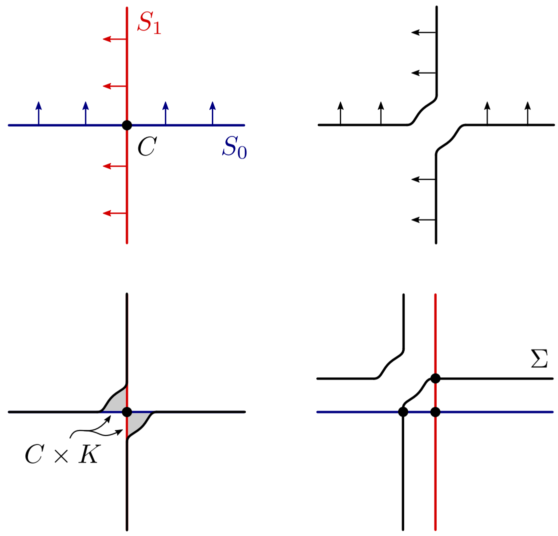



Figure 2.3.1: Performing oriented surgery on $S_0$ and $S_1$. Top left: the local picture of $S_0 \cup S_1$ in a neighborhood of $C$, with framings of the $S_i$ indicated by arrows. Top right: replacing a small neighborhood of $C$ with a pair of ramps. The induced framing on the new hypersurface is illustrated. Bottom left: The shaded region corresponding to $C \times K$ shows that the new hypersurface represents the class $[S_0] + [S_1]$. Bottom right: properly isotoping this hypersurface along its framing yields the oriented surgery $\Sigma$.

Intuitively, "the components of $C$ are distributed among the $T_m$". In particular, for some $m$ we have $N_m \leq \frac{|C|}{2}$ and thus $T_m$ satisfies our claim.

The existence of $N_0, N_1$ is a consequence of the observation that near each component $c$ of $C$, each of the two sheets of $\Sigma$ produced locally by the oriented surgery belongs to a different $T_m$. Indeed, in $\Gamma$, the component of $M \setminus\!\!\setminus \Sigma$ that touches both sheets of $\Sigma$ has one of them as an incoming edge, and the other as an outgoing edge. It is then plainly on display on the bottom right of Figure 2.3.1, that $c$ gives rise to either:

- one component in each of the $T_0 \cap S_i$ and no component in either of the $T_1 \cap S_i$ (if the sheet of $\Sigma$ on the bottom right belongs to $T_0$), or

- one component in each of the $T_1 \cap S_i$ and no component in either of the $T_0 \cap S_i$ (if the sheet of $\Sigma$ on the bottom right belongs to $T_1$).

Each $N_m$ is then the number of components of $C$ that contribute to one (hence both) of the $T_m \cap S_i$. □

*Proof of Theorem 2.3.3.* Let $S_0, S_1$ be a (not necessarily transverse) pair of hypersurfaces representing $\phi$. We produce a new properly embedded hypersurface $S_1'$ by pushing-off $S_1$ along its normal bundle, and perturbing it slightly to make it transverse to $S_0$, while keeping it disjoint from $S_1$ (this uses Proposition 2.2.2). Now apply Proposition 2.3.4 to $S_0$ and $S_1'$ to conclude that

$$d^\dagger_{\mathcal{S}}(S_0, S_1) \leq d^\dagger_{\mathcal{S}}(S_0, S_1') + d^\dagger_{\mathcal{S}}(S_1', S_1) \leq 2\,|S_0 \cap S_1'| + 1,$$

which proves the theorem. □

## 2.4 Thurston norm-realizing surfaces in 3-manifolds

We now study a variation of the simplicial complex from the previous section, where we consider only certain surfaces representing 2-homology classes in irreducible and boundary-irreducible oriented compact smooth 3-manifolds (see Definition 2.4.1 below). These surfaces are, in a sense, most efficient: they realize the Thurston norm and have no homologically trivial parts (Definition 2.4.3). Our goal is to show that restricting the complex from the previous section to the Thurston norm-realizing surfaces for a homology class still results in a connected complex. This will be accomplished simply by adjusting the proof of Proposition 2.3.4.

We begin by recalling some standard terminology.

**Definition 2.4.1.** Let $S$ be a compact smooth surface.

- A properly embedded circle in $S$ is **inessential** if it bounds an embedded disc in $S$. Otherwise, it is **essential**.
- A properly embedded arc in $S$ is **inessential** if, together with an arc in $\partial S$, it bounds an embedded disc in $S$. Otherwise, it is **inessential**.

Let $M$ be a compact smooth 3-manifold.

- $M$ is **irreducible** if every embedded 2-sphere in $M$ bounds an embedded 3-ball.
- An embedded circle in $\partial M$ is called a **meridian** if it is essential in $\partial M$ but bounds a properly embedded disc in $M$.
- $M$ is said to be **boundary-irreducible** if it contains no meridians.

Let $S$ be a properly embedded compact surface in $M$.

- A **compressing disc** for $S$ is a disc $D$ embedded in $M$ as a submanifold, with interior disjoint from $S$, and whose boundary is either:
  - an essential circle in $S$, or
  - the union of an essential arc in $S$ and an embedded arc in $\partial M$ (in which case $D$ is a submanifold with corner).

  We also demand that $D$ intersect $S$ transversely.
- If $S$ has a compressing disc, then $S$ is called **compressible**; otherwise it is **incompressible**.

Note that if $S \subset M$ as above is a sphere or a disc, then $S$ is automatically incompressible. We also collect the following observation.

**Lemma 2.4.2** (Incompressibility via connected components)**.** *A properly embedded compact surface $S$ in a compact smooth 3-manifold $M$ is incompressible if and only if all its components are incompressible.*

*Proof.* Clearly, if $S$ is compressible with compressing disc $D$, then the component of $S$ that intersects $\partial D$ also has $D$ as a compressing disc.

Conversely, suppose $S_0$ is a component of $S$ that is compressible. A compressing disc $D$ for $S_0$ may fail to be a compressing disc for $S$ because its interior may intersect other components of $S$. In that case, we first perturb $D$ slightly to make it transverse to $S$, and then look at an intersection $\gamma$ with $S$ that is innermost in $D$. Let $D' \subseteq D$ be a disc bounded by $\gamma$ (possibly together with an arc in $\partial S$). If $\gamma$ is an essential curve or arc of $S$, then $D'$ is a compressing disc for $S$ and we are done. Otherwise, one can modify $D$ by replacing $D'$ with a parallel copy of a disc $D_S \subset S$ witnessing that $\gamma$ is inessential. The interior of this new compressing disc for $S_0$ has fewer intersections with $S$, so an inductive argument finishes the proof. $\square$

Throughout the remainder of this section, $M$ will denote an irreducible and boundary-irreducible oriented compact smooth 3-manifold.

**Definition 2.4.3.** Given an orientable compact surface $S$, we define the non-negative integer

$$\chi_-(S) := \sum_{C \text{ component of } S} \max\{0, -\chi(C)\},$$

where $\chi$ is the Euler characteristic.

For a homology class $\phi \in \mathrm{H}_2(M, \partial M)$, the **Thurston norm** of $\phi$, denoted $\|\phi\|_M$, is the minimal value of $\chi_-(S)$, over all properly embedded surfaces $S \subset M$ representing $\phi$. Such a surface $S$ is said to be **Thurston norm-realizing** if it realizes this minimum (that is, if $\|[S]\|_M = \chi_-(S)$) and no union of components of $S$ represents the zero class in $\mathrm{H}_2(M, \partial M)$.

It is well-known that $\|\cdot\|_M$ extends to a norm on $\mathrm{H}_2(M, \partial M; \mathbb{R})$, which was first observed by Thurston [Thu86, Theorem 1]. We now collect some straightforward facts about Thurston norm-realizing surfaces:

1. The only Thurston norm-realizing surface for the class $0 \in \mathrm{H}_2(M, \partial M)$ is the empty surface.
2. If a properly embedded surface $S \subset M$ satisfies $\|[S]\|_M = \chi_-(S)$, one can produce from $S$ a Thurston norm-realizing surface simply by discarding a maximal null-homologous union of components of $S$. Each discarded component is necessarily of non-negative Euler characteristic.
3. The fact that $M$ is irreducible and boundary-irreducible implies that properly embedded spheres and discs are null-homologous, so no component of a Thurston norm-realizing surface in $M$ is a sphere or a disc.

The next property requires a bit more thought, so we promote it to a lemma:

**Lemma 2.4.4** (Incompressibility of Thurston norm-realizing surfaces)**.** *Every Thurston norm-realizing surface $S \subset M$ is incompressible.*

*Proof.* Suppose for contradiction that $D$ is a compressing disc for $S$. We modify $S$ by removing a small open neighborhood of $\partial D$ and capping the resulting boundary components with two discs parallel to $D$. After smoothening, the newly-formed surface $S'$ is homologous to $S$ and satisfies $\chi(S') = \chi(S) + 2$. Since $S$ is Thurston norm-realizing, this increase in $\chi$ cannot amount to a decrease in $\chi_-$, so $\partial D$ intersects a compressible component $C$ of $S$ with non-negative Euler characteristic. But spheres and discs are always incompressible, so $C$ must be a torus or an annulus. Modifying $C$ by the surgery along $D$ just described shows that $C$ is homologous to a sphere or a pair of discs, hence null-homologous. This is not allowed by $S$ being Thurston norm-realizing, so we ruled out all possibilities for $C$, and thus $D$ cannot exist. $\square$

We now introduce the main result in this section.

**Definition 2.4.5.** Given a homology class $\phi \in \mathrm{H}_2(M, \partial M)$, we define $\mathcal{T}^\dagger(M, \phi)$ to be the full subcomplex of $\mathcal{S}^\dagger(M, \phi)$ spanned by the vertices that are Thurston norm-realizing surfaces.

**Theorem 2.4.6** (Connectedness of $\mathcal{T}^\dagger(M,\phi)$)**.** *Let $M$ be an irreducible and boundary-irreducible oriented compact smooth 3-manifold, and let $\phi \in \mathrm{H}_2(M, \partial M)$. Then $\mathcal{T}^\dagger(M,\phi)$ is connected.*

In order to prove Theorem 2.4.6, we will establish a distance bound similar to the one given in Proposition 2.3.4. The corresponding statement in this setting is the following, where $d^\dagger_{\mathcal{T}}$ denotes the path length distance in the 1-skeleton of $\mathcal{T}^\dagger(M,\phi)$:

**Proposition 2.4.7** (Distance bound on surfaces intersecting essentially)**.** *Let $S_0, S_1 \subset M$ be a transverse pair of Thurston norm-realizing surfaces representing $\phi$, and assume moreover that each component of $S_0 \cap S_1$ is essential in both $S_0$ and $S_1$. If $k$ is a non-negative integer with $|S_0 \cap S_1| < 2^k$, then $d^\dagger_{\mathcal{T}}(S_1, S_2) \leq 2^k$.*

*Proof.* The proof structure is the same as that of Proposition 2.3.4, where we induct over $k$. Again we will write $C := S_0 \cap S_1$. The case $k = 0$ is immediate.

For the induction step, we will use a similar argument to the one given earlier, where a new surface $T$ was constructed by decomposing an oriented surgery of $S_0$ and $S_1$. We need only ensure that $T$ is Thurston norm-realizing and satisfies the additional requirement that for each $i \in \{0,1\}$, every component of $T \cap S_i$ is essential in both $T$ and $S_i$.

Performing oriented surgery on the $S_i$ via the three-step procedure described in the proof of Proposition 2.3.4 yields an oriented surface $\Sigma_0$ representing the class $2\phi$. For our proof, however, we need an additional step in the construction:

4. Remove a maximal null-homologous union of components of $\Sigma_0$.

Denote the resulting surface by $\Sigma$. Note that irreducibility and boundary-irreducibility of $M$ imply that Step 4 removes every sphere or disc component of $\Sigma_0$ – although in fact we will soon see that none were present in $\Sigma_0$ to begin with.

We make three observations concerning $\Sigma_0$ and $\Sigma$:

1. We have $\chi(\Sigma_0) = \chi(S_0 \sqcup S_1)$. Indeed, as an abstract surface, $\Sigma_0$ can be constructed from the disjoint union $S_0 \sqcup S_1$ by cutting off small neighborhoods of both copies of $C$, and gluing them back along the newly formed boundary (Figure 2.4.1). This does not alter the Euler characteristic.

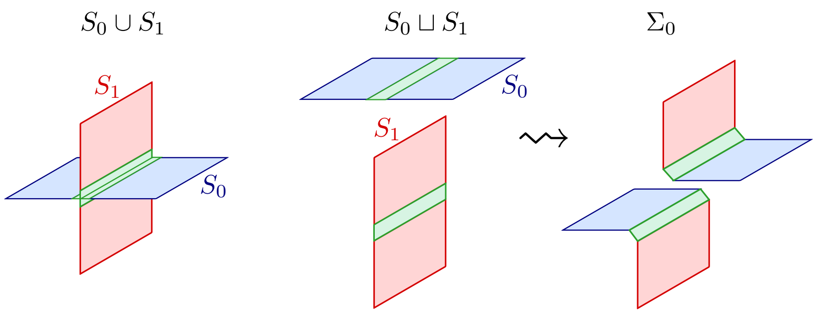



Figure 2.4.1: Constructing $\Sigma_0$ as an abstract surface by performing surgery on $S_0 \sqcup S_1$.

2. The surface $\Sigma_0$ has no sphere or disc components, so $\chi(\Sigma_0) \leq \chi(\Sigma)$. To see this, consider the “seams” in $\Sigma_0$ that result from surgery along $C$. Explicitly, these seams correspond, in the notation of page 44, to the connected components of $C \times \left\{\pm\left(\frac{1}{2}, -\frac{1}{2}\right)\right\} \subset C \times R$. Since there are no sphere or disc components in either of the $S_i$, any sphere or disc component in $\Sigma_0$ would have been produced

during surgery, and thus have a seam. An innermost seam would then correspond to a component of $C$ that is inessential in one of the $S_i$. Such components are excluded by assumption.

3. For each $i \in \{1, 2\}$, every component $\gamma$ of $\Sigma_0 \cap S_i$ is essential in both $\Sigma_0$ and $S_i$. Indeed, $\gamma$ is parallel in $S_i$ to a component of $C$, and hence essential in $S_i$. Moreover, $\gamma$ is parallel in $\Sigma_0$ to a seam, so if it were inessential in $\Sigma_0$, then $\Sigma_0$ would have an inessential seam. As every innermost inessential seam arises from a component of $C$ that is inessential in $S_0$ or $S_1$, we conclude $\gamma$ is essential in $\Sigma_0$. This observation remains valid when we replace $\Sigma_0$ with $\Sigma$.

Now, the same argument as in the proof of Proposition 2.4.7 shows that $\Sigma$ is the disjoint union of two surfaces $T_0, T_1$, each representing the class $\phi$. Step 4 in the construction of $\Sigma$ ensures that no $T_m$ contains a union of null-homologous surfaces. Hence, to prove that the $T_m$ are Thurston norm-realizing, we need only argue that both satisfy $\chi_-(T_m) = \|\phi\|_M$.

Consider the following sequence of (in)equalities:

$$\begin{aligned}
\chi_-(T_0) + \chi_-(T_1) &= -\chi(T_0) - \chi(T_1) && \text{(no spheres or discs in } \Sigma) \\
&= -\chi(\Sigma) \\
&\leq -\chi(\Sigma_0) && \text{(Observation 2)} \\
&= -\chi(S_0 \sqcup S_1) && \text{(Observation 1)} \\
&= -\chi(S_0) - \chi(S_1) \\
&= \chi_-(S_0) + \chi_-(S_1) && \text{(no spheres or discs in the } S_i) \\
&= 2\|\phi\|_M && \text{(the } S_i \text{ are Thurston norm-realizing).}
\end{aligned}$$

Since we cannot have $\chi_-(T_m) < \|\phi\|_M$ for either $m$, this shows $\chi_-(T_0) = \chi_-(T_1) = \|\phi\|_M$.

The final step of the proof of Proposition 2.3.4 consisted of observing that each component of $C$ contributes with one component to both $T_0 \cap S_i$ and none to either $T_1 \cap S_i$, or vice-versa. In view of Step 4 of the construction of $\Sigma$, this observation should now be adapted to: each component of $C$ contributes with *at most* one component to both $T_0 \cap S_i$ and none to either $T_1 \cap S_i$, or vice-versa. This change does not affect the conclusion that for some $T \in \{T_0, T_1\}$ we have $|T \cap S_0| = |T \cap S_1| \leq \frac{|C|}{2}$.

The reduction to the induction hypothesis then works as before, provided that for each $i \in \{0, 1\}$, every component of $T \cap S_i$ is essential in both $T$ and $S_i$. But this is a direct consequence of Observation 3 above. $\square$

Before using Proposition 2.4.7 to prove Theorem 2.4.6, we need an auxiliary result to reduce us to the case where all intersections between the $S_i$ are essential in both $S_i$. Note that whenever $S_0, S_1$ are transverse properly embedded surfaces in $M$ that are incompressible, each component of $S_0 \cap S_1$ that is inessential in one of the $S_i$ is automatically inessential in the other as well. We will thus simply call a component **essential** or **inessential** accordingly, and we will denote by $\mathrm{ess}(S_0, S_1)$ the number of essential components.

**Lemma 2.4.8** (Removing inessential intersections)**.** *Let $S_0, S_1$ be a transverse pair of properly embedded incompressible surfaces in $M$, and suppose $S_0 \cap S_1$ has an inessential component. Then there exists a properly embedded surface $S_1' \subset M$ such that:*

- *$S_1'$ is properly isotopic to and disjoint from $S_1$,*
- *$S_0$ and $S_1'$ are a transverse pair,*

- $|S_0 \cap S_1'| < |S_0 \cap S_1|$, *and*
- $\mathrm{ess}(S_0, S_1') = \mathrm{ess}(S_0, S_1)$.

We point out that we will not need the fourth item in the current section, but it will be useful later, when proving Corollary 2.6.4.

*Proof.* Let $\gamma$ be an inessential component of $S_0 \cap S_1$, and suppose further that it is innermost in $S_0$ – that is, it cuts off a disc in $S_0$ with interior disjoint from $S_1$.

We consider first the case where $\gamma$ is a circle (Figure 2.4.2, left). For each $i \in \{0, 1\}$, let $D_i \subset S_i$ be the disc it bounds in the relevant surfaces (where $D_0$ has interior disjoint from $S_1$). Since the $D_i$ intersect only at $\gamma$, they form an embedded sphere in $M$, which by irreducibility of $M$ bounds a ball $B$. Note that the interior of $B$ is disjoint from $S_1$: indeed, $S_1$ is disjoint from the interior of $D_0$ (because $D_0$ is innermost), and any component of $S_1$ contained in the interior of $B$ would be compressible or a sphere. We can thus properly isotope $S_1$ by pushing it in the direction of its normal bundle that points towards the interior of $B$, and then use a proper isotopy supported in a small neighborhood of the parallel copy $D_1'$ of $D_1$ to push $D_1'$ across $B$ and past $D_0$. For the resulting surface $S_1'$, the intersection $S_0 \cap S_1'$ is comprised precisely of one parallel copy of each component of $S_0 \cap S_1$ away from $D_1$. In particular, $\gamma$ does not contribute to $S_0 \cap S_1'$, and so $|S_0 \cap S_1'| < |S_0 \cap S_1|$. As any other intersections getting removed are contained in $D_1$, they are inessential, and so we also have $\mathrm{ess}(S_0, S_1') = \mathrm{ess}(S_0, S_1)$.

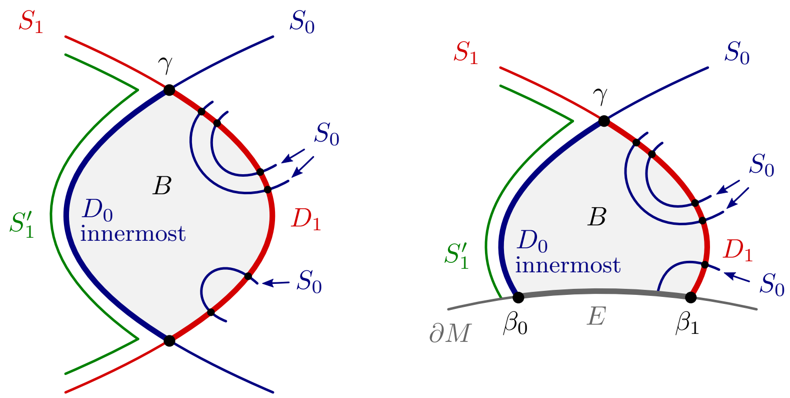



Figure 2.4.2: Using incompressibility of $S_0$ and $S_1$ to remove an inessential intersection $\gamma$ that is innermost in $S_1$. We depict the cases where $\gamma$ is a circle (left) and where it is an arc (right).

If $\gamma$ is an arc, one proceeds in analogous fashion (Figure 2.4.2, right). Since $\gamma$ is inessential in both $S_i$, we have:

- a disc $D_0 \subset S_0$ jointly bounded by $\gamma$ and by an arc $\beta_0 \subset \partial S_0$, such that the interior of $D_0$ is disjoint from $S_1$, and
- a disc $D_1 \subset S_1$ jointly bounded by $\alpha$ and by an arc $\beta_1 \subset \partial S_1$.

Since $D_0 \cup D_1$ is a properly embedded disc in $M$ with boundary $\beta_0 \cup \beta_1$, boundary-irreducibility of $\partial M$ guarantees we also have:

- a disc $E \subset \partial M$ bounded by $\beta_0 \cup \beta_1$.

Irreducibility of $M$ again provides a 3-ball $B$ with interior disjoint from $S_0$, and whose boundary is $D_0 \cup D_1 \cup E$. We push $S_1$ off of itself in the direction of $B$ and use $B$ to properly isotope $D_0$ through $D_1$. Again, we have $|S_0 \cap S_1'| < |S_0 \cap S_1|$ and $\mathrm{ess}(S_0, S_1') = \mathrm{ess}(S_0, S_1)$. □

*Proof of Theorem 2.4.6.* Let $S_0, S_1$ be Thurston norm-realizing surfaces for $\phi$, which we aim to show are connected by a path in the 1-skeleton of $\mathcal{T}^\dagger(M,\phi)$. We may assume that $S_0, S_1$ form a transverse pair – if not, take a parallel copy of $S_1$ and perturb it using Proposition 2.2.2 to produce another Thurston norm-realizing surface for $\phi$ disjoint from $S_1$ and forming a transverse pair with $S_0$.

Since the $S_i$ are incompressible by Lemma 2.4.4, we know that either all components of $S_0 \cap S_1$ are essential in both $S_i$, or, by Lemma 2.4.8, there is a surface $S_1'$ disjoint from $S_1$, forming a transverse pair with $S_0$, and such that $|S_0 \cap S_1'| < |S_0 \cap S_1|$. Repeating this argument with $S_1'$ in place of $S_1$ enough times, we eventually find a surface $S_1^\circ$ connected by a path to $S_1$ and whose intersections with $S_0$ are all essential in both $S_0$ and $S_1^\circ$. Now we apply Proposition 2.4.7 to $S_0$ and $S_1^\circ$, which finishes the construction of a path from $S_0$ to $S_1$. □

## 2.5 The complex $\mathcal{S}^\dagger(M,\phi)$ is simply connected

For this section, we return to the setting where $M$ is an oriented compact smooth manifold of arbitrary dimension $n$, and $\phi \in \mathrm{H}_{n-1}(M, \partial M)$ is a codimension-1 homology class. We will expand the techniques used in the proof of Theorem 2.3.3 to establish the following result:

**Theorem 2.5.1** (Simple connectedness of $\mathcal{S}^\dagger(M,\phi)$)**.** *Let $M$ be an oriented compact smooth $n$-manifold and let $\phi \in \mathrm{H}_{n-1}(M,\partial M)$. Then the complex $\mathcal{S}^\dagger(M,\phi)$ is simply connected.*

Similarly to the previous sections, we will start by proving a weaker version of Theorem 2.5.1 that deals only with collections of hypersurfaces that are transverse, in the sense of Definition 2.2.1. Subcomplexes of $\mathcal{S}^\dagger(M,\phi)$ will be called **transverse** if their set of vertices is transverse (extending the terminology established in Section 2.2 for finite subsets of $\mathrm{V}(\mathcal{S}^\dagger(M,\phi))$).

Before presenting this weaker statement (Proposition 2.5.4), we remind the reader of the notions of a simplicial cone, and of a subdivision of a simplicial complex.

**Definition 2.5.2.** Let $S$ be a simplicial complex. A **simplicial cone** of $S$ is a simplicial complex whose vertex set is obtained from $\mathrm{V}(S)$ by adding one new element $v \notin \mathrm{V}(S)$, and whose $k$-simplices are the $k$-simplices of $S$ and the $\sigma \cup \{v\}$, with $\sigma$ a $(k-1)$-simplex of $S$. The new vertex $v$ is called the **cone point**.

Of course all simplicial cones of $S$ are isomorphic via a unique isomorphism restricting to the identity on $S$, so we may talk about "the" simplicial cone.

**Definition 2.5.3.** A **subdivision** of a simplicial complex $S$ is a simplicial complex $T$ with $\mathrm{V}(S) \subseteq \mathrm{V}(T)$, such that there is a homeomorphism $f\colon |S| \to |T|$ satisfying:

- on the (geometrical realization of the) 0-skeleton of $S$, the map $f$ restricts to the inclusion $\mathrm{V}(S) \hookrightarrow \mathrm{V}(T)$, and
- for each geometric simplex $|\tau| := \mathrm{Hull}(\tau) \subseteq |T|$ of $T$, the pre-image $f^{-1}(|\tau|)$ is contained in some geometric simplex $|\sigma|$ of $S$.

**Proposition 2.5.4** (Contractibility of 1-subcomplexes with transverse edges)**.** *For each transverse 1-dimensional subcomplex $P$ of $\mathcal{S}^\dagger(M,\phi)$, the inclusion $P \hookrightarrow \mathcal{S}^\dagger(M,\phi)$ extends to some subdivision of the simplicial cone of $P$.*

Proving Proposition 2.5.4 will take most of the present section. It would be a straightforward affair if we were able to find a vertex $T$ in $\mathcal{S}^\dagger(M,\phi)$ such that for each simplex $\sigma$

of $P$, the set $\sigma \cup \{T\}$ is a simplex of $\mathcal{S}^\dagger(M,\phi)$. There is however no reason to expect such a hypersurface $T$ to exist, so we will instead just pick some $T$ (satisfying a transversality assumption) and ask how far the $\sigma \cup \{T\}$ are from being simplices. This approach relies on a family of notions of complexity for simplices in the cone of $P$, or, more generally, for simplices in a complex $Q$ with $\mathrm{V}(Q)$ a "nice enough" subset of $\mathrm{V}(\mathcal{S}^\dagger(M,\phi))$.

**Definition 2.5.5.** Let $Q$ be a nonempty finite simplicial complex such that $\mathrm{V}(Q) \subset \mathrm{V}(\mathcal{S}^\dagger(M,\phi))$ and every simplex of $Q$ is a transverse set. For each $l \in \mathbb{N}_{\geq 1}$:

- The ***l*-complexity** of a $k$-simplex $\sigma = \{S_0, \dots, S_k\}$ of $Q$ is

$$\langle \sigma \rangle_l := \begin{cases} 0 & \text{if } k < l, \\ |S_0 \cap \ldots \cap S_l| & \text{if } k = l, \\ \text{the maximal } l\text{-complexity among the } l\text{-dimensional faces of } \sigma & \text{if } k > l. \end{cases}$$

- The ***l*-complexity** of $Q$ is

$$\langle Q \rangle_l := \max\{\langle \sigma \rangle_l \in \mathbb{N} \mid \sigma \text{ is a simplex of } Q\}.$$

Note that if $\langle Q \rangle_l = 0$ for a certain $l$, then $\langle Q \rangle_m = 0$ for all $m \geq l$. Moreover, $Q$ is a subcomplex of $\mathcal{S}^\dagger(M,\phi)$ precisely when $\langle Q \rangle_1 = 0$.

The definition we just gave is more general than necessary for our purposes. Indeed, as the cone of $P$ mentioned in the preceding discussion is 2-dimensional, we will only make use of the notions of 1- and 2-complexity.

**Proposition 2.5.6** (Reducing 2-complexity)**.** *Let $Q$ be a finite 2-dimensional simplicial complex with $\mathrm{V}(Q) \subset \mathrm{V}(\mathcal{S}^\dagger(M,\phi))$, and whose simplices are transverse. If $\langle Q \rangle_2 > 0$, then there is a subdivision $Q'$ of $Q$ with $\mathrm{V}(Q') \subset \mathrm{V}(\mathcal{S}^\dagger(M,\phi))$ and transverse simplices, such that $\langle Q' \rangle_2 < \langle Q \rangle_2$.*

*Proof.* The proof scheme is similar to that of Proposition 2.3.4. The first step will be to construct, for each 2-simplex $\sigma = \{S_0, S_1, S_2\}$ of $Q$ of maximal 2-complexity $N$, an oriented surgery $\Sigma$ of the $S_i$. Of course we have to clarify what this means for a set of three hypersurfaces. We will then decompose $\Sigma$ as a disjoint union of three hypersurfaces $T_0, T_1, T_2$, each representing $\phi$. Finally, we will use a counting argument to see that adding one of the $T_m$ to the vertex set of $Q$ allows us to subdivide $\sigma$ into three triangles of smaller 2-complexity than $\sigma$.

As before, the construction of the oriented surgery $\Sigma$ starts with the union $S_0 \cup S_1 \cup S_2$, which we wish to modify near points where the $S_i$ meet. However, this time we have to consider not only the local model near the intersection of precisely two hypersurfaces described in the proof of Proposition 2.3.4, but also the neighborhoods of triple points.

Let $P_0, P_1, P_2$ be the three coordinate planes in $\mathbb{R}^3$. Denoting $C := S_0 \cap S_1 \cap S_2$, the fact that all involved hypersurfaces are oriented implies that there is a neighborhood $U$ of $C$ in $M$ that is diffeomorphic to $C \times \mathbb{R}^3$, via a diffeomorphism identifying $S_i \cap U$ with $C \times P_i$ in a way that preserves all framings (Figure 2.5.1, top left).

Whereas the "ramp" construction in the proof of Proposition 2.3.4 replaces each double intersection with two sheets (Figure 2.3.1, top right), its analogue for triple intersections gives rise to three sheets (Figure 2.5.1, top right). Performing this modification near triple points, the ramp construction near double points, and then pushing everything in the direction of the framings of the $S_i$ yields the oriented surgery $\Sigma$.The resulting $\Sigma$ represents the homology class $3\phi$ and, together with the $S_i$, forms a transverse set.

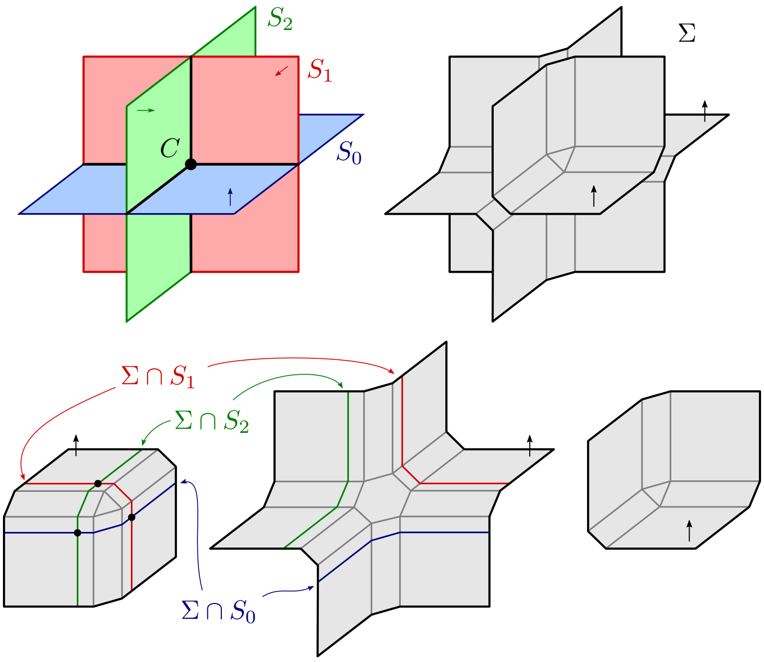



Figure 2.5.1: The oriented surgery construction near triple points. Top left: the local picture of $S_0 \cup S_1 \cup S_2$ near $C$, with framings of the $S_i$ indicated by arrows. Top right: the oriented surgery $\Sigma$ of the $S_i$ near $C$. Bottom: the three sheets of $\Sigma$ formed near $C$, and their intersections with the $S_i$. The bottom sheet has exactly one triple intersection with each two among the $S_i$, the middle sheet intersects each $S_i$ but forming no triple points, and the top sheet is disjoint from all $S_i$.

The oriented surgery $\Sigma$ can be decomposed as the disjoint union of three properly embedded hypersurfaces $T_0, T_1, T_2$, each representing $\phi$: As in the proof of Proposition 2.3.4, choose a map $f\colon M \to \mathbb{S}^1$ for which $f^{-1}(1) = \Sigma$. Regarding $\phi$ as a 1-dimensional cohomology class, the map $f$ is a classifying map for $3\phi$, and so it factors through the 3-fold cover $\mathbb{S}^1 \to \mathbb{S}^1$, $z \mapsto z^3$. Denoting by $g$ the lifted map, and writing $\omega := \mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}$, we may take $T_0 := g^{-1}(1)$, $T_1 := g^{-1}(\omega)$ and $T_2 := g^{-1}(\omega^2)$.

Now we observe that near each component of $C$, each of the three sheets of $\Sigma$ belongs to a different $T_m$. Indeed, there is an arc in $M$ from the positive side of the bottom sheet of $\Sigma$ to the negative side of the middle sheet, and otherwise disjoint from $\Sigma$. Similarly, there is an arc from the positive side of the middle sheet to the negative side of the top sheet. Thus, it follows from the construction of $T_0, T_1, T_2$ that if the bottom sheet belongs to $T_m$, then the middle sheet belongs to $T_{m+1}$ and the top sheet to $T_{m+2}$ (with indices modulo 3). At the bottom of Figure 2.5.1, we also see that the lowest sheet forms one triple intersection with each two of the $S_i$, whereas the other two sheets form no triple intersections. Hence, the total number $N$ of triple intersections among the $S_i$ decomposes as a sum $N = N_0 + N_1 + N_2$ of non-negative integers, where for each $m \in \{0,1,2\}$ we have

$$N_m = |T_m \cap S_0 \cap S_1| = |T_m \cap S_0 \cap S_2| = |T_m \cap S_1 \cap S_2|.$$

Some $m$ must therefore satisfy $N_m \leq \frac{N}{3}$, so let $T := T_m$.

We produce the desired subdivision $Q'$ of $Q$ as follows: for each triangle $\sigma$ of $Q$ with $\langle \sigma \rangle_2 = N$, add the vertex $T$ just described to $\mathrm{V}(Q)$, and replace $\sigma$ with the triangles

$\{T, S_0, S_1\}$, $\{T, S_0, S_2\}$ and $\{T, S_1, S_2\}$ (as well as the necessary edges), as illustrated in Figure 2.5.2.

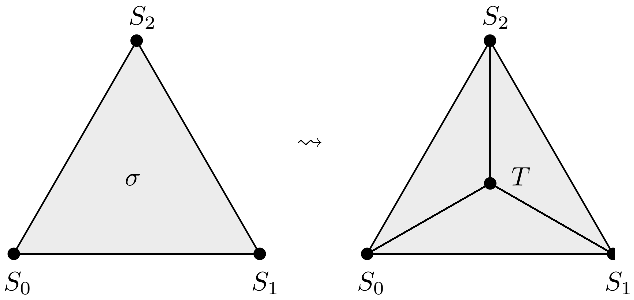



Figure 2.5.2: $Q$ is subdivided into $Q'$ by replacing each triangle $\sigma$ of 2-complexity $N$ by three triangles with 2-complexity at most $\frac{N}{3}$.

Each of the new triangles has 2-complexity at most $\frac{N}{3} < N$, and so $\langle Q' \rangle_2 < \langle Q \rangle$. $\square$

We point out that there seems to be no obstacle to generalizing the above argument beyond complexes of dimension 2. More concretely, let $k \in \mathbb{N}_{\geq 1}$, and let $Q$ be a finite $k$-dimensional simplicial complex with $\mathrm{V}(Q) \subset \mathrm{V}(\mathcal{S}^\dagger(M, \phi))$ and transverse simplices, such that $\langle Q \rangle_k > 0$. Then $Q$ can be subdivided into $Q'$ with $\mathrm{V}(Q') \subset \mathrm{V}(\mathcal{S}^\dagger(M, \phi))$ and transverse simplices, such that $\langle Q' \rangle_k < \langle Q \rangle_k$. Indeed, each application of the construction and decomposition of an oriented surgery of the vertices in a $k$-simplex of $Q$ with maximal $k$-complexity $N$ would subdivide that simplex into $k+1$ simplices of complexity at most $\frac{N}{k+1}$. The main technical annoyance is in showing that the local models such as the one in Figure 2.5.1, which for small $k$ fit our low-dimensional pictures, behave as one would expect when $k$ is larger. As such a statement is unnecessary for our purposes, we will not pursue these details.

**Proposition 2.5.7** (Reducing 1-complexity)**.** *Let $Q$ be a finite 2-dimensional simplicial complex with $\mathrm{V}(Q) \subset \mathrm{V}(\mathcal{S}^\dagger(M, \phi))$, and whose simplices are transverse. Suppose $\langle Q \rangle_2 = 0$ and $\langle Q \rangle_1 > 0$. Then there is a subdivision $Q'$ of $Q$ with $\mathrm{V}(Q') \subset \mathrm{V}(\mathcal{S}^\dagger(M, \phi))$ and transverse simplices, such that $\langle Q' \rangle_2 = 0$ and $\langle Q' \rangle_1 < \langle Q \rangle_1$.*

*Proof.* For each simplex in $Q$ of maximal 1-complexity $N$ (whether it is an edge or a triangle), we perform the oriented surgery construction described in the proofs of Propositions 2.3.4 and Proposition 2.5.6 on its vertex set. Note that in the case of three-fold oriented surgeries, there are no triple points, by our assumption that $\langle Q \rangle_2 = 0$. It will be crucial for our intersection count that when performing the push-off step of the oriented surgeries, the 3-fold surgeries (that is, the ones originating from triangles) be pushed farther than the two-fold ones (that is, coming from edges), as illustrated in Figure 2.5.3.

We wish to decompose these oriented surgeries, producing new hypersurfaces representing $\phi$ to be added to $\mathrm{V}(Q')$. Let us begin with the edges.

If $\{S_0, S_1\}$ is an edge of $Q$ with complexity $N$, then the corresponding oriented surgery $\Sigma_{01}$ represents the class $2\phi$. The argument given in the proof of Proposition 2.3.4 shows that one can write $\Sigma_{01} = T_{01}^0 \sqcup T_{01}^1$ with each $T_{01}^m$ representing $\phi$, and moreover, for some $m \in \{0, 1\}$, we have

$$|T_{01}^m \cap S_0| = |T_{01}^m \cap S_1| \leq \frac{N}{2}.$$

We take $T_{01}$ to be a $T_{01}^m$ satisfying the above condition.

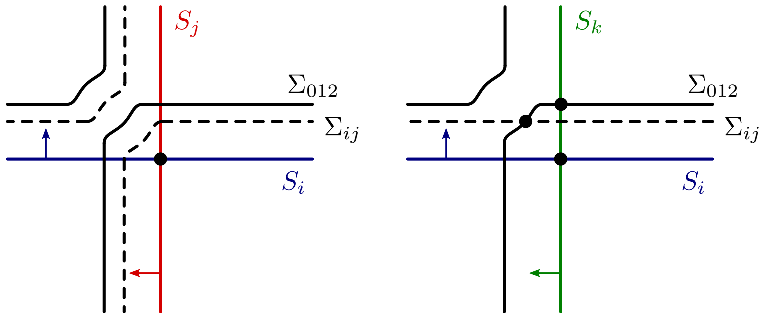



Figure 2.5.3: The oriented surgery $\Sigma_{012}$ at a triangle $\{S_0, S_1, S_2\}$ of $Q$, and the oriented surgery $\Sigma_{ij}$ at one of its sides $\{S_i, S_j\}$. Note that $\Sigma_{012}$ should lie farther from the original surfaces than each $\Sigma_{ij}$. We depict the intersection pattern between $\Sigma_{012}$ and $\Sigma_{ij}$ near $S_i \cap S_j$ (left) and near the intersection of $S_i$ with the third surface $S_k$ (right).

The hypersurfaces constructed in this manner over all edges of $Q$ with complexity $N$ are now used for subdividing the 1-skeleton of $Q$: each such edge $\{S_0, S_1\}$ is replaced by the edges $\{S_0, T_{01}\}$ and $\{S_1, T_{01}\}$. Thus this subdivision of the 1-skeleton of $Q$ has 1-complexity strictly lower than $N$; our goal is to extend it to all of $Q$.

We argue as in the proof of Proposition 2.5.6 to claim that for each triangle $\sigma := \{S_0, S_1, S_2\}$ of $Q$ with $\langle\sigma\rangle_1 = N$, the oriented surgery $\Sigma_{012}$ constructed above decomposes as

$$\Sigma_{012} = T^0_{012} \sqcup T^1_{012} \sqcup T^2_{012},$$

with each $T^m_{012}$ representing $\phi$. Our goal now is to extend the subdivision of the 1-skeleton of $Q$ to all of $Q$ by choosing an appropriate $m \in \{0,1,2\}$ and adding $T_{012} := T^m_{012}$ to $\mathrm{V}(Q')$. The triangles $\sigma$ will then be replaced with the simplices $\tau \cup \{T_{012}\}$, where $\tau$ is a simplex of the subdivision of the boundary of $\sigma$. This is illustrated in Figure 2.5.4.

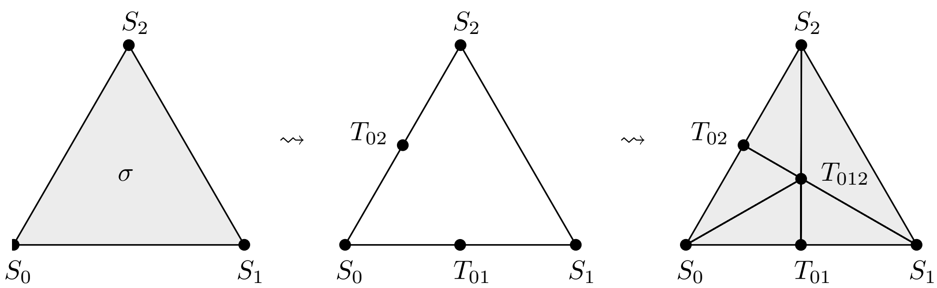



Figure 2.5.4: Subdividing the simplices in $Q$ of maximal 1-complexity $N$. Left: a triangle $\sigma = \{S_0, S_1, S_2\}$ with $\langle\sigma\rangle_1 = N$; for this example, we assume $\{S_0, S_1\}$ and $\{S_0, S_2\}$ are its sides with 1-complexity $N$. Middle: we begin by subdividing the 1-skeleton of $Q$ by adding "midpoints" to the edges of maximal complexity. Right: the subdivision of the 1-skeleton is extended to all of $Q$ by replacing each triangle $\sigma$ with the simplices $\tau \cup \{T_{012}\}$, with $\tau$ ranging over the simplices in the subdivided boundary of $\sigma$.

The proof would be complete if we could show that for each triangle $\sigma = \{S_0, S_1, S_2\}$ of $Q$, there is a choice of $m \in \{0,1,2\}$ such that all triangles in the subdivision given by $T_{012} := T^m_{012}$, as explained above, have 1-complexity strictly less than $N$. We will prove a weaker assertion, which will nevertheless suffice:

**Claim.** *There is $m \in \{0, 1, 2\}$ such that all triangles in the subdivision of $\sigma$ given by $T_{012} := T^m_{012}$ have 1-complexity at most $N$. Moreover, if some side of $\sigma$ has 1-complexity strictly less than $N$, then $m$ can be chosen so that all triangles in the subdivision have 1-complexity strictly less than $N$.*

Before proving this claim, we explain how it implies the statement of Proposition 2.5.7. If all triangles of $Q$ have a side with 1-complexity strictly less than $N$, then the conclusion follows immediately. Suppose then that there exists a triangle $\sigma$ whose sides all have 1-complexity $N$, so the claim does not guarantee a strict decrease in 1-complexity. The key observation is that this will no longer be true of the triangles formed when subdividing $\sigma$. Indeed, all these triangles have an edge in the subdivided boundary of $\sigma$, and those have 1-complexity strictly less than $N$. Hence, repeating the whole procedure of subdividing edges and triangles of 1-complexity $N$, the resulting subdivision $Q'$ will satisfy $\langle Q' \rangle_1 < N$.

*Proof of the claim.* All we need to do is show that for some $m$, all newly-formed edges in the subdivision of $\sigma$ satisfy the claimed control on 1-complexity. As already observed, independently of the choice of $m$, all edges in the subdivided boundary of $\sigma$ have 1-complexity strictly less than $N$. We therefore need only show that for some $m$, the edges containing $T^m_{012}$ satisfy the asserted bound on 1-complexity. The claim thus follows from the following two statements, which we will now show:

1. Independently of the choice of $m$, the 1-complexity of each newly-formed "short edge" is bounded by that of the opposite "long edge". Formally: let $m \in \{0, 1, 2\}$, let $\{S_i, S_j\}$ be a side of $\sigma$ with complexity $N$, and let $\{i, j, k\} = \{0, 1, 2\}$ (that is, the $i, j, k$ are three distinct indices). Then we have
$$|T^m_{012} \cap T_{ij}| \leq |T^m_{012} \cap S_k|.$$
2. For some $m$, the newly-formed long edges satisfy the claimed bound on 1-complexity. In other words, there exists $m \in \{0, 1, 2\}$ such that for all $i \in \{0, 1, 2\}$ we have
$$|T^m_{012} \cap S_i| \leq N,$$
with strict inequality if some side of $\sigma$ has 1-complexity strictly less than $N$.

For the first item: clearly $T^m_{012}$ only intersects $T_{ij}$ near points where two among $S_i$, $S_j$, $S_k$ meet. Thus the proof of this assertion is almost entirely contained in Figure 2.5.3. Indeed, the left side of that figure shows that near $S_i \cap S_j$ there are no components of $T^m_{012} \cap T_{ij}$, and on the right we see that each component of $S_i \cap S_k$ contributes with at most one component to $T^m_{012} \cap T_{ij}$. Explicitly, there is a contribution precisely if the sheet of $\Sigma_{ij}$ parallel to $S_i$ belongs to $T_{ij}$ and the lower sheet of $\Sigma_{012}$ belongs to $T^m_{012}$. But when this happens, this component of $S_i \cap S_k$ also contributes with one component to $T^m_{012} \cap S_k$. The behavior near components of $S_j \cap S_k$ is obviously similar. Overall, we conclude that components of $T^m_{012} \cap T_{ij}$ correspond injectively to components of $T^m_{012} \cap S_k$, and so $|T^m_{012} \cap T_{ij}| \leq |T^m_{012} \cap S_k|$.

For the second item: Let $\{i, j, k\} = \{0, 1, 2\}$. As in the proof of Proposition 2.3.4, the crucial observation is that each component of $S_i \cap S_j$ contributes with exactly one component to $T^m_{012} \cap S_i$ and one to $T^m_{012} \cap S_j$, for precisely one $m \in \{0, 1, 2\}$ (refer back to Figure 2.3.1, bottom right). Denoting this number of contributions by $N^m_{ij}$, we thus have
$$N^0_{ij} + N^1_{ij} + N^2_{ij} = |S_i \cap S_j| \leq N.$$

On the other hand, $|T^m_{012} \cap S_i|$ is the sum of the contributions from $S_i \cap S_j$ and from $S_i \cap S_k$. In other words, the 1-complexity of $\{T^m_{012}, S_i\}$ is given by
$$|T^m_{012} \cap S_i| = N^m_{ij} + N^m_{ik}.$$

Let us organize all these numbers into a grid, each column corresponding to a choice of $m$, and each row to a side of $\sigma$:

$$\begin{matrix} N^0_{01} & N^1_{01} & N^2_{01} \\ N^0_{02} & N^1_{02} & N^2_{02} \\ N^0_{12} & N^1_{12} & N^2_{12} \end{matrix}$$

The entries in each row add up to at most $N$, so the sum of all entries on the grid is at most $3N$, and thus some column $m$ adds up to at most $N$. Since the numbers $|T^m_{012} \cap S_i|$ are precisely the sums of pairs of entries in the $m$-th column, this $m$ satisfies the first part of item 2. Under the additional hypothesis that some side of $\sigma$ has 1-complexity strictly less than $N$, the corresponding row adds up to strictly less than $N$, and so the sum of all entries in the grid is strictly less than $3N$. The same argument then yields $m$ satisfying the stronger conclusion.

With the claim justified, the proof of Proposition 2.5.7 is complete. □

Having established Propositions 2.5.6 and 2.5.7, proving Proposition 2.5.4 is straightforward:

*Proof of Proposition 2.5.4.* Choose any vertex $T$ of $\mathcal{S}^\dagger(M,\phi)$, adjusted using Proposition 2.2.2 so that $T \notin \mathrm{V}(P)$ and the set $\mathrm{V}(P) \cup \{T\}$ is transverse. For $Q$ the (2-dimensional) simplicial cone of $P$ with cone point $T$, it suffices to show that the inclusion of $P \hookrightarrow \mathcal{S}^\dagger(M,\phi)$ extends to some subdivision of $Q$. By an iterated application of Proposition 2.5.6, we may subdivide $Q$ into a simplicial complex $Q'$ with $\mathrm{V}(Q') \subset \mathrm{V}(\mathcal{S}^\dagger(M,\phi))$ and transverse simplices, such that $\langle Q' \rangle_2 = 0$. Then an iterated application of Proposition 2.5.7 subdivides $Q'$ into $Q''$ again with $\mathrm{V}(Q'') \subset \mathrm{V}(\mathcal{S}^\dagger(M,\phi))$ and transverse simplices, such that $\langle Q'' \rangle_1 = 0$. By definition this means that all simplices of $Q''$ are also simplices of $\mathcal{S}^\dagger(M,\phi)$, so the inclusion $\mathrm{V}(Q'') \hookrightarrow \mathrm{V}(\mathcal{S}^\dagger(M,\phi))$ is a map of simplicial complexes. □

Before establishing simple connectedness of $\mathcal{S}^\dagger(M,\phi)$, we need an additional observation that reduces us to the setting where the involved hypersurfaces form a transverse set:

**Lemma 2.5.8** (Perturbing to transverse subcomplexes)**.** *Every finite subcomplex $P$ of $\mathcal{S}^\dagger(M,\phi)$ is isotopic to an isomorphic subcomplex with transverse vertex set.*

*Proof.* The lemma is a consequence of the following statement:

**Claim.** *Let $\mathcal{U} \subset \mathrm{V}(P)$ be a transverse subset and let $S \in \mathrm{V}(P) \setminus \mathcal{U}$. Then there exists a hypersurface $S' \in \mathrm{V}(\mathcal{S}^\dagger(M,\phi)) \setminus \mathrm{V}(P)$ such that:*

- *the set $\mathcal{U} \cup \{S'\}$ is transverse, and*
- *the simplicial complex $P'$ obtained from $P$ by replacing $S$ with $S'$ (in $\mathrm{V}(P)$ and in all simplices containing $S$) is a subcomplex of $\mathcal{S}^\dagger(M,\phi)$ isotopic to $P$.*

If we prove this claim, then starting with $\mathcal{U} = \emptyset$, we can iteratively enlarge $\mathcal{U}$ by replacing each vertex $S$ of $P$ by $S'$ and adding $S'$ to $\mathcal{U}$, ultimately reaching a subcomplex of $\mathcal{S}^\dagger(M,\phi)$ isotopic to $P$, whose vertex set is the transverse set $\mathcal{U}$.

*Proof of the Claim.* Let $S_0$ be a push-off of $S$ disjoint from all the hypersurfaces in $\mathrm{V}(P)$ that are disjoint from $S$. We construct $S'$ by applying Proposition 2.2.2 to $S_0$ and the set $\mathcal{U}$. Since $S'$ can be chosen to live in an arbitrarily small neighborhood of $S_0$, we can take $S'$ disjoint from $S$ and from all hypersurfaces in $\mathrm{V}(P)$ that are disjoint from $S$. Hence, for every simplex $\sigma$ of $P$ containing $S$, the set $\sigma' := \sigma \cup \{S'\} \setminus \{S\}$ is a simplex of $\mathcal{S}^\dagger(M, \phi)$, and thus $P'$ is a subcomplex of $\mathcal{S}^\dagger(M, \phi)$.

Moreover, $S'$ being disjoint from $S$ implies that every such $\sigma$, also $\sigma \cup \{S'\}$ is a simplex of $\mathcal{S}^\dagger(M, \phi)$. Therefore, the path from $S$ to $S'$ along the edge $\{S, S'\}$ of $\mathcal{S}^\dagger(M, \phi)$ extends to an isotopy from each $\sigma$ to $\sigma'$, fixing the remaining vertices of $\sigma$. We extend this as the constant isotopy away from simplices containing $S$, producing an isotopy from $P$ to $P'$.

Having proved the claim, Lemma 2.5.8 follows as explained above. □

Theorem 2.5.1 now follows directly:

*Proof of Theorem 2.5.1.* By cellular approximation, in order to prove simple connectedness of $\mathcal{S}^\dagger(M, \phi)$, it suffices to show that the inclusion $P \hookrightarrow \mathcal{S}^\dagger(M, \phi)$ of every finite 1-subcomplex $P$ is null-homotopic. Lemma 2.5.8 reduces us to the case where $P$ has transverse vertex set, and Proposition 2.5.4 shows that every such $P$ is null-homotopic in $\mathcal{S}^\dagger(M, \phi)$. □

## 2.6 Dropping the daggers

We now discuss the implications of the results in the preceding sections to the complex $\mathcal{S}(M, \phi)$, defined similarly to $\mathcal{S}^\dagger(M, \phi)$, except that everything is taken "up to proper isotopy".

**Definition 2.6.1.** Let $M$ be an oriented compact smooth $n$-manifold and let $\phi \in \mathrm{H}_{n-1}(M, \partial M)$. We denote by $\mathcal{S}(M, \phi)$ the simplicial complex defined as follows:

- The vertices are the (possibly disconnected) oriented smooth properly embedded hypersurfaces in $M$ representing $\phi$, up to smooth proper isotopy.
- A set of $k+1$ isotopy classes forms a $k$-simplex if those classes admit representatives that are pairwise disjoint.

There is an obvious map of simplicial complexes $p\colon \mathcal{S}^\dagger(M, \phi) \to \mathcal{S}(M, \phi)$ sending each vertex to its isotopy class. This map is clearly surjective on simplices of all dimensions.

If $S_0, S_1 \subset M$ are hypersurfaces representing $\phi$, we denote by $d_{\mathcal{S}}(S_0, S_1)$ the path-length distance in the 1-skeleton of $\mathcal{S}(M, \phi)$ between their isotopy classes. From the results in Section 2.3 we easily deduce the following corollary:

**Corollary 2.6.2** (Connectedness of $\mathcal{S}(M, \phi)$)**.** *Let $M$ be an oriented compact smooth $n$-manifold and let $\phi \in \mathrm{H}_{n-1}(M, \partial M)$. Then $\mathcal{S}(M, \phi)$ is connected. Moreover, if $S_0, S_1$ are transverse representatives of two vertices of $\mathcal{S}(M, \phi)$ and $k \in \mathbb{N}$ satisfies $|S_0 \cap S_1| < 2^k$, then $d_{\mathcal{S}}(S_0, S_1) \leq 2^k$.*

*Proof.* Since $p\colon \mathcal{S}^\dagger(M, \phi) \to \mathcal{S}(M, \phi)$ is surjective on vertices, Theorem 2.3.3 immediately implies the first part. The second is a direct consequence of Proposition 2.3.4. □

Similarly, for the 3-dimensional setting, we consider the following simplicial complex:

**Definition 2.6.3.** Let $M$ be an irreducible and boundary-irreducible 3-manifold and let $\phi \in \mathrm{H}_2(M, \partial M)$. We define $\mathcal{T}(M, \phi)$ to be the full subcomplex of $\mathcal{S}(M, \phi)$ spanned by the isotopy classes of Thurston norm-realizing surfaces.

Let $d_{\mathcal{T}}(S_0, S_1)$ denote the distance between the classes of surfaces $S_0, S_1$ in the 1-skeleton of $\mathcal{T}(M, \phi)$. Restricting $p$ to $\mathcal{T}^\dagger(M, \phi)$ yields a surjective map $\mathcal{T}^\dagger(M, \phi) \to \mathcal{T}(M, \phi)$, and we have:

**Corollary 2.6.4** (Connectedness of $\mathcal{T}(M, \phi)$)**.** *Let $M$ be an irreducible and boundary-irreducible oriented compact smooth 3-manifold, and let $\phi \in \mathrm{H}_2(M, \partial M)$. Then $\mathcal{T}(M, \phi)$ is connected. Moreover, if $S_0, S_1$ are transverse representatives of vertices of $\mathcal{T}(M, \phi)$ and $k \in \mathbb{N}$ satisfies $\mathrm{ess}(S_0, S_1) < 2^k$, then $d_{\mathcal{T}}(S_0, S_1) \leq 2^k$.*

We remind the reader that by Lemma 2.4.4, the $S_i$ are incompressible and thus each component of $S_0 \cap S_1$ is essential either in both or in neither of them. The quantity $\mathrm{ess}(S_0, S_1)$ denotes the number of essential such components.

*Proof.* Surjectivity of the map $\mathcal{T}^\dagger(M, \phi) \to \mathcal{T}(M, \phi)$ and the fact that $\mathcal{T}^\dagger(M, \phi)$ is connected (Theorem 2.4.6) immediately imply the first part. For the distance bound: by an iterated application of Lemma 2.4.8, we may assume $S_0$ and $S_1$ intersect only essentially (without changing $\mathrm{ess}(S_0, S_1)$). After this reduction, we have $|S_0 \cap S_1| = \mathrm{ess}(S_0, S_1) < 2^k$ and the statement follows from Proposition 2.4.7. ☐

We now turn to the question of whether simple connectedness of $\mathcal{S}^\dagger(M, \phi)$, established as Theorem 2.5.1, descends to $\mathcal{S}(M, \phi)$. Answering this question will require additional topological input, and we obtain an answer only when $M$ has dimension 2.

**Corollary 2.6.5** (Simple connectedness in the case of surfaces)**.** *Let $M$ be an oriented compact smooth surface and let $\phi \in \mathrm{H}_1(M, \partial M)$. Then $\mathcal{S}(M, \phi)$ is simply connected.*

The remainder of this section is dedicated to proving Corollary 2.6.5. Throughout, $M$ will denote an oriented compact smooth surface, and $\phi \in \mathrm{H}_1(M, \partial M)$ a 1-dimensional homology class.

The key to transferring simple connectedness of $\mathcal{S}^\dagger(M, \phi)$ to $\mathcal{S}(M, \phi)$ is the following assertion.

**Lemma 2.6.6** (Paths within an isotopy class)**.** *Let $S, T$ be a transverse pair of oriented properly embedded 1-dimensional submanifolds of $M$. If $S$ and $T$ are in the same proper isotopy class, then there exists a sequence*

$$S = S_0, S_1, \ldots, S_k = T$$

*of oriented properly embedded 1-dimensional submanifolds of $M$ all in the same proper isotopy class, such that for each $i \in \{1, \ldots, k\}$, we have $S_{i-1} \cap S_i = \emptyset$.*

*Moreover, if $S$ and $T$ are part of a transverse family $\mathcal{U}$ of properly embedded 1-dimensional submanifolds of $M$, then the $S_i$ can be chosen so that $\mathcal{U} \cup \{S_1, \ldots, S_{k-1}\}$ is transverse.*

Our proof of Lemma 2.6.6 relies on the bigon criterion (Theorem 2.6.8 below), a tool available specifically for manifolds of dimension 2.

**Definition 2.6.7.** Let $M$ be a compact smooth surface, and $\sigma, \tau$ a transverse pair of properly embedded connected 1-submanifolds of $M$ (so each of $\sigma, \tau$ is either a circle or an arc).

- The submanifolds $\sigma, \tau$ are said to be in **minimal position** if they cannot be properly isotoped to submanifolds $\sigma', \tau'$, respectively, such that $|\sigma' \cap \tau'| < |\sigma \cap \tau|$.
- We say that $\sigma, \tau$ form a **bigon** (Figure 2.6.1, left) if there exist two distinct points $p, q \in \sigma \cap \tau$ and arcs $\alpha \subset \sigma$, $\beta \subset \tau$ connecting $p$ and $q$, such that $\alpha \cup \beta$ is a circle (with corners) bounding a disc in $M$.

- We say that $\sigma, \tau$ form a **half-bigon** (Figure 2.6.1, right) if there exist
  - points $p \in \sigma \cap \tau$, $q_\sigma \in \sigma \cap \partial M$, $q_\tau \in \tau \cap \partial M$,
  - an arc $\alpha \subset \sigma$ from $p$ to $q_\sigma$,
  - an arc $\beta \subset \tau$ from $p$ to $q_\tau$, and
  - an arc $\gamma \subset \partial M$ from $q_\sigma$ to $q_\tau$,

  such that $\alpha \cup \beta \cup \gamma$ is a circle (with corners) bounding a disc in $M$.

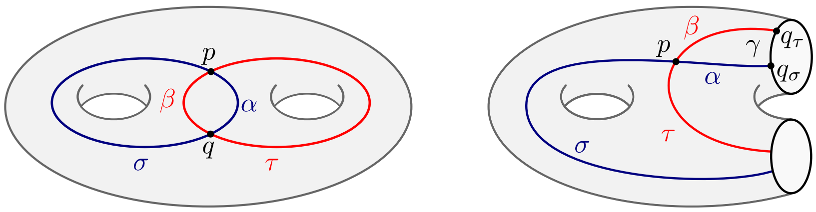



Figure 2.6.1: Example of a bigon (left) and a half-bigon (right).

If $\sigma$ and $\tau$ form a bigon or a half-bigon, then they are certainly not in minimal position: indeed, after choosing a (half-)bigon that is innermost (meaning, for which the disc in the definition is innermost), we may use it to isotope a small neighborhood of the arc $\alpha$ past $\beta$, and then push everything slightly off of $\sigma$ in the direction away from the bigon. This produces a new 1-submanifold $\sigma'$ properly isotopic to $\sigma$ and having fewer intersections with $\tau$ (Figure 2.6.2). The bigon criterion is a converse to this statement.

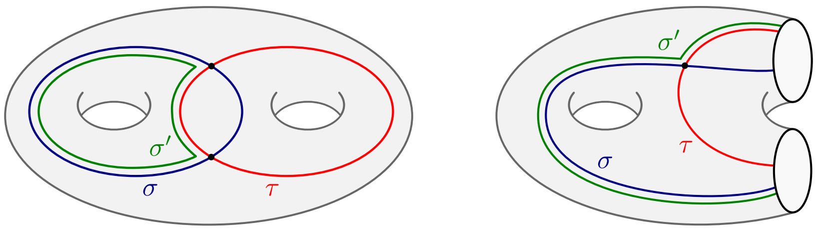



Figure 2.6.2: If two curves $\sigma, \tau$ form a bigon, one can produce a new curve $\sigma'$ that is isotopic to $\sigma$ and has fewer intersections with $\tau$ (left). Similarly for a half-bigon (right).

**Theorem 2.6.8** (The Bigon Criterion)**.** *Let $M$ be an oriented compact smooth surface, and let $\sigma, \tau$ be a transverse pair of properly embedded connected 1-submanifolds of $M$. If $\sigma$ and $\tau$ are not in minimal position, then they form a bigon or a half-bigon.*

For a proof of the bigon criterion in the closed case, see the book of Farb and Margalit [FM12, Proposition 1.6]. The same argument can be adapted to the case of surfaces with boundary.

*Proof of Lemma 2.6.6.* Suppose $\sigma, \tau$ are components of $S, T$, respectively, that have non-empty intersection. We first note that $\sigma, \tau$ are not in minimal position: indeed, if we properly isotope $S$ to $T$ and then push it slightly along the positive direction of the normal bundle of $T$, then we will have isotoped $S$ to be disjoint from $T$, and thus also $\sigma$ to be disjoint from $\tau$. Hence, by the bigon criterion, $\sigma, \tau$ form a (half-)bigon.

By the Diffeotopy Extension Theorem [Wal68, Theorem 2.4.6], the isotopy from $\sigma$ to $\sigma'$ illustrated in Figure 2.6.2 can then be extended to an ambient isotopy of $M$ supported in a small neighborhood of the union of $\sigma$ and the (half-)bigon. This ambient isotopy restricts to a proper isotopy from $S$ to a proper 1-submanifold $S_1$ disjoint from $S$. This submanifold $S_1$ is transverse to $T$ and satisfies $|S_1 \cap T| < |S \cap T|$. Applying Proposition 2.2.2, we may if necessary perturb $S_1$ to make $\mathcal{U} \cup \{S_1\}$ a transverse set. Since $S_1$ is transverse to $T$, a small enough perturbation will not change the topology of $S_1 \cap T$, and in particular no new intersections between $S_1$ and $T$ are formed.

This procedure can be iterated until all intersections are removed. $\square$

With this extra topological input, the problem of showing that finite 1-subcomplexes of $\mathcal{S}(M, \phi)$ are null-homotopic can be "lifted along $p$ to $\mathcal{S}^\dagger(M, \phi)$":

**Lemma 2.6.9** (Lifting 1-subcomplexes)**.** *Let $M$ be an oriented compact smooth surface and let $\phi \in \mathrm{H}_1(M, \partial M)$. Any finite 1-subcomplex $P$ of $\mathcal{S}(M, \phi)$ is the $p$-image of a transverse 1-subcomplex $\widetilde{P}$ of $\widetilde{\mathcal{S}}(M, \phi)$ such that the restriction $p|_{\widetilde{P}} \colon \widetilde{P} \to P$ is a homotopy equivalence.*

*Proof.* Since $p$ is surjective on simplices, each edge of $P$ may be lifted to an edge in $\mathcal{S}^\dagger(M, \phi)$ (but adjacent edges of $P$ might not lift to adjacent edges of $\mathcal{S}^\dagger(M, \phi)$). Moreover, if we lift one edge at a time and always apply Proposition 2.2.2 to the vertices in the lifts, we can guarantee that the union of the lifted edges is a transverse subcomplex of $\mathcal{S}^\dagger(M, \phi)$.

Whenever two edges of $P$ share a vertex that is lifted to two distinct vertices of $\mathcal{S}^\dagger(M, \phi)$, Lemma 2.6.6 provides a path contained in $p^{-1}(v)$ that joins them. Applying this lemma enough times (and always keeping everything transverse), we can construct, for each vertex $v$ of $P$, a tree in $p^{-1}(v)$ connecting the various lifts of $v$. Take $\widetilde{P}$ to be the finite 1-subcomplex of $\mathcal{S}^\dagger(M, \phi)$ comprised of the edge lifts and these trees (Figure 2.6.3). Since $p$ acts on $\widetilde{P}$ by collapsing each tree to a point, we conclude $p|_{\widetilde{P}}$ is a homotopy equivalence. $\square$

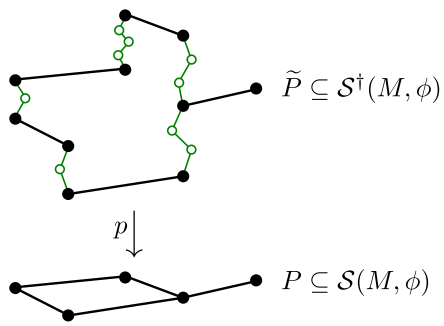



Figure 2.6.3: Lifting a finite 1-subcomplex $P$ in $\mathcal{S}(M, \phi)$ to $\mathcal{S}^\dagger(M, \phi)$. We first lift edges using surjectivity of $p$ (solid dots, thick edges), and then use Lemma 2.6.6 to construct trees (hollow dots, thin edges) connecting distinct pre-images of all vertices.

*Proof of Corollary 2.6.5.* We will show that the inclusion $P \hookrightarrow \mathcal{S}(M, \phi)$ of every finite 1-subcomplex $P$ is null-homotopic. By Lemma 2.6.9, there is a transverse 1-subcomplex $\widetilde{P} \subset \mathcal{S}^\dagger(M, \phi)$ making the following diagram commute:

$$\begin{tikzcd} \widetilde{P} \arrow[r, hook] \arrow[d, "p|_{\widetilde{P}}"] & \mathcal{S}^\dagger(M, \phi) \arrow[d, "p"] \\ P \arrow[r, hook] & \mathcal{S}(M, \phi) \end{tikzcd},$$

and for which the restriction $p|_{\widetilde{P}}$ is a homotopy equivalence.

Let $g\colon P \to \widetilde{P}$ be a homotopy inverse (at the level of topological realizations), and consider its induced map on cones $\mathrm{C}g\colon \mathrm{C}P \to \mathrm{C}\widetilde{P}$. By Proposition 2.5.4, at the level of topological realizations of simplices, the inclusion $\widetilde{P} \hookrightarrow \mathcal{S}^\dagger(M,\phi)$ factors as

$$\widetilde{P} \hookrightarrow \mathrm{C}\widetilde{P} \xrightarrow{h} \mathcal{S}^\dagger(M,\phi),$$

for some continuous map $h$. Hence, the following diagram (where all simplicial complexes are to be understood as their topological realizations) commutes up to homotopy:

$$\begin{tikzcd}
\widetilde{P} \arrow[r, hook] \arrow[rr, hook, bend left] & \mathrm{C}\widetilde{P} \arrow[r, "h"] & \mathcal{S}^\dagger(M,\phi) \arrow[d, "p"] \\
P \arrow[u, "g"] \arrow[r, hook] \arrow[rr, hook, bend right] & \mathrm{C}P \arrow[u, "\mathrm{C}g"'] & \mathcal{S}(M,\phi)
\end{tikzcd}$$

This shows that the inclusion $P \hookrightarrow \mathcal{S}(M,\phi)$ is homotopic to a map factoring through $P \hookrightarrow \mathrm{C}P$, and thus it is null-homotopic. □

We finish this section by briefly commenting on the only obstacle to extending Corollary 2.6.5 to manifolds $M$ of higher dimension. It all boils down to proving Lemma 2.6.6 in higher dimensions. In other words, one would need an affirmative answer to the following question:

**Question 2.6.10.** Let $S, T$ be a transverse pair of oriented properly embedded hypersurfaces in an oriented compact smooth manifold $M$, and suppose $S, T$ are properly isotopic. Does there exist a sequence

$$S = S_0, S_1, \ldots, S_k = T$$

of oriented properly embedded hypersurfaces in $M$, all in the same proper isotopy class, such that $S_{i-1} \cap S_i = \emptyset$ for each $i \in \{1, \ldots, k\}$?

Our proof in the 2-dimensional case relies on the bigon criterion (Theorem 2.6.8), whose straightforward generalization to dimensions greater than 2 is easily seen to be false. We have however not been able to use counterexamples to give a negative answer to Question 2.6.10. If such counterexamples do exist, we expect that answering this question would be a difficult task, since one would presumably need an invariant that distinguishes between codimension-1 submanifolds in the same isotopy class.

## 2.7 Co-oriented regular graphs in 2-complexes

We have been dealing with hypersurfaces representing a fixed codimension-1 homology class, but through Poincaré Duality one may as well think of them as representing a 1-dimensional cohomology class. Turaev has described a way of representing 1-dimensional cohomology classes $\phi \in \mathrm{H}^1(X, \partial X)$ in certain 2-dimensional CW-complexes $X$ (relative to their "boundary subspaces" $\partial X$) by embedded graphs $\Gamma$ satisfying some regularity conditions [Tur02]. In this short section, we briefly comment on extensions of the previous results in this chapter to that context.

More concretely, Turaev treats finite CW-complexes $X$ of dimension 2 that are locally homeomorphic to a cone over a graph, and the embedded graphs $\Gamma$ are required to have closed tubular neighborhoods $U \cong \Gamma \times [-1,1]$ disjoint from $\partial X$ (see Sections 1.1 and 1.2

of his paper for details). Such an embedded graph $\Gamma$ is called **regular**. A regular graph $\Gamma$, together with the choice of a component of $U \setminus \Gamma$ near each component of $\Gamma$ (called a **co-orientation**), determines a continuous map $(X, \partial X) \to (\mathbb{S}^1, \{-1\})$, and thus an element in $\mathrm{H}^1(X, \partial X)$ (the construction of this map is detailed in Turaev's Section 1.2). Conversely, Turaev shows that all cohomology classes can be obtained in this fashion.

To begin, we ponder whether any two co-oriented regular graphs for a fixed element $\phi \in \mathrm{H}^1(X, \partial X)$ can be connected through sequentially disjoint graphs – in other words, if the simplicial complex $\mathcal{S}^\dagger(X, \phi)$ (whose definition is the straightforward adaptation of Definition 2.3.2) is connected. And indeed, all main steps in the proof of Proposition 2.3.4 can be translated to this setting, provided one uses the appropriate notion of transversality – see the proof of Lemma 1.4 in Turaev's article. The co-orientations play the role of the framings of hypersurfaces, and in fact, an analogue of the oriented sum construction is briefly described therein. The fact that any two vertices $\Gamma_0, \Gamma_1$ of $\mathcal{S}^\dagger(X, \phi)$ are connected by a path then follows as in Theorem 2.3.3 by taking a disjoint parallel copy of $\Gamma_1$ and perturbing it to be transverse to $\Gamma_0$ (Turaev uses a similar argument in proving his Lemma 1.4).

Regular co-oriented graphs are then used to define a norm on $\mathrm{H}^1(X, \partial X; \mathbb{R})$ in much the same way one uses surfaces to define the Thurston norm on $\mathrm{H}_2(M, \partial M; \mathbb{R})$, for $M$ an irreducible and boundary-irreducible oriented compact smooth 3-manifold. Explicitly, for a regular graph $\Gamma \subset X$, he writes $\chi_-(\Gamma) := -\chi(\Gamma)$, defines the norm $\|\phi\|_X$ of a class $\phi \in \mathrm{H}^1(X, \partial X)$ to be the minimal $\chi_-(\Gamma)$ over regular graphs $\Gamma$ representing $\phi$, and then extends $\|\cdot\|_X$ to $\mathrm{H}^1(X, \partial X; \mathbb{R})$. The regularity assumption on $\Gamma$ implies that all its components have non-positive Euler-characteristic, making the definition of $\chi_-$ slightly less cumbersome than in the 3-manifold case. We call $\|\cdot\|_X$ the **Turaev norm**.

As in Section 2.4, we can again consider the subcomplex of $\mathcal{S}^\dagger(X, \phi)$ spanned by regular graphs that realize the Turaev norm of $\phi$, and for which no union of components represents the zero class. A happy coincidence dictates we also denote this complex by $\mathcal{T}^\dagger(X, \phi)$. When adapted to this setting, the proof of Proposition 2.4.7 not only carries over, but actually becomes simpler: we no longer need to worry about ensuring that the oriented surgery produces no components in $\Sigma_0$ of positive Euler characteristic, since all regular graphs satisfy $\chi \leq 0$ [Tur02, p. 139]. Concretely, this makes the step where we used Observation 2 become immediate. Moreover, it precludes the need for an analogue of Lemma 2.4.8 in adapting the proof of Theorem 2.4.6 to show that $\mathcal{T}^\dagger(X, \phi)$ is connected.

As our proof of Theorem 2.5.1 requires no additional topological input, only a more methodical approach to counting intersections between oriented surgeries, Theorem 2.5.1 can be adapted to show $\mathcal{S}^\dagger(X, \phi)$ is simply connected. In fact, as we are in the 2-dimensional setting, transversality rules out triple points, obviating the need for an analogue of Proposition 2.5.6.

Regarding the results in Section 2.6, analogues of Corollaries 2.6.2 and 2.6.4 should also hold for the simplicial complexes $\mathcal{S}(X, \phi)$ and $\mathcal{T}(X, \phi)$, respectively, where regular graphs are taken up to isotopy, with essentially the same arguments. We are however aware of no replacement for the bigon criterion for 2-complexes that would yield an analogue of Lemma 2.6.9. This seems to be the only obstacle in adapting the proof of Theorem 2.6.5 to show that $\mathcal{S}(X, \phi)$ is simply connected.

# Chapter 3

# Stable integral simplicial volume of $3$-manifolds

This shorter chapter presents the proof given in joint work with Daniel Fauser, Clara Löh, and Marco Moraschini [Fau+21] that oriented, compact, connected non-elliptic, prime 3-manifolds with (possibly empty) toroidal boundary satisfy integral approximation of simplicial volume (Theorem 1 in the article, stated as Theorem 3.2.4 below). Rather than presenting the argument in full detail, we aim at giving a bird's eye view of the proof, contextualizing it within an introduction to simplicial volume, with a section dedicated to each of the main tools used, highlighting its role in the overall program. In order to keep the exposition concise we will omit or sketch some of the more technically involved arguments.

## Outline of this chapter

Section 3.1 presents the classical notion of simplicial volume of an oriented manifold, and its integral counterpart, along with some well-known properties. In Section 3.2, we introduce the "stable integral simplicial volume", whose definition is closer to integral simplicial volume, but whose behavior with respect to finite coverings mirrors that of classical simplicial volume. This prompts the question of when stable integral simplicial volume matches simplicial volume, and leads to the statement of the main result of this chapter, Theorem 3.2.4.

The strategy for proving Theorem 3.2.4 relies on the JSJ decomposition of irreducible 3-manifolds. In Section 3.3 we take a detour to give an overview of prime and JSJ decompositions of 3-manifolds.

In Section 3.4, we recall a family of variants of simplicial volume taking as additional input a "parameter space" borrowed from ergodic theory. We also define the infimum over all parameter spaces, the "integral foliated simplicial volume". A further "boundary-controlled" refinement is introduced in Section 3.5, allowing us to establish additivity properties with respect to gluing manifolds along toroidal boundary components – the relevant use case for us being the JSJ pieces.

The boundary-controlled integral foliated simplicial volume of hyperbolic JSJ pieces will be computed by looking at their interiors, and this requires generalizing the notion of simplicial volume to open manifolds. This is done in Section 3.6. The integral foliated simplicial volumes of open hyperbolic 3-manifolds of finite volume are then computed yielding a new proportionality principle for open manifolds, which we discuss in Section 3.7.

In Section 3.8, we discuss profinite completions of 3-manifold groups, and how they provide a parameter space whose corresponding simplicial volume serves as a proxy for stable integral simplicial volume. Using additional input from ergodic group theory, we see in Section 3.9 that for the hyperbolic JSJ pieces of an oriented compact connected 3-manifold, this parameter space realizes the integral foliated simplicial volume.

At this point, all pieces are in place for assembling the proof of Theorem 3.2.4, which we do in Section 3.10. In Section 3.11 we discuss some of its implications to homology growth along towers of finite covers.

## 3.1 Classical notions of simplicial volume

If $M$ is an oriented closed connected manifold of dimension 2, the Euler characteristic $\chi(M)$ is as good an integer-valued invariant as one could hope for – it fully classifies such surfaces. The situation is of course drastically different when one moves up to dimension 3 (or indeed any odd dimension), where all closed manifolds have vanishing Euler characteristic. However, Gromov introduced a real-valued invariant for oriented closed connected manifolds containing as much information as the Euler characteristic in dimension 2, but also sensitive to the topology of 3-manifolds – the simplicial volume [Thu80, Chapter 6] [Gro82].

Given a topological space $X$ and a subspace $X_0 \subseteq X$, denote by $|\cdot|_1$ the $\ell^1$-norms on each vector space $\mathrm{C}_k(X, X_0; \mathbb{R})$ of real $k$-chains with respect to the basis given by the singular $k$-simplices with image not contained in $X_0$. We use the same notation for the induced semi-norm on the homology $\mathbb{R}$-vector spaces $\mathrm{H}_k(X, X_0; \mathbb{R})$.

**Definition 3.1.1.** If $M$ is an oriented compact $n$-manifold and $[M]_\mathbb{R} \in \mathrm{H}_n(M, \partial M; \mathbb{R})$ is its real fundamental class, then the **simplicial volume** of $M$ is $\|M\| := |[M]_\mathbb{R}|_1$. More explicitly,

$$\|M\| = \inf\{|c|_1 \in \mathbb{R}_{>0} \mid c \in \mathrm{C}_n(M, \partial M; \mathbb{R}) \text{ is a real } n\text{-cycle representing } [M]_\mathbb{R}\}.$$

This definition can be generalized to non-compact manifolds using locally finite homology. This will be discussed in Section 3.6.

Intuitively, simplicial volume can be interpreted as an answer to the question "What is the smallest amount of simplices needed to construct $M$?". This imagery is even more vivid when one considers the following variant of simplicial volume, defined in terms of *integral* coefficients:

**Definition 3.1.2.** Let $M$ be an oriented compact $n$-manifold. If $[M] \in \mathrm{H}_n(M, \partial M)$ is its (integral) fundamental class, then the **integral simplicial volume** of $M$ is

$$\|M\|_\mathbb{Z} := \min\{|c|_1 \in \mathbb{Z}_{>0} \mid c \in \mathrm{C}_n(M, \partial M) \text{ is an integral } n\text{-cycle representing } [M]\}.$$

Since any integral fundamental cycle can be interpreted as a real fundamental cycle of the same $\ell^1$-norm, the definitions directly imply that one always has $\|M\| \leq \|M\|_\mathbb{Z}$.

Even if its definition might seem more artificial, classical simplicial volume is a much more flexible tool than its integral counterpart. For example, its behavior with respect to maps between manifolds is more predictable:

**Proposition 3.1.3** (Simplicial volume respects mapping degrees)**.** *If $f\colon N \to M$ is a continuous map of degree $d$ between oriented compact connected $n$-manifolds, then $|d| \cdot \|M\| \leq \|N\|$. If $f$ is a covering map, then equality holds.*

*Proof.* This is vacuously true for $d = 0$. Otherwise, note that each real fundamental cycle $c$ for $N$ is pushed forward by $f$ to a cycle $f_\#(c)$ representing the class $d[M]_\mathbb{R}$ with at most the same norm as $c$. Hence, for $c$ of norm $\|N\| + |d| \cdot \epsilon$ (with $\epsilon \geq 0$), the real fundamental cycle $\frac{1}{d}f_\#(c)$ for $M$ has norm bounded above by $\frac{1}{|d|}\|N\| + \epsilon$. Thus $\|M\| \leq \frac{1}{|d|}\|N\|$.

In case $f$ is a covering map, one shows the converse inequality by considering the "transfer" chain map

$$\begin{aligned} \tau \colon \mathrm{C}_*(M, \partial M; \mathbb{R}) &\to \mathrm{C}_*(N, \partial N; \mathbb{R}) \\ \sigma &\mapsto \sum_{\widetilde{\sigma} \text{ lift of } \sigma} \widetilde{\sigma}. \end{aligned}$$

Now, $\tau$ maps each real fundamental cycle $c$ of $M$, or its negative $-c$, to a fundamental cycle of $N$ with norm at most $|d| \cdot |c|_1$. By choosing $c$ of norm $\|M\| + \frac{\epsilon}{|d|}$, we get a fundamental cycle for $N$ with norm not exceeding $|d| \cdot \|M\| + \epsilon$. Hence $\|N\| \leq |d| \cdot \|M\|$. □

Note that the same transfer principle can also be applied to the integral simplicial volume, so if $N$ is a $d$-sheeted cover of $M$, we have $\|N\|_\mathbb{Z} \leq d \cdot \|M\|_\mathbb{Z}$.

Whereas it is obvious that the integral simplicial volume of a non-empty oriented closed connected manifold is always at least 1, its (classical) simplicial volume may very well vanish. Proposition 3.1.3 immediately implies that this happens, for example, whenever $M$ admits a self-map of degree $d$ with $|d| \geq 2$, which is the case for spheres of positive dimension and tori. For manifolds possessing a hyperbolic structure, however, simplicial volume is remarkably sensitive, owing to the following property, due to Thurston [Thu80, Theorem 6.2.2] and Gromov [Gro82, Section 0.4]; see Löh's diploma thesis [Löh05, Chapter 5] for a detailed exposition of Thurston's proof.

**Theorem 3.1.4** (Proportionality principle). *Let $M, N$ be non-empty oriented closed connected Riemannian manifolds with isometric universal covers. Then*

$$\frac{\|M\|}{\mathrm{vol}(M)} = \frac{\|N\|}{\mathrm{vol}(N)}.$$

In the case of closed hyperbolic surfaces $S$, it is known that $\|S\| = -2\chi(S)$ [Gro82, p. 9], so by the Gauß-Bonnet Theorem, which tells us that $\mathrm{vol}(S) = -2\pi\chi(S)$, the above ratio is $\frac{1}{\pi}$, with $\pi$ being the area of the ideal hyperbolic triangle. Note that in this case, simplicial volume contains as much information as the Euler characteristic.

For closed hyperbolic 3-manifolds, the ratio is $\frac{1}{\nu_3}$, with $\nu_3$ the volume of the regular ideal tetrahedron, and a similar statement holds for higher dimensions [Gro82, pp.11–12]. This shows in particular that the Riemannian volume of an oriented closed hyperbolic manifold is a homotopy invariant, a shadow of the stronger Mostow's Rigidity Theorem [Thu80, Theorem 5.7.2]. By Poincaré Duality, oriented closed manifolds of odd dimension have vanishing Euler characteristic, so in this case simplicial volume is a much more interesting invariant. Gromov has asked whether, for oriented closed connected aspherical manifolds, vanishing of simplicial volume implies vanishing of the Euler characteristic, and this question remains open [Gro93, Section 8.$\mathrm{A}_4$]. See the recent preprint of Löh, Moraschini and Raptis for an in-depth discussion of this problem [LMR21].

## 3.2 Integral approximation

There is an intermediate variant of simplicial volume, still defined in terms of integral fundamental classes, but enjoying nicer properties with respect to finite coverings:

**Definition 3.2.1.** The **stable integral simplicial volume** of an oriented compact connected manifold $M$ is defined as

$$\|M\|_{\mathbb{Z}}^{\infty} := \inf \left\{ \frac{\|N\|_{\mathbb{Z}}}{d} \in \mathbb{Q}_{>0} \,\middle|\, N \text{ is a } d\text{-sheeted covering of } M \right\}.$$

It follows directly from the definition that the stable integral simplicial volume behaves similarly to simplicial volume with respect to covering maps of finite index:

**Proposition 3.2.2** (Stable integral simplicial volume respects index of coverings)**.** *If $N \to M$ is a covering map of finite index $d$ between oriented closed connected manifolds, then $d \cdot \|M\|_{\mathbb{Z}}^{\infty} = \|N\|_{\mathbb{Z}}^{\infty}$.*

Stable integral simplicial volume does indeed fit between simplicial volume and integral simplicial volume:

$$\|M\| \leq \|M\|_{\mathbb{Z}}^{\infty} \leq \|M\|_{\mathbb{Z}},$$

the first inequality following from the fact that a $d$-sheeted covering map has degree $\pm d$, and the second by considering the trivial covering of $M$ by itself. Moreover, either inequality can be strict or an equality. For example, since spheres $\mathbb{S}^n$ of dimension $n \geq 2$ have no non-trivial coverings, their stable integral simplicial volume matches the integral simplicial volume (and we have already seen their simplicial volume vanishes): $\|\mathbb{S}^n\| < \|\mathbb{S}^n\|_{\mathbb{Z}}^{\infty} = \|\mathbb{S}^n\|_{\mathbb{Z}}$. On the other hand, manifolds $M$ that have self-coverings of index at least 2, such as tori, have vanishing stable integral simplicial volume and positive integral simplicial volume, so $\|M\| = \|M\|_{\mathbb{Z}}^{\infty} < \|M\|_{\mathbb{Z}}$. This chapter is devoted to the question of what 3-manifolds satisfy this equality.

**Definition 3.2.3.** An oriented compact connected manifold $M$ is said to satisfy **integral approximation of simplicial volume** if its stable integral simplicial volume agrees with the classical simplicial volume: $\|M\|_{\mathbb{Z}}^{\infty} = \|M\|$.

Classical arguments [Gro82, p. 9] show that for oriented closed connected surfaces $S$ of negative Euler characteristic, we have $\|S\|_{\mathbb{Z}}^{\infty} \leq 2\,|\chi(S)| \leq \|S\|$. By the previous discussion these must actually be equalities, and so we conclude that all oriented closed connected surfaces of positive genus satisfy integral approximation.

In this chapter, we discuss the following theorem:

**Theorem 3.2.4** (Integral approximation for non-elliptic prime 3-manifolds [Fau+21, Theorem 1])**.** *Let $M$ be an oriented compact connected 3-manifold with toroidal boundary. If $M$ is prime and not covered by $\mathbb{S}^3$, then $M$ satisfies integral approximation of simplicial volume.*

The value of $\|M\| = \|M\|_{\mathbb{Z}}^{\infty}$ has a convenient expression in terms of the JSJ decomposition, which we present later (Proposition 3.3.9).

Here, the "primality" condition means that $M$ admits no non-trivial decomposition as a connected sum; see Definition 3.3.1 below. And it is an essential hypothesis: by taking a connected sum of any oriented closed connected hyperbolic 3-manifold with sufficiently-many 3-tori, one obtains 3-manifolds not satisfying integral approximation [Fau+21, Corollary 3].

Theorem 3.2.4 had already been established for some classes of oriented closed connected 3-manifolds. In particular, the following partial results were known, and are used in its proof.

**Theorem 3.2.5** (Integral approximation for closed hyperbolic 3-manifolds [Fri+16, Theorem 1.7])**.** *Every oriented closed connected hyperbolic 3-manifold $M$ satisfies integral approximation. In other words,*

$$\|M\|_{\mathbb{Z}}^{\infty} = \frac{\mathrm{vol}(M)}{\nu_3}.$$

The situation is drastically different in higher dimensions: for every closed hyperbolic manifold of dimension at least 4, we have a strict inequality $\|M\| < \|M\|_{\mathbb{Z}}^{\infty}$; see the paper of Francaviglia, Frigerio and Martelli for a more refined statement [FFM12, Theorem 0.5].

**Theorem 3.2.6** (Integral approximation for Seifert-fibered 3-manifolds [LP16, Proposition 8.1])**.** *If $M$ is an oriented compact connected Seifert-fibered 3-manifold with infinite fundamental group, then $\|M\|_{\mathbb{Z}}^{\infty} = 0$.*

This vanishing result has been generalized to manifolds in every dimension with residually finite fundamental group (see Definition 3.8.2 below) that are obtained from manifolds with "tame $\mathbb{S}^1$-structures" by gluing along boundary tori [FFL19]. Of course since the stable integral simplicial volume is an upper bound for classical simplicial volume, we also recover, in such contexts, the already known fact that $\|M\| = 0$ [Thu80, Corollary 6.5.3].

The relevance of hyperbolic and Seifert-fibered manifolds in 3-manifold topology, and specifically to our program, will be made apparent in the next section.

## 3.3 Decompositions of 3-manifolds

There is a classical theory of decomposition of compact 3-manifolds along 2-spheres and tori into pieces that are either Seifert-fibered or whose interior admits a complete hyperbolic metric of finite volume. With this in mind, one might attempt to assemble statements about integral approximation for hyperbolic and for Seifert-fibered manifolds, such as Theorems 3.2.5 and 3.2.6, into statements for more general classes. That is indeed a guiding principle for the proof of Theorem 3.2.4, so in this section we give a quick overview of these decomposition results.

Typically, the highest level decomposition one considers when studying 3-manifolds is the prime decomposition, also sometimes called the Kneser-Milnor decomposition. We will not make use of it, but present its statement nevertheless for the sake of completeness; see for example the books of Aschenbrenner-Friedl-Wilton [AFW15, Section 1.2] or Martelli [Mar16, Section 9.2] for a more detailed discussion with further references.

**Definition 3.3.1.** A non-empty oriented 3-manifold $M$ is called **prime** if it is not homeomorphic to a 3-sphere, and for every decomposition as a connected sum $M \cong M_1 \# M_2$, one of the $M_i$ is homeomorphic to a 3-sphere.

It is important to contrast this notion with that of an **irreducible** 3-manifold, that is, one where every embedded 2-sphere bounds a 3-ball. Whereas every irreducible 3-manifold besides the 3-sphere is easily seen to be prime, there is one exception to the converse implication: an oriented compact connected 3-manifold that is prime but not irreducible is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$ [Hem87, Lemma 3.13].

**Theorem 3.3.2** (Prime Decomposition)**.** *Every non-empty oriented compact connected 3-manifold $M \ncong \mathbb{S}^3$ without spherical boundary components admits a connected sum decomposition*

$$M \cong M_1 \# \dots \# M_m$$

*into prime pieces $M_1, \dots, M_m$. The number $m$ of summands and their oriented homeomorphism types are unique.*

After one has cut a 3-manifold along spheres into its prime summands, the next step is to cut it along tori. This "JSJ decomposition", named after Jaco, Shalen and Johannson, is a central ingredient of the proof of Theorem 3.2.4. Again, the reader is pointed to

Aschenbrenner-Friedl-Wilton's compendium [AFW15, Section 1.6] or Martelli's book [Mar16, Section 11.5] for a more detailed exposition with further references and historical context.

The pieces in the JSJ decomposition come in two (possibly overlapping) types – they are either atoroidal or Seifert-fibered:

**Definition 3.3.3.** A 3-manifold $M$ is **atoroidal** if every $\pi_1$-injective map $\mathbb{S}^1 \times \mathbb{S}^1 \to M$ can be homotoped to $\partial M$.

**Seifert-fibered** manifolds are 3-manifolds admitting a certain type of foliation into circles, but despite them being fully classified, the precise definition is somewhat intricate. As we will not make explicit use of it, we point the interested reader to other sources [AFW15, Section 1.5] [Mar16, Chapter 10].

**Theorem 3.3.4** (JSJ Decomposition)**.** *Let $M$ be an oriented compact connected 3-manifold with toroidal boundary. If $M$ is irreducible, then there is a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \dots, T_m \subset M$ such that each of the pieces obtained by cutting $M$ along the union $\bigcup_{i=1}^m T_i$ is atoroidal or Seifert-fibered. Up to isotopy, there is a unique such collection of tori with minimal number of components.*

The JSJ decomposition theorem is traditionally stated for irreducible 3-manifolds, but of course it also holds for $\mathbb{S}^1 \times \mathbb{S}^2$, which is both atoroidal and Seifert-fibered.

It will be useful to note the following fact.

**Lemma 3.3.5** (JSJ pieces are $\pi_1$-injective)**.** *if $M$ is an irreducible oriented compact connected 3-manifold with toroidal boundary, then for each JSJ piece $N$ of $M$, the canonical map $N \to M$ is $\pi_1$-injective.*

*Proof.* More generally, if $M_0$ is the result of cutting $M$ along any incompressible two-sided embedded surface $S \subset M$, then the canonical map $M_0 \to M$ is $\pi_1$-injective in each of the (one or two) components of $M_0$: indeed, the fact that $S$ is incompressible in $M$ implies that both components $S_1, S_2 \subseteq \partial M_0$ resulting from $S$ are incompressible in $M_0$. By the Loop Theorem [AFW15, Theorem 1.3.1], it follows that both inclusions $S_1, S_2 \hookrightarrow M_0$ are $\pi_1$-injective. Since $\pi_1(M)$ can be recovered from the fundamental groups of the components of $M_0$ via an amalgamated product or an HNN extension, it follows that the map $M_0 \to M$ is $\pi_1$-injective.

As JSJ tori are incompressible and two-sided, the lemma follows by inductively applying this observation. □

The following statement will allow us to assume that the non-Seifert-fibered pieces in the JSJ decomposition of $M$ admit a complete finite-volume hyperbolic metric in their interior. We will call such JSJ pieces **hyperbolic**, but warn the reader that the hyperbolic metric does not extend to the boundary tori. These tori should rather be thought of as "cusps at infinity".

**Proposition 3.3.6** (Atoroidal JSJ pieces are Seifert-fibered or hyperbolic)**.** *If $M$ is an oriented compact connected 3-manifold with toroidal boundary that is prime and not covered by $\mathbb{S}^3$, then the atoroidal pieces in its JSJ decomposition are either Seifert-fibered or hyperbolic.*

*Proof.* The Hyperbolization Theorem [AFW15, Theorem 1.7.5] ensures that the atoroidal pieces with infinite fundamental group are hyperbolic or Seifert-fibered. Hence, it suffices to show that $M$ has no JSJ pieces with finite fundamental group. This is the content of the following lemma. □

**Lemma 3.3.7** (JSJ pieces have infinite $\pi_1$). *If $M$ is an oriented compact connected hyperbolic 3-manifold with toroidal boundary that is prime and not covered by $\mathbb{S}^3$, then all its JSJ pieces have infinite fundamental group.*

*Proof.* If $M \cong \mathbb{S}^1 \times \mathbb{S}^2$, then $M$ has only one JSJ piece and the lemma holds. Hence we may assume $M$ is irreducible.

If $M$ has a compressible boundary torus, then one can see by looking at a neighborhood of that torus and a compressing disc that $M \cong \mathbb{S}^1 \times \mathbb{D}^2$. In that case $M$ again has only one JSJ piece and the theorem holds. Hence we may assume $M$ has incompressible boundary, and of course any such boundary tori are also incompressible in whatever JSJ piece they lay. Similarly, each JSJ torus is incompressible in $M$, and so incompressible in every JSJ piece it bounds.

Since incompressible boundary tori in 3-manifolds are always $\pi_1$-injective by the loop theorem [AFW15, Theorem 1.3.1], we conclude that every JSJ piece of $M$ with finite fundamental group is closed, and in particular, the only piece in the JSJ decomposition. By the Elliptization Theorem [AFW15, Theorem 1.7.3], every closed 3-manifold with finite fundamental group is covered by $\mathbb{S}^3$. Hence $\pi_1(M)$ is infinite. □

The next proposition shows that JSJ decompositions provide the correct viewpoint for understanding the simplicial volume of oriented closed connected 3-manifolds with toroidal boundary.

**Definition 3.3.8.** Let $M$ be an oriented compact connected 3-manifold with toroidal boundary, consider the pieces in the prime decomposition of $M$, and cut them along their JSJ tori. The sum of the volumes of the interiors of all hyperbolic pieces will be denoted by $\operatorname{hypvol}(M)$.

This quantity is well-defined because the homeomorphism types of the prime summands of $M$ and of their JSJ pieces are unique, and moreover, the hyperbolic metric on the interior of each JSJ piece is unique up to isometry, a consequence of Mostow Rigidity [Thu80, Theorem 5.7.2].

**Proposition 3.3.9** (Simplicial volume of 3-manifolds). *If $M$ is an oriented compact connected 3-manifold with toroidal boundary, then*

$$\|M\| = \frac{\operatorname{hypvol}(M)}{\nu_3}.$$

*Proof.* If $M$ admits a complete finite-volume hyperbolic metric in its interior (so $M$ is prime and the only piece in its JSJ decomposition), then Thurston showed that $\|M\| = \frac{\operatorname{vol}(\operatorname{int}(M))}{\nu_3}$ [Thu80, Lemma 6.5.4]. On the other hand, if $M$ is Seifert-fibered we have $\|M\| = 0$ [Thu80, Corollary 6.5.3].

These results can be assembled into a proof of Proposition 3.3.9 using the Gromov Additivity Theorem, which has established in a restricted setting in Gromov's seminal article [Gro82, Section 3.5]; see the work of Bucher, Burger, Frigerio, Iozzi, Pagliantini, and Pozzetti for a more detailed and general account [Buc+14, Theorem 1.3]. The relevant consequences in our setting are that for 3-manifolds with toroidal boundary, simplicial volume is additive under connected sum, and under gluing along incompressible boundary tori. □

We point out that this slick proof is not historically faithful. In particular, the inequality $\|M\| \leq \frac{\operatorname{hypvol}(M)}{\nu_3}$ is originally due to Soma. He built upon Thurston's results to show that a certain upper bound for $\|M\|$ which we introduce later, the "simplicial volume with boundary control" $\|M\|_\partial$ (Definition 3.5.1), satisfies $\|M\|_\partial = \frac{\operatorname{hypvol}(M)}{\nu_3}$ [Som81, Lemma 2(i)]. This approach in terms of a "boundary-controlled" version of simplicial volume is in fact closer to the strategy we will use in proving Theorem 3.2.4.

## 3.4 Parameterized and integral foliated simplicial volume

As a proxy for computing stable integral simplicial volume, we will use yet another variant of simplicial volume. Before defining it, we need to introduce homology with normed local coefficient systems.

**Definition 3.4.1.** Let $X$ be a topological space. A **normed local coefficient system** on $X$ is a functor from the fundamental groupoid $\Pi(X)$ of $X$ to the category of normed abelian groups (with norm non-increasing homomorphisms).

Given a topological space $X$ with a normed local coefficient system $L$, we define its **singular chain complex with local coefficients** in $L$ as the chain complex of normed abelian groups

$$\mathrm{C}_k(X;L) := \bigoplus_\sigma L(\sigma[0]),$$

for all $k \in \mathbb{N}$, where $\sigma$ ranges over the singular simplices $\Delta^k \to X$ and $\sigma[0] := \sigma(e_0)$. Elements $a$ of the summand $L(\sigma[0])$ indexed by a singular simplex $\sigma$ will be denoted by $a \cdot \sigma$. We equip $\mathrm{C}_k(X;L)$ with the $\ell^1$-norm $|\sum_\sigma a_\sigma \cdot \sigma|_1 := \sum_\sigma \|a_\sigma\|$. The boundary operators $\mathrm{C}_k(X;L) \to \mathrm{C}_{k-1}(X;L)$ are given by

$$a_\sigma \cdot \sigma \mapsto L(\sigma[0,1])(a_\sigma) \cdot \partial_0 \sigma + \sum_{i=1}^k (-1)^i a_\sigma \cdot \partial_i \sigma,$$

where $\sigma[0,1]$ denotes the homotopy class of the path given by the $\sigma$-image of the line segment $[e_0, e_1]$. This construction extends to pairs $(X, X_0)$ of topological spaces by defining $\mathrm{C}_k(X, X_0; L) := \mathrm{C}_k(X;L)/\mathrm{C}_k(X_0; L|_{\Pi(X_0)})$ with the induced norm (one easily checks that $\mathrm{C}_k(X_0; L|_{\Pi(X_0)})$ is closed in $\mathrm{C}_k(X;L)$), and taking the induced differentials on the quotients.

**Definition 3.4.2.** Let $(X, X_0)$ be a pair of topological spaces and $L$ a normed local coefficient system on $X$. Then the $k$-th **homology** of $(X, X_0)$ with coefficients in $L$ is the abelian group

$$\mathrm{H}_k(X, X_0; L) := \mathrm{H}_k(\mathrm{C}_*(X, X_0; L))$$

together with the induced semi-norm $|\cdot|_1$.

Note that if $X$ is path-connected and $L$ is a normed local coefficient system on $X$, then the coefficient modules $L(x)$ at each point $x \in X$ are all isometrically isomorphic, and $L$ keeps track of a collection of isomorphisms between any two $L(x), L(y)$. In particular, for each basepoint $x_0 \in X$, the local system $L$ yields a right action of $\pi_1(X, x_0)$ on $L(x_0)$ (the fact that this is a right, rather than left action is due to the mismatch in the standard conventions of concatenating elements in the fundamental group from left to right, but composing morphisms in a category from right to left). This right action can be used to compute homology of $X$ with twisted coefficients in $L(x_0)$, which is then canonically isometrically isomorphic to homology of $X$ with coefficients in $L$. Ultimately this is a reflection of the fact that the inclusion of $\pi_1(X, x_0)$, regarded as a one-point category, into $\Pi(X)$ is an equivalence of categories [Fau+21, Remark 2.6]. We prefer to use homology with local coefficients over twisted homology in order to avoid keeping track of basepoints.

The normed local coefficient systems of relevance to us come from measured group theory, and we will need the following terminology.

**Definition 3.4.3.** A **Polish** space is a topological space that admits a complete metric and is separable. A measurable space that is isomorphic to a Polish space with its Borel $\sigma$-algebra is called a **standard Borel space**. When equipped with a probability measure, it is called a **standard Borel probability space**.

Let $G$ be a groupoid. A **standard $G$-space** is a contravariant functor from $G$ to the category of standard Borel probability spaces and measure-preserving functions.

Our local coefficients are now obtained as the $\mathbb{Z}$-modules of essentially bounded functions on standard Borel probability spaces:

**Definition 3.4.4.** For a topological space $X$ and a standard $\Pi(X)$-space $\alpha$, the **associated normed local coefficient system** $\mathrm{L}^\infty(\alpha, \mathbb{Z})$ is the postcomposition of $\alpha$ with the contravariant "dualizing functor" $\mathrm{L}^\infty(-, \mathbb{Z})$: This functor associates to each standard Borel probability space $\Omega$ the normed abelian group of essentially bounded functions $\Omega \to \mathbb{Z}$ (with the essential supremum norm), and makes measure-preserving maps act by precomposition.

For every pair $(X, X_0)$, the normed local coefficient system with $\mathbb{Z}$ at every point and the identity at every path class recovers integral homology, and given any standard $\Pi(X)$-space $\alpha$, it embeds into $\mathrm{L}^\infty(\alpha, \mathbb{Z})$ as the constant functions. Thus there are canonical maps $\mathrm{H}_k(X, X_0) \to \mathrm{H}_k(X, X_0; \mathrm{L}^\infty(\alpha, \mathbb{Z}))$. Now, when we consider a pair $(M, \partial M)$, where $M$ is an oriented compact manifold, this allows us to define the **$\alpha$-parameterized fundamental class** $[M]^\alpha$ of $M$ as the image of the integral fundamental class $[M]$ under this map.

**Definition 3.4.5** ([Sch05, Definition 5.25])**.** Given a standard $\Pi(M)$-space $\alpha$ on an oriented compact manifold $M$, the **$\alpha$-parameterized simplicial volume** of $M$ is the norm of its $\alpha$-parameterized fundamental class: $\|M\|^\alpha := |[M]^\alpha|_1$.

The infimum of $\|M\|^\alpha$ over all standard $\Pi(M)$-spaces $\alpha$ is called the **integral foliated simplicial volume** of $M$, and denoted by $|M|$.

Parameterized simplicial volumes are useful because with an appropriate choice of parameter space, we can recover stable integral simplicial volume, as we shall see in Theorem 3.8.5 below. Yet the local nature of the coefficient systems makes parameterized simplicial volume suited to arguments involving gluing manifolds along boundary components, as we intend to do with the JSJ pieces. Assembling such estimates will however require a finer version which also takes into account the norm of the boundary of parameterized fundamental cycles. This is the focus of the next section.

## 3.5 Simplicial volume with boundary control

We wish to assemble simplicial volume estimates for the JSJ pieces in a prime 3-manifold into estimates for the whole manifold, and this gluing procedure will require us to control the norms of the boundaries of fundamental cycles. This suggests minimizing the norm of real fundamental chains with $\epsilon$-controlled norm of the boundary, and then making $\epsilon$ arbitrarily small, an idea dating back to the work of Thurston [Thu80, Section 6.5].

**Definition 3.5.1.** Let $M$ be an oriented compact manifold. The **boundary-controlled simplicial volume** of $M$ is the quantity

$$\|M\|_\partial := \sup_{\epsilon>0} \inf\{|c|_1 \in \mathbb{R}_{>0} \mid c \text{ is a real fundamental cycle for } M \text{ with } |\partial c|_1 \leq \epsilon\},$$

with the convention that $\inf \emptyset = +\infty$.

Note that $\|M\|_\partial$ may very well be infinite, namely whenever $\|\partial M\| \neq 0$.

We will not make explicit use of boundary control on classical simplicial volume, but rather of its counterpart in the parameterized setting:

**Definition 3.5.2.** Let $M$ be an oriented compact manifold and $\alpha$ a standard $\Pi(M)$-space. The **boundary-controlled $\alpha$-parameterized simplicial volume** of $M$ is

$$\|M\|_\partial^\alpha := \sup_{\epsilon>0} \inf\{|c|_1 \in \mathbb{R}_{>0} \mid c \text{ is an } \alpha\text{-parameterized fundamental cycle for } M \\ \text{with } |\partial c|_1 \leq \epsilon\}.$$

The infimum of $\|M\|_\partial^\alpha$ over all $\alpha$, denoted by $|M|_\partial$, is the **boundary-controlled integral foliated simplicial volume** of $M$.

It follows directly from the definition that for every $\alpha$, we have $\|M\|_\partial^\alpha \geq \|M\|^\alpha$, so also $|M|_\partial \geq |M|$. When $M$ has empty boundary, these are of course equalities.

For manifolds of dimension $n \geq 2$, the boundary controlled version of parameterized simplicial volume enjoys a useful additivity property with respect to gluing along toroidal boundary components. For us the relevant use-case will of course be when $n = 3$ and the tori being glued give rise to the JSJ tori of resulting 3-manifold. This property is the content of the next proposition.

**Definition 3.5.3.** For a topological space $X$, a standard $\Pi(X)$-space $\alpha$ is said to be **essentially free** if for every basepoint $x_0 \in X$, the action of $\pi_1(X, x_0)$ on $\alpha(x_0)$ is essentially free, that is, every non-trivial element of $\pi_1(X, x_0)$ acts with only a null-set of fixed points.

**Proposition 3.5.4** (Gluing estimate [Fau+21, Propositions 6.4 and 6.5])**.** *Let $M$ be an oriented compact $n$-manifold with $n \geq 2$, let $\alpha$ be an essentially free standard $\Pi(M)$-space, and let $T \subset M$ be a $\pi_1$-injective $(n-1)$-torus embedded in the interior of $M$. Denote by $M_0$ the manifold $M \setminus\!\!\setminus T$ obtained by cutting $M$ along $T$, and by $T_1, T_2$ the components of $\partial M_0$ resulting from $T$. Then*

$$\|M\|_\partial^\alpha \leq \|M_0\|_\partial^{\alpha_0},$$

*where $\alpha_0$ is the precomposition of $\alpha$ with the map of fundamental groupoids $\Pi(M_0) \to \Pi(M)$ induced from the canonical map $M_0 \to M$.*

Note that we put no restriction on whether $T$ is separating in $M$.

The proof of Proposition 3.5.4 will use a result establishing that for tori $T$ (in any dimension), essential freeness of a $\Pi(T)$-space $\alpha$ allows one to find small $\alpha$-parameterized $(k+1)$-chains witnessing null-homologous $k$-chains [FL21, Theorem 1.3]. The translation of this statement from the setting of twisted coefficients to that of local coefficients reads as follows.

**Theorem 3.5.5** (Uniform boundary condition for tori)**.** *Let $T$ be a torus (of any dimension), $\alpha$ an essentially free standard $\Pi(T)$-space, and fix $k \in \mathbb{N}$. Then there exists a constant $K \in \mathbb{R}_{>0}$ such that for every null-homologous $\alpha$-parameterized $k$-chain $c \in \mathrm{C}_k(T; \mathrm{L}^\infty(\alpha))$, there is a $(k+1)$-chain $b \in \mathrm{C}_{k+1}(T; \mathrm{L}^\infty(\alpha))$ with $\partial b = c$ and $|b|_1 \leq K|c|_1$.*

*Proof of Proposition 3.5.4.* The $\pi_1$-injectivity assumption implies that the restriction $\alpha|_T$ of $\alpha$ to $T$ inherits essential freeness from $\alpha$. So let $K$ be a constant for $\alpha|_T$ as in Theorem 3.5.5, with $k = n - 1$.

If $\|M_0\|_\partial^{\alpha_0} = +\infty$, there is nothing to prove. Otherwise, fix $\epsilon, \epsilon_\partial > 0$ and let $c_0$ be an $\alpha_0$-parameterized fundamental cycle for $M_0$ such that

$$|c_0|_1 \leq \|M_0\|_\partial^{\alpha_0} + \epsilon \qquad \text{and} \qquad |\partial c_0|_1 \leq \epsilon_\partial.$$

Moreover, denote by $c$ the $\alpha$-parameterized chain for $(M, \partial M)$ obtained by pushing $c_0$ forward along $M_0 \to M$ (in particular, $|c|_1 \leq |c_0|_1$).

For each $i \in \{1,2\}$, the simplices of $\partial c_0$ supported in $T_i$ assemble to an $\alpha_0|_{T_i}$-parameterized fundamental cycle of $T_i$. Its push-forward $t_i$ along $T_i \to T$ is an $\alpha|_T$-parameterized fundamental cycle for some orientation of $T$. Since the maps $T_i \to T$ induce opposite orientations of $T$, we have in fact that the $\alpha|_T$-parameterized chain $t_1+t_2$ is null-homologous. Moreover, $|t_1+t_2|_1 \leq \epsilon_\partial$. Hence, there is an $\alpha_T$-parameterized $n$-chain $b$ for $T$ with

$$\partial b = t_1 + t_2 \qquad \text{and} \qquad |b|_1 \leq K\,|t_1 + t_2|_1 \leq K\epsilon_\partial.$$

The $\alpha$-parameterized $n$-chain $c^+ := c - b$ for $(M, \partial M)$ is then a cycle, which one can show represents $[M]^\alpha$ (using a local criterion for recognizing fundamental cycles [FFL19, Proposition 3.13]). Moreover, we have

$$|\partial c^+|_1 \leq \epsilon_\partial \qquad \text{and} \qquad |c^+|_1 \leq |c|_1 + |b|_1 \leq \|M_0\|_\partial^{\alpha_0} + \epsilon + K\epsilon_\partial.$$

By making $\epsilon_\partial$ and $\epsilon$ arbitrarily small, we conclude $\|M\|_\partial^\alpha \leq \|M_0\|_\partial^{\alpha_0}$. □

**Corollary 3.5.6** (Sub-additivity of parameterized simplicial volume along JSJ decompositions)**.** *Let $M$ be a prime oriented compact 3-manifold with toroidal boundary, and let $\alpha$ be an essentially free standard $\Pi(M)$-space. Then*

$$\|M\|_\partial^\alpha \leq \sum_{N \text{ JSJ piece of } M} \|N\|_\partial^{\alpha_N},$$

*where $\alpha_N$ denotes the pullback of $\alpha$ along the canonical map $N \to M$.*

*Proof.* One applies Proposition 3.5.4 iteratively over the JSJ tori of $M$. For this inductive argument to work, we need only note that after cutting along a JSJ torus $T$ of $M$, each component of the resulting $M_0 := M \setminus\!\setminus T$ maps $\pi_1$-injectively to $M$ (by Lemma 3.3.5), and so the pull-back of $\alpha$ to $M_0$ is essentially free. □

## 3.6 Non-compact manifolds and locally finite homology

In this section we discuss how several variants of simplicial volume can be generalized to non-compact manifolds. This will be useful for finding an upper bound for the boundary-controlled integral foliated simplicial volume of the hyperbolic pieces in the JSJ decomposition of prime 3-manifolds.

One first needs a notion of fundamental class for non-compact manifolds, which is made available by the theory of locally finite homology. This is a modification of the usual singular homology, generalizing the theory on compact topological spaces. The construction of locally finite homology with real coefficients is sketched in Gromov's paper [Gro82, Section 0.2], and Löh's doctoral dissertation gives a detailed account [Löh07, Section 5.1.1]. The ideas are essentially the same as for local coefficients, so here we describe only the latter, less standard setting.

Let $L$ be a normed local coefficient system on a topological space $X$, and for every $k \in \mathbb{N}$, consider the product $\prod_\sigma L(\sigma[0])$, where $\sigma$ ranges over the continuous maps $\Delta^k \to X$. We denote its (possibly infinite) sequences $(a_\sigma)_\sigma$ by $\sum_\sigma a_\sigma \cdot \sigma$. Such a sequence is called a **locally finite chain** if for every compact subset $K \subseteq X$, only finitely many simplices $\sigma$ with $a_\sigma \neq 0$ have image intersecting $K$. The submodules $\mathrm{C}_k^{\mathrm{lf}}(X; L)$ of locally finite chains assemble to a chain complex of normed abelian groups (with differentials defined as in Section 3.4), and carry a (possibly infinite-valued) $l^1$-norm induced by $L$. The homology groups $\mathrm{H}_k^{\mathrm{lf}}(X; L)$ of this chain complex are then called its **locally finite homology** with $L$-coefficients, and they carry an induced (possibly infinite) semi-norm.

When $W$ is a (possibly non-compact) oriented $n$-manifold without boundary, locally finite homology with $\mathbb{Z}$-coefficients provides a notion of fundamental class $[W] \in \mathrm{H}_n^{\mathrm{lf}}(W)$ generalizing the familiar construction in the compact setting. Given a standard $\Pi(W)$-space $\alpha$, we define the $\alpha$-parameterized fundamental class $[W]^\alpha$ of $W$ to be the image of $[W]$ under the map $\mathrm{H}_n^{\mathrm{lf}}(W) \to \mathrm{H}_n^{\mathrm{lf}}(W; \mathrm{L}^\infty(\alpha; \mathbb{Z}))$ induced from the inclusion of $\mathbb{Z}$ as constant functions.

**Definition 3.6.1.** Let $W$ be a (not necessarily compact) oriented connected $n$-manifold without boundary, and let $\alpha$ be a standard $\Pi(W)$-space. The **$\alpha$-parameterized simplicial volume** of $W$ is

$$\|W\|^\alpha := |[W]^\alpha|_1.$$

The infimum over all standard $\Pi(W)$-spaces $\alpha$ is the **integral foliated simplicial volume** of $W$, denoted by $|W|$.

The reason for introducing parameterized simplicial volume for open manifolds on the way to proving Theorem 3.2.4 is two-fold: first, it gives an upper bound for boundary-controlled parameterized simplicial volume, and second, we can compute it for finite-volume hyperbolic 3-manifolds. We discuss the first of these statements now, and dedicate the next section to the second.

**Proposition 3.6.2** (Gaining boundary control [Fau+21, Proposition 3.14])**.** *Let $M$ be an oriented compact connected $n$-manifold, let $W := M \setminus \partial M$ be its interior, and let $\alpha$ be a standard $\Pi(M)$-space. Then*

$$\|M\|_\partial^\alpha \le \|W\|^\alpha,$$

*where the occurrence of $\alpha$ on the right-hand side should be interpreted as its restriction to $\Pi(W)$. Hence also $|M|_\partial \le |W|$.*

*Proof sketch.* An analogous statement holds for classical simplicial volume [Löh07, Proposition 5.12], and its proof transfers directly to our setup. We may assume $\|W\|^\alpha < \infty$; otherwise there is nothing to prove. The idea is to exhaust $W$ by a sequence of submanifolds $M_1 \subset M_2 \subset \ldots \subset W$ homeomorphic to $M$, obtained by removing ever-smaller open collar neighborhoods of $\partial M$. One then truncates an efficient $\alpha$-parameterized fundamental cycle for $W$ by discarding simplices disjoint from $M_r$, with $r$ large enough. Collapsing the remaining (finitely many) simplices onto $M_r$ yields a fundamental cycle for $M_r$ (hence $M$) with smaller norm, and also small boundary. □

## 3.7 Proportionality for open hyperbolic manifolds

In this section we give an overview of the computation of integral foliated simplicial volume of *open* hyperbolic 3-manifolds.

**Theorem 3.7.1** (Integral foliated simplicial volume of open hyperbolic 3-manifolds [Fau+21, Theorem 5])**.** *Let $W$ be an oriented complete connected finite-volume hyperbolic 3-manifold. Then*

$$|W| = \frac{\mathrm{vol}(W)}{\nu_3},$$

*where $\nu_3$ is the volume of the regular ideal hyperbolic tetrahedron.*

The reader is pointed to the article for the proof of the inequality "$\geq$", as we will not need it. The crucial direction for our purposes "$\leq$" hinges on the following result, valid in all dimensions.

**Theorem 3.7.2** (Upper bound via integral simplicial volume [Fau+21, Theorem 4.1])**.** *Let $W, M$ be non-empty complete oriented connected hyperbolic $n$-manifolds, with $W$ of finite volume and $M$ closed. Then*

$$\frac{|W|}{\operatorname{vol}(W)} \leq \frac{\|M\|_{\mathbb{Z}}}{\operatorname{vol}(M)}.$$

We will comment on Theorem 3.7.2 after using it to prove the relevant inequality of Theorem 3.7.1.

*Proof of Theorem 3.7.1 ($\leq$).* To establish $|W| \leq \frac{\operatorname{vol}(W)}{\nu_3}$, choose any oriented closed connected hyperbolic 3-manifold $M$. For every covering $N$ of $M$ with finite index $d_N$, Theorem 3.7.2 gives

$$\frac{|W|}{\operatorname{vol}(W)} \leq \frac{\|N\|_{\mathbb{Z}}}{\operatorname{vol}(N)} = \frac{\|N\|_{\mathbb{Z}}}{d_N \operatorname{vol}(M)}.$$

Taking the infimum of $\frac{\|N\|_{\mathbb{Z}}}{d_N}$ over all finite covers of $M$ yields the stable integral simplicial volume of $M$:

$$\frac{|W|}{\operatorname{vol}(W)} \leq \frac{\|M\|_{\mathbb{Z}}^{\infty}}{\operatorname{vol}(M)}.$$

The right-hand side now equals $\frac{1}{\nu_3}$ by integral approximation for closed hyperbolic 3-manifolds (Theorem 3.2.5), finishing the proof. $\square$

The proof of Theorem 3.7.2 is too technically involved to present here, so we give only a sketch and direct the interested reader to the article for details [Fau+21, Theorem 4.1].

*Proof sketch of Theorem 3.7.2.* Regard $\mathbb{H}^n$ as the common universal cover of $M$ and $W$. One considers on $W$ the standard $\Pi(W)$-space $\alpha_W$ induced from the obvious left $\pi_1(W)$-action on $\operatorname{Isom}^+(\mathbb{H}^n)/\pi_1(M)$, which carries a standard probability measure inherited from the suitably normalized Haar measure on $\operatorname{Isom}^+(\mathbb{H}^n)$. We then compare the integral simplicial volume of $M$ with the $\alpha_W$-parameterized simplicial volume of $W$.

To do so, we construct a map between the relevant chain complexes

$$\begin{aligned} \varphi_* \colon \mathrm{C}_*(M) &\to \mathrm{C}_*(W; \mathrm{L}^\infty(\alpha_W, \mathbb{Z})) \\ \sigma &\mapsto \sum_{\varrho} f_{\sigma,\varrho} \cdot \varrho, \end{aligned}$$

called the "discrete smearing map". It takes simplices $\sigma$ of $M$ to linear combinations of simplices $\varrho$ of $W$ from a pre-selected collection with useful properties – namely, the $\varrho$ lift to geodesic simplices in $\mathbb{H}^3$ with vertices in a prescribed discrete subset. To define the coefficient functions $f_{\sigma,\varrho} \in \mathrm{L}^\infty(\operatorname{Isom}^+(\mathbb{H}^n)/\pi_1(M), \mathbb{Z})$, we first choose suitable lifts $\widetilde{\sigma}, \widetilde{\varrho}$ of $\sigma, \varrho$ to $\mathbb{H}^n$. The function $f_{\sigma,\varrho}$ is then defined on each coset $g\,\pi_1(M) \in \operatorname{Isom}^+(\mathbb{H}^n)/\pi_1(M)$ as the number of $\pi_1(M)$-translates of $\widetilde{\sigma}$ that are moved "close" to $\widetilde{\varrho}$ by $g$. The definition of the $\varphi_k$ turns out to be independent of all choices, and they do indeed assemble to a chain map [Fau+21, Lemma 4.9].

A short computation [Fau+21, Lemma 4.11] shows that the norm of all maps $\varphi_k$ is bounded from above by the following quotient of measures:

$$\|\varphi_k\| \leq \frac{\mu_W(\pi_1(W)\backslash \operatorname{Isom}^+(\mathbb{H}^n))}{\mu_M(\operatorname{Isom}^+(\mathbb{H}^n)/\pi_1(M))},$$

where $\mu_W$ and $\mu_M$ are both induced by a common Haar measure on $\mathrm{Isom}^+(\mathbb{H}^n)$. This quotient in turn equals $\frac{\mathrm{vol}(W)}{\mathrm{vol}(M)}$. Then, using an integration criterion for detecting parameterized fundamental cycles [Fau+21, Lemma 3.13], one sees that the map induced by $\varphi_*$ on $n$-th homology takes $[M]$ to $[W]^{\alpha_W}$ [Fau+21, Proposition 4.10]. Hence, $\|W\|^{\alpha_W} \leq \frac{\mathrm{vol}(W)}{\mathrm{vol}(M)}\|M\|_{\mathbb{Z}}$, finishing the proof. $\square$

## 3.8 Profinite completions of 3-manifold groups

In our study of integral approximation for oriented compact connected 3-manifolds $M$, a certain standard $\Pi(M)$-space, denoted $\widehat{\Pi(M)}$, will play a crucial role. Its construction is based on the profinite completion of $\pi_1(M)$, so in this section we will introduce profinite completions of groups, discuss their particular relevance in 3-manifold topology, and lay out the properties of $\widehat{\Pi(M)}$ that justify its prominence in our program.

**Definition 3.8.1.** The **profinite completion** of a group $G$ is the topological group obtained as the projective limit of all finite quotients of $G$ together with the canonical projections:

$$\widehat{G} := \varprojlim_{H \underset{\text{f.i.}}{\trianglelefteq} G} G/H,$$

where for each finite index normal subgroup $H$, the quotient $G/H$ is equipped with the discrete topology.

The profinite completion $\widehat{G}$ can be explicitly constructed as the closure of the image of the map $G \to \prod_{H \underset{\text{f.i.}}{\trianglelefteq} G} G/H$ induced by the projections of $G$ onto its finite quotients. The topology on $\widehat{G}$ is compact, Hausdorff and totally disconnected [RZ00, Theorem 1.1.12], and the canonical map $G \to \widehat{G}$ has dense image [RZ00, Lemma 1.1.7]. If $G$ is finite, this map is an isomorphism of discrete groups, so $\widehat{G}$ is uninteresting in that case. Moreover, the profinite completion construction is functorial: every group homomorphism $G_1 \to G_2$ induces a continuous homomorphism $\widehat{G_1} \to \widehat{G_2}$. More generally, $\widehat{G}$ is determined by the property that every homomorphism from $G$ to a profinite group extends uniquely to a continuous homomorphism out of $\widehat{G}$ [RZ00, Lemma 3.2.1].

One can think of $\widehat{G}$ as packaging the information about all finite quotients of $G$. When $G$ is finitely generated, this intuition is made precise by means of a classical result of Dixon, Formalnek, Poland and Ribes [Dix+82], stating that two discrete groups $G_1, G_2$ have the same set of isomorphism classes of finite quotients precisely if their profinite completions $\widehat{G_1}, \widehat{G_2}$ are isomorphic as topological groups.

Profinite completions are particularly useful when studying groups whose elements can all be detected on some finite quotient:

**Definition 3.8.2.** A group $G$ is **residually finite** if for every non-trivial element $g \in G$, there is a finite index normal subgroup $H \underset{\text{f.i.}}{\trianglelefteq} G$ such that the quotient map $G \twoheadrightarrow G/H$ does not have $g$ in its kernel. Equivalently, if the canonical homomorphism $G \to \widehat{G}$ is injective.

Of particular relevance to us is the following fact:

**Theorem 3.8.3** (Residual finiteness of 3-manifold groups)**.** *Every compact 3-manifold has residually finite fundamental group.*

This theorem follows from work of Hempel [Hem87], Thurston [Thu82, Theorem 3.3], and the Geometrization Theorem (which was fully established in 2003; see the text of

Aschenbrenner-Friedl-Wilton [AFW15, Section 1.7] for a historical overview and further references).

Since the profinite completion of a group $G$ is a compact Hausdorff topological group, it carries a Haar probability measure, and the canonical map $G \to \widehat{G}$ gives a measure-preserving $G$-action on $\widehat{G}$. This allows us to define the following standard $\Pi$-space:

**Definition 3.8.4.** For a topological space $X$, we denote by $\widehat{\Pi(X)}$ the standard $\Pi(X)$-space given as follows:

- at each point $x \in X$, we take the standard probability space $\widehat{\pi_1(X, x)}$,
- to each homotopy class of paths in $X$ relative endpoints $\gamma\colon x \to y$, we associate the measure-preserving isomorphism $\widehat{\pi_1(X, y)} \to \widehat{\pi_1(X, x)}$ induced from the group isomorphism

$$\begin{aligned} \pi_1(X, y) &\to \pi_1(X, x) \\ \alpha &\mapsto \gamma * \alpha * \gamma^{-1} \end{aligned}$$

  that conjugates loops at $y$ with $\gamma$.

The first important property of $\widehat{\Pi(M)}$ that we will make use of is that for every oriented compact connected manifold $M$ (in any dimension), its $\widehat{\Pi(M)}$-parameterized simplicial volume can be used as a replacement for stable integral simplicial volume:

**Theorem 3.8.5** (Stable integral simplicial volume via profinite completions)**.** *For every oriented compact connected manifold $M$, we have*

$$\|M\|_{\mathbb{Z}}^{\infty} = \|M\|^{\widehat{\Pi(M)}}.$$

This proposition was proved in the closed case in terms of standard $\pi_1(M)$-spaces, rather than standard $\Pi(M)$-spaces, by Löh and Pagliantini [LP16, Theorem 6.6 and Remark 6.7], but the same arguments work in our setting.

We will also need the following fact:

**Lemma 3.8.6** (Essential freeness)**.** *If $M$ is an oriented compact connected 3-manifold, then $\widehat{\Pi(M)}$ is essentially free.*

*Proof.* Residual finiteness of $\pi_1(M)$ (Theorem 3.8.3) directly implies that for each point $x \in M$, the action of $\pi_1(M, x)$ on $\widehat{\pi_1(M, x)}$ is in fact free. □

This lemma will be useful because it allows us to apply Corollary 3.5.6 to $\widehat{\Pi(M)}$.

The remaining pieces of our program rely on the notion of weak containment of standard $\Pi(M)$-spaces, to which we devote the next section. Here again the choice of $\widehat{\Pi(M)}$ as a standard $\Pi(M)$-space will play a central role.

## 3.9 Weak containment

We now introduce a relationship between parameter spaces over a fixed group(oid), which will ultimately allow us to compare $\widehat{\Pi(M)}$ to all other standard $\Pi(M)$-spaces.

**Definition 3.9.1** ([Kec10, Section 10(C)])**.** Let $\Gamma$ be a countable group and let $(X, \mu)$, $(Y, \nu)$ be standard $\Gamma$-spaces. We say $X$ is **weakly contained** in $Y$, and write $X \prec Y$, if for every $\epsilon > 0$ and finite subset $F \subset \Gamma$, we have: for every finite collection of Borel subsets $A_1, \ldots, A_m \subseteq X$ there exist corresponding Borel subsets $B_1, \ldots, B_m \subseteq Y$ such that

$$\forall i \in 1, \ldots m \quad \forall g \in F \quad |\mu(gA_i \cap A_i) - \nu(gB_i \cap B_i)| \leq \epsilon.$$

If $G$ is a groupoid with countable automorphism groups and $\alpha, \beta$ are standard $G$-spaces, we say $\alpha$ is **weakly contained** in $\beta$ (written $\alpha \prec \beta$) if for every object $x$ of $G$, we have a weak containment of $\mathrm{Aut}_G(x)$-spaces $\alpha(x) \prec \beta(x)$.

The relevance of this notion to our goals is contained in the following proposition.

**Proposition 3.9.2** (Monotonicity with respect to weak containment)**.** *If $M$ is an oriented compact connected manifold with infinite fundamental group and $\alpha, \beta$ are essentially free standard $\Pi(M)$-spaces satisfying $\alpha \prec \beta$, then*

$$\|M\|_\partial^\beta \leq \|M\|_\partial^\alpha.$$

The original proof, in the language of standard $\pi_1(M)$-spaces, was given by Frigerio, Löh, Pagliantini and Sauer [Fri+16, Theorem 3.3]. A translation to the setting of standard $\Pi(M)$-spaces is also available [Fau+21, Appendix 1].

**Proposition 3.9.3** (Reduction to hyperbolic pieces)**.** *Let $M$ be an irreducible oriented compact connected 3-manifold with toroidal boundary, and assume $M$ is not covered by $\mathbb{S}^3$. For each piece $N$ of the JSJ decomposition of $M$, we have:*

1. *if $N$ is Seifert-fibered, then $\|N\|_\partial^{\widehat{\Pi(M)}} = 0$,*
2. *if $N$ is hyperbolic, then $\|N\|_\partial^{\widehat{\Pi(M)}} \leq \|N\|_\partial^{\widehat{\Pi(N)}}$.*

*Here, both occurrences of $\widehat{\Pi(M)}$ should be interpreted as its pull-back along $N \to M$.*

*Proof.* We use the fact that for each basepoint $x_0 \in N$, the canonical map $\pi_1\widehat{(N, x_0)} \to \pi_1\widehat{(M, x_0)}$ is injective [WZ10, Theorem A], making $\pi_1\widehat{(N, x_0)}$ a closed subgroup of $\pi_1\widehat{(M, x_0)}$. Wilton and Zalesskii phrase this as the statement that the profinite topology on $\pi_1(M)$ is **efficient** with respect to its decomposition as a graph of groups induced from the JSJ splitting. In the given reference, they assume that $M$ is closed, but the result also holds in the case of non-empty toroidal boundary [AFW15, Section 3.2 (C.35)].

This implies that $\pi_1\widehat{(M, x_0)}$ is, as a standard $\pi_1(N, x_0)$-space, isomorphic to the product of $\pi_1\widehat{(N, x_0)}$ with a probability space carrying a trivial $\pi_1(N, x_0)$-action [GM17, Example 12]. In the language of Gheysens and Monod, $\pi_1\widehat{(M, x_0)}$ is an **amplification** of $\pi_1\widehat{(N, x_0)}$.

From this description, it directly follows that we have a weak containment of standard $\pi_1(N, x_0)$-spaces $\pi_1\widehat{(N, x_0)} \prec \pi_1\widehat{(M, x_0)}$. At the level of $\Pi(N)$-spaces, we see that $\widehat{\Pi(N)} \prec \widehat{\Pi(M)}$. Now by monotonicity of parameterized simplicial volume with respect to weak containment of parameter spaces (Proposition 3.9.2), we conclude that, in both the Seifert-fibered and the hyperbolic case, $\|N\|_\partial^{\widehat{\Pi(M)}} \leq \|N\|_\partial^{\widehat{\Pi(N)}}$. This proves the second item, and reduces the first to showing that when $N$ is Seifert-fibered, we have $\|N\|_\partial^{\widehat{\Pi(N)}} = 0$.

To do this, note that $N$ must have infinite fundamental group by Lemma 3.3.7, and so by Theorem 3.2.6 we have $\|N\|_\mathbb{Z}^\infty = 0$. Then, by Theorem 3.8.5 we obtain $\|N\|^{\widehat{\Pi(N)}} = 0$. This vanishing transfers to the boundary-controlled version, as it would for any standard $\Pi(N)$-space $\alpha$, because the boundary of an $\alpha$-parameterized $k$-chain $c$ always has norm bounded above by $(k+1)|c|_1$. In particular, every $\epsilon$-small $\alpha$-parameterized fundamental cycle for $N$ automatically has $4\epsilon$-small boundary. We conclude $\|N\|_\partial^{\widehat{\Pi(N)}} = 0$, as desired. $\square$

The final, crucial property of the profinite completion $\widehat{\pi_1(M)}$ that we will use is that it fully captures the dynamical properties of $\pi_1(M)$, in the following sense.

**Definition 3.9.4.** For a topological space $X$, a standard $\Pi(X)$-space $\alpha$ is said to be **ergodic** if for every basepoint $x_0 \in X$, the action of $\pi_1(X, x_0)$ on $\alpha(x_0)$ is ergodic, that is, every $\pi_1(X, x_0)$-invariant measurable subset $A \subseteq \alpha(x_0)$ has either zero or full measure.

**Proposition 3.9.5** ($\widehat{\Pi(M)}$ weakly contains everything)**.** *Let $M$ be an oriented compact connected 3-manifold with toroidal boundary whose interior admits a complete finite-volume hyperbolic metric. Then every ergodic standard $\Pi(M)$-space is weakly contained in $\widehat{\Pi(M)}$.*

In the language of standard $\pi_1(M)$-spaces, this property of $\pi_1(M)$ is called **EMD*** in the literature. We refer to the article for the detailed proof [Fau+21, Proposition 5.2], which is based on results of Kechris [Kec12], as well as Bowen and Tucker-Drob [BT13].

*Proof sketch.* The idea is to first use Agol's Virtual Fibering Theorem [Ago13; FK14] to express $\pi_1(M)$ as a finite-index subgroup of some semidirect product $\Sigma \rtimes \mathbb{Z}$, with $\Sigma$ a finitely generated free or surface group. Such a group $\Sigma$ satisfies another property "MD", which is equivalent to EMD* for residually finite groups. Then one invokes inheritance properties to transfer MD (and hence EMD*) from $\Sigma$ to $\pi_1(M)$. $\square$

**Corollary 3.9.6** (Integral foliated simplicial volume via profinite completions)**.** *Let $M$ be an oriented compact connected 3-manifold with toroidal boundary, and whose interior admits a complete finite-volume hyperbolic metric. Then*

$$|M|_\partial = \|M\|_\partial^{\widehat{\Pi(M)}}.$$

*Proof.* The inequality "$\leq$" holds by definition of boundary-controlled integral foliated simplicial volume.

For the other direction, the idea is to use Proposition 3.9.5 in tandem with monotonicity of parameterized simplicial volume with respect to weak containment of $\Pi(M)$-spaces (Proposition 3.9.2). For that to work, however, one first needs to show that $|M|_\partial$ can be approximated arbitrarily well by $\|M\|_\partial^\alpha$ with $\alpha$ *ergodic*.

Löh and Pagliantini have shown (in the language of twisted, rather than local coefficients) [LP16, Proposition 4.17] that if $M$ is closed (and independently of its dimension $n$), we have

$$|M| = \inf\{\|M\|^\alpha \mid \alpha \text{ is an ergodic standard } \Pi(M)\text{-space}\}.$$

Their proof makes no use of the closedness assumption and would work verbatim for the compact case (without boundary control). We now sketch how to adjust their proof to get the boundary-controlled version.

In their notation, start with an $\epsilon$-efficient $(X, \mu)$-parameterized fundamental cycle $c$ with $\epsilon$-small boundary, that is:

$$c = \sum_{j=0}^{k} f_j \otimes \sigma_j \qquad \text{with boundary} \qquad \partial c = \sum_{l=0}^{m} g_l \otimes \tau_l,$$

such that

$$|c|_1 = \sum_{j=0}^{k} \int_X |f_j| \leq \|M\|_\partial^{(X,\mu)} + \epsilon \qquad \text{and} \qquad |\partial c|_1 = \sum_{l=0}^{m} \int_X |g_l| \leq \epsilon.$$

Then, apply their ergodic decomposition argument to the function $f := \sum_{j=0}^{k} |f_j| + \sum_{l=0}^{m} |g_l|$. This yields an ergodic measure $\mu_p$ on $X$ such that $c$, when regarded as an $(X, \mu_p)$-parameterized fundamental cycle, satisfies

$$|M|_\partial \leq |c|_1 + |\partial c|_1 \leq \|M\|_\partial^{(X,\mu)} + 2\epsilon.$$

By choosing parameter spaces $(X, \mu)$ with $\|M\|_\partial^{(X,\mu)}$ close to $|M|_\partial$ and small enough $\epsilon$, one thus finds arbitrarily efficient ergodic parameter spaces. □

## 3.10 Proof of the main theorem

We are now ready to assemble all the theory developed so far into a proof of the main theorem of this chapter, establishing integral approximation for all oriented compact connected 3-manifolds with toroidal boundary that are prime and not covered by $\mathbb{S}^3$.

*Proof of Theorem 3.2.4.* Since stable integral simplicial volume is an upper bound for classical simplicial volume, as explained on page 68, the inequality $\|M\| \leq \|M\|_{\mathbb{Z}}^{\infty}$ is straightforward.

For the converse direction, we see that

$$\begin{aligned}
\|M\|_{\mathbb{Z}}^{\infty} &= \|M\|^{\widehat{\Pi(M)}} && \text{(Theorem 3.8.5)}\\
&\leq \|M\|_\partial^{\widehat{\Pi(M)}} \\
&\leq \sum_{N \text{ JSJ piece of } M} \|N\|_\partial^{\widehat{\Pi(M)}} && \text{(Lemma 3.8.6 and Corollary 3.5.6)}\\
&\leq \sum_{N \text{ hyperbolic piece of } M} \|N\|_\partial^{\widehat{\Pi(N)}} && \text{(Proposition 3.9.3)}\\
&= \sum_{N \text{ hyperbolic piece of } M} |N|_\partial && \text{(Corollary 3.9.6)}\\
&\leq \sum_{N \text{ hyperbolic piece of } M} |\operatorname{int}(N)| && \text{(Proposition 3.6.2)}\\
&= \sum_{N \text{ hyperbolic piece of } M} \frac{\operatorname{vol}(\operatorname{int}(N))}{\nu_3} && \text{(Theorem 3.7.1)}\\
&= \frac{\operatorname{hypvol}(M)}{\nu_3}.
\end{aligned}$$

Proposition 3.3.9 tells us this is precisely the value of $\|M\|$, so we are done. □

## 3.11 Bounding homology growth

We conclude by briefly commenting on how simplicial volume can be used to control the growth of homology along sequences of finite covers of a manifold, and the consequences of Theorem 3.2.4 in this context.

We will first observe that via Poincaré Duality, integral simplicial volume provides an upper bound for all (rational) Betti numbers of an oriented closed manifold $M$ of any dimension. Afterwards, we will use this to relate $\ell^2$-Betti numbers to stable integral simplicial volume. Weaker versions of these estimates have been stated by Gromov [Gro07, Section 5.38], but we follow the proofs given by Löh [Löh20, Section 6.4.2].

**Proposition 3.11.1** (Integral simplicial volume bounds Betti numbers)**.** *If $M$ is an oriented closed manifold, then for all $k \in \mathbb{N}$ we have*

$$\mathrm{b}_k(M) \leq \|M\|_{\mathbb{Z}}.$$

*Proof.* Let $c = \sum_{i=0}^{m} a_i \sigma_i$ be an integral fundamental cycle for $M$ (with all $a_i$ non-zero and all $n$-simplices $\sigma_i$ distinct). Then Poincaré Duality, together with the explicit formula for the cap product on singular (co)chains, implies that all classes in $\mathrm{H}_k(M, \mathbb{Q})$ are represented by $k$-cycles that are linear combinations of the $k$-simplices spanned by the last $k+1$ vertices of each $\sigma_i$. Hence $\mathrm{H}_k(M, \mathbb{Q})$ is a quotient of the subspace of cycles in $\mathrm{C}_k(M; \mathbb{Q})$ in the $\mathbb{Q}$-span of these $k$-simplices, which are $m$ in number. Therefore

$$\mathrm{b}_k(M) = \dim_{\mathbb{Q}} \mathrm{H}_k(M; \mathbb{Q}) \leq m \leq |c|_1,$$

and since this holds for all integral fundamental cycles, we conclude $\mathrm{b}_k(M) \leq \|M\|_{\mathbb{Z}}$. □

This inequality is paralleled by the stable integral simplicial volume, which gives an upper bound for the $\ell^2$-Betti numbers, provided that $M$ has residually finite fundamental group.

**Proposition 3.11.2** (Stable integral simplicial volume bounds $\ell^2$-Betti numbers)**.** *If $M$ is an oriented closed connected manifold with $\pi_1(M)$ residually finite, then for all $k \in \mathbb{N}$ we have*

$$\mathrm{b}_k^{(2)}(M) \leq \|M\|_{\mathbb{Z}}^{\infty}.$$

*Proof.* The hypothesis that $\pi_1(M)$ is residually finite allows us to use Lück's Approximation Theorem [Lüc94] [Kam19, Theorem 1.6] to express the $k$-th $\ell^2$-Betti number of $M$ as

$$\mathrm{b}_k^{(2)}(M) = \lim_{i\to\infty} \frac{\mathrm{b}_k(M_i)}{d_i},$$

where $\ldots \to M_1 \to M_0 = M$ is any sequence of finite-index coverings of $M$, each $M_i$ with index $d_i$, and with $\bigcap_{i\in\mathbb{N}} \pi_1(M_i) = 1$ (in other words, the $M_i$ form a **residual tower** for $M$).

On the other hand, by the transfer principle for integral simplicial volume, the sequence $\left(\frac{\|M_i\|_{\mathbb{Z}}}{d_i}\right)_{i\in\mathbb{N}}$ is non-increasing, and so it is convergent with

$$\lim_{i\to\infty} \frac{\|M_i\|_{\mathbb{Z}}}{d_i} = \inf_{i\in\mathbb{N}} \frac{\|M_i\|_{\mathbb{Z}}}{d_i}.$$

Using Proposition 3.11.1 we obtain

$$\mathrm{b}_k^{(2)}(M) \leq \inf_{i\in\mathbb{N}} \frac{\|M_i\|_{\mathbb{Z}}}{d_i}.$$

Since every finite cover of $M$ can be completed to a residual tower, we can take the infimum over all finite covers to conclude $\mathrm{b}_k^{(2)}(M) \leq \|M\|_{\mathbb{Z}}^{\infty}$. □

Note that if $M$ is an oriented compact connected manifold with $\pi_1(M)$ residually finite and satisfying $\|M\|_{\mathbb{Z}}^{\infty} = 0$ (and hence integral approximation), then all its $\ell^2$-Betti numbers vanish. As $\chi(M)$ is the alternating sum of the $\ell^2$-Betti numbers [Lüc02, Theorem 1.35(2)] [Kam19, Theorem 3.19], it follows that $\chi(M) = 0$, and so if $M$ is aspherical, it is a positive example for Gromov's question on page 67.

Proposition 3.11.2 illustrates how one can use stable integral simplicial volume to control the asymptotic growth of the rank of homology along finite coverings of a manifold. However, the computation given by Theorem 3.2.4 provides no new information, as the $\ell^2$-Betti numbers of prime oriented compact connected 3-manifolds with toroidal boundary and infinite fundamental group were already known to vanish [LL95].

If we shift our attention to the growth of the torsion part of homology, we can deduce the following result:

**Corollary 3.11.3** (Homology torsion growth via hypvol)**.** *Let $M$ be an oriented closed connected 3-manifold that is prime and not covered by $\mathbb{S}^3$. Moreover, let $\ldots \to M_1 \to M_0 = M$ be a tower of finite coverings for $M$, with $d_i$ the degree of $M_i \to M$, such that the quotients $\frac{\|M_i\|_{\mathbb{Z}}}{d_i}$ converge to $\|M\|_{\mathbb{Z}}^{\infty}$. Then we have*

$$\limsup_{i\to\infty} \frac{\log|\mathrm{tors}(\mathrm{H}_1(M_i))|}{d_i} \leq 12\log(2)\frac{\mathrm{hypvol}(M)}{\nu_3}.$$

We remind the reader that for oriented closed 3-manifolds, homology in dimensions 0, 2 and 3 is torsion-free. We also point out that a stronger estimate had been previously established by Lê [Lê18], with the better constant $\frac{1}{6\pi}$ instead of our $\frac{12\log(2)}{\nu_3}$.

*Proof.* A theorem of Sauer [Sau16, Theorem 3.2] tells us that for an oriented closed manifold $N$ of any dimension $n$, and for every $k \in \mathbb{N}$, we have

$$\log|\mathrm{tors}(\mathrm{H}_k(N))| \leq \log(n+1)\binom{n+1}{k+1}\|N\|_{\mathbb{Z}}.$$

Applying this formula with $k = 1$ to the $M_i$, dividing by the $d_i$, and then letting $i$ tend to infinity yields

$$\limsup_{i\to\infty} \frac{\log|\mathrm{tors}(\mathrm{H}_1(M_i))|}{d_i} \leq 12\log(2)\|M\|_{\mathbb{Z}}^{\infty}.$$

The computation of $\|M\|_{\mathbb{Z}}^{\infty}$ given by Theorem 3.2.4 finishes the proof. □

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