The multiphase Muskat problem in two dimensions

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Abstract

In this thesis we study the two-dimensional multiphase Muskat problem which describes the dynamics in a three fluids system located in a vertical porous medium under the effect of gravity. This topic is of great interest in many applications, but also from a mathematical point of view.

In the first part of the thesis we consider the multiphase Muskat problem with equal viscosities. As a first result, we prove that the velocity field in each fluid phase is explicitly identified in terms of contour integrals described by the functions that parameterize the interfaces between the fluids. Based on this feature, we express the classical formulation of the multiphase Muskat problem as a nonlinear and nonlocal evolution equation for the pair of functions that parameterize the free interfaces. After showing that this problem is parabolic in the regime where the fluids are arranged vertically according to their density, with the fluids possessing larger densities located below, we then establish the local well-posedness of the problem together with two parabolic smoothing properties. Moreover, for solutions which are not global, but bounded in the phase spaces, we exclude the occurrence of squirt singularities, that is the fluid interfaces cannot touch along a curve segment when time elapses.

In the second part of the thesis we establish the first local well-posedness result for the multiphase Muskat problem with general viscosities. Employing underlying Rellich identities in the regime where the viscosities are ordered and a Neumann series argument when the fluids are not ordered according to their viscosities, we show again that the velocity is given by contour integrals which involve the pair of functions that parameterize the free interfaces. As a new feature these integrals comprise a density function which depends nonlocally on this pair. This enables us to formulate the problem as a nonlinear and nonlocal evolution problem for the functions that parameterize the fluid interfaces, which is of parabolic type in the regime where the classical Rayleigh–Taylor condition is satisfied at each interface. This aspect is the main ingredient in the proof of the local well-posedness result, and we establish parabolic smoothing properties for the solutions to the problem also in this setting.

Zusammenfassung

In dieser Arbeit studieren wir das zweidimensionale, mehrphasige Muskat Problem, welches die Dynamik eines Systems, das aus drei Fluiden besteht, in einem vertikalen porösen Medium unter dem Einfluss von Gravitation beschreibt. Dieses Thema ist in vielen Anwendungen, aber auch aus mathematischer Sicht von großem Interesse.

Im ersten Teil der Arbeit beschäftigen wir uns mit dem mehrphasigen Muskat Problem mit gleichwertigen Viskositäten. Als erstes Ergebnis beweisen wir, dass das Geschwindigkeitsfeld in jeder Fluidphase explizit mittels Konturintegralen, die durch die Funktionen beschrieben werden, die die Oberflächen zwischen den Fluiden parametrisieren, gegeben ist. Basierend auf dieser Eigenschaft drücken wir die klassische Formulierung des mehrphasigen Muskat Problems als eine nicht-lineare und nicht-lokale Evolutionsgleichung für das Funktionenpaar, das die freien Oberflächen parametrisiert, aus. Nachdem wir zeigen, dass das Problem in dem Fall, in dem die Fluide vertikal geordnet sind nach deren Dichte, mit den Fluiden mit größerer Dichte weiter unten gelegen, parabolisch ist, beweisen wir die lokale Wohlgestelltheit des Problems zusammen mit zwei parabolischen Glättungseffekten. Des Weiteren schließen wir für Lösungen, die nicht global existieren, aber im Phasenraum beschränkt sind, das Auftreten von Squirt-Singularitäten

aus, das heißt die Fluidoberflächen können sich nicht entlang eines Kurvenabschnittes berühren.

Im zweiten Teil der Arbeit beweisen wir das erste lokale Wohlgestelltheitsresultat für das mehrphasige Muskat Problem mit allgemeinen Viskositäten. Durch Ausnutzen von zugrundeliegenden Rellich-Identitäten in dem Fall, in dem die Viskositäten geordnet sind, und mit Hilfe eines Neumann-Reihen Arguments, in dem Fall, in dem die Fluide nicht nach deren Viskositäten geordnet sind, zeigen wir wieder, dass die Geschwindigkeit mittels Konturintegralen beschrieben werden kann, welche das Funktionenpaar, das die freien Oberflächen parametrisiert, enthalten. Eine neue Eigenschaft ist nun, dass diese Integrale eine Dichtefunktion, die nicht-lokal von diesem Paar abhängt, beinhaltet. Das ermöglicht uns das Problem als nicht-lineares und nicht-lokales Evolutionsproblem für die Funktionen, die die Fluidoberflächen parametrisieren, zu reformulieren. Dieses Evolutionsproblem ist parabolisch in der offenen Menge, welche durch die klassischen Rayleigh–Taylor Bedingungen identifiziert ist. Diese Eigenschaft ist Hauptbestandteil des Beweises des lokalen Wohlgestelltheitsresultates, und wir beweisen auch in dieser Konfiguration für die Lösungen des Problems parabolische Glättungseffekte.

Contents

1. Introduction

1.1. Flows in porous media

Porous media consist, for example, of sand stones which are sedimentary rocks composed of sand-size grains of minerals, rocks, and organic material. These stones posses pores which are typically filled with a fluid. Usually crude oil reserves are located in such porous media.

The interest in flows in porous media reaches widely from science to engineering technology. Only to name a few applications, theory of such flows is used in petroleum extraction, in the matter of CO_2-storage and micro-flows. A prominent example is the Hele-Shaw flow which describes the dynamics in a system of one or more fluids located between two – mostly parallel – transparent plates with very small distance in between, see [53, 54, 80]. This makes the Hele-Shaw flow important for micro-flow applications, see [72, 82]. This flow has been named after the English mechanical and automobile engineer Henry-Selby Hele-Shaw. A recent review of analysis and numerics for one-phase Hele-Shaw models can be found in [67].

The still growing demand for gas and oil together with a decrease of new reservoirs discovered leads inevitably to the necessity of enhanced petroleum extraction methods. Primary recovery is mostly applied to newly exploited oil or gas fields as it is driven by natural mechanisms and ordinary pumps. In this primary recovery phase, the pressure in the oil reservoirs is very high and the oil can be extracted by the oil pumps (and extraction wells). On a second stage – secondary recovery – after the underground pressure decreased such that the oil is no longer naturally forced to the surface, one injects certain fluids in order to increase the pressure again and arrange a flow upwards, which can have a major impact on the amount of production, cf. [87]. Common methods in this phase are water injection, gas injection, and gas lift. Enhanced – or tertiary – oil recovery then improves the flow by decreasing the viscosity of the oil by heating it up, which recently receives some attention, see e.g. [34].

The methods of secondary oil and gas recovery, naturally lead to the topic of CO_2-storage, cf. [5]. The idea is to use compressed CO_2 in order to increase the pressure within the oil field. The advantage is of course that one reduces the CO_2 in the atmosphere provided the injected CO_2 stays in the depleted oil/gas reservoir for as long as possible. Work on the maximization of both, the extracted petroleum and the stored CO_2, can be found for example in [50, 55, 87], where also the impact of this research field is shown even in highest political spheres.

From a mathematical point of view, the interest in flows in porous media dates back to the 19th century when the mathematical foundations for such models have been set by the French engineer Henry Darcy, see [35]. He derived an empirical law, the so called Darcy's law, which is a linear relation that links the velocity of the fluid to the gradient of its pressure. Darcy's law is the main constitutive equation used to describe Hele-Shaw flows, cf. [13, 53, 54]. Many experimental studies show, that such problems are modeled well with Darcy's law, see e.g. [88]. Also mathematical research gives rise to consider Darcy's law, since it arises as the result of a homogenization process starting from many different initial situations. For example Tartar in [84] has rigorously proved that the incompressible steady Stoke's equations converge to a Darcy law when the porous medium is homogenized. Ten years later, the article [58] considers more generally the compressible Navier–Stokes equation in one dimension with statistical assumptions on the porous medium, and the authors obtain after the homogenization again Darcy's law.

In order to model the encroachment of water into a porous medium filled with oil, the American engineer Morris Muskat had also used Darcy's law to describe the dynamics in this two-phase system, cf. [70]. The mathematical model, the so called Muskat problem, is the standard model for two-phase fluid flows in porous media. The Muskat problem and its one-phase version, the Hele-Shaw problem, have received much interest during the decades also in the field of mathematics and are also currently the subject of many studies. Both mathematical models are moving boundary problems that describe the evolution of the fluid domains and of the velocity and the pressure in each fluid phase. We will give some references in the following subsections.

Mathematical research on the classical two-phase Muskat problem

Despite the fact that the Muskat problem has been proposed already in 1934, the first local well-posedness result has been established in 1996, see [90], in the case when the fluid phases are both bounded and surface tension effects are not taken into account. Classical solutions for the Hele-Shaw problem, which is even older, were proven – to our best knowledge first – in [44], after weak solutions were shown to exist previously in [38].

In [90], but also in numerous studies thereafter, the well-posedness of the Muskat problem is established under the assumption that the Rayleigh–Taylor condition is satisfied at the free interface between the fluids. Introduced with the linear theory in [81], the Rayleigh–Taylor condition is a sign restriction on the jump of the pressure gradient in normal direction at the free interface, see Section 7.2. This intricate relation is equivalent, in the setting when the fluids have equal viscosities and are arranged vertically, to the condition that the fluid located below has a larger density. This is sometimes called in the literature the stable case, e.g. [22]. Since in this thesis we neglect surface tension effects, the Rayleigh–Taylor condition will also play an important role in our analysis. In the regime where the Rayleigh–Taylor condition is not satisfied in the sense that it holds with the reverse inequality sign, physical experiments show that in this unstable situation fingering patterns can occur, see e.g. [41, 73], and the mathematical model is ill-posed, see [33, 41, 61, 83].

Concerning well-posedness of the Muskat problem there are numerous studies which addressed this issue in several physical scenarios and with various methods. In [90] the author uses Newton's method to construct local solutions, the analyses in [1, 23, 40–44, 59–62] are based either on the abstract quasilinear parabolic theory presented in [8], see also [65], respectively [76], or on the fully nonlinear parabolicity theory due to Lunardi, see [57]. The well-posedness results in [2–4, 9, 10, 15, 24, 31–33, 46, 52, 71] are established by using the energy method, while the authors of [83] employ methods from complex analysis and a version of the Cauchy-Kowalevsky theorem. In the paper [12] the authors use a fixed point argument to construct local solutions for bounded initial geometry which posses a sharp corner.

A further topic of interest is to identify the equilibria of the problem and to study their stability properties, see [40, 41, 47, 62, 65, 76, 77]. At this point the authors take advantage of the availability of principles of linearized stability corresponding to the parabolic theories presented in [57], respectively [8, 65], and [76, 78, 79].

In the stable case discussed above, it is shown in the papers [17, 24–26, 36, 48, 60, 74, 83, 89] that for 'medium size' initial data the solutions to the Muskat problem are global in time. In this regime there are also studies which consider the singularity formation for non-global solutions, e.g. [20–22, 27, 28, 30, 51, 83]. Quiet recently in [18, 19], by using the method of convex integration, the authors have constructed in the unstable regime so called mixing solutions which posses a diffuse interface region where the two fluids mix up and the density is in this region no longer constant.

We also point out that the Muskat problem without surface tension effects is the singular limit or the Muskat problem with surface tension when the surface tension coefficient vanishes. This property has been rigorously established in the papers [9, 46].

The multiphase Muskat problem: State of the art

In this thesis we study the dynamics in a two-dimensional fluid system that consists of three immiscible and incompressible fluids which are located in a homogeneous porous medium. The mathematical model, the so called multiphase Muskat problem, uses Darcy's law as the main constitutive equation, and is presented in detail in Section 1.2 below. Despite the many applications, see [5, 6, 14], this model has only been recently investigated from a mathematical point of view and only in the very particular situation when the three fluids have equal viscosities. More precisely, in [51], the authors considered the multiphase Muskat problem in $\mathbb{R}^3$ in the stable regime where the fluids are ordered according to their densities, with the fluids possessing larger densities located below, and formulated the model as an evolution problem for the functions that parameterize the sharp interfaces that separate the fluid phases. By using energy estimates they then show that the problem is locally well-posed in $H^k(\mathbb{R}^2)$ with $k \geq 4$.

A similar situation was considered in the papers [71, 75] in $\mathbb{R}^2$, but this research considers one fluid phase as being air at uniform pressure. There are, to our best knowledge, no other results on the well-posedness of the multiphase Muskat problem.

Concerning singularity formation for the multiphase Muskat problem, the authors of [51] have shown that solutions which do not exist globally, but remain bounded in $\mathrm{BUC}^{1+\alpha}(\mathbb{R}^2)$, with $\alpha \in (0,1)$, do not form squirt singularities, that is the fluid interfaces cannot touch along a curve segment when the time elapses. Furthermore, for non-global solutions in a two-dimensional setting it is shown in [49] that uniform bounds on the curvature of the interfaces prevents also the occurrence of splash singularities, that is single point collisions of the interface. The situation is different in the framework of the one-phase Muskat problem where splash singularities are one of the blow-up mechanisms, see [22, 30], while squirt singularities cannot occur [29, 30].

Goals

In view of the existing results on the multiphase Muskat problem, one of the main goals of this thesis is to establish the first local well-posedness result for the multiphase Muskat problem with general viscosities. We emphasize that, by a scaling invariance argument, one may identify the Sobolev space $H^{3/2}(\mathbb{R})^2$ as a critical space for this two-dimensional multiphase Muskat problem, see Remark 2.2 and [1, 74]. Therefore, our goal is to establish the well-posedness of the multiphase Muskat problem in all subcritical phase spaces $H^r(\mathbb{R})^2$, with $r > 3/2$. The corresponding local well-posedness result is presented in Theorem 2.1 in the stable case when the fluids have equal viscosities, and in Theorem 2.4 in the general case when no restrictions on the viscosities are imposed. Moreover, we prove in Proposition 2.3 that also in the two-dimensional setting solutions which are not global but bounded in the phase space cannot form squirt singularities despite the fact that the interfaces intersect, at least at one point, in the limit when the time elapses. In Theorem 2.1 and Theorem 2.4 we establish, additionally to the local well-posedness result, two parabolic smoothing properties where we improve the regularity of the solutions provided by the local well-posendess results. We will give a more detailed overview of our results in Section 2.2.

1.2. The classical formulation of the multiphase Muskat problem

We now present the classical formulation of the multidimensional Muskat problem which is a model for the motion of three incompressible and immiscible fluids with positive constant densities

$$\rho_3 > \rho_2 > \rho_1$$

in a vertical porous medium. In our setting the porous medium is homogeneous with permeability constant $k > 0$, see [45] for descriptions of permeability, the flow is two-dimensional, and the fluid phases cover the entire plane $\mathbb{R}^2$. Moreover, the fluids are assumed to be separated at each time instant $t \geq 0$ by sharp interfaces which are parameterized as graphs over the real line, that is

$$\Gamma_f^{c_\infty}(t) := \{(x, c_\infty + f(t,x)) \,:\, x \in \mathbb{R}\} \qquad \text{and} \qquad \Gamma_h(t) := \{(x, h(t,x)) \,:\, x \in \mathbb{R}\},$$

where c_∞ is a fixed positive constant. We restrict our study to the nondegenerate situation where the distance between the graphs $\Gamma_f^{c_\infty}(t)$ and $\Gamma_h(t)$ is positive during the flow. Moreover, the fluids layers are arranged according to their density. More precisely, the fluid with density ρ_i

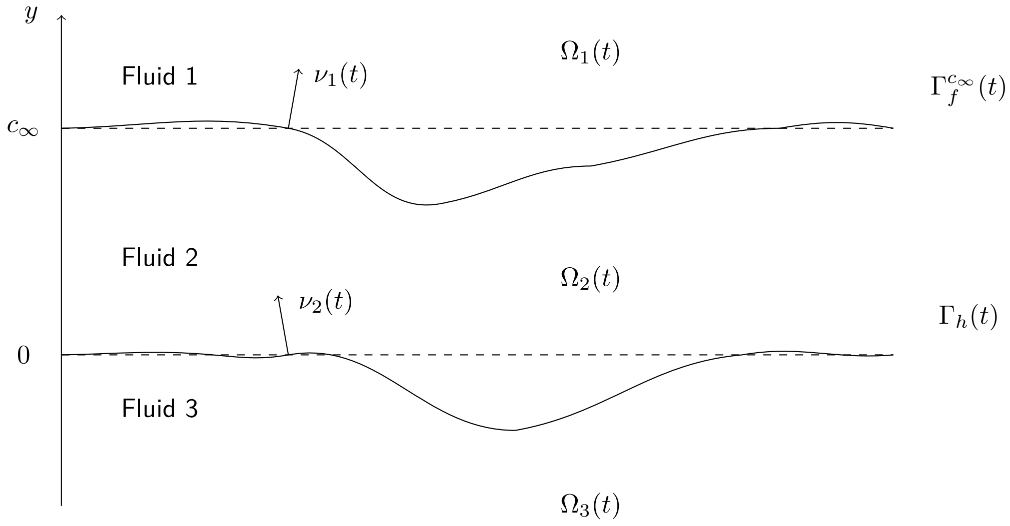

Figure 1.1.: The geometrical configuration of the multiphase Muskat problem

is located at $\Omega_i(t) \subset \mathbb{R}^2$, $1 \leq i \leq 3$, where

$$\begin{aligned}\Omega_1(t) &:= \{(x, y) \in \mathbb{R}^2 : y > c_\infty + f(t, x)\},\\ \Omega_2(t) &:= \{(x, y) \in \mathbb{R}^2 : h(t, x) < y < c_\infty + f(t, x)\},\\ \Omega_3(t) &:= \{(x, y) : y < h(t, x)\},\end{aligned}$$

see Figure 1.1.

Since the fluids are incompressible and the flow occurs at low Reynolds numbers, cf. [13], in the fluid layers the motion is governed by the following equations

$$\left.\begin{aligned} v_i(t) &= -\frac{k}{\mu_i}\big(\nabla p_i(t) + (0, \rho_i g)\big),\\ \operatorname{div} v_i(t) &= 0 \end{aligned}\right\} \qquad \text{in } \Omega_i(t),\ 1 \leq i \leq 3, \tag{1.1a}$$

where μ_i, $p_i(t)$, and $v_i(t) := (v_i^1(t), v_i^2(t))$ is the viscosity, pressure, and velocity, respectively, of the fluid located in $\Omega_i(t)$. The positive constant g is the acceleration due to Earth's gravity. The equation $(1.1\text{a})_1$ is Darcy's law which is, cf., e.g., [13], the standard model for flows in porous media and it enjoys three important properties which we would like to point out. Firstly, an increase of permeability or a decrease of viscosity leads to higher velocity amplitudes. Secondly, the negative sign in front of the pressure gradient forces flow from regions with higher to regions with lower pressure. Thirdly, the additive term $-(0, kg(\rho_i/\mu_i))$ describes the fact that, under the influence of gravity, fluid particles move downwards. Moreover, the equation $(1.1\text{a})_2$ can be derived from conservation of mass together with the assumption of constant density within each fluid layer.

Since we neglect surface tension effects, the equations (1.1a) are supplemented by the following (natural) boundary conditions

$$\left.\begin{aligned} p_i(t) &= p_{i+1}(t),\\ \langle v_i(t)|\nu_i(t)\rangle &= \langle v_{i+1}(t)|\nu_i(t)\rangle \end{aligned}\right\} \qquad \text{on } \partial\Omega_i(t) \cap \partial\Omega_{i+1}(t),\ i = 1, 2, \tag{1.1b}$$

where $\nu_i(t)$ is the unit normal at $\partial\Omega_i(t) \cap \partial\Omega_{i+1}(t)$ pointing into $\Omega_i(t)$, cf. Figure 1.1, and $\langle \cdot | \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^2$.

Additionally, the flow should satisfy the far-field boundary conditions

$$\left.\begin{aligned} v_i(t,x,y) &\to 0 \qquad \text{for } |(x,y)| \to \infty,\ 1 \le i \le 3, \\ f^2(t,x) + h^2(t,x) &\to 0 \qquad \text{for } |x| \to \infty, \end{aligned}\right\} \tag{1.1c}$$

that is far away from the origin the flow is nearly stationary.

In addition, the normal velocity of the interfaces equals the normal component of the velocity field at the free boundary, which is expressed by

$$\left.\begin{aligned} \partial_t f(t) &= \langle v_1(t) | (-\partial_x f(t), 1)\rangle \qquad \text{on } \Gamma_f^{c_\infty}(t), \\ \partial_t h(t) &= \langle v_2(t) | (-\partial_x h(t), 1)\rangle \qquad \text{on } \Gamma_h(t). \end{aligned}\right\} \tag{1.1d}$$

The equations (1.1a)-(1.1d) should hold for each $t \ge 0$.

Finally, the interfaces are assumed to be known initially

$$(f,h)(0,\cdot) = (f_0, h_0). \tag{1.1e}$$

We call the closed system (1.1) the multiphase Muskat problem.

We conclude this section by defining what is meant by a solution to the multiphase Muskat problem (1.1).

Definition 1.1. Let the initial data $X_0 = (f_0, h_0)\colon \mathbb{R} \to \mathbb{R}^2$ be a continuously differentiable function which vanishes at infinity and satisfies

$$\inf(c_\infty + f_0 - h_0) > 0.$$

We call the tuple $(v_1, v_2, v_3, p_1, p_2, p_3, X)$, with $X = (f,h)$, a *classical solution to the multiphase Muskat problem* (1.1), if there exists $T \in (0,\infty]$ such that

- $p_i(t,\cdot) \in \mathrm{BUC}(\Omega_i(t)) \cap \mathrm{C}^2(\Omega_i(t))$ for all $t \in [0,T)$, $i = 1,2,3$,
- $v_i(t,\cdot) \in \mathrm{BUC}(\Omega_i(t), \mathbb{R}^2) \cap \mathrm{C}^1(\Omega_i(t), \mathbb{R}^2)$ for all $t \in [0,T)$, $i = 1,2,3$,
- $f, h \in \mathrm{C}^1([0,T) \times \mathbb{R})$,
- $\inf(c_\infty + f(t,\cdot) - h(t,\cdot)) > 0, \qquad$ for all $t \in [0,T)$,

and (1.1) is satisfied pointwise for all time instants $t \in [0,T)$.

We point out that our well-posedness results stated in Section 2.2 provide solutions that enjoy even better regularity properties than stated in this definition. In our analysis we prove that at each time instant $t \ge 0$, the velocities and the pressures are uniquely determined by $X(t)$. Therefore, we shall also refer only to X as being the solution to (1.1).

1.3. Notation

We introduce in this section some notation which will be used throughout the thesis.

Given k, $n \in \mathbb{N}$ and an open set $\Omega \subset \mathbb{R}^n$, we denote by $\mathrm{C}^k(\Omega)$ the space consisting of real- or complex-valued k-time continuously differentiable functions on Ω, and $\mathrm{UC}^k(\Omega)$ is the subspace of $\mathrm{C}^k(\Omega)$ collecting functions with uniformly continuous derivatives up to order k. Moreover, $\mathrm{BUC}^k(\Omega)$ is the Banach space of functions with bounded and uniformly continuous derivatives up to order k. Furthermore, given $\alpha \in (0,1)$, we set

$$\mathrm{BUC}^{k+\alpha}(\Omega) := \Big\{ f \in \mathrm{BUC}^k(\Omega) \,:\, [\partial^\beta f]_\alpha := \sup_{x \ne y} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x-y|^\alpha} < \infty \ \ \forall\, |\beta| = k \Big\}, \tag{1.2}$$

which is, in the case of $k=0$, called the Hölder space with exponent α. Additionally, we set

$$\mathrm{C}^\infty(\Omega) \coloneqq \bigcap_{k\in\mathbb{N}} \mathrm{C}^k(\Omega) \qquad \text{and} \qquad \mathrm{BUC}^\infty(\Omega) \coloneqq \bigcap_{k\in\mathbb{N}} \mathrm{BUC}^k(\Omega),$$

and we define $\mathrm{C}_0^\infty(\Omega)$ to be the subset of $\mathrm{C}^\infty(\Omega)$ that contains only functions with compact support.

Moreover, if $\Gamma \subset \mathbb{R}^2$ is a continuous curve, we define $\mathrm{BUC}(\Gamma)$ to be the space of bounded and uniformly continuous functions mapping Γ to $\mathbb{R}$ and

$$\mathrm{BUC}^\alpha(\Gamma) \coloneqq \Big\{\varphi \in \mathrm{BUC}(\Gamma) : [f]_\alpha \coloneqq \sup_{\substack{\zeta,\xi\in\Gamma\\ \zeta\neq\xi}} \frac{|f(\zeta)-f(\xi)|}{|\zeta-\xi|^\alpha} < \infty\Big\}. \tag{1.3}$$

Given $T>0$ and a Banach space X, let $B((0,T],X)$ denote the Banach space of all bounded functions from $(0,T]$ into X. The Banach space $\mathrm{C}_\alpha^\alpha((0,T],X)$ is then defined as

$$\mathrm{C}_\alpha^\alpha((0,T],X) \coloneqq \Big\{f \in B((0,T],X) \,:\, \|f\|_{\mathrm{C}_\alpha^\alpha} \coloneqq \|f\|_\infty + \sup_{s\neq t} \frac{\|t^\alpha f(t) - s^\alpha f(s)\|_X}{|t-s|^\alpha} < \infty\Big\}. \tag{1.4}$$

Given Banach spaces X and Y, the set $\mathrm{C}^{1-}(X,Y)$ consists of all locally Lipschitz maps from X to Y and the space $\mathrm{C}^\infty(X,Y)$ is defined as the set of all smooth mappings from X to Y.

The Banach space of all bounded, linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$, and $\mathrm{Isom}(X,Y)$ is the open subset of $\mathcal{L}(X,Y)$ that contains only invertible operators. Furthermore, we write $A \in \mathcal{L}^k(X,Y)$ if $A:\ X^k \to Y$ is k-linear and bounded, whereas $A \in \mathcal{L}^k_{\mathrm{sym}}(X,Y)$ specifies that $A \in \mathcal{L}^k(X,Y)$ is additionally symmetric.

Moreover, we adopt some notation from [7, Section I.1.2] in order to describe that an operator is the generator of an analytic semigroup. Let $(X_0, \|\cdot\|_0)$ and $(X_1, \|\cdot\|_1)$ be Banach spaces, where X_1 is densely embedded into X_0. If $A \in \mathcal{L}(X_1, X_0)$ is the (infinitesimal) generator of an analytic semigroup we write

$$-A \in \mathcal{H}(X_1, X_0). \tag{1.5}$$

In order to abbreviate formulas, we further introduce the following notation

$$\begin{aligned}
\delta_{[x,s]} f &\coloneqq f(x) - f(x-s),\\
\delta_{[x,s]} X &\coloneqq c_\infty + f(x) - h(x-s),\\
\delta'_{[x,s]} X &\coloneqq h(x) - c_\infty - f(x-s),\\
\bar\delta_{[x,s]} X &\coloneqq f(x) - h(x-s),\\
\bar\delta'_{[x,s]} X &\coloneqq h(x) - f(x-s)
\end{aligned} \tag{1.6}$$

for all $x, s \in \mathbb{R}$ and $X = (f,h)$, where $f,h\colon \mathbb{R}\to\mathbb{R}$.

2. Structure of the thesis and the main results

In order to give the reader a guideline for this thesis, we now include some remarks on its structure, cf. Section 2.1, and formulate our main results, see Section 2.2, which will be proven in the following Chapters 3–7.

2.1. Structure of the thesis

We now give an overview of this thesis. After emphasizing the importance of flows in porous media, see Chapter 1, and especially of the multiphase Muskat problem, cf. Section 1.1, we present the classical formulation of the multiphase Muskat problem as the system (1.1). The main results of the thesis are collected below in Section 2.2, where we establish the local well-posedness of the multiphase Muskat problem first in the particular case when the fluids have equal viscosities, see Theorem 2.1, and, then in the general case when there is no restriction on the viscosities, see Theorem 2.4. Moreover, in Proposition 2.3 we prove in the case when the fluids have equal viscosities that squirt singularities cannot occur for solutions that are bounded in the phase space and which are not global.

The remaining of the thesis is devoted to the proof of these results and is divided into two parts. In Part I we establish our main results in the case of equal viscosities:

$$\mu \coloneqq \mu_1 = \mu_2 = \mu_3 > 0,$$

whereas in Part II we consider the general case:

$$\mu_1, \mu_2, \mu_3 > 0$$

and we prove Theorem 2.4.

In Part I we start the investigation of the multiphase Muskat problem with equal viscosities in Chapter 3 by expressing the classical formulation (1.1) as a nonlinear, nonlocal, and strongly coupled evolution problem of the form

$$\frac{dX(t)}{dt} = \Phi(X(t)), \quad t \geq 0, \qquad X(0, \cdot) = X_0 \coloneqq (f_0, h_0), \tag{2.1}$$

provided that the variable $X(t) \coloneqq (f(t), h(t))$ belongs to

$$\mathcal{O}_r \coloneqq \{(f, h) \in H^r(\mathbb{R})^2 : \inf(c_\infty + f - h) > 0\}, \qquad r \in (3/2, 2)$$

at each time instant $t \geq 0$. The operator Φ arises by the relations (1.1d). This reformulation is achieved by identifying the velocity and pressure fields in terms of the functions f and h, cf. Theorem 3.1 in Section 3.1, with the help of results from Appendix A. The coupling terms that occur are of highest order, since the highest (first) spatial derivatives of f and h appear in the same terms, which brings new difficulties compared to the two-phase problem.

Thereafter, in Chapter 4, we decompose the nonlocal operator $\Phi(X)$ in terms of X and certain (singular) integral operators for which we then establish several important mapping properties. We conclude this chapter with Corollary 4.6 which provides the smoothness of Φ as an operator from $\mathcal{O}_r$ to $H^{r-1}(\mathbb{R})^2$.

The analysis in Chapter 5 is devoted to the proofs of Theorem 2.1 and Proposition 2.3 which are presented in Section 5.2 and Section 5.3, respectively. An important preliminary step is provided in Section 5.1 where we prove that the Fréchet derivative $\partial\Phi(X)$ generates an analytic semigroup in $\mathcal{L}(H^{r-1}(\mathbb{R})^2)$ for each $X \in \mathcal{O}_r$. This property identifies the Muskat problem (2.1) as a parabolic evolution problem and enables us to use abstract parabolic theory due to Lunardi, see [57], in the proof of the main results.

In Part II we consider the general case when the assumption of equal viscosities is dropped. The first main goal is to show that again the velocity and the pressure are identified at each time instant t by $X(t)$, see Theorem 6.6 in Section 6.3.

In contrast to the analysis in Part I, in the case of general viscosities the velocity is expressed again as a contour integral, but now this integral involves a density function $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2)$, which is determined as the solution to a linear equation, that comprises the adjoint of the double layer potential operator for Laplace's equation for the hypersurface $\Gamma_f^{c_\infty} \cup \Gamma_h$. The unique solvability of this linear equation is studied in Section 6.2, see Theorem 6.4, where the open set $\mathcal{U}_r$, of all pairs $X \in \mathcal{O}_r$ for which this equation is uniquely solvable, is characterized.

In Chapter 7 we first take advantage of Theorem 6.6 to reformulate in Section 6.3 the multiphase Muskat problem (1.1) as a nonlinear and nonlocal equation of the same form as in the previous case, cf. (2.1). Then, in Section 7.2 we establish the smoothness of Φ as a mapping from $\mathcal{U}_r$ to $H^{r-1}(\mathbb{R})^2$, see Corollary 7.3. Furthermore, we show that the Rayleigh–Taylor condition can be expressed in terms of $\Phi(X)$ and we identify in Section 7.2 an open subset $\mathcal{V}_r$ of $\mathcal{U}_r$ which consists only of pairs for which the Rayleigh–Taylor conditions are satisfied. As a second important result in Section 7.2, we prove in Theorem 7.4 that, for each $X \in \mathcal{V}_r$, the Fréchet derivative $\partial\Phi(X)$ generates an analytical semigroup in $\mathcal{L}(H^{r-1}(\mathbb{R})^2)$. This parabolicity property, the smoothness result in Corollary 7.3, and the abstract parabolic theory presented in [57] are the key ingredients in the proof of our main result Theorem 2.4, which we provide in Section 7.3.

In Appendix A we extend the classical Plemelj formula and the Privalov theorem to the setting of unbounded graphs, which is relevant for the analysis in Part I and Part II. Moreover, in Appendix B we collect results for a particular family of singular integral operators that are used to define the operator Φ, see (2.1). Finally, in Appendix C we recall some basic properties of the Hilbert transform and of two different truncations of it.

2.2. Main results

We now present the main results of this thesis. As mentioned before, given $r \in (3/2, 2)$ we define

$$\mathcal{O}_r := \{X = (f, h) \in H^r(\mathbb{R})^2 : \inf(c_\infty + f - h) > 0\}.$$

The first main result in Part I is formulated in Theorem 2.1 below, where we establish the local well-posedness of the multiphase Muskat problem together with two parabolic smoothing properties for the local solutions to (1.1). Furthermore, we describe the behavior of solutions, which are not global, in the limit when the time variable approaches the maximal existence time.

Theorem 2.1. *Let $r \in (3/2, 2)$ and $\mu_1 = \mu_2 = \mu_3$. Given $X_0 \in \mathcal{O}_r$, the multiphase Muskat problem* (1.1) *possesses a unique maximal solution $X := X(\,\cdot\,; X_0)$ such that*

$$X \in \mathrm{C}([0, T^+), \mathcal{O}_r) \cap \mathrm{C}^1([0, T^+), H^{r-1}(\mathbb{R})^2),$$

with $T^+ = T^+(X_0) \in (0, \infty]$ denoting the maximal time of existence. Furthermore, the associated velocities and pressures satisfy

- $v_i(t) \in \mathrm{BUC}(\Omega_i(t)) \cap \mathrm{C}^\infty(\Omega_i(t))$, $p_i(t) \in \mathrm{UC}^1(\Omega_i(t)) \cap \mathrm{C}^\infty(\Omega_i(t))$ *for* $1 \leq i \leq 3$,
- $[x \mapsto v_i(t, x, c_\infty + f(t, x))] \in H^{r-1}(\mathbb{R})^2$ *for* $1 \leq i \leq 2$,
- $[x \mapsto v_i(t, x, h(t, x))] \in H^{r-1}(\mathbb{R})^2$ *for* $2 \leq i \leq 3$

for each $0 \leq t < T^+$.

Moreover, we have:

(i) *The solution depends continuously on the initial data;*

(ii) *Given* $k \in \mathbb{N}$, *we have* $X \in \mathrm{C}^\infty((0,T^+)\times\mathbb{R},\mathbb{R}^2)\cap \mathrm{C}^\infty((0,T^+),H^k(\mathbb{R})^2)$;

(iii) *If* $T^+<\infty$, *then*

$$\sup_{t\in[0,T^+)}\|X(t)\|_{H^r(\mathbb{R})}=\infty \qquad \text{or} \qquad \liminf_{t\to T^+}\operatorname{dist}(\Gamma_f^{c_\infty}(t),\Gamma_h(t))=0.$$

With respect to the choice of the index $r\in(3/2,2)$ of the Sobolev spaces in Theorem 2.1 (and also in Theorem 2.4 below) we add the following remark.

Remark 2.2. If X is a classical solution to (1.1), then, given $\lambda>0$, also

$$X_\lambda(t,x):=\lambda^{-1}X(\lambda t,\lambda x),\qquad t\geq 0,\ x\in\mathbb{R},$$

is a solution to (1.1) (with initial datum $\lambda^{-1}X_0$). This property identifies $H^{3/2}(\mathbb{R})^2$ as a critical space for the evolution problem (1.1). Therefore, our result in Theorem 2.1 (and in Theorem 2.4 below) covers all subcritical $L_2(\mathbb{R})$-based Sobolev spaces $H^r(\mathbb{R}^2)$ with $r\in(3/2,2)$.

The next proposition states, for bounded solutions which do not exist globally in time, that the fluid interfaces intersect in at least one point along a sequence $(t_n)_{n\in\mathbb{N}}$ with limit T^+. Moreover, using the same strategy as in [27, 32], we exclude in this case that the interfaces collapse along a curve segment.

Proposition 2.3. *Let* $X\in \mathrm{C}([0,T^+),\mathcal{O}_r)\cap \mathrm{C}^1([0,T^+),H^{r-1}(\mathbb{R}))$ *be a maximal solution to* (1.1) *with* $T^+<\infty$ *and such that* $\sup_{t\in[0,T^+)}\|X(t)\|_{H^r}<\infty$. *Then, there exists* $x_0\in\mathbb{R}$ *with the property that*

$$\inf_{t\in[0,T^+)}(c_\infty+f(t,x_0)-h(t,x_0))=0. \tag{2.2}$$

Moreover, for each x_0 *satisfying* (2.2) *and for each* $\delta>0$, *we have*

$$\inf_{t\in[0,T^+)}\sup_{\{|x-x_0|\leq\delta\}}(c_\infty+f(t,x)-h(t,x))>0.$$

The results of Part I have recently been published in [16].

The main result of Part II, where we consider the case of different viscosities, is stated in Theorem 2.4 below where we establish the first local well-posedness result in this general situation and, besides, we provide the same parabolic smoothing properties as in the previous case of equal viscosities. In Theorem 2.4 the open subset $\mathcal{V}_r\subset\mathcal{U}_r$ is, as mentioned before in Section 2.1, identified by the Rayleigh–Taylor condition and the unique solvability of the linear equation for the density $\bar\omega$, and is introduced in (7.10).

Theorem 2.4. *Let* $r\in(3/2,2)$. *Given* $X_0\in\mathcal{V}_r$, *the multiphase Muskat problem* (1.1) *possesses a unique maximal solution* $X:=X(\,\cdot\,;X_0)$ *such that*

$$X\in \mathrm{C}([0,T^+),\mathcal{V}_r)\cap \mathrm{C}^1([0,T^+),H^{r-1}(\mathbb{R})^2),$$

where $T^+=T^+(X_0)\in(0,\infty]$ *denotes the maximal time of existence, and the associated velocities and pressures satisfy for each* $0\leq t<T^+$

- $v_i(t)\in \mathrm{BUC}(\Omega_i(t))\cap \mathrm{C}^\infty(\Omega_i(t))$, $p_i(t)\in \mathrm{UC}^1(\Omega_i(t))\cap \mathrm{C}^\infty(\Omega_i(t))$ *for* $1\leq i\leq 3$,
- $[x\mapsto v_i(t,x,c_\infty+f(t,x))]\in H^{r-1}(\mathbb{R})^2$ *for* $1\leq i\leq 2$,
- $[x\mapsto v_i(t,x,h(t,x))]\in H^{r-1}(\mathbb{R})^2$ *for* $2\leq i\leq 3$.

Moreover, we have:

(i) *The solution depends continuously on the initial data;*

(ii) *Given* $k\in\mathbb{N}$, *we have* $X\in \mathrm{C}^\infty((0,T^+)\times\mathbb{R},\mathbb{R}^2)\cap \mathrm{C}^\infty((0,T^+),H^k(\mathbb{R})^2)$.

Part I.

The multiphase Muskat problem with equal viscosities

In this first part of the thesis we restrict our attention to the particular case where the three fluids have equal viscosity, which we denote by

$$\mu := \mu_1 = \mu_2 = \mu_3 > 0.$$

We prove in Chapter 3 that the velocity field

$$v = v_1|_{\Omega_1} + v_2|_{\Omega_2} + v_3|_{\Omega_3}$$

can be expressed as the contour integral

$$\begin{aligned} v(x,y) = {} & \frac{1}{\pi}\int_{\mathbb{R}} \frac{\big[(x,y)-(s,c_\infty+f(s))\big]^\perp}{|(x,y)-(s,c_\infty+f(s))|^2}\bar\omega_1(s)\,ds \\ & + \frac{1}{\pi}\int_{\mathbb{R}} \frac{\big[(x,y)-(s,h(s))\big]^\perp}{|(x,y)-(s,h(s))|^2}\bar\omega_2(s)\,ds, \qquad (x,y)\in\mathbb{R}^2\setminus\big(\Gamma_f^{c_\infty}\cup\Gamma_h\big), \end{aligned}$$

where $(a,b)^\perp := (-b,a)$ and the density $\bar\omega = (\bar\omega_1,\bar\omega_2)$ is given by

$$\bar\omega = \big(\Theta_1 f', \Theta_2 h'\big)$$

with

$$\Theta_1 = kg\frac{\rho_1-\rho_2}{\mu_1+\mu_2} \qquad \text{and} \qquad \Theta_2 = kg\frac{\rho_2-\rho_3}{\mu_2+\mu_3},$$

see Theorem 3.1. This representation is then used in Section 3.2 to formulate the multiphase Muskat problem as a parabolic evolution equation for the unknown $X = (f,h)$ in the following way

$$\frac{dX(t)}{dt} = \Phi(X(t)), \quad t\geq 0, \qquad X(0,\cdot) = X_0 := (f_0,h_0).$$

The nonlinear operator $\Phi := (\Phi_1,\Phi_2)$ is defined by

$$\begin{aligned} \Phi_1(X) &:= \langle v_1|_{\Gamma_f^{c_\infty}} \mid (-f',1)\rangle, \\ \Phi_2(X) &:= \langle v_2|_{\Gamma_h} \mid (-h',1)\rangle, \end{aligned}$$

where the traces of the velocity at the free boundaries are computed by making use of the results presented in Appendix A.

In Chapter 4 we prove, together with results from Appendix B and Appendix C, that the nonlinear and nonlocal operator Φ depends smoothly on the variable X, cf. Corollary 4.6.

In Chapter 5 we provide the proofs of our main results in Theorem 2.1 and Proposition 2.3. As a preliminary step we show in Theorem 5.10 that the Fréchet derivative of Φ is the generator of an analytic semigroup. This property identifies the multiphase Muskat problem as a parabolic evolution problem and enables us to use abstract results due to A. Lunardi [57] in the proofs of our main results.

3. The contour integral formulation

A remarkable property of (1.1) is the fact that the partial differential equations (1.1a)-(1.1c) are linear with respect to the unknowns v_i and p_i, $i = 1, 2, 3$. This property enables us to identify the velocity field in terms of the a priori unknown functions f and h by means of contour integrals. Such an approach has been followed in the context of the Muskat problem at least at formal level, already in the 1980's, cf. [37], and it is also one of the main steps in our analysis.

For the clarity of the exposition we omit here the time dependence and write $(\cdot)'$ for the x-derivative of functions that depend only on x.

In Theorem 3.1 below we provide, under suitable regularity constraints, an explicit formula for the velocity field in terms of the variable $X := (f, h)$. Our approach generalizes the one followed in [60] in the context of the two-phase Muskat problem and strongly relies on results from Part III Appendices.

3.1. The fixed time problem

As a first step, we fix the interfaces in the full model (1.1), which means, that we fix geometry and time. In advance, we can find explicit formulas for the velocity and pressure fields in terms of the functions parameterizing the interfaces. Thus, we reduce the number of unknowns to only $X = (f, h)$. This also means that the so called fixed time problem, see (3.1), as a subsystem of the full model (1.1), has a unique solution, if we consider $X = (f, h)$ given. We present this result in the following theorem.

Theorem 3.1. *Let $r \in (3/2, 2)$, $c_\infty > 0$, and $f,\, h \in H^r(\mathbb{R})$ with $\inf(c_\infty + f - h) > 0$ be given. Then the boundary value problem*

$$\left.\begin{array}{rcll} v_i &=& -\dfrac{k}{\mu}\big(\nabla p_i + (0, \rho_i g)\big) & \text{in } \Omega_i,\ 1 \le i \le 3, \\ \operatorname{div} v_i &=& 0 & \text{in } \Omega_i,\ 1 \le i \le 3, \\ p_i &=& p_{i+1} & \text{on } \partial\Omega_i \cap \partial\Omega_{i+1},\ i = 1,\, 2, \\ \langle v_i | \nu_i \rangle &=& \langle v_{i+1} | \nu_i \rangle & \text{on } \partial\Omega_i \cap \partial\Omega_{i+1},\ i = 1,\, 2, \\ v_i(x,y) &\to& 0 & \text{for } |(x,y)| \to \infty,\ 1 \le i \le 3 \end{array}\right\} \tag{3.1}$$

has a unique solution[1] $(v_1, v_2, v_3, p_1, p_2, p_3)$ *with*

$$v_i \in \mathrm{BUC}(\Omega_i) \cap \mathrm{C}^\infty(\Omega_i) \qquad \text{and} \qquad p_i \in \mathrm{UC}^1(\Omega_i) \cap \mathrm{C}^\infty(\Omega_i), \qquad 1 \le i \le 3.$$

Moreover, setting $v := v_1 \mathbf{1}_{\Omega_1} + v_2 \mathbf{1}_{\Omega_2} + v_3 \mathbf{1}_{\Omega_3}$, it holds for $z := (x, y) \in \mathbb{R}^2 \setminus (\Gamma_h \cup \Gamma_f^{c_\infty})$ that

$$v(z) = \frac{\Theta_1}{\pi} \int_{\mathbb{R}} \frac{(c_\infty + f(s) - y, x - s)}{(x-s)^2 + (y - c_\infty - f(s))^2} f'(s)\, ds + \frac{\Theta_2}{\pi} \int_{\mathbb{R}} \frac{(h(s) - y, x - s)}{(x-s)^2 + (y - h(s))^2} h'(s)\, ds, \tag{3.2}$$

with constants

$$\Theta_1 := kg \frac{\rho_1 - \rho_2}{2\mu} \qquad \text{and} \qquad \Theta_2 := kg \frac{\rho_2 - \rho_3}{2\mu}. \tag{3.3}$$

Proof. We devise the proof into two steps.

[1] The pressures (p_1, p_2, p_3) are unique up to the same additive constant.

Existence. Let $v : \mathbb{R}^2 \setminus (\Gamma_h \cup \Gamma_f^{c_\infty}) \to \mathbb{R}^2$ be given by (3.2) and set $v_i := v|_{\Omega_i}$, $1 \le i \le 3$. In the notation from Appendix III, see (A.4), it holds that

$$v(z) = 2\Theta_1 v(f)[f'](z - (0, c_\infty)) + 2\Theta_2 v(h)[h'](z), \qquad z \in \mathbb{R}^2 \setminus (\Gamma_h \cup \Gamma_f^{c_\infty}).$$

Then $v \in \mathrm{C}^\infty(\mathbb{R}^2 \setminus (\Gamma_h \cup \Gamma_f^{c_\infty}))$ and, according to Theorem A.6, we also have $v_i \in \mathrm{BUC}^{r-3/2}(\Omega_i)$ for $1 \le i \le 3$. Moreover, Lemma A.7 yields that $(3.1)_2$ and $(3.1)_5$ hold true. In view of Lemma A.4 we further get

$$\begin{aligned} v_i(x, c_\infty + f(x)) = {}& \frac{\Theta_1}{\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{(-\delta_{[x,s]} f, s)}{s^2 + (\delta_{[x,s]} f)^2} f'(x-s)\, ds + \frac{\Theta_2}{\pi} \int_{\mathbb{R}} \frac{(-\delta_{[x,s]} X, s)}{s^2 + (\delta_{[x,s]} X)^2} h'(x-s)\, ds \\ & + (-1)^i \Theta_1 \frac{f'(1, f')}{1 + f'^2}(x), \quad i = 1, 2, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} v_i(x, h(x)) = {}& \frac{\Theta_1}{\pi} \int_{\mathbb{R}} \frac{(-\delta'_{[x,s]} X, s)}{s^2 + (\delta'_{[x,s]} X)^2} f'(x-s)\, ds + \frac{\Theta_2}{\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{(-\delta_{[x,s]} h, s)}{s^2 + (\delta_{[x,s]} h)^2} h'(x-s)\, ds \\ & + (-1)^{i+1} \Theta_2 \frac{h'(1, h')}{1 + h'^2}(x), \quad i = 2, 3, \end{aligned} \tag{3.5}$$

where PV is the principal value, see A.2. The formulas (3.4) and (3.5) lead to

$$\begin{aligned} \langle v_1(x, f(x) + c_\infty) \mid \nu_1 \rangle &= \frac{1}{\sqrt{1 + f'(x)^2}} \Big(\frac{\Theta_1}{\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{f'(x)\delta_{[x,s]} f + s}{s^2 + (\delta_{[x,s]} f)^2} f'(x-s)\, ds \\ &\qquad + \frac{\Theta_2}{\pi} \int_{\mathbb{R}} \frac{f'(x)\delta_{[x,s]} X + s}{s^2 + (\delta_{[x,s]} X)^2} h'(x-s)\, ds \Big) \\ &= \langle v_2(x, f(x) + c_\infty) \mid \nu_1 \rangle, \end{aligned}$$

and

$$\begin{aligned} \langle v_2(x, h(x)) \mid \nu_2 \rangle &= \frac{1}{\sqrt{1 + h'(x)^2}} \Big(\frac{\Theta_1}{\pi} \int_{\mathbb{R}} \frac{h'(x)\delta'_{[x,s]} X + s}{s^2 + (\delta'_{[x,s]} X)^2} f'(x-s)\, ds \\ &\qquad + \frac{\Theta_2}{\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{h'(x)\delta_{[x,s]} h + s}{s^2 + (\delta_{[x,s]} h)^2} h'(x-s)\, ds \Big) \\ &= \langle v_3(x, h(x)) \mid \nu_2 \rangle, \end{aligned}$$

which proves the validity of $(3.1)_4$.

We next define pressures $p_i : \Omega_i \to \mathbb{R}$, $1 \le i \le 3$, via the formula

$$p_i(x, y) := -\frac{\mu}{k} \Big(\int_0^x \langle v_i(s, d_i(s)) | (1, d_i'(s)) \rangle\, ds + \int_{d_i(x)}^y v_i^2(x, s)\, ds \Big) - \rho_i g y + c_i, \tag{3.6}$$

where $v_i := (v_i^1, v_i^2)$, $c_i \in \mathbb{R}$ are constants, and with

$$d_1 := \|f\|_\infty + c_\infty + 1, \qquad d_2 := \frac{1}{2}(c_\infty + f + h), \qquad d_3 := -\|h\|_\infty - 1.$$

Taking advantage of $\partial_y v^1 = \partial_x v^2$ in $\mathbb{R}^2 \setminus (\Gamma_h \cup \Gamma_f^{c_\infty})$, cf. Lemma A.7, we deduce that $p_i \in \mathrm{C}^1(\Omega_i)$ by the computations

$$\begin{aligned}
\partial_x p_i(x,y) &= -\frac{\mu}{k}\Big(\langle v_i(x,d_i(x)) \mid (1,d_i'(x))\rangle + \int_{d_i(x)}^{y} \partial_x v_i^2(x,s)ds - v_i^2(x,y)d_i'(x)\Big) \\
&= -\frac{\mu}{k}\Big(v_i^1(x,d_i(x)) + \int_{d_i(x)}^{y} \partial_y v_i^1(x,s)ds\Big) \\
&= -\frac{\mu}{k} v_i^1(x,y)
\end{aligned}$$

and

$$\partial_y p_i(x,y) = -\frac{\mu}{k} v_i^2(x,y) - \rho_i g.$$

Indeed this shows also that $(3.1)_1$ is satisfied and the regularity properties established for v_i now imply that $p_i \in \mathrm{UC}^1(\Omega_i) \cap \mathrm{C}^\infty(\Omega_i)$, $1 \le i \le 3$.

Using $(3.1)_1$ and (3.4)-(3.5), it follows that

$$\begin{aligned}
\frac{d}{dx}(p_2 - p_1)(x, f(x) + c_\infty) &= \partial_x(p_2 - p_1)(x, f(x) + c_\infty) + \partial_y(p_2 - p_1)(x, f(x) + c_\infty) f'(x) \\
&= -\frac{\mu}{k}\langle (v_2 - v_1)(x, f(x) + c_\infty) \mid (1, f'(x))\rangle - (\rho_2 - \rho_1) g f'(x) \\
&= -\frac{\mu}{k}\Theta_1\Big(\frac{f'(x)}{1 + f'(x)^2} + \frac{f'(x)^3}{1 + f'(x)^2}\Big) - (\rho_2 - \rho_1) g f'(x) \\
&= (-\frac{\mu}{k}\Theta_1 - (\rho_2 - \rho_1) g) f'(x) = 0.
\end{aligned}$$

Analogously one computes that

$$\frac{d}{dx}(p_3 - p_2)(x, h(x)) = 0.$$

Hence, $(p_2 - p_1)|_{\Gamma_f^{c_\infty}}$ and $(p_3 - p_2)|_{\Gamma_h}$ are constants and thus, for a suitable choice of c_i, we may achieve that $(3.1)_3$ are satisfied. Therewith we established the existence of at least a solution to (3.1).

Uniqueness. In order to prove the uniqueness of a solution to (3.1) we first note, that given two solutions $(v_i, p_i), (\tilde{v}_i, \tilde{p}_i), i = 1, 2, 3$, the difference $(V_i := v_i - \tilde{v}_i, P_i := p_i - \tilde{p}_i), i = 1, 2, 3$ solves the system (3.1) if we only neglect the Earth's gravity g. It remains to show that the system (3.1) has, when setting the gravity constant g equal to zero, only the trivial solutions defined by $V = (V^1, V^2) = 0$ and $P = c \in \mathbb{R}$. To begin, we note that $(3.1)_1$ implies $\partial_y V_i^1 - \partial_x V_i^2 = 0$ in Ω_i for $1 \le i \le 3$. Moreover, combining $(3.1)_1$ and $(3.1)_3$ we get

$$V_1|_{\Gamma_f^{c_\infty}} = \nabla P_1|_{\Gamma_f^{c_\infty}} = \nabla P_2|_{\Gamma_f^{c_\infty}} = V_2|_{\Gamma_f^{c_\infty}},$$

which implies $V \in \mathrm{BUC}(\mathbb{R}^2)$. Stokes' theorem then yields

$$\partial_y V^1 - \partial_x V^2 = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{3.7}$$

We next set $\Psi := \psi_1 \mathbf{1}_{\overline{\Omega_1}} + \psi_2 \mathbf{1}_{\overline{\Omega_2}} + \psi_3 \mathbf{1}_{\overline{\Omega_3}}$, where $\psi_i : \overline{\Omega_i} \to \mathbb{R}$ are given by

$$\begin{aligned}
\psi_i(x,y) &:= \int_{h(x)}^{y} V^1(x,s)\,ds - \int_0^x \langle V(s,h(s)) | (-h'(s), 1)\rangle\,ds, \qquad i = 2,\ 3, \\
\psi_1(x,y) &:= \int_{c_\infty + f(x)}^{y} V^1(x,s)\,ds + \psi_2(x, c_\infty + f(x)).
\end{aligned}$$

Using Lemma 3.2 below, we obtain that $\Psi \in \mathrm{C}(\mathbb{R}^2)$, with $\nabla\psi_i = (-V^2, V^1)$ in $\mathcal{D}'(\Omega_i)$, $1 \le i \le 3$. As a direct consequence we get $\psi_i \in \mathrm{UC}^1(\Omega_i)$ for $1 \le i \le 3$. Additionally, $\nabla\Psi \in \mathcal{D}'(\mathbb{R}^2)$ belongs to $\mathrm{BUC}(\mathbb{R}^2)$, hence $\Psi \in \mathrm{UC}^1(\mathbb{R}^2)$. Therefore, given $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^2)$, we have

$$\langle \Delta\Psi, \varphi\rangle = \int_{\mathbb{R}^2} \Psi\Delta\varphi\, dz = -\int_{\mathbb{R}^2} \langle \nabla\Psi|\nabla\varphi\rangle\, dz = \int_{\mathbb{R}^2} \langle (V^2, -V^1)|\nabla\varphi\rangle\, dz = \langle \partial_y V^1 - \partial_x V^2, \varphi\rangle,$$

and (3.7) then yields $\Delta\Psi = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Consequently, Ψ is the real part of a holomorphic function $u : \mathbb{C} \to \mathbb{C}$. Since u' is holomorphic too and $u' = (\partial_x\Psi, -\partial_y\Psi) = -(V^2, V^1)$ is bounded, cf. $(3.1)_5$, Liouville's theorem yields $u' = 0$, hence $V = 0$. Moreover, in view of $(3.1)_1$, we now obtain that $\nabla P = 0$ in $\mathbb{R}^2$, meaning that P is constant in $\mathbb{R}^2$. This completes our arguments. □

In order to prove uniquness of solutions to (3.1), we needed the following auxiliary lemma.

Lemma 3.2. *We define* $\psi_i : \overline{\Omega_i} \to \mathbb{R}$ *by*

$$\psi_i(x, y) := \int_{h(x)}^{y} V^1(x, s)\, ds - \int_0^x \langle V(s, h(s))|(-h'(s), 1)\rangle\, ds, \qquad i = 2, 3,$$

$$\psi_1(x, y) := \int_{c_\infty + f(x)}^{y} V^1(x, s)\, ds + \psi_2(x, c_\infty + f(x)),$$

where $V := V_1\mathbf{1}_{\Omega_1} + V_2\mathbf{1}_{\Omega_2} + V_3\mathbf{1}_{\Omega_3}$. *Let* $V_i \in \mathrm{BUC}(\Omega_i) \cap \mathrm{C}^\infty(\Omega_i)$, $1 \le i \le 3$, *satisfy* $\operatorname{div} V_i = 0$ *in* Ω_i, *as well as*

$$\langle V_1|\nu_1\rangle = \langle V_2|\nu_1\rangle \quad \text{and} \quad \langle V_2|\nu_2\rangle = \langle V_3|\nu_2\rangle. \tag{3.8}$$

Then, $\Psi := \psi_1\mathbf{1}_{\overline{\Omega_1}} + \psi_2\mathbf{1}_{\overline{\Omega_2}} + \psi_3\mathbf{1}_{\overline{\Omega_3}} \in \mathrm{C}(\mathbb{R}^2)$ *and*

$$\nabla\psi_i = (-V^2, V^1) \quad \text{in } \mathcal{D}'(\Omega_i),\ i = 1, 2, 3.$$

Proof. It follows from (3.8) that $\Psi \in \mathrm{C}(\mathbb{R}^2)$.
Step 1: We prove that

$$\nabla\psi_2 = (-V^2, V^1) \text{ in } \mathcal{D}'(\Omega_2).$$

To this end we first note that

$$\partial_y\psi_2 = V_2^1 \quad \text{in } \Omega_2$$

in a classical sense, hence also in $\mathcal{D}'(\Omega_2)$.

We next show that

$$\partial_x\psi_2 = -V_2^2 \quad \text{in } \mathcal{D}'(\Omega_2).$$

The regularity of V_2 does not allow to simply interchange differentiation and integration. Our arguments are based to a large extend on the fact that $\operatorname{div} V_2 = 0$ in Ω_2. To start, let $\varphi \in \mathcal{D}(\Omega_2)$ and $\varepsilon > 0$ such that

$$\operatorname{supp}\varphi \subset \{(x, y) \in \mathbb{R}^2 : h(x) + \varepsilon < y < c_\infty + f(x) - \varepsilon\} =: \Omega_2^\varepsilon.$$

Defining

$$\phi(x, s) := \int_s^{c_\infty + f(x)} \varphi(x, y)\, dy, \quad (x, s) \in \Omega_2,$$

we obtain $\phi \in \mathrm{BUC}^\infty(\Omega_2)$ and $\phi|_{\Gamma_f^{c_\infty}} = 0$. Furthermore,

$$\langle \partial_x \psi_2 \mid \varphi \rangle = -\int_{\Omega_2} \psi_2 \partial_x \varphi \, ds = T_1 + T_2,$$

where

$$T_1 = -\int_{\Omega_2} \left(\int_{h(x)}^{y} V^1(x,s)\, ds \right) \partial_x \varphi(x,y)\, d(x,y),$$
$$T_2 = \int_{\Omega_2} \left(\int_0^x \langle V(s,h(s)) \mid (-h'(s),1)\rangle \, ds \right) \partial_x \varphi(x,y)\, d(x,y).$$

Concerning T_1, we have

$$\begin{aligned} T_1 &= -\int_{\mathbb{R}} \left(\int_{h(x)}^{c_\infty + f(x)} \left(\int_{h(x)}^{y} V^1(x,s) \partial_x \varphi(x,y)\, ds \right) dy \right) dx \\ &= -\int_{\mathbb{R}} \left(\int_{h(x)}^{c_\infty + f(x)} \left(\int_{s}^{c_\infty + f(x)} V^1(x,s) \partial_x \varphi(x,y)\, dy \right) ds \right) dx \\ &= -\int_{\Omega_2} \left(\int_{s}^{c_\infty + f(x)} \partial_x \varphi(x,y)\, dy \right) V^1(x,s)\, d(x,s) \\ &= -\int_{\Omega_2} (V^1 \partial_x \phi)(x,s)\, d(x,s). \end{aligned}$$

Using the fact that $\operatorname{div} V_2 = 0$ in Ω_2, we further compute

$$\begin{aligned} T_1 &= -\int_{\Omega_2} \partial_x(\phi V^1)(x,s) - \phi(x,s)\partial_x V^1(x,s)\, d(x,s) \\ &= -\int_{\Omega_2} \partial_x(\phi V^1)(x,s) + \phi(x,s)\partial_s V^2(x,s)\, d(x,s) \\ &= -\int_{\Omega_2} \partial_x(\phi V^1)(x,s) + \partial_s \left(\phi V^2\right)(x,s) - V^2(x,s)\partial_s \phi(x,s)\, d(x,s) \\ &= -\int_{\Omega_2} (V^2 \varphi)(x,s)\, d(x,s) - \int_{\Omega_2} \operatorname{div}(\phi V)(x,s)\, d(x,s). \end{aligned}$$

Again taking advantage of $\operatorname{div} V_2 = 0$ in Ω_2 and of the fact that $\varphi(x, c_\infty + f(x)) f'(x) = 0$, since $\operatorname{supp} \varphi$ is a compact subset of Ω_2, Stoke's theorem lead us to

$$\begin{aligned} \int_{\Omega_2} \operatorname{div}(\phi V)(x,s)\, d(x,s) &= \lim_{\varepsilon \searrow 0} \int_{\Omega_2^\varepsilon} \operatorname{div}(\phi V)(x,s)\, d(x,s) \\ &= \lim_{\varepsilon \searrow 0} \Big(\int_{\mathbb{R}} \langle (\phi V)(x, c_\infty + f(x) - \varepsilon) \mid (-f'(x),1)\rangle\, dx \\ &\qquad - \int_{\mathbb{R}} \langle (\phi V)(x, h(x) + \varepsilon) \mid (-h'(x),1)\rangle\, dx \Big) \\ &= -\int_{\mathbb{R}} \langle (\phi V)(x,h(x)) \mid (-h'(x),1)\rangle\, dx, \end{aligned}$$

which gives

$$T_1 = -\int_{\Omega_2} (V^2 \varphi)(x,s)\, d(x,s) - \int_{\mathbb{R}} \langle (\phi V)(x,h(x)) \mid (-h'(x),1)\rangle\, dx. \tag{3.9}$$

Moreover, since $\operatorname{supp}\varphi$ is a compact subset of Ω_2, Stokes' theorem implies that

$$\begin{aligned}
T_2 &= \int_{\Omega_2} \partial_x \Big(\varphi(x,y) \int_0^x \langle V(s,h(s)) \mid (-h'(s),1)\rangle \, ds \Big) \, d(x,y) \\
&\quad + \int_{\Omega_2} \langle V(x,h(x)) \mid (-h'(x),1)\rangle \varphi(x,y) \, d(x,y) \\
&= \int_{\mathbb{R}} \Big(\int_{h(x)}^{c_\infty + f(x)} \varphi(x,y) \, dy \Big) \langle V(x,h(x)) \mid (-h'(x),1)\rangle \, dx \\
&= \int_{\mathbb{R}} \langle (\phi V)(x,h(x)) \mid (-h'(x),1)\rangle \, dx.
\end{aligned}$$

Together with (3.9) it follows that

$$\langle \partial_x \psi_2 \mid \varphi \rangle = \langle -V^2 \mid \varphi \rangle, \quad \varphi \in \mathcal{D}(\Omega_2),$$

hence indeed

$$\nabla \psi_2 = (-V^2, V^1) \quad \text{in } \mathcal{D}'(\Omega_2).$$

Step 2. Here we prove that

$$\nabla \psi_1 = (-V^2, V^1) \quad \text{in } \mathcal{D}'(\Omega_1).$$

Again noticing that $\partial_y \psi_1 = V^1$ holds in a classical sense, and therefore also in $\mathcal{D}'(\Omega_1)$, it remains to show that

$$\partial_x \psi_1 = -V^2 \quad \text{in } \Omega_1.$$

Let $\varphi \in \mathcal{D}(\Omega_1)$ be given. For $\varepsilon > 0$ and $N \in \mathbb{N}$ we define

$$\begin{aligned}
\Omega_1^N &:= \{(x,y) \in \mathbb{R}^2 : c_\infty + f(x) < y < N\}, \\
\Omega_1^{\varepsilon,N} &:= \{(x,y) \in \mathbb{R}^2 : c_\infty + f(x) + \varepsilon < y < N\}.
\end{aligned}$$

In the following ε and N are chosen such that $\operatorname{supp}\varphi \subset \Omega_1^{\varepsilon,N}$. Then it holds

$$\begin{aligned}
\langle \partial_x \psi_1 \mid \varphi \rangle &= - \int_{\Omega_1} \psi_1(x,y) \partial_x \varphi(x,y) \, d(x,y) \\
&= - \int_{\Omega_1^N} \partial_x \varphi(x,y) \left(\int_{c_\infty + f(x)}^y V^1(x,s) \, ds \right) d(x,y) \\
&\quad - \int_{\Omega_1^N} \psi_2(x, c_\infty + f(x)) \partial_x \varphi(x,y) \, d(x,y) \\
&=: T_1 + T_2,
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= - \int_{\mathbb{R}} \left(\int_{c_\infty + f(x)}^N \partial_x \varphi(x,y) \left(\int_{c_\infty + f(x)}^y V^1(x,s) \, ds \right) dy \right) dx \\
&= - \int_{\mathbb{R}} \left(\int_{c_\infty + f(x)}^N V^1(x,s) \left(\int_s^N \partial_x \varphi(x,y) \, dy \right) ds \right) dx \\
&= - \int_{\Omega_1^N} V^1(x,s) \partial_x \left(\int_s^N \varphi(x,y) \, dy \right) d(x,s).
\end{aligned}$$

Setting

$$\phi(x,s) := \int_s^N \varphi(x,y)\,dy,$$

we obtain that $\phi \in \mathrm{BUC}^\infty(\Omega_1)$ and $\phi(x,N) = 0$ for all $x \in \mathbb{R}$. Hence, by taking advantage of $\operatorname{div} V_1 = 0$, it holds that

$$\begin{aligned} T_1 &= -\int_{\Omega_1^N} \partial_x(V^1\phi)(x,s) - ((\partial_x V^1)\phi)(x,s)\,d(x,s) \\ &= -\int_{\Omega_1^N} \partial_x(V^1\phi)(x,s) + \partial_s(V^2\phi)(x,s) - (V^2\partial_s\phi)(x,s)\,d(x,s) \\ &= -\int_{\Omega_1^N} (V^2\varphi)(x,s) + \operatorname{div}(V\phi)(x,s)\,d(x,s), \end{aligned}$$

with

$$\begin{aligned} \int_{\Omega_1^N} \operatorname{div}(V\phi)(x,s)\,d(x,s) &= \lim_{\varepsilon\searrow 0}\int_{\Omega_1^{\varepsilon,N}} \operatorname{div}(V\phi)(x,s)\,d(x,s) \\ &= \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}} \langle (V\phi)(x, c_\infty + f(x) + \varepsilon) \mid (-f'(x),1)\rangle\,dx \\ &= \int_{\mathbb{R}} \langle (V\phi)(x, c_\infty + f(x)) \mid (-f'(x),1)\rangle\,dx, \end{aligned}$$

by Stoke's Theorem. Thus,

$$T_1 = -\int_{\Omega_1^N} (V^2\varphi)(x,s)\,d(x,s) - \int_{\mathbb{R}} \langle (V\phi)(x, c_\infty + f(x)) \mid (-f'(x),1)\rangle\,dx.$$

Before further proceeding, let us first infer from a classical result, c.f. [11, Satz 6.24], that since $\psi_2, \partial_x\psi_2$, and $\partial_y\psi_2$ belong to $\mathrm{BUC}(\Omega_2)$, see Step 1, we know that $\psi_2 \in \mathrm{BUC}^1(\Omega_2)$. Using the fact that $\varphi = 0$ on $\partial\Omega_2^N$, we now obtain in view of Stoke's theorem that

$$\begin{aligned} T_2 &= -\int_{\Omega_1^N} \partial_x\left(\psi_2(x, c_\infty + f(x))\varphi(x,y)\right) - \partial_x(\psi_2(x, c_\infty + f(x)))\varphi(x,y)\,d(x,y) \\ &= -\int_{\Omega_1^N} \langle \nabla\psi_2(x, c_\infty + f(x)) \mid (1, f'(x))\rangle \partial_y\phi(x,y)\,d(x,y) \\ &= -\int_{\Omega_1^N} \langle (-V_2^2, V_2^1)(x, c_\infty + f(x)) \mid (1, f'(x))\rangle \partial_y\phi(x,y)\,d(x,y) \\ &= \int_{\mathbb{R}} \langle (V_1\phi)(x, c_\infty + f(x)) \mid (-f'(x),1)\rangle\,dx. \end{aligned}$$

In the last step we used the identity

$$\frac{1}{(1+f'^2)^{1/2}}\langle (-V_2^2|_{\Gamma_f^{c_\infty}}, V_2^1|_{\Gamma_f^{c_\infty}}) \mid (1,f')\rangle = -\langle V_2|_{\Gamma_f^{c_\infty}} \mid \nu_1\rangle = -\langle V_1|_{\Gamma_f^{c_\infty}} \mid \nu_1\rangle,$$

see (3.8). Therefore,

$$\partial_x\psi_1 = -V^2 \quad \text{in } \mathcal{D}'(\Omega_1),$$

and we obtain

$$\nabla\psi_1 = (-V^2, V^1) \quad \text{in } \mathcal{D}'(\Omega_1).$$

Step 3. As a last step we prove that

$$\nabla\psi_3 = (-V^2, V^1) \quad \text{in } \mathcal{D}'(\Omega_3).$$

Noticing that $\partial_y\psi_3 = V^1$ in a classical sense, and thus also in $\mathcal{D}'(\Omega_3)$, it remains to prove that

$$\partial_x\psi_3 = -V^2 \quad \text{in } \mathcal{D}'(\Omega_3).$$

Therefore, let $\varphi \in \mathcal{D}(\Omega_3)$ be given and choose $\varepsilon > 0$ as well as $N \in \mathbb{N}$ such that the sets

$$\begin{aligned} \Omega_3^N &:= \{(x,y) \in \mathbb{R}^2 : h(x) > y > -N\}, \\ \Omega_3^{\varepsilon,N} &:= \{(x,y) \in \mathbb{R}^2 : h(x) - \varepsilon > y > -N\} \end{aligned}$$

satisfy $\operatorname{supp}\varphi \subset \Omega_3^{\varepsilon,N} \subset \Omega_3^N$. Then

$$\phi(x,s) := \int_{-N}^{s} \varphi(x,y)\,dy, \quad (x,y) \in \Omega_3^N$$

fulfills $\phi \in \mathrm{BUC}^\infty(\Omega_3^N)$ and $\phi|_{\{y=-N\}} = 0$. We now compute

$$\begin{aligned} \langle \partial_x\psi_3 \mid \varphi\rangle &= -\int_{\Omega_3} \psi_3(x,y)\partial_x\varphi(x,y)\,d(x,y) \\ &= -\int_{\Omega_3^N} \partial_x\varphi(x,y)\left(\int_{h(x)}^{y} V^1(x,s)\,ds\right) d(x,y) \\ &\quad + \int_{\Omega_3^N} \partial_x\varphi(x,y)\left(\int_0^x \langle V(s,h(s)) \mid (-h'(s),1)\rangle\,ds\right) d(x,y) \\ &=: T_1 + T_2, \end{aligned}$$

where, in view of $\operatorname{div} V_3 = 0$ in Ω_3, it holds that

$$\begin{aligned} T_1 &= \int_{\mathbb{R}} \left(\int_{-N}^{h(x)} \left(\int_{-N}^{s} V^1(x,s)\partial_x\varphi(x,y)\,dy\right) ds\right) dx \\ &= \int_{\Omega_3^N} V^1(x,s)\partial_x\phi(x,s)\,d(x,s) \\ &= \int_{\Omega_3^N} \partial_x(V^1(x,s)\phi(x,s)) - \phi(x,s)\partial_x V^1(x,s)\,d(x,s) \\ &= \int_{\Omega_3^N} \partial_x(\phi V^1)(x,s) + \partial_s(\phi V^2)(x,s) - V^2(x,s)\partial_s\phi(x,s)\,d(x,s) \\ &= \int_{\Omega_3^N} \operatorname{div}(\phi V)(x,s) - (V^2\varphi)(x,s)\,d(x,s). \end{aligned}$$

Moreover, Stoke's theorem together with $\operatorname{div} V_3 = 0$ in Ω_3 implies that

$$\begin{aligned} \int_{\Omega_3^N} \operatorname{div}(\phi V)(x,s)\,d(x,s) &= \lim_{\varepsilon\searrow 0} \int_{\Omega_3^{\varepsilon,N}} \operatorname{div}(\phi V)(x,s)\,d(x,s) \\ &= \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}} \langle (\phi V)(x,h(x)-\varepsilon) \mid (-h'(x),1)\rangle\,dx \\ &= \int_{\mathbb{R}} \langle (\phi V)(x,h(x)) \mid (-h'(x),1)\rangle\,dx \end{aligned}$$

which gives

$$T_1 = \int_{\mathbb{R}} \langle (\phi V)(x, h(x)) \mid (-h'(x), 1) \rangle \, dx - \int_{\Omega_3} (V^2 \varphi)(x, s) \, d(x, s). \tag{3.10}$$

Concerning the remaining Term T_2, we use the fact that $\varphi = 0$ on $\partial\Omega_3^N$ together with Stoke's theorem to obtain

$$\begin{aligned} T_2 = & \int_{\Omega_3^N} \partial_x \left(\varphi(x, y) \left(\int_0^x \langle V(s, h(s)) \mid (-h'(s), 1) \rangle \, ds \right) \right) d(x, y) \\ & - \int_{\Omega_3^N} \varphi(x, y) \langle V(x, h(x)) \mid (-h'(x), 1) \rangle \, d(x, y) \\ = & - \int_{\Omega_3^N} \partial_y \left(\phi(x, y) \langle V(x, h(x)) \mid (-h'(x), 1) \rangle \right) d(x, y) \\ = & - \int_{\mathbb{R}} \langle (\phi V)(x, h(x)) \mid (-h'(x), 1) \rangle \, dx. \end{aligned}$$

Hence,

$$\partial_x \psi_3 = -V^2 \quad \text{in } \mathcal{D}'(\Omega_3),$$

and we conclude that

$$\nabla \psi_3 = (-V^2, V^1) \quad \text{in } \mathcal{D}'(\Omega_3).$$

□

3.2. The abstract evolution problem

Theorem 3.1 motivates us to define the set

$$\mathcal{O}_r := \{(f, h) \in H^r(\mathbb{R})^2 : \inf(c_\infty + f - h) > 0\}, \tag{3.11}$$

where $c_\infty > 0$ and $r \in (3/2, 2)$ are fixed. If (f, h, v, p) is a solution to (1.1) such that $(f(t), h(t))$ belongs to $\mathcal{O}_r$ for all time instants t in the definition interval of the solution, Theorem 3.1 identifies $v(t)$ according to the formula (3.2). Recalling the equations (1.1d) for the motion of the interfaces, we are now in a position to formulate the classical Muskat problem (1.1) as an autonomous evolution problem with $X = (f, h) \in \mathcal{O}_r$ as the unknown, which has the form

$$\frac{dX(t)}{dt} = \Phi(X(t)), \quad t \geq 0, \qquad X(0, \cdot) = X_0 := (f_0, h_0). \tag{3.12}$$

The nonlinear operator $\Phi := (\Phi_1, \Phi_2) \colon \mathcal{O}_r \to H^{r-1}(\mathbb{R})^2$ is defined by

$$\begin{aligned} \Phi_1(X) :=& \langle v_1|_{\Gamma_f^{c_\infty}} \mid (-f', 1) \rangle, \\ \Phi_2(X) :=& \langle v_2|_{\Gamma_h} \mid (-h', 1) \rangle, \end{aligned} \tag{3.13}$$

where $v_1|_{\Gamma_f^{c_\infty}}$ and $v_2|_{\Gamma_h}$ are identified in (3.4) and (3.5). Using these explicit formulas for the traces of the velocities, we arrive at the following expressions for the components of Φ

$$\begin{aligned}\Phi_1(X) &= \frac{\Theta_1}{\pi}\mathrm{PV}\int_{\mathbb{R}} \frac{s+f'(x)\delta_{[x,s]}f}{s^2+(\delta_{[x,s]}f)^2} f'(x-s)\,ds \\ &\quad + \frac{\Theta_2}{\pi}\int_{\mathbb{R}} \frac{s+f'(x)\delta_{[x,s]}X}{s^2+(\delta_{[x,s]}X)^2} h'(x-s)\,ds,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\Phi_2(X) &= \frac{\Theta_1}{\pi}\int_{\mathbb{R}} \frac{s+h'(x)\delta'_{[x,s]}X}{s^2+(\delta'_{[x,s]}X)^2} f'(x-s)\,ds \\ &\quad + \frac{\Theta_2}{\pi}\mathrm{PV}\int_{\mathbb{R}} \frac{s+h'(x)\delta_{[x,s]}h}{s^2+(\delta_{[x,s]}h)^2} h'(x-s)\,ds.\end{aligned} \tag{3.15}$$

We emphasize, that the problem is strongly coupled since the highest derivatives of f and h appear in the same terms. Another observation is that some of the appearing integrals are singular in the sense that the principal value is needed in order to have the integrals well-defined. In order to establish our results announced in Theorem 2.1 and Proposition 2.3, we prove in Chapter 4 and Chapter 5 that the nonlinear and nonlocal operator Φ is well-defined, smooth, and that its Fréchet derivative $\partial\Phi(X)$ is, for each $X \in \mathcal{O}_r$, the generator of an analytic semigroup in $\mathcal{L}(H^{r-1}(\mathbb{R})^2)$. With respect to the latter property, we mention that in this case we regard $\partial\Phi(X)$ as an unbounded operator in $H^{r-1}(\mathbb{R})^2$ with definition domain $H^r(\mathbb{R})^2$. Having established these fundamental properties, we can use abstract results on nonlinear, parabolic equations from [57] to prove Theorem 2.1.

4. Mapping properties

In order to establish the main goal of this chapter, the smoothness property

$$\Phi \in \mathrm{C}^\infty(\mathcal{O}_r, H^{r-1}(\mathbb{R})^2), \tag{4.1}$$

see Corollary 4.6 below, we have a more abstract view on the integrals that define (3.14) and (3.15). More precisely, we introduce five families of integral operators, see (4.2) and (4.4) below, which are later used to decompose the operators Φ_1 and Φ_2 in terms which involve only these integral operators and the unknown $X = (f, h)$, see (4.5) and (4.6). To begin, given Lipschitz continuous maps $u_1, \ldots, u_m, v_1, \ldots, v_n \colon \mathbb{R} \to \mathbb{R}$ and $\bar\omega \in L_2(\mathbb{R})$, we define the operator $B_{n,m}$, $n, m \in \mathbb{N}$ by the formula

$$B_{n,m}(u_1, \ldots, u_m)[v_1, \ldots, v_n, \bar\omega](x) := \frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}} \frac{\prod_{i=1}^n \big(\delta_{[x,s]}u_i/s\big)}{\prod_{i=1}^m \big[1 + \big(\delta_{[x,s]}v_i/s\big)^2\big]} \frac{\bar\omega(x-s)}{s}\, ds. \tag{4.2}$$

Here we used the notation introduced in (1.6) and PV is the principal value taken at 0, see (A.2). In order to keep formulas short, we further introduce

$$B^0_{n,m}(u)[\bar\omega] := B_{n,m}(u, \ldots, u)[u, \ldots, u, \bar\omega], \tag{4.3}$$

where $u \colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, see also (B.16). These operators have been introduced in [61] and [60] in the context of the two-phase Muskat problem. Important properties for this family of singular integral operators have been also established in [1] and [64]. We collect these results in Appendix B. It is worth mentioning that these operators are singular and that $B^0_{n,m} = H$, $n, m \in \mathbb{N}$, where H is the Hilbert transform, see Appendix C.

Moreover, corresponding to the coupling terms in the representations (4.5) and (4.6), we introduce four more classes of integral operators. To this end we define the set

$$\mathcal{O} := \{(f, h) \in W^1_\infty(\mathbb{R})^2 : \inf(c_\infty + f - h) > 0\}.$$

The former defined set $\mathcal{O}_r$, see (3.11), is obviously a subset of $\mathcal{O}$, since $H^r(\mathbb{R})$ embeds continuously into $W^1_\infty(\mathbb{R})$. Actually, since $c_\infty > 0$, it holds that

$$\mathcal{O}_r = \mathcal{O} \cap H^r(\mathbb{R})^2.$$

Given $1 \le m \in \mathbb{N}$ and $X_i := (f_i, h_i) \in \mathcal{O}$, $1 \le i \le m$, we set

$$\begin{aligned}
C_m(X_1, \ldots, X_m)[\bar\omega](x) &:= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\bar\omega(x-s)}{\prod_{i=1}^m \big[s^2 + (\delta_{[x,s]}X_i)^2\big]}\, ds,\\
C'_m(X_1, \ldots, X_m)[\bar\omega](x) &:= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\bar\omega(x-s)}{\prod_{i=1}^m \big[s^2 + (\delta'_{[x,s]}X_i)^2\big]}\, ds,\\
D_m(X_1, \ldots, X_m)[\bar\omega](x) &:= \frac{1}{\pi}\int_{\mathbb{R}} \frac{s\bar\omega(x-s)}{\prod_{i=1}^m \big[s^2 + (\delta_{[x,s]}X_i)^2\big]}\, ds,\\
D'_m(X_1, \ldots, X_m)[\bar\omega](x) &:= \frac{1}{\pi}\int_{\mathbb{R}} \frac{s\bar\omega(x-s)}{\prod_{i=1}^m \big[s^2 + (\delta'_{[x,s]}X_i)^2\big]}\, ds
\end{aligned} \tag{4.4}$$

for $\bar\omega \in L_2(\mathbb{R})$ and $x \in \mathbb{R}$, where we used again the notation introduced in (1.6). In view of $\inf(c_\infty + f_i - h_i) > 0$ for $1 \le i \le m$, these operators are no longer singular, see for example the proof of Lemma 4.1.

We are now in a position to represent the evolution operator Φ, see (3.13) - (3.15), in terms of $B_{n,m}$, C_m, C'_m, D_m and D'_m, for suitable n, $m \in \mathbb{N}$. Given $X = (f, h) \in \mathcal{O}_r$, we have

$$\begin{aligned} \Phi_1(X) &= \Theta_1 \left(B^1_{0,1}(f) + f' B^0_{1,1}(f)\right)[f'] \\ &\quad + \Theta_2 \left((c_\infty + f) f' C_1(X)[h'] - f' C_1(X)[hh'] + D_1(X)[h']\right), \end{aligned} \tag{4.5}$$

$$\begin{aligned} \Phi_2(X) &= \Theta_2 \left(B^1_{0,1}(h) + h' B^0_{1,1}(h)\right)[h'] \\ &\quad + \Theta_1 \left((h - c_\infty) h' C'_1(X)[f'] - h' C'_1(X)[ff'] + D'_1(X)[f']\right), \end{aligned} \tag{4.6}$$

where the constants Θ_1 and Θ_2 are defined in (3.3).

In order to prove similar results for the operators C_m, C'_m, D_m and D'_m, as the ones established for the operators $B_{n,m}$ in Appendix B, we next investigate some mapping properties of them, and in consequence show the smoothness property (4.1).

4.1. Mapping properties of the operators C_m, C'_m, D_m, D'_m

We now prove in Lemma 4.1 and Lemma 4.2 below that the operators C_m, C'_m, D_m, and D'_m are bounded operators in $L_2(\mathbb{R})$ (with respect to the linear variable $\bar{\omega}$) and moreover, they are Lipschitz continuous with respect to the variables $X_1, \ldots, X_m \in \mathcal{O}$. In Lemma 4.4 we prove that in fact the operators map the domain $L_2(\mathbb{R})$ continuously to $H^1(\mathbb{R})$ (again with a Lipschitz continuous dependence of $X_1, \ldots, X_m \in \mathcal{O}$.) Finally, in Section 4.2 we integrate these operators in a more general family of integral operators and then prove that

$$\Big[X \mapsto E_m(X, \ldots, X) \colon \mathcal{O}_r \to \mathcal{L}(L_2(\mathbb{R}), H^1(\mathbb{R}))\Big] \tag{4.7}$$

is a smooth map for all $E_m \in \{C_m, C'_m, D_m, D'_m\}$, see Lemma 4.5.

Estimates in L_2

The integral operators C_m and C'_m are now estimated, for given $X = (f, h) \in \mathcal{O}$, in the L_2-norm, which gives a basis for further mapping properties.

Lemma 4.1. *Given $1 \le m \in \mathbb{N}$ and $X_i := (f_i, h_i) \in \mathcal{O}$, $1 \le i \le m$, we set*

$$c_0 := \min_{1 \le i \le m} \inf(c_\infty + f_i - h_i). \tag{4.8}$$

Then, there exists a constant C that depends only on m, c_0, and $\max_{1 \le i \le m} \|h'_i\|_\infty$ such that

$$\|C_m(X_1, \ldots, X_m)[\bar{\omega}]\|_2 + \|C'_m(X_1, \ldots, X_m)[\bar{\omega}]\|_2 \le C \|\bar{\omega}\|_2, \quad \bar{\omega} \in L_2(\mathbb{R}). \tag{4.9}$$

Proof. We set

$$\delta := \frac{c_0}{2(\max_{1 \le i \le m} \|h'_i\|_\infty + 1)}.$$

Given $x \in \mathbb{R}$ and $|s| < \delta$, it then holds that

$$\begin{aligned} \min\{|\delta_{[x,s]} X_i|, |\delta'_{[x,s]} X_i|\} &\ge \min\{|f_i(x) + c_\infty - h_i(x)|, |f_i(x-s) + c_\infty - h_i(x-s)|\} \\ &\quad - |h_i(x) - h_i(x-s)| \\ &\ge \inf(c_\infty + f_i - h_i) - \delta \|h'_i\|_\infty \\ &\ge c_0/2 \end{aligned}$$

for $1 \le i \le m$. Therefore we get, by making use of Minkowski's integral inequality, that

$$\begin{aligned}
\|C_m(X_1,\ldots,X_m)[\bar\omega]\|_2 &= \frac{1}{\pi}\Big(\int_{\mathbb{R}}\Big|\int_{\mathbb{R}}\frac{\bar\omega(x-s)}{\prod_{i=1}^m[s^2+(\delta_{[x,s]}X_i)^2]}\,ds\Big|^2\,dx\Big)^{1/2}\\
&\le \int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\Big|\frac{\bar\omega(x-s)}{\prod_{i=1}^m[s^2+(\delta_{[x,s]}X_i)^2]}\Big|^2\,dx\Big)^{1/2}\,ds\\
&\le \Big(\frac{2}{c_0}\Big)^{2m}\int_{\{|s|<\delta\}}\|\bar\omega\|_2\,ds+\int_{\{|s|>\delta\}}\frac{\|\bar\omega\|_2}{s^{2m}}\,ds\\
&\le C\|\bar\omega\|_2.
\end{aligned}$$

The estimate concerning C'_m can be obtained by similar arguments and (4.9) follows. □

Extending the results of Lemma 4.1 to the context of the operators D_m and D'_m with $m \ge 2$ is not difficult, but the case $m = 1$ is more subtle and requires a different strategy. When considering D_1 and D'_1, we use the fact that the kernels behave for large s similarly as that of the truncated Hilbert transform $H_\delta : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, with $\delta > 0$, which is defined by

$$H_\delta[\bar\omega](x) := \frac{1}{\pi}\int_{\{|s|>\delta\}}\frac{\bar\omega(x-s)}{s}\,ds, \quad x \in \mathbb{R},$$

see Appendis C. In this appendix we identify H_δ as being a Fourier multiplier and we prove the following bound on its L_2-norm:

$$\|H_\delta\|_{\mathcal{L}(L_2(\mathbb{R}))} \le 2, \tag{4.10}$$

see Corollary C.5. The estimate (4.10) is an important argument in the proof of Lemma 4.2 below.

Lemma 4.2. *Given* $1 \le m \in \mathbb{N}$ *and* $X_i := (f_i, h_i) \in \mathcal{O}$, $1 \le i \le m$, *let* $c_0 > 0$ *be the constant defined in* (4.8). *Then, there exists a constant* C *that depends only on* r, m, c_0, *and* $\max_{1\le i\le m}\|X_i\|_{W^1_\infty}$ *such that*

$$\|D_m(X_1,\ldots,X_m)[\bar\omega]\|_2 + \|D'_m(X_1,\ldots,X_m)[\bar\omega]\|_2 \le C\|\bar\omega\|_2, \qquad \bar\omega \in L_2(\mathbb{R}). \tag{4.11}$$

Proof. As in the proof of Lemma 4.1 let

$$\delta := \frac{c_0}{2(\max_{1\le i\le m}\|h'_i\|_\infty + 1)}. \tag{4.12}$$

Hence, for given $x \in \mathbb{R}$ and $|s| < \delta$, it then again holds that

$$\min\{|\delta_{[x,s]}X_i|,\ |\delta'_{[x,s]}X_i|\} \ge c_0/2 \tag{4.13}$$

for $1 \le i \le m$.

We start with the case, where $m > 1$. With the help of Minkowski's integral inequality, it follows that

$$\begin{aligned}
\|D_m(X_1,\ldots,X_m)[\bar\omega]\|_2 &= \frac{1}{\pi}\Big(\int_{\mathbb{R}}\Big|\int_{\mathbb{R}}\frac{s\bar\omega(x-s)}{\prod_{i=1}^m[s^2+(\delta_{[x,s]}X_i)^2]}\,ds\Big|^2\,dx\Big)^{1/2}\\
&\le \int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\Big|\frac{s\bar\omega(x-s)}{\prod_{i=1}^m[s^2+(\delta_{[x,s]}X_i)^2]}\Big|^2\,dx\Big)^{1/2}\,ds\\
&\le \delta\Big(\frac{2}{c_0}\Big)^{2m}\int_{\{|s|<\delta\}}\|\bar\omega\|_2\,ds+\int_{\{|s|>\delta\}}\frac{\|\bar\omega\|_2}{s^{2m-1}}\,ds\\
&\le C\|\bar\omega\|_2.
\end{aligned}$$

The analogous estimate of D'_m may be concluded by similar arguments.

We now focus on the operator D_1. Let therefore δ be defined as in (4.12) (with $m = 1$) and set

$$I(x,s) := \frac{s\bar\omega(x-s)}{s^2 + (\delta_{[x,s]}X_1)^2} = \Big(1 - \frac{(\delta_{[x,s]}X_1/s)^2}{1 + (\delta_{[x,s]}X_1/s)^2}\Big)\frac{\bar\omega(x-s)}{s}, \qquad x,\, s \in \mathbb{R},\, s \neq 0.$$

Thus, we have for $x \in \mathbb{R}$ that

$$\begin{aligned}|D_1(X_1)[\bar\omega](x)| &\leq \Big|\int_{\mathbb{R}} \frac{s\bar\omega(x-s)}{\prod_{i=1}^m \big[s^2 + (\delta_{[x,s]}X_i)^2\big]}\, ds\Big| \\ &\leq \int_{\{|s|<\delta\}} |I(x,s)|\, ds + |H_\delta[\bar\omega](x)| + \int_{\{|s|>\delta\}} \frac{(\delta_{[x,s]}X_1/s)^2}{1 + (\delta_{[x,s]}X_1/s)^2}\Big|\frac{\bar\omega(x-s)}{s}\Big|\, ds.\end{aligned}$$

Since

$$|\delta_{[x,s]}X_1| \geq c_0/2 \quad \text{ for } |s| \leq \delta,$$

see (4.13), Minkowski's integral inequality yields

$$\Big\|\int_{\{|s|<\delta\}} |I(\cdot,s)|\, ds\Big\|_2 \leq \int_{\{|s|<\delta\}} \Big(\int_{\mathbb{R}} |I(x,s)|^2\, dx\Big)^{1/2} ds \leq \frac{8\delta^2}{c_0^2}\|\bar\omega\|_2.$$

Moreover, taking into account that

$$|\delta_{[\cdot,s]}X_1| \leq c_\infty + \|f_1\|_\infty + \|h_1\|_\infty,$$

Minkowski's integral inequality leads to

$$\begin{aligned}\Big\|\int_{\{|s|>\delta\}} \frac{(\delta_{[\cdot,s]}X_1/s)^2}{1 + (\delta_{[\cdot,s]}X_1/s)^2}\Big|\frac{\bar\omega(\cdot-s)}{s}\Big|\, ds\Big\|_2 &\leq (c_\infty + \|f_1\|_\infty + \|h_1\|_\infty)^2 \int_{\{|s|>\delta\}} \frac{\|\bar\omega\|_2}{|s|^3}\, ds \\ &\leq C\|\bar\omega\|_2.\end{aligned}$$

Recalling (4.10), we conclude that (4.11) is satisfied. □

We are now in a position to conclude from Lemma 4.1 and Lemma 4.2 that the operators C_m, C'_m, D_m and D'_m are indeed Lipschitz continuous with respect to the nonlinear arguments $X_1, \ldots, X_m$.

Corollary 4.3. *Given $1 \leq m \in \mathbb{N}$, it holds that*

$$C_m,\, D_m,\, C'_m,\, D'_m \in \mathrm{C}^{1-}(\mathcal{O}^m, \mathcal{L}(L_2(\mathbb{R}))). \tag{4.14}$$

Proof. Let $X_i = (f_i, h_i)$, $\tilde X_i = (\widetilde f_i, \widetilde h_i) \in \mathcal{O}$, $1 \leq i \leq m$, and $\bar\omega \in L_2(\mathbb{R})$. It then follows with $E_m \in \{C_m,\, D_m\}$ that

$$\begin{aligned}&E_m(X_1,\ldots,X_m)[\bar\omega] - E_m(\widetilde X_1,\ldots,\widetilde X_m)[\bar\omega] \\ &= \sum_{j=1}^m \Big((2c_\infty + \widetilde f_j + f_j)(\widetilde f_j - f_j)E_{m+1}(\widetilde X_1,\ldots,\widetilde X_j, X_j,\ldots,X_m)[\bar\omega] \\ &\qquad - (\widetilde f_j - f_j)E_{m+1}(\widetilde X_1,\ldots,\widetilde X_j, X_j,\ldots,X_m)[(\widetilde h_j + h_j)\bar\omega] \\ &\qquad - (2c_\infty + \widetilde f_j + f_j)E_{m+1}(\widetilde X_1,\ldots,\widetilde X_j, X_j,\ldots,X_m)[(\widetilde h_j - h_j)\bar\omega] \\ &\qquad + E_{m+1}(\widetilde X_1,\ldots,\widetilde X_j, X_j,\ldots,X_m)[(\widetilde h_j^2 - h_j^2)\bar\omega]\Big),\end{aligned} \tag{4.15}$$

respectively, for $E'_m \in \{C'_m,\ D'_m\}$, that

$$
\begin{aligned}
&E'_m(X_1,\dots,X_m)[\bar\omega] - E'_m(\widetilde X_1,\dots,\widetilde X_m)[\bar\omega] \\
&\quad = \sum_{j=1}^m \Big((\widetilde h_j^2 - h_j^2) E'_{m+1}(\widetilde X_1,\dots,\widetilde X_j, X_j,\dots,X_m)[\bar\omega] \\
&\qquad\qquad - (\widetilde h_j - h_j) E'_{m+1}(\widetilde X_1,\dots,\widetilde X_j, X_j,\dots,X_m)[(2c_\infty + \widetilde f_j + f_j)\bar\omega] \\
&\qquad\qquad - (\widetilde h_j + h_j) E'_{m+1}(\widetilde X_1,\dots,\widetilde X_j, X_j,\dots,X_m)[(\widetilde f_j - f_j)\bar\omega] \\
&\qquad\qquad + E'_{m+1}(\widetilde X_1,\dots,\widetilde X_j, X_j,\dots,X_m)[(2c_\infty + \widetilde f_j + f_j)(\widetilde f_j - f_j)\bar\omega]\Big).
\end{aligned}
\tag{4.16}
$$

Applying the L_2-estimates (4.9) and (4.11), we obtain the Lipschitz continuity (4.14). □

Estimates in H^1

In order to strengthen the results above, we improve the estimates on the operators C_m, C'_m, D_m and D'_m and show, that they indeed map, for given $X_i \in \mathcal{O}_r$, $1 \le i \le m$, the definition domain $L_2(\mathbb{R})$ into $H^1(\mathbb{R})$.

Lemma 4.4. *Given $1 \le m \in \mathbb{N}$, $r \in (3/2, 2)$, and $X_i := (f_i, h_i) \in \mathcal{O}_r$, $1 \le i \le m$, let $c_0 > 0$ be the constant defined in (4.8). Given $E_m \in \{C_m,\ D_m\}$, there exists a positive constant C that depends only on r, m, c_0, and $\max_{1\le i\le m} \|X_i\|_{H^r}$ such that*

$$
\|E_m(X_1,\dots,X_m)[\bar\omega]\|_{H^1} + \|E'_m(X_1,\dots,X_m)[\bar\omega]\|_{H^1} \le C\|\bar\omega\|_2, \qquad \bar\omega \in L_2(\mathbb{R}). \tag{4.17}
$$

Moreover, C_m, D_m, C'_m, $D'_m \in \mathrm{C}^{1-}(\mathcal{O}_r^m, \mathcal{L}(L_2(\mathbb{R}), H^1(\mathbb{R})))$.

Proof. Let τ_ξ denote the right shift operator

$$
\tau_\xi := \big[\varphi \mapsto \varphi(\cdot - \xi)\big], \quad \xi \in \mathbb{R}.
$$

We first assume $\bar\omega \in \mathrm{C}^\infty(\mathbb{R})$. Given $0 \ne \xi \in \mathbb{R}$, it holds that

$$
\begin{aligned}
&\frac{1}{\xi}(E_m(X_1,\dots,X_m)[\bar\omega] - \tau_\xi(E_m(X_1,\dots,X_m)[\bar\omega])) \\
&\qquad = E_m(X_1,\dots,X_m)\Big[\frac{\bar\omega - \tau_\xi\bar\omega}{\xi}\Big] \\
&\qquad\quad + \frac{1}{\xi}(E_m(X_1,\dots,X_m)[\tau_\xi\bar\omega] - (E_m(\tau_\xi X_1,\dots,\tau_\xi X_m)[\tau_\xi\bar\omega]))
\end{aligned}
$$

and the formula (4.15) leads us to

$$
\begin{aligned}
&\frac{1}{\xi}(E_m(X_1,\dots,X_m)[\tau_\xi\bar\omega] - (E_m(\tau_\xi X_1,\dots,\tau_\xi X_m)[\tau_\xi\bar\omega])) \\
&\quad = \sum_{j=1}^m \Big((2c_\infty + \tau_\xi f_j + f_j)\frac{\tau_\xi f_j - f_j}{\xi} E_{m+1}(\tau_\xi X_1,\dots,\tau_\xi X_j, X_j,\dots,X_m)[\tau_\xi\bar\omega] \\
&\qquad\qquad - \frac{\tau_\xi f_j - f_j}{\xi} E_{m+1}(\widetilde X_1,\dots,\widetilde X_j, X_j,\dots,X_m)[(\tau_\xi h_j + h_j)\tau_\xi\bar\omega] \\
&\qquad\qquad - (2c_\infty + \tau_\xi f_j + f_j) E_{m+1}(\tau_\xi X_1,\dots,\tau_\xi X_j, X_j,\dots,X_m)\Big[\frac{\tau_\xi h_j - h_j}{\xi}\tau_\xi\bar\omega\Big] \\
&\qquad\qquad + E_{m+1}(\tau_\xi X_1,\dots,\tau_\xi X_j, X_j,\dots,X_m)\Big[\frac{(\tau_\xi h_j)^2 - h_j^2}{\xi}\tau_\xi\bar\omega\Big]\Big).
\end{aligned}
$$

In view of Corollary 4.3 we can pass to the limit $\xi \to 0$ and conclude that

$$E_m(X_1,\dots,X_m)[\bar\omega] \in H^1(\mathbb{R}),$$

with

$$\begin{aligned}(E_m(X_1,\dots,X_m)[\bar\omega])' = E_m(X_1,\dots,X_m)[\bar\omega'] \\ - 2\sum_{j=1}^m \Big((c_\infty+f_j)f_j' E_{m+1}(X_1,\dots,X_m,X_j)[\bar\omega] \\ - f_j' E_{m+1}(X_1,\dots,X_m,X_j)[h_j\bar\omega] \\ - (c_\infty+f_j)E_{m+1}(X_1,\dots,X_m,X_j)[h_j'\bar\omega] \\ + E_{m+1}(X_1,\dots,X_m,X_j)[h_jh_j'\bar\omega]\Big).\end{aligned}$$

Except for the term $E_m(X_1,\dots,X_m)[\bar\omega']$, one can use a standard density argument to obtain that the right-hand side is well-defined as long as $\bar\omega \in L_2(\mathbb{R})$. Moreover, using integration by parts, we can rewrite

$$\begin{aligned}C_m(X_1,\dots,X_m)[\bar\omega'](x) &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{-\partial_s(\bar\omega(x-s))}{\prod_{i=1}^m (s^2+(\delta_{[x,s]}X_i)^2)}\,ds \\ &= \frac{-2}{\pi}\sum_{j=1}^m \int_{\mathbb{R}} \frac{\bar\omega(x-s)(s+(\delta_{[x,s]}X_j)h_j'(x-s))}{\prod_{i=1}^m (s^2+(\delta_{[x,s]}X_i)^2)(s^2+(\delta_{[x,s]}X_j)^2)}\,ds \\ &= -2\sum_{j=1}^m \Big(D_{m+1}(X_1,\dots,X_m,X_j)[\bar\omega](x) \\ &\qquad + (c_\infty+f_j(x))C_{m+1}(X_1,\dots,X_m,X_j)[h_j'\bar\omega](x) \\ &\qquad - C_{m+1}(X_1,\dots,X_m,X_j)[h_jh_j'\bar\omega](x)\Big),\end{aligned}$$

and respectively

$$\begin{aligned}D_m(X_1,\dots,X_m)[\bar\omega'] &= C_m(X_1,\dots,X_m)[\bar\omega] \\ &\quad \frac{-2}{\pi}\sum_{j=1}^m \int_{\mathbb{R}} \frac{s\bar\omega(x-s)(s+(\delta_{[x,s]}X_j)h_j'(x-s))}{\prod_{i=1}^m (s^2+(\delta_{[x,s]}X_i)^2)(s^2+(\delta_{[x,s]}X_j)^2)}\,ds \\ &= (1-2m)C_m(X_1,\dots,X_m)[\bar\omega] \\ &\quad - 2\sum_{j=1}^m \Big((c_\infty+f_j)D_{m+1}(X_1,\dots,X_m,X_j)[h_j'\bar\omega](x) \\ &\qquad - D_{m+1}(X_1,\dots,X_m,X_j)[h_jh_j'\bar\omega](x) \\ &\qquad - (c_\infty+f_j)^2 C_{m+1}(X_1,\dots,X_m,X_j)[\bar\omega](x) \\ &\qquad - C_{m+1}(X_1,\dots,X_m,X_j)[h_j^2\bar\omega](x) \\ &\qquad + 2(c_\infty+f_j)C_{m+1}(X_1,\dots,X_m,X_j)[h_j\bar\omega](x)\Big)\end{aligned}$$

for $x \in \mathbb{R}$. Combining the last three identities with the L_2-estimates from Lemma 4.1 and Lemma 4.2, we can use a standard density argument in order to deduce that (4.17) holds for E_m.

The claim for the operator E'_m follows by similar arguments.

Finally, the Lipschitz continuity property is obtained from (4.17) together with (4.15) and (4.16). □

4.2. Smoothness of Φ

The next main step on our way of establishing the smoothness of Φ, see (4.1), is to show that

$$\Big[X \mapsto E_m(X,\dots,X)\Big] : \mathcal{O}_r \to \mathcal{L}(L_2(\mathbb{R}), H^1(\mathbb{R})), \quad m \in \mathbb{N},$$

is smooth for $E_m \in \{C_m, C'_m, D_m, D'_m\}$, cf. (4.7). To this end we integrate these operators in a larger family of integral operators for which we prove that they are differentiable and the Fréchet derivative is again an element of this family. In this way we establish the desired smoothness property.

Given n, m, $p \in \mathbb{N}$, $m \geq 1$, $X_i \in \mathcal{O}_r$, $1 \leq i \leq m+p$, $Y_i \in H^r(\mathbb{R})^2$, $1 \leq i \leq n$, $\bar\omega \in L_2(\mathbb{R})$, and $E \in \{C, D\}$, we use the notation from (1.6) to define

$$C_{n,m,p}(X_1,\dots,X_{m+p})[Y_1,\dots,Y_n,\bar\omega](x) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{\bar\omega(x-s)\big(\prod_{i=m+1}^{m+p}\delta_{[x,s]}X_i\big)\prod_{i=1}^{n}\overline{\delta}_{[x,s]}Y_i}{\prod_{i=1}^{m}\big[s^2+(\delta_{[x,s]}X_i)^2\big]}\,ds,$$

$$D_{n,m,p}(X_1,\dots,X_{m+p})[Y_1,\dots,Y_n,\bar\omega](x) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{s\bar\omega(x-s)\big(\prod_{i=m+1}^{m+p}\delta_{[x,s]}X_i\big)\prod_{i=1}^{n}\overline{\delta}_{[x,s]}Y_i}{\prod_{i=1}^{m}\big[s^2+(\delta_{[x,s]}X_i)^2\big]}\,ds,$$

$$C'_{n,m,p}(X_1,\dots,X_{m+p})[Y_1,\dots,Y_n,\bar\omega](x) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{\bar\omega(x-s)\big(\prod_{i=m+1}^{m+p}\delta'_{[x,s]}X_i\big)\prod_{i=1}^{n}\overline{\delta}'_{[x,s]}Y_i}{\prod_{i=1}^{m}\big[s^2+(\delta'_{[x,s]}X_i)^2\big]}\,ds,$$

$$D'_{n,m,p}(X_1,\dots,X_{m+p})[Y_1,\dots,Y_n,\bar\omega](x) := \frac{1}{\pi}\int_{\mathbb{R}} \frac{s\bar\omega(x-s)\big(\prod_{i=m+1}^{m+p}\delta'_{[x,s]}X_i\big)\prod_{i=1}^{n}\overline{\delta}'_{[x,s]}Y_i}{\prod_{i=1}^{m}\big[s^2+(\delta'_{[x,s]}X_i)^2\big]}\,ds$$

for $x \in \mathbb{R}$. We note that the latter formulas extend our previous notation introduced in (4.4), because, given $E \in \{C,\, D,\, C',\, D'\}$ and $m \geq 1$, the following operators coincide

$$E_{0,m,0}(X_1,\dots,X_m) = E_m(X_1,\dots,X_m).$$

Setting $Y_i = (u_i, v_i)$, for $1 \leq i \leq n$, it holds, in view of the algebraic relation

$$\prod_{i=1}^{n}(a_i+b_i) = \sum_{S\subset\{1,\dots,n\}}\Big[\Big(\prod_{j\in S}a_j\Big)\Big(\prod_{j\in S^c}b_j\Big)\Big],$$

that

$$\begin{aligned}
&E_{n,m,p}(X_1,\dots,X_{m+p})[Y_1,\dots,Y_n,\bar\omega]\\
&\quad= \sum_{S\subset\{1,\dots,n\}}(-1)^{|S^c|}\Big(\prod_{j\in S}u_j\Big)E_{0,m,p}(X_1,\dots,X_{m+p})\Big[\bar\omega\prod_{j\in S^c}v_j\Big],\\
&E'_{n,m,p}(X_1,\dots,X_{m+p})[Y_1,\dots,Y_n,\bar\omega]\\
&\quad= \sum_{S\subset\{1,\dots,n\}}(-1)^{|S^c|}\Big(\prod_{j\in S}v_j\Big)E'_{0,m,p}(X_1,\dots,X_{m+p})\Big[\bar\omega\prod_{j\in S^c}u_j\Big],
\end{aligned}$$

where for each $S \subset \{1,\dots,n\}$ we set $S^c := \{1,\dots,n\}\setminus S$ to be the complement of S.

Furthermore, letting $X_j := (f_j, h_j)$, $m+1 \le j \le m+p$, it holds that

$$\begin{aligned}
&E_{0,m,p}(X_1,\ldots,X_{m+p})[\bar\omega]\\
&\quad= \sum_{S\subset\{m+1,\ldots,m+p\}} (-1)^{|S^c|}\Big(\prod_{j\in S}(c_\infty+f_j)\Big)E_m(X_1,\ldots,X_m)\Big[\bar\omega\prod_{j\in S^c}h_j\Big],\\
&E'_{0,m,p}(X_1,\ldots,X_{m+p})[\bar\omega]\\
&\quad= \sum_{S\subset\{m+1,\ldots,m+p\}} (-1)^{|S^c|}\Big(\prod_{j\in S}(h_j-c_\infty)\Big)E'_m(X_1,\ldots,X_m)\Big[\bar\omega\prod_{j\in S^c}f_j\Big].
\end{aligned}$$

Recalling the H^1-estimate (4.17), we deduce for $E \in \{C, C', D, D'\}$, that

$$\|E_{n,m,p}(X_1,\ldots,X_{m+p})[Y_1,\ldots,Y_n,\cdot]\|_{\mathcal{L}(L_2(\mathbb{R}),H^1(\mathbb{R}))} \le C\prod_{i=1}^n \|Y_i\|_{H^r}, \tag{4.18}$$

with C is independent of Y_i, $1\le i\le n$.

Finally, given $E \in \{C, C', D, D'\}$, $n, m, p \in \mathbb{N}$, $m \ge 1$, $Y_i \in H^r(\mathbb{R})^2$, $1 \le i \le n$, and $X \in \mathcal{O}_r$, we define

$$E^n_{m,p}(X)[Y_1,\ldots,Y_n] := E_{n,m,p}(X,\ldots,X)[Y_1,\ldots,Y_n,\cdot] \in \mathcal{L}(L_2(\mathbb{R}),H^1(\mathbb{R})).$$

The estimate (4.18) shows that

$$E^n_{m,p}\colon \mathcal{O}_r \to \mathcal{L}^n_{\mathrm{sym}}(H^r(\mathbb{R})^2, \mathcal{L}(L_2(\mathbb{R}),H^1(\mathbb{R})))^1.$$

In the next lemma we establish the Fréchet differentiability of $E^n_{m,p}$.

Lemma 4.5. *Given $n, m, p \in \mathbb{N}$, $m \ge 1$, and $X \in \mathcal{O}_r$, the operator $E^n_{m,p}$ is differentiable in X and its Fréchet derivative is given by*

$$\partial E^n_{m,p}(X)[Y][Y_1,\ldots,Y_n] = pE^{n+1}_{m,p-1}(X)[Y_1,\ldots,Y_n,Y] - 2mE^{n+1}_{m+1,p+1}(X)[Y_1,\ldots,Y_n,Y] \tag{4.19}$$

for $Y, Y_1,\ldots,Y_n \in H^r(\mathbb{R})^2$. Consequently, for each $n, m, p\in\mathbb{N}$, $m\ge 1$, we have

$$E^n_{m,p} \in \mathrm{C}^\infty(\mathcal{O}_r, \mathcal{L}^n_{\mathrm{sym}}(H^r(\mathbb{R})^2, \mathcal{L}(L_2(\mathbb{R}),H^1(\mathbb{R})))). \tag{4.20}$$

Proof. Setting

$$\begin{aligned}
R(X,Y)[Y_1,\ldots,Y_n] :=\ & E^n_{m,p}(X+Y)[Y_1,\ldots,Y_n] - E^n_{m,p}(X)[Y_1,\ldots,Y_n]\\
&- pE^{n+1}_{m,p-1}(X)[Y_1,\ldots,Y_n,Y] + 2mE^{n+1}_{m+1,p+1}(X)[Y_1,\ldots,Y_n,Y],
\end{aligned}$$

elementary algebraic manipulations lead us to the following identity

$$R(X,Y)[Y_1,\ldots,Y_n] = \sum_{j=0}^{p-1}(p-j-1)R_{1,j} - \sum_{j=0}^{m-1}R_{2,j} + \sum_{j=0}^{m-1}\sum_{l=0}^{m-j-1}2R_{3,j,l},$$

[1] If $n = 0$ we identify $\mathcal{L}^n_{\mathrm{sym}}(H^r(\mathbb{R})^2, \mathcal{L}(L_2(\mathbb{R}),H^1(\mathbb{R})))$ with $\mathcal{L}(L_2(\mathbb{R}),H^1(\mathbb{R}))$.

where

$$\begin{aligned}
R_{1,j} &= E_{n+2,m,p-2}(\underbrace{X+Y,\ldots,X+Y}_{m},\underbrace{X+Y,\ldots,X+Y}_{j},\underbrace{X,\ldots,X}_{p-2-j})[Y_1,\ldots,Y_n,Y,Y],\\
R_{2,j} &= (1+2p)E_{n+2,m+1,p}(\underbrace{X+Y,\ldots,X+Y}_{m-j},\underbrace{X,\ldots,X}_{j+1},\underbrace{X,\ldots,X}_{p})[Y_1,\ldots,Y_n,Y,Y]\\
&\quad + pE_{n+3,m+1,p-1}(\underbrace{X+Y,\ldots,X+Y}_{m-j},\underbrace{X,\ldots,X}_{j+1},\underbrace{X,\ldots,X}_{p-1})[Y_1,\ldots,Y_n,Y,Y,Y],\\
R_{3,j,l} &= 2E_{n+2,m+2,p+2}(\underbrace{X+Y,\ldots,X+Y}_{m-j-l},\underbrace{X,\ldots,X}_{j+l+2},\underbrace{X,\ldots,X}_{p+2})[Y_1,\ldots,Y_n,Y,Y]\\
&\quad + E_{n+3,m+2,p+1}(\underbrace{X+Y,\ldots,X+Y}_{m-j-l},\underbrace{X,\ldots,X}_{j+l+2},\underbrace{X,\ldots,X}_{p+1})[Y_1,\ldots,Y_n,Y,Y,Y].
\end{aligned}$$

Hence, for all Y sufficiently close to X in $H^r(\mathbb{R})^2$, it follows from Lemma 4.4, by arguing as in the derivation of (4.18), that

$$\|R(X,Y)[Y_1,\ldots,Y_n]\|_{\mathcal{L}(L_2(\mathbb{R}),H^1(\mathbb{R}))} \leq C\|Y\|_{H^r}^2 \prod_{i=1}^{n} \|Y_i\|_{H^r},$$

and (4.19) follows.

Moreover, using (4.19) iteratively, we also obtain (4.20). ☐

In view of the smoothness results for the operators $B_{n,m}$, established in Appendis B, and for the operators C_m, C'_m, D_m, and D'_m, we can now infer the following corollary, which establishes the main goal of this chapter.

Corollary 4.6. *It holds that*

$$\Phi \in \mathrm{C}^\infty(\mathcal{O}_r, H^{r-1}(\mathbb{R})^2).$$

Proof. The claim follows, due to the embedding $H^1(\mathbb{R}) \hookrightarrow H^{r-1}(\mathbb{R})$ and the algebra property of $H^{r-1}(\mathbb{R})$, from the representations (4.5) and (4.6) of Φ, together with the Lemma B.9 and Lemma 4.5. ☐

5. Proof of Theorem 2.1 and Proposition 2.3

In this chapter we establish the proofs of our main results, see Theorem 2.1 and Proposition 2.3, for the multiphase Muskat problem with equal viscosities.

The first main goal of this chapter is to identify the Fréchet derivative $\partial\Phi(X), X \in \mathcal{O}_r$, of Φ and to prove that it generates an analytic semigroup in $\mathcal{L}(H^{r-1}(\mathbb{R})^2)$, see Theorem 5.10. This property identifies the multiphase Muskat problem as a parabolic evolution equation and, in view of Corollary 4.6, we may use abstract results on parabolic evolution equations from [57] when establishing Theorem 2.1. The proofs of Theorem 2.1 and Proposition 2.3 are presented in Section 5.2 and Section 5.3, respectively.

5.1. The generator property

Establishing the parabolicity of (3.12) is the main goal of this section, which will be achieved by showing the generator property

$$-\partial\Phi(X) \in \mathcal{H}(H^r(\mathbb{R})^2, H^{r-1}(\mathbb{R})^2), \qquad X \in \mathcal{O}_r. \tag{5.1}$$

As shown in [7, Section I.1.2], if $(X_0, \|\cdot\|_0)$ and $(X_1, \|\cdot\|_1)$ are Banach spaces, where X_1 is densely embedded into X_0 and for $A \in \mathcal{L}(X_1, X_0)$ there exist constants $\kappa \geq 1$ and $\omega > 0$ such that

(i) $\omega - A \in \mathrm{Isom}(X_1, X_0)$, and

(ii) $|\lambda| \cdot \|x\|_0 + \|x\|_1 \leq \kappa \|(\lambda - A)x\|_0$ for all $x \in E_1$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq \omega$,

then

$$-A \in \mathcal{H}(X_1, X_0). \tag{5.2}$$

In the context of the problem (3.12), we choose $X_1 := H^r(\mathbb{R})^2$, $X_0 := H^{r-1}(\mathbb{R})^2$ with the corresponding Sobolev-norms. In order to establish (5.1), we represent the Fréchet derivative $\partial\Phi(X)$ as the matrix operator

$$\partial\Phi(X) = \begin{pmatrix} \partial_f \Phi_1(X) & \partial_h \Phi_1(X) \\ \partial_f \Phi_2(X) & \partial_h \Phi_2(X) \end{pmatrix} \in \mathcal{L}(H^r(\mathbb{R})^2, H^{r-1}(\mathbb{R})^2).$$

We now recall a classical result on the generator property for 2×2-matrix operators, which can be found in a slightly more precise version in [7, Theorem I.1.6.1].

Theorem 5.1. *Let $(X_0, \|\cdot\|_0)$ and $(X_1, \|\cdot\|_1)$ be Banach spaces, where X_1 is densely embedded into X_0. Let further $A_{ij} \in \mathcal{L}(X_1, X_0),\ 1 \leq i, j \leq 2$ satisfy*

$$-A_{ii} \in \mathcal{H}(X_1, X_0), \quad i = 1, 2.$$

Assume moreover that, for each $\nu > 0$, there exists $K = K(\nu) > 0$ such that

$$\|A_{12}x\|_0 \leq \nu \|x\|_1 + K\|x\|_0 \quad \text{for all } x \in X_1. \tag{5.3}$$

Then the operator

$$A := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathcal{L}(X_1^2, X_0^2),$$

satisfies

$$-A \in \mathcal{H}(X_1^2, X_0^2).$$

In order to show that the assumptions of Theorem 5.1 hold true in our setting, we recall some results on the Fréchet derivatives of C_m, D_m, C'_m, D'_m, see (4.19), and of $B_{n,m}$, see Lemma B.10. Given $Y = (u, v) \in H^r(\mathbb{R})^2$ and $\bar{\omega} \in H^{r-1}(\mathbb{R})$, it holds that

$$\partial B^0_{n,1}(f)[u][\bar{\omega}] = nB_{n,1}(f)[f, \ldots, f, u, \bar{\omega}] - 2B_{n+2,2}(f, f)[f, \ldots, f, u, \bar{\omega}], \tag{5.4}$$

$$\partial E_1(X)[Y] = -2E^1_{2,1}(X)[Y] = 2E^0_{2,1}(X)[v\,\cdot] - 2uE^0_{2,1}(X), \tag{5.5}$$

$$\partial E'_1(X)[Y] = -2E'^1_{2,1}(X)[Y] = 2E'^0_{2,1}(X)[u\cdot] - 2vE'^0_{2,1}(X) \tag{5.6}$$

where $E \in \{C, D\}$ and $n \in \mathbb{N}$.

We first show that the off-diagonal entry $\partial_h \Phi_1(X)$ can be considered as a perturbation in the sense of (5.3), see Lemma 5.2.

Lemma 5.2. *Let $X \in \mathcal{O}_r$ and $r \in (3/2, 2)$. Then, given $\nu > 0$, there exists a positive constant $K = K(\nu)$ such that*

$$\|\partial_h \Phi_1(X)[v]\|_{H^{r-1}} \leq \nu\|v\|_{H^r} + K\|v\|_{H^{r-1}} \qquad \text{for all } v \in H^r(\mathbb{R}). \tag{5.7}$$

Proof. From the representation (4.5) and the relation (5.5) we infer that

$$\begin{aligned}\partial_h \Phi_1(X)[v] = \Theta_2\big((c_\infty + f)f'\big(C_1(X)[v'] + 2C^0_{2,1}(X)[vh']\big)\\ - f'\big(C_1(X)[hv' + vh'] + 2C^0_{2,1}(X)[vhh']\big)\\ + D_1(X)[v'] + 2D^0_{2,1}(X)[vh']\big), \quad v \in H^r(\mathbb{R}).\end{aligned}$$

Recalling (4.18), we conclude that

$$\|\partial_h \Phi_1(X)[v]\|_{H^{r-1}} \leq \|\partial_h \Phi_1(X)[v]\|_{H^1} \leq C\|v\|_{H^1}.$$

Let $\tau \in (r-1, r)$ be given. Using complex interpolation theory, e.g. [86, Chapter 2.4], it holds for $\theta := 1 - r + \tau \in (0, 1)$ that

$$H^\tau(\mathbb{R}) = \big[H^{r-1}(\mathbb{R}), H^r(\mathbb{R})\big]_\theta,$$

is the complex interpolation space between $H^{r-1}(\mathbb{R})$ and $H^r(\mathbb{R})$ for the parameter θ. In consequence, for given $\nu > 0$ and $u \in H^r(\mathbb{R})$, it holds, thanks to Young's inequality, that

$$\|u\|_{H^\tau} \leq C\left(\frac{\nu}{C\theta}\|u\|_{H^r}\right)^\theta \left(\left(\frac{C\theta}{\nu}\right)^{\theta/(1-\theta)} \|u\|_{H^{r-1}}\right)^{1-\theta} \leq \nu\|u\|_{H^r} + K\|u\|_{H^{r-1}}, \tag{5.8}$$

where $K = K(\nu) > 0$. Thus, (5.7) holds true. □

In view of Theorem 5.1 and Lemma 5.2, it remains to show that the diagonal entries $\partial_f \Phi_1(X)$ and $\partial_h \Phi_2(X)$ are generators of analytic semigroups, that is

$$-\partial_f \Phi_1(X) \in \mathcal{H}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R})), \tag{5.9}$$

$$-\partial_h \Phi_2(X) \in \mathcal{H}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R})). \tag{5.10}$$

We start with proving (5.9). In order to shorten formulas we introduce the linear operator

$$\mathbb{B}(u) := B^0_{0,1}(u) + u'B^0_{1,1}(u), \quad u \in H^r(\mathbb{R}). \tag{5.11}$$

Therefore, we conclude from (4.5) with the help of (5.4) and (5.5), that

$$\begin{aligned}\partial_f\Phi_1(X)[u] &= \Theta_1\Big(B_{0,1}(f)[u'] - 2B_{2,2}(f,f)[f,u,f'] + u'B_{1,1}(f)[f,f'] \\ &\qquad + f'(B_{1,1}(f)[f,u'] + B_{1,1}(f)[u,f'] - 2B_{3,2}(f,f)[f,f,u,f'])\Big) \\ &\quad + \Theta_2\Big(uf'C_1(X)[h'] + (c_\infty+f)u'C_1(X)[h'] - 2(c_\infty+f)f'uC^0_{2,1}(X)[h'] \\ &\qquad - u'C_1(X)[hh'] + 2uf'C^0_{2,1}(X)[hh'] - 2uD^0_{2,1}(X)[h']\Big) \\ &= \Theta_1(\mathbb{B}(f)[u'] + \partial\mathbb{B}(f)[u][f']) + a_1(X)u' + T_{\mathrm{lot},1}[u]\end{aligned}$$

for $u \in H^r(\mathbb{R})$, where

$$a_1(X) := \Theta_2\big((c_\infty+f)C_1(X)[h'] - C_1(X)[hh']\big) \in H^1(\mathbb{R})$$

and

$$\begin{aligned}T_{\mathrm{lot},1}[u] := \Theta_2\Big(&uf'C_1(X)[h'] - 2(c_\infty+f)f'uC^0_{2,1}(X)[h'] \\ &+ 2uf'C^0_{2,1}(X)[u] - 2uD^0_{2,1}(X)[h']\Big).\end{aligned}$$

Now we define a continuous path $\Psi_1 : [0,1] \to \mathcal{L}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$ with starting point a Fourier multiplier and endpoint $\partial_f\Phi_1(X)$ in the following way:

$$\Psi_1(\tau)[u] := \Theta_1\big(\mathbb{B}(\tau f)[u'] + \tau\partial\mathbb{B}(f)[u][f']\big) + \tau a_1(X)u' + \tau T_{\mathrm{lot},1}[u] \quad \text{for } \tau \in [0,1]. \tag{5.12}$$

Remark 5.3.

(i) Letting H denote the Hilbert transform, see Appendix C for more details, we have

$$\Psi_1(0) = \Theta_1\mathbb{B}(0)\circ(d/dx) = \Theta_1 H\circ(d/dx) = \Theta_1\Big(-\frac{d^2}{dx^2}\Big)^{1/2}.$$

(ii) Following Lemma C.6, we get that $\Psi_1(0)$ is a Fourier multiplier with symbol

$$[\xi \mapsto \Theta_1|\xi|].$$

(iii) It holds that $\Psi_1(1) = \partial_f\Phi_1(X)$.

(iv) In view of (4.18), it holds that

$$\|T_{\mathrm{lot},1}[u]\|_{H^{r-1}} \le C\|u\|_{H^1} \qquad \text{for all } u \in H^r(\mathbb{R}). \tag{5.13}$$

The next step is to locally approximate the operator $\Psi_1(\tau)$ uniformly in τ by some Fourier multipliers. First, we introduce some auxiliary tools. Given $\varepsilon \in (0,1)$ and $N = N(\varepsilon) \in \mathbb{N}$, we define a so-called ε-localization family, that is a family

$$\{\pi_j^\varepsilon \,:\, -N+1 \le j \le N\} \subset \mathrm{C}^\infty(\mathbb{R},[0,1]),$$

such that

- supp π_j^ε is an interval of length ε for all $|j| \le N-1$, supp $\pi_N^\varepsilon \subset (-\infty,-1/\varepsilon] \cup [1/\varepsilon,\infty)$;
- $\pi_j^\varepsilon \cdot \pi_l^\varepsilon = 0$ if $[|j-l| \ge 2,\ \max\{|j|,|l|\} \le N-1]$ or $[|l| \le N-2,\ j = N]$;
- $\sum_{j=-N+1}^{N}(\pi_j^\varepsilon)^2 = 1$;
- $\|(\pi_j^\varepsilon)^{(k)}\|_\infty \le C\varepsilon^{-k}$ for all $k \in \mathbb{N}, -N+1 \le j \le N$.

To each finite ε-localization family we associate a second family

$$\{\chi_j^\varepsilon \,:\, -N+1 \le j \le N\} \subset \mathrm{C}^\infty(\mathbb{R}, [0,1])$$

such that

- $\chi_j^\varepsilon = 1$ on supp π_j^ε, supp $\chi_N^\varepsilon \subset [|x| \ge 1/\varepsilon - \varepsilon]$;
- supp χ_j^ε is an interval of length 3ε and with the same midpoint as supp π_j^ε, $|j| \le N-1$.

A first observation on finite ε-localization families is the following lemma.

Lemma 5.4. *Given $\varepsilon \in (0,1)$ and a finite ε-localization family, the mapping*

$$\left[h \mapsto \sum\nolimits_{j=-N+1}^{N} \|\pi_j^\varepsilon h\|_{H^r}\right] : H^r(\mathbb{R}) \to [0,\infty)$$

defines a norm equivalent to the standard norm on $H^r(\mathbb{R})$ for all $r \ge 0$.

Proof. Let
$$[r] := \max\{n \in \mathbb{N} : n \le r\}.$$
Then, since each $a \in \mathrm{C}^{[r]+1}(\mathbb{R})$ is a pointwise multiplier in $H^r(\mathbb{R})$, that is

$$\|ah\|_{H^r} \le C\|a\|_{\mathrm{C}^{[r]+1}}\|h\|_{H^r}, \quad a \in \mathrm{C}^{[r]+1}(\mathbb{R}), h \in H^r(\mathbb{R}),$$

cf. [86, Section 2.2.3], we have

$$\sum_{j=-N+1}^{N} \|\pi_j^\varepsilon h\|_{H^r} \le C\left(\sum_{j=-N+1}^{N} \|\pi_j^\varepsilon\|_{\mathrm{C}^{[r]+1}}\right)\|h\|_{H^r} \le C\|h\|_{H^r},$$

where C only depends on ε, r, and N. Moreover,

$$\|h\|_{H^r} = \left\|\sum_{j=-N+1}^{N} (\pi_j^\varepsilon)^2 h\right\|_{H^r} \le C\sum_{j=-N+1}^{N} \|\pi_j^\varepsilon\|_{\mathrm{C}^{[r]+1}} \cdot \|\pi_j^\varepsilon h\|_{H^r} \le C\sum_{j=-N+1}^{N} \|\pi_j^\varepsilon h\|_{H^r},$$

where again the constant C only depends on ε, r, and N. □

We are now in a position to locally approximate the path Ψ_1.

Lemma 5.5. *Let $X \in \mathcal{O}_r$ be fixed and choose $r' \in (3/2, r)$. Given $\nu > 0$, there exist $\varepsilon \in (0,1)$, a finite ε-localization family $\{\pi_j^\varepsilon \,:\, -N+1 \le j \le N\}$, a positive constant $K = K(\varepsilon, X)$, and bounded operators $\mathbb{A}_{j,\tau}^1 \in \mathcal{L}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$, $j \in \{-N+1, \ldots, N\}$, $\tau \in [0,1]$, such that*

$$\|\pi_j^\varepsilon \Psi_1(\tau)[u] - \mathbb{A}_{j,\tau}^1[\pi_j^\varepsilon u]\|_{H^{r-1}} \le \nu\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}} \tag{5.14}$$

for all $-N+1 \le j \le N$, $\tau \in [0,1]$, and $u \in H^r(\mathbb{R})$. The operators $\mathbb{A}_{j,\tau}^1$ are defined by

$$\mathbb{A}_{j,\tau}^1 := \alpha_{\tau,1}(x_j^\varepsilon)\Big(-\frac{d^2}{dx^2}\Big)^{1/2} + \beta_{\tau,1}(x_j^\varepsilon)\frac{d}{dx}, \quad |j| \le N-1, \qquad \mathbb{A}_{N,\tau}^1 := \Theta_1\Big(-\frac{d^2}{dx^2}\Big)^{1/2},$$

where $x_j^\varepsilon \in$ supp π_j^ε, $|j| \le N-1$, and with functions $\alpha_{\tau,1}$, $\beta_{\tau,1}$ given by

$$\alpha_{\tau,1} := \frac{1+(1-\tau)f'^2}{1+f'^2}\Theta_1 \qquad \textit{and} \qquad \beta_{\tau,1} := \tau\Theta_1 B_{1,1}^0(f)[f'] + \tau a_1(X).$$

Proof. In the following constants denoted by C do not depend on ε, while K stands for constants that depend on ε. We first consider the case where $|j| \le N-1$. Recalling (5.12), it holds that

$$\begin{aligned}\|\pi_j^\varepsilon \Psi_1(\tau)[u] - \mathbb{A}^1_{j,\tau}[\pi_j^\varepsilon u]\|_{H^{r-1}} &\le \Theta_1 \|\pi_j^\varepsilon \mathbb{B}(\tau f)[u'] - H[(\pi_j^\varepsilon u)']\|_{H^{r-1}}\\ &\quad + \Theta_1 \Big\| \tau \pi_j^\varepsilon \partial\mathbb{B}(f)[u][f'] + \tau \frac{f'^2(x_j^\varepsilon)}{1+f'^2(x_j^\varepsilon)} H[(\pi_j^\varepsilon u)']\\ &\qquad - \tau B^0_{1,1}(f)[f'](x_j^\varepsilon)(\pi_j^\varepsilon u)'\Big\|_{H^{r-1}}\\ &\quad + \|\pi_j^\varepsilon \tau a_1(X)u' - \tau a_1(X)(x_j^\varepsilon)(\pi_j^\varepsilon u)'\|_{H^{r-1}}\\ &\quad + \|\pi_j^\varepsilon \tau T_{\mathrm{lot},1}[u]\|_{H^{r-1}},\end{aligned}$$

for $u \in H^r(\mathbb{R})$. The estimate (5.13) leads us to

$$\|\pi_j^\varepsilon \tau T_{\mathrm{lot},1}[u]\|_{H^{r-1}} \le K\|u\|_{H^{r'}}.$$

Furthermore, in view of Lemma B.13 and recalling the definition (5.11) of $\mathbb{B}(\tau f)$, we have

$$\begin{aligned}\|\pi_j^\varepsilon \mathbb{B}(\tau f)[u'] - H[(\pi_j^\varepsilon u)']\|_{H^{r-1}} &\le \Big\|\pi_j^\varepsilon B_{0,1}(\tau f)[u'] - \frac{1}{1+(\tau f'(x_j^\varepsilon))^2} H[\pi_j^\varepsilon u']\Big\|_{H^{r-1}}\\ &\quad + \Big\|\pi_j^\varepsilon \tau f'(B^0_{1,1}(\tau f)[u'] - \frac{(\tau f'(x_j^\varepsilon))^2}{1+(\tau f'(x_j^\varepsilon))^2} H[\pi_j^\varepsilon u']\Big\|_{H^{r-1}}\\ &\quad + K\|u\|_{H^1}\\ &\le \frac{\nu}{3\Theta_1}\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}},\end{aligned}$$

provided that ε is chosen sufficiently small. Moreover, Theorem B.7 implies that

$$\begin{aligned}&\Big\| -2B_{2,2}(f,f)[f,u,f'] + f'B_{1,1}(f)[u,f'] - 2f'B_{3,2}(f,f)[f,f,u,f']\\ &\quad - f'\big(-2B^0_{1,2}(f)[u'] + f'B_{0,1}(f)[u'] - 2f'B^0_{2,2}(f)[u']\big)\Big\|_{H^{r-1}} \le C\|u\|_{H^{r'}}\end{aligned} \tag{5.15}$$

for all $u \in H^r(\mathbb{R})$. Thus, invoking again Lemma B.13, it holds that

$$\begin{aligned}&\Big\|\tau\pi_j^\varepsilon \partial\mathbb{B}(f)[u][f'] + \tau\frac{f'^2(x_j^\varepsilon)}{1+f'^2(x_j^\varepsilon)} H[(\pi_j^\varepsilon u)'] - \tau B^0_{1,1}(f)[f'](x_j^\varepsilon)(\pi_j^\varepsilon u)'\Big\|_{H^{r-1}}\\ &\le 2\Big\|\pi_j^\varepsilon f' B^0_{1,2}(f)[u'] - \frac{f'^2(x_j^\varepsilon)}{[1+f'^2(x_j^\varepsilon)]^2} H[\pi_j^\varepsilon u']\Big\|_{H^{r-1}}\\ &\quad + \big\|\big(B^0_{1,1}(f)[f'] - B^0_{1,1}(f)[f'](x_j^\varepsilon)\big)\pi_j^\varepsilon u'\big\|_{H^{r-1}}\\ &\quad + \Big\|\pi_j^\varepsilon f'^2 B_{0,1}(f)[u'] - \frac{f'^2(x_j^\varepsilon)}{1+f'^2(x_j^\varepsilon)} H[\pi_j^\varepsilon u']\Big\|_{H^{r-1}}\\ &\quad + 2\Big\|\pi_j^\varepsilon f'^2 B^0_{2,2}(f)[u']) - \frac{f'^4(x_j^\varepsilon)}{[1+f'^2(x_j^\varepsilon)]^2} H[\pi_j^\varepsilon u']\Big\|_{H^{r-1}} + K\|u\|_{H^{r'}}\\ &\le \big\|\big(B^0_{1,1}(f)[f'] - B^0_{1,1}(f)[f'](x_j^\varepsilon)\big)(\pi_j^\varepsilon u')\big\|_{H^{r-1}}\\ &\quad + \frac{\nu}{6\Theta_1}\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}}\end{aligned}$$

for $u \in H^r(\mathbb{R})$. Furthermore, using (B.18), together with the relation

$$\chi_j^\varepsilon \pi_j^\varepsilon = \pi_j^\varepsilon, \qquad |j| \le N-1,$$

we conclude

$$\begin{aligned}\big\|\big(B^0_{1,1}(f)[f']-&B^0_{1,1}(f)[f'](x^\varepsilon_j)\big)\pi^\varepsilon_j u'\big\|_{H^{r-1}}\\&=\big\|\chi^\varepsilon_j\left(B^0_{1,1}(f)[f']-B^0_{1,1}(f)[f'](x^\varepsilon_j)\right)\pi^\varepsilon_j u'\big\|_{H^{r-1}}\\&\leq C\,\big\|\chi^\varepsilon_j\left(B^0_{1,1}(f)[f']-B^0_{1,1}(f)[f'](x^\varepsilon_j)\right)\big\|_\infty\|\pi^\varepsilon_j u'\|_{H^{r-1}}\\&\quad+C\,\big\|\chi^\varepsilon_j\left(B^0_{1,1}(f)[f']-B^0_{1,1}(f)[f'](x^\varepsilon_j)\right)\big\|_{H^{r-1}}\|\pi^\varepsilon_j u'\|_\infty\\&\leq\frac{\nu}{6\Theta_1}\|\pi^\varepsilon_j u\|_{H^r}+K\|u\|_{H^{r'}}\end{aligned}$$

for $\varepsilon>0$ sufficiently small and $u\in H^r(\mathbb{R})$. Here we used that $u'\in H^{r'-1}(\mathbb{R})$ is bounded and $B^0_{1,1}(\tau f)[\tau f']\in H^{r-1}(\mathbb{R})\hookrightarrow \mathrm{BUC}^{r-3/2}(\mathbb{R})$. Following similar arguments we obtain, in view of $a_1(X)\in H^1(\mathbb{R})$, that

$$\begin{aligned}\|\pi^\varepsilon_j\tau a_1(X)u'-\tau a_1(X)(x^\varepsilon_j)(\pi^\varepsilon_j u)'\|_{H^{r-1}}&\leq\|\chi^\varepsilon_j(a_1(X)-a_1(X)(x^\varepsilon_j))(\pi^\varepsilon_j u')\|_{H^{r-1}}\\&\leq C\|\chi^\varepsilon_j(a_1(X)-a_1(X)(x^\varepsilon_j))\|_\infty\|\pi^\varepsilon_j u'\|_{H^{r-1}}\\&\quad+C\|\chi^\varepsilon_j(a_1(X)-a_1(X)(x^\varepsilon_j))\|_{H^{r-1}}\|\pi^\varepsilon_j u'\|_\infty\\&\leq\frac{\nu}{3}\|\pi^\varepsilon_j u\|_{H^r}+K\|u\|_{H^{r'}},\end{aligned}$$

which leads us to (5.14) for $|j|\leq N-1$, $\tau\in[0,1]$, and all $u\in H^r(\mathbb{R})$, provided that $\varepsilon>0$ is sufficiently small.

Now we consider the second case, when $j=N$. We have

$$\begin{aligned}\|\pi^\varepsilon_N\Psi_1(\tau)[u]-\mathbb{A}^1_{N,\tau}[\pi^\varepsilon_N u]\|_{H^{r-1}}\leq\Theta_1&\big\|\pi^\varepsilon_N(\mathbb{B}(\tau f)[u']+\tau\partial\mathbb{B}(f)[u][f'])-H[(\pi^\varepsilon_N u)']\big\|_{H^{r-1}}\\&+\|\pi^\varepsilon_N\tau a_1(X)u'\|_{H^{r-1}}+\|\pi^\varepsilon_N\tau T_{\mathrm{lot},1}[u]\|_{H^{r-1}},\end{aligned}$$

where, with the help of (5.13) we obtain

$$\|\pi^\varepsilon_N\tau T_{\mathrm{lot},1}[u]\|_{H^{r-1}}\leq K\|u\|_{H^{r'}},\quad u\in H^r(\mathbb{R}).$$

Moreover, recalling (5.15), we have

$$\begin{aligned}\big\|\pi^\varepsilon_N(\mathbb{B}(\tau f)[u']+&\tau\partial\mathbb{B}(f)[u][f'])-H[(\pi^\varepsilon_N u)']\big\|_{H^{r-1}}\\&\leq\big\|\pi^\varepsilon_N B_{0,1}(\tau f)[u']-H[\pi^\varepsilon_N u']\big\|_{H^{r-1}}+\big\|\pi^\varepsilon_N\tau f'B^0_{1,1}(\tau f)[u']\big\|_{H^{r-1}}\\&\quad+2\big\|\pi^\varepsilon_N f'B^0_{1,2}(f)[u']\big\|_{H^{r-1}}+\big\|\pi^\varepsilon_N u'B^0_{1,1}(f)[f']\big\|_{H^{r-1}}\\&\quad+\big\|\pi^\varepsilon_N f'^2B_{0,1}(f)[u']\big\|_{H^{r-1}}+2\big\|\pi^\varepsilon_N f'^2B^0_{2,2}(f)[u']\big\|_{H^{r-1}}+K\|u\|_{H^{r'}}\end{aligned}$$

for all $u\in H^r(\mathbb{R})$. In virtue of Lemma B.14 we have

$$\begin{aligned}\big\|\pi^\varepsilon_N\tau f'B^0_{1,1}(\tau f)[u']\big\|_{H^{r-1}}+2\big\|\pi^\varepsilon_N f'B^0_{1,2}(f)[u']\big\|_{H^{r-1}}+\big\|\pi^\varepsilon_N f'^2B_{0,1}(f)[u']\big\|_{H^{r-1}}&\\+2\big\|\pi^\varepsilon_N f'^2B^0_{2,2}(f)[u']\big\|_{H^{r-1}}\leq\frac{\nu}{3\Theta_1}\|\pi^\varepsilon_j u\|_{H^r}+K\|u\|_{H^{r'}},&\end{aligned}$$

and Lemma B.15 yields

$$\Big\|\pi^\varepsilon_N B_{0,1}(\tau f)[u']-H[\pi^\varepsilon_N u']\Big\|_{H^{r-1}}\leq\frac{\nu}{12\Theta_1}\|\pi^\varepsilon_j u\|_{H^r}+K\|u\|_{H^{r'}}$$

for all $u\in H^r(\mathbb{R})$ and $\tau\in[0,1]$, provided that ε is sufficiently small. Using the formula

$$\chi^\varepsilon_N\pi^\varepsilon_N=\pi^\varepsilon_N$$

and (B.18), we get for $\varepsilon > 0$ sufficiently small that

$$\begin{aligned}\left\|\pi_N^\varepsilon u' B_{1,1}^0(f)[f']\right\|_{H^{r-1}} &\le C\left\|\chi_N^\varepsilon B_{1,1}^0(f)[f']\right\|_\infty \|\pi_N^\varepsilon u'\|_{H^{r-1}} \\ &\quad + C\left\|\chi_N^\varepsilon B_{1,1}^0(f)[f']\right\|_{H^{r-1}} \|\pi_N^\varepsilon u'\|_\infty \\ &\le \frac{\nu}{12\Theta_1}\|\pi_N^\varepsilon u'\|_{H^{r-1}} + K\|u\|_{H^{r'}} \\ &\le \frac{\nu}{12\Theta_1}\|\pi_N^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}},\end{aligned}$$

since $B_{1,1}^0(f)[f']$ vanishes at infinity, see (A.19). Similar arguments show that

$$\|\pi_N^\varepsilon \tau a_1(X)u'\|_{H^{r-1}} \le \frac{\nu}{2}\|\pi_N^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}},$$

if ε is sufficiently small and the estimate (5.14) follows. □

In order to show (5.10), we follow the same approach and thus compute in view of (5.4) and (5.5) that

$$\begin{aligned}\partial_h\Phi_2(X)[v] &= \Theta_2\Big(B_{0,1}(h)[v'] - 2B_{2,2}(h,h)[h,v,h'] + v'B_{1,1}(h)[h,h'] \\ &\qquad + h'(B_{1,1}(h)[v,h'] + B_{1,1}(h)[v,h'] - 2B_{3,2}(h,h)[h,h,v,h'])\Big) \\ &\quad + \Theta_1\Big(vh'C_1'(X)[f'] + (h-c_\infty)v'C_1'(X)[f'] - 2h'(h-c_\infty)vC_{2,1}'^0(X)[f'] \\ &\qquad - v'C_1'(X)[ff'] + 2vh'C_{2,1}'^0(X)[ff'] - 2vD_{2,1}'^0(X)[f']\Big) \\ &= \Theta_2(\mathbb{B}(h)[v'] + \partial\mathbb{B}(h)[v][h']) + a_2(X)v' + T_{\mathrm{lot},2}[v]\end{aligned}$$

for $v \in H^r(\mathbb{R})$, where

$$a_2(X) := \Theta_1\big((h-c_\infty)C_1'(X)[f'] - C_1'(X)[ff']\big) \in H^1(\mathbb{R})$$

and

$$\begin{aligned}T_{\mathrm{lot},2} := \Theta_1\Big(&vh'C_1'(X)[f'] - 2(h-c_\infty)h'vC_{2,1}'^0(X)[f'] \\ &+ 2vh'C_{2,1}'^0(X)[ff'] - 2vD_{2,1}'^0(X)[f']\Big)\end{aligned}$$

Moreover, define the path $\Psi_2\colon [0,1] \to \mathcal{L}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$ by

$$\Psi_2(\tau) := \Theta_2\big(\mathbb{B}(\tau h)[v'] + \tau\partial\mathbb{B}(h)[v][h']\big) + \tau a_2(X)v' + \tau T_{\mathrm{lot},2}[v],$$

which enjoys similar properties than the path Ψ_1, stated in the following Remark 5.6.

Remark 5.6.

(i) Letting H denote the Hilbert transform, see Appendix C for more details, we have

$$\Psi_2(0) = \Theta_2\mathbb{B}(0)\circ(d/dx) = \Theta_2 H\circ(d/dx) = \Theta_2\Big(-\frac{d^2}{dx^2}\Big)^{1/2}.$$

(ii) Following Lemma C.6, we get that $\Psi_2(0)$ is a Fourier multiplier with symbol

$$[\xi \mapsto \Theta_2|\xi|].$$

(iii) It holds that $\Psi_2(1) = \partial_h\Phi_2(X)$.

(iv) In view of (4.18) it holds that

$$\|T_{\mathrm{lot},2}[v]\|_{H^{r-1}} \leq C\|v\|_{H^1} \qquad \text{for all } v \in H^r(\mathbb{R}). \tag{5.16}$$

On account of the proof of Lemma 5.5, it is not hard to infer the following result on Ψ_2.

Lemma 5.7. *Let $X \in \mathcal{O}_r$ be fixed and choose $r' \in (3/2, r)$. Given $\nu > 0$, there exist $\varepsilon \in (0,1)$, a finite ε-localization family $\{\pi_j^\varepsilon : -N+1 \leq j \leq N\}$, a positive constant $K = K(\varepsilon, X)$, and bounded operators $\mathbb{A}_{j,\tau}^2 \in \mathcal{L}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$, $j \in \{-N+1, \ldots, N\}$ and $\tau \in [0,1]$, such that*

$$\|\pi_j^\varepsilon \Psi_2(\tau)[v] - \mathbb{A}_{j,\tau}^2[\pi_j^\varepsilon v]\|_{H^{r-1}} \leq \nu\|\pi_j^\varepsilon v\|_{H^r} + K\|v\|_{H^{r'}} \tag{5.17}$$

for all $-N+1 \leq j \leq N$, $\tau \in [0,1]$, and $v \in H^r(\mathbb{R})$. The operators $\mathbb{A}_{j,\tau}^2$ are defined by

$$\mathbb{A}_{j,\tau}^2 := \alpha_{\tau,2}(x_j^\varepsilon)\Big(-\frac{d^2}{dx^2}\Big)^{1/2} + \beta_{\tau,2}(x_j^\varepsilon)\frac{d}{dx}, \quad |j| \leq N-1, \qquad \mathbb{A}_{N,\tau}^2 := \Theta_2\Big(-\frac{d^2}{dx^2}\Big)^{1/2},$$

where $x_j^\varepsilon \in \operatorname{supp} \pi_j^\varepsilon$, $|j| \leq N-1$, and with functions $\alpha_{\tau,2}$, $\beta_{\tau,2}$ given by

$$\alpha_{\tau,2} := \frac{1+(1-\tau)h'^2}{1+h'^2}\Theta_2 \qquad \text{and} \qquad \beta_{\tau,2} := \tau\Theta_2 B_{1,1}^0(h)[h'] + \tau a_2(X).$$

Proof. The proof is identical to that of Lemma 5.5 and therefore we omit it. □

We now show that the Fourier multipliers $\mathbb{A}_{j,\tau}^k$, $\tau \in [0,1]$, $-N+1 \leq j \leq N$, and $k = 1,2$, identified in the Lemma 5.5 and Lemma 5.7 are generators of analytic semigroups in $\mathcal{L}(H^{r-1}(\mathbb{R}))$ that is

$$-\mathbb{A}_{j,\tau}^k \in \mathcal{H}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R})), \tag{5.18}$$

cf. (5.2). Moreover, as shown in Lemma 5.8 below and the subsequent discussion, these operators satisfy estimates, which are uniform with respect to $\tau \in [0,1]$, $j \in -N+1, \ldots, N$, and $k = 1,2$.

Lemma 5.8. *Given $\eta \in (0,1)$, $\alpha \in [\eta, 1/\eta]$ and $|\beta| \leq 1/\eta$, there exists $\kappa_0 = \kappa_0(\eta) \geq 1$ such that the Fourier multiplier*

$$\mathbb{A}_{\alpha,\beta} := -\alpha\left(-\frac{d^2}{dx^2}\right)^{1/2} + \beta\frac{d}{dx},$$

satisfies

$$\lambda - \mathbb{A}_{\alpha,\beta} \in \operatorname{Isom}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R})), \tag{5.19}$$

$$\kappa_0\|(\lambda - \mathbb{A}_{\alpha,\beta})[\bar\omega]\|_{H^{r-1}} \geq |\lambda| \cdot \|\bar\omega\|_{H^{r-1}} + \|\bar\omega\|_{H^r} \tag{5.20}$$

for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 1$ and $\bar\omega \in H^r(\mathbb{R})$.

Proof. First we note that $\mathbb{A}_{\alpha,\beta}$ is a Fourier multiplier with symbol

$$m(\zeta) := -\alpha|\zeta| + i\beta\zeta, \quad \zeta \in \mathbb{R}.$$

For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 1$, we define the operator $R(\lambda, \mathbb{A}_{\alpha,\beta})$ by the formula

$$\mathcal{F}(R(\lambda, \mathbb{A}_{\alpha,\beta})[\bar\omega]) = \frac{1}{\lambda - m}\mathcal{F}\bar\omega, \quad \bar\omega \in H^{r-1}(\mathbb{R}).$$

This operator belongs to $\mathcal{L}(H^{r-1}(\mathbb{R}), H^r(\mathbb{R}))$ since

$$\begin{aligned}
\|R(\lambda, \mathbb{A}_{\alpha,\beta})\bar{\omega}\|_{H^r}^2 &= \int_{\mathbb{R}} (1+|\zeta|^2)^r \frac{|\mathcal{F}\bar{\omega}(\zeta)|^2}{|\lambda - m(\zeta)|^2}\, d\zeta \\
&= \int_{\mathbb{R}} (1+|\zeta|^2)^r \frac{|\mathcal{F}\bar{\omega}(\zeta)|^2}{(\operatorname{Re}\lambda + \alpha|\zeta|)^2 + (\operatorname{Im}\lambda - \beta\zeta)^2}\, d\zeta \\
&\leq \frac{1}{\min\{1,\alpha\}} \int_{\mathbb{R}} \frac{(1+|\zeta|^2)^r}{(1+|\zeta|^2)} |\mathcal{F}\bar{\omega}(\zeta)|^2\, d\zeta \\
&\leq \frac{1}{\eta} \int_{\mathbb{R}} (1+|\zeta|^2)^{r-1} |\mathcal{F}\bar{\omega}(\zeta)|^2\, d\zeta \\
&= \frac{1}{\eta} \|\bar{\omega}\|_{H^{r-1}}^2
\end{aligned}$$

for $\bar{\omega} \in H^{r-1}(\mathbb{R})$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 1$, and is therefore the inverse of $\lambda - \mathbb{A}_{\alpha,\beta}$. This leads us to (5.19). Moreover, another consequence of this estimate is the relation

$$\|\left(\lambda - \mathbb{A}_{\alpha,\beta}\right)[\bar{\omega}]\|_{H^{r-1}} \geq \eta^{1/2}\|\bar{\omega}\|_{H^r}, \quad \bar{\omega} \in H^r(\mathbb{R}),\ \lambda \in \mathbb{C},\ \operatorname{Re}\lambda \geq 1.$$

Furthermore, given $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 1$ and $\zeta \in \mathbb{R}$, it holds in view of

$$(\operatorname{Im}\lambda)^2 \leq 2(\operatorname{Im}\lambda - \beta\zeta)^2 + 2(\beta\zeta)^2$$

that

$$\begin{aligned}
\frac{|\lambda|^2}{|\lambda - m(\zeta)|^2} &\leq \frac{(\operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2}{(\operatorname{Re}\lambda + \alpha|\zeta|)^2 + (\operatorname{Im}\lambda - \beta\zeta)^2} \\
&\leq \frac{(\operatorname{Re}\lambda)^2}{(\operatorname{Re}\lambda + \alpha|\zeta|)^2} + \frac{2(\operatorname{Im}\lambda - \beta\zeta)^2 + 2(\beta\zeta)^2}{(\operatorname{Re}\lambda + \alpha|\zeta|)^2 + (\operatorname{Im}\lambda - \beta\zeta)^2} \\
&\leq 1 + 2\frac{(\operatorname{Im}\lambda - \beta\zeta)^2}{(\operatorname{Re}\lambda + \alpha|\zeta|)^2 + (\operatorname{Im}\lambda - \beta\zeta)^2} + 2\frac{\beta^2\zeta^2}{(\operatorname{Re}\lambda + \alpha|\zeta|)^2} \\
&\leq 3 + 2\frac{\beta^2}{\alpha^2} \\
&\leq 3 + \frac{2}{\eta^4}.
\end{aligned}$$

In virtue of this estimate we get

$$\begin{aligned}
|\lambda|^2 \cdot \|\bar{\omega}\|_{H^{r-1}}^2 &= \int_{\mathbb{R}} \frac{|\lambda|^2}{|\lambda - m(\zeta)|^2} |\lambda - m(\zeta)|^2 (1+|\zeta|^2)^{r-1} |\mathcal{F}\bar{\omega}(\zeta)|^2\, d\zeta \\
&\leq (3 + \frac{2}{\eta^4}) \int_{\mathbb{R}} |\lambda - m(\zeta)|^2 (1+|\zeta|^2)^{r-1} |\mathcal{F}\bar{\omega}(\zeta)|^2\, d\zeta \\
&= (3 + \frac{2}{\eta^4}) \|(\lambda - \mathbb{A}_{\alpha,\beta})[\bar{\omega}]\|_{H^{r-1}}^2.
\end{aligned}$$

Therewith we established also (5.20) and the proof is complete. □

We point out that since $X \in \mathcal{O}_r$, there exists a constant $\eta = \eta(X) \in (0,1)$ such that

$$\eta \leq \alpha_{\tau,k} \leq \frac{1}{\eta} \quad \text{and} \quad \|\beta_{\tau,k}\|_\infty \leq \frac{1}{\eta} \qquad \text{for all } \tau \in [0,1],\ k = 1,2. \tag{5.21}$$

In particular, the operators $\mathbb{A}_{j,\tau}^k$, $\tau \in [0,1]$, $j = -N+1, \ldots, N$, $k = 1,2$, satisfy the properties (5.19) and (5.20) of Lemma 5.8.

We are now in a position to prove the generator properties for $\partial_f \Phi_1(X)$ and for $\partial_h \Phi_2(X)$, see Proposition (5.9). The proof relies to a large extend on the previous results established in

Lemma 5.5, Lemma 5.7, and Lemma 5.8.

Proposition 5.9. *Given $X \in \mathcal{O}_r$ it holds that*

$$-\partial_f \Phi_1(X) \in \mathcal{H}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$$

and

$$-\partial_h \Phi_2(X) \in \mathcal{H}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R})).$$

Proof. Let κ_0 be the constant determined in Lemma 5.8 for η as identified in (5.21) and define $\nu := 1/2\kappa_0$. Choosing $r' \in (3/2, r)$, Lemma 5.5 ensures there exists a finite ε-localization family $\{\pi_j^\varepsilon : -N+1 \leq j \leq N\}$, for some $\varepsilon > 0$, and a constant $K = K(\varepsilon, X)$ satisfying

$$2\kappa_0 \|\pi_j^\varepsilon \Psi_1(\tau)[u] - \mathbb{A}_{j,\tau}^1[\pi_j^\varepsilon u]\|_{H^{r-1}} \leq \|\pi_j^\varepsilon u\|_{H^r} + 2\kappa_0 K \|u\|_{H^{r'}}$$

for all $\tau \in [0,1]$, $j \in \{-N+1, \ldots, N\}$, and $u \in H^r(\mathbb{R})$. Moreover, the estimate (5.20) implies that

$$2\kappa_0 \|(\lambda - \mathbb{A}_{j,\tau}^1)[\pi_j^\varepsilon u]\|_{H^{r-1}} \geq 2|\lambda| \cdot \|\pi_j^\varepsilon u\|_{H^{r-1}} + 2\|\pi_j^\varepsilon u\|_{H^r}$$

for all $j \in \{-N+1, \ldots, N\}$, $\tau \in [0,1]$, $\operatorname{Re}\lambda \geq 1$, and $u \in H^r(\mathbb{R})$. This leads to

$$\begin{aligned} 2\kappa_0 \|\pi_j^\varepsilon (\lambda - \Psi_1(\tau))[u]\|_{H^{r-1}} &\geq 2\kappa_0 \|(\lambda - \mathbb{A}_{j,\tau}^1)[\pi_j^\varepsilon u]\|_{H^{r-1}} \\ &\quad - 2\kappa_0 \|\pi_j^\varepsilon \Psi_1(\tau)[u] - \mathbb{A}_{j,\tau}^1[\pi_j^\varepsilon u]\|_{H^{r-1}} \\ &\geq 2|\lambda| \cdot \|\pi_j^\varepsilon u\|_{H^{r-1}} + \|\pi_j^\varepsilon u\|_{H^r} - 2\kappa_0 K \|u\|_{H^{r'}} \end{aligned}$$

for all $j \in \{-N+1, \ldots, N\}$, $\tau \in [0,1]$, $\operatorname{Re}\lambda \geq 1$, and $u \in H^r(\mathbb{R})$. Summing up over all indices $j \in \{-N+1, \ldots, N\}$, Lemma 5.4 and (5.8) (with $\tau = r'$) imply there exist constants $\kappa = \kappa(X) \geq 1$ and $\omega = \omega(X) \geq 1$ such that

$$\kappa \|(\lambda - \Psi_1(\tau))[u]\|_{H^{r-1}} \geq |\lambda| \cdot \|u\|_{H^{r-1}} + \|u\|_{H^r} \tag{5.22}$$

for all $\tau \in [0,1]$, $\operatorname{Re}\lambda \geq \omega$, and $u \in H^r(\mathbb{R})$.

We note that by (5.19) and Remark 5.3 it holds that

$$(\omega - \Psi_1(\tau))\,|_{\tau=0} = \omega + \Theta_1 \left(-\frac{d^2}{dx^2}\right)^{1/2} \in \operatorname{Isom}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R})). \tag{5.23}$$

In view of (5.22) and (5.23), the method of continuity, see e.g. [7, Theorem I.1.1.1], yields

$$\omega - \partial_f \Phi_1(X) \in \operatorname{Isom}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R})). \tag{5.24}$$

The properties (5.22) (with $\tau = 1$) and (5.24) lead to $-\partial_f \Phi_1 \in \mathcal{H}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$.

The second claim concerning $\partial_h \Phi_2(X)$ follows, due to Lemma 5.7 and Lemma 5.8 with $k = 2$, by arguing along the same lines. □

We can finally prove (5.1).

Theorem 5.10. *Let $X \in \mathcal{O}_r$. Then,*

$$-\partial\Phi(X) \in \mathcal{H}(H^r(\mathbb{R})^2, H^{r-1}(\mathbb{R})^2).$$

Proof. The claim follows from Theorem 5.1, Lemma 5.2, and Proposition 5.9. □

5.2. The well-posedness result

In view of Chapter 4, we know that Φ is mapping the set $\mathcal{O}_r$ smoothly to $H^{r-1}(\mathbb{R})^2$, see (4.1). Moreover, we have shown that, given $X \in \mathcal{O}_r$, the operator $\partial\Phi(X)$ is the generator of an analytic semigroup, see (5.1). These two properties will be the main ingredients when we establish the well-posedness of the multiphase Muskat problem (1.1) with the help of abstract theory from [57].

Proof of Theorem 2.1. The properties (4.1) and (5.1) enable us to use the abstract parabolic theory from [57, Chapter 8] in the context of the evolution problem (1.1). More precisely, given $X_0 \in \mathcal{O}_r$, [57, Theorem 8.1.1] implies there exists a time $T > 0$ and a solution $X = X(\cdot; X_0)$ to (1.1) such that

$$X \in \mathrm{C}([0,T], \mathcal{O}_r) \cap \mathrm{C}^1([0,T], H^{r-1}(\mathbb{R})^2) \cap \mathrm{C}^{\alpha}_{\alpha}((0,T], H^r(\mathbb{R})^2)$$

for some $\alpha \in (0,1)$, see. (1.4) for the definition of $\mathrm{C}^{\alpha}_{\alpha}(0,T], H^r(\mathbb{R})^2)$. Since Φ is independent of time, this property holds for all $\alpha \in (0,1)$. Moreover, this solution is unique within

$$\bigcup_{\alpha\in(0,1)} \mathrm{C}^{\alpha}_{\alpha}((0,T], H^r(\mathbb{R})^2) \cap \mathrm{C}([0,T], \mathcal{O}_r) \cap \mathrm{C}^1([0,T], H^{r-1}(\mathbb{R})^2).$$

In order to prove that the solution is unique within $\mathrm{C}([0,T], \mathcal{O}_r) \cap \mathrm{C}^1([0,T], H^{r-1}(\mathbb{R})^2)$, as we claimed in Theorem 2.1, we assume by contradiction that there exist two different solutions $X_i : [0,T] \to \mathcal{O}_r$, with $i = 1, 2$, to (1.1) corresponding to the same initial data, that is

$$X_1(0) = X_2(0) \quad \text{and} \quad X_1(t) \neq X_2(t) \quad \text{for all } t \in (0,T].$$

Let $r' \in (3/2, r)$ be arbitrary and set $\alpha := r - r' \in (0,1)$. The mean value theorem together with the inequality $\|a\|_{H^{r'}} \leq \|a\|^{\alpha}_{H^{r-1}} \|a\|^{1-\alpha}_{H^r}$, $a \in H^r(\mathbb{R})$, see (5.8), yields the existence of a constant $C > 0$ such that

$$\begin{aligned}
\|X_i(t) - X_i(s)\|_{H^{r'}} &\leq \|X_i(t) - X_i(s)\|^{\alpha}_{H^{r-1}} \cdot \|X_i(t) - X_i(s)\|^{1-\alpha}_{H^r} \\
&\leq \Big(2 \max_{[0,T]} \|X_i\|_{H^r}\Big)^{1-\alpha} \Big(\max_{[0,T]} \Big\|\frac{dX_i}{dt}\Big\|_{H^{r-1}} |t-s|\Big)^{\alpha} \\
&\leq C|t-s|^{\alpha}, \qquad s,\, t \in [0,T],\, i = 1,\, 2,
\end{aligned}$$

which means

$$X_i \in \mathrm{C}^{\alpha}([0,T], H^{r'}(\mathbb{R})^2) \hookrightarrow \mathrm{C}^{\alpha}_{\alpha}((0,T], H^{r'}(\mathbb{R})^2). \tag{5.25}$$

The abstract result [57, Theorem 8.1.1] applied again in the context of (1.1) with r replaced by r' ensures that $X_1 = X_2$ in $[0,T]$, which contradicts our assumption. Arguing as in [57, Section 8.2], this unique solution can be extended up to a maximal existence time $T^+ = T^+(X_0)$.

In order to conclude (i), we make use of [57, Proposition 8.2.3], which ensures the continuous dependence of the solution on the initial data.

We next prove (ii) with a parameter trick successfully applied also to other problems, cf., e.g. [43, 60]. Let $\lambda = (\lambda_1, \lambda_2) \in (0,\infty) \times \mathbb{R}$ and $X = X(\cdot, X_0)$ be a maximal solution to (3.12). We then set

$$X_\lambda(t,x) := X(\lambda_1 t, x + \lambda_2 t), \quad x \in \mathbb{R}, t \in [0, T_{+,\lambda}),$$

where $T_{+,\lambda} := T_+/\lambda_1$. Now we note that $X_\lambda \in \mathrm{C}([0,T_{+,\lambda}), \mathcal{O}_r) \cap \mathrm{C}^1([0,T_{+,\lambda}), H^{r-1}(\mathbb{R})^2)$ is a solution to

$$\frac{dX}{dt} = \Psi(X,\lambda), \quad t \geq 0, \quad X(0) = X_0,$$

where $\Psi\colon \mathcal{O}_r \times (0,\infty) \times \mathbb{R} \to H^{r-1}(\mathbb{R})$ is defined by

$$\Psi(X,\lambda) := \lambda_1 \Phi(X) + \lambda_2 \frac{dX}{dx}. \tag{5.26}$$

In virtue of (4.1), it holds that $\Psi \in \mathrm{C}^\infty(\mathcal{O}_r \times (0,\infty) \times \mathbb{R}, H^{r-1}(\mathbb{R})^2)$. Given $X_0 \in \mathcal{O}_r$ and $\lambda \in (0,\infty) \times \mathbb{R}$, the partial derivative of Ψ with respect to X is given by

$$\partial_X \Psi(X_0, \lambda) = \lambda_1 \partial\Phi(X_0) + \lambda_2 \frac{d}{dx}.$$

Since the operator d/dx corresponds to a diagonal 2×2-matrix whose non-trivial entries are Fourier multipliers with symbol $[\zeta \mapsto i\zeta]$, we obtain, by arguing as in the proof of Proposition 5.9, that

$$-\partial_X \Psi(X_0, \lambda) \in \mathcal{H}(H^r(\mathbb{R})^2, H^{r-1}(\mathbb{R})^2)$$

for all $(X_0, \lambda) \in \mathcal{O}_r \times (0,\infty) \times \mathbb{R}$. According to [57, Theorem 8.1.1] and arguing as in the first part of the proof, for each $(X_0, \lambda) \in \mathcal{O}_r \times (0,\infty) \times \mathbb{R}$ there is a unique maximal solution X to (5.26) with

$$X = X(\cdot; X_0, \lambda) \in \mathrm{C}([0, \tilde{T}_+), \mathcal{O}_r) \cap \mathrm{C}^1([0, \tilde{T}_+), H^{r-1}(\mathbb{R})^2),$$

where $\tilde{T}_+ = T_+(X_0, \lambda)$ is the maximal existence time. Using [57, Corollary 8.3.8], we conclude that the set

$$\Omega := \{(t, X_0, \lambda) : (X_0, \lambda) \in \mathcal{O}_r \times (0,\infty) \times \mathbb{R}, 0 < t < T_+(X_0, \lambda)\}$$

is open and

$$[(t, X_0, \lambda) \mapsto X(t; X_0, \lambda)] \in \mathrm{C}^\infty(\Omega, \mathcal{O}_r).$$

By uniqueness of the solution to (5.26), we have

$$T_+(X_0, \lambda) = \frac{T_+(X_0)}{\lambda_1} \quad \text{and} \quad X(t, X_0, \lambda) = X_\lambda(t), \quad t \in [0, T_+(X_0)/\lambda_1).$$

Given $0 < t_0 < T_+(X_0)$, we may choose $\delta > 0$ such that for all λ within the disc $D_\delta((1,0))$ it holds that $0 < t_0 < T_+(X_0, \lambda)$, and thus

$$[\lambda \mapsto X_\lambda(t_0)] \in \mathrm{C}^\infty(D_\delta((1,0)), H^r(\mathbb{R})^2).$$

Using that $\partial_{\lambda_2} X_\lambda(t_0) = t_0 \partial_x X(\lambda_1 t_0, \cdot + \lambda_2 t_0)$ we obtain

$$[\lambda \mapsto \partial_x X(\lambda_1 t_0, \cdot + \lambda_2 t_0)] : D_\delta((1,0)) \to H^r(\mathbb{R})^2$$

is smooth and thus

$$[\lambda \mapsto X(\lambda_1 t_0, \cdot + \lambda_2 t_0) = X_\lambda(t_0)] : D_\delta((1,0)) \to H^{r+1}(\mathbb{R})^2$$

is smooth, too. In conclusion we get that

$$[t \mapsto X(t)] \in \mathrm{C}^\infty((t_0 - \delta, t_0 + \delta), H^{r+1}(\mathbb{R})^2).$$

Iterating these arguments leads to

$$[t \mapsto X(t)] \in \mathrm{C}^\infty((0, T_+(X_0)), H^k(\mathbb{R})^2) \qquad \text{for all } k \in \mathbb{N}.$$

Given $x_0 \in \mathbb{R}$, the mapping $[Y \mapsto Y(x_0)] : H^r(\mathbb{R})^2 \to \mathbb{R}$ is linear, hence

$$[\lambda \mapsto X(\lambda_1 t_0, x_0 + \lambda_2 t_0)] : D_\delta((1,0)) \to \mathbb{R}^2$$

is smooth. Moreover, the mapping $\varphi : D_\varepsilon((t_0, x_0)) \to D_\delta((1,0))$ with

$$\varphi(t, x) := \left(\frac{t}{t_0}, \frac{x - x_0}{t_0}\right)$$

is well-defined and, for $\varepsilon > 0$ sufficiently small, is smooth too. Composed with the previous function, this shows that

$$[(t,x) \mapsto X(t,x)] : D_\varepsilon((t_0,x_0)) \to \mathbb{R}^2$$

is smooth.

To prove (iii) we assume there exists a maximal solution $X = X(\cdot; X_0)$ to (1.1) with $T^+ < \infty$ and such that

$$\sup_{t\in[0,T^+)} \|X(t)\|_{H^r} < \infty \qquad \text{and} \qquad \liminf_{t\to T_+} \operatorname{dist}(\Gamma_f^{c_\infty}(t), \Gamma_h(t)) = c_0 > 0.$$

Arguing as above, we deduce for some fixed $r' \in (3/2, r)$, that $X : [0,T^+) \to \mathcal{O}_{r'}$ is Hölder continuous, see (5.25). Applying [57, Theorem 8.1.1] to (1.1) (with r replaced by r'), we may extend the solution X to an interval $[0,T')$ with $T^+ < T'$ and such that

$$X \in \mathrm{C}([0,T'), \mathcal{O}_{r'}) \cap \mathrm{C}^1([0,T'), H^{r'-1}(\mathbb{R})^2).$$

Moreover, the parabolic smoothing property established at (ii) (with r replaced by r') implies that $X \in \mathrm{C}^1((0,T'), H^r(\mathbb{R})^2)$, which contradicts the maximality property of X. This completes the proof. □

5.3. Excluding squirt singularities

The next result shows, for bounded solutions with $T^+ < \infty$, that the fluid interfaces intersect in at least one point along a sequence $t_n \to T^+$. Moreover, using the same strategy as in [27, 32], we exclude for such solutions that the two fluid interfaces collapse along a curve segment.

Proof of Proposition 2.3. Let $M > 0$ be chosen such that $\|X(t)\|_{H^r} \le M$ for all times $t \in [0,T^+)$. In view of (1.1d) and Lemma A.5, we deduce from this uniform bound that there exists a constant $C > 0$ such that

$$\Big\|\frac{dX(t)}{dt}\Big\|_\infty \le C(1+M^4), \qquad t\in[0,T^+).$$

Therefore the derivative of the operator $X^2 := (f^2, h^2) \in \mathrm{C}^1([0,T^+), L_2(\mathbb{R})^2)$, as well as the operator $X^2 : [0,T) \to H^r(\mathbb{R})^2$ are bounded. Since for fixed $r' \in (3/2, r)$ it holds that

$$\|a\|_{H^{r'}} \le \|a\|_2^{1-r'/r}\|a\|_{H^r}^{r'/r}, \quad a \in H^r(\mathbb{R}),$$

we conclude

$$\begin{aligned}\|X^2(t) - X^2(s)\|_{H^{r'}} &\le \|X^2(t) - X^2(s)\|_2^{1-r'/r} \cdot \|X^2(t) - X^2(s)\|_{H^r}^{r'/r} \\ &\le \Big(2\sup_{[0,T^+)} \|X^2\|_{H^r}\Big)^{r'/r} \Big(\sup_{[0,T^+)}\Big\|\frac{dX^2}{dt}\Big\|_2 |t-s|\Big)^{1-r'/r} \\ &\le C|t-s|^{1-r'/r}, \qquad s,\, t\in[0,T^+),\end{aligned}$$

and thus

$$X^2 \in \mathrm{BUC}^{1-r'/r}([0,T^+), \mathcal{O}_{r'}).$$

Hence, there exists $X_* \in H^{r'}(\mathbb{R})^2$ such that $X^2(t) \to X_* = (f_*, h_*)$ in $H^{r'}(\mathbb{R})^2$ for $t \to T^+$.

Taking advantage of the property

$$\liminf_{t\to T^+} \operatorname{dist}(\Gamma_f^{c_\infty}(t), \Gamma_h(t)) = 0,$$

which holds due to Theorem 2.1 (iii), there exists a sequence in time $(t_n)_{n\in\mathbb{N}}$ with $t_n \nearrow T^+$ and a sequence in space $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$c_\infty + f(t_n, x_n) - h(t_n, x_n) \to 0 \qquad \text{for } n \to \infty. \tag{5.27}$$

The next step is to show that $(x_n)_{n\in\mathbb{N}}$ is bounded. To this end, we use the $H^{r'}(\mathbb{R})^2$-convergence of $X^2(t) \to X_*$ and the fact that

$$X_* \in H^{r'}(\mathbb{R})^2 \subset \{(f,h) \in \mathrm{C}(\mathbb{R})^2 : f(x), h(x) \to 0 \text{ for } |x| \to \infty\}$$

to conclude, that there exists $n_0 \in \mathbb{N}$ such that

$$|f(t_n, x)| + |h(t_n, x)| < c_\infty/2 \quad \text{ for all } \quad n \geq n_0,\ |x| \geq n_0.$$

This property yields, with the help of (5.27), that $(x_n)_{n\in\mathbb{N}}$ is indeed bounded.

After passing to a subsequence, we note that $x_n \to x_0$ for $n \to \infty$ in $\mathbb{R}$. Since X_* is continuous and $X^2(t_n)$ converges to X_* in $H^{r'}(\mathbb{R})^2$ we obtain

$$X(t_n, x_n) \to (\sqrt{f_*}(x_0), \sqrt{h_*}(x_0)).$$

The relation (5.27) now yields

$$c_\infty + \sqrt{f_*}(x_0) = \sqrt{h_*}(x_0).$$

Finally, the convergence $X^2(t_n, x_0) \to X_*(x_0)$ and the latter identity show that

$$c_\infty + f(t_n, x_0) - h(t_n, x_0) \to 0,$$

and the desired claim

$$\liminf_{t\to T^+} (c_\infty + f(t, x_0) - h(t, x_0)) = 0$$

follows at once.

In order to prove the second claim we argue by contradiction and assume there exists $x_0 \in \mathbb{R}$ and $\delta > 0$ such that

$$\liminf_{t\to T^+} \sup_{\{|x-x_0|\leq\delta\}} (c_\infty + f(t, x) - h(t, x)) = 0.$$

By invariance of (1.1) under horizontal translations, we assume without loss of generality that $x_0 = 0$. This ensures the existence of a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \nearrow T^+$ for $n \to \infty$ such that

$$c_\infty + f(t_n) - h(t_n) \to 0 \qquad \text{in } L_\infty([-\delta, \delta]). \tag{5.28}$$

Recalling Lemma A.5, we find a positive constant $c_1 = c_1(M)$ such that $v_2(t) = (v_2^1(t), v_2^2(t))$ satisfies

$$\sup_{t\in[0,T^+)} \|v_2(t)\|_{L_\infty(\Omega_2(t))} \leq c_1. \tag{5.29}$$

Given $t \in (T^+ - \delta/c_1, T^+)$, we set

$$R(t) \coloneqq \delta + c_1(t - T^+).$$

Then R is a positive, smooth, and increasing function with $R(t) \to \delta$ for $t \to T^+$. Furthermore, we define the differentiable function

$$S(t) \coloneqq \int_{-R(t)}^{R(t)} (c_\infty + f(t, x) - h(t, x))\, dx, \qquad t \in (T^+ - \delta/c_1, T^+).$$

Note that $S(t)$ is the surface area between $\Gamma_f^{c_\infty}(t)$ and $\Gamma_h(t)$ and the vertical lines $\{x = \pm R(t)\}$. Let $n_0 \in \mathbb{N}$ be fixed such that $t_n > T^+ - \delta/c_1$ for all $n \geq n_0$. On the one hand, $S(t_n) > 0$ for all $n \geq n_0$. Moreover, (5.28) implies that

$$S(t_n) \to 0 \quad \text{for } n \to \infty.$$

On the other hand, given $t \in (T^+ - \delta/c_1, T^+)$, we compute

$$\begin{aligned} S'(t) &= [c_\infty + f(t, R(t)) - h(t, R(t))]\, R'(t) \\ &\quad + [c_\infty + f(t, -R(t)) - h(t, -R(t))]\, R'(t) \\ &\quad + \int_{-R(t)}^{R(t)} \partial_t f(t,x) - \partial_t h(t,x)\, dx. \end{aligned}$$

Furthermore, Stokes' theorem and $\operatorname{div} v_2(t) = 0$ in $\Omega_2(t)$ lead to

$$\begin{aligned} \int_{-R(t)}^{R(t)} \partial_t f(t,x) - \partial_t h(t,x)\, dx &= \int_{-R(t)}^{R(t)} |(1, f'(t,x))| \cdot \langle v_2|_{\Gamma_f^{c_\infty}(t)} \mid \nu_1\rangle(t,x)\, dx \\ &\quad - \int_{-R(t)}^{R(t)} |(1, h'(t,x))| \cdot \langle v_2|_{\Gamma_h(t)} \mid \nu_2\rangle(t,x)\, dx \\ &= \int_{h(t,-R(t))}^{c_\infty + f(t,-R(t))} v_2^1(t, -R(t), y)\, dy \\ &\quad - \int_{h(t,R(t))}^{c_\infty + f(t,R(t))} v_2^1(t, R(t), y)\, dy. \end{aligned}$$

We conclude that

$$S'(t) = \int_{h(t,R(t))}^{c_\infty + f(t,R(t))} (R'(t) - v_2^1(t, R(t), y))\, dy + \int_{h(t,-R(t))}^{c_\infty + f(t,-R(t))} (R'(t) + v_2^1(t, -R(t), y))\, dy,$$

and further

$$R'(t) \mp v_2^1(t, \pm R(t), y) \geq c_1 - \sup_{t \in [0,T^+)} \|v_2(t)\|_{L_\infty(\Omega_2(t))} \geq 0.$$

Consequently, $S(t_n) \geq S(t_{n_0}) > 0$ for all $n \geq n_0$, which contradicts $S(t_n) \to 0$ for $n \to \infty$. Hence, our assumption was false and the proof is complete. □

Part II.

The multiphase Muskat problem with general viscosities

In this second part of the thesis we consider the multiphase Muskat problem (1.1) with general viscosities $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$. We prove in Section 6.3 that the velocity field

$$v = v_1|_{\Omega_1} + v_2|_{\Omega_2} + v_3|_{\Omega_3}$$

can be expressed as the contour integral

$$\begin{aligned} v(x,y) &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\big[(x,y)-(s,c_\infty+f(s))\big]^\perp}{|(x,y)-(s,c_\infty+f(s))|^2}\bar\omega_1(s)\,ds \\ &\quad + \frac{1}{\pi}\int_{\mathbb{R}} \frac{\big[(x,y)-(s,h(s))\big]^\perp}{|(x,y)-(s,h(s))|^2}\bar\omega_2(s)\,ds, \qquad (x,y)\in\mathbb{R}^2\setminus\big(\Gamma_f^{c_\infty}\cup\Gamma_h\big), \end{aligned}$$

where $(a,b)^\perp := (-b,a)$. The densities are no longer defined explicitly in terms of f and h, but solve the equation

$$(1 - A_\mu \mathcal{A}(X))[\bar\omega] = \Theta X'$$

where $\mathcal{A}(X)$, defined in (6.2) below, is the adjoint of the double layer potential for Laplace's equation corresponding to the hypersurface $\Gamma_f^{c_\infty}\cup\Gamma_h$,

$$A_\mu := \operatorname{diag}(a_\mu^1, a_\mu^2) \qquad \text{with} \qquad a_\mu^1 := \frac{\mu_1-\mu_2}{\mu_1+\mu_2}, \quad a_\mu^2 := \frac{\mu_2-\mu_3}{\mu_2+\mu_3},$$

and

$$\Theta := \operatorname{diag}(\Theta_1,\Theta_2) \qquad \text{with} \qquad \Theta_1 := kg\frac{\rho_1-\rho_2}{\mu_1+\mu_2}, \quad \Theta_2 := kg\frac{\rho_2-\rho_3}{\mu_2+\mu_3}.$$

We consider the invertibility of the operator $1 - A_\mu\mathcal{A}(X)$ in Section 6.1 and Section 6.2. Thereafter, we are in a position to establish the validity of the above formula for the velocity. In Chapter 7, we then reformulate the multiphase Muskat problem with general viscosities as a nonlinear evolution problem,

$$\frac{dX(t)}{dt} = \Phi(X(t)), \quad t \geq 0, \qquad X(0) = X_0,$$

see Section 7.1, and prove that Φ is a smooth operator whose Fréchet derivative generates an analytic semigroup, see Section 7.2, provided that the Rayleigh–Taylor conditions are satisfied at each interface, cf. (7.8). Then, using abstract parabolic semigroup theory, we establish the proof of Theorem 2.4, which provides the well-posedness of the Muskat problem with general viscosities, see Section 7.3.

6. The fixed time problem

In this chapter, we establish the solvability of the fixed time problem (1.1a)–(1.1c), where the time variable t is fixed and the function $X = X(t) = (f(t), h(t))$, which parameterizes the interfaces between the fluids, is given, see Theorem 6.6. As a first step, we introduce in Section 6.1 some integral operators for which we establish several important properties. In Section 6.2, we then investigate the invertibility of the operator $1 - A_\mu \mathcal{A}(X)$, see Theorem 6.4, which is the crucial step in the proof of Theorem 6.6.

6.1. Some integral operators

In the study of the multiphase Muskat problem with general viscosities we are confronted with certain integral operators, which we now introduce and represent in terms of the integral operators studied in Chapter 4 and Appendix B.

Given $u \in W^1_\infty(\mathbb{R})$ and

$$X = (f, h) \in \mathcal{O} = \{(f, h) \in W^1_\infty(\mathbb{R})^2 : \inf(c_\infty + f - h) > 0\},$$

we define the following operators

$$\begin{aligned}
\mathbb{A}(u)[\bar\omega](x) &:= \frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}} \frac{su'(x) - \delta_{[x,s]}u}{s^2 + (\delta_{[x,s]}u)^2}\bar\omega(x-s)\,ds,\\
\mathbb{B}(u)[\bar\omega](x) &:= \frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}} \frac{s + u'(x)\delta_{[x,s]}u}{s^2 + (\delta_{[x,s]}u)^2}\bar\omega(x-s)\,ds,\\
S(X)[\bar\omega](x) &:= \frac{1}{\pi}\int_{\mathbb{R}} \frac{sf'(x) - \delta_{[x,s]}X}{s^2 + (\delta_{[x,s]}X)^2}\bar\omega(x-s)\,ds,\\
S'(X)[\bar\omega](x) &:= \frac{1}{\pi}\int_{\mathbb{R}} \frac{sh'(x) - \delta'_{[x,s]}X}{s^2 + (\delta'_{[x,s]}X)^2}\bar\omega(x-s)\,ds,\\
T(X)[\bar\omega](x) &:= \frac{1}{\pi}\int_{\mathbb{R}} \frac{s + f'(x)\delta_{[x,s]}X}{s^2 + (\delta_{[x,s]}X)^2}\bar\omega(x-s)\,ds,\\
T'(X)[\bar\omega](x) &:= \frac{1}{\pi}\int_{\mathbb{R}} \frac{s + h'(x)\delta'_{[x,s]}X}{s^2 + (\delta'_{[x,s]}X)^2}\bar\omega(x-s)\,ds,
\end{aligned} \tag{6.1}$$

where $x \in \mathbb{R}$, PV is, as usual, the principal value and the notation of (1.6) is used. We enclose some remarks on the operators $\mathbb{A}$ and $\mathbb{B}$.

Remark 6.1.

- The operators $\mathbb{A}(u)$ and $\mathbb{B}(u)$ have been introduced in [61] in the study of the two-phase Muskat problem.
- It holds that $\mathbb{A}(0) = 0$ and $\mathbb{B}(0) = H$, where H denotes the Hilbert transform, see Appendix C.

Furthermore, we define for given $X \in \mathcal{O}$ the linear operators

$$\mathcal{A}(X) := (\mathcal{A}_1(X), \mathcal{A}_2(X)) \qquad \text{and} \qquad \mathcal{B}(X) := (\mathcal{B}_1(X), \mathcal{B}_2(X)),$$

where

$$\mathcal{A}_1(X)[\bar\omega] := \mathbb{A}(f)[\bar\omega_1] + S(X)[\bar\omega_2], \qquad \mathcal{A}_2(X)[\bar\omega] := S'(X)[\bar\omega_1] + \mathbb{A}(h)[\bar\omega_2],$$
$$\mathcal{B}_1(X)[\bar\omega] := \mathbb{B}(f)[\bar\omega_1] + T(X)[\bar\omega_2], \qquad \mathcal{B}_2(X)[\bar\omega] := T'(X)[\bar\omega_1] + \mathbb{B}(h)[\bar\omega_2],$$

for $\bar\omega := (\bar\omega_1, \bar\omega_2) \in L_2(\mathbb{R})^2$, or equivalently

$$\mathcal{A}(X) := \begin{pmatrix} \mathbb{A}(f) & S(X) \\ S'(X) & \mathbb{A}(h) \end{pmatrix} \quad \text{and} \quad \mathcal{B}(X) := \begin{pmatrix} \mathbb{B}(f) & T(X) \\ T'(X) & \mathbb{B}(h) \end{pmatrix}. \tag{6.2}$$

The aforementioned operators can be represented as combinations of operators of types discussed in Chapter 4 and Appendix B, which allows us to conclude several important properties.

Lemma 6.2. *It holds that*

$$[u \mapsto \mathbb{A}(u)],\ [u \mapsto \mathbb{B}(u)] \in \mathrm{C}^{1-}(W^1_\infty(\mathbb{R}), \mathcal{L}(L_2(\mathbb{R}))), \tag{6.3}$$
$$S,\ S',\ T,\ T' \in \mathrm{C}^{1-}(\mathcal{O}, \mathcal{L}(L_2(\mathbb{R}))), \tag{6.4}$$
$$[X \mapsto \mathcal{A}(X)],\ [X \mapsto \mathcal{B}(X)] \in \mathrm{C}^{1-}(\mathcal{O}, \mathcal{L}(L_2(\mathbb{R})^2)), \tag{6.5}$$

and, given $X \in \mathcal{O}_r = \mathcal{O} \cap H^r(\mathbb{R})^2$, $r \in (3/2, 2)$, we further have

$$S(X),\ S'(X),\ T(X),\ T'(X) \in \mathcal{L}(L_2(\mathbb{R}), H^{r-1}(\mathbb{R})), \tag{6.6}$$
$$\mathcal{A}(X),\ \mathcal{B}(X) \in \mathcal{L}(L_2(\mathbb{R})^2) \cap \mathcal{L}(H^{r-1}(\mathbb{R})^2). \tag{6.7}$$

Moreover, if $u \in \mathrm{BUC}^1(\mathbb{R})$, we have

$$\lambda - \mathbb{A}(u) \in \mathrm{Isom}(L_2(\mathbb{R})), \qquad \lambda \in \mathbb{R},\ |\lambda| \geq 1, \tag{6.8}$$

and, if additionally $u \in H^r(\mathbb{R})$, $r > 3/2$, it holds that

$$\mathbb{A}(u),\ \mathbb{B}(u) \in \mathcal{L}(H^{r-1}(\mathbb{R})), \tag{6.9}$$
$$\lambda - \mathbb{A}(u) \in \mathrm{Isom}(H^{r-1}(\mathbb{R})), \qquad \lambda \in \mathbb{R},\ |\lambda| \geq 1. \tag{6.10}$$

Proof. The properties (6.3) and (6.9) follow by the representations

$$\mathbb{A}(u) = u'B_{0,1}(u) - B_{1,1}(u)[u,\cdot] \qquad \text{and} \qquad \mathbb{B}(u) = B_{0,1}(u) + u'B_{1,1}(u)[u,\cdot], \tag{6.11}$$

together with the results of Theorem B.3 and Theorem B.5. Moreover, Corollary 4.3, Lemma 4.4, and the identities

$$\begin{aligned} S(X) &= f'D_1(X) - (c_\infty + f)C_1(X) + C_1(X)[h\cdot], \\ S'(X) &= h'D_1'(X) + (c_\infty - h)C_1'(X) + C_1'(X)[f\cdot], \\ T(X) &= D_1(X) + (c_\infty + f)f'C_1(X) - f'C_1(X)[h\cdot], \\ T'(X) &= D_1'(X) - (c_\infty - h)h'C_1'(X) - h'C_1'(X)[f\cdot] \end{aligned} \tag{6.12}$$

imply (6.4) and (6.6).

Using the now proven results (6.3), (6.4), (6.6), and (6.9), we conclude in view of the definitions of $\mathcal{A}$ and $\mathcal{B}$, see (6.2), that (6.5) and (6.7) hold true.

The property (6.8) is proven in [61, Theorem 3.5], whereas (6.10) is established in [1, Theorem 5]. □

6.2. The unique solvability of equation (6.13)

In this section we investigate the unique solvability of the equation

$$(1 - A_\mu \mathcal{A}(X))[\bar{\omega}] = \Theta X'. \tag{6.13}$$

This property is established in Theorem 6.4 below. In the analysis we need to differentiate between the case when the viscosities are ordered, that is

$$(\mu_2 - \mu_1)(\mu_3 - \mu_2) \geq 0$$

as in this case we can use some underlying Rellich identities in the proof, and the case when the viscosities satisfy the inequality

$$(\mu_2 - \mu_1)(\mu_3 - \mu_2) < 0,$$

when a Neumann series argument is employed.

Given $X \in \mathcal{O}_r, r \in (3/2, 2)$, and $A := \operatorname{diag}(a_1, a_2) \in \mathbb{R}^{2\times 2}$ with $\max\{|a_1|, |a_2|\} < 1$, an essential point in our analysis is to study the invertibility of the bounded operator $1 - A\mathcal{A}(X)$ in the Banach algebra $\mathcal{L}(H^{r-1}(\mathbb{R})^2)$, see Theorem 6.4. We thus consider a slightly more general version of (6.13), that is

$$(1 - A\mathcal{A}(X))[\bar{\omega}] = F, \tag{6.14}$$

or equivalently the linear system

$$\left.\begin{aligned} \bar{\omega}_1 - a_1\mathcal{A}_1(X)[\bar{\omega}] = F_1, \\ \bar{\omega}_2 - a_2\mathcal{A}_2(X)[\bar{\omega}] = F_2. \end{aligned}\right\}$$

We look for a solution $\bar{\omega} := (\bar{\omega}_1, \bar{\omega}_2) \in H^{r-1}(\mathbb{R})^2$to a given right-hand side $F := (F_1, F_2) \in H^{r-1}(\mathbb{R})^2$.

We include some remarks on the operators $\mathcal{A}(X)$ and $\mathbb{A}(f)$, $X = (f, h) \in \mathcal{O}_r$ and we discuss an example, where, in a limiting case, the equation (6.14) is not solvable (not even in $L_2(\mathbb{R})$).

Remark 6.3.

(i) The operator $\mathbb{A}(f)$ is the adjoint of the double layer potential for Laplace's equation associated to the hypersurface $\Gamma_f^{c_\infty}$ and the operator $\mathcal{A}(X)$ is its generalization to the hypersurface $\Gamma_f^{c_\infty} \cup \Gamma_h$. As already mentioned, $\lambda - \mathbb{A}(f)$ (and $\lambda - \mathcal{A}(X)$, cf. Corollary 6.5 below) is an L_2-isomorphism for all $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$. This property is essential in the study of the two-phase Muskat problem, see [31, 61, 62].

In the multiphase setting we need to address the invertibility of the operator $1 - A\mathcal{A}(X)$ where $A := \operatorname{diag}(a_1, a_2) \in \mathbb{R}^{2\times 2}$ satisfies $\max\{|a_1|, |a_2|\} < 1$, see Theorem 6.4. This is a major deviation from the two-phase setting considered in [31, 61, 62] because herein we consider the operator $\mathcal{A}(X)$ multiplied by a matrix which has in general different entries (with possibly opposite sign). However, on the base of the Rellich identities (6.26) it is possible to show that $1 - A\mathcal{A}(X)$ is an isomorphism whenever its entries have the same sign. If $a_1 a_2 < 0$, we can invert this operator only under a smallness assumption on the W^1_∞−norm of X, and it is not clear to us whether this smallness assumption can be dropped (as shown in (ii), $1 - A\mathcal{A}(X)$ is not invertible if $A := \operatorname{diag}(1, -1)$ and $X = 0$).

(ii) Let $A := \operatorname{diag}(1, -1)$. Then $1 - A\mathcal{A}(0)$ is not invertible in $\mathcal{L}(L_2(\mathbb{R})^2)$ (or $\mathcal{L}(H^{r-1}(\mathbb{R})^2)$).

Proof of Remark 6.3 (ii). Let $F := (F_1, 0) \in H^{r-1}(\mathbb{R})^2$ be chosen such that $\mathcal{F}F_1 \in \mathrm{C}_0^\infty(\mathbb{R})$ is a function, which is constant 1 on the interval $[-1, 1]$ (where $\mathcal{F}$ denotes the Fourier transform). If $\bar{\omega} := (\bar{\omega}_1, \bar{\omega}_2) \in L_2(\mathbb{R})^2$ satisfies $(1 - A\mathcal{A}(0))[\bar{\omega}] = F$, we arrive in virtue of $\mathbb{A}(0) = 0$ and

$$S(0)[\bar{\omega}_i] = -S'(0)[\bar{\omega}_i] = -\varphi * \bar{\omega}_i, \quad i = 1, 2, \tag{6.15}$$

where

$$\varphi(s) := \frac{1}{\pi}\frac{c_\infty}{s^2+c_\infty^2}, \quad s \in \mathbb{R}, \tag{6.16}$$

at the system

$$\left.\begin{aligned} \bar{\omega}_1 - \varphi * \bar{\omega}_2 &= F_1, \\ \bar{\omega}_2 - \varphi * \bar{\omega}_1 &= 0. \end{aligned}\right\}$$

Combining these equations we get

$$\bar{\omega}_1 - \varphi * \varphi * \bar{\omega}_1 = F_1,$$

where we can apply the Fourier transform on both sides and the convolution theorem to obtain

$$\mathcal{F}\bar{\omega}_1 = \frac{\mathcal{F}F_1}{1-2\pi(\mathcal{F}\varphi)^2}.$$

We are now focusing on $\mathcal{F}\varphi$. Therefore we set

$$\psi(s) := e^{-c_\infty|s|}, \quad s \in \mathbb{R}.$$

Then ψ belongs to $L_2(\mathbb{R})$ and it holds that

$$\begin{aligned} \sqrt{2\pi}\mathcal{F}\psi(\zeta) &= \int_{\mathbb{R}} e^{-(c_\infty|s|+is\zeta)}\,ds \\ &= \int_{-\infty}^{0} e^{s(c_\infty-i\zeta)}\,ds + \int_0^\infty e^{-s(c_\infty+i\zeta)}\,ds \\ &= \frac{1}{c_\infty-i\zeta} + \frac{1}{c_\infty+i\zeta} \\ &= 2\frac{c_\infty}{c_\infty^2+\zeta^2} = 2\pi\varphi(\zeta), \quad \zeta \in \mathbb{R}, \end{aligned}$$

which gives

$$2\pi(\mathcal{F}\varphi)^2 = \psi^2.$$

Observing that

$$\frac{1-\psi^2(\zeta)}{|\zeta|} \to 2c_\infty \quad \text{for } |\zeta| \to 0,$$

we conclude that $\mathcal{F}\bar{\omega}_1$ is not square-integrable in a neighborhood of 0 and does therefore not belong to $L_2(\mathbb{R})$. Thus, $1 - A\mathcal{A}(0)$ is not surjective, and this proves the claim. □

In Theorem 6.4 below we characterize, in dependence of A, the set of pairs $X \in \mathcal{O}_r$ for which (6.14) is uniquely solvable.

Theorem 6.4. *Let $r \in (3/2, 2)$ and $A := \operatorname{diag}(a_1, a_2) \in \mathbb{R}^{2\times 2}$ with $\max\{|a_1|, |a_2|\} < 1$ be given. Then, $\mathcal{U}_r := \mathcal{U} \cap H^r(\mathbb{R})^2$, where*

$$\mathcal{U} := \{X = (f,h) \in \mathrm{BUC}^1(\mathbb{R})^2 \,:\, \inf(c_\infty + f - h) > 0 \text{ and } 1 - T_A(X) \in \mathrm{Isom}(L_2(\mathbb{R}))\}$$

and

$$T_A(X) := a_1a_2(1-a_1\mathbb{A}(f))^{-1}S(X)(1-a_2\mathbb{A}(h))^{-1}S'(X),$$

is a nonempty open subset of $\mathcal{O}_r$ and $1 - A\mathcal{A}(X) \in \mathrm{Isom}(H^{r-1}(\mathbb{R})^2)$ for all $X \in \mathcal{U}_r$.

Moreover, it holds that:

(a) If $a_1a_2 \geq 0$, then $\mathcal{U}_r = \mathcal{O}_r$;

(b) If $a_1a_2 < 0$, there exists a constant $\sigma = \sigma(A) > 0$ such that

$$\{X \in \mathcal{O}_r \,:\, \|X\|_{W^1_\infty} < \sigma\} \subset \mathcal{U}_r.$$

In particular, $\mathcal{U}_r$ is an unbounded subset of $H^r(\mathbb{R})^2$.

Proof. Let $X = (f, h) \in \mathcal{O}_r$ be given. We first note that the invertibility of the operator $1 - A\mathcal{A}(X)$ in $\mathcal{L}(H^{r-1}(\mathbb{R})^2)$ is equivalent to the unique solvability of the system

$$\left.\begin{aligned} \bar\omega_1 - a_1\mathbb{A}_1(f)[\bar\omega_1] - a_1S(X)[\bar\omega_2] = F_1,\\ \bar\omega_2 - a_2S'(X)[\bar\omega_1] - a_2\mathbb{A}_2(h)[\bar\omega_2] = F_2 \end{aligned}\right\} \tag{6.17}$$

in $H^{r-1}(\mathbb{R})^2$ for each $F := (F_1, F_2) \in H^{r-1}(\mathbb{R})^2$, see (6.2) and (6.14). Since $1 - a_1\mathbb{A}(f)$ and $1 - a_2\mathbb{A}(h)$ are invertible in $\mathcal{L}(H^{r-1}(\mathbb{R}))$, cf. (6.10), we are able to compute

$$\bar\omega_2 =(1 - a_2\mathbb{A}_2(h))^{-1}(F_2 + a_2S'(X)[\bar\omega_1])$$

and thus

$$\begin{aligned}(1 - a_1\mathbb{A}_1(f))[\bar\omega_1] - a_1a_2S(X)(1 - a_2\mathbb{A}_2(h))^{-1}S'(X)[\bar\omega_1]\\ = F_1 + a_1S(X)(1 - a_2\mathbb{A}_2(h))^{-1}[F_2].\end{aligned}$$

We conclude that the system (6.17) is equivalent to the following equation for $\bar\omega_1$ in $H^{r-1}(\mathbb{R})$:

$$(1 - T_A(X))[\bar\omega_1] = (1 - a_1\mathbb{A}(f))^{-1}[F_1 + a_1S(X)(1 - a_2(h))^{-1}[F_2]]. \tag{6.18}$$

Hence, if $X \in \mathcal{U}_r$, then (6.18) has a unique solution $\bar\omega_1 \in L_2(\mathbb{R})$. Additionally it follows from (6.6), (6.9), (6.10) and (6.18) that $\bar\omega_1 \in H^{r-1}(\mathbb{R})$. Hence, $1 - A\mathcal{A}(X) \in \mathrm{Isom}(H^{r-1}(\mathbb{R})^2)$ for all $X \in \mathcal{U}_r$.

From (6.3), (6.4), (6.8), and the smoothness of the mapping

$$[T_A \mapsto T_A^{-1}] : \mathrm{Isom}(L_2(\mathbb{R})) \to \mathrm{Isom}(L_2(\mathbb{R})),$$

we obtain that

$$[X \mapsto T_A(X)] \in \mathrm{C}^{1-}(\{X \in \mathrm{C}^1(\mathbb{R})^2 \,:\, \inf(c_\infty + f - h) > 0\}, \mathcal{L}(L_2(\mathbb{R}))). \tag{6.19}$$

Therefore $\mathcal{U}_r$ is an open subset of $\mathcal{O}_r$.

In order to show that $\mathcal{U}_r$ is not empty, we observe that

$$T_A(0) = a_1a_2S(0)S'(0).$$

Recalling that $-S(0) = S'(0)$ and $S'(0)[\bar\omega_2] = \varphi * \bar\omega_2$, see (6.15) and (6.16), we deduce in view of $\|\varphi\|_1 = 1$, that

$$\begin{aligned}\|S(0)[\bar\omega_2]\|_2 &= \|S'(0)[\bar\omega_2]\|_2\\ &= \|\varphi * \bar\omega_2\|_2\\ &\leq \|\varphi\|_1\|\bar\omega_2\|_2\\ &= \|\bar\omega_2\|_2.\end{aligned}$$

Therewith we get that

$$\|T_A(0)\|_{\mathcal{L}(L_2(\mathbb{R}))} \leq |a_1a_2| < 1.$$

Hence, $1-T_A(0)$ is invertible as the inverse is given through the Neumann series of $T_A(0)$ and thus $0\in\mathcal{U}_r$. Recalling (6.19), we may find $\sigma=\sigma(A)>0$ such that $\{X\in\mathcal{O}_r : \|X\|_{W^1_\infty}<\sigma\}\subset\mathcal{U}_r$.

It remains to show that if $a_1a_2\geq 0$, we have that $\mathcal{U}_r=\mathcal{O}_r$. If $a_1a_2=0$, then $T_A(X)=0$ and therefore $\mathcal{O}_r=\mathcal{U}_r$. The proof in the case $a_1a_2>0$ is more involved. The crucial step is to establish that $1-A\mathcal{A}(X)\in\mathrm{Isom}(L_2(\mathbb{R})^2)$ for all $X\in\mathcal{O}_r$. To this end we consider the unbounded operator $1-A\mathcal{A}(X)\in\mathcal{L}(L_2(\mathbb{R})^2)$, where A is viewed now as a parameter matrix, and prove the existence of a constant $C=C(\|X'\|_\infty)>0$ such that

$$\|(1-A\mathcal{A}(X))[\bar\omega]\|_2\geq Cm(A)\|\bar\omega\|_2 \quad \text{for all } \bar\omega\in L_2(\mathbb{R})^2, \tag{6.20}$$

where

$$m(A):=\min\left\{\frac{1+a_1}{|a_1|},\frac{1-a_2}{|a_2|},|a_1|(1-a_1),|a_2|(1+a_2)\right\}>0.$$

We note that for $|a_1|,|a_2|$ small enough to ensure that $\|A\mathcal{A}(X)\|_{\mathcal{L}(L_2(\mathbb{R})^2)}<1$, it holds that the operator $1-A\mathcal{A}(X)\in\mathcal{L}(L_2(\mathbb{R})^2)$ is invertible by using again a Neumann series argument. The method of continuity, cf. [7, Proposition I.1.1.1], together with the estimate (6.20) then implies that $1-A\mathcal{A}(X)\in\mathrm{Isom}(L_2(\mathbb{R})^2)$, provided for all $A=\mathrm{diag}\,(a_1,a_2)$ with $\max\{|a_1|,|a_2|\}<1$. If $F\in H^{r-1}(\mathbb{R})^2$ additionally, we may use the fact that

$$S(X)[\bar\omega_2],S'(X)[\bar\omega_1]\in H^{r-1}(\mathbb{R}),$$

see (6.6), to conclude from (6.17), in view of (6.10), that

$$\bar\omega:=(1-A\mathcal{A}(X))^{-1}[F]\in H^{r-1}(\mathbb{R})^2.$$

This shows that indeed $1-A\mathcal{A}(X)\in\mathrm{Isom}(H^{r-1}(\mathbb{R})^2)$ for all $X\in\mathcal{O}_r$, which implies that $\mathcal{U}_r=\mathcal{O}_r$.

We now proceed with the proof of (6.20). Given $X=(f,h)\in\mathrm{C}_0^\infty(\mathbb{R})^2$ with $c_\infty+f>h$ and $\bar\omega:=(\bar\omega_1,\bar\omega_2)\in\mathrm{C}_0^\infty(\mathbb{R})^2$, let $v:=(v_1,v_2)$ be given by (see equation (A.4))

$$\begin{aligned}v(z)&:=2v(f)[\bar\omega_1](z-(0,c_\infty))+2v(h)[\bar\omega_2](z)\\&=\frac{1}{\pi}\left(\int_{\mathbb{R}}\frac{(c_\infty+f(s)-y,x-s)}{(x-s)^2+(y-c_\infty-f(s))^2}\bar\omega_1(s)\,ds+\int_{\mathbb{R}}\frac{(h(s)-y,x-s)}{(x-s)^2+(y-h(s))^2}\bar\omega_2(s)\,ds\right)\end{aligned}$$

for $z=(x,y)\in\mathbb{R}^2\setminus(\Gamma_f^{c_\infty}\cap\Gamma_h)$. We next infer from the results in Appendix A, in particular from Theorem A.6, that

$$v_i:=v|_{\Omega_i}\in\mathrm{BUC}(\Omega_i)\cap\mathrm{C}^\infty(\Omega_i),\quad 1\leq i\leq 3,$$

and that

$$\begin{aligned}v_i(x,c_\infty+f(x))&=\frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}}\frac{(-\delta_{[x,s]}f,s)}{s^2+(\delta_{[x,s]}f)^2}\bar\omega_1(x-s)\,ds+\frac{1}{\pi}\int_{\mathbb{R}}\frac{(-\delta_{[x,s]}X,s)}{s^2+(\delta_{[x,s]}X)^2}\bar\omega_2(x-s)\,ds\\&\quad+(-1)^i\frac{\bar\omega_1(1,f')}{1+f'^2}(x),\quad i=1,\,2,\end{aligned}\tag{6.21}$$

and

$$\begin{aligned}v_i(x,h(x))&=\frac{1}{\pi}\int_{\mathbb{R}}\frac{(-\delta'_{[x,s]}X,s)}{s^2+(\delta'_{[x,s]}X)^2}\bar\omega_1(x-s)\,ds+\frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}}\frac{(-\delta_{[x,s]}h,s)}{s^2+(\delta_{[x,s]}h)^2}\bar\omega_2(x-s)\,ds\\&\quad+(-1)^{i+1}\frac{\bar\omega_2(1,h')}{1+h'^2}(x),\quad i=2,\,3\end{aligned}\tag{6.22}$$

for $x\in\mathbb{R}$. In consequence, the normal and tangential traces of v are related to the operators $\mathcal{A}(X)$

and $\mathcal{B}(X)$ defined in (6.2), since

$$\begin{aligned}\mathcal{A}_1(X)[\bar\omega](x) &= \langle v_i(x, c_\infty + f(x)) \mid (1, f'(x))\rangle + (-1)^{1+i}\bar\omega_1(x), \quad i = 1,2,\\ \mathcal{A}_2(X)[\bar\omega](x) &= \langle v_i(x, h(x)) \mid (1, h'(x))\rangle + (-1)^{i}\bar\omega_2(x), \quad i = 2,3,\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_1(X)[\bar\omega](x) &= \langle v_i(x, c_\infty + f(x)) \mid (-f'(x), 1)\rangle \quad i = 1,2,\\ \mathcal{B}_2(X)[\bar\omega](x) &= \langle v_i(x, h(x)) \mid (-h'(x), 1)\rangle \quad i = 2,3.\end{aligned}$$

From (A.17) we know that

$$\partial_x v_i^1 + \partial_y v_i^2 = 0 = \partial_y v_i^1 - \partial_x v_i^2 \quad \text{in } \Omega_i,\ 1 \le i \le 3.$$

Thus, we can conclude

$$\int_{\Omega_i} \operatorname{div}\begin{pmatrix} 2v_i^1 v_i^2 \\ (v_i^2)^2 - (v_i^1)^2\end{pmatrix} dz = 2\int_{\Omega_i} v_i^1(\partial_x v_i^2 - \partial_y v_i^1) + v_i^2(\partial_x v_i^1 + \partial_y v_i^2)\, dz = 0 \tag{6.23}$$

for $i = 1,2,3$. We now choose a family $(\varphi_n)_{n\in\mathbb{N}} \in \mathrm{C}_0^\infty(\mathbb{R}^2)$ with the properties that $\varphi = 1$ on $B_n(0)$ and $\varphi = 0$ in $\mathbb{R}^2 \setminus B_{n+1}(0)$, while $\sup_n \|\nabla\varphi_n\|_\infty < \infty$. Here $B_n(0)$ is the ball centered in 0 with radius n. Using Lemma 6.2 together with the representations (6.24) and (6.25), equation (6.23), Lebesgue's dominated convergence, and Stokes' theorem, we obtain

$$\begin{aligned}&\int_{\Gamma_f} \left\langle \begin{pmatrix} 2v_2^1 v_2^2 \\ (v_2^2)^2 - (v_2^1)^2\end{pmatrix} \middle| \nu_1 \right\rangle d\sigma_1 - \int_{\Gamma_h} \left\langle \begin{pmatrix} 2v_2^1 v_2^2 \\ (v_2^2)^2 - (v_2^1)^2\end{pmatrix} \middle| \nu_2 \right\rangle d\sigma_2\\ &= \lim_{n\to\infty}\left(\int_{\Gamma_f} \varphi_n \left\langle \begin{pmatrix} 2v_2^1 v_2^2 \\ (v_2^2)^2 - (v_2^1)^2\end{pmatrix} \middle| \nu_1 \right\rangle d\sigma_1 + \int_{\Gamma_h} \varphi_n \left\langle \begin{pmatrix} 2v_2^1 v_2^2 \\ (v_2^2)^2 - (v_2^1)^2\end{pmatrix} \middle| -\nu_2 \right\rangle d\sigma_2\right)\\ &= \lim_{n\to\infty}\int_{\Omega_2} \operatorname{div}\left(\varphi_n \begin{pmatrix} 2v_2^1 v_2^2 \\ (v_2^2)^2 - (v_2^1)^2\end{pmatrix}\right) dz\\ &= \lim_{n\to\infty}\int_{\Omega_2} \left\langle \nabla\varphi_n \middle| \begin{pmatrix} 2v_2^1 v_2^2 \\ (v_2^2)^2 - (v_2^1)^2\end{pmatrix}\right\rangle dz.\end{aligned}$$

Because of Lemma A.8, we further have

$$\begin{aligned}\left|\int_{\Omega_2} \left\langle \nabla\varphi_n \middle| \begin{pmatrix} 2v_2^1 v_2^2 \\ (v_2^2)^2 - (v_2^1)^2\end{pmatrix}\right\rangle dz\right| &\le C\int_{\Omega_2\cap\{n\le|z|\le n+1\}} \frac{1}{|z|^2}\, dz\\ &\le C\int_n^{n+1} \frac{1}{r}\, dr\\ &= C\ln\left(\frac{n+1}{n}\right), \qquad n\in\mathbb{N},\end{aligned}$$

which provides, on account of the above computation, the relation

$$\int_{\Gamma_f} \left\langle \begin{pmatrix} 2v_2^1 v_2^2 \\ (v_2^2)^2 - (v_2^1)^2\end{pmatrix} \middle| \nu_1 \right\rangle d\sigma_1 - \int_{\Gamma_h} \left\langle \begin{pmatrix} 2v_2^1 v_2^2 \\ (v_2^2)^2 - (v_2^1)^2\end{pmatrix} \middle| \nu_2 \right\rangle d\sigma_2 = 0.$$

Analogously we get

$$\begin{aligned}0 &= \int_{\Gamma_f} \left\langle \begin{pmatrix} 2v_1^1 v_1^2 \\ (v_1^2)^2 - (v_1^1)^2\end{pmatrix} \middle| \nu_1 \right\rangle d\sigma_1,\\ 0 &= \int_{\Gamma_h} \left\langle \begin{pmatrix} 2v_3^1 v_3^2 \\ (v_3^2)^2 - (v_3^1)^2\end{pmatrix} \middle| \nu_2 \right\rangle d\sigma_2.\end{aligned}$$

We further compute

$$\left\langle \begin{pmatrix} 2v_i^1 v_i^2 \\ (v_i^2)^2 - (v_i^1)^2 \end{pmatrix} |_{\Gamma_f^{c\infty}} \middle| (-f', 1) \right\rangle = \left(-2f'(v_i^1 v_i^2)|_{\Gamma_f^{c\infty}} + (v_i^2)|_{\Gamma_f^{c\infty}}^2 - (v_i^1)|_{\Gamma_f^{c\infty}}^2 \right)$$
$$= \frac{1}{1+f'^2}\Big(2f'\mathcal{B}_1(X)[\bar\omega](\mathcal{A}_1(X)[\bar\omega] + (-1)^i\bar\omega_1) \tag{6.24}$$
$$+ \mathcal{B}_1(X)[\bar\omega]^2 - (\mathcal{A}_1(X)[\bar\omega] + (-1)^i\bar\omega_1)^2\Big),$$

where $i = 1, 2$ and analogously for $i = 2, 3$ that

$$\left\langle \begin{pmatrix} 2v_i^1 v_i^2 \\ (v_i^2)^2 - (v_i^1)^2 \end{pmatrix} |_{\Gamma_h} \middle| (-h', 1) \right\rangle = \frac{1}{1+h'^2}\Big(2h'\mathcal{B}_2(X)[\bar\omega](\mathcal{A}_2(X)[\bar\omega] + (-1)^{i+1}\bar\omega_2)$$
$$+ \mathcal{B}_2(X)[\bar\omega]^2 - (\mathcal{A}_2(X)[\bar\omega] + (-1)^{i+1}\bar\omega_2)^2\Big). \tag{6.25}$$

Hence, we arrive at the following system of Rellich identities

$$0 = \int_{\mathbb{R}} \frac{1}{1+f'^2}\Big((\mathcal{A}_1(X)[\bar\omega] - \bar\omega_1)^2 - 2f'\mathcal{B}_1(X)[\bar\omega](\mathcal{A}_1(X)[\bar\omega] - \bar\omega_1) - \mathcal{B}_1(X)[\bar\omega]^2\Big)\,dx, \tag{6.26a}$$

$$\begin{aligned} 0 = \int_{\mathbb{R}} & \frac{1}{1+f'^2}\Big((\mathcal{A}_1(X)[\bar\omega] + \bar\omega_1)^2 - 2f'\mathcal{B}_1(X)[\bar\omega](\mathcal{A}_1(X)[\bar\omega] + \bar\omega_1) - \mathcal{B}_1(X)[\bar\omega]^2\Big) \\ & - \frac{1}{1+h'^2}\Big((\mathcal{A}_2(X)[\bar\omega] - \bar\omega_2)^2 - 2h'\mathcal{B}_2(X)[\bar\omega](\mathcal{A}_2(X)[\bar\omega] - \bar\omega_2) - \mathcal{B}_2(X)[\bar\omega]^2\Big)\,dx, \end{aligned} \tag{6.26b}$$

$$0 = \int_{\mathbb{R}} \frac{1}{1+(h')^2}\Big((\mathcal{A}_2(X)[\bar\omega] + \bar\omega_2)^2 - 2h'\mathcal{B}_2(X)[\bar\omega](\mathcal{A}_2(X)[\bar\omega] + \bar\omega_2) - \mathcal{B}_2(X)[\bar\omega]^2\Big)\,dx. \tag{6.26c}$$

Using the algebraic identity

$$|\mathcal{A}_i(X)[\bar\omega] \pm \bar\omega_i|^2 = \frac{(\bar\omega_i - a_i\mathcal{A}_i(X)[\bar\omega])^2 - 2(1\pm a_i)\bar\omega_i(\bar\omega_i - a_i\mathcal{A}_i(X)[\bar\omega]) + (1\pm a_i)^2\bar\omega_i^2}{a_i^2},$$

which holds for $i = 1, 2$, the system (6.26) is equivalent to

$$\begin{aligned} 0 = \int_{\mathbb{R}} \frac{1}{1+f'^2}\Big[& \frac{(\bar\omega_1 - a_1\mathcal{A}_1(X)[\bar\omega])^2 - 2(1-a_1)\bar\omega_1(\bar\omega_1 - a_1\mathcal{A}_1(X)[\bar\omega]) + (1-a_1)^2\bar\omega_1^2}{a_1^2} \\ & - 2f'\mathcal{B}_1(X)[\bar\omega](\mathcal{A}_1(X)[\bar\omega] - \bar\omega_1) - \mathcal{B}_1(X)[\bar\omega]^2\Big]\,dx, \end{aligned} \tag{6.27a}$$

$$\begin{aligned} 0 = \int_{\mathbb{R}} & \frac{1}{1+f'^2}\Big[\frac{(\bar\omega_1 - a_1\mathcal{A}_1(X)[\bar\omega])^2 - 2(1+a_1)\bar\omega_1(\bar\omega_1 - a_1\mathcal{A}_1(X)[\bar\omega]) + (1+a_1)^2\bar\omega_1^2}{a_1^2} \\ & \qquad - 2f'\mathcal{B}_1(X)[\bar\omega](\mathcal{A}_1(X)[\bar\omega] + \bar\omega_1) - \mathcal{B}_1(X)[\bar\omega]^2\Big] \\ & - \frac{1}{1+h'^2}\Big[\frac{(\bar\omega_2 - a_2\mathcal{A}_2(X)[\bar\omega])^2 - 2(1-a_2)\bar\omega_2(\bar\omega_2 - a_2\mathcal{A}_2(X)[\bar\omega]) + (1-a_2)^2\bar\omega_2^2}{a_2^2} \\ & \qquad - 2h'\mathcal{B}_2(X)[\bar\omega](\mathcal{A}_2(X)[\bar\omega] - \bar\omega_2) - \mathcal{B}_2(X)[\bar\omega]^2\Big]\,dx, \end{aligned} \tag{6.27b}$$

$$\begin{aligned} 0 = \int_{\mathbb{R}} \frac{1}{1+h'^2}\Big[& \frac{(\bar\omega_2 - a_2 - \mathcal{A}_2(X)[\bar\omega])^2 - 2(1+a_2)\bar\omega_2(\bar\omega_2 - a_2\mathcal{A}_2(X)[\bar\omega]) + (1+a_2)^2\bar\omega_2^2}{a_2^2} \\ & - 2h'\mathcal{B}_2(X)[\bar\omega](\mathcal{A}_2(X)[\bar\omega] + \bar\omega_2) - \mathcal{B}_2(X)[\bar\omega]^2\Big]\,dx. \end{aligned} \tag{6.27c}$$

We next multiply the first equation (6.27a) by $(1+a_1)$, the second equation (6.27b) by $-(1-a_1)$, and the third identity (6.27c) by $-(1-a_1)(1-a_2)/(1+a_2)$, and then sum up the resulting identities to arrive, after multiplying by $\operatorname{sign}(a_1) = \operatorname{sign}(a_2)$, at

$$
\begin{aligned}
&\int_{\mathbb{R}} \frac{1}{1+f'^2}\Big[\frac{(\bar\omega_1 - a_1\mathcal{A}_1(X)[\bar\omega])^2}{|a_1|(1-a_1)} + \frac{2\operatorname{sign}(a_1)}{1-a_1} f'\mathcal{B}_1(X)[\bar\omega](\bar\omega_1 - a_1\mathcal{A}_1(X)[\bar\omega])\Big]\, dx \\
&\quad + \int_{\mathbb{R}} \frac{1}{1+h'^2}\Big[\frac{(\bar\omega_2 - a_2\mathcal{A}_2(X)[\bar\omega])^2}{|a_2|(1+a_2)} + \frac{2\operatorname{sign}(a_2)}{1+a_2} h'\mathcal{B}_2(X)[\bar\omega](\bar\omega_2 - a_2\mathcal{A}_2(X)[\bar\omega])\Big]\, dx \\
&= \int_{\mathbb{R}} \frac{1}{1+f'^2}\Big[\frac{1+a_1}{|a_1|}\bar\omega_1^2 + \frac{|a_1|}{1-a_1}\mathcal{B}_1(X)[\bar\omega]^2\Big]\, dx \\
&\quad + \int_{\mathbb{R}} \frac{1}{1+h'^2}\Big[\frac{1-a_2}{|a_2|}\bar\omega_2^2 + \frac{|a_2|}{1+a_2}\mathcal{B}_2(X)[\bar\omega])^2\Big]\, dx.
\end{aligned}
\tag{6.28}
$$

Using Young's inequality, we get

$$
\begin{aligned}
\Big|\frac{2\operatorname{sign}(a_1)}{1-a_1} f'\mathcal{B}_1(X)[\bar\omega](\bar\omega_1 - a_1\mathcal{A}_1(X)[\bar\omega])\Big| &\le \frac{2f'^2}{(1-a_1)|a_1|}(\bar\omega_1 - a_1\mathcal{A}_1(X)[\bar\omega])^2 \\
&\quad + \frac{|a_1|}{2(1-a_1)}\mathcal{B}_1(X)[\bar\omega]^2
\end{aligned}
$$

and

$$
\begin{aligned}
\Big|\frac{2\operatorname{sign}(a_2)}{1+a_2} h'\mathcal{B}_2(X)[\bar\omega](\bar\omega_2 + a_2\mathcal{A}_2(X)[\bar\omega])\Big| &\le \frac{2h'^2}{(1+a_2)|a_2|}(\bar\omega_2 - a_2\mathcal{A}_2(X)[\bar\omega])^2 \\
&\quad + \frac{|a_2|}{2(1+a_2)}\mathcal{B}_2(X)[\bar\omega]^2,
\end{aligned}
$$

which enables us to conclude from (6.28) there exists a constant $c = c(\|X'\|_\infty)$ such that

$$
\begin{aligned}
\frac{\|\bar\omega_1 - a_1\mathcal{A}_1(X)[\bar\omega]\|_2^2}{|a_1|(1-a_1)} + \frac{\|\bar\omega_2 - a_2\mathcal{A}_2(X)[\bar\omega]\|_2^2}{|a_2|(1+a_2)} &\ge C\Big(\frac{1+a_1}{|a_1|}\|\bar\omega_1\|_2^2 + \frac{|a_1|}{2(1-a_1)}\|\mathcal{B}_1(X)[\bar\omega]\|_2^2 \\
&\quad + \frac{1-a_2}{|a_2|}\|\bar\omega_2\|_2^2 + \frac{|a_2|}{2(1+a_2)}\|\mathcal{B}_2(X)[\bar\omega]\|_2^2\Big) \\
&\ge C\Big(\frac{1+a_1}{|a_1|}\|\bar\omega_1\|_2^2 + \frac{1-a_2}{|a_2|}\|\bar\omega_2\|_2^2\Big).
\end{aligned}
$$

The desired estimate (6.20) follows now by using a standard density argument in view of (6.5). □

We now conclude this section with a result on the resolvent of $\mathcal{A}(X)$.

Corollary 6.5. *Let $r \in (3/2, 2)$ and $X \in \mathcal{O}_r$ be given. Then*

$$\lambda - \mathcal{A}(X) \in \operatorname{Isom}(L_2(\mathbb{R})^2) \cap \operatorname{Isom}(H^{r-1}(\mathbb{R})^2) \qquad \textit{for all } \lambda \in \mathbb{R} \setminus (-1, 1).$$

Proof. If $|\lambda| > 1$, then the claim follows from Theorem 6.4 (by choosing $a_1 = a_2 = 1/\lambda$). It remains to establish the result for $\lambda \in \{\pm 1\}$. To this end we infer from (6.26), by using Young's and Hölder's inequalities, there exists a constant $C = C(\|X'\|_\infty) \ge 1$ such that

$$
\begin{aligned}
&C^{-1}\|\bar\omega_1 - \mathcal{A}_1(X)[\bar\omega]\|_2 \le \|\mathcal{B}_1(X)[\bar\omega]\|_2 \le C\|\bar\omega_1 - \mathcal{A}_1(X)[\bar\omega]\|_2, \\
&C^{-1}\|\bar\omega_2 + \mathcal{A}_2(X)[\bar\omega]\|_2 \le \|\mathcal{B}_2(X)[\bar\omega]\|_2 \le C\|\bar\omega_2 + \mathcal{A}_2(X)[\bar\omega]\|_2, \\
&C^{-1}\big(\|\bar\omega_1 + \mathcal{A}_1(X)[\bar\omega]\|_2 + \|\mathcal{B}_2(X)[\bar\omega]\|_2\big) \le \|\bar\omega_2 - \mathcal{A}_2(X)[\bar\omega]\|_2 + \|\mathcal{B}_1(X)[\bar\omega]\|_2, \\
&\|\bar\omega_2 - \mathcal{A}_2(X)[\bar\omega]\|_2 + \|\mathcal{B}_1(X)[\bar\omega]\|_2 \le C\big(\|\bar\omega_1 + \mathcal{A}_1(X)[\bar\omega]\|_2 + \|\mathcal{B}_2(X)[\bar\omega]\|_2\big).
\end{aligned}
$$

Combining these relations, we get

$$\begin{aligned}\|\bar\omega\|_2 &\le \|\bar\omega_1 - \mathcal{A}_1(X)[\bar\omega]\|_2 + \|\bar\omega_1 + \mathcal{A}_1(X)[\bar\omega]\|_2 + \|\bar\omega_2 - \mathcal{A}_2(X)[\bar\omega]\|_2 + \|\bar\omega_2 + \mathcal{A}_2(X)[\bar\omega]\|_2\\
&\le C\big(|\mathcal{B}_1(X)[\bar\omega]\|_2 + \|\bar\omega_1 + \mathcal{A}_1(X)[\bar\omega]\|_2 + \|\bar\omega_2 - \mathcal{A}_2(X)[\bar\omega]\|_2 + \|\mathcal{B}_2(X)[\bar\omega]\|_2\big)\\
&\le C\min\{\|\bar\omega_2 - \mathcal{A}_2(X)[\bar\omega]\|_2 + |\mathcal{B}_1(X)[\bar\omega]\|_2, \|\bar\omega_1 + \mathcal{A}_1(X)[\bar\omega]\|_2 + \|\mathcal{B}_2(X)[\bar\omega]\|_2\}\\
&\le C\big(\|\bar\omega_1 \pm \mathcal{A}_1(X)[\bar\omega]\|_2 + \|\bar\omega_2 \pm \mathcal{A}_2(X)[\bar\omega]\|_2\big).\end{aligned}$$

The latter estimates together with (6.20), the continuity property (6.5), the method of continuity [7, Proposition I.1.1.1], and the arguments in the proof of Theorem 6.4 show that the claim indeed holds also for $\lambda \in \{\pm 1\}$. □

6.3. Unique solvability of the fixed time problem

In this section we establish the unique solvability of the system (6.29) when assuming $X \in \mathcal{U}_r$, where $\mathcal{U}_r$ is the open subset of $H^r(\mathbb{R})$ found in Theorem 6.4 for the choice $A := A_\mu$. This shows, in particular, that for classical solutions to (1.1) (in the sense of Definition 1.1), the free interfaces identify at each time instant the velocities and the pressures.

Theorem 6.6. *Let $r \in (3/2, 2)$, $c_\infty > 0$, and $X = (f, h) \in \mathcal{U}_r$, where $\mathcal{U}_r$ is the open subset of $H^r(\mathbb{R})^2$ found in Theorem 6.4 for the choice*

$$A := A_\mu := \operatorname{diag}(a_\mu^1, a_\mu^2) \qquad \textit{with} \qquad a_\mu^1 := \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}, \quad a_\mu^2 := \frac{\mu_2 - \mu_3}{\mu_2 + \mu_3}.$$

Then the boundary value problem

$$\left.\begin{array}{rcll} v_i &=& -\dfrac{k}{\mu_i}\big(\nabla p_i + (0, \rho_i g)\big) & \textit{in } \Omega_i,\ 1 \le i \le 3,\\ \operatorname{div} v_i &=& 0 & \textit{in } \Omega_i,\ 1 \le i \le 3,\\ p_i &=& p_{i+1} & \textit{on } \partial\Omega_i \cap \partial\Omega_{i+1},\ i = 1, 2,\\ \langle v_i | \nu_i\rangle &=& \langle v_{i+1} | \nu_i\rangle & \textit{on } \partial\Omega_i \cap \partial\Omega_{i+1},\ i = 1, 2,\\ v_i(x, y) &\to& 0 & \textit{for } |(x,y)| \to \infty,\ 1 \le i \le 3 \end{array}\right\} \tag{6.29}$$

has a unique solution[1] $(v_1, v_2, v_3, p_1, p_2, p_3)$ *with the properties that*

- $v_i \in \mathrm{BUC}(\Omega_i) \cap \mathrm{C}^\infty(\Omega_i)$, $p_i \in \mathrm{UC}^1(\Omega_i) \cap \mathrm{C}^\infty(\Omega_i)$ *for* $1 \le i \le 3$,
- $[x \mapsto v_i(x, c_\infty + f(x))] \in H^{r-1}(\mathbb{R})$ *for* $1 \le i \le 2$,
- $[x \mapsto v_i(x, h(x))] \in H^{r-1}(\mathbb{R})$ *for* $2 \le i \le 3$.

Moreover, setting $v := v_1\mathbf{1}_{\Omega_1} + v_2\mathbf{1}_{\Omega_2} + v_3\mathbf{1}_{\Omega_3}$, it holds for $z := (x, y) \in \mathbb{R}^2 \setminus (\Gamma_f^{c_\infty} \cup \Gamma_h)$ that

$$v(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{(c_\infty + f(s) - y, x - s)}{(x-s)^2 + (y - c_\infty - f(s))^2}\bar\omega_1(s)\,ds + \frac{1}{\pi}\int_{\mathbb{R}} \frac{(h(s) - y, x - s)}{(x-s)^2 + (y - h(s))^2}\bar\omega_2(s)\,ds, \tag{6.30}$$

where $\bar\omega := (\bar\omega_1, \bar\omega_2) \in H^{r-1}(\mathbb{R})^2$ denotes the unique solution to the equation

$$(1 - A_\mu\mathcal{A}(X))[\bar\omega] = \Theta X', \tag{6.31}$$

cf. (6.13). *The matrix $\Theta := \operatorname{diag}(\Theta_1, \Theta_2)$ is defined by*

$$\Theta_1 := kg\frac{(\rho_1 - \rho_2)}{\mu_1 + \mu_2}, \quad \Theta_2 := kg\frac{(\rho_2 - \rho_3)}{\mu_2 + \mu_3}. \tag{6.32}$$

[1] The pressures p_1, p_2, p_3 are unique only up to the same additive constant.

Proof. We divide the proof into two steps of proving existence and uniqueness of a solution.

Existence. To each pair $\bar\omega = (\bar\omega_1, \bar\omega_2) \in H^{r-1}(\mathbb{R})^2$ we associate the velocity $v := v(X)[\bar\omega]$ which is defined by (6.30) in $\mathbb{R}^2 \setminus (\Gamma_f^{c_\infty} \cup \Gamma_h)$. The results of Lemma A.4 and Lemma A.6 imply that $v_i \in \mathrm{BUC}^{r-3/2}(\Omega_i) \cap \mathrm{C}^\infty(\Omega_i)$ for $1 \le i \le 3$ as well as

$$v_i(x, c_\infty + f(x)) = \frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}} \frac{(-\delta_{[x,s]}f, s)}{s^2 + (\delta_{[x,s]}f)^2}\bar\omega_1(x-s)\,ds + \frac{1}{\pi}\int_{\mathbb{R}} \frac{(-\delta_{[x,s]}X, s)}{s^2 + (\delta_{[x,s]}X)^2}\bar\omega_2(x-s)\,ds$$
$$+ (-1)^i \frac{\bar\omega_1(1, f')}{1+f'^2}(x), \quad i = 1,\, 2, \tag{6.33}$$

and

$$v_i(x, h(x)) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{(-\delta'_{[x,s]}X, s)}{s^2 + (\delta'_{[x,s]}X)^2}\bar\omega_1(x-s)\,ds + \frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}} \frac{(-\delta_{[x,s]}h, s)}{s^2 + (\delta_{[x,s]}h)^2}\bar\omega_2(x-s)\,ds$$
$$+ (-1)^{i+1} \frac{\bar\omega_2(1, h')}{1+h'^2}(x), \quad i = 2,\, 3. \tag{6.34}$$

As a direct consequence of the relations (6.33) and (6.34) we obtain that $(6.29)_4$ holds true. We further infer from Lemma 4.4 and Theorem B.5 that

- $[x \mapsto v_i(x, c_\infty + f(x))] \in H^{r-1}(\mathbb{R})^2$ for $1 \le i \le 2$,
- $[x \mapsto v_i(x, h(x))] \in H^{r-1}(\mathbb{R})^2$ for $2 \le i \le 3$,

because we can represent the traces of the velocities on the interfaces as follows

$$v_i(x, c_\infty + f(x)) = \begin{pmatrix} -B_{1,1}(f)[f, \bar\omega_1](x) - C^0_{1,1}(X)[\bar\omega_2](x) + (-1)^i \dfrac{\bar\omega_1}{1+f'^2}(x) \\ B_{0,1}(f)[\bar\omega_1](x) + D_1(X)[\bar\omega_2](x) + (-1)^i \dfrac{\bar\omega_1 f'}{1+f'^2}(x) \end{pmatrix}, \quad i = 1, 2,$$
$$v_i(x, h(x)) = \begin{pmatrix} -C'^0_{1,1}(X)[\bar\omega_1](x) - B_{1,1}(h)[h, \bar\omega_2](x) + (-1)^{i+1} \dfrac{\bar\omega_2}{1+h'^2}(x) \\ D'_1(X)[\bar\omega_1](x) + B_{0,1}(h)[\bar\omega_2](x) + (-1)^{i+1} \dfrac{\bar\omega_2 h'}{1+h'^2}(x) \end{pmatrix}, \quad i = 2, 3.$$

Moreover, Lemma A.7 implies that $(6.29)_2$ and $(6.29)_5$ are satisfied and that

$$\partial_y v^1 = \partial_x v^2 \qquad \text{in } \mathbb{R}^2 \setminus (\Gamma_h \cup \Gamma_f^{c_\infty}). \tag{6.35}$$

Corresponding to v, we now define pressures $p_i : \Omega_i \to \mathbb{R}$, $1 \le i \le 3$, by the formula

$$p_i(z) := -\frac{\mu_i}{k}\Big(\int_0^x \langle v_i(s, d_i(s)) | (1, d_i'(s))\rangle\,ds + \int_{d_i(x)}^y v_i^2(x,s)\,ds\Big) - \rho_i g y + c_i \tag{6.36}$$

for $z = (x, y) \in \Omega_i$, with $v_i =: (v_i^1, v_i^2)$, constants $c_i \in \mathbb{R}$, and

$$d_1 := \|f\|_\infty + c_\infty + 1, \qquad d_2 := \frac{1}{2}(c_\infty + f + h), \qquad d_3 := -\|h\|_\infty - 1.$$

Using (6.35), we deduce that $p_i \in \mathrm{C}^1(\Omega_i)$ and that $(6.29)_1$ is satisfied. The regularity properties established for v_i together with $(6.29)_1$ show that

$$p_i \in \mathrm{UC}^1(\Omega_i) \cap \mathrm{C}^\infty(\Omega_i), \qquad 1 \le i \le 3.$$

We point out that all equations constituting (6.29), except for $(6.29)_3$, are valid for any choice of $\bar\omega$. We now prove that the dynamic boundary condition $(6.29)_3$ identifies $\bar\omega$ as the unique solution to (6.13). To this end we take advantage of $(6.29)_1$ and (6.33)-(6.34) and compute that

$$\begin{aligned}\frac{d}{dx}\big((p_2-p_1)(x,c_\infty+f(x))\big) &= (\rho_1-\rho_2)gf'(x)+\Big\langle\frac{\mu_1 v_1-\mu_2 v_2}{k}(x,c_\infty+f(x))\Big|(1,f'(x))\Big\rangle\\ &= (\rho_1-\rho_2)gf'(x)-\frac{\mu_1+\mu_2}{k}\bar\omega_1(x)+\frac{\mu_1-\mu_2}{k}\mathcal{A}_1(X)[\bar\omega](x)\\ \frac{d}{dx}\big((p_3-p_2)(x,h(x))\big) &= (\rho_2-\rho_3)gh'(x)+\Big\langle\frac{\mu_2 v_2-\mu_3 v_3}{k}(x,h(x)))\Big|(1,h'(x))\Big\rangle\\ &= (\rho_2-\rho_3)gh'(x)-\frac{\mu_2+\mu_3}{k}\bar\omega_2(x)+\frac{\mu_2-\mu_3}{k}\mathcal{A}_2(X)[\bar\omega](x)\end{aligned}$$

for $x\in\mathbb{R}$. Hence, $(p_2-p_1)|_{\Gamma_f^{c_\infty}}$ and $(p_3-p_2)|_{\Gamma_h}$ are constant functions if and only if $\bar\omega$ is the unique solution to (6.13). In this case we may choose the constants c_i, $1\le i\le 3$, to achieve that $(6.29)_3$ is satisfied. Therewith, we have proven that there exists at least one solution to (6.29).

Uniqueness. In order to establish the uniqueness of the solution, let $(\widetilde v_1,\widetilde v_2,\widetilde v_3,\widetilde p_1,\widetilde p_2,\widetilde p_3)$ be a further solution to (6.29) with the required regularity properties and set

$$\widetilde v := \widetilde v_1\mathbf{1}_{\Omega_1}+\widetilde v_2\mathbf{1}_{\Omega_2}+\widetilde v_3\mathbf{1}_{\Omega_3}.$$

The main step is to show that the function $\bar\omega:=(\bar\omega_1,\bar\omega_2)\in H^{r-1}(\mathbb{R})^2$ given by

$$\begin{aligned}\bar\omega_1(x) &:= \frac12\langle(\widetilde v_2-\widetilde v_1)(x,c_\infty+f(x))|(1,f'(x)),\\ \bar\omega_2(x) &:= \frac12\langle(\widetilde v_3-\widetilde v_2)(x,h(x))|(1,h'(x)),\end{aligned}\tag{6.37}$$

is the unique solution to (6.31) and that $\widetilde v = v$, where $v=v[\bar\omega]$ is defined in (6.30).

To start, we infer from (6.33) and (6.34) that the relations (6.37) remain valid if we replace $\widetilde v$ by v. This, together with $(6.29)_4$, implies that the global field $V:=(V^1,V^2):=\widetilde v - v$ belongs to $\mathrm{BUC}(\mathbb{R}^2)\cap \mathrm{C}^\infty(\Omega_i)$, $1\le i\le 3$. Let $\Psi:=\psi_1\mathbf{1}_{\overline{\Omega_1}}+\psi_2\mathbf{1}_{\overline{\Omega_2}}+\psi_3\mathbf{1}_{\overline{\Omega_3}}$, where $\psi_i:\overline{\Omega_i}\to\mathbb{R}$ are given by

$$\begin{aligned}\psi_i(x,y) &:= \int_{h(x)}^y V^1(x,s)\,ds-\int_0^x\langle V(s,h(s))|(-h'(s),1)\rangle\,ds, \qquad i=2,\,3,\\ \psi_1(x,y) &:= \int_{c_\infty+f(x)}^y V^1(x,s)\,ds+\psi_2(x,c_\infty+f(x)).\end{aligned}$$

In view of Lemma 3.2, we have $\Psi\in \mathrm{C}(\mathbb{R}^2)$ and $\nabla\psi_i=(-V^2,V^1)$ in $\mathcal{D}'(\Omega_i)$. This implies that $\psi_i\in \mathrm{UC}^1(\Omega_i)$, and the (distributional) gradient $\nabla\Psi=(-V^2,V^1)$ belongs to $\mathrm{BUC}(\mathbb{R}^2)^2$. In virtue of the latter property we obtain for given $\varphi\in \mathrm{C}_0^\infty(\mathbb{R}^2)$, that

$$\langle\Delta\Psi,\varphi\rangle = -\int_{\mathbb{R}^2}\langle\nabla\Psi|\nabla\varphi\rangle\,dz=\int_{\mathbb{R}^2}\langle(V^2,-V^1)|\nabla\varphi\rangle\,dz=\langle\partial_y V^1-\partial_x V^2,\varphi\rangle.$$

Moreover, in virtue of $(6.29)_1$, we have $\partial_y V^1-\partial_x V^2=0$ in $\mathcal{D}'(\Omega_i)$ for $1\le i\le 3$, and taking advantage of $V\in\mathrm{BUC}(\mathbb{R}^2)$ we deduce that $\partial_y V^1-\partial_x V^2=0$ in $\mathcal{D}'(\mathbb{R}^2)$. Altogether we conclude that $\Delta\Psi=0$ in $\mathcal{D}'(\mathbb{R}^2)$. Consequently, Ψ is the real part of a holomorphic function $u:\mathbb{C}\to\mathbb{C}$. Since u' is also holomorphic and $u'=(\partial_x\Psi,-\partial_y\Psi)=-(V^2,V^1)$ is bounded, cf. $(6.29)_5$, Liouville's theorem yields $u'=0$, and therefore $V=0$. Hence, $\widetilde v=v[\bar\omega]$, and, as shown in the first part of the proof, $\bar\omega$ has to solve (6.13). Finally, we note that $(6.29)_1$

and $(6.29)_3$ imply that

$$P \coloneqq (p_1 - \widetilde{p}_1)\mathbf{1}_{\overline{\Omega_1}} + (p_2 - \widetilde{p}_2)\mathbf{1}_{\overline{\Omega_2}} + (p_3 - \widetilde{p}_3)\mathbf{1}_{\overline{\Omega_3}}$$

satisfies $\nabla P = 0$ in $\mathbb{R}^2$, meaning that P is constant in $\mathbb{R}^2$. This completes the proof. □

7. Proof of Theorem 2.4

In this chapter we establish the well-posedness of the multiphase Muskat problem with general viscosities. In order to do so, we first use the representation formula for the velocity derived in Theorem 6.6 to represent the multiphase Muskat problem as an evolution problem for the pair $X = (f, h)$, see equation (7.1). Thereafter, we investigate the nonlinearity Φ for which we prove that it depends smoothly on X and that its Fréchet derivative generates an analytic semigroup, under the assumption that the Rayleigh–Taylor condition holds at each interface, see Section 7.2. The proof of the well-posedness result Theorem 2.4 is postponed at the end of the chapter, and uses, besides those two important features of Φ, abstract parabolic theory as presented in [57].

7.1. The contour integral formulation

Let $r \in (3/2, 2)$ be fixed and let $\mathcal{U}_r$ be the open subset of $\mathcal{O}_r$ identified in Theorem 6.4. In view of Theorem 6.4 we conclude that if $X(t) = (f(t), h(t))$ belongs to $\mathcal{U}_r$ at each time instant $t \geq 0$, then the velocity $v(t)$ is identified according to the formulas (6.30) and (6.31). Recalling (1.1d), the multiphase Muskat problem (1.1) can now be recast as a nonlinear and nonlocal evolution problem with nonlinearities expressed in terms of contour integrals

$$\frac{dX(t)}{dt} = \Phi(X(t)), \quad t \geq 0, \qquad X(0) = X_0, \tag{7.1}$$

where $\Phi := (\Phi_1, \Phi_2) \colon \mathcal{U}_r \subset H^r(\mathbb{R})^2 \to H^{r-1}(\mathbb{R})^2$ is given by

$$\Phi(X) := \mathcal{B}(X)[\bar{\omega}]. \tag{7.2}$$

Here, $\mathcal{B}(X)$ is the integral operator introduced in (6.2) and $\bar{\omega} := (\bar{\omega}_1, \bar{\omega}_2) \in H^{r-1}(\mathbb{R})^2$ is the unique solution to the equation

$$(1 - A_\mu \mathcal{A}(X))[\bar{\omega}] = \Theta X',$$

see (6.13), found in Theorem 6.4 (for the choice $A = A_\mu$). Below we prove that the operator Φ is smooth

$$\Phi \in \mathrm{C}^\infty(\mathcal{U}_r, H^{r-1}(\mathbb{R})^2), \tag{7.3}$$

cf. Corollary 7.3. Furthermore, we show in Theorem 7.4 that its Fréchet derivative $\partial\Phi(X)$ generates an analytic semigroup in $\mathcal{L}(H^{r-1}(\mathbb{R})^2)$, that is

$$-\partial\Phi(X) \in \mathcal{H}(H^r(\mathbb{R})^2, H^{r-1}(\mathbb{R})^2). \tag{7.4}$$

As stated in Theorem 7.4, we establish this property for X belonging to the open subset $\mathcal{V}_r$ of $\mathcal{U}_r$, defined in (7.10) below, that consists of the pairs $X \in \mathcal{U}_r$ for which the Rayleigh–Taylor conditions, see (7.8), are satisfied.

7.2. Smoothness and the generator property

In this section we follow two main goals, namely to establish the smoothness of the operator Φ and to show that the Fréchet derivative $\partial\Phi(X)$ generates an analytic semigroup in $\mathcal{L}(H^{r-1}(\mathbb{R})^2)$

at any given point $X \in \mathcal{V}_r$. Here we consider the set $\mathcal{V}_r$, which is the open subset of $\mathcal{U}_r$, that consists of all pairs fulfilling the Rayleigh–Taylor conditions, see (7.8).

Smoothness of Φ

In order to establish the smoothness of Φ, see (7.3), we make use of results from Chapter 4 and Appendix B. We start with showing that the operators $\mathcal{A}$ and $\mathcal{B}$, defined in (6.2), are smooth mappings from $\mathcal{O}_r$ to $\mathcal{L}(H^{r-1}(\mathbb{R})^2)$.

Lemma 7.1. *Given $r \in (3/2, 2)$, it holds that*

$$[X \mapsto \mathcal{A}(X)],\ [X \mapsto \mathcal{B}(X)] \in \mathrm{C}^\infty(\mathcal{O}_r, \mathcal{L}(H^{r-1}(\mathbb{R})^2)). \tag{7.5}$$

Proof. We recall that for $X \in \mathcal{O}_r$ the operators $\mathcal{A}(X) = (\mathcal{A}_1(X), \mathcal{A}_2(X))$ and $\mathcal{B}(X) = (\mathcal{B}_1(X), \mathcal{B}_2(X))$ are given by

$$\mathcal{A}_1(X)[\bar\omega] = \mathbb{A}(f)[\bar\omega_1] + S(X)[\bar\omega_2], \qquad \mathcal{A}_2(X)[\bar\omega](x) = S'(X)[\bar\omega_1] + \mathbb{A}(h)[\bar\omega_2],$$
$$\mathcal{B}_1(X)[\bar\omega] = \mathbb{B}(f)[\bar\omega_1] + T(X)[\bar\omega_2], \qquad \mathcal{B}_2(X)[\bar\omega](x) = T'(X)[\bar\omega_1] + \mathbb{B}(h)[\bar\omega_2]$$

for $\bar\omega = (\bar\omega_1, \bar\omega_2) \in H^{r-1}(\mathbb{R})$, cf. (6.2). Moreover, given $u \in H^{r-1}(\mathbb{R})$, we infer from (6.1) that

$$\mathbb{A}(u) = u' B_{0,1}(u) - B_{1,1}(u)[u, \cdot] \quad \text{and} \quad \mathbb{B}(u) = B_{0,1}(u) + u' B_{1,1}(u)[u, \cdot], \tag{7.6}$$

and Lemma B.9 together with the algebra property of $H^{r-1}(\mathbb{R})$ implies that

$$[u \mapsto \mathbb{A}(u)],\ [u \mapsto \mathbb{B}(u)] \in \mathrm{C}^\infty(H^r(\mathbb{R}), \mathcal{L}(H^{r-1}(\mathbb{R}))). \tag{7.7}$$

Furthermore, recalling (6.12), the smoothness result established in Lemma 4.5, (7.7), the embedding $H^1(\mathbb{R}) \hookrightarrow H^{r-1}(\mathbb{R})$, and the algebra property of $H^{r-1}(\mathbb{R})$ lead us to (7.5). ☐

We next introduce the solution operator corresponding to the equation (6.13).

Lemma 7.2. *Given $r \in (3/2, 2)$ and $X \in \mathcal{U}_r$, let*

$$\bar\omega(X) := \Theta(1 - A_\mu \mathcal{A}(X))^{-1}[X'] \in H^{r-1}(\mathbb{R})^2$$

denote the unique solution to (6.13) found in Theorem 6.4. It then holds

$$\bar\omega \in \mathrm{C}^\infty(\mathcal{U}_r, H^{r-1}(\mathbb{R})^2).$$

Proof. The claim follows from Theorem 6.4 and Lemma 7.1, by using the smoothness of the mapping

$$\left[T \mapsto T^{-1}\right] \colon \operatorname{Isom}(H^{r-1}(\mathbb{R})^2) \to \operatorname{Isom}(H^{r-1}(\mathbb{R})^2).$$

☐

We are now in a position to conclude our first main result of this section.

Corollary 7.3. *Given $r \in (3/2, 2)$, we have*

$$\Phi \in \mathrm{C}^\infty(\mathcal{U}_r, H^{r-1}(\mathbb{R})^2).$$

Proof. In view of the definition of Φ, see (7.2), the smoothness of Φ follows from the same property of the operator $\mathcal{B}(X)$, cf. Lemma 7.1, and of the mapping $\bar\omega$ introduced in Lemma 7.2. ☐

The Rayleigh–Taylor conditions

The Rayleigh–Taylor condition, see e.g. [81, 85], is a sign restriction on the jump of the pressure gradients in normal direction at each interface and it reads

$$\begin{aligned} \partial_{\nu_1}(p_2 - p_1) &< 0 \qquad \text{on } \Gamma_f^{c_\infty}, \\ \partial_{\nu_2}(p_3 - p_2) &< 0 \qquad \text{on } \Gamma_h. \end{aligned} \tag{7.8}$$

Assuming that $X = (f, h) \in \mathcal{U}_r$, we can express the Rayleigh–Taylor condition thanks to Darcy's law, cf. $(1.1\text{a})_1$, and Theorem 6.6 as follows

$$\Theta_1 + a_\mu^1 \Phi_1(X) < 0 \qquad \text{and} \qquad \Theta_2 + a_\mu^2 \Phi_2(X) < 0, \tag{7.9}$$

where $\Theta = \operatorname{diag}(\Theta_1, \Theta_2)$ is the diagonal matrix defined in (6.32). Let

$$\mathcal{V}_r := \{X \in \mathcal{U}_r \,:\, \Theta_1 + a_\mu^1 \Phi_1(X) < 0 \text{ and } \Theta_2 + a_\mu^2 \Phi_2(X) < 0\}. \tag{7.10}$$

Since $\Theta_i < 0$, $i = 1, 2$, and $\Phi(0) = \bar{\omega}(0) = 0$, it follows by using the smoothness of Φ, see Corollary 7.3, that $\mathcal{V}_r$ is a nonempty open subset of $\mathcal{U}_r$. It is worth mentioning that if $A_\mu = 0$, which means that the viscosities are equal, then it holds that

$$\mathcal{V}_r = \mathcal{U}_r = \mathcal{O}_r.$$

Hence, in this particular case, we are in the general setting considered in Part I.

The generator property

The next goal is to show that the evolution problem (7.1) is of parabolic type in $\mathcal{V}_r$, in the sense that $\partial\Phi(X)$ generates an analytic semigroup in $\mathcal{L}(H^{r-1}(\mathbb{R})^2)$ for each $X \in \mathcal{V}_r$, as the next result states.

Theorem 7.4. *Given $r \in (3/2, 2)$ and $X \in \mathcal{V}_r$, we have*

$$-\partial\Phi(X) \in \mathcal{H}(H^r(\mathbb{R})^2, H^{r-1}(\mathbb{R})^2).$$

Proof. The claim follows from Theorem 5.1, Lemma 7.5, and Theorem 7.6 below. □

In order to established the results used in the proof of Theorem 7.4, we fix $r \in (3/2, 2)$ and $X = (f, h) \in \mathcal{V}_r$, and set

$$\bar{\omega} := (\bar{\omega}_1, \bar{\omega}_2) := \bar{\omega}(X), \tag{7.11}$$

see Lemma 7.2. The Fréchet derivative $\partial\Phi(X)$ can be represented as a matrix operator

$$\partial\Phi(X) = \begin{pmatrix} \partial_f \Phi_1(X) & \partial_h \Phi_1(X) \\ \partial_f \Phi_2(X) & \partial_h \Phi_2(X) \end{pmatrix} \in \mathcal{L}(H^r(\mathbb{R})^2, H^{r-1}(\mathbb{R})^2).$$

Our strategy is analogous to the one followed in Part I, that is, we make use of Theorem 5.1 and of the fact that the diagonal entries $\partial_f \Phi_1(X)$ and $\partial_h \Phi_2(X)$ are generators of analytic semigroups in $\mathcal{L}(H^{r-1}(\mathbb{R}))$, see Theorem 7.6, whereas the off-diagonal entry $\partial_h \Phi_1(X)$ is a perturbation, cf. Lemma 7.5 below.

Lemma 7.5. *Let $X \in \mathcal{U}_r$ and $r \in (3/2, 2)$. Then, given $\nu > 0$, there exists a positive constant $K = K(\nu) > 0$ such that*

$$\|\partial_h \Phi_1(X)[v]\|_{H^{r-1}} \le \nu \|v\|_{H^r} + K\|v\|_{H^{r-1}} \qquad \text{for all } v \in H^r(\mathbb{R}). \tag{7.12}$$

Proof. We start by fixing $r' \in (3/2, r)$. Recalling (6.2), we then compute

$$\partial_h \Phi_1(X)[v] = \partial_h \mathcal{B}_1(X)[v][\bar{\omega}] + \mathbb{B}(f)[\partial_h \bar{\omega}_1(X)[v]] + T(X)[\partial_h \bar{\omega}_2(X)[v]] \tag{7.13}$$

for $v \in H^r(\mathbb{R})$, where, in view of $(6.12)_3$, we have

$$\begin{aligned}\partial_h \mathcal{B}_1(X)[v][\bar{\omega}] = \partial_h D_1(X)[v][\bar{\omega}_2] + (c_\infty + f) f' \partial_h C_1(X)[v][\bar{\omega}_2] \\ - f' \partial_h C_1(X)[v][h\bar{\omega}_2] - f' C_1(X)[v\bar{\omega}_2]\end{aligned} \tag{7.14}$$

for all $v \in H^r(\mathbb{R})$. Moreover, differentiation of (6.13) leads to

$$\partial\bar{\omega}(X)[Y] - A_\mu \partial\mathcal{A}(X)[Y][\bar{\omega}(X)] - A_\mu \mathcal{A}(X)[\partial\bar{\omega}(X)[Y]] = \Theta Y', \quad Y \in H^r(\mathbb{R})^2,$$

and thus

$$(1 - A_\mu \mathcal{A}(X))[\partial\bar{\omega}(X)[Y]] = A_\mu \partial\mathcal{A}(X)[Y][\bar{\omega}(X)] + \Theta Y', \quad Y \in H^r(\mathbb{R})^2. \tag{7.15}$$

Using (6.2) and (6.12), it holds that

$$\begin{aligned}\partial_h S(X)[v][\bar{\omega}_2] = f' \partial_h D_1(X)[v][\bar{\omega}_2] + C_1(X)[v\bar{\omega}_2] \\ + \partial_h C_1(X)[v][h\bar{\omega}_2] - (c_\infty + f)\partial_h C_1(X)[v][\bar{\omega}_2]\end{aligned}$$

for $v \in H^r(\mathbb{R})$. From (7.15) we then conclude

$$\begin{aligned}(1 - a_\mu^1 \mathbb{A}(f))[\partial_h \bar{\omega}_1(X)[v]] = a_\mu^1 \big(S(X)[\partial_h \bar{\omega}_2(X)[v]] + f' \partial_h D_1(X)[v][\bar{\omega}_2] + C_1(X)[v\bar{\omega}_2] \\ + \partial_h C_1(X)[v][h\bar{\omega}_2] - (c_\infty + f)\partial_h C_1(X)[v][\bar{\omega}_2]\big).\end{aligned} \tag{7.16}$$

Before proceeding with the estimates, we recall from Lemma 4.5 (with $r = r'$) that

$$\partial_h E_1(X) \in \mathcal{L}(H^{r'}(\mathbb{R}), \mathcal{L}(L_2(\mathbb{R}), H^1(\mathbb{R}))) \qquad \text{for } E \in \{C, C', D, D'\}. \tag{7.17}$$

Moreover, Lemma 7.2 (with $r = r'$) implies that the Fréchet derivative $\partial\bar{\omega}(X)$ satisfies

$$\partial\bar{\omega}(X) \in \mathcal{L}(H^{r'}(\mathbb{R})^2, H^{r'-1}(\mathbb{R})^2). \tag{7.18}$$

Using (7.17) and Lemma 4.5, we infer from (7.14) that

$$\begin{aligned}\|\partial_h \mathcal{B}_1(X)[v][\bar{\omega}]\|_{H^{r-1}} &\leq C(\|\partial_h D_1(X)[v][\bar{\omega}_2]\|_{H^1} + \|\partial_h C_1(X)[v][\bar{\omega}_2]\|_{H^1} \\ &\qquad + \|\partial_h C_1(X)[v][h\bar{\omega}_2]\|_{H^1} + \|C_1(X)[v\bar{\omega}_2]\|_{H^1}) \\ &\leq C\|v\|_{H^{r'}}.\end{aligned} \tag{7.19}$$

Additionally, $(6.12)_3$, Lemma 4.5, and (7.18) lead to

$$\begin{aligned}\|T(X)[\partial_h \bar{\omega}_2(X)[v]]\|_{H^{r-1}} &\leq C(\|D_1(X)[\partial_h \bar{\omega}_2(X)[v]]\|_{H^1} + \|C_1(X)[\partial_h \bar{\omega}_2(X)[v]]\|_{H^1} \\ &\qquad + \|C_1(X)[h\partial_h \bar{\omega}_2(X)[v]]\|_{H^1}) \\ &\leq C\|\partial_h \bar{\omega}_2(X)[v]\|_{L_2} \\ &\leq C\|\partial_h \bar{\omega}_2(X)[v]\|_{H^{r'-1}} \\ &\leq C\|v\|_{H^{r'}}.\end{aligned} \tag{7.20}$$

Arguing similarly as above, it also holds that the right hand-side of (7.16) can be estimated by the same quantity

$$\|(1 - a_\mu^1 \mathbb{A}(f))[\partial_h \bar{\omega}_1(X)[v]]\|_{H^{r-1}} \leq C\|v\|_{H^{r'}}.$$

The isomorphism property (6.10) (with $u = f$ and $\lambda = 1/a^1_\mu$) yields

$$\|\partial_h \bar\omega_1(X)[v]\|_{H^{r-1}} \le C\|v\|_{H^{r'}}.$$

Combining this property with (6.9), we obtain

$$\|\mathbb{B}(f)[\partial_h \bar\omega_1(X)[v]]\|_{H^{r-1}} \le C\|\partial_h \bar\omega_1(X)[v]\|_{H^{r-1}} \le C\|v\|_{H^{r'}}. \tag{7.21}$$

Gathering (7.13) and (7.19)-(7.21), we conclude that

$$\|\partial_h \Phi_1(X)[v]\|_{H^{r-1}} \le C\|v\|_{H^{r'}} \qquad \text{for all } v \in H^r(\mathbb{R}).$$

The inequality (5.8) (where $\tau = r'$) implies now the desired estimate (7.12). □

The estimate (7.12) is the first ingredient in the proof of Theorem 7.4. We prove below that the diagonal entries of $\partial\Phi(X)$ are both analytic generators. As the next result shows, the generator property for $\partial_f\Phi_1(X)$ is established when merely assuming that the Rayleigh–Taylor condition is satisfied at the interface $\Gamma^{c_\infty}_f$, respectively the generation property for $\partial_h\Phi_2(X)$ uses only the Rayleigh–Taylor condition on Γ_h.

Theorem 7.6.

(i) *Assume that* $\Theta_1 + a^1_\mu\Phi_1(X) < 0$. *Then* $-\partial_f\Phi_1(X) \in \mathcal{H}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$.

(ii) *Assume that* $\Theta_2 + a^2_\mu\Phi_2(X) < 0$. *Then* $-\partial_h\Phi_2(X) \in \mathcal{H}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$.

The proof of Theorem 7.6 is postponed to the end of the section as it requires some preparation. To start, we differentiate (7.2) with respect to f to arrive, in view of (6.2), at the formula

$$\partial_f\Phi_1(X)[u] = \partial_f\mathcal{B}_1(X)[u][\bar\omega] + \mathbb{B}(f)[\partial_f\bar\omega_1(X)[u]] + T(X)[\partial_f\bar\omega_2(X)[u]] \tag{7.22}$$

for $u \in H^r(\mathbb{R})$, where, on account of $(6.12)_3$, we have

$$\partial_f\mathcal{B}_1(X)[u][\bar\omega] = \partial\mathbb{B}(f)[u][\bar\omega_1] + a_{1,1}(X)u' + T^1_{\mathrm{lot},1}[u], \tag{7.23}$$

with

$$a_{1,1}(X) := (c_\infty + f)C_1(X)[\bar\omega_2] - C_1(X)[h\bar\omega_2] \in H^1(\mathbb{R}),$$

$$T^1_{\mathrm{lot},1}[u] := \partial_f D_1(X)[u][\bar\omega_2] + (c_\infty + f)f'\partial_f C_1(X)[u][\bar\omega_2] + uf'C_1(X)[\bar\omega_2] - f'\partial_f C_1(X)[u][h\bar\omega_2].$$

The term $T^1_{\mathrm{lot},1}[u]$ is a sum of lower order terms since (7.17) and Lemma 4.5 imply

$$\|T^1_{\mathrm{lot},1}[u]\|_{H^{r-1}} \le C\|u\|_{H^{r'}} \tag{7.24}$$

for all $u \in H^r(\mathbb{R})$. We next differentiate the first component of (6.13) with respect to f, cf. (7.15) to obtain, with the help of $(6.12)_1$, that

$$(1 - a^1_\mu\mathbb{A}(f))[\partial_f\bar\omega_1(X)[u]] = \Theta_1 u' + a^1_\mu\big(\partial\mathbb{A}(f)[u][\bar\omega_1] + a_{2,1}(X)u' + T^2_{\mathrm{lot},1}[u]\big), \tag{7.25}$$

where

$$a_{2,1}(X) := D_1(X)[\bar\omega_2] \in H^1(\mathbb{R}),$$

$$\begin{aligned} T^2_{\mathrm{lot},1}[u] := {} & S(X)[\partial_f\bar\omega_2(f)[u]] + f'\partial_f D_1(X)[u][\bar\omega_2] - (c_\infty + f)\partial_f C_1(X)[u][\bar\omega_2] \\ & - uC_1(X)[\bar\omega_2] + \partial_f C_1(X)[u][h\bar\omega_2]. \end{aligned}$$

Also $T^2_{\mathrm{lot},1}[u]$ is a sum of lower order terms since (6.6), (7.17), (7.18), and Lemma 4.5 combined yield

$$\|T^2_{\mathrm{lot},1}[u]\|_{H^{r-1}} \leq C\|u\|_{H^{r'}} \tag{7.26}$$

for all $u \in H^r(\mathbb{R})$.

We now consider the continuous path $\Psi_1\colon [0,1] \to \mathcal{L}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$ defined by

$$\Psi_1(\tau) := \tau\partial_f\mathcal{B}_1(X)[u][\bar{\omega}] + \mathbb{B}(\tau f)[w_1(\tau)[u]] + \tau T(X)[\partial_f\bar{\omega}_2(X)[u]] \tag{7.27}$$

where $w_1\colon [0,1] \to \mathcal{L}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$ is a continuous path given by

$$\begin{aligned}(1 - a^1_\mu\mathbb{A}(\tau f))[w_1(\tau)[u]] = \Theta_1 u' + a^1_\mu\big(&\tau\partial\mathbb{A}(f)[u][\bar{\omega}_1] + \tau a_{2,1}(X)u' \\ &+ \tau T^2_{\mathrm{lot},1}[u] + (1-\tau)u'\Phi_1(X)\big).\end{aligned} \tag{7.28}$$

With respect to the definitions (7.27) and (7.28) we include the following remarks.

Remark 7.7.

(i) If $\tau = 1$, then $w_1(1) = \partial_f\bar{\omega}_1(X)$ and $\Psi_1(1) = \partial_f\Phi_1(X)$.

(ii) Letting H denote the Hilbert transform, we have

$$w_1(0) = (\Theta_1 + a^1_\mu\Phi_1(X))\frac{d}{dx} \qquad \text{and} \qquad \Psi_1(0) = H[w_1(0)].$$

It is worthwhile to point out that the term $(1-\tau)a^1_\mu u'\Phi_1(X)$ in the definition (7.28) is a term introduced artificially and is very important for the following facts. When $\tau = 0$, we obtain due to this term a negative coefficient function – which is exactly the function from the Rayleigh–Taylor condition on $\Gamma^{c_\infty}_f$ – for the differential operator $w_1(0)$. This aspect is important when establishing the invertibility of $\lambda - \partial_f\Phi_1(X)$ for sufficiently large λ, see the proof of Theorem 7.6 below. Besides, below we localize the operator $\Psi_1(\tau)$ and show that it can locally approximated by certain Fourier multipliers, see Theorem 7.8. Thanks to this artificial term the Fourier multipliers have a coefficient which is the product of a positive function with the function from the Rayleigh–Taylor condition on $\Gamma^{c_\infty}_f$ (both frozen at a certain point), see (7.30)-(7.31) below. These features enable us to show that the Fourier multipliers are generators of analytic semigroups and that they satisfy certain uniform estimates similar to those in Section 5.1, see Lemma 5.8.

(iii) The properties (6.10) and (7.7) (both with $r = r'$) together with (7.26) and (7.28) imply that there exists a constant $C > 0$ such that

$$\|w_1(\tau)[u]\|_{H^{r'-1}} \leq C\|u\|_{H^{r'}}, \qquad u \in H^{r'}(\mathbb{R}),\ \tau \in [0,1]. \tag{7.29}$$

As a further step we locally approximate in Theorem 7.8 below the operator $\Psi_1(\tau)$, $\tau \in [0,1]$, by certain Fourier multipliers $\mathbb{A}^1_{j,\tau}$. To this end we associate with each given $\varepsilon \in (0,1)$, a positive integer $N = N(\varepsilon)$ and a finite ε-localization family

$$\{\pi^\varepsilon_j\ :\ -N+1 \leq j \leq N\},$$

cf. Section 5.1.

The Fourier multipliers $\mathbb{A}^1_{j,\tau}$ mentioned above are defined by

$$\begin{aligned}\mathbb{A}^1_{j,\tau} &:= \mathbb{A}^{\varepsilon,1}_{j,\tau} := \alpha_{\tau,1}(x^\varepsilon_j)\Big(-\frac{d^2}{dx^2}\Big)^{1/2} + \beta_{\tau,1}(x^\varepsilon_j)\frac{d}{dx}, \quad |j| \leq N-1, \\ \mathbb{A}^1_{N,\tau} &:= \mathbb{A}^{\varepsilon,1}_{N,\tau} := \Theta_1\Big(-\frac{d^2}{dx^2}\Big)^{1/2},\end{aligned} \tag{7.30}$$

where

$$\alpha_{\tau,1} := \frac{1+(1-\tau)f'^2}{1+f'^2}(\Theta_1 + a_\mu^1\Phi_1(X)), \qquad \beta_{\tau,1} := \tau B_{1,1}^0(f)[\bar{\omega}_1] + \tau a_{1,1}(X) + \frac{\tau a_\mu^1 \bar{\omega}_1}{1+f'^2}. \quad (7.31)$$

The next results provides an estimate for the localization error and is the main step in the proof of Theorem 7.6 (i). Here we closely follow the proof of Lemma 5.5 and Lemma 5.7, and we benefit from the fact that the operator $\mathbb{B}(f)$ and its Fréchet derivative have been already localized there.

Theorem 7.8. *Let $\nu > 0$ be given and fix $r' \in (3/2, r)$. Then, there exist $\varepsilon \in (0,1)$ and a positive constant $K = K(\varepsilon)$ such that*

$$\|\pi_j^\varepsilon \Psi_1(\tau)[u] - \mathbb{A}_{j,\tau}^1[\pi_j^\varepsilon u]\|_{H^{r-1}} \leq \nu\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}} \quad (7.32)$$

for all $-N+1 \leq j \leq N$, $\tau \in [0,1]$, and $u \in H^r(\mathbb{R})$.

Proof. Let $\varepsilon \in (0,1)$ be given. Let further $\{\pi_j^\varepsilon : -N+1 \leq j \leq N\}$ be a finite ε-localization family and $\{\chi_j^\varepsilon : -N+1 \leq j \leq N\}$ a second family with the following properties:

- $\chi_j^\varepsilon \in C^\infty(\mathbb{R}, [0,1])$ and $\chi_j^\varepsilon = 1$ on supp π_j^ε, $-N+1 \leq j \leq N$;
- supp χ_j^ε is an interval of length 3ε, $|j| \leq N-1$, and supp $\chi_N^\varepsilon \subset \{|x| \geq 1/\varepsilon - \varepsilon\}$.

We denote by C a constant that does not depend on ε, while the constant K may depend on ε. The proof is divided in four main steps.

Step 1: The term $\partial_f \mathcal{B}_1(X)[u][\bar{\omega}]$. Similarly to the proof of Lemma 5.5, we obtain that

$$\begin{aligned}\Big\|\pi_j^\varepsilon \partial\mathbb{B}(f)[u][\bar{\omega}_1] + \frac{\bar{\omega}_1(x_j^\varepsilon)f'(x_j^\varepsilon)}{1+f'^2(x_j^\varepsilon)}H[(\pi_j^\varepsilon u)'] - B_{1,1}^0(f)[\bar{\omega}_1](x_j^\varepsilon)(\pi_j^\varepsilon u)'\Big\|_{H^{r-1}}& \\ \leq \frac{\nu}{4}\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}}&\end{aligned} \quad (7.33)$$

for all $|j| \leq N-1$ and $u \in H^r(\mathbb{R})$, and

$$\|\pi_N^\varepsilon \partial\mathbb{B}(f)[u][\bar{\omega}_1]\|_{H^{r-1}} \leq \frac{\nu}{4}\|\pi_N^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}} \quad (7.34)$$

for all $u \in H^r(\mathbb{R})$, provided that ε is sufficiently small.

Moreover, thanks to $a_{1,1}(X) \in \mathrm{BUC}^{1/2}(\mathbb{R})$, the estimate (B.18) together with the identity $\chi_j^\varepsilon \pi_j^\varepsilon = \pi_j^\varepsilon$, $-N+1 \leq j \leq N$ yields that

$$\begin{aligned}\|\pi_j^\varepsilon a_{1,1}(X)u' - a_{1,1}(X)(x_j^\varepsilon)(\pi_j^\varepsilon u)'\|_{H^{r-1}} &\leq \|\chi_j^\varepsilon\big(a_{1,1}(X) - a_{1,1}(X)(x_j^\varepsilon)\big)(\pi_j^\varepsilon u)'\|_{H^{r-1}} + K\|u\|_{H^{r-1}} \\ &\leq C\|\chi_j^\varepsilon\big(a_{1,1}(X) - a_{1,1}(X)(x_j^\varepsilon)\big)\|_\infty\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}} \\ &\leq \frac{\nu}{4}\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}}, \qquad |j| \leq N-1,\end{aligned} \quad (7.35)$$

if ε is sufficiently small. By taking into account that $a_{1,1}(X)$ vanishes at infinity, one obtains

$$\begin{aligned}\|\pi_N^\varepsilon a_{1,1}(X)u'\|_{H^{r-1}} &\leq \|\chi_N^\varepsilon a_{1,1}(X)(\pi_N^\varepsilon u)'\|_{H^{r-1}} + K\|u\|_{H^{r-1}} \\ &\leq C\|\chi_N^\varepsilon a_{1,1}(X)\|_\infty\|\pi_N^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}} \\ &\leq \frac{\nu}{4}\|\pi_N^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}}\end{aligned} \quad (7.36)$$

for all $u \in H^r(\mathbb{R})$.

Recalling (7.24), we have

$$\|\pi_j^\varepsilon T^1_{\mathrm{lot},1}[u]\|_{H^{r-1}} \leq K\|u\|_{H^{r'}} \tag{7.37}$$

for all $u \in H^r(\mathbb{R})$ and $-N+1 \leq j \leq N$, and therewith we have localized all three summands in the formula (7.23) for $\partial_f \mathcal{B}_1(X)[u][\bar\omega]$.

Step 2: The term $T(X)[\partial_f \bar\omega_2(X)[u]]$. Combining (6.6) and (7.18), we have

$$\begin{aligned}\|\pi_j^\varepsilon T(X)[\partial_f\bar\omega_2(X)[u]]\|_{H^{r-1}} \leq& K\|T(X)[\partial_f\bar\omega_2(X)[u]]\|_{H^{r-1}}\\ \leq& K\|\partial_f\bar\omega_2(X)[u]\|_2\\ \leq& K\|\partial_f\bar\omega_2(X)[u]\|_{H^{r'-1}}\\ \leq& K\|u\|_{H^{r'}}\end{aligned} \tag{7.38}$$

for all $u \in H^r(\mathbb{R})$, $\varepsilon \in (0,1)$, and $-N+1 \leq j \leq N$.

Step 3: The term $\mathbb{B}(\tau f)[w_1(\tau)[u]]$. We divide this step into two substeps.

Step 3a. We prove that there exists a positive constant C_0 such that

$$\|\pi_j^\varepsilon w_1(\tau)[u]\|_{H^{r-1}} \leq C_0\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}} \tag{7.39}$$

for all $\varepsilon \in (0,1)$, $-N+1 \leq j \leq N$, $\tau \in [0,1]$, and $u \in H^r(\mathbb{R})$. To this end we multiply (7.28) by π_j^ε and arrive at

$$\begin{aligned}(1-a_\mu^1\mathbb{A}(\tau f))[\pi_j^\varepsilon w_1(\tau)[u]] = \Theta_1\pi_j^\varepsilon u' &+ a_\mu^1\big(\pi_j^\varepsilon\mathbb{A}(\tau f)[w_1(\tau)[u]] - \mathbb{A}(\tau f)[\pi_j^\varepsilon w_1(\tau)[u]]\big)\\ &+ a_\mu^1\pi_j^\varepsilon\big(\tau\partial\mathbb{A}(f)[u][\bar\omega_1] + \tau a_{2,1}(X)u'\\ &\qquad + \tau T^2_{\mathrm{lot},1}[u] + (1-\tau)u'\Phi_1(X)\big).\end{aligned} \tag{7.40}$$

Taking (6.10) and (7.7) into account, it remains to show that the H^{r-1}-norm of the right side of (7.40) may be estimated by the right side of (7.39). To start, we infer from Lemma B.12 and (7.29) that

$$\|\pi_j^\varepsilon\mathbb{A}(\tau f)[w_1(\tau)[u]] - \mathbb{A}(\tau f)[\pi_j^\varepsilon w_1(\tau)[u]]\|_{H^{r-1}} \leq K\|w_1(\tau)[u]\|_2 \leq K\|u\|_{H^{r'}} \tag{7.41}$$

for all $\varepsilon \in (0,1)$, $-N+1 \leq j \leq N$, $\tau \in [0,1]$, and $u \in H^r(\mathbb{R})$. Furthermore, taking into account that $a_{2,1}(X) \in H^1(\mathbb{R})$ and $\Phi_1(X) \in H^{r-1}(\mathbb{R})$, we have

$$\begin{aligned}\|\Theta_1\pi_j^\varepsilon u' + a_\mu^1\pi_j^\varepsilon(\tau a_{2,1}(X)u' + (1-\tau)u'\Phi_1(X))\|_{H^{r-1}} \leq& C\|\pi_j^\varepsilon u'\|_{H^{r-1}}\\ \leq& C\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}}.\end{aligned} \tag{7.42}$$

Recalling (7.26), we get

$$\|\pi_j^\varepsilon T^2_{\mathrm{lot},1}[u]\|_{H^{r-1}} \leq K\|u\|_{H^{r'}}, \tag{7.43}$$

and it remains to estimate the term $\pi_j^\varepsilon\tau\partial\mathbb{A}(f)[u][\bar\omega_1]$. Due to (5.4) and (6.11) , it holds that

$$\begin{aligned}\partial\mathbb{A}(f)[u][\bar\omega_1] =& u'B_{0,1}(f)[\bar\omega_1] - 2f'B_{2,2}(f,f)[f,u,\bar\omega_1]\\ &- B_{1,1}(f)[u,\bar\omega_1] + 2B_{3,2}(f,f)[f,f,u,\bar\omega_1], \quad u \in H^r(\mathbb{R}).\end{aligned}$$

Invoking Theorem B.7, we deduce that

$$\partial\mathbb{A}(f)[u][\bar\omega_1] = u'B_{0,1}(f)[\bar\omega_1] + \bar\omega_1\big(-2f'B^0_{1,2}(f)[u'] - B^0_{0,1}(f)[u'] + 2B^0_{2,2}(f)[u']\big) + T^3_{\mathrm{lot},1}[u], \tag{7.44}$$

where

$$\begin{aligned}T^3_{\mathrm{lot},1}[u] :=& -2f'B_{2,2}(f,f)[f,u,\bar\omega_1] - B_{1,1}(f)[u,\bar\omega_1] + 2B_{3,2}(f,f)[f,f,u,\bar\omega_1]\\ &- \bar\omega_1\big(-2f'B^0_{1,2}(f)[u'] - B^0_{0,1}(f)[u'] + 2B^0_{2,2}(f)[u']\big)\end{aligned}$$

satisfies

$$\|T^3_{\mathrm{lot},1}[u]\|_{H^{r-1}} \leq C\|u\|_{H^{r'}}. \tag{7.45}$$

Since for $n, m \in \mathbb{N}$ we have

$$\begin{aligned}\|\pi^\varepsilon_j B^0_{n,m}(f)[u']\|_{H^{r-1}} &\leq \|B^0_{n,m}(f)[\pi^\varepsilon_j u']\|_{H^{r-1}} + \|\pi^\varepsilon_j B^0_{n,m}(f)[u'] - B^0_{n,m}(f)[\pi^\varepsilon_j u']\|_{H^{r-1}}\\ &\leq C\|\pi^\varepsilon_j u'\|_{H^{r-1}} + K\|u'\|_2\\ &\leq C\|\pi^\varepsilon_j u\|_{H^r} + K\|u\|_{H^{r'}},\end{aligned}$$

cf. Theorem B.5 and Lemma B.12, we conclude that

$$\|\pi^\varepsilon_j \tau\partial\mathbb{A}(f)[u][\bar\omega_1]\|_{H^{r-1}} \leq C\|\pi^\varepsilon_j u\|_{H^r} + K\|u\|_{H^{r'}}. \tag{7.46}$$

Gathering (7.40)-(7.46), it now follows from (6.10) and (7.7) that (7.39) indeed holds true.

Step 3b. Let C_0 be the constant from (7.39). In view of (6.11), (7.29), and Lemma B.13 (when $|j| \leq N-1$), Lemma B.14 and Lemma B.15 (when $j = N$), respectively, we have for sufficiently small ε that

$$\begin{aligned}\|\pi^\varepsilon_j\mathbb{B}(\tau f)[w_1(\tau)[u]] - H[\pi^\varepsilon_j w_1(\tau)[u]]\|_{H^{r-1}} &\leq \frac{\nu}{4C_0}\|\pi^\varepsilon_j w_1(\tau)[u]\|_{H^{r-1}} + K\|w_1(\tau)[u]\|_{H^{r'-1}}\\ &\leq \frac{\nu}{4}\|\pi^\varepsilon_j u\|_{H^r} + K\|u\|_{H^{r'}}\end{aligned} \tag{7.47}$$

for all $-N+1 \leq j \leq N$, $\tau \in [0,1]$, and $u \in H^r(\mathbb{R})$. We further define

$$\varphi_\tau := \Theta_1 + \tau a^1_\mu a_{2,1}(X) + (1-\tau)a^1_\mu\Phi_1(X) + \tau a^1_\mu B_{0,1}(f)[\bar\omega_1], \qquad \tau \in [0,1].$$

We prove below that if ε is sufficiently small, then

$$\Big\|H[\pi^\varepsilon_j w_1(\tau)[u]] - \varphi_\tau(x^\varepsilon_j)H[(\pi^\varepsilon_j u)'] - \frac{\tau a^1_\mu\bar\omega_1(x^\varepsilon_j)}{1+f'^2(x^\varepsilon_j)}(\pi^\varepsilon_j u)'\Big\|_{H^{r-1}} \leq \frac{\nu}{4}\|\pi^\varepsilon_j u\|_{H^r} + K\|u\|_{H^{r'}} \tag{7.48}$$

for all $|j| \leq N-1$, $\tau \in [0,1]$, and $u \in H^r(\mathbb{R})$, respectively that

$$\|H[\pi^\varepsilon_N w_1(\tau)[u]] - \Theta_1 H[(\pi^\varepsilon_N u)']\|_{H^{r-1}} \leq \frac{\nu}{4}\|\pi^\varepsilon_N u\|_{H^r} + K\|u\|_{H^{r'}} \tag{7.49}$$

for all $\tau \in [0,1]$ and $u \in H^r(\mathbb{R})$. Indeed, due to $H^2 = -\mathrm{id}_{H^{r-1}(\mathbb{R})}$ and $\|H\|_{\mathcal{L}(H^{r-1}(\mathbb{R}))} = 1$, we get

$$\begin{aligned}&\Big\|H[\pi^\varepsilon_j w_1(\tau)[u]] - \varphi_\tau(x^\varepsilon_j)H[(\pi^\varepsilon_j u)'] - \frac{\tau a^1_\mu\bar\omega_1(x^\varepsilon_j)}{1+f'^2(x^\varepsilon_j)}(\pi^\varepsilon_j u)'\Big\|_{H^{r-1}}\\ &\leq \Big\|\pi^\varepsilon_j w_1(\tau)[u] - \varphi_\tau(x^\varepsilon_j)(\pi^\varepsilon_j u)' + \frac{\tau a^1_\mu\bar\omega_1(x^\varepsilon_j)}{1+f'^2(x^\varepsilon_j)}H[(\pi^\varepsilon_j u)']\Big\|_{H^{r-1}}\end{aligned}$$

for $|j| \leq N-1$, respectively

$$\|H[\pi^\varepsilon_N w_1(\tau)[u]] - \Theta_1 H[(\pi^\varepsilon_N u)']\|_{H^{r-1}} \leq \|\pi^\varepsilon_N w_1(\tau)[u] - \Theta_1(\pi^\varepsilon_N u)'\|_{H^{r-1}}.$$

In order to estimate the right-hand sides of the latter two estimates let first $|j| \leq N-1$. Multiplying the equation (7.28) by π_j^ε, we arrive at

$$\pi_j^\varepsilon w_1(\tau)[u] - \varphi_\tau(x_j^\varepsilon)(\pi_j^\varepsilon u)' + \frac{\tau a_\mu^1 \bar\omega_1(x_j^\varepsilon)}{1+f'^2(x_j^\varepsilon)} H[(\pi_j^\varepsilon u)'] = T_1 + T_2 + T_3 + T_4,$$

where

$$\begin{aligned}
T_1 :=& \pi_j^\varepsilon(\Theta_1 + \tau a_\mu^1 a_{2,1}(X) + (1-\tau)a_\mu^1 \Phi_1(X))u' \\
&- (\Theta_1 + \tau a_\mu^1 a_{2,1}(X) + (1-\tau)a_\mu^1 \Phi_1(X))(x_j^\varepsilon)(\pi_j^\varepsilon u)', \\
T_2 :=& \tau a_\mu^1 \Big(\pi_j^\varepsilon \partial\mathbb{A}(f)[u][\bar\omega_1] - B_{0,1}^0(f)[\bar\omega_1](x_j^\varepsilon)(\pi_j^\varepsilon u)' + \frac{\bar\omega_1(x_j^\varepsilon)}{1+f'^2(x_j^\varepsilon)} H[(\pi_j^\varepsilon u)']\Big), \\
T_3 :=& a_\mu^1 \pi_j^\varepsilon \mathbb{A}(\tau f)[w_1(\tau)[u]], \\
T_4 :=& \tau a_\mu^1 \pi_j^\varepsilon T_{\mathrm{lot},1}^2[u].
\end{aligned}$$

Since $a_{2,1}(X)$, $\Phi_1(X) \in \mathrm{BUC}^{r-3/2}(\mathbb{R})$, the arguments used to derive (7.35) together with (7.26) yield

$$\|T_1\|_{H^{r-1}} + \|T_4\|_{H^{r-1}} \leq \frac{\nu}{12}\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}}.$$

Furthermore, recalling (7.44) and (7.45), the arguments used to derive (7.35) and repeated use of Lemma B.13 lead to

$$\|T_2\|_{H^{r-1}} \leq \frac{\nu}{12}\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}}.$$

Finally, combining (6.11), Lemma B.13, (7.29), and (7.39) we get

$$\begin{aligned}
\|T_3\|_{H^{r-1}} \leq& |a_\mu^1| \Big\| \tau f' B_{0,1}^0(\tau f)[w_1(\tau)[u]] - \frac{\tau f'(x_j^\varepsilon)}{1+\tau^2 f'^2(x_j^\varepsilon)} H[\pi_j^\varepsilon w_1(\tau)[u]] \Big\|_{H^{r-1}} \\
&+ |a_\mu^1| \Big\| B_{1,1}^0(\tau f)[w_1(\tau)[u]] - \frac{\tau f'(x_j^\varepsilon)}{1+\tau^2 f'^2(x_j^\varepsilon)} H[\pi_j^\varepsilon w_1(\tau)[u]] \Big\|_{H^{r-1}} \\
\leq& \frac{\nu}{12}\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}},
\end{aligned}$$

provided that ε is sufficiently small. Hence, we establish (7.48).

Let now $j = N$. Multiplying the equation (7.28) by π_N^ε we arrive at

$$\pi_N^\varepsilon w_1(\tau)[u] - \Theta_1(\pi_N^\varepsilon u)' = T_1 + T_2 + T_3 + T_4,$$

where

$$\begin{aligned}
T_1 &:= \pi_N^\varepsilon(\Theta_1 + \tau a_\mu^1 a_{2,1}(X) + (1-\tau)a_\mu^1 \Phi_1(X))u' - \Theta_1(\pi_N^\varepsilon u)', \\
T_2 &:= \tau a_\mu^1 \pi_N^\varepsilon \partial\mathbb{A}(f)[u][\bar\omega_1] + a_\mu^1 \pi_N^\varepsilon \mathbb{A}(\tau f)[w_1(\tau)[u]] + \tau a_\mu^1 \pi_N^\varepsilon T_{\mathrm{lot},1}^2[u].
\end{aligned}$$

Since both $a_{2,1}(X)$ and $\Phi_1(X)$ vanish at infinity, we obtain

$$\|T_1\|_{H^{r-1}} \leq \frac{\nu}{8}\|\pi_N^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}},$$

by arguing as in the derivation of (7.36). Furthermore, the relations (6.11), (7.26), (7.39), (7.44), (7.45), the fact that $B_{0,1}(f)[\bar\omega_1]$ belongs to $H^{r-1}(\mathbb{R})$ and vanishes at infinity, Lemma B.14, and Lemma B.15 lead us to

$$\|T_2\|_{H^{r-1}} \leq \frac{\nu}{8}\|\pi_N^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}},$$

provided that ε is sufficiently small. This proves (7.49).

Combining (7.47) and (7.48), we conclude that if ε is sufficiently small, then

$$\Big\|\pi_j^\varepsilon\mathbb{B}(\tau f)[w_1(\tau)[u]] - \varphi_\tau(x_j^\varepsilon)H[(\pi_j^\varepsilon u)'] - \frac{\tau a_\mu^1\bar\omega_1(x_j^\varepsilon)}{1+f'^2(x_j^\varepsilon)}(\pi_j^\varepsilon u)'\Big\|_{H^{r-1}} \leq \frac{\nu}{2}\|\pi_j^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}} \tag{7.50}$$

for all $|j|\leq N-1$, $\tau\in[0,1]$, and $u\in H^r(\mathbb{R})$. If $j=N$, we conclude from (7.47) and (7.49) that, if ε is sufficiently small, then

$$\|\pi_N^\varepsilon\mathbb{B}(\tau f)[w_1(\tau)[u]] - \Theta_1 H[(\pi_N^\varepsilon u)']\|_{H^{r-1}} \leq \frac{\nu}{2}\|\pi_N^\varepsilon u\|_{H^r} + K\|u\|_{H^{r'}} \tag{7.51}$$

for all $\tau\in[0,1]$ and $u\in H^r(\mathbb{R})$.

Step 4. The desired claim (7.32) follows, in the case $j=N$, directly from (7.34), (7.36), (7.37), (7.38), and (7.51). If $|j|\leq N-1$, we infer from (7.33), (7.35), (7.37), (7.38), and (7.50) that the claim (7.32) holds true, but for the coefficient function $\alpha_{\tau,1}$ defined in (7.31) we obtain the formula

$$\alpha_{\tau,1} = \Theta_1 - \frac{\tau\bar\omega_1 f'}{1+f'^2} + \tau a_\mu^1 a_{2,1}(X) + (1-\tau)a_\mu^1\Phi_1(X) + \tau a_\mu^1 B_{0,1}^0(f)[\bar\omega_1], \qquad \tau\in[0,1].$$

However, recalling the definition (7.11) of $\bar\omega$, the definitions of $a_{i,1}(X)$, $i=1$, 2, the definition (7.2) of $\Phi_1(X)$, and the formulas (6.2), (6.11), and (6.12), we derive the following relations

$$\begin{aligned}\bar\omega_1 &= \Theta_1 f' + a_\mu^1\big(f'B_{0,1}(f)[\bar\omega_1] - B_{1,1}^0(f)[\bar\omega_1] + f'a_{2,1}(X) - a_{1,1}(X)\big),\\ \Phi_1(X) &= B_{0,1}^0(f)[\bar\omega_1] + f'B_{1,1}^0(f)[\bar\omega_1] + f'a_{1,1}(X) + a_{2,1}(X).\end{aligned}$$

Replacing $\bar\omega_1$ in the second term of the formula for $\alpha_{\tau,1}$ by the expression found above, we get that $\alpha_{\tau,1}$ can be indeed expressed as in (7.31), by using the identity for $\Phi_1(X)$. This completes the proof. □

Theorem 7.8 is the essential step in the proof of Theorem 7.6 (i). With respect to Theorem 7.6 (ii), we provide in Theorem 7.10 below a similar approximation result. To start, we differentiate (7.2) with respect to h to arrive, in view of (6.2), at the formula

$$\partial_h\Phi_2(X)[v] = \partial_h\mathcal{B}_2(X)[v][\bar\omega] + \mathbb{B}(h)[\partial_h\bar\omega_2(X)[v]] + T'(X)[\partial_h\bar\omega_1(X)[v]] \tag{7.52}$$

for $v\in H^r(\mathbb{R})$, where, taking $(6.12)_3$ into account, we have

$$\partial_h\mathcal{B}_2(X)[v][\bar\omega] = \partial\mathbb{B}(h)[v][\bar\omega_2] + a_{1,2}(X)v' + T_{\mathrm{lot},2}^1[v], \tag{7.53}$$

with

$$\begin{aligned}&a_{1,2}(X) := -(c_\infty-h)C_1'(X)[\bar\omega_1] - C_1'(X)[f\bar\omega_1]\in H^1(\mathbb{R}),\\ &T_{\mathrm{lot},2}^1[v] := \partial_h D_1'(X)[v][\bar\omega_1] - (c_\infty-h)h'\partial_h C_1'(X)[v][\bar\omega_1] + vh'C_1'(X)[\bar\omega_1] - h'\partial_h C_1'(X)[v][f\bar\omega_1].\end{aligned}$$

The term $T_{\mathrm{lot},2}^1[v]$ is a sum of lower order terms since (7.17) and Lemma 4.5 imply

$$\|T_{\mathrm{lot},2}^1[v]\|_{H^{r-1}} \leq C\|v\|_{H^{r'}}. \tag{7.54}$$

We next differentiate the second component of (6.13) with respect to h, cf. (7.15) to obtain, in view of $(6.12)_2$, that

$$(1-a_\mu^2\mathbb{A}(h))[\partial_h\bar\omega_2(X)[v]] = \Theta_2 v' + a_\mu^2\big(\partial\mathbb{A}(h)[v][\bar\omega_2] + a_{2,2}(X)v' + T_{\mathrm{lot},2}^2[v]\big), \tag{7.55}$$

where

$$\begin{aligned}
a_{2,2}(X) &:= D_1'(X)[\bar{\omega}_1] \in H^1(\mathbb{R}),\\
T^2_{\mathrm{lot},2}[v] &:= S'(X)[\partial_h \bar{\omega}_1(h)[v]] + h'\partial_h D_1'(X)[v][\bar{\omega}_1] + (c_\infty - h)\partial_h C_1'(X)[v][\bar{\omega}_1]\\
&\qquad - vC_1'(X)[\bar{\omega}_1] + \partial_h C_1'(X)[v][f\bar{\omega}_1].
\end{aligned}$$

Also $T^2_{\mathrm{lot},2}[v]$ is a sum of lower order terms since (6.6), (7.17), (7.18), and Lemma 4.5 combined yield

$$\|T^2_{\mathrm{lot},2}[v]\|_{H^{r-1}} \le C\|v\|_{H^{r'}}. \tag{7.56}$$

We now consider the continuous path $\Psi_2\colon [0,1] \to \mathcal{L}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$ defined by

$$\Psi_2(\tau) := \tau\partial_h \mathcal{B}_2(X)[v][\bar{\omega}] + \mathbb{B}(\tau h)[w_2(\tau)[v]] + \tau T'(X)[\partial_h \bar{\omega}_1(X)[v]] \tag{7.57}$$

where $w_2\colon [0,1] \to \mathcal{L}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$ is the continuous path given by

$$\begin{aligned}
(1 - a_\mu^2 \mathbb{A}(\tau h))[w_2(\tau)[v]] = \Theta_2 v' + a_\mu^2\big(&\tau\partial\mathbb{A}(h)[v][\bar{\omega}_2] + \tau a_{2,2}(X)v'\\
&+ \tau T^2_{\mathrm{lot},2}[v] + (1-\tau)v'\Phi_2(X)\big).
\end{aligned}\tag{7.58}$$

With respect to the definitions (7.57) and (7.58) we include the following remarks.

Remark 7.9.

(i) If $\tau = 1$, then $w_2(1) = \partial_h \bar{\omega}_2(X)$ and $\Psi_2(1) = \partial_h \Phi_2(X)$.

(ii) Letting H denote the Hilbert transform, we have

$$w_2(0) = (\Theta_2 + a_\mu^2 \Phi_2(X))\frac{d}{dx} \qquad \text{and} \qquad \Psi_2(0) = H[w_2(0)].$$

It is worthwhile to point out that the term $(1-\tau)a_\mu^2 v'\Phi_2(X)$ in the definition (7.58) is a term introduced artificially and is very important for the following facts. When $\tau = 0$, we obtain due to this term a negative coefficient function – which is exactly the function from the Rayleigh–Taylor condition on Γ_h – for the differential operator $w_2(0)$. This aspect is important when establishing the invertibility of $\lambda - \partial_h \Phi_2(X)$ for sufficiently large λ, see the proof of Theorem 7.6. Besides, we localize the operator $\Psi_2(\tau)$ and show that it can locally approximated by certain Fourier multipliers, see Theorem 7.10 below. Thanks to this artificial term the Fourier multipliers have a coefficient which is the product of a positive function with the function from the Rayleigh–Taylor condition on Γ_h (both frozen at a certain point), see (7.60)-(7.61) below. These features enable us to show that the Fourier multipliers are generators of analytic semigroups and that they satisfy certain uniform estimates, see Lemma 5.8 and (7.63) below.

(iii) The properties (6.9) and (7.7) (both with $r = r'$) together with (7.56) and (7.58) imply that there exists a constant $C > 0$ such that

$$\|w_2(\tau)[v]\|_{H^{r'-1}} \le C\|v\|_{H^{r'}}, \qquad v \in H^{r'}(\mathbb{R}),\ \tau \in [0,1]. \tag{7.59}$$

As a further step we locally approximate in Theorem 7.10 below the operator $\Psi_2(\tau)$, $\tau \in [0,1]$, by certain Fourier multipliers $\mathbb{A}^2_{j,\tau}$, which are defined by

$$\begin{aligned}
\mathbb{A}^2_{j,\tau} &:= \mathbb{A}^{\varepsilon,2}_{j,\tau} := \alpha_{\tau,2}(x_j^\varepsilon)\Big(-\frac{d^2}{dx^2}\Big)^{1/2} + \beta_{\tau,2}(x_j^\varepsilon)\frac{d}{dx}, \quad |j| \le N-1,\\
\mathbb{A}^2_{N,\tau} &:= \mathbb{A}^{\varepsilon,2}_{N,\tau} := \Theta_2\Big(-\frac{d^2}{dx^2}\Big)^{1/2},
\end{aligned}\tag{7.60}$$

where

$$\alpha_{\tau,2} := \frac{1+(1-\tau)h'^2}{1+h'^2}(\Theta_2 + a_\mu^2\Phi_2(X)), \qquad \beta_{\tau,2} := \tau B_{1,1}^0(h)[\bar{\omega}_2] + \tau a_{1,2}(X) + \frac{\tau a_\mu^2 \bar{\omega}_2}{1+h'^2}. \quad (7.61)$$

The next results provides an estimate for the localization error and is the main step in the proof of Theorem 7.6 (ii).

Theorem 7.10. *Let $\nu > 0$ be given and fix $r' \in (3/2, r)$. Then, there exist $\varepsilon \in (0,1)$ and a positive constant $K = K(\varepsilon)$ such that*

$$\|\pi_j^\varepsilon \Psi_2(\tau)[v] - \mathbb{A}_{j,\tau}^2[\pi_j^\varepsilon v]\|_{H^{r-1}} \leq \nu\|\pi_j^\varepsilon v\|_{H^r} + K\|v\|_{H^{r'}} \quad (7.62)$$

for all $-N+1 \leq j \leq N$, $\tau \in [0,1]$, and $v \in H^r(\mathbb{R})$.

Proof. The proof follows by arguing along the same lines as in the proof of Theorem 7.8. □

We now consider the Fourier multipliers defined in (7.30) and (7.60) more closely. The Rayleigh–Taylor conditions, see (7.9), together with the fact that f', h' $\Phi_i(X)$, $\bar{\omega}_i$, $a_{1,i}(X)$, and $B_{1,1}^0(f)[\bar{\omega}_i]$ all belong to $H^{r-1}(\mathbb{R})$ implies there exists $\eta \in (0,1)$ such that the coefficient functions $\alpha_{\tau,i}$ and $\beta_{\tau,i}$ of the Fourier multipliers, cf. (7.31) and (7.61), satisfy

$$\eta \leq -\alpha_{\tau,i} \leq \frac{1}{\eta} \quad \text{and} \quad \|\beta_{\tau,i}\|_\infty \leq \frac{1}{\eta} \qquad \text{for all } \tau \in [0,1],\ i = 1,2. \quad (7.63)$$

Consequently, Lemma 5.8 together with our previous results enable us to obtain the desired generator property for $\partial_f\Phi_1(X)$ and $\partial_h\Phi_2(X)$.

Proof of Theorem 7.6. (i) Let κ_0 be the constant determined in Lemma 5.8 where η is identified in (7.63) and define $\nu := 1/2\kappa_0$. Choosing $r' \in (3/2, r)$, Theorem 7.8 ensures there exist a finite ε-localization family $\{\pi_j^\varepsilon : -N+1 \leq j \leq N\}$, with $\varepsilon \in (0,1)$, and a constant $K = K(\varepsilon, X)$ satisfying

$$2\kappa_0\|\pi_j^\varepsilon \Psi_1(\tau)[u] - \mathbb{A}_{j,\tau}^1[\pi_j^\varepsilon u]\|_{H^{r-1}} \leq \|\pi_j^\varepsilon u\|_{H^r} + 2\kappa_0 K\|u\|_{H^{r'}}$$

for all $\tau \in [0,1]$, $j \in \{-N+1, \ldots, N\}$, and $u \in H^r(\mathbb{R})$. Moreover, the estimate (5.20) implies that

$$2\kappa_0\|(\lambda - \mathbb{A}_{j,\tau}^1)[\pi_j^\varepsilon u]\|_{H^{r-1}} \geq 2|\lambda| \cdot \|\pi_j^\varepsilon u\|_{H^{r-1}} + 2\|\pi_j^\varepsilon u\|_{H^r}$$

for all $j \in \{-N+1, \ldots, N\}$, $\tau \in [0,1]$, $\operatorname{Re}\lambda \geq 1$, and $u \in H^r(\mathbb{R})$. This leads to

$$\begin{aligned} 2\kappa_0\|\pi_j^\varepsilon(\lambda - \Psi_1(\tau))[u]\|_{H^{r-1}} &\geq 2\kappa_0\|(\lambda - \mathbb{A}_{j,\tau}^1)[\pi_j^\varepsilon u]\|_{H^{r-1}} \\ &\quad - 2\kappa_0\|\pi_j^\varepsilon \Psi_1(\tau)[u] - \mathbb{A}_{j,\tau}^1[\pi_j^\varepsilon u]\|_{H^{r-1}} \\ &\geq 2|\lambda| \cdot \|\pi_j^\varepsilon u\|_{H^{r-1}} + \|\pi_j^\varepsilon u\|_{H^r} - 2\kappa_0 K\|u\|_{H^{r'}} \end{aligned}$$

for all $j \in \{-N+1, \ldots, N\}$, $\tau \in [0,1]$, $\operatorname{Re}\lambda \geq 1$, and $u \in H^r(\mathbb{R})$. Summing up over all indices $j \in \{-N+1, \ldots, N\}$, Lemma 5.4 and (5.8) (with $\tau = r'$) imply there exist constants $\kappa = \kappa(X) \geq 1$ and $\omega = \omega(X) \geq 1$ such that

$$\kappa\|(\lambda - \Psi_1(\tau))[u]\|_{H^{r-1}} \geq |\lambda| \cdot \|u\|_{H^{r-1}} + \|u\|_{H^r} \quad (7.64)$$

for all $\tau \in [0,1]$, $\operatorname{Re}\lambda \geq \omega$, and $u \in H^r(\mathbb{R})$.

The Rayleigh–Taylor condition (7.9) for the interface $\Gamma_f^{c_\infty}$, see Remark 7.7, ensures that the function $\Theta_1 + a_\mu^1\Phi_1(X)$ is negative and $\Phi_2(X) \in H^{r-1}(\mathbb{R})$. In view of [1, Proposition 1], we may therefore choose ω sufficiently large to ensure that

$$(\omega - \Psi_1(\tau)) \mid_{\tau=0} = \omega - H\left[(\Theta_1 + a_\mu^1\Phi_1(X))\frac{d}{dx}\right] \in \operatorname{Isom}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R})). \quad (7.65)$$

Thanks to (7.64) and (7.65), the method of continuity, see e.g. [7, Theorem I.1.1.1], yields

$$\omega - \partial_f \Phi_1(X) \in \mathrm{Isom}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R})). \tag{7.66}$$

The properties (7.64) (with $\tau = 1$) and (7.66) lead to $-\partial_f \Phi_1 \in \mathcal{H}(H^r(\mathbb{R}), H^{r-1}(\mathbb{R}))$, cf. (5.2).

(ii) The generator property for $\partial_h \Phi_2(X)$ follows, due to Theorem 7.10, (7.9), and Remark 7.9, by using similar arguments and therefore we omit its proof. □

7.3. The well-posedness result

We finally come to the proof of our main result which, on account of the abstract parabolic theory presented in [57, Chapter 8], is now obtained as a consequence of the smoothness property established in Corollary 7.3 and of the fact that the evolution problem (7.1) is parabolic in $\mathcal{V}_r$, cf. Theorem 7.4.

Proof of Theorem 2.4. Corollary 7.3 and Theorem 7.4 ensure that the assumptions of [57, Theorem 8.1.1] are all satisfied in the context of the evolution problem (7.1). Applying this theorem, we find for each $X \in \mathcal{V}_r$, a local solution $X(\cdot; X_0)$ to (7.1) such that

$$X \in \mathrm{C}([0,T], \mathcal{V}_r) \cap \mathrm{C}^1([0,T], H^{r-1}(\mathbb{R}^2) \cap \mathrm{C}^{\alpha}_{\alpha}((0,T], H^r(\mathbb{R})^2),$$

where $T > 0$ and $\alpha \in (0,1)$ is fixed (but arbitrary).This solution is unique within the set

$$\bigcup_{\beta \in (0,1)} \mathrm{C}^{\beta}_{\beta}((0,T], H^r(\mathbb{R})^2) \cap \mathrm{C}([0,T], \mathcal{V}_r) \cap \mathrm{C}^1([0,T], H^{r-1}(\mathbb{R})^2).$$

As stated in Theorem 2.4, the uniqueness holds true in $\mathrm{C}([0,T], \mathcal{V}_r) \cap \mathrm{C}^1([0,T], H^{r-1}(\mathbb{R})^2)$. Indeed, let $X \in \mathrm{C}([0,T], \mathcal{V}_r) \cap \mathrm{C}^1([0,T], H^{r-1}(\mathbb{R})^2)$ be a solution to (7.1), let $r' \in (3/2, r)$ be fixed, and set $\alpha := r - r' \in (0,1)$. It then holds

$$\|X(t_1) - X(t_2)\|_{H^{r'}} \leq \|X(t_1) - X(t_2)\|^{\alpha}_{H^{r-1}} \|X(t_1) - X(t_2)\|^{1-\alpha}_{H^r} \leq C|t_1 - t_2|^{\alpha}, \quad t_1,\, t_2 \in [0,T],$$

which shows in particular that $X \in \mathrm{C}^{\alpha}_{\alpha}((0,T], H^{r'}(\mathbb{R})^2)$. Applying the uniqueness statement of [57, Theorem 8.1.1] in the context of (7.1) with $\Phi \in \mathrm{C}^{\infty}(\mathcal{V}_{r'}, H^{r'-1}(\mathbb{R})^2)$, now implies the desired uniqueness assertion. This unique solution can be extended up to a maximal existence time $T^+(X_0)$, see [57, Section 8.2]. Moreover, it follows from Theorem 6.6 that the velocities and the pressures enjoy all the properties mentioned in Theorem 2.4.

(i) The continuous dependence of the solution on the initial data follows from [57, Proposition 8.2.3].

(ii) The parabolic smoothing properties follow by using the parameter trick also applied in the proof of Theorem 2.1. The details are identical to those in the proof of Theorem 2.1 and therefore we omit them.

□

Part III.

Appendices

A. An extension of Privalov's Theorem

In this chapter we first present two classical results of complex analysis, which we then extend to our setting. Classically, the theorem of Plemelj considers the smooth boundary of a bounded domain in the plane and describes the values of a function, defined on the interior and exterior of this boundary as a Cauchy-type integral, when it is extended to the boundary. Moreover, according to Privalov's theorem, this extension has even better regularity than only continuity provided that the original function fulfills an additional regularity assumption.

After recalling these results in Section A.1 we then extend them in Section A.2 to the setting of an unbounded graph with bounded slope that separates $\mathbb{R}^2$ in two regions.

A.1. Classical results of Plemelj and Privalov

We first recall the classical result of Plemelj, see e.g. [56]. Let therefore Ω_+ denote a bounded C^1-domain in the complex plane with boundary Γ, which is a closed, positively oriented, and continuously differentiable Jordan-curve with parametrization by arc length γ. Let further Ω_- be the exterior domain $\Omega_- := \mathbb{C} \setminus \overline{\Omega_+}$, see Figure A.1.

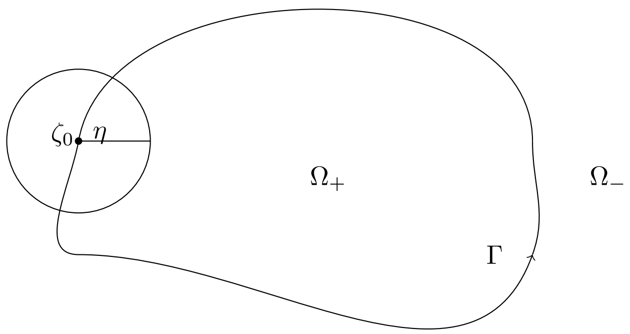

Figure A.1.: Notation

Given a continuous function $\varphi\colon \Gamma \to \mathbb{C}$, classical results show that the mapping $\Phi\colon \mathbb{C} \setminus \Gamma \to \mathbb{C}$ defined by

$$\Phi(z) := \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\zeta)}{\zeta - z} \, d\zeta, \tag{A.1}$$

is analytic in $\mathbb{C} \setminus \Gamma$. Under some additional assumptions on φ, one can extend the map Φ up to Γ. To this end we first introduce some notation: Let $\zeta_0 \in \Gamma$ be fixed and $\eta > 0$. Set Γ_η to be the part of Γ, which is within the ball of radius η around ζ_0, see Figure A.1. Given a function ψ which is integrable on $\Gamma \setminus \Gamma_\eta$ for all η small enough, the Cauchy principal value is defined by

$$\mathrm{PV} \int_\Gamma \psi(\zeta) \, d\zeta := \lim_{\eta \to 0} \int_{\Gamma - \Gamma_\eta} \psi(\zeta) \, d\zeta, \tag{A.2}$$

if the limit exists. Furthermore, given an integrable function $\Psi\colon \Gamma \to \mathbb{C}$, we define

$$\int_\Gamma \Psi(\zeta)\, |d\zeta| := \int_0^{|\Gamma|} \Psi(\gamma(s))|\gamma'(s)|\, ds.$$

Theorem A.1 below presents the Plemelj's formula which gives the values of

$$\lim_{\Omega_\pm \ni z \to \zeta_0} \Phi(z),$$

where $\zeta_0 \in \Gamma$ by means of a principal value integral.

Theorem A.1 (Plemelj's formula)**.** *Assume there exists* $\alpha \in (0,1)$ *such that* $\varphi \in \mathrm{BUC}^\alpha(\Gamma)$*, see* (1.3)*. Then, given* $\zeta_0 \in \Gamma$*, it holds*

$$\lim_{\Omega_\pm \ni z \to \zeta_0} \Phi(z) = \frac{1}{2\pi i} \mathrm{PV} \int_\Gamma \frac{\varphi(\zeta)}{\zeta - \zeta_0}\, d\zeta \pm \frac{1}{2}\varphi(\zeta_0). \tag{A.3}$$

Remark A.2. When Γ is only piecewise C^1 and continuous, the statement (A.3) is still true if Γ is C^1 in a neighborhood of ζ_0.

Under the assumptions of Theorem A.1, the function Φ is not only uniformly continuous on $\Omega_\pm$ but it is also Hölder-continuous with the same exponent as φ. This is also a classical result known as Privalov's Theorem, which can be found for example in [56].

Theorem A.3 (Privalov's Theorem)**.** *Assume there exists* $\alpha \in (0,1)$ *such that* $\varphi \in \mathrm{BUC}^\alpha(\Gamma)$*. Then, the function* Φ *defined in* (A.1) *belongs to* $\mathrm{BUC}^\alpha(\Omega_+)$ *and* $\mathrm{BUC}^\alpha(\Omega_-)$*.*

A.2. An extension to unbounded graphs

In the following we want to extend these results, cf. Theorem A.1 and Theorem A.3, to the setting of unbounded domains, which have boundaries parameterized as graphs. Here we consider a more general situation than in Part I and Part II, as the function f, which describes the (graph) boundary of the domains can actually be unbounded. In order to do so, let $p \in (1,\infty)$ and $\alpha \in (0,1)$, as well as $\bar\omega \in \mathrm{BUC}^\alpha(\mathbb{R}) \cap L_p(\mathbb{R})$ and a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ with $f' \in \mathrm{BUC}^\alpha(\mathbb{R})$ be fixed. We study the function

$$v := v(f)[\bar\omega]\colon \mathbb{R}^2 \setminus \Gamma_f \to \mathbb{R}^2$$

given by the formula

$$v(x,y) := v(f)[\bar\omega](x,y) := \frac{1}{2\pi} \int_\mathbb{R} \frac{(f(s)-y, x-s)}{(x-s)^2 + (y-f(s))^2} \bar\omega(s)\, ds, \tag{A.4}$$

where

$$\Gamma_f := \{(x, f(x)) : x \in \mathbb{R}\}.$$

Defining the function

$$\varphi(\zeta) := -\frac{\bar\omega(1,-f')}{1+f'^2}(s), \quad \zeta = (s, f(s)) \in \Gamma_f,$$

we find for $z = (x, y) \in \mathbb{R}^2 \setminus \Gamma_f$, in view of (A.1), that

$$\begin{aligned} v(z) &= \frac{i}{2\pi} \int_{\mathbb{R}} \frac{x - s + i(y - f(s))}{|x - s + i(y - f(s))|^2} \bar{\omega}(s)\, ds \\ &= -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\bar{\omega}(s)}{x - s - i(y - f(s))} \frac{1 - if'(s)}{1 + f'^2(s)} (1 + if'(s))\, ds \\ &= \overline{\frac{1}{2\pi i} \int_{\Gamma_f} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta} =: \overline{\Phi(z)}. \end{aligned} \tag{A.5}$$

Since f' and $\bar{\omega}$ both belong to $\mathrm{BUC}^{\alpha}(\mathbb{R})$, it holds that $\varphi \in \mathrm{BUC}^{\alpha}(\Gamma_f)$. Moreover, the function $\Phi\colon \mathbb{C} \setminus \Gamma_f \to \mathbb{C}$ defined in (A.5) is holomorphic. As a consequence, v is smooth in $\mathbb{C} \setminus \Gamma_f$.

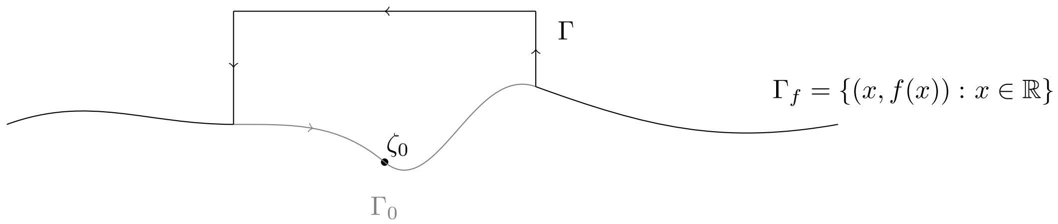

Figure A.2.: The setting in Lemma A.4

Lemma A.4. *Let*

$$\Omega_{\pm} := \{(x, y) \in \mathbb{R}^2 \,:\, \pm(y - f(x)) > 0\}.$$

The restrictions $v_{\pm} := v|_{\Omega_{\pm}} : \Omega_{\pm} \to \mathbb{R}^2$ *of the function* v *defined in* (A.4) *extend continuously up to* Γ_f *and, given* $x \in \mathbb{R}$, *we have*

$$v_{\pm}(x, f(x)) = \frac{1}{2\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{(f(s) - f(x), x - s)}{(x - s)^2 + (f(x) - f(s))^2} \bar{\omega}(s)\, ds \mp \frac{1}{2} \frac{\bar{\omega}(1, f')}{1 + f'^2}(x). \tag{A.6}$$

Proof. Let $\zeta_0 = (x_0, f(x_0)) \in \Gamma_f$ be fixed. First, we focus on the limit

$$\lim_{\Omega_+ \ni z \to \zeta_0} v_+(z).$$

In order to make use of Plemelj's formula, see (A.3), we choose a suitable curve Γ, such that $\zeta_0 \in \Gamma$ and Γ is as in Remark A.2, i.e. a positively oriented, piecewise C^1-Jordan-curve, in the following way. We consider

$$\Gamma_0 := \{(x, f(x)) : |x - x_0| \le 1\}.$$

Let further

$$D := 1 + 2\|f'\|_{\infty} + \max\{f(x_0 - 1), f(x_0 + 1)\}$$

and set $\Gamma_1 \subset \overline{\Omega_+}$ to be the polygonal path defined by the segments

$$\begin{aligned} &[(x_0 + 1, f(x_0 + 1)), (x_0 + 1, D)], \\ &[(x_0 + 1, D), (x_0 - 1, D)], \\ \text{and } &[(x_0 - 1, D), (x_0 - 1, f(x_0 - 1))], \end{aligned}$$

see Figure A.2, oriented counterclockwise. We define the curve $\Gamma := \Gamma_0 + \Gamma_1$, which is closed, piecewise C^1, and oriented counterclockwise.

Additionally, we define $\widetilde{\varphi}\colon \Gamma \to \mathbb{R}$ by

$$\widetilde{\varphi}(\xi) := \begin{cases} \varphi(\xi) & , \quad \xi \in \Gamma_0, \\ \varphi_+ & , \quad \xi = (x_0+1, y),\ f(x_0+1) \le y \le D, \\ \dfrac{(1+x_0-x)\varphi_- + (1+x-x_0)\varphi_+}{2} & , \quad \xi = (x, D),\ |x-x_0| \le 1, \\ \varphi_- & , \quad \xi = (x_0-1, y),\ f(x_0-1) \le y \le D, \end{cases}$$

where $\varphi_\pm := \varphi(x_0 \pm 1, f(x_0 \pm 1))$.

The function $\widetilde{\varphi}$ is obviously continuous and

$$\|\tilde{\varphi}\|_\infty \le \|\varphi\|_\infty \le \|\bar{\omega}\|_\infty. \tag{A.7}$$

In fact, it is not difficult to prove that $\varphi \in \mathrm{BUC}^\alpha(\Gamma)$ and

$$[\tilde{\varphi}]_\alpha \le 2\|\varphi\|_\infty + [\varphi]_\alpha \tag{A.8}$$

holds. Besides, we obtain by the definition that

$$|\Gamma| \le 7(1 + \|f'\|_\infty). \tag{A.9}$$

For $z \in \mathbb{C} \setminus \Gamma_f$ we then have

$$\begin{aligned} \Phi(z) &= \frac{1}{2\pi i} \int_{\Gamma_f} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i} \left(\int_\Gamma \frac{\tilde{\varphi}(\zeta)}{\zeta - z}\, d\zeta - \int_{\Gamma_1} \frac{\tilde{\varphi}(\zeta)}{\zeta - z}\, d\zeta + \int_{\Gamma_f - \Gamma_0} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta \right). \end{aligned}$$

Furthermore, taking the limit $\Omega_+ \ni z \to \zeta_0$, we get with the help of Lebesgue's dominated convergence and the fact that $\zeta_0 = \Gamma_0$ that

$$\lim_{\Omega_+ \ni z \to \zeta_0} \left[\int_{\Gamma_f - \Gamma_0} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta - \int_{\Gamma_1} \frac{\tilde{\varphi}(\zeta)}{\zeta - z}\, d\zeta \right] = \int_{\Gamma_f - \Gamma_0} \frac{\varphi(\zeta)}{\zeta - \zeta_0}\, d\zeta - \int_{\Gamma_1} \frac{\tilde{\varphi}(\zeta)}{\zeta - \zeta_0}\, d\zeta.$$

Remark A.2 further implies that

$$\lim_{\Omega_+ \ni z \to \zeta_0} \frac{1}{2\pi i} \int_\Gamma \frac{\tilde{\varphi}(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \mathrm{PV} \int_\Gamma \frac{\tilde{\varphi}(\zeta)}{\zeta - \zeta_0}\, d\zeta + \frac{1}{2}\varphi(\zeta_0),$$

which in conclusion leads us to

$$\begin{aligned} \lim_{\Omega_+ \ni z \to \zeta_0} \Phi(z) &= \frac{1}{2\pi i} \left(\mathrm{PV} \int_\Gamma \frac{\tilde{\varphi}(\zeta)}{\zeta - \zeta_0}\, d\zeta + \pi i \varphi(\zeta_0) - \int_{\Gamma_1} \frac{\tilde{\varphi}(\zeta)}{\zeta - \zeta_0}\, d\zeta + \int_{\Gamma_f - \Gamma_0} \frac{\varphi(\zeta)}{\zeta - \zeta_0}\, d\zeta \right) \\ &= \frac{1}{2\pi i} \left(\mathrm{PV} \int_{\Gamma_0} \frac{\varphi(\zeta)}{\zeta - \zeta_0}\, d\zeta + \int_{\Gamma_f - \Gamma_0} \frac{\varphi(\zeta)}{\zeta - \zeta_0}\, d\zeta \right) + \frac{1}{2}\varphi(\zeta_0) \\ &= \frac{1}{2\pi i} \mathrm{PV} \int_{\Gamma_f} \frac{\varphi(\zeta)}{\zeta - \zeta_0}\, d\zeta + \frac{1}{2}\varphi(\zeta_0). \end{aligned}$$

This convergence implies that v_+ can indeed be extended continuously in ζ_0 and the value of the extension in ζ_0 is given by formula (A.6).

Arguing similarly we obtain the claim also for v_-. □

We next prove that v, defined in (A.4), can be bounded in $\mathbb{R}^2 \setminus \Gamma_f$ by a constant depending explicitly on the norms of f and $\bar{\omega}$.

Lemma A.5. *There exists a constant C, which is independent of f and $\bar\omega$, such that*

$$\|v\|_\infty \le C(\|\bar\omega\|_p + \|\bar\omega\|_{\mathrm{BUC}^\alpha})(1+\|f'\|_{\mathrm{BUC}^\alpha})^2. \tag{A.10}$$

Proof. We divide the proof in three steps.

Step 1. In this step we provide bounds for the restrictions of $v_\pm$ to Γ_f. Given $x \in \mathbb{R}$, it follows from (A.6) and Hölder's inequality, after a change of variables, that

$$\begin{aligned}
|v_\pm(x,f(x))| &\le \Big|\mathrm{PV}\int_{\mathbb{R}} \frac{(f(x-s)-f(x),s)}{s^2+(f(x)-f(x-s))^2}\bar\omega(x-s)\,ds\Big| + \Big|\frac{\bar\omega(1,f')}{1+f'^2}(x)\Big| \\
&\le \Big|\int_{-1}^1 \frac{(f(x-s)-f(x),s)}{s^2+(f(x-s)-f(s))^2}(\bar\omega(x-s)-\bar\omega(x))\,ds\Big| \\
&\quad + \|\bar\omega\|_\infty \Big|\mathrm{PV}\int_{-1}^1 \frac{(f(x-s)-f(x),s)}{s^2+(f(x)-f(x-s))^2}\,ds\Big| \\
&\quad + \int_{\{|s|>1\}} \Big|\frac{\bar\omega(x-s)}{s}\Big|\,ds + \|\bar\omega\|_\infty \\
&\le \|\bar\omega\|_\infty + C\|\bar\omega\|_p + I_1 + \|\bar\omega\|_\infty I_2,
\end{aligned}$$

with

$$\begin{aligned}
I_1 &:= \Big|\int_{-1}^1 \frac{(f(x-s)-f(x),s)}{s^2+(f(x-s)-f(s))^2}(\bar\omega(x-s)-\bar\omega(x))\,ds\Big|, \\
I_2 &:= \Big|\mathrm{PV}\int_{-1}^1 \frac{(f(x-s)-f(x),s)}{s^2+(f(x)-f(x-s))^2}\,ds\Big|.
\end{aligned}$$

The term I_1 can be estimated as follows

$$\begin{aligned}
I_1 &\le [\bar\omega]_\alpha \int_{-1}^1 \frac{|(f(x-s)-f(x),s)|}{s^2+(f(x)-f(x-s))^2}|s|^\alpha\,ds \\
&\le [\bar\omega]_\alpha \int_{-1}^1 \frac{|s|^\alpha}{|(s,f(x)-f(x-s))|}\,ds \\
&\le [\bar\omega]_\alpha \int_{-1}^1 |s|^{\alpha-1}\,ds \le C[\bar\omega]_\alpha.
\end{aligned}$$

Concerning I_2, we have

$$\begin{aligned}
I_2 &\le \int_0^1 \Big|\frac{(f(x-s)-f(x),s)}{s^2+(f(x)-f(x-s))^2} + \frac{(f(x+s)-f(x),-s)}{s^2+(f(x)-f(x+s))^2}\Big|\,ds \\
&\le \int_0^1 \Big[\frac{s^2|f(x-s)-2f(x)+f(x+s)|}{(s^2+(f(x)-f(x-s))^2)(s^2+(f(x)-f(x+s))^2)} \\
&\qquad + \frac{(f(x-s)-f(x))(f(x)-f(x+s))^2+(f(x+s)-f(x))(f(x)-f(x-s))^2}{(s^2+(f(x)-f(x-s))^2)(s^2+(f(x)-f(x+s))^2)} \\
&\qquad + \frac{s|(f(x)-f(x+s))^2-(f(x)-f(x-s))^2|}{(s^2+(f(x)-f(x-s))^2)(s^2+(f(x)-f(x+s))^2)}\Big]\,ds \\
&\le 3\int_0^1 \frac{|f(x+s)-2f(x)+f(x-s)|}{s^2}\,ds \\
&\le 6[f']_\alpha \int_0^1 |s|^{\alpha-1}\,ds \\
&\le C[f']_\alpha.
\end{aligned}$$

To obtain the second last inequality we have used the mean value theorem to estimate

$$|f(x+s) - 2f(x) + f(x-s)| \leq s|f'(x_2) - f'(x_1)| \leq [f]_\alpha s^{\alpha+1},$$

where $x_1, x_2 \in \mathbb{R}$ satisfy $x - s < x_1 < x < x_2 < x + s$. Gathering these estimates we conclude that

$$\left\| v_\pm|_{\Gamma_f} \right\|_\infty \leq C(\|\bar{\omega}\|_p + \|\bar{\omega}\|_{\mathrm{BUC}^\alpha})(1 + \|f'\|_{\mathrm{BUC}^\alpha}). \tag{A.11}$$

Step 2. Given $z = (x, y) \in \mathbb{R}^2$, we set $d(z) := \operatorname{dist}(z, \Gamma_f)$. We next prove that

$$\sup_{\{1/4 \leq d(z)\}} |v_\pm(z)| \leq C\|\bar{\omega}\|_p. \tag{A.12}$$

Indeed, if $1/4 \leq d(z)$, then $\sqrt{s^2 + (y - f(x-s))^2} \geq \max\{1/4, |s|\}$ for all $s \in \mathbb{R}$ and together with Hölder's inequality we conclude from (A.4) that

$$\begin{aligned} |v(z)| &\leq \int_{\mathbb{R}} \frac{|(f(x-s) - y, s)|}{s^2 + (y - f(x-s))^2} |\bar{\omega}(x-s)| \, ds \\ &\leq \int_{\mathbb{R}} \frac{1}{\max\{1/4, |s|\}} |\bar{\omega}(x-s)| \, ds \\ &\leq 4 \int_{\{|s|<1\}} |\bar{\omega}(x-s)| \, ds + \int_{\{|s|>1\}} \left| \frac{\bar{\omega}(x-s)}{s} \right| ds \leq C\|\bar{\omega}\|_p. \end{aligned}$$

Step 3. In this final step we prove that

$$\sup_{\{0 < d(z) < 1/4\}} |v_\pm(z)| \leq C(\|\bar{\omega}\|_p + \|\bar{\omega}\|_{\mathrm{BUC}^\alpha})(1 + \|f'\|_{\mathrm{BUC}^\alpha})^2. \tag{A.13}$$

We first consider the case when $z \in \Omega_+$. We associate to z a point $z_\Gamma = (x_0, f(x_0)) \in \Gamma_f$ such that

$$d(z) = |z - z_\Gamma| \in (0, 1/4).$$

Let $\Gamma = \Gamma_0 + \Gamma_1$ and $\tilde{\varphi}$ be as defined in the proof of Lemma A.4 (with z_Γ instead of ζ_0). Recalling (A.11), Theorem A.1, and (A.5), we have

$$\begin{aligned} |v_+(z)| &\leq |v_+(z) - v_+(z_\Gamma)| + |v_+(z_\Gamma)| \\ &\leq \Big| \frac{1}{2\pi i} \int_{\Gamma_f} \frac{\varphi(\xi)}{\xi - z} \, d\xi - \frac{1}{2\pi i} \mathrm{PV} \int_{\Gamma_f} \frac{\varphi(\xi)}{\xi - z_\Gamma} \, d\xi - \frac{1}{2} \varphi(z_\Gamma) \Big| + |v_+(z_\Gamma)| \\ &\leq T_1 + T_2 + T_3 + C(\|\bar{\omega}\|_p + \|\bar{\omega}\|_{\mathrm{BUC}^\alpha})(1 + \|f'\|_{\mathrm{BUC}^\alpha}), \end{aligned}$$

where

$$\begin{aligned} T_1 &:= \Big| \int_{\Gamma_f - \Gamma_0} \Big(\frac{\varphi(\xi)}{\xi - z} - \frac{\varphi(\xi)}{\xi - z_\Gamma} \Big) \, d\xi \Big|, \\ T_2 &:= \Big| \int_{\Gamma_1} \Big(\frac{\tilde{\varphi}(\xi)}{\xi - z} - \frac{\tilde{\varphi}(\xi)}{\xi - z_\Gamma} \Big) \, d\xi \Big|, \\ T_3 &:= \Big| \frac{1}{2\pi i} \int_{\Gamma} \frac{\tilde{\varphi}(\xi)}{\xi - z} \, d\xi - \frac{1}{2\pi i} \mathrm{PV} \int_{\Gamma} \frac{\tilde{\varphi}(\xi)}{\xi - z_\Gamma} \, d\xi - \frac{1}{2} \varphi(z_\Gamma) \Big|. \end{aligned}$$

We have

$$\min\{|\xi - z|, \, |\xi - z_\Gamma|\} \geq |\xi - z_\Gamma| - 1/4 \geq 3/4, \quad \xi \in \Gamma_1, \tag{A.14}$$

and (A.7) together with (A.9) yields

$$
\begin{aligned}
T_2 &\le \|\varphi\|_\infty \int_{\Gamma_1} \frac{|z - z_\Gamma|}{|(\xi - z)(\xi - z_\Gamma)|}\, d\xi \\
&\le 2\|\varphi\|_\infty |\Gamma_1| \cdot |z - z_\Gamma| \\
&\le C\|\bar{\omega}\|_\infty (1 + \|f'\|_\infty).
\end{aligned}
$$

Moreover, since $\min\{|\xi - z|, |\xi - z_\Gamma|\} \ge \max\{3/4, |s - x_0|/2\}$ for all $\xi = (s, f(s)) \in \Gamma_f - \Gamma_0$, Hölder's inequality together with the estimate $|\varphi(\xi) \le |\bar{\omega}(s)|$ for $\xi \in \Gamma_f$ leads us to

$$
\begin{aligned}
T_1 &\le |z - z_\Gamma| \int_{\Gamma_f - \Gamma_0} \frac{|\varphi(\xi)|}{|\xi - z\|\xi - z_\Gamma|}\, |d\xi| \\
&\le \frac{8}{3}|z - z_\Gamma| \int_{\{|x_0 - s| > 1\}} \frac{|\bar{\omega}(s)|}{\max\{3/2, |x_0 - s|\}}\, ds \\
&\le \frac{8}{3}\|\bar{\omega}\|_p |z - z_\Gamma| \Big(\int_{\mathbb{R}} \frac{1}{\max\{3/2, |s|\}^{p'}}\, ds \Big)^{1/p'} \\
&\le C\|\bar{\omega}\|_p,
\end{aligned}
$$

where $p' \in (1, \infty)$ is the adjoint exponent to p, that is $p^{-1} + {p'}^{-1} = 1$.

In order to estimate T_3 we first note that

$$
\frac{1}{2\pi i} \int_\Gamma \frac{1}{\xi - z}\, d\xi = 1 \qquad \text{and} \qquad \frac{1}{2\pi i} \mathrm{PV} \int_\Gamma \frac{1}{\xi - z_\Gamma}\, d\xi = \frac{1}{2}. \tag{A.15}
$$

The first relation follows from Cauchy's integral formula. The second identity is a direct consequence of Plemelj's formula, cf. Theorem A.1. Using (A.15), we get

$$
\begin{aligned}
T_3 &= \Big| \frac{1}{2\pi i} \int_\Gamma \frac{\tilde{\varphi}(\xi) - \tilde{\varphi}(z_\Gamma)}{\xi - z}\, d\xi + \frac{\tilde{\varphi}(z_\Gamma)}{2\pi i} \int_\Gamma \frac{1}{\xi - z}\, d\xi \\
&\quad - \frac{1}{2\pi i} \int_\Gamma \frac{\tilde{\varphi}(\xi) - \tilde{\varphi}(z_\Gamma)}{\xi - z_\Gamma}\, d\xi - \frac{\tilde{\varphi}(z_\Gamma)}{2\pi i} \mathrm{PV} \int_\Gamma \frac{1}{\xi - z_\Gamma}\, d\xi - \frac{1}{2}\varphi(z_\Gamma) \Big| \\
&= \Big| \frac{1}{2\pi i} \int_\Gamma \frac{\tilde{\varphi}(\xi) - \tilde{\varphi}(z_\Gamma)}{\xi - z}\, d\xi - \frac{1}{2\pi i} \int_\Gamma \frac{\tilde{\varphi}(\xi) - \tilde{\varphi}(z_\Gamma)}{\xi - z_\Gamma}\, d\xi \Big| \\
&\le T_{3a} + T_{3b},
\end{aligned}
$$

where

$$
T_{3a} := \Big| \int_{\Gamma_1} \frac{(z - z_\Gamma)(\tilde{\varphi}(\xi) - \tilde{\varphi}(z_\Gamma))}{(\xi - z)(\xi - z_\Gamma)}\, d\xi \Big| \quad \text{and} \quad T_{3b} := \Big| \int_{\Gamma_0} \frac{(z - z_\Gamma)(\varphi(\xi) - \varphi(z_\Gamma))}{(\xi - z)(\xi - z_\Gamma)}\, d\xi \Big|.
$$

The estimates (A.7), (A.9), and (A.14) lead us to

$$
T_{3a} \le \frac{32}{9}|z - z_\Gamma\|\|\varphi\|_\infty |\Gamma_1| \le C\|\bar{\omega}\|_\infty (1 + \|f'\|_\infty).
$$

In order to estimate T_{3b} we note that for given $\xi = (s, f(s)) \in \Gamma_0$ it holds

$$
|\xi - z| \ge \min_{\xi \in \Gamma} |\xi - z| = d(z) = |z - z_\Gamma|,
$$

and therefore

$$\begin{aligned}
T_{3b} &\leq [\varphi]_\alpha \int_{\Gamma_0} \frac{|z-z_\Gamma|}{|\xi - z|} \frac{|\xi - z_\Gamma|^\alpha}{|\xi - z_\Gamma|} |d\xi| \\
&\leq (1+\|f'\|_\infty)[\varphi]_\alpha \int_{\{|x_0-s|<1\}} |(s,f(s))-z_\Gamma|^{\alpha-1}\, ds \\
&\leq (1+\|f'\|_\infty)[\varphi]_\alpha \int_{\{|s|<1\}} |s|^{\alpha-1}\, ds \\
&\leq C(1+\|f'\|_\infty)[\varphi]_\alpha.
\end{aligned}$$

Observing now, by an elementary computation, that

$$[\varphi]_\alpha \leq C\|\bar{\omega}\|_{\mathrm{BUC}^\alpha}(1+\|f'\|_{\mathrm{BUC}^\alpha}),$$

the latter arguments show that (A.13) holds for $z \in \Omega_+$. Arguing along the same lines it is easy to see that (A.13) is satisfied also for $z \in \Omega_-$. The claim (A.10) follows now from (A.11)-(A.13). □

We now extend Privalov's theorem A.3 to the setting considered herein, where the contour integral in (A.1) is defined over an unbounded graph in the plane.

Theorem A.6. *The restrictions $v_\pm := v|_{\Omega_\pm}$ of the function v defined in* (A.4) *satisfy*

$$v_\pm \in \mathrm{BUC}^\alpha(\Omega_\pm).$$

Proof. We first establish the Hölder continuity of $v_+ =: (v_+^1, v_+^2)$. We devise the proof in three main steps.

Step 1. Let $z,\ z' \in \overline{\Omega_+}$ satisfy $|z-z'| > 1/8$. Then, according to Lemma A.4 and Lemma A.5, we have

$$|v_+(z)-v_+(z')| \leq 2\|v\|_\infty \leq 16\|v\|_\infty |z-z'|^\alpha \leq C|z-z'|^\alpha.$$

Step 2. Given $z \in \mathbb{R}^2$, we set again $d(z) := \mathrm{dist}\,(z, \Gamma_f)$ and let the point $z_\Gamma \in \Gamma_f$ be defined by the relation $d(z) = |z - z_\Gamma|$. Assume now that $z,\ z' \in \Omega_+$ are chosen such that $|z-z'| \leq 1/8$. Then, letting

$$S_{zz'} := \{(1-t)z + tz' \,:\, t \in [0,1]\}$$

denote the direct line that connects z and z', there exists at least a point $\overline{\zeta} \in S_{zz'}$ such that

$$d(\overline{\zeta}) = |\overline{\zeta} - \overline{\zeta}_\Gamma| = \mathrm{dist}\,(S_{zz'}, \Gamma_f).$$

We distinguish two cases.

Step 2a. If $|z-z'| < |\overline{\zeta} - \overline{\zeta}_\Gamma|$, then $S_{zz'} \subset \Omega_+$. Then we have

$$|v_+(z)-v_+(z')| = |\Phi(z)-\Phi(z')| = \Big| \int_{S_{zz'}} \Phi'(\zeta)\, d\zeta \Big| = \Big| \int_{S_{zz'}} \Big(\frac{1}{2\pi i} \int_{\Gamma_f} \frac{\varphi(\xi)}{(\xi-\zeta)^2}\, d\xi \Big)\, d\zeta \Big|,$$

where Φ is the holomorphic function defined in (A.5). Given $\zeta \in S_{zz'}$, using Cauchy's integral theorem, we get

$$\begin{aligned}
\int_{\Gamma_f} \frac{1}{(\zeta-\xi)^2}\, d\xi &= \lim_{n\to\infty} \int_{\Gamma_f \cap B_n(0)} \frac{1}{(\zeta-\xi)^2}\, d\xi \\
&= \lim_{n\to\infty} \left(\int_{\Gamma_f \cap B_n(0)} \frac{1}{(\zeta-\xi)^2}\, d\xi + \int_{\Gamma_n} \frac{1}{(\zeta-\xi)^2}\, d\xi - \int_{\Gamma_n} \frac{1}{(\zeta-\xi)^2}\, d\xi \right) \\
&= -\lim_{n\to\infty} \int_{\Gamma_n} \frac{1}{(\zeta-\xi)^2}\, d\xi,
\end{aligned}$$

where $\Gamma_n = \partial B_n(0) \cap \overline{\Omega_-}$ is the part of the circle with radius n around the origin within $\overline{\Omega_-}$ oriented negatively. Then, for n sufficiently large we have $|\zeta - \xi| \le n/2$ for all $\xi \in \Gamma_\eta$ and therefore

$$\Big|\int_{\Gamma_n} \frac{1}{(\zeta-\xi)^2}\,d\xi\Big| \le \int_{\Gamma_n} \frac{1}{|\zeta-\xi|^2}\,d|\xi| \\ \le \pi n \frac{4}{n^2} = \frac{4\pi}{n}.$$

Consequently, we obtain

$$\int_{\Gamma_f} \frac{1}{(\zeta-\xi)^2}\,d\xi = 0.$$

Therewith we get

$$\begin{aligned} |v_+(z) - v_+(z')| &= \Big|\int_{S_{zz'}} \Big(\frac{1}{2\pi i}\int_{\Gamma_f} \frac{\varphi(\xi)-\varphi(\zeta_\Gamma)}{(\xi-\zeta)^2}\,d\xi\Big)\,d\zeta\Big| \\ &\le |z-z'| \sup_{\zeta\in S_{zz'}} \Big|\int_{\Gamma_f} \frac{\varphi(\xi)-\varphi(\zeta_\Gamma)}{(\xi-\zeta)^2}\,d\xi\Big| \\ &\le |z-z'|\cdot[\varphi]_\alpha \sup_{\zeta\in S_{zz'}} \int_{\Gamma_f} \frac{|\xi-\zeta_\Gamma|^\alpha}{|\xi-\zeta|^2}\,|d\xi|. \end{aligned}$$

Recalling the definition of ζ_Γ, we have $|\xi-\zeta_\Gamma| \le |\xi-\zeta| + |\zeta-\zeta_\Gamma| \le 2|\xi-\zeta|$ for all $\zeta \in S_{zz'}$ and $\xi \in \Gamma_f$, hence $|\xi-\zeta_\Gamma| + |\zeta-\zeta_\Gamma| \le 3|\xi-\zeta|$. Noticing also that $|z-z'| < |\zeta-\zeta_\Gamma|$ for $\zeta \in S_{zz'}$, we obtain in view of these inequalities

$$\begin{aligned} \int_{\Gamma_f} \frac{|\xi-\zeta_\Gamma|^\alpha}{|\xi-\zeta|^2}\,|d\xi| &\le 9\int_{\Gamma_f} \frac{|\xi-\zeta_\Gamma|^\alpha}{(|\xi-\zeta_\Gamma|+|\zeta-\zeta_\Gamma|)^2}\,|d\xi| \\ &\le 9\int_{\Gamma_f} (|\xi-\zeta_\Gamma|+|\zeta-\zeta_\Gamma|)^{\alpha-2}\,|d\xi| \\ &\le 9(1+\|f'\|_\infty)\int_{\mathbb{R}} (|s|+|\zeta-\zeta_\Gamma|)^{\alpha-2}\,ds \\ &\le C|\zeta-\zeta_\Gamma|^{\alpha-1} \le C|z-z'|^{\alpha-1}, \end{aligned}$$

and therefore

$$|v_+(z) - v_+(z')| \le C|z-z'|^\alpha.$$

Step 2b. We now consider the second case when $|z-z'| \ge |\overline{\zeta} - \overline{\zeta}_\Gamma|$. Since $\overline{\zeta} \in S_{zz'}$ we have

$$\max\{|z-\overline{\zeta}_\Gamma|,\, |z'-\overline{\zeta}_\Gamma|\} \le \max\{|z-\overline{\zeta}|,\, |z'-\overline{\zeta}|\} + |\overline{\zeta}-\overline{\zeta}_\Gamma| \le 2|z-z'| \le \frac{1}{4}.$$

Assuming there exits a constant $C > 0$ such that

$$|v_+(z) - v_+(z_0)| \le C|z-z_0|^\alpha \qquad \forall\, z_0 \in \Gamma_f \text{ and } z \in \overline{\Omega_+} \text{ with } |z-z_0| \le 1/4, \tag{A.16}$$

we then have

$$|v_+(z)-v_+(z')| \le |v_+(z)-v_+(\overline{\zeta}_\Gamma)| + |v_+(z')-v_+(\overline{\zeta}_\Gamma)| \le C\big(|z-\overline{\zeta}_\Gamma|^\alpha + |z'-\overline{\zeta}_\Gamma|^\alpha\big) \le C|z-z'|^\alpha,$$

and the claim then follows.

Step 3. It remains to establish (A.16). Let $z_0 \in \Gamma_f$ and $z \in \Omega_+$ satisfy $|z-z_0| \le 1/4$, and let $\Gamma = \Gamma_0 + \Gamma_1$ and $\tilde{\varphi}$ be as defined in the proof of Lemma A.4. Recalling Theorem A.1, it follows similarly as in Step 3 of the proof of Lemma A.5 that

$$\begin{aligned}
|v_+(z) - v_+(z_0)| &\leq \Big| \int_{\Gamma_f - \Gamma_0} \Big(\frac{\varphi(\xi)}{\xi - z} - \frac{\varphi(\xi)}{\xi - z_0} \Big) \, d\xi \Big| + \Big| \int_{\Gamma_1} \Big(\frac{\tilde{\varphi}(\xi)}{\xi - z} - \frac{\tilde{\varphi}(\xi)}{\xi - z_0} \Big) \, d\xi \Big| \\
&\quad + \Big| \frac{1}{2\pi i} \int_\Gamma \frac{\tilde{\varphi}(\xi)}{\xi - z} \, d\xi - \frac{1}{2\pi i} \mathrm{PV} \int_\Gamma \frac{\tilde{\varphi}(\xi)}{\xi - z_0} \, d\xi - \frac{1}{2} \varphi(z_0) \Big| \\
&=: \sum_{i=1}^{3} T_i,
\end{aligned}$$

with

$$T_1 + T_2 \leq C|z - z_0| \leq C|z - z_0|^\alpha.$$

It remains to estimate the term T_3 which, in view of (A.15), can be written as

$$\begin{aligned}
T_3(z) &:= \Big| \frac{1}{2\pi i} \int_\Gamma \frac{\tilde{\varphi}(\xi)}{\xi - z} \, d\xi - \frac{1}{2\pi i} \mathrm{PV} \int_\Gamma \frac{\tilde{\varphi}(\xi)}{\xi - z_0} \, d\xi - \frac{1}{2} \varphi(z_0) \Big| \\
&= \Big| \frac{1}{2\pi i} \int_\Gamma \frac{\tilde{\varphi}(\xi) - \tilde{\varphi}(z_\Gamma)}{\xi - z} \, d\xi - \frac{1}{2\pi i} \int_\Gamma \frac{\tilde{\varphi}(\xi) - \tilde{\varphi}(z_0)}{\xi - z_0} \, d\xi \\
&\quad - \frac{\varphi(z_0)}{2\pi i} \mathrm{PV} \int_\Gamma \frac{1}{\xi - z_0} \, d\xi + \frac{\varphi(z_\Gamma)}{2\pi i} \int_\Gamma \frac{1}{\xi - z} \, d\xi - \frac{1}{2} \varphi(z_0) \Big| \\
&\leq \sum_{i=1}^{3} S_i,
\end{aligned}$$

where we set

$$\begin{aligned}
S_1 &:= |\varphi(z_\Gamma) - \varphi(z_0)|, \\
S_2 &:= \Big| \int_{\Gamma_1} \Big(\frac{\tilde{\varphi}(\xi) - \tilde{\varphi}(z_\Gamma)}{\xi - z} - \frac{\tilde{\varphi}(\xi) - \tilde{\varphi}(z_0)}{\xi - z_0} \Big) \, d\xi \Big|, \\
S_3 &:= \Big| \int_{\Gamma_0} \Big(\frac{\varphi(\xi) - \varphi(z_\Gamma)}{\xi - z} - \frac{\varphi(\xi) - \varphi(z_0)}{\xi - z_0} \Big) \, d\xi \Big|.
\end{aligned}$$

Noticing that $|z_\Gamma - z_0| \leq |z_\Gamma - z| + |z - z_0| \leq 2|z - z_0|$, we obtain

$$S_1 \leq [\varphi]_\alpha |z_\Gamma - z_0|^\alpha \leq C|z - z_0|^\alpha.$$

Moreover, given $\xi \in \Gamma_1$, we have $\min\{|\xi - z|, |\xi - z_0|\} \geq 3/4$ and together with (A.7) and (A.9) we get

$$\begin{aligned}
S_2 &\leq \int_{\Gamma_1} \Big| \frac{(\tilde{\varphi}(\xi) - \tilde{\varphi}(z_\Gamma))(\xi - z_0) - (\tilde{\varphi}(\xi) - \tilde{\varphi}(z_0))(\xi - z) + \tilde{\varphi}(z_\Gamma) z - \tilde{\varphi}(z_\Gamma) z}{(\xi - z)(\xi - z_0)} \Big| \, |d\xi| \\
&\leq \int_{\Gamma_1} \Big(\Big| \frac{(\tilde{\varphi}(\xi) - \tilde{\varphi}(z_\Gamma))(z - z_0)}{(\xi - z)(\xi - z_0)} \Big| + \Big| \frac{(\tilde{\varphi}(z_0) - \tilde{\varphi}(z_\Gamma))}{(\xi - z_0)} \Big| \Big) \, |d\xi| \\
&\leq \frac{16}{9} \int_{\Gamma_1} (|\tilde{\varphi}(\xi) - \tilde{\varphi}(z_\Gamma)| \cdot |z - z_0| + |\varphi(z_0) - \varphi(z_\Gamma)|) \, |d\xi| \\
&\leq C|\Gamma_1| (\|\varphi\|_\infty |z - z_0| + [\varphi]_\alpha |z_0 - z_\Gamma|^\alpha) \\
&\leq C|z - z_0|^\alpha.
\end{aligned}$$

In order to estimate S_3 we let $\eta := |z - z_0| \in (0, 1/4]$, we set $z_0 =: (x_0, f(x_0))$, and we introduce the curve $\Gamma_\eta \coloneqq \{(s, f(s)) \,:\, |s - x_0| \le 2\eta\}$. It then holds

$$S_3 \le S_{3a} + S_{3b} + S_{3c},$$

where

$$\begin{aligned}
S_{3a} &\coloneqq \Big| \int_{\Gamma_\eta} \Big(\frac{\varphi(\xi) - \varphi(z_\Gamma)}{\xi - z} - \frac{\varphi(\xi) - \varphi(z_0)}{\xi - z_0} \Big)\, d\xi \Big|, \\
S_{3b} &\coloneqq |z - z_0| \cdot \Big| \int_{\Gamma_0 - \Gamma_\eta} \frac{\varphi(\xi) - \varphi(z_\Gamma)}{(\xi - z)(\xi - z_0)}\, d\xi \Big|, \\
S_{3c} &\coloneqq \Big| \int_{\Gamma_0 - \Gamma_\eta} \frac{\varphi(z_0) - \varphi(z_\Gamma)}{\xi - z_0}\, d\xi \Big|.
\end{aligned}$$

The relation $|z - z_\Gamma| \le |z - z_0| = \eta$ implies that $z_\Gamma \in \Gamma_\eta$. Taking also into account that

$$|\xi - z_\Gamma| \le |\xi - z| + |z - z_\Gamma| \le 2|\xi - z| \qquad \text{for } \xi \in \Gamma_f,$$

we have

$$\begin{aligned}
S_{3a} &\le 2 \int_{\Gamma_\eta} \frac{[\varphi]_\alpha |\xi - z_\Gamma|^\alpha}{|\xi - z_\Gamma|} + [\varphi]_\alpha |\xi - z_0|^{\alpha - 1}\, |d\xi| \\
&\le 2[\varphi]_\alpha \Big(\int_{\Gamma_\eta} |\xi - z_\Gamma|^{\alpha-1}\, |d\xi| + \int_{\Gamma_\eta} |\xi - z_0|^{\alpha - 1}\, |d\xi| \Big) \\
&\le C[\varphi]_\alpha (1 + \|f'\|_\infty) \eta^\alpha \\
&\le C|z - z_0|^\alpha.
\end{aligned}$$

Given $\xi \in \Gamma_0 - \Gamma_\eta$, the relation $|\xi - z_0| \ge 2\eta = 2|z - z_0|$ leads us to

$$\begin{aligned}
|\xi - z| &\ge |\xi - z_0| - |z - z_0| \ge \eta = |z - z_0|, \\
2|\xi - z_0| &\ge |\xi - z_0| + |z_0 - z| \ge |\xi - z|, \\
3|\xi - z| &\ge |\xi - z_0| - |z - z_0| + 2|z - z_0| = |\xi - z_0| + |z - z_0|.
\end{aligned}$$

Recalling also that $|\xi - z_\Gamma| \le 2|\xi - z|$, we then obtain

$$\begin{aligned}
S_{3b} &\le 4|z - z_0| \cdot [\varphi]_\alpha \int_{\Gamma_0 - \Gamma_\eta} |\xi - z|^{\alpha - 2}\, |d\xi| \\
&\le C|z - z_0| \cdot [\varphi]_\alpha \int_{\Gamma_0 - \Gamma_\eta} (|\xi - z_0| + |z - z_0|)^{\alpha - 2}\, |d\xi| \\
&\le C[\varphi]_\alpha (1 + \|f'\|_\infty) |z - z_0| (2\eta + |z - z_0|)^{\alpha - 1} \\
&\le C|z - z_0|^\alpha.
\end{aligned}$$

Finally, since $|z_0 - z_\Gamma| \le 2|z - z_0|$, we have

$$S_{3c} \le [\varphi]_\alpha |z_0 - z_\Gamma|^\alpha \Big| \int_{\Gamma_0 - \Gamma_\eta} \frac{1}{\xi - z_0}\, d\xi \Big| \le C|z_0 - z|^\alpha \Big| \int_{\Gamma_0 - \Gamma_\eta} \frac{1}{\xi - z_0}\, d\xi \Big|$$

and, after identifying the real and imaginary parts of the integral, we get

$$\begin{aligned}
\Big|\int_{\Gamma_0-\Gamma_\eta}\frac{1}{\xi-z_0}\,d\xi\Big| &= \Big|\int_{\{2\eta\le|s|\le1\}}\frac{1+if'(x_0+s)}{s+i(f(x_0+s)-f(x_0))}\,ds\Big|\\
&\le \Big|\int_{\{2\eta\le|s|\le1\}}\frac{s+f'(x_0-s)(f(x_0)-f(x_0-s))}{s^2+(f(x_0)-f(x_0-s)^2}\,ds\Big|\\
&\quad+\Big|\int_{\{2\eta\le|s|\le1\}}\frac{sf'(x_0-s)-(f(x_0)-f(x_0-s))}{s^2+(f(x_0)-f(x_0-s)^2}\,ds\Big|\\
&\le \frac{1}{2}\Big|\int_{\{2\eta\le|s|\le1\}}\frac{d}{ds}\Big(\ln(s^2+(f(x_0)-f(x_0-s))^2)\Big)\,ds\Big|\\
&\quad+[f']_\alpha\int_{\{2\eta\le|s|\le1\}}|s|^{\alpha-1}\,ds\Big|\\
&\le \Big|\ln\Big(\frac{(2\eta)^2+(f(x_0)-f(x_0+2\eta))^2}{(2\eta)^2+(f(x_0)-f(x_0-2\eta))^2}\cdot\frac{1+(f(x_0)-f(x_0-1))^2}{1+(f(x_0)-f(x_0+1))^2}\Big)\Big|+C[f']_\alpha\\
&\le 2\ln(1+\|f'\|_\infty^2)+C[f']_\alpha.
\end{aligned}$$

In the second last inequality we have used the mean value theorem to estimate

$$|sf'(x_0-s)-f(x_0)+f(x_0-s)|\le|s|\cdot|f'(x_0-s)-f'(x^*)|\le|s|^{\alpha+1}[f']_\alpha,$$

where x^* lies between x_0 and x_0-s. Hence, we have shown that $S_{3c}\le C|z_0-z|^\alpha$ and the proof is completed for v_+.

The claim for v_- follows with similar arguments. □

We proceed with the following result.

Lemma A.7. *It holds that*

$$\partial_x v^1+\partial_y v^2=0=\partial_y v^1-\partial_x v^2\qquad \text{in } \mathbb{R}^2\setminus\Gamma_f \tag{A.17}$$

and

$$v_\pm(z)\to 0\qquad \text{for } z\in\Omega_\pm \text{ with } |z|\to\infty. \tag{A.18}$$

Proof. The relations (A.17) follow by direct computation:

$$\begin{aligned}
\partial_x v^1(x,y) &= -\frac{1}{2\pi}\int_{\mathbb{R}}\frac{2(x-s)(f(s)-y)}{((x-s)^2+(y-f(s))^2)^2}\bar\omega(s)\,ds=-\partial_y v^2(x,y)\\
\partial_y v^1(x,y) &= \frac{1}{2\pi}\int_{\mathbb{R}}\frac{(y-f(s))^2-(x-s)^2}{((x-s)^2+(y-f(s))^2)^2}\bar\omega(s)\,ds=\partial_x v^2(x,y)
\end{aligned}$$

for all $(x,y)\in\mathbb{R}^2\setminus\Gamma_f$.

We next prove that v_+ vanishes at infinity (the claim for v_- follows by arguing along the same lines). We divide the proof in two steps.

Step 1. We first show that $v_+(x,f(x))\to 0$ for $|x|\to\infty$. Recalling Lemma A.4 and using the notation introduced in (B.16), we write

$$\begin{aligned}
v_+(x,f(x)) &= \frac{1}{2\pi}\mathrm{PV}\int_{\mathbb{R}}\frac{(f(x-s)-f(x),s)}{s^2+(f(x)-f(x-s))^2}\bar\omega(x-s)\,ds-\frac{1}{2}\frac{\bar\omega(1,f')}{1+f'^2}(x)\\
&= \frac{1}{2}\big(-B^0_{1,1}(f)[\bar\omega],B^0_{0,1}(f)[\bar\omega]\big)(x)-\frac{1}{2}\frac{\bar\omega(1,f')}{1+f'^2}(x),\qquad x\in\mathbb{R}.
\end{aligned}$$

Because $\bar{\omega} \in \mathrm{BUC}^\alpha(\mathbb{R}) \cap L_p(\mathbb{R})$, we have that $\bar{\omega}$ vanishes at infinity and thus

$$\Big| -\frac{1}{2}\frac{\bar{\omega}(1,f')}{1+f'^2}(x)\Big| \le |\bar{\omega}(x)| \to 0 \qquad \text{for } |x|\to\infty.$$

We next prove that, given $n,\, m \in \mathbb{N}$, we also have

$$B^0_{n,m}(f)[\bar{\omega}](x) \to 0 \qquad \text{for } |x|\to\infty. \tag{A.19}$$

Let thus $\varepsilon > 0$ be given and choose $N > 0$ such that

$$\|f'\|_\infty^n \|\bar{\omega}\|_p \Big(\frac{2}{(p'-1)N^{p'-1}}\Big)^{1/p'} \le \frac{\varepsilon}{2},$$

where p' is the adjoint exponent to p. This choice together with Hölder's inequality then yields

$$\begin{aligned}
|B^0_{n,m}(f)[\bar{\omega}](x)| &\le \Big|\mathrm{PV}\int_{\mathbb{R}} \frac{\big(\delta_{[x,s]}f/s\big)^n}{\big[1+\big(\delta_{[x,s]}f/s\big)^2\big]^m}\frac{\bar{\omega}(x-s)}{s}\,ds\Big| \\
&\le \Big|\mathrm{PV}\int_{\{|s|\le N\}} \frac{\big(\delta_{[x,s]}f/s\big)^n}{\big[1+\big(\delta_{[x,s]}f/s\big)^2\big]^m}\frac{\bar{\omega}(x-s)}{s}\,ds\Big| \\
&\quad + \int_{\{|s|>N\}} |\delta_{[x,s]}f/s|^n \Big|\frac{\bar{\omega}(x-s)}{s}\Big|\,ds \\
&\le T(x) + \|f'\|_\infty^n\|\bar{\omega}\|_p\Big(\frac{2}{(p'-1)N^{p'-1}}\Big)^{1/p'} \le T(x)+\frac{\varepsilon}{2},
\end{aligned}$$

where

$$T(x) := \Big|\mathrm{PV}\int_{\{|s|\le N\}} \frac{\big(\delta_{[x,s]}f/s\big)^n}{\big[1+\big(\delta_{[x,s]}f/s\big)^2\big]^m}\frac{\bar{\omega}(x-s)}{s}\,ds\Big|, \qquad x\in\mathbb{R}.$$

In order to estimate $T(x)$ we note that

$$\begin{aligned}
T(x) &\le \int_0^N \Big|\frac{\big(\delta_{[x,s]}f/s\big)^n}{\big[1+\big(\delta_{[x,s]}f/s\big)^2\big]^m}\frac{\bar{\omega}(x-s)}{s} + \frac{\big(\delta_{[x,-s]}f/s\big)^n}{\big[1+\big(\delta_{[x,-s]}f/s\big)^2\big]^m}\frac{\bar{\omega}(x+s)}{s}\Big|\,ds \\
&\le C\int_0^N \Big|\frac{\bar{\omega}(x-s)-\bar{\omega}(x+s)}{s}\Big| + |\bar{\omega}(x+s)|\cdot\Big|\frac{f(x+s)-2f(x)+f(x-s)}{s^2}\Big|\,ds,
\end{aligned}$$

with C depending only on n, m, and $\|f'\|_\infty$. Taking into account that $\bar{\omega}$ vanishes at infinity we obtain for $|x| > M$, where $M > N$ is chosen sufficiently large, that

$$\begin{aligned}
T(x) &\le C\Big([\bar{\omega}]_\alpha^{1/2}\int_0^N s^{\alpha/2-1}(\bar{\omega}(x-s)-\bar{\omega}(x+s))^{1/2}\,ds + [f']_\alpha\int_0^N |\bar{\omega}(x+s)|s^{\alpha-1}\,ds\Big) \\
&\le C\big([\bar{\omega}]_\alpha^{1/2}\|\bar{\omega}\|^{1/2}_{L_\infty(\{|x|>M-N\})} + \|\bar{\omega}\|_{L_\infty(\{|x|>M-N\})}\big) \\
&\le \frac{\varepsilon}{2}.
\end{aligned}$$

This establishes (A.19).

Step 2. We now prove that $v_+(z) \to 0$ for $|z|\to\infty$. Let thus $\varepsilon > 0$ be given. From Step 1 we find $x_0 > 0$ such that $|v_+(x,f(x))| \le \varepsilon/2$ for all $|x| \ge x_0$. Given $z=(x,y)\in\Omega_+$, let again $d(z) := \mathrm{dist}\,(z,\Gamma_f) = |z-z_\Gamma|$ with $z_\Gamma \in \Gamma_f$.

Assume first that

$$d(z) \le \delta := \min\{1, \varepsilon/(2(1+[v_+]_\alpha))\}.$$

Let $x_1 := x_0+1$. If $z=(x,y)\in\Omega_+$ satisfies $d(z)\le\delta$ and $|x|\ge x_1$, we may deduce for the corresponding point $z_\Gamma := (x_\Gamma, f(x_\Gamma))$ on Γ_f that $|x_\Gamma| \geq x_0$. Hence, for all such $z \in \Omega_+$, Theorem A.3 leads us to

$$|v_+(z)| = |v_+(z) - v_+(z_\Gamma)| + |v_+(z_\Gamma)| \leq [v_+]_\alpha d(z) + \varepsilon/2 < \varepsilon.$$

Assume now that $d(z) > \delta$. Let $s_0 > 0$ be chosen such that

$$\|\bar{\omega}\|_p \Big(\frac{2}{(p'-1)s_0^{p'-1}} \Big)^{1/p'} \leq \frac{\varepsilon}{2}.$$

It then holds by Hölder's inequality

$$\begin{aligned}
|v_+(z)| &\leq \Big| \int \frac{(f(x-s)-y, s)}{s^2 + (f(x-s)-y)^2} \bar{\omega}(x-s)\, ds \Big| \\
&\leq \int_{\mathbb{R}} \frac{|\bar{\omega}(x-s)|}{\sqrt{s^2 + (y - f(x-s))^2}}\, ds \\
&\leq T(z) + \int_{\{|s|>s_0\}} \frac{|\bar{\omega}(x-s)|}{|s|}\, ds \\
&\leq T(z) + \|\bar{\omega}\|_p \Big(\frac{2}{(p'-1)s_0^{p'-1}} \Big)^{1/p'} \\
&\leq T(z) + \frac{\varepsilon}{2},
\end{aligned}$$

where

$$T(z) := \int_{\{|s|<s_0\}} \frac{|\bar{\omega}(x-s)|}{\sqrt{s^2 + (y - f(x-s))^2}}\, ds, \qquad z = (x,y) \in \Omega_+,\ d(z) > \delta.$$

Let $N > 0$ be chosen such that

$$\frac{4s_0 \|\bar{\omega}\|_\infty}{N} + \frac{2s_0 \|\bar{\omega}\|_{L_\infty(\{|x|\geq N\})}}{\delta} \leq \frac{\varepsilon}{2}$$

and set $M_1 := N + s_0$, $M_2 := N + 2\|f\|_{L_\infty(\{|x|\leq M_1+s_0\})}$, and $M := 2\max\{M_1,\, M_2\}$.

Given $|z| \geq M$, we distinguish two cases.

Step 2a. If $|x| \geq M_1$, then

$$\begin{aligned}
T(z) &\leq \|\bar{\omega}\|_{L_\infty(\{|x|\geq M_1 - s_0\})} \int_{\{|s|<s_0\}} \frac{1}{|y - f(x-s)|}\, ds \\
&\leq \frac{2s_0}{\delta} \|\bar{\omega}\|_{L_\infty(\{|x|\geq M_1 - s_0\})} = \frac{2s_0}{\delta} \|\bar{\omega}\|_{L_\infty(\{|x|\geq N\})} \leq \frac{\varepsilon}{2}.
\end{aligned}$$

Step 2b. If $|x_1| \leq M_1$ and $|y| \geq M_2$, then $|y - f(x-s)| \geq |y/2|$ and therefore

$$T(z) \leq \frac{4s_0}{|y|} \|\bar{\omega}\|_\infty \leq \frac{4s_0}{N} \|\bar{\omega}\|_\infty \leq \frac{\varepsilon}{2}.$$

Hence $|v_+(z)| \leq \varepsilon$ for all $z \in \Omega_+$ that satisfy $d(z) \geq \delta$ and $|z| \geq M$.

To summarize, for all $z \in \Omega_+$ with $|z| \geq \max\{M, x_1 + \|f\|_{L_\infty(\{|x|\leq x_1+1\})} + 1\}$ we have established that $|v_+(z)| \leq \varepsilon$ and this completes the proof. □

We conclude this section with the following result, which we use in Section 6.2 in the proof of Theorem 6.4.

Lemma A.8. *Let $f \in L_\infty(\mathbb{R})$ and $\bar{\omega} \in \mathrm{C}_0^\infty(\mathbb{R})$. Then, there exists a constant $C > 0$ such that the function v defined in* (A.4) *satisfies*

$$|v(z)| \leq \frac{C}{|z|} \quad \textit{for } |z| \to \infty.$$

Proof. Choose $N \in \mathbb{N}$ such that $\operatorname{supp} \bar{\omega} \subset [-N, N]$. Given $z = (x, y) \in \mathbb{C} \setminus \Gamma_f$, it holds that

$$\begin{aligned} |v(x,y)| &\leq \int_{\mathbb{R}} \left| \frac{(f(s) - y, x - s)}{(x-s)^2 + (y - f(s))^2} \bar{\omega}(s) \right| ds \\ &\leq \int_{-N}^{N} \frac{|\bar{\omega}(s)|}{|(x-s)^2 + (y - f(s))^2|^{1/2}} \, ds. \end{aligned}$$

Set $M \coloneqq N + \|f\|_\infty$. Given $z = (x, y) \in \mathbb{C} \setminus \Gamma_f$ with $|z| \geq 4M$, we distinguish two cases:

Case 1: If $|x| \geq |y|$, then $|x| \geq 2M > 2N$ and

$$|x - s| \geq \frac{|x|}{2} + \frac{|x|}{2} - |s| \geq \frac{|x|}{2}.$$

Thus, it holds that

$$|v(z)| \leq \frac{C}{|x|} \int_{-N}^{N} |\bar{\omega}(s)| \, ds \leq C \frac{\|\bar{\omega}\|_1}{|z|},$$

since $|z| \leq \sqrt{2}|x|$.

Case 2: If $|y| \geq |x|$, then $|y| \geq 2M > 2N$ and we obtain

$$|y - f(s)| \geq \frac{|y|}{2} + \frac{|y|}{2} - \|f\|_\infty \geq \frac{|y|}{2}.$$

Therefore,

$$|v(z)| \leq \frac{C}{|y|} \int_{-N}^{N} |\bar{\omega}(s)| \, ds \leq C \frac{\|\bar{\omega}\|_1}{|z|},$$

since $|z| \leq \sqrt{2}|y|$. $\square$

B. Properties of a class of singular integral operators

In this section we study a general class of singular integral operators suited to our approach. They have been introduced only recently, see [60, 61], in the setting of the two-phase Muskat problem. In this Appendix B we collect some results spread out in the papers [1, 60, 61, 63], concerning the mapping properties of these operators and also present a method how to localize the operators $B_{n,m}$, see Section B.2.

Given $n, m \in \mathbb{N}$ and Lipschitz continuous functions $a_1, \ldots, a_m, b_1, \ldots, b_n : \mathbb{R} \to \mathbb{R}$, we introduce the singular integral operators

$$B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \bar{\omega}](x) := \frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}} \frac{\bar{\omega}(x-s)}{s} \frac{\prod_{i=1}^{n}(\delta_{[x,s]}b_i/s)}{\prod_{i=1}^{m}\left[1+(\delta_{[x,s]}a_i/s)^2\right]}\, ds \tag{B.1}$$

for $\bar{\omega} \in L_2(\mathbb{R})$ and $x \in \mathbb{R}$, where we used the notation from (1.6). These operators play an important role in the study of the Muskat problem considered herein, but also are relevant in the setting of the two-phase Stokes problem, cf. [63, 64].

B.1. Mapping properties

Before addressing the boundedness of these operators, we first collect in Lemma B.1 some algebraic properties of this family of operators, which were provided in [1, Lemma 3].

Lemma B.1. *Let $n, m \in \mathbb{N}$ and $a_1, \ldots, a_m, b_1, \ldots, b_n : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous and $\bar{\omega} \in L_2(\mathbb{R})$. It then holds:*

(i) If $n \geq 1$ and additionally $b_1, \varphi \in W^1_\infty(\mathbb{R})$, then

$$\begin{aligned}\varphi B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \bar{\omega}] &- B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \varphi\bar{\omega}]\\ &= b_1 B_{n,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, \varphi, \bar{\omega}] - B_{n,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, \varphi, b_1\bar{\omega}].\end{aligned} \tag{B.2}$$

(ii) If $\tilde{a}_1, \ldots, \tilde{a}_m : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous, then

$$\begin{aligned}B_{n,m}(\tilde{a}_1, \ldots, \tilde{a}_m)&[b_1, \ldots, b_n, \bar{\omega}] - B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \bar{\omega}]\\ &= \sum_{i=1}^{m} B_{n+2,m+1}(\tilde{a}_1, \ldots, \tilde{a}_i, a_i, \ldots, a_m)[b_1, \ldots, b_n, a_i + \tilde{a}_i, a_i - \tilde{a}_i, \bar{\omega}].\end{aligned} \tag{B.3}$$

Proof. The claim *(i)* follows by direct computation:

$$\begin{aligned}\varphi B_{n,m}&(a_1, \ldots, a_m)[b_1, \ldots, b_n, \bar{\omega}] - B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \varphi\bar{\omega}]\\ &= \frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}} (\varphi(x) - \varphi(x-s))\, \frac{b_1(x) - b_1(x-s)}{s} \frac{\bar{\omega}(x-s)}{s} \frac{\prod_{i=2}^{n}(\delta_{[x,s]}b_i/s)}{\prod_{i=1}^{m}\left[1+(\delta_{[x,s]}a_i/s)^2\right]}\, ds\\ &= b_1 B_{n,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, \varphi, \bar{\omega}] - B_{n,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, \varphi, b_1\bar{\omega}].\end{aligned}$$

Concerning *(ii)*, we have

$$\begin{aligned}B_{n,m}&(\tilde{a}_1, \ldots, \tilde{a}_m)[b_1, \ldots, b_n, \bar{\omega}] - B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \bar{\omega}]\\ &= \frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}} \frac{\bar{\omega}(x-s)}{s} \prod_{i=1}^{n}(\delta_{[x,s]}b_i/s) \frac{T(x,s)}{\prod_{i=1}^{m}\left[1+(\delta_{[x,s]}a_i/s)^2\right]\prod_{i=1}^{m}\left[1+(\delta_{[x,s]}\tilde{a}_i/s)^2\right]}\, ds,\end{aligned}$$

where, in view of the (algebraic) formula

$$\prod_{i=1}^{m}\alpha_i-\prod_{i=1}^{m}\beta_i=\sum_{i=1}^{m}\left(\left(\prod_{k=1}^{i-1}\alpha_k\right)(\alpha_i-\beta_i)\left(\prod_{k=i+1}^{m}\beta_k\right)\right), \tag{B.4}$$

we get

$$\begin{aligned}T(x,s)&:=\prod_{i=1}^{m}\left[1+(\delta_{[x,s]}a_i/s)^2\right]-\prod_{i=1}^{m}\left[1+(\delta_{[x,s]}\tilde a_i/s)^2\right]\\&=\sum_{i=1}^{m}\left(\prod_{k=1}^{i-1}(1+(\delta_{[x,s]}a_k/s)^2)\left(\delta_{[x,s]}(a_i+\tilde a_i)/s\right)\left(\delta_{[x,s]}(a_i-\tilde a_i)/s\right)\right.\\&\qquad\left.\prod_{k=i+1}^{m}\left(1+(\delta_{[x,s]}\tilde a_k/s)^2\right)\right).\end{aligned}$$

Hence,

$$\begin{aligned}&B_{n,m}(\tilde a_1,\ldots,\tilde a_m)[b_1,\ldots,b_n,\bar\omega]-B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,\bar\omega]\\&\quad=\sum_{i=1}^{m}\left(\frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}}\frac{\bar\omega(x-s)}{s}\prod_{i=1}^{n}(\delta_{[x,s]}b_i/s)\frac{\left(\delta_{[x,s]}(a_i+\tilde a_i)/s\right)\left(\delta_{[x,s]}(a_i-\tilde a_i)/s\right)}{\prod_{k=1}^{i}\left[1+(\delta_{[x,s]}\tilde a_i/s)^2\right]\prod_{k=i}^{m}\left[1+(\delta_{[x,s]}a_i/s)^2\right]}\,ds\right)\\&\quad=\sum_{i=1}^{m}B_{n+2,m+1}(\tilde a_1,\ldots,\tilde a_i,a_i,\ldots,a_m)[b_1,\ldots,b_n,a_i+\tilde a_i,a_i-\tilde a_i,\bar\omega].\end{aligned}$$

□

An important tool in the analysis of the operators $B_{n,m}$ is the following result of harmonic analysis.

Theorem B.2. *Let $a:\mathbb{R}\to\mathbb{R}$ be a Lipschitz continuous function. For $f\in \mathrm{C}_0^\infty(\mathbb{R})$ let*

$$T_a[f](x):=\mathrm{PV}\int_{\mathbb{R}}\frac{f(x-y)}{y}\exp\left(i\frac{a(x)-a(x-y)}{y}\right)dy.$$

The operator T_a has an extension $T_a\in\mathcal{L}(L_2(\mathbb{R}))$ and it holds that

$$\|T_a\|_{\mathcal{L}(L_2(\mathbb{R}))}\leq C(1+\|a'\|_\infty).$$

This deep result of harmonic analysis has already been established in [69], see also [68] for a weaker version of this result. We emphasize that if $a\in \mathrm{BUC}^\alpha(\mathbb{R})$ for $\alpha>0$, then the principal value exists in the classical sense. If a is merely Lipschitz continuous, then the operator T_a is defined as a suitable series, see [66].

The results in Theorem B.3 was established in [60] and provides the boundedness of the operator $B_{n,m}$ in $L_2(\mathbb{R})$.

Theorem B.3. *Given Lipschitz continuous functions $a_1,\ldots,a_m,b_1,\ldots,b_n:\mathbb{R}\to\mathbb{R}$, there exists a constant C depending only on n,m and $\max_{i=1,\ldots,m}\|a_i'\|_\infty$, such that*

$$\|B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,\cdot]\|_{\mathcal{L}(L_2(\mathbb{R}))}\leq C\prod_{i=1}^{n}\|b_i'\|_\infty.$$

Moreover, $B_{n,m}\in \mathrm{C}^{1-}((W^1_\infty(\mathbb{R}))^m,\mathcal{L}^n_{\mathrm{sym}}(W^1_\infty(\mathbb{R}),\mathcal{L}(L_2(\mathbb{R}))))$.

Proof. It holds that

$$B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,\bar\omega](x)=\frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}}\frac{\bar\omega(x-s)}{s}K(x,s)\,ds,\quad x\in\mathbb{R}$$

where

$$K(x,s) \coloneqq \frac{\prod_{i=1}^{n}(\delta_{[x,s]}b_i/s)}{\prod_{i=1}^{m}\left[1+(\delta_{[x,s]}a_i/s)^2\right]}, \qquad x,s\in\mathbb{R}, s\neq 0.$$

We define the functions $F\colon \mathbb{R}^{n+m}\to\mathbb{R}$ with

$$F(u_1,\dots,u_m,v_1,\dots,v_n) \coloneqq \frac{\prod_{i=1}^{n} v_i}{\prod_{i=1}^{m}\left[1+u_i^2\right]},$$

and $A \coloneqq (a_1,\dots,a_m,b_1,\dots,b_n)\colon \mathbb{R}\to\mathbb{R}^{n+m}$. As a rational function F is clearly smooth. We note that A is Lipschitz continuous because for given $x,y\in\mathbb{R}$ it holds that

$$\begin{aligned}|A(x)-A(y)| &= \left(\sum_{i=1}^{m}|a_i(x)-a_i(y)|^2+\sum_{i=1}^{n}|b_i(x)-b_i(y)|^2\right)^{1/2}\\ &\leq \left(\sum_{i=1}^{m}\|a_i'\|_\infty^2+\sum_{i=1}^{n}\|b_i\|_\infty^2\right)^{1/2}|x-y|\\ &\leq L|x-y|,\end{aligned}$$

where

$$L \coloneqq (n+m\max_{i=1,\dots,m}\|a_i'\|_\infty^2)^{1/2},$$

provided that $\|b_i'\|_\infty \leq 1$ for all $i=1,\dots,n$. Moreover,

$$K(x,s) = \frac{1}{s}F\left(\frac{\delta_{[x,s]}A}{s}\right),$$

and

$$\left|\frac{\delta_{[x,s]}A}{s}\right| \leq \|A'\|_\infty \leq L.$$

In view of the latter property we may find a smooth function $\tilde{F}\colon\mathbb{R}^{n+m}\to\mathbb{R}$, which is $4L$-periodic in each variable and with

$$\tilde{F} = F \qquad \text{on } [-L,L]^{n+m}.$$

Using the periodicity of $\tilde{F}$, we can represent this function by its Fourier series

$$\tilde{F} = \sum_{p\in\mathbb{Z}^{n+m}} \alpha_p e^{i\pi/(2L)\langle p|\cdot\rangle},$$

where $(\alpha_p)_{p\in\mathbb{Z}^{n+m}}$ is a rapidly decaying sequence and $\langle\cdot|\cdot\rangle$ is the standard scalar product on $\mathbb{R}^{n+m}$. Then

$$K(x,s) = \frac{1}{s}F\left(\frac{\delta_{[x,s]}A}{s}\right) = \frac{1}{s}\widetilde{F}\left(\frac{\delta_{[x,s]}A}{s}\right) = \sum_{p\in\mathbb{Z}^{n+m}}\alpha_p K_p(x,s),$$

with

$$K_p(x,s) = \frac{1}{s}\exp\left(\frac{i\pi}{2L}\frac{\delta_{[x,s]}\langle p|A\rangle}{s}\right).$$

According to Theorem B.2, the norm of the singular integral operators A_p defined by the kernels $\alpha_p K_p$ satisfies

$$\|A_p\|_{\mathcal{L}(L_2)} \leq C|\alpha_p|(1+|p|)$$

and, since $(\alpha_p)_{p\in\mathbb{Z}^{n+m}}$ is a rapidly decaying sequence, the series $\sum_{p\in\mathbb{Z}^{n+m}} A_p$ converges absolutely in $\mathcal{L}(L_2(\mathbb{R}))$. Hence,

$$\|B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n]\|_{\mathcal{L}(L_2(\mathbb{R}))} \leq C \sum_{p\in\mathbb{Z}^{n+m}} \|A_p\|_{\mathcal{L}(L_2)} \leq C,$$

where $C = C(n,m,\max_{i=1,\ldots,m}\|a_i'\|_\infty) > 0$. This proves the claim when $\|b_i'\|_\infty \leq 1$. For general Lipschitz continuous b_i, $1 \leq i \leq n$, we use the linearity of $B_{n,m}$ with respect to these arguments. □

In Theorem B.4 we recall the result [63, Lemma 3.1 (iii)], where it is shown that the operator $B_{n,m}$ maps into $L_\infty(\mathbb{R})$ provided the arguments posses additional regularity properties.

Theorem B.4. *Given $r \in (3/2, 2)$ and $\tau \in (1/2, 1)$, it holds that*

$$\|B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,\bar\omega]\|_\infty \leq C\|\bar\omega\|_{H^\tau} \prod_{i=1}^n \|b_i\|_{H^r} \tag{B.5}$$

for all $a_1,\ldots,a_m, b_1,\ldots,b_n \in H^r(\mathbb{R})$ and $\bar\omega \in H^\tau(\mathbb{R})$ with C depending only on n, m, r, τ, and $\max_{i=1,\ldots,m}\|a_i\|_{H^r}$.

Moreover, $B_{n,m} \in \mathrm{C}^{1-}((H^r(\mathbb{R}))^m, \mathcal{L}^{n+1}((H^r(\mathbb{R}))^n \times H^\tau(\mathbb{R}), L_\infty(\mathbb{R})))$.

Proof. We split

$$B_{n,m}(a_1,\ldots,a_m)[b_1,\ldots,b_n,\bar\omega] = \frac{1}{\pi}\Big(T_1 + T_2 + T_3\Big),$$

where

$$\begin{aligned}
T_1(x) &:= -\int_{\{|s|<1\}} \frac{\prod_{i=1}^n(\delta_{[x,s]}b_i/s)}{\prod_{i=1}^m\big[1+(\delta_{[x,s]}a_i/s)^2\big]}\frac{\delta_{[x,s]}\bar\omega}{s}\,ds,\\
T_2(x) &:= \bar\omega(x)\mathrm{PV}\int_{\{|s|<1\}} \frac{1}{s}\frac{\prod_{i=1}^n(\delta_{[x,s]}b_i/s)}{\prod_{i=1}^m\big[1+(\delta_{[x,s]}a_i/s)^2\big]}\,ds,\\
T_3(x) &:= \int_{\{|s|>1\}} \frac{\prod_{i=1}^n(\delta_{[x,s]}b_i/s)}{\prod_{i=1}^m\big[1+(\delta_{[x,s]}a_i/s)^2\big]}\frac{\bar\omega(x-s)}{s}\,ds.
\end{aligned}$$

Since $\bar\omega \in H^\tau(\mathbb{R}) \hookrightarrow \mathrm{BUC}^{\tau-1/2}(\mathbb{R})$ and $H^r(\mathbb{R}) \hookrightarrow \mathrm{BUC}^{r-1/2}(\mathbb{R})$, we have

$$\|T_1\|_\infty \leq \prod_{i=1}^n \|b_i'\|_\infty [\bar\omega]_{\tau-1/2}\int_{\{|s|<1\}} |s|^{\tau-3/2}\,ds \leq C\|\bar\omega\|_{H^\tau}\prod_{i=1}^n\|b_i\|_{H^r}.$$

Moreover, Hölder's inequality leads us to

$$\begin{aligned}
\|T_3\|_\infty &\leq \prod_{i=1}^n \|b_i'\|_\infty \int_{\{|s|>1\}} \frac{|\bar\omega(x-s)|}{|s|}\,ds\\
&\leq \prod_{i=1}^n \|b_i'\|_\infty \|\bar\omega\|_2\Big(\int_{\{|s|>1\}}|s|^{-2}\,ds\Big)^{1/2}\\
&= 2\|\bar\omega\|_2\prod_{i=1}^n\|b_i'\|_\infty.
\end{aligned}$$

Since $H^r(\mathbb{R}) \hookrightarrow \mathrm{BUC}^{r-1/2}(\mathbb{R})$ we have

$$\frac{|u(x+s)-2u(x)+u(x-s)|}{|s|^{r-1/2}} \leq \frac{|u'(\zeta_1)-u'(\zeta_2)|}{|s|^{r-3/2}} \leq [u']_{r-3/2}\frac{|\zeta_1-\zeta_2|^{r-3/2}}{|s|^{r-3/2}} \leq 2[u']_{r-3/2} \tag{B.6}$$

for some $\zeta_1, \zeta_2 \in (x-|s|, x+|s|)$ and all $u \in H^r(\mathbb{R}), x, s \in \mathbb{R}, s \neq 0$. Therefore we obtain by the definition of the principal value, see (A.2), that

$$\begin{aligned}
\|T_2\|_\infty &\le \|\bar\omega\|_\infty \lim_{\varepsilon\searrow 0}\left(\int_\varepsilon^1 \frac{1}{s}\left|\frac{\prod_{i=1}^n(\delta_{[x,s]}b_i/s)}{\prod_{i=1}^m\big[1+(\delta_{[x,s]}a_i/s)^2\big]}+\frac{\prod_{i=1}^n(\delta_{[x,-s]}b_i/-s)}{\prod_{i=1}^m\big[1+(\delta_{[x,-s]}a_i/s)^2\big]}\right|ds\right)\\
&\le \|\bar\omega\|_\infty \lim_{\varepsilon\searrow 0}\left(\int_\varepsilon^1 \frac{1}{s}\Big|\prod_{i=1}^n(\delta_{[x,s]}b_i/s)\prod_{i=1}^m\big[1+(\delta_{[x,-s]}a_i/s)^2\big]\right.\\
&\qquad\left.+\prod_{i=1}^n(\delta_{[x,-s]}b_i/-s)\prod_{i=1}^m\big[1+(\delta_{[x,s]}a_i/s)^2\big]\Big|\,ds\right)\\
&\le \|\bar\omega\|_\infty \lim_{\varepsilon\searrow 0}\left(\int_\varepsilon^1 \frac{1}{s}\Big|\Big[\prod_{i=1}^n(\delta_{[x,s]}b_i/s)+\prod_{i=1}^n(\delta_{[x,-s]}b_i/-s)\Big]\prod_{i=1}^m\big[1+(\delta_{[x,-s]}a_i/s)^2\big]\Big|\right.\\
&\qquad+\frac{1}{s}\Big|\prod_{i=1}^n(\delta_{[x,-s]}b_i/-s)\\
&\qquad\quad\left.\times\Big[\prod_{i=1}^m\big[1+(\delta_{[x,s]}a_i/s)^2-\prod_{i=1}^m\big[1+(\delta_{[x,-s]}a_i/s)^2\big]\big]\Big]\Big|\,ds\right),
\end{aligned}$$

where, recalling (B.4) and (B.6), we have

$$\begin{aligned}
&\frac{1}{s}\Big|\Big[\prod_{i=1}^n(\delta_{[x,s]}b_i/s)+\prod_{i=1}^n(\delta_{[x,-s]}b_i/-s)\Big]\prod_{i=1}^m\big[1+(\delta_{[x,-s]}a_i/s)^2\big]\Big|\\
&\qquad\le(1+\max_{i=1,\dots,m}\|a_i'\|_\infty^2)^m\frac{1}{s^2}\sum_{i=1}^n\left(\prod_{j=1}^{i-1}(|\delta_{[x,s]}b_j/s|)|\delta_{[x,s]}b_i-\delta_{[x,-s]}b_i|\prod_{j=i+1}^n(|\delta_{[x,-s]}b_j/s|)\right)\\
&\qquad\le Cs^{r-5/2}\sum_{i=1}^n\left([b_i']_{r-3/2}\prod_{j=1,j\neq i}^n\|b_j'\|_\infty\right),
\end{aligned}$$

and (see also proof of (B.3))

$$\begin{aligned}
&\frac{1}{s}\Big|\prod_{i=1}^n(\delta_{[x,-s]}b_i/-s)\Big[\prod_{i=1}^m\big[1+(\delta_{[x,s]}a_i/s)^2\big]-\prod_{i=1}^m\big[1+(\delta_{[x,-s]}a_i/s)^2\big]\Big]\Big|\\
&\qquad\le\frac{1}{s}\prod_{i=1}^n\|b_i'\|_\infty\sum_{i=1}^m\left|\prod_{k=1}^{i-1}(1+(\delta_{[x,-s]}a_i/s)^2)(\delta_{[x,s]}a_i/s+\delta_{[x,-s]}a_i/s)\right.\\
&\qquad\qquad\left.\times(\delta_{[x,s]}a_i/s-\delta_{[x,-s]}a_i/s)\prod_{k=i+1}^m(1+(\delta_{[x,s]}a_i/s)^2)\right|\\
&\qquad\le 2s^{r-5/2}\prod_{i=1}^n\|b_i'\|_\infty\sum_{i=1}^m[a_i']_{r-3/2}\Big(\prod_{j=1}^n(1+\|a_j'\|_\infty^2)\Big).
\end{aligned}$$

Hence,

$$\|T_2\|_\infty\le C\|\bar\omega\|_\infty\Big(\sum_{i=1}^n[b_i']_{r-3/2}\prod_{\substack{j=1\\ j\neq i}}^n\|b_j'\|_\infty+\prod_{i=1}^n\|b_i'\|_\infty\Big)\lim_{\varepsilon\searrow 0}\int_\varepsilon^1 s^{r-5/2}\,ds\le C\|\bar\omega\|_{H^r}\prod_{i=1}^n\|b_i\|_{H^r}.$$

Gathering the estimates of T_1, T_2, and T_3 we conclude (B.5). The Lipschitz property is a straightforward consequence of (B.5) and (B.3). □

In Theorem B.5 below, we provide an result established in [1, Lemma 6] concerning the boundedness of the operators $B_{n,m} = B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \cdot]$ in $\mathcal{L}(H^{r-1}(\mathbb{R}))$.

Theorem B.5. *Given $r \in (3/2, 2)$ and $a_1, \ldots, a_m \in H^r(\mathbb{R})$, there exists a constant $C > 0$, depending only on n, m, r, and $\max_{i=1,\ldots,m} \|a_i\|_{H^r}$, such that*

$$\|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \bar{\omega}]\|_{H^{r-1}} \le C\|\bar{\omega}\|_{H^{r-1}} \prod_{i=1}^{n} \|b_i'\|_{H^{r-1}}$$

for all $b_1, \ldots, b_n \in H^r(\mathbb{R})$ and $\bar{\omega} \in H^{r-1}(\mathbb{R})$.

Moreover, $B_{n,m} \in \mathrm{C}^{1-}((H^r(\mathbb{R}))^m, \mathcal{L}^n_{\mathrm{sym}}(H^r(\mathbb{R}), \mathcal{L}(H^{r-1}(\mathbb{R}))))$.

In order to prove Theorem B.5, we need some preparation. First we note that the spaces $H^r(\mathbb{R})$ and $W_2^r(\mathbb{R})$, where $r \in [0, 1)$, have equivalent norms, see e.g. [86, Section 2.3]. To be more precise, we recall that given $r \in [0, 1)$ we have

$$W_2^r(\mathbb{R}) := \{v \in L_2(\mathbb{R}) : [v]_{W_2^r} < \infty\},$$

where

$$[v]_{W_2^r} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|v(x) - v(x+y)|^2}{|y|^{1+2r}}\,dx\right) dy\right)^{1/2} = \left(\int_{\mathbb{R}} \frac{\|v - \tau_y v\|_2^2}{|y|^{1+2r}}\,dy\right)^{1/2}.$$

Here, τ_y is the right translation operator

$$\tau_y := \big[f \mapsto f(\cdot - y)\big].$$

Denoting the Fourier transform of u with $\hat{u}$, we compute for $u \in \mathrm{C}_0^\infty(\mathbb{R})$ that

$$\begin{aligned}
[u]_{W_2^r}^2 &= \int_{\mathbb{R}} \frac{\|u - \tau_y u\|_2^2}{|y|^{1+2r}}\,dy \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|\hat{u}(\zeta) - e^{iy\zeta}\hat{u}(\zeta)|^2}{|y|^{1+2r}}\,d\zeta\right) dy \\
&= \int_{\mathbb{R}} |\hat{u}(\zeta)|^2 \left(\int_{\mathbb{R}} \frac{|e^{iy\zeta} - 1|^2}{|y|^{1+2r}}\,dy\right) d\zeta.
\end{aligned}$$

Changing variables, we find that

$$\begin{aligned}
\int_{\mathbb{R}} \frac{|e^{iy\zeta} - 1|^2}{|y|^{1+2r}}\,dy &= \int_{\mathbb{R}} \left(\frac{|\cos(y\zeta) - 1|^2}{|y|^{1+2r}} + \frac{|\sin(y\zeta)|^2}{|y|^{1+2r}}\right) dy \\
&= |\zeta|^{2r} \int_{\mathbb{R}} \left(\frac{|\cos(t) - 1|^2}{|t|^{1+2r}} + \frac{|\sin(t)|^2}{|t|^{1+2r}}\right) dt, \quad \zeta \in \mathbb{R},
\end{aligned}$$

where

$$\int_{\mathbb{R}} \frac{|\cos(t) - 1|^2}{|t|^{1+2r}} + \frac{|\sin(t)|^2}{|t|^{1+2r}}\,dt =: C < \infty.$$

Indeed, in view of $r \in [0, 1)$, we have

$$\int_{\mathbb{R}} \frac{|\cos(t) - 1|^2}{|t|^{1+2r}} + \frac{|\sin(t)|^2}{|t|^{1+2r}}\,dt \le \int_{-1}^{1} \frac{|t|^2}{|t|^{1+2r}}\,dt + \int_{|t|>1} \frac{5}{|t|^{1+2r}}\,dt < \infty.$$

Therefore

$$[u]_{W_2^r}^2 = C \int_{\mathbb{R}} |\hat{u}(\zeta)|^2 |\zeta|^{2r}\,d\zeta =: C[u]_{H^r}^2, \tag{B.7}$$

and consequently

$$\|u\|_{W_2^r} := \left(\|u\|_2^2 + [u]_{W_2^r}\right)^{1/2} \qquad \text{and} \qquad \|u\|_{H^r} := \left(\int_{\mathbb{R}} |\hat{u}(\zeta)|^2 (|\zeta|^2+1)^r \, d\zeta\right)^{1/2} \tag{B.8}$$

are equivalent norms. The relation (B.7) is used extensively in the arguments below.

In the proof of Theorem B.5, we use the following result established in [1, Lemma 4].

Lemma B.6. *Let $n, m \in \mathbb{N}$ with $n \geq 1$, $r \in (3/2, 2)$, and $\tau \in (5/2 - r, 1)$ be given. Given $a_1, \ldots, a_m \in H^r(\mathbb{R})$, there exists a constant $C > 0$, depending only on n, m, r, τ, and $\max_{i=1,\ldots,m} \|a_i\|_{H^r}$, such that*

$$\|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \bar{\omega}]\|_2 \leq C \|b_1'\|_2 \|\bar{\omega}\|_{H^{r-1}} \prod_{i=2}^n \|b_i'\|_{H^{r-1}} \tag{B.9}$$

and

$$\begin{aligned} \|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \bar{\omega}] - \bar{\omega} B_{n-1,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, b_1']\|_2 \\ \leq C \|b_1\|_{H^\tau} \|\bar{\omega}\|_{H^{r-1}} \prod_{i=2}^n \|b_i'\|_{H^{r-1}} \end{aligned} \tag{B.10}$$

for all $b_1, \ldots, b_n \in H^r(\mathbb{R})$ and $\bar{\omega} \in H^{r-1}(\mathbb{R})$.

Moreover, $B_{n,m} \in \mathrm{C}^{1-}((H^r(\mathbb{R}))^m, \mathcal{L}^{n+1}(H^1(\mathbb{R}) \times (H^r(\mathbb{R}))^{n-1} \times H^{r-1}(\mathbb{R}), L_2(\mathbb{R})))$.

Proof. We may assume that $\bar{\omega} \in \mathrm{C}_0^\infty(\mathbb{R})$. In order to establish (B.9) we use the identity

$$\frac{\partial}{\partial s}\left(\frac{\delta_{[x,s]} h}{s}\right) = \frac{h'(x-s)}{s} - \frac{\delta_{[x,s]} h}{s^2}, \qquad x, s \in \mathbb{R}, s \neq 0,$$

which holds for all differentiable functions $h \colon \mathbb{R} \to \mathbb{R}$. Due to this relation we compute

$$\begin{aligned} B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \bar{\omega}](x) &= \frac{1}{\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{\delta_{[x,s]} b_1}{s^2} \frac{\prod_{i=2}^n (\delta_{[x,s]} b_i / s)}{\prod_{i=1}^m [1 + (\delta_{[x,s]} a_i / s)^2]} \bar{\omega}(x-s) \, ds \\ &= B_{n-1,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, b_1' \bar{\omega}](x) \\ &\quad - \frac{1}{\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{\partial}{\partial s}\left(\frac{\delta_{[x,s]} b_1}{s}\right) \frac{\prod_{i=2}^n (\delta_{[x,s]} b_i / s)}{\prod_{i=1}^m [1 + (\delta_{[x,s]} a_i / s)^2]} \bar{\omega}(x-s) \, ds. \end{aligned}$$

Integration by parts leads to

$$\begin{aligned} &\frac{1}{\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{\partial}{\partial s}\left(\frac{\delta_{[x,s]} b_1}{s}\right) \frac{\prod_{i=2}^n (\delta_{[x,s]} b_i / s)}{\prod_{i=1}^m [1 + (\delta_{[x,s]} a_i / s)^2]} \bar{\omega}(x-s) \, ds \\ &\quad = b_1(x) B_{n-1,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, \bar{\omega}'](x) - B_{n-1,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, b_1 \bar{\omega}'](x) \\ &\qquad + \frac{1}{\pi} \sum_{j=2}^n \int_{\mathbb{R}} K_{1,j}(x,s) \bar{\omega}(x-s) \, ds - \frac{1}{\pi} \sum_{j=1}^m \int_{\mathbb{R}} K_{2,j}(x,s) \bar{\omega}(x-s) \, ds, \end{aligned}$$

where, for given $x \in \mathbb{R}$, $s \neq 0$, we set

$$\begin{aligned} K_{1,j}(x,s) &:= \frac{\prod_{i=2, i\neq j}^n (\delta_{[x,s]} b_i / s)}{\prod_{i=1}^m [1 + (\delta_{[x,s]} a_i / s)^2]} \frac{\delta_{[x,s]} b_j - s b_j'(x-s)}{s^2} \frac{\delta_{[x,s]} b_1}{s}, \\ K_{2,j}(x,s) &:= 2 \frac{\prod_{i=2}^n (\delta_{[x,s]} b_i / s)}{[1 + (\delta_{[x,s]} a_j / s)^2] \prod_{i=1}^m [1 + (\delta_{[x,s]} a_i / s)^2]} \frac{\delta_{[x,s]} a_j - s a_j'(x-s)}{s^2} \frac{\delta_{[x,s]} a_j}{s} \frac{\delta_{[x,s]} b_1}{s}. \end{aligned}$$

Using Minkowski's integral inequality we obtain

$$\left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} K_{1,j}(x,s)\bar{\omega}(x-s)\,ds\right|^2 dx\right)^{1/2} \leq \|\bar{\omega}\|_\infty \int_{\mathbb{R}}\left(\int_{\mathbb{R}} |K_{1,j}(x,s)|^2\,dx\right)^{1/2} ds.$$

Let $\alpha \in \{\tau - 1/2, 1/2\}$. Since $b_1 \in \mathrm{BUC}^\alpha(\mathbb{R})$ we conclude that

$$\left(\int_{\mathbb{R}} |K_{1,j}(x,s)|^2\,dx\right)^{1/2} = \frac{1}{|s|^{3-\alpha}}[b_1]_\alpha \left(\prod_{\substack{i=2\\ i\neq j}} \|b_i'\|_\infty\right)\left(\int_{\mathbb{R}} |\delta_{[x,s]}b_j - sb_j'(x-s)|^2\,dx\right)^{1/2}.$$

Again denoting the right translation operator by

$$\tau_y := \big[f \mapsto f(\cdot - y)\big], \qquad y \in \mathbb{R},$$

we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{1}{|s|^{3-\alpha}}\Big(\int_{\mathbb{R}} |\delta_{[x,s]}b_j - sb_j'(x-s)|^2\,dx\Big)^{1/2} ds \\
&\qquad = \int_{\mathbb{R}} \frac{1}{|s|^{2-\alpha}}\Big(\int_{\mathbb{R}}\Big|\int_0^1 [b_j'(x-(1-t)s) - b_j'(x-s)]\,dt\Big|^2 dx\Big)^{1/2} ds \\
&\qquad \leq \int_0^1 \Big[\int_{\{|s|<1\}} |s|^{\alpha-2}\Big(\int_{\mathbb{R}} |b_j'(x-(1-t)s) - b_j'(x-s)|^2\,dx\Big)^{1/2} ds \\
&\qquad\qquad + \int_{\{|s|>1\}} |s|^{\alpha-2}\Big(\int_{\mathbb{R}} |b_j'(x-(1-t)s)|^2\,dx + |b_j'(x-s)|^2\,dx\Big)^{1/2} ds\Big]\,dt \\
&\qquad \leq \int_0^1 \left(\int_{\{|s|<1\}} \frac{\|b_j' - \tau_{-ts}b_j'\|_2}{|s|^{2-\alpha}}\,ds\right) dt + 2\|b_j'\|_2 \int_{\{|s|>1\}} |s|^{\alpha-2}\,ds \\
&\qquad \leq C\left(\int_{\{|s|<1\}} \frac{1}{|s|^{5-2\alpha-2r}}\,ds\right)^{1/2} [b_j']_{H^{r-1}} + C\|b_j'\|_2 \\
&\qquad \leq C\|b_j'\|_{H^{r-1}},
\end{aligned}$$

where we used (B.7), Fubini's theorem, Minkowski's integral inequality, Hölder's inequality, and the inequality $\alpha + r > 2$. Thus, we find for $2 \leq j \leq n$

$$\left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} K_{1,j}(x,s)\bar{\omega}(x-s)\,ds\right|^2 dx\right)^{1/2} \leq C\|\bar{\omega}\|_\infty\,[b_1]_\alpha \left(\prod_{i=2}^n \|b_i'\|_{H^{r-1}}\right), \tag{B.11}$$

and by similar arguments

$$\left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} K_{2,j}(x,s)\bar{\omega}(x-s)\,ds\right|^2 dx\right)^{1/2} \leq C\|\bar{\omega}\|_\infty\,[b_1]_\alpha \left(\prod_{i=2}^n \|b_i'\|_\infty\right). \tag{B.12}$$

Using the formula

$$\bar{\omega}'(x-s) = \frac{\partial}{\partial s}(\bar{\omega}(x) - \bar{\omega}(x-s))$$

and integration by parts, we have

$$\begin{aligned}b_1(x)B_{n-1,m}(a_1,\dots,a_m)[b_2,\dots,b_n,\bar\omega'](x) - B_{n-1,m}(a_1,\dots,a_m)[b_2,\dots,b_n,b_1\bar\omega'](x)\\ = \frac{1}{\pi}\sum_{j=1}^{n}\int_{\mathbb{R}} K_{3,j}(x,s)\,ds - \frac{2}{\pi}\sum_{j=1}^{m}\int_{\mathbb{R}} K_{4,j}(x,s)\,ds,\end{aligned}$$

where

$$\begin{aligned}K_{3,j} &:= \frac{\prod_{i=1,i\neq j}^{n}\delta_{[x,s]}b_i/s}{\prod_{i=1}^{m}[1+(\delta_{[x,s]}a_i/s)^2]}\frac{\delta_{[x,s]}\bar\omega}{s}\left(\frac{\delta_{[x,s]}b_j}{s} - b_j'(x-s)\right),\\ K_{4,j} &:= \frac{\prod_{i=1}^{n}\delta_{[x,s]}b_i/s}{[1+(\delta_{[x,s]}a_j/s)^2]\prod_{i=1}^{m}[1+(\delta_{[x,s]}a_i/s)^2]}\frac{\delta_{[x,s]}\bar\omega}{s}\frac{\delta_{[x,s]}a_j}{s}\left(\frac{\delta_{[x,s]}a_j}{s} - a_j'(x-s)\right).\end{aligned}$$

We estimate with the help of Minkowski's integral inequality for $2 \le j \le n$ that

$$\left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} K_{3,j}(x,s)\,ds\right|^2 dx\right)^{1/2} \le \int_{\mathbb{R}}\left(\int_{\mathbb{R}}|K_{3,j}(x,s)|^2\,dx\right)^{1/2} ds,$$

and, since $b_1 \in \mathrm{BUC}^\alpha(\mathbb{R})$, that

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|K_{3,j}(x,s)|^2\,dx\right)^{1/2} ds \le 2[b_1]_\alpha\left(\prod_{i=2}^{n}\|b_i'\|_\infty\right)\int_{\mathbb{R}}|s|^{\alpha-2}\left(\int_{\mathbb{R}}|\delta_{[x,s]}\bar\omega|^2\,dx\right)^{1/2} ds.$$

Moreover,

$$\begin{aligned}\int_{\mathbb{R}}|s|^{\alpha-2}\left(\int_{\mathbb{R}}|\bar\omega(x)-\bar\omega(x-s)|^2\,dx\right)^{1/2} ds &= \int_{\mathbb{R}}|s|^{\alpha-2}\|\bar\omega-\tau_s\bar\omega\|_2\,ds\\ &\le 2\int_{\{|s|>1\}}|s|^{\alpha-2}\|\bar\omega\|_2\,ds\\ &\quad+\int_{\{|s|<1\}}|s|^{\alpha-2}\|\bar\omega-\tau_s\bar\omega\|_2\,ds\\ &\le C\|\bar\omega\|_2 + \left(\int_{\{|s|<1\}}\frac{1}{|s|^{5-2\alpha-2r}}\,ds\right)^{1/2}[\bar\omega]_{H^{r-1}}\\ &\le C\|\bar\omega\|_{H^{r-1}}.\end{aligned}$$

We arrive at

$$\left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} K_{3,j}(x,s)\,ds\right|^2 dx\right)^{1/2} \le C\|\bar\omega\|_{H^{r-1}}[b_1]_\alpha\prod_{i=2}^{n}\|b_i'\|_\infty,\quad 2\le j\le n. \tag{B.13}$$

Furthermore,

$$\begin{aligned}B_{n-1,m}(a_1,\dots,a_m)[b_2,\dots,b_n,b_1'\bar\omega](x) - \frac{1}{\pi}\int_{\mathbb{R}} K_{3,1}(x,s)\,ds\\ = \bar\omega(x)B_{n-1,m}(a_1,\dots,a_m)[b_2,\dots,b_n,b_1'](x) - \frac{1}{\pi}\int_{\mathbb{R}} K(x,s)\,ds,\end{aligned}$$

where

$$K(x,s) := \frac{\prod_{i=2}^{n}\delta_{[x,s]}b_i/s}{\prod_{i=1}^{m}[1+(\delta_{[x,s]}a_i/s)^2]}\frac{\delta_{[x,s]}\bar\omega}{s}\frac{\delta_{[x,s]}b_1}{s},\quad x,s\in\mathbb{R}, s\neq 0.$$

Analogously to the estimate (B.13) we conclude for $1 \leq j \leq m$ that

$$\left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} K_{4,j}(x,s)\, ds \right|^2 dx \right)^{1/2} + \left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} K(x,s)\, ds \right|^2 dx \right)^{1/2} \leq C\|\bar{\omega}\|_{H^{r-1}}[b_1]_\alpha \prod_{i=2}^n \|b_i'\|_\infty. \tag{B.14}$$

Further, Theorem B.3 yields

$$\|\bar{\omega} B_{n-1,m}(a_1,\dots,a_m)[b_2,\dots,b_n,b_1']\|_2 \leq C\|\bar{\omega}\|_\infty \|b_1'\|_2 \prod_{i=2}^n \|b_1'\|_\infty. \tag{B.15}$$

From a standard density argument and the estimates (B.11)-(B.15) we can conclude (B.9) and (B.10) since for $\alpha = 1/2$ it holds that $[b_1]_\alpha \leq \|b_1'\|_2$ and for $\alpha = \tau - 1/2$ that $[b_1]_\alpha \leq \|b_1\|_{H^\tau}$. Finally, the Lipschitz continuity is a consequence of the now proven estimate (B.9) and of the property (B.3). □

We are now in a position to prove Theorem B.5.

Proof of Theorem B.5. Keeping $a_1,\dots,a_m, b_1,\dots,b_n,\bar{\omega}$ fixed, we abbreviate

$$B_{n,m} := B_{n,m}(a_1,\dots,a_m)[b_1,\dots,b_n,\bar{\omega}].$$

From Theorem B.3 we know that

$$\|B_{n,m}\|_2 \leq C\|\bar{\omega}\|_2 \prod_{i=1}^n \|b_i'\|_{H^r}.$$

Since the seminorms $[\cdot]_{H^{r-1}}$ and $[\cdot]_{W_2^{r-1}}$ are equivalent, see (B.7), it remains to consider the $W_2^{r-1}(\mathbb{R})$-seminorm of $B_{n,m}$ and, again denoting the right translation operator by

$$\tau_y := \big[f \mapsto f(\cdot - y)\big], \qquad y \in \mathbb{R},$$

we need to estimate the expression

$$[B_{n,m}]^2_{W_2^{r-1}} = \int_{\mathbb{R}} \frac{\|B_{n,m} - \tau_\zeta B_{n,m}\|_2^2}{|\zeta|^{1+2(r-1)}}\, d\zeta.$$

In view of (B.3) it holds that

$$\begin{aligned} B_{n,m} - \tau_\zeta B_{n,m} &= B_{n,m}(a_1,\dots,a_m)[b_1,\dots,b_n,\bar{\omega} - \tau_\zeta\bar{\omega}] \\ &\quad + B_{n,m}(a_1,\dots,a_m)[b_1,\dots,b_n,\tau_\zeta\bar{\omega}] - B_{n,m}(a_1,\dots,a_m)[\tau_\zeta b_1,\dots,\tau_\zeta b_n,\tau_\zeta\bar{\omega}] \\ &\quad + \Big[B_{n,m}(a_1,\dots,a_m) - B_{n,m}(\tau_\zeta a_1,\dots,\tau_\zeta a_m)\Big][\tau_\zeta b_1,\dots,\tau_\zeta b_n,\tau_\zeta\bar{\omega}] \\ &= T_1(\zeta) + T_2(\zeta) + T_3(\zeta), \end{aligned}$$

where

$$\begin{aligned} T_1(\zeta) &:= B_{n,m}(a_1,\dots,a_m)[b_1,\dots,b_n,\bar{\omega} - \tau_\zeta\bar{\omega}], \\ T_2(\zeta) &:= \sum_{i=1}^n B_{n,m}(a_1,\dots,a_m)[\tau_\zeta b_1,\dots,\tau_\zeta b_{i-1}, b_i - \tau_\zeta b_i, b_{i+1},\dots,b_n,\tau_\zeta\bar{\omega}], \\ T_3(\zeta) &:= \sum_{i=1}^m B_{n+2,,m+1}(a_1,\dots,a_i,\tau_\zeta a_i,\dots,\tau_\zeta a_m)[\tau_\zeta b_1,\dots,\tau_\zeta b_n, a_i + \tau_\zeta a_i, a_i - \tau_\zeta a_i, \tau_\zeta\bar{\omega}]. \end{aligned}$$

Moreover,

$$[B_{n,m}]^2_{W_2^{r-1}} \leq 9 \sum_{i=1}^{3} \int_{\mathbb{R}} \frac{\|T_i(\zeta)\|_2^2}{|\zeta|^{1+2(r-1)}}\, d\zeta,$$

and with the help of Theorem B.3 and (B.7) we conclude

$$\begin{aligned}\int_{\mathbb{R}} \frac{\|T_1(\zeta)\|_2^2}{|\zeta|^{1+2(r-1)}}\, d\zeta &\leq C \left(\prod_{i=1}^{n} \|b_i'\|_\infty^2\right) \int_{\mathbb{R}} \frac{\|\bar{\omega} - \tau_\zeta \bar{\omega}\|_2^2}{|\zeta|^{1+2(r-1)}}\, d\zeta \\ &= C \left([\bar{\omega}]_{H^{r-1}} \prod_{i=1}^{n} \|b_i'\|_\infty\right)^2.\end{aligned}$$

In addition we use Theorem B.5, (B.7), and the symmetry of $B_{n,m}$ with respect to each of the arguments b_i, $1 \leq i \leq n$, to obtain

$$\begin{aligned}\int_{\mathbb{R}} \frac{\|T_2(\zeta)\|_2^2}{|\zeta|^{1+2(r-1)}}\, d\zeta &\leq C\|\bar{\omega}\|^2_{H^{r-1}} \sum_{i=1}^{n} \left(\prod_{\substack{j=1\\ j\neq i}}^{n} \|b_j'\|^2_{H^{r-1}} \int_{\mathbb{R}} \frac{\|b_i' - \tau_\zeta b_i'\|_2^2}{|\zeta|^{1+2(r-1)}}\, d\zeta\right) \\ &\leq C \left([\bar{\omega}]_{H^{r-1}} \prod_{i=1}^{n} \|b_i'\|_{H^{r-1}}\right)^2\end{aligned}$$

and by a similar argumentation

$$\int_{\mathbb{R}} \frac{\|T_3(\zeta)\|_2^2}{|\zeta|^{1+2(r-1)}}\, d\zeta \leq C \left([\bar{\omega}]_{H^{r-1}} \prod_{i=1}^{n} \|b_i'\|_{H^{r-1}}\right)^2.$$

Gathering these results, we have established the estimate in Theorem B.5.

Finally, the local Lipschitz continuity property is a consequence of the now proven estimate and the property (B.3). □

We now estimate the commutator type operator considered in (B.10) in the H^{r-1}-norm.

Theorem B.7. *Let $n \in \mathbb{N}$, $n \geq 1$, $3/2 < r' < r < 2$, and $a_1, \ldots, a_m \in H^r(\mathbb{R})$ be given. There exists a constant C, depending only on n, m, r, r', and $\max_{i=1,\ldots,m} \|a_i\|_{H^r}$, such that*

$$\begin{aligned}&\|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \bar{\omega}] - \bar{\omega} B_{n-1,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, b_1']\|_{H^{r-1}} \\ &\qquad\qquad \leq C\|b_1\|_{H^{r'}} \|\bar{\omega}\|_{H^{r-1}} \prod_{i=2}^{n} \|b_i\|_{H^r}\end{aligned}$$

for all $b_1, \ldots, b_n \in H^r(\mathbb{R})$ and $\bar{\omega} \in H^{r-1}(\mathbb{R})$.

Proof. In order to keep formulas short we set

$$T := B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \bar{\omega}] - \bar{\omega} B_{n-1,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, b_1'].$$

Recalling (B.10) (with $\tau \in (5/2 - r, 1)$ arbitrary), it holds that

$$\|T\|_2 \leq C\|b_1\|_{H^{r'}} \|\bar{\omega}\|_{H^{r-1}} \prod_{i=2}^{n} \|b_i\|_{H^r},$$

and it remains to estimate the term

$$[T]^2_{W_2^{r-1}} = \int_{\mathbb{R}} \frac{\|T - \tau_\zeta T\|_2^2}{|\zeta|^{1+2(r-1)}}\, d\zeta.$$

We can rewrite

$$(T - \tau_\zeta T) = T_1(\zeta) + T_2(\zeta) + T_3(\zeta) + T_4(\zeta),$$

where

$$\begin{aligned}
T_1(\zeta) &\coloneqq B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \bar\omega - \tau_\zeta \bar\omega] \\
&\quad - (\bar\omega - \tau_\zeta\bar\omega) B_{n-1,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, b_1'], \\
T_2(\zeta) &\coloneqq B_{n,m}(a_1, \ldots, a_m)[b_1 - \tau_\zeta b_1, \ldots, b_n, \tau_\zeta\bar\omega] \\
&\quad - \tau_\zeta\bar\omega B_{n-1,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, b_1' - \tau_\zeta b_1'], \\
T_3(\zeta) &\coloneqq B_{n,m}(a_1, \ldots, a_m)[\tau_\zeta b_1, b_2 \ldots, b_n, \tau_\zeta\bar\omega] \\
&\quad - B_{n,m}(\tau_\zeta a_1, \ldots, \tau_\zeta a_m)[\tau_\zeta b_1, \ldots, \tau_\zeta b_n, \tau_\zeta\bar\omega], \\
T_4(\zeta) &\coloneqq \tau_\zeta\bar\omega B_{n-1,m}(\tau_\zeta a_1, \ldots, \tau_\zeta a_m)[\tau_\zeta b_2, \ldots, \tau_\zeta b_n, \tau_\zeta b_1'] \\
&\quad - \tau_\zeta\bar\omega B_{n-1,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, \tau_\zeta b_1'].
\end{aligned}$$

Using Theorem B.3 and (B.5) (with $\tau = r' - 1$), we get

$$\begin{aligned}
\|T_1(\zeta)\|_2 &\le \|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \bar\omega - \tau_\zeta\bar\omega]\|_2 \\
&\quad + \|\bar\omega - \tau_\zeta\bar\omega\|_2 \|B_{n-1,m}(a_1, \ldots, a_m)[b_2, \ldots, b_n, b_1']\|_\infty \\
&\le C\|\bar\omega - \tau_\zeta\bar\omega\|_2 \left(\prod_{i=1}^n \|b_i'\|_\infty + \|b_1'\|_{H^{r'-1}} \prod_{i=2}^n \|b_i\|_{H^r} \right) \\
&\le C\|\bar\omega - \tau_\zeta\bar\omega\|_2 \|b_1\|_{H^{r'}} \prod_{i=2}^n \|b_i\|_{H^r}.
\end{aligned}$$

Taking advantage of (B.10) with $\tau = r' - r + 1$ and of Theorem B.3, we obtain

$$\|T_2(\zeta)\|_2 \le C\|b_1 - \tau_\zeta b_1\|_{H^{r'-r+1}} \|\bar\omega\|_{H^{r-1}} \prod_{i=2} \|b_i\|_{H^r}.$$

Recalling (B.3), we can decompose $T_3(\zeta)$ as follows

$$\begin{aligned}
T_3(\zeta) &= \sum_{i=2}^n B_{n,m}(a_1, \ldots, a_m)[\tau_\zeta b_1, \ldots, \tau_\zeta b_{i-1}, b_i - \tau_\zeta b_i, b_{i+1}, \ldots, b_n, \tau_\zeta\bar\omega] \\
&\quad + B_{n,m}(a_1, \ldots, a_m)[\tau_\zeta b_1, \ldots, \tau_\zeta b_n, \tau_\zeta\bar\omega] - B_{n,m}(\tau_\zeta a_1, \ldots, \tau_\zeta a_m)[\tau_\zeta b_1, \ldots, \tau_\zeta b_n, \tau_\zeta\bar\omega] \\
&= \sum_{i=2}^n B_{n,m}(a_1, \ldots, a_m)[\tau_\zeta b_1, \ldots, \tau_\zeta b_{i-1}, b_i - \tau_\zeta b_i, b_{i+1}, \ldots, b_n, \tau_\zeta\bar\omega] \\
&\quad - \sum_{i=1}^m B_{n+2,m+1}(a_1, \ldots, a_i, \tau_\zeta a_i, \ldots, \tau_\zeta a_m)[\tau_\zeta b_1, \ldots, \tau_\zeta b_n, a_i + \tau_\zeta a_i, a_i - \tau_\zeta a_i, \tau_\zeta\bar\omega],
\end{aligned}$$

and similarly

$$T_4(\zeta) = \tau_\zeta \bar\omega \sum_{i=2}^{n} B_{n,m}(a_1,\dots,a_m)[b_2,\dots,b_{i-1},\tau_\zeta b_i - b_i, \tau_\zeta b_{i+1},\dots,\tau_\zeta b_n, \tau_\zeta b_1']$$
$$+ \tau_\zeta \bar\omega \sum_{i=1}^{m} B_{n+1,m+1}(a_1,\dots,a_i,\tau_\zeta a_i,\dots,\tau_\zeta a_m)[\tau_\zeta b_2,\dots,\tau_\zeta b_n, a_i+\tau_\zeta a_i, a_i-\tau_\zeta a_i, \tau_\zeta b_1'].$$

Applying (B.9) with $r = r'$, we conclude

$$\begin{aligned}\|T_3(\zeta)\|_2 &\leq \sum_{i=2}^{n} \|B_{n,m}(a_1,\dots,a_m)[b_i - \tau_\zeta b_i, \tau_\zeta b_1,\dots,\tau_\zeta b_{i-1}, b_{i+1},\dots,b_n,\tau_\zeta\bar\omega]\|_2\\ &\quad+\sum_{i=1}^{m} \|B_{n+2,m+1}(a_1,\dots,a_i,\tau_\zeta a_i,\dots,\tau_\zeta a_m)[a_i-\tau_\zeta a_i, \tau_\zeta b_1,\dots,\tau_\zeta b_n, a_i+\tau_\zeta a_i,\tau_\zeta\bar\omega]\|_2\\ &\leq C\|\bar\omega\|_{H^{r-1}}\|b_1\|_{H^{r'}}\Big[\sum_{i=2}^{n}\Big(\prod_{\substack{j=2\\ j\neq i}}^{n}\|b_j\|_{H^r}\Big)\|b_i-\tau_\zeta b_i\|_{H^1}\\ &\qquad+\Big(\prod_{j=2}^{n}\|b_j\|_{H^r}\Big)\sum_{i=1}^{m}\|a_i-\tau_\zeta a_i\|_{H^1}\Big]\end{aligned}$$

and similarly

$$\|T_4(\zeta)\|_2 \leq C\|\bar\omega\|_\infty\|b_1\|_{H^{r'}}\Big[\sum_{i=2}^{n}\Big(\prod_{\substack{j=2\\ j\neq i}}^{n}\|b_j\|_{H^r}\Big)\|b_i-\tau_\zeta b_i\|_{H^1} + \Big(\prod_{j=2}^{n}\|b_j\|_{H^r}\Big)\sum_{i=1}^{m}\|a_i-\tau_\zeta a_i\|_{H^1}\Big].$$

Altogether we arrive at

$$[T]_{H^{r-1}} \leq C\|\bar\omega\|_{H^{r-1}}\prod_{i=2}^{n}\|b_i\|_{H^r}\left[\|b_1\|_{H^{r'}} + \Big(\int_{\mathbb{R}}\frac{\|b_1-\tau_\zeta b_1\|^2_{H^{r'-r+1}}}{|\zeta|^{1+2(r-1)}}\Big)^{1/2}\right],$$

and Lemma B.8 now implies the claim with the help of (B.7). □

In the proof of Lemma B.7 we used the following auxiliary lemma.

Lemma B.8. *Let $1 < r' < r < 2$. Then there exists a constant $C > 0$ such that*

$$\Big(\int_{\mathbb{R}}\frac{\|b-\tau_\zeta b\|^2_{H^{r'-r+1}}}{|\zeta|^{1+2(r-1)}}d\zeta\Big)^{1/2} \leq C\|b\|_{H^{r'}} \qquad \text{for all } b\in H^{r'}(\mathbb{R}).$$

Proof. In view of (B.7) and the definition of the Sobolev norm it holds that

$$\begin{aligned}\Big(\int_{\mathbb{R}}\frac{\|b-\tau_\zeta b\|^2_{H^{r'-r+1}}}{|\zeta|^{1+2(r-1)}}d\zeta\Big)^{1/2} &\leq C\Big(\int_{\mathbb{R}}\frac{\|b-\tau_\zeta b\|_2^2}{|\zeta|^{1+2(r-1)}}\,d\zeta + \int_{\mathbb{R}}\frac{[b-\tau_\zeta b]^2_{W_2^{r'-r+1}}}{|\zeta|^{1+2(r-1)}}\,d\zeta\Big)\\ &\leq C\|b\|_{H^{r'}} + C\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\|b-\tau_\zeta b-\tau_\xi(b-\tau_\zeta b)\|_2^2}{|\xi|^{1+2(r'-r+1)}|\zeta|^{1+2(r-1)}}\,d\xi\,d\zeta.\end{aligned}$$

It remains to estimate the last term. To this end we note that

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\|b-\tau_\zeta b-\tau_\xi(b-\tau_\zeta b)\|_2^2}{|\xi|^{1+2(r'-r+1)}|\zeta|^{1+2(r-1)}}\,d\xi\,d\zeta &= \int_{\mathbb{R}}\int_{\{|\zeta|<|\xi|\}}\frac{\|b-\tau_\zeta b-\tau_\xi(b-\tau_\zeta b)\|_2^2}{|\xi|^{1+2(r'-r+1)}|\zeta|^{1+2(r-1)}}\,d\xi\,d\zeta\\ &\quad+\int_{\mathbb{R}}\int_{\{|\zeta|>|\xi|\}}\frac{\|b-\tau_\zeta b-\tau_\xi(b-\tau_\zeta b)\|_2^2}{|\xi|^{1+2(r'-r+1)}|\zeta|^{1+2(r-1)}}\,d\xi\,d\zeta\end{aligned}$$

The mean value theorem implies that

$$\|b-\tau_\zeta b-\tau_\xi(b-\tau_\zeta b)\|_2 \leq |\zeta|\cdot\|b'-\tau_\xi b'\|_2 \quad \text{for } \xi,\zeta\in\mathbb{R},$$

and thus

$$\begin{aligned}\int_{\mathbb{R}}\int_{\{|\zeta|<|\xi|\}} &\frac{\|b-\tau_\zeta b-\tau_\xi(b-\tau_\zeta b)\|_2^2}{|\xi|^{1+2(r'-r+1)}|\zeta|^{1+2(r-1)}}\,d\xi\,d\zeta\\ &\leq C\int_{\mathbb{R}}\frac{\|b'-\tau_\xi b'\|_2^2}{|\xi|^{1+2(r'-r+1)}}\Big(\int_{\{|\zeta|<|\xi|\}}\frac{1}{|\zeta|^{1+2(r-2)}}\,d\zeta\Big)\,d\xi\\ &= C\int_{\mathbb{R}}\frac{\|b'-\tau_\xi b'\|_2^2}{|\xi|^{1+2(r'-1)}}\,d\xi\\ &\leq C\|b\|_{H^{r'}}^2.\end{aligned}$$

Moreover, the mean value theorem also leads to

$$\|b-\tau_\zeta b-\tau_\xi(b-\tau_\zeta b)\|_2 \leq |\xi|\cdot\|b'-\tau_\zeta b'\|_2 \quad \text{for } \xi,\zeta\in\mathbb{R},$$

and thus

$$\begin{aligned}\int_{\mathbb{R}}\int_{\{|\zeta|>|\xi|\}} &\frac{\|b-\tau_\zeta b-\tau_\xi(b-\tau_\zeta b)\|_2^2}{|\xi|^{1+2(r'-r+1)}|\zeta|^{1+2(r-1)}}\,d\xi\,d\zeta\\ &\leq C\int_{\mathbb{R}}\frac{\|b'-\tau_\zeta b'\|_2^2}{|\zeta|^{1+2(r-1)}}\Big(\int_{\{|\zeta|>|\xi|\}}\frac{1}{|\xi|^{1+2(r'-r)}}\,d\xi\Big)\,d\zeta\\ &= C\int_{\mathbb{R}}\frac{\|b'-\tau_\zeta b'\|_2^2}{|\zeta|^{1+2(r'-1)}}\,d\zeta\\ &\leq C\|b\|_{H^{r'}}^2.\end{aligned}$$

We conclude the claim. □

In the following we will establish the smoothness of the multilinear singular operators $B^k_{n,m}$, which are defined by

$$\begin{aligned}&B^k_{n,m}\colon H^r(\mathbb{R})\to\mathcal{L}^k_{\mathrm{sym}}(H^r(\mathbb{R}),\mathcal{L}(H^{r-1}(\mathbb{R}))),\\ &B^k_{n,m}(f)[f_1,\ldots,f_k][\bar\omega]:=B_{n+k,m}(f,\ldots,f)[f,\ldots,f,f_1,\ldots,f_k,\bar\omega],\quad \bar\omega\in H^r(\mathbb{R}).\end{aligned}\tag{B.16}$$

Computing their Fréchet derivative, we can establish the smoothness of these operators.

Lemma B.9. *Given $k,n,m\in\mathbb{N}$ and $r\in(3/2,2)$, we have*

$$B^k_{n,m}\in \mathrm{C}^\infty(H^r(\mathbb{R}),\mathcal{L}^k_{\mathrm{sym}}(H^r(\mathbb{R}),\mathcal{L}(H^{r-1}(\mathbb{R})))).\tag{B.17}$$

Lemma B.9 is clearly a consequence of the following lemma, see [63, Lemma C.4], which identifies the Fréchet derivatives of the operators $B^k_{n,m}$.

Lemma B.10. *Given $k,n,m\in\mathbb{N}$, $r\in(3/2,2)$, and $f\in H^r(\mathbb{R})$, the map $B^k_{m,n}$ is Fréchet differentiable at f, and the Fréchet derivative $\partial B^k_{n,m}(f)$ is given by*

$$\partial B^k_{n,m}(f)[g][f_1,\ldots,f_k]=nB^{k+1}_{n-1,m}(f)[f_1,\ldots,f_k,g]-2mB^{k+1}_{n+1,m+1}(f)[f_1,\ldots,f_k,g],$$

for $g,f_1,\ldots,f_k\in H^r(\mathbb{R})$.

Proof. Let $f,g,f_1,\ldots,f_k\in H^r(\mathbb{R})$. Considering the remainder

$$\begin{aligned}R(f,g)[f_1,\ldots,f_k]:=&\,(B^k_{n,m}(f+g)-B^k_{n,m}(f))[f_1,\ldots,f_k]\\ &-nB^{k+1}_{n-1,m}(f)[f_1,\ldots,f_k,g]+2mB^{k+1}_{n+1,m+1}(f)[f_1,\ldots,f_k,g]\end{aligned}$$

we note that

$$\begin{aligned}
&R(f,g)[f_1,\ldots,f_k] \\
&\quad=\sum_{j=0}^{n-2}(n-j-1)B_{n+k,m}(f+g,\ldots,f+g)[\underbrace{f+g,\ldots,f+g}_{j}\underbrace{f,\ldots,f}_{n-j-2},f_1,\ldots,f_k,g,g,\cdot] \\
&\qquad-\sum_{l=0}^{m-1}B_{n+k+2,m+1}(\underbrace{f+g,\ldots,f+g}_{m-l},\underbrace{f,\ldots,f}_{l+1})[f,\ldots,f,n(2f+g)+f,f_1,\ldots,f_k,g,g,\cdot] \\
&\qquad+2\sum_{l=0}^{m-1}\sum_{j=0}^{m-l-1}B_{n+k+4,m+2}(\underbrace{f+g,\ldots,f+g}_{m-l-j},\underbrace{f,\ldots,f}_{l+j+2})[f,\ldots,f,2f+g,f_1,\ldots,f_k,g,g,\cdot].
\end{aligned}$$

Negative upper summation limits represent an empty sum, which is set to be zero. For $\|g\|_{H^r}<1$ we can use Theorem B.5 to conclude that

$$\|R(f,g)[f_1,\ldots,f_k]\|_{\mathcal{L}(H^{r-1}(\mathbb{R}))}\le C\|g\|_{H^r}^2\prod_{i=1}^k\|f_i\|_{H^r},$$

and the claim follows. □

B.2. Localization of the operators $B_{n,m}$

In this section we present some technical results which enable us to locally approximate the operators $B_{n,m}$ in $\mathcal{L}(H^{r-1})$, $r\in(3/2,2)$, by certain Fourier multipliers, see Lemma B.13 - Lemma B.15. These results can be viewed as a generalization of the method of freezing the coefficients of elliptic differential operators and were established in [1]. They are extensively used in Section 5.1 and Section 7.2.

To begin, we establish a basic estimate which will be useful in the further analysis.

Lemma B.11. *Let $r\in(1/2,1)$. Then, there exists $C>0$ such that*

$$\|fg\|_{H^r}\le C\left(\|f\|_\infty\|g\|_{H^r}+\|g\|_\infty\|f\|_{H^r}\right)\qquad\forall f,g\in H^r(\mathbb{R}). \tag{B.18}$$

Proof. We first recall that

$$H^r(\mathbb{R})=W_2^r(\mathbb{R})$$

with equivalent norms $\|\cdot\|_{H^r}$ and $\|\cdot\|_{W_2^r}$, see (B.8). Therefore,

$$\begin{aligned}
\|fg\|_{H^r}&\le C\|fg\|_{W_2^r}\\
&\le C\left(\|fg\|_2+\left(\int_{\mathbb{R}}\frac{\|fg-f\tau_\zeta g\|_2^2+\|f\tau_\zeta g-\tau_\zeta(fg)\|_2^2}{|\zeta|^{1+2r}}\,d\zeta\right)^{\frac12}\right)\\
&\le C\left(\|f\|_\infty\|g\|_2+\|f\|_\infty[g]_{W_2^r}+\|g\|_\infty[f]_{W_2^r}\right)\\
&\le C\left(\|f\|_\infty\|g\|_{H^r}+\|g\|_\infty\|f\|_{H^r}\right)
\end{aligned}$$

for $f,g\in H^r(\mathbb{R})$, and (B.18) follows. □

Given a Lipschitz continuous function $f\colon\mathbb{R}\to\mathbb{R}$ and $n,m\in\mathbb{N}$, let

$$B_{n,m}^0(f):=B_{n,m}(f,\ldots,f)[f,\ldots,f,\cdot]\in\mathcal{L}(L_2(\mathbb{R})),$$

see (B.16). Before localizing the operators $B_{n,m}^0(f)$, we present in Lemma B.12 a commutator result established in [1, Lemma 12]

Lemma B.12. *Let n, $m \in \mathbb{N}$, $r \in (3/2, 2)$, $f \in H^r(\mathbb{R})$, and $\varphi \in \mathrm{BUC}^1(\mathbb{R})$. Then, there exists a constant $K > 0$ that depends only on $n, m, \|\varphi'\|_\infty$, and $\|f\|_{H^r}$ such that*

$$\|\varphi B^0_{n,m}(f)[\bar\omega] - B^0_{n,m}(f)[\varphi\bar\omega]\|_{H^1} \le K\|\bar\omega\|_2$$

for all $\bar\omega \in L_2(\mathbb{R})$.

Proof. We first consider the case where $\bar\omega \in \mathrm{C}^\infty_0(\mathbb{R})$. We abbreviate

$$T \coloneqq \varphi B^0_{n,m}(f)[\bar\omega] - B^0_{n,m}(f)[\varphi\bar\omega]$$

and conclude with Theorem B.3 that

$$\|T\|_2 \le K\|\bar\omega\|_2.$$

Moreover, it holds for given $\zeta \in \mathbb{R}\setminus\{0\}$ that

$$\begin{aligned}\frac{\tau_\zeta T - T}{\zeta} &= \frac{\tau_\zeta\varphi B^0_{n,m}(\tau_\zeta f)[\tau_\zeta\bar\omega] - B^0_{n,m}(\tau_\zeta f)[\tau_\zeta(\varphi\bar\omega)] - \varphi B^0_{n,m}(f)[\bar\omega] + B^0_{n,m}(f)[\varphi\bar\omega]}{\zeta}\\ &= T_1 + T_2 + T_3\\ &\quad + \frac{\varphi\left(B^0_{n,m}(\tau_\zeta f)[\bar\omega] - B^0_{n,m}(f)[\bar\omega]\right) + B^0_{n,m}(f)[\varphi\bar\omega] - B^0_{n,m}(\tau_\zeta f)[\varphi\bar\omega]}{\zeta},\end{aligned}$$

where

$$\begin{aligned}T_1 &\coloneqq \frac{\tau_\zeta\varphi - \varphi}{\zeta} B^0_{n,m}(\tau_\zeta f)[\tau_\zeta\bar\omega],\\ T_2 &\coloneqq \varphi B^0_{n,m}(\tau_\zeta f)\Big[\frac{\tau_\zeta\bar\omega - \bar\omega}{\zeta}\Big],\\ T_3 &\coloneqq -B^0_{n,m}(\tau_\zeta f)\Big[\frac{\tau_\zeta(\varphi\bar\omega) - \varphi\bar\omega}{\zeta}\Big].\end{aligned}$$

Using (B.3), we obtain

$$\begin{aligned}\varphi\left(B^0_{n,m}(\tau_\zeta f)[\bar\omega] - B^0_{n,m}(f)[\bar\omega]\right) &= \varphi\left(B^0_{n,m}(\tau_\zeta f)[\bar\omega] - B_{n,m}(f,\dots,f)[\tau_\zeta f,\dots,\tau_\zeta f,\bar\omega]\right)\\ &\quad + \varphi\left(B_{n,m}(f,\dots,f)[\tau_\zeta f,\dots,\tau_\zeta f,\bar\omega] - B^0_{n,m}(f)[\bar\omega]\right)\\ &= \zeta(T_{4a} + T_{5a}),\end{aligned}$$

where

$$\zeta T_{4a} \coloneqq \varphi\sum_{i=1}^n B_{n,m}(f,\dots,f)[\underbrace{f,\dots,f}_{i},\tau_\zeta f,\dots,\tau_\zeta f,\tau_\zeta f - f,\bar\omega],$$

and

$$\zeta T_{5a} \coloneqq -\varphi\sum_{i=1}^m B_{n+2,m+1}(\underbrace{\tau_\zeta f,\dots,\tau_\zeta f}_{i}, f\dots,f)[\tau_\zeta f,\dots,\tau_\zeta f,\tau_\zeta f + f,\tau_\zeta f - f,\bar\omega].$$

Similarly, it holds that

$$B^0_{n,m}(f)[\varphi\bar\omega] - B^0_{n,m}(\tau_\zeta f)[\varphi\bar\omega] = \zeta(T_{4b} + T_{5b}),$$

with

$$\zeta T_{4b} \coloneqq -\sum_{i=1}^n B_{n,m}(f,\dots,f)[\underbrace{f,\dots,f}_{i},\tau_\zeta f,\dots,\tau_\zeta f,\tau_\zeta f - f,\varphi\bar\omega],$$

and

$$\zeta T_{5b} := \sum_{i=1}^{m} B_{n+2,m+1}(\underbrace{\tau_\zeta f, \dots, \tau_\zeta f}_{i}, f, \dots, f)[\tau_\zeta f, \dots, \tau_\zeta f, \tau_\zeta f + f, \tau_\zeta f - f, \varphi\bar{\omega}].$$

Moreover, (B.2) leads to

$$\begin{aligned} T_4 :=& T_{4a} + T_{4b} \\ =& \frac{\tau_\zeta f - f}{\zeta} \sum_{i=1}^{n} B_{n,m}(f, \dots, f)[\underbrace{f, \dots, f}_{i}, \tau_\zeta f, \dots, \tau_\zeta f, \varphi, \bar{\omega}] \\ &- \sum_{i=1}^{n} B_{n,m}(f, \dots, f)\Big[\underbrace{f, \dots, f}_{i}, \tau_\zeta f, \dots, \tau_\zeta f, \varphi, \frac{\tau_\zeta f - f}{\zeta}\bar{\omega}\Big], \end{aligned}$$

and

$$\begin{aligned} T_5 :=& T_{5a} + T_{5b} \\ =& -\frac{\tau_\zeta f - f}{\zeta} \sum_{i=1}^{m} B_{n+2,m+1}(\underbrace{\tau_\zeta f, \dots, \tau_\zeta f}_{i}, f \dots, f)[\tau_\zeta f, \dots, \tau_\zeta f, \tau_\zeta f + f, \varphi, \bar{\omega}] \\ &+ \sum_{i=1}^{m} B_{n+2,m+1}(\underbrace{\tau_\zeta f, \dots, \tau_\zeta f}_{i}, f, \dots, f)\Big[\tau_\zeta f, \dots, \tau_\zeta f, \tau_\zeta f + f, \varphi, \frac{\tau_\zeta f - f}{\zeta}\bar{\omega}\Big]. \end{aligned}$$

In view of the representation

$$T = T_1 + T_2 + T_3 + T_4 + T_5,$$

Theorem B.3 now implies that the limit $\zeta \to 0$ of $(\tau_\zeta T - T)/\zeta$ exists in $L_2(\mathbb{R})$. Hence, $T \in H^1(\mathbb{R})$ and

$$\begin{aligned} T' = & \varphi' B^0_{n,m}(f)[\bar{\omega}] + \varphi B^0_{n,m}(f)[\bar{\omega}'] - B^0_{n,m}(f)[\varphi\bar{\omega}'] - B^0_{n,m}(f)[\varphi'\bar{\omega}] \\ &+ nf' B_{n,m}(f, \dots, f)[f, \dots, f, \varphi, \bar{\omega}] - nB_{n,m}(f, \dots, f)[f, \dots, f, \varphi, f'\bar{\omega}] \\ &- 2mf' B_{n+2,m+1}(f, \dots, f)[f, \dots, f, \varphi, \bar{\omega}] + 2mB_{n+2,m+1}(f, \dots, f)[f \dots, f, \varphi, f'\bar{\omega}]. \end{aligned}$$

Again using Theorem B.3, we conclude

$$\|T' - \varphi B^0_{n,m}(f)[\bar{\omega}'] - B^0_{n,m}(f)[\varphi\bar{\omega}']\|_2 \le K\|\bar{\omega}\|_2.$$

Thus, it remains to consider

$$T_6 := \varphi B^0_{n,m}(f)[\bar{\omega}'] - B^0_{n,m}(f)[\varphi\bar{\omega}'].$$

Integration by parts leads to

$$\begin{aligned} T_6 =& \int_{\mathbb{R}} \frac{(\delta_{[x,s]}f/s)^n}{[1 + (\delta_{[x,s]}f/s)^2]^m} \frac{\delta_{[x,s]}\varphi}{s} \frac{d}{ds}(-\bar{\omega}(x - s))\, ds \\ =& B^0_{n,m}(f)[\varphi'\bar{\omega}] - B_{n+1,m}(f, \dots, f)[f, \dots, f, \varphi, \bar{\omega}] \\ &+ nB_{n,m}(f, \dots, f)[f, \dots, f, \varphi, f'\bar{\omega}] - nB_{n+1,m}(f, \dots, f)[f, \dots, f, \varphi, \bar{\omega}] \\ &- 2mB_{n+2,m+1}(f, \dots, f)[f, \dots, f, \varphi, f'\bar{\omega}] + 2mB_{n+3,m+1}(f, \dots, f)[f, \dots, f, \varphi, \bar{\omega}], \end{aligned}$$

which gives, due to Theorem B.3, that

$$\|T_6\|_2 \leq C\|\bar\omega\|_2.$$

Altogether we conclude the claim for $\bar\omega \in \mathrm{C}_0^\infty(\mathbb{R})$ and a standard density argument leads us to the claim for general $\bar\omega \in L_2(\mathbb{R})$. □

The next three lemmas describe how to localize the operator $B^0_{n,m}(f)$ (or the product of this operator with a H^{r-1}-function). Before proceeding, we recall the definition of a finite ε-localization family $\{\pi_j^\varepsilon : -N+1 \leq j \leq N\}$ and of the associated family $\{\chi_j^\varepsilon : -N+1 \leq j \leq N\}$ from Section 5.1, where $\varepsilon \in (0,1)$ is arbitrary.

Lemma B.13. *Let n, $m \in \mathbb{N}$, $3/2 < r' < r < 2$, and $\nu \in (0,\infty)$ be given. Let further $f \in H^r(\mathbb{R})$ and $a \in \{1\} \cup H^{r-1}(\mathbb{R})$. For any sufficiently small $\varepsilon \in (0,1)$, there exists a constant K that depends only on ε, n, m, $\|f\|_{H^r}$, and $\|a\|_{H^{r-1}}$ (if $a \neq 1$) such that*

$$\Big\|\pi_j^\varepsilon a B^0_{n,m}(f)[\bar\omega] - \frac{a(x_j^\varepsilon)f'^n(x_j^\varepsilon)}{[1+f'^2(x_j^\varepsilon)]^m}H[\pi_j^\varepsilon\bar\omega]\Big\|_{H^{r-1}} \leq \nu\|\pi_j^\varepsilon\bar\omega\|_{H^{r-1}} + K\|\bar\omega\|_{H^{r'-1}}$$

for all $|j| \leq N-1$ and $\bar\omega \in H^{r-1}(\mathbb{R})$.

Proof. In the following constants denoted by C do not depend on ε, while K stands for constants that depend on ε.

We fix $|j| < N-1$ and write

$$\pi_j^\varepsilon a B^0_{n,m}(f)[\bar\omega] - \frac{a(x_j^\varepsilon)f'^n(x_j^\varepsilon)}{[1+f'^2(x_j^\varepsilon)]^m}H[\pi_j^\varepsilon\bar\omega] = T_1 + a(x_j^\varepsilon)T_2,$$

where

$$\begin{aligned}
T_1 &:= \pi_j^\varepsilon a B^0_{n,m}(f)[\bar\omega] - a(x_j^\varepsilon)B^0_{n,m}(f)[\pi_j^\varepsilon\bar\omega],\\
T_2 &:= B^0_{n,m}(f)[\pi_j^\varepsilon\bar\omega] - \frac{f'^n(x_j^\varepsilon)}{[1+f'^2(x_j^\varepsilon)]^m}H[\pi_j^\varepsilon\bar\omega].
\end{aligned}$$

We consider these terms separately.

The term T_1. Recalling that $\chi_j^\varepsilon\pi_j^\varepsilon = \pi_j^\varepsilon$, we decompose

$$T_1 = T_{1a} + a(x_j^\varepsilon)T_{1b},$$

with

$$\begin{aligned}
T_{1a} &:= \chi_j^\varepsilon(a - a(x_j^\varepsilon))\pi_j^\varepsilon B^0_{n,m}(f)[\bar\omega],\\
T_{1b} &:= \pi_j^\varepsilon B^0_{n,m}(f)[\bar\omega] - B^0_{n,m}(f)[\pi_j^\varepsilon\bar\omega].
\end{aligned}$$

Lemma B.12 now implies

$$\|T_{1b}\|_{H^{r-1}} \leq C\|T_{1b}\|_{H^1} \leq K\|\bar\omega\|_2. \tag{B.19}$$

In view of (B.18), it follows from (B.5) (with $\tau = r'-1$) that

$$\begin{aligned}
\|T_{1a}\|_{H^{r-1}} &\leq C\Big(\|\chi_j^\varepsilon(a - a(x_j^\varepsilon))\|_\infty\|\pi_j^\varepsilon B^0_{n,m}(f)[\bar\omega]\|_{H^{r-1}}\\
&\qquad + \|\chi_j^\varepsilon(a - a(x_j^\varepsilon))\|_{H^{r-1}}\|\pi_j^\varepsilon B^0_{n,m}(f)[\bar\omega]\|_\infty\Big)\\
&\leq C\|\chi_j^\varepsilon(a - a(x_j^\varepsilon))\|_\infty\Big(\|T_{1b}\|_{H^{r-1}} + \|B^0_{n,m}(f)[\pi_j^\varepsilon\bar\omega]\|_{H^{r-1}}\Big) + K\|\bar\omega\|_{H^{r'-1}}.
\end{aligned}$$

Theorem B.5 and (B.19) lead to

$$\begin{aligned}\|T_{1a}\|_{H^{r-1}} &\leq C\|\chi_j^\varepsilon(a-a(x_j^\varepsilon))\|_\infty\|B_{n,m}^0(f)[\pi_j^\varepsilon\bar\omega]\|_{H^{r-1}} + K\|\bar\omega\|_{H^{r'-1}}\\ &\leq \frac{\nu}{2}\|\pi_j^\varepsilon\bar\omega\|_{H^{r-1}} + K\|\bar\omega\|_{H^{r'-1}},\end{aligned}$$

for sufficiently small $\varepsilon\in(0,1)$, and thus

$$\|T_1\|_{H^{r-1}} \leq \frac{\nu}{2}\|\pi_j^\varepsilon\bar\omega\|_{H^{r-1}} + K\|\bar\omega\|_{H^{r'-1}}.$$

The term T_2. Using the identity $\chi_j^\varepsilon\pi_j^\varepsilon=\pi_j^\varepsilon$ once more we write

$$\begin{aligned}T_2 &= B_{n,m}^0(f)[\chi_j^\varepsilon(\pi_j^\varepsilon\bar\omega)] - \frac{f'^n(x_j^\varepsilon)}{[1+f'^2(x_j^\varepsilon)]^m}H[\chi_j^\varepsilon(\pi_j^\varepsilon\bar\omega)]\\ &= T_{2a}+T_{2b},\end{aligned}$$

where

$$\begin{aligned}T_{2a} &:= \frac{f'^n(x_j^\varepsilon)}{[1+f'^2(x_j^\varepsilon)]^m}(\chi_j^\varepsilon H[\pi_j^\varepsilon\bar\omega]-H[\chi_j^\varepsilon(\pi_j^\varepsilon\bar\omega)])\\ &\qquad -(\chi_j^\varepsilon B_{n,m}^0(f)[\pi_j^\varepsilon\bar\omega]-B_{n,m}^0(f)[\chi_j^\varepsilon(\pi_j^\varepsilon\bar\omega)]),\\ T_{2b} &:= \chi_j^\varepsilon B_{n,m}^0(f)[\pi_j^\varepsilon\bar\omega] - \frac{f'^n(x_j^\varepsilon)}{[1+f'^2(x_j^\varepsilon)]^m}\chi_j^\varepsilon H[\pi_j^\varepsilon\bar\omega].\end{aligned}$$

Lemma B.12 yields

$$\|T_{2a}\|_{H^{r-1}} \leq K\|\bar\omega\|_2. \tag{B.20}$$

Noting that Theorem B.3 gives

$$\|T_{2b}\|_2 \leq K\|\bar\omega\|_2, \tag{B.21}$$

it remains, in view of (B.7), to estimate the $W_2^{r-1}(\mathbb{R})$-seminorm of T_{2b}. With the help of the identities

$$f'(x_j^\varepsilon)=\frac{\delta_{[x,s]}(f'(x_j^\varepsilon)\mathrm{id}_\mathbb{R})}{s},\qquad H=B_{0,0},$$

and recalling (B.3), it holds that

$$\begin{aligned}T_{2b} &= \chi_j^\varepsilon\Big(B_{n,m}^0(f)[\pi_j^\varepsilon\bar\omega]-B_{n,m}(f,\ldots,f)[f'(x_j^\varepsilon)\mathrm{id}_\mathbb{R},\ldots,f'(x_j^\varepsilon)\mathrm{id}_\mathbb{R},\pi_j^\varepsilon]\\ &\qquad + B_{n,m}(f,\ldots,f)[f'(x_j^\varepsilon)\mathrm{id}_\mathbb{R},\ldots,f'(x_j^\varepsilon)\mathrm{id}_\mathbb{R},\pi_j^\varepsilon]-B_{n,m}^0(f'(x_j^\varepsilon)\mathrm{id}_\mathbb{R})[\pi_j^\varepsilon\bar\omega]\Big)\\ &= \sum_{k=0}^{n-1}(f'(x_j^\varepsilon))^{n-k-1}\chi_j^\varepsilon B_{k+1,m}(f,\ldots,f)[f,\ldots,f,f-f'(x_j^\varepsilon)\mathrm{id}_\mathbb{R},\pi_j^\varepsilon\bar\omega]\\ &\quad -\sum_{k=0}^{m-1}\frac{f'^n(x_j^\varepsilon)}{[1+f'^2(x_j^\varepsilon)]^{m-k}}\chi_j^\varepsilon B_{2,k+1}(f,\ldots,f)[f-f'(x_j^\varepsilon)\mathrm{id}_\mathbb{R},f+f'(x_j^\varepsilon)\mathrm{id}_\mathbb{R},\pi_j^\varepsilon\bar\omega].\end{aligned}$$

Let

$$B_k := \chi_j^\varepsilon B_{k+1,m}(f,\ldots,f)[f,\ldots,f,f-f'(x_j^\varepsilon)\mathrm{id}_\mathbb{R},\pi_j^\varepsilon\bar\omega]$$

for $0\leq k\leq n-1$. In order to estimate the W_2^{r-1}-seminorm of B_k we note that for $\zeta\in\mathbb{R}$ it holds that

$$\frac{\delta_{[x,s]}\tau_\zeta\mathrm{id}_\mathbb{R}}{s}=1,\qquad x,s\in\mathbb{R},$$

and write

$$\begin{aligned}B_k - \tau_\zeta B_k =&(\chi_j^\varepsilon - \tau_\zeta\chi_j^\varepsilon)\tau_\zeta B_{k+1,m}(f,\dots,f)[f,\dots,f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon\bar\omega]\\ &+\chi_j^\varepsilon\big(B_{k+1,m}(f,\dots,f)[f,\dots,f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon\bar\omega-\tau_\zeta(\pi_j^\varepsilon\bar\omega)]\\ &\qquad+B_{k+1,m}(f,\dots,f)[f,\dots,f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\tau_\zeta(\pi_j^\varepsilon\bar\omega)]\\ &\qquad-B_{k+1,m}(\tau_\zeta f,\dots,\tau_\zeta f)[\tau_\zeta f,\dots,\tau_\zeta f,\tau_\zeta f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\tau_\zeta(\pi_j^\varepsilon\bar\omega)]\big)\\ =&B_{kA}+B_{kB}+\chi_j^\varepsilon B_{kC},\end{aligned}$$

where

$$\begin{aligned}B_{kA}&:=(\chi_j^\varepsilon - \tau_\zeta\chi_j^\varepsilon)\tau_\zeta B_{k+1,m}(f,\dots,f)[f,\dots,f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon\bar\omega],\\ B_{kB}&:=\chi_j^\varepsilon B_{k+1,m}(f,\dots,f)[f,\dots,f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon\bar\omega-\tau_\zeta(\pi_j^\varepsilon\bar\omega)],\\ B_{kC}&:=B_{k+1,m}(f,\dots,f)[f,\dots,f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\tau_\zeta(\pi_j^\varepsilon\bar\omega)]\\ &\qquad-B_{k+1,m}(\tau_\zeta f,\dots,\tau_\zeta f)[\tau_\zeta f,\dots,\tau_\zeta f,\tau_\zeta f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\tau_\zeta(\pi_j^\varepsilon\bar\omega)].\end{aligned}$$

In view of (B.3) it holds

$$\begin{aligned}B_{kC} =& B_{k+1,m}(f,\dots,f)[f,\dots,f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\tau_\zeta(\pi_j^\varepsilon\bar\omega)]\\ &-B_{k+1,m}(f,\dots,f)[\tau_\zeta f,\dots,\tau_\zeta f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\tau_\zeta(\pi_j^\varepsilon\bar\omega)]\\ &+B_{k+1,m}(f,\dots,f)[\tau_\zeta f,\dots,\tau_\zeta f,f-\tau_\zeta f,\tau_\zeta(\pi_j^\varepsilon\bar\omega)]\\ &+B_{k+1,m}(f,\dots,f)[\tau_\zeta f,\dots,\tau_\zeta f,\tau_\zeta f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\tau_\zeta(\pi_j^\varepsilon\bar\omega)]\\ &-B_{k+1,m}(\tau_\zeta f,\dots,\tau_\zeta f)[\tau_\zeta f,\dots,\tau_\zeta f,\tau_\zeta f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\tau_\zeta(\pi_j^\varepsilon\bar\omega)]\\ =&\sum_{i=1}^k B_{k+1,m}(f,\dots,f)[\underbrace{\tau_\zeta f,\dots,\tau_\zeta f}_{i-1},f-\tau_\zeta f,f,\dots,f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\tau_\zeta(\pi_j^\varepsilon\bar\omega)]\\ &+B_{k+1,m}(f,\dots,f)[\tau_\zeta f,\dots,\tau_\zeta f,f-\tau_\zeta f,\tau_\zeta(\pi_j^\varepsilon\bar\omega)]\\ &+\sum_{i=1}^m \widetilde{B}^i_{k+3,m+1}[\tau_\zeta f,\dots,\tau_\zeta f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\tau_\zeta f+f,\tau_\zeta f-f,\tau_\zeta(\pi_j^\varepsilon\bar\omega)],\end{aligned}$$

where

$$\widetilde{B}^i_{k+3,m+1}:=B_{k+3,m+1}(\underbrace{f,\dots,f}_{i},\tau_\zeta f,\dots,\tau_\zeta f).$$

The estimate (B.5) with $\tau=r'-1$ yields

$$\begin{aligned}\|B_{kA}\|_2 &\leq K\|\chi_j^\varepsilon-\tau_\zeta\chi_j^\varepsilon\|_2\|B^0_{k+1,m}(f)[\pi_j^\varepsilon\bar\omega]-f'(x_j^\varepsilon)B^0_{k,m}(f)[\pi_j^\varepsilon\bar\omega]\|_\infty\\ &\leq K\|\chi_j^\varepsilon-\tau_\zeta\chi_j^\varepsilon\|_2\|\pi_j^\varepsilon\bar\omega\|_{H^{r'-1}}\\ &\leq K\|\chi_j^\varepsilon-\tau_\zeta\chi_j^\varepsilon\|_2\|\bar\omega\|_{H^{r'-1}}.\end{aligned}$$

Next we define the Lipschitz continuous function F by $F=f$ on $\operatorname{supp}\chi_j^\varepsilon$ and $F'=f'(x_j^\varepsilon)$ on $\mathbb{R}\setminus\operatorname{supp}\chi_j^\varepsilon$. If $|\zeta|\geq\varepsilon$, then Theorem B.3 shows

$$\begin{aligned}\|B_{kB}\|_2 &\leq \|\chi_j^\varepsilon(B^0_{k+1,m}(f)[\pi_j^\varepsilon\bar\omega-\tau_\zeta(\pi_j^\varepsilon\bar\omega)]-f'(x_j^\varepsilon)B^0_{k,m}(f)[\pi_j^\varepsilon\bar\omega-\tau_\zeta(\pi_j^\varepsilon\bar\omega)])\|_2\\ &\leq K\|\chi_j^\varepsilon(\pi_j^\varepsilon\bar\omega-\tau_\zeta(\pi_j^\varepsilon\bar\omega))\|_2\\ &\leq K\|\bar\omega\|_2.\end{aligned}$$

If $|\zeta|<\varepsilon$, then

$$\zeta+\operatorname{supp}\pi_j^\varepsilon\subset\operatorname{supp}\chi_j^\varepsilon$$

and thus the defining properties of F and Theorem B.3 lead to

$$\begin{aligned}\|B_{kB}\|_2 &= \|\chi_j^\varepsilon B_{k+1,m}(f,\dots,f)[f,\dots,f,F-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon\bar\omega-\tau_\zeta(\pi_j^\varepsilon\bar\omega)]\|_2\\ &\le C\|F'-f'(x_j^\varepsilon)\|_{L_\infty(\mathrm{supp}\chi_j^\varepsilon)}\|\pi_j^\varepsilon\bar\omega-\tau_\zeta(\pi_j^\varepsilon\bar\omega)\|_2\\ &\le \frac{\nu}{12(n+1)C_0^n}\|\pi_j^\varepsilon\bar\omega-\tau_\zeta(\pi_j^\varepsilon\bar\omega)\|_2,\end{aligned}$$

provided that ε is sufficiently small, where $C_0 \coloneqq 1+\|a\|_\infty+\|f'\|_\infty$.

Finally, (B.9) (with $r=r'$) gives

$$\|\chi_j^\varepsilon B_{kC}\|_2 \le K\|f'-\tau_\zeta f'\|_2\|\pi_j^\varepsilon\bar\omega\|_{H^{r'-1}} \le K\|f'-\tau_\zeta f'\|_2\|\bar\omega\|_{H^{r'-1}}.$$

Alltogether we conclude

$$[B_k]_{W_2^{r-1}} \le \frac{\nu}{4(n+1)C_0^n}\|\pi_j^\varepsilon\bar\omega\|_{H^{r-1}}+K\|\bar\omega\|_{H^{r'-1}}.$$

Similarly as above we obtain

$$\begin{aligned}&[\chi_j^\varepsilon B_{k+1,m}(f,\dots,f)[f,\dots,f,f-f'(x_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon\bar\omega]]_{W_2^{r-1}}\\ &\qquad\qquad\le \frac{\nu}{4(m+1)C_0^{n+1}}\|\pi_j^\varepsilon\bar\omega\|_{H^{r-1}}+K\|\bar\omega\|_{H^{r'-1}},\end{aligned}$$

provided that ε is sufficiently small. Recalling (B.21), we conclude for such ε that

$$\|T_{2b}\|_{W_2^{r-1}} \le \frac{\nu}{2C_0}\|\pi_j^\varepsilon\bar\omega\|_{H^{r-1}}+K\|\bar\omega\|_{H^{r'-1}},$$

and together with (B.20) and the first part we establish the claim. □

Lemma B.14 and Lemma B.15 extend the result of Lemma B.13 to the case $j=N$.

Lemma B.14. *Let n, $m\in\mathbb{N}$, $3/2<r'<r<2$, and $\nu\in(0,\infty)$ be given. Let further $f\in H^r(\mathbb{R})$ and $a\in H^{r-1}(\mathbb{R})$. For any sufficiently small $\varepsilon\in(0,1)$, there exists a constant K that depends only on ε, n, m, $\|f\|_{H^r}$, and $\|a\|_{H^{r-1}}$ such that*

$$\|\pi_N^\varepsilon aB_{n,m}^0(f)[\bar\omega]\|_{H^{r-1}} \le \nu\|\pi_N^\varepsilon\bar\omega\|_{H^{r-1}}+K\|\bar\omega\|_{H^{r'-1}}$$

for all $\bar\omega\in H^{r-1}(\mathbb{R})$.

Proof. In view of

$$\chi_N^\varepsilon\pi_N^\varepsilon=\pi_N^\varepsilon,$$

it holds that

$$\pi_N^\varepsilon aB_{n,m}^0(f)[\bar\omega]=T_1+T_2,$$

with

$$\begin{aligned}T_1 &\coloneqq \chi_N^\varepsilon a\big(\pi_N^\varepsilon B_{n,m}^0(f)[\bar\omega]-B_{n,m}^0(f)[\pi_N^\varepsilon\bar\omega]\big),\\ T_2 &\coloneqq \chi_N^\varepsilon aB_{n,m}^0(f)[\pi_N^\varepsilon\bar\omega].\end{aligned}$$

Lemma B.12 shows that

$$\|T_1\|_{H^{r-1}} \le \|T_1\|_{H^1} \le K\|\bar\omega\|_2.$$

Furthermore, if ε is sufficiently small, then

$$\|a\|_{L_\infty(\mathrm{supp}\,\chi_N^\varepsilon)}<\nu.$$

Thus, (B.18) together with Theorem B.5 and (B.5) (where $\tau = r' - 1$) implies

$$\begin{aligned}\|T_2\|_{H^{r-1}} &\le C\Big(\|\chi_N^\varepsilon a\|_\infty \|B_{n,m}^0(f)[\pi_N^\varepsilon\bar\omega]\|_{H^{r-1}} \\ &\qquad + \|\chi_N^\varepsilon a\|_{H^{r-1}}\|B_{n,m}^0(f)[\pi_N^\varepsilon\bar\omega]\|_\infty\Big) \\ &\le C\|\chi_N^\varepsilon a\|_\infty \|\pi_N^\varepsilon\bar\omega\|_{H^{r-1}} + K\|\pi_N^\varepsilon\bar\omega\|_{H^{r'-1}} \\ &\le \nu\|\pi_N^\varepsilon\bar\omega\|_{H^{r-1}} + K\|\bar\omega\|_{H^{r'-1}}.\end{aligned}$$

This completes our arguments. □

Lemma B.15 is the counterpart of Lemma B.14 in the case when $a = 1$.

Lemma B.15. *Let n, $m \in \mathbb{N}$, $3/2 < r' < r < 2$, and $\nu \in (0,\infty)$ be given. Let further $f \in H^r(\mathbb{R})$. For any sufficiently small $\varepsilon \in (0,1)$, there exists a constant K that depends only on ε, n, m, and $\|f\|_{H^r}$ such that*

$$\|\pi_N^\varepsilon B_{0,m}^0(f)[\bar\omega] - H[\pi_N^\varepsilon\bar\omega]\|_{H^{r-1}} \le \nu\|\pi_N^\varepsilon\bar\omega\|_{H^{r-1}} + K\|\bar\omega\|_{H^{r'-1}}$$

and

$$\|\pi_N^\varepsilon B_{n,m}^0(f)[\bar\omega]\|_{H^{r-1}} \le \nu\|\pi_N^\varepsilon\bar\omega\|_{H^{r-1}} + K\|\bar\omega\|_{H^{r'-1}}, \qquad n \ge 1,$$

for all $\bar\omega \in H^{r-1}(\mathbb{R})$.

Proof. First, we consider the case where $n = 0$. Then

$$\pi_N^\varepsilon B_{0,m}^0(f)[\bar\omega] - H[\pi_N^\varepsilon\bar\omega] = T_a + T_b + T_c,$$

where

$$\begin{aligned}T_a &:= \pi_N^\varepsilon B_{0,m}^0(f)[\bar\omega] - B_{0,m}^0(f)[\pi_N^\varepsilon\bar\omega],\\ T_b &:= \chi_N^\varepsilon H[\pi_N^\varepsilon\bar\omega] - H[\chi_N^\varepsilon(\pi_N^\varepsilon\bar\omega)] - \big(\chi_N^\varepsilon B_{0,m}^0(f)[\pi_N^\varepsilon\bar\omega] - B_{0,m}^0(f)[\chi_N^\varepsilon(\pi_N^\varepsilon\bar\omega)]\big),\\ T_c &:= \chi_N^\varepsilon\big(B_{0,m}(f)[\pi_N^\varepsilon\bar\omega] - H[\pi_N^\varepsilon\bar\omega]\big).\end{aligned}$$

Lemma B.12 shows that

$$\|T_a\|_{H^{r-1}} + \|T_b\|_{H^{r-1}} \le K\|\bar\omega\|_2.$$

We now consider in view of $H = B_{0,0}$ and of (B.3) the term

$$T_c = -\sum_{k=0}^{m-1}\chi_N^\varepsilon B_{2,m-k}^0(f)[\pi_N^\varepsilon\bar\omega]$$

and set

$$B_k := \chi_N^\varepsilon B_{2,m-k}^0(f)[\pi_N^\varepsilon\bar\omega], \qquad 0 \le k \le m-1.$$

On account of Theorem B.3 we have

$$\|T_c\|_2 \le K\|\bar\omega\|_2.$$

In order to estimate the W_2^{r-1}-seminorm of B_k, we write for $\zeta \in \mathbb{R}$

$$B_k - \tau_\zeta B_k = B_{kA} + B_{kB} + \chi_N^\varepsilon B_{kC},$$

with

$$\begin{aligned} B_{kA} &:= (\chi_N^\varepsilon - \tau_\zeta \chi_N^\varepsilon)\tau_\zeta B^0_{2,m-k}(f)[\pi_N^\varepsilon \bar\omega], \\ B_{kB} &:= \chi_N^\varepsilon B^0_{2,m-k}(f)[\pi_N^\varepsilon \bar\omega - \tau_\zeta(\pi_N^\varepsilon \bar\omega)], \\ B_{kC} &:= B^0_{2,m-k}(f)[\tau_\zeta(\pi_N^\varepsilon \bar\omega)] \\ &\quad - B^0_{2,m-k}(\tau_\zeta f)[\tau_\zeta(\pi_N^\varepsilon \bar\omega)]. \end{aligned}$$

The estimate (B.5) with $\tau = r' - 1$ yields

$$\begin{aligned} \|B_{kA}\|_2 &\le K\|\chi_N^\varepsilon - \tau_\zeta\chi_N^\varepsilon\|_2 \|B^0_{2,m-k}(f)[\pi_N^\varepsilon\bar\omega]\|_\infty \\ &\le K\|\chi_N^\varepsilon - \tau_\zeta\chi_N^\varepsilon\|_2 \|\pi_N^\varepsilon\bar\omega\|_{H^{r'-1}} \\ &\le K\|\chi_N^\varepsilon - \tau_\zeta\chi_N^\varepsilon\|_2 \|\bar\omega\|_{H^{r'-1}}. \end{aligned}$$

We define F as the Lipschitz continuous function with the properties $F(x) = f(x)$ if $|x| \ge 1/\varepsilon - \varepsilon$ and which is linear elsewhere. Then, Theorem B.3 yields for $|\zeta| \ge \varepsilon$ that

$$\begin{aligned} \|B_{kB}\|_2 &\le \|B^0_{2,m-k}(f)[\chi_N^\varepsilon(\pi_N^\varepsilon\bar\omega - \tau_\zeta(\pi_N^\varepsilon\bar\omega))]\|_2 \\ &\le K\|\chi_N^\varepsilon(\pi_N^\varepsilon\bar\omega - \tau_\zeta(\pi_N^\varepsilon\bar\omega))\|_2 \\ &\le K\|\bar\omega\|_2. \end{aligned}$$

If $|\zeta| < \varepsilon$, it holds that

$$\zeta + \operatorname{supp}\pi_j^\varepsilon \subset \operatorname{supp}\chi_j^\varepsilon$$

and the defining properties of F and Theorem B.3 lead to

$$\begin{aligned} \|B_{kB}\|_2 &= \|\chi_N^\varepsilon B_{2,m-k}(f,\dots,f)[F,F,\pi_N^\varepsilon\bar\omega - \tau_\zeta(\pi_N^\varepsilon\bar\omega)]\|_2 \\ &\le C\|F'\|_\infty \|\pi_N^\varepsilon\bar\omega - \tau_\zeta(\pi_N^\varepsilon\bar\omega)\|_2 \\ &\le \frac{\nu}{3(m+1)}\|\pi_N^\varepsilon\bar\omega - \tau_\zeta(\pi_N^\varepsilon\bar\omega)\|_2, \end{aligned}$$

provided that ε is sufficiently small. To obtain the last estimate we have used the following property

$$\|F'\|_\infty \to 0 \qquad \text{for} \qquad \varepsilon \to 0.$$

Moreover, (B.3) yields

$$\begin{aligned} B_{kC} &= B^0_{2,m-k}(f)[\tau_\zeta(\pi_N^\varepsilon\bar\omega)] - B_{2,m-k}(f,\dots,f)[\tau_\zeta f, \tau_\zeta f, \tau_\zeta(\pi_N^\varepsilon\bar\omega)] \\ &\quad + B_{2,m-k}(f,\dots,f)[\tau_\zeta f, \tau_\zeta f, \tau_\zeta(\pi_N^\varepsilon\bar\omega)] - B^0_{2,m-k}(\tau_\zeta f)[\tau_\zeta(\pi_N^\varepsilon\bar\omega)] \\ &= B_{2,m-k}(f,\dots,f)[f - \tau_\zeta f, f, \tau_\zeta(\pi_N^\varepsilon\bar\omega)] + B_{2,m-k}(f,\dots,f)[\tau_\zeta f, f - \tau_\zeta f, \tau_\zeta(\pi_N^\varepsilon\bar\omega)] \\ &\quad + \sum_{i=1}^{m-k} B_{4,m-k+1}(\underbrace{f,\dots,f}_{i}, \tau_\zeta f, \dots, \tau_\zeta f)[\tau_\zeta f, \tau_\zeta f, \tau_\zeta f + f, \tau_\zeta f - f, \tau_\zeta(\pi_N^\varepsilon\bar\omega)]. \end{aligned}$$

Finally, (B.9) (with $r = r'$) gives

$$\|\chi_N^\varepsilon B_{kC}\|_2 \le K\|f' - \tau_\zeta f'\|_2 \|\pi_N^\varepsilon\bar\omega\|_{H^{r'-1}} \le K\|f' - \tau_\zeta f'\|_2 \|\bar\omega\|_{H^{r'-1}}.$$

Altogether we conclude

$$[B_k]_{W_2^{r-1}} \le \frac{\nu}{3(m+1)}\|\pi_N^\varepsilon\bar\omega\|_{H^{r-1}} + K\|\bar\omega\|_{H^{r'-1}}.$$

□

C. The Hilbert and the truncated Hilbert transform

In this section we first represent the truncated Hilbert transform as a Fourier multiplier and then investigate its limits. This will lead us to the classical Hilbert transform for which we then prove some basic properties.

Definition C.1 (Truncated Hilbert Transform)**.** Let $f \in L_2(\mathbb{R})$. Then, for $0 < \delta < \eta < \infty$, we set

$$H_{\delta,\eta} f(x) := \frac{1}{\pi} \int_{\{\delta<|y|<\eta\}} \frac{f(x-y)}{y}\, dy, \quad x \in \mathbb{R}.$$

The truncated Hilbert transform $H_{\delta,\eta}$ is pointwise well-defined due to Hölder's inequality. Furthermore, $H_{\delta,\eta}$ is a continuous operator on $L_2(\mathbb{R})$, since by Minkowski's integral inequality we have

$$\begin{aligned} \pi \|H_{\delta,\eta} f\|_2 &= \left(\int_{\mathbb{R}} \left| \int_{\{\delta<|y|<\eta\}} \frac{f(x-y)}{y}\, dy \right|^2 dx \right)^{\frac{1}{2}} \\ &\leq \int_{\{\delta<|y|<\eta\}} \left(\int_{\mathbb{R}} \left| \frac{f(x-y)}{y} \right|^2 dx \right)^{\frac{1}{2}} dy \\ &\leq \frac{2(\eta-\delta)}{\delta} \|f\|_2. \end{aligned}$$

In Lemma C.2 we prove that $H_{\delta,\eta}$ is a Fourier multiplier and we compute its symbol. Below $\mathcal{F}f = \hat{f}$ denotes again the Fourier transform of the function f.

Lemma C.2. *Let $f \in L_2(\mathbb{R})$. Then*

$$\mathcal{F}(H_{\delta,\eta} f)(\xi) = \left(-\frac{2}{\pi} i \operatorname{sign}(\xi) \int_{\delta|\xi|}^{\eta|\xi|} \frac{\sin(t)}{t}\, dt \right) \mathcal{F}f(\xi), \quad \xi \in \mathbb{R}. \tag{C.1}$$

Proof. Let $\xi \in \mathbb{R}$ and $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ be given. Fubini's theorem then gives

$$\begin{aligned} \pi\sqrt{2\pi}\mathcal{F}\left(H_{\delta,\eta} f\right)(\xi) &= \int_{\{\delta<|y|<\eta\}} \int_{\mathbb{R}} \frac{f(x-y)e^{-i\xi x}}{y}\, dx\, dy \\ &= \int_{\{\delta<|y|<\eta\}} \frac{1}{y} \int_{\mathbb{R}} f(z) e^{-i\xi(z+y)}\, dz\, dy \\ &= \sqrt{2\pi}\left(\mathcal{F}f(\xi)\right) \int_{\{\delta<|y|<\eta\}} \frac{e^{-i\xi y}}{y} dy, \end{aligned}$$

where

$$\int_{\{\delta<|y|<\eta\}} \frac{e^{-i\xi y}}{y} dy = \int_\delta^\eta \frac{e^{-i\xi y} - e^{i\xi y}}{y}\, dy = -2i \int_\delta^\eta \frac{\sin(\xi y)}{y}\, dy = -2i \operatorname{sign}(\xi) \int_{\delta|\xi|}^{\eta|\xi|} \frac{\sin t}{t}\, dt.$$

We now recall that $[t \mapsto \sin(t)/t]$ is the sinc-function, which is integrable and smooth on the real line. Since $\mathcal{F}$ and $H_{\delta,\eta}$ are continuous operators, the equation (C.1) holds even for $f \in L_2(\mathbb{R})$. □

Now we study the symbol of this Fourier multiplier.

Lemma C.3. *For all* $0 \le a < b \le \infty$ *it holds*

$$\left| \int_a^b \frac{\sin(t)}{t}\, dt \right| \le \pi.$$

Proof. Assume first that $b \le \pi/2$. Then, since the sinc-function is bounded by 1, we get

$$\left| \int_a^b \frac{\sin(t)}{t}\, dt \right| \le \int_0^{\pi/2} 1\, dt = \pi/2.$$

Now, assuming that $b > \pi/2$, we can estimate

$$\begin{aligned} \left| \int_a^b \frac{\sin(t)}{t}\, dt \right| &\le \left| \int_0^{\pi/2} \frac{\sin(t)}{t}\, dt \right| + \left| \int_{\pi/2}^b \frac{\sin(t)}{t}\, dt \right| \\ &\le \frac{\pi}{2} + \left| \frac{\cos(b)}{b} + \int_{\pi/2}^b \frac{\cos(t)}{t^2}\, dt \right| \\ &\le \frac{\pi}{2} + \frac{2}{\pi} + \int_{\pi/2}^\infty t^{-2}\, dt < \pi. \end{aligned}$$

□

In the following we want to present some convergence results concerning the truncated Hilbert transform and its limits.

Lemma C.4. *Let* $f \in L_2(\mathbb{R})$ *and* $0 < \delta < \eta$. *Then*

$$H_{\delta,\eta} f \to H_\delta f \quad \text{for } \eta \to \infty \quad \text{in } L_2(\mathbb{R}),$$

where

$$H_\delta f(x) := \frac{1}{\pi} \int_{\{\delta < |y|\}} \frac{f(x-y)}{y}\, dy, \quad x \in \mathbb{R}.$$

Moreover, H_δ *is the Fourier multiplier given by*

$$\mathcal{F}\left(H_\delta f\right)(\xi) = \left(-\frac{2}{\pi} i \operatorname{sign}(\xi) \int_{\delta|\xi|}^\infty \frac{\sin(t)}{t}\, dt \right) \mathcal{F}(f)(\xi), \quad \xi \in \mathbb{R}.$$

Proof. Note that H_δ is pointwise well-defined due to Hölder's inequality. Let now $f \in L_2(\mathbb{R})$ and $x \in \mathbb{R}$. Then, using Hölders inequality, we have

$$|H_{\delta,\eta} f(x) - H_\delta f(x)| \le \int_\delta^\infty \left| \frac{f(x-y) - f(x+y)}{y} \right| \chi_{[\eta,\infty]}(y)\, dy \le C \|f\|_2 \eta^{-1/2} \to 0$$

for $\eta \to \infty$. Thus,

$$H_{\delta,\eta} f \to H_\delta f \quad \text{pointwise} \quad \text{for } \eta \to \infty.$$

Let now $A_\delta \in \mathcal{L}(L_2(\mathbb{R}))$ be the Fourier multiplier

$$\mathcal{F}(A_\delta f)(\xi) := \left(-\frac{2}{\pi} i \operatorname{sign}(\xi) \int_{\delta|\xi|}^\infty \frac{\sin(t)}{t}\, dt \right) \mathcal{F}(f)(\xi), \quad f \in L_2(\mathbb{R}), \xi \in \mathbb{R}.$$

Lemma C.3 implies that A is well-defined and, using Lebesgue's dominated convergence theorem together with Plancherel's theorem, we get

$$\|H_{\delta,\eta} f - A_\delta f\|_2 = \|(\mathcal{F} H_{\delta,\eta} - \mathcal{F} A_\delta) f\|_2 \to 0$$

for $\eta \to \infty$. Therefore there exists a subsequence $\left(H_{\delta,\eta_j} f\right)_{j\in\mathbb{N}}$ such that

$$H_{\delta,\eta_j} f \to A_\delta f \quad \text{for } j \to \infty \quad \text{pointwise almost everywhere.}$$

In conclusion,

$$A_\delta f = H_\delta f,$$

and the claim follows. □

As a direct consequence of Lemma C.3 and Lemma C.4 we obtain the following result.

Corollary C.5. *Given* $0 < \delta < \eta < \infty$*, we have*

$$\|H_\delta\|_{\mathcal{L}(L_2(\mathbb{R}))} \leq 2 \qquad \textit{and} \qquad \|H_{\delta,\eta}\|_{\mathcal{L}(L_2(\mathbb{R}))} \leq 2.$$

Next we consider the limit $\delta \searrow 0$ of H_δ. Given a rapidly decaying function f, that is $f \in \mathcal{S}(\mathbb{R})$, we define the Hilbert transform of f by

$$Hf(x) := \frac{1}{\pi}\mathrm{PV}\int_{\mathbb{R}} \frac{f(x-y)}{y}\,dy, \quad x \in \mathbb{R},$$

which is pointwise well-defined due to

$$\begin{aligned}
\pi Hf(x) &= \lim_{\delta\searrow 0}\int_{\{\delta<|y|<\infty\}} \frac{f(x-y)}{y}\,dy \\
&= \lim_{\delta\searrow 0}\int_{\delta}^{\infty} \frac{f(x-y)-f(x+y)}{y}\,dy \\
&= \int_0^{\infty} \frac{f(x-y)-f(x+y)}{y}\,dy.
\end{aligned}$$

Hence,

$$\lim_{\delta\searrow 0} H_\delta f = Hf \quad \text{pointwise,}$$

provided that $f \in \mathcal{S}(\mathbb{R})$. In Lemma C.6 below we prove that the convergence actually holds in $L_2(\mathbb{R})$ and that H is a Fourier multiplier.

Lemma C.6. *The Hilbert transform* H *extends to a continuous operator on* $L_2(\mathbb{R})$*. Moreover, given* $f \in L_2(\mathbb{R})$*, we have*

$$\mathcal{F}(Hf)(\xi) = -i\,\mathrm{sign}(\xi)\mathcal{F}f(\xi), \quad \xi \in \mathbb{R}.$$

It further holds that

$$\lim_{\delta\searrow 0} H_\delta f = Hf \quad \textit{in } L_2(\mathbb{R}).$$

Proof. We follow the same strategy as in the previous Lemma. Thus, let $f \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$. Then, letting $\delta \searrow 0$, we get

$$|H_\delta f(x) - Hf(x)| \leq \int_0^{\infty} \left|\frac{f(x-y)-f(x+y)}{y}\right| \chi_{[0,\delta]}(y)\,dy \to 0.$$

Thus,

$$H_\delta f \to Hf \quad \text{pointwise.}$$

Moreover, we define the Fourier multiplier $B \in \mathcal{L}(L_2(\mathbb{R}))$ by

$$\mathcal{F}(Bf)(\xi) := -i\,\mathrm{sign}(\xi)\mathcal{F}(f)(\xi), \quad f \in L_2(\mathbb{R}), \xi \in \mathbb{R}.$$

Since

$$\int_0^\infty \frac{\sin(t)}{t}\,dt = \frac{\pi}{2},$$

Lemma C.3, Lemma C.4, and Lebesgue's dominated convergence theorem yield

$$\|H_\delta f - Bf\|_{L_2(\mathbb{R})}^2 = \int_{\mathbb{R}} \left|-\frac{2}{\pi} i\,\mathrm{sign}(\xi)\right|^2 \left|\int_0^{\delta|\xi|} \frac{\sin(t)}{t}\,dt\right|^2 |\mathcal{F}(f)(\xi)|^2\,d\xi \to 0$$

for $\delta \searrow 0$. Therefore there exists a subsequence $\left(H_{\delta_j} f\right)_{j\in\mathbb{N}}$ such that

$$H_{\delta_j} f \to Bf \quad \text{for } j \to \infty \quad \text{pointwise almost everywhere.}$$

In conclusion, $Bf = Hf$ for all $f \in \mathcal{S}(\mathbb{R})$, and the claim follows. □

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