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Cauchy problems on Lorentzian manifolds with parallel vector and spinor fields

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Contents

Introduction

This thesis is about Cauchy problems on Lorentzian manifolds, in particular we start with the setup of a Lorentzian manifold with a parallel null vector field. When we restrict this vector field to a spacelike Cauchy surface, we obtain a solution of a constraint equation, which we will call the Riemannian constraint equation, see Equation (1.1) for more details. In the following we can ask whether a Riemannian manifold equipped with a nowhere vanishing vector field, which satisfies the constraint equation, can be extended to a Lorentzian manifold and carries a parallel null vector.

Lorentzian manifolds with parallel null vectors are of interest to us, since they have special holonomy. Recently a lot of work was carried out in the field of special holonomy, for example the work of Helga Baum, Thomas Leistner and Andree Lischewski, see [17], [28] or [21]. The holonomy group is an important tool in geometry, which is given by parallel transports along arbitrary loops with respect to a connection on a vector bundle, e. g. the Levi-Civita connection on the tangent bundle. The applications for holonomy range from Cauchy problems in Riemannian geometry (see [6]) up to physics (see [10] or [15]). In string theory physicists consider parallel null spinors and similar objects which are invariant under the spin representation of the holonomy group and therefore such a Lorentzian manifold has special holonomy.

In particular we care about the case of globally hyperbolic Lorentzian manifolds which carry a parallel null vector field. We can show that these objects have a very special local structure, i. e. they are a foliated by Riemannian submanifolds of the form

$$(I \times \mathcal{F}, g = u^{-2}ds^2 + h_s),$$

where I is an intervall and $(\mathcal{F}, h_s)$ a smooth family of Riemannian manifolds. Moreover, these foliation enable us to construct a broad class of special holonomies by encoding it into a flow equation of these foliations and their families of Riemannian metrics.

Finally, we consider the spinorial setting of Lorentzian manifolds and show that if we restrict a parallel null spinor to a spacelike Cauchy surface we obtain an imaginary W-Killing spinor, the so called spin constraints. A lot of work was done in the real case, the Killing spinors, see [6], [16] for origins and [9] for recent considerations. We will construct a solution for the spinorial Cauchy problem corresponding to the spin constraint equations similar to the first case, by reducing them to the Riemannian constraint equation (see Equation (1.1)) with the help of the construction of the Dirac current and the associated screen bundle. Thomas Leistner could show that the screen bundle of these Lorentzian manifolds carries the main part of the Lorentzian holonomy and is given by a Riemannian holonomy group, see [21].

We now describe the structure of this thesis. In the first three chapters we yield the proof of Theorem 1.1, i. e. the solution for the Cauchy problem corresponding to the Riemannian constraint equation. We call this Cauchy problem the Riemannian Cauchy problem. Moreover the solution metric is of the special form of Equation (1.2).

In Chapter 1 we start with the setup of a Lorentzian manifold with a spacelike Cauchy surface and parallel null vector field and establish locally a system of evolution equations for the metric, the dual of the parallel vector field and the Ricci curvature. Such a system enables us to construct a local solution for the Riemannian Cauchy problem.

In Chapter 2 we state a list of initial data for such equations to write down a well-posed problem, which can be solved. In the following we rewrite the previously obtained set of evolution equations as a symmetric hyperbolic system. The reformulation has the improvement that we have a uniqueness and existence theory for such a system. Therefore we obtain a local solution if we state the suitable initial data. Unfortunately we end up with a bunch of data which depends on our initial choices, e. g. a background metric.

In Chapter 3 we finish the remaining parts of the extension of the Riemannian Cauchy problem. We show the vanishing of the correlation between the background metric and the previously constructed Lorentzian metric. In accordance with that we have to establish a coupled system of PDEs for a list of data which is derived from the correlation and the vector field obtained from the local solution. Moreover we prove the existence of a parallel null vector field. The global solution is given by a simple gluing procedure of the local solutions and we have to exploit this construction, when we show the globally hyperbolicity.

In Chapter 4 we construct a solution for the Riemannian Cauchy problem following (some) ideas by Piotr Chrusciel. Initially we consider a weaker version of the system corresponding to the Riemannian constraint equation. The main idea boils down to the fact, that we express the ambient Lorentzian metric in terms of hypersurface data. The crucial ingredient will be the existence of a parallel null vector field, which is a Killing vector field. The flow of this Killing vector field is an isometry and enforces the time independence of the metric in a special chart. Therefore we are able to write down a solution for the Riemannian Cauchy problem with adapted initial conditions. In Chapter 4 we show that we can deform such a solution to a metric of the desired form in Theorem 1.1. For this purpose we construct a diffeomorphism of the Lorentzian manifold that fixes the hypersurface and pulls back the solution of our adapted Cauchy problem to a Lorentzian metric of the desired form.

One of the major goals of this thesis was to summarize the proof from [28], as presented in Chapters 1 to 3. During the writing process, a comment by Chrusciel indicated a proof for an adapted Cauchy problem corresponding to the Riemannian constraint equation presented in Chapter 4. The proof of the modified Cauchy problem seems more elegant to us, as it is very short and easy to verify, and it avoids technical analytical tools. Moreover it yields a straightforward solution for the original problem of finding a Lorentzian manifold that extends a Riemannian manifold equipped with a nowhere vanishing vector field which satisfies the Riemannian constraint equation. The advantage of the proof presented in Chapters 1 to 3 is that it puts the Riemannian constraint equation in a context similar to other Cauchy problems, e. g. to Cauchy problems for the Einstein equation in general relativity and thus it shows the strength of the involved analytical tools.

In Chapter 5 we provide another important ingredient for the construction of Lorentzian metrics with special holonomy in the following chapter. We show that the Riemannian manifold satisfying the constraint equation is foliated by Riemannian submanifolds of the form $(\{t\} \times \mathcal{F}, h_s)_{t \in I}$. In order to obtain locally the foliation, we use the Frobenius theorem. The corresponding distribution is given as the orthogonal complement of some nowhere vanishing vector field U. The integrability of this distribution is a consequence of the symmetry of the $(0, 2)$-tensor $\nabla U^\flat$.

In accordance with this fact we give a characterization of special holonomy of the ambient Lorentzian manifolds in terms of flow equations of the foliated Riemannian submanifold and their family of Riemannian metrics in Chapter 6. The special form of the Riemannian submanifolds coming from the foliation enables us to translate to a condition of special holonomy in terms of flow equations for the family of metrics h_s.

Finally we consider in Chapter 7 the case of Lorentzian spin manifolds equipped with a parallel null spinor and an analogous question as before, i. e. the spin constraint equations obtained by a restriction of a parallel spinor to a hypersurface. We will construct an extension for the spinorial Cauchy problem corresponding to the spin constraint equations.

1 The constraint equation

We start with a simple observation. Let $(\overline{M}, \overline{g})$ be a globally hyperbolic Lorentzian manifold with a spacelike Cauchy surface (M, g) and Weingarten map W. Moreover there exists a parallel null vector field V on $\overline{M}$. Let T be a time orientation, then we can decompose V along T as $uT - U$, with $U \in \Gamma(TM)$ and a function $u \in C^\infty(M)$. When we restrict $\nabla^{\overline{g}} V$ to the hypersurface M we obtain

$$\begin{aligned} 0 = \pi^{TM}(\nabla^{\overline{g}}_X \underbrace{V}_{uT-U}) &= \pi^{TM}((\partial_X u) \underbrace{T}_{=0, \text{ on } TM} + u\nabla^{\overline{g}}_X T - \nabla^{\overline{g}}_X U) \\ &= -uW(X) - \nabla^g_X U, \end{aligned}$$

for all $X \in TM$. This implies the constraint equation $\nabla^g U + uW = 0$ on M and we can ask the reversed question if we can extend a Riemannian manifold that satisfies this constraint to a Lorentzian manifold with parallel null vector field. We call the constraint equation also Riemannian constraint equation, with focus on the spin constraint equation that we will consider in Chapter 7.

Thus we want to prove the following theorem, see [28, Thm 1].

Theorem 1.1: Let (M, g) be a Riemannian manifold with a nowhere vanishing vector field U and a g-symmetric endomorphism W solving

$$\nabla^g U + uW = 0, \tag{1.1}$$

where $u = \sqrt{g(U, U)}$. Moreover, let $\lambda \in C^\infty(M, \mathbb{R})$ be a positive function.

Then there exists a neighbourhood $\overline{M}$ of M in $\mathbb{R} \times M$ and a Lorentzian metric

$$\overline{g} = -\tilde{\lambda}^2 dt^2 + g_t, \tag{1.2}$$

where g_t is a smooth family of Riemannian metrics on M and $\tilde{\lambda}$ is a positive function on $\overline{M}$ with the following conditions:

$$g_0 = g \text{ and } \tilde{\lambda}_{|M} = \lambda, \tag{1.3}$$

and the vector field U extends to a parallel null vector field on $(\overline{M}, \overline{g})$. Moreover, the Lorentzian manifold $(\overline{M}, \overline{g})$ can be chosen globally hyperbolic with spacelike Cauchy surface M and Weingarten map W.

We divide the proof in three steps.

1) We derive a list of local evolution equations in the unkowns given by the metric $\overline{g}$, the dual of the parallel null vector field $\alpha = V^\flat$ and the Ricci curvature $Z = \overline{\mathrm{Ric}} := \mathrm{Ric}^{\overline{g}}$ and solve these equations. These evolution equations depend on a choice of a background metric. This is done in Section 1.2 and Chapter 2.
2) In the second step, we show that the quantities E and $\overline{\nabla} V$ vanish, where E depends on the background metric and is morally speaking the correlation of the original metric and the background metric. This is done in Sections 3.1 and 3.2.
3) In the last step, we show that the local construction globalizes and that the solution can be chosen globally hyperbolic. This is done in Section 3.3.

1.1 Symmetric hyperbolic systems

We need the solution theory for a symmetric hyperbolic systems, because the involved equations in the following sections will be of this type, see [1, section 3.7, page 141]. We call a system

$$A^0(t,x,w)\partial_t w = \sum_i A^i(t,x,w)\partial_i w + b(t,x,w), \tag{1.4}$$

a **symmetric hyperbolic system** if it msatisfies the following:

- The functions A^ν and b are smooth functions of type $U \subset \mathbb{R}^{n+1\times n+1} \to \mathbb{R}^n$ and $\tilde{U} \subset \mathbb{R}^{n+1} \to \mathbb{R}^N$ for open sets $U, \tilde{U}$.
- The matrices A^ν are symmetric.
- The matrix A^0 is strictly positive definite, i. e. there is a constant $c > 0$ such that $A^0 \geq c \cdot \mathrm{id}$ holds.

We have a uniqueness and existence result for the Cauchy problem of a symmetric hyperbolic system, see [27, page 360].

In the future part of this thesis it is convenient to state a global version of a symmetric hyperbolic systems. Therefore we define a (global) symmetric hyperbolic system in the following way.

Definition 1.2: Let N be a time-oriented Lorentzian manifold and $E \to N$ be a vector bundle with connection ∇ and compatible metric $\langle\cdot,\cdot\rangle$. Then we call a first order differential operator $P\colon \Gamma(E) \to \Gamma(E)$ a (global) **symmetric hyperbolic system** if we have for all $x \in N$ the following statements:

- The principial symbol $\sigma(P,\xi)_x$ is symmetric for all $\xi \in T^*N$ w. r. t. to the induced bundle metric on π^*E_x with $\pi\colon T^*N \to N$.
- The bilinearform $\langle \sigma(P,\tau)\cdot, \cdot\rangle$ on π^*E is positive definite for all future-directed timelike covectors $\tau \in T_x^*N$.

In the following we will consider an important example for a global symmetric hyperbolic system.

Lemma 1.3: Let N be a time-oriented Lorentzian manifold with a vector bundle $E \to N$ equipped with a connection ∇ and a compatible metric $\langle\cdot,\cdot\rangle$. Moreover, we have a nowhere vanishing, future-directed causal vector field V on N. Then the first order differential operator $P\colon \Gamma(E) \to \Gamma(E), e \mapsto \nabla_V e$ is a symmetric hyperbolic system.

Proof of Lemma 1.3.
Let $\nabla\colon \Gamma(E) \to \Gamma(T^*N \otimes E)$ be a covariant derivative on an arbitary vector bundle $E \to M$. This is a differential operator of order 1. Indeed, let $f \in C^\infty(N,\mathbb{R})$ with $f(x) = 0, d_x f = \xi \in T_x^*N$ and $e \in E_x$ with an extended vector field $\tilde{e}$, s. t. $\tilde{e}(x) = e$, then we compute the principial symbol:

$$\sigma(P = \nabla, \xi)_x e = \nabla(f\tilde{e})_x = \underbrace{d_x f}_{=\xi} \otimes \underbrace{\tilde{e}(x)}_{=e} + \underbrace{f(x)}_{=0} \nabla(\tilde{e})_x = \xi \otimes e.$$

Let E be as above, then $P = \nabla_V$ is a first order differential operator with principial symbol $\sigma(P,\xi)\colon \pi^*E \to \pi^*E, v \mapsto \xi(V)v$. The symmetry of the principial symbol is clear, since the principial symbol is the multiplication with a smooth function.

Let τ as in Definition 1.2, then we have $\tau(V) > 0$, because V is nowhere vanishing, future-directed causal and τ is given by a future directed timelike vectorfield $\tilde{\tau}$, s. t. $\tau(V) = \langle\tilde{\tau}, V\rangle > 0$. Hence

$$\langle \sigma(P,\tau)A, A\rangle = \underbrace{\tau(V)}_{>0} \|A\|^2 > 0$$

for all $A \in \Gamma(\pi^*E)$ non-zero. ■

However, we will often consider the Cauchy problem for a section $A \in \Gamma(E)$ corresponding to the symmetric hyperbolic system $P\colon \Gamma(E) \to \Gamma(E), e \mapsto \nabla_V e$, i. e.

$$\begin{cases} P(A) = 0 & \text{on } N, \\ \quad A = 0 & \text{on } M, \end{cases} \tag{1.5}$$

where M is a spacelike Cauchy surface of the Lorentzian manifold N and V is a nowhere vanishing, parallel null vector field[1]. Similar to the local case we have an existence and uniqueness result for global symmetric hyperbolic systems, see [1, Corollary 3.7.6] and [1, Theorem 3.7.7].

1.2 The evolution equations

In this part we want to derive local evolution equations for the unkowns $(\overline{g}, \alpha, Z)$. We assume that we have an extension of the Riemannian manifold (M, g, U) like in the conclusion of the Theorem 1.1 or to be precise:

- A Lorentzian manifold $(\overline{M}, \overline{g})$ as an open neighbourhood of $\{0\} \times M$ in $\mathbb{R} \times M$ with an isometric embedding $M \hookrightarrow \overline{M}$ w. r. t. the Weingarten map W.
- A parallel null vector field V on $\overline{M}$ that extends U, i. e. V is of the form $uT - U = u(T - N)$, where T is a time orientation of $\overline{M}$, $u = \sqrt{g(U,U)}$ and $N = \frac{1}{u}U$.
- The Lorentzian metric $\overline{g}$ is given by $-\tilde{\lambda}^2 dt^2 + g_t$, where the lapse function is $\tilde{\lambda} = \sqrt{-\overline{g}(\partial_t, \partial_t)}$ and g_t is a family of Riemannian metrics with $g_0 = g$.

1.2.1 Conventions

In the following sections we will use the $\bar{\cdot}$ notation for all the data and structure on the Lorentzian manifold $\overline{M}$, e. g. the metric $\overline{g}$, the connection $\overline{\nabla}$, the Riemannian curvature $\overline{R}$ and the Ricci curvature $\overline{\mathrm{Ric}}$. However a Lorentzian metric is by definition of signature $(1, n)$ or in other words $(-, +, \dots, +)$. The greek indices $\mu, \nu, \sigma, \dots$ and the first latin indices a, b, c, d range over $0, \dots, n$ and the latin indices $i, j, k, l, \dots$ range over $1, \dots, n$. The symbol ε_μ denotes -1 if $\mu = 0$ and $+1$ else. Let $T_{\mu\nu}$ be a tensor on a manifold, then we write $T_{(\mu\nu)}$ for the symmetrization of T, i. e. $T_{(\mu\nu)} := \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$. We denote the 0-coordinate in a Lorentzian manifolds sometimes as a time variable, i. e. $\partial_0 = \partial_t$. We define the Ricci curvature as the contractions of the Riemannian curvature w. r. t. the first and third index, i. e. $\mathrm{Ric} = \sum_\mu \varepsilon_\mu R(e_\mu, \cdot, e_\mu, \cdot)$ for a local orthonormal frame $\{e_\mu\}_\mu$ of the tangent bundle.

[1] We do not need the assumption of future-direction on V, since the vector field is nowhere vanishing and therefore either future-directed or past-directed and therefore we consider the system $P = \nabla_{-V}$ for past-direction.

1.2.2 The evolution equation for Z

At the beginning we have the fact that the parallel vector field V annihilates the Riemannian curvature (i. e. $\overline{R}(V,\cdot,\cdot,\cdot) = 0$) and by the definition also the Ricci curvature, i. e.

$$\overline{\mathrm{Ric}}(V,\cdot) = \sum_\mu \varepsilon_\mu \overline{R}(e_\mu, V, e_\mu, \cdot) = 0.$$

The fact that the Riemannian tensor $\overline{R}$ is annihiliated by the parallel vector field V leads to the vanishing of $\overline{\nabla}_V \overline{\mathrm{Ric}}$, because by the product rule we have

$$\begin{aligned}
(\overline{\nabla}_V \overline{\mathrm{Ric}})(X,Y) &= \sum_\mu \varepsilon_\mu (\overline{\nabla}_V \overline{R})(e_\mu, X, e_\mu, Y) \\
&\overset{\text{second Bianchi id.}}{=} -\sum_\mu \varepsilon_\mu \Big((\overline{\nabla}_\mu \overline{R})(X, V, e_\mu, Y) + (\overline{\nabla}_X \overline{R})(V, e_\mu, e_\mu, Y) \Big) \\
&= \sum_\mu \varepsilon_\mu \Big(-\partial_\mu (\overline{R}(X, V, e_\mu, Y)) + \overline{R}(\overline{\nabla}_\mu X, V, e_\mu, Y) + \overline{R}(X, \overline{\nabla}_\mu V, e_\mu, Y) \\
&\quad + \overline{R}(X, V, \overline{\nabla}_\mu e_\mu, Y) + \overline{R}(X, V, e_\mu, \overline{\nabla}_\mu Y) - \partial_X (\overline{R}(V, e_\mu, e_\mu, Y)) \\
&\quad + \overline{R}(\overline{\nabla}_X V, e_\mu, e_\mu, Y) + \overline{R}(V, \overline{\nabla}_X e_\mu, e_\mu, Y) + \overline{R}(V, e_\mu, \overline{\nabla}_X e_\mu, Y) \\
&\quad + \overline{R}(V, e_\mu, e_\mu, \overline{\nabla}_X Y) \Big) \overset{(*)}{=} 0,
\end{aligned}$$

where $\{e_\mu\}_\mu$ is a generalised orthonormal frame of $T\overline{M}$ and $X, Y \in T\overline{M}$. We used at $(*)$, that $\overline{\nabla} V$ and $\overline{R}(V,\cdot,\cdot,\cdot)$ vanish.

In the following we will treat the Ricci curvature as a new artifical variable Z. With the previous observation by hand and the local version of the covariant derivative of a bilinear form[2]

$$\nabla_{\partial_t} Z_{\mu\nu} = \partial_t Z_{\mu\nu} - Z_{\gamma\mu}\Gamma^\gamma_{0\nu} - Z_{\gamma\nu}\Gamma^\gamma_{0\mu},$$

we write $\overline{\nabla}_V Z = 0$ as an evolution equation. At first we rewrite

$$\overline{\nabla}_V Z = 0 \Leftrightarrow \overline{\nabla}_{\partial_t} Z = \tilde{\lambda} \overline{\nabla}_N Z,$$

then we combine the two previous results to obtain:

$$\partial_t Z_{kl} = \tilde{\lambda} N^i \partial_i Z_{kl} + Z_{jl}\Gamma^j_{tk} + Z_{kj}\Gamma^j_{tl} - \tilde{\lambda} N^i \left(Z_{jl}\Gamma^j_{ik} + Z_{kj}\Gamma^j_{il} \right) \tag{1.6}$$

We only state the equation above for $k, l > 0$, because by $Z(T,\cdot) = Z(N,\cdot)$, the equations for $\partial_t Z_{0\mu}$ are given in terms of $\partial_t Z_{kl}$ and Z_{kl}, therefore redundant.

[2]This follows directly from the definition of the induced connection on tensor bundles, i. e. product rule.

1.2.3 The evolution equation for $\overline{g}$

If we want to apply the theory of symmetric hyperbolic systems directly to the Ricci curvature, then we are confronted with the problem of the invariance of the Ricci curvature under the group action of $\mathrm{Diff}(\overline{M})$. A diffeomorphism invariant operator cannot be elliptic.

Let g be a Ricci-flat metric on the toy example $M := \mathbb{R}^{n+1}$ and ϕ be a diffeomorphism of M, s. t. $\phi_\Sigma = \mathrm{id}$ and $d\phi_\Sigma = \mathrm{id}$ holds, where Σ is a Cauchy surface w. l. o. g. $\Sigma = \{x^0 = t = 0\}$. By assumption of the metric g we have that $\phi^* g$ is also Ricci-flat, because $\mathrm{Ric}^{\phi^* g} = \phi^*(\mathrm{Ric}^g) = 0$ holds. In other words, the pullback metric $\phi^* g$ will be a solution of the Einstein vaccum equation $\mathrm{Ric}^g = 0$, by the assumption that $\phi_\Sigma = \mathrm{id}$ and $d\phi_\Sigma = \mathrm{id}$ holds. But this is a contradiction for the uniqueness result of symmetric hyperbolic systems. Thus we can not write the usual Ricci operator as a symmetric hyperbolic system.

We break the gauge invariance of the diffeomorphism group on the Lorentzian manifold $\overline{M}$, by working with a background metric

$$h := -\lambda^2 dt^2 + g. \tag{1.7}$$

Moreover we define

$$\begin{aligned} F(X) &:= \quad \mathrm{tr}^{\overline{g}}(\overline{g}(\nabla^h_\cdot \cdot, X)), \\ E(X) &:= -\,\mathrm{tr}^{\overline{g}}(\overline{g}(A(\cdot,\cdot), X)), \end{aligned}$$

where $A(X,Y) := \nabla^{\overline{g}}_X Y - \nabla^h_X Y$. The tensor E is called the correlation between $\overline{g}$ and the backgrounnd metric h. Additionally we use the following representation of the Ricci curvature:

$$\overline{\mathrm{Ric}}_{\mu\nu} = -\frac{1}{2}\overline{g}^{\alpha\beta}\partial_\alpha\partial_\beta\overline{g}_{\mu\nu} + \overline{\nabla}_{(\mu}\Gamma_{\nu)} + \overline{g}^{\alpha\beta}\overline{g}^{\gamma\delta}\left[\Gamma_{\alpha\mu\gamma}\Gamma_{\beta\nu\delta} + \Gamma_{\alpha\mu\gamma}\Gamma_{\beta\delta\nu} + \Gamma_{\alpha\nu\gamma}\Gamma_{\beta\delta\mu}\right].$$

Where $\Gamma_\mu = \overline{g}^{\alpha\beta}\Gamma_{\alpha\beta\mu}$ and $\overline{\nabla}_\mu\Gamma_\nu = \partial_\mu\Gamma_\nu - \overline{g}^{\alpha\beta}\overline{g}^{\gamma\delta}\Gamma_{\mu\nu\alpha}\Gamma_{\gamma\delta\beta}$, see [14, Appendix D, page 422] for more details.

Using the definition of E, we set

$$\overline{\mathrm{Ric}} = Z - \mathrm{Sym}(\overline{\nabla}E) \tag{1.8}$$

as the new evolution equation for the metric $\overline{g}$, which is locally of the form

$$\begin{aligned} Z_{\mu\nu} = \overline{\mathrm{Ric}}_{\mu\nu} + \overline{\nabla}_{(\mu}E_{\nu)} = &-\frac{1}{2}\overline{g}^{\alpha\beta}\partial_\mu\partial_\nu\overline{g}_{\alpha\beta} + \overline{\nabla}_{(\mu}F_{\nu)} \\ &+ \overline{g}^{\alpha\beta}\overline{g}^{\gamma\delta}\left[\Gamma_{\alpha\mu\gamma}\Gamma_{\beta\nu\delta} + \Gamma_{\alpha\mu\gamma}\Gamma_{\beta\delta\nu} + \Gamma_{\alpha\nu\gamma}\Gamma_{\beta\delta\mu}\right]. \end{aligned} \tag{1.9}$$

This form has the improvement, that it only depends in its leading term on the second derivatives of the metric $\overline{g}$, since we replaced the $\overline{\nabla}_{(\mu}\Gamma_{\nu)}$ part with the $\overline{g}$ independent part $\overline{\nabla}_{(\mu}F_{\nu)}$. We will use this in the later sections for the reformulation into a symmetric hyperbolic system.

1.2.4 The evolution equation for α

Let V be a parallel null vector field, then we want an evolution equation for the dual 1-form $\alpha = V^\flat$ given by the dual of V. By the parallelity of V we have

$$(d + \delta^{\overline{g}})\alpha = (c \circ \overline{\nabla})\alpha = \sum_\mu c(e_\mu^\flat)(\overline{\nabla}_{e_\mu} V)^\flat = 0,$$

where the first equality comes from the following lemma.

Lemma 1.4: Let (M, g) be a Riemannian manifold, then we have for all forms $\alpha \in \Omega^*(M)$ the identity

$$(d + \delta^g)\alpha = (c \circ \nabla)\alpha, \tag{1.10}$$

where $c\colon T^*M \to \mathrm{End}(\Lambda^*T^*M)$ is the Clifford multiplication[3] on the form bundle Λ^*T^*M given by $c(\alpha)\omega = \alpha \wedge \omega - \iota_\alpha\omega$. Where $\iota_\alpha\omega$ is defined by

$$\iota_\alpha(v_1 \wedge \ldots \wedge v_k) \coloneqq \sum_{l=1}^k (-1)^{l+1} g(\alpha^\sharp, v_l^\sharp) v_1 \wedge \ldots \wedge \hat{v}_l \wedge \ldots \wedge v_k,$$

on elementary forms and extends linear on Λ^*T^*M.

Proof.
Let $\omega \in \Gamma(\Lambda^k T^*M)$. Then we can write $\nabla w \in \Gamma(T^*M \otimes \Lambda^k T^*M)$ as $\sum_l e_l^\flat \otimes \nabla_l \omega$, where $\{e_l\}_l$ is a local orthonormal frame of T^*M. We can plug this into the operator $c \circ \nabla$ and obtain:

$$c \circ \nabla\omega = c\left(\sum_l e_l^\flat \otimes \nabla_l \omega\right) = \sum_l e_l^\flat \wedge \nabla_l\omega - \underbrace{\sum_l \iota_{e_l} \nabla_l \omega}_{=-\delta^g\omega}.$$

[3] With respect to an identification we also write $c\colon T^*M \otimes \Lambda^*T^*M \to \Lambda^*T^*M$ for the Clifford multiplication.

Let $X_1, \dots, X_{k+1} \in \Gamma(TM)$, then we look at the remaining expression:

$$\begin{aligned}
\sum_l (e_l^\flat \wedge \nabla_l \omega)(X_1, \dots, X_{k+1}) &= \frac{1}{(k+1)!} \sum_{\sigma,l} \operatorname{sgn}(\sigma) g(X_{\sigma(1)}, e_l)(\nabla_l \omega)(X_{\sigma(2)}, \dots, X_{\sigma(k+1)}) \\
&= \frac{1}{(k+1)!} \sum_{\sigma} \operatorname{sgn}(\sigma)(\nabla_{X_{\sigma(1)}} \omega)(X_{\sigma(2)}, \dots, X_{\sigma(k+1)}) \\
&= \frac{1}{(k+1)!} \sum_{\sigma} \operatorname{sgn}(\sigma)(\nabla \omega)(X_{\sigma(1)}, \dots, X_{\sigma(k+1)}) \\
&= \operatorname{Alt}(\nabla \omega)(X_1, \dots, X_{k+1}) \\
&= d\omega(X_1, \dots, X_{k+1})
\end{aligned}$$

Where we used that the antisymmetrization operator Alt *is given by*

$$\operatorname{Alt}(\beta)(X_1, \dots, X_n) := \frac{1}{n!} \sum_{\sigma} \operatorname{sgn}(\sigma) \beta(X_1, \dots, X_n),$$

for all tensors $\beta \in (T^*M)^{\otimes n}$ and the summation runs over all permutations σ of $\{1, \dots, n\}$. In the last step we use the well-known fact that the antisymmetrization of the Levi-Civita connection is the Cartan differential on forms. ■

We take a generalised orthonormal frame $\{s_0, \dots, s_n\}$ by the Gram-Schmidt process from the coordinate vector fields $\partial_0, \dots, \partial_n$ and identify (as in the proof before) the form $\nabla\alpha \in \Gamma(T^*M \otimes \Lambda^*M)$ with $-s_0^\flat \otimes \overline{\nabla}_{s_0}\alpha + \sum_l s_l^\flat \otimes \overline{\nabla}_{s_l}\alpha$ and apply the Clifford multiplication. This yields

$$0 = -c(s_0)\overline{\nabla}_{s_0}\alpha + \sum_l c(s_l)\overline{\nabla}_{s_l}\alpha. \tag{1.11}$$

We multiply this expression with $c(s_0)$ and use the Clifford relation of the Clifford multiplication

$$c(X)c(Y) + c(Y)c(X) = -2\overline{g}(X, Y) \cdot 1$$

to obtain

$$0 = -\underbrace{c(s_0)c(s_0)}_{=-\overline{g}(s_0,s_0)=+1} \overline{\nabla}_{s_0}\alpha + \sum_l c(s_0)c(s_l)\overline{\nabla}_{s_l}\alpha$$

and thus

$$\overline{\nabla}_{s_0}\alpha = \sum_l c(s_0)c(s_l)\overline{\nabla}_{s_l}\alpha. \tag{1.12}$$

However, we obtained the basis $s_0, \ldots, s_n$ by the Gram-Schmidt process, hence we have coefficients ξ given by

$$s_0 = \frac{1}{\bar{\tilde{\lambda}}} \partial_t \text{ and } s_i = \sum_{\mu=0}^{n} \xi_i^\mu [\overline{g}] \partial_\mu.$$

Now we fix a bundle chart $x\colon U \to V$ of $\overline{M}$ of the bundle $\Lambda^* \mathbb{R}^{1,n}$ and identify in this trivialization the fiber of the form bundle $\Lambda^* \mathbb{R}^{1,n}$ with the vector space $\mathbb{R}^{2^{n+1}}$. In this identification we can write the covariant derivative as $\overline{\nabla}_{\partial_\mu} = \partial_\mu + \Gamma$, where Γ is an endomorphism on forms. The exact form of Γ is not relevant for the arguments in the following[4]. Thus, we plug in the expansion of the basis s_i with respect to the coefficients ξ^μ and the representation of the covariant derivative to rewrite Equation (1.12) as an evolution equation:

$$\begin{aligned}\left(\frac{1}{\bar{\tilde{\lambda}}} - \sum_l \xi_l^0 c(s_0) c(s_l)\right) \partial_t \alpha = &\sum_i \left(\sum_l \xi_l^i c(s_0) c(s_l)\right) \partial_i \alpha \\ &+ \left(\frac{1}{\bar{\tilde{\lambda}}} - \sum_l \xi_l^0 c(s_0) c(s_l)\right) \Gamma \alpha + \sum_{k,l} \xi_l^k c(s_0) c(s_l) \Gamma_k \alpha.\end{aligned} \tag{1.13}$$

[4]In general we can deduce from the product rule for the Levi-Civita connection: $\nabla_c T_{b_1 \ldots b_n} = \partial_c T_{b_1 \ldots b_n} - \sum_{i=1}^n \Gamma^d_{b_i c} T_{b_1 \ldots b_{i-1}, d, b_{i+1} \ldots b_n}$

2 The constraint equation as a symmetric hyperbolic system

In the last section we had considered a parallel vector field and derived a list of evolution equations for quantities corresponding to the parallel vector field and the Lorentzian metric. The next step will be to write down initial data for this equations in order to state a well-posed Cauchy problem for a symmetric hyperbolic system.

2.1 Initial data

Let $x = (x^\mu) = (t = x^0, x^i)$ be a chart of $\overline{M}$, then we introduce new variables $\overline{g}_{\mu\nu,i} \coloneqq \partial_i \overline{g}_{\mu\nu}$ and $\kappa_{\mu\nu} \coloneqq \partial_t \overline{g}_{\mu\nu}$.

We have to specify the initial data for the list $(\overline{g}_{\mu\nu}, \overline{g}_{\mu\nu,i}, \kappa_{\mu\nu}, \alpha, Z_{ij})$ to ensure that the involved equations (PDEs) are well-posed and hyperbolic. We state the complete list of the initial data here and explain the origins in the following remark:

$$\overline{g}_{|t=0} = h_{|t=0} \tag{2.1}$$

$$\overline{g}_{\mu\nu,i_{|t=0}} = (\partial_i \overline{g}_{\mu\nu})_{|t=0} = (\partial_i h_{\mu\nu})_{|t=0} \tag{2.2}$$

$$\kappa_{\mu\nu_{|t=0}} = \begin{cases} -2\lambda^2 (F_0)_{|t=0} + 2\lambda^3 \operatorname{tr}^g W, & \text{if } (\mu,\nu) = (0,0) \\ 0, & \text{if } (\mu,\nu) = (i,0),(0,j) \\ -2\lambda W_{ij,} & \text{if } (\mu,\nu) = (i,j) \end{cases} \tag{2.3}$$

$$\alpha_{|t=0} = h\left(\frac{u}{\lambda}\partial_t - U, \cdot\right)_{|t=0} \tag{2.4}$$

$$Z(N,\cdot)_{|t=0} = d \operatorname{tr}^g W + \delta^g W \tag{2.5}$$

$$\begin{aligned} Z(X,Y)_{|t=0} = {} & \operatorname{Ric}(X,Y) + R(X,N,N,Y) + W^2(X,Y) - W(X,Y) \operatorname{tr}^g W \\ & - W(X,N)W(Y,N) + W(X,Y)W(N,N) \text{ for } X, Y \in N^\perp \end{aligned} \tag{2.6}$$

Remark 2.1: We will explain the origins of the initial conditions for the evolution equations derived in the previous chapter. The first one is the vanishing of the correlation E between the Lorentzian metric $\overline{g}$ and the background metric h. The second is the desired extension of the vector field U to a parallel null vector field:

Equation (2.3): The vector $T = \frac{1}{\lambda}\partial_t$ is a time orientation for the background metric h and if we consider the following equations restricted to M for $\{e_i\}_i$ a frame of TM:

$$\begin{aligned}
\partial_t \overline{g}(e_i, e_j) &= \overline{g}(\underbrace{\overline{\nabla}_t e_i}_{=\overline{\nabla}_i \partial_t,\text{ because } [\partial_t, e_i]=0}, e_j) + \overline{g}(e_i, \overline{\nabla}_t e_j) \\
&= \overline{g}(\underbrace{\overline{\nabla}_i(\partial_t)}_{\overline{\nabla}_i(\lambda T) = (\partial_i \lambda) T + \lambda \overline{\nabla}_i T}, e_j) + \overline{g}(e_i, \overline{\nabla}_j \partial_t) \\
&= -\lambda \left[\overline{g}(W(e_i), e_j) + \overline{g}(e_i, W(e_j))\right] \\
&= -2\lambda W_{ij}.
\end{aligned}$$

Thus $\kappa_{ij_{|t=0}} = (\partial_t \overline{g}_{ij})_{|t=0} = -2\lambda W_{ij}$. The two other cases ($(\mu,\nu) = (0,i), (i,j)$) follow from the fact that the quantity E_μ has to vanish on $(\overline{M}, \overline{g})$, hence we show the following claim.

Claim: Under the assumption of the Equation (2.1), Equation (2.2) and the case $(\mu,\nu) = (i,j)$ the vanishing of $E_{\mu_{|t=0}} = 0$ is equivalent to the system

$$\begin{aligned}
\kappa_{00_{|t=0}} &= -2\lambda^2 F_{0_{|t=0}} + 2\lambda^3 \operatorname{tr}^g W \\
\kappa_{0i_{|t=0}} &= -\lambda^2 \left[-F_{i_{|t=0}} + \partial_i \log \lambda + \frac{1}{2} g^{jk}(2\partial_j g_{ki} - \partial_i g_{jk}). \right]
\end{aligned}$$

Proof.
Recall the definition of the expression $F_\mu = \overline{g}_{\mu\gamma}\overline{g}^{\alpha\beta}\tilde{\Gamma}_{\alpha\beta}^{\ \ \gamma}$ and restrict E_0 to $t = 0$:

$$\begin{aligned}
E_{0_{|t=0}} - F_{0_{|t=0}} &= -\overline{g}_{0\gamma}\overline{g}^{\alpha\beta}\Gamma_{\alpha\beta}^{\ \ \gamma} \\
&= \lambda^2 \overline{g}^{\alpha\beta}\Gamma_{\alpha\beta}^{\ \ 0} \\
&= \frac{\lambda^2}{2}\overline{g}^{\alpha\beta}\overline{g}^{0\gamma}[2\partial_\alpha \overline{g}_{\beta\gamma} - \partial_\gamma \overline{g}_{\alpha\beta}] \\
&= -\overline{g}^{\alpha\beta}\partial_\alpha \overline{g}_{\beta 0} + \frac{1}{2}\overline{g}^{\alpha\beta}\partial_0 \overline{g}_{\alpha\beta} \\
&= -g^{ij}\underbrace{(\partial_i \overline{g}_{0j})}_{=0} + \frac{1}{\lambda^2}\underbrace{\partial_0 \overline{g}_{00}}_{=\kappa_{00}} - \frac{1}{2\lambda^2}\kappa_{00} + \frac{1}{2} g^{ij}\partial_0 \overline{g}_{ij} \\
&= \frac{1}{2\lambda^2}\kappa_{00} + \underbrace{g^{ij}\kappa_{ij}}_{=-2\lambda \operatorname{tr}^g W}
\end{aligned}$$

Where we used that Equations (2.1) *and* (2.2) *holds. Equivalently we can write this as*

$$\kappa_{00_{|t=0}} = \lambda^2 E_{0_{|t=0}} - \lambda^2 F_{0_{|t=0}} + \lambda^3 \operatorname{tr}^g W.$$

We consider also the second equation for the system, where all expressions are again restricted to $t = 0$*:*

$$\begin{aligned}
E_{i_{|t=0}} - F_{i_{|t=0}} &= -\overline{g}_{i\gamma}\overline{g}^{\alpha\beta}\Gamma_{\alpha\beta}^{\ \ \gamma} \\
&= -g_{ij}\overline{g}^{\alpha\beta}\Gamma_{\alpha\beta}^{\ \ j} \\
&= -\frac{1}{2}g_{ij}\overline{g}^{\alpha\beta}\overline{g}^{j\gamma}\left[2\partial_\alpha\overline{g}_{\beta\gamma} - \partial_\gamma\overline{g}_{\alpha\beta}\right] \\
&= -\frac{1}{2}\underbrace{g_{ij}g^{jk}}_{\delta_i^k}\overline{g}^{\alpha\beta}\left[2\partial_\alpha\overline{g}_{\beta k} - \partial_k\overline{g}_{\alpha\beta}\right] \\
&= -\frac{1}{2}\underbrace{\overline{g}^{\alpha\beta}}_{\overline{g}^{00}+g^{jk}}\left[2\partial_\alpha\overline{g}_{\beta i} - \partial_i\overline{g}_{\alpha\beta}\right] \\
&= -\frac{1}{\lambda^2}(\underbrace{\partial_0\overline{g}_{0i}}_{=\kappa_{0i}} - \frac{1}{2}\underbrace{\partial_i\overline{g}_{00}}_{-\partial_i\lambda^2}) - \frac{1}{2}g^{jk}(2\partial_j g_{ki} - \partial_i g_{jk})
\end{aligned}$$

We can write the previous equivalently by:

$$\kappa_{0i_{|t=0}} = -\lambda^2\left[E_{i_{|t=0}} - F_{i_{|t=0}} + \partial_i\log\lambda + \frac{1}{2}g^{jk}(2\partial_j g_{ki} - \partial_i g_{jk})\right].$$

■

Finally, when we plug in the special form of the background metric into the second equation of the system we obtain the vanishing of the expression κ_{0i} at $t = 0$ or in other words: $(F_i)_{|t=0} = \partial_i \log\lambda + \frac{1}{2}g^{jk}(2\partial_j g_{ki} - \partial_i g_{jk})$.

Equation (2.4): As mentioned before we like to extend the vector field U on M to a parallel null vector field V, therefore we express V on the hypersurface ($t = 0$) in hypersurface data namely the right hand side of Equation (2.4).

Equation (2.5): Here we will use the fact that E should vanish on M. This has the consequence that the tensor Z coincides with $\overline{\mathrm{Ric}}$ on M:

$$Z(N,\cdot)_{|M} = \overline{\mathrm{Ric}}(N,\cdot)_{|M} \overset{\mathrm{Ric}(V,\cdot)=0}{=} \overline{\mathrm{Ric}}(T,\cdot)_{|M} = d\operatorname{tr}^g W + \delta^g W.$$

Where the last equality follows from the Mainardi equation, see [25, page 115, Proposition 33].

Equation (2.6): Similar to the previous step. Let $e_0, \dots, e_n$ be a local generalised orthogonal frame for a neighbourhood of a point in $(\overline{M}, \overline{g})$, where e_0 and e_n are given by T and N. Let $X, Y \in TM$, then we have:

$$\begin{aligned} Z(X,Y)_{|M} = \overline{\mathrm{Ric}}(X,Y)_{|M} &= \sum_{i=0}^{n} \overline{g}(e_i,e_i)\overline{R}(e_i,X,e_i,Y) \\ &= -\underbrace{\overline{R}(T,X,T,Y)}_{=\overline{R}(N,X,N,Y)} + \sum_{i=1}^{n} \overline{R}(e_i,X,e_i,Y) \\ &= \sum_{i=1}^{n-1} \overline{R}(e_i,X,e_i,Y). \end{aligned}$$

At this step we use the Gauß equation, see [25, page 100, Theorem 5]:

$$\overline{R}(A,B,C,D) = R(A,B,C,D) - W(A,C)W(B,D) + W(A,D)W(B,C),$$

where $A, B, C, D \in TM$. Then we get

$$\begin{aligned} &= \sum_{i=1}^{n-1} R(e_i,X,e_i,Y) - W(e_i,e_i)W(X,Y) + W(e_i,X)W(e_i,Y) \\ &= \mathrm{Ric}(X,Y) - R(N,X,N,Y) + W(N,N)W(X,Y) \\ &\quad - W(X,Y)\,\mathrm{tr}^g W - W(N,X)W(N,Y) + W^2(X,Y). \end{aligned}$$

With this initial data we are able to rewrite the initial evolution equations locally as symmetric hyperbolic system and obtain a well-posed Cauchy problem.

Proposition 2.2: Under the assumptions of Theorem 1.1, there exists for any point $p \in M$ an open neighbourhood $\mathcal{V}_p$ of p in $\mathbb{R} \times M$, s. t. the coupled system of equations

$$(EQ) = \begin{cases} (d + \delta^{\overline{g}})\alpha & = 0 \\ \nabla^{\overline{g}}_V Z & = 0 \\ \mathrm{Ric}^{\overline{g}} & = Z - \mathrm{Sym}(\nabla^{\overline{g}} E) \end{cases}$$

for the unkowns $(\alpha, \overline{g}, Z)$[1] on $\mathcal{V}_p$ is locally equivalent to a symmetric hyperbolic system, provided the initial data Equations (2.1) to (2.6) holds. The objects U, V, E, u above are given in terms of the unkowns $(\alpha, \overline{g}, Z)$ in the following way: $V = \alpha_1^{\sharp(\overline{g})}$,

[1]Recall: α is a form, $\overline{g}$ is Lorentzian metric and Z is a symmetric bilinear form.

$U = -\pi_{TM}(V)$, $u = \sqrt{\overline{g}(U,U)}$, $E(X) = -\operatorname{tr}^{\overline{g}}(\overline{g}(A(\cdot,\cdot),X))$, $A(X,Y) = \nabla^{\overline{g}}_X Y - \nabla^h_X Y$ and h is the background metric $-\lambda^2 dt^2 + g$ on $\mathbb{R} \times M$.

Proof.
We fix a point $p \in M$ and a coordinate neighbourhood $\mathcal{V}_p$ of p in $\mathbb{R} \times M$ and write $x = (x^\mu) = (x^0 = t, x^i)$. The unkowns $(\alpha, \overline{g}, Z)$ should vary over $\Omega^*(\mathcal{V}_p)$, $\mathcal{G}_p$ and $\Gamma(\bigodot^2 T\mathcal{V}_p)$, where $\mathcal{G}_p$ is the open subset of Lorentzian metrics on $\mathcal{V}_p$ given by

$$\mathcal{G}_p := \{\overline{g} \mid \overline{g}(\partial_t,\partial_t) < 0, \overline{g}(\operatorname{grad}^{\overline{g}} t, \operatorname{grad}^{\overline{g}} t) < 0, \overline{g}_{TM\otimes TM} \textit{ Riemannian metric}\}. \quad (2.7)$$

Indeed, $\mathcal{G}_p$ is not empty since it contains h: We have $\operatorname{grad}^h t = -\frac{1}{\lambda^2}\partial_t$ and therefore $h(\operatorname{grad}^h t, \operatorname{grad}^h t) = -\frac{1}{\lambda^2} < 0$. Furthermore the restriction of the background metric h on M is by construction positive definite, since it is g.

It remains to show that the system (EQ) is locally a symmetric hyperbolic system (see Equation (1.4)). We notice that the objects V, U, N, u depends on $\overline{g}$ and α. In particular, $\tilde{\lambda}$ only depends on the metric $\overline{g}$.

$\overline{\nabla}_V Z = 0$*: This equation is by Equation (1.6) locally given by*

$$A_1^0(t,x)\partial_t(Z_{kl})_{k,l>0} = A_1^i(t,x,\overline{g},\alpha)\partial_i(Z_{kl})_{k,l>0} + b_1(t,x,\overline{g},\partial\overline{g},\alpha,Z), \quad (2.8)$$

where we set

$$A_1^0(t,x) = \mathrm{id} \textit{ and } A_1^i(t,x,\overline{g},\alpha) = \tilde{\lambda}N^i\,\mathrm{id}$$
$$(b_1(t,x,\overline{g},\partial\overline{g},\alpha,Z))_{kl} = Z_{jl}\Gamma^j_{tk} + Z_{kj}\Gamma^j_{tl} - \tilde{\lambda}N^i\left(Z_{jl}\Gamma^j_{ik} + Z_{kj}\Gamma^j_{il}\right).$$

This is a symmetric hyperbolic system, since the matrices A_1^0, A_1^i are symmetric and $A_1^0(t,x)$ is positive definite.

$(d + \delta^{\overline{g}})\alpha = 0$*: We have a system for α of the form:*

$$A_2^0(t,x,\overline{g})\partial_t\alpha = A_2^i(t,x,\overline{g})\partial_i\alpha + b_2(t,x,\overline{g},\partial\overline{g},\alpha). \quad (2.9)$$

Where the matrices and the inhomogenous part b_2 can be read off from Equation (1.13). We have to check that the matrices are symmetric w. r. t. the metric. Indeed, its enough to show that the endomorphisms $c(s_0)c(s_i)$ are symmetric. We can reduces this claim to the following lemma.

Lemma 2.3: We consider $\mathbb{R}^{1,n}$ with an orientation induced by the standard orthonormal basis. Let $e_0, \ldots, e_n$ be a positive orthonormal frame of $\mathbb{R}^{1,n}$. Then we induce a scalar product $\langle\cdot,\cdot\rangle$ on $\Lambda^*\mathbb{R}^{1,n}$ by the rule that $\{e^{i_1} \wedge \ldots \wedge$

$e^{i_k} \mid k = 0, \ldots, n\}$ in $\Lambda^*\mathbb{R}^{1,n}$ is positive orthonormal basis for $\langle\cdot,\cdot\rangle$. Then we have

$$\langle c(e_\mu)\alpha,\beta\rangle = -\varepsilon_\mu \langle \alpha, c(e_\mu)\beta\rangle,$$

for all $\alpha, \beta \in \Lambda^*\mathbb{R}^{1,n}$. Where $\varepsilon_0 = -1$, $\varepsilon_{i>0} = 1$ and $c(X) = X^\flat \wedge -\iota_X$ is the Clifford multiplication on forms.

Proof.
W. l. o. g. let $\alpha = e^{i_1} \wedge \ldots \wedge e^{i_k}$, $\beta = e^{j_1} \wedge \ldots \wedge e^{j_l}$ where $i_1 < \ldots < i_k, j_1 < \ldots < j_l$ and check the four possible cases by hand, i. e. $\mu \in$ or $\notin \{i_1, \ldots, i_k\}$ and $\mu \in$ or $\notin \{j_1, \ldots, j_l\}$. This shows the claim. ■

The previous lemma shows that the matrices are symmetric, since

$$\begin{aligned}\langle c(s_0)c(s_i)\alpha,\beta\rangle = -\varepsilon_0\langle c(s_i),c(s_0)\beta\rangle = \varepsilon_0\varepsilon_i\langle\alpha,c(s_i)c(s_0)\beta\rangle &= \underbrace{-\varepsilon_0\varepsilon_i}_{=1}\langle\alpha,c(s_0)c(s_i)\beta\rangle \\ &= \langle\alpha,c(s_0)c(s_i)\beta\rangle\end{aligned}$$

holds. Moreover, the matrix A_2^0 is for $\overline{g} = h$ given by $\frac{1}{\lambda}\,\mathrm{id}$, thus positive definite and also positive definite on a neighbourhood of h.

$\overline{\mathrm{Ric}} = Z - \mathrm{Sym}(\overline{\nabla}E)$*: In this last case we use the trick that we introduce new variables $\kappa_{\mu\nu} := \partial_t \overline{g}_{\mu\nu}$ and $\overline{g}_{\mu\nu,i} := \partial_i \overline{g}_{\mu\nu}$ and rewrite the system with respect to this new variables. We start with Equation* (1.9)*:*

$$Z_{\mu\nu} = -\frac{1}{2}\overline{g}^{\alpha\beta}\partial_\alpha\partial_\beta\overline{g}_{\mu\nu} + \overline{\nabla}_{(\mu}F_{\nu)} + H_{\mu\nu},$$

where we set $H_{\mu\nu} := \overline{g}^{\alpha\beta}\overline{g}^{\gamma\delta}\left[\Gamma_{\alpha\mu\gamma}\Gamma_{\beta\nu\delta} + \Gamma_{\alpha\mu\gamma}\Gamma_{\beta\delta\nu} + \Gamma_{\alpha\nu\gamma}\Gamma_{\beta\delta\mu}\right]$. Then we rewrite this system in the following way:

$$\partial_t\overline{g}_{\mu\nu} = \kappa_{\mu\nu} \tag{2.10}$$
$$\overline{g}^{ij}\partial_t\overline{g}_{\mu\nu,i} = \overline{g}^{ij}\partial_i\kappa_{\mu\nu} \tag{2.11}$$
$$-\overline{g}^{00}\partial_t\kappa_{\mu\nu} = 2\overline{g}^{0j}\partial_j\kappa_{\mu\nu} + \overline{g}^{ij}\partial_j\overline{g}_{\mu\nu,i} - 2\overline{\nabla}_{(\mu}F_{\nu)} - 2H_{\mu\nu} + 2Z_{\mu\nu} \tag{2.12}$$

Indeed, this new system is equivalent to the previous one under the assumption of the initial data on $\overline{g}_{\mu\nu}$ and $\overline{g}_{\mu\nu,i}$. We have:

$$\overline{g}_{\mu\nu,i} = \partial_i\overline{g}_{\mu\nu} = \partial_i\kappa_{\mu\nu}$$

on M. Let $(\overline{g}_{\mu\nu}, \overline{g}_{\mu\nu,i}, \kappa_{\mu\nu})$ be a solution of the new system above, then by the initial condition of $\overline{g}$, we know that $\overline{g}^{ij}$ is invertible for a small neighbourhood around p, hence Equation (2.11) can be written in the form

$$0 = \partial_t \overline{g}_{\mu\nu,i} - \partial_i \underbrace{\kappa_{\mu\nu}}_{\partial_t \overline{g}_{\mu\nu}} = \partial_t \left(\overline{g}_{\mu\nu,i} - \partial_i \overline{g}_{\mu\nu} \right).$$

However, we know by the additional assumption that $\overline{g}_{\mu\nu,i} = \partial_i \overline{g}_{\mu\nu}$ holds on M and by the previous identity for any time. The Equation (2.12) is exactly the same equation as the initial one, hence we have a equivalent formulation for the Equation (1.8). We can write the new system as a symmetric hyperbolic system:

$$A_3^0(t,x,\overline{g})\partial_t(\overline{g}_{\mu\nu}, \overline{g}_{\mu\nu,i}, \kappa_{\mu\nu}) = \sum_l A_3^l(t,x,\overline{g})\partial_l(\overline{g}_{\mu\nu}, \overline{g}_{\mu\nu,i}, \kappa_{\mu\nu}) + b_3(t,x,\overline{g},\partial\overline{g},\kappa,Z) \tag{2.13}$$

where the matrices are given by

$$A_3^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \overline{g}^{ij} & 0 \\ 0 & 0 & -\overline{g}^{00} \end{pmatrix} \text{ and } A_3^i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \overline{g}^{ij} \\ 0 & \overline{g}^{ij} & 2\overline{g}^{0j} \end{pmatrix}$$

and the inhomogenous part by $b_3 = (0, 0, -2\overline{\nabla}_{(\mu} F_{\nu)} - 2H_{\mu\nu} + 2Z_{\mu\nu})$. It is evident that the matrices are symmetric and A_3^0 is positive definite, hence we have a symmetric hyperbolic system.

We proceed that we can write (EQ) locally as a symmetric hyperbolic system when we combine Equations (2.8), (2.9) and (2.13). This shows the claim. ■

2.2 Solution of the local system

We have shown in the last section that we can write the coupled system of equations for the unkowns $(\alpha, \overline{g}, Z)$ locally as a symmetric hyperbolic system in the objects $(\overline{g}_{\mu\nu}, \overline{g}_{\mu\nu,i}, \kappa_{\mu\nu}, \alpha, Z_{kl})$. By the uniqueness and existence result from the theory of symmetric hyperbolic system (see [27]), we have a local solution on a neighbourhood $\mathcal{U}_p$. This solution enables us to construct a metric $\overline{g} = \overline{g}_{\mu\nu} dx^\mu dx^\nu$, a form α and a symmetric tensor $Z = Z_{kl} dx^k dx^l$ [2] on the open set $\mathcal{U}_p$. The solution $\overline{g}$ on M is equal to h, hence we can restrict to the neighbourhood, s. t. $\overline{g}$ is of Lorentzian signature. The bilinearform Z is up to now a section of $\Gamma(\mathcal{U}_p, T^*M \otimes T^*M)$ and we

[2] Indeed, this tensor is symmetric by uniqueness of the solution and the symmetric initial conditions.

want to extend it to $T^*\mathcal{U}_p$. We do this with respect to the splitting $T\mathcal{U}_p = \mathbb{R}V \oplus TM_p$, i. e. $Z(V, \cdot) = 0$. Furthermore we restrict $\mathcal{U}_p$, s. t. $M_p = M \cap \mathcal{U}_p$ is a Cauchy surface in $\mathcal{U}_p$. Finally we define the vector field $T = \frac{1}{\lambda}\partial_t$ and check easily that this is a time orientation for $\mathcal{U}_p$. We obtained a solution

$$\omega_p = (\overline{g}^{\,\mathcal{U}_p}, \alpha^{\mathcal{U}_p}, Z^{\mathcal{U}_p})$$

on a neighbourhood of a point $p \in M$.

3 Global solution and the proof of Theorem 1.1

3.1 Properties of E and $\overline{\nabla}V$

In the last part we obtained a local solution $(\alpha, \overline{g}, Z)$ for the system (EQ) on an open neighbourhood $\mathcal{U}_p$ of $p \in M$ from the local reformulation of the constraint equation. In this section we want to derive properties for this local solution. Up to now, the form α is a general element in $\Omega^*(\mathcal{U}_p)$, but in the end we want an 1-form to consider the dual vector field. Thus we show the following lemma.

Lemma 3.1: Let $(\alpha, \overline{g}, Z)$ be the local solution on $\mathcal{U}_p$ of the local reformulation of (EQ) as constructed in the previous section and assume the initial condition $\alpha_{|M} = h\left(\frac{u}{\lambda}\partial_t - U, \cdot\right)_{|M}$ as in Proposition 2.2, then α is an 1-form.

For the proof of the previous lemma we need the notion of a **normally hyperbolic operator**, see [1, chapter 2].

Definition 3.2: Let E be a vector bundle over a Lorentzian manifold $\overline{M}$ equipped with a bundle metric $\langle \cdot,\cdot \rangle$. An operator $P\colon \Gamma(E) \to \Gamma(E)$ on this bundle of order 2 is called **normally hyperbolic** if the principial symbol P is of the form

$$\sigma(P, \xi) = -\langle \xi,\xi \rangle \, \mathrm{id}_{\pi^* E},$$

for all $\xi \in T^*\overline{M}$. In particular, if we choose coordinates $(x^\mu) = (x^0, \ldots, x^n)$, we can write the operator P locally as:

$$P(x) = \sum_{\mu\nu} -\overline{g}^{\mu\nu}(x)\partial_\mu\partial_\nu + \sum_{\mu} A_\mu(x)\partial_\mu + B(x)$$

Example 3.3: The Hodge-Laplacian $\Delta^{HL} = (d + \delta^{\overline{g}})^2$ on a Lorentzian manifold $(\overline{M}, \overline{g})$ is a normally hyperbolic operator. Indeed, we compute the principial symbol of the Cartan differential, i. e. $P = d\colon \Omega^k(M) \to \Omega^{k+1}(M), \alpha \mapsto d\alpha$. Let

$f \in C^\infty(M)$ be a smooth function and x be a point of the Riemannian manifold (M,g), s. t. $f(x) = 0$ and $d_x f = \xi$ holds. Moreover, we have a k-form α with extension $\tilde{\alpha}$ to a section in $\Omega^k(M)$ such that $\tilde{\alpha}(x) = \alpha$ holds, then we compute:

$$\sigma(P = d, \xi)_x \tilde{\alpha} = d(f\tilde{\alpha})_x = (d_x f) \wedge \tilde{\alpha}(x) + f(x) d_x \tilde{\alpha} = \xi \wedge \alpha.$$

Therefore the principial symbol is $\sigma(d, \xi) = \xi \wedge \cdot$. It is immediate that the principial symbol of Δ is given by

$$\sigma(\Delta, \xi) = \sigma((d + \delta^g)^2, \xi) = \sigma(d + \delta^g, \xi)^2 = (\sigma(d,\xi) - \sigma(d,\xi)^*)^2 = c(\xi)^2 = -\langle \xi,\xi \rangle \cdot \mathrm{id}\,.$$

Where we used the exchange rule of the principial symbol with the adjoint, i. e. $\sigma(P^*, \xi) = -\sigma(P,\xi)^*$, and the composition rule $\sigma(P \circ Q, \xi) = \sigma(P, \xi) \circ \sigma(Q, \xi)$. Finally we know that the adjoint of the wedge ($\xi \wedge \cdot$) is given by the insertion of a covector, i. e. $(\xi \wedge \cdot)^* = \iota_\xi$. It is also possible to use the identity $(d + \delta^g) = c \circ \nabla$ from Lemma 1.4 and the previous comments on the principial symbol.

We are able to proof the lemma with the previous consideration.

Proof of Lemma 3.1.
In the first step we decompose the form α degreewise i. e. $\alpha = \alpha_0 + \ldots + \alpha_n \in \Omega^*(\mathcal{U}_p) = \bigoplus_{k=0}^{n+1} \Omega^k(\mathcal{U}_p)$. We know that $\Delta^{HL}\alpha = (d + \overline{\delta^g})^2 \alpha$ vanish by construction of α and as well $\Delta^{HL}\alpha_k$ for every k, since Δ^{HL} preserves the degree of the forms. Now we can consider the following well-posed normally hyperbolic system

$$\begin{cases} \Delta^{HL}\alpha_k &= 0, \quad \textit{on } \mathcal{U}_p \\ \alpha_k &= 0, \quad \textit{on } M_p \\ (\overline{\nabla}_{\partial_t}\alpha_k) &= 0, \quad \textit{on } M_p \end{cases}$$

for each $k \neq 1$ on $M_p = M \cap \mathcal{U}_p$. Indeed, when we use Equation (1.12)*, then we observe for $k \neq 1$ the following:*

$$\left(\overline{\nabla}_{\partial_t}\alpha_k\right)_{|M_p} = \sum_l c(\partial_t) c(s_l) \overline{\nabla}_l \alpha_{k|M_p} = 0.$$

Where we used that the derivative $\overline{\nabla}_l \alpha_k$ vanish on M, since we derive in TM_p direction and α_k already vanish on M_p. The vanishing of α_k on M_p is exactly the Equation (2.4)*. Thus both initial conditions are satisfied and we can conclude by the uniqueness result for normally hyperbolic system (see [8, Corollary 3.2.4,page 76]) that α_k vanish for $k \neq 1$. Here we needed the previous noticed fact that M_p is a Cauchy surface of $\mathcal{U}_p$.*

■

The previous lemma yields the simplified solution $(\overline{g}, \alpha = \alpha_1, Z)$ and we can construct the vector field $V = \alpha_1^{\flat(\overline{g})}$ and decompose V along the time orientation T in the form $uT - U = u(T - N)$ by the initial condition in Equation (2.4). The data $(\overline{g}, V, Z)$ satisfies by construction the following equations:

$$\overline{\mathrm{Ric}} = Z - \mathrm{Sym}(\overline{\nabla}E), \tag{3.1}$$

$$(d + \delta^{\overline{g}})V^{\flat} = 0, \tag{3.2}$$

$$\overline{\nabla}_V Z(A, B) = 0 \text{ for all } A, B \in TM, \tag{3.3}$$

$$Z(V, \cdot) = 0. \tag{3.4}$$

For the sake of completeness and latter usage we will write down the local expressions of the equations above:

$$\overline{\mathrm{Ric}}_{ab} = Z_{ab} - \frac{1}{2}\left(\overline{\nabla}_a E_b + \overline{\nabla}_b E_a\right), \tag{3.5}$$

$$(d + \delta^{\overline{g}})V_a = 0, \tag{3.6}$$

$$V^c \overline{\nabla}_c Z_{ij} = 0, \tag{3.7}$$

$$V^c Z_{ca} = 0. \tag{3.8}$$

The next goal is to show the vanishing of E and $\overline{\nabla}V$. The idea is to write down a coupled system of a normally hyperbolic system and symmetric hyperbolic system for the data that involves $E, \overline{\nabla}V$ and terms that are derivatives and contractions of E and $\overline{\nabla}V$. Then we will write this coupled system locally as a symmetric hyperbolic system and show that E and $\overline{\nabla}V$ vanish. Here we will use the initial conditions (Equations (2.1) to (2.6)).

In the following we will consider a bunch of tensors of different types, thus we collect them in a single vector field:

$$\eta = (\overline{\nabla}V, E, \overline{\nabla}_V E, (\overline{\nabla}E)(V)).$$

This is a section of the bundle

$$\mathcal{E} := (T^*\mathcal{U}_p \otimes T\mathcal{U}_p) \oplus T^*\mathcal{U}_p \oplus T^*\mathcal{U}_p \oplus T^*\mathcal{U}_p,$$

which we equip with the induced covariant Laplace operator $\Delta = \overline{\nabla}^*\overline{\nabla}$ acting in each summand. We need furthermore the tensor $\xi = \delta^{\overline{g}}L$, where L is given by $Z - \frac{1}{2}(\mathrm{tr}^{\overline{g}} Z)\overline{g}$. We want to show the following proposition.

Proposition 3.4: Let Δ be the induced covariant Laplacian on $\mathcal{E}$, (which acts diagonally on the sections of $\mathcal{E}$) then there exist linear operators F, H on $\mathcal{E}$, such that

$$\Delta\eta = F(\eta, \overline{\nabla}\eta, \xi), \tag{3.9}$$
$$\overline{\nabla}_V \xi = H(\eta, \overline{\nabla}\eta) \tag{3.10}$$

holds.

The first step in this proof is to derive the coupled system in the data $\overline{\nabla}V$, E, $\overline{\nabla}_V E$, $(\overline{\nabla}E)(V)$ and $\delta^{\overline{g}}L$. The following lemmata involves the following definition.

Definition 3.5: Let $A, B, C_r \in \Gamma(T^{*,*}M)$ be tensors on a Riemannian manifold (M, g) for $r = 1, \dots, k$. Then we write

$$A \equiv B \mod (C_1, \dots, C_k),$$

if there exists a linear function $F\colon T^{*,*}M^{\otimes N} \to T^{*,*}M$ in the variables C and all possible contractions of elements C_r w. r. t. the metric g, which also depends on the Riemannian metric g, such that $A - B = F(C, \mathcal{C}C)$ holds, where $\mathcal{C}C$ denotes the list of all contractions of all elements of C.

The following example will be of great importance in future.

Example 3.6: We want to prove $\overline{\nabla}_V Z \equiv 0 \mod (\overline{\nabla}V)$.

We decompose V in the form $u(T - N)$ and rewrite this as $T = \frac{1}{u}V + N$. Let $X = A + aT$ and $Y = B + bT$ be general elements in $T\mathcal{U}_p = TM_p \oplus \mathbb{R}T$, then we calculate:

$$(\overline{\nabla}_V Z)(X, Y) = \underbrace{(\overline{\nabla}_V Z)(A, B)}_{=0,\text{Equation (3.4)}} + b(\overline{\nabla}_V Z)(A, T) + a(\overline{\nabla}_V Z)(T, B) + ab(\overline{\nabla}_V Z)(T, T)$$

So we consider the expressions $(\overline{\nabla}_V Z)(T, T)$ and $(\overline{\nabla}_V Z)(T, X)$ for $X \in TM_p$:

$$\begin{aligned}
(\overline{\nabla}_V Z)(T, T) &= \frac{1}{u^2}(\overline{\nabla}_V Z)(V, V) + \frac{2}{u}(\overline{\nabla}_V Z)(V, N) + \underbrace{(\overline{\nabla}_V Z)(N, N)}_{=0} \\
&= \frac{1}{u^2}\left(\overline{\nabla}_V \underbrace{(Z(V, V))}_{=0} - 2\underbrace{Z(\overline{\nabla}_V V, V)}_{=0}\right) \\
&\quad + \frac{2}{u}\left(\overline{\nabla}_V \underbrace{(Z(V, N))}_{=0} - Z(\overline{\nabla}_V V, N) - \underbrace{Z(V, \overline{\nabla}_V N)}_{=0}\right) \\
&= -\frac{2}{u} Z(\overline{\nabla}_V V, N).
\end{aligned}$$

So we have shown that $(\nabla_V Z)(T,T) \equiv 0 \mod (\overline{\nabla}V)$ holds.

Similar for $(\overline{\nabla}_V Z)(V,X) \equiv 0 \mod (\overline{\nabla}V)$:

$$
\begin{aligned}
(\overline{\nabla}_V Z)(T,X) &= \frac{1}{u}(\overline{\nabla}_V Z)(V,X) + \underbrace{(\overline{\nabla}_V Z)(N,X)}_{=0} \\
&= \frac{1}{u}\Big[\partial_V(Z(V,X)) - Z(\overline{\nabla}_V V, X) - Z(V, \overline{\nabla}_V X)\Big] \\
&= -\frac{1}{u} Z(\overline{\nabla}_V V, X).
\end{aligned}
$$

In the following we will use frequently the commutator of covariant derivatives on arbitary tensors, see [7, page 286].

Lemma 3.7: Let $T^{k_1\ldots k_p}_{l_1\ldots l_q}$ be a tensor of type (p,q), then the curvature of this tensor in terms of the usual Riemannian curvature is given by

$$
[\nabla_a, \nabla_b]\, T^{k_1\ldots k_p}_{l_1\ldots l_q} = \sum_{r=1}^{p} \overline{R}_{abm}{}^{k_r} T^{k_1\ldots k_{r-1},m,k_{r+1}\ldots k_p}_{l_1\ldots l_q} - \sum_{s=1}^{q} \overline{R}_{abl_s}{}^{m} T^{k_1\ldots k_p}_{l_1\ldots l_{s-1},m,l_{s+1}\ldots l_q}. \tag{3.11}
$$

In particular the frequently used cases are

$$
[\nabla_a, \nabla_b] X^c = \overline{R}_{ab}{}^{c}{}_{d} X^d \tag{3.12}
$$

$$
\text{and } [\nabla_a, \nabla_b] T_c{}^d = \overline{R}_{abm}{}^{d} T_c{}^m - \overline{R}_{abc}{}^{m} T_m{}^d. \tag{3.13}
$$

As indicated in Proposition 3.4, we have to compute the Laplacians of the sections η and $\delta^{\overline{g}}L$ in linear terms of $\eta, \overline{\nabla}\eta$ and $\delta^{\overline{g}}L$. We state the first result.

Lemma 3.8: The derivative $\overline{\nabla}V$ satisfies:

$$
\Delta\overline{\nabla}V \equiv 0 \mod \left(\overline{\nabla}V, \overline{\nabla}E, (\overline{\nabla}\overline{\nabla}E)(V), \overline{\nabla}\overline{\nabla}_V E, \overline{\nabla}_V \overline{\nabla}E\right).
$$

Proof.
We plug $\overline{\nabla}V$ into the Laplacian:

$$\begin{aligned}
\Delta\overline{\nabla}_a V^b &= \overline{\nabla}^c \underbrace{\overline{\nabla}_c\overline{\nabla}_a}_{\textit{commute}} V^b \overset{\textit{Equation } (3.12)}{=} \overline{\nabla}^c \left[\overline{\nabla}_a\overline{\nabla}_c V^b + \overline{R}_{ca\ \ d}^{\ \ \ b} V^d\right] \\
&= \underbrace{\overline{\nabla}^c\overline{\nabla}_a}_{\textit{commute}} \overline{\nabla}_c V^b + V^d \overline{\nabla}^c \overline{R}_{ca\ \ d}^{\ \ \ b} + \overline{R}_{ca\ \ d}^{\ \ \ b} \overline{\nabla}^c V^d \\
&\overset{\textit{Equation } (3.13)}{=} \overline{\nabla}_a\overline{\nabla}^c\overline{\nabla}_c V^b - \overline{\mathrm{Ric}}_a^{\ c}\overline{\nabla}_c V^b + \overline{R}^{d\ b}_{\ a\ c}\overline{\nabla}_d V^c + V^d\overline{\nabla}^c\overline{R}_{ca\ \ d}^{\ \ \ b} + \overline{R}_{ca\ \ d}^{\ \ \ b}\overline{\nabla}^c V^d \\
&\equiv \underbrace{\overline{\nabla}_a\overline{\nabla}^c\overline{\nabla}_c V^b}_{(*)} + \underbrace{V^d\overline{\nabla}^c\overline{R}_{ca\ \ d}^{\ \ \ b}}_{(**)} \mod \left(\overline{\nabla}V\right)
\end{aligned}$$

We have an identity between the two Laplacians $\Delta^{HL} := (d+\delta^{\overline{g}})^2$ and $\Delta := \overline{\nabla}^a\overline{\nabla}_a$ given by

$$\Delta^{HL} = \Delta + \overline{\mathrm{Ric}}. \tag{3.14}$$

This is a so called Weitzenböck formula. By the Equation (3.6) *we have*

$$0 = \Delta^{HL} V^b = \overline{\nabla}^a\overline{\nabla}_a V^b + \overline{\mathrm{Ric}}^b_{\ a} V^a \tag{3.15}$$

and can write $(*)$ as:

$$\begin{aligned}
\overline{\nabla}_a \Delta V^b &\overset{\textit{Equation } (3.15)}{=} -\overline{\nabla}_a\left[\overline{\mathrm{Ric}}^b_{\ c} V^c\right] = -V^c\overline{\nabla}_a\overline{\mathrm{Ric}}^b_{\ c} - \overline{\mathrm{Ric}}^b_{\ c}\overline{\nabla}_a V^c \\
&\overset{\textit{Equation } (3.5)}{=} -V^c\overline{\nabla}_a Z^b_{\ c} + \frac{1}{2}V^c\left[\overline{\nabla}_a\overline{\nabla}^b E_c + \overline{\nabla}_a\overline{\nabla}_c E^b\right] - \overline{\mathrm{Ric}}^b_{\ c}\overline{\nabla}_a V^c \\
&\equiv 0 \mod \left(\overline{\nabla}V, \overline{\nabla}E, (\overline{\nabla}\overline{\nabla}E)(V), (\overline{\nabla}\overline{\nabla}_V E)\right)
\end{aligned}$$

Where we used that the expression $V^c\overline{\nabla}_a Z^b_{\ c}$ is equivalently given by:

$$\begin{aligned}
V^c\overline{\nabla}_a Z^b_{\ c} = \overline{\nabla}_a \underbrace{[Z^b_{\ c}V^c]}_{=0,\textit{by Equation } (3.8)} - Z^b_{\ c}\overline{\nabla}_a V^c &= \left(\overline{\mathrm{Ric}}^b_{\ c} + \frac{1}{2}(\overline{\nabla}^b E_c + \overline{\nabla}_c E^b)\right)\overline{\nabla}_a V^c \\
&\equiv 0 \mod (\overline{\nabla}V, \overline{\nabla}E)
\end{aligned}$$

Finally we consider the expression $(**)$:

$$\begin{aligned}
V^d\overline{\nabla}^a\overline{R}_{ac\ \ d}^{\ \ \ b} &= -V^d\left[\overline{\nabla}^b\overline{R}_{d\ ac}^{\ a} + \overline{\nabla}_d\overline{R}^{ab}_{\ \ ac}\right] \\
&\overset{(\alpha)}{=} V^d\left[\overline{\nabla}^b\overline{\mathrm{Ric}}_{da} - \overline{\nabla}_d\overline{\mathrm{Ric}}_{bc}\right] \\
&= V^d\left[\overline{\nabla}^b Z_{da} - \frac{1}{2}\overline{\nabla}^b[\overline{\nabla}_d E_a + \overline{\nabla}_a E_d] - \overline{\nabla}_d Z_{bc} + \frac{1}{2}\overline{\nabla}_d[\overline{\nabla}_b E_c + \overline{\nabla}_c E_b]\right] \\
&\equiv 0 \mod \left(\overline{\nabla}V, \overline{\nabla}E, (\overline{\nabla}\overline{\nabla}E)(V), \overline{\nabla}_V\overline{\nabla}E, \overline{\nabla}\overline{\nabla}_V E\right).
\end{aligned}$$

Where we used at (α) the second Bianchi identity: $\overline{\nabla}_a\overline{R}_{bcde} + \overline{\nabla}_b\overline{R}_{cade} + \overline{\nabla}_c\overline{R}_{abde} = 0$.

■

Lemma 3.9: The 1-form $(\overline{\nabla}E)(V)$ satisfies:

$$\Delta((\overline{\nabla}E)(V)) \equiv 0 \mod \left(\overline{\nabla}V, \overline{\nabla}\overline{\nabla}V, E, \overline{\nabla}E\right).$$

Proof.
We calculate:

$$\begin{aligned}(\Delta((\overline{\nabla}E)(V)))_a &= \overline{\nabla}^c\overline{\nabla}_c(V^b\overline{\nabla}_aE_b) \\ &= \underbrace{(\overline{\nabla}^c\overline{\nabla}_cV^b)(\overline{\nabla}_aE_b)}_{(\alpha)} + 2\underbrace{(\overline{\nabla}_cV^b)\overline{\nabla}^c\overline{\nabla}_aE_b}_{(\beta)} + \underbrace{V^b\overline{\nabla}^c\overline{\nabla}_c\overline{\nabla}_aE_b}_{(\gamma)}.\end{aligned}$$

(α): *Here we use Equation (3.14) to obtain:*

$$\Delta V_b = \underbrace{\Delta^{HL}V_b}_{=0} - \overline{\mathrm{Ric}}_b{}^aV_a \equiv 0 \mod (\overline{\nabla}V).$$

(β): *Similar to the previous case.*

(γ): *We start with*

$$V^b\Delta\overline{\nabla}_aE_b = V^b\overline{\nabla}^c\overline{\nabla}_c\overline{\nabla}_aE_b \equiv V^b\overline{\nabla}_a\Delta E_b \mod (\overline{\nabla}V, \overline{\nabla}\overline{\nabla}V, E, \overline{\nabla}E), \quad (3.16)$$

where we used that if we pass by the covariant derivative $\overline{\nabla}_a$ along the Laplacian we obtain curvature expressions depending on E.

Now we have to show the identity $\Delta E_b \equiv \delta L_b \mod E$, where $G(Z) := Z - \frac{1}{2}\operatorname{tr}(Z)\overline{g}$ is the Einstein tensor of Z and $L := G(Z)$. We know by the vanishing divergence of the Einstein tensor $G(\overline{\mathrm{Ric}})$, that we have

$$0 = \delta G(\overline{\mathrm{Ric}}) = \delta G(Z - \mathrm{Sym}(\overline{\nabla}E)) = \delta G(Z) - \delta G(\overline{\nabla}E) = \delta L - \delta G(\overline{\nabla}E)$$

and moreover

$$\delta G(\overline{\nabla}E)_a = \overline{\nabla}_b\left[\frac{1}{2}(\overline{\nabla}^bE_a + \overline{\nabla}_aE^b) - \frac{1}{2}(\overline{\nabla}_bE^b)\right] = \frac{1}{2}\left[\underbrace{\overline{\nabla}^b\overline{\nabla}_b}_{=\Delta}E_a + \underbrace{\overline{\nabla}_a\overline{\nabla}_bE^b - \overline{\nabla}_b\overline{\nabla}_aE^b}_{=-\overline{\mathrm{Ric}}_{ab}E^b}\right]$$

and thus

$$0 = \delta L_a - \frac{1}{2}\Delta E_a - \frac{1}{2}\overline{\mathrm{Ric}}_{ab}E^b. \quad (3.17)$$

We conclude $\Delta E_a \equiv \delta L_a \mod E$. *The Equation* (3.16) *reads as*

$$V^b\overline{\nabla}_a\Delta E_b \equiv V^b\overline{\nabla}_a(\delta L)_b \mod (\overline{\nabla}V, E, \overline{\nabla}E)$$

and we look at the expression $V^b\delta L_b$ *and obtain*

$$\begin{aligned} V^b\delta L_b &= V^b\overline{\nabla}_a Z^a_{\ b} - \frac{1}{2}V^b\overline{\nabla}_b Z_c^{\ c} \\ &= \overline{\nabla}_a \underbrace{(V^b Z^a_{\ b})}_{=0} - \overline{\nabla}_a V^b Z^a_{\ b} - \frac{1}{2}V^b\overline{\nabla}_b Z_c^{\ c} \equiv 0 \mod \overline{\nabla}V. \end{aligned} \tag{3.18}$$

Finally we have shown all remaining cases ($(\alpha), (\beta), (\gamma)$) *and conclude the claim.* ■

Lemma 3.10: The 1-form $\overline{\nabla}_V\delta L$ satisfies:

$$\overline{\nabla}_V\delta L \equiv 0 \mod (\overline{\nabla}V, \overline{\nabla\nabla}V, \overline{\nabla}E).$$

Proof.
We compute with the definition of L*:*

$$\begin{aligned} V^b\overline{\nabla}_b\delta L_a &= V^b\overline{\nabla}_b\overline{\nabla}_c Z^c_{\ a} - \frac{1}{2}\underbrace{V^b\overline{\nabla}_b\overline{\nabla}_a Z_c^{\ c}}_{\equiv 0 \mod \overline{\nabla}V} \\ &\equiv V^b\overline{\nabla}_b\overline{\nabla}_c Z^c_{\ a} \mod \overline{\nabla}V \\ &= V^b\overline{\nabla}^c\overline{\nabla}_b Z_{ca} - \overline{R}^c_{\ bc}{}^d Z_{da} - \overline{R}^c_{\ ba}{}^d Z_{cd} \\ &= \underbrace{V^b\overline{\nabla}^c\overline{\nabla}_b Z_{ca}}_{(\alpha)} - \underbrace{V^b\overline{\mathrm{Ric}}_b{}^d Z_{da}}_{(\beta)} - \underbrace{V^b\overline{R}^c_{\ ba}{}^d Z_{cd}}_{(\gamma)} \end{aligned}$$

and look at the remaining cases:

(α)*: We already know that* $\overline{\nabla}_V Z \equiv 0 \mod \overline{\nabla}V$*, hence* $\overline{\nabla\nabla}_V Z \equiv 0 \mod (\overline{\nabla}V, \overline{\nabla\nabla}V)$.

(β)*: This expression only depends on the first derivative of* V*, hence* $(\beta) \equiv 0 \mod \overline{\nabla}V$.

(γ)*: We argue similar to the previous case. The expression* (γ) *only depends on the first and second derivative of* $\overline{\nabla}V$ *and thus is equivalent to* $0 \mod (\overline{\nabla}V, \overline{\nabla\nabla}V, \overline{\nabla}E)$.

■

3.2 Vanishing of E and $\overline{\nabla}V$

In this section we want to show the vanishing of $\overline{\nabla}V$ and E, in particular the proof of Proposition 3.4. We will use the existence and uniqueness result from the theory of symmetric hyperbolic systems.

Proof of Proposition 3.4.
Let η be given by $(\overline{\nabla}V, E, \overline{\nabla}_V E, (\overline{\nabla}E)(V))$, then $\Delta\eta$ is entrywise linear in the data $\eta, \overline{\nabla}\eta$ and ξ, because of Lemma 3.8, Lemma 3.9 and Lemma 3.10:

$\Delta\overline{\nabla}V$*: Here we know by Lemma 3.8, that*

$$\Delta\overline{\nabla}V \equiv 0 \mod (\overline{\nabla}V, \overline{\nabla}E, (\overline{\nabla}\overline{\nabla}E)(V), \overline{\nabla}\overline{\nabla}_V E, \overline{\nabla}_V\overline{\nabla}E)$$

holds. All the expressions in the brackets are part of η or $\overline{\nabla}\eta$.

ΔE*: In the proof of Lemma 3.9 we have shown $\Delta E \equiv \delta L = \xi \mod E$.*

$\Delta\overline{\nabla}_V E$*: Here we have $\Delta\overline{\nabla}_V E \equiv \overline{\nabla}_V\xi \equiv 0 \mod (\overline{\nabla}V, \overline{\nabla}\overline{\nabla}V\overline{\nabla}E)$ by Lemma 3.10.*

$\Delta(\overline{\nabla}E)(V)$*: Also by Lemma 3.9 we have: $\Delta(\overline{\nabla}E)(V) \equiv 0 \mod (\overline{\nabla}V, \overline{\nabla}\overline{\nabla}V, \overline{\nabla}E)$.*

$\overline{\nabla}_V\xi$*: This is also clear by Lemma 3.10.*

However we have constructed linear operators F and H, which depend linear on the data $\eta, \overline{\nabla}\eta$ and ξ as indicated in the claim. ■

Now we trivialize the bundle $\mathcal{E}$ on a neighbourhood $U \subset \overline{M}$, w. l. o. g. $U = \mathcal{U}_p$ with coordinates $(x^0, \dots, x^n)$ and consider for arbitary sections η and ξ the coupled system of Equations (3.9) and (3.10) as an operator which acts on η, ξ and view this as a **normally hyperbolic** operator.

We will use the fact that the operator $P = \Delta - F(\cdot, \overline{\nabla}\cdot, \xi)$ is a normally hyperbolic operator for each ξ and write Equations (3.9) and (3.10) locally as a symmetric hyperbolic system.

Proposition 3.11: We can write Equations (3.9) and (3.10) locally on $\overline{M}$ as a symmetric hyperbolic system, under the assumption of the initial Equations (2.1) to (2.6).

Proof.
We trivialize the bundle $\mathcal{E}$ in an open neighbourhood U of a point, i. e. $\mathcal{E}_{|U} \cong U \times \mathbb{R}^N$ where $N = (n+1)^2 + 3(n+1)$ and $\dim(\overline{M}) = n+1$. We can write the operator $P := -\Delta + F(\cdot, \overline{\nabla}\cdot, \xi) \colon \Gamma(\mathcal{E}_{|U}) \to \Gamma(\mathcal{E}_{|U})$ for fixed ξ locally as

$$P(x) = \sum_{\mu\nu} -\overline{g}^{\mu\nu}(x)\partial_\mu\partial_\nu + \sum_\mu A_\mu(x)\partial_\mu + B(x),$$

because in Proposition 3.4 we have shown that the function F is linear in the correct arguments, hence does not change the principial symbol of P. Let $A = 0, \ldots, N$ and define the new variables $\kappa_A := \partial_t \eta_A$ and $\eta_{A,i} := \partial_i \eta_A$. Then the system $P\eta = 0$ is equivalent to

$$\begin{cases} \partial_t \eta_A & = \kappa_A \\ \overline{g}^{ij} \partial_t \eta_{A,i} & = \overline{g}^{ij} \partial_i \kappa_A \\ -\overline{g}^{00} \partial_t \kappa_A & = 2\overline{g}^{0j} \partial_i \kappa_A + \overline{g}^{ij} \partial_j \eta_{A,i} + (A^0)_A^{\ B} \kappa_B + (A^i)_A^{\ B} \eta_{B,i} + (B)_A^{\ B} \eta_B, \end{cases}$$

where we used the same arguments as in the proof of Proposition 2.2. When we set

$$A_1^0(t,x) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \overline{g}^{ij} & 0 \\ 0 & 0 & -\overline{g}^{00} \end{pmatrix}, \quad (b_1)_A(t,x,\eta,\partial\eta,\xi) := (A^0)_A^{\ B} \kappa_B + (A^i)_A^{\ B} \eta_{B,i} + (B)_A^{\ B} \eta_B$$

$$A_1^i(t,x) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{g}^{ij} & \overline{g}^{i0} \\ 0 & \overline{g}^{0j} & \overline{g}^{ij} \end{pmatrix},$$

then we can write the system above in the form of a symmetric hyperbolic system:

$$A_1^0(t,x) \begin{pmatrix} \eta_A \\ \eta_{A,i} \\ \kappa_A \end{pmatrix} = \sum_l A_1^l(t,x,\overline{g}) \partial_l \begin{pmatrix} \eta_A \\ \eta_{A,i} \\ \kappa_A \end{pmatrix} + b_1(t,x,\eta,\partial\eta,\xi)$$

Indeed, this is a symmetric hyperbolic system, since A_1^0 is positive definite and A_1^μ is symmetric.

When we look at Equation (3.10) *and identify $\overline{\nabla}_\mu = \partial_\mu + \Gamma$ as before, then we can write this equations equivalently as:*

$$\overline{\nabla}_V \xi = H(\eta, \overline{\nabla}\eta) \Leftrightarrow \partial_t \xi = \sqrt{-\overline{g}(\partial_t, \partial_t)} N^i \partial_i \xi + b_2(t,x,\eta,\partial\eta,\xi)$$

Where we used $V = u(T - N)$ and that H only depends linear on the quantites $\eta, \overline{\nabla}\eta$. we define similarly to the previous case, $A_2^0(t,x) = \mathrm{id}$, $A_2^l(t,x,\overline{g},\lambda)$ and b_2 as above. This is obviously a symmetric hyperbolic system.

Finally, when we patch these two systems together we obtain a bigger symmetric hyperbolic system and hence the claim. ■

The previous proposition shows that the data η and ξ satisfies locally a symmetric hyperbolic system. In the next step we will use the uniqueness result for these systems to show that the data vanish on the whole manifold $\mathcal{U}_p$.

Proposition 3.12: If we assume Equations (2.1) to (2.6), then the initial data for η, ξ vanish on M, i. e.

$$\overline{g}(V,V)_{|M} = 0, \qquad \eta_{|M} = 0,$$
$$\overline{\nabla}_T \eta_{|M} = 0, \qquad \xi_{|M} = 0.$$

Where η,ξ are given by $(\overline{\nabla} V, E, \overline{\nabla}_V E, (\overline{\nabla} E)(V))$ and $\xi = \delta^{\overline{g}} L$.

Proof.
In the following steps all equations are restricted to M, but we do not write this down explicitly.

$\overline{g}(V,V)_{|M} = 0$*: The initial data for V is given by $V_{|M} = uT - U$ (see Equation (2.4)) and now we can plug in:*

$$\begin{aligned}\overline{g}(V,V) &= u^2 \underbrace{\overline{g}(T,T)}_{=-1} - 2u \underbrace{\overline{g}(T,U)}_{=0} + \underbrace{\overline{g}(U,U)}_{=u^2} \\ &= -u^2 + u^2 = 0\end{aligned}$$

$\overline{\nabla} V_{|M} = 0$*: The initial data for V is given by $V_{|M} = uT - U$ and when we use the constraint equation for U, W (see Equation (1.1)) we obtain:*

$$\pi_{TM}(\overline{\nabla}_X V) = \overline{\nabla}_X (uT - U)_{|M} = \underbrace{(\partial_X u)T}_{=0, T_{|M}=0} + u\overline{\nabla}_X T - \nabla_X U = -uW(X) - \nabla_X U = 0$$

and then $\pi_{TM}(\overline{\nabla}_X V)$ vanish on M for $X \in TM$. When we derive $\overline{g}(V,V)$ along $X \in TM$ and restrict to M, we obtain:

$$0 = \frac{1}{2}\partial_X \overline{g}(V,V) = \overline{g}(\overline{\nabla}_X V, V) = u\overline{g}(\overline{\nabla}_X V, T) - \underbrace{\overline{g}(\overline{\nabla}_X V, U)}_{=0}$$

and thus $\overline{g}(\overline{\nabla}_X V, T) = 0$. We obtain $\overline{\nabla}_X V = 0$ on M. In the next step we have to show $\overline{\nabla}_T V = 0$ on M. We start with Equation (3.2):

$$0 = (d + \delta^{\overline{g}})V^\flat = (c \circ \overline{\nabla})(V^\flat)$$

and obtain $c(T)(\overline{\nabla}_T V^\flat) = 0$ by the previous case. When we multiply with $c(T)$, we obtain

$$0 = c(T)c(T)(\overline{\nabla}_T V^\flat) = +(\overline{\nabla}_T V^\flat)$$

and have $\overline{\nabla}_T V = 0$. Finally $\overline{\nabla} V$ vanish on M.

$E_{|M}=0$*: This is clear by the previous Remark 2.1.*

$\overline{\nabla}E_{|M},\overline{\nabla}_V E_{|M},(\overline{\nabla}E)(V)_{|M}=0$*: By the previous point it is clear that* $\overline{\nabla}_X E$ *vanish on* M *for* $X\in TM$*. So we have to show that* $\overline{\nabla}_T E$ *vanish on* M*. We rewrite the Einstein tensor* $\overline{G}$ *(given by* $\overline{\mathrm{Ric}}-\frac{1}{2}\overline{\mathrm{scal}}\cdot\overline{g}$*) with the help of Equation* (3.1) *in the following way:*

$$\begin{aligned}\overline{G}_{ab}&=\overline{\mathrm{Ric}}_{ab}-\frac{1}{2}\overline{\mathrm{Ric}}_c{}^c\overline{g}_{ab}\\&=Z_{ab}-\overline{\nabla}_{(a}E_{b)}-\frac{1}{2}(Z_c{}^c-\overline{\nabla}_cE^c)\overline{g}_{ab}.\end{aligned}\tag{3.19}$$

On the other hand we have the well-known hypersurface formulas on M*:*

$$\begin{aligned}\overline{G}(T,T)&=\frac{1}{2}\left(\mathrm{scal}-\mathrm{tr}^g(W^2)+(\mathrm{tr}^g W)^2\right),\\\overline{G}(T,X)&=(d\,\mathrm{tr}^g W)(X)+(\delta^g W)(X),\end{aligned}$$

for $X\in TM$*. These are consequences of the Codazzi and Mainardi equations, see [25]. When we combine these formulas for* $(ab)=(0i)$*, we obtain*

$$\overline{G}_{0i}=\overline{\nabla}_iW_k{}^k-\overline{\nabla}_kW^k_{\ i}=Z_{0i}-\frac{1}{2}(\overline{\nabla}_0E_i+\overline{\nabla}_iE_0),\tag{3.20}$$

because $\overline{g}_{0i}=h_{0i}=0$ *holds on* M*. Since* $Z(T,X)=Z(N,X)+\frac{1}{u}\underbrace{Z(V,X)}_{=0,\textit{Equation } (3.4)}$ *holds, we have*

$$-\frac{1}{2}(\overline{\nabla}_0E_i+\overline{\nabla}_iE_0)=\overline{\nabla}_iW^k_{\ k}-\overline{\nabla}_kW^k_{\ i}-Z_{0i}=\overline{\nabla}_iW^k_{\ k}-\overline{\nabla}_kW^k_{\ i}-N^jZ_{ij}=0.$$

Where the expression vanish by the Equation (2.5)*. We know already that* $\overline{\nabla}_iE_0$ *vanish on* M *and hence by the above argumentation also* $\overline{\nabla}_0E_i=0$ *on* M*. Finally we have to consider the term* $\overline{\nabla}_0E_0$*.*

Therefore we consider again the Equation (3.19) *for* $(ab)=(00)$ *to obtain*

$$\begin{aligned}\frac{1}{2}(\mathrm{scal}-W_{ij}W^{ij}+(W_k{}^k)^2)=\overline{G}_{00}&=Z_{00}-\overline{\nabla}_0E_0-\frac{1}{2}\underbrace{\overline{g}_{00}}_{=-1}(Z_c{}^c-\overline{\nabla}_cE^c)\\&=Z_{00}+\frac{1}{2}(\underbrace{Z_0{}^0}_{=-Z_{00}}+Z_k{}^k)-\overline{\nabla}_0E_0-\frac{1}{2}(\overline{\nabla}_0E^0+\overline{\nabla}_kE^k)\\&=\frac{1}{2}Z_{00}-\frac{1}{2}\overline{\nabla}_0E^0+\frac{1}{2}Z_k{}^k-\underbrace{\overline{\nabla}_kE^k}_{=0}.\end{aligned}$$

When we use the simple rewriting $Z_{00} = Z(T,T) = Z(N,N) + \frac{2}{u}Z(N,V) + Z(V,V) = N^iN^jZ_{ij}$*, we obtain*

$$\overline{\nabla}_0 E_0 = N^iN^jZ_{ij} + Z_k{}^k - (\mathrm{scal} - W_{ij}W^{ij} + (W_k{}^k)^2). \tag{3.21}$$

The last step in this part is to show that the previous term vanish on M*. We recall the connection between* u, U, N *and* W*:*

$$N_i = \frac{1}{u}U_i, \qquad u^2 = g(U,U), \qquad \nabla_i U_j + uW_{ij} = 0.$$

We derive the second equation in direction ∂_i*, to obtain*

$$u\nabla_i u = \frac{1}{2}\nabla_i(u^2) = g(\nabla_i U, U) = -ug(W(e_i), U)$$

and rewrite this as $\nabla_i u = -uN^kW_{ik}$*, where we used the equations above for* u, U *and* W*. With this connection we will rewrite the first initial data (see Equation* (2.5)*) on* Z*:*

$$\begin{aligned}
Z(N,N) &= N^iN^jZ_{ij} = N^i\nabla_i W_j{}^j - N^i\nabla^j W_{ij} \\
&= N^i\nabla_i\left[-\frac{1}{u}\nabla_j U^j\right] - N^i\nabla^j\left[-\frac{1}{u}\nabla_i U_j\right] \\
&= N^i\nabla_i\left(-\frac{1}{u}\right)\nabla_j U^j - \frac{1}{u}N^i\nabla_i\nabla_j U^j - N^i\nabla^j\left(-\frac{1}{u}\right)\nabla_i U^j + \frac{1}{u}N^i\nabla_j\nabla_i U^j \\
&= \frac{1}{u^2}N^i\nabla_i\nabla_j U^j - \frac{1}{u}N^i\nabla_i\nabla_j U^j - \frac{1}{u^2}N^i\nabla^j u\nabla_i U^j + \frac{1}{u}N^i\nabla_j\nabla_i U^j \\
&= -\frac{1}{u}N^iN^kW_{ik}\nabla_j U^j - \frac{1}{u}N^i\left(\nabla_i\nabla_j - \nabla_j\nabla_i\right)U^j + \frac{1}{u}N^iN^kW^j_{\ k}\nabla_i U_j \\
&= N^iN^kW_{ik}W_j{}^j - N^iN^kR_{ij}{}^j{}_k - N^iN^kW_{ij}W^j_{\ k} \\
&= -N^iN^k\left[W_{ij}W^j_{\ k} - W_{ik}W_j{}^j - \mathrm{Ric}_{ik}\right] \\
&= -W^2(N,N) + W(N,N)\,\mathrm{tr}^g W + \mathrm{Ric}(N,N)
\end{aligned}$$

Now we pick an orthonormal basis of TM *of the form* $e_1, \ldots, e_{n-1}, e_n$*, with* $e_n = N$*. Indeed, this is orthogonal, since* $g(N,N) = \frac{1}{u^2}g(U,U) = 1$ *holds. We rewrite the* *expression* $Z_k^{\ k} + N^i N^j Z_{ij}$ *in the following way:*

$$
\begin{aligned}
&Z(N,N) + \sum_{i=1}^{n} Z(e_i,e_i) = 2Z(N,N) + \sum_{i=1}^{n-1} Z(e_i,e_i) \\
&= -2W^2(N,N) + 2W(N,N)\operatorname{tr}^g W + 2\operatorname{Ric}(N,N) + \sum_{i=1}^{n-1} \operatorname{Ric}(e_i,e_i) \\
&\quad - R(e_i,N,e_i,N) - W^2(e_i,e_i) + W(e_i,e_i)\operatorname{tr}^g W + W(e_i,N)^2 - W(e_i,e_i)W(N,N) \\
&= \operatorname{Ric}(N,N) + \sum_{i=1}^{n-1} \operatorname{Ric}(e_i,e_i)) + \operatorname{Ric}(N,N) - \sum_{i=1}^{n-1} R(e_i,N,e_i,N) + W(N,N)\operatorname{tr}^g W \\
&\quad + (\operatorname{tr}^g W)^2 - W^2(N,N) - \operatorname{tr}^g(W^2) + \sum_{i=1}^{n-1} W(e_i,N)^2 - W(e_i,e_i)W(N,N) \\
&= \operatorname{scal} + (\operatorname{tr}^g W)^2 - (\operatorname{tr}^g W^2) + \underbrace{\sum_{i=1}^{n} W(e_i,N)^2 - W^2(N,N)}_{=0}.
\end{aligned}
$$

This concludes that the right hand side of Equation (3.21) *vanish and hence* $\overline{\nabla}_a E_{|M} = 0$. *In particular* $\overline{\nabla} E_{|M} = 0$ *,* $\overline{\nabla}_V E_{|M} = 0$ *and* $(\overline{\nabla} E)(V)_{|M} = 0$.

$\overline{\nabla}\overline{\nabla} V_{|M} = 0$*: We already know that* $\overline{\nabla} V$ *vanish on* M*, hence* $\overline{\nabla}_i \overline{\nabla}_a V^b_{|M} = 0$. *In the first case we show that* $\overline{\nabla}_0 \overline{\nabla}_i V^b$ *vanish on* M*, i. e.* $a = i$*, or equivalently:*

$$\overline{\nabla}_0 \overline{\nabla}_i V^b = \underbrace{\overline{\nabla}_i \overline{\nabla}_0 V^b}_{=0} + \overline{R}_{0i\ a}^{\ \ b} V^a. \tag{3.22}$$

Consider the term $\overline{R}_{0abc} V^b$ *for* $(b,c) = (i,j)$ *and* $(b,c) = (0,j)$*:*

$$\overline{R}_{0aij} V^a = -\overline{R}_{ija0} V^a = 0,$$

by $\overline{\nabla}_i \overline{\nabla}_a V_{|M} = 0$ *and*

$$\overline{R}_{0aj}^{\ \ \ 0} V^a = -\overline{\operatorname{Ric}}_{aj} V^a + \overline{R}^{k}_{\ akj} V^a = 0,$$

where $\overline{\operatorname{Ric}}_{ab} V^a$ *vanish on* M *by Equation* (3.1) *and Equation* (3.4)*. This concludes* $\overline{R}_{0abc} V^a = 0$ *on* M.

We will use this fact to rewrite the condition of the vanishing of Equation (3.22) *by the following: We will use the indices* r, s *that have the range* $1, \dots, n-1$ *and set* $i = n$ *for the vector field* N*. We can restrict the indices* i, j *in* $\overline{R}_{ia0j} V^a$ *to* $i, j \neq n$*, because of the following calculation:*

$$\overline{R}_{na0j} V^a = \overline{R}(\underbrace{N}_{T - \frac{1}{u} V}, V, T, E_j) = \overline{R}(T, V, T, E_j) = \overline{R}_{0a0j} V^a = 0$$

and conclude

$$\begin{aligned}\overline{R}_{ia0n}V^a &= \overline{R}(e_i, V, T, N) = -\frac{1}{u}\overline{R}(e_i, V, T, V) = -\frac{1}{u}\overline{R}_{ib0a}V^aV^b \\ &= -\frac{1}{u}\underbrace{\overline{R}_{0aib}V^a}_{=0}V^b = 0.\end{aligned}$$

We consider:

$$\begin{aligned}\overline{R}_{ra0s}V^a &\overset{V^0=u, V^i=-uN^i}{=} u\overline{R}_{r00s} - uN^i\overline{R}_{ri0s} \\ &= u\overline{R}_{r0s}{}^0 - uN^iN^j\overline{R}_{rijs} - \underbrace{\overline{R}_{rias}V^a}_{=0}N^i \\ &= u\overline{\mathrm{Ric}}_{rs} - u\overline{R}_{ris}{}^i - uN^iN^j\overline{R}_{rijs}.\end{aligned}$$

Since we know that $\overline{\nabla}E$ vanish on M and the Equation (3.1) *holds, it follows:*

$$Z_{rs} = N^iN^j\overline{R}_{rijs} + \overline{R}_{ris}{}^i. \tag{3.23}$$

When we use the Gauß equation in its local form

$$\overline{R}_{ijkl} = R_{ijkl} - W_{ik}W_{lj} + W_{il}W_{kj}$$

to rewrite the Equation (3.23) *and replace all the data on $\mathcal{U}_p$ by data on M, we obtain exactly second initial condition on Z, i. e. Equation* (2.6) *and hence we have shown that $\overline{\nabla}_a\overline{\nabla}_iV$ vanish on M.*

In the last step we consider $\overline{\nabla}_0\overline{\nabla}_0V$ on M. The conclusion follows from the fact that $\overline{\mathrm{Ric}}_{ab}V^a$ vanish on M and thus

$$0 = \Delta^{HL}V^b = \Delta V^b + \underbrace{\overline{\mathrm{Ric}}_a{}^bV^a}_{=0} = \overline{\nabla}_c\overline{\nabla}^cV^b = \underbrace{\overline{\nabla}_i\overline{\nabla}^iV^b}_{=0} - \overline{\nabla}_0\overline{\nabla}_0V^b.$$

Where we used the identity $\overline{\nabla}_i\overline{\nabla}_jV^b = 0$ on M. This shows that $\overline{\nabla}\overline{\nabla}V$ vanish on M.

$\delta^{\overline{g}}L_{|M} = 0$: *We start with the expression $\overline{\mathrm{Ric}}_{bc} = Z_{bc} - \overline{\nabla}_{(b}E_{c)}$ and the fact that $\overline{\nabla}_a(V^aZ_{bc})$ vanish by Example 3.6 and the vanishing of $\overline{\nabla}V$. This yields:*

$$\begin{aligned}V^a\overline{\nabla}_a\overline{\mathrm{Ric}}_{bc} &= V^a\overline{\nabla}_a\overline{\nabla}_{(b}E_{c)} = \frac{1}{2}V^a\left(\overline{\nabla}_a\overline{\nabla}_bE_c + \overline{\nabla}_a\overline{\nabla}_cE_b\right) \\ &= \frac{1}{2}V^a\Big(\overline{\nabla}_a\overline{\nabla}_bE_c - \overline{\nabla}_b\overline{\nabla}_aE_c + \overline{\nabla}_a\overline{\nabla}_cE_b \\ &\quad -\overline{\nabla}_c\overline{\nabla}_aE_b + \overline{\nabla}_b\overline{\nabla}_aE_c + \overline{\nabla}_c\overline{\nabla}_aE_b\Big) \\ &= V^aR_{a(bc)}{}^dE_d + V^a\overline{\nabla}_{(b}\overline{\nabla}_aE_{c)}.\end{aligned}$$

Now we set $(b,c)=(0,j)$ in the previous equation to obtain

$$
\begin{aligned}
V^a\overline{\nabla}_a\overline{\mathrm{Ric}}_{0j} &= \underbrace{V^a\overline{R}_{a(0j)}{}^d}_{=0} E_d + V^a\overline{\nabla}_{(0}\overline{\nabla}_a E_{j)} \\
&= \frac{1}{2}V^a\left(\overline{\nabla}_0\overline{\nabla}_a E_j + \underbrace{\overline{\nabla}_j\overline{\nabla}_a E_0}_{=0,\overline{\nabla}E=0 \text{ on } M}\right) \\
&= \frac{1}{2}V^a\overline{\nabla}_0\overline{\nabla}_a E_j \overset{V^0=u,V^i=-uN^i}{=} \frac{u}{2}\overline{\nabla}_0\overline{\nabla}_0 E_j - \frac{u}{2}\underbrace{N^i\overline{\nabla}_0\overline{\nabla}_i E_j}_{=(*)}.
\end{aligned}
$$

The term $(*)$ vanish by the following computation:

$$
N^i\overline{\nabla}_0\overline{\nabla}_i E_j = N^i\underbrace{\overline{\nabla}_i\overline{\nabla}_0 E_j}_{=0} + N^i R_{0ij}{}^d E_d = \overline{R}(T,N,e_j,E^\sharp) = \overline{R}(T,T-\frac{1}{u}V,e_j,E^\sharp) = 0.
$$

On the other hand we can rewrite the curvature expression $V^a\overline{\nabla}_a\overline{\mathrm{Ric}}_{0j}$ with the second Bianchi identity as

$$
V^a\overline{\nabla}_a\overline{\mathrm{Ric}}_{0j} = -V^a\overline{\nabla}_a\overline{R}_{ji0}{}^i = V^a(\overline{\nabla}_j\overline{R}_{ia0}{}^i + \overline{\nabla}_i\overline{R}_{aj0}{}^i)
$$

and the last two terms vanish because $\overline{R}_{ia0}{}^iV^a$ vanish on M and hence the derivative in M direction. We can conclude that $\overline{\nabla}_0\overline{\nabla}_0 E_j$ vanish on M and by Equation (3.17) *we have $\Delta E_i - \overline{\mathrm{Ric}}(E^\sharp,\cdot)_i = 2\delta^{\overline{g}}L_i$, hence $\delta^{\overline{g}}L_i = 0$ on M. The time component of $\delta^{\overline{g}}L$ is given by*

$$
\delta^{\overline{g}}L_0 = \delta^{\overline{g}}L(\underbrace{T}_{\frac{1}{u}V+N}) = \frac{1}{u}V^a(\delta^{\overline{g}}L)_a + N^i\underbrace{\delta^{\overline{g}}L_i}_{=0}
$$

and vanish by the fact that the Equation (3.18) *is linear in the vanishing term $\overline{\nabla}V$ and we already know that $\overline{\nabla}V$ vanish on M. We conclude that $\delta^{\overline{g}}L$ vanish on M.*

$\overline{\nabla}_T\overline{\nabla}_V E_{|M} = 0$***:*** *In the previous step we have shown that $V^a\overline{\nabla}_0\overline{\nabla}_a E_i$ vanish. It remains to show that $V^a\overline{\nabla}_0\overline{\nabla}_a E_0$ vanish on M. We use again the fact that ΔE_0 vanish and that we have the identity*

$$
0 = \Delta E_0 = \overline{\nabla}_c\overline{\nabla}^c E_0 = \overline{\nabla}_0\overline{\nabla}^0 E_0 + \underbrace{\overline{\nabla}_i\overline{\nabla}^i E_0}_{=0}.
$$

This concludes that $\overline{\nabla}_0\overline{\nabla}_0 E_0$ vanish and thus $\overline{\nabla}_0\overline{\nabla}_0 E_b = 0$. Finally:

$$
\begin{aligned}
V^a\overline{\nabla}_0\overline{\nabla}_a E_0 &= u\overline{\nabla}_0\overline{\nabla}_0 E_0 - uN^i\overline{\nabla}_0\overline{\nabla}_i E_0 = N^i\overline{\nabla}_i\overline{\nabla}_0 E_0 + N^i\overline{R}_{0i0}{}^d E_d \\
&= \overline{R}(T,N,T,E^\sharp) = \overline{R}(T,T-\frac{1}{u}V,T,E^\sharp) = 0.
\end{aligned}
$$

So we had considered all different cases and the conclusion follows. ■

Finally we can conclude the following corollary.

Corollary 3.13: The data E and $\overline{\nabla}V$ vanish on $\mathcal{U}_p$.

Proof.
We recapitulate the previous steps: We have shown that η and ξ satisfies locally a symmetric hyperbolic system (see Proposition 3.11) and the initial data vanish on M (see Proposition 3.12), thus (by the uniqueness) all components of η and ξ vanish, in particular E and $\overline{\nabla}V$. ■

We can summarizes the previous results. For every Riemannian manifold (M, g) equipped with a nowhere vanishing vector field U and endomorphism W, which satisfies Equation (1.1), we obtained a time-oriented Lorentzian manifold $(\mathcal{U}_p, \overline{g}^{\,\mathcal{U}_p})$ around every point $p \in M$ with a parallel null vector field V. This construction depends on the background metric h that we have chosen on the background $\mathbb{R} \times M$, but it depends uniquely on this choice. The vanishing of E gives us the direct correlation between the two metrics $\overline{g}$ and h.

3.3 Global solution and shape of the Lorentzian metric

In the last section we constructed a local solution for the Riemannian Cauchy problem, which we obtained by a local reformulation of the constraint equation. But now we want to globalize this construction. Let p be a point in M, then we write the solution of the Cauchy problem on a neighbourhood of this point as $\omega_p = (\overline{g}^{\,\mathcal{U}_p}, V^{\mathcal{U}_p}, Z^{\mathcal{U}_p})$, where $\mathcal{U}_p$ is the globally hyperbolic Lorentzian manifold which we obtained as a solution. Let q be another point of M with solution $\omega_q = (\overline{g}^{\,\mathcal{U}_q}, V^{\mathcal{U}_q}, Z^{\mathcal{U}_q})$, such that $\mathcal{U}_p \cap \mathcal{U}_q \neq \emptyset$ holds. Since ω_p and ω_q are solutions of the local reformulation of (EQ) and this system has a unique solution, they coincide on the intersection, i. e.

$$w_{p|\mathcal{U}_p \cap\, \mathcal{U}_q} = w_{q|\mathcal{U}_p \cap\, \mathcal{U}_q}.$$

Hence we can construct the global solution as the union of all local solutions, i.e.

$$\overline{M} = \bigcup_{p \in M} \mathcal{U}_p \subset \mathbb{R} \times M.$$

Moreover, the data $w = (\overline{g}, V, Z)$ is given by a union of all local solution. This gives a well-defined Lorentzian metric $\overline{g}$, since each local solution $\overline{g}^{\mathcal{U}_p}$ is a solution in the space $\mathcal{G}_p$ (see Equation (2.7)) and the spacelike hypersurface (M, g) embeds into $(\overline{M}, \overline{g})$ with the Weingarten map W. Indeed, the Weingarten map of the inclusion is given by W. We have to show that

$$\overline{\nabla}_X Y - \nabla_X Y = -W(X, Y)T$$

holds for all $X, Y \in TM$ and T is the time orientation given by $T = \frac{1}{u}(U + V)$. It is clear that the difference of the connections vanish in TM direction and thus it remains to consider the projection in V direction.

$$\begin{aligned} \overline{g}(\overline{\nabla}_X Y - \nabla_X Y, V) &= \overline{g}(\overline{\nabla}_X Y, V) - \overline{g}(\nabla_X Y, V) \\ &= \partial_X \overline{g}(V, X) - \overline{g}(Y, \underbrace{\overline{\nabla}_X V}_{=0, \overline{\nabla} V = 0}) + g(\nabla_X Y, U) \\ &= -\partial_X g(Y, U) + \partial_X g(Y, U) - g(\nabla_X U, Y) \\ &= -g(\nabla_X U, Y) = uW(X, Y). \end{aligned}$$

In the last step we used the Riemannian constraint equation. Now we are able to write the difference of the connections with respect to a generalised local orthonormal frame (T, e_i) of $T\overline{M}$:

$$\begin{aligned} \overline{\nabla}_X Y - \nabla_X Y &= -\overline{g}(\overline{\nabla}_X Y - \nabla_X Y, T)T + \sum_i \overline{g}(\overline{\nabla}_X Y - \nabla_X Y, e_i)e_i \\ &= -\frac{1}{u}\overline{g}(\overline{\nabla}_X Y - \nabla_X Y, V)T \\ &= -\frac{1}{u}uW(X, Y)T = -W(X, Y)T. \end{aligned}$$

Thus the Weingarten map is given by W. Finally, we have a parallel null vector field V as desired.

3.3.1 M is a Cauchy surface of $\overline{M}$

In this part, we have to show that (M, g) embeds into $(\overline{M}, \overline{g})$ as a Cauchy surface, i.e. every inextendible timelike C^1-curve intersects (M, g) exactly once. So let $\gamma \colon I \to \overline{M}$ be such a curve, then there exists a time $t_p \in I$ where the curve goes through $\mathcal{U}_p$, i.e. $\gamma(t_p) \in \mathcal{U}_p$. W. l. o. g. we consider the curve which is γ restricted to this globally hyperbolic Lorentzian manifold $\mathcal{U}_p$. Indeed, we only have to consider the curve in the subset $\mathcal{U}_p$ of $\overline{M}$, since we can choose a countable covering of $\overline{M}$ by $\mathcal{U}_p$'s. If the curve intersects the hypersurface M more than once, we can choose a subcovering of the covering above and obtain a contradiction by the timelikeness of the curve γ. Finally we have to show that the curve hits the hypersurface exactly once. So let γ be the curve that lies in $\mathcal{U}_p$, then we can consider the splitting of the curve

$$\gamma = (\gamma_t, \gamma_M),$$

which we get by the splitting of ambient Lorentzian manifold $\overline{M} \subset \mathbb{R} \times M$. We have $\overline{g}(\dot{\gamma}, \dot{\gamma}) < 0$ by the timelikeness of the curve. Thus

$$\overline{g}(\dot{\gamma_M}, \dot{\gamma_M}) + 2\overline{g}(\dot{\gamma_t}\partial_t, \dot{\gamma_M}) + \overline{g}(\dot{\gamma_t}\partial_t, \dot{\gamma_t}\partial_t) < 0. \tag{3.24}$$

When we think back to the construction of the local Lorentzian metric $\overline{g}$, we obtain a contradiction if we assume that there exists a time $t' \in I$ such that $\dot{\gamma_t}(t')$ vanish. By Equation (3.24) we obtain the condition $\overline{g}^{\,\mathcal{U}_p}(\dot{\gamma_M}(t'), \dot{\gamma_M}(t')) < 0$ and see that this is a contradiction by the construction of $\overline{g}$, since the spatial part of the Lorentzian metric is positive, see Equation (2.7).

Hence the map $\dot{\gamma_t}$ is strictly monotone and by assumption on the hypersurface M_p we see that γ hits M exactly once. So have shown that the hypersurface M is a Cauchy surface of $\overline{M}$.

3.3.2 The metric $\overline{g}$ is of the form $-\tilde{\lambda}^2 dt^2 + g_t$

Finally we have to show that our solution $(\overline{M}, \overline{g})$ is of the form $(\overline{M}, -\tilde{\lambda}^2 dt^2 + g_t)$ (up to a diffeomorphism). On the first hand we consider the projection map $t\colon \overline{M} \to \mathbb{R}, (t, x) \mapsto t$ and the corresponding gradient vector field $\mathrm{grad}^{\overline{g}}(t)$. We consider two derived objects for this gradient vector field: The rescaled vector field

$$F := \frac{1}{dt(\mathrm{grad}^{\overline{g}}(t))} \cdot \mathrm{grad}^{\overline{g}}(t)$$

and the flow ϕ of F. The vector field F is well-defined, since we have

$$dt(\mathrm{grad}^{\overline{g}}(t)) = \overline{g}(\mathrm{grad}^{\overline{g}}(t), \mathrm{grad}^{\overline{g}}(t)) < 0,$$

by Equation (2.7). We can prove the following lemma.

Lemma 3.14: The flow ϕ of F sends level sets to level sets, i. e. we have $\phi_s(p) \in M_{t+s}$ for any $p \in M_t := \{t\} \times M$. Moreover, the map

$$\begin{aligned} \psi\colon \mathcal{U} := \{(t, x) \in \overline{M} \mid \phi_t(x) \text{ exists}\} &\to \overline{M} \\ (t, x) &\mapsto \phi_t(x) \end{aligned}$$

is a well-defined diffeomorphism.

Proof.
Let p be a fixed point in M_t for any $t \in \mathbb{R}$ such that $\phi_t(p)$ exists. Then we define the map $f(s) := t(\phi_s(p))$ for any s such that this expression is well-defined and check the identity $f(s) = s + t$ for f. It is immediate, when we compute the derivative of the function f:

$$f'(s) = d_{\phi_s(p)}t(d\phi_s(p)) = d_{\phi_s(p)}t(F) = 1$$

and hence we have $f(s) = s \cdot f'(0) + f(0) = s + t$. So a point $p \in \overline{M}$ is by defintion in a level set M_t if $t(p) = t$ holds. We have shown that $p \in M_t$ is map to $\phi_s(p) \in M_{t+s}$, i. e. sends level sets to level sets.

We know by the previous part, that the hypersurface M_0 is a Cauchy surface, hence the curve $\gamma\colon I \to \overline{M}, s \mapsto \phi_s(p)$ hits M_0 exactly once. Therefore exists a unique $\tau(p) \in \mathbb{R}$ such that $\phi_{-\tau(p)}(p) \in M_0$ holds. So we can define the invers map of ψ with the help of $\tau(p)$:

$$\begin{aligned}\psi^{-1}\colon \overline{M} &\to \mathcal{U}\\ p &\mapsto (\tau(p), \mathsf{proj}^M(\phi_{-\tau(p)}(p)))\end{aligned}$$

This is obviously a well-defined smooth map and by the previous argument also an invers of ψ. This shows the claim. ■

To conclude the proof of Theorem 1.1 we consider the pullback metric $\psi^*\overline{g}$ on $\mathcal{U}$. Initially we compute the derivative of the map ψ at a point $p = (s, x) \in \mathcal{U}$ and obtain

$$(d_p\psi)(\partial_t) = F_{|\psi_s(x)},$$

by the definition of the flow ψ of F. Thus we can compute the pullback metric $\psi^*\overline{g}$ with the help of the previous identity:

$$\begin{aligned}(\psi^*\overline{g})_{(s,x)}(\partial_t, \partial_t) &= \overline{g}_{\psi_s(x)}(F, F) = \frac{1}{\overline{g}(\mathrm{grad}^{\overline{g}}(t), \mathrm{grad}^{\overline{g}}(t))} < 0\\ (\psi^*\overline{g})_{(s,x)}(\partial_t, X) &= \overline{g}_{\psi_s(x)}(F, d_{(s,x)}\psi(X)) = 0\end{aligned}$$

Where X is an element of TM_s, which is orthogonal to the gradient $\mathrm{grad}^{\overline{g}}(t)$. Hence we can write the pullback metric as

$$\psi^*\overline{g} = -\tilde{\lambda}^2 dt^2 + g_t,$$

where $\tilde{\lambda} := \left[-\overline{g}(\mathrm{grad}^{\overline{g}}(t), \mathrm{grad}^{\overline{g}}(t))\right]^{-\frac{1}{2}}$ and g_t is a family of Riemannian metrics which comes from the pullback construction and is given by $g_t(X, Y) := (\psi^*\overline{g})(X, Y)$.

Finally we have to show Equation (1.3): Since the diffeomorphism ψ restricts to the identity on the level set M_0 we obtain $g_0 = g$. Furthermore we have

$$\tilde{\lambda}_{|M} = \left[-h(\mathrm{grad}^h(t), \mathrm{grad}^h(t))_{|M}\right]^{-\frac{1}{2}} = \lambda_{|M},$$

where we used the definition of the background metric, i. e. Equation (1.7). We conclude for the proof of Theorem 1.1, that we pass from $(\overline{M}, \overline{g})$ to the isomorphic Lorentzian manifold $(\mathcal{U}, \psi^*\overline{g})$ and obtain everything as in the claim. This shows the Theorem 1.1.

4 Alternative ansatz for the constraint equation

In the last sections we constructed an extension $(\overline{M}, \overline{g})$ for a Riemannian manifold (M, g) with nowhere vanishing vector field U, a g-symmetric endomorphism W and the function $u = \sqrt{g(U,U)}$, which satisfies

$$\nabla U + uW = 0.$$

In this chapter we want to show an alternative way to solve the Cauchy problem corresponding to Equation (1.1) to obtain a Lorentzian manifold $(\overline{M}, \overline{g})$ with parallel null vector field V, which extends U and has (M, g) as a Cauchy surface. We follow some ideas from a discussion with Piotr Chrusciel, see [26].

The basic idea is to choose suitable coordinates on a tubular neighbourhood of M in $\overline{M}$ and express the Lorentzian metric $\overline{g}$ in terms of data on M. In this way we can omit the involved PDEs, that we considered in the previous sections and reduce everything to simple manipulation of metrics.

Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold with parallel null vector field V and let $\mathcal{U}$ be the tubular neighbourhood of M in $\overline{M}$, where the flow ϕ_t^V of the vector field V is well-defined. This flow is an isometry, since V is a Killing vector field, i. e. we have to show that the Lie derivative of the metric $\overline{g}$ vanish along V: Let $X, Y \in T\overline{M}$, then we consider:

$$\begin{aligned}(\mathcal{L}_V \overline{g})(X, Y) &= \partial_V(\overline{g}(X, Y)) - \overline{g}([V, X], Y) - \overline{g}(X, [V, Y]) \\ &= \overline{g}(\overline{\nabla}_V X, Y) + \overline{g}(X, \overline{\nabla}_V Y) - \overline{g}(\overline{\nabla}_V X - \overline{\nabla}_X V, Y) - \overline{g}(X, \overline{\nabla}_V Y - \overline{\nabla}_Y V) \\ &= \overline{g}(\overline{\nabla}_X V, Y) + \overline{g}(X, \overline{\nabla}_Y V) = 0.\end{aligned}$$

Thus the Killing vector field V generates an isometric flow ϕ_t^V[1].

[1]The arguments that follow do not depend on the fact that V is parallel. The crucial ingredient is that the generated flow of V is an isometry, thus we could restrict to the assumption that we start with a Killing vector field V.

However we can extend the nowhere vanishing vector field $V = \frac{\partial}{\partial t}$ to a frame $\frac{\partial}{\partial t}, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ of $T\mathcal{U}$ with x a fixed chart of M. Then we can write the metric $\overline{g}$ in these coordinates in the general form:

$$\overline{g} = -\lambda_t dt^2 + \alpha_t \odot dt + g_t \tag{4.1}$$

Where λ_t, α_t and g_t depend smoothly on the time coordinate t and $\odot$ denotes the symmetric tensor product, i. e. $\alpha \odot \beta := \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$. When we act on the metric by the flow ϕ_s^V, we obtain the shifted version of the metric for sufficent small s:

$$(\phi_s^V)^* \overline{g} = -\lambda_{t+s} dt^2 + \alpha_{s+t} \odot dt + g_{t+s}.$$

But we know that the flow acts as an isometry and thus fixes the metric $\overline{g}$. We have the identity

$$(\phi_s^V \overline{g}) = \overline{g}$$

for all s in a small intervall where ϕ_s^V is defined, hence by the basis representation of $dt, dx^1, \dots, dx^n$ the coefficients in Equation (4.1) are time independent. So we have

$$\overline{g} = -\lambda dt^2 + \alpha \odot dt + g,$$

because the metric restricts on $\{t = 0\} = M$ to the metric g.

When we use that the vector field V is lightlike, we obtain

$$0 = \overline{g}(V, V) = -\lambda + \underbrace{\alpha(V)}_{=0} dt(V) + \underbrace{g(V, V)}_{=0, \text{ since } g(V, \cdot) = 0} = -\lambda$$

and thus λ vanish. Let $X \in TM$, then we consider the TM-projection:

$$\overline{g}(V, X) = (\alpha \odot dt)(V, X) + g(V, X) = \frac{1}{2}(\alpha(V)dt(X) + \alpha(X)dt(V)) = \frac{1}{2}\alpha(X)$$

and thus

$$\alpha(X) = 2\overline{g}(V, X) \overset{\overline{g}(T,X)=0}{=} -2\overline{g}(U, X) = -2g(U, X).$$

Finally we have $\alpha = -2U^\flat$.

We conclude that we can express the metric $\overline{g}$ in the tubular neighbourhood $\mathcal{U}$ in data of M:

$$\overline{g} = -2U^\flat \odot dt + g \text{ on } \mathcal{U}.$$

Now we are able to construct a solution for a similar Cauchy problem as in Theorem 1.1.

Proposition 4.1: Let (M, g) be a Riemannian manifold with a nowhere vanishing vector field U, a g-symmetric endomorphism W and the function $u = \sqrt{g(U, U)}$, which satisfies

$$\nabla U + uW = 0.$$

Then there exists a solution $(\overline{M}, \overline{g}, V, T)$, given by a Lorentzian manifold $(\overline{M}, \overline{g})$ as a open subset of $\mathbb{R} \times M$ with Cauchy surface M, a parallel null vector field V and a time orientation T, of the following system:

$$\begin{cases} \nabla^{\overline{g}} V = 0, & \text{on } \overline{M} \\ \quad V = -U, & \text{on } M. \end{cases} \tag{4.2}$$

Moreover, the Weingarten map of the inclusion $M \hookrightarrow \overline{M}$ is given by the map W.

Remark 4.2: The second equation of Equation (4.2) is an identity of sections in tangent bundle TM or in other words: We have an orthogonal decomposition of $T\overline{M} = \mathbb{R}T \oplus TM$ and therefore a projection $\pi^{TM,T}$ onto the TM part. The second equation of Equation (4.2) is an identity with respect to this projection.

Proof of Proposition 4.1.
Let (M, g) be the Riemannian manifold equipped with a nowhere vanishing vector field U, a g-symmetric endomorphism W and the function $\sqrt{g(U, U)}$, which satisfies $\nabla U + uW = 0$. We consider $\overline{M} = \mathbb{R} \times M$ equipped with the metric

$$\overline{g} = -2U^\flat \odot dt + g,$$

where t corresponds to the first coordinate of $\overline{M}$. Moreover, we define $V := \partial_t$ as a section of $T\overline{M} \to \overline{M}$ and have to prove that this nowhere vanishing vector field is parallel and null.

*V **is null:** We plug V into the metric:*

$$\begin{aligned} \overline{g}(V, V) &= -2(U^\flat \odot dt)(V, V) + g(V, V) \\ &= -2(g(V, U)) + g(V, V) = 0, \end{aligned}$$

since $g(V, \cdot) = 0$. Now we define a time orientation of $\overline{M}$ by $T := \frac{1}{u}(V + U)$ and check

$$\begin{aligned} \overline{g}(T, T) &= -2g(T, U)dt(T) + g(T, T) = \frac{1}{u^2}(-2g(U, U) + g(V + U, V + U)) \\ &= \frac{1}{u^2}(-2u^2 + u^2) = -1. \end{aligned}$$

Thus T is a time orientation.

V ***is parallel:*** *We use the Koszul formula for the metric $\overline{g}$:*

$$\begin{aligned}2\overline{g}(\overline{\nabla}_X V, Y) &= X(\overline{g}(V,Y)) + V(\overline{g}(X,Y)) - Y(\overline{g}(V,X)) \\ &\quad + \overline{g}([X,V],Y) - \overline{g}([X,Y],V) - \overline{g}([V,Y],X).\end{aligned}$$

for all $X, Y \in T\overline{M} = \mathbb{R}V \oplus TM$ and check all different cases. We start with $\overline{\nabla}_V V$ on the V-projection:

$$2\overline{g}(\overline{\nabla}_V V, V) = V\overline{g}(V,V) = 0$$

We go on with $\overline{\nabla}_V V$ on the other part $Y \in TM$:

$$\begin{aligned}2\overline{g}(\overline{\nabla}_V V, Y) &= V\overline{g}(V,Y) + V\overline{g}(V,Y) - Y\overline{g}(V,V) \\ &\quad + \overline{g}([V,V],Y) - \overline{g}([V,Y],V) - \overline{g}([V,Y],V) \\ &= 2\left(V\overline{g}(V,Y) - \overline{g}([V,Y],V)\right) \\ &= 2\left(\overline{g}([V,V],Y) + \overline{g}([V,Y],V) - \overline{g}([V,Y],V)\right) = 0\end{aligned}$$

Thus $\overline{\nabla}_V V$ vanish. In the next step we want to show that $\overline{\nabla}_X V$ vanish for all $X \in TM$.

$$\begin{aligned}2\overline{g}(\overline{\nabla}_X V, V) &= X\overline{g}(V,V) + V\overline{g}(X,V) - V\overline{g}(V,X) \\ &\quad + \overline{g}([X,V],V) - \overline{g}([X,V],V) - \overline{g}([V,V],X) = 0\end{aligned}$$

Finally we consider $X, Y \in TM$:

$$\begin{aligned}2\overline{g}(\overline{\nabla}_X V, Y) &= X\overline{g}(V,Y) + V\overline{g}(X,Y) - Y\overline{g}(V,X) \\ &\quad + \overline{g}([X,V],Y) - \underbrace{\overline{g}([X,Y],V)}_{=-g([X,Y],U)} - \overline{g}([V,Y],X) \\ &= -Xg(U,Y) + Yg(U,X) + \overline{g}([V,X],Y) \\ &\quad + \overline{g}(X,[V,Y]) + \overline{g}([X,V],Y) - \overline{g}([V,Y],X) - g([X,Y],U) \\ &= -g(\nabla_X U, Y) + g(\nabla_Y U, X) + g([X,Y] - \nabla_X Y + \nabla_Y X, U) = 0\end{aligned}$$

Where we used that $\nabla U = -uW$ is g-symmetric by assumption. We conclude that V is a parallel null vector field.

Cauchy problem is solved by $(\overline{M}, \overline{g}, V, T)$***:*** *We have to show that the above constructed vector field V satisfies the Equation* (4.2)*. The first part of Equation* (4.2) *follows directly from the calculation above. The second part of Equation* (4.2) *is a consequence of the decomposition $V = uT - U$ in $T\overline{M} = \mathbb{R}T \oplus TM$. When we use the projection $\pi^{TM,T}\colon T\overline{M} = \mathbb{R}T \oplus TM \to TM, X \mapsto X + \overline{g}(T,X)T$, we obtain:*

$$\pi^{TM,M}(V) = V + \overline{g}(T,V)T = V - uT = -U.$$

W ***is the second fundamental form of*** $(M, g) \hookrightarrow (\overline{M}, \overline{g})$***:*** *We deduce that* M *is Cauchy surface in* $\overline{M}$*, since all inextendible timelike curves intersect* M *exactly ones. This follows in an analogous way as in Section 3.3.1. In particular we can do the same calculation as in Section 3.3 to show that the Weingarten map is given by* W*.*

■

In the previous proof, we constructed a Lorentzian manifold with a metric of the form $\overline{g} = -2U^\flat \odot dt + g$. In accordance with the statement of Theorem 1.1 we are interested in the existence of a diffeomorphism ϕ, such that the pullback $\phi^*\overline{g}$ is of the form like in the conclusion of Theorem 1.1, i. e. the metric is of the form $-\tilde{\lambda}^2 dt^2 + g_t$ and $\tilde{\lambda}, g_t$ restricts on the hypersurface to the stated objects λ, g.

In the following we will need the extension of a tensor on T^*M to a tensor on $T^*(M \times \mathbb{R})$. So let T be a tensor in $(T_x^*M)^{\otimes n}$ for a point $x \in M$, then we define the **extension** $E(T) \in (T^*_{(x,0)}M \times \mathbb{R})^{\otimes n}$ as the following:

$$E(T)(\tilde{X}_1, \dots, \tilde{X}_n) := T(X_1, \dots, X_n)$$

for all $\tilde{X}_i = X_i + a_i\partial_t \in T_x^*M \times T_0\mathbb{R}$. This construction is unnatural in the sense, that it is not compatible with pullback.

In the next theorem we will prove the existence of a suitable diffeomorphism or in other words: We find a diffeomorphism, which deforms the solution metric from the last proof into the form $-\tilde{\lambda}^2 dt^2 + g_t$ and moreover fixes the Cauchy surface.

Theorem 4.3: Let (N, h) be a time-oriented Lorentzian manifold with a compact spacelike hypersurface M and $\lambda \colon M \to (0, \infty)$ is a smooth function. Then we can construct the following objects:

- An $\varepsilon > 0$.
- A function $\tilde{\lambda} \colon M \times (-\varepsilon, \varepsilon) \to \mathbb{R}$.
- A map $\phi \colon M \times (-\varepsilon, \varepsilon) \to N$.
- A smooth family of Riemannian metrics $(g_t)_{t \in (-\varepsilon, \varepsilon)}$.

These objects satisfying the following statements:

- We have $\tilde{\lambda}(x, 0) = \lambda(x)$ for all $x \in M$ and $g_0 = g$, where $g := (M \hookrightarrow N)^*h$.
- The map ϕ is a diffeomorphism on its image and fixes the hypersurface, i. e. $\phi(x, 0) = x$ for $x \in M$.

- The pullback of h along ϕ is given by:

$$\phi^* h = -\tilde{\lambda}^2 dt^2 + E(g_t)$$

Proof.
At the beginning we have (by compactness and time orientation) a timelike normal vector field T (i. e. $h(T,T) = -1$) given on a neighbourhood $M \times (-\varepsilon, \varepsilon)$ of M in N, which is orthogonal to the tangent space $T_x M$ for every point $x \in M$. Additionally we consider the flow of the vector field $Z(x,t) = \lambda(x)T(x,t)$, defined by:

$$\frac{d\phi_t^Z}{dt}(x) = Z_{\phi_t^Z(x)} = \lambda(x)T(\phi_t^Z(x)), \text{ for all } (x,t) \in N \times (-\varepsilon, \varepsilon)$$
$$\phi_0^Z = \mathrm{id}$$

We write ψ for the restricted flow to the open neighbourhood $M \times (-\varepsilon, \varepsilon)$, i. e. $\psi(x,t) = \phi_t^Z(x)$ and $\psi(x,0) = x$ for $x \in M$ (We choose an ε small enough s. t. the identification is possible). When we consider the pullback of the metric h along the diffeomorphism ψ, we obtain the general form:

$$\psi^*(h) = -\lambda^2 dt^2 + 2E(X_t^\flat) \odot dt + E(\tilde{g}_t).$$

Where X_t is a time dependent family of vector fields on M or in other words a section $M \times (-\varepsilon, \varepsilon) \to TM$. The spacelike part of $\psi^* h$ is given by a family of Riemannian metric $\tilde{g}_t$ on M, i. e. $\tilde{g}_t(\partial_{x_i}, \partial_{x_j}) = (\psi^* h)(\partial_{x_i}, \partial_{x_j})$ for a frame $(\partial_{x_i})_i$ of TM. This is true, since $\psi_* \partial_{x_i}$ is tangent to TM by the orthogonality $T_x \perp T_x M$ for all $x \in M$ and the flow equation for Z. The lapse function of $\psi^* h$ is given by the initial λ, since $\psi_* \partial_t = \lambda T$ holds by the flow equation of ψ and therefore we have

$$(\psi^* h)(\partial_t, \partial_t) = h(\psi_* \partial_t, \psi_* \partial_t) = \lambda^2 h(T,T) = -\lambda^2.$$

We like to have a diffeomorphism $\phi\colon M \times (-\varepsilon, \varepsilon) \to M \times (-\varepsilon, \varepsilon)$ which satisfies an identity of the form $\phi^*(h) = -\tilde{\lambda}^2 dt^2 + E(g_t)$ for suitable $\tilde{\lambda}$ and g_t. In accordance with that we consider the flow $\rho_t := \phi_t^{-X_t}$ of the time dependent vector field $-X_t$, which satisfies:

$$\frac{d}{dt}(\rho_t)_{(y,t)} = -(X_t)_{\rho_t(y,t)}, \text{ for } (y,t) \in N \times (-\varepsilon, \varepsilon).$$
$$\rho_0 = \mathrm{id}$$

Now we are able to write down all the necessary steps for the construction of the desired diffeomorphism ϕ:

1) *First we define the diffeomorphism* $\Theta\colon M \times (-\varepsilon, \varepsilon) \to M \times (-\varepsilon, \varepsilon), (x, t) \mapsto (\theta_t(x), t)$, *where* θ_t *is given by the invers of the flow* ρ_t, *i. e.* $\theta_t := \rho_t^{-1}$. *We check that this map fixes the hypersurface* M. *Let* $x \in M$, *then we have* $\Theta(x, 0) = (\theta_0(x), 0) = (x, 0)$, *by the flow equation for* ρ_t.

2) *In the second step we define the vector field* $V_t := \frac{d}{dt}\theta_t$ *and check that this satisfies the identity* $V_t = (\theta_t)_* X_t$. *Indeed this is true, since we can derive the identity* $\rho_t \circ \theta_t = \mathrm{id}$ *and use the flow equation for* ρ_t.

3) *In the third step we define the Lorentzian metric* $H = -\tilde{\lambda}^2 dt^2 + g_t$ *on* $M \times (-\varepsilon, \varepsilon)$, *where* g_t *and* $\tilde{\lambda}$ *are given by* $\rho_t^* \tilde{g}_t$ *and* $\theta_t^* \left[\lambda^2 - g_t(V_t, V_t)\right]^{\frac{1}{2}}$. *The smooth function* $\tilde{\lambda}$ *is well-defined, since* X_t *vanish for* $t = 0$ *and therefore* V_t *by the previous step. This has the consequence that the expression* $\lambda^2 - g_t(V_t, V_t)$ *is positive in a small neighbourhood of* $\{t = 0\}$ *in* $M \times (-\varepsilon, \varepsilon)$, *since* λ *is positive by assumption.*

4) *In the next step we check that the diffeomorphism* Θ *pulls back the metric* $H = -\tilde{\lambda}^2 dt^2 + g_t$ *to* $\psi^* h$, *i. e.* $\Theta^* H = \psi^* h$: *Let* $(\partial_{x_i})_i$ *be a local frame of* TM, *then we check the spatial part of the metric* $\Theta^* H$:

$$(\Theta^* H)(\partial_{x_i}, \partial_{x_j}) = H(\Theta_* \partial_{x_i}, \Theta_* \partial_{x_j}) \overset{(\alpha)}{=} (\theta_t^* g_t)(\partial_{x_i}, \partial_{x_j}) = \tilde{g}_t(\partial_{x_i}, \partial_{x_j})$$

Where we used at (α) *that* $\Theta_* \partial_{x_i}$ *is tangential to* TM, *since by the definition of* $\Theta(x, t) = (\theta_t(x), t)$ *there is no* ∂_t *shift. The next step we look at the mixed part of* $T(M \times (-\varepsilon, \varepsilon))$:

$$\begin{aligned}(\Theta^* H)(\partial_t, \partial_{x_i}) &= H(\Theta_* \partial_t, \Theta_* \partial_{x_i}) \\ &\overset{(\beta)}{=} H(\partial_t + V_t, (\theta_t)_* \partial_{x_i}) \\ &= H((\theta_t)_* X_t, (\theta_t)_* \partial_{x_i}) \\ &= g_t((\theta_t)_* X_t, (\theta_t)_* \partial_{x_i}) \\ &= (\theta^* g_t)(X_t, \partial_{x_i}) \\ &= \tilde{g}_t(X_t, \partial_{x_i})\end{aligned}$$

Where we used at (β) *that* $\Theta_* \partial_t$ *is given by* $\partial_t + V_t$. *This follows directly from the definition of* $\Theta(x, t) = (\theta_t(x), t)$ *and* $V_t = \frac{d}{dt}\theta_t$. *In the last step we look at the time part of metric* $\Theta^* H$:

$$\begin{aligned}(\Theta^* H)(\partial_t, \partial_t) &= H(\Theta_* \partial_t, \Theta_* \partial_t) \\ &= H(\partial_t + V_t, \partial_t + V_t) \\ &= -(\tilde{\lambda} \circ \theta_t)^2 + g_t(V_t, V_t) = -\lambda^2\end{aligned}$$

Where we used in the last step the definition of λ. Now we are able to write down the global form of the Lorentzian metric $\Theta^* H$:

$$\Theta^* H = -\lambda^2 dt^2 + 2dt \odot E(X_t^\flat) + E(\tilde{g}_t)$$

This shows the identity $\Theta^* H = \psi^* h$

5) *In the last step we set $\phi = \psi \circ \Theta^{-1}$ and check that this does the job:*

$$\phi^* h = (\psi \circ \Theta^{-1})^* h = \Theta_* \circ \psi^* h = \Theta_* \Theta^* H = H.$$

Therefore we have the desired identity $\phi^* h = -\tilde{\lambda}^2 dt^2 + E(g_t)$. It remains to show that the objects $\tilde{\lambda}$ and g_t restrict on M to the objects λ and g. We know that g_t is given by $\rho_t^* \tilde{g}_t$. When we restrict this object to $t = 0$ we obtain

$$g_{t=0} = (\underbrace{\rho_{t=0}}_{=\mathrm{id}})^* \tilde{g}_{t=0} = \tilde{g}_{t=0} = g,$$

where we used in the last step that ψ fixes the hypersurface. The metric $\tilde{g}_0$ coincides with h on M and therefore $\tilde{g}_0$ is given by g. Finally we show that $\tilde{\lambda}$ restricts to λ on M:

$$\tilde{\lambda}(x, 0) = \underbrace{(\theta_{t=0})^*}_{=\mathrm{id}} \left[\lambda^2 - g_{t=0}(V_{t=0}, V_{t=0})\right]^{\frac{1}{2}} \overset{V_{t=0}=0}{=} \lambda.$$

We conclude, that we have shown all the statements from the theorem. ■

Remark 4.4: The previous theorem only mention compact hypersurfaces, but one can prove a similar result for non-compact hypersurfaces. So let (N, h) be a time-oriented Lorentzian manifold with a non-compact spacelike hypersurface M. We choose a tubular neighbourhood $\mathcal{U}$ of M in N and a compact exhaustion of M, i. e. there is a family of compact subspaces $(K_i)_{i\in\mathbb{N}}$ of M such that $\bigcup_{i\in\mathbb{N}} K_i = M$ and $K_i \subset \mathrm{Int}(K_{i+1})$ holds. Now we can apply the previous theorem to the sequence of Lorentzian manifolds $(K_i \times (-\varepsilon_i, \varepsilon_i), h_i)$ coming from the compact exhaustion and the tubular neighbhourhood. We obtain a family of diffeomorphisms ϕ_i such that the pullback metrics $\phi_i^* h_i$ is of the desired form $\tilde{\lambda}_i^2 dt^2 + (g_i)_t$. Finally we consider the metric $\overline{g} = \phi^* h$, where ϕ is given by a suitable diffeomorphism, which is given by a compatible sequence of diffeomorphism ϕ_i.

5 Local structure of manifolds with the constraint equation

In the last sections we constructed an extension for a Riemannian manifold with a nowhere vanishing vector field which satisfies the constraint Equation (5.1). Now we want to show that these manifolds are foliated by Riemannian submanifolds, see [28, Theorem 2].

Theorem 5.1: Let (M, g) be a Riemannian manifold, U a nowhere vanishing vector field on M and W a g-symmetric endomorphism, s. t.

$$\nabla U + uW = 0 \tag{5.1}$$

holds, where $u = \sqrt{g(U, U)}$. Then (M, g) is locally isometric to $(I \times \mathcal{F}, u^{-2}ds^2 + h_s)$, where h_s is a smooth family of Riemannian metrics on $\mathcal{F}$ and I a real interval. We can write W as

$$\begin{pmatrix} \partial_s \left(\frac{1}{u}\right) & \operatorname{grad}^{h_s} \frac{1}{u} \\ d\left(\frac{1}{u}\right) & -\frac{u}{2}\mathcal{L}_{\partial_s} h_s \end{pmatrix}$$

w. r. t. the local isometry.

Proof.
We start with the fact that the $(0, 2)$*-tensor* $A := \nabla U^\flat$ *is by Equation* (5.1) *symmetric, since* W *is symmetric. The symmetry of* W *is equivalent to the closedness of* $U^\flat$ *as an* 1*-form, since we have*

$$\begin{aligned}(dU^\flat)(X, Y) &= X(g(U, Y)) - Y(g(U, X)) - g(U, [X, Y]) \\ &= g(\nabla_X U, Y) - g(\nabla_Y U, X) + g(U, \underbrace{\nabla_X Y - \nabla_Y X - [X, Y]}_{=0}) \\ &= A(X, Y) - A(Y, X) = 0.\end{aligned}$$

Let $x_0 \in M$ *be an arbitary point, then there exists by the Poincare lemma an open neighbourhood* $\mathcal{V}$ *of* p *in* M *and a smooth function* $z \colon \mathcal{V} \to \mathbb{R}$*, s. t.* $U^\flat = dz$ *on* $\mathcal{V}$ *and* $z(x_0) = 0$ *holds. Using this construction we consider the distribution* $\ker(dz)$ *on* $\mathcal{V}$*. We* *obtain the integretability of the corresponding distribution given by $U^\perp$, since the 1-form $U^\perp$ is closed. Indeed, let $X, Y \in U^\perp$ then we have:*

$$0 = (dU^\flat)(X,Y) = X(\underbrace{g(U,Y)}_{=0}) - Y(\underbrace{g(U,X)}_{=0}) - g(U,[X,Y])$$

and hence $[X,Y] \in U^\perp$. In particular $U^\perp$ is involutive and by the Frobenius theorem integrable. The integrable manifolds of the distribution $U^\perp$ are given by the level sets $\mathcal{U}_c := z^{-1}(c)$ for all $c \in \mathbb{R}$. Indeed, let $\gamma\colon (-\varepsilon,\varepsilon) \to \mathcal{U}_c$ be a smooth curve with $\gamma(0) = x \in \mathcal{U}_c$ and $\dot\gamma(0) = v \in T_x\mathcal{U}_c$, then we compute

$$0 = \frac{d}{dt}_{|t=0} \underbrace{z(\gamma(t))}_{=c} = (d_{\gamma(0)}z)(\dot\gamma(0)) = (d_x z)(v)$$

and hence $v \in \ker(dz)_x = U_x^\perp$.

Now we consider the flow ϕ of $Z = \frac{1}{u^2}U$ on $(-\varepsilon,\varepsilon) \times \mathcal{W}$, where $\mathcal{W}$ is an open neighbourhood of x_0 in $\mathcal{V}$ which is chosen geodesic centered around x_0, i. e. the set is given by $B_r(x_0)$.

The maps

$$\begin{aligned} \psi\colon (-\varepsilon,\varepsilon) \times \mathcal{U}_0 &\leftrightarrow \mathcal{H} := \{y \in \mathcal{W} \mid |z(y)| < \varepsilon\} \colon \tilde\psi \\ (s,x) &\mapsto \phi_s(x) \\ (z(y), \phi_{-z(y)}(y)) &\mapsfrom y \end{aligned}$$

are the desired diffeomorphisms. At first we show that the flow sends level sets to level sets:

$$\frac{d}{ds}_{|s=0} z(x) + s = 1 \overset{(*)}{=} (d_x z)(Z_{\phi_0(x)}) = (d_{\phi_0(x)}z)(\dot\phi_0(x)) = \frac{d}{ds}_{|s=0} z(\phi_s(x)).$$

However, this implies $z(\phi_s(x)) = z(x) + s$, because $z(\phi_0(x)) = z(x) = z(x) + 0$ holds. At $(*)$ we used that

$$dz(Z) = g(\operatorname{grad}^g z, Z) \overset{Z=\frac{1}{u^2}U}{=} = \frac{1}{g(U,U)} g(\operatorname{grad}^g z, \operatorname{grad}^g z) \overset{U=\operatorname{grad}^g z \text{ on } \mathcal{V}}{=} 1$$

holds. We have to check that ψ is a diffeomorphism with invers $\tilde\psi$:

Well-definedness: *Check $\psi(s,x) \in \mathcal{H}$, this is clear by: $|z(\phi_s(x))| = |\underbrace{z(x)}_{=0, x\in\mathcal{U}_0} + s| = |s| < \varepsilon$. Now Check $\tilde\psi(y) \in (-\varepsilon,\varepsilon) \times \mathcal{U}_0$: $z(\phi_{-z(y)}(y)) = z(y) + (-z(y)) = 0$.*

$\psi \circ \tilde{\psi} = \mathrm{id}$*: Let $y \in \mathcal{H}$, then we have*

$$\psi(\tilde{\psi}(y)) = \psi(z(y), \phi_{-z(y)}(y)) = \phi_{z(y)}(\phi_{-z(y)}(y)) = \phi_{z(y)-z(y)}(y) = \phi_0(y) = y.$$

$\tilde{\psi} \circ \psi = \mathrm{id}$*: Let $(s, x) \in (-\varepsilon, \varepsilon) \times \mathcal{U}_0$, then we have*

$$\begin{aligned}\tilde{\psi}(\psi(s, x)) &= \tilde{\psi}(\phi_s(x)) \\ &= (z(\phi_s(x)), \phi_{-z(\phi_s(x))}(\phi_s(x))) \\ &= (z(x) + s, \phi_{-z(x)-s}(\phi_s(x)) \\ &\overset{z(x)=0}{=} (s, x).\end{aligned}$$

Set $\mathcal{F} \coloneqq \mathcal{U}_0$. Let $(s, x) \in (-\varepsilon, \varepsilon) \times \mathcal{F}$ and $\gamma(t) = (s + t, x)$ be a curve with $\gamma(0) = (s, x)$ and $\dot{\gamma}(0) = \partial_s$, then we have

$$(d_{(s,x)}\psi)(\partial_s) = \frac{d}{dt}_{|t=0}(\psi \circ \gamma)(t) = \frac{d}{dt}_{|t=0}\phi_{s+t}(x) = \dot{\phi}_s(x) = Z_{\phi_s(x)}.$$

Now we look at the pullback metric $\psi^*(g)$ on $(-\varepsilon, \varepsilon) \times \mathcal{F}$:

$$\begin{aligned}(\psi^* g)_{(s,x)}(\partial_s, \partial_s) &= g_{\psi(s,x)}((d_{(s,x)}\psi)(\partial_s), (d_{(s,x)}\psi)(\partial_s)) = (g(Z, Z))_{\phi_s(x)} = \left(\frac{1}{u^2}\right)_{\phi_s(x)} \\ (\psi^* g)_{(s,x)}(\partial_s, X) &= g_{\phi_s(x)}(Z_{\phi_s(x)}, (d_{(s,x)}\psi)(X)) = 0\end{aligned}$$

Where we used that Z is a multiple of U, $d\psi$ maps level sets to sets, which are given by $U^\perp$ and X is orthogonal to U. If we set $(h_s(X, Y))_{(s,x)} \coloneqq (\psi^* g)_{(s,x)}(X, Y)$, then we have the metric $h = \frac{1}{u^2}ds^2 + h_s$ on $I \times \mathcal{F}$ and we can express the g-symmetric endomorphism W in this identification in the following way:

$$\begin{aligned}(\psi^* W)_{(s,x)}(\partial_s, \partial_s) &= W_{\phi_s(x)}(Z_{\phi_s(x)}, Z_{\phi_s(x)}) \\ &= -\frac{1}{u} g_{\phi_s(x)}(\nabla^g_Z U, Z) \\ &= -\frac{1}{u^5} g_{\phi_s(x)}(\nabla^g_U U, U) \\ &= -\frac{1}{2u^5}\partial_U(u^2)_{|\psi(s,x)} \\ &= -\frac{1}{u^2}(\partial_s u)_{|\psi(s,x)} = \partial_s\left(\frac{1}{u}\right)_{|\psi(s,x)}\end{aligned}$$

Where we used the identification $\psi_*(\partial_s) = Z$. However, we have

$$\begin{aligned}
(\psi^* g)_{(s,x)}(\partial_s, X) &= W_{\phi_s(x)}(Z, \psi_*(X)) \\
&= -\frac{1}{u} g_{\phi_s(x)}(\nabla^g_Z U, \psi_*(X)) \\
&= -\frac{1}{u^3} g_{\phi_s(x)}(\nabla^g_U U, \psi_*(X)) \\
&= -\frac{1}{u^2}(\psi_*(X))\sqrt{g(U,U)} = (\psi_*(X))_{(s,x)}\left(\frac{1}{u}\right)
\end{aligned}$$

on the mixed part for $X \in T\mathcal{F}$. Let $X, Y \in T\mathcal{F}$, then we have

$$\begin{aligned}
(\psi^* W)_{(s,x)}(X,Y) &= W_{\phi_s(x)}(\psi_*(X), \psi_*(Y)) \\
&= -\frac{1}{u} g_{\phi_s(x)}(\nabla^g_{\psi_*(X)} \underbrace{U}_{=u^2 Z}, \psi_*(Y)) \\
&\overset{g(Z,\psi_*(X))=0}{=} -u g_{\phi_s(x)}(\nabla_{\psi_*(X)} Z, \psi_*(Y)) \\
&= -\frac{u}{2}\left[(g_{\phi_s(x)}(\nabla_{\psi_*(X)} Z, \psi_*(Y)) + g_{\phi_s(x)}(\nabla_{\psi_*(Y)} Z, \psi_*(X))\right] \\
&= -\frac{u}{2}(\mathcal{L}_Z g)(\psi_*(X), \psi_*(Y)) \\
&= -\frac{u}{2}(\mathcal{L}_{\psi_*(\partial_s)} \psi_*(h))(\psi_*(X), \psi_*(Y)) \\
&= -\frac{u}{2}\psi_*\left((\mathcal{L}_{\partial_s}.h)(X,Y)\right)
\end{aligned}$$

This implies that we can write W in the identification of ψ like:

$$\begin{pmatrix} \partial_s\left(\frac{1}{u}\right) & \mathrm{grad}^{h_s}\frac{1}{u} \\ d\left(\frac{1}{u}\right) & -\frac{u}{2}\mathcal{L}_{\partial_s} h_s \end{pmatrix}$$

■

6 Lorentzian manifolds with special holonomy

One of the motivations of the construction of Lorentzian manifold with a parallel null vector fields is the fact that these manifolds have special holonomy. In this section we give a short overview on the subject holonomy, major theorems and the special situation in the Lorentzian case. We will be guided by [2] and [29].

6.1 The holonomy group

Let $E \to M$ be a vector bundle with a connection ∇ over a manifold M. In general this connection is not induced by a metric. Let γ be a piecewise smooth curve in M with start point $p \in M$ and v be a vector in E_p. Then we can consider a system for a vector field along a curve:

$$\begin{cases} \frac{\nabla}{dt}\xi &= 0 \\ \xi_{|t=0} &= v. \end{cases} \tag{6.1}$$

The solution of the system exists, since we can trivialize the bundle along the path γ with finitely many trivializations and patch the solutions over any of these trivializations together, where the existence of the solutions on any of these trivial parts is guaranted by Picard-Lindelöf. The existence of the vector field ξ along the curve γ induces a linear map

$$\begin{aligned} \mathcal{P}_\gamma^\nabla : E_{\gamma(0)} &\to E_{\gamma(1)} \\ v &\mapsto \xi(1), \end{aligned}$$

where ξ is the solution of Equation (6.1) for v as the initial vector. This map is the so called **parallel transport** along γ. We can encode geometric information about our manifold M in the group of all parallel transports of curves that start and end in a single point $p \in M$, this is the **holonomy group** of ∇ at a point p.

Definition 6.1: Let (M, ∇) be a manifold with connection ∇ on a vector bundle $E \to M$, then we define the **holonomy group of** (M, ∇) by

$$\mathrm{Hol}(M, \nabla)_p := \{\mathcal{P}_\gamma^\nabla \mid \gamma \text{ piecewise smooth curve with } \gamma(0) = \gamma(1) = p\}.$$

The group multiplication is given by the composition of maps. The so called **connected holonomy group** is given by

$$\mathrm{Hol}^0(M, \nabla)_p := \{\mathcal{P}_\gamma^\nabla \mid \gamma \text{ piecewise smooth curve with } \gamma(0) = \gamma(1) = p, \gamma \text{ contractible}\}.$$

If the connection is induced by a metric g, we write $\mathrm{Hol}^{(0)}(M, g)_p := \mathrm{Hol}^{(0)}(M, \nabla^g)_p$.

Indeed this is a group by composition, since we can patch the solutions of Equation (6.1) together and obtain again a parallel transport. Moreover, the inverse is given by the parallel transport along the reversed curve. However the holonomy group is a subgroup of the isomorphisms of E_p and thus is equipped with a representation

$$\rho\colon \mathrm{Hol}(M, \nabla)_p \to \mathrm{GL}(E_p).$$

The holonomy group characterizes special geometry of the underlying manifold, e. g. existence of an orientation or existence of nice metrics.

Remark 6.2: Let (M^n, g) be a connected Riemannian manifold, then we can state useful characterizations of special geometry, see [23]:

(1) The manifold is orientable if and only if the holonomy group $\mathrm{Hol}(M, g)$ lies in $\mathsf{SO}(n)$.

(2) The manifold (M, g) is called **Kähler**, if there exists a compatible, almost complex structure $J\colon TM \to TM$, i. e. $J^* = -J$ and $J^2 = -\mathrm{id}$ holds, and a symplectic 2-form ω, s. t.

$$g = \omega(J\cdot, \cdot)$$

holds and the objects J and ω are parallel w. r. t. the induced Levi-Civita connection ∇^g. Now there is a characterization of Kähler in terms of the holonomy: The manifold (M, g) is Kähler if and only if $\mathrm{Hol}(M, g) \subset \mathsf{U}(\frac{n}{2})$.

Beside the characterizations of geometric structure one can ask whether what kind of subgroups of $\mathrm{GL}(\mathbb{R}^n)$ appear as holonomy groups of manifolds with arbitary connections. The following result gives an exhaustive answer to that question, see [19].

Theorem 6.3 (Hano,Ozeki '55): Let $n \geq 2$. Any closed Lie subgroup of $\mathrm{GL}(\mathbb{R}^n)$ can be realised as a holonomy group of a linear connection on $\mathbb{R}^n$ (in general with torsion).

If we are only interested in the restricted holonomy group, we can assume that the manifold is simply-connected.

Lemma 6.4: Let (M, g) be a connected semi-Riemannian manifold and $\pi\colon (\tilde{M}, \tilde{g}) \to (M, g)$ be its universal Riemannian covering, then π induces an isomorphism on the restricted holonomy groups. Moreover, holonomy groups for different points are conjugated to each other.

Proof.
Let $\tilde{\gamma}\colon [a, b] \to \tilde{M}$ be a path in $\tilde{M}$ with start point $\tilde{x}$, then we know by assumption that π is a local isometry and we have

$$\mathcal{P}^{\tilde{g}}_{\tilde{\gamma}} = (d_{\tilde{\gamma}(b)}\pi)^{-1} \circ \mathcal{P}^{g}_{\gamma} \circ d_{\tilde{\gamma}(a)}\pi \tag{6.2}$$

where $\gamma := \pi \circ \tilde{\gamma}$. Now we know that any path $\gamma\colon [a, b] \to M$, which is contractible, can be lifted to the universal covering. Thus by Equation (6.2) *we have*

$$\mathrm{Hol}_{\tilde{x}}(\tilde{M}, \tilde{g}) = \mathrm{Hol}^0_{\tilde{x}}(\tilde{M}, \tilde{g}) = (d_{\tilde{x}}\pi)^{-1} \circ \mathrm{Hol}_x(M, g) \circ (d_{\tilde{x}}\pi) \cong \mathrm{Hol}_x(M, g),$$

where the last isomorphism comes from the fact that $d_x\pi$ is an isomorphism.

Let $p, q \in M$ be arbitary points and $\gamma : p \rightsquigarrow q$ be a curve, then we have an isomorphism

$$\begin{aligned} \mathrm{Hol}_p(M, g) &\to \mathcal{P}^g_{\gamma^{-1}} \circ \mathrm{Hol}_q(M, g) \circ \mathcal{P}^g_{\gamma} \\ \mathcal{P}^g_{\delta} &\mapsto \mathcal{P}^g_{\gamma^{-1}} \circ \mathcal{P}^g_{\delta} \circ \mathcal{P}^g_{\gamma} \end{aligned}$$

of groups. ■

The lemma tells us that we can drop the point x in the notation $\mathrm{Hol}(M, g)_x$.

If the holonomy group is induced by a metric connection (i. e. ∇ is a connection on $E \to M$ with a compatible metric), then the holonomy group acts isometrically on the fiber of the vector bundle, hence the representation is of the form

$$\mathrm{Hol}(M, \nabla)_x \to \mathsf{O}(E_x, g_x).$$

In general it is a nontrivial task to compute the holonomy group of a connection, but we can compute its Lie algebra in terms of the curvature of the connection ∇, see [2, page 125, Satz 4.5].

Theorem 6.5 (Ambrose-Singer Theorem)**:** Let $(E, \nabla) \to M$ be a bundle with connection and $p \in M$ be a point, then the Lie algebra of the holonomy group $\mathrm{Hol}(M, \nabla)_p$ is given by:

$$\mathfrak{hol}(M, \nabla)_p = \operatorname{span}\left\{ \mathcal{P}_\gamma^{-1} \circ R^\nabla_{\gamma(1)}(X, Y) \circ \mathcal{P}_\gamma \;\middle|\; \begin{array}{r} \gamma\colon [0,1] \to M \text{ piecewise smooth curve} \\ \text{with } \gamma(0) = p, X, Y \in T_{\gamma(1)}M \end{array} \right\}$$

Beside the computation of the holonomy Lie algebra and hence the holonomy group, we can characterize parallel objects of a bundle in terms of holonomy invariant objects, see [2, Satz 4.8].

Theorem 6.6 (Holonomy Principle)**:** Let (M, g) be semi-Riemannian manifold, p be a point of M and $\mathcal{T}$ be a tensor bundle over M with induced Levi-Civita connection ∇^g, i. e. subbundle of $T^{*,*}M = \bigoplus_k (T^*M)^{\otimes k} \oplus (TM)^{\otimes k}$. Then we have a bijection

$$\{\phi \in \Gamma(\mathcal{T}) \mid \phi \text{ is parallel}\} \longrightarrow \{v \in \mathcal{T}_x \mid \mathrm{Hol}(M, g)v = v\}$$
$$\phi \mapsto \phi_x,$$

or in other words: There is an one-to-one correspondence between parallel objects of the tensor bundle and holonomy invariant tensors of a fibre of $\mathcal{T}$.

The last important aspect of the holonomy group is a characterization of local isometric splittings of our semi-Riemannian manifold (M, g). In accordance with that we need the notion irreducible and indecomposable representations.

Definition 6.7: Let (M, g) be a semi-Riemannian manifold, $x \in M$ be a point and let $\rho\colon \mathrm{Hol}(M, g)_x \to O(T_xM, g_x)$ be the holonomy representation.

- We call (M, g) or $\mathrm{Hol}(M, g)$ **irreducible**, if the representation ρ is irreducible, i. e. there is no proper invariant subspace of T_xM.
- We call (M, g) or $\mathrm{Hol}(M, g)$ **indecomposable**, if there is no proper, non-degenerated invariant subspace of T_xM.

It is clear that every irreducible manifold is indecomposable. Let (M, g) be a semi-Riemannian manifold and $E \subset T_pM$ be a proper non-degenerated invariant subspace, then we can construct the so called holonomy distribution given by

$$H\colon p \in M \mapsto \mathcal{P}^g_\gamma(E) \subset T_pM,$$

where $\gamma\colon q \rightsquigarrow p$ is an arbitary path. Indeed, this distribution is well-defined, because if we choose another path $\gamma'\colon q \rightsquigarrow p$, then by the holonomy invariance of E we obtain $\mathcal{P}^g_{\gamma^{-1}\gamma'}(E) = E$ and hence $\mathcal{P}^g_\gamma(E) = \mathcal{P}^g_{\gamma'}(E)$. The crucial property of this distribution is the involutivity and the maximal connected integrable manifolds are submanifolds of our initial semi-Riemannian manifold. Thus by the Frobenius theorem we have a local metric splitting.

Theorem 6.8 (De Rahm-Wu decomposition): Let (M, g) be a semi-Riemannian manifold, p be a point of M and $E \subset T_pM$ be a proper non-degnerated holonomy invariant subspace. Then for any point $q \in M$ exists an open neighbourhood U_q, such that U_q splits isometrically, i. e.

$$(U_q, g_{q|U_q}) \cong (U_1, g_1) \times (U_2, g_2).$$

Moreover, the holonomy group $\mathrm{Hol}^0_p(M, g)$ is isomorphic to $\mathsf{O}(E) \times \mathsf{O}(E^\perp)$. If the semi-Riemannian manifold (M, g) is simply-connected and geodesically complete, then (M, g) is isometric to a product of simply-connected and geodesically complete semi-Riemannian manifolds, i. e.

$$(M, g) \cong (M_0, g_0) \times (M_1, g_1) \times \ldots \times (M_k, g_k),$$

where (M_0, g_0) is a flat semi-Riemannian manifold and each (M_i, g_i) for $i \geq 1$ is indecomposable and non-flat.

The previous decomposition theorem enables us to reduce the question of the structure of the Lorentzian holonomy groups to the study of indecomposable ones.

6.2 Decomposition of a Lorentzian manifold

In this part we want to restrict our point of view to the class of Lorentzian manifolds and the corresponding holonomy groups. In the first instance we describe the general situation. There is a classification of irreducible, not local-symmetric manifolds first described by Berger, see [5].

Theorem 6.9: Let (M, g) be a simply-connected, irreducible semi-Riemannian manifold with signature (p, q), which is not local-symmetric. Then the holonomy group is up to conjugation in $\mathsf{O}(p, q)$ isomorphic to $\mathsf{SO}^0(p, q)$ or one of the following groups in the tabular:

Dimension	Signature	Holonomy group
$n \geq 4$	$(2r, 2s)$	$\mathsf{U}(r,s)$ or $\mathsf{SU}(r,s)$
$n \geq 4$	(p,p)	$\mathsf{SO}(p,\mathbb{C})$
$2n \geq 8$	$(4r, 4s)$	$\mathsf{Sp}(r,s)$ or $\mathsf{Sp}(r,s) \cdot \mathsf{Sp}(1)$
$2n \geq 8$	$(2r, 2s)$	$\mathsf{Sp}(r,\mathbb{R}) \cdot \mathsf{SL}(2,\mathbb{R})$
$2n \geq 16$	$(4r, 4r)$	$\mathsf{Sp}(r,\mathbb{C}) \cdot \mathsf{SL}(2,\mathbb{C})$
7	$(4,3)$	$G^*_{2(2)}$
14	$(7,7)$	$G_2^{\mathbb{C}}$
8	$(4,4)$	$\mathsf{Spin}(4,3)$
16	$(8,8)$	$\mathsf{Spin}(7,\mathbb{C})$

The situation in the Lorentzian case is special, because of the special property of the Lie group $\mathsf{SO}^0(1,n)$, see [11].

Theorem 6.10 (Di Scala, Olmos '01)**:** If H is a connected subgroup of $\mathsf{SO}^0(1,n)$ which acts irreducible on $\mathbb{R}^{1,n}$, then H is already $\mathsf{SO}^0(1,n)$, i. e. $H = \mathsf{SO}^0(1,n)$.

We can adapt the De-Rahm-Wu decomposition to the Lorentzian case with the help of Theorem 6.10.

Theorem 6.11: Let (M,g) be a simply-connected, geodesically complete Lorentzian manifold , then there exists a global isometric decomposition, i. e.

$$(M,g) \cong (M_0, g_0) \times (M_1, g_1) \times \ldots \times (M_k, g_k),$$

where each (M_i, g_i) for $i \geq 1$ is either flat or irreducible and (M_0, g_0) is one of the following:

(1): The space $(\mathbb{R}, -dt^2)$.

(2): An irreducible Lorentzian manifold with holonomy $\mathrm{Hol}^0(M_0, g_0)$ isomorphic to $\mathsf{SO}^0(M_0, g_0)$.

(3): An indecomposable, non-irreducible Lorentzian manifold.

Thus we are interested in indecomposable, non-irreducible Lorentzian manifolds. In the following we describe an important example, see [2, Bsp. 5.5].

Example 6.12: In this example we will construct a Lorentzian manifold (M,g) which is indecomposable, but non-irreducible. Let $(M,g) = (\mathbb{R} \times \mathbb{R} \times F, g)$ with coordinates (v, u, x), where $x = (x^1, \ldots, x^n)$ is a chart of the Riemannian manifold (F, h). The metric g is given by

$$g = 2dv \odot du + f du^2 + h,$$

where $f \in C^\infty(\mathbb{R} \times F)$ is a smooth function in the variables (u, x). We consider the following frame

$$e_- := \frac{\partial}{\partial v}, \qquad e_+ := -\frac{f}{2}\frac{\partial}{\partial v} + \frac{\partial}{\partial u}, \qquad s_i := \frac{\partial}{\partial x^i}.$$

Where $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ is an orthonormal frame of the chart x. Moreover the frame is a Witt basis, i. e. $g(e_+, e_+) = g(e_-, e_-) = 0, g(e_-, e_+) = 1$ and $g(s_i, s_j) = \delta_{ij}$. In particular the metric g in this frame is of the form

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & E_n \end{pmatrix}.$$

The only nontrivial commutator of this frame is given by $[e_+, s_i] = \frac{1}{2}s_i(f)e_-$, since $e_-(f) = 0$ holds by assumption. We claim that the description of the Levi-Civita connection of g in the frame $\{e_+, e_-, s_i\}$ is given by the following tabular:

$\nabla^g_X Y$	e_-	e_+	s_j
e_-	0	0	0
e_+	0	$-\frac{1}{2}\operatorname{grad}^h f$	$\frac{1}{2}s_j(f)e_-$
s_i	0	0	$\nabla^h_{s_i} s_j$

Table 6.1: Levi-Civita connection of (M, g)

In the following calculation we will use the Koszul formula for the Levi-Civita connection, see [25, Theorem 11 page 61]:

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ & + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X) \end{aligned}$$

for all $X, Y, Z \in TM$.

$\nabla e_- = 0$: If we set $Y = e_-$ in the Koszul formula and let vary X, Z over the set $\{e_-, e_+, s_i\}$, then the first three expressions vanish, because each $g(\cdot, \cdot)$ is constant in this frame, thus

$$2g(\nabla_X e_-, Z) = \underbrace{g([X, e_-], Z)}_{=0} - g([X, Z], e_-) - \underbrace{g([e_-, Z], X)}_{=0}.$$

Where we used that $[e_-, X]$ vanish for all $X \in \{e_-, e_+, s_i\}$. The only nowhere vanishing expressions can be only one of the cases $(X, Z) = (e_+, s_i), (s_i, e_+)$, but every commutator $[X, Z]$ is of the form we_- for a function w. Thus $g(e_-, e_-)$ vanish and hence $\nabla e_- = 0$.

$\nabla_{e_-} e_+ = 0$: This follows from torsionfreeness of the Levi-Civita connection ∇^g and the previous case: $\nabla_{e_-} e_+ = \nabla_{e_+} e_- + [e_-, e_+] = 0$

$\nabla_{e_+} e_+ = -\frac{1}{2} \operatorname{grad}^F f$: We will use again the Koszul formula, to obtain:

$$2g(\nabla_{e_+} e_+, Z) = g([e_+, e_+], Z) - g([e_+, Z], e_+) - g([e_+, Z], e_+) = -2g([e_+, Z], e_+)$$

Let $Z = s_i$, then we have $g(\nabla_{e_+} e_+, s_i) = -g([e_+, s_i], e_+) = -\frac{1}{2} s_i(f)$ and thus $\nabla_{e_+} e_+ = -\frac{1}{2} \operatorname{grad}^F f$.

$\nabla_{e_-} s_j = 0$: Again torsionfreeness:

$$\nabla_{e_-} s_j = \nabla_{s_j} e_- + [e_-, s_j] = 0$$

$\nabla_{e_+} s_j = \frac{1}{2} s_j(f) e_-$: Same as before:

$$2g(\nabla_{e_+} s_j, Z) = g([e_+, s_j], Z) - g([e_+, Z], s_j) - g([s_j, Z], e_+)$$

Let $Z = e_-$, then all commutators vanish. Let $Z = e_+$, then

$$= 0 + 0 + g([e_+, s_j], e_+) = \frac{1}{2} s_j(f).$$

And in the last case $Z = s_j$ we have $g(e_-, s_i) = 0$. Thus $\nabla_{e_+} s_j = \frac{1}{2} s_j(f) e_-$.

$\nabla_{s_i} e_+ = 0$: Torsionfreeness:

$$\nabla_{s_i} e_+ = \nabla_{e_+} s_i - [e_+, s_i] = \frac{1}{2} s_i(f) e_- - \frac{1}{2} s_i(f) e_- = 0.$$

$\nabla^g_{s_i} s_j = \nabla^h_{s_i} s_j$: This is clear by the construction of g.

Beside the Levi-Civita connection we are interested in the curvature of g. Therefore we claim that the description of the Riemannian curvature of g in the frame $\{e_+, e_-, s_i\}$ is given by the following tabular:

(X,Y,Z)	$R^g(X,Y)Z$
(s_i,s_j,s_k)	$R^h(s_i,s_j)s_k$
(e_+,s_j,s_k)	$-\frac{1}{2}\operatorname{Hess}^h(f)(s_j,s_k)e_-$
(s_i,s_j,e_+)	0
(e_+,s_j,e_+)	$\frac{1}{2}\nabla^h_{s_j}(\operatorname{grad}^h f)$

Table 6.2: Curvature of the Levi-Civita connection of (M,g)

These are all the interesting curvature terms, because by the parallelity of e_- all expressions of the form $R^g(X,Y)Z$ vanish if one of X,Y,Z is e_-.

$R^g(s_i,s_j)s_k$: This is clear, because the connections ∇^g and ∇^h coincide on the frame of (F,h).

$R^g(e_+,s_j)s_k$: We use the entries of Table 6.1 and the definition of the Hessian of f, given by $\operatorname{Hess}(f)(X,Y) = (\nabla df)(X,Y) = XY(f) - (\nabla_X Y)(f)$ for $X,Y \in TM$, to obtain:

$$\begin{aligned}
R^g(e_+,s_j)s_k &= \nabla^g_{e_+}\nabla^g_{s_j}s_k - \nabla^g_{s_j}\nabla^g_{e_+}s_k - \nabla^g_{[e_+,s_j]}s_k \\
&= (\Gamma^h)^l_{jk}\nabla^g_{e_+}s_l - \nabla^g_{s_j}\left(\frac{1}{2}s_k(f)e_-\right) - \frac{1}{2}s_j(f)\underbrace{\nabla^g_{e_-}s_k}_{=0} \\
&= (\Gamma^h)^l_{jk}\frac{1}{2}s_l(f)e_- - \frac{1}{2}s_j(s_k(f))e_- \\
&= -\frac{1}{2}\left(s_js_k(f) - (\nabla^h_{s_j}s_k)(f)\right)e_- \\
&= -\frac{1}{2}\operatorname{Hess}^h(f)(s_j,s_k)e_-
\end{aligned}$$

Where we used that $(\Gamma^h)^l_{jk}$ only depends on the x coordinates, thus the ∂_{e_+} derivative vanish.

$R^g(s_i,s_j)e_+$: We calculate:

$$R^g(s_i,s_j)e_+ = \nabla^g_{s_i}\nabla^g_{s_j}e_+ - \nabla^g_{s_j}\nabla^g_{s_i}e_+ - \nabla^g_{[s_i,s_j]}e_+ = 0$$

Where we used that $[s_i,s_j]$ and the derivative $\nabla_{s_j}e_+$ vanish.

$R^g(e_+,s_j)e_+$: We calculate:

$$\begin{aligned}
R^g(e_+,s_j)e_+ &= \nabla^g_{e_+}\nabla^g_{s_j}e_+ - \nabla^g_{s_j}\nabla^g_{e_+}e_+ - \nabla^g_{[e_+,s_j]}e_+ \\
&= \frac{1}{2}\nabla^h_{s_j}(\operatorname{grad}^h f) - \frac{1}{2}s_j(f)\underbrace{\nabla^g_{e_-}e_+}_{=0} = \frac{1}{2}\nabla^h_{s_j}(\operatorname{grad}^h f)
\end{aligned}$$

Now we are in the situation to determine the holonomy group of (M, g). We want to prove:

$$\mathrm{Hol}(M,g)_{\tilde{p}} = \mathrm{Hol}(F,h)_p \ltimes \mathbb{R}^n$$

Where we set $\tilde{p} = (0,0,p) \in M$ with $p \in F$. We have to consider parallel transport $\mathcal{P}^g_\gamma$ along curves $\gamma(\tau) = (v(\tau), u(\tau), \delta(\tau))$ closed at $\tilde{p}$. Let $X(\tau)$ be an arbitary vector field along γ, then we can use the frame (e_-, e_+, s_i) to write it as $a(\tau)e_-(\tau) + b(\tau)e_+(\tau) + Y(\tau)$, where $e_-(\tau), e_+(\tau)$ are the vector fields transported parallel along γ with start vectors e_-, e_+ and $Y(\tau)$ lies in $T\mathcal{F}$. In general we have the following local formula for the covariant derivative along a curve:

$$\frac{\nabla}{d\tau}X(\tau) = \left(\dot{X}(\tau)^\mu + \dot{\gamma}^\nu(\tau)X^\sigma(\tau)\Gamma^\mu_{\nu\sigma}(\gamma(\tau))\right)\frac{\partial}{\partial x^\mu}.$$

When we use the previous identity and the special form of the vector field $X(\tau)$, we obtain

$$\frac{\nabla}{d\tau}X(\tau) = \left(\dot{a} + \frac{1}{2}u(\tau)Y(f)\right)e_-(\tau) + \dot{b}e_+(\tau) + \nabla^h_{\dot{\delta}}Y(\tau) - \frac{1}{2}b(\tau)\dot{u}(\tau)\,\mathrm{grad}^h f. \quad (6.3)$$

We have to compute the parallel transport of our initial frame e_-, e_+, s_j along γ. We claim that these are given by:

$$\mathcal{P}^g_\gamma(e_-)_{\tilde{p}} = (e_-)_{\tilde{p}}, \quad (6.4)$$

$$\mathcal{P}^g_\gamma(s_j)_{\tilde{p}} = -\frac{1}{2}\left(\int_0^1 \dot{u}(s)df(s_j(s))ds\right)\cdot(e_-)_{\tilde{p}} + \mathcal{P}^h_\delta(s_j)_p, \quad (6.5)$$

$$\mathcal{P}^g_\gamma(e_+)_{\tilde{p}} = -\frac{1}{2}\left(\int_0^1 \dot{u}(s)df(V)ds\right)\cdot(e_-)_{\tilde{p}} + (e_+)_{\tilde{p}} + V(1). \quad (6.6)$$

The Equation (6.4) comes from the fact that e_- is parallel. The Equations (6.5) and (6.6) are directly solutions of Equation (6.3), by integrating the system:

$$\begin{cases} \frac{\nabla}{d\tau}X(\tau) & = 0 \\ X(0) & = e_+ \text{ or } s_j, \end{cases}$$

where V in Equation (6.5) is a solution of the ODE $\nabla^h_{\dot{\delta}}V = \frac{1}{2}\dot{u}(\tau)\,\mathrm{grad}^h f$ with $V(0) = 0$. In other words, we can write a parallel transport $\mathcal{P}^g_\gamma$ in the basis e_-, s_i, e_+ as a matrix in the form:

$$\mathcal{P}^g_\gamma = \begin{pmatrix} 1 & \left(-\frac{1}{2}\left(\int_0^1 \dot{u}(s)df(s_i(s))ds\right)\right)^T_i & -\frac{1}{2}\left(\int_0^1 \dot{u}(s)df(V)ds\right) \\ 0 & \mathcal{P}^h_\delta & V(1) \\ 0 & 0 & 1 \end{pmatrix} \quad (6.7)$$

The Equation (6.4) shows that the subspace $L := (\mathbb{R}e_-)_{\tilde{p}} \subset T_{\tilde{p}}M$ is holonomy invariant and by the lightlikeness of e_- also degenerated, hence (M,g) is **non-irreducible**. On the other hand we will show that any invariant subspace contains L and thus is degenerated.

Let $\mathcal{P}^g_\gamma \in \mathrm{Hol}(M,g)$ be a parallel transport, then we have by Equation (6.4) an inclusion of the holonomy group into the stabilisator of e_-

$$\mathrm{Hol}(M,g)_{\tilde{p}} \subset \mathsf{O}(T_{\tilde{p}}M, g_{\tilde{p}})_{e_-} = \left\{ \begin{pmatrix} 1 & v^T & -\frac{1}{2}|v|^2 \\ 0 & A & -Av \\ 0 & 0 & 1 \end{pmatrix} \mid v \in \mathbb{R}^n, A \in \mathsf{O}(n) \right\}, \tag{6.8}$$

where we write the matrices in the basis $e_-, s_1, \dots, s_n, e_+$. We will show that any nontrivial invariant subspace is degenerated and hence the holonomy is indecomposable. Let U be a nontrivial holonomy invariant subspace of $T_{\tilde{p}}M$, then we want to show that e_- lies in U. At first we consider the holonomy representation $\rho\colon \mathrm{Hol}(M,g)_{\tilde{p}} \to \mathsf{O}(T_{\tilde{p}}M, g_{\tilde{p}})$ and obtain a representation of the Lie algebra of the holonomy group into the skew-symmetric endomorphism:

$$d\rho\colon \mathfrak{hol}(M,g)_{\tilde{p}} \to \mathfrak{o}(T_{\tilde{p}}M, g_{\tilde{p}}).$$

We consider an ideal I of $\mathfrak{hol}(M,g)_{\tilde{p}}$ given by

$$I := \left\{ \begin{pmatrix} 0 & x^T & 0 \\ 0 & 0 & -x \\ 0 & 0 & 0 \end{pmatrix} \mid x \in \mathbb{R}^n \right\} \subset \mathfrak{hol}(M,g)_{\tilde{p}} \subset \left\{ \begin{pmatrix} 0 & x^T & 0 \\ 0 & B & -x \\ 0 & 0 & 0 \end{pmatrix} \middle| \begin{array}{l} x \in \mathbb{R}^n, \\ B \in \mathfrak{o}(T_{\tilde{p}}M, g_{\tilde{p}})_p \end{array} \right\}. \tag{6.9}$$

This ideal leads again to a representation on $T_{\tilde{p}}M$, in particular it stabilizes the invariant subspace U. Indeed, this is an ideal in the Lie algebra $\mathfrak{hol}(M,g)$ by the following observation: Let $f_0 := f(0,\cdot)$ be non-degenerated at the point $p \in F$, i. e. the Hessian is non-degenerated at the point p and moreover the Hessian should be diagonalized by the frame $s_1, \dots, s_n$, i. e.

$$\mathrm{Hess}(f_0)^h(s_i, s_j) = \lambda_i \delta_{ij}$$

or equivalently given by

$$\nabla^h_{s_j} \operatorname{grad}^h f_0 = (\mathrm{Hess}^h(f_0)(s_j,\cdot))^{\flat(h)} = \lambda_j s_j$$

for all $i, j \in \{1, \dots, n\}$ and $\lambda_i \neq 0$. Then the Table 6.2 leads to the identification of the curvature endomorphism $R^g(e_+, s_j)$ in terms of the function f.

Let $X, Y \in T_{\tilde{p}}M$ and we identify the curvature $R^g(X,Y)$ as a form in $\Lambda^2(T_{\tilde{p}}M)$. Moreover, we consider the generalised orthonormal basis of $\Lambda^2(T_{\tilde{p}}M)$ given by $s_v \wedge s_u, s_v \wedge s_j, s_u \wedge s_j, s_i \wedge s_j$. Where we have the local generalised orthonormal frame s_v, s_u, s_i, with $s_v = \frac{1}{\sqrt{2}}(e_- - e_+)$,$s_u = \frac{1}{\sqrt{2}}(e_- + e_+)$. Then we can develop the curvature expression in the new basis and calculate:

$$\begin{aligned}
R^g(e_+, s_j) &= -g(R^g(e_+, s_j)s_v, s_u)s_v \wedge s_u - g(R^g(e_+, s_j)s_v, s_k)s_v \wedge s_k \\
&\quad + g(R^g(e_+, s_j)s_u, s_k)s_u \wedge s_k + g(R^g(e_+, s_j)s_k, s_l)s_k \wedge s_l \\
&= -g(R^g(e_+, s_j)\frac{1}{\sqrt{2}}(e_- - e_+), \frac{1}{\sqrt{2}}(e_- + e_+))e_- \wedge e_+ \\
&\quad - g(R^g(e_+, s_j)\frac{1}{\sqrt{2}}(e_- - e_+), s_k)s_v \wedge s_k + g(R^g(e_+, s_j)\frac{1}{\sqrt{2}}(e_- + e_+)s_k)s_u \wedge s_k \\
&\quad + g(R^g(e_+, s_j)s_k, s_l)s_k \wedge s_l \\
&= -\frac{1}{2\sqrt{2}}s_v \wedge (\nabla^h_{s_j} \operatorname{grad}^h f) + \frac{1}{2\sqrt{2}}s_u \wedge (\nabla^h_{s_j} \operatorname{grad}^h f) \\
&= \frac{1}{4}e_+ \wedge (\nabla^h_{s_j} \operatorname{grad}^h f_0) \\
&= \frac{1}{4}\lambda_j e_+ \wedge s_j = \frac{1}{4}\begin{pmatrix} 0 & -\lambda_j s_j^T & 0 \\ 0 & 0 & \lambda_j s_j \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{hol}(M,g)_{\tilde{p}}
\end{aligned}$$

Where we used the Ambrose-Singer theorem, which states that the holonomy Lie algebra is generated by curvature terms, see Theorem 6.5. The ideal generated by the terms $R^g(e_+, s_j)$ is exactly I.

Now let v be a non-zero vector in U, we write v in the basis e_-, s_i, e_+ as (a, w_0, b) and consider different cases.

$b = 0$: If w_0 vanish, then we have $v = (a, 0, 0) \in U$ and we are finished, because we assumed that v is non-zero and thus $e_- \in U$. In other words U contains a lightlike vector and therefore is a degenerated subspace. If the vector w_0 is non-zero, we can act on v by an element of the ideal I:

$$\begin{pmatrix} 0 & -\left(-\frac{w_0}{\|w_0\|^2}\right)^t & 0 \\ 0 & 0 & \left(-\frac{w_0}{\|w_0\|^2}\right) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ w_0 \\ 0 \end{pmatrix} = (1,0,0)^t = e_- \in U.$$

$b \neq 0$: So we have $v = (a, w_0, b)$ and can assume $b = 1$, because we can multiply by a scalar, since we are in the vector space U. Now we consider the case $w_0 = 0$

and act again on v by an element of the ideal:

$$\begin{pmatrix} 0 & -{e_i}^t & 0 \\ 0 & 0 & e_i \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix} = (0, s_i, 0)^t.$$

Thus we have $s_i \in U$ and can act again on this element and obtain with the same argument as before that $e_- \in U$. Now let $w_0 \neq 0$, then we can do the same trick and obtain $(a + \|w_0\|^2, 0, 1) \in U$. As in the previous case we obtain $s_i \in U$ and hence $e_- \in U$.

This shows that all invariant subspaces are degenerated and therefore the holonomy is **indecomposable**.

A consequence of Equation (6.7) is the general form of a parallel transport:

$$\mathcal{P}^g_\gamma = \begin{pmatrix} 1 & * & * \\ 0 & \mathcal{P}^h_\delta & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus we had proved that the holonomy group is exactly of the form $\mathrm{Hol}(F, h) \ltimes \mathbb{R}^n$.

6.3 The definition of the screen bundle and its properties

Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold with parallel null vector field V. Then we can consider the screen bundle of V given by

$$S := V^\perp / V \to \overline{M}.$$

This bundle is well-defined, since $V \subset V^\perp$ or in other words V is lightlike. On this bundle exists a connection and a compatible metric induced by the structure on $T\overline{M}$:

$$\begin{aligned} \overline{g}([X], [Y]) &:= \overline{g}(X, Y) \\ \nabla^S_Z [Y] &:= [\nabla^{\overline{g}}_Z Y] \end{aligned}$$

for all $X, Y \in V^\perp$ and $Z \in T\overline{M}$. Indeed, this is well-defined: Let $X - \tilde{X} = \alpha V, Y - \tilde{Y} = \beta V$ with $X, \tilde{X}, Y, \tilde{Y} \in V^\perp$, then we have

$$\begin{aligned} \overline{g}([X], [Y]) = \overline{g}(X, Y) &= \overline{g}(\tilde{X} + \alpha V, \tilde{Y} + \beta V) \\ &= \overline{g}(\tilde{X}, \tilde{Y}) + \beta \underbrace{\overline{g}(X, V)}_{=0, \tilde{X} \in V^\perp} + \alpha \underbrace{\overline{g}(V, \tilde{Y})}_{=0, \tilde{Y} \in V^\perp} + \alpha\beta \underbrace{\overline{g}(V, V)}_{=0, V \text{ is null}} \\ &= \overline{g}([\tilde{X}], [\tilde{Y}]) \end{aligned}$$

and

$$\nabla^S_X[Y] = [\nabla^{\overline{g}}_X Y] = [\nabla^{\overline{g}}_X(\tilde{Y} + \beta V)] = [\nabla^{\overline{g}}_X \tilde{Y} + (\partial_X \beta)V + \underbrace{\beta \nabla^{\overline{g}}_X V}_{=0, V \text{ is parallel}}]$$
$$\overset{[X+V]=[X]}{=} [\nabla^{\overline{g}}_X \tilde{Y}] = \nabla^S_X[\tilde{Y}].$$

The screen bundle is in general not a subbundle of the tangent bundle, but in the situation of Theorem 1.1 we can prove such an identification.

Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold with a parallel null vector field V, which is a solution of Theorem 1.1, i. e. the vector field V is decomposable as $uT - U$, where T is the time orientation of $\overline{M}$. In this situation we can identify the screen bundle as a subbundle of the tangent bundle of $\overline{M}$.

Lemma 6.13: The bundle map

$$(T^\perp \cap V^\perp \subset T\overline{M}, \mathrm{pr}^*\,\overline{g}, \mathrm{pr}\circ\overline{\nabla}) \overset{\psi}{\to} (S = V^\perp/V, g^S, \nabla^S)$$
$$Y \mapsto [Y]$$

is an isomorphism of vector bundles, which preserves the metric and connection.

Proof.
We check that the map is fiberwise an isomorphism. Let $\overline{M} = \mathbb{R}^{1,n}$, $V = (1,1,0,\dots,0)$, $T = (1,0,\dots,0)$ and $\overline{g} = \mathrm{diag}(-1,1,\dots,1)$. W. l. o. g. we have $T^\perp = \langle e_1,\dots,e_n\rangle$ and $V^\perp = \langle v = e_0 + e_1, e_2, \dots, e_n\rangle$, where $e_0, e_1, \dots, e_n$ is the standard generalised orthonormal basis of $\mathbb{R}^{1,n}$. This implies $T^\perp \cap V^\perp = \langle e_2,\dots,e_n\rangle \overset{\cong}{\to} V^\perp/V = (\langle v, e_1, \dots, e_n\rangle)/(\langle v\rangle)$. The preservation of the metric and connection follows directly from the observation that the bundle metric and connection on $T^\perp \cap V^\perp$ are the pulls back under the map ψ. ■

We want to consider the holonomy group of a time-oriented Lorentzian manifold with a parallel null vector field. These parallel objects gives us a reduction of the holonomy group.

Proposition 6.14: Let $(\overline{M}, \overline{g})$ be a time-oriented Lorentzian manifold with a parallel null vector field V, then we have

$$\mathrm{Hol}(\overline{M}, \overline{g}) \subset \mathsf{SO}(1,n)_V \subset \mathsf{SO}(1,n),$$

where $\mathsf{SO}(1,n)_V$ is the stabiliser of V. Furthermore, we can identify this stabiliser with

$$\mathsf{SO}(1,n)_V \overset{\cong}{\to} \mathsf{SO}(n) \ltimes \mathbb{R}^n = \left\{ \begin{pmatrix} 1 & v^T & -\frac{1}{2}\|v\|^2 \\ 0 & A & -Av \\ 0 & 0 & 1 \end{pmatrix} \middle| \, A \in \mathsf{SO}(n), v \in \mathbb{R}^n \right\}.$$

Moreover, we have the identification of holonomy groups:

$$\mathrm{Hol}(S, \nabla^S) \cong \pi^{\mathsf{SO}(n)}\,\mathrm{Hol}(\overline{M}, \overline{g}) \tag{6.10}$$

Proof.
The first statement follows immediately from the holonomy principle or in other words: A parallel object is fixed under parallel transport. The identification of the stabiliser with the group $\mathsf{SO}(n) \ltimes \mathbb{R}^n$ can be seen by dimension counting and the following calculation:

W. l. o. g. let $(\overline{M}, \overline{g}) = (\mathbb{R}^{1,n}, \overline{g} = \mathrm{diag}(-1,1,\dots,1))$, with the Witt basis $(s_-, s_1, \dots, s_{n-1}, s_+)$, such that $\overline{g}$ in this basis is given by

$$(\overline{g}(s_\alpha, s_\beta))_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & E_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then we set $V = s_-$ as the lightlike vector and see that an arbitary element $(A, v) \in \mathsf{SO}(n) \ltimes \mathbb{R}^n$ given by a matrix as in the claim fixes this vector v. By dimension counting of $\mathsf{SO}(1,n)_V$ and $\mathsf{SO}(n) \ltimes \mathbb{R}^n$ and the fact that both Lie groups are closed we have an diffeomorphism.

The identification of the holonomy groups follows directly by the identification of connections: $\nabla^S = \pi \circ \overline{\nabla}_{|S}$. ∎

6.4 Special holonomy and the constraint equation

A consequence of the proposition in the previous section is that the $\mathsf{SO}(n)$-factor of the semi-direct product $\mathsf{SO}(n) \ltimes \mathbb{R}^n$ of the holonomy group $\mathrm{Hol}(\overline{M}, \overline{g})$ is determined by the screen holonomy. When we want to construct Lorentzian metrics with special holonomy, we can consider the Lorentzian metrics that we obtained in Theorem 1.1, since they carry a parallel null vector. The goal of the following theorem is to translate the special holonomy into analytical conditions on the hypersurface. The idea is to find suitable PDEs on the foliation that we obtained in Theorem 5.1.

Theorem 6.15: Let $(M, g, U, W) = (I \times \mathcal{F}, g = u^{-2}ds^2 + h_s, U = u^2\partial_s, W)$ be a Riemannian manifold as in Theorem 5.1 that satisfies the Riemannian constraint equation and let $(\overline{M}, \overline{g}, V)$ be a solution of Riemannian Cauchy problem for (M, g, U, W) as in Theorem 1.1. Moreover let $G = \pi^{\mathsf{SO}(n)} \operatorname{Hol}(\overline{M}, \overline{g})$ be the screen holonomy induced by the parallel null vector field V, then for all $k, l \geq 0$ the following statements are equivalent:

i) There exists a tensor $\sigma \in T^{k,l}\mathbb{R}^n$, such that the screen holonomy lies in the stabilizer of σ, i. e. $G \subset \mathsf{SO}(n)_\sigma$.

ii) There exists a ∇^{h_s}-parallel family η_s in $\Gamma(T\mathcal{F}^{k,l})$, which solves

$$\dot{\eta}_s := \mathcal{L}_{\partial_s}\eta_s = -\frac{1}{2}\dot{h}_s^\sharp \circ \eta_s.$$

Where $\dot{h}_s^\sharp \circ$ denotes the following action of endomorphisms $f \in \operatorname{End}(T\mathcal{F})$ on arbitary tensors $T \in T^{k,l}\mathcal{F}$:

$$(f \circ T)(X_1, \ldots, X_r) := f(T(X_1, \ldots, X_r)) - \sum_{m=1}^{r} T(X_1, \ldots, f(X_m), \ldots, X_r)$$

for all $X_1, \ldots, X_r \in T\mathcal{F}$. Moreover:

1) There are proper subgroups H_1, H_2 of $\mathsf{SO}(n)$ such that $G \subset H_1 \times H_2$ holds if and only if there is a local metric splitting $(\mathcal{F}, h_s) \cong (\mathcal{F}_1 \times \mathcal{F}_2, h_s^1 \oplus h_s^2)$ with the condition $\operatorname{Hol}(\mathcal{F}_i, h_s^i) \subset H_i$ for $i = 1, 2$.

2) If the screen holonomy G is contained in one of the special holonomy groups $\mathsf{SU}(m), \mathsf{Sp}(k), G_2, \mathsf{Spin}(7)$ or the trivial group, then the family of metrics h_s satisfies the conditions of the following table:

$\dim(\mathcal{F})$	Condition on $\mathcal{F}$	$\operatorname{Hol}(\overline{M}, \overline{g}) \subset$
$2m$	$(\mathcal{F}, \omega_s, J_s, h_s = \omega_s(J_s\cdot, \cdot))$, Ricci-flat Kähler, $\dot{J}_s = -\frac{1}{2}\dot{h}_s^\sharp \circ J_s, \delta^{h_s}(\dot{h}_s) = 0$	$\mathsf{SU}(m) \ltimes \mathbb{R}^{2m}$
$4m$	$(\mathcal{F}, \omega_s^i, J_s^i, h_s = \omega_s^i(J_s^i\cdot, \cdot))_{i=1,2,3}$, hyper-Kähler, $\dot{J}_s^i = -\frac{1}{2}\dot{h_s}^\sharp \circ J_s^i$	$\mathsf{Sp}(m) \ltimes \mathbb{R}^{4m}$
7	$(\mathcal{F}, \varphi_s \in \Omega^3(\mathcal{F}, h_s)), G_2$ metrics, $\dot{\varphi}_s = -\frac{1}{2}\dot{h_s}^\sharp \circ \varphi_s$	$G_2 \ltimes \mathbb{R}^7$
8	$(\mathcal{F}, \psi_s \in \Omega^4(\mathcal{F}), h_s = h_s(\psi_s)), \mathsf{Spin}(7)$ metrics, $\dot{\psi}_s = -\frac{1}{2}\dot{h_s}^\sharp \circ \psi_s$	$\mathsf{Spin}(7) \ltimes \mathbb{R}^8$
n	h_s flat	$\mathbb{R}^n$

Table 6.3: Characterization of special screen holonomy in terms of flow equations

For the proof of this theorem we have to consider the following bundles of rank $n-1$:

$$\begin{tikzcd}
(T\mathcal{F}, \nabla^{h_s}) \arrow[d] & (U^\perp, \nabla^\perp) = (S_{|M}, \nabla^S_{|M}) \arrow[d] & (S, \nabla^S) \arrow[d] \\
\mathcal{F} \arrow[r, hook] & M \arrow[r, hook] & \overline{M}
\end{tikzcd}$$

In accordance with the desired translation of the screen holonomy and analytical conditions on $T\mathcal{F}$, we have to compare the connections and curvatures of the involved bundles.

Lemma 6.16: Let $(\overline{M}, \overline{g}, V)$ be a Lorentzian manifold as in Theorem 6.15. Then we have the following identities of structures on $(T\overline{M}, \overline{\nabla})$ and (S, ∇^S):

- $\nabla^S_X \sigma = \overline{\nabla}_X \sigma - \frac{1}{u}\overline{g}(\sigma, \overline{\nabla}_X T)V$
- $R^S(X,Y)\sigma = \pi^S \overline{R}(X,Y)\sigma$
- $(\nabla^S_X R^S)(Y,Z) + (\nabla^S_Y R^S)(Z,X) + (\nabla^S_Z R^S)(X,Y) = 0$

for all $\sigma \in S$ and $X, Y, Z \in T\overline{M}$.

Proof.
We know we can write the orthogonal projection $\pi^S : T\overline{M} \to S$ as

$$\pi^S = \mathrm{id} + \frac{1}{u}\overline{g}(\cdot, T)V.$$

One immediately sees by Lemma 6.13 the following:

$$\begin{aligned}
\nabla^S_X \sigma = \pi^S \overline{\nabla}_X \sigma &= \overline{\nabla}_X \sigma + \frac{1}{u}\overline{g}(\overline{\nabla}_X \sigma, T)V \\
&= \overline{\nabla}_X \sigma + \frac{1}{u}\left(\partial_X(\underbrace{\overline{g}(\sigma, T)}_{=0, \sigma \in S = T^\perp \cap V^\perp}) - \overline{g}(\sigma, \overline{\nabla}_X T)\right)V \\
&= \overline{\nabla}_X \sigma - \frac{1}{u}\overline{g}(\sigma, \overline{\nabla}_X T)V.
\end{aligned}$$

In the next step will use the fact that the map π^S is a projection, i. e. $\pi^S \circ \pi^S = \pi^S$, and that it commutes with the screen covariant derivative, since

$$\begin{aligned}
[\nabla^S, \pi^S]\sigma &= \nabla^S \underbrace{(\pi^S\sigma)}_{=\sigma\in S} - \pi^S \underbrace{\nabla^S}_{\pi^S\overline{\nabla}} \sigma \\
&= \nabla^S\sigma - \underbrace{\pi^S \circ \pi^S}_{\pi^S} \overline{\nabla}\sigma \\
&= \nabla^S\sigma - \nabla^S\sigma = 0,
\end{aligned}$$

for all $\sigma \in \Gamma(S)$. Using this observation, we calculate:

$$\begin{aligned}
R^S(X,Y)\sigma &= \nabla^S_X\nabla^S_Y\sigma - \nabla^S_Y\nabla^S_X\sigma - \nabla^S_{[X,Y]}\sigma \\
&= \pi^S\left[\overline{\nabla}_X\overline{\nabla}_Y\sigma - \overline{\nabla}_Y\overline{\nabla}_X\sigma - \overline{\nabla}_{[X,Y]}\sigma\right] \\
&= \pi^S\overline{R}(X,Y)\sigma
\end{aligned}$$

and

$$\begin{aligned}
(\nabla^S_X R^S)(Y,Z) &\overset{\textit{Definition}}{=} \nabla^S_X(R^S(Y,Z)) - R^S(\overline{\nabla}_X Y, Z) - R^S(Y, \overline{\nabla}_X Z) \\
&= \nabla^S(\pi^S \circ \overline{R}(Y,Z)) - \pi^S \circ \overline{R}(\overline{\nabla}_X Y, Z) - \pi^S \circ \overline{R}(Y, \overline{\nabla}_X Z)) \\
&= \pi^S\left(\overline{\nabla}_X(\overline{R}(Y,Z)) - \overline{R}(\overline{\nabla}_X Y, Z) - \overline{R}(Y, \overline{\nabla}_X Z)\right) \\
&= \pi^S(\overline{\nabla}_X\overline{R})(Y,Z).
\end{aligned}$$

Where we used that the commutator $[\nabla^S, \pi^S]$ vanish also on the bundle $S \otimes S^*$, by the induced operators on this tensor product and the dual space[1]. The alternating sum of this expression is zero by the second Bianchi identity of the usual curvature operator $\overline{R}$. This yields the Bianchi identity for the curvature R^S and therefore the claim.

■

In the next step we observe that we have an one-to-one correspondence between the parallel sections of the bundles $S \to \overline{M}$ and $S_{|M} \to M$.

Proposition 6.17: Let $(\overline{M}, \overline{g}, V)$ be a Lorentzian manifold as in Theorem 6.15. There exists a bijection $\psi\colon \Gamma_{||}(S_{|M}) \to \Gamma_{||}(S)$, where $\Gamma_{||}(E)$ denotes the parallel sections of a vector bundle $E \to M$ with connection ∇.

Proof.
We define the map ψ in the following way: Let σ be a section of $S_{|M} \to M$, then we can extend this to a section of $S \to \overline{M}$ by parallel transport along the flow lines of V in $\overline{M}$.

[1] dual with respect to $\overline{g}^S$

We write $\bar{\sigma}$ for this extended map. We have to show that this construction restricts to a bijection on the corresponding parallel sections.

Let $\sigma \in \Gamma_{||}(S_{|M})$ be a parallel section of $S_{|M}$, then we have to show that $\bar{\sigma}$ is again parallel. By definition we know that $\nabla^S_V \bar{\sigma}$ vanish, because $\bar{\sigma}$ is given by the parallel transport along the flow lines of V. Let $X \in TM$, then it remains to show the vanishing of $\nabla^S_X \bar{\sigma}$.

We consider the bundle $H := (T^\perp)^* \otimes S \to \overline{M}$ and the following operator

$$\begin{aligned} P \colon \Gamma(H) &\to \Gamma(H) \\ A &\mapsto \nabla^S_V A. \end{aligned}$$

We have shown that this operator is a symmetric hyperbolic system (see Lemma 1.3). Thus it remains to show that the section $A(X) := \nabla^S_X \bar{\sigma}$ of H is a solution of the Cauchy problem

$$\begin{cases} P(A) = 0, & \textit{on } \overline{M} \\ \quad A \;= 0, & \textit{on } M. \end{cases}$$

Let $X \in T^\perp = TM$, then we calculate:

$$\begin{aligned} P(A)(X) &= (\nabla^S_V A)(X) \\ &= \nabla^S_V (A(X)) - A(\underbrace{\overline{\nabla}_V X}_{=[V,X],\ \textit{since}\ \overline{\nabla} V = 0}) \\ &= \nabla^S_V \nabla^S_X \bar{\sigma} - \nabla^S_{[V,X]} \bar{\sigma} \\ &= R^S(V,X)\bar{\sigma} + \nabla^S_X \nabla^S_V \bar{\sigma} = 0. \end{aligned}$$

Where we used that V is parallel and $\nabla^S_V \bar{\sigma}$ vanish. On the other hand we can restrict A to M and obtain for all $X \in T^\perp$:

$$\begin{aligned} A(X)_{|M} = (\nabla^S_X \bar{\sigma})_{|M} &\overset{\textit{Lemma 6.16}}{=} (\overline{\nabla}_X \bar{\sigma} - \frac{1}{u} \bar{g}(\bar{\sigma}, \overline{\nabla}_X T) V)_{|M} \\ &= (\overline{\nabla}_X \bar{\sigma} + \frac{1}{u} \bar{g}(\overline{\nabla}_X \bar{\sigma}, T) V)_{|M} \\ &= (\mathsf{proj}^\perp \circ \overline{\nabla}_X \bar{\sigma})_{|M} \\ &= (\mathsf{proj}^\perp \circ \nabla^g_X \sigma) = \nabla^\perp_X \sigma = 0, \end{aligned}$$

where $\mathsf{proj}^\perp$ is the projection on the subbundle $U^\perp = S_{|M}$ and $\nabla^\perp \sigma$ vanish by assumption on σ. Thus we have shown that the section A solves the Cauchy problem and thus vanish by uniqueness and linearity. The previous arguments show that the map ψ is well-defined. Now it is clear that this map is injective, since the extended sections coincide on M and extends by the same procedure to $\overline{M}$. Furthermore follows the surjectivity, since with the same argument as before a parallel section of S restricts to a parallel section of $S_{|M}$ and this is exactly the preimage of this section under the map ψ, hence we have a bijection. ■

Corollary 6.18: Let $(\overline{M}, \overline{g}, V)$ be a Lorentzian manifold as in Theorem 6.15. Then there exists a bijection $\psi^{r,s} \colon \Gamma_{||}((S_{|M})^{r,s}) \to \Gamma_{||}(S^{r,s})$ for any $r, s \geq 0$, where $E^{r,s} \to M$ is the tensor vector bundle for $E \to M$ with connection ∇, given by $E^{r,s} := (E^*)^{\otimes r} \otimes E^{\otimes s}$ with induced connection.

Proof.
We extend the map ψ from the previous lemma to a bijection of the corresponding tensor bundle or in other words we repeat the proof of the previous lemma with the corresponding tensor bundles. ∎

We want to compare the different connections on the bundles $U^\perp$ and $T\mathcal{F}$. In the first step we observe that a section $Z \in \Gamma(U^\perp)$ is given by a smooth family of sections $(Z_s)_s \in \Gamma(T\mathcal{F})$, since the ∂_s component of Z vanish by assumption and hence we can write a general section $Z = a\partial_s + X_s$ simplified as $Z = X_s$. Where the vector field X_s takes values in $T\mathcal{F}$ with coefficients depending also on the parameter $s \in I$ or in other words: It is a section in $\Gamma(I \times \mathcal{F} \to T\mathcal{F})$.

Now we derive the section $Z = (Z_s)_s$ in the direction ∂_s w. r. t. the connection ∇^S and obtain, by the usage of the Koszul formula, the following identity:

$$\begin{aligned}
2g(\nabla^S_{\partial_s} Z, X) &= \partial_s(g(Z, X)) + Z(g(\partial_s, X)) - X(g(\partial_s, Z)) \\
&\quad + g([\partial_s, Z], X) - g([\partial_s, X], Z) - g([Z, X], \partial_s) \\
&= (\mathcal{L}_{\partial_s} g)(Z, X) + g([\partial_s, Z], X) + g([\partial_s, Z], X) \\
&\quad + g([\partial_s, X], Z) + g([\partial_s, Z], X) - g([\partial_s, X], Z) \\
&= (\mathcal{L}_{\partial_s} h_s)(Z, X) + 2g([\partial_s, Z], X)
\end{aligned}$$

Here we used that $[Z, X] \in T\mathcal{F}$ holds and the identity $(\mathcal{L}_{\partial_s} g)(X, Y) = (\mathcal{L}_{\partial_s} h_s)(X, Y)$ for all $X, Y, Z \in T\mathcal{F}$. Furthermore, we can dualize this equation and obtain

$$\nabla^\perp_{\partial_s} Z = [\partial_s, Z] + \frac{1}{2} \dot{h_s}^\sharp (Z). \tag{6.11}$$

However, we can do the same calculation in direction of $T\mathcal{F}$ and obtain the identity $\nabla^\perp_X Z = \nabla^{h_s}_X Z$.

If we have a family of 1-forms $\omega_s \in \Gamma(T^*\mathcal{F})$, then we can consider

$$\begin{aligned}
(\nabla^\perp_{\partial_s} \omega_s)(X) &= \partial_s(\omega_s(X)) - \omega_s(\nabla^\perp_{\partial_s} X) \\
&= (\mathcal{L}_{\partial_s} \omega_s)(X) - \frac{1}{2} \omega_s(\dot{h_s}^\sharp (X))
\end{aligned}$$

for all $X \in T\mathcal{F}$. Also we can dualize this and obtain

$$\nabla^\perp_{\partial_s} \omega_s = \dot{\omega_s} + \frac{1}{2} \dot{h_s}^\sharp \circ \omega_s. \tag{6.12}$$

The previous calculations generalizes to higher tensors and we are able to put everything together.

Proposition 6.19: Let (M, g, U, W) be as in Theorem 6.15 and $p, q \geq 0$, then there is an one-to-one correspondence between the following objects:

$i)$ A section $\bar{\omega} \in \Gamma(S^{p,q})$ which is ∇^S-parallel.

$ii)$ A section $\omega \in \Gamma((U^\perp)^{p,q})$ which is $\nabla^\perp$-parallel.

$iii)$ A smooth family of sections $\omega_s \in \Gamma(T\mathcal{F}^{p,q})$ which satisfies for all $s \in I$ the following:

$$\nabla^{h_s}_X \omega_s = 0, \text{ for all } X \in T\mathcal{F} \tag{6.13}$$

$$\dot{\omega_s} = -\frac{1}{2}\dot{h_s}^\sharp \circ \omega_s. \tag{6.14}$$

Proof.
The proof is a recapitulation of the previous statements. By Corollary 6.18 we have the one-to-one correspondence between $i)$ and $ii)$. Furthermore, if we have such a parallel section, then we observed in Equations (6.11) and (6.12) and $\nabla^\perp_X Z = \nabla^{h_s}_X Z$ for all $X \in T\mathcal{F}$, that these statements are equivalent. ■

Now we can prove the Theorem 6.15.

Proof of Theorem 6.15.
We have the Lorentzian manifold $(\overline{M}, \overline{g})$ as a solution of the Riemannian Cauchy problem on (M, g). Additionally we have a parallel null vector field V which restricts the holonomy group of $(\overline{M}, \overline{g})$ in such a way that the screen holonomy $G = \pi^{\mathsf{SO}(n)}(\mathrm{Hol}(\overline{M}, \overline{g}))$ is by Equation (6.10) given by the holonomy of the screen bundle (S, ∇^S).

On the other hand the holonomy principle yields an one-to-one correspondence between the parallel sections of the screen bundle $(S^{p,q}, \nabla^S)$ and tensors of $T^{p,q}\mathbb{R}^n$ which are stable under the action of the holonomy group. Hence by the previous Proposition 6.19 we know that if we have a $\sigma \in T^{p,q}\mathbb{R}^n$ such that $G \subset \mathsf{SO}(n)_\sigma$, then there exists a corresponding parallel section $\eta \in \Gamma(S^{p,q})$ and thus a family of η_s of ∇^{h_s}-parallel sections which satisfies the flow equation

$$\dot{\eta_s} = -\frac{1}{2}\dot{h_s}^\sharp \circ \eta_s.$$

We start the proof with (2) and have to show the analytic conditions on the underlying Riemannian manifold $\mathcal{F}$ if the screen holonomy G is contained in one of the special holonomy cases, i. e. $G \subset \mathsf{SU}(m), \mathsf{Sp}(m), G_2, \mathsf{Spin}(7)$ or trivial.

Let G be contained in $\mathsf{U}(m)$ *for* $\dim \overline{M} = n = 2m$ *even-dimensional. Then by Equation* (6.13)*, we know that* $\mathrm{Hol}(\mathcal{F}, h_s)$ *is also contained in* $\mathsf{U}(m)$ *and hence by a characterization of complex geometry (see [23, Prop 17.2]), there exists smooth families of almost complex structures J_s and symplectic forms ω_s on $(\mathcal{F}, h_s)$, s. t. J_s, ω_s are parallel w. r. t. the Levi-Civita connection of $h_s = \omega_s(J_s\cdot, \cdot)$. The Proposition 6.19 tells us that the parallelity of J_s and ω_s corresponds to the flow equations:*

$$\dot{J_s} = -\frac{1}{2}\dot{h_s}^{\sharp} \circ J_s \tag{6.15}$$

$$\dot{\omega_s} = -\frac{1}{2}\dot{h_s}^{\sharp} \circ \omega_s \tag{6.16}$$

Thus we have the equivalence of $\mathrm{Hol}(S, \nabla^S) \subset \mathsf{U}(m)$ *and that* $\mathrm{Hol}(\mathcal{F}, h_s)$ *lies in* $\mathsf{U}(m)$ *and the smooth families J_s, ω_s satisfies Equation* (6.15) *and Equation* (6.16)*.*

$G \subset \mathsf{Sp}(4m)$*: Analog to the previous consideration, we have that* $\mathrm{Hol}(\mathcal{F}, h_s)$ *lies in* $\mathsf{Sp}(4m)$ *and thus there exists families of almost complex structures J_s^i with $J_s^i J_s^j = \varepsilon_{ijk} J_s^k$ for all $i, j, k \in \{1, 2, 3\}$ which satisfying the corresponding flow equations, because it is parallel w. r. t. h_s.*

$G \subset G_2$*: There exists a smooth family of a stable 3-form φ_s, which satisfies the flow equation as in Theorem 6.15.*

$G \subset \mathsf{Spin}(7)$*: Again, there exists a family of 4-forms ψ_s, which is ∇^{h_s}-parallel and thus satisfies a flow equation.*

$G \subset \mathsf{SU}(m)$*: This is the tricky case.*

On the first hand we use the previous consideration and see that there has to exist smooth families of complex structures J_s and symplectic forms ω_s which are h_s-parallel. By Proposition 6.19 we know that this induces a complex structure on $U^\perp \to M$ and by parallel transport a parallel complex structure J^S on the screen bundle (S, ∇^S).

On the level of Lie algebras, the inclusion $\mathfrak{hol}(S, \nabla^S) \subset \mathfrak{su}(m)$ is equivalent to $\mathfrak{hol}(S, \nabla^S) \subset \mathfrak{u}(m)$ and that

$$\mathrm{tr}(J^S \circ A) = 0 \tag{6.17}$$

holds for all $A \in \mathfrak{hol}(S, \nabla^S) \subset \mathrm{End}(T_pS)$ and a fixed point $p \in \overline{M}$. This follows directly from the identification of the holonomy algebra and its inclusion into the endomorphisms of T_pS and the fact that

$$\mathrm{tr}^{\mathbb{R}}(J \circ A^{\mathbb{R}}) = 2i\, \mathrm{tr}^{\mathbb{C}}(A)$$

holds for all $A \in \mathfrak{hol}(S, \nabla^S) \subset \mathfrak{u}(m)$, since these A are skew-symmetric. Here we use the $\mathbb{R}$-algebra homomorphism:

$$\begin{aligned} (\cdot)^{\mathbb{R}} \colon \mathbb{C}^{n\times n} &\to \mathbb{R}^{2n\times 2n} \\ A = A_x + iA_y &\mapsto \begin{pmatrix} A_x & -A_y \\ A_y & A_x \end{pmatrix}. \end{aligned}$$

The goal of the following argumentation is to translate the trace condition (Equation (6.17)) into the analytic statement $\delta^{h_s}(\dot{h_s}) = 0$.

We start with the observation that by the Ambrose-Singer theorem for the holonomy algebra $\mathfrak{hol}(S, \nabla^S)_p$, we can write each element A of this algebra as sums, products and compositions of curvature terms of the form

$$\mathcal{P}_\gamma^{-1} \circ R^S(X, Y) \circ \mathcal{P}_\gamma, \tag{6.18}$$

where $\gamma\colon [0,1] \to \overline{M}$ is a curve with startpoint p and $X, Y \in T_{\gamma(1)}\overline{M}$. Thus we consider the trace condition for an A of the form of Equation (6.18).

Since we know that a parallel section commutes with parallel transport, we can write Equation (6.17) as

$$\operatorname{tr}(J^S \circ R^S(X,Y)) = 0 \textit{ for all } X, Y \in \Gamma(T\overline{M}).$$

Additionally, we can translate the previous statement by the decomposition $T\overline{M} = \mathbb{R}V \oplus TM$ and the fact that V annihilates the curvature into the form:

$$\operatorname{tr}(J^S \circ R^S(X,Y)) = 0 \textit{ for all } X, Y \in \Gamma(T^\perp = TM)$$

When we want to achieve an analytical characterization of the previous statement, we have to consider the trace expression as a section of a form bundle, i. e. $C \in \Gamma(\overline{M}, \Lambda^2 T^\perp)$, given by $C(X,Y) := \operatorname{tr}(J^S \circ R^S(X,Y))$ for all $X, Y \in T^\perp$. Where we equip the bundle $\Lambda^2 T^\perp$ with the induced Levi-Civita connection coming from $T\overline{M}$ and write $\overline{\nabla}$ for this connection.

Now we use thate we have a symmetric hyperbolic system on this new bundle, given by $\overline{\nabla}_V \colon \Gamma(\Lambda^2 T^\perp) \to \Gamma(\Lambda^2 T^\perp)$ as we have proven in Lemma 1.3. In order to derive a Cauchy problem like in Equation (1.5) we will show the vanishing of $\overline{\nabla}_V C$ for all $X, Y \in T\overline{M}$:

$$\begin{aligned} (\overline{\nabla}_V C)(X,Y) &\overset{(\alpha)}{=} \operatorname{tr}(J^S \circ (\overline{\nabla}^S_V R^S)(X,Y)) \\ &\overset{\textit{Lemma 6.16}}{=} -\operatorname{tr}(J^S \circ (\nabla^S_X R^S)(Y,V)) - \operatorname{tr}(J^S \circ (\nabla^S_Y R^S)(V,X)) \\ &\overset{\overline{\nabla}V=0}{=} 0 \end{aligned}$$

Where we used at (α) *that* J^S *is parallel and the Bianchi identity for the curvature* R^S*.*

Thus we have the equivalence of $C = 0$ *and* $C_{|M} = 0$ *by the uniqueness of the corresponding symmetric hyperbolic system* $\overline{\nabla}_V$*. In the next step we express the vanishing of* $C_{|M}$ *in terms of data on* M*. Let* J *be the induced complex structure on the bundle* $S_{|M} \to M$*, let* $X, Y \in TM$ *and* $\{s_i\}_i$ *be a local orthonormal frame of* $S_{|M}$*, then we have:*

$$\begin{aligned}
-\operatorname{tr}(J \circ R^S(X,Y))_{|M} &= -\sum_i g(J^S(R^S(X,Y)s_i), s_i) \\
&= \sum_i g(R^S(X,Y)s_i, J^S(s_i)) \\
&= \sum_i \overline{R}(X,Y,s_i,J^S(s_i)) \\
&\overset{(*)}{=} \sum_i R(X,Y,s_i,J(s_i)) - W(X,s_i)W(Y,J(s_i)) \\
&\quad + W(X,J(s_i))W(Y,s_i) \\
&= -\operatorname{tr}(J \circ R(X,Y)) - W(Y,J(W(X))) + W(X,J(W(Y)))
\end{aligned}$$

Where we used at $(*)$ *the Gauß equation for a spacelike hypersurfaces in a Lorentzian manifold:*

$$\begin{aligned}
\overline{R}(X_1,X_2,X_3,X_4) = R(X_1,X_2,X_3,X_4) &- W(X_1,X_3)W(X_2,W_4) \\
&+ W(X_1,X_4)W(X_2,X_3)
\end{aligned} \tag{6.19}$$

for all $X_1, X_2, X_3, X_4 \in TM$*. Thus the statement* $C_{|M} = 0$ *is equivalent to*

$$\operatorname{tr}(J \circ R(X,Y)) = -W(Y,J(W(X))) + W(X,J(W(Y))) \tag{6.20}$$

for all $X, Y \in TM$*. We can write the left hand side of Equation* (6.20) *in terms of data of* $(\mathcal{F}, h_s)$*, when we use the Riemannian Gauß equation*[2]*:*

$$\begin{aligned}
R(X_1,X_2,X_3,X_4) = R_s(X_1,X_2,X_3,X_4) &+ W(X_1,X_3)W(X_2,W_4) \\
&- W(X_1,X_4)W(X_2,X_3)
\end{aligned}$$

for all $X_1, X_2, X_3, X_4 \in T\mathcal{F}$*. However, let* $X, Y \in T\mathcal{F}$ *and plug in the Riemannian*

[2] Attention: There is a different sign as in Equation (6.19)!

Gauß equation into left hand side of Equation (6.20) *and obtain:*

$$
\begin{aligned}
\operatorname{tr}(J \circ R(X,Y)) &\overset{X,Y\in T\mathcal{F}}{=} \operatorname{tr}(J_s \circ R_s(X,Y)) \\
&= -\sum_i h_s(R_s(X,Y)s_i, J_s(s_i)) \\
&\overset{(*)}{=} -\sum_i R_s(X,Y,J_s(s_i),s_i) \\
&\overset{1.\ \textit{Bianchi identity}}{=} \sum_i R_s(Y,J_s(s_i),X,s_i) + R_s(J_s(s_i),X,Y,s_i) \\
&= -\operatorname{Ric}_s(X,J_s(Y)) + \operatorname{Ric}_s(J_s(X),Y) = -2\operatorname{Ric}_s(X,J_s(Y)) = 0
\end{aligned}
$$

Where $\{s_i\}_i$ *is a local orthonormal frame of* $T\mathcal{F}$ *and we used that the holonomy group of* $(\mathcal{F},h_s)$ *is contained in* $\mathsf{SU}(\frac{n}{2})$ *and thus* $(\mathcal{F},h_s)$ *is Ricci-flat, see [23, page 121, Theorem 17.5]. Additionally we used at* $(*)$ *that* J_s *is* ∇^{h_s}*-parallel. Indeed, we can show that* $R_s(J_s(X_1),X_2,X_3,X_4) = -R_s(X_1,J_s(X_2),X_3,X_4)$ *holds for all* $X_1,X_2,X_3,X_4 \in T\mathcal{F}$ *by the simple calculation:*

$$
\begin{aligned}
R_s(J_s(X_1),X_2,X_3,X_4) &= R_s(X_3,X_4,J_s(X_1),X_2) \\
&= h_s(R_s(X_3,X_4)J_s(X_1),X_2) \\
&= h_s\left((\nabla^{h_s}_{X_3}\nabla^{h_s}_{X_4} - \nabla^{h_s}_{X_4}\nabla^{h_s}_{X_3} - \nabla^{h_s}_{[X_3,X_4]})J(X_1),X_2\right) \\
&\overset{\nabla J=0}{=} h_s\left(J_s(R_s(X_3,X_4)X_1),X_2\right) \\
&\overset{(\beta)}{=} -R_s(X_3,X_4,X_1,J_s(X_2)) = -R_s(X_1,J_s(X_2),X_3,X_4)
\end{aligned}
$$

Where we used at (β) *that* $h_s(J_s(X),Y) = -h_s(X,J_s(Y))$ *holds for all* $X,Y \in T\mathcal{F}$*, since* J_s *and* h_s *are compatible.*

It remains to evaluate the left hand side of Equation (6.20) *on the mixed part of* $TM = \mathbb{R}\partial_s \oplus T\mathcal{F}$*. We can achieve this with the help of the Codazzi Equation (see [25, page 115, 34 Corallary]):*

$$R(X_1,\tilde{T},X_2,X_3) = (\nabla^{h_s}_{X_2}W_s)(X_3,X_1) - (\nabla^{h_s}_{X_3}W_s)(X_2,X_1)$$

for all $X_1,X_2,X_3 \in T\mathcal{F}$ *and* $\tilde{T} = u\partial_s$ *is a normal vector field for the hypersurface* $\mathcal{F} \subset M$*. Moreover, the* h_s*-symmetric tensor* W_s *is second fundamental form of the embedding of* $(\mathcal{F},h_s)$ *into* (M,g) *and is given by*

$$\nabla^g_X Y - \nabla^{h_s}_X Y = W_s(X,Y)\cdot\tilde{T}$$

for all $X,Y \in T\mathcal{F}$.

Let $X \in T\mathcal{F}$ and plug (∂_s, X) into Equation (6.20) *to obtain*

$$-(\delta^{h_s}\dot{h}_s)(J_s(X)) - 2g(\mathrm{grad}^g\left(\frac{1}{u}\right), J(W(X)).$$

on the right hand side and $-2g(\mathrm{grad}^g\left(\frac{1}{u}\right), J(W(X))$ on the left hand side, where we used the identification of the Weingarten map with

$$\begin{pmatrix} \partial_s\left(\frac{1}{u}\right) & \mathrm{grad}^{h_s}\frac{1}{u} \\ d\left(\frac{1}{u}\right) & -\frac{u}{2}\mathcal{L}_{\partial_s}h_s \end{pmatrix}.$$

Since J_s is an isomorphism, we can write Equation (6.20) *equivalently as*

$$(\delta^{h_s}\dot{h}_s) = 0.$$

Therefore we translated the condition of special holonomy $\mathsf{SU}(m)$ *into the analytic condition above.*

flat case: *Finally we have to show the flat case. At first we show the splitting case, i. e. let H_1, H_2 be subgroups of* $\mathsf{SO}(n)$ *such that*

$$\mathrm{Hol}(S, \nabla^S) = G \subset H_1 \times H_2 \subset \mathsf{SO}(n)$$

holds, then there exists a nontrivial, decomposable and ∇^S-parallel form of the screen bundle (S, ∇^S). By the holonomy principle and the de Rahm-Wu Theorem we have a splitting

$$(\mathcal{F}, h_s) \cong (\mathcal{F}_1 \times \mathcal{F}_2, h_s^1 \oplus h_s^2),$$

where $\mathrm{Hol}(\mathcal{F}_i, h_s^i) \subset H_i$ and additonally the equation

$$\mathcal{L}_{\partial_s}\mathrm{vol}^{h_s^i} = -\frac{1}{2}\dot{h}_s^{i,\sharp} \circ \mathrm{vol}^{h_s^i}$$

holds. But one can show by a simple calculation that this is always satisfied. Now the flat case follows directly by an iterated splitting into trivial holonomy representations.

This shows the claim. ■

7 Spin geometry of Lorentzian manifolds

In this chapter we will consider a Lorentzian spin manifold which carries a parallel null spinor. Analogous to the first observation in Chapter 1 we can restrict the parallel null spinor to a spacelike hypersurface and obtain a constraint on this hypersurface. In the following we assume a general knowledge about spin geometry, in particular the defintion of a spin structure on a semi-Riemannian manifold, Clifford modules and its representation and the induced spinor bundle. The reader can consult [3], [18], [20] or [12, page 102].

7.1 Sesquilinearforms on a spinor bundle

Let $\Sigma^g N$ be the spinor bundle of a Lorentzian spin manifold (N, g) of dimension n, then there exists two different non-degenerated sesquilinearforms on the spinor bundle, that we will need in later sections. We give here a convenient construction and list of properties of these two. We follow [3, section 1.5, page 67].

We write Δ for the unique irreducible $\mathbb{C}\mathrm{l}(n,1)$-module if n is even and Δ_+ for the irreducible $\mathbb{C}\mathrm{l}(n,1)$-module which commutes with the volume element $\omega = i^{\frac{n(n+1)}{2}} e_1 \cdot \ldots \cdot e_n$ if n is odd. We denote by $\{e_i\}_i$ the standard basis of $\mathbb{R}^n$.

We start with the standard sesquilinearform on $\Delta_{(+)}$ given by

$$(v, w) \coloneqq \sum_{k=1}^{2^m} v_k \bar{w}_k,$$

where $v, w \in \Delta_{(+)} = \mathbb{C}^{2^m}$ and $m = \lfloor \frac{n}{2} \rfloor$. On the other hand we can also define

$$\langle v,w \rangle \coloneqq (v, w)_{e_1} \coloneqq (e_1 \cdot v, w)$$

for all $v, w \in \Delta_{(+)}$ and where $\{e_i\}_i$ denotes the standard basis of $\mathbb{R}^n$. We can prove important properties of these two forms.

Proposition 7.1: The scalar product $(\cdot, \cdot)$ satisfies the following properties:

(1) It is invariant under the maximal-compact subgroup K of $\mathsf{Pin}(n,1)$, i. e.

$$K = \left\{ y_1 \cdot \ldots \cdot y_r \cdot x_1 \cdot \ldots \cdot x_s \in \mathbb{C}\mathrm{l}(n,1) \;\middle|\; \begin{array}{r} r, s \in \mathbb{N}, \text{ and for all } 1 \leq i \leq r, 1 \leq j \leq s: \\ \langle y_i, y_i \rangle_{n,1} = -1, y_i \in \operatorname{span}(e_1) \text{ and} \\ \langle x_j, x_j \rangle_{n,1} = 1, x_j \in \operatorname{span}(e_2, \ldots, e_n) \end{array} \right\}.$$

(2) We have the identity

$$(x \cdot v, w) + (v, \theta(x) \cdot w) = 0,$$

for all $x \in \mathbb{R}^n$ and $v, w \in \Delta_{(+)}$, where the map $\theta\colon \mathbb{R}^n \to \mathbb{R}^n$ is the reflection along the hypersurface $\operatorname{span}(e_2, \ldots, e_n)$.

Proof.
In the following we will need an explicit form of the Clifford multiplication: Let

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},$$

then we define the following maps:

$$\begin{aligned} \phi_{n=2m}\colon\ \mathbb{C}l(n,1) &\to \operatorname{Mat}(\mathbb{C}, 2^m) \\ e_{2j-1} &\mapsto \tau_{2j-1} W^{\otimes(j-1)} \otimes U \otimes E^{\otimes(m-j)} \\ e_{2j} &\mapsto \qquad W^{\otimes(j-1)} \otimes V \otimes E^{\otimes(m-j)} \end{aligned}$$

and in the odd case

$$\begin{aligned} \phi_{n=2m+1}\colon\ \mathbb{C}l(n,1) &\to \operatorname{Mat}(\mathbb{C}, 2^m) \oplus \operatorname{Mat}(\mathbb{C}, 2^m) \\ e_j &\mapsto (\phi_{2m}(e_j), \phi_{2m}(e_j)), \text{ if } j \leq n-1 \\ e_n &\mapsto (iW^{\otimes m}, -iW^{\otimes m}). \end{aligned}$$

Where $\otimes$ denotes the Kronecker product of matrices, i. e. $A \otimes B := (a_{ij}B)_{i,j}$, which is associative and τ_j is given by i if $j = 1$ and 1 otherwise. Let $\hat{\phi}_n$ be given as ϕ_n if n is even and $\mathsf{proj}_1 \circ \phi_n$ if n is odd, then we prove the following identity:

$$\hat{\phi}_n(e_j)^T = -\varepsilon_j \overline{\hat{\phi}_n(e_j)},$$

for all $1 \leq j \leq n$. We see immediately that the Kronecker product preserves complex conjuagtion and transposing, i. e. $\overline{A \otimes B} = \overline{A} \otimes \overline{B}$ and $(A \otimes B)^T = A^T \otimes B^T$. Now we consider four different cases:

$n = 2m, j = 2k-1$***: We have***

$$\begin{aligned}\phi_n\hat{(e_j)}^T &= \left[\tau_{2k-1}W^{\otimes(k-1)} \otimes U \otimes E^{\otimes(m-k)}\right]^T \\ &= \tau_{2k-1}W^{\otimes(k-1)} \otimes \underbrace{U^T}_{=-U} \otimes E^{\otimes(m-k)} \\ &= -\varepsilon_k\overline{\tau_{2k-1}}W^{\otimes(k-1)} \otimes U \otimes E^{\otimes(m-k)} \\ &= -\varepsilon_j\overline{\hat{\phi}_n(e_j)}.\end{aligned}$$

$n = 2m, j = 2k$***: We have***

$$\begin{aligned}\hat{\phi}_n(e_j)^T &= \left[\tau_{2k-1}W^{\otimes(k-1)} \otimes V \otimes E^{\otimes(m-k)}\right]^T \\ &= \tau_{2k-1}W^{\otimes(k-1)} \otimes \underbrace{V}_{-\overline{V}} \otimes E^{\otimes(m-k)} \\ &= -\overline{\hat{\phi}_n(e_j)}.\end{aligned}$$

$n = 2m+1$
$1 \leq j \leq n-1$***: It follows from the definition of*** $\hat{\phi}_n(e_j)$ ***and the previous case.***

$n = 2m+1, j = n$***: We calculate:***

$$\begin{aligned}\hat{\phi}_n(e_n)^T = (iW^{\otimes m})^T = iW^{\otimes m} &= -\overline{(iW^{\otimes m})} \\ &= -\varepsilon_n\overline{\hat{\phi}_n(e_n)}.\end{aligned}$$

Thus we have proven the first statement. Now let $v, w \in \Delta_{(+)}$***,*** $x \in \operatorname{span}(e_2, \ldots, e_n)$ ***and*** $y \in \operatorname{span}(e_1)$***, then we have:***

$$(y \cdot v, w) = (\hat{\phi}_n(y)v, w) = (\hat{\phi}_n(y)v)^T\overline{w} = v^T(\hat{\phi}_n(y))^T\overline{w} = +v^T\overline{\hat{\phi}_n(y)w} = (v, y \cdot w)$$

and

$$(x \cdot v, w) = (\hat{\phi}_n(x)v, w) = (\hat{\phi}_n(x)v)^T\overline{w} = v^T(\hat{\phi}_n(x))^T\overline{w} = -v^T\overline{\hat{\phi}_n(x)w} = -(v, x \cdot w)$$

Hence we have proven (2)***. Let*** $k = y_1 \cdot \ldots \cdot y_r \cdot x_1 \cdot \ldots \cdot x_s$ ***be a general element in*** K***, then we consider:***

$$\begin{aligned}(k \cdot v, k \cdot w) &= (y_1 \cdot \ldots \cdot y_r \cdot x_1 \cdot \ldots \cdot x_s \cdot v, y_1 \cdot \ldots \cdot y_r \cdot x_1 \cdot \ldots \cdot x_s \cdot w) \\ &= (-1)^s(v, x_s \cdot \ldots \cdot x_1 \cdot y_r \cdot \ldots \cdot y_1 \cdot y_1 \cdot \ldots \cdot y_r \cdot x_1 \cdot \ldots \cdot x_s \cdot w) \\ &= (-1)^s \prod_{i=1}^{s}(-\langle x_i,x_i\rangle) \prod_{j=1}^{r}(-\langle y_j,y_j\rangle) \cdot (v, w) \\ &= (-1)^{2s}(v, w) = (v, w)\end{aligned}$$

This shows the invariance of the scalar product and hence the claim.

■

The obvious disadvantage of the scalar product $(\cdot,\cdot)$ is the absence of a nice compatibility with the Clifford multiplication. Because of that we consider the adapted sesquilinearform $\langle\cdot,\cdot\rangle$. But we have to pay the price for this, as we will see in Proposition 7.3.

Proposition 7.2: The sesquilinearform $\langle\cdot,\cdot\rangle$ satisfies the following properties:

(1) It is non-degenerated and of split signature, i. e. its signature is $(2^{m-1}, 2^{m-1})$.

(2) It is invariant under the action of $\mathsf{Spin}^0(n,1)$.

(3) We have

$$\langle x \cdot v, w\rangle = \langle v, x \cdot w\rangle$$

for all $x \in \mathbb{R}^n$ and $v, w \in \Delta_{(+)}$.

Proof.
The sesquilinearform $\langle\cdot,\cdot\rangle$ is determined by the complex matrix $\hat{\phi}_n(e_1)$, which is of the form

$$\begin{pmatrix} 0 & iE_{2^{m-1}} \\ -iE_{2^{m-1}} & 0 \end{pmatrix}$$

and has the eigenvalues ± 1 of multiplicity 2^{m-1}. Moreover, we have

$$\overline{\langle v, w\rangle} = \overline{(\hat{\phi}(e_1)v, w)} = (w, \hat{\phi}_n(e_1)v) = (\hat{\phi}_n(e_1)w, v) = \langle w, v\rangle$$

and this shows (1). Let $x = x_1 + x_2$ be given in terms of the decomposition of $\mathbb{C}^n = \operatorname{span}(e_1) \oplus \operatorname{span}(e_2, \ldots, e_n)$. Then we have

$$\begin{aligned}
\langle x \cdot v, w\rangle &= \langle x_1 \cdot v, w\rangle + \langle x_2 \cdot v, w\rangle \\
&= (e_1 \cdot x_1 \cdot v, w) + (e_1 \cdot x_2 \cdot v, w) \\
&= (x_1 \cdot e_1 \cdot v, w) - (x_2 \cdot e_1 \cdot v, w) \\
&= (e_1 \cdot v, x_1 \cdot w) + (e_1 \cdot v, x_2 \cdot w) \\
&= (e_1 \cdot v, x \cdot w) = \langle v, x \cdot w\rangle.
\end{aligned}$$

Now let a be a general element in $\mathsf{Spin}^+(n,1)$, i. e. $a = a_1 \cdot \ldots a_{2i}$ where $\langle a_j, a_j\rangle_{n,1} = \pm 1$ for all j and $a_1 \cdot \ldots \cdot a_{2i} \cdot a_{2i} \cdot \ldots \cdot a_1 = 1$ holds, then we have

$$\langle a \cdot v, a \cdot w\rangle = (-1)^{2i} \langle v, a_1 \cdot \ldots \cdot a_{2i} \cdot a_{2i} \cdot \ldots \cdot a_1 \cdot w\rangle = \langle v, w\rangle,$$

where we used the previous computation. We conclude that we have shown (2) *and* (3).

■

By the invariance of these sesquilinearform, we can extend these to the spinor bundle $\Sigma^g N$ of a Lorentzian spin manifold. Moreover, these sesquilinearforms have optimal properties as we will see in the next proposition.

Proposition 7.3: We have the following non-existence statements:

(1) There cannot exist a $\mathsf{Spin}(n,1)$-invariant, non-degenerated sesquilinearform on $\Delta_{(+)}$.

(2) There cannot exist a $\mathsf{Spin}^0(n,1)$-invariant scalar product $\Delta_{(+)}$.

(3) There cannot exist a scalar product $\langle\langle\cdot,\cdot\rangle\rangle$ on $\Delta_{(+)}$ and a constant $c \in \mathbb{C}$, s. t.

$$\langle\langle x \cdot v, w\rangle\rangle = c\langle\langle v, x \cdot w\rangle\rangle$$

holds for all $x \in \mathbb{R}^n$, $v, w \in \Delta_{(+)}$.

Proof.
We write $\rho\colon \mathsf{Spin}(n,1) \to \mathrm{GL}(\Delta_{(+)})$ *for the irreducible spin representation. When* $\langle\langle\cdot,\cdot\rangle\rangle$ *is a non-degenerated sesquilinearform, which is invariant under the action of* $\mathsf{Spin}(n,1)$*, then we can derive the equation*

$$\langle\langle \rho(a)v, \rho(a)w\rangle\rangle = \langle\langle v, w\rangle\rangle$$

at the neutral element $1 \in \mathsf{Spin}(n,1)$ *and obtain*

$$\langle\langle d\rho(A)(v), w\rangle\rangle = -\langle\langle v, d\rho(A)(w)\rangle\rangle,$$

for all elements A *in the Lie algebra* $\mathfrak{spin}(n,1)$*, that is generated by the elements* $e_i e_j$ *for* $i < j$*. Thus we have*

$$\langle\langle e_1 e_j v, w\rangle\rangle = -\langle\langle v, e_1 e_j w\rangle\rangle$$

for all $v, w \in \Delta_{(+)}$ *and* $1 < j$*. Now let* a *be a general element of* $\mathsf{Spin}^0(n,1)$*, i. e.* $a = a_1 \cdot \ldots \cdot a_{2i}$ *and* $\langle a_k, a_k\rangle = \pm 1$*, then we have by definition*

$$e_1 \cdot e_j \cdot a \in \mathsf{Spin}(n,1).$$

Thus

$$\begin{aligned}\langle\langle v, w\rangle\rangle &\overset{\textit{assumption}}{=} \langle\langle e_1 e_j a v, e_1 e_j a w\rangle\rangle = -\langle\langle a \cdot v, e_1 e_j e_1 e_j a \cdot w\rangle\rangle \\ &= -\langle\langle a \cdot v, a \cdot w\rangle\rangle = -\langle\langle v, w\rangle\rangle\end{aligned}$$

and this is of course a contradiction to the assumption.

For (2) *we consider a* $\mathsf{Spin}^0(n,1)$*-invariant scalar product* $\langle\langle\cdot,\cdot\rangle\rangle$*. Let* $1 < j$ *and* $v \in \Delta_{(+)}$*, then we consider*

$$0 \leq \langle\langle e_1 e_j \cdot v, e_1 e_j \cdot v\rangle\rangle = -\langle\langle v, e_1 e_j e_1 e_j \cdot v\rangle\rangle = -\langle\langle v,v\rangle\rangle \leq 0,$$

which is a contradiction to the assumption.

For the last part of the claim, we assume that we have a scalar product $\langle\langle\cdot,\cdot\rangle\rangle$*, which satisfies*

$$\langle\langle x \cdot v, w\rangle\rangle = c\langle\langle v, x \cdot w\rangle\rangle.$$

Now we set $A := (\langle\langle e_i, e_j\rangle\rangle)_{i,j}$ *as the matrix that determines the scalar product*$\langle\langle\cdot,\cdot\rangle\rangle$*, i. e. we have* $\langle\langle v,w\rangle\rangle = (v, Aw)$ *by definition. Let* $i = 1, \ldots, n$ *be an index, then we can use the previous to obtain the following:*

$$-\varepsilon_i(v, e_i \cdot Aw) = (e_i \cdot v, Aw) = \langle\langle e_i \cdot v, w\rangle\rangle = c\langle\langle v, e_i \cdot w\rangle\rangle = c(v, Ae_i \cdot w)$$

and thus $cA\phi_n(e_i) = -\varepsilon_i \phi_n(e_i)A$ *holds for all* i*. When we apply the determinante on this identity, we obtain*

$$c \det(A\phi_n(e_i)) = -\varepsilon_i \det(\phi_n(e_i)A)$$

and thus

$$c = -\varepsilon_i$$

for all i*, since* $\langle\langle\cdot,\cdot\rangle\rangle$ *is positive definite and hence* $\det(A) > 0$ *and* $\det(\phi_n(e_i)) \neq 0$ *by definition. Hence we have a contradiction:*

$$1 = -\varepsilon_1 = c = -\varepsilon_2 = -1$$

■

7.2 The Dirac current and its properties

Let (N, g) be a time-oriented, spin manifold with a metric $g \in \mathrm{Sym}^2(T^*N)$ of the signature (p,q), see [29, Def 7]. We want to introduce a vector field on N associated to a spinor.

Definition 7.4: Let $\varphi \in \Gamma(\Sigma N)$ be a spinor, then we call the vector field V_φ, given by

$$g(V_\varphi, X) = i^{p+1}\langle \varphi, X \cdot \varphi\rangle$$

for all $X \in \Gamma(TN)$, the **Dirac current** of φ. In otherwords, the (a priori complex) vector field V_φ is induced by the 1-form $i^{p+1}\langle \varphi, (\cdot)\cdot\varphi\rangle \in \Omega^1(N, \mathbb{C})$ via the isomorphism $g\colon T_{\mathbb{C}}N \to T^*_{\mathbb{C}}N, X \mapsto g(X, \cdot)$.

In the later part of this thesis we need some properties of the Dirac current, so we prove the following proposition.

Proposition 7.5 (Properties Dirac current)**:** Let $\varphi \in \Gamma(\Sigma N)$ be a spinor and V_φ the induced Dirac current. Then we have the following statements:

(1) The Dirac current is real-valued.

(2) If φ is parallel, then also V_φ.

(3) The zero sets of φ and V_φ coincide.

(4) We have $g(V_\varphi, V_\varphi) \leq 0$.

Now let (N, g) be a metric of Lorentzian signature.

(5) If φ is nowhere vanishing, then: $V_\varphi \cdot \varphi = 0$ if and only if φ is lightlike, i. e. V_φ is lightlike.

(6) If φ is parallel and lightlike, then there exists a function $f \in C^\infty(N)$, s. t.

$$\mathrm{Ric}^g = f V_\varphi^\flat \otimes V_\varphi^\flat, \qquad V_\varphi(f) = 0,$$

in particular $\mathrm{scal}^g = 0$.

Proof.

(1) This is a simple calculation:

$$\begin{aligned}\overline{g(V_\varphi, X)} = \overline{i^{p+1}\langle \varphi, X \cdot \varphi\rangle} = (-i)^{p+1}\langle X \cdot \varphi, \varphi\rangle &= (-i)^{p+1}(-1)^{p+1}\langle \varphi, X \cdot \varphi\rangle \\ &= g(V_\varphi, X)\end{aligned}$$

where we used the general slide property of sesquilinearform:

$$\langle X \cdot \varphi, \psi\rangle = (-1)^{p+1}\langle \varphi, X \cdot \psi\rangle.$$

(2) *We rewrite the expression* $g(\nabla_X \varphi, Y)$ *for all* $X, Y \in \Gamma(TN)$*:*

$$\begin{aligned} g(\nabla_X V_\varphi, Y) &= X(g(V_\varphi, Y)) - g(V_\varphi, \nabla_X Y) \\ &= X(i^{p+1}\langle \varphi, Y \cdot \varphi\rangle) - i^{p+1}\langle \varphi, \nabla_X Y \cdot \varphi\rangle \\ &= i^{p+1}(\langle \nabla_X \varphi, Y \cdot \varphi\rangle + \langle \varphi, \nabla_X (Y \cdot \varphi)\rangle - \langle \varphi, \nabla_X Y \cdot \varphi\rangle) \\ &= i^{p+1}((-1)^{p+1}\langle Y \cdot \nabla_X \varphi, \varphi\rangle + \langle \varphi, Y \cdot \nabla_X \varphi\rangle) \\ &= 2\,\mathrm{Re}((-i)^{p+1}\langle Y \cdot \nabla_X \varphi, \varphi\rangle). \end{aligned}$$

So if φ *is parallel, then also* V_φ*.*

(3) *We choose a time orientation* $\xi \in \Gamma(TN)$ *of* N*, then we have*

$$g(V_\varphi, \xi) = -\langle \varphi, \xi \cdot \varphi\rangle = -\langle \xi \cdot \varphi, \varphi\rangle = -(\varphi, \varphi) \leq 0, \tag{7.1}$$

since the scalar product $(\cdot,\cdot)_\xi$ *is positive definite. If* $V_\varphi(x)$ *vanish at a point* $x \in \mathrm{zero}(V_\varphi)$*, then by Equation* (7.1) *also* $\varphi(x) = 0$*. On the other hand, let* $\varphi(x) = 0$*, then we see with the Definition 7.4 also* $V_\varphi(x) = 0$*.*

(4) *At this point, we use the fact that a time orientation* ξ *gives us a decomposition of the tangent bundle* $TN = \mathbb{R}\xi \oplus \xi^\perp$*, hence*

$$V_\varphi = \alpha\xi + Z$$

with $\alpha = -g(V_\varphi, \xi), Z \in \xi^\perp$*. Here is* $\alpha > 0$*, because of the Equation* (7.1) *and the assumption that* φ *is nowhere vanishing . We want to show* $g(V_\varphi, V_\varphi) \leq 0$ *and consider different cases.*

$x \in \mathrm{zero}(V_\varphi)$*: This is obvious:* $g\left(V_\varphi(x), V_\varphi(x)\right) = 0 \leq 0$*.*

$x \in \mathrm{zero}(Z)$*: We have by Equation* (7.1)*:*

$$\begin{aligned} g(V_\varphi, V_\varphi)(x) = g\left(-g(V_\varphi,\xi)\xi, -g(V_\varphi,\xi)\xi\right)(x) &= g(V_\varphi,\xi)^2\, \underbrace{g(\xi,\xi)}_{=-1}(x) \\ &= -g(V_\varphi,\xi)^2(x) \leq 0. \end{aligned}$$

$x \in \tilde{N} := N \setminus \{\mathrm{zero}(V_\varphi) \cup \mathrm{zero}(Z)\}$ *We are allowed to write down the following vector field on* $\Gamma(T\tilde{N})$*:*

$$N := -\frac{Z}{\sqrt{g(Z,Z)}} \in \xi^\perp$$

We can decompose V_φ *w. r. t. the new vector field* N*:*

$$V_\varphi = \alpha\xi + \beta N$$

where $\alpha = -g(V_\varphi, \xi) > 0$ *and* $\beta = g(V_\varphi, N) < 0$. *Because:*

$$g(V_\varphi, N) = g(\alpha\xi + \beta N, N) \overset{g(\xi,N)=0}{=} \beta g(N,N) = \beta \frac{1}{g(Z,Z)} \cdot g(Z,Z) = \beta.$$

We obtain a second decomposition, with the following eigenspace decomposition of the endomorphism $\xi \cdot N \cdot \colon \Sigma N_{|\tilde{N}} \to \Sigma N_{|\tilde{N}}$. *The only eigenvalues of this endomorphisms are* ± 1, *because:*

$$(\xi \cdot N) \cdot (\xi \cdot N) \cdot \overset{g(\xi,N)=0}{=} - \underbrace{\xi \cdot \xi}_{=-g(\xi,\xi)=1} \cdot \underbrace{N \cdot N}_{=-1} \cdot = \mathrm{id}_{\Sigma N_{|\tilde{N}}} \cdot$$

In particular we have an orthogonal (w. r. t. to $(\cdot,\cdot)_\xi$*) eigenvalue decomposition* $\varphi = \varphi_+ + \varphi_- \in \Sigma N_{|\tilde{N}} = \Sigma_+ \oplus \Sigma_-$. *There is a way to express the coefficients* α, β *in terms of* $\varphi_\pm$*:*

$$\begin{aligned}
\alpha &= -g(V_\varphi, \xi) = \langle \xi \cdot \varphi, \varphi \rangle = (\varphi, \varphi)_\xi = \|\varphi_+\|_\xi^2 + \|\varphi_+\|_\xi^2 \\
\beta &= g(V_\varphi, N) = -\langle N \cdot \varphi, \varphi \rangle = -\langle \xi \cdot \xi \cdot N \cdot \varphi, \varphi \rangle = -(\xi \cdot N \cdot \underbrace{\varphi}_{=\varphi_+ + \varphi_-}, \varphi)_\xi \\
&= -(\varphi_+, \varphi)_\xi + (\varphi_-, \varphi)_\xi = \|\varphi_-\|_\xi^2 - \|\varphi_+\|_\xi^2.
\end{aligned}$$

With this at hand we can consider $g(V_\varphi, V_\varphi)$*:*

$$\begin{aligned}
g(V_\varphi, V_\varphi) = g(\xi,\xi)\alpha^2 + g(N,N)\beta^2 &= -\alpha^2 + \beta^2 \qquad (7.2) \\
&= -\left(\|\varphi_+\|_\xi^2 + \|\varphi_-\|_\xi^2\right)^2 + \left(\|\varphi_+\|_\xi^2 - \|\varphi_-\|_\xi^2\right)^2 \\
&= -4\|\varphi_+\|_\xi^2 \cdot \|\varphi_-\|_\xi^2 \leq 0.
\end{aligned}$$

(5) *We consider again the decompositions* $V_\varphi = \alpha\xi + \beta N$ *and* $\varphi = \varphi_+ + \varphi_-$, *then we have the following chain of equivalences:*

$$\begin{aligned}
\varphi \text{ is lightlike } &:\Longleftrightarrow V_\varphi \text{ is lightlike} \\
&\overset{\text{Definition}}{\Longleftrightarrow} g(V_\varphi, V_\varphi) = 0 \\
&\overset{\text{Equation (7.2)}}{\Longleftrightarrow} -\alpha^2 + \beta^2 = 0 \\
&\overset{\beta<0,}{\Longleftrightarrow} \alpha = -\beta > 0 \\
&\Longleftrightarrow \|\varphi_+\|_\xi^2 + \|\varphi_-\|_\xi^2 = \|\varphi_+\|_\xi^2 - \|\varphi_-\|_\xi^2 \\
&\Longleftrightarrow \varphi_- = 0 \\
&\Longleftrightarrow \xi \cdot \varphi = N \cdot \varphi \\
&\Longleftrightarrow V_\varphi \cdot \varphi = \alpha\xi \cdot \varphi + \beta \underbrace{N \cdot \varphi}_{=\xi\cdot\varphi} = \underbrace{(\alpha + \beta)}_{=0} \xi \cdot \varphi = 0
\end{aligned}$$

(6) *On the first hand we need a relation between the spin and Riemannian curvature. Let $R^{\Sigma N}$ be the spin curvature and R^g the Riemannian curvature. Moreover let $s_0, \dots, s_n$ be an orthogonal frame of TN, then we have:*

$$\sum_j \varepsilon_j s_j \cdot R^{\Sigma N}(X, s_j)\varphi = -\frac{1}{2}\operatorname{Ric}^g(X)\cdot\varphi, \tag{7.3}$$

$$\sum_j \varepsilon_j s_j \cdot \operatorname{Ric}^g(s_j)\cdot\varphi = -\operatorname{scal}^g\varphi \tag{7.4}$$

for all $X \in TN$. We know that the local formula for the spin curvature is given by: $R^{\Sigma N}(X,Y)\phi = \frac{1}{2}\sum_{k<l}\varepsilon_k\varepsilon_l R^g(X,Y,s_k,s_l)s_k\cdot s_l\cdot\phi$. So we have:

$$\begin{aligned}\sum_j \varepsilon_j s_j \cdot R^{\Sigma N}(X, s_j)\varphi &= \frac{1}{2}\sum_{j;k<l} R^g(X, s_j, s_k, s_l)\varepsilon_{jkl} s_j\cdot s_k\cdot s_l\cdot\varphi \\ &= \frac{1}{4}\sum_{j,k,l}\varepsilon_{jkl}R^g(X, s_j, s_k, s_l)s_j\cdot s_k\cdot s_l\cdot\varphi\end{aligned}$$

The sum above runs over the index set $\{j,k,l = 0,\dots,n \mid k \neq l\}$, since the curvature term vanish for $k = l$. Now we split the sum w. r. t. to the decomposition $\{j,k,l \mid k\neq l\} = \{j\neq k\neq l\neq j\}\dot\cup\{j = k\neq l\}\dot\cup\{j = l\neq k\}$ to obtain:

$$\frac{1}{4}\left[\sum_{j\neq k\neq l\neq j}\varepsilon_{jkl}R^g(X, s_j, s_k, s_l)s_j\cdot s_k\cdot s_l\cdot\varphi + \sum_{j=k;l}\varepsilon_{jkl}R^g(X, s_k, s_k, s_l)s_k\cdot s_k\cdot s_l\cdot\varphi\right.$$
$$\left. + \sum_{j=l;k}\varepsilon_{jkl}R^g(X, s_l, s_k, s_l)s_l\cdot s_k\cdot s_l\cdot\varphi\right]$$

and

$$\begin{aligned}&= \frac{1}{4}\left[-\sum_{k,l}\varepsilon_{kl}R^g(s_k, X, s_l, s_k)s_l\cdot\varphi + \sum_{k,l}\varepsilon_{kl}(-1)R^g(s_k, X, s_l, s_k)s_l\cdot\varphi\right] \\ &= -\frac{1}{2}\sum_l\left[\sum_k\varepsilon_{kl}R^g(s_k, X, s_l, s_k)\right]s_l\cdot\varphi \\ &= -\frac{1}{2}\sum_l\varepsilon_l\operatorname{Ric}^g(X, s_l)s_l\cdot\varphi \\ &= -\frac{1}{2}\operatorname{Ric}^g(X)\cdot\varphi.\end{aligned}$$

and

$$\begin{aligned}\sum_j \varepsilon_j s_j \cdot \mathrm{Ric}^g(s_j) \cdot \varphi &= \sum_{j,k} \varepsilon_{jk} \,\mathrm{Ric}^g(s_j, s_k) s_j \cdot s_k \cdot \varphi \\ &= \sum_k \mathrm{Ric}^g(s_k, s_k) \underbrace{s_k \cdot s_k}_{=-\varepsilon_k} \varphi = -\,\mathrm{scal}^g \varphi\end{aligned}$$

If φ is lightlike, parallel and nowhere vanishing, we conclude from Equations (7.3) and (7.4) that

$$\begin{aligned}\mathrm{Ric}^g(X) \cdot \varphi &= 0, \\ \mathrm{scal}^g &= 0\end{aligned}$$

holds. Moreover, we obtain

$$\begin{aligned}\mathrm{Ric}^g(X) \cdot \mathrm{Ric}^g(X) \cdot \varphi &= -g(\mathrm{Ric}^g(X), \mathrm{Ric}^g(X))\varphi = 0 \\ \textit{and } (\mathrm{Ric}^g(X) \cdot V_\varphi + V_\varphi \cdot \mathrm{Ric}^g(X)) \cdot \varphi &= -2g(V_\varphi, \mathrm{Ric}^g(X))\varphi = 0,\end{aligned}$$

where the last line is true, because of the fact that $g(V_\phi, \mathrm{Ric}(X)) = -\langle \mathrm{Ric}(X) \cdot \phi, \phi \rangle = 0$ holds. So we have $g(V_\varphi, V_\varphi) = g(V_\varphi, \mathrm{Ric}^g(X)) = g(\mathrm{Ric}^g(X), \mathrm{Ric}^g(X)) = 0$. Therefore the two vector fields V_φ and $\mathrm{Ric}^g(X)$ are linear dependent, hence there exist an 1-form ω, s. t. $\mathrm{Ric}^g(X) = \omega(X) V_\phi$. Lets write $T := \frac{\xi}{g(\xi, V_\varphi)}$ and check $g(V_\varphi, T) = 1$: We see now:

$$\omega(X) = \omega(X) g(V_\varphi, T) = \mathrm{Ric}(X, T) = \mathrm{Ric}(T, X) = \omega(T) g(V_\varphi, X). \qquad (7.5)$$

We can now write the Ricci curvature as the following:

$$\mathrm{Ric}(X, Y) = \omega(X) g(V_\varphi, Y) \overset{\textit{Equation } (7.5)}{=} \omega(T) g(V_\varphi, X) g(V_\varphi, Y) = (f V_\varphi^\flat \otimes V_\varphi^\flat)(X, Y)$$

for all $X, Y \in TN$ and we set $f := \omega(T)$.

We will need the divergence of a $(p+1, 0)$-tensor A, which is given by:

$$\delta(A)(X_1, \ldots, X_p) := \sum_i (\nabla_{e_i} A)(e_i, X_1, \ldots, X_p)$$

for all $X_1, \ldots, X_p \in TM$. In the last step we want to show that $V_\varphi(f)$ vanish. We use the well-known identity $d\,\mathrm{scal} = \frac{1}{2}\delta(\mathrm{Ric})$ and the vanishing of the scalar curvature *to obtain:*

$$\begin{aligned}
0 = \delta(\mathrm{Ric}) &= \delta(fV_\varphi^\flat \otimes V_\varphi^\flat) = \sum_i \nabla_i(fV_\varphi^\flat \otimes V_\varphi^\flat)(X, e_i) \\
&= \sum_i \partial_i(fV_\varphi^\flat(X)V_\varphi^\flat(e_i)) - fV_\varphi^\flat(\nabla_i X)V_\varphi^\flat(e_i) - fV_\varphi^\flat(X)V_\varphi^\flat(\nabla_i e_i) \\
&= \sum_i (\partial_i f)V_\varphi^\flat(X)V_\varphi^\flat(e_i) + f(g(\nabla_i V_\varphi) + g(V_\varphi, \nabla_i X))g(V_\varphi, e_i) \\
&\quad + fg(V_\varphi, X)(g(\nabla_i V_\varphi, e_i) + g(V_\varphi, \nabla_i e_i)) \\
&\quad - fg(V_\varphi, \nabla_i X)g(V_\varphi, e_i) - fg(V_\varphi, X)g(V_\varphi, \nabla_i e_i) \\
&= V_\varphi^\flat(X)\sum_i (\partial_i f)V_\varphi^\flat(e_i) + f\sum_i g(\nabla_i V_\varphi, X)g(V_\varphi, e_i) \\
&\quad + fV_\varphi^\flat(X)\sum_i g(\nabla_i V_\varphi, e_i) \\
&= V_\varphi(f)V_\varphi^\flat(X) + f(\nabla_{V_\varphi} V_\varphi)^\flat(X) + fV_\varphi^\flat(X)\delta(V_\varphi)
\end{aligned}$$

By assumption V_φ is parallel, hence $V_\varphi(f)$ vanish. This shows the proposition.

■

Remark 7.6: Let (M, g) be a Riemannian spin manifold with a nowhere vanishing parallel spinor ϕ, then by Equation (7.3) we have

$$0 = \sum_j s_j \cdot \underbrace{R^{\Sigma M}(X, s_j)\phi}_{=0} = -\frac{1}{2}\,\mathrm{Ric}^g(X) \cdot \phi,$$

since a parallel section annihilates the curvature. If we apply the Clifford multiplication of $\mathrm{Ric}^g(X)$ to the previous identity, we obtain

$$0 = \mathrm{Ric}(X) \cdot \mathrm{Ric}(X) \cdot \phi = -g(\mathrm{Ric}(X), \mathrm{Ric}(X))\phi.$$

By assumption that we have a nowhere vanishing spinor, we can conclude that a Riemannian manifold with a parallel spinor needs to be Ricci-flat. An example for a Lorentzian manifold with a parallel spinor, that is not Ricci-flat, is given by a special case of Example 6.12. We consider $(N, h) = (\mathbb{R}^{n+2}, h = g_\lambda)$ with metric

$$g_\lambda = \begin{pmatrix} 0 & 1 & 0 \\ 1 & f & 0 \\ 0 & 0 & E_n \end{pmatrix}$$

and $f(x) = \sum_{j=1}^n \lambda_j x_j^2$ and $\lambda \in \mathbb{R}^n$. We compute the Ricci curvature as follows: Recall that we have two different frames of interest:

$$e_- = \frac{\partial}{\partial v} \qquad e_+ = -\frac{f}{2}\frac{\partial}{\partial v} + \frac{\partial}{\partial u} \qquad s_i = \frac{\partial}{\partial x^i}$$

and

$$s_v = \frac{1}{\sqrt{2}}(e_- - e_+) \qquad s_u = \frac{1}{\sqrt{2}}(e_- + e_+) \qquad s_i = s_i.$$

where we have $g(e_-, e_-) = g(e_+, e_+) = 0, g(e_-, e_+) = 1$ and s_v, s_u, s_i is a generalised orthonormal basis, s. t. $g(s_\mu, s_\nu) = \varepsilon_\mu \delta_{\mu\nu}$ and $\varepsilon_{\mu=v} = -1, \varepsilon_{\mu=u,i} = 1$.

We compute the Ricci curvature with help of the generalised orthonormal basis:

$$\begin{aligned}
\mathrm{Ric}^g(X,Y) &= \sum_\mu \varepsilon_\mu g(R^g(s_\mu, X)Y, s_\mu) \\
&= -g(R^g(s_v, X)Y, s_v) + g(R^g(s_u, X)Y, s_u) + \sum_i g(R^g(s_i, X)Y, s_i) \\
&= -\frac{1}{2} g(R^g(e_+, X)Y, e_+) + \frac{1}{2} g(R^g(e_+, X)Y, e_+) + \sum_i g(R^g(s_i, X)Y, s_i) \\
&= \sum_i g(R^g(s_i, X)Y, s_i)
\end{aligned}$$

Where we used that the curvature vanish if we plug in the parallel vector e_-. We have three cases $(X,Y) = (e_+, e_+), (e_+, s_k), (s_j, s_k)$:

$(X,Y) = (e_+, e_+)$: We will use again Table 6.2 and calculate:

$$\begin{aligned}
\sum_i g(R^g(s_i, e_+)e_+, s_i) &= -\frac{1}{2} \sum_i g(\nabla^h_{s_i} \operatorname{grad}^h f, s_i) \\
&= -\frac{1}{2} \sum_i \partial_{s_i}(g(\operatorname{grad}^h f, s_i)) - g(\underbrace{\nabla^h_{s_i} s_i}_{=0,\text{ since } h \text{ is flat}}, \operatorname{grad}^h f) \\
&= -\frac{1}{2} \sum_i \partial_i \partial_i f = -\sum_i \lambda_i
\end{aligned}$$

$(X,Y) = (e_+, s_k)$: We have

$$\sum_i g(R^g(s_i, e_+)s_k, s_i) = \frac{1}{2} \sum_i \mathrm{Hess}^h(f)(s_i, s_k) \underbrace{g(e_-, s_i)}_{=0} = 0.$$

$(X,Y) = (s_j, s_k)$: We have

$$\sum_i g(R^g(s_i, s_j)s_k, s_i) = 0$$

since h is flat.

Thus we can write the Ricci curvature as $\mathrm{Ric}^g = -(\sum_i \lambda_i) e_-^\flat \otimes e_-^\flat$ and of course this is non-zero for generic λ. We recognize this curvature expression with the general form of part $6)$ of Proposition 7.5.

7.3 Spin constraints

Let $(\overline{M}, \overline{g})$ be a Lorentzian spin manifold with a spacelike hypersurface (M, g), a fixed orientation and the embedding of the hypersurface has trivial normal bundle, given by a vector field T. Then we can equip the hypersurface with a compatible spin structure coming from $(\overline{M}, \overline{g})$. We consider an embedding of frame bundles over M

$$\begin{aligned} \iota\colon P_{\mathsf{SO}(n)}M &\to P_{\mathsf{SO}^0(n,1)}\overline{M}_{|M} \\ (s_1, \ldots, s_n) &\mapsto (T, s_1, \ldots, s_n). \end{aligned}$$

and define the induced spin structure on M by the pullback $\iota^*\left(P_{\mathsf{Spin}^0(n,1)}\overline{M}_{|M}\right)$. There is a connection between the two different Clifford multiplications on ΣM and $\Sigma\overline{M}$ given by: $X \cdot \varphi = iT \star X \star \phi_{|M}$ for all $X \in TM$. We write $X \cdot \varphi$ for the Clifford multiplication on ΣM and $X \star \varphi$ on $\Sigma\overline{M}$. The reader can consult [9, section 3] for a more detailed construction.

However, when we have a parallel null spinor on a Lorentzian manifold, then we can restrict to the hypersurface and obtain constraints on the hypersurface.

Proposition 7.7 (spin constraints): Let $(\overline{M}, \overline{g})$ be a Lorentzian spin manifold, $(M, g) \subset (\overline{M}, \overline{g})$ be a spacelike hypersurface with a future-directed timelike unit normal field T. Moreover we have a lightlike parallel spinor ϕ on $(\overline{M}, \overline{g})$. The restricted spinor $\varphi = \phi_{|M}$ satisfies the following equations on (M, g):

$$\begin{cases} \nabla^{\Sigma M}_X \varphi &= \frac{i}{2} W(X) \cdot \varphi \\ U_\varphi \cdot \varphi &= i u_\varphi \varphi \end{cases} \tag{7.6}$$

for all $X \in \Gamma(TM)$, the so called **spin constraints** for φ, also called imaginery W-Killing spinor. Where U_φ is the Dirac current of φ, $u_\varphi = \|U_\varphi\|_g = \|\varphi\|^2$ and W is the Weingarten map from the embedding $(M, g) \hookrightarrow (\overline{M}, \overline{g})$.

Proof.
Let $s_1, \ldots, s_n$ of M be an orthonormal frame and $X \in \Gamma(TM)$. When we keep in mind the identification of spinor bundles of M and $\overline{M}$ as in [9, section 3], then we have $T, s_1, \ldots, s_n$ as an orthonormal frame of $T\overline{M}$. Set $e_0 = T, e_i = s_i$. Now we can consider

the spin covariant derivative of ϕ in direction X:

$$\begin{aligned}
\nabla_X^{\Sigma\overline{M}}\phi &= X(\phi) + \frac{1}{2}\sum_{\mu<\nu}\varepsilon_\mu\varepsilon_\nu\overline{g}(\nabla_X^{\overline{g}}e_\mu, e_\nu)e_\mu \star e_\nu \star \phi \\
&= X(\phi) - \frac{1}{2}\sum_{\mu=0;l>0}\overline{g}(\nabla_X^{\overline{g}}T, s_l)T \star s_l \star \phi + \frac{1}{2}\sum_{0<k<l}\overline{g}(\nabla_X^{\overline{g}}s_k, s_l)s_k \star s_l \star \phi \\
&= X(\phi) + \frac{i}{2}\sum_{l>0}\overline{g}(\underbrace{\nabla_X^{\overline{g}}T}_{=-W(X)}, s_l)(iT \star s_l) \star \phi \\
&\quad + \frac{1}{2}\sum_{0<k<l}\overline{g}(\nabla_X s_k, s_l)(iT \star s_k) \star (iT \star s_l) \star \phi
\end{aligned}$$

If we restrict the previous expression to the hypersurface M, we obtain:

$$\begin{aligned}
&= X(\varphi) - \frac{i}{2}\sum_j g(W(X), s_j)s_j \cdot \varphi + \frac{1}{2}\sum_{k<j} g(\nabla_X^g s_k, s_j)s_k \cdot s_j \cdot \varphi \\
&= \nabla_X^{\Sigma M}\varphi - \frac{i}{2}W(X) \cdot \varphi.
\end{aligned}$$

Where we used the identification of the Clifford multiplications $X \cdot \varphi = iT \star X \star \phi_{|M}$ and the fact that the Weingarten map is given as: $W(X) = -\pi^{TM}\overline{\nabla}_X T$ for $X \in TM$.

In the next step we decompose V_ϕ on M as $uT - U$, where $U = \pi^{TM}(-V_\phi)_{|M}$ and $u = \sqrt{g(U,U)}$. The equivalence in Proposition 7.5 gives us the vanishing of $V_\phi \cdot \phi$. So we obtain by multiplication of T the following:

$$0 = T \star V_\phi \star \phi_{|M} = T \star (uT - U) \star \phi_{|M} = u\varphi + i\underbrace{(iT \star U \star \phi)_{|M}}_{U\cdot\varphi} = u\varphi + iU \cdot \varphi.$$

Hence the algebraic condition $U \cdot \varphi = iu\varphi$. Finally we consider:

$$u_\varphi^2 = g(U_\varphi, U_\varphi) = i\langle\varphi, U_\varphi \cdot \varphi\rangle = i(-i)u_\varphi\langle\varphi,\varphi\rangle$$

and thus $u_\varphi = \|\varphi\|^2$, which shows the last part of the statement. ■

Analogous to the question of the Riemannian constraint equation and the previous result, we can ask if existence of a solution of the spin constraint requires the existence of an extension of the initial Riemannian manifold into a Lorentzian spin manifold with parallel spinor, which restricts to the spin constraints on the initial Riemannian manifold.

So we want to prove the following theorem, see [22, Thm 1].

Theorem 7.8: Let (M, g) be Riemannian, spin manifold with a nowhere vanishing imaginary W-Killing spinor φ, i. e. φ satisfies

$$\begin{cases} \nabla_X^{\Sigma M}\varphi & = \frac{i}{2}W(X)\cdot\varphi \\ U_\varphi\cdot\varphi & = iu_\varphi\varphi \end{cases}$$

for all $X \in \Gamma(TM)$ and some g-symmetric endomorphism W on M. Where $u_\varphi = \langle\varphi,\varphi\rangle$ and U_φ is the Dirac current of φ. Then there exists an open neighborhood $\overline{M}$ of $M = \{0\} \times M$ in $\mathbb{R} \times M$ and a unique Lorentzian metric $\overline{g}$ on $\overline{M}$ such that:

(1) The manifold $(\overline{M}, \overline{g})$ is spin and admits a parallel null spinor ϕ.

(2) We have $\phi_{|M} = \varphi$ and $\overline{g}_{|M} = g$.

In particular (M, g) embeds into $(\overline{M}, \overline{g})$ as a spacelike Cauchy hypersurface with Weingarten map W .

In order to prove the previous theorem we ascribe the W-Killing spinor φ to the solution of the Riemannian constraint equation for the Dirac current V_φ for φ.

Proposition 7.9: Let (M, g) be a Riemannian spin manifold with a spinor ϕ and a g-symmetric endomorphism W, which satisfies the spin constraints (Equation (7.6)), then the induced Dirac current U_ϕ of ϕ satisfies the Equation (5.1), i. e. the Riemannian constraint equation.

Proof.
We derive the expression $g(U_\phi, Y) = i\langle\phi,Y\cdot\phi\rangle$ *and consider:*

$$\begin{aligned} g(\nabla_X U_\phi, Y) &= \partial_X g(U_\phi, Y) - g(U_\phi, \nabla_X Y) \\ &= \partial_X(i\langle\phi,Y\cdot\phi\rangle) - i\langle\phi,\nabla_X Y\cdot\phi\rangle \\ &= i\langle\nabla_X\phi,Y\cdot\phi\rangle + \cancel{i\langle\phi,\nabla_X Y\cdot\phi\rangle} + i\langle\phi,Y\cdot\nabla_X\phi\rangle - \cancel{i\langle\phi,\nabla_X Y\cdot\phi\rangle} \\ &= i\langle\frac{i}{2}W(X)\cdot\phi,Y\cdot\phi\rangle + i\langle\phi,\frac{i}{2}Y\cdot W(X)\cdot\phi\rangle \\ &= \frac{1}{2}\langle\phi,(W(X)\cdot Y + Y\cdot W(X)\cdot\phi\rangle = g(-\|\phi\|^2 W(X), Y) \end{aligned}$$

and hence $\nabla U_\phi + \|\phi\|^2 W = 0$. *Finally we have to show that* $u_\phi = \sqrt{g(U_\phi, U_\phi)}$ *holds:*

$$g(U_\phi, U_\phi) = i\langle\phi,U_\phi\cdot\phi\rangle = i\langle U_\phi,iu_\phi\phi\rangle = i\cdot(-i)u_\phi\|\phi\|^2 = u_\phi^2.$$

Where we used that the sesquilinearform $\langle\cdot,\cdot\rangle$ *is complex antilinear in the second argument, the definition* $u_\phi := \langle\phi,\phi\rangle$ *and Proposition 7.7.* ■

Proof of Theorem 7.8.
In the first step we use the Theorem 1.1, because the corresponding Dirac current U_φ for the imaginery W-Killing spinor φ satisfies the Riemannian constraint equation (see Equation (5.1)*), as we have proved in Proposition 7.9.*

Now we have a globally hyperbolic extension $(\overline{M}, \overline{g})$ of our data and a parallel, lightlike vector field V on $\overline{M}$, we can extend φ along the flow lines of V to the whole of $\overline{M}$ to a spinor ϕ. It remains to show that ϕ is parallel and V coincides with the Dirac current V_ϕ. The spin structure of $(\overline{M}, \overline{g})$ is the extension of the spin structure of (M, g) given the semi-Riemannian cylinder construction as in [9, 3. section].

ϕ ***is parallel:***

By the definition it is clear that $\nabla^{\Sigma\overline{M}}_V \phi$ vanish, because we had defined the spinor ϕ by parallel transport along the flow lines of V.

Consider now an arbitary vector field $X \in \partial_t^\perp$ and show that $\nabla^{\Sigma\overline{M}}_X \phi$ vanish. We define the section $A \in \Gamma((\partial_t^\perp)^* \otimes \Sigma\overline{M})$ given by $A(X) := \nabla^{\Sigma\overline{M}}_X \phi$ and show that A satisfies a symmetric hyperbolic system.

For this purpose we consider the differential operator of order 1*, given by*

$$\begin{aligned} P = \nabla_V^{T^*\overline{M}\otimes\Sigma\overline{M}} &: \Gamma(T^*\overline{M} \otimes \Sigma\overline{M}) \to \Gamma(T^*\overline{M} \otimes \Sigma\overline{M}) \\ A \mapsto \nabla_V A &:= \Big(X \mapsto (\nabla_V A)(X) = \nabla_V^{\Sigma\overline{M}}(A(X)) - A(\nabla_V^{\overline{g}} X)\Big) \end{aligned}$$

and we have to show that this operator is a symmetric hyperbolic system. But we already proved that in Lemma 1.3. In the following we have to show that we have a Cauchy problem for $A(X) = \nabla^{\Sigma\overline{M}}_X \phi$:

$$\begin{cases} P(A) = 0 & \text{on } \overline{M} \\ \quad A = 0 & \text{on } M \end{cases} \tag{7.7}$$

The first step is to show the vanishing of $P(A)$ for the initial data $A(X) = \nabla^{\Sigma\overline{M}}_X \phi$. Let $X \in \Gamma(T\overline{M})$, then we calculate:

$$\begin{aligned} P(A)(X) = (\nabla_V^{T^*\overline{M}\otimes\Sigma\overline{M}} A)(X) &= \nabla_V^{\Sigma\overline{M}}(A(X)) - A(\nabla_V^{\overline{g}} X) \\ &= \nabla_V^{\Sigma\overline{M}} \nabla_X^{\Sigma\overline{M}} \phi - \nabla^{\Sigma\overline{M}}_{\nabla_V^{\overline{g}} X} \phi \\ &= R^{\Sigma\overline{M}}(V, X)\phi + \nabla_X^{\Sigma\overline{M}} \underbrace{\nabla_V^{\Sigma\overline{M}}\phi}_{=0,\ \textit{assumption}} - \nabla^{\Sigma\overline{M}}_{\underbrace{\nabla_X^{\overline{g}} V}_{=0, V\ \textit{parallel}}} \phi \\ &= \frac{1}{2}\sum_{k<l} \varepsilon_k \varepsilon_l \underbrace{R^{\overline{g}}(V, X, s_k, s_l)}_{=0} s_k \cdot s_l \cdot \phi = 0 \end{aligned} \tag{7.8}$$

Where we used in the last step that V annihilates the curvature $R^{\overline{g}}$, since V is parallel. Now we want to use the uniqueness result from [1, Corollary 3.7.6] for the Equation (7.7). Where the first equations in Equation (7.7) holds through Equation (7.8). The second part of Equation (7.7) holds because of the following observation: Let $X \in \Gamma(T\overline{M})$ be a vector field, then we have a splitting of X in a V- and a TM-part. The V-part vanish by construction. Let $X \in \Gamma(TM)$, then we have

$$A(X)_{|M} = (\nabla^{\Sigma\overline{M}}_X \phi)_{|M} \overset{\textit{Proof of Proposition 7.7}}{=} \nabla^{\Sigma M}_X \varphi - \frac{i}{2} W(X) \cdot \varphi \overset{\textit{constraint equations}}{=} 0.$$

Hence the uniqueness result for the Cauchy problem of symmetric hyperbolic systems gives us the unique solution $A = 0$. Therefore the spinor ϕ is parallel.

V is the Dirac current of ϕ:

We can decompose the parallel null vector field V as $uT - U$, moreover we have $T = \frac{1}{\lambda}\partial_t$ and $N = \frac{1}{u}U$. This global vector fields reduce the $\mathsf{SO}(n,1)$ *frame bundle P of $\overline{M}$ to a* $\mathsf{SO}(n-1)$ *frame bundle $\tilde{P}$, since they are parallel. Let $\tilde{P}$ be the* $\mathsf{SO}(n-1)$*-reduction of the frame bundle P, then we have an induced* $\mathsf{Spin}(n-1)$*-principial bundle Q.*

Claim: Let P be a G-principial bundle, Q be a H-principial bundle, R be a $\tilde{H}$-principial bundle and we have an inclusion $G \hookrightarrow H$ and a group morphism $\lambda\colon \tilde{H} \to H$. Moreover $\pi_1 : P \to Q$ and $\pi_2 : R \to Q$ are reductions in the sense of [2, Defintion 2.11]. Then there exists a $\lambda^{-1}(G)$-principial bundle S, s. t.

$$\begin{array}{ccc} S & \xrightarrow{\pi} & R \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ P & \xrightarrow{\pi_1} & Q \end{array}$$

commutes and π is a reduction of principial bundles.

The proof of that claim is simple. Set $S := \pi_2^{-1}(\pi_1(P))$ and show that this S is a $\lambda^{-1}(G)$-principial bundle, but this is clear by the pullback construction.

Now we have the identification of spinor bundles $\Sigma\overline{M} = \hat{S} \otimes \Delta_{1,1}$. The Clifford multiplication of T, N and an arbitary $X \in \operatorname{span}(T,N)^{\perp}$ on $\psi \otimes u(\varepsilon)$ is given by:

$$T \cdot (\psi \otimes u(\varepsilon)) = -\psi \otimes u(-\varepsilon) \tag{7.9}$$

$$N \cdot (\psi \otimes u(\varepsilon)) = \varepsilon\psi \otimes u(-\varepsilon) \tag{7.10}$$

$$X \cdot (\psi \otimes u(\varepsilon)) = -\varepsilon(X \cdot \psi) \otimes u(\varepsilon) \tag{7.11}$$

Where we used the basis of $\Delta_{1,1} = \mathbb{C}^2$*:* $\{u(\varepsilon) = \begin{pmatrix} 1 \\ -\varepsilon i \end{pmatrix} \mid \varepsilon \in \{\pm 1\}\}$*. Now let* $\psi = \psi_1 \otimes u(\varepsilon) + \psi_{-1} \otimes u(-\varepsilon) \in \Gamma(\Sigma\overline{M})$ *be an arbitary spinor, then we have a chain of equivalences:*

$$V \cdot \psi = 0 \overset{V=u(T-N)}{\iff} T \cdot \psi = N \cdot \psi \overset{T \cdot T = 1}{\iff} T \cdot N \cdot \psi = \psi \tag{7.12}$$
$$\overset{\text{Equation (7.9),Equation (7.10)}}{\iff} \psi = \psi_{-1} \otimes u(-1)$$

In the next step we want to show that the product $V \cdot \phi$ *vanish on* $\overline{M}$*. Here we consider again a symmetric hyperbolic system, given by* $P = \nabla_V^{\Sigma\overline{M}} : \Gamma(\Sigma\overline{M}) \to \Gamma(\Sigma\overline{M})$ *and the corresponding Cauchy problem for* $V \cdot \phi$*:*

$$\begin{cases} P(V \cdot \phi) = 0 & \textit{on } \overline{M}, \\ (V \cdot \phi) = 0 & \textit{on } M. \end{cases} \tag{7.13}$$

The first part of Equation (7.13) *is given by*

$$P(V \cdot \phi) = \nabla_V^{\Sigma\overline{M}}(V \cdot \phi) = (\nabla_V^{\overline{g}} V) \cdot \phi + V \cdot (\nabla_V^{\Sigma\overline{M}} \phi) = 0,$$

where we used that both fields V, ϕ *are parallel. The second part of Equation* (7.13) *is a consequence of the second part of the spin constraint equations: We have* $U \cdot \varphi = iu\varphi$ *on* $\Gamma(\Sigma M)$*. The induced Clifford multiplication on* $\Gamma(\Sigma\overline{M})$ *results in*

$$iuT \star N \star \phi_{|M} = iT \star U \star \phi_{|M} = U \cdot \varphi \overset{\text{spin constraints}}{=} iu\varphi = iu\phi_{|M}$$

and hence $T \star N \star \phi_{|M} = \phi_{|M}$*. The chain of equivalences in Equation* (7.12) *gives us the second part of the symmetric hyperbolic system and again with the uniqueness and linearity of that system we obtain the desired result:* $V \cdot \phi$ *vanish on* $\overline{M}$*.*

At this point we want to compare the parallel null vector fields V *and the Dirac current* V_ϕ *of* ϕ*. Let's write* V_ϕ *as a linear combination* $V_\phi = \alpha T + \beta N + \sum_k \gamma_k X_k$ *for* T, N *and* $X_k \in \operatorname{span}(T, N)^\perp$*. The coefficients are given by:*

$$-\alpha = \alpha g(T, T) = g(V_\phi, T) = -\langle T \cdot \phi, \phi \rangle = -(\phi, \phi)_T = -\|\phi\|_T^2$$

and

$$\begin{aligned} \beta = g(V_\phi, N) &= -\langle N \cdot \phi, \phi \rangle \\ &= -\langle T \cdot T \cdot N \cdot \phi, \phi \rangle \\ &= -(T \cdot N \cdot \phi, \phi)_T \\ &\overset{\text{Equation (7.12)}}{=} -(\phi, \phi)_T \\ &= -\|\phi\|_T^2 \end{aligned}$$

and

$$g(V_\phi, X_k) = -\langle X_k \cdot \phi,\phi\rangle = -(T \cdot X_k \cdot \phi, \phi)_T \overset{\text{Equation (7.12)},\phi=\phi_{-1}\otimes u(-1)}{=} -((X_k \cdot \phi) \otimes u(1), \phi_{-1} \otimes u(-1)) = 0.$$

Hence we have

$$V_\phi = \alpha T + \beta N = \|\phi\|_T^2 T - \|\phi\|_T^2 N = \|\phi\|_T^2 (T - N) = \frac{\|\phi\|_T^2}{u} V.$$

Now we use the fact that both vector fields V_ϕ, V are parallel along the t-lines in $\overline{M}$, hence there exists a smooth function $c \in C^\infty(\overline{M})$ independent of t-part, s. t. $u(t,x) = c(x)\|\phi(t,x)\|_T^2$. We consider now: $u_0(x) = u(0,x) = \langle\varphi(x),\varphi(x)\rangle = \|\varphi(x)\|_T^2 = \|\phi(0,x)\|_T^2$ and hence $c(x) = 1$.

The last argument shows that V and the Dirac current V_ϕ coincide. ∎

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Ich habe die Arbeit selbstständig verfasst, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und bisher keiner anderen Prüfungsbehörde vorgelegt. Außerdem bestätige ich hiermit, dass die vorgelegten Druckexemplare und die vorgelegte elektronische Version der Arbeit identisch sind und dass ich von den in §26 Abs. 6 vorgesehenen Rechtsfolgen Kenntnis habe.

Unterschrift: ______________________