Homogeneous Causal Fermion Systems

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Zusammenfassung

Kausale Fermionsysteme ermöglichen die mathematische Formulierung relativistischer Quantentheorie. Zentrales Thema der vorliegenden Arbeit ist eine spezielle Form kausaler Fermionsysteme: *homogene kausale Fermionsysteme*. Die Homogenität dieser Systeme liegt in dem Sachverhalt begründet, dass der Fermionische Projektor lediglich von der Differenz zweier Raumzeit-Punkte abhängt. Ziel der vorliegenden Arbeit ist eine eingehende Untersuchung homogener kausaler Fermionsysteme. Von vorrangigem Interesse ist die Einführung einer Symmetrie kausaler Fermionsysteme sowie deren Implikationen auf die Definition des Fermionischen Projektors und die Implementierung negativ definiter Maße. Mithilfe von Gruppentheorie und Spektralanalysis lassen sich unter geeigneten Annahmen interessante Erkenntnisse gewinnen. Von großer Bedeutung ist darüber hinaus die Rekonstruktion kausaler Fermionsysteme, insbesondere die des zugehörigen Hilbertraumes. Es zeigt sich, dass ausgehend von der Gruppe der Translationen im $\mathbb{R}^4$, einem indefiniten Innenproduktraum und einem negativ definiten Maß unter Zuhilfenahme maßtheoretischer sowie funktionalanalytischer Methoden ein kausales Fermionsystem rekonstruiert werden kann. Die Art der Rekonstruktion des Hilbertraums des kausalen Fermionsystems ist dabei nicht eindeutig festgelegt, sondern ist auf unterschiedliche Weisen möglich. Abschließendes Thema der vorliegenden Arbeit ist die Frage nach der Existenz von Minimierern der kausalen Wirkung. Lässt sich die kausale Wirkung unter Variation negativ definiter Maße minimieren? Nach Einführung der Problemstellung und zentraler Begriffe soll zunächst gezeigt werden, dass negativ definite Maße existieren, für welche die kausale Wirkung endliche Werte annimmt. Hinsichtlich der Variationsrechnung von vorrangigem Interesse ist die Fragestellung, ob sich aus einer Minimalfolge negativ definiter Maße eine konvergente Teilfolge auswählen lässt, deren Grenzwert einen Minimierer der kausalen Wirkung darstellt. Dieser Aspekt ist nicht endgültig geklärt und stellt einen Ausgangspunkt für weitere Untersuchungen zu diesem Thema dar.

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1. Homogeneous Causal Fermion Systems

1.1. Introduction

In nonrelativistic quantum mechanics, particles are described by Schrödinger's equation. In relativistic quantum mechanics, the description of a particle depends on its spin. Particles with half-integer spin, including quarks and leptons, are called fermions. They are described by the Dirac equation.[1] The solution space of the free Dirac equation in Minkowski space is of special interest to us for the following fact: Considering a subspace of the solution space of the Dirac equation, introducing a suitable scalar product on this subspace, and taking the completion, one obtains a Hilbert space as explained in [FGS12]. Moreover, one may even derive a socalled *causal fermion system.* As stated in [FGS12], causal fermion systems provide a general mathematical framework for the formulation of relativistic quantum theory. The theory of causal fermion systems has evolved to an approach to fundamental physics and is a candidate for a unified physical theory. Causal fermion systems will be of crucial interest throughout the present work and are defined as follows:

Definition 1.1 (Causal fermion system).
Consider a triple $(\mathscr{H}, \mathscr{F}, \rho)$ consisting of the following structures:

i) Let $\mathscr{H}$ be a separable complex Hilbert space endowed with a scalar product $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}$ and a socalled *spin dimension* $n \in \mathbb{N}$.

ii) Let $\mathscr{F} \subset \mathrm{L}(\mathscr{H})$ be the set of all self-adjoint operators on $\mathscr{H}$ of finite rank, which have at most n positive and at most n negative eigenvalues.

iii) Finally, let ρ be a positive measure defined on the Borel σ-algebra of $\mathscr{F}$. The measure ρ is called *universal measure*.[2]

[1] See e. g. [Gri08] or [Sha94].

[2] Regular Borel measures are of special interest for causal fermion systems, see [Fin16]. Hence we will consider ρ to be a Borel measure all over the present work.

Then, we refer to $(\mathscr{H}, \mathscr{F}, \rho)$ as a *causal fermion system.* See [Fin16].

The present work deals with a particular kind of causal fermion systems, namely socalled homogeneous causal fermion systems. Given a homogeneous causal fermion system, several questions arise: Is it possible to introduce a symmetry which leaves the universal measure invariant? How can one reduce this symmetry? Which implications yields such a symmetry regarding the fermionic projector? Using some group theoretical properties and adding some spectral theory, we will see that there can be drawn some quite interesting conclusions. As mentioned above, in [FGS12] a causal fermion system is constructed based on a subspace of the solution space of the Dirac equation. This leads to the question: May one reconstruct the original causal fermion system starting with a finite-dimensional subspace of the Hilbert space together with a suitable operator-valued measure? From this socalled reconstruction problem a further question arises: Is there only one way of reconstructing the causal fermion system, or do different methods of reconstruction exist? As we will see in chapter 4, there are several procedures which give back the original causal fermion system. However, there is still one important issue left: As described in [Fin16], minimizing the causal action is of particular interest in the theory of causal fermion systems. Consider a minimizing sequence for the causal action which satisfies some constraining conditions. Can one choose a convergent subsequence such that its limit is a minimizer of the causal action? What is the connection between the symmetry of a causal fermion system and the existence of minimizers of the causal action? These questions due to the calculus of variations will be addressed to in chapter 5. Throughout the present work, socalled negative definite measures are of particular interest, leading to measure theoretical considerations. Moreover, integration theory and functional analysis on Hilbert spaces will play an important role.

The above overview of the present work shall be enriched by some more technical details. To this aim, the reader has to be familiar with the basic definitions concerning homogeneous causal fermion systems which are in general not part of common textbooks. Instead, most of the stated definitions and results are due to Prof. Finster's research. A general overview of and a profound introduction into the topic of causal fermion systems can be found in Prof. Finster's latest book, *The Continuum Limit of Causal Fermion Systems*, see [Fin16]. In the following, we will first introduce some fundamental definitions and explain what homogeneous causal fermion systems are. Afterwards, we will give a more technical overview of the contents and the aims of the present work. The reader interested in a non-technical introduction is referred to [FK15].

1.2. Basic Definitions

The postponed definitions shall be stated without any further comment in order to get a rough idea of the most important definitions of the work. The following terms are supposed to be fundamental all over this work.

In the following, let $(\mathscr{H}, \mathscr{F}, \rho)$ be a causal fermion system. In order to explain what homogeneous causal fermion systems are, the basic structures shall be introduced. First, we define *space-time* M as the support of the universal measure,

$$M := \operatorname{supp} \rho.$$

For every $x \in M$, the corresponding *spin space* S_x is defined by

$$S_x := x(\mathscr{H}).$$

For $x \in M$, let $\pi_x \colon \mathscr{H} \to S_x$ be the *orthogonal projection* onto the spin space S_x. Then, for $x, y \in M$ the *kernel of the fermionic projector* is in general defined as

$$P(x,y) := \pi_x y \colon S_y \to S_x.$$

Based on this definition, we may assume that our causal fermion system is *homogeneous* is the sense that the kernel of the fermionic projector $P(x,y)$ only depends on the difference vector $\xi := y - x$. One last important definition due to causal fermion systems is the following:

Definition 1.2 (**Symmetry of a causal fermion system**).
Let $(\mathscr{H}, \mathscr{F}, \rho)$ be a causal fermion system. Then, a *symmetry of a causal fermion system* is a group $\mathscr{G}$ together with a unitary representation U on $\mathscr{H}$ which leaves the universal measure invariant, i. e.

$$\rho\left(U_g \Omega U_g^{-1}\right) = \rho\left(\Omega\right)$$

for all $g \in \mathscr{G}$ and all measurable $\Omega \subset \mathscr{F}$.

Next, we consider a finite-dimensional indefinite inner product space $(V, \prec \cdot | \cdot \succ)$ as defined in the appendix, definitions A.11 and A.12. Assume a vector space $\mathscr{G} \simeq \mathbb{R}^4$ endowed with a Haar measure μ, and let $\mathscr{G}^*$ be its dual. To $\mathscr{G}^*$ we refer to as *momentum space*. Then, we introduce one more important definition:

Definition 1.3 (**Negative definite measure**).
Let $(V, \prec \cdot \mid \cdot \succ)$ be a finite-dimensional indefinite inner product space. Consider a regular Borel measure ν on $\mathscr{G}^*$ taking values in $\mathrm{L}(V)$ with the following properties:

i) For every $u \in V$, the measure $\mathrm{d} \prec u \mid \nu u \succ$ is a finite real measure.

ii) For every Borel set $\Omega \subset \mathscr{G}^*$, the operator $-\nu(\Omega) \in \mathrm{L}(V)$ is positive, i. e.

$$\prec u \mid (-\nu(\Omega))\, u \succ \geq 0 \qquad \text{for all } u \in V.$$

Then ν is called a *negative definite measure* on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$.

Using negative definite measures, we may give an alternative definition of the kernel of the fermionic projector: Let V be a finite-dimensional indefinite inner product space, and let ν be a negative definite measure on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$. In the homogeneous setting, for $\xi \in \mathscr{G}$ we define the *kernel of the fermionic projector* as an endomorphism on V,

$$P[\nu](\xi)\colon V \to V, \qquad P[\nu](\xi) := \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi}\, \mathrm{d}\nu(k).$$

For $\xi \in \mathscr{G}$, this leads to the definition of the *closed chain* $A[\nu](\xi)\colon V \to V$,

$$A[\nu](\xi) := P[\nu](\xi)\, P[\nu](-\xi).$$

Then, we can define the *spectral weight* of a linear operator $A\colon V \to V$ as the sum of the absolute values of its eigenvalues,

$$|A| := \sum_{i=1}^{\dim V} |\lambda_i|\,.$$

Taken the last two definitions together, we define the *Lagrangian* $\mathcal{L}$ by

$$\mathcal{L}[\nu](\xi) := \left|A[\nu](\xi)^2\right| - \frac{1}{2n}\, |A[\nu](\xi)|^2\,,$$

and finally we introduce the *causal action* $\mathcal{S}$ by

$$\mathcal{S}[\nu] := \int_{\mathscr{G}} \mathcal{L}[\nu](\xi)\, \mathrm{d}\mu(\xi).$$

1.3. Overview

After introducing these fundamental definitions, we can give a more detailed overview of this work. The aim of the present work is to scrutinize homogeneous causal fermion systems, in particular regarding the fermionic projector, the reconstruction of causal fermion systems, and the existence of minimizers of the causal action.

First of all, we want to introduce a symmetry of a causal fermion system. What implications may be derived from such a symmetry? To this question we address in chapter 2: Assume a causal fermion system endowed with a symmetry of a causal fermion system. Moreover, consider the group of translations $(\mathscr{G}, +)$ and let $\mathscr{G}^*$ be its dual space. Then, the definition of a suitable transitive group action from $\mathscr{G}$ onto space-time M yields the identification of spin spaces S_x for $x \in M$. What is the connection to negative definite measures? As we will see, application of spectral theory allows to construct a negative definite measure on $\mathscr{G}^*$ with values in $\mathrm{L}(S_x)$. To this aim we prove the existence of an operator-valued spectral measure E_k which will be part of the negative definite measure.

Next, we want to consider the reverse situation: Assume there is no causal fermion system given. Instead, let $x \in \mathscr{G}$, let $V := S_x$ be a finite-dimensional indefinite inner product space endowed with a spin scalar product $\prec \cdot | \cdot \succ := \prec \cdot | \cdot \succ_x$, and let ν be a negative definite measure on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$. The natural question arising of this setting is: Is it possible to reconstruct the original causal fermion system $(\mathscr{H}, \mathscr{F}, \rho)$? Moreover, let E_k be the above operator-valued spectral measure on $\mathscr{G}$. Is it even possible to get back the spectral measure E_k? These questions will be addressed to in chapter 3.

In chapter 4 we shall work out an alternative way of reconstructing the causal fermion system. Let $(V, \prec \cdot | \cdot \succ)$ be a finite-dimensional indefinite inner product space. How to reconstruct the causal fermion system beginning with functions $\chi \colon \mathscr{G}^* \to V$ in momentum space? The main idea will be to start with suitable wave functions in momentum space in order to obtain the original Hilbert space. However, the crucial question is: Are those reconstructions equivalent? Are there in addition any advantages due to this alternative reconstruction?

Finally, in chapter 5 we will deal with the existence of minimizers of the causal action. This chapter is motivated by [Fin10, Chapter 4]: The main result of this article is the proof of a theorem which yields the existence of convergent subsequences of minimizing sequences of the causal action. However, the theorem in [Fin10] is proven for compact subsets in momentum space only. Is it possible to generalize this result for the whole momentum space $\mathscr{G}^*$? As will be shown below, this question is not yet answered. Problems and ideas will be presented and discussed in chapter 5.

Finally, some well-known but important definitions and theorems are stated in the appendix.

2. Reduction of the Symmetry Group

2.1. Preliminaries

The goal of this chapter is to introduce a symmetry of a causal fermion system. Afterwards, the implications of such a symmetry shall be investigated. In particular, its application due to the kernel of the fermionic projector will be of special interest to us. At the end of this chapter, we want to deal with the construction of a negative definite measure. For completeness of each chapter and a better understanding of the procedure, we will recall all necessary definitions. All over this chapter, let $(\mathscr{H}, \mathscr{F}, \rho)$ be a causal fermion system, defined as follows:

Definition 2.1 (Causal fermion system).
Consider a triple $(\mathscr{H}, \mathscr{F}, \rho)$ consisting of the following structures:

i) Let $\mathscr{H}$ be a separable complex Hilbert space endowed with a scalar product $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}$ and a socalled *spin dimension* $n \in \mathbb{N}$.

ii) Let $\mathscr{F} \subset \mathrm{L}(\mathscr{H})$ be the set of all self-adjoint operators on $\mathscr{H}$ of finite rank, which have at most n positive and at most n negative eigenvalues.

iii) Finally, let ρ be a positive regular Borel measure defined on the Borel σ-algebra of $\mathscr{F}$. The measure ρ is called *universal measure.*

Then, we refer to $(\mathscr{H}, \mathscr{F}, \rho)$ as a *causal fermion system.*[1] See [Fin16].

We first have to prove that causal fermion systems are indeed well-defined which will be done in remark 2.2. For a Banach space X, we let $\mathrm{L}(X) := \mathrm{L}(X; X)$ be the set of linear and continuous operators onto X as usual.

Remark 2.2. Consider a causal fermion system $(\mathscr{H}, \mathscr{F}, \rho)$. Since $\mathscr{H}$ is a Hilbert space, the space of linear operators $\mathrm{L}(\mathscr{H}) := \mathrm{L}(\mathscr{H}; \mathscr{H})$ is a Banach space[2] with

[1] Actually, we refer to $(\mathscr{H}, \mathscr{F}, \rho)$ as a *causal fermion system in particle representation*, see [FGS12, Definition 1.2].

[2] Moreover, $\mathrm{L}(\mathscr{H})$ is even a Banach algebra, see [Alt06].

respect to the operator norm $\|\cdot\|_{L(\mathscr{H})}$ according to [Alt06]. The operator norm is defined as usual by

$$\|A\|_{L(\mathscr{H})} := \sup_{\substack{u \in \mathscr{H} \\ \|u\|_{\mathscr{H}} \le 1}} \|Au\|_{\mathscr{H}} \qquad (A \in L(\mathscr{H})),$$

where $\|\cdot\|_{\mathscr{H}}$ is the norm with respect to the scalar product $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}$ on $\mathscr{H}$. Since each norm induces a metric, we may define open subsets of $L(\mathscr{H})$. Thus, the operator norm induces a topology on $L(\mathscr{H})$ which we denote by $\mathfrak{O}_{L(\mathscr{H})}$. Elements in $\mathfrak{O}_{L(\mathscr{H})}$ are called open sets, and $(L(\mathscr{H}), \mathfrak{O}_{L(\mathscr{H})})$ is called a topological space, see [Alt06]. Since $\mathscr{F}$ is a subset of $L(\mathscr{H})$, we can introduce the *relative topology* on $\mathscr{F}$ induced by $\mathfrak{O}_{L(\mathscr{H})}$. The relative topology on $\mathscr{F}$ is given by

$$\mathfrak{O}_{\mathscr{F}} := \{U \cap \mathscr{F} : U \in \mathfrak{O}_{L(\mathscr{H})}\}$$

and defines a topology on $\mathscr{F}$, see e. g. [Fol99]. Then, $(\mathscr{F}, \mathfrak{O}_{\mathscr{F}})$ is also a topological space. Elements of $\mathfrak{O}_{\mathscr{F}}$ are called *open sets*, and $\mathscr{B}(\mathscr{F}) := \sigma(\mathfrak{O}_{\mathscr{F}})$ denotes the Borel σ-algebra generated by $\mathfrak{O}_{\mathscr{F}}$. The pair $(\mathscr{F}, \mathscr{B}(\mathscr{F}))$ is called a measurable space, the sets in $\mathscr{B}(\mathscr{F})$ are called *measurable sets* or Borel sets, see [Els11]. Hence, the universal measure ρ is well-defined on $\mathscr{B}(\mathscr{F})$,

$$\rho\colon \mathscr{B}(\mathscr{F}) \to [0, \infty],$$

and the triple $(\mathscr{F}, \mathscr{B}(\mathscr{F}), \rho)$ is referred to as a measure space. In summary, causal fermion systems are well-defined.

2.2. Symmetry of a Causal Fermion System

In the following, let $(\mathscr{H}, \mathscr{F}, \rho)$ be a causal fermion system, endowed with a symmetry of a causal fermion system which is defined as follows:

Definition 2.3 (**Symmetry of a causal fermion system**).
A *symmetry of a causal fermion system* is a group $\mathscr{G}$ together with a unitary representation U on $\mathscr{H}$ (where $\mathscr{H}$ may be infinite-dimensional) which leaves the universal measure invariant, i. e.

$$\rho\,(U_g \Omega U_g^{-1}) = \rho\,(\Omega)$$

for all $g \in \mathscr{G}$ and all measurable $\Omega \subset \mathscr{F}$.

Considering the group $\mathscr{G} = \{0\}$ together with a unitary representation $U\colon \mathscr{G} \to \mathscr{H}$, $U(0) = \mathrm{Id}$, then $(\mathscr{G}, U)$ is obviously a trivial symmetry of a causal fermion system

$(\mathscr{H}, \mathscr{F}, \rho)$ which always exists.

After introducing the symmetry of a causal fermion system, naturally one question arises: For which reason do we consider a symmetry of a causal fermion system? This question shall be answered immediately: There are multiple benefits due to such a symmetry. First, the symmetry of a causal fermion system leaves space-time M invariant and it allows to define a faithful group action on M. Assuming the group action to be transitive, it even permits the identification of the spin spaces. Moreover, we will obtain an alternative representation of the kernel of the fermionic projector. Finally, this procedure will lead to the definition of negative definite measures. Thus, the symmetry of a causal fermion system becomes a central tool in what follows.

In order to prove that the symmetry of a causal fermion system is well-defined, we need the definition of a unitary representation:

Definition 2.4 (Unitary representation).
Let H be a Hilbert space and let $\mathrm{U}(H)$ be the group of unitary operators on H. Then, a *unitary representation* of a group G is a group homomorphism[3]

$$U \colon G \to \mathrm{U}(H).$$

For any $v \in H$, the mapping

$$G \to H, \qquad g \mapsto U(g)\, v \tag{2.1}$$

is assumed to be continuous with respect to the strong operator topology. See [Fol95].

Remark 2.5. In section 2.7 we will need the fact that continuity of the mapping (2.1) implies continuity of the one-parameter group $t \mapsto U(t)$ for all $t \in \mathscr{G}$.

Since $\mathscr{H}$ is a complex Hilbert space, the set of unitary operators $\mathrm{U}(\mathscr{H})$ forms a multiplicative group according to [Heu82]. We consider the unitary representation

$$U \colon \mathscr{G} \to \mathrm{U}(\mathscr{H}), \qquad g \mapsto U_g := U(g),$$

which leaves the universal measure invariant in the sense of definition 2.3, i. e.

$$\rho\,(U_g\, \Omega\, U_g^{-1}) = \rho\,(\Omega)$$

[3]That is, $U \colon G \to \mathrm{U}(H)$ satisfies $U(xy) = U(x)\, U(y)$ and $U(x^{-1}) = U(x)^{-1}$ for any $x, y \in G$.

for all $g \in \mathscr{G}$ and all measurable sets $\Omega \in \mathscr{B}(\mathscr{F})$. Since U is a group homomorphism by definition of a unitary representation, the following identity holds:

$$U_e = U(e) = U(gg^{-1}) = U(g)U(g^{-1}) = U(g)U(g)^{-1} = \text{Id}, \tag{2.2}$$

where e denotes the neutral element of the group $\mathscr{G}$.

Now we can prove that the symmetry of a causal fermion system is well-defined:

Lemma 2.6. *The symmetry of a causal fermion system is well-defined in the sense that $U_g \Omega U_g^{-1}$ is a measurable subset of $\mathscr{F}$ for all $g \in \mathscr{G}$ and any measurable set $\Omega \subset \mathscr{F}$.*

Proof. Let $\Omega \subset \mathscr{F}$ be a measurable subset of $\mathscr{F}$. For every $g \in \mathscr{G}$, the operator $U_g \colon \mathscr{H} \to \mathscr{H}$ is an endomorphism onto $\mathscr{H}$, and the unitary mappings U_g, U_g^{-1} are linear and bounded (see e. g. [Rud91]). Obviously, each unitary operator is continuous: Let $x_k \to_{k\to\infty} x$ in $\mathscr{H}$. Since every unitary operator is isometric, we obtain

$$\|Ux_k - Ux\|_{\mathscr{H}} = \|U(x_k - x)\|_{\mathscr{H}} = \|x_k - x\|_{\mathscr{H}} \to_{k\to\infty} 0.$$

In order to show that the symmetry of a causal fermion system is well-defined, we have to prove that for any $A \in \Omega$ the linear map

$$U_g \, A \, U_g{}^{-1} \colon \mathscr{H} \to \mathscr{H}$$

is a self-adjoint endomorphism for all $g \in \mathscr{G}$. Since $\Omega \in \mathscr{B}(\mathscr{F})$ is a set of self-adjoint operators onto $\mathscr{H}$, every $A \in \Omega$ is a self-adjoint operator. For any unitary operator U holds $U^{-1} = U^*$ by definition A.24. This yields for all $x, y \in \mathscr{H}$

$$\begin{aligned}\langle U_g \, A \, U_g{}^{-1} x \mid y \rangle &= \langle A \, U_g{}^{-1} x \mid U_g{}^* y \rangle = \langle U_g{}^{-1} x \mid A^* \, U_g{}^* y \rangle = \langle x \mid U_g \, A^* \, U_g{}^* y \rangle \\ &= \langle x \mid U_g \, A \, U_g{}^{-1} y \rangle.\end{aligned}$$

The Hellinger-Toeplitz theorem[4] implies that $U_g \, A \, U_g{}^{-1}$ is continuous and self-adjoint for any $A \in \Omega$, $g \in \mathscr{G}$, see [Wer11, Satz V.5.5]. Next, the signature of any operator $A \in \Omega$ is given by (n, n). Since unitary operators leave the eigenvalues of A unchanged,

$$\det\left(\lambda - U_g \, A \, U_g^{-1}\right) = \det\left(U_g \, (\lambda - A) \, U_g^{-1}\right) = \det\left(\lambda - A\right),$$

[4]Remark: It is not necessary to apply the Hellinger-Toeplitz theorem in this case in order to prove self-adjointness.

the signature of $U_g A U_g^{-1}$ is also given by (n,n). Hence, $U_g \Omega U_g^{-1} \subset \mathscr{F}$ for all $g \in \mathscr{G}$. It remains to show that $U_g \Omega U_g^{-1}$ is a *measurable* subset of $\mathscr{F}$, i. e.

$$U_g \Omega U_g^{-1} \in \mathscr{B}(\mathscr{F}).$$

To this aim, consider the map $\phi = \phi_g$ for any $g \in \mathscr{G}$ given by

$$\phi \colon \mathscr{F} \to \mathscr{F}, \qquad A \mapsto U_g A U_g^{-1}.$$

We claim that ϕ is continuous: Let $A, (A_k)_{k\in\mathbb{N}} \subset \mathscr{F}$ such that $\|A - A_k\|_{\mathrm{L}(\mathscr{H})} \to_{k\to\infty} 0$. Since being unitary, U_g and U_g^{-1} are linear and bounded. The fact that $\mathrm{L}(\mathscr{H})$ is a Banach algebra implies

$$\begin{aligned}\|\phi(A) - \phi(A_k)\|_{\mathrm{L}(\mathscr{H})} &= \left\|U_g A U_g^{-1} - U_g A_k U_g^{-1}\right\|_{\mathrm{L}(\mathscr{H})} = \left\|U_g (A - A_k) U_g^{-1}\right\|_{\mathrm{L}(\mathscr{H})} \\ &\leq \left\|U_g\right\|_{\mathrm{L}(\mathscr{H})} \|A - A_k\|_{\mathrm{L}(\mathscr{H})} \left\|U_g^{-1}\right\|_{\mathrm{L}(\mathscr{H})} \to_{k\to\infty} 0.\end{aligned}$$

Thus, ϕ is continuous. In particular, ϕ is measurable by [Els11].[5] According to [Fin16], the set $U_g \Omega U_g^{-1}$ is measurable since its pre-image

$$\phi^{-1}\left(U_g \Omega U_g^{-1}\right) = \phi^{-1}(\phi(\Omega)) = \Omega$$

as an element of $\mathscr{B}(\mathscr{F})$ is measurable. We conclude that the given symmetry of a causal fermion system is well-defined in the above sense. □

2.3. Space-Time

Next, we want to introduce space-time. For this reason, consider the topological space $(\mathscr{F}, \mathfrak{O}_{\mathscr{F}})$. Elements in $\mathfrak{O}_{\mathscr{F}}$ are called open sets, and generate the Borel σ-algebra $\mathscr{B}(\mathscr{F}) = \sigma(\mathfrak{O}_{\mathscr{F}})$. A subset $\Omega \subset \mathscr{F}$ is called *closed* if its complement is open, that is $\Omega^{\mathrm{c}} \in \mathfrak{O}_{\mathscr{F}}$. Since $\mathscr{B}(\mathscr{F})$ is a σ-algebra, for any $\Omega \in \mathscr{B}(\mathscr{F})$ holds $\Omega^{\mathrm{c}} \in \mathscr{B}(\mathscr{F})$. This gives rise to the following definition:

Definition 2.7 (**Support of a measure**).
The support of a measure is defined as the complement of the largest open set of measure zero, i. e.

$$\operatorname{supp} \rho := \mathscr{F} \setminus \bigcup \{\Omega \subset \mathscr{F} \text{ open} \colon \rho(\Omega) = 0\}.$$

By definition, it is a closed set. See [Fin16].

[5] Let $(X, \mathfrak{A})$, $(Y, \mathfrak{B})$ be measurable spaces. A function $f \colon X \to Y$ is called *measurable* if $f^{-1}(\mathfrak{B}) \subset \mathfrak{A}$. In particular, every continuous function is measurable. See [Els11, Definition III.1.1] and [Els11, Korollar III.1.4].

A variety of measure theoretical textbooks defines the support of Radon measures in analogy to definition 2.7. Since the universal measure ρ is a regular Borel measure by definition of causal fermion systems 2.1, definition 2.7 can be applied to the universal measure. However, the above definition 2.7 may indeed be applied to more general measures, too. The importance of the support of a measure is reflected in the following definition:

Definition 2.8 (**Space-time**).
We define *space-time* as usual as the support of the universal measure ρ, i. e.

$$M := \operatorname{supp}(\rho).$$

Remark 2.9. Note that elements in space-time M are self-adjoint operators.

Lemma 2.10. *Let $(\mathscr{H}, \mathscr{F}, \rho)$ be a causal fermion system endowed with a symmetry of a causal fermion system, $(\mathscr{G}, U)$. Then, the unitary representation U leaves space-time M invariant in the sense that $U_g M U_g^{-1} = M$ for any $g \in \mathscr{G}$. Moreover, $\rho(U_g M U_g^{-1}) = \rho(M)$ holds for all $g \in \mathscr{G}$.*

Proof. First note $M \subset \mathscr{F}$. Defining the relative topology $\mathfrak{O}_M$ on M by

$$\mathfrak{O}_M = \{U \cap M \colon U \in \mathfrak{O}_{\mathscr{F}}\}, \tag{2.3}$$

we obtain open subsets of M. Open sets in $\mathfrak{O}_{\mathrm{L}(\mathscr{H})}$ may be defined with respect to the operator norm: For any $x \in M$, the set

$$B_\varepsilon(x) = \left\{ y \in \mathrm{L}(\mathscr{H}) \colon \|x - y\|_{\mathrm{L}(\mathscr{H})} < \varepsilon \right\} \in \mathfrak{O}_{\mathrm{L}(\mathscr{H})}$$

is an open set in $\mathrm{L}(\mathscr{H})$ for all $\varepsilon > 0$. Considering $B_\varepsilon^{\mathscr{F}}(x) := B_\varepsilon(x) \cap \mathscr{F} \in \mathfrak{O}_{\mathscr{F}}$ yields

$$B_\varepsilon^{\mathscr{F}}(x) = \left\{ y \in \mathscr{F} \colon \|x - y\|_{\mathrm{L}(\mathscr{H})} < \varepsilon \right\} \in \mathfrak{O}_{\mathscr{F}}.$$

In particular, $B_\varepsilon^{\mathscr{F}}(x) \in \mathscr{B}(\mathscr{F})$. By definition of space-time M, for any open set $\Omega \in \mathfrak{O}_M$ holds $\rho(\Omega) > 0$. By $B_\varepsilon^{\mathscr{F}}(x) \cap M$ we obtain open sets in $\mathfrak{O}_M$ for any $\varepsilon > 0$ according to (2.3). Since $B_\varepsilon^{\mathscr{F}}(x) \cap M \in \mathfrak{O}_M$, we have $\rho(B_\varepsilon^{\mathscr{F}}(x)) > 0$ for all $\varepsilon > 0$.

Next, consider the symmetry of a causal fermion system, $(\mathscr{G}, U)$. Then, U_g is unitary for all $g \in \mathscr{G}$. In general, for any unitary operator $U \in \mathrm{L}(\mathscr{H})$ holds

$$\|Uy\|_{\mathrm{L}(\mathscr{H})} = \sup_{\|u\| \le 1} \|Uyu\|_{\mathscr{H}} = \sup_{\|u\| \le 1} \|yu\|_{\mathscr{H}} = \|y\|_{\mathrm{L}(\mathscr{H})}$$

for all $y \in \mathrm{L}(\mathscr{H})$ since U is an isometry. Moreover, for any unitary operator U holds $\|U\|_{\mathrm{L}(\mathscr{H})} = 1$, as we obtain by

$$\|U\|_{\mathrm{L}(\mathscr{H})} = \sup_{\|v\|_{\mathscr{H}}=1} \|Uv\|_{\mathscr{H}} = \sup_{\|v\|_{\mathscr{H}}=1} \|v\|_{\mathscr{H}} = 1. \tag{2.4}$$

If U is unitary, U^{-1} is also unitary. Hence, for arbitrary $y \in \mathrm{L}(\mathscr{H})$ we achieve

$$\left\|yU^{-1}\right\|_{\mathrm{L}(\mathscr{H})} \leq \|y\|_{\mathrm{L}(\mathscr{H})} \left\|U^{-1}\right\|_{\mathrm{L}(\mathscr{H})} \overset{(2.4)}{=} \|y\|_{\mathrm{L}(\mathscr{H})}. \tag{2.5}$$

Conversely, consider $v \in \mathscr{H}$ with $\|v\|_{\mathscr{H}} = 1$. Then, $\|Uv\|_{\mathscr{H}} = 1$ implies

$$\|yv\|_{\mathscr{H}} = \left\|yU^{-1}Uv\right\|_{\mathscr{H}} \leq \sup_{\|Uv\|_{\mathscr{H}}=1} \left\|yU^{-1}Uv\right\|_{\mathscr{H}} \leq \sup_{\|w\|_{\mathscr{H}}=1} \left\|yU^{-1}w\right\|_{\mathscr{H}} = \left\|yU^{-1}\right\|_{\mathrm{L}(\mathscr{H})}.$$

This yields

$$\sup_{\|v\|_{\mathscr{H}}=1} \|yv\|_{\mathscr{H}} \leq \left\|yU^{-1}\right\|_{\mathrm{L}(\mathscr{H})}$$

for all $y \in \mathrm{L}(\mathscr{H})$, which proves

$$\|y\|_{\mathrm{L}(\mathscr{H})} \leq \left\|yU^{-1}\right\|_{\mathrm{L}(\mathscr{H})}. \tag{2.6}$$

Finally, for $y \in \mathrm{L}(\mathscr{H})$ by the inequations (2.5) and (2.6) we obtain

$$\left\|yU^{-1}\right\|_{\mathrm{L}(\mathscr{H})} = \|y\|_{\mathrm{L}(\mathscr{H})}$$

for any unitary operator U. This proves

$$\begin{aligned}
UB_\varepsilon^{\mathscr{F}}(x)U^{-1} &= U\left\{y \in \mathscr{F}:\ \|x-y\|_{\mathrm{L}(\mathscr{H})} < \varepsilon\right\} U^{-1} \\
&= \left\{UyU^{-1} \in \mathscr{F}:\ \|x-y\|_{\mathrm{L}(\mathscr{H})} < \varepsilon\right\} \\
&= \left\{y \in \mathscr{F}:\ \left\|x - U^{-1}yU\right\|_{\mathrm{L}(\mathscr{H})} < \varepsilon\right\} \\
&= \left\{y \in \mathscr{F}:\ \left\|U(x - U^{-1}yU)U^{-1}\right\|_{\mathrm{L}(\mathscr{H})} < \varepsilon\right\} \\
&= \left\{y \in \mathscr{F}:\ \left\|UxU^{-1} - y\right\|_{\mathrm{L}(\mathscr{H})} < \varepsilon\right\} \\
&= B_\varepsilon^{\mathscr{F}}(UxU^{-1})
\end{aligned}$$

for any $\varepsilon > 0$, $x \in M$. In particular, for all $x \in M$ and $\varepsilon > 0$ holds

$$\rho(UB_\varepsilon^{\mathscr{F}}(x)U^{-1}) > 0 \quad \Longleftrightarrow \quad \rho(B_\varepsilon^{\mathscr{F}}(UxU^{-1})) > 0.$$

Since $(\mathscr{G}, U)$ is a symmetry of a causal fermion system, for the measurable set $B_\varepsilon^{\mathscr{F}}(x) \in \mathfrak{O}_{\mathscr{F}}$ we achieve $\rho(U_g^{-1}B_\varepsilon^{\mathscr{F}}(x)U_g) = \rho(B_\varepsilon^{\mathscr{F}}(x))$ for all $g \in \mathscr{G}$. Thus, we even obtain the following equivalence:

$$\rho(B_\varepsilon^{\mathscr{F}}(x)) > 0 \quad \Longleftrightarrow \quad \rho(U_gB_\varepsilon^{\mathscr{F}}(x)U_g^{-1}) > 0 \quad \Longleftrightarrow \quad \rho(B_\varepsilon^{\mathscr{F}}(U_gxU_g^{-1})) > 0$$

for all $x \in M$, $g \in \mathscr{G}$, $\varepsilon > 0$. This yields the desired result for any $g \in \mathscr{G}$:

$$\begin{aligned} x \in M \quad &\iff \quad \rho(B_\varepsilon^{\mathscr{F}}(x)) > 0 \quad \text{for all } \varepsilon > 0 \\ &\iff \quad \rho(U_g B_\varepsilon^{\mathscr{F}}(x) U_g^{-1}) > 0 \quad \text{for all } \varepsilon > 0 \\ &\iff \quad \rho(B_\varepsilon^{\mathscr{F}}(U_g x U_g^{-1})) > 0 \quad \text{for all } \varepsilon > 0 \\ &\iff \quad x \in U_g M U_g^{-1}, \end{aligned}$$

proving $M = U_g M U_g^{-1}$ for all $g \in \mathscr{G}$. In particular, $\rho(M) = \rho(U_g M U_g^{-1})$ holds for any $g \in \mathscr{G}$. Hence, the universal measure leaves space-time M invariant. □

Due to the symmetry of a causal fermion system, we want to introduce a group action on space-time M as presented in the next section.

2.4. Group Action

Let $(\mathscr{G}, U)$ again be the symmetry of the causal fermion system $(\mathscr{H}, \mathscr{F}, \rho)$, and let M denote space-time.

Remember, that a group G is said to *act on* a set X if there is a function

$$G \times X \to X, \qquad (g, x) \mapsto g \cdot x$$

such that $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all $x \in X$, $g, h \in G$, where $e \in G$ denotes the identical element, see definition A.5.

Then, we can introduce a group action on space-time M as follows:

Lemma 2.11. *The induced mapping*

$$T : \mathscr{G} \times M \to M, \qquad (g, x) \mapsto U_g \, x \, U_g^{-1}$$

is an action *of* $\mathscr{G}$ *on* M*.*

Remark 2.12. Moreover, lemma 2.11 naturally induces the map $T_g : M \to M$ with $T_g := T(g, \cdot)$ for any $g \in \mathscr{G}$.

Proof of lemma 2.11. It is easy to see that $T : \mathscr{G} \times M \to M$, $(g, x) \mapsto U_g \, x \, U_g^{-1}$ is a group action:

i) Since $U_e = \mathrm{Id}$ according to (2.2), we obtain the first condition,

$$T_e x = T(e, x) = U_e \, x \, U_e^{-1} = \mathrm{Id} \, x \, \mathrm{Id}^{-1} = x.$$

ii) The second condition is given as follows:

$$T_{g_2}\left(T_{g_1}x\right) \equiv T(g_2, T(g_1, x)) = U_{g_2}\,U_{g_1}\,x\,U_{g_1}^{-1}\,U_{g_2}^{-1} = U_{g_2}\,U_{g_1}\,x\left(U_{g_2}\,U_{g_1}\right)^{-1}$$
$$= U_{g_2 g_1}\,x\left(U_{g_2 g_1}\right)^{-1} = T(g_2\,g_1, x) \equiv \left(T_{g_2 g_1}\right)x.$$

Hence, $T\colon \mathscr{G} \times M \to M$ is a group action. □

Furthermore, for the map $T_g\colon M \to M$ with $g \in \mathscr{G}$ the following general statement holds:

Proposition 2.13. *Let G be a group which acts on a set X. Then, for $g \in G$ the mapping*

$$T_g\colon X \to X, \qquad x \mapsto T_g(x) := g \cdot x$$

is a bijection *with the inverse mapping $T_{g^{-1}}$.*

Proof. We have

$$T_{g^{-1}}(T_g(x)) = T_{g^{-1}}(g \cdot x) = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = x,$$

and $T_g \circ T_{g^{-1}} = \mathrm{Id}$, respectively. See [Gub10, Definition 1.6.1]. □

Our goal is to identify spin spaces, as we will see in section 2.6. To this aim, we introduce the definition of the symmetry group:

Definition 2.14 (Symmetry group).
Let X be a set and let $M(X)$ be the set of all mappings $f\colon X \to X$. Consider a set M, together with an associative operation "·" and a neutral element $e \in M$. We define

$$M^* := \left\{a \in M\colon \exists\, a^{-1} \in M \text{ with } a \cdot a^{-1} = a^{-1} \cdot a = e\right\}.$$

Then M^* is a group. Moreover, $M(X)^*$ is the set of bijections onto X. We refer to

$$\mathrm{S}(X) := M(X)^*$$

as the *symmetry group* on X. See [Gub10].

Now we can apply the previous results to our setting: Considering $X = M$, $g \in \mathscr{G}$, and the mapping $T_g := T(g, \cdot)$, proposition 2.13 yields a bijection

$$T_g\colon M \to M, \qquad x \mapsto T_g\,x := U_g\,x\,U_g^{-1}. \tag{2.7}$$

Let $\mathrm{S}(M)$ be the symmetry group on M. Defining the group homomorphism

$$\varphi\colon \mathscr{G} \to \mathrm{S}(M), \qquad g \mapsto \varphi(g) := T_g, \tag{2.8}$$

we obtain a mapping from $\mathscr{G}$ into the symmetry group $\mathrm{S}(M)$. As we will see, this mapping is even injective. Therefore, we need the following definition:

Definition 2.15. An action of a group G on a set X is called *faithful* if and only if $T_g = \mathrm{Id}$ only holds for $g = e$. See [Gub10].

In what follows, we may assume without loss of generality that the action of $\mathscr{G}$ on M is faithful (otherwise, consider the group $(\mathscr{G}/\mathscr{N}, +)$ instead). Then, the kernel of the above group homomorphism $\varphi\colon \mathscr{G} \to \mathrm{S}(M)$ is given by $\ker(\varphi) = \{e\}$. Equivalently, $g \mapsto \varphi(g) = T_g$ is injective according to definition A.4. Thus, the map $T\colon \mathscr{G} \times M \to M$ induces an injection of $\mathscr{G}$ into the symmetry group $\mathrm{S}(M)$. If we furthermore assume that the group action is also *transitive* (see definition A.6), we may identify spin spaces S_x and S_y for any $x, y \in M$ as we will see below.

2.5. Factor Groups

Before identifying spin spaces, we have to justify that we may assume the group action to be faithful. So, what if the group action of $\mathscr{G}$ on space-time M is not faithful? Then, we may form *quotient groups* (or *factor groups*, respectively) which indeed act faithfully on M. In order to define quotient groups, we need socalled *normal subgroups* ("Normalteiler") which are defined as follows:

Definition 2.16 (Normal subgroup).
Let G be a group. A subgroup N of G is called a *normal subgroup* if

$$gNg^{-1} = N$$

holds for all $g \in \mathscr{G}$, i. e. $gng^{-1} \in N$ for all $n \in N$, $g \in G$. See [Gub10].

Lemma 2.17. *Let $\varphi\colon G_1 \to G_2$ be a group homomorphism between groups G_1 and G_2. Then, $\ker(\varphi)$ is a normal subgroup of G_1.*

Proof. The proof can be found in [Wol11, Satz 2.13]. □

We consider the group homomorphism $\varphi\colon \mathscr{G} \to \mathrm{S}(M)$, $g \mapsto T_g$ as defined in (2.8). Obviously, the identity element of the symmetry group is given by $\mathrm{Id} \in \mathrm{S}(M)$. The definition of the kernel yields

$$\mathscr{N} := \left\{g \in \mathscr{G} : T_g = \varphi(g) = \mathrm{Id}\right\} = \ker(\varphi).$$

Then, $\mathscr{N}$ is a normal subgroup of $\mathscr{G}$ by lemma 2.17. Hence we can form the quotient group $\mathscr{G}/\mathscr{N}$ which is actually a group, too, see e. g. [Bos06a]. In particular, the group action $T\colon \mathscr{G}/\mathscr{N} \times M \to M$ is faithful. Generally spoken: If the group action $T\colon \mathscr{G} \times M \to M$ is not faithful, we may go over to the new group

$$(\mathscr{G}/\mathscr{N}, +) \qquad \text{with} \quad \mathscr{N} := \{g \in \mathscr{G} : T_g = \mathrm{Id}\} = \ker(\varphi),$$

on which the group action

$$T: \mathscr{G}/\mathscr{N} \times M \to M, \qquad (\overline{g}, x) \mapsto U_{\overline{g}} x U_{\overline{g}}^{-1}$$

is faithful. This ensures: Since $(\mathscr{G}/\mathscr{N}, U)$ is still a symmetry of the causal fermion system, we may assume without loss of generality the group action $\mathscr{G}$ on M being faithful. The group $\mathscr{G}$ in this context is a mere tool to identify the spin spaces as well as to introduce the kernel of the fermionic projector as an endomorphism on spin spaces. However, in order to realize these ideas we need to impose one more assumption:

Assumption 2.18.
We assume that the action of $\mathscr{G}$ on M is *transitive* (see definition A.6).[6]

Under this additional assumption 2.18, the group homomorphism (2.8) even yields a bijection between $\mathscr{G}$ and the symmetry group $\mathrm{S}(M)$. Furthermore, we achieve the following lemma:

Lemma 2.19. *Let $(\mathscr{G}, U)$ be the symmetry of a causal fermion system, and let M denote space-time. Under assumption 2.18, the following statements hold:*

i) To every $x, y \in M$ there is a unique $\xi \in \mathscr{G}$ such that $y = T_\xi x$.

ii) For a given $x \in M$ there is a bijection $T(\cdot, x): \mathscr{G} \to M$. Moreover, we have

$$y = T_\xi x = U_\xi \, x \, U_\xi^{-1}. \tag{2.9}$$

Thus we obtain the identification $M \simeq \mathscr{G}$.

iii) Additionally, there is a bijection $T_\xi : M \to M$ for any $\xi \in \mathscr{G}$.

Proof. In order to prove this lemma, we will make use of the fact that the group action of $\mathscr{G}$ on M is faithful and transitive:

i) For $x, y \in M$, transitivity yields the existence of a $\xi \in \mathscr{G}$ such that $y = T_\xi x$. Since $\mathscr{G}$ acts faithfully, we obtain uniqueness: For $\xi, \zeta \in \mathscr{G}$, we have

$$\begin{aligned} T_\xi x = T_\zeta x \quad &\Longleftrightarrow \quad T_{\zeta^{-1}} T_\xi x = x \\ &\Longleftrightarrow \quad \varphi(\zeta^{-1}) \varphi(\xi) = \mathrm{Id} \\ &\Longleftrightarrow \quad \varphi(\zeta^{-1} \xi) = \mathrm{Id} \\ &\Longleftrightarrow \quad \zeta^{-1} \xi = e \\ &\Longleftrightarrow \quad \xi = \zeta, \end{aligned}$$

since $\varphi(\xi), \varphi(\zeta) \in \mathrm{S}(M)$ being bijections, $\varphi(\zeta^{-1})$ being the inverse mapping and the group homomorphism $g \mapsto \varphi(g)$ being injective.

[6] By contrast to acting faithfully, we have to *postulate* the group action being transitive.

ii) Fixing $x \in M$ and varying y, transitivity of the group action yields the existence of a $\xi = \xi(y) \in \mathscr{G}$ such that $T_\xi x = y$, proving surjectivity. Since T acts faithfully, ξ is unique. This proves injectivity. Hence, the map $T(\cdot, x)\colon \mathscr{G} \to M$ is a bijection, yielding the identification $\mathscr{G} \simeq M$. Using (2.7), i. e. $T_\xi x = U_\xi x U_\xi^{-1}$, we obtain formula (2.9):

$$y = T_\xi x = U_\xi \, x \, U_\xi^{-1}.$$

iii) The bijection $T_\xi\colon M \to M$ for any $\xi \in \mathscr{G}$ follows immediately by proposition 2.13.

□

Lemma 2.19 is the main result in order to identify spin spaces. Nevertheless, it is important that the group $\mathscr{G}$ acts faithfully and transitively on space-time M as the proof of lemma 2.19 shows. Later on, we would like to make use of momentum space $\mathscr{G}^*$ as the dual space of $\mathscr{G}$. However, by [Gub11] we need a vector space in order to define the algebraic dual space. For this reason, $\mathscr{G}$ has not only to be an arbitrary group but also a vector space. As we will see below, we may assume $\mathscr{G}$ being a vector space *and* the group action being faithful. This shall be explained in more detail: If $\mathscr{G}$ is a vector space and W is a subspace of $\mathscr{G}$, then $\mathscr{G}/W$ is also a vector space (see e. g. [Bos06b]). For the given group homomorphism $\varphi\colon \mathscr{G} \to \mathrm{S}(M)$, the set $\mathscr{N} = \ker(\varphi)$ is a subspace of $\mathscr{G}$. Hence, $\mathscr{G}/\mathscr{N}$ is also a vector space. In particular, $(\mathscr{G}/\mathscr{N}, +)$ is an abelian group which acts faithfully on space-time M as seen above. Moreover, on $\mathscr{G}/\mathscr{N}$ we may define the algebraic dual space. We will focus on this in more detail in section 2.7. We now come to the identification of spin spaces.

2.6. Identification of Spin Spaces

Assume a causal fermion system $(\mathscr{H}, \mathscr{F}, \rho)$ endowed with a symmetry $(\mathscr{G}, U)$. In the previous section we have shown that lemma 2.19 holds if the group action $T\colon \mathscr{G} \times M \to M$, $(g, x) \mapsto U_g \, x \, U_g^{-1}$ is faithful and transitive. Assuming these two properties of the group action, it is convenient to identify the spin spaces. For this reason, let us first introduce the definition of a spin space:

Definition 2.20 (**Spin space**).
For any self-adjoint operator $x \in M$, the corresponding *spin space* $(S_x, \prec \cdot \,|\, \cdot \succ_x)$ is defined by

$$S_x := x(\mathscr{H}), \qquad \prec \cdot \,|\, \cdot \succ_x \; := -\langle \cdot \,|\, x \, \cdot \rangle_{\mathscr{H}} \Big|_{S_x \times S_x}.$$

Then, $\prec \cdot \mid \cdot \succ_x$ is referred to as a *spin scalar product.*

Remark 2.21 (**Indefinite inner product space**).
By definition of causal fermion systems, the Hilbert space $\mathscr{H}$ is a complex vector space. Consider $x \in M$. By definition of space-time, x is a self-adjoint linear operator on $\mathscr{H}$ with at most n positive and at most n negative eigenvalues. Hence, the spin space $S_x := x(\mathscr{H})$ is an at most $2n$-dimensional subspace of $\mathscr{H}$: Let $x(u), x(v) \in S_x$. Then, $x(u) + x(v) = x(u + v) \in S_x$. The remaining vector space axioms follow immediately. Since $x \in M$ is of finite rank, S_x is a finite-dimensional vector space. We prove that $\prec \cdot \mid \cdot \succ_x$ is an indefinite inner product: Linearity in the second argument holds for each scalar product by definition. Furthermore, antisymmetry is satisfied for all $u, v \in S_x$ since each scalar product is antisymmetric and $x \in M$ is self-adjoint:

$$\prec u \mid v \succ_x = \langle u \mid xv \rangle_{\mathscr{H}} = \overline{\langle xv \mid u \rangle_{\mathscr{H}}} = \overline{\langle v \mid xu \rangle_{\mathscr{H}}} = \overline{\prec v \mid u \succ_x}.$$

Non-degeneracy: If $0 = \prec u \mid v \succ_x = \langle u \mid xv \rangle_{\mathscr{H}}$ holds for all $v \in S_x$, then we have $0 = \langle xu \mid v \rangle_{\mathscr{H}}$ for all $v \in S_x$. Choosing $v = xu \in S_x$, we obtain $0 = \langle xu \mid xu \rangle_{\mathscr{H}}$. Since every scalar product is positive definite by definition, we obtain $xu = 0$. Hence, $u = 0$. According to definition A.11, the spin scalar product is indeed an indefinite inner product. Definition A.12 yields that $(S_x, \prec \cdot \mid \cdot \succ_x)$ is an indefinite inner product space.

Now we make use of lemma 2.19 in order to identify spin spaces:

Lemma 2.22 (**Identification of spin spaces**).
Let $(\mathscr{G}, U)$ be the symmetry of a causal fermion system $(\mathscr{H}, \mathscr{F}, \rho)$. Take assumption 2.18 for granted. Then, for $x, y \in M$ and $\xi = \xi(x, y) \in \mathscr{G}$, the mapping U_ξ gives an isomorphism of corresponding spin spaces,

$$U_\xi \colon S_x \to S_y.$$

Fixing the space-time point x, for $y \in M$ we obtain the identification

$$S_y \simeq S_x, \qquad \textit{given by} \quad S_y = U_\xi S_x. \tag{2.10}$$

Proof. Consider arbitrary $x, y \in M$ and a fixed $\xi = \xi(x, y) \in \mathscr{G}$, and let $z \in S_y$ be arbitrary. According to the definition of a spin space, there is an element $\omega \in \mathscr{H}$ such that $y(\omega) = z$. By formula (2.9), i. e. $y = T_\xi x = U_\xi \, x \, U_\xi^{-1}$, we obtain

$$z = y(\omega) = U_\xi \, x \, U_\xi^{-1}(\omega)$$

and thus surjectivity, since $x\,U_\xi^{-1}(\omega)$ is an element of $x(\mathscr{H}) = S_x$. Injectivity of U_ξ immediately results from injectivity of unitary operators. Hence, $U_\xi \colon S_x \to S_y$ is an isomorphism. Since U_ξ is an isometry, this isomorphism between the spin spaces S_x and S_y is even isometric. □

Remark 2.23. Lemma 2.22 proves: For given $x, y \in M$, a corresponding $\xi \in \mathscr{G}$ exists such that U_ξ is an isometric isomorphism between the corresponding spin spaces S_x, S_y. Considering bijections $U_\xi \colon S_x \to S_y$ and $U_\zeta \colon S_x \to S_z$, we achieve the following diagram:

$$\begin{array}{ccc} S_x & \xrightarrow{U_\xi} & S_y \\ & \searrow^{U_\zeta} & \\ & & S_z \end{array}$$

In particular, $S_x \simeq S_y \simeq S_z$. Thus, the choice of the spin spaces does not depend on a space-time point $x \in M$ since we may change the spin space by a suitable isomorphism.

By identification of spin spaces, we obtain a new definition of the kernel of the fermionic projector as an endomorphism onto a chosen spin space, as we will see in the next section.

2.7. Fermionic Projector

Let $(\mathscr{G}, U)$ be a symmetry of the causal fermion system $(\mathscr{H}, \mathscr{F}, \rho)$. We now restrict attention to a specific group: the group of translations in the four-dimensional space,

$$\mathscr{G} = (\mathbb{R}^4, +)$$

(most of the constructions also work for other groups, but we restrict attention to the situation which is most interesting for us). We choose the group of translations since in the homogeneous setting only the difference of two space-time points is of interest. On $\mathscr{G}$ we may define a real vector space structure as usual. Hence we may introduce *momentum space* $\mathscr{G}^* \simeq \mathbb{R}^4$ as the algebraic dual space of $\mathscr{G}$. Denoting the canonical basis of $\mathbb{R}^4$ by $(e_0, \ldots, e_3)$, the four operators $U_{e_0}, \ldots, U_{e_3}$ are unitary by definition of a unitary representation and mutually commute: Using the fact that U is a group homomorphism,

$$U_{e_i}\,U_{e_j} = U(e_i)\,U(e_j) = U(e_i + e_j) = U(e_j + e_i) = U(e_j)\,U(e_i) = U_{e_j}\,U_{e_i}$$

holds for $i, j = 0, \ldots, 3$. Furthermore, for all $t \in \mathbb{R}$, $i = 0, \ldots, 3$, the set $\{U_{te_i} : t \in \mathbb{R}\}$ is a strongly continuous one-parameter unitary group: By definition of a unitary representation, $U(te_i)$ is a unitary operator for any $t \in \mathbb{R}$. Since U is a group homomorphism, we obtain

$$U_{(s+t)\,e_i} = U((s+t)\,e_i) = U(se_i + te_i) = U(se_i)\,U(te_i) = U_{se_i}\,U_{te_i}$$

for all $s, t \in \mathbb{R}$, $i = 0, \ldots, 3$. For any $v \in \mathscr{H}$, the mapping $\mathscr{G} \ni g \mapsto U(g)\,v \in \mathscr{H}$ is continuous by definition of a unitary representation. Hence, we have

$$\lim_{h\to 0} U((t+h)\,e_i)\,v = \lim_{h\to 0} U(te_i)\,U(he_i)\,v = U(te_i)\,U(0)\,v = U(te_i)\,v$$

for all $t \in \mathbb{R}$, $i = 0, \ldots, 3$. Therefore, the set $\{U_{te_i} : t \in \mathbb{R}\}$ is a strongly continuous one-parameter unitary group according to definition A.26.
Our goal is now to achieve a spectral representation of the unitary operator $U(t)$ with $t \in \mathbb{R}^4$. In order to apply theorem A.27, we have to show: For any $t \in \mathbb{R}^4$, the mapping $t \mapsto U(t) = U(t_1, \ldots, t_n)$ is a strongly continuous map of $\mathbb{R}^4$ into the group of unitary operators on a separable Hilbert space $\mathscr{H}$ satisfying $U(t+s) = U(t)U(s)$ for all $s, t \in \mathbb{R}^4$ and $U(0) = \mathrm{Id}$. Considering arbitrary $t = \sum_{i=0}^{3} t_i e_i \in \mathscr{G}$, the group homomorphism property of U shows that

$$U(t) = U(\sum_{i=0}^{3} t_i e_i) = \prod_{i=0}^{3} U(t_i e_i).$$

Hence, $U(t)$ is unitary by definition A.24:

$$U(t)^* = \left(\prod_{i=0}^{3} U(t_i e_i)\right)^* = \prod_{i=0}^{3} U(t_i e_i)^* = \prod_{i=0}^{3} U(t_i e_i)^{-1} = \left(\prod_{i=0}^{3} U(t_i e_i)\right)^{-1} = U(t)^{-1},$$

since the operators $U(t_i e_i)$ mutually commute. Using (2.2), $U(0) = \mathrm{Id}$ follows immediately. Application of theorem A.27 yields the existence of a projection-valued measure E_k on $\mathbb{R}^4$ which allows the spectral representation

$$\langle u \mid U(\xi)\,v\rangle_{\mathscr{H}} = \int_{\mathbb{R}^4} \mathrm{e}^{\mathrm{i}\xi\cdot k}\,\mathrm{d}\langle u \mid E_k\,v\rangle_{\mathscr{H}}$$

for all $u, v \in \mathscr{H}$, $\xi \in \mathscr{G}$. Considering $\xi = (t, x, y, z) \in \mathscr{G}$, we obtain the symbolically written spectral representation

$$U_{(t,x,y,z)} = \int_{\mathbb{R}^4} \mathrm{e}^{\mathrm{i}(k_0 t + k_1 x + k_2 y + k_3 z)}\,\mathrm{d}E_k. \tag{2.11}$$

Using the Einstein convention, we write (2.11) in the shorter form

$$U_\xi = \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k_j \xi^j}\,\mathrm{d}E_k \tag{2.12}$$

for any $\xi = (t, x, y, z) \in \mathscr{G}$ which is *symbolically written* in the following sense: For all $\psi, \phi \in \mathscr{H}$, $\xi \in \mathscr{G}$ holds

$$\langle \psi \mid U_\xi \, \phi \rangle_{\mathscr{H}} = \langle \psi \mid \left(\int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i} k_j \xi^j} \, \mathrm{d}E_k \right) \phi \rangle_{\mathscr{H}} \overset{(A.2)}{\coloneqq} \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i} k_j \xi^j} \, \mathrm{d}\langle \psi \mid E_k \, \phi \rangle_{\mathscr{H}}$$

according to definition A.15. Instead of using a scalar product, we consider duality by $k(\xi) = \sum_{i=0}^{3} k_i \xi_i$ for any $k \in \mathscr{G}^*$, $\xi \in \mathscr{G}$.

In order to introduce the kernel of the fermionic projector, we mention the following definition of a projection:

Definition 2.24 (Projection).
Let H be a Hilbert space. If $P \in \mathrm{L}(H)$ and $P^2 = P$, then P is called a *projection*. If in addition $P = P^*$, then P is called an *orthogonal projection*. See e. g. [RS80].

Remark 2.25. Each *projection-valued measure* P_Ω is an orthogonal projection. For $\phi \in H$, $(\phi, P_\Omega \, \phi)$ is a well-defined Borel measure which we denote by $\mathrm{d}(\phi, P_\lambda \, \phi)$. The complex measure $\mathrm{d}(\phi, P_\lambda \, \psi)$ is defined by polarization. See [RS80].

We then introduce the orthogonal projection onto a spin space:

Definition 2.26 (Orthogonal projection).
For $x \in M$, let $\pi_x \colon \mathscr{H} \to S_x$ be the *orthogonal projection* from $\mathscr{H}$ onto the spin space $S_x \coloneqq x(\mathscr{H})$.

Since $x(\mathscr{H})$ is a finite-dimensional subspace of $\mathscr{H}$ and thus closed by theorem A.14, theorem 2.27 below yields the existence of such an orthogonal projection from $\mathscr{H}$ onto $x(\mathscr{H})$. Hence, the orthogonal projection π_x is well-defined.

Theorem 2.27. *Let $U \neq \{0\}$ be a closed subspace of a Hilbert space H. Then, there is a linear projection $P_U \colon H \to U$, and we have $H = U \oplus U^\perp$. P_U is called orthogonal projection onto U.*

Proof. See [Wer11, Theorem V.3.4]. □

We now come to the important definition of the kernel of the fermionic projector:

Definition 2.28 (Kernel of the fermionic projector).
For $x, y \in M$, we define the *kernel of the fermionic projector* by

$$P(x, y) \coloneqq \pi_x \, y \colon S_y \to S_x. \tag{2.13}$$

Note the implicit restriction of the kernel of the fermionic projector $P(x,y)$ on the spin space S_y for any $x,y \in M$ according to definition 2.28. Using the identification of the spin spaces and formula (2.10), i. e. $S_y = U_\xi S_x$, where $\xi = \xi(x,y) \in \mathscr{G}$, we may define the *kernel of the fermionic projector* for any $x,y \in M$ as an endomorphism onto S_x by

$$P(x,y) := \pi_x\, y\, U_\xi \colon S_x \to S_x.$$

Since $x \in M$ is arbitrary, the kernel of the fermionic projector does not depend on the choice of x. Using formula (2.9), i. e. $y = U_\xi\, x\, U_\xi^{-1}$, we obtain

$$P(x,y) = \pi_x\, y\, U_\xi = \pi_x\, U_\xi\, U_\xi^{-1}\, y\, U_\xi \overset{(2.9)}{=} \pi_x\, U_\xi\, x. \tag{2.14}$$

Now we can put in the spectral representation $U_\xi = \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k_j \xi^j}\, \mathrm{d}E_k$ according to (2.12). Thus, by (2.14) we obtain in accordance with definition A.15

$$\begin{aligned}
\langle u \mid P(x,y)\, v\rangle_{\mathscr{H}} &= \langle u \mid \pi_x\, U_\xi\, x\, v\rangle_{\mathscr{H}} = \langle u \mid \pi_x \left(\int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k_j \xi^j}\, \mathrm{d}E_k \right) x\, v\rangle_{\mathscr{H}} \\
&= \langle \pi_x\, u \mid \left(\int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k_j \xi^j}\, \mathrm{d}E_k \right) x\, v\rangle_{\mathscr{H}} \\
&= \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k_j \xi^j}\, \mathrm{d}\langle \pi_x\, u \mid E_k\, x\, v\rangle_{\mathscr{H}} \\
&= \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k_j \xi^j}\, \mathrm{d}\langle u \mid (\pi_x\, E_k\, x)\, v\rangle_{\mathscr{H}}
\end{aligned}$$

for all $u, v \in S_x$. As we shall see in section 2.8, it is justified to write

$$\langle u \mid P(x,y)\, v\rangle_{\mathscr{H}} = \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k_j \xi^j}\, \mathrm{d}\langle u \mid (\pi_x\, E_k\, x)\, v\rangle_{\mathscr{H}} = \langle u \mid \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k_j \xi^j}\, \mathrm{d}(\pi_x\, E_k\, x)\, v\rangle_{\mathscr{H}}$$

for all $x, y \in M$, $u, v \in S_x$, or written symbolically:

$$P(x,y) = \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k_j \xi^j}\, (\pi_x\, \mathrm{d}E_k\, x). \tag{2.15}$$

Formula (2.15) very much resembles the Fourier representation used as an ansatz for homogeneous causal fermion systems as considered in [Fin10].[7] But now this ansatz has been *derived* by imposing the action of a symmetry group. Apart from being more satisfying, the advantage of this new procedure is that it is more satisfying *and* has the potential of being generalized to other symmetry groups (like for example the isometry group $\mathbb{R} \times S^3$). The obvious questions are:

[7]This ansatz was given in [Fin10]: For a negative definite measure ν and a bounded set $\hat{K}$ in momentum space, the *kernel* of the fermionic projector was introduced as

$$P(\xi) = \int_{\hat{K}} \mathrm{e}^{\mathrm{i}\langle p, \xi\rangle}\, \mathrm{d}\nu(p), \qquad \text{where } \xi \equiv y - x.$$

(1) What is the connection between $\mathrm{d}(\pi_x E_k x)$ and negative definite measures as considered in [Fin10], *Causal variational principles on measure spaces*?

(2) Suppose we are only given the spin space $(S_x, \prec \cdot | \cdot \succ_x)$ and the measure $(\pi_x \,\mathrm{d}E_k\, x)$ on S_x. Is it possible to reconstruct the whole operator-valued measure $\mathrm{d}E_k$ on $\mathscr{H}$?

These questions will be addressed to in the following.

2.8. Negative Definite Measures

The aim of this section is to justify formula (2.15), and to answer the first question. Therefore, we finally come to the definition of negative definite measures:

Definition 2.29 (**Negative definite measure**).
Let $(V, \prec \cdot | \cdot \succ)$ be a finite-dimensional indefinite inner product space. Consider a regular Borel measure ν on $\mathscr{G}^*$ taking values in $\mathrm{L}(V)$ with the following properties:

i) For every $u \in V$, the measure $\mathrm{d} \prec u \,|\, \nu u \succ$ is a finite real measure.

ii) For every Borel set $\Omega \subset \mathscr{G}^*$, the operator $-\nu(\Omega) \in \mathrm{L}(V)$ is positive, i. e.

$$\prec u \,|\, (-\nu(\Omega))\, u \succ \geq 0 \qquad \text{for all } u \in V. \tag{2.16}$$

Then ν is called a *negative definite measure* on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$.

For completeness and in accordance with [Fin16], a *regular* Borel measure is a measure on the Borel sets with the property that it is continuous under approximations by compact sets from inside and by open sets from outside. An operator-valued measure is called a regular Borel measure if each component is a regular Borel measure. Then we can give the answer to the first question:

Proposition 2.30. *The measure $\pi_x \,\mathrm{d}E_k\, x$ is a negative definite measure on $\mathscr{G}^*$ with values in $\mathrm{L}(S_x)$ for any $x \in M$.*

Proof. By definition of the spin scalar product,

$$\prec u \,|\, v \succ_x := -\langle u \,|\, x\, v \rangle_{\mathscr{H}}$$

holds for all $u, v \in S_x$. Considering the above operator-valued spectral measure E_k and arbitrary measurable sets $\Omega \in \mathscr{B}(\mathscr{G}^*)$, the fact that $x \in M$ is self-adjoint and π_x as an orthogonal projection onto $x(\mathscr{H})$ is also self-adjoint yields

$$\begin{aligned} \prec u \mid (\pi_x E_k(\Omega) x)\, v \succ_x &= -\langle u \mid x\,(\pi_x E_k(\Omega) x)\, v \rangle_{\mathscr{H}} \\ &= -\langle x\, u \mid \pi_x E_k(\Omega) x\, v \rangle_{\mathscr{H}} \\ &= -\langle \pi_x x\, u \mid E_k(\Omega) x\, v \rangle_{\mathscr{H}} \\ &= -\langle x\, u \mid E_k(\Omega) x\, v \rangle_{\mathscr{H}} \end{aligned}$$

for all $u, v \in S_x$. Now one sees immediately that the measure $\pi_x\, \mathrm{d}E_k\, x$ is negative definite: For any Borel set $\Omega \in \mathscr{B}(\mathscr{G}^*)$, the spectral measure $E_k(\Omega)$ is an orthogonal projection. This implies $E_k(\Omega)^2 = E_k(\Omega) = E_k(\Omega)^*$ and thus yields the inequality

$$\begin{aligned} \prec u \mid -(\pi_x E_k(\Omega) x)\, u \succ_x &= \langle x\, u \mid E_k(\Omega) x\, u \rangle_{\mathscr{H}} \\ &= \langle x\, u \mid E_k(\Omega)^2 x\, u \rangle_{\mathscr{H}} \\ &= \langle E_k(\Omega) x\, u \mid E_k(\Omega) x\, u \rangle_{\mathscr{H}} \geq 0 \end{aligned}$$

for any $u \in V$ since $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}$ is a scalar product. Considering the restriction

$$\pi_x\, \mathrm{d}E_k(\Omega)\, x\big|_{S_x} : S_x \to S_x$$

for any $\Omega \in \mathscr{B}(\mathscr{G}^*)$, the measure $\pi_x\, \mathrm{d}E_k\, x$ only takes values in $\mathrm{L}(S_x)$. □

Thus, proposition 2.30 points out a connection between the spectral measure E_k and negative definite measures. Let us finally come back to formula (2.15). Considering $x \in M$, proposition 2.30 shows that $\pi_x\, \mathrm{d}E_k\, x$ is a negative definite measure on $\mathscr{G}^*$ with values in $\mathrm{L}(S_x)$. Hence, it is justified to write

$$\begin{aligned} \langle u \mid P(x,y)\, v \rangle_{\mathscr{H}} &= \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k_j \xi^j}\, \mathrm{d}\langle \pi_x\, u \mid E_k\, x\, v \rangle_{\mathscr{H}} \\ &= \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k_j \xi^j}\, \mathrm{d}\langle u \mid (\pi_x\, E_k\, x)\, v \rangle_{\mathscr{H}} \\ &\overset{(A.3)}{=} \langle u \mid \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k_j \xi^j}\, (\pi_x\, \mathrm{d}E_k\, x)\, v \rangle_{\mathscr{H}} \end{aligned}$$

with $u, v \in S_x$, $y \in M$. The latter is defined as operator-valued integration according to definition A.16. Thus, the above explanations give an answer to the first question. As we have seen, it is possible to construct a negative definite measure on the dual space $\mathscr{G}^*$ with values in $\mathrm{L}(S_x)$ for a given space-time point $x \in M$ which concludes this chapter. With the second question regarding the reconstruction of a causal fermion system we shall deal in the next chapter.

3. Reconstruction of the Causal Fermion System

3.1. Introduction

In the previous chapter, two questions were of special interest to us. The first question was already answered in chapter 2. The remaining second question is:

> Suppose we are only given an indefinite inner product space $(V, \prec \cdot \mid \cdot \succ)$, the group of translations $\mathscr{G} = (\mathbb{R}^4, +)$, and a negative definite measure ν on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$. Is it possible to reconstruct the original causal fermion system $(\mathscr{H}, \mathscr{F}, \rho)$? Is it even possible to reconstruct the whole operator-valued measure $\mathrm{d}E_k$ on $\mathscr{H}$?

This question shall be answered in this chapter. Our first goal is to reconstruct the original causal fermion system. Let $\mathscr{G} = (\mathbb{R}^4, +)$ be the group of translations, and let momentum space $\mathscr{G}^*$ be its dual. Then, for any fixed $x \in \mathscr{G}$, we introduce a finite-dimensional spin space V and a negative definite measure ν on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$,

$$(V, \prec \cdot \mid \cdot \succ) := (S_x, \prec \cdot \mid \cdot \succ_x) \qquad \text{and} \qquad \mathrm{d}\nu_k := \pi_x \, \mathrm{d}E_k x. \tag{3.1}$$

The construction should give us back the causal fermion system we started with. That is what we understand by "reconstruction" of the causal fermion system. Before beginning, we note that our reconstruction can determine the universal measure only up to a constant factor. This is obvious, because this constant does not show up in the data given for the reconstruction.

3.2. Wave Functions

Fundamental part of the reconstruction of the original causal fermion system are wave functions which shall be introduced right now: In general, we define *wave functions* as a mapping $\psi\colon \mathscr{G} \to V$. In order to define continuity, we point out that we may choose a basis of the vector space V and thus define a (non-canonical) norm on V. Since V is finite-dimensional, all norms on V are equivalent, see e.g. [Wer11, Satz I.2.5]. Then, continuity of a wave function is defined as usual: For $x, y \in \mathscr{G}$, a wave function $\psi\colon \mathscr{G} \to V$ is said to be *continuous* if $|x - y| \to 0$ in $\mathscr{G}$ implies $\|\psi(x) - \psi(y)\|_V \to 0$. Since all norms are equivalent, the definition of continuity is independent of the chosen norm.

Moreover, we define *derivatives* of wave functions $\psi\colon \mathscr{G} \to V$ as usual: If

$$\psi'(t) := \lim_{h\to 0} \frac{\psi(t) + \psi(t+h)}{h}$$

exists in V for all $t \in \mathscr{G}$, we denote by $\psi'\colon \mathscr{G} \to V$ the first derivative of ψ. If $\psi'\colon \mathscr{G} \to V$ exists and is continuous, we call the wave function ψ *continuously differentiable*. As usual, we define the m-th derivative of ψ by $\psi^{(m)} := (\psi^{(m-1)})'$. If $\psi^{(j)}$ is continuously differentiable for all $j = 0, \ldots, m-1$ and if $\psi^{(m)}$ is continuous, we refer to ψ as an m-times continuously differentiable wave function. By $C^m(\mathscr{G}; V)$ we denote the space of m-times continuously differentiable wave functions. Furthermore, let

$$C^\infty(\mathscr{G}; V) := \bigcap_{m=1}^{\infty} C^m(\mathscr{G}; V)$$

be the space of smooth wave functions. If $C_c^m(\mathscr{G}; V)$ denotes the space of m-times continuously differentiable wave functions with compact support, we define analogously the space of smooth wave functions with compact support,

$$C_c^\infty(\mathscr{G}; V) := \bigcap_{m=1}^{\infty} C_c^m(\mathscr{G}; V).$$

Obviously, derivatives of wave functions are again wave functions. Let $\mathscr{K}$ be the set of all wave functions $\psi\colon \mathscr{G} \to V$. Then, $\mathscr{K}_0 := C_c(\mathscr{G}; V)$ denotes the space of continuous wave functions $\psi\colon \mathscr{G} \to V$ with compact support. Let $\psi, \phi \in \mathscr{K}_0$. Defining $(\psi + \phi)(x) := \psi(x) + \phi(x)$, $(\lambda\psi)(x) := \lambda\psi(x)$ for all $x \in \mathscr{G}$, $\lambda \in \mathbb{C}$, $\mathscr{K}_0$ forms a complex vector space, since $\phi + \psi \in \mathscr{K}_0$ is again a continuous wave function with compact support.

Next, on the group $\mathscr{G} = (\mathbb{R}^4, +)$ we consider a Haar measure $\mu\colon \mathscr{B}(\mathscr{G}) \to [0, \infty]$. Note the existence of a Haar measure on $\mathscr{G}$, since $(\mathbb{R}^4, +)$ is a locally compact abelian topological group by [Els11]. By [Hal95, Theorem B, p. 254] there exists at least one Haar measure in every locally compact topological group. This allows us to consider a Haar measure μ on $\mathscr{G}$, which is a positive constant times the Lebesgue measure on $\mathbb{R}^4$.

Then, we introduce the *inner product* $<\cdot\,|\,\cdot>\colon \mathscr{K}_0 \times \mathscr{K}_0 \to \mathbb{C}$,

$$<\psi\,|\,\phi> := \int_{\mathscr{G}} \prec \psi(x)\,|\,\phi(x) \succ \, \mathrm{d}\mu(x) \qquad (\psi, \phi \in \mathscr{K}_0). \tag{3.2}$$

Since every Haar measure of a compact set is finite by lemma A.10, we obtain $\mu(\operatorname{supp}\psi) < \infty$ for any $\psi \in \mathscr{K}_0$. Hence the integral is finite and thus the inner product (3.2) is well-defined for continuous wave functions with compact support.

3.3. The Fermionic Projector

In order to define the fermionic projector, we will need the next lemma:

Lemma 3.1. *Let X, Y be normed $\mathbb{K}$-vector spaces, and let $T\colon X \to Y$ be a linear operator. Then, the following statements are equivalent:*

i) T *is continuous.*

ii) $\|T\|_{\mathrm{L}(X;Y)} := \sup_{\|x\|_X \leq 1} \|Tx\|_Y < \infty$.

iii) There is a constant C such that $\|Tx\|_Y \leq C\,\|x\|_X$ for all $x \in X$.

Proof. See [Alt06, Lemma 3.3.1]. □

In a first step, we introduce the kernel of the fermionic projector as follows:

Definition 3.2 (Kernel of the fermionic projector).
Consider an indefinite inner product space $(V, \prec\cdot\,|\,\cdot\succ)$, and let ν be a negative definite measure on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$. Then, for $x, y \in \mathscr{G}$ we define the *kernel of the fermionic projector* by

$$P(x,y)\colon V \to V, \qquad P(x,y) := \int_{\mathscr{G}^*} e^{\mathrm{i}k(y-x)}\,\mathrm{d}\nu(k). \tag{3.3}$$

The above integral is *written symbolically* in the sense that

$$\prec u\,|\,P(x,y)\,v \succ = \prec u\,|\int_{\mathscr{G}^*} e^{\mathrm{i}k(y-x)}\,\mathrm{d}\nu(k)\,v \succ \overset{(A.3)}{:=} \int_{\mathscr{G}^*} e^{\mathrm{i}k(y-x)}\,\mathrm{d} \prec u\,|\,\nu(k)\,v \succ$$

for all $u, v \in V$, $x, y \in \mathscr{G}$. Since $\mathrm{d} \prec u\,|\,\nu(k)\,v \succ$ is a bounded complex measure for all $u, v \in V$, $P(x,y)$ is well-defined on the whole vector space V.

By definition of an indefinite inner product space, V is a finite-dimensional complex vector space. Since $V := S_x$ as defined above is a subspace of dimension at most $2n$, let $\dim V = 2n$.[1] Hence, $V \simeq \mathbb{C}^{2n}$. Let $\langle \cdot \mid \cdot \rangle_{\mathbb{C}^{2n}} : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$ denote the standard scalar product on $\mathbb{C}^{2n}$, given by

$$\langle u \mid v \rangle_{\mathbb{C}^{2n}} := \sum_{i=1}^{2n} \overline{u}_i \, v_i$$

for all $u, v \in \mathbb{C}^{2n}$. For simplicity, we write $\langle \cdot \mid \cdot \rangle := \langle \cdot \mid \cdot \rangle_{\mathbb{C}^{2n}}$. In the following, consider a *pseudo-orthonormal basis* $(e_i)_{i=1,\dots,2n}$ of V for which $\prec \cdot \mid \cdot \succ$ is represented by

$$\prec u \mid v \succ = \langle u \mid S \, v \rangle \tag{3.4}$$

for all $u, v \in V$ and $S = \mathrm{diag}(1, \dots, 1, -1, \dots, -1)$.

Then, the kernel of the fermionic projector has the following properties:

Lemma 3.3. *For any $x, y \in \mathscr{G}$, the kernel of the fermionic projector is a linear and continuous operator, i. e. $P(x, y) \in \mathrm{L}(V)$ where $\mathrm{L}(V)$ is considered as the set of linear and continuous operators onto V.*

Proof. Consider arbitrary $x, y \in \mathscr{G}$. In order to prove linearity of $P(x, y)$, we make use of the linearity of the integrator according to definition A.29. For any $u, v, w \in V$, $\alpha \in \mathbb{C}$ we thus obtain

$$\begin{aligned}
\prec u \mid P(x,y)\,(\alpha\, v + w) \succ &= \prec u \mid \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)} \, \mathrm{d}\nu(k)\,(\alpha\, v + w) \succ \\
&= \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)} \, \mathrm{d} \prec u \mid \nu(k)\,(\alpha\, v + w) \succ \\
&= \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)} \, \mathrm{d} \prec u \mid \alpha\, \nu(k)\, v + \nu(k)\, w \succ \\
&= \int_{\mathscr{G}^*} \alpha\, \mathrm{e}^{\mathrm{i}k(y-x)} \, \mathrm{d} \prec u \mid \nu(k)\, v \succ \; + \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)} \, \mathrm{d} \prec u \mid \nu(k)\, w \succ \\
&= \prec u \mid \int_{\mathscr{G}^*} \alpha\, \mathrm{e}^{\mathrm{i}k(y-x)} \, \mathrm{d}\nu(k)\, v \succ \; + \prec u \mid \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)} \, \mathrm{d}\nu(k)\, w \succ \\
&= \prec u \mid \alpha\, P(x,y)\, v \succ \; + \prec u \mid P(x,y)\, w \succ \\
&= \prec u \mid \alpha\, P(x,y)\, v + P(x,y)\, w \succ .
\end{aligned}$$

[1] According to [Fin16, Definition 1.1.5], a space-time point $x \in M$ is said to be *regular* if $\dim S_x = 2n$. Otherwise, the space-time point is called *singular*.

This proves linearity of $P(x, y)$. In order to verify $P(x, y) \in \mathrm{L}(V)$, it remains to confirm continuity of $P(x, y)$. Since $P(x, y)$ is linear, by lemma 3.1 it is enough to prove boundedness with respect to the operator norm on $\mathrm{L}(V)$: To this aim, we consider the components of $P(x, y)$: Since

$$P(x,y)\, e_j = \begin{pmatrix} P(x,y)_{1,1} & \dots & P(x,y)_{1,2n} \\ \vdots & \ddots & \vdots \\ P(x,y)_{2n,1} & \dots & P(x,y)_{2n,2n} \end{pmatrix} e_j = \begin{pmatrix} \sum_{k=1}^{2n} P(x,y)_{1,k}\,(e_j)_k \\ \vdots \\ \sum_{k=1}^{2n} P(x,y)_{2n,k}\,(e_j)_k \end{pmatrix} = \begin{pmatrix} P(x,y)_{1,j} \\ \vdots \\ P(x,y)_{2n,j} \end{pmatrix}$$

for any $x, y \in \mathscr{G}$, $j = 1, \dots, 2n$, we have

$$P(x,y)_{ij} = \langle e_i \mid \left(P(x,y)_{1,j}, \dots, P(x,y)_{2n,j}\right)^{\mathrm{T}} \rangle = \langle e_i \mid P(x,y)\, e_j \rangle = \prec e_i \mid S\, P(x,y)\, e_j \succ .$$

Thus we obtain

$$\begin{aligned}
\|P(x,y)\|_{\mathrm{L}(V)} &:= \sup_{\|u\|\le 1} \|P(x,y)\, u\|_V \\
&= \sup_{\|u\|\le 1} \left\| \left(\sum_{k=1}^{2n} P(x,y)_{1,k}\, u_k, \dots, \sum_{k=1}^{2n} P(x,y)_{2n,k}\, u_k \right)^{\mathrm{T}} \right\|_V \\
&= \sup_{\|u\|\le 1} \left\| \sum_{k=1}^{2n} \left(P(x,y)_{1,k}\, u_k, \dots, P(x,y)_{2n,k}\, u_k\right)^{\mathrm{T}} \right\|_V \\
&\le \sup_{\|u\|\le 1} \sum_{k=1}^{2n} \left\| \left(P(x,y)_{1,k}\, u_k, \dots, P(x,y)_{2n,k}\, u_k\right)^{\mathrm{T}} \right\|_V \\
&= \sup_{\|u\|\le 1} \sum_{k=1}^{2n} \left\| \sum_{i=1}^{2n} P(x,y)_{i,k}\, u_k\, e_i \right\|_V \\
&\le \sup_{\|u\|\le 1} \sum_{k=1}^{2n} \sum_{i=1}^{2n} \left\| P(x,y)_{i,k}\, u_k\, e_i \right\|_V \\
&\le \sup_{\|u\|\le 1} \sum_{k=1}^{2n} \sum_{i=1}^{2n} \left| P(x,y)_{i,k} \right| \, |u_k| \, \|e_i\|_V \\
&\le \sum_{k=1}^{2n} \sum_{i=1}^{2n} \left| P(x,y)_{i,k} \right|
\end{aligned}$$

for all $x, y \in \mathscr{G}$. This proves the estimation

$$\|P(x,y)\|_{\mathrm{L}(V)} \le \sum_{k=1}^{2n} \sum_{i=1}^{2n} \left| P(x,y)_{i,k} \right| . \tag{3.5}$$

Hence it is enough to prove boundedness of all components $P(x, y)_{i,k}$:

$$\left| P(x,y)_{i,k} \right| = \left| \prec e_k \mid S\, P(x,y)\, e_i \succ \right|$$

$$
\begin{aligned}
&= \left| < S\, e_k \mid \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d}\nu(k)\, e_i > \right| \\
&= \left| \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d} < S\, e_k \mid \nu(k)\, e_i > \right| \\
&= \left| \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d} < s_k\, e_k \mid \nu(k)\, e_i > \right| \\
&= |s_k| \left| \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d} < e_k \mid \nu(k)\, e_i > \right| \\
&\leq \| < e_k \mid \nu(\cdot)\, e_i > \| (\mathscr{G}^*),
\end{aligned}
$$

where $\| < e_k \mid \nu(\cdot)\, e_i > \| (\mathscr{G}^*)$ denotes the total variation of the complex measure $\mathrm{d} < e_k \mid \nu\, e_i >$ for all $i, k = 1, \ldots, 2n$. Since the total variation of any bounded complex measure is finite, we obtain

$$
\left| P(x,y)_{i,k} \right| \leq \| < e_k \mid \nu(\cdot)\, e_i > \| (\mathscr{G}^*) < \infty \tag{3.6}
$$

for all $i, k = 1, \ldots, 2n$. This proves boundedness of the linear operator $P(x,y)$ and hence continuity, i. e. $P(x,y) \in \mathrm{L}(V)$ for all $x, y \in \mathscr{G}$. Additionally, the kernel of the fermionic projector is well-defined. □

Naturally, the *kernel* of the fermionic projector motivates the following definition:

Definition 3.4 (**Fermionic projector**).
Let $\mathscr{K}_0$ be the set of continuous wave functions with compact support. Then, we define the *fermionic projector* by

$$
P \colon \mathscr{K}_0 \to \mathscr{K}, \qquad (P\phi)(x) := \int_{\mathscr{G}} P(x,y)\, \phi(y)\, \mathrm{d}\mu(y), \tag{3.7}
$$

where μ denotes the Haar measure on $\mathscr{G}$.

Since $P(x,y) \colon V \to V$ and $\phi(y) \in V$, we obtain $P(x,y)\, \phi(y) \in V$ for all $x, y \in \mathscr{G}$. Hence, $(P\phi)(x) \in V$ and $P\phi$ is again a wave function. For this reason, the above integral is to be considered as a Bochner integral. Since V is a finite-dimensional vector space endowed with a norm, it is a Banach space according to theorem A.14. However, we have to show that $\mathscr{G} \ni y \mapsto P(x,y)\, \phi(y) \in V$ is indeed Bochner integrable. For this reason, consider

$$
f \colon \mathscr{G} \to V, \qquad f(y) := P(x,y)\, \phi(y) \in V.
$$

By theorem A.21, we have to prove that f is μ-measurable and $\int_{\mathscr{G}} \|f\|\, \mathrm{d}\mu < \infty$. Using the above estimations (3.5) and (3.6), we obtain for all $x, y \in \mathscr{G}$

$$
\sup_{y \in K} \|P(x,y)\|_{\mathrm{L}(V)} \leq \sup_{y \in K} \sum_{i,k=1}^{2n} \left| P(x,y)_{i,k} \right| \leq \sup_{y \in K} \sum_{i,k=1}^{2n} \| < e_k \mid \nu(\cdot)\, e_i > \| (\mathscr{G}^*) < \infty. \tag{3.8}
$$

Hence, $P(x,y)$ is uniformly bounded by a positive constant $C = C(\nu)$. This yields

$$\begin{aligned}\int_{\mathscr{G}} \|P(x,y)\,\phi(y)\|_V \,\mathrm{d}\mu(y) &\le \int_{\mathscr{G}} \underbrace{\|P(x,y)\|_{\mathrm{L}(V)}}_{\le\, C(\nu)\,<\,\infty} \|\phi(y)\|_V \,\mathrm{d}\mu(y) \\ &\le C(\nu) \sup_{y \in K} \|\phi(y)\|_V \,\mu(K) < \infty,\end{aligned} \tag{3.9}$$

where K denotes the compact support of ϕ. Hence, $\int_{\mathscr{G}} \|f\|\,\mathrm{d}\mu < \infty$. It remains to prove μ-measurability of f. A function f with values in a Banach space is said to be μ-measurable if there exists a sequence of finitely-valued functions strongly convergent to f almost everywhere. Since $V \simeq \mathbb{C}^{2n} \simeq \mathbb{R}^{4n}$, consider $f = (f_1, \dots, f_{4n})$ with real valued functions $f_i \colon \mathscr{G} \to \mathbb{R}$ for all $i = 1, \dots, 4n$. Let $y_j \to y$ in $\mathscr{G}$. Then,

$$\begin{aligned}\left\|P(x,y_j) - P(x,y)\right\|_{\mathrm{L}(V)} &= \left\| \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y_j - x)}\,\mathrm{d}\nu(k) - \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\,\mathrm{d}\nu(k) \right\|_{\mathrm{L}(V)} \\ &= \left\| \int_{\mathscr{G}^*} \left(\mathrm{e}^{\mathrm{i}k(y_j - x)} - \mathrm{e}^{\mathrm{i}k(y-x)}\right) \mathrm{d}\nu(k) \right\|_{\mathrm{L}(V)} \\ &\le \sup_{k \in \mathscr{G}^*} \left|\mathrm{e}^{\mathrm{i}k(y_j - x)} - \mathrm{e}^{\mathrm{i}k(y-x)}\right| \|\nu\|(\mathscr{G}^*) \to_{j\to\infty} 0,\end{aligned}$$

since the total variation of bounded measures is finite and $\exp\colon \mathbb{C} \to \mathbb{C}$ is continuous. Hence, $f = P(x,\cdot)\,\phi$ is continuous. In particular, each f_i is continuous for all $i = 1, \dots, 4n$ and thus measurable. According to [Els11, Korollar III.4.14], there is a sequence of step functions $g_j^{(i)}$ such that $g_j^{(i)} \to_{j\to\infty} f_i$ for all $i = 1, \dots, 4n$. Let $g_j = (g_j^{(1)}, \dots, g_j^{(4n)})$. Then, $g_j \colon \mathscr{G} \to V$ is a finitely-valued function with values in a Banach space V such that $g_j \to f$ almost everywhere. Hence, f is μ-measurable, and $\mathscr{G} \ni y \mapsto P(x,y)\,\phi(y) \in V$ is Bochner integrable by theorem A.21. Therefore, the Bochner integral is well-defined by [DU77], and thus, the fermionic projector is well-defined on $\mathscr{K}_0$. Linearity of the Bochner integral and $P(x,y) \in \mathrm{L}(V)$ imply linearity of the fermionic projector P.

3.4. Symmetry of the Fermionic Projector

Next, we want to show that the fermionic projector P is symmetric with respect to the inner product $< \cdot \mid \cdot > \colon \mathscr{K}_0 \times \mathscr{K}_0 \to \mathbb{C}$,

$$< \psi \mid \phi > := \int_{\mathscr{G}} \prec \psi(x) \mid \phi(x) \succ \,\mathrm{d}\mu(x) \qquad (\psi, \phi \in \mathscr{K}_0),$$

i.e. for all $\psi, \phi \in \mathscr{K}_0$ holds

$$< P\psi \mid \phi > = < \psi \mid P\phi > .$$

To this aim and in order to construct the Hilbert space $\mathscr{H}$, we will need the following definition:

Definition 3.5. Let $(V, \prec \cdot \mid \cdot \succ)$ be a finite-dimensional indefinite inner product space and let $A\colon V \to V$ be a linear operator. We say that A is *positive* if

$$\prec u \mid Au \succ \, \geq 0 \qquad \text{for all } u \in V.$$

Furthermore, we define the *adjoint* A^* be the relation

$$\prec u \mid Av \succ \, = \, \prec A^* u \mid v \succ \qquad \text{for all } u, v \in V.$$

A linear operator A is said to be *symmetric* (or *self-adjoint*) if $A^* = A$.

Lemma 3.6. *Let $(V, \prec \cdot \mid \cdot \succ)$ be a finite-dimensional indefinite inner product space and let $A\colon V \to V$ be a positive linear operator. Then A is symmetric.*

Proof. Since A is positive, we have $0 \leq \, \prec u \mid Au \succ \, = \overline{\prec Au \mid u \succ} = \, \prec Au \mid u \succ$ for any $u \in V$ by antisymmetry of an indefinite inner product. Then,

$$\prec A^* u \mid u \succ \, = \, \prec u \mid Au \succ \, = \overline{\prec Au \mid u \succ} = \, \prec Au \mid u \succ$$

holds for all $u \in V$. Hence $\prec (A - A^*) u \mid u \succ \, = 0$ for all $u \in V$. By representation (3.4) we obtain $\langle S(A - A^*) u \mid u \rangle = 0$ for all $u \in V$. Application of [Lan87, Theorem VII.2.4] implies $A = A^*$. □

Since every positive (or negative, respectively) linear operator is symmetric by lemma 3.6, for all $u, v \in V$, $x, y \in \mathscr{G}$ we obtain

$$\begin{aligned}
\prec u \mid P(x,y)\, v \succ \; &= \; \prec u \mid \int_{\mathscr{G}^*} e^{ik(y-x)}\, d\nu(k)\, v \succ \overset{(A.3)}{:=} \int_{\mathscr{G}^*} e^{ik(y-x)}\, d \prec u \mid \nu(k)\, v \succ \\
&= \overline{\overline{\int_{\mathscr{G}^*} e^{ik(y-x)}\, d \prec u \mid \nu(k)\, v \succ}} = \overline{\int_{\mathscr{G}^*} \overline{e^{ik(y-x)}}\, d\overline{\prec u \mid \nu(k)\, v \succ}} \\
&= \overline{\int_{\mathscr{G}^*} e^{ik(x-y)}\, d \prec \nu(k)\, v \mid u \succ} = \overline{\int_{\mathscr{G}^*} e^{ik(x-y)}\, d \prec v \mid \nu(k)\, u \succ} \\
&\overset{(A.3)}{=} \overline{\prec v \mid \int_{\mathscr{G}^*} e^{ik(x-y)}\, d\nu(k)\, u \succ} = \; \prec \int_{\mathscr{G}^*} e^{ik(x-y)}\, d\nu(k)\, u \mid v \succ \\
&= \; \prec P(y,x)\, u \mid v \succ \; .
\end{aligned} \tag{3.10}$$

This proves $P(x,y)^* = P(y,x)$ with respect to the indefinite inner product $\prec \cdot \mid \cdot \succ$ for all $x, y \in \mathscr{G}$. In calculation (3.10) we made use of the complex conjugation of an integral as shown in section A.6. Integration of (3.10) yields

$$\int_{\mathscr{G}} \int_{\mathscr{G}} d\mu(y)\, d\mu(x) \prec u \mid P(x,y)\, v \succ \, = \int_{\mathscr{G}} \int_{\mathscr{G}} d\mu(x)\, d\mu(y) \prec P(y,x)\, u \mid v \succ$$

for all $u, v \in V$. Moreover, by antisymmetry of the spin scalar product, by its linearity in the second argument, and by Hille's theorem (A.7), we obtain in general

$$\begin{aligned}\int_{\mathscr{G}} \prec \phi(x) \mid u \succ \, \mathrm{d}\mu(x) &= \int_{\mathscr{G}} \overline{\prec u \mid \phi(x) \succ} \, \mathrm{d}\mu(x) = \overline{\int_{\mathscr{G}} \prec u \mid \phi(x) \succ \, \mathrm{d}\mu(x)} \\ &\overset{(A.7)}{=} \overline{\prec u \mid \int_{\mathscr{G}} \phi(x) \, \mathrm{d}\mu(x) \succ} = \prec \int_{\mathscr{G}} \phi(x) \, \mathrm{d}\mu(x) \mid u \succ\end{aligned} \tag{3.11}$$

for all $u \in V$, $\phi \in \mathscr{K}_0$. This allows us to switch the integral unchanged between the outside of the spin scalar product and its first argument, since we may immediately draw the integral into the second argument by Hille's theorem.

In order to show symmetry of the fermionic projector, let $\psi, \phi \in \mathscr{K}_0$. Then, Schwarz inequality implies

$$\begin{aligned}&\int_{\mathscr{G}} \int_{\mathscr{G}} |\prec P(x,y)\, \psi(y) \mid \phi(x) \succ| \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x) \\ &= \int_{\mathscr{G}} \int_{\mathscr{G}} |\langle P(x,y)\, \psi(y) \mid S\, \phi(x) \rangle| \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x) \\ &\leq \sup_{x,y \in \mathscr{G}} \|P(x,y)\|_{\mathrm{L}(V)} \, \|S\|_{\mathrm{L}(V)} \, \|\psi(x)\|_V \, \|\phi(y)\|_V \, \mu(\operatorname{supp} \psi) \, \mu(\operatorname{supp} \phi) < \infty\end{aligned}$$

since $P(x,y)$ is uniformly bounded for all $x, y \in \mathscr{G}$ according to (3.8). This allows us to apply of Fubini's theorem. Thus, calculation (3.10) above and application of Fubini's theorem yield

$$\begin{aligned}< P\psi \mid \phi > &= \int_{\mathscr{G}} \prec (P\psi)(x) \mid \phi(x) \succ \, \mathrm{d}\mu(x) \\ &\overset{(3.3)}{=} \int_{\mathscr{G}} \prec \int_{\mathscr{G}} P(x,y)\, \psi(y) \, \mathrm{d}\mu(y) \mid \phi(x) \succ \, \mathrm{d}\mu(x) \\ &\overset{(3.11)}{=} \int_{\mathscr{G}} \int_{\mathscr{G}} \prec P(x,y)\, \psi(y) \mid \phi(x) \succ \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(x) \\ &\overset{(3.10)}{=} \int_{\mathscr{G}} \int_{\mathscr{G}} \prec \psi(y) \mid P(y,x)\, \phi(x) \succ \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \\ &= \int_{\mathscr{G}} \prec \psi(y) \mid \int_{\mathscr{G}} P(y,x)\, \phi(x) \, \mathrm{d}\mu(x) \succ \mathrm{d}\mu(y) \\ &= \int_{\mathscr{G}} \prec \psi(y) \mid (P\phi)(y) \succ \mathrm{d}\mu(y) \\ &= < \psi \mid P\phi > .\end{aligned}$$

This proves symmetry of the fermionic projector. Furthermore, by calculation (3.10) we even obtain symmetry of the kernel of the fermionic projector $P(x,y)^* =$

$P(y,x)$ for all $x, y \in \mathscr{G}$ in the sense that

$$\prec P(y,x)\, u \mid v \succ \, = \, \prec u \mid P(x,y)\, v \succ$$

holds for all $u, v \in V$.

3.5. A Positive Semi-Definite Sesquilinear Form

Our goal is to reconstruct the corresponding Hilbert space $\mathscr{H}$ endowed with a scalar product $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}$. The basic idea of the procedure is to introduce a positive semi-definite sesquilinear form on a suitable vector space using the inner product $< \cdot \mid \cdot >: \mathscr{K}_0 \times \mathscr{K}_0 \to \mathbb{C}$ defined by (3.2). In a first step we want to demonstrate that $< \cdot \mid P \cdot >$ is a *negative* semi-definite sesquilinear form on $\mathscr{K}_0$. Therefore, we will define the Fourier transform of a wave function. For technical reasons due to semi-linearity of the indefinite inner product $\prec \cdot \mid \cdot \succ$ in the first component, we define the *Fourier transform* of a wave function $\psi \in \mathscr{K}_0$ according to [FR13] by

$$\widehat{\psi}\colon \mathscr{G}^* \to V, \qquad \widehat{\psi}(k) := \int_{\mathscr{G}} \psi(x)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}x. \tag{3.12}$$

The Fourier transform $\widehat{\psi}$ is well-defined since continuous functions with compact support are bounded. Thus, for any $k \in \mathscr{G}^*$ we have

$$\int_{\mathscr{G}} \left\| \psi(x)\, \mathrm{e}^{\mathrm{i}kx} \right\|_V \, \mathrm{d}x \leq \sup_{x \in \mathscr{G}} \|\psi(x)\|_V \, \left| \operatorname{supp} \psi \right| < \infty. \tag{3.13}$$

Hence, the continuous function $\widehat{\psi}$ is Bochner integrable by theorem A.21. We consider $\widehat{\psi}$ as a wave function in momentum space. In terms of the Haar measure μ on $\mathscr{G}$ as a multiple of the Lebesgue measure, i. e. $\mathrm{d}\mu(x) = c_0\, \mathrm{d}x$ with a positive constant c_0, we may rewrite (3.12) as

$$c_0\, \widehat{\psi}(k) = \int_{\mathscr{G}} \psi(x)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}\mu(x) \qquad (\psi \in \mathscr{K}_0).$$

In order to show that $< \cdot \mid P \cdot >$ is a negative semi-definite sesquilinear form on $\mathscr{K}_0$, we consider the following identity for arbitrary $\Omega \in \mathscr{B}(\mathscr{G}^*)$, $x \in \mathscr{G}$, and wave functions $\psi \in \mathscr{K}_0$:

$$(\nu(\Omega)\, \psi(x))_j = \left(\left(\sum_{i=1}^{2n} \nu(\Omega)_{1,i}\, \psi(x)_i, \ldots, \sum_{i=1}^{2n} \nu(k)_{2n,i}\, \psi(x)_i \right)^{\mathrm{T}} \right)_j = \sum_{i=1}^{2n} \nu(\Omega)_{j,i}\, \psi(x)_i.$$

Furthermore, we will make use of the calculation

$$(S\nu(\Omega)\, \psi(y))_j = \left(\left(\sum_{r=1}^{2n} (S\nu(\Omega))_{s,r} \left(\sum_{m=1}^{2n} \psi(y)_m\, e_m \right)_r \right)_{s\,=\,1,\ldots,2n} \right)_j$$

$$= \sum_{r=1}^{2n} (S\nu(\Omega))_{j,r} \left(\sum_{m=1}^{2n} \psi(y)_m \, e_m \right)_r = \sum_{r=1}^{2n} s_j \, \nu(\Omega)_{j,r} \left(\sum_{m=1}^{2n} \psi(y)_m \, e_m \right)_r$$

$$= s_j \sum_{r=1}^{2n} \nu(\Omega)_{j,r} \, \psi(y)_r$$

for all $\psi \in \mathscr{K}_0$, $y \in \mathscr{G}$, $j = 1, \ldots, 2n$, since

$$\begin{pmatrix} s_{1,1} & \ldots & s_{1,2n} \\ \vdots & \ddots & \vdots \\ s_{2n,1} & \ldots & s_{2n,2n} \end{pmatrix} \begin{pmatrix} \nu(\Omega)_{1,1} & \ldots & \nu(\Omega)_{1,2n} \\ \vdots & \ddots & \vdots \\ \nu(\Omega)_{2n,1} & \ldots & \nu(\Omega)_{2n,2n} \end{pmatrix} \begin{pmatrix} \psi(y)_1 \\ \vdots \\ \psi(y)_{2n} \end{pmatrix}$$

$$= \begin{pmatrix} s_{1,1} & \ldots & s_{1,2n} \\ \vdots & \ddots & \vdots \\ s_{2n,1} & \ldots & s_{2n,2n} \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{2n} \nu(\Omega)_{1,j} \, \psi(y)_j \\ \vdots \\ \sum_{j=1}^{2n} \nu(\Omega)_{2n,j} \, \psi(y)_j \end{pmatrix} = \begin{pmatrix} s_{1,1} \sum_{j=1}^{2n} \nu(\Omega)_{1,j} \, \psi(y)_j \\ \vdots \\ s_{2n,2n} \sum_{j=1}^{2n} \nu(\Omega)_{2n,j} \, \psi(y)_j \end{pmatrix}.$$

Considering the general expression

$$\begin{aligned} \prec u \mid \nu(\Omega) \, w \succ &= \sum_{j=1}^{2n} \overline{u}_j \prec e_j \mid \nu(\Omega) \, w \succ = \sum_{j=1}^{2n} \overline{u}_j \, \langle e_j \mid S \, \nu(\Omega) \, w \rangle \\ &= \sum_{i,j=1}^{2n} \overline{u}_j \, \overline{(e_j)}_i \, (S \, \nu(\Omega) \, w)_i = \sum_{j=1}^{2n} \overline{u}_j \, (S \, \nu(\Omega) \, w)_j \\ &= \sum_{j=1}^{2n} \overline{u}_j \, s_j \sum_{r=1}^{2n} \nu(\Omega)_{j,r} \, w_r = \sum_{i,j=1}^{2n} s_j \, \overline{u}_j \, w_i \, \nu(\Omega)_{j,i} \end{aligned}$$

for any $u, w \in V$, $\Omega \in \mathscr{B}(\mathscr{G}^*)$, we obtain the following *definition* of integration:

Definition 3.7. Let $(V, \prec \cdot \mid \cdot \succ)$ be a finite-dimensional indefinite inner product space, and let ν be a negative definite measure on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$. For any $\psi, \phi \in \mathscr{K}_0$, $f \in C_b(\mathscr{G}^*)$, we define integration in the following way:

$$\int_{\mathscr{G}^*} f(k) \, \mathrm{d} \prec \psi(k) \mid \nu(k) \, \phi(k) \succ := \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} f(k) \, \overline{\psi(k)}_j \, \phi(k)_i \, \mathrm{d}\nu(k)_{j,i}. \tag{3.14}$$

Then, (3.14) is obviously well-defined in the common sense since wave functions $\psi, \phi \in C_c(\mathscr{G}; V)$ as well as f are bounded and $\nu_{i,j}$ being a bounded complex measure for every $i, j = 1, \ldots, 2n$. Note that the left-hand side in (3.14) is merely a kind of *notation* for the right-hand side in (3.14) rather than a profound definition. Nevertheless, we thus obtain

$$c_0^2 \int_{\mathscr{G}^*} \mathrm{d} \prec \widehat{\psi}(k) \mid \nu(k) \, \widehat{\phi}(k) \succ$$

$$\overset{(3.14)}{=} c_0^2 \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \overline{\widehat{\psi}(k)}_j \, \widehat{\phi}(k)_i \, \mathrm{d}\nu(k)_{j,i}$$
$$= c_0^2 \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \overline{\left(\int_{\mathscr{G}} \psi(x)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}x\right)}_j \left(\int_{\mathscr{G}} \phi(y)\, \mathrm{e}^{\mathrm{i}ky}\, \mathrm{d}y\right)_i \mathrm{d}\nu(k)_{j,i}$$
$$= \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \overline{\left(\int_{\mathscr{G}} \psi(x)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}\mu(x)\right)}_j \left(\int_{\mathscr{G}} \phi(y)\, \mathrm{e}^{\mathrm{i}ky}\, \mathrm{d}\mu(y)\right)_i \mathrm{d}\nu(k)_{j,i}$$
$$= \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \int_{\mathscr{G}} \overline{\psi(x)}_j\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}\mu(x) \int_{\mathscr{G}} \phi(y)_i\, \mathrm{e}^{\mathrm{i}ky}\, \mathrm{d}\mu(y)\, \mathrm{d}\nu(k)_{j,i}$$

for all $\psi, \phi \in \mathscr{K}_0$, which proves the following expression:

$$\begin{aligned} & c_0^2 \int_{\mathscr{G}^*} \mathrm{d} \prec \widehat{\psi}(k) \mid \nu(k)\, \widehat{\phi}(k) \succ \\ & = \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \int_{\mathscr{G}} \overline{\psi(x)}_j\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}\mu(x) \int_{\mathscr{G}} \phi(y)_i\, \mathrm{e}^{\mathrm{i}ky}\, \mathrm{d}\mu(y)\, \mathrm{d}\nu(k)_{j,i}. \end{aligned} \tag{3.15}$$

In order to achieve a positive semi-definite sesquilinear form on $\mathscr{K}_0$, we prove that $< \cdot \mid P \cdot >$ is negative semi-definite:

Lemma 3.8. *The sesquilinear form $< \cdot \mid P \cdot >: \mathscr{K}_0 \times \mathscr{K}_0 \to \mathbb{C}$ is negative semi-definite.*

Proof. Applying Fubini's theorem A.19, for all $\psi \in \mathscr{K}_0$ we obtain the estimation

$$\begin{aligned} < \psi \mid P\psi > &= \int_{\mathscr{G}} \mathrm{d}\mu(x) \prec \psi(x) \mid (P\psi)(x)\, \mathrm{d}\mu(y) \succ \\ &= \int_{\mathscr{G}} \mathrm{d}\mu(x) \prec \psi(x) \mid \int_{\mathscr{G}} P(x,y)\, \psi(y)\, \mathrm{d}\mu(y) \succ \\ &\overset{(A.7)}{=} \int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \prec \psi(x) \mid P(x,y)\, \psi(y) \succ \\ &= \int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \prec \psi(x) \mid \left(\int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d}\nu(k)\right) \psi(y) \succ \\ &= \int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d} \prec \psi(x) \mid \nu(k)\, \psi(y) \succ \\ &= \int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d}\langle \psi(x) \mid S\nu(k)\, \psi(y) \rangle \\ &= \int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d}\left(\sum_{j=1}^{2n} \overline{\psi(x)}_j\, (S\nu(k)\, \psi(y))_j\right) \\ &= \int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d}\left(\sum_{j=1}^{2n} \overline{\psi(x)}_j \sum_{i=1}^{2n} s_j\, \nu(k)_{j,i}\, \psi(y)_i\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{2n} \int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d}\left(\overline{\psi(x)}_j \sum_{i=1}^{2n} \nu(k)_{j,i}\, s_j\, \psi(y)_i \right) \\
&= \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d}\left(\nu(k)_{j,i}\, \psi(y)_i\, \overline{\psi(x)}_j \right) \\
&= \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \psi(y)_i\, \overline{\psi(x)}_j\, \mathrm{d}\nu(k)_{j,i} \\
&\overset{(A.6)}{=} \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}^*} \mathrm{d}\nu(k)_{j,i} \int_{\mathscr{G}} \mathrm{d}\mu(y)\, \mathrm{e}^{\mathrm{i}k(y-x)}\, \psi(y)_i\, \overline{\psi(x)}_j \\
&\overset{(A.6)}{=} \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \mathrm{d}\nu(k)_{j,i} \int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y)\, \mathrm{e}^{\mathrm{i}ky}\, \mathrm{e}^{-\mathrm{i}kx}\, \psi(y)_i\, \overline{\psi(x)}_j \\
&= \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \left(\int_{\mathscr{G}} \overline{\psi(x)}\, \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}\mu(x) \right)_j \left(\int_{\mathscr{G}} \psi(y)\, \mathrm{e}^{\mathrm{i}ky} \mathrm{d}\mu(y) \right)_i \mathrm{d}\nu(k)_{j,i} \\
&\overset{(3.15)}{=} c_0^2 \int_{\mathscr{G}^*} \mathrm{d} \prec \widehat{\psi}(k) \mid \nu(k)\, \widehat{\psi}(k) \succ \\
&\leq 0.
\end{aligned}
$$

Negativity of the above expression is obtained in the following way: Because of $\widehat{\psi}(k) \in V$ for every $k \in \mathscr{G}^*$, we have

$$
\begin{aligned}
&\int_{\mathscr{G}^*} \mathrm{d} \prec \widehat{\psi}(k) \mid \nu(k)\, \widehat{\psi}(k) \succ \\
&= \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \overline{\left(\int_{\mathscr{G}} \psi(x)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}\mu(x) \right)_j} \left(\int_{\mathscr{G}} \psi(y)\, \mathrm{e}^{\mathrm{i}ky}\, \mathrm{d}\mu(y) \right)_i \mathrm{d}\nu(k)_{j,i} \\
&\leq \sup_{u \in V} \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \overline{u}_j\, u_i\, \mathrm{d}\nu(k)_{j,i} \\
&= \sup_{u \in V} \sum_{i,j=1}^{2n} s_j\, \overline{u}_j\, u_i\, \mathrm{d}\nu(\mathscr{G}^*)_{j,i} \\
&= \sup_{u \in V} \prec u \mid \nu(\mathscr{G}^*)\, u \succ \\
&\leq 0
\end{aligned}
$$

by definition of the negative definite measure ν. This proves the lemma. □

If we can confirm that the application of Fubini's theorem is justified the above calculation yields

$$
\prec \psi \mid P\phi \succ = c_0^2 \int_{\mathscr{G}^*} \mathrm{d} \prec \widehat{\psi}(k) \mid \nu(k)\, \widehat{\phi}(k) \succ \tag{3.16}
$$

for all $\psi, \phi \in \mathscr{K}_0$. The latter integral is to be understood in the sense of (3.14), i. e.

$$\int_{\mathscr{G}^*} \mathrm{d} \prec \widehat{\psi}(k) \mid \nu(k)\widehat{\phi}(k) \succ = \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \overline{\widehat{\psi}(k)}_j \, \widehat{\phi}(k)_i \, \mathrm{d}\nu(k)_{j,i}$$

for any $\psi, \phi \in \mathscr{K}_0$. Since

$$\begin{aligned} \sup_{k \in \mathscr{G}^*} \sum_{i,j=1}^{2n} \left|\overline{\widehat{\psi}(k)}_j\right| \left|\widehat{\phi}(k)_i\right| &= \sup_{k \in \mathscr{G}^*} \sum_{i,j=1}^{2n} \left|\int_{\mathscr{G}} \overline{\psi(x)}_j \, \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}x\right| \left|\int_{\mathscr{G}} \phi(y)_i \, \mathrm{e}^{\mathrm{i}ky} \, \mathrm{d}y\right| \\ &\leq \sum_{i,j=1}^{2n} \sup_{x,y \in \mathscr{G}} \left|\overline{\psi(x)}_j\right| |\phi(y)_i| \left|\operatorname{supp}\psi\right| \left|\operatorname{supp}\psi\right| < \infty, \end{aligned}$$

the above expression (3.16) is well-defined for all $\psi, \phi \in \mathscr{K}_0$.

It remains to show that we may indeed make use of Fubini's theorem for arbitrary $\psi, \phi \in \mathscr{K}_0$ in the following steps:

$$\begin{aligned} &\int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)} \, \phi(y)_i \, \overline{\psi(x)}_j \, \mathrm{d}\nu(k)_{j,i} \\ &\overset{\text{Fubini}}{=} \int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}^*} \mathrm{d}\nu(k)_{j,i} \int_{\mathscr{G}} \mathrm{d}\mu(y) \, \mathrm{e}^{\mathrm{i}k(y-x)} \, \phi(y)_i \, \overline{\psi(x)}_j \\ &\overset{\text{Fubini}}{=} \int_{\mathscr{G}^*} \mathrm{d}\nu(k)_{j,i} \int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \, \mathrm{e}^{\mathrm{i}k(y-x)} \, \phi(y)_i \, \overline{\psi(x)}_j. \end{aligned}$$

By [RS80] it suffices to show that

$$\left|\int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \left|\mathrm{e}^{\mathrm{i}k(y-x)} \, \phi(y)_i \, \overline{\psi(x)}_j\right| \, \mathrm{d}\nu(k)_{j,i}\right| < \infty$$

for all $\psi, \phi \in \mathscr{K}_0$. To this aim, we will split up the integral in the following way:

$$\begin{aligned} &\int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \left|\mathrm{e}^{\mathrm{i}k(y-x)} \, \phi(y)_i \, \overline{\psi(x)}_j\right| \, \mathrm{d}\nu(k)_{j,i} \\ &= \int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \left|\mathrm{e}^{\mathrm{i}k(y-x)} \, \phi(y)_i \, \overline{\psi(x)}_j\right| \, \mathrm{d}\left(\operatorname{Re}\nu(k)_{j,i}\right) \\ &+ \mathrm{i} \int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \left|\mathrm{e}^{\mathrm{i}k(y-x)} \, \phi(y)_i \, \overline{\psi(x)}_j\right| \, \mathrm{d}\left(\operatorname{Im}\nu(k)_{j,i}\right). \end{aligned}$$

Then, for all $\psi, \phi \in \mathscr{K}_0$ we obtain

$$\begin{aligned} &\left|\int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \left|\mathrm{e}^{\mathrm{i}k(y-x)} \, \phi(y)_i \, \overline{\psi(x)}_j\right| \, \mathrm{d}\left(\operatorname{Re}\nu(k)_{j,i}\right)\right| \\ &= \left|\int_{K} \mathrm{d}\mu(x) \int_{K} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \left|\phi(y)_i \, \overline{\psi(x)}_j\right| \, \mathrm{d}\left(\operatorname{Re}\nu(k)_{j,i}\right)\right| \end{aligned}$$

$$
\begin{aligned}
&\le \sup_{y\in\mathscr{G}} |\phi(y)_i| \sup_{x\in\mathscr{G}} \left|\overline{\psi(x)}_j\right| \left| \int_{\operatorname{supp}\psi} \mathrm{d}\mu(x) \int_{\operatorname{supp}\phi} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \mathrm{d}\left(\operatorname{Re}\nu(k)_{j,i}\right) \right| \\
&= \sup_{y\in\mathscr{G}} |\phi(y)_i| \sup_{x\in\mathscr{G}} \left|\overline{\psi(x)}_j\right| \mu(\operatorname{supp}\psi)\, \mu(\operatorname{supp}\phi) \left| \left(\operatorname{Re}\nu(\mathscr{G}^*)_{j,i}\right) \right| \\
&< \infty
\end{aligned}
$$

since a Haar measure of a compact set is finite by theorem A.10, $\operatorname{Re}\nu_{j,i}$ is a finite real measure, and continuous functions with compact support are bounded. Analogously, we obtain the corresponding result for the imaginary part and thus we achieve

$$
\left| \int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \left| \mathrm{e}^{\mathrm{i}k(y-x)}\, \phi(y)_i\, \overline{\psi(x)}_j \right| \, \mathrm{d}\nu(k)_{j,i} \right| < \infty
$$

for all $\psi, \phi \in \mathscr{K}_0$. Hence, we may apply Fubini's theorem in lemma 3.8. Therefore, $< \cdot \mid (-P) \cdot >$ is a positive semi-definite sesquilinear form on $\mathscr{K}_0$ which will play a crucial role in order to obtain a scalar product and thus a Hilbert space. This will be shown in the next section.

3.6. Construction of the Hilbert Space

In order to construct a Hilbert space $\mathscr{H}$ endowed with a corresponding scalar product $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}$, we proceed as follows: Let $\mathscr{H}_0 := P(\mathscr{K}_0)$. First, we claim that $\mathscr{H}_0$ is a complex vector space. Consider $P\psi, P\phi \in \mathscr{H}_0$, i. e. $\psi, \phi \in \mathscr{K}_0$. Since $\mathscr{K}_0$ is a vector space, $\psi + \phi \in \mathscr{K}_0$. By linearity of the fermionic projector, we have $P\psi + P\phi = P(\psi + \phi) \in \mathscr{H}_0$. Defining

$$
(P\psi + P\phi)(x) := (P\psi)(x) + (P\phi)(x), \qquad (\lambda P\psi)(x) := \lambda (P\psi)(x)
$$

for all $x \in \mathscr{G}$, $\lambda \in \mathbb{C}$, the vector space axioms are obviously satisfied. Hence, $\mathscr{H}_0$ is a vector space.

Thus by $\langle \cdot \mid \cdot \rangle_{\mathscr{H}_0} \colon \mathscr{H}_0 \times \mathscr{H}_0 \to \mathbb{C}$,

$$
\langle P\psi \mid P\phi \rangle_{\mathscr{H}_0} := < \psi \mid (-P)\, \phi > \qquad (\psi, \phi \in \mathscr{K}_0) \tag{3.17}
$$

we may introduce a positive semi-definite sesquilinear form on $\mathscr{H}_0$ in accordance with calculation (3.16) above. Using the definition of the fermionic projector we may rewrite this positive semi-definite sesquilinear form also as

$$
\langle P\psi \mid P\phi \rangle_{\mathscr{H}_0} \overset{(3.17)}{=} < \psi \mid (-P)\, \phi >
$$

$$\stackrel{(3.2)}{=} \int_{\mathscr{G}} \prec \psi(x) \mid (-P\phi)(x) \succ \, \mathrm{d}\mu(x)$$
$$\stackrel{(3.7)}{=} -\int_{\mathscr{G}} \mathrm{d}\mu(x) \int_{\mathscr{G}} \mathrm{d}\mu(y) \prec \psi(x) \mid P(x,y)\,\phi(y) \succ$$

for any $\phi, \psi \in \mathscr{K}_0$, respectively. The idea in order to obtain a Hilbert space $\mathscr{H}$ is to choose a suitable subspace of $\mathscr{H}_0$ on which the sesquilinear form (3.17) is *positive definite*. For this reason, we proceed as follows:

Consider the *nullspace* $\mathscr{N}$ of $\mathscr{H}_0$ which is given by

$$\mathscr{N} = \{\Psi \in \mathscr{H}_0 \colon \langle \Psi \mid \Psi \rangle_{\mathscr{H}_0} = 0\},$$

or, equivalently,

$$\mathscr{N} = \{P\psi \colon \psi \in \mathscr{K}_0 \text{ with } \langle P\psi \mid P\psi \rangle_{\mathscr{H}_0} = < \psi \mid (-P)\,\psi > = 0\}.$$

The idea is to obtain a suitable subspace of $\mathscr{H}_0$ by dividing out the nullspace. To this aim, we have to show that $\mathscr{N}$ is a vector space and thus a subspace of $\mathscr{H}_0$.

Lemma 3.9. *The nullspace $\mathscr{N}$ is a subspace of $\mathscr{H}_0$.*

Proof. In order to prove this lemma, we have to verify the common vector space axioms. The most difficult step is to prove closure of the nullspace in the sense that the sum of any two elements of $\mathscr{N}$ is again an element of the nullspace. The other axioms follow easily from the vector space structure of $\mathscr{H}_0$. Considering $P\psi, P\phi \in \mathscr{N}$, we have to show that $P\psi + P\phi \in \mathscr{N}$, i. e.

$$< (\psi + \phi) \mid (-P)(\psi + \phi) > = \langle P(\psi + \phi) \mid P(\psi + \phi) \rangle_{\mathscr{H}_0} = 0.$$

We know that $< \psi \mid (-P)\,\psi > = 0$ and $< \phi \mid (-P)\,\phi > = 0$. Since $\mathscr{K}_0$ is a vector space, we have $\psi \pm \phi \in \mathscr{K}_0$ for all $\psi, \phi \in \mathscr{K}_0$. Using that $< \cdot \mid (-P) \cdot >$ is a positive semi-definite sesquilinear form on $\mathscr{K}_0$ by (3.16), for all $P\psi, P\phi \in \mathscr{N}$ holds

$$\begin{aligned}
0 \;&\leq\; < (\psi + \phi) \mid (-P)\,(\psi + \phi) > \\
&\stackrel{(3.2)}{=} \int_{\mathscr{G}} \prec (\psi + \phi)(x) \mid (-P)\,(\psi + \phi)(x) \succ \, \mathrm{d}\mu(x) \\
&= \int_{\mathscr{G}} \prec \psi(x) \mid (-P\psi)\,(x) \succ \, \mathrm{d}\mu(x) + \int_{\mathscr{G}} \prec \psi(x) \mid (-P\phi)\,(x) \succ \, \mathrm{d}\mu(x) \\
&\quad + \int_{\mathscr{G}} \prec \phi(x) \mid (-P\psi)\,(x) \succ \, \mathrm{d}\mu(x) + \int_{\mathscr{G}} \prec \phi(x) \mid (-P\phi)\,(x) \succ \, \mathrm{d}\mu(x) \\
&= \underbrace{< \psi \mid (-P)\,\psi >}_{=0} + \int_{\mathscr{G}} \prec \psi(x) \mid (-P\phi)\,(x) \succ \, \mathrm{d}\mu(x)
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{\mathscr{G}} \prec \phi(x) \mid (-P\psi)(x) \succ \, \mathrm{d}\mu(x) + \underbrace{< \phi \mid (-P)\,\phi >}_{=0} \\
&= \int_{\mathscr{G}} \prec \psi(x) \mid (-P\phi)(x) \succ \, \mathrm{d}\mu(x) + \int_{\mathscr{G}} \prec \phi(x) \mid (-P\psi)(x) \succ \, \mathrm{d}\mu(x) \\
&= \; < \psi \mid (-P)\,\phi > \; + \; < \phi \mid (-P)\,\psi > .
\end{aligned}
$$

Analogously, for all $P\psi,\, P\phi \in \mathscr{N}$ we obtain

$$
0 \le < (\psi - \phi) \mid (-P)\,(\psi - \phi) > = -\,(< \psi \mid (-P)\,\phi > \; + \; < \phi \mid (-P)\,\psi >)
$$

and hence $< \psi \mid (-P)\,\phi > + < \phi \mid (-P)\,\psi > \le 0$. We conclude that

$$
< \psi \mid (-P)\,\phi > + < \phi \mid (-P)\,\psi > = 0
$$

holds for all $P\psi,\, P\phi \in \mathscr{N}$. This proves

$$
\begin{aligned}
0 &= < \psi \mid (-P)\,\phi > + < \phi \mid (-P)\,\psi > \\
&= < \psi \mid (-P)\,\phi > + < \phi \mid (-P)\,\psi > + < \psi \mid (-P)\,\psi > + < \phi \mid (-P)\,\phi > \\
&= < (\psi + \phi) \mid (-P)\,(\psi + \phi) >
\end{aligned}
$$

for all $P\psi,\, P\phi \in \mathscr{N}$ and thus $P\psi + P\phi \in \mathscr{N}$. Since the remaining vector space axioms are easy to see, the nullspace $\mathscr{N}$ is a complex vector space. In particular, it is a subspace of $\mathscr{H}_0$. □

The previous lemma shows that $\mathscr{N}$ is a subspace of $\mathscr{H}_0$. Therefore, $\mathscr{H}_0/\mathscr{N}$ again is a vector space on which the sesquilinear form $< \cdot \mid (-P) \cdot >$ is *positive definite.* Antisymmetry of the spin scalar product implies that the sesquilinear form

$$
\langle \cdot \mid \cdot \rangle_{\mathscr{I}} := \langle \cdot \mid \cdot \rangle_{\mathscr{H}_0} \big|_{\mathscr{I} \times \mathscr{I}}
$$

is even a *scalar product* on $\mathscr{I} := \mathscr{H}_0/\mathscr{N}$:

Lemma 3.10. *Let* $\mathscr{I} := \mathscr{H}_0/\mathscr{N}$. *Then,*

$$
\langle \cdot \mid \cdot \rangle_{\mathscr{I}} : \mathscr{I} \times \mathscr{I} \to \mathbb{C}, \qquad \langle P\psi \mid P\phi \rangle_{\mathscr{I}} := < \psi \mid (-P)\,\phi >
$$

defines a scalar product *for any* $P\psi, P\phi \in \mathscr{I}$.

Proof. The sesquilinear form $\langle \cdot \mid \cdot \rangle_{\mathscr{I}}$ defines a scalar product on $\mathscr{I}$ if the following axioms are satisfied:

i) Linearity in the second argument,

ii) antisymmetry,[2]

iii) positive definiteness.

We shall prove these axioms of a scalar product in the following:

i) Linearity in the second argument follows immediately from linearity of the indefinite inner product in the second argument and from linearity of the integral.

ii) We use antisymmetry of the indefinite inner product and symmetry of the fermionic projector with respect to the inner product to show antisymmetry of the positive definite sesquilinear form: For all $P\psi, P\phi \in \mathscr{I}$ holds

$$\begin{aligned}\langle P\psi \mid P\phi\rangle_{\mathscr{I}} &= <\psi \mid (-P)\,\phi> \\ &= \int_{\mathscr{G}} \prec \psi(x) \mid (-P\phi)\,(x) \succ \,\mathrm{d}\mu(x) \\ &= \int_{\mathscr{G}} \overline{\prec (-P\phi)\,(x) \mid \psi(x) \succ}\,\mathrm{d}\mu(x) \\ &= \overline{\int_{\mathscr{G}} \prec (-P\phi)\,(x) \mid \psi(x) \succ \,\mathrm{d}\mu(x)} \\ &= \overline{<(-P)\,\phi \mid \psi>} \\ &= \overline{<\phi \mid (-P)\,\psi>} \\ &= \overline{\langle P\phi \mid P\psi\rangle}_{\mathscr{I}},\end{aligned}$$

yielding antisymmetry.

iii) We have $\langle P\psi \mid P\psi\rangle_{\mathscr{H}_0} = <\psi \mid (-P)\,\psi> \geq 0$ for all $P\psi \in \mathscr{H}_0$, and by dividing out the nullspace $\mathscr{N}$, $\langle P\psi \mid P\psi\rangle_{\mathscr{I}} = <\psi \mid (-P)\,\psi> = 0$ holds for $P\psi \in \mathscr{I}$ if and only if $\psi = 0$. This proves positive definiteness.

Hence, $\langle\cdot \mid \cdot\rangle_{\mathscr{I}}$ is a scalar product on the space $\mathscr{H}_0/\mathscr{N}$. □

In order to obtain the corresponding Hilbert space $\mathscr{H}$, we will finally need the following statement:

[2]Remark: Throughout this work, we denote by antisymmetry the property $\prec u \mid v \succ = \overline{\prec v \mid u \succ}$ or $\langle u \mid v\rangle = \overline{\langle v \mid u\rangle}$ for any $u, v \in V$, respectively, according e.g. to [GLR05]. Note that physicists do have a different understanding of antisymmetry.

Theorem 3.11. *Let* $(H, \langle \cdot \mid \cdot \rangle_H)$ *be a (not necessarily complete) inner product space. Then, there is a Hilbert space* $\tilde{H}$ *such that* H *is a dense subspace of* $\tilde{H}$*, and the inner product of* H *is induced by the inner product of* $\tilde{H}$*. Moreover, the Hilbert space* $\tilde{H}$ *is unique up to isomorphisms.*

Proof. See [Heu82, Satz 41.5]. □

Since $(\mathscr{I}, \langle \cdot \mid \cdot \rangle_{\mathscr{I}})$ is an inner product space, we may take the completion of $\mathscr{I}$. According to theorem 3.11, we obtain a Hilbert space which shall be denoted by $\mathscr{H} := \overline{\mathscr{I}}$, where the corresponding scalar product is given by $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}$. In particular, $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}\big|_{\mathscr{I} \times \mathscr{I}} = \langle \cdot \mid \cdot \rangle_{\mathscr{I}}$. Thus we obtain the Hilbert space $\mathscr{H}$ of our causal fermion system. To summarize our procedure: Defining a positive definite sesquilinear form on a suitable space $\mathscr{H}_0$, dividing out the nullspace, and taking the completion gives a Hilbert space $(\mathscr{H}, \langle \cdot \mid \cdot \rangle_{\mathscr{H}})$ which is unique up to isomorphisms.[3] The next step in order to obtain a causal fermion system is to reconstruct the universal measure ρ.

3.7. Local Correlation Matrices

The aim of the chapter is to reconstruct the original causal fermion system $(\mathscr{H}, \mathscr{F}, \rho)$. As shown above, we may define a positive semi-definite sesquilinear form on a suitable space $\mathscr{H}_0$. Thus, we obtain a Hilbert space $(\mathscr{H}, \langle \cdot \mid \cdot \rangle_{\mathscr{H}})$. Next, we introduce $\mathscr{F} \subset \mathrm{L}(\mathscr{H})$ as the set of all self-adjoint linear operators on $\mathscr{H}$ of finite rank which have at most n positive and at most n negative eigenvalues. In the following, let $\mathscr{I} = \mathscr{H}_0 / \mathscr{N}$ again denote the dense subspace of $\mathscr{H}$. In order to reconstruct the universal measure ρ, we define *local correlation matrices* $F \colon \mathscr{G} \to \mathrm{L}(\mathscr{I})$,

$$\langle \psi \mid F(x)\, \phi \rangle_{\mathscr{H}} := \prec \psi(x) \mid \phi(x) \succ \qquad (\psi, \phi \in \mathscr{I}). \tag{3.18}$$

For $\psi, \phi \in \mathscr{I}$, these operators obviously are well-defined. Moreover, $F(x)$ is self-adjoint with respect to $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}$ by theorem A.25 since

$$\begin{aligned} \langle \psi \mid F(x)\, \phi \rangle_{\mathscr{H}} &= \prec \psi(x) \mid \phi(x) \succ = \overline{\prec \phi(x) \mid \psi(x) \succ} = \overline{\langle \phi \mid F(x)\, \psi \rangle}_{\mathscr{H}} \\ &= \langle F(x)\, \psi \mid \phi \rangle_{\mathscr{H}} \end{aligned}$$

[3] Alternatively, one could consider the positive semi-definite sesquilinear form $\prec \cdot \mid (-P) \cdot \succ$ on $\mathscr{K}_0$. Dividing out the nullspace $\mathscr{N}_0 := \{\psi \in \mathscr{K}_0 \colon \prec \psi \mid (-P)\, \psi \succ = 0\}$, we obtain $\mathscr{I}_0 := \mathscr{K}_0 / \mathscr{N}_0$, on which $\langle \cdot \mid \cdot \rangle_{\mathscr{I}_0} := \prec \cdot \mid (-P) \cdot \succ$ defines a scalar product. In analogy to the above, we obtain a Hilbert space $\mathscr{H} := \overline{\mathscr{K}_0 / \mathscr{N}_0}$ with a scalar product $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}$. Note the similarity to the equivalent construction of the Hilbert space $\mathscr{H}$ as presented in chapter 4.

for any $\phi, \psi \in \mathscr{I}$. However, we would like F to be a mapping from $\mathscr{G}$ to $\mathscr{F}$ such that $F(x) \in \mathrm{L}(\mathscr{H})$ operates on the whole Hilbert space $\mathscr{H}$ instead of its dense subspace $\mathscr{I}$ only. To this aim, we want to define a continuation of the operator F by $F_f : \mathscr{G} \to \mathrm{L}(\mathscr{H})$. In a second step we prove $F_f(x) \in \mathscr{F}$ for all $x \in \mathscr{G}$. First, we make use of the following definition of a densely defined operator:

Definition 3.12. If X_0 is a dense subspace of a normed vector space X, then $T \in \mathrm{L}(X_0, Y)$ is called *densely defined* in X. We refer to $T_f \in \mathrm{L}(X, Y)$ with $T_f x = Tx$ for all $x \in X_0$ as a *continuation* of T on X. See [Bal09].

From definition 3.12 we may deduce the following lemma:

Lemma 3.13. *Let Y be a Banach space, and let T be densely defined in X. Then, there is a unique continuation T_f of T on X, and it holds* $\|T_f\| = \|T\|$.

Proof. See [Bal09]. □

Remark 3.14. Let X be a normed vector space and $T \in \mathrm{L}(X)$. Let Y be the completion of X. Then, Y is a Banach space containing a dense subspace which is isometrically isomorphic to X and which we identify with X. Moreover, T is densely defined in Y and may be continued on the whole space Y. See [Bal09].

Then, by the following theorem we obtain the desired result:

Theorem 3.15. *Let $F : \mathscr{G} \to \mathrm{L}(\mathscr{I})$ be defined by* (3.18). *Then, there is a unique continuation $F_f : \mathscr{G} \to \mathscr{F}$ such that $F_f(x)\big|_{\mathscr{I}} = F(x)$ for all $x \in \mathscr{G}$.*

Proof. By definition of the Hilbert space $\mathscr{H}$, $\mathscr{H}_0/\mathscr{N}$ is a dense subspace of $\mathscr{H}$. Let $X_0 := \mathscr{H}_0/\mathscr{N}$, $Y := \mathscr{H}$, $X := \mathscr{H}$, and consider $F(x) \in \mathrm{L}(X_0, Y)$ for an arbitrary $x \in \mathscr{G}$. Applying lemma 3.13, there is a unique continuation

$$F_f(x) \in \mathrm{L}(X, Y) = \mathrm{L}(\mathscr{H}) \qquad (x \in \mathscr{G}).$$

In order to prove self-adjointness of $F_f(x)$, i. e.

$$\langle F_f(x)\,\phi \mid \psi\rangle_{\mathscr{H}} = \langle \phi \mid F_f(x)\,\psi\rangle_{\mathscr{H}} \qquad (\phi, \psi \in \mathscr{H}),$$

we want to show that

$$\langle F_f(x)\,\psi \mid \psi\rangle_{\mathscr{H}} = \langle \psi \mid F_f(x)\,\psi\rangle_{\mathscr{H}} \qquad (\psi \in \mathscr{H})$$

holds, implying $\langle F_f(x)\,\psi \mid \psi\rangle_{\mathscr{H}} = \overline{\langle F_f(x)\,\psi \mid \psi\rangle_{\mathscr{H}}} \in \mathbb{R}$ for all $\psi \in \mathscr{H}$. Then, we may apply [Wer11, Satz V.5.6] in order to obtain self-adjointness of $F_f(x)$. Since the operator $F_f(x)\big|_{\mathscr{I}} = F(x)$ is self-adjoint on $\mathscr{I} := \mathscr{H}_0/\mathscr{N}$, it suffices to consider an arbitrary $\psi \in \mathscr{H} \setminus \mathscr{I}$. Since $\mathscr{I}$ is a dense subspace of $\mathscr{H}$, there is a sequence $(\psi_k)_{k\in\mathbb{N}}$ in $\mathscr{I}$ such that $\psi_k \to_{k\to\infty} \psi$ in $\mathscr{H}$. Thus by Schwarz inequality we obtain

$$
\begin{aligned}
&\left|\langle F_f(x)\,\psi \mid \psi\rangle_{\mathscr{H}} - \langle \psi \mid F_f(x)\,\psi\rangle_{\mathscr{H}}\right| \\
&\le \left|\langle F_f(x)\,\psi \mid \psi\rangle_{\mathscr{H}} - \langle F_f(x)\,\psi_k \mid \psi\rangle_{\mathscr{H}}\right| + \left|\langle F_f(x)\,\psi_k \mid \psi\rangle_{\mathscr{H}} - \langle \psi \mid F_f(x)\,\psi_k\rangle_{\mathscr{H}}\right| \\
&\qquad + \left|\langle \psi \mid F_f(x)\,\psi_k\rangle_{\mathscr{H}} - \langle \psi \mid F_f(x)\,\psi\rangle_{\mathscr{H}}\right| \\
&\le\; 2\left(\left\|F_f(x)\right\|_{\mathrm{L}(\mathscr{H})}\, \|\psi_k - \psi\|_{\mathscr{H}}\, \|\psi\|_{\mathscr{H}}\right) \\
&\qquad + \left|\langle F_f(x)\,\psi_k \mid \psi\rangle_{\mathscr{H}} - \langle F_f(x)\,\psi_k \mid \psi_k\rangle_{\mathscr{H}} + \langle F_f(x)\,\psi_k \mid \psi_k\rangle_{\mathscr{H}} - \langle \psi \mid F_f(x)\,\psi_k\rangle_{\mathscr{H}}\right| \\
&\le\; 2\left(\left\|F_f(x)\right\|_{\mathrm{L}(\mathscr{H})}\, \|\psi_k - \psi\|_{\mathscr{H}}\, \|\psi\|_{\mathscr{H}}\right) \\
&\qquad + \left|\langle F_f(x)\,\psi_k \mid \psi - \psi_k\rangle_{\mathscr{H}}\right| + \left|\langle \psi_k - \psi \mid F_f(x)\,\psi_k\rangle_{\mathscr{H}}\right| \\
&\le 2\left(\left\|F_f(x)\right\|_{\mathrm{L}(\mathscr{H})}\, \|\psi_k - \psi\|_{\mathscr{H}}\, \|\psi\|_{\mathscr{H}}\right) + 2\left(\left\|F_f(x)\right\|_{\mathrm{L}(\mathscr{H})}\, \|\psi_k - \psi\|_{\mathscr{H}}\, \|\psi_k\|_{\mathscr{H}}\right) \to_{k\to\infty} 0.
\end{aligned}
$$

Hence, $F_f(x)$ is indeed self-adjoint. By definition of the given spin scalar product $\prec \cdot \mid \cdot \succ$, the signature of $F_f(x)$ is (n,n), that is $F_f(x)$ has at most n positive and at most n negative eigenvalues. This gives a map $F \equiv F_f \colon \mathscr{G} \to \mathscr{F}$. □

Remark 3.16. In the following, $F\colon \mathscr{G} \to \mathscr{F}$ denotes the continuation of the operator defined by (3.18).

Using these local correlation matrices, we may define the universal measure ρ. We say that a subset $\Omega \subset \mathscr{F}$ is measurable if and only if its pre-image $F^{-1}(\Omega)$ is a measurable subset of $\mathscr{G}$. For this reason, the universal measure ρ is defined as usual by

$$\rho := F_*\mu,$$

i. e. $\rho(\Omega) := \mu(F^{-1}(\Omega))$ for all measurable subsets $\Omega \subset \mathscr{F}$. This construction is referred to as the *push-forward measure.* By identifying elements of $\mathscr{G}$ with the same image, we may consider the subset $F(\mathscr{G}) \subset \mathscr{F}$ as our space-time. By straightforward inspection one verifies that performing the above construction and choosing (3.1), one gets back the original causal fermion system $(\mathscr{H}, \mathscr{F}, \rho)$. This is explained in detail in [FGS12], *Causal fermion systems: A quantum space-time emerging from an action principle.* After the reconstruction of our original causal fermion system $(\mathscr{H}, \mathscr{F}, \rho)$, we come to the last part of the second question: Is it possible to achieve the operator-valued spectral measure $\mathrm{d}E_k$? This remaining step will be discussed in the next chapter.

3.8. Symmetry of the Causal Fermion System

The idea of reconstructing the spectral measure $\mathrm{d}E_k$ is to proceed in analogy to chapter 2. To this aim, we first have to prove the existence of a symmetry of the reconstructed causal fermion system $(\mathscr{H}, \mathscr{F}, \rho)$. For this reason, let us recall the definition of a symmetry of a causal fermion system:

Definition 3.17. A symmetry of a causal fermion system is a group $\mathscr{G}$ together with a unitary representation U on $\mathscr{H}$ which leaves the universal measure invariant, i. e.

$$\rho\,(U_g\,\Omega\,U_g^{-1}) = \rho\,(\Omega)$$

for all $g \in \mathscr{G}$ and all measurable $\Omega \subset \mathscr{F}$.

Naturally, we consider the group of translations $\mathscr{G} = (\mathbb{R}^4, +)$. To achieve a symmetry, we have to show the existence of a unitary representation according to definition 2.4. Therefore, the basic idea is to define a unitary representation $U : \mathscr{G} \to \mathrm{L}(\mathscr{H})$ such that $(P\phi)(x+g) = U_{-g}\,(P\phi)(x)$ holds for all $x, g \in \mathscr{G}$, $\phi \in \mathscr{K}_0$. Under this condition, we obtain $F(x+g) = U_g F(x) U_g^{-1}$ for any $x, g \in \mathscr{G}$ and thus $\rho(U_g \Omega U_g^{-1}) = \rho(\Omega)$ for any $g \in \mathscr{G}$, $\Omega \subset \mathscr{F}$ measurable, proving the existence of a symmetry of a causal fermion system. If the group action of $\mathscr{G}$ on space-time M is in addition transitive, we may proceed as in chapter 2 to reconstruct the operator-valued spectral measure E_k.

The main difficulty is that we are given only an arbitrary negative definite measure ν. In order to illustrate a possible procedure *assume* the existence of a spectral measure E such that the kernel of the fermionic projector is given by

$$P(x,y) = \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\,\mathrm{d}E(k)$$

for all $x, y \in \mathscr{G}$. Then, the fermionic projector is represented by

$$(P\phi)(x) = \int_{\mathscr{G}} P(x,y)\,\phi(y)\,\mathrm{d}\mu(y) = \int_{\mathscr{G}} \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\,\mathrm{d}E(k)\,\phi(y)\,\mathrm{d}\mu(y)$$

with $\phi \in \mathscr{K}_0$. Using the fact that the equation

$$\left(\int_{\mathbb{R}^n} f(\lambda)\,\mathrm{d}E_\lambda\right)\left(\int_{\mathbb{R}^n} g(\lambda')\,\mathrm{d}E_{\lambda'}\right) = \int_{\mathbb{R}^n} f(\lambda)\,g(\lambda)\,\mathrm{d}E_\lambda$$

holds for any spectral measure E on $\mathbb{R}^n$ and $f, g \in \mathscr{B}(\mathbb{R}^n)$ by [Fin13, Theorem 7.3.10] for any $n \in \mathbb{N}$, we obtain the following expression:

$$(P\phi)(g+x) = \int_{\mathscr{G}} \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-(x+g))}\,\mathrm{d}E(k)\,\phi(y)\,\mathrm{d}\mu(y)$$

$$= \int_{\mathscr{G}} \int_{\mathscr{G}^*} \mathrm{e}^{-\mathrm{i}kg} \,\mathrm{d}E(k) \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)} \,\mathrm{d}E(k)\, \phi(y) \,\mathrm{d}\mu(y)$$

for all $x, g \in \mathscr{G}$, $\phi \in \mathscr{K}_0$. Then, for any $g \in \mathscr{G}$ we introduce unitary operators $U(g)$ by the mapping

$$\mathscr{G} \ni g \mapsto U(g) \in \mathrm{L}(\mathscr{H}), \qquad U(g) := \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}kg} \,\mathrm{d}E(k).$$

In order to prove that these operators are indeed unitary, we make use of the fact that

$$\left(\int_{\mathbb{R}^n} f(\lambda) \,\mathrm{d}E_\lambda \right)^* = \left(\int_{\mathbb{R}^n} \overline{f(\lambda)} \,\mathrm{d}E_\lambda \right)$$

holds for any $f \in \mathscr{B}(\mathbb{R}^n)$ according to [Fin13, Theorem 7.3.9]. Thus we obtain

$$U_g^* = \left(\int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}kg} \,\mathrm{d}E(k) \right)^* = \int_{\mathscr{G}^*} \overline{\mathrm{e}^{\mathrm{i}kg}} \,\mathrm{d}E(k) = \int_{\mathscr{G}^*} \mathrm{e}^{-\mathrm{i}kg} \,\mathrm{d}E(k) = U_{-g}$$

as well as

$$U_g \, U_{-g} = \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}kg} \,\mathrm{d}E(k) \int_{\mathscr{G}^*} \mathrm{e}^{-\mathrm{i}kg} \,\mathrm{d}E(k) = \int_{\mathscr{G}^*} \mathrm{d}E(k) = \mathrm{Id}$$

for all $g \in \mathscr{G}$, where $U_g := U(g)$. The latter expression holds by [Fin13, Definition 7.3.6] for any operator-valued spectral measure E, since $E_{\mathbb{R}^n} = \mathrm{Id}$. Analogously, $U_{-g}U_g = \mathrm{Id}$, yielding $U_g^* = U_{-g} = U_g^{-1}$ for all $g \in \mathscr{G}$. According to definition A.24, $U(g)$ is unitary for any $g \in \mathscr{G}$. Moreover, $gh = g + h$ implies

$$\begin{aligned} U(gh) &= \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(gh)} \,\mathrm{d}E(k) = \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}kg} \,\mathrm{e}^{\mathrm{i}kh} \,\mathrm{d}E(k) = \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}kg} \,\mathrm{d}E(k) \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k'h} \,\mathrm{d}E(k') \\ &= U(g)U(h) \end{aligned}$$

for any $g, h \in \mathscr{G}$. Hence, U is a group homomorphism and thus a unitary representation. Using this unitary representation, by Hille's theorem (A.7) we obtain

$$\begin{aligned} (P\phi)(g + x) &= \int_{\mathscr{G}} \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-(x+g))} \,\mathrm{d}E(k)\, \phi(y) \,\mathrm{d}\mu(y) \\ &= \int_{\mathscr{G}} \int_{\mathscr{G}^*} \mathrm{e}^{-\mathrm{i}kg} \,\mathrm{d}E(k) \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)} \,\mathrm{d}E(k)\, \phi(y) \,\mathrm{d}\mu(y) \\ &= \int_{\mathscr{G}} U_{-g} \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)} \,\mathrm{d}E(k)\, \phi(y) \,\mathrm{d}\mu(y) \\ &\overset{(A.7)}{=} U_{-g} \int_{\mathscr{G}} \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)} \,\mathrm{d}E(k)\, \phi(y) \,\mathrm{d}\mu(y) \\ &= U_{-g}\, (P\phi)(x) \end{aligned}$$

for any wave function $\phi \in \mathscr{K}_0$, $g, x \in \mathscr{G}$. Hence, for wave functions $\psi, \phi \in \mathscr{H}_0/\mathscr{N}$, $g, x \in \mathscr{G}$, we finally achieve the following calculation:

$$\begin{aligned}\langle \phi \mid F(x+g)\,\psi\rangle &= \prec \phi(x+g) \mid \psi(x+g) \succ = \prec U_{-g}\,\phi(x) \mid U_{-g}\,\psi(x) \succ \\ &= \langle U_{-g}\,\phi \mid F(x)U_{-g}\,\psi\rangle = \langle \phi \mid U_{-g}^{*}F(x)U_{-g}\,\psi\rangle \\ &= \langle \phi \mid U_{-g}^{-1}F(x)U_{g}^{-1}\,\psi\rangle = \langle \phi \mid U_{g}F(x)U_{g}^{-1}\,\psi\rangle.\end{aligned}$$

This proves $F(x+g) = U_g F(x) U_g^{-1}$ for any $x, g \in \mathscr{G}$, implying

$$gx = g + x = F^{-1}(U_g F(x) U_g^{-1})$$

for all $x, g \in \mathscr{G}$. Next, let Ω be a measurable subset of $\mathscr{F}$. Without loss of generality, we may assume that $\Omega \subset F(\mathscr{G})$ (otherwise consider $\Omega \cap F(\mathscr{G})$). Let $A \in F(\mathscr{G})$. Since identifying elements in $\mathscr{G}$ with the same image, there exists a unique x_A such that $F(x_A) = A$. Defining $g\Omega := \{gx\colon x \in \Omega\}$, where $gx = g + x$ for any $g, x \in \mathscr{G}$, we have

$$\bigcup_{A\in\Omega} gx_A = \bigcup_{A\in\Omega}(g + x_A) = g + \bigcup_{A\in\Omega} x_A = g\bigcup_{A\in\Omega} x_A.$$

Then, for any measurable subset $\Omega \subset \mathscr{F}$ and all $g \in \mathscr{G}$ holds

$$\begin{aligned}\rho(U_g\Omega U_g^{-1}) &= \rho(U_g\bigcup_{A\in\Omega} F(x_A)U_g^{-1}) = \rho(\bigcup_{A\in\Omega} U_gF(x_A)U_g^{-1}) = \mu(F^{-1}(\bigcup_{A\in\Omega} U_gF(x_A)U_g^{-1})) \\ &= \mu(\bigcup_{A\in\Omega} F^{-1}(U_gF(x_A)U_g^{-1})) = \mu(\bigcup_{A\in\Omega} gx_A) = \mu(g\bigcup_{A\in\Omega} x_A) \\ &= \mu(\bigcup_{A\in\Omega} x_A) = \mu(\bigcup_{A\in\Omega} F^{-1}(F(x_A))) = \mu(F^{-1}(\bigcup_{A\in\Omega} F(x_A))) \\ &= \rho(\Omega).\end{aligned}$$

Hence, $(\mathscr{G}, U)$ is a symmetry of a causal fermion system. However, the above considerations depend on the representation of the kernel of the fermionic projector in terms of a spectral measure E. For this reason, the task in order to get back the whole operator-valued spectral measure is still to be solved, which may be due to further research concerning this problem.

4. Equivalent Construction of the Hilbert Space

We now come to the question if there are different ways of constructing the Hilbert space giving back the original causal fermion system. The goal of this chapter is to present an equivalent construction of the Hilbert space in order to make the point of construction clearer. Are there any advantages due to this method?

4.1. Preliminaries

The attempt to reconstruct the Hilbert space $\mathscr{H}$ in an alternative way is based on the same assumptions: Consider a finite-dimensional vector space V endowed with a spin scalar product $\prec \cdot \mid \cdot \succ$. Let $\mathscr{G} \simeq \mathbb{R}^4$, and let $\mathscr{G}^* \simeq \mathbb{R}^4$ be its dual. Moreover, assume a negative definite measure $\nu \colon \mathscr{B}(\mathscr{G}^*) \to \mathrm{L}(V)$. The main idea in the new setting is to *start with functions in momentum space*,

$$\chi \colon \mathscr{G}^* \to V.$$

The question arises, what properties these functions χ in momentum space should have in order to obtain an equivalent construction of the Hilbert space. Since we are going to apply Fubini's theorem later on, we need to assume functions $\chi \in \mathscr{H}_0$ for a suitable vector space $\mathscr{H}_0$. As shown above (see section 3.2), we may define continuity and derivatives of wave functions $\psi \colon \mathscr{G} \to V$. Analogously, we can define the space of smooth wave functions with compact support in momentum space, $C_c^\infty(\mathscr{G}^*; V)$. Moreover, we may even define the space of Schwartz functions on $\mathscr{G}^*$ with values in V, denoted by $\mathscr{S}(\mathscr{G}^*; V)$. Usually, complex-valued Schwartz functions are defined by

$$\mathscr{S}(\mathbb{R}^n; \mathbb{C}) := \left\{ f \in C^\infty(\mathbb{R}^n; \mathbb{C}) \colon \lim_{|\alpha| \to \infty} x^\alpha D^\beta f(x) = 0 \;\forall \alpha, \beta \in \mathbb{N}_0^n \right\},$$

see e.g. [Wer11, Definition V.2.3]. Analogously, due to $V \simeq \mathbb{C}^{2n}$ we obtain

$$\mathscr{S}(\mathscr{G}^*; V) \simeq \mathscr{S}(\mathscr{G}^*; \mathbb{C}^{2n}) \simeq \mathscr{S}(\mathscr{G}^*; \mathbb{C})^{2n}.$$

Hence, every $f \in \mathscr{S}(\mathscr{G}^*; V)$ may be considered as $f = (f_1, \dots, f_{2n})$ with components $f_i \in \mathscr{S}(\mathscr{G}^*; \mathbb{C})$ for all $i = 1, \dots, 2n$. Then,

$$C_c^\infty(\mathscr{G}^*; V) \subset \mathscr{S}(\mathscr{G}^*; V).$$

Moreover, denoting the space of Schwartz functions on $\mathscr{G}$ by $\mathscr{S}(\mathscr{G}; V)$, the Fourier transform

$$\mathfrak{F} : \mathscr{S}(\mathscr{G}; V) \to \mathscr{S}(\mathscr{G}^*; V), \qquad f \to \widehat{f} := \mathfrak{F}[f],$$

is an isomorphism according to well-known results. Since $C_c^\infty(\mathscr{G}; V) \subset \mathscr{S}(\mathscr{G}; V)$, we introduce $\mathscr{Z} \subset \mathscr{S}(\mathscr{G}^*; V)$ by

$$\mathscr{Z} := \mathfrak{F}\left(C_c^\infty(\mathscr{G}; V)\right).$$

Moreover, we introduce the set $\mathscr{Q} \subset \mathscr{S}(\mathscr{G}^*; V)$ by

$$\mathscr{Q} = \mathfrak{F}\left(\mathscr{S}(\mathscr{G}; V) \setminus C_c^\infty(\mathscr{G}; V)\right) \cup \{0\}.$$

Then, for all $\chi \in \mathscr{Q}$ holds $\mathfrak{F}^{-1}[\chi] \notin C_c^\infty(\mathscr{G}; V) \setminus \{0\}$. Thus we obtain

$$\mathscr{S}(\mathscr{G}^*; V) = \mathscr{Z} \oplus \mathscr{Q},$$

where $\mathscr{Z}, \mathscr{Q}$ are vector spaces.[1] Hence, $\mathscr{H}_0 := \mathscr{S}(\mathscr{G}^*; V)/\mathscr{Q}$ is also a vector space, and $\mathfrak{F}^{-1}[\chi] \in C_c^\infty(\mathscr{G}; V)$ for all $\chi \in \mathscr{H}_0$. Then, $\mathscr{H}_0$ is the desired suitable vector space which will be considered in the following.

In order to construct a Hilbert space we first introduce a positive semi-definite sesquilinear form on the space $\mathscr{H}_0$ by

$$\langle \chi \mid \tilde{\chi} \rangle_{\mathscr{H}_0} := -c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \chi(k) \mid \nu(k) \tilde{\chi}(k) \succ \qquad (\chi, \tilde{\chi} \in \mathscr{H}_0) \tag{4.1}$$

where c_1 shall be a positive constant.[2] We add the following remark regarding the integral in formula (4.1).

Remark 4.1. Consider the pseudo-orthonormal basis $(e_i)_{i=1,\dots,2n}$ of the indefinite inner product space V which allows the representation

$$\prec u \mid v \succ = \langle u \mid S v \rangle \qquad (u, v \in V)$$

[1] $\mathscr{Z}, \mathscr{Q}$ are vector spaces since $\mathfrak{F}$ is linear and $C_c^\infty(\mathscr{G}; V)$ is a vector space. Let $\mathfrak{F}[\chi], \mathfrak{F}[\tilde{\chi}] \in \mathscr{Q}$. Then, $\mathfrak{F}[\chi] + \mathfrak{F}[\tilde{\chi}] = \mathfrak{F}[\chi + \tilde{\chi}] \in \mathscr{Q}$ because of $\chi + \tilde{\chi} \in \mathscr{S}(\mathscr{G}; V) \setminus C_c^\infty(\mathscr{G}; V) \cup \{0\}$.

[2] Note the analogy of the procedure to section 3.6 as well as the analogy of definition (4.1) to the above result (3.16).

according to (3.4), where $S = \mathbf{diag}(1, \dots, 1, -1, \dots, -1)$, and let $s_j := S_{j,j}$ for all $j = 1, \dots, 2n$. In analogy to definition 3.7, we introduce the following notation of integration with respect to $\prec \cdot \mid \cdot \succ$,

$$\int_{\mathscr{G}^*} f(k)\,\mathrm{d} \prec \chi(k) \mid \nu(k)\tilde{\chi}(k) \succ := \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} f(k)\overline{\chi(k)}_j \tilde{\chi}(k)_i \,\mathrm{d}\nu(k)_{j,i} \tag{4.2}$$

for any $\chi, \tilde{\chi} \in \mathscr{H}_0$, $f \in C_b(\mathscr{G}^*)$. Then, definition (4.2) is in accordance with (3.14). Furthermore, (4.2) is obviously well-defined since components of negative definite measures are bounded complex measures, and Schwartz functions in $\mathscr{H}_0$ are bounded.

Using (4.2), the sesquilinear form (4.1) is indeed positive semi-definite as the following estimation for any $\chi \in \mathscr{H}_0$ shows:

$$\begin{aligned}
\langle \chi \mid \chi \rangle_{\mathscr{H}_0} &\overset{(4.1)}{=} -c_1 \int_{\mathscr{G}} \mathrm{d} \prec \chi(k) \mid \nu(k)\chi(k) \succ = -c_1 \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \overline{\chi(k)}_j \chi(k)_i \,\mathrm{d}\nu(k)_{j,i} \\
&\geq -c_1 \inf_{u \in \mathscr{H}} \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \overline{u}_j\, u_i \,\mathrm{d}\nu(k)_{j,i} = -c_1 \inf_{u \in \mathscr{H}} \sum_{i,j=1}^{2n} s_j\, \overline{u}_j\, u_i\, \nu(\mathscr{G}^*)_{j,i} \\
&= -c_1 \inf_{u \in \mathscr{H}} \prec u \mid \nu(\mathscr{G}^*)\, u \succ = c_1 \inf_{u \in \mathscr{H}} \prec u \mid -\nu(\mathscr{G}^*)\, u \succ \\
&\geq 0.
\end{aligned}$$

Since $\mathscr{H}_0 \subset \mathscr{S}(\mathscr{G}^*; V)$, we may define a function $\chi \colon \mathscr{G} \to V$ as the inverse Fourier transform of $\widehat{\chi} \in \mathscr{H}_0$ by

$$\widehat{\chi}(k) = \int_{\mathscr{G}} \mathrm{e}^{\mathrm{i}kx} \chi(x)\,\mathrm{d}\mu(x),$$

where we use the fact that $\mathfrak{F} : \mathscr{S}(\mathscr{G}; V) \to \mathscr{S}(\mathscr{G}^*; V)$, $f \mapsto \widehat{f} := \mathfrak{F}[f]$ is an isomorphism. This is according to [FR13] and due to linearity of the indefinite inner product in the second argument. Note that $\chi = \mathfrak{F}^{-1}[\widehat{\chi}] \in C_c^\infty(\mathscr{G}; V)$. Continuity and the existence of a compact support imply measurability and boundedness of these functions.

Finally it remains to show that the sesquilinear form (4.1) is well-defined on $\mathscr{H}_0$:

Lemma 4.2. *The sesquilinear form* (4.1) *is well-defined on* $\mathscr{H}_0$.

Proof. Well-definition of the sesquilinear form (4.1) follows easily from the above definition (4.2):

$$\left|\langle \chi \mid \tilde{\chi} \rangle_{\mathscr{H}_0}\right| = \left| -\int_{\mathscr{G}^*} \mathrm{d} \prec \chi(k) \mid \nu(k)\tilde{\chi}(k) \succ \right| = \left| -\sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \overline{\psi(k)}_j\, \phi(k)_i \,\mathrm{d}\nu(k)_{j,i} \right|$$

$$\leq \sum_{i,j=1}^{2n} \left| \int_{\mathscr{G}^*} \overline{\chi(k)}_j \, \tilde{\chi}(k)_i \, \mathrm{d}\nu(k)_{j,i} \right| \leq \sum_{i,j=1}^{2n} \sup_{x,\, y \,\in\, \mathscr{G}^*} \left|\overline{\chi(x)}_j\right| \, |\tilde{\chi}(y)_i| \, \left\|\nu_{j,i}\right\| (\mathscr{G}^*)$$

$$< \infty,$$

since Schwartz functions in $\mathscr{H}_0$ are bounded, and the total variation of bounded measures is finite. Hence, the positive semi-definite sesquilinear form $\langle \cdot \mid \cdot \rangle_{\mathscr{H}_0}$ is well-defined on $\mathscr{H}_0$. □

4.2. Construction of the Hilbert Space

The procedure of constructing the Hilbert space is similar to section 3.6. In order to achieve the Hilbert space $\mathscr{H}$, we consider the *nullspace* $\mathscr{N}$ of $\mathscr{H}_0$, which is given by

$$\mathscr{N} = \{\chi \in \mathscr{H}_0 \colon \langle \chi \mid \chi \rangle_{\mathscr{H}_0} = 0\}\,.$$

Since $\langle \cdot \mid \cdot \rangle_{\mathscr{H}_0}$ is only a positive *semi*-definite sesquilinear form on $\mathscr{H}_0$, the idea is to divide out the nullspace $\mathscr{N}$ in order to obtain a positive *definite* sesquilinear form on $\mathscr{H}_0/\mathscr{N}$.

Lemma 4.3. *The nullspace* $\mathscr{N}$ *is a subspace of* $\mathscr{H}_0$.

Proof. The most difficult step in order to show that $\mathscr{N}$ is a subspace of $\mathscr{H}_0$ is to prove that the sum of any two elements of $\mathscr{N}$ is again an element of $\mathscr{N}$. The other vector space axioms are easy to see. Linearity of the integrator follows according to definition (4.2). On the one hand, for any $\chi, \tilde{\chi} \in \mathscr{N}$ we thus obtain

$$\begin{aligned}
0 &\leq \langle \chi + \tilde{\chi} \mid \chi + \tilde{\chi} \rangle_{\mathscr{H}_0} \\
&= -c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec (\chi + \tilde{\chi})(k) \mid \nu(k)\, (\chi + \tilde{\chi})(k) \succ \\
&= -c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \chi(k) \mid \nu(k) \chi(k) \succ - c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \tilde{\chi}(k) \mid \nu(k) \chi(k) \succ \\
&\quad - c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \chi(k) \mid \nu(k) \tilde{\chi}(k) \succ - c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \tilde{\chi}(k) \mid \nu(k) \tilde{\chi}(k) \succ \\
&= \underbrace{\langle \chi \mid \chi \rangle_{\mathscr{H}_0}}_{=0} - c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \tilde{\chi}(k) \mid \nu(k) \chi(k) \succ \\
&\quad - c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \chi(k) \mid \nu(k) \tilde{\chi}(k) \succ - \underbrace{\langle \tilde{\chi} \mid \tilde{\chi} \rangle_{\mathscr{H}_0}}_{=0} \\
&= -c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \tilde{\chi}(k) \mid \nu(k) \chi(k) \succ - c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \chi(k) \mid \nu(k) \tilde{\chi}(k) \succ\,.
\end{aligned}$$

On the other hand, we have

$$0 \leq \langle \chi - \tilde{\chi} \mid \chi - \tilde{\chi} \rangle_{\mathscr{H}_0} \;=\; c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \tilde{\chi}(k) \mid \nu(k)\chi(k) \succ \; + c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \chi(k) \mid \nu(k)\tilde{\chi}(k) \succ$$

for all $\chi, \tilde{\chi} \in \mathscr{N}$. The latter inequality implies

$$-c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \tilde{\chi}(k) \mid \nu(k)\chi(k) \succ \; - c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \chi(k) \mid \nu(k)\tilde{\chi}(k) \succ \leq 0$$

for all $\chi, \tilde{\chi} \in \mathscr{N}$. Taking both inequalities together, we achieve

$$\langle \chi + \tilde{\chi} \mid \chi + \tilde{\chi} \rangle_{\mathscr{H}_0} = -c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \tilde{\chi}(k) \mid \nu(k)\chi(k) \succ \; - c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \chi(k) \mid \nu(k)\tilde{\chi}(k) \succ = 0,$$

proving $\chi + \tilde{\chi} \in \mathscr{N}$ for all $\chi, \tilde{\chi} \in \mathscr{N}$. This concludes the lemma. □

Hence, $\mathscr{N}$ is a subspace of $\mathscr{H}_0$, and $\mathscr{H}_0/\mathscr{N}$ is again a vector space on which the sesquilinear form $\langle \cdot \mid \cdot \rangle_{\mathscr{H}_0}$ is positive definite. In fact, $\langle \cdot \mid \cdot \rangle_{\mathscr{H}_0}$ forms a scalar product on $\mathscr{H}_0/\mathscr{N}$ as the next lemma shows.

Lemma 4.4. *The positive definite sesquilinear form*

$$\langle \cdot \mid \cdot \rangle_{\mathscr{H}_0} : \mathscr{H}_0/\mathscr{N} \times \mathscr{H}_0/\mathscr{N} \to \mathbb{C}$$

defines a scalar product.

Proof. We have to prove that $\langle \cdot \mid \cdot \rangle_{\mathscr{H}_0}$ satisfies the following axioms:

i) Positivity: After dividing out the nullspace $\mathscr{N}$, the conditions $\langle \chi \mid \chi \rangle_{\mathscr{H}_0} \geq 0$ and $\langle \chi \mid \chi \rangle_{\mathscr{H}_0} = 0 \Leftrightarrow \chi = 0$ obviously hold for any $\chi \in \mathscr{H}_0/\mathscr{N}$.

ii) Antisymmetry: Assume $\Omega \in \mathscr{B}(\mathscr{G}^*)$. Using the fact that

$$\prec e_i \mid \nu(\Omega)\, e_j \succ = \overline{\prec \nu(\Omega)\, e_j \mid e_i \succ} = \overline{\prec e_j \mid \nu(\Omega)\, e_i \succ}$$

where

$$\prec e_i \mid \nu(\Omega)\, e_j \succ = \langle e_i \mid S\,\nu(\Omega)\, e_j \rangle = \langle s_i\, e_i \mid (\nu(\Omega)_{1,j}, \ldots, \nu(\Omega)_{2n,j} \rangle = s_i\, \nu(\Omega)_{i,j},$$

by $s_i^2 = 1$ for any $i = 1, \ldots, 2n$ we obtain

$$\overline{\nu(\Omega)}_{i,j} = s_i\, s_j\, \nu(\Omega)_{j,i} \tag{4.3}$$

for all $\Omega \in \mathscr{B}(\mathscr{G}^*)$, $i, j = 1, \ldots, 2n$. According to definition (4.2), we have

$$\langle \chi \mid \tilde{\chi} \rangle_{\mathscr{H}_0} \;=\; -c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \chi(k) \mid \nu(k)\tilde{\chi}(k) \succ$$

$$
\begin{aligned}
&\overset{(4.2)}{=} -c_1 \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \overline{\chi(k)}_j \tilde{\chi}(k)_i \, \nu(k)_{j,i} \\
&= \overline{-c_1 \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \chi(k)_j \overline{\tilde{\chi}(k)}_i \, \overline{\nu(k)}_{j,i}} \\
&\overset{(4.3)}{=} \overline{-c_1 \sum_{i,j=1}^{2n} s_i \int_{\mathscr{G}^*} \chi(k)_j \overline{\tilde{\chi}(k)}_i \, \nu(k)_{i,j}} \\
&\overset{(4.2)}{=} \overline{-c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \tilde{\chi}(k) \mid \nu(k) \chi(k) \succ} \\
&= \overline{\langle \tilde{\chi} \mid \chi \rangle}_{\mathscr{H}_0}
\end{aligned}
$$

for all $\chi, \tilde{\chi} \in \mathscr{H}_0/\mathscr{N}$, yielding antisymmetry.

iii) Linearity in the second argument: For any $\chi, \tilde{\chi}, \psi \in \mathscr{H}_0/\mathscr{N}$, $\alpha \in \mathbb{C}$ holds

$$
\begin{aligned}
&\langle \chi \mid \alpha\psi + \tilde{\chi} \rangle_{\mathscr{H}_0} \\
&= -c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \chi(k) \mid \nu(k) \, (\alpha\psi(k) + \tilde{\chi}(k)) \succ \\
&= -c_1 \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \overline{\chi(k)}_j \, (\alpha\psi(k) + \tilde{\chi}(k))_i \, \nu(k)_{j,i} \\
&= -c_1 \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \overline{\chi(k)}_j \, (\alpha\psi(k))_i \, \nu(k)_{j,i} - c_1 \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \overline{\chi(k)}_j \tilde{\chi}(k)_i \, \nu(k)_{j,i} \\
&= -\alpha \, c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \chi(k) \mid \nu(k) \, \psi(k) \succ - c_1 \int_{\mathscr{G}^*} \mathrm{d} \prec \chi(k) \mid \nu(k) \tilde{\chi}(k) \succ \\
&= \alpha \, \langle \chi \mid \psi \rangle_{\mathscr{H}_0} + \langle \chi \mid \tilde{\chi} \rangle_{\mathscr{H}_0}.
\end{aligned}
$$

Hence, $\langle \cdot \mid \cdot \rangle_{\mathscr{H}_0}$ is a scalar product on $\mathscr{H}_0/\mathscr{N}$. □

Then, the following procedure gives back the Hilbert space: As lemma 4.4 shows, the space $(\mathscr{H}_0/\mathscr{N}, \langle \cdot \mid \cdot \rangle_{\mathscr{H}_0})$ forms an inner product space. Completion of the inner product space $\mathscr{H}_0/\mathscr{N}$ with respect to the scalar product $\langle \cdot \mid \cdot \rangle_{\mathscr{H}_0}$ according to theorem 3.11 gives a Hilbert space $\mathscr{H}$, endowed with a scalar product $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}$. In particular holds $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}\big|_{\mathscr{H}_0 \times \mathscr{H}_0} = \langle \cdot \mid \cdot \rangle_{\mathscr{H}_0}$. Elements in $\mathscr{H}$ can be thought of as the physical wave functions in momentum space.

4.3. Introduction of Wave Functions

Next, we introduce wave functions in position space. Considering Schwartz functions $\chi \in \mathscr{H}_0$, we define wave functions $\psi\colon \mathscr{G} \to V$ by vector-valued integration,

$$\psi(x) := \int_{\mathscr{G}^*} \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}\nu(k)\, \chi(k) \qquad (x \in \mathscr{G}), \tag{4.4}$$

i. e. for any $u \in V$ and $x \in \mathscr{G}$ holds

$$\prec u \mid \psi(x) \succ = \int_{\mathscr{G}^*} \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d} \prec u \mid \nu(k)\, \chi(k) \succ .$$

By definition (4.2), the above expression has to be considered as

$$\begin{aligned} \prec u \mid \psi(x) \succ &= \int_{\mathscr{G}^*} \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d} \prec u \mid \nu(k)\, \chi(k) \succ \\ &\overset{(4.2)}{=} \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \mathrm{e}^{-\mathrm{i}kx}\, \overline{u}_j\, \chi(k)_i\, \mathrm{d}\nu(k)_{j,i} \\ &= \sum_{i,j=1}^{2n} s_j\, \overline{u}_j \int_{\mathscr{G}^*} \mathrm{e}^{-\mathrm{i}kx}\, \chi(k)_i\, \mathrm{d}\nu(k)_{j,i} \end{aligned}$$

for any $u \in V$, $x \in \mathscr{G}$. Hence, wave functions are well-defined since $u \in V$ is arbitrary, Schwartz functions in $\mathscr{H}_0$ are bounded, and $\nu(\mathscr{G}^*) \in \mathrm{L}(V)$ is a bounded operator.

Assuming (4.4) and formally interchanging integrals, for $x \in \mathscr{G}$ the definition of the fermionic projector (3.3) yields

$$\begin{aligned} (P\chi)(x) &= \int_{\mathscr{G}} P(x,y)\, \chi(y)\, \mathrm{d}\mu(y) \\ &= \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d}\nu(k)\, \chi(y) \\ &= \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}\nu(k)\, \mathrm{e}^{\mathrm{i}ky}\, \chi(y) \\ &\overset{!}{=} \int_{\mathscr{G}^*} \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}\nu(k) \int_{\mathscr{G}} \mathrm{d}\mu(y)\, \mathrm{e}^{\mathrm{i}ky}\, \chi(y) \\ &= \int_{\mathscr{G}^*} \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}\nu(k)\, \widehat{\chi}(k) \\ &= \psi(x). \end{aligned} \tag{4.5}$$

In the following we will show that it is justified to interchange the integrals in the above way by application of Fubini's theorem. If calculation (4.5) can be verified, the definition of wave functions by (4.4) yields that these wave functions are in the range of the fermionic projector. Considering wave functions $\widehat{\chi} \in \mathscr{H}_0$, we have $\chi \in C_c^\infty(\mathscr{G}; V) \subset \mathscr{K}_0$. Hence, we may indeed apply the fermionic projector according to (3.7).

4.4. Application of Fubini's Theorem

In order to show equivalence of the construction of the Hilbert space, we first have to justify why we may switch the integrals in (4.5) in the following way:

$$\int_{\mathscr{G}^*} \mathrm{d}\nu(k) \int_{\mathscr{G}} \chi(y)\, \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d}\mu(y) \overset{!}{=} \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \mathrm{d}\nu(k)\, \chi(y)\, \mathrm{e}^{\mathrm{i}k(y-x)}.$$

To this aim, we will use the definition of vector-valued integration (A.4), Fubini's theorem (A.6), and Hille's theorem (A.7). Considering the pseudo-orthonormal basis $(e_i)_{i=1,\ldots,2n}$ which allows representation (3.4), we obtain

$$\begin{aligned}
&\prec e_m \mid \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d}\nu(k)\, \chi(y) \succ \\
&\overset{(A.7)}{=} \int_{\mathscr{G}} \mathrm{d}\mu(y) \prec e_m \mid \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d}\nu(k)\, \chi(y) \succ \\
&\overset{(A.4)}{=} \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d} \prec e_m \mid \nu(k)\, \chi(y) \succ \\
&= \int_{\mathscr{G}} \mathrm{d}\mu(y) \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \overline{(e_m)}_j\, \chi(y)_i\, \mathrm{d}\nu(k)_{j,i} \\
&= \sum_{i=1}^{2n} s_m \int_{\mathscr{G}} \mathrm{d}\mu(y) \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \chi(y)_i\, \mathrm{d}\nu(k)_{m,i} \\
&\overset{(A.6)}{=} \sum_{i=1}^{2n} s_m \int_{\mathscr{G}^*} \mathrm{d}\nu(k)_{m,i} \int_{\mathscr{G}} \mathrm{e}^{\mathrm{i}k(y-x)}\, \chi(y)_i\, \mathrm{d}\mu(y) \\
&= \sum_{i=1}^{2n} s_m \int_{\mathscr{G}^*} \int_{\mathscr{G}} \mathrm{d}\mu(y)\, \mathrm{e}^{\mathrm{i}k(x-y)}\, \chi(y)_i\, \mathrm{d}\nu(k)_{m,i} \\
&= \sum_{i,j=1}^{2n} s_j \int_{\mathscr{G}^*} \overline{(e_m)}_j \left(\int_{\mathscr{G}} \mathrm{d}\mu(y)\, \mathrm{e}^{\mathrm{i}k(x-y)}\, \chi(y) \right)_i \mathrm{d}\nu(k)_{j,i} \\
&\overset{(4.2)}{=} \int_{\mathscr{G}^*} \mathrm{d} \prec e_m \mid \nu(k) \int_{\mathscr{G}} \mathrm{d}\mu(y)\, \mathrm{e}^{\mathrm{i}k(x-y)}\, \chi(y) \succ \\
&\overset{(A.4)}{=} \prec e_m \mid \int_{\mathscr{G}^*} \mathrm{d}\nu(k) \int_{\mathscr{G}} \mathrm{d}\mu(y)\, \mathrm{e}^{\mathrm{i}k(x-y)}\, \chi(y) \succ
\end{aligned}$$

for any $m = 1, \ldots, 2n$. By non-degeneracy of an indefinite inner product we obtain (4.5).

In order to apply Fubini's theorem, we split up the complex measure $\mathrm{d}\nu_{ij}$ into its real and its imaginary part for any $i, j = 1, \dots, 2n$,

$$\begin{aligned}
&\int_{\mathcal{G}} \int_{\mathcal{G}^*} \left| \mathrm{e}^{\mathrm{i}k(y-x)}\, s_{ii}\, \chi(y)_j \right| \, \mathrm{d}\nu(k)_{ij} \, \mathrm{d}\mu(y) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}^*} \left| \chi(y)_j \right| \, \mathrm{d}\left(\mathrm{Re}\, \nu(k)_{ij} \right) \, \mathrm{d}\mu(y) + \mathrm{i} \int_{\mathcal{G}} \int_{\mathcal{G}^*} \left| \chi(y)_j \right| \, \mathrm{d}\left(\mathrm{Im}\, \nu(k)_{ij} \right) \, \mathrm{d}\mu(y).
\end{aligned}$$

Considering $\widehat{\chi} \in \mathscr{H}_0$, we obtain $\chi \in C_c^\infty(\mathcal{G}; V)$. Hence,

$$\int_{\mathcal{G}} \int_{\mathcal{G}^*} \left| \chi(y)_j \right| \, \mathrm{d}\left(\mathrm{Re}\, \nu(k)_{ij} \right) \, \mathrm{d}\mu(y) \leq \left| \mathrm{Re}\, \nu(\mathcal{G}^*)_{ij} \right| \sup_{x \in \operatorname{supp}\chi} \left| \chi(x)_j \right| \mu(\operatorname{supp}\chi) < \infty$$

and

$$\int_{\mathcal{G}} \int_{\mathcal{G}^*} \left| \chi(y)_j \right| \, \mathrm{d}\left(\mathrm{Im}\, \nu(k)_{ij} \right) \, \mathrm{d}\mu(y) \leq \left| \mathrm{Im}\, \nu(\mathcal{G}^*)_{ij} \right| \sup_{x \in \operatorname{supp}\chi} \left| \chi(x)_j \right| \mu(\operatorname{supp}\chi) < \infty,$$

since $\operatorname{supp}\chi$ is compact, μ is a Haar measure, and $\mathrm{d}(\mathrm{Re}\, \nu_{ij})$, $\mathrm{d}(\mathrm{Im}\, \nu_{ij})$ are bounded real measures. This allows to apply Fubini's theorem and thus to switch the integrals with respect to the complex measure $\mathrm{d}\nu_{ij}$ for any $i, j = 1, \dots, 2n$. Considering arbitrary $u, v \in V$ and using

$$\begin{aligned}
&\prec u \mid \int_{\mathcal{G}} \int_{\mathcal{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)} \, \mathrm{d}\nu(k) \, \chi(y) \, \mathrm{d}\mu(y) \, v \succ \\
&= \sum_{i=1}^{2n} \sum_{j=1}^{2n} \overline{u_i}\, v_j \prec e_i \mid \int_{\mathcal{G}} \int_{\mathcal{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)} \, \mathrm{d}\nu(k) \, \chi(y) \, \mathrm{d}\mu(y) \, e_j \succ
\end{aligned}$$

where $u = \sum_{i=1}^{2n} u_i\, e_i$, $v = \sum_{j=1}^{2n} v_j\, e_j$, we obtain

$$\prec u \mid \int_{\mathcal{G}} \int_{\mathcal{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)} \, \mathrm{d}\nu(k) \, \chi(y) \, \mathrm{d}\mu(y) \, v \succ = \prec u \mid \int_{\mathcal{G}^*} \int_{\mathcal{G}} \mathrm{d}\mu(y) \, \mathrm{e}^{\mathrm{i}k(y-x)} \, \mathrm{d}\nu(k) \, \chi(y) \, v \succ$$

for all $u, v \in V$. As explained in the appendix, every bounded linear operator between finite-dimensional normed vector spaces is closed. Hence, we may also apply Hille's theorem. This justifies to switch the integrals.

In summary, for $x \in \mathcal{G}$ we immediately obtain the above calculation (4.5), proving

$$(P\chi)(x) := \int_{\mathcal{G}} P(x, y) \, \chi(y) \, \mathrm{d}\mu(y) = \int_{\mathcal{G}^*} \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}\nu(k) \, \widehat{\chi}(k) = \psi(x)$$

for any $\chi \in C_c^\infty(\mathcal{G}; V)$.

4.5. Equivalence of the Construction

We finally deal with the question concerning the equivalence of both performed constructions of the Hilbert space $\mathscr{H}$. In section 3.6 as well as in section 4.2, a Hilbert space $(\mathscr{H}, \langle \cdot \mid \cdot \rangle_{\mathscr{H}})$ was constructed. In the following we want to prove that both constructions are equivalent. If this can be shown, the way of constructing the Hilbert space doesn't matter. To make clear the difference between the vector spaces $\mathscr{H}_0$ used in both constructions, we denote them by

(i) $\mathscr{H}_0^{\text{old}} := P(\mathscr{K}_0) = P(C_c(\mathscr{G}; V))$,

(ii) $\mathscr{H}_0^{\text{new}} := \mathscr{S}(\mathscr{G}^*; V)/\mathscr{Q}$.

After dividing out the nullspace $\mathscr{N}$, the corresponding Hilbert space was defined as the closure of the quotient space, $\mathscr{H} := \overline{\mathscr{H}_0/\mathscr{N}}$.

In the first setting, we consider continuous wave functions with compact support $\psi, \phi \in \mathscr{K}_0 = C_c(\mathscr{G}^*; V)$ and let $\mathscr{H}_0^{\text{old}} = P(\mathscr{K}_0)$. Because of $\psi \in C_c(\mathscr{G}; V) \subset L^1(\mathscr{G}; V)$, we may define the Fourier transform of ψ by

$$\widehat{\psi}(k) := \int_{\mathscr{G}} \psi(x)\, e^{ixk}\, dx$$

according to [FR13]. By formula (3.16), for all $\psi, \phi \in \mathscr{K}_0$ we obtain

$$< \psi \mid P\phi > = c_0^2 \int_{\mathscr{G}^*} d \prec \widehat{\psi}(k) \mid \nu(k)\,\widehat{\phi}(k) \succ .$$

Together with definition (3.17), i. e.

$$\langle P\psi \mid P\phi \rangle_{\mathscr{H}}^{\text{old}} := < \psi \mid (-P)\,\phi > = - < \psi \mid P\phi > \qquad (\psi, \phi \in \mathscr{K}_0),$$

we obtain

$$\langle P\psi \mid P\phi \rangle_{\mathscr{H}}^{\text{old}} = -\, c_0^2 \int_{\mathscr{G}^*} d \prec \widehat{\psi}(k) \mid \nu(k)\widehat{\phi}(k) \succ \tag{4.6}$$

for all $\psi, \phi \in \mathscr{K}_0$. The question is: What is the connection between (4.6) and the new definition (4.1),

$$\langle \widehat{\chi} \mid \widehat{\tilde{\chi}} \rangle_{\mathscr{H}}^{\text{new}} := -c_1 \int_{\mathscr{G}^*} d \prec \widehat{\chi}(k) \mid \nu(k)\widehat{\tilde{\chi}}(k) \succ$$

with $\chi, \tilde{\chi} \in \mathscr{H}_0^{\text{new}}$? Choosing $c_1 = c_0^2$, we obtain agreement in the following sense:

$$\langle \widehat{\chi} \mid \widehat{\tilde{\chi}} \rangle_{\mathscr{H}}^{\text{new}} = -\, c_1 \int_{\mathscr{G}^*} d \prec \widehat{\chi}(k) \mid \nu(k)\widehat{\tilde{\chi}}(k) \succ$$

$$= -c_0^2 \int_{\mathscr{G}^*} \mathrm{d} \prec \widehat{\chi}(k) \mid \nu(k) \widehat{\tilde{\chi}}(k) \succ \overset{(4.6)}{=} \langle P\chi \mid P\tilde{\chi} \rangle_{\mathscr{H}}^{\mathrm{old}}$$

for all $\widehat{\chi}, \widehat{\tilde{\chi}} \in \mathscr{H}_0^{\mathrm{new}}$. Application of equation (4.5) proves $P\chi = \psi \in \mathscr{H}_0^{\mathrm{old}}$ for any $\chi \in C_c^\infty(\mathscr{G}; V)$. This yields agreement in the following sense:

$$\langle \widehat{\chi} \mid \widehat{\tilde{\chi}} \rangle_{\mathscr{H}}^{\mathrm{new}} = \langle P\chi \mid P\tilde{\chi} \rangle_{\mathscr{H}}^{\mathrm{old}} = \langle \psi \mid \tilde{\psi} \rangle_{\mathscr{H}}^{\mathrm{old}}. \tag{4.7}$$

The first ones are *wave functions in momentum space*, the second ones are *wave functions in position space*. By definition of $\mathscr{H}_0^{\mathrm{new}}$, $\mathfrak{F}^{-1}[\mathscr{H}_0^{\mathrm{new}}]$ is a subset of $\mathscr{K}_0$. Therefore, $\widehat{\chi} \in \mathscr{H}_0^{\mathrm{new}}$ implies $\chi \in \mathscr{K}_0$. For this reason, equation (4.6) holds for any χ with $\widehat{\chi} \in \mathscr{H}_0^{\mathrm{new}}$. Thus, equation (4.7) yields agreement of both sesquilinear forms for all $\widehat{\chi}, \widehat{\tilde{\chi}} \in \mathscr{H}_0^{\mathrm{new}}$. Dividing out the nullspace $\mathscr{N}^{\mathrm{new}}$, the sesquilinear form $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}^{\mathrm{new}}$ defines a scalar product on $\mathscr{H}_0^{\mathrm{new}} / \mathscr{N}^{\mathrm{new}}$. By theorem 3.11, the closure of $\mathscr{H}_0^{\mathrm{new}} / \mathscr{N}^{\mathrm{new}}$ with respect to $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}^{\mathrm{new}}$ is a Hilbert space. In order to obtain equivalence of both constructions, assume $\widehat{\chi} \in \mathscr{H}_0^{\mathrm{new}} / \mathscr{N}^{\mathrm{new}}$. Then, $\mathfrak{F}^{-1}[\widehat{\chi}] \in C_c^\infty(\mathscr{G}; V)$ and $P\mathfrak{F}^{-1}[\widehat{\chi}] \in \mathscr{H}_0^{\mathrm{old}}$. Additionally, agreement of both sesquilinear forms in accordance with (4.7) yields

$$\langle P\mathfrak{F}^{-1}[\widehat{\chi}] \mid P\mathfrak{F}^{-1}[\widehat{\chi}] \rangle_{\mathscr{H}}^{\mathrm{old}} = \langle \widehat{\chi} \mid \widehat{\chi} \rangle_{\mathscr{H}}^{\mathrm{new}}$$

for any $\widehat{\chi} \in \mathscr{H}_0^{\mathrm{new}} / \mathscr{N}^{\mathrm{new}}$. Since $\mathfrak{F}[0] = 0$, we obtain

$$\begin{aligned} \langle P\mathfrak{F}^{-1}[\widehat{\chi}] \mid P\mathfrak{F}^{-1}[\widehat{\chi}] \rangle_{\mathscr{H}}^{\mathrm{old}} = 0 \quad &\Longleftrightarrow \quad \langle \widehat{\chi} \mid \widehat{\chi} \rangle_{\mathscr{H}}^{\mathrm{new}} = 0 \quad \Longleftrightarrow \quad \widehat{\chi} = 0 \\ &\Longleftrightarrow \quad P\mathfrak{F}^{-1}[\widehat{\chi}] = 0, \end{aligned}$$

implying

$$P\left(\mathfrak{F}^{-1}\left[\mathscr{H}_0^{\mathrm{new}} / \mathscr{N}^{\mathrm{new}}\right]\right) \subset \mathscr{H}_0^{\mathrm{old}} / \mathscr{N}^{\mathrm{old}}.$$

Since $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}^{\mathrm{old}}$ defines a scalar product on $\mathscr{H}_0^{\mathrm{old}} / \mathscr{N}^{\mathrm{old}}$, it is in particular a scalar product on $P(\mathfrak{F}^{-1}[\mathscr{H}_0^{\mathrm{new}} / \mathscr{N}^{\mathrm{new}}])$. By theorem 3.11, the closure of $P(\mathfrak{F}^{-1}[\mathscr{H}_0^{\mathrm{new}} / \mathscr{N}^{\mathrm{new}}])$ with respect to $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}^{\mathrm{old}}$ is also a Hilbert space. According to [Heu82, Satz 41.5], the closure of not necessarily complete inner product spaces is unique up to isomorphisms. Hence we obtain equivalence of the Hilbert spaces,

$$\mathscr{H}^{\mathrm{old}} := \overline{\mathscr{H}_0^{\mathrm{old}} / \mathscr{N}^{\mathrm{old}}} \simeq \overline{P\left(\mathfrak{F}^{-1}\left[\mathscr{H}_0^{\mathrm{new}} / \mathscr{N}^{\mathrm{new}}\right]\right)} \simeq \overline{\mathscr{H}_0^{\mathrm{new}} / \mathscr{N}^{\mathrm{new}}} =: \mathscr{H}^{\mathrm{new}}.$$

Thus, $(\mathscr{H}, \langle \cdot \mid \cdot \rangle_{\mathscr{H}})$ represents the closure with respect to both scalar products. In particular, both scalar products on $\mathscr{H}_0 / \mathscr{N}$ are induced by $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}$.

Remark 4.5. The equivalence of the construction of the Hilbert space $\mathscr{H}$ shows that it suffices to consider smooth wave functions ψ with compact support in the dense subset $C_c^\infty(\mathscr{G}; V)$ of $\mathscr{K}_0$ in the first setting.

The advantage of the new method could be due to the fact that it is not necessary to make a detour by defining the fermionic projector. Thus, the second method seems to be a more direct way of constructing the Hilbert space $\mathscr{H}$. Defining local correlation matrices as above, we get back our original causal fermion system $(\mathscr{H}, \mathscr{F}, \rho)$.

5. Existence of Minimizers

The goal of the last chapter is to discuss the existence of minimizers of the causal action. To this aim, we will first introduce operator-valued measures and negative definite measures in particular. Then, we will state some basic definitions leading to the causal action principle. Finally, some ideas will be presented which could prove the existence of minimizers.

5.1. Operator-Valued Measures

Let $(V, \prec \cdot \mid \cdot \succ)$ be a finite-dimensional indefinite inner product space. Assume $\mathscr{G} = (\mathbb{R}^4, +)$ to be the group of translations as usual and let $\mathscr{G}^*$ be its dual. First, we want to introduce operator-valued measures with values in $\mathrm{L}(V)$ as a generalization of negative definite measures:

Definition 5.1 (Operator-valued measures with values in L(V)).
Let $(V, \prec \cdot \mid \cdot \succ)$ be a finite-dimensional indefinite inner product space. Consider a regular Borel measure ν on $\mathscr{G}^*$ taking values in $\mathrm{L}(V)$ with the following property: For every $u \in V$, the measure $\mathrm{d} \prec u \mid \nu u \succ$ is a finite real measure. Then we refer to ν as an *operator-valued measure* on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$.

Remark 5.2. We call an operator-valued measure ν a *regular Borel measure* if all components of ν are regular Borel measures.

Corollary 5.3. *The set of operator-valued measures on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$ forms a metric space which we denote by $\mathrm{OVM}(\mathscr{G}^*; \mathrm{L}(V))$.*

Proof. Let $\mathrm{OVM}(\mathscr{G}^*; \mathrm{L}(V))$ be the set of operator-valued measures on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$. Considering $\nu, \rho \in \mathrm{OVM}(\mathscr{G}^*; \mathrm{L}(V))$, $\lambda \in \mathbb{R}$, $\Omega \in \mathscr{B}(\mathscr{G}^*)$, we define the pointwise operations

$$(\nu + \rho)(\Omega) := \mu(\Omega) + \rho(\Omega), \qquad (\lambda\nu)(\Omega) := \lambda\nu(\Omega).$$

First, we have to prove that $\mathrm{OVM}(\mathscr{G}^*; \mathrm{L}(V))$ is an abelian group. To this aim we have check the group axioms: It is easy to see that $\mathrm{OVM}(\mathscr{G}^*; \mathrm{L}(V))$ is closed with respect to addition. Commutativity and associativity are clear. The zero measure represents the neutral element of additivity, whereas $-\nu$ is the additive inverse with respect to ν. The remaining vector space axioms are doubtlessly fulfilled. Thus, $\mathrm{OVM}(\mathscr{G}^*; \mathrm{L}(V))$ is a real vector space.

Next, we define a norm on the real vector space $\mathrm{OVM}(\mathscr{G}^*; \mathrm{L}(V))$ according to [Fin10]: Let $(e_i)_i$ be a basis of V and let $\nu \in \mathrm{OVM}(\mathscr{G}^*; \mathrm{L}(V))$. Then,

$$\|\nu\| := \sum_{i,j=1}^{\dim V} \left\| \mathrm{d} \prec e_i \mid \nu\, e_j \succ \right\| := \sum_{i,j=1}^{\dim V} \left| \mathrm{d} \prec e_i \mid \nu\, e_j \succ \right| (\mathscr{G}^*), \tag{5.1}$$

where $\left| \mathrm{d} \prec e_i \mid \nu\, e_j \succ \right|(A)$ with $A \in \mathscr{B}(\mathscr{G}^*)$ denotes the variation of the complex measure $\mathrm{d} \prec e_i \mid \nu\, e_j \succ$. The total variation, given by

$$\left\| \mathrm{d} \prec e_i \mid \nu\, e_j \succ \right\| := \left| \mathrm{d} \prec e_i \mid \nu\, e_j \succ \right| (\mathscr{G}^*),$$

defines a norm on the space of complex measures which becomes a complex Banach space with respect to this norm, see [Els11]. Hence, (5.1) defines a norm on $\mathrm{OVM}(\mathscr{G}^*; \mathrm{L}(V))$:

i) $\|\nu\| \geq 0$ and $\|\nu\| = 0 \Leftrightarrow \left\| \mathrm{d} \prec e_i \mid \nu\, e_j \succ \right\|$ for all $i, j = 1, \ldots, \dim V \Leftrightarrow \nu = 0$.

ii) For all $\alpha \in \mathbb{R}$ (diagonal entries have to stay real measures) holds

$$\|\alpha\nu\| = \sum_{i,j=1}^{\dim V} \left| \mathrm{d} \prec e_i \mid \alpha\nu\, e_j \succ \right| (\mathscr{G}^*) = |\alpha| \sum_{i,j=1}^{\dim V} \left| \mathrm{d} \prec e_i \mid \nu\, e_j \succ \right| (\mathscr{G}^*) = |\alpha|\, \|\nu\| \,.$$

iii) The triangular inequation is obtained by

$$\begin{aligned} \|\nu_1 + \nu_2\| &= \sum_{i,j=1}^{\dim V} \left\| \mathrm{d} \prec e_i \mid (\nu_1 + \nu_2)\, e_j \succ \right\| \\ &\leq \sum_{i,j=1}^{\dim V} \left\| \mathrm{d} \prec e_i \mid \nu_1\, e_j \succ \right\| + \sum_{i,j=1}^{\dim V} \left\| \mathrm{d} \prec e_i \mid \nu_2\, e_j \succ \right\| = \|\nu_1\| + \|\nu_2\| \end{aligned}$$

for all $\nu_1, \nu_2 \in \mathrm{OVM}(\mathscr{G}^*; \mathrm{L}(V))$.

Since each norm induces a metric by [Alt06], the vector space $\mathrm{OVM}(\mathscr{G}^*; \mathrm{L}(V))$ forms a metric space. □

Remark 5.4. Using the metric induced by (5.1), we may define open subsets of $\mathrm{OVM}(\mathscr{G}^*; \mathrm{L}(V))$. Thus, $(\mathrm{OVM}(\mathscr{G}^*; \mathrm{L}(V)), \mathfrak{O}_{\mathrm{OVM}(\mathscr{G}^*;\mathrm{L}(V))})$ even forms a topological space. If the topology $\mathfrak{O}_{\mathrm{OVM}(\mathscr{G}^*;\mathrm{L}(V))}$ is understood, we shall simply refer to the topological space $\mathrm{OVM}(\mathscr{G}^*; \mathrm{L}(V))$, see [Fol99, Chapter 4].

Let us now briefly recall the definition of a negative definite measure:

Definition 5.5 (Negative definite measure).
Let $(V, \prec \cdot \mid \cdot \succ)$ be a finite-dimensional indefinite inner product space. Consider a regular Borel measure ν on $\mathscr{G}^*$ taking values in $\mathrm{L}(V)$ with the following properties:

i) For every $u \in V$, the measure $\mathrm{d} \prec u \mid \nu u \succ$ is a finite real measure.

ii) For every Borel set $\Omega \subset \mathscr{G}^*$, the operator $-\nu(\Omega) \in \mathrm{L}(V)$ is positive, i. e.

$$\prec u \mid (-\nu(\Omega))\, u \succ \geq 0 \qquad \text{for all } u \in V.$$

Then ν is called a *negative definite measure* on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$.

Remark 5.6. Let $\mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V))$ denote the set of negative definite measures on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$. Then, $\mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V))$ is obviously a subset of the vector space $\mathrm{OVM}(\mathscr{G}^*; \mathrm{L}(V))$. However, $\mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V))$ is *not* a vector space: Let $\nu \in \mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V))$. Then, $-\nu \notin \mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V))$, as the following calculation for any $u \in V$ easily confirms:

$$\prec u \mid -(-\nu)\, u \succ = - \prec u \mid (-\nu)\, u \succ \leq 0.$$

Nonetheless, the set $\mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V)) \subset \mathrm{OVM}(\mathscr{G}^*; \mathrm{L}(V))$ of negative definite measures forms a topological space: Consider the relative topology

$$\mathfrak{O}_{\mathrm{NDM}(\mathscr{G}^*;\mathrm{L}(V))} := \left\{ U \cap \mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V)) \colon\, U \in \mathfrak{O}_{\mathrm{OVM}(\mathscr{G}^*;\mathrm{L}(V))} \right\}.$$

Then, $\mathfrak{O}_{\mathrm{NDM}(\mathscr{G}^*;\mathrm{L}(V))}$ is a topology on $\mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V))$, and $\mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V))$ can be referred to as a topological space, see e. g. [Els11] or [Fol99]. We may also consider the subset of negative definite measures with compact support which shall be denoted by $\mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V))_{\mathrm{c}} \subset \mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V))$. Furthermore, for any $\Omega \in \mathscr{B}(\mathscr{G}^*)$ the trace of the operator $\nu(\Omega) \in \mathrm{L}(V)$ can be determined. The set of operator-valued measures with values in $\mathrm{L}(V)$ satisfying the trace constraint $\mathrm{Tr}(\nu(\mathscr{G}^*)) = a$ shall be denoted by $\mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V))^a$. Defining relative topologies, these sets also become topological spaces. For this reason, we may consider variational problems on $\mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V))$ and its subsets.

5.2. Basic Definitions

In the following, consider the group of translations $\mathscr{G} = (\mathbb{R}^4, +)$, and let $\mathscr{G}^* \simeq \mathbb{R}^4$ be its dual space. Let μ be the Haar measure on $\mathscr{G}$. Since the Haar measure on $\mathscr{G}$

is only a multiple of the Lebesgue measure, the total space-time volume $\mu(\mathscr{G})$ is infinite, i. e. $\mu(\mathscr{G}) = \infty$. Furthermore, assume a finite-dimensional indefinite inner product space $(V, \prec \cdot \mid \cdot \succ)$, and let ν be a negative definite measure on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$. Recall definition (3.3) of the *kernel of the fermionic projector*

$$P(x,y)\colon V \to V, \qquad P(x,y) := \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k(y-x)}\, \mathrm{d}\nu(k)$$

for any $x, y \in \mathscr{G}$. We also write $P[\nu](x,y)$ in order to emphasize that the kernel of the fermionic projector depends on the negative definite measure ν. Considering *homogeneous* causal fermion systems, only the difference of two space-time points $x, y \in \mathscr{G}$ matters. For this reason, we define the kernel of the fermionic projector depending on $\xi = y - x \in \mathscr{G}$ by

$$P(\xi)\colon V \to V, \qquad P(\xi) := \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi}\, \mathrm{d}\nu(k).$$

Here, for any $k \in \mathscr{G}^*$ and arbitrary $\xi \in \mathscr{G}$ we define duality by $k\,\xi := \sum_{i=0}^{3} k_i\, \xi_i$. In order to introduce eigenvalues of an operator we focus on endomorphisms on the vector space V. To this aim, for $\xi \in \mathscr{G}$ we define the *closed chain* $A(\xi)\colon V \to V$,

$$A(\xi) := P(\xi)\, P(-\xi).$$

Again, we also write $A[\nu](\xi)$ in order to make clear the dependence of the closed chain on the negative definite measure ν. Considering a finite-dimensional vector space V, we define the *spectral weight* of a linear operator $A\colon V \to V$ by

$$|A| := \sum_{i=1}^{\dim V} |\lambda_i|\,,$$

where $(\lambda_i)_{i=1,\dots,2n}$ shall denote the eigenvalues of the operator A counted with algebraic multiplicity.

In the following, consider a finite-dimensional indefinite inner product space V and let ν be a negative definite measure on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$. Then, for $\xi \in \mathscr{G}$ we introduce the *Lagrangian* $\mathcal{L}$ as

$$\mathcal{L}[\nu](\xi) := \left|A[\nu](\xi)^2\right| - \frac{1}{2n}\, |A[\nu](\xi)|^2\,.$$

The functional $\mathcal{T}$ shall be introduced by

$$\mathcal{T}[\nu] := \int_{\mathscr{G}} |A[\nu](\xi)|^2\; \mathrm{d}\mu(\xi).$$

Finally, the *causal action* $\mathcal{S}$ is defined as

$$\mathcal{S}[\nu] := \int_{\mathscr{G}} \mathcal{L}[\nu](\xi)\, \mathrm{d}\mu(\xi).$$

The goal is to minimize the causal action $\mathcal{S}$ by varying the negative definite measure ν within a suitable subset of $\mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V))$. We refer to this procedure as the socalled *causal action principle*. In order to exclude trivial minimizers, we want to fix the trace of $\nu(\mathscr{G}^*)$, considering the *trace constraint*

$$\operatorname{Tr} \nu(\mathscr{G}^*) = 1. \tag{5.2}$$

Moreover, we assume a *boundedness constraint* in the sense that the functional $\mathcal{T}$ is uniformly bounded, supposing a positive constant $C > 0$ such that

$$\mathcal{T}[\nu] \leq C. \tag{5.3}$$

Note that the causal action principle crucially depends on the eigenvalues of the closed chain $A[\nu](\xi)$ for $\xi \in \mathscr{G}$ and a negative definite measure ν.

In the following, we only consider negative definite measures with compact support which satisfy trace constraint (5.2). The set of such measures shall be denoted by $\mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V))^1_{\mathrm{c}}$. Therefore, we assume the causal action $\mathcal{S}$ being a functional

$$\mathcal{S} \colon \mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V))^1_{\mathrm{c}} \to \mathbb{R} \cup \{\infty\}, \qquad \nu \mapsto \mathcal{S}[\nu] := \int_{\mathscr{G}} \mathcal{L}[\nu](\xi)\, \mathrm{d}\mu(\xi). \tag{5.4}$$

This is the causal action principle which shall be considered in the remaining sections. First of all, in the next section we want to show that the causal action principle (5.4) is well-posed in the sense that we may prove the existence of negative definite measures $\nu \in \mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V))^1_{\mathrm{c}}$ such that $\mathcal{S}[\nu]$ is finite.

5.3. Well-Posedness of the Minimizing Problem

In order to prove that the causal action principle (5.4) is well-posed, we show that there are negative definite measures with compact support satisfying the above trace constraint (5.2) such that the causal action takes finite values. Considering the boundedness constraint (5.3), we conclude

$$\mathcal{S}[\nu] = \int_{\mathscr{G}} \mathcal{L}[\nu](\xi)\, \mathrm{d}\mu(\xi) = \int_{\mathscr{G}} \left| A[\nu](\xi)^2 \right| - \frac{1}{2n} \left| A[\nu](\xi) \right|^2 \mathrm{d}\mu(\xi)$$

$$\geq \int_{\mathscr{G}} \left|A[\nu](\xi)^2\right| \, \mathrm{d}\mu(\xi) - \frac{C}{2n} \geq -\frac{C}{2n}.$$

Hence, the causal action is bounded from below. Well-posedness of the causal action principle implies that the causal action $\mathcal{S}[\nu_\ell]$ converges to a finite value for each minimizing sequence $(\nu_\ell)_{\ell\in\mathbb{N}}$.

Remark 5.7. In the example below, we consider the Lebesgue-Borel measure $\beta^p = \lambda^p\big|_{\mathscr{B}(\mathbb{R}^p)}$ as the restriction of the Lebesgue measure on the Borel σ-algebra $\mathscr{B}^p = \mathscr{B}(\mathbb{R}^p)$, see [Els11]. According to [Els11], the Lebesgue-Borel measure $\beta^p \colon \mathscr{B}^p \to [0,\infty]$ is a Radon measure on $\mathscr{B}^p$ which is by definition an inner regular Borel measure. Moreover, the following statement holds: If X is a σ-compact Hausdorff space, then every Radon measure on X is regular, see [Els11, Korollar VIII.1.13]. Obviously, $\mathbb{R}^p$ is a σ-compact Hausdorff space since we may consider $\mathbb{R}^p = \bigcup_{R\in\mathbb{N}} \overline{B_R(0)}$ as a countable union of compact sets. For this reason, the Radon measure β^p is a regular Borel measure on $\mathscr{B}^p$.

Example 5.8 (**Illustrative example**).
This example proves the existence of negative definite measures with compact support which satisfy trace constraint (5.2) and for which the causal action $\mathcal{S}$ takes finite values. In a first step, for all $\ell \in \mathbb{N}$, $i = 1,\ldots,2n$, we define the measures $\eta_\ell^{(i)} \colon \mathscr{B}(\mathscr{G}^*) \to \mathbb{R}$ with compact support $\operatorname{supp}\eta_\ell^{(i)} = [-\ell,\ell]^4$ in the following way:

$$\eta_\ell^{(i)} := \begin{cases} -\frac{1}{n} \left|[-\ell,\ell]^4\right|^{-1} \beta^4\big|_{[-\ell,\ell]^4} & i = 1,\ldots,n, \\ \frac{2}{n} \left|[-\ell,\ell]^4\right|^{-1} \beta^4\big|_{[-\ell,\ell]^4} & i = n+1,\ldots,2n, \end{cases}$$

where $\mathrm{d}\beta^4(k) = \mathrm{d}k_0\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3$ denotes the Lebesgue-Borel measure on $\mathbb{R}^4$. For any $\Omega \in \mathscr{B}(\mathscr{G}^*)$, the restriction $\beta^4\big|_{[-\ell,\ell]^4}$ shall be defined by

$$\beta^4\big|_{[-\ell,\ell]^4}(\Omega) := \begin{cases} 0 & \text{if } \Omega \subset \left([-\ell,\ell]^4\right)^{\mathrm{c}}, \\ \beta^4\left([-\ell,\ell]^4 \cap \Omega\right) & \text{otherwise.} \end{cases}$$

In particular holds

$$\eta_\ell^{(i)}(\mathscr{G}^*) = \begin{cases} -\frac{1}{n} \left|[-\ell,\ell]^4\right|^{-1} \beta^4\big|_{[-\ell,\ell]^4}(\mathscr{G}^*) = -\frac{1}{n} & \text{for } i = 1,\ldots,n, \\ \frac{2}{n} \left|[-\ell,\ell]^4\right|^{-1} \beta^4\big|_{[-\ell,\ell]^4}(\mathscr{G}^*) = \frac{2}{n} & \text{for } i = n+1,\ldots,2n. \end{cases}$$

Furthermore, we introduce positive constants $\mathrm{C}_\ell^{(i)} > 0$ by

$$\mathrm{C}_\ell^{(i)} := \begin{cases} \frac{1}{n} \left|[-\ell,\ell]^4\right|^{-1} & \text{for } i = 1,\ldots,n, \\ \frac{2}{n} \left|[-\ell,\ell]^4\right|^{-1} & \text{for } i = n+1,\ldots,2n. \end{cases} \tag{5.5}$$

Then, we define operator-valued measures

$$\nu_\ell \colon \mathscr{B}(\mathscr{G}^*) \to \mathrm{L}(V), \qquad \nu_\ell := \operatorname{diag}\left(\eta_\ell^{(1)}, \ldots, \eta_\ell^{(n)}, \eta_\ell^{(n+1)}, \ldots, \eta_\ell^{(2n)}\right).$$

We want to show that ν_ℓ is a negative definite measure for every $\ell \in \mathbb{N}$. For any $\ell \in \mathbb{N}$, $\Omega \in \mathscr{B}(\mathscr{G}^*)$, and $u \in V$ holds

$$\begin{aligned}
\prec u \mid -\nu_\ell(\Omega)\, u \succ &= \langle u \mid -S\, \nu_\ell(\Omega)\, u\rangle = \langle u \mid \operatorname{diag}(-1, \ldots, -1, 1, \ldots, 1)\, \nu_\ell(\Omega)\, u\rangle \\
&= \langle u \mid \operatorname{diag}\left(-\eta_\ell^{(1)}(\Omega), \ldots, -\eta_\ell^{(n)}(\Omega), \eta_\ell^{(n+1)}(\Omega), \ldots, 2\eta_\ell^{(2n)}(\Omega)\right)\, u\rangle \\
&= \langle u \mid \operatorname{diag}\left(\mathrm{C}_\ell^{(1)}\, \beta^4\big|_{[-\ell,\ell]^4}(\Omega), \ldots, \mathrm{C}_\ell^{(2n)}\, \beta^4\big|_{[-\ell,\ell]^4}(\Omega)\right)\, u\rangle \\
&= \langle u \mid \left(\mathrm{C}_\ell^{(1)}\, \beta^4\big|_{[-\ell,\ell]^4}(\Omega)\, u_1, \ldots, \mathrm{C}_\ell^{(2n)}\, \beta^4\big|_{[-\ell,\ell]^4}(\Omega)\, u_{2n}\right)^{\mathrm{T}}\rangle \\
&= \sum_{i=1}^{2n} \mathrm{C}_\ell^{(i)}\, \beta^4\big|_{[-\ell,\ell]^4}(\Omega)\, \overline{u}_i\, u_i \\
&= \sum_{i=1}^{2n} \mathrm{C}_\ell^{(i)}\, \beta^4\big|_{[-\ell,\ell]^4}(\Omega)\; |u_i|^2 \geq 0.
\end{aligned}$$

This proves positivity of the operators $-\nu_\ell(\Omega)$ for all $\ell \in \mathbb{N}$. In order to show that $\mathrm{d} \prec u \mid \nu_\ell\, u \succ$ is a finite real measure for all $u \in V$, let $(e_i)_{i=1,\ldots,2n}$ be the pseudo-orthonormal basis which allows representation (3.4). Using

$$\begin{aligned}
\langle e_i \mid S\, \nu_\ell\, e_j\rangle &= \langle e_i \mid S\, \left((\nu_\ell)_{1,j}, \ldots, (\nu_\ell)_{2n,j}\right)^{\mathrm{T}}\rangle \\
&= \langle e_i \mid \left(s_1\, (\nu_\ell)_{1,j}, \ldots, s_n\, (\nu_\ell)_{n,j}, s_{n+1}\, (\nu_\ell)_{n+1,j}, \ldots, s_{2n}\, (\nu_\ell)_{2n,j}\right)^{\mathrm{T}}\rangle \\
&= \sum_{k=1}^{2n} \overline{(e_i)}_k\, s_k\, (\nu_\ell)_{k,j} = s_i\, (\nu_\ell)_{i,j}
\end{aligned}$$

for any $i, j = 1, \ldots, 2n$, for arbitrary $u \in V$ we obtain

$$\begin{aligned}
\prec u \mid \nu_\ell\, u \succ &= \prec \sum_{i=1}^{2n} u_i\, e_i \mid \nu_\ell\, (\sum_{j=1}^{2n} u_j\, e_j) \succ = \sum_{i,j=1}^{2n} \overline{u_i}\, u_j \prec e_i \mid \nu_\ell\, e_j \succ \\
&= \sum_{i,j=1}^{2n} \overline{u_i}\, u_j\, \langle e_i \mid S\, \nu_\ell\, e_j\rangle = \sum_{i,j=1}^{2n} \overline{u_i}\, u_j\, s_i\, (\nu_\ell)_{i,j} \\
&= \sum_{i=1}^{2n} |u_i|^2\, s_i\, \eta_\ell^{(i)}.
\end{aligned}$$

Since $\eta_\ell^{(i)}$ is a finite real measure for all $i = 1, \ldots, 2n$, so $\mathrm{d} \prec u \mid \nu_\ell \, u \succ$ is. In particular, the measure ν_ℓ is locally finite and hence a Borel measure by definition. Remark 5.7 implies regularity of the measure ν_ℓ for all $\ell \in \mathbb{N}$. Therefore, ν_ℓ is a negative definite measure for all $\ell \in \mathbb{N}$. Moreover, ν_ℓ satisfies trace constraint (5.2) for all $\ell \in \mathbb{N}$ since

$$\operatorname{Tr} \nu_\ell(\mathscr{G}^*) = \sum_{i=1}^{2n} \nu_\ell(\mathscr{G}^*)_{i,i} = \sum_{i=1}^{2n} \eta_\ell^{(i)}(\mathscr{G}^*) = \sum_{i=1}^{n} \left(-\frac{1}{n}\right) + \sum_{i=n+1}^{2n} \frac{2}{n} = -1 + 2 = 1.$$

It remains to show that $\mathcal{S}[\nu_\ell] < \infty$ for all $\ell \in \mathbb{N}$. To this aim, let us first consider the kernel of the fermionic projector

$$P[\nu_\ell](\xi) \colon V \to V, \qquad P[\nu_\ell](\xi) := \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi} \, \mathrm{d}\nu_\ell(k)$$

for any $\xi \in \mathscr{G}$. Since the operator $\nu_\ell(\Omega) \in \mathrm{L}(V)$ may be represented as a diagonal matrix by definition of ν_ℓ, the kernel of the fermionic projector $P[\nu_\ell](\xi)$ may also be represented as a diagonal matrix. For $i, j = 1, \ldots, 2n$, its components are given by

$$\begin{aligned}(P[\nu_\ell](\xi))_{ij} &= \langle e_i \mid P[\nu_\ell](\xi) \, e_j \rangle = \langle e_i \mid \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi} \, \mathrm{d}\nu_\ell(k) \, e_j \rangle = \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi} \, \mathrm{d}\langle e_i \mid \nu_\ell(k) \, e_j \rangle \\ &= \delta_{ij} \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi} \, \mathrm{d}\eta_\ell^{(i)}(k).\end{aligned}$$

Hence, the off-diagonal entries of $P[\nu_\ell](\xi)$ are zero for any $\xi \in \mathscr{G}$. Since $\eta_\ell^{(i)}$ are real measures for all $\ell \in \mathbb{N}$, we obtain the following representation for the closed chain $A[\nu_\ell](\xi)$:

$$\begin{aligned}(A[\nu_\ell](\xi))_{ij} &= (P[\nu_\ell](\xi) \, P[\nu_\ell](-\xi))_{ij} = \sum_{m=1}^{2n} P[\nu_\ell](\xi)_{im} \, P[\nu_\ell](-\xi)_{mj} \\ &= \sum_{m=1}^{2n} \delta_{im} \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi} \, \mathrm{d}\eta_\ell^{(i)}(k) \, \delta_{mj} \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi} \, \mathrm{d}\eta_\ell^{(j)}(k) \\ &= \delta_{ij} \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi} \, \mathrm{d}\eta_\ell^{(i)}(k) \int_{\mathscr{G}^*} \mathrm{e}^{-\mathrm{i}k\xi} \, \mathrm{d}\eta_\ell^{(i)}(k) \\ &= \delta_{ij} \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi} \, \mathrm{d}\eta_\ell^{(i)}(k) \, \overline{\int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi} \, \mathrm{d}\eta_\ell^{(i)}(k)} \\ &= \delta_{ij} \left| \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi} \, \mathrm{d}\eta_\ell^{(i)}(k) \right|^2.\end{aligned}$$

In particular, $A[\nu_\ell](\xi)$ is a diagonal matrix. Therefore, the eigenvalues $\lambda_i^{(\ell)}(\xi)$ of the closed chain $A[\nu_\ell](\xi)$ are given by its diagonal entries, i. e.

$$\lambda_i^{(\ell)}(\xi) = \left| \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi} \, \mathrm{d}\eta_\ell^{(i)}(k) \right|^2 = \left| \int_{\mathscr{G}^*} (\cos(k\xi) + \mathrm{i} \sin(k\xi)) \, \mathrm{d}\eta_\ell^{(i)}(k) \right|^2$$

$$
\begin{aligned}
&= \left| \int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) + \mathrm{i} \int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right|^2 \\
&= \left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2
\end{aligned}
$$

for all $i = 1, \ldots, 2n$. Hence, for $\xi \in \mathscr{G}$ the spectral weight of $A[\nu_\ell](\xi)$ is given by

$$
|A[\nu_\ell](\xi)| = \sum_{i=1}^{2n} \left| \lambda_i^{(\ell)}(\xi) \right| = \sum_{i=1}^{2n} \left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 .
$$

Moreover, $A[\nu_\ell](\xi)^2$ is also a diagonal matrix with eigenvalues $\lambda_i^{(\ell)}(\xi)^2$, $i = 1, \ldots, 2n$. Thus, for any $\xi \in \mathscr{G}$ we obtain

$$
\left| A[\nu_\ell](\xi)^2 \right| = \sum_{i=1}^{2n} \left| \lambda_i^{(\ell)}(\xi)^2 \right| = \sum_{i=1}^{2n} \left(\left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 \right)^2 .
$$

In order to prove that the causal action takes finite values for every negative definite measure ν_ℓ, we consider the following inequality:

$$
\begin{aligned}
\mathcal{S}[\nu_\ell] &= \int_{\mathscr{G}} \mathcal{L}[\nu_\ell](\xi)\, \mathrm{d}\mu(\xi) \\
&= \int_{\mathscr{G}} \left(\left| A[\nu_\ell](\xi)^2 \right| - \frac{1}{2n} \left| A[\nu_\ell](\xi) \right|^2 \right) \mathrm{d}\mu(\xi) \\
&= \int_{\mathscr{G}} \left| A[\nu_\ell](\xi)^2 \right|\, \mathrm{d}\mu(\xi) - \frac{1}{2n} \int_{\mathscr{G}} \left| A[\nu_\ell](\xi) \right|^2\, \mathrm{d}\mu(\xi) \\
&\leq \int_{\mathscr{G}} \left| A[\nu_\ell](\xi)^2 \right|\, \mathrm{d}\mu(\xi) + \frac{1}{2n} \int_{\mathscr{G}} \left| A[\nu_\ell](\xi) \right|^2\, \mathrm{d}\mu(\xi) \\
&= \sum_{i=1}^{2n} \int_{\mathscr{G}} \left(\left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 \right)^2 \mathrm{d}\mu(\xi) \\
&\quad + \frac{1}{2n} \int_{\mathscr{G}} \left(\sum_{i=1}^{2n} \left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 \right)^2 \mathrm{d}\mu(\xi).
\end{aligned}
$$

Considering the calculation

$$
\begin{aligned}
&\left(\sum_{i=1}^{2n} \left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 \right)^2 \\
&= \sum_{i,j=1}^{2n} \left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 \left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(j)}(k) \right)^2 \\
&+ \sum_{i,j=1}^{2n} \left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(j)}(k) \right)^2
\end{aligned}
$$

$$+\sum_{i,j=1}^{2n}\left(\int_{\mathscr{G}^*}\sin(k\xi)\,\mathrm{d}\eta_\ell^{(i)}(k)\right)^2\left(\int_{\mathscr{G}^*}\cos(k\xi)\,\mathrm{d}\eta_\ell^{(j)}(k)\right)^2$$
$$+\sum_{i,j=1}^{2n}\left(\int_{\mathscr{G}^*}\sin(k\xi)\,\mathrm{d}\eta_\ell^{(i)}(k)\right)^2\left(\int_{\mathscr{G}^*}\sin(k\xi)\,\mathrm{d}\eta_\ell^{(j)}(k)\right)^2,$$

it suffices to show that each summand is bounded from above. To this aim, we have to prove the existence of a constant $C > 0$ such that for all $i, j = 1, \dots, 2n$ holds

$$\int_{\mathscr{G}}\left(\left(\int_{\mathscr{G}^*}\cos(k\xi)\,\mathrm{d}\eta_\ell^{(i)}(k)\right)^2+\left(\int_{\mathscr{G}^*}\sin(k\xi)\,\mathrm{d}\eta_\ell^{(i)}(k)\right)^2\right)^2\mathrm{d}\mu(\xi)\le C,$$
$$\int_{\mathscr{G}}\left(\int_{\mathscr{G}^*}\cos(k\xi)\,\mathrm{d}\eta_\ell^{(i)}(k)\right)^2\left(\int_{\mathscr{G}^*}\cos(k\xi)\,\mathrm{d}\eta_\ell^{(j)}(k)\right)^2\mathrm{d}\mu(\xi)\le C,$$
$$\int_{\mathscr{G}}\left(\int_{\mathscr{G}^*}\cos(k\xi)\,\mathrm{d}\eta_\ell^{(i)}(k)\right)^2\left(\int_{\mathscr{G}^*}\sin(k\xi)\,\mathrm{d}\eta_\ell^{(j)}(k)\right)^2\mathrm{d}\mu(\xi)\le C,$$
$$\int_{\mathscr{G}}\left(\int_{\mathscr{G}^*}\sin(k\xi)\,\mathrm{d}\eta_\ell^{(i)}(k)\right)^2\left(\int_{\mathscr{G}^*}\sin(k\xi)\,\mathrm{d}\eta_\ell^{(j)}(k)\right)^2\mathrm{d}\mu(\xi)\le C.$$

Using the addition theorems[1]

$$\cos(z+w) = \cos z\cos w - \sin z\sin w,$$
$$\sin(z+w) = \sin z\cos w + \cos z\sin w$$

for any $z, w \in \mathbb{C}$, for all $u, v, w, z \in \mathbb{C}$ we achieve the general expression

$$\begin{aligned}
&\cos(u+v+w+z) = \cos(u)\cos(v+w+z) - \sin(u)\sin(v+w+z)\\
&= \cos(u)\,(\cos(v)\cos(w+z) - \sin(v)\sin(w+z))\\
&- \sin(u)\,(\sin(v)\cos(w+z) + \cos(v)\sin(w+z))\\
&= \cos(u)\,(\cos(v)\,(\cos(w)\cos(z) - \sin(w)\sin(z)) - \sin(v)\,(\sin(w)\cos(z) + \cos(w)\sin(z)))\\
&- \sin(u)\,(\sin(v)\,(\cos(w)\cos(z) - \sin(w)\sin(z)) + \cos(v)\,(\sin(w)\cos(z) + \cos(w)\sin(z)))\\
&= \cos(u)\,(\cos(v)\cos(w)\cos(z) - \cos(v)\sin(w)\sin(z) - \sin(v)\sin(w)\cos(z) - \sin(v)\cos(w)\sin(z))\\
&- \sin(u)\,(\sin(v)\cos(w)\cos(z) - \sin(v)\sin(w)\sin(z) + \cos(v)\sin(w)\cos(z) + \cos(v)\cos(w)\sin(z))\\
&= \cos(u)\cos(v)\cos(w)\cos(z) - \cos(u)\cos(v)\sin(w)\sin(z)\\
&- \cos(u)\sin(v)\sin(w)\cos(z) - \cos(u)\sin(v)\cos(w)\sin(z)\\
&- \sin(u)\sin(v)\cos(w)\cos(z) + \sin(u)\sin(v)\sin(w)\sin(z)\\
&- \sin(u)\cos(v)\sin(w)\cos(z) - \sin(u)\cos(v)\cos(w)\sin(z).
\end{aligned}$$

[1] See e. g. [AE02, Satz III.6.3].

Similarly, we obtain the following expression:

$$
\begin{aligned}
&\sin(u+v+w+z) = \sin(u)\cos(v+w+z) + \cos(u)\sin(v+w+z) \\
&= \sin(u)\,(\cos(v)\cos(w+z) - \sin(v)\sin(w+z)) \\
&+ \cos(u)\,(\sin(v)\cos(w+z) + \cos(v)\sin(w+z)) \\
&= \sin(u)\,(\cos(v)\,(\cos(w)\cos(z) - \sin(w)\sin(z)) - \sin(v)\,(\sin(w)\cos(z) + \cos(w)\sin(z))) \\
&+ \cos(u)\,(\sin(v)\,(\cos(w)\cos(z) - \sin(w)\sin(z)) + \cos(v)\,(\sin(w)\cos(z) + \cos(w)\sin(z))) \\
&= \sin(u)\,(\cos(v)\cos(w)\cos(z) - \cos(v)\sin(w)\sin(z) - \sin(v)\sin(w)\cos(z) + \sin(v)\cos(w)\sin(z)) \\
&+ \cos(u)\,(\sin(v)\cos(w)\cos(z) - \sin(v)\sin(w)\sin(z) + \cos(v)\sin(w)\cos(z) + \cos(v)\cos(w)\sin(z)) \\
&= \sin(u)\cos(v)\cos(w)\cos(z) - \sin(u)\cos(v)\sin(w)\sin(z) \\
&- \sin(u)\sin(v)\sin(w)\cos(z) + \sin(u)\sin(v)\cos(w)\sin(z) \\
&+ \cos(u)\sin(v)\cos(w)\cos(z) - \cos(u)\sin(v)\sin(w)\sin(z) \\
&+ \cos(u)\cos(v)\sin(w)\cos(z) + \cos(u)\cos(v)\cos(w)\sin(z)
\end{aligned}
$$

for all $u, v, w, z \in \mathbb{C}$.

Let $\mathrm{C}_\ell^{(i)}$ be the multiplicative constant of the Borel measure as defined in (5.5), i. e.

$$\eta_\ell^{(i)} = -s_i\, \mathrm{C}_\ell^{(i)}\, \beta^4\big|_{[-\ell,\ell]^4} \qquad (i = 1, \ldots, 2n)$$

where $s_i = S_{i,i}$ with $S = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$. Choosing $u = k_0\,\xi_0$, $v = k_1\,\xi_1$, $w = k_2\,\xi_2$, and $z = k_3\,\xi_3$ with $k \in \mathscr{G}^*$, $\xi \in \mathscr{G}$, we obtain for all $i = 1, \ldots, 2n$

$$
\begin{aligned}
&\frac{1}{\mathrm{C}_\ell^{(i)}} \int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) = \int_{(-\ell,\ell)^4} \cos(k\xi)\, \mathrm{d}k_0\, \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3 \\
&= \int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \cos(\sum_{i=0}^{3} k_i\,\xi_i)\, \mathrm{d}k_0\, \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3 \\
&= \int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \cos(k_0\,\xi_0)\cos(k_1\,\xi_1)\cos(k_2\,\xi_2)\cos(k_3\,\xi_3)\, \mathrm{d}k_0\, \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3 \\
&- \int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \cos(k_0\,\xi_0)\cos(k_1\,\xi_1)\sin(k_2\,\xi_2)\sin(k_3\,\xi_3)\, \mathrm{d}k_0\, \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3 \\
&- \int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \cos(k_0\,\xi_0)\sin(k_1\,\xi_1)\sin(k_2\,\xi_2)\cos(k_3\,\xi_3)\, \mathrm{d}k_0\, \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3 \\
&- \int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \cos(k_0\,\xi_0)\sin(k_1\,\xi_1)\cos(k_2\,\xi_2)\sin(k_3\,\xi_3)\, \mathrm{d}k_0\, \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3 \\
&- \int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \sin(k_0\,\xi_0)\sin(k_1\,\xi_1)\cos(k_2\,\xi_2)\cos(k_3\,\xi_3)\, \mathrm{d}k_0\, \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \sin(k_0\,\xi_0)\sin(k_1\,\xi_1)\sin(k_2\,\xi_2)\sin(k_3\,\xi_3)\,\mathrm{d}k_0\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\\
&- \int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \sin(k_0\,\xi_0)\cos(k_1\,\xi_1)\sin(k_2\,\xi_2)\cos(k_3\,\xi_3)\,\mathrm{d}k_0\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\\
&- \int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \sin(k_0\,\xi_0)\cos(k_1\,\xi_1)\cos(k_2\,\xi_2)\sin(k_3\,\xi_3)\,\mathrm{d}k_0\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\\
&= \int_{-\ell}^{\ell}\cos(k_0\,\xi_0)\,\mathrm{d}k_0 \int_{-\ell}^{\ell}\cos(k_1\,\xi_1)\,\mathrm{d}k_1 \int_{-\ell}^{\ell}\cos(k_2\,\xi_2)\,\mathrm{d}k_2 \int_{-\ell}^{\ell}\cos(k_3\,\xi_3)\,\mathrm{d}k_3\\
&- \int_{-\ell}^{\ell}\cos(k_0\,\xi_0)\,\mathrm{d}k_0 \int_{-\ell}^{\ell}\cos(k_1\,\xi_1)\,\mathrm{d}k_1 \int_{-\ell}^{\ell}\sin(k_2\,\xi_2)\,\mathrm{d}k_2 \int_{-\ell}^{\ell}\sin(k_3\,\xi_3)\,\mathrm{d}k_3\\
&- \int_{-\ell}^{\ell}\cos(k_0\,\xi_0)\,\mathrm{d}k_0 \int_{-\ell}^{\ell}\sin(k_1\,\xi_1)\,\mathrm{d}k_1 \int_{-\ell}^{\ell}\sin(k_2\,\xi_2)\,\mathrm{d}k_2 \int_{-\ell}^{\ell}\cos(k_3\,\xi_3)\,\mathrm{d}k_3\\
&- \int_{-\ell}^{\ell}\cos(k_0\,\xi_0)\,\mathrm{d}k_0 \int_{-\ell}^{\ell}\sin(k_1\,\xi_1)\,\mathrm{d}k_1 \int_{-\ell}^{\ell}\cos(k_2\,\xi_2)\,\mathrm{d}k_2 \int_{-\ell}^{\ell}\sin(k_3\,\xi_3)\,\mathrm{d}k_3\\
&- \int_{-\ell}^{\ell}\sin(k_0\,\xi_0)\,\mathrm{d}k_0 \int_{-\ell}^{\ell}\sin(k_1\,\xi_1)\,\mathrm{d}k_1 \int_{-\ell}^{\ell}\cos(k_2\,\xi_2)\,\mathrm{d}k_2 \int_{-\ell}^{\ell}\cos(k_3\,\xi_3)\,\mathrm{d}k_3\\
&+ \int_{-\ell}^{\ell}\sin(k_0\,\xi_0)\,\mathrm{d}k_0 \int_{-\ell}^{\ell}\sin(k_1\,\xi_1)\,\mathrm{d}k_1 \int_{-\ell}^{\ell}\sin(k_2\,\xi_2)\,\mathrm{d}k_2 \int_{-\ell}^{\ell}\sin(k_3\,\xi_3)\,\mathrm{d}k_3\\
&- \int_{-\ell}^{\ell}\sin(k_0\,\xi_0)\,\mathrm{d}k_0 \int_{-\ell}^{\ell}\cos(k_1\,\xi_1)\,\mathrm{d}k_1 \int_{-\ell}^{\ell}\sin(k_2\,\xi_2)\,\mathrm{d}k_2 \int_{-\ell}^{\ell}\cos(k_3\,\xi_3)\,\mathrm{d}k_3\\
&- \int_{-\ell}^{\ell}\sin(k_0\,\xi_0)\,\mathrm{d}k_0 \int_{-\ell}^{\ell}\cos(k_1\,\xi_1)\,\mathrm{d}k_1 \int_{-\ell}^{\ell}\cos(k_2\,\xi_2)\,\mathrm{d}k_2 \int_{-\ell}^{\ell}\sin(k_3\,\xi_3)\,\mathrm{d}k_3.
\end{aligned}
$$

Considering

$$\int_{\mathscr{G}^*} \sin(k\xi)\,\mathrm{d}\eta_\ell^{(i)}(k) = \mathrm{C}_\ell^{(i)} \int_{(-\ell,\ell)^4} \sin(k\xi)\,\mathrm{d}k_0\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3$$

instead, we will obtain a similar result. For simplicity, we will omit the superscripts in $\mathrm{C}_\ell^{(i)}$ and $\eta_\ell^{(i)}$ in the following, writing C_ℓ and η_ℓ instead. The substitution rule is given by[2]

$$\int_a^b f(\phi(x))\,\phi'(x)\,\mathrm{d}x = \int_{\phi(a)}^{\phi(b)} f(y)\,\mathrm{d}y.$$

In our case, we have $\phi(k_i) = \xi_i\,k_i$ with $\phi'(k_i) = \xi_i$ independent of k_i, yielding

$$\int_{-\ell}^{\ell} \cos(k_i\,\xi_i)\,\xi_i\,\mathrm{d}k_i = \int_{-\ell\xi_i}^{\ell\xi_i} \cos(k_i')\,\mathrm{d}k_i'.$$

Considering $\xi_i \in \mathbb{R} \setminus [-\pi, \pi]$, we may divide by $\xi_i \neq 0$ obtaining the inequality

$$\int_{-\ell}^{\ell} \cos(k_i\,\xi_i)\,\mathrm{d}k_i = \frac{1}{\xi_i}\int_{-\ell\xi_i}^{\ell\xi_i} \cos(k_i')\,\mathrm{d}k_i' \leq \frac{1}{\xi_i}\int_{-\pi/2}^{\pi/2} \cos(k_i')\,\mathrm{d}k_i' = \frac{2}{\xi_i}.$$

[2]See e. g. [AE06, Theorem 5.1].

Analogously, we achieve

$$\int_{-\ell}^{\ell} \sin(k_i\,\xi_i)\,\mathrm{d}k_i = \frac{1}{\xi_i}\int_{-\ell\xi_i}^{\ell\xi_i} \sin(k_i')\,\mathrm{d}k_i' \leq \frac{1}{\xi_i}\int_0^{\pi} \sin(k_i')\,\mathrm{d}k_i' = \frac{2}{\xi_i}$$

since $\ell\xi_i \geq \pi$ for all $\ell \in \mathbb{N}$ and[3]

$$\int_{-\pi/2}^{\pi/2} \cos(x)\,\mathrm{d}x = \int_0^{\pi} \sin(x)\,\mathrm{d}x = 2.$$

Hence the absolute value of the above expression yields the following inequality:

$$\left|\int_{\mathscr{G}^*} \cos(k\xi)\,\mathrm{d}\eta_\ell(k)\right| \leq 8\,\mathrm{C}_\ell \prod_{i=0}^{3} \frac{2}{|\xi_i|} \leq 2^3 \cdot 2^4\,\mathrm{C}_\ell \prod_{i=0}^{3} \frac{1}{|\xi_i|} = 2^7\,\mathrm{C}_\ell \prod_{i=0}^{3} \frac{1}{|\xi_i|}.$$

Squaring the above term yields

$$\left(\int_{\mathscr{G}^*} \cos(k\xi)\,\mathrm{d}\eta_\ell(k)\right)^2 = \left(\left|\int_{\mathscr{G}^*} \cos(k\xi)\,\mathrm{d}\eta_\ell(k)\right|\right)^2 \leq 2^{14}\,\mathrm{C}_\ell^2 \prod_{i=0}^{3} \frac{1}{|\xi_i|^2}. \tag{5.6}$$

Analogously, we obtain the following inequality:

$$\left(\int_{\mathscr{G}^*} \sin(k\xi)\,\mathrm{d}\eta_\ell(k)\right)^2 = \left(\left|\int_{\mathscr{G}^*} \sin(k\xi)\,\mathrm{d}\eta_\ell(k)\right|\right)^2 \leq 2^{14}\,\mathrm{C}_\ell^2 \prod_{i=0}^{3} \frac{1}{|\xi_i|^2}. \tag{5.7}$$

Taken together, we achieve

$$\left(\int_{\mathscr{G}^*} \cos(k\xi)\,\mathrm{d}\eta_\ell(k)\right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi)\,\mathrm{d}\eta_\ell(k)\right)^2 \leq 2^{15}\,\mathrm{C}_\ell^2 \prod_{i=0}^{3} \frac{1}{|\xi_i|^2}. \tag{5.8}$$

Squaring the last expression leads to the following inequality:

$$\left(\left(\int_{\mathscr{G}^*} \cos(k\xi)\,\mathrm{d}\eta_\ell(k)\right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi)\,\mathrm{d}\eta_\ell(k)\right)^2\right)^2 \leq \left(2^{15}\,\mathrm{C}_\ell^2 \prod_{i=0}^{3} \frac{1}{|\xi_i|^2}\right)^2 = 2^{30}\,\mathrm{C}_\ell^4 \prod_{i=0}^{3} \frac{1}{|\xi_i|^4}$$

[3]The integrals can be computed as follows:

$$\int_{-\pi/2}^{\pi/2} \cos(x)\,\mathrm{d}x = [\sin(x)]_{-\pi/2}^{\pi/2} = 2\,\sin(\pi/2) = 2$$

and

$$\int_0^{\pi} \sin(x)\,\mathrm{d}x = [-\cos(x)]_0^{\pi} = -\cos(\pi) + \cos(0) = 1 + 1 = 2.$$

for all $\xi \in \mathscr{G} \setminus [-\pi,\pi]^4$. Considering the product instead, by (5.6) and (5.7) we obtain

$$\begin{aligned}
&\left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k)\right)^2 \left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(j)}(k)\right)^2 \\
&\leq \left(2^{14} \left(\mathrm{C}_\ell^{(i)}\right)^2 \prod_{m=0}^{3} \frac{1}{|\xi_m|^2}\right)\left(2^{14} \left(\mathrm{C}_\ell^{(j)}\right)^2 \prod_{m=0}^{3} \frac{1}{|\xi_m|^2}\right) \\
&= 2^{28} \left(\mathrm{C}_\ell^{(i)}\right)^2 \left(\mathrm{C}_\ell^{(j)}\right)^2 \left(\prod_{m=0}^{3} \frac{1}{|\xi_m|^2}\right)^2 = 2^{28} \left(\mathrm{C}_\ell^{(i)}\right)^2 \left(\mathrm{C}_\ell^{(j)}\right)^2 \prod_{m=0}^{3} \frac{1}{|\xi_m|^4}
\end{aligned} \tag{5.9}$$

for all $i, j = 1, \ldots, 2n$. Analogously, we achieve

$$\left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k)\right)^2 \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(j)}(k)\right)^2 \leq 2^{28} \left(\mathrm{C}_\ell^{(i)}\right)^2 \left(\mathrm{C}_\ell^{(j)}\right)^2 \prod_{m=0}^{3} \frac{1}{|\xi_m|^4} \tag{5.10}$$

and

$$\left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k)\right)^2 \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(j)}(k)\right)^2 \leq 2^{28} \left(\mathrm{C}_\ell^{(i)}\right)^2 \left(\mathrm{C}_\ell^{(j)}\right)^2 \prod_{m=0}^{3} \frac{1}{|\xi_m|^4}. \tag{5.11}$$

Considering the general calculation

$$\int_\pi^\infty \frac{1}{x^4}\, \mathrm{d}x = \lim_{\substack{b\to\infty \\ a\to\pi}} [-\frac{1}{3}\frac{1}{x^3}]_a^b = \lim_{\substack{b\to\infty \\ a\to\pi}} \left(-\frac{1}{3}\frac{1}{b^3} + \frac{1}{3}\frac{1}{a^3}\right) = \frac{1}{3\pi^3},$$

integrating over $\mathscr{G} \setminus [-\pi,\pi]^4$ leads to the following result:

$$\begin{aligned}
&\int_{\mathscr{G}\setminus[-\pi,\pi]^4} \left(\left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell(k)\right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell(k)\right)^2\right)^2 \mathrm{d}\mu(\xi) \\
&\leq 2^{30}\, \mathrm{C}_\ell^4\, \mathrm{C}_\mu \int_{\mathbb{R}\setminus[-\pi,\pi]} \int_{\mathbb{R}\setminus[-\pi,\pi]} \int_{\mathbb{R}\setminus[-\pi,\pi]} \int_{\mathbb{R}\setminus[-\pi,\pi]} \prod_{i=0}^{3} \frac{1}{|\xi_i|^4}\, \mathrm{d}\xi_0\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\xi_3 \\
&= 2^{30}\, \mathrm{C}_\ell^4\, \mathrm{C}_\mu \prod_{i=0}^{3} \int_{\mathbb{R}\setminus[-\pi,\pi]} \frac{1}{|\xi_i|^4}\, \mathrm{d}\xi_i = 2^4 \cdot 2^{30}\, \mathrm{C}_\ell^4\, \mathrm{C}_\mu \prod_{i=0}^{3} \int_\pi^\infty \frac{1}{|\xi_i|^4}\, \mathrm{d}\xi_i \\
&= \frac{2^{34}}{(3\pi^3)^4}\, \mathrm{C}_\ell^4\, \mathrm{C}_\mu,
\end{aligned}$$

where the multiplicative factor C_μ shows up due to the fact that the Haar measure μ is a multiple of the Lebesgue measure, i. e. $\mathrm{d}\mu = \mathrm{C}_\mu\, \mathrm{d}\lambda^4$. Boundedness of the remaining integral can be proven as follows:

$$\int_{[-\pi,\pi]^4} \left(\left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell(k)\right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell(k)\right)^2\right)^2 \mathrm{d}\mu(\xi)$$

$$
\begin{aligned}
&\leq \int_{[-\pi,\pi]^4} \left(|\eta_\ell(\mathscr{G}^*)|^2 + |\eta_\ell(\mathscr{G}^*)|^2\right)^2 \, \mathrm{d}\mu(\xi) \\
&= 4 \int_{[-\pi,\pi]^4} |\eta_\ell(\mathscr{G}^*)|^4 \, \mathrm{d}\mu(\xi) \\
&= 4 \, |\eta_\ell(\mathscr{G}^*)|^4 \, \mu([-\pi,\pi]^4) < \infty,
\end{aligned}
$$

since the Haar measure of compact sets is finite. Considering the product, by (5.9) we obtain for $i, j = 1, \ldots, 2n$

$$
\begin{aligned}
&\int_{\mathscr{G}\setminus[-\pi,\pi]^4} \left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k)\right)^2 \left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(j)}(k)\right)^2 \mathrm{d}\mu(\xi) \\
&\leq 2^{28} \left(\mathrm{C}_\ell^{(i)}\right)^2 \left(\mathrm{C}_\ell^{(j)}\right)^2 \int_{\mathscr{G}\setminus[-\pi,\pi]^4} \prod_{m=0}^{3} \frac{1}{|\xi_m|^4} \, \mathrm{d}\mu(\xi) \\
&= 2^{32} \left(\mathrm{C}_\ell^{(i)}\right)^2 \left(\mathrm{C}_\ell^{(j)}\right)^2 \mathrm{C}_\mu \prod_{m=0}^{3} \int_{\pi}^{\infty} \frac{1}{|\xi_m|^4} \, \mathrm{d}\xi_m \\
&= \frac{2^{32}}{(3\pi^3)^4} \left(\mathrm{C}_\ell^{(i)}\right)^2 \left(\mathrm{C}_\ell^{(j)}\right)^2 \mathrm{C}_\mu.
\end{aligned}
$$

Analogously, by (5.10) and (5.11) we have

$$
\int_{\mathscr{G}\setminus[-\pi,\pi]^4} \left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k)\right)^2 \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(j)}(k)\right)^2 \mathrm{d}\mu(\xi) \leq \frac{2^{32}}{(3\pi^3)^4} \left(\mathrm{C}_\ell^{(i)}\right)^2 \left(\mathrm{C}_\ell^{(j)}\right)^2 \mathrm{C}_\mu
$$

and

$$
\int_{\mathscr{G}\setminus[-\pi,\pi]^4} \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k)\right)^2 \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(j)}(k)\right)^2 \mathrm{d}\mu(\xi) \leq \frac{2^{32}}{(3\pi^3)^4} \left(\mathrm{C}_\ell^{(i)}\right)^2 \left(\mathrm{C}_\ell^{(j)}\right)^2 \mathrm{C}_\mu.
$$

Calculation of the remaining integral yields for all $i, j = 1, \ldots, 2n$

$$
\begin{aligned}
&\int_{[-\pi,\pi]^4} \left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k)\right)^2 \left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(j)}(k)\right)^2 \mathrm{d}\mu(\xi) \\
&\leq \int_{[-\pi,\pi]^4} \eta_\ell^{(i)}(\mathscr{G}^*)^2\, \eta_\ell^{(j)}(\mathscr{G}^*)^2 \, \mathrm{d}\mu(\xi) = \eta_\ell^{(i)}(\mathscr{G}^*)^2\, \eta_\ell^{(j)}(\mathscr{G}^*)^2 \, \mu\left([-\pi,\pi]^4\right) < \infty
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\int_{[-\pi,\pi]^4} \left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k)\right)^2 \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(j)}(k)\right)^2 \mathrm{d}\mu(\xi) \\
&\quad \leq \eta_\ell^{(i)}(\mathscr{G}^*)^2\, \eta_\ell^{(j)}(\mathscr{G}^*)^2 \, \mu\left([-\pi,\pi]^4\right) < \infty
\end{aligned}
$$

and

$$
\int_{[-\pi,\pi]^4} \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k)\right)^2 \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(j)}(k)\right)^2 \mathrm{d}\mu(\xi)
$$

$$\leq \eta_\ell^{(i)}(\mathscr{G}^*)^2\, \eta_\ell^{(j)}(\mathscr{G}^*)^2\, \mu\left([-\pi,\pi]^4\right) < \infty.$$

Taken together, we finally achieve the following estimation:

$$\begin{aligned}
\mathcal{S}[\nu_\ell] \leq{}& \sum_{i=1}^{2n} \int_{\mathscr{G}} \left(\left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 \right)^2 \mathrm{d}\mu(\xi) \\
&+ \frac{1}{2n} \int_{\mathscr{G}} \left(\sum_{i=1}^{2n} \left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\eta_\ell^{(i)}(k) \right)^2 \right)^2 \mathrm{d}\mu(\xi) \\
\leq{}& \sum_{i=1}^{2n} \left(\frac{2^{34}}{3^4\, \pi^{12}} \left(\mathrm{C}_\ell^{(i)}\right)^4 \mathrm{C}_\mu + 4 \left|\eta_\ell^{(i)}(\mathscr{G}^*)\right|^4 \mu([-\pi,\pi]^4) \right) \\
&+ \frac{4}{2n} \sum_{i,j=1}^{2n} \left(\frac{2^{32}}{(3\pi^3)^4} \left(\mathrm{C}_\ell^{(i)}\right)^2 \left(\mathrm{C}_\ell^{(j)}\right)^2 \mathrm{C}_\mu + \eta_\ell^{(i)}(\mathscr{G}^*)^2\, \eta_\ell^{(j)}(\mathscr{G}^*)^2\, \mu\left([-\pi,\pi]^4\right) \right) \\
<{}& \infty.
\end{aligned}$$

Hence, the causal action is finite for all $\ell \in \mathbb{N}$. This concludes the example.

Remark 5.9. Example 5.8 illustrates the existence of negative definite measures with compact support satisfying trace constraint (5.2), for which the causal action takes finite values. Therefore, the causal action converges towards a finite value for any minimizing sequence. This proves the causal action principle (5.4) being well-posed.

5.4. Transformation of Negative Definite Measures

Example 5.8 shows that there are negative definite measures with compact support which satisfy trace constraint (5.2) such that the causal action is indeed finite. The crucial question is: Is it possible to prove that the causal action diverges if the sequence of negative definite measures is not uniformly bounded? Then, every minimizing sequence would be uniformly bounded so that we may apply Banach-Alaoglu's or Prochorov's theorem which yield the existence of a weak-* or weak convergent subsequence, respectively. For this reason the idea could be to construct a contradiction: Assume there was a minimizing sequence of negative definite measures with compact support satisfying trace constraint (5.2) which diverges at least in one component, i. e. there is at least one $i = 1, \ldots, 2n$ such that

$$|\nu_\ell(\mathscr{G}^*)_{ii}| \to_{\ell\to\infty} \infty.$$

The hope is to show that this condition is sufficient for the causal action to diverge, $\mathcal{S}[\nu_\ell] \to_{\ell\to\infty} \infty$. If this holds for arbitrary divergent sequences, we may conclude that minimizing sequences contain a convergent subsequence.

In order to prove such a divergence condition we have the possibility to transform the sequence of negative definite measures such that its support contains zero. To this aim, let us first state the following lemma:

Lemma 5.10. *Let ν be a negative definite measure on $\mathscr{G}^*$ and let $q \in \mathscr{G}^*$. Then, the transformation*

$$\nu \mapsto T_q\nu, \qquad (T_q\nu)(\Omega) := \nu(\Omega - q) \qquad (\Omega \in \mathscr{B}(\mathscr{G}^*))$$

leaves the Lagrangian $\mathcal{L}$, the causal action $\mathcal{S}$, and the functional $\mathcal{T}$ unchanged.

Proof. It suffices to prove that $\mathcal{L}[T_q\nu] = \mathcal{L}[\nu]$ for any $q \in \mathscr{G}^*$. To this aim, we want to show that the transformation $\nu \mapsto T_q\nu$ leaves the closed chain A unchanged, which implies that $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{L}$ stay also unchanged. Considering

$$P[\nu](\xi) = \int_{\mathscr{G}^*} e^{ik\xi}\, d\nu(k)$$

for any $\xi \in \mathscr{G}$, on the one hand we obtain

$$\begin{aligned} P[T_q\nu](\xi) &= \int_{\mathscr{G}^*} e^{ik\xi}\, dT_q\nu(k) = \int_{\mathscr{G}^*} e^{ik\xi}\, d\nu(k-q) = \int_{\mathscr{G}^*} e^{i(k+q)\xi}\, d\nu(k) \\ &= \int_{\mathscr{G}^*} e^{ik\xi}\, e^{iq\xi}\, d\nu(k) = e^{iq\xi} \int_{\mathscr{G}^*} e^{ik\xi}\, d\nu(k) = e^{iq\xi}\, P[\nu](\xi) \end{aligned}$$

for all $\xi \in \mathscr{G}$. Analogously we have

$$P[T_q\nu](-\xi) = e^{-iq\xi}\, P[\nu](-\xi) \qquad (\xi \in \mathscr{G}).$$

Considering the closed chain $A(\xi) := P(\xi)\, P(-\xi)$ for any $\xi \in \mathscr{G}$, we obtain

$$\begin{aligned} A[T_q\nu](\xi) &:= P[T_q\nu](\xi)\, P[T_q\nu](-\xi) = e^{iq\xi}\, P[\nu](\xi)\, e^{-iq\xi}\, P[\nu](-\xi) \\ &= P[\nu](\xi)\, P[\nu](-\xi) = A[\nu](\xi). \end{aligned}$$

Hence, any translation of the negative definite measure ν by $q \in \mathscr{G}^*$ leaves the closed chain $A(\xi)$ unchanged, and thus the Lagrangian $\mathcal{L}$,

$$\begin{aligned} \mathcal{L}[T_q\nu](\xi) &= \left|A[T_q\nu](\xi)^2\right| - \frac{1}{2n}\left|A[T_q\nu](\xi)\right|^2 \\ &= \left|A[\nu](\xi)^2\right| - \frac{1}{2n}\, |A[\nu](\xi)|^2 = \mathcal{L}[\nu](\xi), \end{aligned}$$

the causal action $\mathcal{S}$,

$$\mathcal{S}[T_q\nu] = \int_{\mathscr{G}} \mathcal{L}[T_q\nu](\xi)\,\mathrm{d}\mu(\xi) = \int_{\mathscr{G}} \mathcal{L}[\nu](\xi)\,\mathrm{d}\mu(\xi) = \mathcal{S}[\nu],$$

and the functional $\mathcal{T}$:

$$\mathcal{T}[T_q\nu] = \int_{\mathscr{G}} \left|A[T_q\nu](\xi)\right|^2\,\mathrm{d}\mu(\xi) = \int_{\mathscr{G}} |A[\nu](\xi)|^2\,\mathrm{d}\mu(\xi) = \mathcal{T}[\nu].$$

This concludes the proof. □

Remark 5.11. Let $(\nu_\ell)_{\ell\in\mathbb{N}}$ be a minimizing sequence for the causal action $\mathcal{S}$. By lemma 5.10, we may assume without loss of generality that $0 \in \operatorname{supp}\nu_\ell$ for all $\ell \in \mathbb{N}$. Otherwise, one can transform the compact support of the negative definite measure in a suitable way which leaves the functionals $\mathcal{L}$, $\mathcal{S}$, and $\mathcal{T}$ unchanged. Moreover, we may assume that

$$\operatorname{diam}(\operatorname{supp}\nu_\ell) \geq \ell \qquad \text{for all } \ell \in \mathbb{N}.$$

If the last inequality was not true, the support of ν_ℓ would be uniformly bounded for all $\ell \in \mathbb{N}$. Then we could find a compact set $\hat{K} \subset \mathscr{G}^*$ such that $\operatorname{supp}\nu_\ell \subset \hat{K}$ for all $\ell \in \mathbb{N}$. However, this case was already treated in [Fin10, Theorem 4.2].

5.5. Existence of Minimizers

As noted above, the causal action principle crucially depends on the eigenvalues of the closed chain. If the operator $\nu(\mathscr{G}^*)$ can be represented as a diagonal matrix, then its diagonal entries are exactly its eigenvalues. The following lemma 5.12 allows us to make a statement about the eigenvalues of a positive operator onto a finite-dimensional vector space V:

Lemma 5.12. *Let $(V, \prec \cdot \mid \cdot \succ)$ be a $2n$-dimensional indefinite inner product space and let B be a positive linear operator on V, i. e. for all $u \in V$ holds $\prec u \mid B u \succ \geq 0$. Then, for any $\varepsilon > 0$ there is a unitary transformation U on $(V, \prec \cdot \mid \cdot \succ)$ such that*

$$UBU^{-1} = -\operatorname{diag}(\nu_1, \ldots, \nu_{2n}) + \Delta B, \tag{5.12}$$

where the real parameters ν_i can be ordered as

$$\nu_1 \leq \ldots \leq \nu_n \leq 0 \leq \nu_{n+1} \leq \ldots \leq \nu_{2n} \tag{5.13}$$

and $\|\Delta B\| < \varepsilon$.

Proof. See [Fin10, Lemma 4.4]. □

The idea in order to prove the existence of convergent subsequences of minimizing sequences for the causal action is to show that minimizing sequences of negative definite measures are uniformly bounded. Using lemma 5.12, the vague hope is to prove the following conjecture:

Conjecture 5.13. *Let V be a $2n$-dimensional indefinite inner product space. Let $(\nu_\ell)_{\ell\in\mathbb{N}} \subset \mathrm{NDM}(\mathscr{G}^*;\mathrm{L}(V))^1_{\mathrm{c}}$ be a minimizing sequence for the causal action principle* (5.4)*, satisfying trace constraint* (5.2) *and boundedness constraint* (5.3)*. Then, the sequence of negative definite measures $(\nu_\ell)_{\ell\in\mathbb{N}}$ is uniformly bounded.*

Sketch of proving ideas: The idea in order to apply Banach-Alaoglu's or Prochorov's theorem is to prove that the total variation of a minimizing sequence of negative definite measures $(\nu_\ell)_{\ell\in\mathbb{N}}$ is uniformly bounded. Assuming boundedness constraint (5.3), the causal action is bounded from below for any negative definite measure ν,

$$\begin{aligned}
\mathcal{S}[\nu] &= \int_{\mathscr{G}} \mathcal{L}[\nu](\xi)\,\mathrm{d}\mu(\xi) = \int_{\mathscr{G}} \left(\left|A[\nu](\xi)^2\right| - \frac{1}{2n}\left|A[\nu](\xi)\right|^2 \right) \mathrm{d}\mu(\xi) \\
&= \int_{\mathscr{G}} \left|A[\nu](\xi)^2\right| \,\mathrm{d}\mu(\xi) - \int_{\mathscr{G}} \frac{1}{2n}\left|A[\nu](\xi)\right|^2 \,\mathrm{d}\mu(\xi) \\
&\geq \int_{\mathscr{G}} \left|A[\nu](\xi)^2\right| \,\mathrm{d}\mu(\xi) - \frac{\mathrm{C}}{2n} \geq -\frac{\mathrm{C}}{2n}.
\end{aligned}$$

Due to example 5.8, the causal action $\mathcal{S}[\nu_\ell]$ converges to a finite value for any minimizing sequence $(\nu_\ell)_{\ell\in\mathbb{N}}$. Hence, the expression $\int_{\mathscr{G}} \left|A[\nu_\ell](\xi)^2\right| \mathrm{d}\mu(\xi)$ is bounded for any minimizing sequence $(\nu_\ell)_{\ell\in\mathbb{N}}$. Denoting the eigenvalues of the squared closed chain $A[\nu_\ell](\cdot)^2$ by $\lambda_i^{(\ell)}(\cdot)^2$ for any $\ell\in\mathbb{N}$, $i = 1,\dots,2n$, we thus may conclude

$$L^1(\mathscr{G}) \ni \left|A[\nu_\ell](\cdot)^2\right| = \sum_{i=1}^{2n} \left|\lambda_i^{(\ell)}(\cdot)^2\right|.$$

We even obtain

$$\left|\lambda_i^{(\ell)}(\cdot)^2\right| \in L^1(\mathscr{G})$$

for all $i = 1,\dots,2n$. However, it is not clear how deduce the total variation of the negative definite measures ν_ℓ being uniformly bounded.

The alternative way of proving conjecture 5.13 is to obtain a contradiction. To this aim, assume the minimizing sequence of negative definite measures *not* to be uniformly bounded, i. e. assume a subsequence such that

$$\|\nu_\ell\| \to_{\ell\to\infty} \infty. \tag{5.14}$$

Since $(\nu_\ell)_{\ell\in\mathbb{N}} \subset \mathrm{NDM}(\mathscr{G}^*; \mathrm{L}(V))^1_{\mathrm{c}}$ is by assumption a minimizing sequence, there is a positive constant $\mathrm{M} > 0$ and an $\ell_0 \in \mathbb{N}$ such that

$$\mathrm{M} \geq \mathcal{S}[\nu_\ell] \tag{5.15}$$

for all $\ell \geq \ell_0$ according to example 5.8 (without loss of generality consider $\ell_0 = 0$). The crucial question is: Is condition (5.14) sufficient to prove

$$\int_{\mathscr{G}} \left| A[\nu_\ell](\xi)^2 \right| \, \mathrm{d}\mu(\xi) \to_{\ell\to\infty} \infty? \tag{5.16}$$

Then, (5.16) would contradict condition (5.15), proving the desired result.

In order to extract a convergent subsequence of any minimizing sequence $(\nu_\ell)_{\ell\in\mathbb{N}}$, we have to show that assumption (5.14) implies condition (5.16). To this aim, we have to examine the closed chain $\left| A[\nu_\ell](\xi)^2 \right|$ for any $\xi \in \mathscr{G}$ in more detail. In order to prove condition (5.16), it suffices the integrand to decay too slowly as $|\xi| \to \infty$. Since the spectral weight of an operator is given by the sum of the absolute values of its eigenvalues, one could try to make a statement regarding the eigenvalues of $A[\nu_\ell](\xi)^2$. The easiest case would be $A[\nu_\ell](\xi)^2$ being a diagonal matrix. However, the form of $A[\nu_\ell](\xi)^2$ crucially depends on the negative definite measure ν_ℓ. Nonetheless, some ideas shall be explained right now.

For simplicity, *assume* $(\nu_\ell)_{ij} = 0$ for all $i, j = 1, \dots, 2n$, $i \neq j$. Then, the kernel of the fermionic projector $P[\nu_\ell](\xi)$ may be represented as a diagonal matrix for any $\xi \in \mathscr{G}$. Thus, the closed chain $A[\nu_\ell](\xi) = P[\nu_\ell](\xi)\, P[\nu_\ell](-\xi)$ as well as $A[\nu_\ell](\xi)^2$ are diagonal matrices for any $\xi \in \mathscr{G}$. The eigenvalues are given by the diagonal entries. Since $(\nu_\ell)_{ii}$ is a real measure for all $i = 1, \dots, 2n$, $\ell \in \mathbb{N}$, we achieve

$$\begin{aligned}
(A[\nu_\ell](\xi))_{ii} &= (P[\nu_\ell](\xi)\, P[\nu_\ell](-\xi))_{ii} = \sum_{k=1}^{2n} P[\nu_\ell](\xi)_{ik}\, P[\nu_\ell](-\xi)_{ki} = P[\nu_\ell](\xi)_{ii}\, P[\nu_\ell](-\xi)_{ii} \\
&= \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi} \, \mathrm{d}\nu_\ell(k)_{ii} \int_{\mathscr{G}^*} \mathrm{e}^{-\mathrm{i}k\xi} \, \mathrm{d}\nu_\ell(k)_{ii} = \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi} \, \mathrm{d}\nu_\ell(k)_{ii} \overline{\int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi} \, \mathrm{d}\nu_\ell(k)_{ii}} \\
&= \left| \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi} \, \mathrm{d}\nu_\ell(k)_{ii} \right|^2 = \left(\int_{\mathscr{G}^*} \cos(k\xi) \, \mathrm{d}\nu_\ell(k)_{ii} \right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi) \, \mathrm{d}\nu_\ell(k)_{ii} \right)^2 .
\end{aligned}$$

We immediately obtain

$$\begin{aligned}
(A[\nu_\ell](\xi)^2)_{ii} &= \left(\left| \int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi} \, \mathrm{d}\nu_\ell(k)_{ii} \right|^2 \right)^2 \\
&= \left(\left(\int_{\mathscr{G}^*} \cos(k\xi) \, \mathrm{d}\nu_\ell(k)_{ii} \right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi) \, \mathrm{d}\nu_\ell(k)_{ii} \right)^2 \right)^2
\end{aligned}$$

for all $i = 1, \dots, 2n$. Considering the spectral weight of $A[\nu_\ell](\xi)^2$, the latter identity yields

$$\begin{aligned}\left|A[\nu_\ell](\xi)^2\right| &= \sum_{i=1}^{2n} \left|(A[\nu_\ell](\xi)^2)_{ii}\right| \\ &= \sum_{i=1}^{2n} \left(\left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\nu_\ell(k)_{ii} \right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\nu_\ell(k)_{ii} \right)^2 \right)^2\end{aligned}$$

for any $\xi \in \mathscr{G}$. We finally obtain the following inequality:

$$\begin{aligned}\mathcal{S}[\nu_\ell] &\geq \int_{\mathscr{G}} \left|A[\nu](\xi)^2\right| \,\mathrm{d}\mu(\xi) - \frac{\mathrm{C}}{2n} \\ &= \sum_{i=1}^{2n} \int_{\mathscr{G}} \left(\left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\nu_\ell(k)_{ii} \right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\nu_\ell(k)_{ii} \right)^2 \right)^2 \mathrm{d}\mu(\xi) - \frac{\mathrm{C}}{2n}.\end{aligned}$$

The remaining decisive question is if assumption (5.14) suffices to prove divergence of the causal action, i. e.

$$\int_{\mathscr{G}} \left(\left(\int_{\mathscr{G}^*} \cos(k\xi)\, \mathrm{d}\nu_\ell(k)_{ii} \right)^2 + \left(\int_{\mathscr{G}^*} \sin(k\xi)\, \mathrm{d}\nu_\ell(k)_{ii} \right)^2 \right)^2 \mathrm{d}\mu(\xi) \overset{?}{\to}_{\ell\to\infty} \infty.$$

Considering a minimizing sequence of *general* negative definite measures, one could try to apply lemma 5.12 in order to obtain a similar situation. By definition of a negative definite measure, $-\nu_\ell(\mathscr{G}^*)$ is a positive operator for any $\ell \in \mathbb{N}$ with respect to $\prec \cdot \mid \cdot \succ$. According to lemma 5.12 we may diagonalize these operators up to an arbitrary small error term: For any $\varepsilon > 0$, $\ell \in \mathbb{N}$, there exists a unitary operator $U_\ell = U_\ell^{(\varepsilon)}$ on V such that

$$-\nu_\ell^{(\varepsilon)}(\mathscr{G}^*) := U_\ell\,(-\nu_\ell(\mathscr{G}^*))\, U_\ell^{-1} = -\operatorname{diag}(\kappa_1^{(\ell)}, \dots, \kappa_{2n}^{(\ell)}) + \Delta(-\nu_\ell(\mathscr{G}^*)),$$

where $\|\Delta(-\nu_\ell(\mathscr{G}^*))\|_{\mathrm{L}(V)} < \varepsilon$. Defining $(\nu_\ell^{(\varepsilon)})_{\ell\in\mathbb{N}}$ by

$$\nu_\ell^{(\varepsilon)} := U_\ell\, \nu_\ell\, U_\ell^{-1},$$

we obtain a sequence of negative definite measures: Obviously, $\nu_\ell^{(\varepsilon)}(\Omega) \in \mathrm{L}(V)$ for all $\ell \in \mathbb{N}$, $\varepsilon > 0$. For any $u \in V$, $\ell \in \mathbb{N}$, $\varepsilon > 0$,

$$\mathrm{d} \prec u \mid \nu_\ell^{(\varepsilon)} u \succ = \mathrm{d} \prec u \mid U_\ell\, \nu_\ell\, U_\ell^{-1} u \succ = \mathrm{d} \prec U_\ell^{-1} u \mid \nu_\ell\, U_\ell^{-1} u \succ$$

is a finite real measure, since $\mathrm{d} \prec u \mid \nu_\ell\, u \succ$ is a finite real measure for any $u \in V$, $\ell \in \mathbb{N}$. Furthermore,

$$\prec u \mid -\nu_\ell^{(\varepsilon)}(\Omega)\, u \succ = \prec u \mid U_\ell\,(-\nu_\ell(\Omega))\, U_\ell^{-1} u \succ = \prec U_\ell^{-1} u \mid -\nu_\ell(\Omega)\, U_\ell^{-1} u \succ \geq 0$$

holds for any $\Omega \in \mathscr{B}(\mathscr{G}^*)$. Hence, $(\nu_\ell^{(\varepsilon)})_{\ell\in\mathbb{N}}$ is a sequence of negative definite measures for any $\varepsilon > 0$. Additionally, trace and eigenvalues of the operators $\nu_\ell(\mathscr{G}^*)$ and $\nu_\ell^{(\varepsilon)}(\mathscr{G}^*)$ coincide: Multiplicativity of the determinant yields

$$\begin{aligned}\det\left(\kappa^{(\ell)} - \nu_\ell^{(\varepsilon)}(\mathscr{G}^*)\right) &= \det\left(\kappa^{(\ell)} - U_\ell \nu_\ell(\mathscr{G}^*) U_\ell^{-1}\right)\\ &= \det\left(U_\ell \kappa^{(\ell)} U_\ell^{-1} - U_\ell \nu_\ell(\mathscr{G}^*) U_\ell^{-1}\right)\\ &= \det\left(U_\ell \left(\kappa^{(\ell)} - \nu_\ell(\mathscr{G}^*)\right) U_\ell^{-1}\right)\\ &= \det\left(\kappa^{(\ell)} - \nu_\ell(\mathscr{G}^*)\right).\end{aligned}$$

Therefore, the eigenvalues of $\nu_\ell^{(\varepsilon)}(\mathscr{G}^*)$ are identical to those of $\nu_\ell(\mathscr{G}^*)$. Moreover, trace constraint (5.2) is still satisfied as the following calculation proves:

$$\mathrm{Tr}(\nu_\ell^{(\varepsilon)}(\mathscr{G}^*)) = \mathrm{Tr}(U_\ell \nu_\ell(\mathscr{G}^*) U_\ell^{-1}) = \mathrm{Tr}(\nu_\ell(\mathscr{G}^*)) = 1.$$

Even $\operatorname{supp} \nu_\ell^{(\varepsilon)} = \operatorname{supp} \nu_\ell$ holds for all $\ell \in \mathbb{N}$, $\varepsilon > 0$.

The crystallizing problem of the proposed procedure seems to be: The idea of applying lemma 5.12 is to diagonalize the kernel of the fermionic projector, $P[\nu_\ell^{(\varepsilon)}](\xi)$, and the closed chain, $A[\nu_\ell^{(\varepsilon)}](\xi)$, for any $\xi \in \mathscr{G}$ up to an arbitrary small error term. Since $\nu_\ell^{(\varepsilon)}$ is a negative definite measure for every $\ell \in \mathbb{N}$, each component $(\nu_\ell^{(\varepsilon)})_{ij}$ is a bounded complex measure for every $\ell \in \mathbb{N}$, $i, j = 1, \ldots, 2n$. The set of complex-valued measures together with the total variation $\|\cdot\|$ forms a complex Banach space according to [Els11]. Furthermore, the total variation of bounded measures is finite,

$$\left\|(\nu_\ell^{(\varepsilon)})_{ij}\right\| < \infty \qquad (i, j = 1, \ldots, 2n). \tag{5.17}$$

However, we may not deduce $\left\|(\nu_\ell^{(\varepsilon)})_{ij}\right\| < \varepsilon$ for any $i \neq j$. Hence, we cannot conclude the off-diagonal entries of the fermionic projector or of the closed chain to be arbitrary small since

$$\left|P[\nu_\ell^{(\varepsilon)}](\xi)_{ij}\right| = \left|\int_{\mathscr{G}^*} \mathrm{e}^{\mathrm{i}k\xi}\, \mathrm{d}\nu_\ell^{(\varepsilon)}(k)_{ij}\right| \leq \left\|(\nu_\ell^{(\varepsilon)})_{ij}\right\|.$$

For this reason, conjecture 5.13 still remains to be proven, if possible. □

If conjecture 5.13 can be proven, minimizing sequences of the causal action were uniformly bounded. As explained in the next section, this could be sufficient in order to prove the existence of a convergent subsequence. The limit of this convergent subsequence could be a non-trivial minimizer of the causal action.

5.6. Discussion of Problems and Ideas

This last section shall summarize the ideas how to proceed in order to prove the existence of non-trivial minimizers for the causal action principle (5.4). The procedure to achieve the desired result could be due to the following steps:

i) Let $(\nu_\ell)_{\ell\in\mathbb{N}}$ be a minimizing sequence for the causal action $\mathcal{S}$ with compact support $\operatorname{supp}\nu_\ell$ for any $\ell \in \mathbb{N}$, satisfying trace constraint (5.2) as well as boundedness constraint (5.3).

ii) The idea is to prove the existence of a convergent subsequence of $(\nu_\ell)_{\ell\in\mathbb{N}}$ using Banach-Alaoglu's theorem (see e. g. [RS80, Theorem IV.21]) or Prochorov's theorem (see e. g. [Els11, Korollar VIII.4.4.23]), respectively. To this aim, we have to prove uniformly boundedness

$$\|\nu_\ell\| \leq C$$

for all $\ell \in \mathbb{N}$ and a positive constant $C > 0$.

iii) The previous condition is equivalent to

$$\left\|(\nu_\ell)_{i,j}\right\| \leq C$$

for all $\ell \in \mathbb{N}$ and all $i, j = 1, \ldots, 2n$ according to (5.1).

iv) The latter inequality implies

$$\left|\nu_\ell\big|_{B_R(0)}(\mathscr{G}^*)_{i,i}\right| \leq C$$

for any $i = 1, \ldots, 2n$, $R \in \mathbb{N}$, $\ell \in \mathbb{N}$.

v) Considering the complement, we obtain

$$\left|\nu_\ell(\mathscr{G}^* \setminus B_R(0))_{i,i}\right| \to_{R\to\infty} 0$$

for all $\ell \in \mathbb{N}$ and $i = 1, \ldots, 2n$, proving the decay of the negative definite measure. Maybe the reverse situation could lead to the proof of uniformly boundedness.

vi) *If* Banach-Alaoglu's or Prochorov's theorem can be applied, then it is possible to extract a weak-* or weak convergent subsequence of $(\nu_\ell)_{\ell\in\mathbb{N}}$, respectively. Its limit shall be denoted by ν.

vii) *In the case of continuity of the trace* we obtain

$$\begin{aligned}\operatorname{Tr}\nu(\mathscr{G}^*) &= \operatorname{Tr}\left(\int_{\mathscr{G}^*} 1\,\mathrm{d}\nu\right) = \operatorname{Tr}\left(< 1, \nu >\right) = \operatorname{Tr}\left(\lim_{\ell\to\infty} < 1, \nu_\ell >\right) \\ &= \operatorname{Tr}\left(\lim_{\ell\to\infty}\int_{\mathscr{G}^*} 1\,\mathrm{d}\nu_\ell\right) = \operatorname{Tr}\left(\lim_{\ell\to\infty}\nu_\ell(\mathscr{G}^*)\right) \stackrel{?}{=} \lim_{\ell\to\infty}\operatorname{Tr}\nu_\ell(\mathscr{G}^*) \\ &= \lim_{\ell\to\infty} 1 = 1.\end{aligned}$$

viii) Granting the previous step to be true, the measure ν is in particular non-trivial.

ix) Using Fatou's lemma A.5 in analogy to [Fin10, Proof of Theorem 4.2], the limit ν is a minimizer of the causal action $\mathcal{S}$ and satisfies trace constraint (5.2) as well as bondedness constraint (5.3).

Hence, proving the uniformly boundedness of the minimizing sequence according to conjecture 5.13 is of crucial interest regarding further considerations. However, uniformly boundedness maybe could be shown in an alternative way due to the minimizing property of the considered minimizing sequence.

6. Summary

Finally, the results of the present work shall shortly be summarized.

- Starting with a causal fermion system endowed with a symmetry and assuming a transitive group action on space-time M, there is a spectral measure E_k which allows to construct a negative definite measure ν by $\nu := \pi_x E_k x$ for any space-time point $x \in M$.

- In contrary, starting with a finite-dimensional indefinite inner product space $(V, \prec \cdot \mid \cdot \succ)$ and a negative definite measure ν on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$ it is possible to reconstruct the original causal fermion system. Maybe it is even possible to prove that the causal fermion is endowed with a symmetry. Defining a transitive group action, a spectral measure may be obtained which characterizes the negative definite measure.

- Nevertheless, there is not a unique way of constructing the original causal fermion system, as chapter 4 shows. The equivalent construction makes the point of reconstructing the Hilbert space of the causal fermion system clearer.

- The crucial question if a minimizing sequence of the causal action contains a convergent subsequence is still to be answered. In chapter 5 some ideas are presented how to prove the conjecture a minimizing sequence to be uniformly bounded. However, somewhat more research will be necessary regarding this topic.

A. Appendix

A.1. Group Theory

Definition A.1 (**Group**).
A *group* is a pair $(G, \cdot)$ consisting of a set G and an operation $\cdot\colon G \times G \to G$ with $(a, b) \mapsto a \cdot b$ which satisfies the following conditions:

i) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity).

ii) $\exists\, e \in G$ with $a \cdot e = e \cdot a = a$ (identical element).

iii) $\forall\, a \in G\colon \exists\, a^{-1} \in G$ with $a \cdot a^{-1} = a^{-1} \cdot a = e$ (inverse element).

A group G with an operation " $\cdot$ " is called *abelian* if it is commutative, i. e.

$$a \cdot b = b \cdot a \quad \text{for all} \quad a, b \in G.$$

See [Gub10] and [Wol11].

Definition A.2 (**Group homomorphism**).
A *group homomorphism* is a map $\varphi\colon G_1 \to G_2$ between two groups $(G_1, \cdot)$ and $(G_2, *)$, such that

$$\varphi(a \cdot b) = \varphi(a) * \varphi(b)$$

for any $a, b \in G_1$. The composition of two group homomorphisms is a group homomorphism again. See [Gub10] and [Bos06a].

Remark A.3. Let $\varphi\colon G_1 \to G_2$ be a group homomorphism between two groups, G_1, G_2. Then, the identity

$$\varphi(g)^{-1} = \varphi(g^{-1}) \tag{A.1}$$

holds for all $g \in G_1$. Additionally, $\varphi(e_1) = e_2$ holds for any group homomorphism $\varphi\colon G_1 \to G_2$. See e. g. [Bos06a].

Definition A.4 (**Kernel**).
Let $\varphi\colon G_1 \to G_2$ be a group homomorphism. We define the *kernel* of φ by

$$\ker(\varphi) := \varphi^{-1}(e_2) = \{a \in G_1 : \varphi(a) = e_2\}.$$

Then, $\ker(\varphi)$ is a subgroup of G_1 and $\varphi(G_1)$ is a subgroup of G_2. The group homomorphism φ is injective if and only if $\ker(\varphi) = \{e_1\}$. See [Gub10].

Definition A.5 (**Group action**).
Consider a group G and a set X. A group G is said to *act on* the set X if there exists a map

$$\phi\colon G \times X \to X, \qquad (g, x) \mapsto g \cdot x \in X$$

satisfying the following properties:

i) $e \cdot x = x$,

ii) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$

for all $g_1, g_2 \in G$, $x \in X$. See [Gub10] and [Jac85].

Definition A.6 (**Transitivity**).
Giving a group action $G \times X \to X$, the G-*orbit* of $x_0 \in X$ is $G \cdot x_0 = \{g \cdot x_0 : g \in G\}$. We say that G acts *transitively* on X if there is just one orbit, that is, if $G \cdot x = X$ for some $x \in X$ (and hence for every $x \in X$). Equivalently, for any $x, y \in X$ there exists a $g \in G$ such that $g \cdot x = y$. See [Gub10] and [Jac85].

A.2. Topology

Definition A.7 (**Topological group**).
A *topological group* is a group X which is a Hausdorff space such that the transformation $X \times X \to X$, $(x, y) \mapsto x^{-1}y$ is continuous. Important examples of such groups are $\mathbb{R}^n$ or $\mathbb{C}$. See [Hal95].

Definition A.8 (**Locally compact space**).
A topological space X is called *locally compact* if every point of X has a neighborhood whose closure is compact. See e. g. [Hal95].

Definition A.9 (**Haar measure**).
Suppose that G is a locally compact group. A Borel measure μ on G is called left-invariant if $\mu(xE) = \mu(E)$ for all $x \in G$ and $E \in \mathscr{B}(G)$. A left *Haar measure* on G is a non-zero left-invariant Radon measure μ on G. See [Fol99].

A Haar measure has the following properties:

Lemma A.10 (**Properties of a Haar measure**).
If G is a locally compact topological Hausdorff group, there is a left-invariant Radon measure $\mu\colon \mathscr{B}(G) \to [0, \infty]$, $\mu \neq 0$, uniquely determined up to a positive factor. μ is called a left Haar measure *on G and has the following properties:*

i) $\mu(aB) = \mu(B)$ *for all* $a \in G$, $B \in \mathscr{B}(G)$.

ii) $\mu(K) < \infty$ *for all* $K \subset G$ *compact.*

iii) $\mu(B) = \sup\{\mu(K)\colon K \subset B \text{ compact}\}$ *for all* $B \in \mathscr{B}(G)$.

iv) $\mu(U) > 0$ *for every* $U \subset G$ *open,* $U \neq \emptyset$.

v) $0 < \mu(U) < \infty$ *for every relatively compact open subset* $U \subset G$.

Proof. See [Els11]. □

A.3. Indefinite Inner Product Spaces

Definition A.11 (**Indefinite inner product**).
A function $\prec \cdot \mid \cdot \succ : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is called an *indefinite inner product* if the following conditions hold:

i) $\prec y \mid \alpha x_1 + \beta x_2 \succ = \alpha \prec y \mid x_1 \succ + \beta \prec y \mid x_2 \succ$ for all x_1, x_2, $y \in \mathbb{C}^n$, $\alpha, \beta \in \mathbb{C}$ (linearity in the second argument).

ii) $\prec x \mid y \succ = \overline{\prec y \mid x \succ}$ for all x, $y \in \mathbb{C}^n$ (antisymmetry).

iii) $\prec x \mid y \succ = 0$ for all $y \in \mathbb{C}^n \implies x = 0$ (non-degeneracy).

See [GLR05, Definition 2.1].

Definition A.12 (**Indefinite inner product space**).
Let V be a finite-dimensional complex vector space, endowed with an indefinite inner product $\prec \cdot \mid \cdot \succ$. Then, $(V, \prec \cdot \mid \cdot \succ)$ is called an *indefinite inner product space.*[1] On V we may define the signature (p, q), where p and q shall denote the maximal dimensions of positive or negative subspaces of V, respectively. See e. g. [Fin07].

[1] *Remark*: Since $\prec \cdot \mid \cdot \succ$ is in general not positive, it is in general *no* scalar product.

Lemma A.13 (**Schwarz inequality**).
Let $(V, \prec \cdot \mid \cdot \succ)$ be an indefinite inner product space. Let $B\colon V \to V$ be a positive operator, i. e. $\prec u \mid B u \succ \geq 0$ for all $u \in V$. Then,

$$|\prec u \mid B v \succ| \leq \sqrt{|\prec u \mid B u \succ|}\ \sqrt{|\prec v \mid B v \succ|}$$

for all $u, v \in V$.

Proof. See [Fin07, Lemma 4.1]. □

Theorem A.14. *Every finite-dimensional normed vector space is complete and thus a Banach space. All norms are equivalent. Every linear map between finite-dimensional vector spaces is continuous.*

Proof. See [Bal09]. □

A.4. Measure and Integration Theory

Definition A.15 (**Projection-valued integration**).
Let H be a Hilbert space and $u, v \in H$. Let E be a spectral measure on $\mathbb{R}^n$ and $f \in \mathscr{B}(\mathbb{R}^n)$. We write

$$\int_{\mathbb{R}^n} f(\lambda)\, \mathrm{d}\langle u \mid E_\lambda v\rangle =: \langle u \mid \left(\int_{\mathbb{R}^n} f(\lambda)\, \mathrm{d}E_\lambda \right) v\rangle \tag{A.2}$$

and will use this to define *projector-valued integration* in $\mathrm{L}(H)$. See [Fin13, Lemma 7.3.8].

In analogy to definition A.15, we define integration in $\mathrm{L}(V)$ in the following way:

Definition A.16 (**Operator-valued integration**).
Let $(V, \prec \cdot \mid \cdot \succ)$ be an indefinite inner product space and let $\nu\colon \mathscr{G}^* \to \mathrm{L}(V)$ be a negative definite measure. Let $f\colon \mathscr{G}^* \to \mathbb{C}$ be a measurable bounded function. For $u, v \in V$, we write

$$\prec u \mid \left(\int_{\mathscr{G}^*} f(k)\, \mathrm{d}\nu(k) \right) v \succ := \int_{\mathscr{G}^*} f(k)\, \mathrm{d} \prec u \mid \nu(k)\, v \succ \tag{A.3}$$

in order to define *operator-valued integration* in $\mathrm{L}(V)$.

The right-hand side of equation (A.3) is well-defined, since f is a bounded function and $\mathrm{d} \prec u \mid \nu w \succ$ is a bounded measure for any $u, w \in V$. The similar definition of vector-valued integration is given in the following:

Definition A.17 (**Vector-valued integration**).
Let $(V, \prec \cdot \mid \cdot \succ)$ be an indefinite inner product space and let $\nu\colon \mathscr{G}^* \to \mathrm{L}(V)$ be a negative definite measure. Let $f\colon \mathscr{G}^* \to \mathbb{C}$, $\chi\colon \mathscr{G}^* \to V$ be measurable bounded functions. For arbitrary $u \in V$, we define $\int_{\mathscr{G}^*} f(k)\, \mathrm{d}\nu(k)\chi(k) \in V$ in the following way:

$$\prec u \mid \int_{\mathscr{G}^*} f(k)\, \mathrm{d}\nu(k)\chi(k) \succ := \int_{\mathscr{G}^*} f(k)\, \mathrm{d} \prec u \mid \nu(k)\chi(k) \succ . \tag{A.4}$$

To (A.4) we refer to as *vector-valued integration* in V.

The integral on the right-hand side in equation (A.4) is defined by formula (3.14) in definition 3.7. For completeness, some frequently used measure theoretical results shall be stated in the following.

Lemma A.18 (**Fatou's lemma**).
Let (X, Σ, μ) be a measure space. If $f_n\colon X \to [0, \infty]$ is measurable for each integer n, then

$$\int_X \left(\liminf_{n\to\infty} f_n \right) \mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X f_n \, \mathrm{d}\mu. \tag{A.5}$$

Proof. See [Rud70]. □

Analogously, Fatou's lemma implies

$$\limsup_{n\to\infty} \int_X f_n \, \mathrm{d}\mu \leq \int_X \left(\limsup_{n\to\infty} f_n \right) \mathrm{d}\mu.$$

Theorem A.19 (**Fubini's theorem**).
Let f be a measurable function on $M \times N$. Let μ be a measure on M, ν a measure on N. Then

$$\int_M \left(\int_N |f(m,n)| \, \mathrm{d}\nu(n) \right) \mathrm{d}\mu(m) < \infty$$

if and only if

$$\int_N \left(\int_M |f(m,n)| \, \mathrm{d}\mu(m) \right) \mathrm{d}\nu(n) < \infty$$

and if one (and thus both) of these integrals is finite, then

$$\int_M \left(\int_N f(m,n)\, \mathrm{d}\nu(n) \right) \mathrm{d}\mu(m) = \int_N \left(\int_M f(m,n)\, \mathrm{d}\mu(m) \right) \mathrm{d}\nu(n). \tag{A.6}$$

Proof. See [RS80, Theorem I.21]. □

For the definition of Bochner integrals we refer to [DU77] and [Yos80]. However, the following theorems are of crucial interest regarding Bochner integrals:

Theorem A.20. *Let $(S, \mathscr{B}, \mu)$ be a measure space. A strongly $\mathscr{B}$-measurable function $x(s)$ is Bochner μ-integrable if and only if $\|x(s)\|$ is μ-integrable. A function $x(s)$ with values in a Banach space X is said to be strongly $\mathscr{B}$-measurable if there exists a sequence of finitely-valued functions strongly convergent to $x(s)$ μ-a. e. on S. See [Yos80].*

Theorem A.21. *Let (Ω, Σ, μ) be a measure space and let X be a Banach space. A μ-measurable function $f\colon \Omega \to X$ is Bochner integrable if and only if $\int_\Omega \|f\| \, \mathrm{d}\mu < \infty$.*

Proof. See [DU77] and [Yos80]. □

In order to state one further important theorem, we cite the following definition:

Definition A.22. Let E, F be normed vector spaces. Let D be a subspace of E, and let $A\colon D \to F$ be a linear operator. Then, A is called a *closed* operator if $x_n \to x$ in E and $Ax_n \to y$ in F imply $x \in D$ and $Ax = y$. See [Heu82].

Hence, any bounded linear operator $A\colon V \to V$ onto a finite-dimensional vector space V is closed, since V is a Banach space by theorem A.14 and thus closed, and

$$\|Ax - y\|_V = \|Ax - Ax_n + Ax_n - y\|_V \leq \|Ax - Ax_n\|_V + \|Ax_n - y\|_V \to_{n\to\infty} 0.$$

In particular, $\nu(\mathscr{G}^*) \in \mathrm{L}(V)$ is a closed operator for any negative definite measure ν on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$.

Theorem A.23 (**Hille**).
Let (Ω, Σ, μ) be a finite measure space, and let X be a Banach space. Let T be a closed linear operator defined inside X and having values in a Banach space Y. If f and Tf are Bochner integrable with respect to μ, then

$$T\left(\int_E f \, \mathrm{d}\mu\right) = \int_E Tf \, \mathrm{d}\mu \tag{A.7}$$

for all $E \in \Sigma$.

Proof. See [DU77, Theorem 6]. □

A.5. Hilbert Space Results

Definition A.24 (**Unitary operator**).
Let H be a Hilbert space. An operator $U\colon H \to H$ is called *unitary* if $\|Ux\| = \|x\|$ for all $x \in H$. Moreover, a linear operator $U\colon H \to H$ is unitary if and only if $U^{-1} = U^*$, see [Lax02] and [Yos80]. The latter property can be used to define unitary operators, see e. g. [Lan93, Chapter XVIII].

Theorem A.25 (**Hellinger-Toeplitz**).
Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. If a linear mapping $T\colon H \to H$ satisfies the symmetry condition

$$\langle Tx, y\rangle = \langle x, Ty\rangle$$

for all $x, y \in H$, then T is continuous and self-adjoint.

Proof. See [Wer11, Satz V.5.5]. □

Definition A.26 (**Strongly continuous one-parameter unitary group**).
Let H be a separable Hilbert space. An operator-valued function $U(t)$ on H is called a *strongly continuous one-parameter unitary group* if

i) $U(t)$ is a unitary operator for each $t \in \mathbb{R}$,

ii) $U(t+s) = U(t)\,U(s)$ for all $s, t \in \mathbb{R}$ and

iii) $U(t)\phi \to U(t_0)\phi$ for $\phi \in H$ and $t \to t_0$.

See [RS80].

We even have the following theorem which will be used in chapter 2:

Theorem A.27. *Let $t \mapsto U(t) = U(t_1, \ldots, t_n)$ be a strongly continuous map of $\mathbb{R}^n$ into the unitary operators on a separable Hilbert space H satisfying $U(t+s) = U(t)U(s)$ and $U(0) = \mathrm{Id}$. Let D be the set of finite linear combinations of vectors of the form*

$$\phi_f = \int_{\mathbb{R}^n} f(t)U(t)\,\phi\,\mathrm{d}t, \qquad \phi \in H, \quad f \in C_0^\infty(\mathbb{R}^n).$$

Then D is a domain of essential self-adjointness for each of the generators A_j of the one-parameter subgroups $U(0,0,\ldots,t_j,\ldots,0)$, each $A_j\colon D \to D$ and the A_j commute, $j = 1,\ldots,n$. Furthermore, there is a projection-valued measure P_Ω on $\mathbb{R}^n$ so that

$$(\phi, U(t)\,\psi) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}t\cdot\lambda}\,\mathrm{d}(\phi, P_\lambda\psi)$$

for all $\phi, \psi \in H$.

Proof. See [RS80, Theorem VIII.12]. □

A.6. Complex Conjugation of an Integral

In order to determine the complex conjugation of an integral, we first introduce the following polarization formula:

Lemma A.28 (**Polarization formula**).
Let V be a $\mathbb{C}$-vector space and $\mathrm{b}\colon V \times V \to \mathbb{C}$ *a sesquilinear form. Defining its corresponding quadratic form* $\mathrm{q}\colon V \to \mathbb{R}$ *by* $\mathrm{q}(v) := \mathrm{b}(v, v)$*, for sesquilinear forms being linear in the first argument we obtain the* polarization formula

$$\mathrm{b}(v, w) = \frac{1}{4}(\mathrm{q}(v + w) - \mathrm{q}(v - w)) + \frac{\mathrm{i}}{4}(\mathrm{q}(v + \mathrm{i}w) - \mathrm{q}(v - \mathrm{i}w)) \tag{A.8}$$

for any $v, w \in V$. See e. g. [GLR05].

As usual, complex conjugation of a complex number $z = a + \mathrm{i}\,b \in \mathbb{C}$ is given by

$$\overline{z} = \overline{a + \mathrm{i}\,b} = a - \mathrm{i}\,b \in \mathbb{C}.$$

For sesquilinear forms $\mathrm{b}'\colon V \times V \to \mathbb{C}$ being linear in the second argument, we obtain by polarization formula (A.8)

$$\mathrm{b}'(w, v) = \overline{\mathrm{b}'(v, w)} = \frac{1}{4}(\mathrm{q}(v + w) - \mathrm{q}(v - w)) - \frac{\mathrm{i}}{4}(\mathrm{q}(v + \mathrm{i}w) - \mathrm{q}(v - \mathrm{i}w)) \tag{A.9}$$

for any $u, v, \in V$.

Assume a measurable bounded function f on $\mathscr{G}^*$ with values in a finite-dimensional $\mathbb{C}$-vector space V. Furthermore, let $\prec \cdot \mid \cdot \succ\colon V \times V \to \mathbb{C}$ be an indefinite inner product on V. Let ν be a negative definite measure on $\mathscr{G}^*$ with values in $\mathrm{L}(V)$ such that $\mathrm{d} \prec u \mid \nu\, u \succ$ is a real-valued finite Borel measure for all $u \in V$. Using polarization formula (A.8), we achieve a complex-valued bounded Borel measure $\mathrm{d} \prec u \mid \nu\, w \succ$ for any $u, w \in V$. By symmetry of positive linear operators, $\mathrm{d} \prec \nu\, u \mid w \succ$ is also a complex-valued bounded Borel measure for any $u, w \in V$. We now consider integration with respect to the complex Borel measure $\mathrm{d} \prec u \mid \nu\, v \succ$ for any $u, v \in V$.

Definition A.29 (**Linearity of the integrator**).
We *define* linearity of the integrator of an integral in the following way:

$$\int f \,\mathrm{d} \prec \alpha\, u \mid \nu(k)\, v \succ \;:= \overline{\alpha} \int f \,\mathrm{d} \prec u \mid \nu(k)\, v \succ,$$
$$\int f \,\mathrm{d} \prec u \mid \alpha\, \nu(k)\, v \succ \;:= \alpha \int f \,\mathrm{d} \prec u \mid \nu(k)\, v \succ$$

for any $\alpha \in \mathbb{C}$, and

$$\int f \,\mathrm{d} \prec \nu(k)\, u \mid v + w \succ \;:=\; \int f \,\mathrm{d} \prec \nu(k)\, u \mid v \succ + \int f \,\mathrm{d} \prec \nu(k)\, u \mid w \succ,$$

$$\int f \,\mathrm{d} \prec u + v \mid \nu(k)\, w \succ \;:=\; \int f \,\mathrm{d} \prec u \mid \nu(k)\, w \succ + \int f \,\mathrm{d} \prec v \mid \nu(k)\, w \succ$$

for any $u, v, w \in V$.

Lemma A.30. *Consider a finite-dimensional indefinite inner product space* $(V, \prec \cdot \mid \cdot \succ)$*. Let* f *be a bounded complex-valued function on* $\mathscr{G}^*$ *and let* ν *be a negative definite measure on* $\mathscr{G}^*$ *with values in* $\mathrm{L}(V)$*. Then, for all* $u, v \in V$ *holds*

$$\overline{\int f(k) \,\mathrm{d} \prec u \mid \nu(k)\, v \succ} = \int \overline{f(k)} \,\mathrm{d}\, \overline{\prec u \mid \nu(k)\, v \succ}.$$

Proof. Considering the integral $\int f(k) \,\mathrm{d} \prec u \mid \nu(k)\, v \succ$ and writing f instead of $f(k)$, we obtain by linearity of the integrator and polarization formula (A.9)

$$\begin{aligned}
&\int f \,\mathrm{d} \prec u \mid \nu(k)\, v \succ = \\
&= \frac{1}{4}\Big(\int f \,\mathrm{d} \prec \nu(k)\, v + u \mid \nu(k)\, v + u \succ - \int f \,\mathrm{d} \prec \nu(k)\, v - u \mid \nu(k)\, v - u \succ\Big) \\
&- \frac{\mathrm{i}}{4}\Big(\int f \,\mathrm{d} \prec \nu(k)\, v + \mathrm{i}\, u \mid \nu(k)\, v + \mathrm{i}\, u \succ - \int f \,\mathrm{d} \prec \nu(k)\, v - \mathrm{i}\, u \mid \nu(k)\, v - \mathrm{i}\, u \succ\Big)
\end{aligned}$$

for any $u, v \in V$. Considering a *real*-valued function f, the above formula yields

$$\begin{aligned}
&\overline{\int f \,\mathrm{d} \prec u \mid \nu(k)\, v \succ} \;= \\
&= \overline{\frac{1}{4}\Big(\int f \,\mathrm{d} \prec \nu(k)\, v + u \mid \nu(k)\, v + u \succ - \int f \,\mathrm{d} \prec \nu(k)\, v - u \mid \nu(k)\, v - u \succ\Big)} \\
&\overline{- \frac{\mathrm{i}}{4}\Big(\int f \,\mathrm{d} \prec \nu(k)\, v + \mathrm{i}\, u \mid \nu(k)\, v + \mathrm{i}\, u \succ - \int f \,\mathrm{d} \prec \nu(k)\, v - \mathrm{i}\, u \mid \nu(k)\, v - \mathrm{i}\, u \succ\Big)} \\
&= \frac{1}{4}\Big(\int f \,\mathrm{d} \prec \nu(k)\, v + u \mid \nu(k)\, v + u \succ - \int f \,\mathrm{d} \prec \nu(k)\, v - u \mid \nu(k)\, v - u \succ\Big) \\
&+ \frac{\mathrm{i}}{4}\Big(\int f \,\mathrm{d} \prec \nu(k)\, v + \mathrm{i}\, u \mid \nu(k)\, v + \mathrm{i}\, u \succ - \int f \,\mathrm{d} \prec \nu(k)\, v - \mathrm{i}\, u \mid \nu(k)\, v - \mathrm{i}\, u \succ\Big) \\
&= \frac{1}{4}\Big(\int f \,\mathrm{d} \prec u + \nu(k)\, v \mid u + \nu(k)\, v \succ - (-1)^2 \int f \,\mathrm{d} \prec u - \nu(k)\, v \mid u - \nu(k)\, v \succ\Big) \\
&+ \frac{\mathrm{i}}{4}\Big(\int f \,\mathrm{d} \prec \mathrm{i}\,(-\mathrm{i}\,\nu(k)\, v + u) \mid \mathrm{i}\,(-\mathrm{i}\,\nu(k)\, v + u) \succ - \int f \,\mathrm{d} \prec \mathrm{i}\,(-\mathrm{i}\,\nu(k)\, v - u) \mid \mathrm{i}\,(-\mathrm{i}\,\nu(k)\, v - u) \succ\Big) \\
&= \frac{1}{4}\Big(\int f \,\mathrm{d} \prec u + \nu(k)\, v \mid u + \nu(k)\, v \succ - \int f \,\mathrm{d} \prec u - \nu(k)\, v \mid u - \nu(k)\, v \succ\Big)
\end{aligned}$$

$$+\frac{\mathrm{i}}{4}\left((-\mathrm{i}^2)\int f\,\mathrm{d} \prec u-\mathrm{i}\,\nu(k)\,v \mid u-\mathrm{i}\,\nu(k)\,v \succ -(-\mathrm{i}^2)\int f\,\mathrm{d} \prec -\mathrm{i}\,\nu(k)\,v-u \mid -\mathrm{i}\,\nu(k)\,v-u \succ\right)$$
$$=\frac{1}{4}\left(\int f\,\mathrm{d} \prec u+\nu(k)\,v \mid u+\nu(k)\,v \succ -\int f\,\mathrm{d} \prec u-\nu(k)\,v \mid u-\nu(k)\,v \succ\right)$$
$$+\frac{\mathrm{i}}{4}\left(\int f\,\mathrm{d} \prec u-\mathrm{i}\,\nu(k)\,v \mid u-\mathrm{i}\,\nu(k)\,v \succ -(-1)^2\int f\,\mathrm{d} \prec \mathrm{i}\,\nu(k)\,v+u \mid \mathrm{i}\,\nu(k)\,v+u \succ\right)$$
$$=\frac{1}{4}\left(\int f\,\mathrm{d} \prec u+\nu(k)\,v \mid u+\nu(k)\,v \succ -\int f\,\mathrm{d} \prec u-\nu(k)\,v \mid u-\nu(k)\,v \succ\right)$$
$$-\frac{\mathrm{i}}{4}\left(\int f\,\mathrm{d} \prec u+\mathrm{i}\,\nu(k)\,v \mid u+\mathrm{i}\,\nu(k)\,v \succ -\int f\,\mathrm{d} \prec u-\mathrm{i}\,\nu(k)\,v \mid u-\mathrm{i}\,\nu(k)\,v \succ\right)$$
$$=\int f\,\mathrm{d} \prec \nu(k)\,v \mid u \succ$$
$$=\int f\,\mathrm{d}\overline{\prec u \mid \nu(k)\,v \succ}.$$

If f is a *complex*-valued function, we obtain

$$\begin{aligned}
\overline{\int f\,\mathrm{d} \prec \nu(k)\,v \mid u \succ} &= \overline{\int (\mathrm{Re}f+\mathrm{i}\,\mathrm{Im}f)\,\mathrm{d} \prec \nu(k)\,v \mid u \succ} \\
&= \overline{\int \mathrm{Re}f\,\mathrm{d} \prec \nu(k)\,v \mid u \succ + \int \mathrm{i}\,\mathrm{Im}f\,\mathrm{d} \prec \nu(k)\,v \mid u \succ} \\
&= \overline{\int \mathrm{Re}f\,\mathrm{d} \prec \nu(k)\,v \mid u \succ - \int \mathrm{Im}f\,\mathrm{d} \prec \mathrm{i}\,\nu(k)\,v \mid u \succ} \\
&= \overline{\int \mathrm{Re}f\,\mathrm{d} \prec \nu(k)\,v \mid u \succ} - \overline{\int \mathrm{Im}f\,\mathrm{d} \prec \mathrm{i}\,\nu(k)\,v \mid u \succ} \\
&= \int \mathrm{Re}f\,\mathrm{d}\overline{\prec \nu(k)\,v \mid u \succ} - \int \mathrm{Im}f\,\mathrm{d}\overline{\prec \mathrm{i}\,\nu(k)\,v \mid u \succ} \\
&= \int \mathrm{Re}f\,\mathrm{d} \prec u \mid \nu(k)\,v \succ - \int \mathrm{Im}f\,\mathrm{d} \prec u \mid \mathrm{i}\,\nu(k)\,v \succ \\
&= \int \mathrm{Re}f\,\mathrm{d} \prec u \mid \nu(k)\,v \succ - \int \mathrm{i}\,\mathrm{Im}f\,\mathrm{d} \prec u \mid \nu(k)\,v \succ \\
&= \int (\mathrm{Re}f-\mathrm{i}\,\mathrm{Im}f)\,\mathrm{d} \prec u \mid \nu(k)\,v \succ \\
&= \int \overline{f}\,\mathrm{d} \prec u \mid \nu(k)\,v \succ = \int \overline{f}\,\mathrm{d}\overline{\prec \nu(k)\,v \mid u \succ}.
\end{aligned}$$

Hence, the complex conjugation of an integral with a complex-valued integrand and a complex-valued integrator is given by conjugating both integrand and integrator. The symbol “d” will not be conjugated. □

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Eigenständigkeitserklärung

Ich habe die Arbeit selbstständig verfasst, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und bisher keiner anderen Prüfungsbehörde vorgelegt.

Regensburg, den 27. Februar 2017

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