The Rise of Social Sentiment and Payment for Order Flow: New Implications for Non-Fundamental Information in Financial Markets

Dissertation zur Erlangung des Grades eines Doktors der Wirtschaftswissenschaft

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1 Introduction

"Three great forces rule the world: stupidity, fear, and greed."

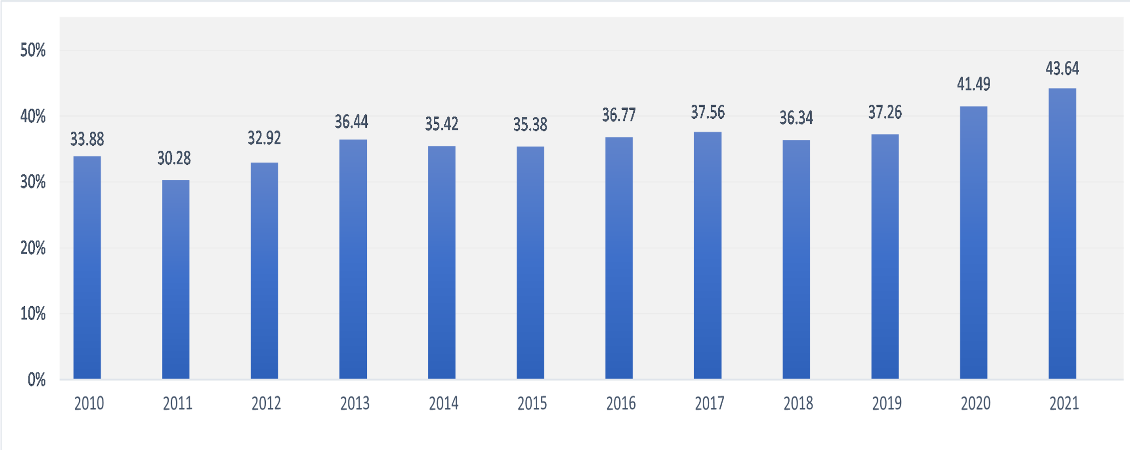
Albert Einstein

Having been pronounced dead by researchers a decade ago (see, e.g., Davis, 2009), retail investing has recently experienced a remarkable comeback. Relying on an estimate from *Bloomberg Intelligence*, the *Wall Street Journal* reports that retail investing was responsible for 20% of the total trading volume in U.S. equity markets in 2020, compared to only 10% in 2010. Strikingly, more than half of this surge occurred solely in 2020.[1] Other sources report an even sharper rise. According to *Citadel Securities*, retail investing's share of the U.S. total trading volume in that year added up to approximately 25%.[2] In light of this immense growth, U.S. online broker *Charles Schwab* labels the group of retail traders that has started to invest since 2020 as "Generation Investor" or "Generation I."[3]

Evidence of the emergence of Generation I can also be found in other parts of the world. According to the data of *Deutsches Aktieninstitut*, approximately 2.7 million Germans started to invest in stocks in 2020, which constituted by far the largest increase in the last 20 years.[4] In South Korea, retail investing's share of the total trading volume in stock markets rose from 48% in 2019 to 65% in 2020.[5]

There are several converging reasons that facilitated the appearance of Generation I. Due to a lack of investment alternatives caused by the ongoing low-interest phase in most industrialized countries, private households have begun to turn their attention to stock markets in search of returns on their savings. The decline in trading commissions, accompanied by the rise of stock message boards and of easy-to-use trading applications for mobile devices, which have led to the "gamification" of investing, has additionally contributed to the surge in retail trading. Perhaps the most important factor is the COVID-19 pandemic. The sharp fall in equity prices at the beginning of the pandemic in March 2020 along with the stay-at-home orders provided new market participants with attractive entry opportunities as well as with the extra time necessary to start investing. Alluding to the pandemic-induced trend of investing from home, the financial press has labeled the trading activity of Generation I as "kitchen table trading."[6]

Figure 1.1: Share of total trading volume reported to FINRA TRFs

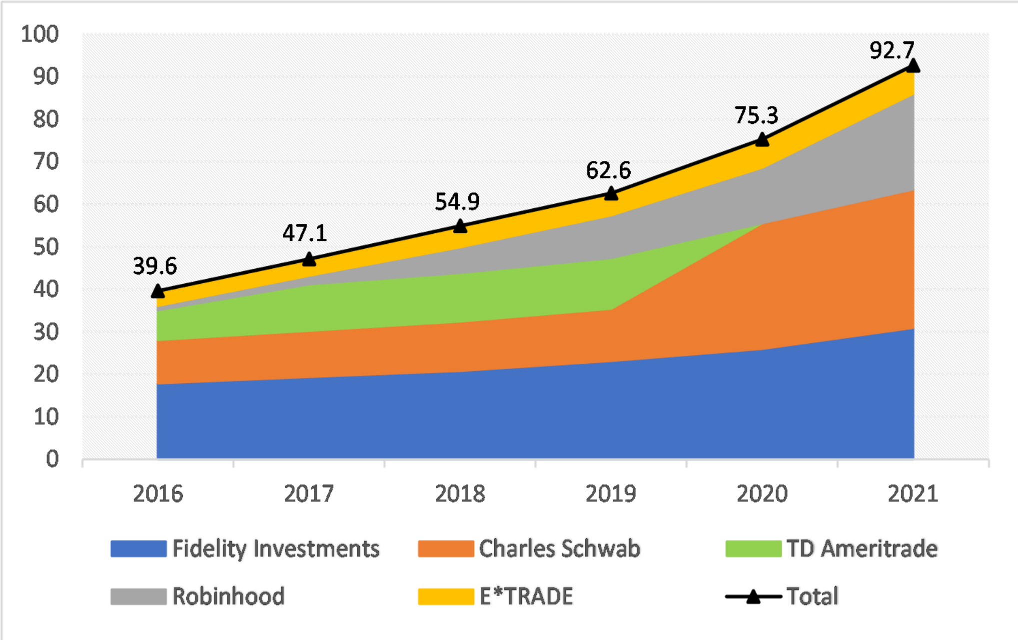


Source: Own calculations based on data from Cboe Global Markets.

Further numerical evidence of the increasing influence of retail investing in the U.S., albeit of a suggestive nature, can be found by inspecting the evolution of the share of total U.S. equity trading volume that is reported to a so-called "Trade Reporting Facility" (TRF). Transactions that are not executed on official exchanges such as NASDAQ and NYSE must be reported to one of the TRFs to enhance market transparency. The TRFs are operated by the Financial Industry Regulatory Authority (FINRA). Since brokers and market makers execute a considerable amount of retail traders' orders in off-exchange trading, a rise in the share of total trading volume reported to the FINRA TRFs can be seen as equivalent to a rise in retail trading.[7] Figure 1.1 depicts the evolution of the FINRA TRF percentage volume from 2010 to 2021. While oscillating around a value of 36% between 2010 and 2019, the share of total equity trading volume reported to the FINRA TRFs exceeded 40% in 2020 and further climbed to more than 43% in 2021, representing the rising impact of retail investing. Notably, a new monthly high of 47.18% was reached in January 2021.

The emergence of Generation I has, moreover, contributed to unprecedented growth in the number of retail accounts administered by the U.S. online brokerage sector. New players such as the often-debated fintech company *Robinhood*, which went public in July 2021, compete with established online brokers such as *TD Ameritrade*, *Charles Schwab*, *Fidelty Investments*, and *E*TRADE* - often referred to as the "Big Four brokerages" - for the growing mass of customers. As shown in Figure 1.2, the number of retail trading accounts at the Big Four brokerages and their aspiring competitor *Robinhood* has increased in every year since 2016, with the surge since 2020 being particularly pronounced. In 2020 alone, the five displayed online brokers gained approximately 13 million new retail clients. This trend further intensified in

Figure 1.2: Number of retail investor accounts administered by selected U.S. online brokers (in millions)



Source: Data on *TD Ameritrade* are from its 2019 annual form 10-K and its 2018 annual report; data on *Charles Schwab* are from its 2020 annual report and its Q3 2021 earnings report; data on *Fidelity Investments* are from its 2018 shareholder update, its 2020 highlights report, and its Q3 2021 business update; collected data on *Robinhood* are taken from Statista[9]; data on *E*TRADE* are from its report on the full year 2016 results, its 2019 annual form 10-K, and *The Wall Street Journal*.[10]

Note: The data for 2021 are as of September 30.

Note: *Charles Schwab* took over *TD Ameritrade* in October 2020. The numbers of *Charles Schwab* in 2020 and 2021 also comprise the former accounts of *TD Ameritrade*.

2021, when more than 17 million new retail investors opened a brokerage account at the five named companies during the first nine months of the year.

The recent global influx of retail traders has also brought forward new competitors in the European online brokerage sector, which aim at imitating the business model of their U.S. counterparts. The German start-up company and online broker *Trade Republic*, for example, which is also called the "European Robinhood", tries to attract private investors by jumping on the bandwagon of commission-free trading. As of May 2021, *Trade Republic* has more than one million clients.[8] Overall, there is a remarkable amount of evidence that retail investing has become significantly more important in financial markets in recent years.

Dumb money and non-fundamental information. When trying to classify investors who participate in financial markets, financial experts typically distinguish between so-called "smart money" and so-called "dumb money." Smart money refers to the investment of sophisticated, experienced, and rational traders whose professions are linked to financial business. Institutional investors such as mutual funds, hedge funds, pension funds, banks, and insurance companies, which dispose of a wide range of analysts and financial data, are often assumed to invest smart money. The central characteristic attributed to smart-money investors is that they possess valuable information related to the fundamental value of an asset. In finance, the fundamental or intrinsic value of an asset is usually defined as the net present value of all future cash flows linked to the asset, such as dividends. Based on their fundamental information, smart-money investors evaluate the existing mispricing in the market and adequately choose their investment position to counteract the mispricing and profit from it.

In contrast, "dumb money" represents investments of unsophisticated, inexperienced, and irrational traders whose professions are not linked to financial business. These market participants invest on the basis of emotions, heuristics, and wrong beliefs rather than on the basis of robust fundamental information. Borrowing from Albert Einstein's quote at the beginning of this Introductory Chapter, one could alternatively say that dumb-money investors are guided by "stupidity, fear, and greed." The greatest source of dumb money comes from retail investing.

In line with the importance of retail investors or dumb money in financial markets, smart-money investors no longer exclusively focus on gleaning fundamental information. More particularly, there is a remarkable amount of evidence that a second type of information has attracted smart-money investors' attention, even before the emergence of Generation I: so-called *non-fundamental* information. Generally, if professional investors collect non-fundamental information, they aim to determine how dumb-money investors behave in the market and what their demand is.

Anecdotal evidence of non-fundamental information can be found in the financial press. In an article titled *"The Smart Way to Follow 'Dumb Money'"*, the *Wall Street Journal* reports that "analysts try to draw smart conclusions from watching so-called 'dumb money' slosh around the market."[11] Farboodi and Veldkamp (2020) provide more specific suggestive evidence by relying on numbers from the *Lipper TASS Database* between 1995 and 2015. They find that hedge funds are shifting away from fundamental analysis, whereas other strategies that include non-fundamental analysis are gaining in importance. More specifically, the assets under management per fund that rely on fundamental analysis have decreased by approximately 50% since 2000. In contrast, the assets under management per fund that focus on quantitative or mixed analysis have nearly quadrupled within the same period. As of 2015, quantitative and mixed analysis have become as important as fundamental analysis to hedge funds. Farboodi and Veldkamp (2020) provide additional suggestive evidence of the importance of non-fundamental analysis by inspecting *Google* search data between 2004 and 2016. They show that the total search volume for "fundamental analysis" fell by approximately 50% within the given period. In strong contrast, the total search volume involving "order flow" nearly tripled, surpassing that of "fundamental analysis" as of 2016.

Since the recent boom in retail investing, additional pieces of evidence undoubtedly show that the interest of the financial world in non-fundamental information has surged even more. In an article from March 2021, the *Financial Times* emphasizes that

> *"[h]aving demonstrated an ability to move markets, retail traders are now a community of market participants that savvy investors want to understand and plug in to their own trading models. The flows are now large enough to count."*[12]

This quote underscores that data on retail investing or dumb money have become increasingly popular with smart-money investors. In light of the increased demand for non-fundamental information, the financial research company *Vanda Research* has developed a method to track retail investors' net purchases of U.S. stocks on a daily basis and sells these data to professional traders such as hedge funds and banks.[13]

Apart from purchasing aggregated data on retail trading from specialized firms, there are several other ways of obtaining non-fundamental information. Some sophisticated investors rely on so-called technical analysis, which entails analyzing price trends and shifts in trading volume to detect uninformed trades. Anecdotal evidence of this technique is provided by Shleifer and Summers (1990, p. 26), who state that "[m]arket professionals spend considerable resources tracking price trends, volume, short interest, odd lot volume, [...] and numerous other gauges of demand for equities." More importantly, the recent rise in retail trading has led to the manifestation of *two* sources of non-fundamental information in financial markets that deserve special attention: social sentiment and payment for order flow.

Social sentiment. The establishment of social media platforms in everyday life has resulted in the emergence of new sources of large amounts of unstructured data, labeled as "big data." Big data linked to the financial world stem, e.g., from finance blogs, search queries, and stock message boards. Stock message boards, in particular, have become a common tool mainly used by the growing mass of retail investors to share opinions, emotions, and concrete investment positions. The temporarily sharp increase in the value of the *GameStop* stock from $20 to $480 in January 2021 is perhaps the most famous example of the importance of stock message boards for financial market movements. At the beginning of 2021, retail investors coordinated their buy orders for *GameStop* stocks via stock message boards such as *WallStreetBets*, deliberately inflicting severe loses on hedge funds that had speculated on falling prices by going short in the *GameStop* stock.[14]

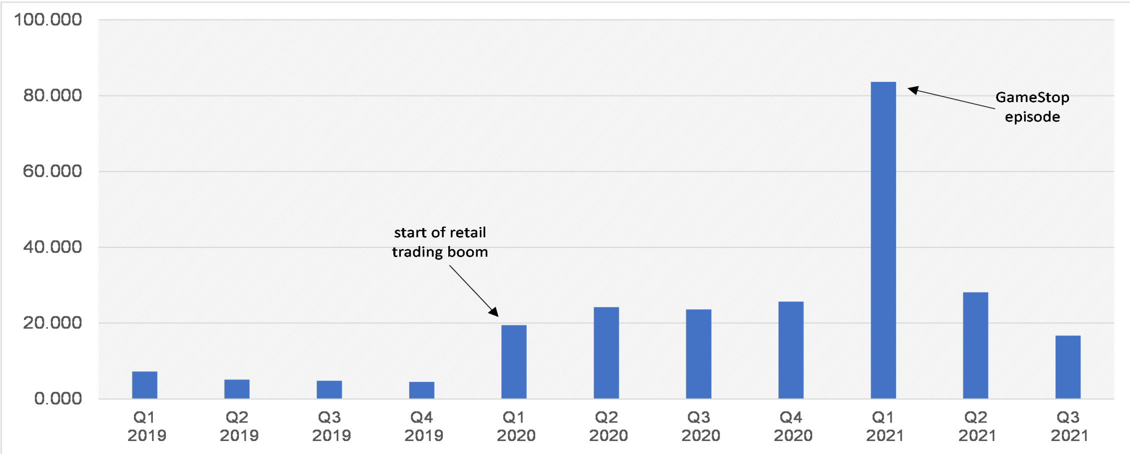
Remarkably, the happenings around *GameStop* gave rise to a new class of stocks, so-called "meme stocks." Meme stocks are characterized by going viral online among retail investors, followed by a significant rise in prices. Further important examples of meme stocks are *AMC Entertainment* and *Blackberry*, which experienced notable growth rates in mid-2021 after drawing retail investors' attention on stock message boards.[15] Under the impression of such new developments, Goldstein et al. (2021) consider big data and its consequences for financial markets to be a major area for future research in the field of financial economics.

The immense volatility of the *GameStop* stock also aroused attention in the political world. In 2021, the *U.S. House Committee on Financial Services* held several virtual hearings on the topic "*Game Stopped? Who Wins and Loses When Short Sellers, Social Media, and Retail Investors Collide*", in which persons involved, financial experts, and officials were invited to give a statement on the occurrences and outline possible implications for financial markets and policy measures.

When analyzing the increased popularity of stock message boards in more detail, the forum *WallStreetBets*, which has experienced immense growth in user statistics since the appearance of meme stocks, stands out. Having counted fewer than 800,000 users in 2019, *WallStreetBets* had approximately 1.75 million registered users in 2020. Even more strikingly, it added more than 8.5 million new users to a total of 10.6 million users in 2021, representing an increase of more than 500% year over year.[16] The increased impact of *WallStreetBets* becomes even more visible when looking at Figure 1.3, which depicts the average daily number of comments and posts made on the platform from 2019 to 2021 on a quarterly basis. Ever since the emergence of Generation I, the average daily user contributions on *WallStreetBets* significantly surpass those from 2019. Due to the *GameStop* episode in January 2021, a temporary high of more than 80,000 average daily user contributions was reached in the first quarter of 2021. Other stock message boards such as *StockTwits* also registered a sharp increase in activity as a consequence of the retail trading boom. In 2021, the platform counted with more than five million users and more than seven million monthly messages, compared to two million users and four million monthly messages in 2019.[17,18]

Having recognized the growing influence of social media on financial markets, financial analysts and professional investors have started to make use of advances in artificial intelligence technology such as text analytics and machine learning to elaborate the big data contained in stock message boards and other related sources (see also OECD, Business and Finance Outlook 2021[19]). The resulting *social sentiment indicators*, which try to capture social media users' contemporaneous

Figure 1.3: Number of average daily user contributions on *WallStreetBets*



Source: Own calculations based on data from Subreddit Stats.

mood or sentiment, serve as a new source of non-fundamental information. As a consequence, a new trading strategy called *social sentiment investing* has emerged. As the term suggests, social sentiment investing aims at evaluating and capitalizing on the sentiment contained in social media platforms.

A recent comment of Alexis Goldstein, Senior Policy Analyst at *Americans for Financial Reform*, from her testimony before the *U.S. House Committee on Financial Services* from March 17, 2021, highlights the importance of social sentiment investing among professional investors:

> *"My time on Wall Street [...] showed me that major institutional players guard information about their own positions, while simultaneously spending large sums of time and resources trying to glean the positions of their competitors [...]. Thousands of users of the WallStreetBets subreddit posting their positions and their future plans for those positions is a source of data that major Wall Street players will mine for information. Many will likely have created software to extract and analyze the content of the posts, and made, trading decisions based on it."*[20]

This statement is backed by several other pieces of evidence that verify the significance of social sentiment investing in financial markets. The business news channel *CNBC* reports that hedge funds purchase data on social sentiment from specialized firms and include it in their investment strategies.[21] For instance, the companies *Quiver Quantitative* and *ExtractAlpha* track social sentiment from stock message boards and finance blogs and sell their information to hedge funds.[22] Remarkably, there are also ways for retail investors themselves to profit from social sentiment. The company *Social Market Analytics* offers so-called "S-scores" to retail investors, which try to numerically characterize the current social sentiment regarding a stock.[23] The investment management company *VanEck* provides retail investors (and also professional investors) with another possibility for engaging in social sentiment investing. In March 2021, it launched the first social sentiment ETF, labeled "BUZZ", which selects its portfolio composition on the basis of sentiment from several social media platforms.[24]

Payment for order flow. The second source of non-fundamental information, which deserves special attention, is related to the recent structural change in the U.S. online brokerage sector toward commission-free trading. The rapid growth of *Robinhood* along with its offer of zero trading fees has forced the Big Four brokerages, viz., *Fidelity Investments*, *Charles Schwab*, *TD Ameritrade*, and *E*TRADE*, into intense price competition, which resulted in the successive elimination of their trading fees for private investors by the end of 2019.[25,26]

This development has obliged U.S. online brokers to find new ways to generate revenue. Currently, customers of commission-free online brokers pay with data rather than money. *Robinhood* and other online brokers generate revenue by routing their customer orders to third parties in the market, mainly to large high-frequency traders or electronic market makers such as *Citadel Securities* and *Virtu Americas*, which are also referred to as "wholesalers." This practice is known as "payment for order flow" (PFOF). The wholesalers then match and execute the received orders and try to profit from the bid-ask spread.

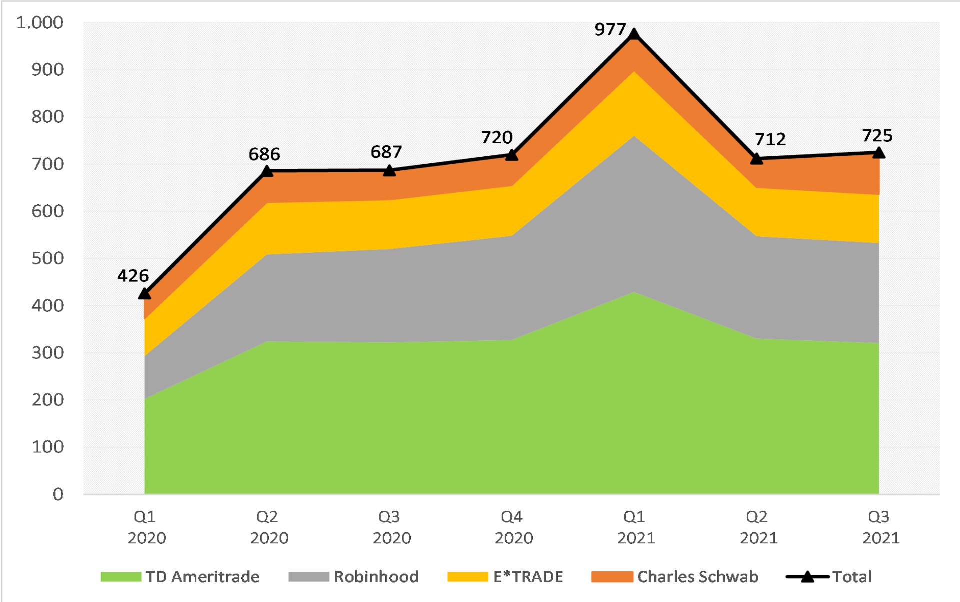
While PFOF has been present in U.S. financial markets since the 1980s (see Parlour and Rajan, 2003), it did not receive much public attention before the recent boom in retail trading. Since then, PFOF has been intensely and controversially debated, as critics of this practice doubt that wholesalers execute retail investors' orders at the best available price for them. This is why the U.S. Securities and Exchange Commission (SEC) has put regulating PFOF on the top of its agenda.[27]

Notably, there is a second important way that wholesalers profit from engaging in PFOF. In his testimony before the *U.S. House Committee on Financial Services* from May 6, 2021, SEC chairman Gary Gensler emphasized that

> "*[i]n addition, the wholesalers get valuable information from this order flow that other market participants get with a delay, if at all. In many aspects of the economy, from social media to search engines, access to data is a growing competitive advantage. Our capital markets are no different.*"[28]

Thus, apart from simply matching and executing retail investors' orders, wholesalers can additionally profit from information about retail traders' order flow when they trade on their own account. Since retail trading represents a significant share of the total trading volume, such information is undoubtedly valuable in financial markets.

Figure 1.4: Net payments received through payment for order flow by selected U.S. online brokers (in million USD)



Source: Own calculations based on data from the online brokers' company 606 reports.

Except for *Fidelity Investments*, all members of the Big Four brokerages raise money by engaging in PFOF.[29] Since the beginning of 2020, the SEC has obliged online brokers to disclose their net payments generated by PFOF on a quarterly basis. Figure 1.4 illustrates the evolution of the amount of money raised through PFOF by *Robinhood*, *Charles Schwab*, *TD Ameritrade*, and *E*TRADE* in 2020 and 2021. Following a sharp increase of more than 60% from the first to the second quarter of 2020, the joint PFOF-related revenue of the four listed U.S. online brokers reached a temporary plateau of approximately $700 million. In the first quarter of 2021, the figures climbed to an all-time high of nearly $1 billion. Since then, joint revenue has declined from its record value, but still remains at a much higher level than in the pre-2020 era.

The boom in PFOF becomes even more visible by comparing the numbers in Figure 1.4 to estimations for 2019. Based on an approximation from *Alphacution*, *Yahoo! Finance* reports that the joint PFOF-related revenue of the four listed online brokers added up to less than $900 million in all of 2019, which falls short of the amount generated in the first quarter of 2021 alone.[30] Thus, one can state that the boom in retail investing and online brokerage has also resulted in a boom in the availability of non-fundamental information in financial markets, caused by PFOF.

Thesis goal and structure. As indicated in this Introductory Chapter, due to the recent surge in retail investing, non-fundamental information has gained enormous importance among professional investors in financial markets. In particular, the two latest and most striking examples of the increasing availability and usage of non-fundamental information, social sentiment and PFOF, should draw economists' attention.

Retail traders that coordinate their stock market activity on social media are capable of generating bubbles in financial markets, as was impressively witnessed during the *GameStop* episode. One important question that naturally arises in this context is how professional investors' trading behavior that tries to capitalize on social sentiment affects the stock price bubbles induced by retail trading. That is, does social sentiment investing tend to counteract or amplify such bubbles? Put differently, does social sentiment investing generally drive prices closer toward their fundamental value, thereby raising their efficiency, or does the opposite hold true?

Considering PFOF, we have seen that the growing popularity of this practice has significantly increased the availability of information about retail order flow in financial markets. More specifically, as a consequence of PFOF, financial markets are populated by several wholesalers that observe different retail traders' order flows. This fact gives rise to several questions: how do wholesalers use their non-fundamental information when trading in financial markets, and how do the different wholesalers interact? What effect does their trading behavior exert on important properties of financial markets such as price efficiency and adverse selection?

The aim of this thesis is to contribute to the theoretical literature on non-fundamental information by investigating the listed research questions. The theoretical framework for the ensuing analyses follows the competitive noisy rational expectations equilibrium (REE) framework in the spirit of Grossman and Stiglitz (1980) (henceforth: GS 1980), Hellwig (1980), and others.

The remainder of this thesis is structured as follows: Chapter 2 provides a literature review, which is divided into two parts. First, to highlight the importance of non-fundamental information from an academic perspective, the literature review describes the intense debate on the determinants of asset prices in the field of financial economics. Second, it briefly outlines the origins of the competitive noisy REE framework and reviews the existing theoretical literature on non-fundamental information. Chapter 3 turns to the phenomenon of social sentiment investing and explores the effect of this new trading strategy on price efficiency. Chapter 4 investigates the strategic interactions between different professional traders that glean non-fundamental information by engaging in PFOF. Building on the theoretical results, some implications regarding the effects of PFOF are derived. Chapter 5 summarizes the main results and points to possible directions for future research. Model proofs and additional technical material are presented in the Appendix.

2 Literature Review

"As economics began to stress mathematical models, economists found that the simplest models to solve were those that assumed everyone in the economy was rational. This is similar to doing physics without bothering with the messy bits caused by friction."

Richard Thaler, 2009

All the anecdotal and empirical evidence listed in the Introductory Chapter underscores the (increasing) importance of non-fundamental information in real financial markets. When trying to understand its importance from an academic point of view, it is indispensable to precisely review the academic debate over what factors drive asset prices, starting with the "efficient market hypothesis" (EMH). This task is carried out in Section 2.1. To set the stage for the theoretical analyses in Chapters 3 and 4, Section 2.2 sketches the origins of the competitive noisy REE framework and gives a detailed overview of the existing theoretical contributions on non-fundamental information.

2.1 From Efficient Markets to Noise Trading

In the 1970s, the EMH, mainly shaped by the work of financial economist Eugene F. Fama, constituted the dominant view among researchers in the field of asset pricing theory. Fama (1965b, p. 56) describes an efficient market as follows:

> *"An 'efficient' market is defined as a market where there are large numbers of rational, profit-maximizers actively competing, with each trying to predict future market values of individual securities [...].*
> *In an efficient market, competition among the many intelligent participants leads to a situation where, at any point in time, actual prices of individual securities already reflect the effects of information based both on events that have already occurred and on events which, as of now, the market expects to take place in the future. In other words, in an efficient market at any point in time the actual price of a security will be a good estimate of its intrinsic value."*

According to Fama's (1965b) definition, an efficient market is characterized by a large number of rational, sophisticated market participants competing with each other. As a consequence of this competition, all available payoff-relevant information is incorporated into asset prices, which makes them reflect their intrinsic or fundamental value at all times. Fama (1965b) further states that due to an efficient market, asset prices follow a random walk, i.e., changes in asset prices are unpredictable, which implies that past price changes do not reveal any information about future price movements. Since rational investors' trading behavior leads prices to reflect their fundamental value "at any point in time", prices only change when new information arrives in the market. As the arrival of new information is unpredictable, changes in asset prices are unpredictable as well (see also Malkiel, 2003).

The EMH is thus based on three central elements: (i) rational market participants in competition, (ii) reflection of fundamentals in asset prices, and (iii), as a consequence of (ii), unpredictable asset price movements.

In related contributions, Fama provides explicit evidence of his EMH. Fama (1965a) shows that the correlations of price changes in stocks belonging to the Dow Jones Industrial Average between 1957 and 1962 do not significantly differ from zero, which implies unpredictable asset price movements. Furthermore, Fama (1970) offers a comprehensive survey of the existing theoretical and empirical literature on the randomness of asset price fluctuations, such as Bachelier (1900), Samuelson (1965), Mandelbrot (1966), and Jensen (1968). Due to the notable amount of evidence of the random walk behavior of asset prices, the EMH developed into the dominant paradigm among financial economists. A comment made by Jensen (1978, p. 96) perhaps best illustrates the general attitude of the academic community toward the EMH during that time:

> *"[i]n the literature of finance, accounting, and the economics of uncertainty, the Efficient Market Hypothesis is accepted as a fact of life, and a scholar who purports to model behavior in a manner which violates it faces a difficult task of justification."*

Challenges to the EMH. After its development in the 1960s and its rise to become the dominant view within the field of asset pricing theory in the 1970s, an era of increasing criticism of the EMH dawned. One central element of the EMH that has been heavily criticized ever since is the assumed rationality of individuals. In this context, the pioneering work of behavioral economists Daniel Kahneman and Amos Tversky is often mentioned.[1] Although their studies did not initially aim at attacking the EMH, they are often cited by financial economists who doubt the full rationality of individuals (see, e.g., Shleifer, 2000, and Barberis and Thaler, 2003). In the 1970s, during the dominant era of the EMH, Kahneman and Tversky provided extensive evidence of the irrational behavior of individuals when making decisions or judgments under uncertainty. Tversky and Kahneman (1971) and Kahneman and Tversky (1972) show that individuals view small samples drawn from a population as more representative than standard probability theory would predict. Kahneman and Tversky label this judgment bias the "law of small numbers." Further empirical evidence shows that individuals rely on personal experiences and other instantly available connections in their memories when assessing the probability of a state or an event, which can lead to estimation biases (see Tversky and Kahneman, 1973). People additionally draw on other heuristics, such as anchoring (see Tversky and Kahneman, 1974). Perhaps most importantly to the field of financial economics, Kahneman and Tversky (1973) prove that people sometimes ignore prior probabilities and the accuracy of new information when updating their expectations about uncertain outcomes. In other words, when forming conditional expectations, people tend to make errors that are irreconcilable with Bayes' rule. The work of Kahneman and Tversky, thus, leads financial researchers to suggest that (some) individuals' investment decisions lack full rationality.

A second important point of critique relates to the evidence that is used to confirm the EMH. Notably, the early evidence of the EMH focuses exclusively on the unpredictability of asset price movements and not on the reflection of fundamentals in prices. Nevertheless, some contributions, such as Fama (1965a), infer from evidence of random asset price fluctuations that prices indeed reflect their fundamental value. Shiller (1984, p. 459), however, calls this conclusion "one of the most remarkable errors in the history of economic thought." He emphasizes that empirical evidence of asset prices varying randomly does *not* necessarily imply that prices reflect their fundamental value.

Following this critique, the relevant literature stresses two different components of the EMH, viz., "no free lunch" and "prices are right" (see, e.g., Barberis and Thaler, 2003). According to the no-free-lunch principle of the EMH, efficient markets are viewed as those where price changes are unpredictable and, thus, any investment strategy is as good as all other ones. The prices-are-right aspect of the EMH implies that in an efficient market, asset prices reflect their intrinsic or fundamental value at any time. Hence, according to this distinction, prices can be "wrong" in the sense that they do not reflect fundamentals, but they can still be unpredictable. Interestingly, the two different versions of market efficiency, as widely used currently in academia, contrast with the initial definition of an efficient market elaborated by Fama (1965b), which inseparably linked the unpredictability of price movements to the reflection of fundamentals in prices.

Following Fama's (1965a) evidence on uncorrelated price movements, the EMH in the spirit of "no free lunch" continues to receive broad support from empirical researchers. There is extensive evidence that professional fund managers are incapable of beating the market, which is interpreted as a result of the unpredictability of price movements. The first comprehensive study in this field dates back to Jensen (1968), who assesses the profitability of 115 selected actively managed U.S. mutual funds between 1945 and 1964 and shows that none of them was able to beat the market. Other more recent examples of the underperformance of mutual funds with respect to passive market portfolios can be found in Rubinstein (2001), Malkiel (2005), French (2008), and Busse et al. (2014). Pástor and Vorsatz (2020) provide recent evidence of approximately 4,000 actively managed U.S. equity mutual funds that perform significantly worse than the S&P 500 index during the COVID-19 crisis.

In contrast, the EMH in the spirit of "prices are right" has been exposed to sharp critique from empirical researchers since the 1980s. Pioneering work in this field goes back to Shiller (1981) and LeRoy and Porter (1981). Shiller (1981) calculates the fundamental value, defined as the present value of all future expected dividends, of the S&P 500 index between 1871 and 1979 and of the Dow Jones Industrial Average index between 1928 and 1979. He shows that the two indices are much more volatile than their underlying fundamental values in the given periods, which clearly contradicts the prices-are-right aspect of the EMH. Mankiw et al. (1985) and West (1988) confirm Shiller's (1981) results by applying different volatility tests to the data set he initially used.

Along similar lines, LeRoy and Porter (1981) prove for the period between 1955 and 1973 that the real value of the S&P 500 index contrasts with the present value relation (i.e., the efficient market model). Like Shiller (1981), LeRoy and Porter (1981) conclude that real stock prices vary much more than the efficient market model would predict, which yields another piece of empirical evidence of the excess volatility of stock prices.

Roll (1988) measures the explanatory power of factors that shape a stock's fundamentals, such as macroeconomic news and public firm-related news, for stock returns. Even after including returns on similar stocks as an additional explanatory variable, he finds that such factors only account for approximately 35% on average in monthly price variations of stocks that were traded on the New York Stock Exchange and on the American Stock Exchange between 1982 and 1987. When looking at the daily returns, the results are even more striking. The expounded factors only explain, on average, approximately 20% of total stock price variations between 1982 and 1986.

Cutler et al. (1989) provide another empirical study that suggests that stock prices are not exclusively driven by fundamentals. The authors estimate that macroeconomic news explain less than one-fifth of the total monthly variation in aggregated U.S. equity prices between 1926 and 1985. Excess volatility can also be found in markets for other securities, such as options (see, e.g., Stein, 1989).

GS 1980 develop an important theoretical challenge to the prices-are-right component of the EMH. In their seminal work, they prove that in a competitive market, prices cannot fully reflect all available fundamental information at any time whenever gleaning fundamental information is linked to (physical) costs. In an efficient market, traders are unable to receive any compensation for gathering costly fundamental information, as there is no mispricing that could be profited from. This absence of profit opportunities eliminates any incentive to collect costly information. Instead, in a competitive and efficient market, investors prefer to free-ride on the public market price because all available fundamental information can be costlessly inferred from it. Thus, if the price fully reflects fundamentals, no trader gleans costly information. In this situation, however, the price cannot reflect any fundamental information because there is no one collecting information and bringing it into the price. As a consequence, it is impossible for a fully efficient price to prevail if gathering fundamental information is costly. This famous result is known as the Grossman-Stiglitz paradox.

Investor sentiment and noise trading. The empirical and theoretical challenges to market efficiency in the sense of "prices are right" presented above suggest that there must be other determinants apart from fundamentals that shape asset prices. As already outlined, empirical evidence unequivocally states that individuals behave irrationally when making judgments or decisions under uncertainty. Building on this important observation, the theory of *investor sentiment* has emerged as a response to the detected anomalies in financial markets. Shleifer (2000, p. 24) describes it as "the theory of how real-world investors actually form their beliefs and valuations, and more generally their demands for securities." Put differently, the theory of investor sentiment aims to include empirically observed irrational behavioral patterns in traders' investment decisions in financial markets.

As a consequence of this approach, microfounded asset pricing models began to distinguish between two classes of investors: rational, utility-maximizing traders (i.e., smart money or professional investors) and less sophisticated traders exposed to wrong or biased beliefs, maximization errors, and other forms of irrationality (i.e., dumb money or retail investors). These irrational traders are often called *noise traders*. The relevant literature lists many different characterizations and behavioral patterns of noise traders. Glosten and Milgrom (1985, p. 77) view noise traders as investors who trade for exogenous liquidity needs, which "may arise from predictable life cycle needs or from less predictable events such as job promotions or unemployment, deaths or disabilities, or myriad other causes." Black (1986, p. 531) describes noise trading as "trading on noise as if it were information."

A remarkable amount of the theoretical asset pricing literature models noise trader demand as an exogenously given, random component that is independent of fundamentals (e.g., Kyle, 1985; Danthine and Moresi, 1993; Han and Yang, 2013; Easley et al., 2016; Arnold and Zelzner, 2020; Banerjee et al., 2021; Xue and Zheng, 2021). Another strand models noise traders as investors who naively extrapolate past price trends, so-called "positive feedback traders." Pioneering work in this field dates back to Cutler et al. (1990) and De Long et al. (1990b). More recent contributions include Barberis and Shleifer (2003), Arnold and Brunner (2015), and Barberis et al. (2015, 2018).[2]

In De Long et al. (1990a), noise traders correctly maximize their expected utility, but form biased beliefs about future prices. More specifically, they misperceive the expected return by a random error term. Similar approaches can be found in Hirshleifer (2006), Yan (2010), and Yang and Li (2013). Noise traders in Mendel and Shleifer (2012) and Banerjee and Green (2015) differ from those in De Long et al. (1990a) in the way that they form completely wrong beliefs about future prices rather than only biased beliefs. The boundedly rational traders in Mondria et al. (2021) misinterpret the market price's information about fundamentals.

By pointing to the importance of irrational behavior in financial markets, investor sentiment or noise trading can contribute to giving an explanation for the listed challenges to the prices-are-right component of the EMH. The irrational behavior of some investors could indeed (partially) solve the excess-volatility puzzle proposed by Shiller (1981) and LeRoy and Porter (1981). The existence of noise traders can also provide a solution for the Grossman-Stiglitz paradox: inefficient prices resulting from noise trading generate profit opportunities for rational investors, which can incentivize them to gather costly fundamental information. Noise trading makes prices *partially* rather than *fully* reflect fundamental information. Thus, it is impossible for rational, uninformed traders to infer all available fundamental information from observing the market price. In fact, noise trading can establish a competitive equilibrium with costly fundamental information and a positive number of informed, rational investors (see GS 1980).

Limits of arbitrage. Notably, the advocates of efficient or "right" prices do not per se deny that irrational investors exist. Instead, they argue that this type of trader cannot have a persistent influence on asset prices. Fama (1965a) states that rational investors would immediately and fully exploit the profit opportunities generated by irrational investors and, by that, keep prices efficient. This argument is in line with Friedman's (1953) early argument that rational traders or "speculators" act as stabilizers by buying low and selling high.

Thus, investor sentiment or noise trading alone cannot provide a reason for why prices should be inefficient in the presence of rational market participants. Instead, there is a second important foundation needed to explain inefficient prices: *limits of arbitrage*. According to the standard textbook definition given by Sharpe and

Gordon (1990, p. 795), arbitrage is characterized as the "simultaneous purchase and sale of the same, or essentially similar, security in two different markets for advantageously different prices." In a more practical manner, Shleifer and Summers (1990, p. 20) describe arbitrage "as trading by fully rational investors not subject to [...] sentiment." As suggested by the former definition, arbitrage is costless and without any risk from a theoretical point of view. Under such conditions, rational traders or "arbitrageurs" are able to offset any influence stemming from noise traders and keep prices efficient. In reality, however, there are several factors that make arbitrage costly and risky and, therefore, limited.

Arbitrageurs can be exposed to *fundamental risk*. This type of risk encompasses any uncertainty associated with the fundamental value of an asset. Fundamentals can be shaped by future news that is unpredictable today, thereby creating risk. This risk, however, is not problematic if perfect substitutes for an asset exist, which can be used to hedge. In the absence of such substitutes, arbitrageurs face real fundamental risk. If rational traders are additionally risk-averse, their limited risk-bearing capacity constrains their trading position, and prices turn out to be inefficient in the presence of noise trading (see Barberis and Thaler, 2003). The combination of fundamental risk, unavailable substitutes, and risk aversion leads to inefficient prices in, e.g., GS 1980 and Campbell and Kyle (1993).

Another important source of risk that can pave the way for inefficient prices comes from noise traders themselves. Irrational traders can worsen existing mispricing in the short run and, thus, generate temporary losses for arbitrageurs. As shown by De Long et al. (1990a), *noise trader risk* flanked by risk aversion and short trading horizons of arbitrageurs can make prices inefficient, even in the absence of fundamental risk.

In another pioneering article, Shleifer and Vishny (1997) identify *capital constraints* as a further reason for limits of arbitrage. They point to the fact that many real-world arbitrageurs depend on the capital of other, less sophisticated investors that evaluate arbitrageurs' performance on the basis of short-term gains and losses. Arbitrageurs exposed to noise trader risk can be most constrained in correcting existing mispricing exactly when prices sharply deviate from fundamentals. If arbitrageurs invest their clients' money in an underpriced asset and noise traders become even more "bearish" about the asset, external investors may interpret arbitrageurs' short-term losses as an expression of their lack of expertise. As a result, they withdraw their capital, which leaves arbitrageurs with the smallest financial resources when their profit opportunities are highest. Interestingly, Shleifer and Vishny (1997) provide an incomplete theoretical characterization of their seminal model. Arnold (2009) carries out the full theoretical analysis.

Building on Shleifer and Vishny (1997), the subsequent literature identifies some further reasons that can limit arbitrageurs' capital or trading position. In Gromb and Vayanos (2002, 2018), arbitrageurs need to collateralize their investment, which restricts the capital they can deploy. In Dow et al. (2021), arbitrageurs can only trade a limited amount of shares. Another strand of the theoretical literature imposes direct costs on arbitrageurs if they wish to trade. These costs include a fixed market entry cost as in Allen and Gale (1994) or costs that are proportional to the number of traded shares as in Isaenko (2015) and to the transaction price as in Buss and Dumas (2019).

Abreu and Brunnermeier (2002, 2003) identify a possible *synchronization risk*, which can limit arbitrage. They assume that arbitrageurs sequentially, rather than simultaneously, realize mispricing in the market and need to coordinate the timing of their investment decision to successfully counteract it. A lack of synchronization can make deviations from fundamentals persistent even if rational traders could deploy sufficient capital.

The large theoretical literature on limits of arbitrage is backed by abundant empirical evidence (see, e.g., Barberis and Thaler, 2003, and Gromb and Vayanos, 2010, for comprehensive overviews). One famous example is given by Froot and Dabora (1999), who investigate the relative pricing of the stocks of *Royal Dutch* and *Shell Transport*. In 1907, the two companies agreed to merge and to distribute future earnings on a 60:40 basis without giving up their legal entity. If both stocks were correctly priced in line with this type of merge, the stock price of *Royal Dutch* should be 1.5 times the stock price of *Shell Transport*. Froot and Dabora (1999), however, show that the relative mispricing of *Royal Dutch* varied from 10% to 40% between 1980 and 1994, which provides a clear piece of evidence of limited arbitrage.

Another frequently mentioned instance is index inclusion. Shleifer (1986) finds abnormal returns on average of approximately 3% for stocks between 1976 and 1983 after announcing that they would be included in the S&P 500 index. Notably, these abnormal returns did not vanish during the following trading days. Other empirical studies that identify an index inclusion effect are provided by Harris and Gurel (1986) and Beneish and Whaley (1996), among others. Wurgler and Zhuravskaya (2002) show that the price jump after index inclusion is highest for stocks that do not have any close substitutes and are, hence, characterized by high arbitrage risk.

Brunnermeier and Nagel (2004) and Griffin et al. (2011) provide empirical evidence of the model of Abreu and Brunnermeier (2002, 2003). They show that most hedge funds were reluctant to bet against the tech bubble in the NASDAQ index between January 1997 and March 2000, until a joint selling effort of theirs made the bubble eventually burst.

Behavioral finance and measuring noise trading. As illustrated, the two central foundations "investor sentiment" and "limits of arbitrage" can jointly explain why prices persistently deviate from fundamentals. Shleifer (2000, p. 24) stresses the indispensable connection between the two concepts to justify inefficient prices:

> *"If arbitrage is unlimited, then arbitrageurs accommodate the uninformed shifts in demand as well as make sure that news is incorporated into prices quickly and correctly. Markets then remain efficient even when many investors are irrational. Without investor sentiment, there are no disturbances to efficient prices in the first place, so prices do not deviate from efficiency."*

The field that draws on investor sentiment and the limits of arbitrage to show that prices can be inefficient is called *behavioral finance*. After its development in the early 1990s, behavioral finance quickly became the dominant approach in financial economics. Ever since the emergence of behavioral finance, there has been little doubt among financial researchers that some traders in financial markets act irrationally and contribute to inefficient prices. Due to the persistent influence of noise trading on prices, non-fundamental information is unequivocally helpful for rational traders to better understand market movements. Interestingly, there is an intense debate in the academic community about how one can actually gauge investor sentiment or noise trading and obtain non-fundamental information.

The relevant empirical literature distinguishes between three types of measurement: two traditional ones and one more modern approach. The first traditional type relies on direct *market data*. Classical work in this field dates back to Lee et al. (1991), who try to gauge investor sentiment through closed-end fund discounts. In a comprehensive and influential study, Baker and Wurgler (2006) construct a sentiment index based on six market indicators, including trading volume, closed-end fund discount, and dividend premium. Others proxy for noise trading with mutual fund flows (e.g., Frazzini and Lamont, 2008; Lou, 2012; Akbas et al., 2015).

More importantly for this thesis, noise trading is also measured by directly drawing on retail investor transaction data from brokers (e.g., Kumar and Lee, 2006; Barber et al., 2009; Foucault et al., 2011; Peress and Schmidt, 2019, 2021). In light of recent developments, Barber et al. (2021), Eaton et al. (2021), and Ozik et al. (2021) rely on data from the online broker *Robinhood* and show that the trading platform mostly attracts inexperienced investors.

The second traditional method, which is less relevant for this thesis, takes on data from *surveys*. Early works by Fisher and Statman (2003), Charoenrook (2005), and Lemmon and Portniaguina (2006) draw on the "University of Michigan Consumer Sentiment Index" and the "Conference Board Consumer Confidence Index", which evaluate household surveys, to proxy for investor sentiment. Qiu and Welch (2006) make use of the "Survey of Investor Sentiment" by *UBS/Gallup*. Amromin and Sharpe (2014), Greenwood and Shleifer (2014), and Banchit et al. (2020) constitute other, more recent examples that rely on indices based on survey data.

However, the two traditional types of measurements of investor sentiment or noise trading are exposed to criticism that questions their validity. Da et al. (2015, p. 2) state that measures based on aggregate market data "have the disadvantage of being the equilibrium outcome of many economic forces other than investor sentiment." Considering survey measures, Zhou (2018, p. 248) emphasizes that participants of such surveys "may not respond, and those who respond may not have an incentive to tell the truth." The third, non-traditional type of measuring investor sentiment, which gained enormous popularity among empirical researchers in the last decade, takes data from *publicly available media content* such as internet search results, blogs, and stock message boards. By that, it is closely related to the concept of social sentiment. When assessing the validity of these measurements, Zhou (2018, p. 250) notes that "[i]n comparison with market- and survey-based measures, [...] measures based on textual analysis perform better by far."

Pioneering work in this area dates back to Wysocki (1998), who shows that the cumulative stock message posting volume on *Yahoo! Finance* predicts shifts in next-day stock returns and trading volume. Similarly, Antweiler and Frank (2004) and Das and Chen (2007) prove that the sentiment derived from *Yahoo! Finance* is linked to volatility and trading volume in financial markets. Karabulut (2013) and Siganos et al. (2014) show that a rise in *Facebook*'s Gross National Happiness Index is positively correlated with changes in the next day's stock market returns and future trading volume.

Da et al. (2011) identify a positive correlation between *Google*'s search volume index (SVI) and stock returns during the next two weeks. In an extension of their previous work, Da et al. (2015) measure investor sentiment with the so-called Financial and Economics Attitudes Revealed by Search (FEARS) index, which is based on the SVI. The authors show that changes in the FEARS index predict stock market returns in the next two days. More recently, Desagre and D'Hondt (2021) uncover a positive relationship between the SVI and retail investors' trading activity.

The social media platform *Twitter* along its stock message board *StockTwits* constitutes another often used source to proxy for investor sentiment. Sul et al. (2017) derive that the contemporaneous sentiment contained in the tweets from *StockTwits* predicts future prices. Along similar lines, Duz and Tas (2020) conclude that the content of *StockTwits* has predictive power for short-term price movements and shifts in trading volume. Ngo and Nguyen (2021) show that the public sentiment contained in tweets was related to the V-shaped behavior of asset prices at the beginning of the COVID-19 pandemic.[3]

2.2 The Competitive Noisy REE Framework

Having highlighted the importance of non-fundamental information from an academic perspective, the second part of the literature review addresses the competitive noisy REE framework, which forms the basis for the subsequent theoretical analyses in Chapters 3 and 4. The competitive noisy REE framework builds upon five main assumptions: (i) stochastic asset payoffs, (ii) a fully competitive economy, (iii) some source of "noise" in the market, (iv) exponential utility with constant absolute risk aversion (CARA), and (v) random variables that follow a multivariate normal distribution. The foundations of the framework date back to the seminal contribution of GS 1980.[4] According to *Google Scholar*, as of November 2021, it was cited more than 10,000 times, thereby making it one of the most influential papers ever published in the *American Economic Review*.

In the setup of GS 1980, a discrete mass of rational agents characterized by CARA utility functions trades one riskless and one risky asset in a static, competitive financial market. The risky asset pays off its unknown fundamental value, which consists of two independent random components, one period ahead. Rational traders can acquire unbiased information about the first fundamental component at a fixed cost. Through trading, their private information is factored into the price. The portion of rational traders that remains uninformed tries to infer informed agents' fundamental information from observing the market price. To prevent the price from being fully revealing, the authors assume the supply of the risky asset to be random. The randomness of supply injects noise into the market price, which confronts uninformed traders with a signal extraction problem: they do not know whether a high price is due to high fundamentals or to low supply. Thus, noise in the asset price can incentivize (some) rational traders to acquire fundamental information. Without noise, a competitive equilibrium with a positive portion of informed traders would not exist. This result leads to the formulation of the famous Grossman-Stiglitz paradox, which states that in a competitive economy, prices cannot fully reflect fundamentals if gathering fundamental information is costly.

GS 1980 further assume that all random variables in their model are jointly normally distributed. The combination of CARA utility and normal random variables delivers an analytically tractable setting, which has become known as the CARA-normal framework in the literature. The equilibrium that GS 1980 derive from this setting is an REE. This implies that rational agents' conjecture about the behavior of the asset price is self-fulfilling, i.e., the price function conjectured by agents needs to coincide with the "real" function in the economy that results from market clearing. This makes agents' expectations rational. GS 1980 show that equilibrium is unique and characterized by strategic substitutability in acquiring fundamental information. The reason for this result lies in the positive correlation between the mass of informed traders and the degree of price efficiency. As more agents with fundamental information enter the market, the price more accurately reflects fundamentals. This, in return, lowers the incentive for uninformed traders to acquire costly information and makes them prefer to free-ride on the public market price instead.

Although playing a pioneering role in financial economics, the seminal model of GS 1980 is not free of criticism. Hellwig (1980, p. 478) identifies a conceptual weakness that leads him to call the rational agents in GS 1980 "slightly schizophrenic." Since the number of traders in GS 1980 is finite, a single trader's behavior indeed influences the market price. However, GS 1980 neglect this phenomenon in their setup, as all agents are assumed to take the price as given. Hellwig (1980) resolves this issue by assuming a "large market" with a continuum of agents. In such an economy, each trader is infinitesimally "small" and price-taking behavior is optimal.

Diamond and Verrecchia (1981) deliver another important contribution to the field. Similar to Hellwig (1980), they model the risky asset's fundamental payoff as a single random component rather than as two independent components, as in GS 1980. Each rational trader is assumed to observe a private, noisy signal about fundamentals. In this model, contrary to GS 1980, informed traders glean fundamental information from observing the market price. Since their own private information is not perfectly precise, informed agents have an incentive to infer further information about fundamentals from the price, which aggregates traders' private fundamental information. The authors show that the resulting REE is unique with the same characteristics as in GS 1980. Verrecchia (1982) extends the model of Diamond and Verrecchia (1981) by introducing an information acquisition stage.

As opposed to GS 1980 and Hellwig (1980), Diamond and Verrecchia (1981) and Verrecchia (1982) provide an explicit economic interpretation for the noisiness of asset supply. They relate it to random shocks to agents' individual asset endowments, which jointly determine aggregate supply. Since individual endowments are random, total asset supply is also random.

Subsequent work in the field develops a different interpretation for the randomness of asset supply by linking it to the concept of liquidity traders or noise traders. Allen (1984), who conducts a welfare analysis in the framework of GS 1980, explicitly attributes the random fluctuations in asset supply to the behavior of liquidity traders.[5] Similarly, Admati (1985, p. 632), who considers a multi-asset economy, states that movements in asset supply might "be caused by some trade of a nonspeculative nature (such as for life-cycle or liquidity reasons), or by some traders lacking perfect knowledge of the market structure." Since then, noise trading has become a common explanation for random fluctuations in asset supply in the competitive noisy REE framework (see also Vives, 2008, Chapter 4).

Singleton (1986), Brown and Jennings (1989), and Grundy and McNichols (1989) deliver other pioneering contributions by transferring the originally static setup into a dynamic setup. In such a setting, the risky asset is traded multiple times before it pays off its fundamental value. Thus, agents are concerned with predicting future prices rather than fundamentals only. Traders are modeled either as short-lived agents in terms of overlapping generations (OLG) or as long-lived agents (LLA). In the former case, there are different generations of agents, each of which trades at one date only. The LLA variant implies that the same agents trade at all dates.

Literature on non-fundamental information. As previously shown, there exist two alternative interpretations for random fluctuations in asset supply in the competitive noisy REE framework: random shocks to agents' individual asset endowments and noise trading. Following the field of behavioral finance outlined in Section 2.1, we adopt the noise trader interpretation in this thesis.[6] When analyzing the effect of non-fundamental information, this is without loss of generality. Using information about aggregate supply contained in random asset endowments delivers very similar results to making use of information about noise trader demand (see, e.g., Ganguli and Yang, 2009, and Manzano and Vives, 2011). Because of this similarity, the literature dealing with random asset endowments and with information about noise trading in the competitive noisy REE framework is reviewed in the rest of this chapter. For the sake of simplicity, both types of information are labeled "non-fundamental information" in the following discussion.

There exists a small but growing body of literature that explores the effects of non-fundamental information in the competitive noisy REE framework.[7] In an early contribution, Gennotte and Leland (1990) assume that a fixed portion of rational traders observes a part of the noisy asset supply, which is driven by liquidity traders. The authors focus on explaining stock market crashes that can occur due to unobserved shifts in supply. Following an unperceived increase in supply, uninformed traders might misinterpret the ensuing fall in prices as bad fundamental news received by informed traders, which makes them reduce their demand as a consequence. This exacerbates even more the initial fall in prices and can lead to a stock market crash.

Ganguli and Yang (2009) build on Diamond and Verrecchia (1981) and consider a static economy where rational traders are endowed with a random amount of the model's risky asset. Each endowment is characterized by a common and an idiosyncratic component. Thus, an agent's individual endowment yields valuable information about the unknown aggregate supply in the economy. The existence of private non-fundamental information can lead to two self-fulfilling equilibria in the financial market.[8] The two equilibria differ in the effect that a rise in the mass of fundamentally informed traders exerts on price efficiency. In the first equilibrium, which Ganguli and Yang (2009) label SUB-REE, a rise in the mass of fundamentally informed traders increases price efficiency, as in the unique equilibrium of GS 1980 and Diamond and Verrecchia (1981). In the second equilibrium, which the authors label COM-REE, more fundamentally informed traders *decrease* price efficiency. However, price efficiency is always higher in both equilibria than in the respective economy without non-fundamental information.

The fact that more informed trading can reduce price efficiency paves the way for complementarities in fundamental information acquisition. Ganguli and Yang (2009) show that acquiring fundamental information is always a complement in the COM-REE. Contrary to GS 1980 and Diamond and Verrecchia (1981), a rise in the mass of fundamentally informed traders makes price efficiency decrease in this equilibrium. Therefore, as more traders acquire fundamental information, the incentive for others to do so increases due to a less efficient price.

Another striking difference compared to GS 1980 and Diamond and Verrecchia (1981) is that too much informed trading can lead to a market breakdown. If information is too precise or the mass of informed traders is too large, trading stops, and the two equilibria vanish. The reason for this is intensifying adverse selection in the financial market induced by increasing information asymmetry. This phenomenon will be reviewed in more detail in Chapter 4. Ganguli and Yang (2009) extend the basic model by attributing random changes in asset supply to the behavior of noise traders and by allowing for the simultaneous acquisition of private fundamental and private non-fundamental information. The authors show that acquiring both types of information simultaneously can be a complement in the two equilibria of the model.

Manzano and Vives (2011) build on the model of Ganguli and Yang (2009) by introducing correlations among the error terms in agents' private fundamental signals. In this scenario, the existence of an equilibrium is always guaranteed, and there are up to three equilibria possible. Multiple equilibria arise if an increase in price efficiency makes traders rely more on their private information. The authors show that acquiring fundamental information is a strategic substitute (resp., a strategic complement) at the two extreme equilibria (resp., at the intermediate equilibrium). Manzano and Vives (2011) also analyze the stability of the equilibria, which is highly controversial in a static setup. They find that the equilibrium that is characterized by strategic complementarity in information acquisition is unstable, whereas the other two equilibria with strategic substitutability are stable. Using a similar argument, Manzano and Vives (2011) consider the COM-REE of Ganguli and Yang (2009) to be unstable.

Zeng et al. (2018) modify the setup of Ganguli and Yang (2009) by modeling fundamentally and non-fundamentally informed investors as separate groups. At the information acquisition stage, they show that the equilibrium mass of non-fundamentally informed traders is negatively correlated with that of fundamentally informed investors (holding the costs of acquiring information constant).

Marmora and Rytchkov (2018) investigate the effects of non-fundamental information on price efficiency in more detail. In an economy where agents are endowed with diverse prior information about the asset's fundamental value, the authors assign a fixed information processing capacity to rational traders that they can use to produce private fundamental and private non-fundamental information. Marmora and Rytchkov (2018) derive that agents tend to specialize in information acquisition. Those with precise prior information about fundamentals focus on the acquisition of fundamental information. Those with imprecise prior information switch to acquiring non-fundamental information. Marmora and Rytchkov (2018) prove that acquiring non-fundamental information unequivocally benefits price efficiency. This result is not trivial in their setup as the introduction of non-fundamental information acquisition exerts two counteracting effects. On the one hand, it makes some investors reallocate their fixed information processing capacity and produce less private fundamental information, which harms price efficiency. On the other hand, non-fundamental information allows rational traders to partly offset the influence of noise trader demand on the market price, thereby increasing price efficiency. Marmora and Rytchkov (2018) show that the latter, positive effect unambiguously dominates in their model.

There also exist some models that explore the effects of non-fundamental information in a dynamic financial market. Spiegel (1998) analyzes an infinite-horizon OLG economy with multiple risky assets in which agents are endowed with uncertain amounts of these assets. The coordination among short-lived traders can lead to two self-fulfilling equilibria with different levels of price volatility. Watanabe (2008) extends Spiegel's (1998) model to the case of private fundamental information with heterogeneous precision.

Farboodi and Veldkamp (2020) also consider an infinite-horizon OLG economy and analyze the effects coming from information about contemporaneous noise trader demand. Similar to Marmora and Rytchkov (2018), they assign a data constraint to traders to process current private fundamental and current private non-fundamental information. There is technological progress over time, i.e., agents can process more information as time advances. Farboodi and Veldkamp (2020) show that non-fundamental information increases price efficiency in a dynamic context, although it crowds out fundamental information under a data constraint.

Implementing dynamic frameworks additionally allows an analysis of the effects coming from information about *future* noise shocks. Traders who try to predict future prices clearly have an incentive to gather information about future noise trading (since it affects future prices). Cespa and Vives (2012, 2015) follow up on this idea by exploring the impact of persistent noise trader demand in three-period

LLA and OLG models, respectively. If noise shocks across periods are correlated, rational traders can use the current price to infer information about the current noise shock, which then yields information about the next period's noise shock. Thus, in a dynamic context, the market price can also be used to infer valuable information about noise given that noise shocks are correlated across periods. In this sense, the market price plays a dual role: it conveys fundamental as well as non-fundamental information. Cespa and Vives (2012, 2015) show that persistent noise trading can generate multiple equilibria in financial markets and identify an equilibrium that is characterized by high price efficiency. Concerning the model variant with OLG of investors, this finding challenges the widespread view that short-term trading contributes to inefficient prices.

In a three-period LLA model, Avdis (2016) explores the consequences of correlated noise shocks at the information acquisition stage. He shows that acquiring fundamental information can be a complement in such a setting. The reason for this is that more fundamentally informed trading makes the price more informative about fundamentals and thus *less* informative about noise. Hence, more fundamentally informed trading makes it more difficult for uninformed traders to infer non-fundamental information from the price. As a consequence, the incentive for uninformed traders to acquire fundamental information can increase since it helps to more accurately extract information about noise from the market price.

3 Social Sentiment Investing and Price Efficiency

"Developments in machine learning, data analytics, and natural language processing have allowed sophisticated investors to monitor various forms of public communication to see relationships between words and prices. This practice, called sentiment analysis, has picked up steam in the last couple of years, and it has grown to include online communities."

SEC chairman Gary Gensler, May 2021

This chapter is based on Arnold and Russ (2021). It deals with the first source of non-fundamental information that has experienced increased importance in financial markets: social sentiment. The rising popularity of social media platforms and, in particular, stock message boards such as *WallStreetBets* has opened up new possibilities for retail traders to coordinate their stock market activity and move prices. The *GameStop* episode in January 2021 gave rise to the emergence of a new class of stocks that attracts retail investors' attention on social media, known as meme stocks. Advances in artificial intelligence technology allow professional investors to capitalize on information contained in stock message boards by engaging in social sentiment investing, i.e., by forming investment strategies on the basis of social sentiment derived from stock message boards, which yields information about retail traders' behavior in financial markets.

The aim of this chapter is to investigate the ensuing impact of social sentiment investing on one of the central metrics of financial markets: price efficiency. Does social sentiment investing generally drive stock prices closer to fundamentals and make them more efficient, thereby counteracting the bubbles induced by retail trading, or does the opposite hold true?

In the spirit of behavioral finance, we interpret retail traders as noise traders. Because of the remarkable validity of sentiment measures stemming from textual analysis (see Zhou, 2018, Subsection 4.3), social sentiment indeed provides a reasonable proxy for retail or noise trading. Importantly, as outlined in Section 2.1, empirical researchers identify a positive link between sentiment derived from social media platforms and future prices. Thus, due to this *predictive* power, social sentiment serves not only as a measure for current but also for *future* noise trader behavior, which affects *future* prices. Since the big data used to gauge social sentiment stem from the same publicly available internet sources, we assume that social sentiment yields noisy, *public* signals about current and future noise trader demand.

We integrate these signals into the canonical dynamic REE framework in the spirit of Singleton (1986) and Brown and Jennings (1989) and assess the impact of information about future noise trading on contemporaneous price efficiency. The ensuing dynamic analysis distinguishes between the three-period OLG and LLA setups. The two models have a unique equilibrium that can be computed in closed form. We show that introducing public information about *future* noise trader demand potentially *harms* current price efficiency, both in the OLG and LLA models. Thus, current price efficiency tends to be higher if public information about future noise trader demand is absent, implying a potentially *negative* effect of social sentiment investing on price efficiency. This finding is consistent with Goldstein et al.'s (2021, p. 3222) conjecture that "although big data provides more information for sophisticated players such as institutional investors and firms, the impact of big data may not always be positive."

Moreover, our result sharply contrasts with the outcomes of three related parts of the theoretical literature that explore the effects induced by information about contemporaneous noise in static setups (see Ganguli and Yang, 2009; Manzano and Vives, 2011; Marmora and Rytchkov, 2018; Zeng et al., 2018), by information about contemporaneous noise in a dynamic model (see Farboodi and Veldkamp, 2020), and by public information about fundamentals in a dynamic setup (see Gao, 2008). In all cited contributions, *any* level of precision *unequivocally* delivers *higher* price efficiency than *zero* precision. Hence, price efficiency is unambiguously higher in the presence than in the absence of the respective type of information. In the models of this chapter, by contrast, price efficiency can be *lower* in the *presence* of a public signal about future noise trader demand.

The driving force behind our result is an additional mechanism in the dynamic models that is not present in the respective static benchmark. In the static economy, by gauging social sentiment, agents can only glean public information about *contemporaneous* noise trader demand. Thus, price efficiency is shaped by two components. First, it is affected by current noise trader demand, whose influence is mitigated if rational agents trade more aggressively on private fundamental information or more aggressively against the public signal about current noise trading. Second, price efficiency is influenced by the common error term inherent in the public signal about current noise trading. More aggressive trading against the public signal amplifies the negative impact of this common error term on price efficiency.

If the precision of the public signal about current noise trader demand increases, price efficiency is affected in two opposite ways. For one thing, a more precise signal about current noise means that rational agents offset more of the noise trader demand, which reduces its influence on the market price relative to fundamentals. For another, a higher precision amplifies the detrimental impact of the public signal's common error term on price efficiency. In the static model version, the former effect unequivocally dominates, which makes public information about current noise raise price efficiency.

In the dynamic setup, by gauging social sentiment, rational traders can additionally glean information about *future* noise trader demand. Hence, price efficiency at the early date is influenced by the same two components as in the static version and by a third component, which represents the impact of the signal about future noise trading. Since this signal is uncorrelated with fundamentals and current noise trading, its introduction injects a new source of noise into the price, moving it away from fundamentals.

In the LLA model, if public information about current noise is absent, *any* level of precision of the signal about future noise *unequivocally* yields *lower* price efficiency than *zero* precision. If information about current noise is unavailable, the two components that are also present in the static setting are independent of the precision of the signal about future noise. Hence, changes in signal precision only affect the component that determines the influence of the public signal about future noise on price efficiency. Zero precision yields maximum price efficiency since this is the only finite value for which traders ignore the signal.

Moreover, a perfectly precise signal generally leads to lower price efficiency than a completely imprecise signal. The component exclusive to the dynamic model drops out in both limiting cases so that price efficiency is determined by the same two components as in the static version. The result is driven by the fact that agents trade more aggressively against their public signal about *current* noise if the precision of the signal about *future* noise switches from infinity to zero. As in the static version, the stabilizing impact coming from offsetting more current noise dominates the destabilizing effect induced by amplifying the impact of the common error term of the public signal about contemporaneous noise. Strikingly, parameterizations of the model even exist such that price efficiency is *monotonically decreasing* in the precision of the public signal about future noise trader demand.

In the OLG model, price efficiency is generally lower than in the LLA variant (with identical model parameters). Nevertheless, public information about future noise is less likely to harm price efficiency. Allen et al. (2006) show that in short-trading economies, prices are driven by higher-order expectations about fundamentals rather than by actual expectations about fundamentals. Allen et al. (2006) label this phenomenon the "Keynesian beauty contest" (KBC) in financial markets.[1] Additionally, the authors show that rational agents underweight private fundamental information when exhibiting a short trading horizon. However, in the OLG model of this chapter, short-lived agents trade more aggressively on private fundamental information as the signal about future noise gains in precision. A more precise *public* signal about future noise trader demand makes date-2 rational traders offset more of the date-2 noise trader demand, thereby driving the date-2 price closer to fundamentals. This leads date-1 rational traders to trade more aggressively on private fundamental information, which boosts price efficiency. Nevertheless, public information about future noise can still be detrimental to price efficiency. As in the LLA model, it can even happen that price efficiency is a *monotonically decreasing* function of signal precision.

Our findings contribute to two strands of the theoretical literature. First, they add to the field that explores the impact of related types of information on price efficiency in the competitive noisy REE framework. As shown in Section 2.2, several papers investigate the effects of information about contemporaneous noise in static models. The common result is that more precise information about current noise is conducive to price efficiency in a stable equilibrium. More precise non-fundamental information can lead to a fall in price efficiency in unstable equilibria. However, price efficiency is *unequivocally* higher in *all* equilibria in the presence than in the absence of non-fundamental information (see Ganguli and Yang, 2009; Manzano and Vives, 2011; Marmora and Rytchkov, 2018; Zeng et al. 2018). In the models of this chapter, by contrast, price efficiency can be *lower* in the *presence* of information about future noise in the unique equilibrium.

Building on Allen et al. (2006), Gao (2008) investigates the influence of public information about fundamentals in an OLG economy. He shows that more precise public fundamental information unambiguously raises price efficiency, even though the KBC intensifies. Our analysis shows that the same does not hold true for public *non*-fundamental information. Farboodi and Veldkamp (2020) turn their attention to private information about contemporaneous noise in a dynamic model populated by OLG of investors. Similar to Marmora and Rytchkov (2018), traders can process a limited amount of private fundamental and non-fundamental information. Farboodi and Veldkamp (2020) show that non-fundamental information unequivocally raises price efficiency, although it can crowd out fundamental information. Again, this contrasts with our finding that the effect of non-fundamental information can be detrimental to price efficiency.

Second, our results relate to the strand of the theoretical literature concerned with rational destabilization of prices. In a pioneering contribution, De Long et al. (1990b) show that rational traders can drive prices away from fundamentals if noise traders are modeled as trend-chasing positive feedback traders (Arnold and Brunner, 2015, however, show that the scope for destabilizing rational speculation shrinks as the number of trading dates increases). In a setup à la Kyle (1985), which entails risk-neutral investors and strategic behavior, Madrigal (1996) and Yang and Zhu (2017) show that the existence of a non-fundamental speculator can harm price efficiency. Abreu and Brunnermeier (2002, 2003) develop a model in which rational investors temporarily contribute to the growth of a bubble until coordinated selling pressure eventually makes it burst. In a recent paper, using the Kyle (1985) setup, Sadzik and Woolnough (2021) show that a rational trader with information about a persistent component of noise trader demand can act destabilizing by amplifying noise traders' impact on prices. Our outcomes add to this field by demonstrating that rational traders can destabilize prices by trading on non-fundamental information in a dynamic *competitive* economy.

The remainder of this chapter is structured as follows: as a benchmark for the subsequent dynamic analysis, Section 3.1 analyzes the static setup with a public signal about noise. Sections 3.2 and 3.3 turn to the dynamic OLG and LLA models, respectively. Section 3.4 provides a brief comparison of the two dynamic model variants.

3.1 The Static Model

To set the stage for the following dynamic analysis, this section develops a static benchmark with a public signal about current noise trading. In line with the common result in the literature, we show that non-fundamental information is unambiguously conducive to price efficiency in a static setting.

3.1.1 Model Assumptions

Consider a financial market in which a riskless asset and a risky asset are traded. The riskless asset (i.e., a bond) can be traded without any supply restrictions and serves as a numeraire in the market. For the sake of simplicity, its safe return is normalized to zero. The risky asset (i.e., a stock) is in zero net supply and trades at market price P at date 1, which will be endogenized below. At date 2, the risky asset pays off its fundamental value given by $\theta \sim N(0, \tau_\theta^{-1})$. The parameter τ_θ measures the prior precision of θ, which is the inverse of its variance.

There exists a continuum of rational investors indexed by the interval $[0, 1]$ in the financial market. Without loss of generality, agents' initial wealth is normalized to zero. The terminal wealth of agent i is given by $\pi_i = (\theta - P)\, D_i$, where D_i represents agent i's demand for the risky asset. A rational agent derives utility from consuming her final wealth. More specifically, each rational trader is characterized by the utility function $U(\pi_i) = -\exp(-\delta^{-1}\pi_i)$. The parameter δ (> 0) measures agents' identical degree of risk tolerance, the inverse of which corresponds to their degree of risk aversion. Moreover, there exist noise traders in the market, whose demand for the risky asset is given by $s \sim N(0, \tau_s^{-1})$. Since we do not explicitly model a feedback effect from social sentiment to noise trader demand (see Semenova and Winkler, 2021), we assume that the exogenous demand s already involves such possible interactions.

Rational agent i is endowed with a noisy private signal about θ given by $x_i = \theta + \epsilon_i$, where $\epsilon_i \sim$ i.i.d. $N(0, \tau_\epsilon^{-1})$. Since the signal x_i results from the sum of two normally distributed random variables, it is normally distributed too (see Appendix B.1.2). As the economy is assumed to be competitive, all rational traders are price takers and, thus, observe the market price P. What distinguishes the model from standard competitive noisy REE models in the spirit of GS 1980, Hellwig (1980), and Diamond and Verrecchia (1981) is that rational traders additionally glean a public signal related to noise trader demand, which stems from gauging social sentiment:

$$Y = s + \eta, \quad \eta \sim N(0, \tau_\eta^{-1}).$$

Consequently, the normally distributed signal Y opens up the possibility of social sentiment investing, where η stands for the common error term inherent in the signal. The random variables θ, ϵ_i, s, and η are assumed to be jointly normal and pairwise independent for all $i \in [0, 1]$.

3.1.2 Equilibrium Determination

The equilibrium we derive in this subsection is a linear REE. Since the linear REE constitutes the central equilibrium concept in this thesis, we provide an extensive step-by-step derivation below. The price function that prevails in the linear REE is obtained via a conjecture-and-verify approach. That is, we first conjecture a specific function of P and verify later on that P is indeed determined by this function in equilibrium. Assume that all rational agents conjecture the price to be linear in θ, s, and Y:

$$P = a\,\theta + b\,s - c\,Y, \tag{3.1}$$

for constants a, b, and c. The coefficients in (3.1) are assumed to be common knowledge across all rational traders. Since P is determined by sums of and differences between linear transformations of independent normal random variables (recall that Y can be decomposed in s and η), it is also normally distributed (see Appendix B.1

for a formal proof). Let $I_i = (x_i, P, Y)$ further denote the vector representing agent i's information set.

Definition (linear REE): Price function (3.1) and asset demands D_i, $i \in [0,1]$, are a *linear REE* if

(i) D_i maximizes expected utility $\mathrm{E}\left[U(\pi_i)\,|\,I_i\right]$ for all $i \in [0,1]$,

(ii) and the market for the risky asset clears, i.e., $\int_0^1 D_i\, di + s = 0$.

We derive the linear REE by making use of a four-step procedure based on Brunnermeier (2001, Chapter 3).

Step 1: updating an agent's beliefs about θ. A rational agent uses her gathered information to update her prior beliefs about the fundamental asset value θ. A central feature of competitive noisy REE models is that traders also use the market price to glean information about θ. Since the market price aggregates all private and public information that investors dispose of, it serves rational traders as a (noisy) transmitter of aggregate fundamental information. Recall that $I_i = (P, x_i, Y)$. The first two conditional moments of θ are then given by $\mathrm{E}\,(\theta\,|\,P, x_i, Y)$ and $\mathrm{Var}\,(\theta\,|\,P, x_i, Y)$. Conditional on the normal random variables P, x_i, and Y, the fundamental asset value θ is still normally distributed. This conclusion can be drawn from the projection theorem, which determines the distribution of normal random variables conditional on other normal variables. A formal derivation of this theorem can be found in Appendix B.2.1.

Since the signal Y is uncorrelated with fundamentals but correlated with the non-fundamental components in price function (3.1), it can be combined with P to form a single signal about θ. More precisely, the signals P and Y can be united as follows:

$$P^* \equiv \frac{P + cY}{a} - \frac{1}{\rho}\mathrm{E}\,(s\,|\,Y) = \theta + \frac{1}{\rho}\left[s - \mathrm{E}\,(s\,|\,Y)\right], \tag{3.2}$$

where $\rho \equiv a/b$. As agents know Y, they can extract all of its direct influence on P from function (3.1). Since Y is correlated with noise trader demand, rational agents can further use it to extract noise from the price stemming from s. Formally, the decomposition method used in (3.2) follows from the projection theorem (see Appendix B.2.2 for a derivation).

Note that the variance of the noise term in P^* (i.e., $\mathrm{Var}\,(P^*\,|\,\theta)$) is given by

$$\begin{aligned}\mathrm{Var}\,(P^*\,|\,\theta) &= \mathrm{Var}\left\{\frac{1}{\rho}[s - \mathrm{E}\,(s\,|\,Y)]\right\}\\ &= \frac{1}{\rho^2}\mathrm{Var}\left(s - \frac{\tau_\eta}{\tau_s + \tau_\eta}Y\right)\end{aligned}$$

$$= \frac{1}{\rho^2} \mathrm{Var}\left[\left(1 - \frac{\tau_\eta}{\tau_s + \tau_\eta}\right) s - \frac{\tau_\eta}{\tau_s + \tau_\eta}\eta\right]$$

$$= \frac{1}{\rho^2}\left[\frac{\tau_s^2}{(\tau_s + \tau_\eta)^2}\frac{1}{\tau_s} + \frac{\tau_\eta^2}{(\tau_s + \tau_\eta)^2}\frac{1}{\tau_\eta}\right]$$

$$= \frac{1}{\rho^2(\tau_s + \tau_\eta)}, \tag{3.3}$$

where $\mathrm{E}\,(s \mid Y) = \tau_\eta Y/(\tau_s + \tau_\eta)$ also follows from the projection theorem (see Appendix B.2). Equation (3.3) shows that the non-fundamental signal Y reduces the variance of the noise term in P^* compared to the case where the signal is absent (i.e., $\tau_\eta = 0$). A rational agent uses her public non-fundamental information to extract noise inherent in the market price, making the price a more precise signal about θ than in the absence of Y.

As illustrated, each rational trader possesses two signals (i.e., x_i and P^*) to update her prior beliefs about θ. Since the noise terms in x_i and P^* are uncorrelated, the first two conditional moments of θ are

$$\mathrm{E}\,(\theta \mid x_i, P^*) = \frac{\tau_\epsilon\, x_i + \rho^2(\tau_s + \tau_\eta)\, P^*}{\tau_\theta + \tau_\epsilon + \rho^2(\tau_s + \tau_\eta)},$$

$$\mathrm{Var}\,(\theta \mid x_i, P^*) = \frac{1}{\tau_\theta + \tau_\epsilon + \rho^2(\tau_s + \tau_\eta)}.$$

These formulas can also be deduced from the projection theorem. A derivation can be found in Appendix B.2.2. An agent's updated expectation about the unknown fundamental asset value is a weighted sum of all signals that belong to her information set (as the prior mean of θ is normalized to zero, an agent does not put any explicit weight on it). The weights are given by the precision of a particular signal in relation to the sum of the precisions of all signals and the prior precision of the fundamental asset value. As the precision of the private signal x_i or the price signal P^* rises, a trader puts more weight on this signal relative to the other one when forming her updated expectations about θ.

The conditional variance of θ is determined by the inverse of the sum of its prior precision and the precisions of the observed signals. The higher the sum of the three precisions, the lower the residual uncertainty about θ an agent faces.

Step 2: determining the demand for the risky asset. A rational trader chooses her demand for the risky asset D_i by maximizing conditional expected utility. Since an agent's final wealth $\pi_i = (\theta - P)D_i$ results from the difference between linear transformations of normal random variables (taking D_i as given), it is also normally distributed. Conditional on an agent's information set I_i, final wealth is still normally distributed. Thus, conditional expected utility becomes

$$\mathrm{E}\left[-\exp\left(-\frac{\pi_i}{\delta}\right)\big|I_i\right] = -\exp\left\{-\frac{1}{\delta}\left[\mathrm{E}\left(\pi_i \mid I_i\right) - \frac{1}{2\delta}\mathrm{Var}\left(\pi_i \mid I_i\right)\right]\right\}. \tag{3.4}$$

The proof can be found in Appendix B.3.1. The conditional moments of π_i can be further developed as follows:

$$\mathrm{E}\left(\pi_i \mid I_i\right) = \left[\mathrm{E}\left(\theta \mid I_i\right) - P\right] D_i, \tag{3.5}$$

$$\mathrm{Var}\left(\pi_i \mid I_i\right) = \mathrm{Var}\left(\theta \mid I_i\right) D_i^2. \tag{3.6}$$

Plugging (3.5) and (3.6) into (3.4) delivers

$$\mathrm{E}\left[-\exp\left(-\frac{\pi_i}{\delta}\right)\big|I_i\right] = -\exp\left(-\frac{1}{\delta}\left\{\left[\mathrm{E}\left(\theta \mid I_i\right) - P\right] D_i - \frac{1}{2\delta}\mathrm{Var}\left(\theta \mid I_i\right) D_i^2\right\}\right). \tag{3.7}$$

Note that the term in curly brackets in (3.7) stands for the conditional certainty equivalent of an agent's risky final wealth, CE_{I_i} say. It is well known that an agent assigns to the certainty equivalent the same utility level as she expects to achieve through her risky final wealth. Conditional on her information set, this yields $U(CE_{I_i}) = \mathrm{E}\left[U(\pi_i) \mid I_i\right]$. Recalling an agent's exponential utility function, we obtain

$$\begin{aligned} -\exp\left(-\frac{CE_{I_i}}{\delta}\right) &= \mathrm{E}\left[-\exp\left(-\frac{\pi_i}{\delta}\right)\big|I_i\right] \\ &= -\exp\left(-\frac{1}{\delta}\left\{\left[\mathrm{E}\left(\theta \mid I_i\right) - P\right] D_i - \frac{1}{2\delta}\mathrm{Var}\left(\theta \mid I_i\right) D_i^2\right\}\right). \end{aligned}$$

Consequently, as indicated above, the conditional certainty equivalent is

$$CE_{I_i} = \left[\mathrm{E}\left(\theta \mid I_i\right) - P\right] D_i - \frac{1}{2\delta}\mathrm{Var}\left(\theta \mid I_i\right) D_i^2. \tag{3.8}$$

Since

$$\frac{\partial}{\partial D_i}\mathrm{E}\left[-\exp\left(-\frac{\pi_i}{\delta}\right)\big|I_i\right] \;\propto\; \frac{\partial\, CE_{I_i}}{\partial D_i},$$

maximizing an agent's conditional expected utility is equivalent to maximizing the respective conditional certainty equivalent CE_{I_i}. The first-order condition of (3.8) in D_i is

$$\frac{\partial\, CE_{I_i}}{\partial D_i} = \mathrm{E}\left(\theta \mid I_i\right) - P - \frac{1}{\delta}\mathrm{Var}\left(\theta \mid I_i\right) D_i = 0. \tag{3.9}$$

Rearranging terms in (3.9) delivers an agent's optimal demand for the risky asset:

$$D_i = \delta \frac{\mathrm{E}(\theta \mid I_i) - P}{\mathrm{Var}(\theta \mid I_i)}. \tag{3.10}$$

Inspecting (3.9) shows that the condition for a maximum is met (i.e., $\partial^2 CE_{I_i}/\partial D_i^2 < 0$). According to (3.10), an agent takes a long (resp., short) position in the risky asset if her updated expectations about θ exceed (resp., are inferior to) the market price. As agents are assumed to be risk-averse, demand is constrained by the conditional variance of fundamentals. A higher (resp., lower) residual uncertainty about θ leads to lower (resp., higher) demand in absolute terms. Lower (resp., higher) risk tolerance indicated by a smaller (resp., larger) δ exerts the same influence.

Since $\mathrm{E}(\theta \mid I_i)$ is a linear function of x_i and P^* and $\mathrm{Var}(\theta \mid I_i)$ is non-random, an agent's demand is linear in x_i, P^*, and P. The exact demand for the risky asset depends on the concrete realisations of the random variables. By (3.2), P^* is a linear function of θ, s, and Y. Thus, agent i's demand can be written as a linear function of x_i, θ, s, Y, and P:

$$\begin{aligned} D_i &= \delta \frac{\mathrm{E}(\theta \mid I_i) - P}{\mathrm{Var}(\theta \mid I_i)} \\ &= \delta\tau_\epsilon x_i + \delta\rho^2(\tau_s + \tau_\eta) P^* - \delta[\tau_\theta + \tau_\epsilon + \rho^2(\tau_s + \tau_\eta)]P \\ &= \delta\tau_\epsilon x_i + \delta\rho^2(\tau_s + \tau_\eta) \left[\theta + \frac{1}{\rho}\left(s - \frac{\tau_\eta}{\tau_s + \tau_\eta} Y\right)\right] - \delta[\tau_\theta + \tau_\epsilon + \rho^2(\tau_s + \tau_\eta)]P \\ &= \delta\tau_\epsilon x_i + \delta\rho^2(\tau_s + \tau_\eta)\theta + \delta\rho(\tau_s + \tau_\eta)s - \delta\rho\tau_\eta Y - \delta[\tau_\theta + \tau_\epsilon + \rho^2(\tau_s + \tau_\eta)]P. \end{aligned} \tag{3.11}$$

By (3.11), a rational agent trades *against* the public non-fundamental signal Y (i.e., $\partial D_i/\partial Y < 0$). Recall that a rational trader uses Y to extract noise from the market price (see (3.2)). Hence, holding the market price constant, a higher value of Y predicts lower fundamentals by indicating stronger noise trading. This, in return, makes the agent reduce her demand. Inversely, a decline in the value of Y implies weaker noise trading and, assuming an unchanged price, higher fundamentals. Consequently, the agent raises her demand. In other words, a rational trader follows a contrarian strategy with regard to non-fundamental information.

Step 3: imposing market clearing. Marketing clearing implies that the aggregated demand for the risky asset coming from rational agents and noise traders equals the asset's zero net supply. This endogenously determines the market price. Formally,

$$\int_0^1 D_i \, di + s = 0. \tag{3.12}$$

Using (3.11), rational agents' aggregated demand becomes

$$\begin{aligned}\int_0^1 D_i\,di &= \int_0^1 \{\delta\tau_\epsilon\,x_i + \delta\rho^2(\tau_s+\tau_\eta)\theta + \delta\rho(\tau_s+\tau_\eta)s - \delta\rho\tau_\eta Y \\ &\quad - \delta[\tau_\theta+\tau_\epsilon+\rho^2(\tau_s+\tau_\eta)]P\}di \\ &= \delta\tau_\epsilon \int_0^1 x_i\,di + \delta\rho^2(\tau_s+\tau_\eta)\theta + \delta\rho(\tau_s+\tau_\eta)s - \delta\rho\tau_\eta Y \\ &\quad - \delta[\tau_\theta+\tau_\epsilon+\rho^2(\tau_s+\tau_\eta)]P.\end{aligned} \tag{3.13}$$

The integral in (3.13) can be solved as follows:

$$\int_0^1 x_i\,di = \int_0^1 (\theta+\epsilon_i)\,di = \theta + \int_0^1 \epsilon_i\,di. \tag{3.14}$$

By the strong law of large numbers, the value of the integral converges almost surely to the mean of the random variable ϵ_i (see Vives, 2008, Technical Appendix):[2]

$$\int_0^1 \epsilon_i\,di \;\rightarrow\; \mathrm{E}\,(\epsilon_i) = 0. \tag{3.15}$$

By (3.14) and (3.15), (3.13) becomes

$$\int_0^1 D_i\,di = [\delta\tau_\epsilon + \delta\rho^2(\tau_s+\tau_\eta)]\theta + \delta\rho(\tau_s+\tau_\eta)s - \delta\rho\tau_\eta Y - \delta[\tau_\theta+\tau_\epsilon+\rho^2(\tau_s+\tau_\eta)]P. \tag{3.16}$$

Plugging (3.16) into (3.12) and solving for P gives

$$[\delta\tau_\epsilon + \delta\rho^2(\tau_s+\tau_\eta)]\theta + \delta\rho(\tau_s+\tau_\eta)s - \delta\rho\tau_\eta Y - \delta[\tau_\theta+\tau_\epsilon+\rho^2(\tau_s+\tau_\eta)]P + s = 0$$

$$\Leftrightarrow P = \frac{\tau_\epsilon+\rho^2(\tau_s+\tau_\eta)}{\tau_\theta+\tau_\epsilon+\rho^2(\tau_s+\tau_\eta)}\theta + \frac{1+\delta\rho(\tau_s+\tau_\eta)}{\delta[\tau_\theta+\tau_\epsilon+\rho^2(\tau_s+\tau_\eta)]}s - \frac{\rho\tau_\eta}{\tau_\theta+\tau_\epsilon+\rho^2(\tau_s+\tau_\eta)}Y. \tag{3.17}$$

According to (3.17), the market price can indeed be represented by a linear function of θ, s, and Y, as conjectured in (3.1).

Step 4: invoking rational expectations. One of the central characteristics of an REE is that the coefficients in the agents' conjectured price function coincide with those in the market-clearing price function. This makes agents' expectations rational. Comparing the coefficients in (3.1) with those in (3.17) immediately yields

$$a = \frac{\tau_\epsilon+\rho^2(\tau_s+\tau_\eta)}{\tau_\theta+\tau_\epsilon+\rho^2(\tau_s+\tau_\eta)},$$

$$b = \frac{1 + \delta\rho(\tau_s + \tau_\eta)}{\delta[\tau_\theta + \tau_\epsilon + \rho^2(\tau_s + \tau_\eta)]},$$

$$c = \frac{\rho\tau_\eta}{\tau_\theta + \tau_\epsilon + \rho^2(\tau_s + \tau_\eta)}.$$

Recall that $\rho \equiv a/b$. By (3.17), we obtain

$$\rho = \frac{\delta\tau_\epsilon + \delta\rho^2(\tau_s + \tau_\eta)}{1 + \delta\rho(\tau_s + \tau_\eta)}$$

$$\Leftrightarrow \rho[1 + \delta\rho(\tau_s + \tau_\eta)] = \delta\tau_\epsilon + \delta\rho^2(\tau_s + \tau_\eta)$$

$$\Leftrightarrow \rho = \delta\tau_\epsilon.$$

Proposition 3.1. *There exists a unique linear REE, in which*

$$a = \frac{\tau_\epsilon + \rho^2(\tau_s + \tau_\eta)}{\tau_\theta + \tau_\epsilon + \rho^2(\tau_s + \tau_\eta)},$$

$$b = \frac{1 + \delta\rho(\tau_s + \tau_\eta)}{\delta[\tau_\theta + \tau_\epsilon + \rho^2(\tau_s + \tau_\eta)]},$$

$$c = \frac{\rho\,\tau_\eta}{\tau_\theta + \tau_\epsilon + \rho^2(\tau_s + \tau_\eta)},$$

$$\rho = \delta\tau_\epsilon.$$

The unique linear REE provides a simple closed-form solution for the coefficients a, b, and c. Furthermore, from (3.11), we can conclude that

$$\int_0^1 \frac{\partial D_i}{\partial x_i}\, di = \delta\tau_\epsilon = \rho,$$

which implies that ρ indicates how aggressively agents trade on their private fundamental signals (i.e., ρ measures rational traders' aggregate trading intensity on private fundamental information). This result can also be obtained in a different way. By (3.16), agents' aggregate demand is a linear function of θ, s, Y, and P. Thus, in general form,

$$\bar{D} \equiv \int_0^1 D_i\, di = \frac{\partial \bar{D}}{\partial \theta}\,\theta + \frac{\partial \bar{D}}{\partial s}\,s - \left|\frac{\partial \bar{D}}{\partial Y}\right|\,Y - \left|\frac{\partial \bar{D}}{\partial P}\right|\,P.$$

Imposing market clearing and solving for P delivers

$$\frac{\partial \bar{D}}{\partial \theta}\,\theta + \frac{\partial \bar{D}}{\partial s}\,s - \left|\frac{\partial \bar{D}}{\partial Y}\right|\,Y - \left|\frac{\partial \bar{D}}{\partial P}\right|\,P + s = 0$$

$$\Leftrightarrow P = \underbrace{\frac{\partial \bar{D}/\partial \theta}{|\partial \bar{D}/\partial P|}}_{a} \theta + \underbrace{\frac{1 + \partial \bar{D}/\partial s}{|\partial \bar{D}/\partial P|}}_{b} s - \underbrace{\frac{|\partial \bar{D}/\partial Y|}{|\partial \bar{D}/\partial P|}}_{c} Y. \tag{3.18}$$

Hence,

$$\rho \equiv \frac{a}{b} = \frac{\partial \bar{D}/\partial \theta}{1 + \partial \bar{D}/\partial s}.$$

By (3.11) and (3.16), we can express the two trading intensities $\partial \bar{D}/\partial \theta$ and $\partial \bar{D}/\partial s$ as

$$\frac{\partial \bar{D}}{\partial \theta} = \int_0^1 \frac{\partial D_i}{\partial x_i}\, di + \int_0^1 \frac{\partial D_i}{\partial \theta}\, di = \int_0^1 \frac{\partial D_i}{\partial x_i}\, di + \delta \rho^2 (\tau_s + \tau_\eta),$$

$$\frac{\partial \bar{D}}{\partial s} = \int_0^1 \frac{\partial D_i}{\partial s}\, di = \delta \rho (\tau_s + \tau_\eta).$$

This gives

$$\rho = \frac{\int_0^1 (\partial D_i/\partial x_i)\, di + \delta \rho^2 (\tau_s + \tau_\eta)}{1 + \delta \rho (\tau_s + \tau_\eta)},$$

which, after solving for ρ, also shows that $\rho = \int_0^1 (\partial D_i/\partial x_i)\, di = \delta \tau_\epsilon$.

3.1.3 Price Efficiency

Having derived the linear REE, we turn to our main object of interest: price efficiency. Price efficiency indicates how accurately the market price reflects the asset's underlying fundamental value. Following Hayek's (1945) early argument that prices aggregate the dispersed private information of market participants, high price efficiency is often considered to be desirable. In this view, an accurate market price provides agents inside and also outside of financial markets with valuable information that they can use to make more informed investment or policy decisions (see, e.g., Fama and Miller, 1972, Chapter 8). Consequently, higher price efficiency in financial markets is assumed to translate into higher real efficiency in the economy. Formally, we denote

$$\text{price efficiency} \equiv \frac{1}{\operatorname{Var}(\theta \mid P)},$$

which, according to Goldstein and Yang (2017, Subsubsection 2.3.2), represents the common definition of price efficiency in the literature. A rise in $\operatorname{Var}^{-1}(\theta \mid P)$ means that the market price becomes a more precise signal about the fundamental asset value, corresponding with higher price efficiency. By that, high price efficiency is also closely related to the prices-are-right formulation of the EMH (see Section 2.1).

Remarkably, a recent strand of the theoretical literature shows that the nexus between price efficiency and real efficiency is not as close as originally supposed (for a survey of the literature see Bond et al., 2012, and Goldstein and Yang, 2017, Section 4). The relevant contributions explicitly model real decision-makers, such as firm managers, capital providers, and governments, that rely on information conveyed by asset prices. By endogenizing the real decision-makers' investment or policy decisions, these models show that higher price efficiency does not per se translate into higher real efficiency. Nevertheless, since our model focuses on a financial market only, we consider price efficiency to be the relevant efficiency measure (as, e.g., in Gao, 2008).

Additionally, one can motivate our analysis of price efficiency by adding a set of measure zero of rational investors to the model with no signal except the asset price (as in reality it is unlikely that each trader possesses valuable information about the sentiment contained in social media platforms). Then, higher price efficiency allows these traders to make more informed investment decisions. By continuity, our results on price efficiency carry over to the model variant that encompasses a positive but sufficiently small mass of such investors.

Note that observing P is informationally equivalent to observing

$$P^{**} \equiv \frac{P}{a} = \theta + \frac{1}{\rho}s - \frac{c}{a}Y. \tag{3.19}$$

The variance of the noise term in (3.19) is given by

$$\begin{aligned}
\operatorname{Var}(P^{**} \mid \theta) &= \operatorname{Var}\left(\frac{1}{\rho}s - \frac{c}{a}Y\right) \\
&= \operatorname{Var}\left[\left(\frac{1}{\rho} - \frac{c}{a}\right)s\right] + \operatorname{Var}\left(-\frac{c}{a}\,\eta\right) \\
&= \left(\frac{1}{\rho} - \frac{c}{a}\right)^2 \frac{1}{\tau_s} + \left(\frac{c}{a}\right)^2 \frac{1}{\tau_\eta}.
\end{aligned}$$

Using the bivariate case of the projection theorem, we eventually obtain our measure of price efficiency:

$$\begin{aligned}
\operatorname{Var}(\theta \mid P^{**}) &= \frac{1}{\tau_\theta + \operatorname{Var}^{-1}(P^{**} \mid \theta)} \\
\Leftrightarrow \operatorname{Var}^{-1}(\theta \mid P^{**}) &= \tau_\theta + \left[\left(\frac{1}{\rho} - \frac{c}{a}\right)^2 \frac{1}{\tau_s} + \left(\frac{c}{a}\right)^2 \frac{1}{\tau_\eta}\right]^{-1}. \qquad (3.20)
\end{aligned}$$

The first term in square brackets in (3.20) measures the impact of noise trader demand on price efficiency. We call it the "CON (COntemporaneous Noise trading)" effect in what follows. The impact of demand volatility (i.e., $1/\tau_s$) is attenuated when rational agents trade more aggressively on private fundamental information (i.e., ρ rises) or when the ratio c/a increases. According to (3.18), c/a stands for rational agents' trading intensity against the public non-fundamental signal Y relative to their trading intensity on fundamentals θ. The higher c/a, the more aggressively rational traders trade against the non-fundamental signal (relative to trading on fundamentals), mitigating the impact of noise trader demand.

The second term in square brackets in (3.20) captures the impact of the common error component η, which is inherent in agents' public non-fundamental information, on price efficiency. We label this term as the "COMESCON (COMmon Error in the Signal about COntemporaneous Noise trading)" effect. More aggressive trading against public non-fundamental information plays a double-edged role with regard to price efficiency. For one thing, by the CON effect, the impact of noise trader demand is alleviated, which boosts price efficiency. For another, by the COMESCON effect, the impact of the common error term in the signal Y is amplified. This harms price efficiency. Thus, at first glance, the ensuing influence of Y on price efficiency seems ambiguous.

Recalling the results contained in Proposition 3.1, further computations yield

$$
\begin{aligned}
\mathrm{Var}^{-1}(\theta \mid P^{**}) &= \tau_\theta + \left\{ \left[\frac{1}{\rho} - \frac{\delta\tau_\eta}{1+\delta\rho(\tau_s+\tau_\eta)} \right]^2 \frac{1}{\tau_s} + \left[\frac{\delta\tau_\eta}{1+\delta\rho(\tau_s+\tau_\eta)} \right]^2 \frac{1}{\tau_\eta} \right\}^{-1} \\
&= \tau_\theta + \left\{ \frac{(1+\delta\rho\tau_s)^2}{\rho^2[1+\delta\rho(\tau_s+\tau_\eta)]^2\tau_s} + \frac{\delta^2\tau_\eta}{[1+\delta\rho(\tau_s+\tau_\eta)]^2} \right\}^{-1} \\
&= \tau_\theta + \left\{ \frac{1+\delta^2\rho^2\tau_s^2 + 2\delta\rho\tau_s + \delta^2\rho^2\tau_\eta\tau_s}{\rho^2[1+\delta\rho(\tau_s+\tau_\eta)]^2\tau_s} \right\}^{-1} \\
&= \tau_\theta + \frac{\mathrm{A}^2\rho^2\tau_s}{1+\delta\rho\tau_s(\mathrm{A}+1)},
\end{aligned}
\tag{3.21}
$$

where $\mathrm{A} \equiv 1 + \delta\rho(\tau_s + \tau_\eta)$. Comparative-statics analysis of (3.21) with respect to τ_η gives

$$
\begin{aligned}
\frac{\partial \left[\mathrm{Var}^{-1}(\theta \mid P^{**})\right]}{\partial \tau_\eta} &= \frac{2A\delta\rho^3\tau_s[1+\delta\rho\tau_s(\mathrm{A}+1)] - A^2\delta^2\rho\tau_s^2}{[1+\delta\rho\tau_s(\mathrm{A}+1)]^2} \\
&= \frac{\mathrm{A}\delta\rho^3\tau_s[2+\delta\rho\tau_s(\mathrm{A}+2)]}{[1+\delta\rho\tau_s(\mathrm{A}+1)]^2} > 0.
\end{aligned}
\tag{3.22}
$$

Equation (3.22) unequivocally demonstrates that public non-fundamental information is conducive to price efficiency in a static setup. Thus, even though the common error term η injects an additional source of noise into the price, the public signal Y raises price efficiency by counteracting noise trader demand. This result adds to the findings of Ganguli and Yang (2009), Manzano and Vives (2011), and Zeng et al. (2018).

3.2 Dynamic Setup - Overlapping Generations

The following sections leave the common static setup and incorporate public information about contemporaneous and *future* noise trader demand into the canonical dynamic REE model in the spirit of Singleton (1986) and Brown and Jennings (1989). In this section, we analyze the model variant with OLG of investors, before we turn to the version with LLA in Section 3.3. The dynamic models show that the precision of a non-fundamental signal loses the unequivocally positive role for price efficiency it plays in the static version. In particular, we show that more information about future noise trader demand can move the current price away from fundamentals, implying a potentially negative effect of social sentiment investing on price efficiency.

In the OLG setup, the impact of public information about future noise trader demand is two-edged. For one thing, the public information injects an additional source of noise into the current price, which reduces its efficiency. For another, public information about date-2 noise trading drives the date-2 price toward fundamentals, allowing date-1 agents to trade more aggressively on private fundamental information. This boosts date-1 price efficiency. Due to the latter effect, public information about future noise trader demand is less likely to harm price efficiency in the OLG setup than in the LLA model.

In the absence of information about current noise trader demand, price efficiency in the OLG variant is higher with a perfectly precise signal about future noise trader demand than with no signal at all. Nevertheless, public information about future noise trader demand can reduce price efficiency in this case if it is sufficiently imprecise. In the presence of information about contemporaneous noise trader demand, price efficiency can be lower with a perfectly precise signal about future noise trader demand than with no signal. More strikingly, the relationship between signal precision and price efficiency can be even *monotonically decreasing*. Thus, on the basis of the results of the OLG model, public information about future noise trader demand can indeed *harm* price efficiency. This in turn indicates a potentially negative effect of social sentiment investing on price efficiency.

3.2.1 Model Assumptions

The financial market consists of a riskless asset and a risky asset that show almost the same characteristics as in the static version (see Subsection 3.1.1). The only difference is that the risky asset now pays off its random fundamental value θ at date 3. This turns date 2 into an additional trading date. The risky asset is traded at price P_1 at date 1 and at price P_2 at date 2. At each of the two trading dates, a continuum of rational traders indexed by the interval $[0,1]$ is born. Each generation is assumed to live for one period only. The first generation enters the market at date 1 and unwinds its position at date 2, thereby exhibiting a short trading horizon. The second generation enters the market at date 2 and lives till date 3.

The final wealth of agent i belonging to the first generation is $\pi_{1i} = (P_2 - P_1)D_{1i}$. D_{1i} stands for agent i's demand for the risky asset at date 1. Analogously, the final wealth of a second-generation rational trader is given by $\pi_{2i} = (\theta - P_2)D_{2i}$, where D_{2i} represents agent i's demand for the risky asset at date 2. Each rational trader is characterized by the utility function $U(\pi_{ti}) = -\exp(-\delta^{-1}\pi_{ti})$, for $t = 1, 2$. The parameter δ (> 0) measures agents' identical degree of risk tolerance.

Noise trader demand is exogenous and given by $s_1 \sim N(0, \tau_{s_1}^{-1})$ at date 1 and by $s_2 \sim N(0, \tau_{s_2}^{-1})$ at date 2. As, e.g., in Allen et al. (2006), Gao (2008), Farboodi and Veldkamp (2020), and Farboodi et al. (2021), we model noise trading as transient. That is, s_1 and s_2 are assumed to be independent of each other.[3]

Each rational investor is endowed with a noisy private fundamental signal $x_{ti} = \theta + \epsilon_{ti}$, where $\epsilon_{ti} \sim$ i.i.d. $N(0, \tau_{\epsilon}^{-1})$, for $t = 1, 2$. Rational agents are able to observe current and past prices. The first generation observes P_1, the second generation knows P_1 and P_2. Additionally, all rational traders glean public signals related to date-1 and date-2 noise trader demand:

$$Y_t = s_t + \eta_t, \quad t = 1, 2,$$

with $\eta_t \sim N(0, \tau_{\eta_t}^{-1})$. By gauging social sentiment at date 1, rational traders gain valuable information about current and also future noise trading. Consequently, social sentiment allows rational traders to form an investment strategy based on information about how noise traders will act in the near future. This assumption is motivated by and consistent with the strong predictive power of social sentiment for future prices and stock returns (see the relevant literature cited in Section 2.1). Thus, agent i's information set at date 1 is $I_{1i} = (x_{1i}, P_1, Y_1, Y_2)$. At date 2, we have $I_{2i} = (x_{2i}, P_1, P_2, Y_1, Y_2)$. The random variables θ, ϵ_{ti}, s_t, and η_t are assumed to be jointly normal and pairwise independent for $t = 1, 2$ and for all $i \in [0, 1]$.

3.2.2 Equilibrium Determination

Analogous to the static version, assume that rational agents conjecture the following linear price functions:

$$P_1 = a_1\theta + b_1 s_1 - c_{11}Y_1 + c_{12}Y_2, \tag{3.23}$$

$$P_2 = a_2\theta + b_2 s_2 - c_{21}Y_1 - c_{22}Y_2 + d_2 P_1, \tag{3.24}$$

for constants a_t, b_t, c_{t1}, c_{t2} ($t = 1, 2$), and d_2. Since P_1 and P_2 are determined by sums of and differences between linear transformations of normal random variables, they are (jointly) normally distributed.

Definition (linear dynamic REE with OLG of investors): Price functions (3.23) and (3.24) and asset demands D_{ti} ($t = 1, 2$, $i \in [0, 1]$) are a *linear dynamic REE with OLG of investors* if

(i) D_{ti} maximizes date-t expected utility $\mathrm{E}[U(\pi_{ti}) \mid I_{ti}]$ ($t = 1, 2$) for all $i \in [0, 1]$,

(ii) and the market for the risky asset clears at both trading dates, i.e., $\int_0^1 D_{ti}\, di + s_t = 0$ ($t = 1, 2$).

The utility-maximizing demand functions in the OLG economy are

$$D_{2i} = \delta\, \frac{\mathrm{E}(\theta \mid I_{2i}) - P_2}{\mathrm{Var}(\theta \mid I_{2i})}, \tag{3.25}$$

$$D_{1i} = \delta\, \frac{\mathrm{E}(P_2 \mid I_{1i}) - P_1}{\mathrm{Var}(P_2 \mid I_{1i})}. \tag{3.26}$$

Since a date-2 agent's utility and wealth functions follow the same form as in the static model (cf. Subsections 3.1.1 and 3.2.1), maximizing date-2 expected utility in the OLG setup works analogously to maximizing expected utility in the static benchmark. This immediately yields date-2 demand function (3.25). At date 1, investors are concerned with predicting the date-2 price rather than fundamentals, as they will unwind their position at the beginning of date 2. Given that P_2 and P_1 are jointly normally distributed, a date-1 investor's final wealth $\pi_{1i} = (P_2 - P_1)D_{1i}$ follows a normal distribution too. Thus, maximizing date-1 expected utility again works analogously to maximizing expected utility in the static model, which gives date-1 demand function (3.26).

According to (3.25), a rational trader goes long (resp., short) in the risky asset at date 2 if her expectations about the fundamental asset value exceed (resp., are inferior to) the date-2 price. Her demand is constrained by the residual uncertainty about fundamentals, represented by the conditional variance of θ. Analogously, a date-1 investor takes a long (resp., short) position whenever her expectations about the date-2 price exceed (resp., are inferior to) the date-1 price. The residual uncertainty about P_2 limits her demand.

The general form of asset prices. Before deriving the specific price functions in the linear dynamic REE, we propose a general determination of P_1 and P_2, which is similar to that of Cespa and Vives (2015), to clearly identify what factors influence asset prices under short-term trading. Using the date-2 market-clearing condition, P_2 can be expressed in general form as

$$\int_0^1 D_{2i}\,di + s_2 = 0$$

$$\Leftrightarrow \int_0^1 \frac{\delta[\mathrm{E}(\theta \,|\, I_{2i}) - P_2]}{\mathrm{Var}(\theta \,|\, I_{2i})}\,di + s_2 = 0$$

$$\Leftrightarrow P_2 = \bar{\mathrm{E}}_2(\theta) + \frac{\mathrm{Var}(\theta \,|\, I_{2i})}{\delta}\,s_2, \tag{3.27}$$

with $\bar{\mathrm{E}}_2(\theta) \equiv \int_0^1 \mathrm{E}(\theta \,|\, I_{2i})\,di$. According to (3.27), the date-2 price is a function of date-2 investors' average expectations about the fundamental asset value and date-2 noise trader demand, whose influence is adjusted by rational agents' residual uncertainty about fundamentals. Since agents are assumed to be risk-averse, the residual uncertainty about θ prevents them from fully absorbing noise trader demand. As long as the fundamental asset value entails risk, noise traders influence the date-2 price in equilibrium. By imposing market clearing at date 1 and recalling (3.27), P_1 can be written as

$$\int_0^1 D_{1i}\,di + s_1 = 0$$

$$\Leftrightarrow \int_0^1 \frac{\delta[\mathrm{E}(P_2 \,|\, I_{1i}) - P_1]}{\mathrm{Var}(P_2 \,|\, I_{1i})}\,di + s_1 = 0$$

$$\Leftrightarrow P_1 = \bar{\mathrm{E}}_1(P_2) + \frac{\mathrm{Var}(P_2 \,|\, I_{1i})}{\delta}\,s_1$$

$$\Leftrightarrow P_1 = \bar{\mathrm{E}}_1\left[\bar{\mathrm{E}}_2(\theta) + \frac{\mathrm{Var}(\theta \,|\, I_{2i})}{\delta}\,s_2\right] + \frac{\mathrm{Var}\,(P_2 \,|\, I_{1i})}{\delta}\,s_1$$

$$\Leftrightarrow P_1 = \underbrace{\bar{\mathrm{E}}_1\left[\bar{\mathrm{E}}_2(\theta)\right]}_{\substack{Keynesian\,beauty\\ contest}} + \frac{\mathrm{Var}(\theta \,|\, I_{2i})}{\delta}\underbrace{\bar{\mathrm{E}}_1\,(s_2)}_{\substack{forecasting\\ noise}} + \frac{\mathrm{Var}(P_2 \,|\, I_{1i})}{\delta}\,s_1, \tag{3.28}$$

with $\bar{\mathrm{E}}_1(P_2) \equiv \int_0^1 \mathrm{E}(P_2 \,|\, I_{1i})\,di$. By (3.28), the date-1 price is influenced by date-1 investors' average expectations about the date-2 price and date-1 noise trader demand. As P_2 is a function of date-2 investors' average expectations about the fundamental asset value, forecasting the date-2 price entails forecasting date-2 investors' average expectations about fundamentals. This fact describes the KBC in financial markets, as date-1 agents need to form higher-order expectations about fundamentals when predicting P_2. Allen et al. (2006) show that the law of iterated expectations does not hold for average expectations and that prices characterized by a KBC overweight public information (and underweight average private fundamental information) relative to investors' average expectations about fundamentals. Since date-1 agents know that date-2 agents will observe the same public information as they do, public information is extraordinarily helpful for predicting date-2 traders' average expectations about fundamentals. As a consequence, date-1 traders' average expectations about date-2 traders' average expectations about fundamentals put excessive weight on public information compared to date-1 traders' average expectations about fundamentals. This makes the date-1 price put excessive weight on public information too, further implying that the date-1 price is systematically farther away from fundamentals than date-1 investors' average expectations about fundamentals.

However, according to (3.28), the date-1 price is also influenced by date-1 investors' average expectations about date-2 noise trader demand, as long as date-2 rational traders face uncertainty about the asset's fundamentals (i.e., $\mathrm{Var}(\theta \mid I_{2i}) \neq 0$). In Allen et al. (2006), noise trading is transient with mean zero (as in our model) and date-1 rational traders do not glean any signal related to s_2. Thus, in the setup of Allen et al. (2006), average expectations about date-2 noise trader demand do not influence P_1.

Cespa and Vives (2015) show that Allen et al.'s (2006) seminal result does not have to hold true if noise trading is correlated across periods. In this case, the date-1 price is also determined by investors' average expectations about future noise trading. In Cespa and Vives (2015), date-1 rational traders infer information about contemporaneous noise trading from the date-1 price that can be used to predict future noise trading. This additional usage of information contained in the date-1 price can reverse the outcome of Allen et al. (2006), and the date-1 price in equilibrium may *underweight* public information, which moves it systematically closer to fundamentals than date-1 investors' average expectations about fundamentals.

In contrast to Allen et al. (2006) and Cespa and Vives (2015), we focus on how well the date-1 price reflects fundamentals *as a whole*, and not in comparison to investors' average expectations about fundamentals. This means that the effect of any noise on the price needs to be explicitly taken into account. In our setup with transient noise trading, as we will see below, date-1 rational traders' average expectations about s_2 are influenced by public information about future noise trading only (and not by information contained in the date-1 price). As opposed to Cespa and Vives (2015), investors' average expectations about s_2 add a component to the date-1 price that represents pure noise. This is the main driver of the results of our model, which will be presented in Subsection 3.2.3.

The specific form of asset prices. Having outlined the general form of asset prices, we now derive the specific expressions of the coefficients in price functions (3.23) and (3.24). Recall that $I_{1i} = (x_{1i}, P_1, Y_1, Y_2)$ and $I_{2i} = (x_{2i}, P_1, P_2, Y_1, Y_2)$. Thus, all rational agents of both generations can disentangle the information contained in the date-1 price, given by (3.23), as follows:

$$\begin{aligned} P_1^* \equiv \frac{P_1 + c_{11}Y_1 - c_{12}Y_2}{a_1} - \frac{1}{\rho_1}\mathrm{E}(s_1 \,|\, Y_1) &= \theta + \frac{1}{\rho_1}\left[s_1 - \mathrm{E}(s_1 \,|\, Y_1)\right] \\ &= \theta + \frac{1}{\rho_1}\left(s_1 - \frac{\tau_{\eta_1}}{\tau_{s_1} + \tau_{\eta_1}}Y_1\right), \end{aligned}$$

with $\rho_1 \equiv a_1/b_1$. The way the signal Y_1 is used to extract noise inherent in P_1 follows the method that has already been applied in equation (3.2) of the static setup (see Appendix B.2.2 for a formal derivation). Note that rational traders cannot make use of Y_2 to extract noise from the market price induced by s_1, because Y_2 and s_1 are uncorrelated. Thus, P_1^* is a signal about θ with precision $\rho_1^2(\tau_{s_1} + \tau_{\eta_1})$ (i.e., $\mathrm{Var}^{-1}(P_1^* \,|\, \theta) = \rho_1^2(\tau_{s_1} + \tau_{\eta_1})$). Since the computations to obtain $\mathrm{Var}^{-1}(P_1^* \,|\, \theta)$ are carried out analogously to those of the static version (see Subsection 3.1.2), they are omitted at this point. Moreover, we see that P_1 does not provide any additional information about future noise trader demand s_2 that goes beyond the information conveyed by the public signal Y_2. This justifies why agents do not rely on information contained in the date-1 price when predicting future noise trading.

Using all available information, the date-2 market price, given in (3.24), turns into the following signal about fundamentals for the date-2 investors:

$$\begin{aligned} P_2^* \equiv \frac{P_2 + c_{21}Y_1 + c_{22}Y_2 - d_2 P_1}{a_2} - \frac{1}{\rho_2}\mathrm{E}(s_2 \,|\, Y_2) &= \theta + \frac{1}{\rho_2}\left[s_2 - \mathrm{E}(s_2 \,|\, Y_2)\right] \\ &= \theta + \frac{1}{\rho_2}\left(s_2 - \frac{\tau_{\eta_2}}{\tau_{s_2} + \tau_{\eta_2}}Y_2\right), \end{aligned}$$

with $\rho_2 \equiv a_2/b_2$. At date 2, the non-fundamental signal Y_2 is indeed correlated with the noise trader shock that affects the market price. Hence, rational agents use Y_2 to extract noise from the date-2 price injected by s_2. Since date-2 agents are concerned with predicting θ only, they exclusively use Y_2 to counteract noise coming from s_2. Thus, P_2^* is a signal about θ with precision $\rho_2^2(\tau_{s_2} + \tau_{\eta_2})$.

Using P_2^*, P_1^*, and x_{2i}, we can determine agent i's updated beliefs about fundamentals at date 2, which are plugged into demand function (3.25). Then, we impose market clearing at date 2 and solve for the equilibrium function of P_2, whose coefficients are matched with those in (3.24). By updating date-1 beliefs about fundamentals using P_1^* and x_{1i} and about future noise trading using Y_2, we obtain agent i's conditional beliefs about the date-2 price function (3.24), which are substituted into demand function (3.26). Imposing market clearing at date 1 and solving for P_1 delivers the date-1 price function in equilibrium, whose coefficients are matched with those in (3.23). This yields:

Proposition 3.2. *There exists a unique linear dynamic REE with OLG of investors, in which*

$$a_1 = \frac{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})(\Delta + \tau_\epsilon) + \tau_\epsilon\left[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})\right]}{\Delta\left[\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\right]},$$

$$b_1 = \frac{a_1}{\rho_1},$$

$$c_{11} = \frac{(\Delta + \tau_\epsilon)\rho_1\tau_{\eta_1}}{\Delta[\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]},$$

$$c_{12} = \frac{\tau_{\eta_2}}{\delta\Delta(\tau_{s_2} + \tau_{\eta_2})},$$

$$a_2 = \frac{\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})}{\Delta},$$

$$b_2 = \frac{1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})}{\delta\Delta},$$

$$c_{21} = \frac{-\rho_1^2(\tau_{s_1} + \tau_{\eta_1})\frac{c_{11}}{a_1} + \rho_1\tau_{\eta_1}}{\Delta},$$

$$c_{22} = \frac{\rho_2\tau_{\eta_2} + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\frac{c_{12}}{a_1}}{\Delta},$$

$$d_2 = \frac{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})}{a_1\Delta},$$

$$\Delta \equiv \tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1}) + \rho_2^2(\tau_{s_2} + \tau_{\eta_2}),$$

$$\rho_1 \equiv \frac{a_1}{b_1} = \frac{\delta^3\tau_\epsilon^2(\tau_{s_2} + \tau_{\eta_2})}{1 + \delta^2\tau_\epsilon(\tau_{s_2} + \tau_{\eta_2})},$$

$$\rho_2 \equiv \frac{a_2}{b_2} = \delta\tau_\epsilon.$$

The proof can be found in Appendix A. Analogous to the static model of Section 3.1, the linear dynamic REE with OLG of investors is unique and given in closed form. We see that P_1 reacts positively to changes in the non-fundamental signal Y_2 (i.e., $c_{12} > 0$). This is due to the fact that date-1 agents raise their demand if they observe a higher Y_2. A higher Y_2 indicates higher noise trader demand at date 2 and, thus, a higher price at date 2. Hence, rational traders front-run the higher date-2 noise trader demand by purchasing more shares of the risky asset at date 1.

In a static context, agents are exclusively concerned with predicting the risky fundamental asset value. As a consequence, any non-fundamental information is used to extract noise inherent in the market price, and rational traders follow a contrarian strategy with respect to non-fundamental information (see (3.11)). A higher non-fundamental signal ceteris paribus indicates more noise in the price and, thus, a lower fundamental asset value. This leads agents to demand less shares of the risky asset. In a static setup, it is only rational to trade *against* noise. In a dynamic setup, by contrast, agents also have to predict the next trading date's noise trader demand. For them, it can be completely rational to trade *on* noise.

As date-2 agents are concerned with predicting the fundamental asset value only, they clearly trade against noise coming from s_2. This leads to a negative impact of Y_2 on P_2 (i.e., $c_{22} > 0$, which, by (3.24), implies a negative correlation between Y_2 and P_2). The non-fundamental signal Y_2 enables date-2 rational traders to extract noise inherent in P_2 and get a more precise signal about θ. Thus, date-1 and date-2 rational traders use the public signal Y_2 in diametrically opposite ways.

Additionally, both generations of rational investors trade against the public signal Y_1. Similar to the static context, all rational agents use Y_1 to extract noise from P_1 and predict fundamentals more accurately. It is rational for both generations to trade *against* Y_1. This leads to a negative effect of Y_1 on P_1 and P_2.

Furthermore, from $\mathrm{E}(\theta \mid I_{1i})$, given in Appendix A, and P_1^*, we can compute date-1 rational investors' average expectations about fundamentals:

$$\bar{\mathrm{E}}_1(\theta) = \frac{\tau_\epsilon \int_0^1 x_{1i}\, di + \rho_1^2(\tau_{s_1} + \tau_{\eta_1}) \left[\theta + \frac{1}{\rho_1}\left(s_1 - \frac{\tau_{\eta_1}}{\tau_{s_1} + \tau_{\eta_1}} Y_1\right)\right]}{\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})}$$

$$= \frac{[\tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]\theta + \rho_1(\tau_{s_1} + \tau_{\eta_1})s_1 - \rho_1 \tau_{\eta_1} Y_1}{\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})},$$

where $\int_0^1 x_{1i}\, di = \theta$ follows from the strong law of large numbers. Recall that noise trading is transient and date-1 rational agents' average expectations about s_2 are a function of the public signal about future noise trading only (see also the proof of Proposition 3.2 in Appendix A). Thus, in line with Allen et al. (2006), the date-1 price puts excessive weight on public information Y_1 compared to investors' average expectations about fundamentals:

$$c_{11} = \frac{(\Delta + \tau_\epsilon)\rho_1 \tau_{\eta_1}}{\Delta[\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]} > \text{weight to } Y_1 \text{ in } \bar{\mathrm{E}}_1(\theta) = \frac{\rho_1 \tau_{\eta_1}}{\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})}.$$

As opposed to Y_1, date-1 and date-2 agents use the public signal Y_2 in different ways. At date 2, Y_2 helps investors to extract noise from the date-2 price and forecast fundamentals more precisely. At date 1, by contrast, traders use Y_2 to predict future noise, and not fundamentals. As a consequence, date-1 investors' average expectations about fundamentals are independent of the public signal Y_2. This leads to the conclusion that Allen et al.'s (2006) result cannot be applied to public information about *future* noise trading in our setup.

Furthermore, the coefficients of the price functions in the linear REE are influenced by ρ_1 and ρ_2, which measure, analogous to the static model, rational agents' trading intensity on private fundamental information at dates 1 and 2. It can be clearly seen that $\rho_1 < \rho_2$, i.e., in an OLG economy with transient noise trading, date-1 agents underweight private fundamental information relative to date-2 agents (see also Allen et al., 2006). Along the proof of Proposition 3.2 in Appendix A, we show that an agent's demand function at dates 1 and 2 can be expressed as

$$D_{2i} = \delta\tau_\epsilon\, x_{2i} + \delta\rho_1^2(\tau_{s_1} + \tau_{\eta_1})\, P_1^* + \delta\rho_2^2(\tau_{s_2} + \tau_{\eta_2})P_2^* - \frac{\delta}{\mathrm{Var}(\theta \mid I_{2i})}P_2, \tag{3.29}$$

$$\begin{aligned} D_{1i} = {} & \frac{a_2 \mathrm{Var}(\theta \mid I_{1i})}{\mathrm{Var}(P_2 \mid I_{1i})}\left[\delta\tau_\epsilon\, x_{1i} + \delta\rho_1^2(\tau_{s_1} + \tau_{\eta_1})P_1^*\right] \\ & + \frac{\delta}{\mathrm{Var}(P_2 \mid I_{1i})}\left[b_2\, \mathrm{E}(s_2 \mid Y_2) - c_{21}Y_1 - c_{22}Y_2 + d_2 P_1\right] - \frac{\delta}{\mathrm{Var}(P_2 \mid I_{1i})}P_1. \end{aligned} \tag{3.30}$$

At date 2, an agent possesses three signals to predict fundamentals (i.e., x_{2i}, P_1^*, P_2^*). The more precise a signal, the more aggressively the agent trades on it (see (3.29)). The first summand of a date-1 agent's demand function in (3.30) comprises the two signals used to predict fundamentals at date 1 (i.e., x_{1i} and P_1^*). Recalling the expression of ρ_1 from Proposition 3.2 and that $\rho_1 = \int_0^1 (\partial D_{1i}/\partial x_{1i})\, di$, we can conclude that

$$\frac{a_2 \mathrm{Var}(\theta \mid I_{1i})}{\mathrm{Var}(P_2 \mid I_{1i})} = \frac{\delta^2\tau_\epsilon(\tau_{s_2} + \tau_{\eta_2})}{1 + \delta^2\tau_\epsilon(\tau_{s_2} + \tau_{\eta_2})} < 1.$$

Hence, due to her short trading horizon, a date-1 agent trades less aggressively than a date-2 agent not only on private fundamental information, but also on the price signal P_1^* (which contains Y_1). The second summand in (3.30) represents an agent's prediction regarding the other components that shape the date-2 price apart from the fundamental asset value. The first term in square brackets stands for a date-1 agent's incentive to forecast future noise trader demand and front-run it. The other three terms are linked to forecasting date-2 agents' average expectations about fundamentals, as date-2 agents use the signals Y_1, Y_2, and P_1 to predict fundamentals.

Furthermore, note that

$$\frac{\partial \rho_1}{\partial \tau_{\eta_2}} = \frac{\delta^3 \tau_\epsilon^2}{[1 + \delta^2 \tau_\epsilon (\tau_{s_2} + \tau_{\eta_2})]^2} > 0.$$

Thus, a more precise public signal about date-2 noise trader demand s_2 makes date-1 agents trade more aggressively on private fundamental information. The reason for this is the following: as Y_2 becomes more precise, date-2 rational agents trade more aggressively *against* the signal and offset more of the date-2 noise trader demand. This drives the date-2 price toward fundamentals. Put differently, as Y_2 gains in precision, forecasting P_2 comes closer to forecasting θ only. This alleviates the impact of date-1 rational agents' short trading horizon and makes them trade more aggressively on private fundamental information.

Table 3.1: Coefficients of price functions as $\tau_{\eta_2} \to \infty$

ρ_1	$\delta \tau_\epsilon$
ρ_2	$\delta \tau_\epsilon$
a_1	$\frac{\tau_\epsilon + \delta^2 \tau_\epsilon^2 (\tau_{s_1} + \tau_{\eta_1})}{\tau_\theta + \tau_\epsilon + \delta^2 \tau_\epsilon^2 (\tau_{s_1} + \tau_{\eta_1})}$
b_1	$\frac{1 + \delta^2 \tau_\epsilon (\tau_{s_1} + \tau_{\eta_1})}{\delta[\tau_\theta + \tau_\epsilon + \delta^2 \tau_\epsilon^2 (\tau_{s_1} + \tau_{\eta_1})]}$
c_{11}	$\frac{\delta \tau_\epsilon \tau_{\eta_1}}{\tau_\theta + \tau_\epsilon + \delta^2 \tau_\epsilon^2 (\tau_{s_1} + \tau_{\eta_1})}$
c_{12}	0
a_2	1
b_2	$\frac{1}{\delta \tau_\epsilon}$
c_{21}	0
c_{22}	$\frac{1}{\delta \tau_\epsilon}$
d_2	0

For the sake of illustrating how the signal Y_2 influences the financial market, we additionally analyze the limiting case where Y_2 is perfectly precise (i.e., $\tau_{\eta_2} \to \infty$). Table 3.1 displays the coefficients of price functions (3.23) and (3.24) in this extreme scenario. The respective values follow straightforwardly from the results in Proposition 3.2. As $\tau_{\eta_2} \to \infty$, ρ_1 approaches ρ_2. This means that date-1 agents trade as aggressively as date-2 agents on private fundamental information. If Y_2

reveals s_2, date-2 rational traders can perfectly infer θ from the signal P_2^* (i.e., the precision of the signal P_2^* diverges to infinity). This allows date-2 rational agents to fully absorb date-2 noise trader demand (i.e., $b_2 = c_{22} = 1/\delta\tau_\epsilon$). In this case, date-2 noise traders have no influence on P_2. Moreover, Y_1 and P_1 have no influence on P_2 either (i.e., $c_{21} = d_2 = 0$). Since the signal P_2^* is already perfectly precise, date-2 rational traders do not put any weight on the signal P_1^* (which contains Y_1) when forming expectations about θ. Thus, as $\tau_{\eta_2} \to \infty$, P_2 constitutes a fully efficient price in the spirit of the prices-are-right formulation of the EMH (see Section 2.1), and predicting P_2 equals predicting θ. As a consequence, date-1 agents trade with the same intensity as date-2 agents on private fundamental information.

From Table 3.1, we also see that Y_2 turns out to be useless for date-1 rational traders in the limiting case (i.e., $c_{12} = 0$). As Y_2 becomes perfectly precise, noise trading s_2 exerts no impact on P_2. Thus, as of date 1, any information related to date-2 noise trader demand is redundant. The results contained in Table 3.1 are, furthermore, useful when investigating the impact of Y_2 on price efficiency. This task will be carried out in the next subsection.

3.2.3 Price Efficiency

In the static setup of Section 3.1, we have analyzed the effect of public information about *contemporaneous* noise trader demand on price efficiency. In line with the results of the existing literature on non-fundamental information in static models, public information about current noise trading unequivocally benefits price efficiency. The dynamic model, by contrast, allows us to assess the effect of public information about *future* noise trader demand on price efficiency. As seen in Subsection 3.2.2, the date-1 price only is influenced by average expectations about future noise trading. At date 2, rational traders have to forecast fundamentals, since the risky asset pays off its fundamental value at date 3. As of date 2, there is no future noise trading. Therefore, the date-2 price is *not* influenced by average expectations about *future* noise trading. This motivates why we focus on the efficiency of the date-1 price in the following analysis. Indeed, public information about future noise trading can move the date-1 price away from fundamentals. That is, there are scenarios in which the date-1 asset price would be closer to fundamentals if public information about future noise trader demand did not exist. Recalling price function (3.23), observing P_1 is informationally equivalent to observing

$$
\begin{aligned}
P_1^{**} \equiv \frac{P_1}{a_1} &= \theta + \frac{1}{\rho_1} s_1 - \frac{c_{11}}{a_1} Y_1 + \frac{c_{12}}{a_1} Y_2 \\
&= \theta + \left(\frac{1}{\rho_1} - \frac{c_{11}}{a_1} \right) s_1 - \frac{c_{11}}{a_1} \eta_1 + \frac{c_{12}}{a_1} (s_2 + \eta_2). \qquad (3.31)
\end{aligned}
$$

Thus, by using the projection theorem, date-1 price efficiency is given by

$$\mathrm{Var}^{-1}(\theta \,|\, P_1^{**}) = \tau_\theta + \left[\left(\frac{1}{\rho_1} - \frac{c_{11}}{a_1}\right)^2 \frac{1}{\tau_{s_1}} + \left(\frac{c_{11}}{a_1}\right)^2 \frac{1}{\tau_{\eta_1}} + \left(\frac{c_{12}}{a_1}\right)^2 \left(\frac{1}{\tau_{s_2}} + \frac{1}{\tau_{\eta_2}}\right)\right]^{-1}. \tag{3.32}$$

According to (3.32), date-1 price efficiency is determined by the interplay of three terms. The first two terms in square brackets correspond to the CON and COMESCON effects, which have already been identified in the static model (cf. (3.20)). They capture the effects of current noise trading and the common error term of the public signal Y_1. The third term in square brackets in (3.32) is unique to the dynamic model. It expresses the impact of public information about future noise trader demand on current price efficiency. Due to date-1 agents' incentive to front-run date-2 noise trading, this term adds a new source of noise to the date-1 price. We call this component the "COMSFUN (COMmon Signal about FUture Noise trading)" effect. The detrimental impact of the COMSFUN effect is greater when date-1 agents' aggregate trading intensity on the public signal Y_2 rises relative to their trading intensity on fundamentals θ (i.e., c_{12}/a_1 increases).

In what follows, we assess how introducing public information about future noise trader demand influences price efficiency. In particular, we aim to show that P_1 can be less efficient in the presence than in the absence of the signal Y_2. We first compare the cases where Y_2 is completely imprecise (i.e., $\tau_{\eta_2} = 0$) and perfectly precise (i.e., $\tau_{\eta_2} \to \infty$), given that information about contemporaneous noise trading is available (i.e., $\tau_{\eta_1} > 0$) or unavailable (i.e., $\tau_{\eta_1} = 0$). The results are summarized in the following proposition (with the proof delegated to Appendix A):

Proposition 3.3.

(a) Let $\tau_{\eta_1} = 0$.

Then, $\mathrm{Var}^{-1}(\theta \,|\, P_1^{**})$ *is smaller for* $\tau_{\eta_2} = 0$ *than as* $\tau_{\eta_2} \to \infty$.

(b) Let $\tau_{\eta_1} > 0$.

Then, $\mathrm{Var}^{-1}(\theta \,|\, P_1^{**})$ *can be greater or smaller for* $\tau_{\eta_2} = 0$ *than as* $\tau_{\eta_2} \to \infty$.

Part (a) in Proposition 3.3 considers the special case where noise traders' activity on social media does not yield any valuable or processable information about their contemporaneous behavior (i.e., $\tau_{\eta_1} = 0$). In this situation, the COMESCON effect vanishes and the CON effect boils down to $1/(\rho_1^2 \tau_{s_1})$ (see the proof of Proposition 3.3 in Appendix A). Thus, price efficiency becomes

$$\mathrm{Var}^{-1}(\theta \,|\, P_1^{**}) = \tau_\theta + \left[\frac{1}{\rho_1^2 \tau_{s_1}} + \left(\frac{c_{12}}{a_1}\right)^2 \left(\frac{1}{\tau_{s_2}} + \frac{1}{\tau_{\eta_2}}\right)\right]^{-1}.$$

If $\tau_{\eta_2} = 0$, the signal about future noise trading is useless for predicting future noise. This leads to $c_{12}/a_1 = 0$, and the COMSFUN effect disappears. As a consequence, the CON effect only determines price efficiency. For $\tau_{\eta_2} = 0$, the CON effect is most pronounced, as agents trade weakly on private fundamental information in this case (i.e., ρ_1 is relatively small). As $\tau_{\eta_2} \to \infty$, agents perfectly know s_2 by observing Y_2. Nevertheless, the COMSFUN effect vanishes in this situation too. If Y_2 perfectly reveals s_2, date-2 rational traders offset all noise inherent in the date-2 price. P_2 equals θ and noise trading does not shape the date-2 price anymore. This makes Y_2 useless for forecasting P_2 as of date 1. Hence, date-1 price efficiency is again solely shaped by the CON effect, which is least pronounced as $\tau_{\eta_2} \to \infty$. This justifies why date-1 price efficiency is higher as $\tau_{\eta_2} \to \infty$ than for $\tau_{\eta_2} = 0$.

Part (b) in Proposition 3.3 shows that the relationship in part (a) can be reversed if information about contemporaneous noise is available. Whenever $\tau_{\eta_1} > 0$, price efficiency can indeed be higher if information about future noise is absent. As before, the COMSFUN effect disappears in both limiting cases. Thus, price efficiency is shaped by the CON and COMESCON effects only:

$$\mathrm{Var}^{-1}(\theta \,|\, P_1^{**}) = \tau_\theta + \left[\left(\frac{1}{\rho_1} - \frac{c_{11}}{a_1} \right)^2 \frac{1}{\tau_{s_1}} + \left(\frac{c_{11}}{a_1} \right)^2 \frac{1}{\tau_{\eta_1}} \right]^{-1}.$$

As $\tau_{\eta_2} \to \infty$, ρ_1 is still greater than for $\tau_{\eta_2} = 0$, which weakens the CON effect. Nevertheless, the CON effect can be *more* pronounced as $\tau_{\eta_2} \to \infty$ (see the proof of Proposition 3.3 in Appendix A). If $\tau_{\eta_1} > 0$, the CON effect is also influenced by rational agents' trading intensity against the signal about current noise trading Y_1 (through the ratio c_{11}/a_1). Similar to the static model, agents' aggregate demand at date 1 can be expressed as a linear function of θ, s_1, Y_1, Y_2, and P_1 (cf. also (3.30)). Denote $D_1 \equiv \int_0^1 D_{1i}\, di$. Then, in general form,

$$\frac{c_{11}}{a_1} = \frac{|\partial D_1/\partial Y_1|}{\partial D_1/\partial \theta},$$

which indicates rational agents' trading intensity against the public signal Y_1 relative to their trading intensity on fundamentals. By inspecting a date-1 agent's demand function in (3.30) and recalling that P_1^* is a linear function of θ, s_1, and Y_1, we obtain

$$\left| \frac{\partial D_1}{\partial Y_1} \right| = \frac{a_2 \mathrm{Var}(\theta \,|\, I_{1i})}{\mathrm{Var}(P_2 \,|\, I_{1i})} \delta \rho_1 \tau_{\eta_1} + \frac{\delta}{\mathrm{Var}(P_2 \,|\, I_{1i})} c_{21}, \tag{3.33}$$

$$\frac{\partial D_1}{\partial \theta} = \frac{a_2 \mathrm{Var}(\theta \,|\, I_{1i})}{\mathrm{Var}(P_2 \,|\, I_{1i})} [\delta \tau_\epsilon + \delta \rho_1^2 (\tau_{s_1} + \tau_{\eta_1})]. \tag{3.34}$$

According to (3.33), date-1 agents use Y_1 in two different ways. For one thing, they use Y_1 together with P_1 to predict fundamentals, represented by the first term in

(3.33). For another, date-1 agents know that date-2 traders also observe the public signal Y_1 and use it to extract noise from P_1. Thus, predicting P_2 entails predicting Y_1. This is represented by the second term in (3.33).

As $\tau_{\eta_2} \to \infty$, there are two competing effects. On the one hand, as indicated by the first summand in (3.33), agents trade more aggressively on P_1^* and, thus, more aggressively against Y_1 than for $\tau_{\eta_2} = 0$ (recall that $a_2 \mathrm{Var}(\theta \mid I_{1i})/\mathrm{Var}(P_2 \mid I_{1i})$ is increasing in τ_{η_2}). On the other hand, date-2 agents do not rely on the signal Y_1 anymore when forecasting fundamentals (i.e., $c_{21} = 0$), and the second summand in (3.33) vanishes. This makes date-1 agents trade less aggressively against Y_1. It is ambiguous which effect dominates and, thus, if agents trade more or less aggressively against Y_1 as $\tau_{\eta_2} \to \infty$ than for $\tau_{\eta_2} = 0$ (see the proof of Proposition 3.3 in Appendix A).

According to (3.34), rational agents unequivocally trade more aggressively on fundamentals as $\tau_{\eta_2} \to \infty$ than for $\tau_{\eta_2} = 0$. Hence, it can happen that date-1 agents offset less contemporaneous noise as $\tau_{\eta_2} \to \infty$ than for $\tau_{\eta_2} = 0$ (i.e., $|\partial D_1/\partial Y_1|$ falls) and c_{11}/a_1 decreases. Then, the CON effect intensifies (resp., is mitigated) exactly if the fall in c_{11}/a_1 is stronger (resp., weaker) than the rise in ρ_1 in absolute terms.

Analogously, the COMESCON effect can also be greater or smaller as $\tau_{\eta_2} \to \infty$ than for $\tau_{\eta_2} = 0$ (depending on whether c_{11}/a_1 increases or decreases). In total, it can happen that the sum of the CON and COMESCON effects rises as τ_{η_2} switches from zero to infinity, which explains why price efficiency can be lower as $\tau_{\eta_2} \to \infty$ than for $\tau_{\eta_2} = 0$.

Next, we once again turn to the special case where information about current noise trading is unavailable (i.e., $\tau_{\eta_1} = 0$). As stated in Proposition 3.3, a perfectly precise signal about future noise trading boosts price efficiency compared to a completely imprecise signal. Nevertheless, sufficiently small values of precision can be detrimental to price efficiency:

Proposition 3.4. *Let $\tau_{\eta_1} = 0$. $\partial[\mathrm{Var}^{-1}(\theta \mid P_1^{**})]/\partial \tau_{\eta_2} < 0$ for $\tau_{\eta_2} = 0$ exactly if*

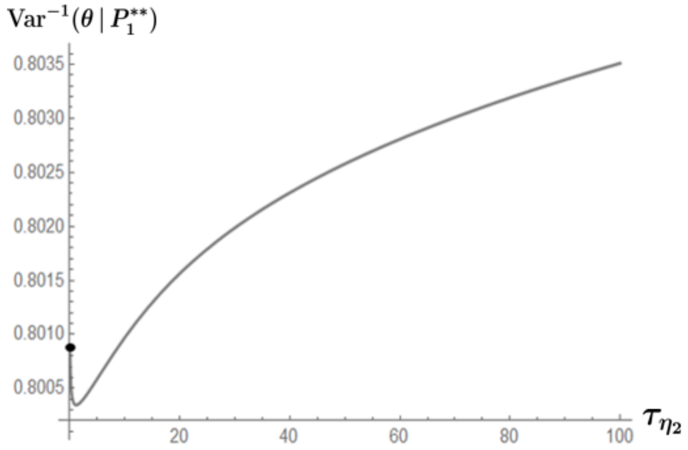
$$\frac{2}{\delta \tau_\epsilon^2 \rho_{10} \tau_{s_1}} < \frac{(\tau_\theta + \tau_\epsilon + \rho_{10}^2 \tau_{s_1})^2}{[\rho_{10}^2 \tau_{s_1}(\tau_\theta + \tau_\epsilon + \rho_{10}^2 \tau_{s_1}) + (\rho_{10}^2 \tau_{s_1} + \tau_\epsilon)(\tau_\epsilon + \rho_2^2 \tau_{s_2})]^2},$$

where

$$\rho_{10} = \frac{\delta^3 \tau_\epsilon^2 \tau_{s_2}}{1 + \delta^2 \tau_\epsilon \tau_{s_2}}.$$

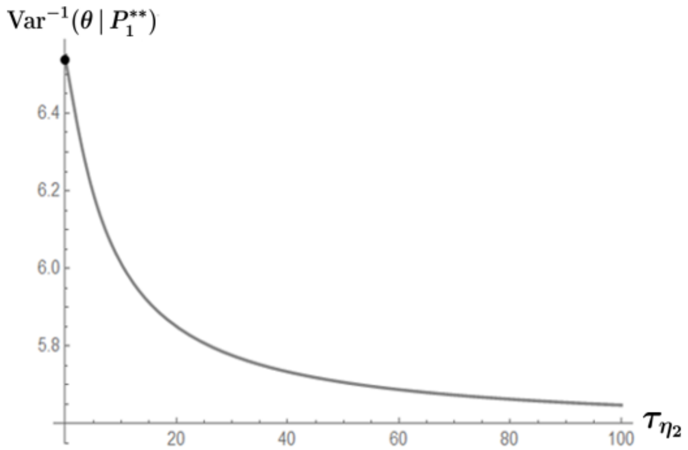
The proof can be found in Appendix A. Note that the above inequality could be solved for a unique τ_θ. Suppose τ_{η_2} rises, starting from zero. The term on the left-hand side of the inequality measures the ensuing change in the CON effect, which is induced by the increase in ρ_1. More aggressive trading on private fundamental information

Figure 3.1: Date-1 price efficiency in the OLG model (1)



Parameters: $\tau_{\eta_1} = 3.5$, $\tau_{s_1} = 2$, $\tau_{s_2} = 3$, $\tau_\epsilon = 0.01$, $\tau_\theta = 0.8$, $\delta = 4$

Figure 3.2: Date-1 price efficiency in the OLG model (2)



Parameters: $\tau_{\eta_1} = 2.5$, $\tau_{s_1} = 0.01$, $\tau_{s_2} = 3.5$, $\tau_\epsilon = 0.8$, $\tau_\theta = 4$, $\delta = 2$

mitigates the influence of current noise on the price relative to fundamentals. This benefits price efficiency through a fall in the CON effect. The term on the right-hand side represents the impact on the COMSFUN effect. The change is positive, which means that the COMSFUN effect intensifies. This harms price efficiency. The condition in the proposition is satisfied if the destabilizing impact coming from the rise in the COMSFUN effect dominates the stabilizing impact resulting from the fall in the CON effect. In this case, for sufficiently small values of τ_{η_2}, the date-1 price would be more efficient if information about future noise did not exist.

The fact that public information about future noise trader demand can reduce price efficiency for small values of precision carries over to the case with information about current noise trading (i.e., $\tau_{\eta_1} > 0$). The numerical example in Figure 3.1

shows that the signal about future noise harms price efficiency in the presence of information about current noise as long as it is sufficiently imprecise. More strikingly, the numerical example plotted in Figure 3.2 illustrates that combinations of the exogenous model parameters exist for which the relationship between signal precision and price efficiency is even *monotonically decreasing*. In total, the propositions and figures of this subsection demonstrate that public information about future noise trader demand can indeed *reduce* price efficiency.

Joint price efficiency. As outlined, the novel insights of our model are related to the impact of information about *future* noise trader demand, expressed by the signal Y_2. Nevertheless, since Y_2 is public, it can also be observed at date 2 and, consequently, affects date-2 price efficiency. However, analogous to the static model, date-2 agents use Y_2 to counteract rather than to predict noise trading, which benefits date-2 price efficiency. To account for the different usage of the public signal Y_2 at dates 1 and 2, we additionally investigate its impact on the *joint* efficiency of P_1 and P_2. This allows us to assess whether the potentially detrimental impact of Y_2 on date-1 price efficiency carries over to the case of the joint efficiency of *both* prices.

From price function (3.24), conditional on P_1, observing P_2 is informationally equivalent to observing

$$
\begin{aligned}
P_2^{**} \equiv \frac{P_2 - d_2 P_1}{a_2} &= \theta + \frac{1}{\rho_2} s_2 - \frac{c_{21}}{a_2} Y_1 - \frac{c_{22}}{a_2} Y_2 \\
&= \theta + \left(\frac{1}{\rho_2} - \frac{c_{22}}{a_2} \right) s_2 - \frac{c_{22}}{a_2} \eta_2 - \frac{c_{21}}{a_2} (s_1 + \eta_1). \qquad (3.35)
\end{aligned}
$$

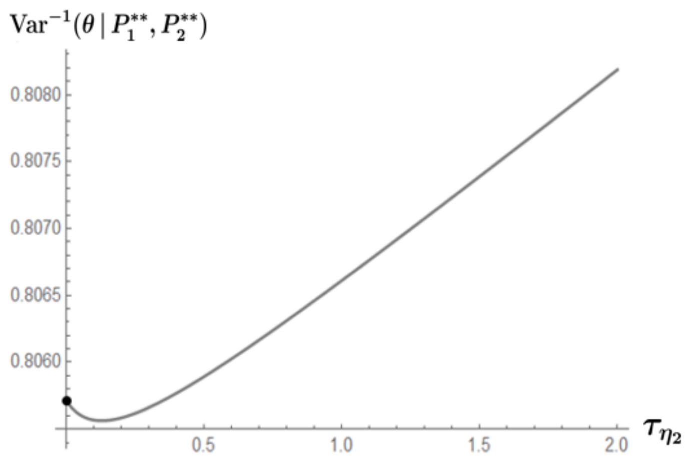
Then, by (3.31) and (3.35), we can determine the joint efficiency of P_1 and P_2. It is stated in the next proposition (with the proof delegated to Appendix A):

Proposition 3.5. *Joint price efficiency is given by*

$$
\operatorname{Var}^{-1}(\theta \mid P_1^{**}, P_2^{**}) = \left\{ \tau_\theta^{-1} - \tau_\theta^{-2} \frac{\operatorname{Var}(P_1^{**}) + \operatorname{Var}(P_2^{**}) - 2 \operatorname{Cov}(P_1^{**}, P_2^{**})}{\operatorname{Var}(P_1^{**}) \operatorname{Var}(P_2^{**}) - [\operatorname{Cov}(P_1^{**}, P_2^{**})]^2} \right\}^{-1}.
$$

The characterizations of $\operatorname{Var}(P_1^{**})$, $\operatorname{Var}(P_2^{**})$, and $\operatorname{Cov}(P_1^{**}, P_2^{**})$ can also be found in the proof in Appendix A. We can immediately conclude that joint price efficiency diverges to infinity as $\tau_{\eta_2} \to \infty$. This follows from the fact that the date-2 price is fully efficient (i.e., is equal to θ) if the signal Y_2 is perfectly precise (see also the coefficients of the price functions displayed in Table 3.1). Thus, in the limiting case, one can infer the undistorted value of θ from observing the date-2 price. This is equivalent to joint price efficiency diverging to infinity. The complexity of the expression of joint price efficiency precludes the derivation of further analytical results. Therefore, we draw on a numerical example to show that introducing the

Figure 3.3: Joint price efficiency in the OLG model



Parameters: $\tau_{\eta_1} = 3.5$, $\tau_{s_1} = 2$, $\tau_{s_2} = 3$, $\tau_\epsilon = 0.01$, $\tau_\theta = 0.8$, $\delta = 4$

signal Y_2 can harm the joint efficiency of P_1 and P_2 as long as its precision is low enough. As can be seen in Figure 3.3, joint price efficiency is decreasing in the precision of the signal about date-2 noise trader demand for sufficiently small values of precision. This proves that the potentially harmful effect of Y_2 also applies, albeit to a lesser extent, to the case of joint price efficiency.

3.3 Dynamic Setup - Long-Lived Agents

This section relaxes the OLG assumption and models rational traders as LLA, i.e., as agents who trade at *both* trading dates. In the LLA model, public information about future noise and, thus, social sentiment investing are *more* likely to reduce price efficiency than in the model with OLG of investors. This is mainly due to the fact that, contrary to the OLG model, agents' trading intensity on private fundamental information at date 1 does *not* depend on the precision of the signal about date-2 noise. The possibility of trading again at date 2 allows agents to partially hedge against unfavorable price movements between dates 1 and 2. In equilibrium, this reduces risk and, as opposed to the OLG model, traders do not underweight private fundamental information at date 1. Thus, higher signal precision does *not* make agents trade more aggressively on private fundamental information. The absence of this stabilizing effect raises the probability of public information about future noise harming price efficiency. In particular, there are two important differences in the LLA model compared to the outcomes of the OLG model. First, if information about current noise is absent, *zero* precision implies *maximum* price efficiency. Second, in the presence of information about contemporaneous noise, price efficiency is *unequivocally* lower if the signal is perfectly precise than if it is completely imprecise.

Additionally, the relationship between signal precision and price efficiency can even be *monotonically decreasing*, as in the OLG variant.

3.3.1 Model Assumptions

Consider a dynamic financial market with one riskless and one risky asset that exhibit the same properties as in the OLG model of Section 3.2. The financial market is populated by a continuum of long-lived, rational agents indexed by the interval $[0, 1]$ and noise traders. Each rational agent $i \in [0, 1]$ now trades at both trading dates. Her final wealth is $\pi_i = (\theta - P_2)D_{2i} + (P_2 - P_1)D_{1i}$ and she is again characterized by a CARA utility function of the form $U(\pi_i) = -\exp(-\delta^{-1}\pi_i)$, where δ (> 0) measures agents' identical degree of risk tolerance. Noise trading is transient and given by the random variable s_t at date t, for $t = 1, 2$.

At date 1, each rational investor observes three signals: a private fundamental signal $x_i = \theta + \epsilon_i$ and two public signals related to contemporaneous and future noise trader demand $Y_t = s_t + \eta_t$, for $t = 1, 2$, which stem from gauging social sentiment. Additionally, rational traders observe P_1 at date 1 and P_1 as well as P_2 at date 2. This gives $I_{1i} = (x_i, P_1, Y_1, Y_2)$ and $I_{2i} = (x_i, P_1, P_2, Y_1, Y_2)$. The exogenous random variables are again jointly normally distributed and pairwise independent with zero means and the notation for variances already used in the OLG model (see Subsection 3.2.1).

3.3.2 Equilibrium Determination

Long-lived agents conjecture the same linear price functions (3.23) and (3.24) as in the model with OLG of investors, which means that P_1 and P_2 are again jointly normally distributed. Equilibrium further entails maximizing expected utility and market clearing at both trading dates:

Definition (linear dynamic REE with LLA): Price functions (3.23) and (3.24) and asset demands D_{ti} $(t = 1, 2,\ i \in [0, 1])$ are a *linear dynamic REE with LLA* if

(i) D_{2i} maximizes date-2 expected utility $\mathrm{E}[U(\pi_i) \,|\, I_{2i}]$ for all $i \in [0, 1]$,

(ii) D_{1i} maximizes date-1 expected utility $\mathrm{E}[U(\pi_i) \,|\, I_{1i}]$ given D_{2i} for all $i \in [0, 1]$,

(iii) and the market for the risky asset clears at both trading dates, i.e., $\int_0^1 D_{ti}\, di + s_t = 0$, $t = 1, 2$.

The following proposition states a rational agent's utility-maximizing demand functions (with the proof delegated to Appendix A):

Proposition 3.6. *A rational agent's demand functions in the LLA model are*

$$D_{2i} = \delta \, \frac{\mathrm{E}(\theta \,|\, I_{2i}) - P_2}{\mathrm{Var}(\theta \,|\, I_{2i})}, \tag{3.36}$$

$$D_{1i} = \delta \, \frac{\mathrm{E}(P_2 \,|\, I_{1i}) - P_1}{\mathrm{Var}(P_2 \,|\, I_{1i})(1 - Corr^2)} - \delta h \, \frac{\mathrm{E}(\theta - P_2 \,|\, I_{1i})}{\mathrm{Var}(P_2 \,|\, I_{1i})(1 - Corr^2)}, \tag{3.37}$$

where

$$h \equiv \frac{\mathrm{Cov}(P_2, \theta - P_2 \,|\, I_{1i})}{\mathrm{Var}(\theta - P_2 \,|\, I_{1i})},$$

$$Corr = \frac{\mathrm{Cov}(P_2, \theta - P_2 \,|\, I_{1i})}{\sqrt{\mathrm{Var}(P_2 \,|\, I_{1i}) \, \mathrm{Var}(\theta - P_2 \,|\, I_{1i})}}.$$

The general form of an agent's date-2 demand function, given by (3.36), equals that of the OLG setup (see (3.25)), as the risky asset pays off its fundamental value θ in both models at date 3. However, date-1 demand in the LLA model differs from that of the OLG model. The first term in (3.37) represents an agent's incentive to speculate on short-term returns. The sign of this term depends on the sign of the expected myopic return at date 2. If agents expect prices to rise (resp., to fall), this component is positive (resp., negative). Since agents are long-lived, they trade again at date 2. This fact is taken into account when forming demand at date 1. Trading at date 2 partially serves as a hedge against unfavorable price movements between dates 1 and 2, which already creates hedging demand at date 1. This is represented by the second term in (3.37). The sign of the hedging component depends on the interplay of two terms: the conditional covariance of the date-2 price and the date-3 return (i.e., $\mathrm{Cov}(P_2, \theta - P_2 \,|\, I_{1i})$) and the expected date-3 return conditional on date-1 information (i.e., $\mathrm{E}(\theta - P_2 \,|\, I_{1i})$). The conditional covariance is positive if the date-2 price is mainly driven by fundamentals rather than by (partly) unknown noise coming from s_2. In this situation, a high P_2 is linked with a high return at date 3 (i.e., a high $\theta - P_2$) by indicating strong fundamentals. Inversely, the conditional covariance is negative if P_2 is strongly shaped by unknown noise. In this case, a high P_2 implies a low return at date 3 (i.e., a low $\theta - P_2$) by indicating high noise trader demand.

If the conditional covariance is positive (i.e., $h > 0$) and the expected date-3 return $\mathrm{E}(\theta - P_2 \,|\, I_{1i})$ is positive (resp., negative), hedging demand is negative (resp., positive). A rise (resp., a fall) in prices between dates 1 and 2 then predicts a higher (resp., lower) date-3 return. Since the date-3 return is expected to be positive (resp., negative) as of date 1, a rise (resp., a fall) in prices predicts an even more positive (resp., even more negative) date-3 return. Thus, traders can profit by taking a long (resp., short) position at date 2. The opportunity to benefit from rising (resp., falling) prices by taking a long (resp., short) position at date 2 compensates traders for not having benefited from date-1 asset purchases (resp., sells). This justifies the negative (resp., positive) hedging demand at date 1.

Analogously, assume the conditional covariance to be negative (i.e., $h < 0$) and the expected date-3 return $\mathrm{E}(\theta - P_2 \,|\, I_{1i})$ to be positive (resp., negative). This leads to a positive (resp., negative) hedging demand. A fall (resp., a rise) in prices between dates 1 and 2 then predicts a rise (resp., a fall) in the date-3 return. As the date-3 return is expected to be positive (resp., negative) as of date 1, a fall (resp., a rise) in prices predicts an even more positive (resp., even more negative) return at date 3 and traders can profit by going long (resp., short) at date 2. The opportunity to benefit from falling (resp., rising) prices by taking a long (resp., short) position at date 2 compensates traders for not having benefited from date-1 asset sells (resp., purchases). This justifies the positive (resp., negative) date-1 hedging demand.

Agents' ability to hedge reduces risk in equilibrium. The uncertainty rational traders face at date 1, expressed by the denominator in (3.37), is clearly smaller than the residual uncertainty about the date-2 price, $\mathrm{Var}(P_2 \,|I_{1i})$, and depends negatively on the squared correlation between P_2 and the date-3 return $\theta - P_2$ conditional on date-1 information. The higher the squared correlation, i.e., the stronger the link between P_2 and the date-3 return $\theta - P_2$, the more effectively agents can hedge by trading again at date 2. This results in higher risk reduction at date 1.

The general form of asset prices. As in the OLG model, we expound a general determination of asset prices before considering the specific price functions in the linear dynamic REE. Since the general form of the date-2 demand function in (3.36) does not differ from that of the OLG model, given in (3.25), imposing market clearing at date 2 and solving for the date-2 price delivers

$$P_2 = \bar{\mathrm{E}}_2(\theta) + \frac{\mathrm{Var}(\theta \,|\, I_{2i})}{\delta}\, s_2.$$

The date-2 price in the LLA model is still a function of rational investors' date-2 average expectations about fundamentals and date-2 noise trader demand. By imposing market clearing at date 1, P_1 can be written as

$$\int_0^1 D_{1i}\, di + s_1 = 0$$

$$\Leftrightarrow \int_0^1 \delta\, \frac{\mathrm{E}(P_2 \,|\, I_{1i}) - P_1}{\mathrm{Var}(P_2 \,|\, I_{1i})(1 - Corr^2)} - \delta h\, \frac{\mathrm{E}(\theta - P_2 \,|\, I_{1i})}{\mathrm{Var}(P_2 \,|\, I_{1i})(1 - Corr^2)}\, di + s_1 = 0$$

$$\Leftrightarrow P_1 = (1 + h)\, \bar{\mathrm{E}}_1(P_2) - h\, \bar{\mathrm{E}}_1(\theta) + \frac{\mathrm{Var}(P_2 \,|\, I_{1i})(1 - Corr^2)}{\delta}\, s_1$$

$$\Leftrightarrow P_1 = (1+h)\,\bar{\mathrm{E}}_1\left[\bar{\mathrm{E}}_2(\theta) + \frac{\mathrm{Var}(\theta|I_{2i})}{\delta}\,s_2\right] - h\,\bar{\mathrm{E}}_1(\theta) + \frac{\mathrm{Var}(P_2\,|\,I_{1i})(1-Corr^2)}{\delta}\,s_1$$

$$\begin{aligned}\Leftrightarrow P_1 = {} & (1+h)\,\bar{\mathrm{E}}_1\left[\bar{\mathrm{E}}_2(\theta)\right] - h\,\bar{\mathrm{E}}_1(\theta) + (1+h)\frac{\mathrm{Var}(\theta\,|\,I_{2i})}{\delta}\,\bar{\mathrm{E}}_1(s_2) \\ & + \frac{\mathrm{Var}(P_2\,|\,I_{1i})(1-Corr^2)}{\delta}\,s_1.\end{aligned} \tag{3.38}$$

The first term in (3.38) shows that in the LLA model, the date-1 price is also characterized by a KBC. Notably, since agents directly forecast fundamentals at date 1 when forming their hedging demand, the date-1 price is also shaped by investors' date-1 average expectations about fundamentals. This is represented by the second term in (3.38). Nevertheless, Cespa and Vives (2012) demonstrate that in an LLA model with transient noise trading, the date-1 price overweights public information and is systematically farther away from fundamentals than investors' date-1 average expectations about fundamentals. However, the authors prove that the inverse result can hold true if noise trading has a persistent element. When looking at our LLA setup, rational traders' date-1 average expectations about future noise trading s_2 are again influenced by the signal Y_2 only. As in the OLG model, agents' incentive to front-run date-2 noise trader demand injects additional noise into the date-1 price, which moves it away from fundamentals.

The specific form of asset prices. Now, we derive the specific solutions for the coefficients in price functions (3.23) and (3.24). Recall that $I_{1i} = (x_i, P_1, Y_1, Y_2)$ and $I_{2i} = (x_i, P_1, P_2, Y_1, Y_2)$. Thus, analogous to the OLG model, (P_1^*, x_i) and (P_1^*, P_2^*, x_i) convey the same information about fundamentals as I_{1i} and I_{2i} (cf. Subsection 3.2.2). After updating agent i's date-2 beliefs about fundamentals, plugging them into demand function (3.36), and imposing market clearing at date 2, we obtain the function of P_2 in equilibrium. The resulting coefficients are, then, matched with those in (3.24). Using agent i's date-1 information set, we can determine her updated beliefs about fundamentals and future noise trading, which enter demand function (3.37). We proceed to impose market clearing at date 1, allowing us to determine the equilibrium function of P_1. Again, we match the resulting coefficients with those in (3.23). This yields:

Proposition 3.7. *There exists a unique linear dynamic REE with LLA, in which*

$$a_1 = \frac{[\tau_\epsilon + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})]\,\Gamma_1\Gamma_2\Delta b_2^2 + (1-a_2)\Gamma_1\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}{\Delta[(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2]},$$

$$b_1 = \frac{a_1}{\rho_1},$$

$$c_{11} = a_1 \frac{\rho_1 \tau_{\eta_1} \left(1 + \frac{1-a_2}{\Delta b_2^2 \Gamma_2}\right)}{\tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1}) \left(1 + \frac{1-a_2}{\Delta b_2^2 \Gamma_2}\right)},$$

$$c_{12} = a_1 \frac{\frac{\delta \tau_{\eta_2}[\tau_\theta + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]}{[1+\delta\rho_2(\tau_{s_2} + \tau_{\eta_2})]^2}}{\tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1}) \left(1 + \frac{1-a_2}{b_2^2 \Gamma_2 \Delta}\right)},$$

$$a_2 = \frac{\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})}{\Delta},$$

$$b_2 = \frac{1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})}{\delta\Delta},$$

$$c_{21} = \frac{-\rho_1^2(\tau_{s_1} + \tau_{\eta_1})\frac{c_{11}}{a_1} + \rho_1\tau_{\eta_1}}{\Delta},$$

$$c_{22} = \frac{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})\frac{c_{12}}{a_1} + \rho_2\tau_{\eta_2}}{\Delta},$$

$$d_2 = \frac{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})}{a_1\Delta},$$

$$\Gamma_1 \equiv \left[\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\right]^{-1},$$

$$\Gamma_2 \equiv (\tau_{s_2} + \tau_{\eta_2})^{-1},$$

$$\Delta \equiv \tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1}) + \rho_2^2(\tau_{s_2} + \tau_{\eta_2}),$$

$$\rho_2 \equiv \frac{a_2}{b_2} = \delta\tau_\epsilon,$$

$$\rho_1 \equiv \frac{a_1}{b_1} = \delta\tau_\epsilon.$$

The proof is contained in Appendix A. Although more complex than in the OLG model, the coefficients in the price functions (3.23) and (3.24) can also be determined in closed form in the LLA setup. P_1 still reacts positively to changes in Y_2 and negatively to changes in Y_1. P_2 is again negatively related to both public signals about noise trading. The most striking difference compared to the equilibrium in the OLG model is the endogenous value of ρ_1, which measures how aggressively agents trade on private fundamental information at date 1. In the variant with OLG of investors, agents underweight private fundamental information at date 1 relative to date 2 (i.e., $\rho_1 < \rho_2$). If agents are long-lived, by contrast, ρ_1 equals ρ_2. This implies that agents trade with the same intensity on private fundamental information at both trading dates (which is also independent of the precision of the signal about date-2 noise). The reason for this lies in the hedging possibilities that date-2 trading provides to rational investors. As outlined, in equilibrium, the ability to hedge reduces risk, allowing agents at date 1 to trade as aggressively on private fundamental information as at date 2.

Along the proof of Proposition 3.7 in Appendix A, we show that an agent's demand function at dates 1 and 2 can be written as

$$D_{2i} = \delta\tau_\epsilon\, x_i + \delta\rho_1^2(\tau_{s_1} + \tau_{\eta_1})\, P_1^* + \delta\rho_2^2(\tau_{s_2} + \tau_{\eta_2})P_2^* - \frac{\delta}{\mathrm{Var}(\theta \mid I_{2i})} P_2, \tag{3.39}$$

$$D_{1i} = \delta\tau_\epsilon\, x_i + \delta\rho_1^2(\tau_{s_1} + \tau_{\eta_1})P_1^* + \frac{\delta(1-a_2)}{b_2^2\Gamma_2}\left[b_2\,\mathrm{E}(s_2 \mid Y_2) - c_{21}Y_1 - c_{22}Y_2 + d_2P_1\right]$$
$$- \frac{\delta}{\mathrm{Var}(P_2 \mid I_{1i})(1 - Corr^2)} P_1. \tag{3.40}$$

An agent's demand for the risky asset at date 2 is given in (3.39). It has the same form as in the OLG model (cf. (3.29)). In strong contrast to the OLG setup, the demand function in (3.40) shows that at date 1, an agent trades as aggressively on her two signals used to predict fundamentals (i.e., x_i and P_1^*) as at date 2. This is due to her ability to hedge. The third summand in (3.40) is linked to an agent's engagement in short-term speculation.

Moreover, we can conclude without any further calculations that the LLA and OLG models coincide as $\tau_{\eta_2} \to \infty$. In the limiting case, the signal P_2^* is perfectly precise and rational traders can observe the unbiased value of θ at date 2, making them fully absorb date-2 noise trader demand. This leads P_2 to equal θ. In this situation, as in the OLG model, agents are concerned with predicting fundamentals only at date 1. Since rational agents' date-1 information sets are identical in the OLG and LLA variants, the two models are equal as $\tau_{\eta_2} \to \infty$ (see Table 3.1 for the exact coefficients of the price functions).

3.3.3 Price Efficiency

In this subsection, we explore the impact of public information about future noise trading on price efficiency in the LLA model and demonstrate that the potentially detrimental effect is *more* likely to occur than in the OLG model. Analogous to the OLG setup, the efficiency of P_1 is given by

$$\mathrm{Var}^{-1}(\theta \mid P_1^{**}) = \tau_\theta + \left[\left(\frac{1}{\rho_1} - \frac{c_{11}}{a_1}\right)^2 \frac{1}{\tau_{s_1}} + \left(\frac{c_{11}}{a_1}\right)^2 \frac{1}{\tau_{\eta_1}} + \left(\frac{c_{12}}{a_1}\right)^2 \left(\frac{1}{\tau_{s_2}} + \frac{1}{\tau_{\eta_2}}\right)\right]^{-1},$$

where $P_1^{**} \equiv P_1/a_1$ (see (3.31)) and ρ_1, a_1, c_{11}, and c_{12} are now taken from Proposition 3.7 instead of Proposition 3.2. In the LLA model, the known CON, COMESCON, and COMSFUN effects still determine date-1 price efficiency.

First, we assess the impact of information about future noise in the special case where valuable information about contemporaneous noise trader demand is unavailable to rational traders (i.e., $\tau_{\eta_1} = 0$). In this reduced setting, zero precision yields maximum price efficiency:

Proposition 3.8. *Let* $\tau_{\eta_1} = 0$. *Then,*

(a) $\mathrm{Var}^{-1}(\theta \,|\, P_1^{**})$ *is greater for* $\tau_{\eta_2} = 0$ *than for any finite* $\tau_{\eta_2} > 0$.

(b) $\mathrm{Var}^{-1}(\theta \,|\, P_1^{**})$ *is U-shaped in* τ_{η_2}.

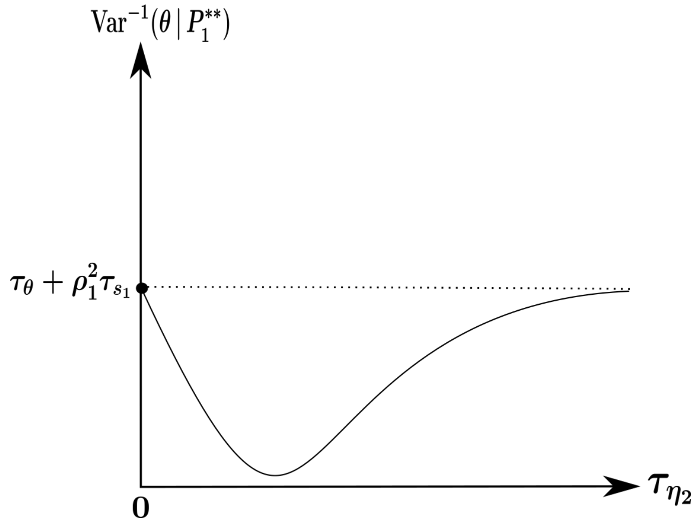
The proof can be found in Appendix A. If $\tau_{\eta_1} = 0$, we have $c_{11}/a_1 = 0$ (see Proposition 3.7) so that the COMESCON effect vanishes and the CON effect reduces to $1/(\rho_1^2 \tau_{s_1})$. Consequently, price efficiency is given by

$$\mathrm{Var}^{-1}(\theta \,|\, P_1^{**}) = \tau_\theta + \left[\frac{1}{\rho_1^2 \tau_{s_1}} + \left(\frac{c_{12}}{a_1} \right)^2 \left(\frac{1}{\tau_{s_2}} + \frac{1}{\tau_{\eta_2}} \right) \right]^{-1}.$$

In strong contrast to the OLG model, ρ_1 and, therefore, the CON effect are independent of τ_{η_2} in the LLA variant. Moreover, we know that $c_{12}/a_1 = 0$ for $\tau_{\eta_2} = 0$ and as $\tau_{\eta_2} \to \infty$ (see Proposition 3.7 and Table 3.1), meaning that the COMSFUN effect vanishes in both limiting cases. P_1 is, thus, equally efficient for $\tau_{\eta_2} = 0$ and as $\tau_{\eta_2} \to \infty$. Since the COMSFUN effect is positive for any $\tau_{\eta_2} > 0$, $\mathrm{Var}^{-1}(\theta \,|\, P_1^{**})$ is greater for $\tau_{\eta_2} = 0$ than for any $\tau_{\eta_2} > 0$. Hence, according to part (a) in Proposition 3.8, *any* level of precision *unequivocally* yields *lower* price efficiency than *zero* precision. This outcome is diametrically opposed to related findings on the effects of information about contemporaneous noise in static and dynamic setups (e.g., Ganguli and Yang, 2009; Manzano and Vives, 2011; Farboodi and Veldkamp, 2020) and of public information about fundamentals in a dynamic setup (see Gao, 2008).

More specifically, as stated in part (b) in Proposition 3.8, the relationship between date-1 price efficiency and τ_{η_2} is U-shaped (see also Figure 3.4). As τ_{η_2} rises, the weight traders put on the signal about future noise relative to fundamentals, expressed by the ratio c_{12}/a_1, is influenced by two counteracting effects. For one thing, as Y_2 predicts future noise trader demand more precisely, agents trade more aggressively on the signal when forming their demand. For another, a more precise Y_2 implies that at date 2, rational traders offset more of the date-2 noise trader demand. This reduces the noise in the date-2 price, making Y_2 less useful for predicting P_2 at date 1 (cf. also (3.40)).

Figure 3.4: Date-1 price efficiency for $\tau_{\eta_1} = 0$ in the LLA model



When the signal is imprecise (i.e., τ_{η_2} is small), the destabilizing effect dominates and agents trade more aggressively on the signal Y_2 relative to fundamentals as τ_{η_2} increases. Consequently, price efficiency falls. However, there exists a point where the stabilizing effect takes over and agents trade less aggressively on Y_2 at date 1 as the signal gains further in precision. In the limit, if the signal about date-2 noise trading is perfectly precise, rational traders offset all noise inherent in the date-2 price. As a consequence, Y_2 is useless for forecasting P_2. In this situation, P_1 is not influenced by Y_2 (i.e., $c_{12} = 0$) and is, thus, as efficient as when the signal about future noise is completely imprecise.

Next, we turn to the general case with valuable information about current noise trading (i.e., $\tau_{\eta_1} > 0$). Recall from Proposition 3.3 that in the OLG variant, price efficiency can be higher or lower for $\tau_{\eta_2} = 0$ than as $\tau_{\eta_2} \to \infty$. In the LLA model, by contrast, price efficiency is *unequivocally lower* if the signal about future noise is perfectly precise:

Proposition 3.9. $\mathrm{Var}^{-1}(\theta \mid P_1^{**})$ *is greater for* $\tau_{\eta_2} = 0$ *than as* $\tau_{\eta_2} \to \infty$.

The proof is delegated to Appendix A. As already known, the COMSFUN effect disappears in both limiting scenarios and price efficiency is shaped by the CON and COMESCON effects only:

$$\mathrm{Var}^{-1}(\theta \mid P_1^{**}) = \tau_\theta + \left[\left(\frac{1}{\rho_1} - \frac{c_{11}}{a_1} \right)^2 \frac{1}{\tau_{s_1}} + \left(\frac{c_{11}}{a_1} \right)^2 \frac{1}{\tau_{\eta_1}} \right]^{-1}.$$

From Proposition 3.7, we know that ρ_1 is independent of τ_{η_2}. Thus, the result in Proposition 3.9 is driven by the ratio c_{11}/a_1, which indicates rational agents' trading intensity against the public signal about contemporaneous noise trading relative to their trading intensity on fundamentals (i.e., $|\partial D_1/\partial Y_1|/(\partial D_1/\partial \theta)$). By (3.40), we obtain

$$\left|\frac{\partial D_1}{\partial Y_1}\right| = \delta\rho_1\tau_{\eta_1} + \frac{\delta(1-a_2)}{b_2^2\Gamma_2}c_{21}, \tag{3.41}$$

$$\frac{\partial D_1}{\partial \theta} = \delta\tau_\epsilon + \delta\rho_1^2(\tau_{s_1} + \tau_{\eta_1}). \tag{3.42}$$

Along the proof of Proposition 3.9 in Appendix A, we show that c_{11}/a_1 is greater for $\tau_{\eta_2} = 0$ than as $\tau_{\eta_2} \to \infty$. This implies that agents trade more aggressively against contemporaneous noise compared to trading on fundamentals when the signal about future noise is completely imprecise. The explanation is the following: in sharp contrast to the OLG setup, long-lived agents do *not* underweight the two signals x_i and P_1^*, which are used to predict fundamentals at date 1 (cf. (3.30) and (3.40)). Thus, as illustrated in (3.42), agents' aggregate response to fundamentals is independent of τ_{η_2}. Hence, changes in c_{11}/a_1 are driven by changes in $|\partial D_1/\partial Y_1|$.

Analogous to the OLG model, long-lived traders use the signal about current noise Y_1 in two ways. On the one hand, Y_1 is contained in P_1^* and used to extract noise inherent in the date-1 market price. In the OLG setup, as τ_{η_2} switches from zero to infinity, agents trade more aggressively on P_1^* and, thus, more aggressively against Y_1 (see (3.33)). In the LLA model, by contrast, this effect is absent, as τ_{η_2} does *not* influence how aggressively agents trade on P_1^* (see the first term in (3.41)). On the other hand, since the public signal Y_1 is also observable at date 2, forecasting P_2 entails forecasting Y_1. Thus, Y_1 directly helps to predict P_2, which is represented by the second term in (3.41). If Y_2 is perfectly precise (i.e., $\tau_{\eta_2} \to \infty$), θ can be observed at date 2 by disentangling the information conveyed by P_2. Consequently, rational traders at date 2 do not use Y_1 to predict fundamentals (i.e., $c_{21} = 0$). As of date 1, this makes Y_1 less useful for predicting P_2 and traders put less weight on the signal when forming date-1 demand than for $\tau_{\eta_2} = 0$ (i.e., the second term in (3.41) vanishes). This explains why $|\partial D_1/\partial Y_1|$ and c_{11}/a_1 are unequivocally larger for $\tau_{\eta_2} = 0$ than as $\tau_{\eta_2} \to \infty$.

The fact that c_{11}/a_1 is greater in the absence of information about future noise trading implies that the CON effect is less pronounced for $\tau_{\eta_2} = 0$ than as $\tau_{\eta_2} \to \infty$, raising price efficiency. The COMESCON effect, by contrast, is clearly more pronounced for $\tau_{\eta_2} = 0$, which harms price efficiency. Thus, as τ_{η_2} switches from infinity to zero, the impact of more aggressive trading against Y_1 on price efficiency is two-edged. Nevertheless, the result in Proposition 3.9 shows that the stabilizing impact coming from the reduction in the CON effect dominates the destabilizing impact generated by the increase in the COMESCON effect.

Thus, information about future noise unambiguously harms price efficiency for sufficiently high values of precision. Lastly, we prove that it can reduce price efficiency also for low values of precision:

Proposition 3.10. $\partial[\mathrm{Var}^{-1}(\theta \mid P_1^{**})]/\partial\tau_{\eta_2} < 0$ *for* $\tau_{\eta_2} = 0$ *exactly if*

$$\frac{2\tau_{\eta_1}\tau_\epsilon^2(1-\delta\rho_2\tau_{s_2})}{\tau_{s_1}[\tau_\epsilon + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi_{02})]} < \frac{\tau_\theta + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})}{1+\delta\rho_2\tau_{s_2}},$$

where

$$\phi_{02} \equiv \frac{\delta^2\tau_{s_2}[\tau_\theta + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})]}{(1+\delta\rho_2\tau_{s_2})^2}.$$

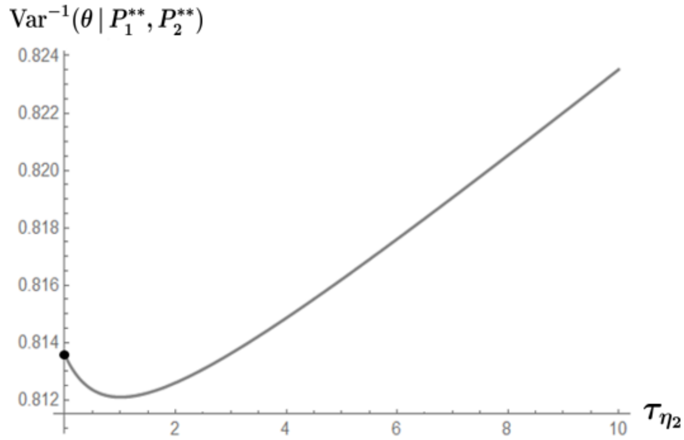
The proof can be found in Appendix A. The term on the right-hand side of the inequality in Proposition 3.10 measures the change in the COMSFUN effect induced by the rise in τ_{η_2}, starting from $\tau_{\eta_2} = 0$. Since the change is positive, the COMSFUN effect intensifies and the ensuing impact on price efficiency is negative. The expression on the left-hand side stands for the combined impact on the CON and COMESCON effects. Along the proof of the proposition in Appendix A, we show that agents' aggregate response to changes in the signal about contemporaneous noise trader demand Y_1 becomes less pronounced (i.e., c_{11}/a_1 decreases) when $\delta\rho_2\tau_{s_2}$ exceeds unity. In this case, the increase in the CON effect exceeds the reduction in the COMESCON effect in absolute terms. Consequently, the ensuing impact on the CON and COMESCON effects also harms price efficiency and the condition in Proposition 3.10 is certainly satisfied.

If $\delta\rho_2\tau_{s_2} < 1$, agents trade more aggressively against Y_1 as τ_{η_2} rises, starting from zero (i.e., c_{11}/a_1 increases). As the stabilizing impact due to the reduction in the CON effect is more pronounced than the destabilizing impact linked to the increase in the COMESCON effect, more aggressive trading against Y_1 is conducive to price efficiency. In this scenario, date-1 price efficiency falls exactly if the destabilizing impact coming from the rise in the COMSFUN effect is stronger than the stabilizing impact resulting from the combination of the CON and COMESCON effects.

Joint price efficiency. Having pointed out a potentially negative effect of information about future noise trading on current price efficiency, we eventually consider the joint efficiency of both prices. In particular, we show that the public signal Y_2 can harm the joint efficiency of P_1 and P_2 in the LLA model as well. Following the identical general form of the equilibrium price functions in the LLA and OLG models, joint price efficiency in the LLA setup is given by the same expression as in the OLG variant (see Proposition 3.5):

$$\mathrm{Var}^{-1}(\theta \mid P_1^{**}, P_2^{**}) = \left\{\tau_\theta^{-1} - \tau_\theta^{-2}\frac{\mathrm{Var}(P_1^{**}) + \mathrm{Var}(P_2^{**}) - 2\,\mathrm{Cov}(P_1^{**}, P_2^{**})}{\mathrm{Var}(P_1^{**})\,\mathrm{Var}(P_2^{**}) - [\mathrm{Cov}(P_1^{**}, P_2^{**})]^2}\right\}^{-1}.$$

Figure 3.5: Joint price efficiency in the LLA model



Parameters: $\tau_{\eta_1} = 3.5$, $\tau_{s_1} = 2$, $\tau_{s_2} = 3$, $\tau_\epsilon = 0.01$, $\tau_\theta = 0.8$, $\delta = 4$

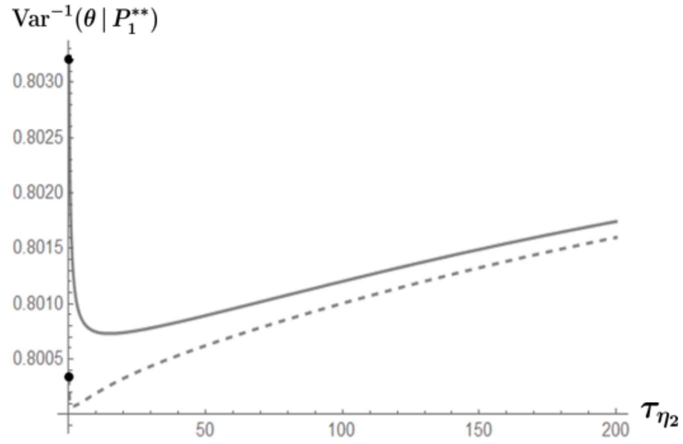
The coefficients determining $\mathrm{Var}\left(P_1^{**}\right)$, $\mathrm{Var}\left(P_2^{**}\right)$, and $\mathrm{Cov}\left(P_1^{**}, P_2^{**}\right)$ are now taken from Proposition 3.7 instead of Proposition 3.2. As in the OLG model, joint price efficiency diverges to infinity as $\tau_{\eta_2} \to \infty$, since P_2 is fully efficient in this case. The numerical example depicted in Figure 3.5, furthermore, proves that introducing the signal Y_2 can reduce the joint efficiency of P_1 and P_2 for sufficiently small values of precision. That is, there can exist values of τ_{η_2} for which joint price efficiency is lower than for $\tau_{\eta_2} = 0$. This confirms the potentially negative impact of information about date-2 noise trader demand on price efficiency in the LLA setup.

3.4 Model Comparison

Subsections 3.2.3 and 3.3.3 investigate the influence of public information about date-2 noise trader demand on price efficiency in the OLG and LLA models, respectively. In this section, we conduct a brief, direct comparison of price efficiency in both setups. Numerical analysis shows that date-1 price efficiency is generally lower in the OLG model than in the LLA variant (with identical model parameters). Since both models are equal as $\tau_{\eta_2} \to \infty$ (see Subsection 3.3.2), the marginal impact of increases in signal precision is more positive in the OLG setup.

The numerical example in Figure 3.6 compares date-1 price efficiency in the two models for the special case where $\tau_{\eta_1} = 0$. In this situation, a perfectly precise signal about future noise yields higher price efficiency than no signal at all in the OLG variant (see Proposition 3.3). An increase in signal precision drives the date-2 price closer to fundamentals and alleviates the impact of date-1 agents' short trading horizon, making them trade more aggressively on private fundamental information (i.e., ρ_1 rises). This benefits price efficiency. Nevertheless, price efficiency can be

Figure 3.6: Date-1 price efficiency in the LLA and OLG models (1)

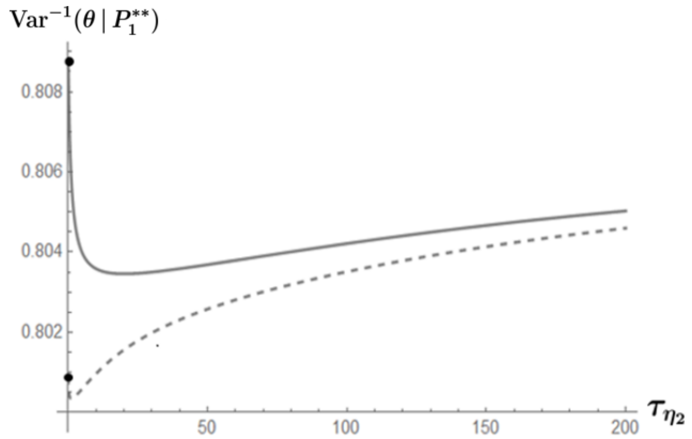


Parameters: $\tau_{\eta_1} = 0$, $\tau_{s_1} = 2$, $\tau_{s_2} = 3$, $\tau_\epsilon = 0.01$, $\tau_\theta = 0.8$, $\delta = 4$.

Note: The solid (resp., dashed) curve corresponds to the LLA (resp., OLG) model.

harmed for sufficiently small values of precision (see Proposition 3.4). In the LLA model, in strong contrast, ρ_1 is independent of τ_{η_2} and zero precision unequivocally yields maximum price efficiency (see Proposition 3.8).

Figure 3.7: Date-1 price efficiency in the LLA and OLG models (2)

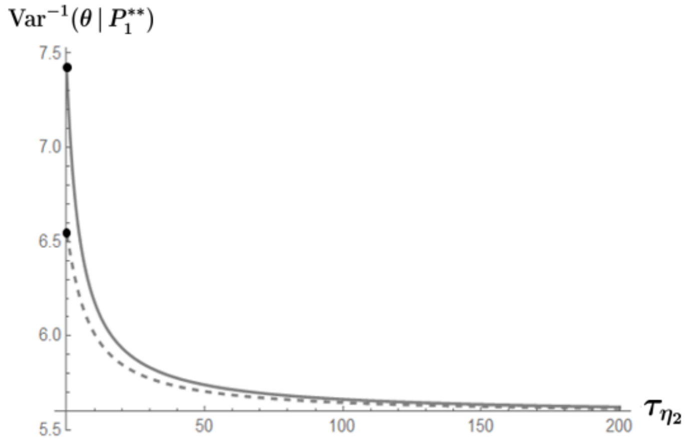


Parameters: $\tau_{\eta_1} = 3.5$, $\tau_{s_1} = 2$, $\tau_{s_2} = 3$, $\tau_\epsilon = 0.01$, $\tau_\theta = 0.8$, $\delta = 4$.

Note: The solid (resp., dashed) curve corresponds to the LLA (resp., OLG) model.

If information about current noise is available (i.e., $\tau_{\eta_1} > 0$), price efficiency can be lower for a perfectly precise signal about future noise than for no signal at all in the OLG model (see Proposition 3.3). Again, information about future noise trading can harm current price efficiency for small values of precision (see Figure 3.7). In the LLA setup, date-1 price efficiency is *unambiguously* lower as $\tau_{\eta_2} \to \infty$ than for $\tau_{\eta_2} = 0$ (see Proposition 3.9). In fact, Figure 3.7 shows that in the LLA model, zero

Figure 3.8: Date-1 price efficiency in the LLA and OLG models (3)



Parameters: $\tau_{\eta_1} = 2.5$, $\tau_{s_1} = 0.01$, $\tau_{s_2} = 3.5$, $\tau_\epsilon = 0.8$, $\tau_\theta = 4$, $\delta = 2$.

Note: The solid (resp., dashed) curve corresponds to the LLA (resp., OLG) model.

precision can also yield maximum price efficiency in the presence of information about current noise. Even more strikingly, the example in Figure 3.8 demonstrates that parameter values exist such that price efficiency is *monotonically decreasing* in signal precision in *both* models.

The numerical illustrations in Figures 3.3 and 3.5, moreover, prove that the signal Y_2 can reduce the joint efficiency of P_1 and P_2 in both model variants. As in the case of date-1 price efficiency alone, the potentially negative impact of Y_2 is more pronounced in the LLA model (i.e., joint price efficiency falls for a larger value range of τ_{η_2} in Figure 3.5 than in Figure 3.3).

To sum up, based on the results derived in this chapter, public information about future noise trading, obtained from gauging social sentiment, may *harm* price efficiency in both the LLA and OLG models. Our findings challenge the conventional wisdom that non-fundamental information is unequivocally conducive to the efficiency of prices. The fact that the detrimental impact is more likely to occur in the LLA setup should put additional emphasis on the possible negative effect of social sentiment investing on price efficiency. Due to the high frequency of trading in financial markets, the LLA model seems to be more relevant than the OLG setup.

4 Payment for Order Flow and Multidimensional Noise

"How can a broker, charged with the duty of getting its clients the best available prices, possibly do so by selling that client's orders to amazingly sophisticated high-frequency trading firms, who in turn will make billions of dollars trading against these orders?"

Sal Arnuk, co-founder of agency broker Themis Trading, 2021

The following chapter is based on Russ (2022). It focuses on the second recent observation related to the rising importance of non-fundamental information in financial markets: PFOF. As outlined in the Introductory Chapter, the surge in retail investing has significantly contributed to a boom in the U.S. online brokerage sector. Increased competition between the major online brokers, moreover, resulted in the successive elimination of trading fees for private investors. In search of alternative sources of revenue, *Robinhood*, *Charles Schwab*, *TD Ameritrade*, *E*TRADE*, and others amplified the use of PFOF. As seen in Figure 1.4, the joint PFOF-related revenue of the four named online brokers grew immensely in 2020 and 2021, also compared to the year before. Thus, the recent boom in retail investing has led not only to significant growth in the online brokerage sector, but also to an increase in available information about retail order flow in financial markets (see also SEC chairman Gary Gensler's quote on p. 8 in the Introductory Chapter).

In particular, due to PFOF, financial markets are populated by *different* professional traders who observe *different* components of the whole order flow linked to retail investing in the market. The aim of this chapter is to analyze the interactions that emerge among these diversely informed traders and the ensuing consequences for important properties of financial markets such as price efficiency and adverse selection.

Interpreting retail investors as noise traders makes the competitive noisy REE framework a suitable framework for the analysis of the expounded research task. Nevertheless, there is one important modification that we need to make to the standard framework in order to investigate the interactions between the different professional traders engaged in PFOF. The vast majority of the existing literature assumes noise to be one-dimensional. The demand coming from noise traders is summed up in a single random variable (see also Chapter 3). This common assumption, however, does not account for the fact that there can exist sophisticated traders in the market who possess unbiased knowledge of a *part* of the whole demand stemming from noise traders. That is, rather than gleaning information about the whole order flow linked to noise trading, these investors *precisely* know the demand of *some* noise traders in the market. Of course, there exist other noise traders, whose orders they do not observe. Thus, we need to extend the competitive noisy REE framework to the case where noise is not one- but *multidimensional*. This means that the market price is affected by more than one noise factor in equilibrium.

This chapter analyzes the cases with two- and three-dimensional noise. The model with two-dimensional noise and two different groups of noise-informed traders reveals several types of complementarities in traders' interactions that cannot be studied in the classical one-dimensional setup in the spirit of Ganguli and Yang (2009). Additionally, it highlights several important differences compared to a setup with two-dimensional fundamentals à la Goldstein and Yang (2015) (henceforth: GY 2015). At the trading stage, an inference augmentation effect leads to complementarities in trading against *different* types of noise. In GY 2015, in sharp contrast, a similar effect favors strategic substitutabilities in trading on different fundamentals. At the information acquisition stage, acquiring information about the same noise component can be a complement, whereas acquiring information about the same fundamental is unequivocally a substitute, even if fundamentals are multidimensional (see GY 2015). The two-dimensional noise setup further allows us to analyze the strategic interactions in the acquisition of information about *different* noise components. Thus, multidimensionality of noise permits to assess whether cross-complementarities or cross-substitutabilities in non-fundamental information acquisition exist, which is particularly interesting in the light of PFOF. As we show, this new type of interaction can also be characterized by complementarities.

The newly identified strong complementarity in trading against different types of noise can lead to multiple equilibria in the financial market, which exhibit, if noise is two-dimensional, similar properties to those of Ganguli and Yang (2009). If noise is three-dimensional, by contrast, some new equilibrium properties arise that have not been recognized by the literature so far. Additionally, the three-dimensional noise model uncovers a complementarity in non-fundamental information acquisition that can exist even if equilibrium is unique. If noise is three-dimensional but non-fundamental information only two-dimensional, equilibrium is unique. Nevertheless, acquiring information about *different* noise components can still be a complement. This insight sheds new light on the relationship between non-fundamental information and multiple equilibria in generating complementarities. It shows that complementarities in the acquisition of specific types of non-fundamental information can also exist in the absence of multiple equilibria.

Perhaps most importantly, the three-dimensional noise model uncovers a negative correlation between the dimensionality of noise and the severity of adverse selection in financial markets. In Ganguli and Yang (2009) and also in the two-dimensional noise setup, strong informed trading intensifies the adverse selection problem in financial markets, which can lead to a market breakdown. Interestingly, in the three-dimensional noise case, a market breakdown is less likely to occur than in the two-dimensional case. The higher the dimensionality of noise, the smaller the informational advantage obtained from observing a single noise component. Thus, adverse selection is weaker if noise is characterized by a high dimensionality. This mitigates the possibility of a market breakdown.

As in GS 1980 and Diamond and Verrecchia (1981), we consider a static competitive economy. In such an environment, agents use non-fundamental information to extract noise from the market price and gain a more precise signal about fundamentals out of it. Whenever non-fundamentally informed traders observe a high noise trader demand, they expect the price to be noisy and, thus, fundamentals to be low, making them reduce their demand. If they observe low noise trader demand, fundamentals are expected to be high, and they raise their demand. Hence, rational agents follow a contrarian strategy with respect to non-fundamental information. They trade against noise traders and, therefore, mitigate noise traders' influence on the price relative to fundamentals. This is why non-fundamental information unambiguously raises price efficiency in a static setup (see also Chapter 3).

At the trading stage, more aggressive trading against noise by one group encourages other groups to do the same. This is due to an inference augmentation effect. More aggressive trading against one type of noise makes the market price react less strongly to it relative to fundamentals. Hence, all rational traders that do not observe this specific type of noise benefit from a more informative price signal. As a consequence, they trade more aggressively on this signal. Since all noise-informed agents use their non-fundamental information jointly with the market price to infer information about fundamentals, more aggressive trading on their price signals implies more aggressive trading *against* the types of noise they observe.

The results on complementarities at the information acquisition stage are driven by the fact that a rise in the mass of one noise-informed group affects not only how this group but also how other non-fundamentally informed groups trade against the types of noise that they know. Thus, more non-fundamentally informed traders change not only price efficiency as a whole but also the residual uncertainty about fundamentals each specific noise-informed group faces (which crucially depends on how aggressively other noise-informed groups trade against the observed types of noise).

The theoretical results yield three implications for the usage of non-fundamental information obtained through PFOF in financial markets. First, since noise-informed agents trade against noise and their interactions are characterized by complementarities, PFOF should be conducive to price efficiency. Second, complementarities in acquiring information about *different* noise components predict an increase in the amount of non-fundamental information obtained through PFOF in the market. Third, as higher dimensionality of noise weakens adverse selection and mitigates the possibility of a market breakdown, information about retail investor demand gained by engaging in PFOF should be sufficiently dispersed among professional traders. These three implications will be discussed in more detail in Section 4.6.

Our model results relate to three strands of the theoretical literature. The first strand deals with the effects of non-fundamental information in the competitive noisy REE framework. In the relevant contributions, noise is either one-dimensional (e.g., Ganguli and Yang, 2009; Manzano and Vives, 2011; Marmora and Rytchkov, 2018; Farboodi and Veldkamp, 2020) or two-dimensional (e.g., Gennotte and Leland, 1990, Cespa and Vives, 2012, 2015; Avdis, 2016), whereas non-fundamental information is always one-dimensional. This chapter, by contrast, considers the case where both noise and non-fundamental information are *multidimensional*. As already outlined, this yields several new insights that cannot be obtained in a one-dimensional setup.

Second, this chapter contributes to the strand of the theoretical literature that deals with adverse selection in financial markets and potential market breakdown. A common result in the relevant settings is that precise fundamental information obtained by insiders can lead to a market collapse (e.g., Bhattacharya and Spiegel, 1991; Spiegel and Subrahmanyam, 1992; Bhattacharya et al., 1995; Vayanos and Wang, 2009). In Medrano and Vives (2004), the probability of a market breakdown additionally rises as informed investors become more risk-tolerant. Similar to the cited literature, adverse selection intensifies in the models of this chapter as informed trading becomes more pronounced, which can produce a market failure. The novel contribution lies in uncovering the negative relationship between the intensity of adverse selection and the dimensionality of noise. This points to the important fact that the severity of adverse selection and, thus, the probability of a market breakdown are significantly reduced as the dimensionality of noise rises.

Third, our results relate to the literature on complementarities in traders' interactions in a competitive market environment. Complementarities in fundamental information acquisition can, e.g., occur when (i) some traders receive information earlier than other ones (Hirshleifer et al., 1994), (ii) information costs are endogenous (Veldkamp, 2006), (iii) traders derive utility from comparing their wealth to the average wealth in the economy (Garcia and Strobl, 2011), (iv) agents' investment opportunities differ (Goldstein et al., 2014), (v) the noisy asset supply is correlated across periods (Avdis, 2016), or (vi) traders are characterized by different private evaluations regarding the value of an asset (Rahi and Zigrand, 2018).

In a setup closely related to ours, Ganguli and Yang (2009) show in an environment characterized by one-dimensional noise that the existence of private non-fundamental information can lead to complementarities in the acquisition of fundamental information. The reason for this is that in the presence of private non-fundamental, information more fundamentally informed trading can make the price *less* informative about fundamentals, increasing the incentive for others to acquire fundamental information. Moreover, the authors demonstrate that acquiring a fixed bundle of private fundamental and private non-fundamental information can be a complement.

As already pointed out, the models of the present chapter identify new types of complementarities that cannot be analyzed in the setup of Ganguli and Yang (2009), viz., complementarities in trading against *different* types of noise and in acquiring information about *different* noise components. Additionally, the three-dimensional noise setup points to new equilibrium properties. In Ganguli and Yang (2009), a rise in the mass of informed traders unequivocally increases efficiency in one equilibrium, while decreasing efficiency in the other one. In the three-dimensional noise model, by contrast, more informed traders can raise efficiency in *both* equilibria of the model. Furthermore, acquiring information about *different* noise components can still be a complement even if equilibrium is unique. In Ganguli and Yang (2009) as well as in Manzano and Vives (2011), complementarities in information acquisition and multiple equilibria are closely linked with each other. The three-dimensional noise model, by contrast, points to a type of non-fundamental information whose acquisition can be characterized by complementarities even in the absence of equilibrium multiplicity.

In another related paper, GY 2015 extend the seminal setup of GS 1980 by modeling different rational traders that are informed about different fundamentals, which jointly determine the "fair" value of a stock. The authors show that different agents' trades on different fundamentals can be complements due to an uncertainty reduction effect. GY 2015 additionally identify an inference augmentation effect, which favors strategic substitutability in trading on different fundamentals. The two effects will be explained in more detail in Section 4.3.

When concerning multidimensional *non*-fundamental information, we also identify an inference augmentation effect that, however, works in the opposite direction and induces *complementarities* in trading against different types of noise. While the inference augmentation effect prevents equilibrium multiplicity in GY 2015 by favoring substitutabilities in trading, it is responsible for generating multiple equilibria in our models. Moreover, although fundamental information is multidimensional, acquiring information about the same fundamental is always a substitute in GY 2015. However, the authors prove that acquiring information about different fundamentals can be a complement. As already mentioned, both acquiring information about the same and about different noise components can be characterized by complementarities.

The remainder of this chapter is organized as follows: Section 4.1 describes the model with two-dimensional noise. Section 4.2 derives its equilibrium in the financial market. Section 4.3 focuses on traders' interactions at the trading stage. In Section 4.4, we derive the equilibrium at the information acquisition stage and explore the respective interactions. Section 4.5 analyzes the model with three-dimensional noise. On the basis of the theoretical results, Section 4.6 discusses some implications for the increased usage of PFOF in financial markets.

4.1 Model Assumptions

The financial market consists of one riskless asset and one risky asset. The riskless asset (i.e., a bond) is in unlimited supply and serves as a numeraire in the market. Its safe return is normalized to zero. The risky asset (i.e., a stock) is in zero net supply and is traded at market price P at date 1. At date 2, it pays off its fundamental value $\theta \sim N(0, \tau_\theta^{-1})$. There are six different types of traders in the financial market, two of which stand for noise traders with exogenous demands $x_1 \sim N(0, \tau_x^{-1})$ and $x_2 \sim N(0, \tau_x^{-1})$, respectively.[1] Moreover, there exist two sets of non-fundamentally informed agents indexed by the intervals $[0, \lambda_1]$ and $[0, \lambda_2]$. Each trader $n_1 \in [0, \lambda_1]$ observes x_1. Each trader $n_2 \in [0, \lambda_2]$ knows x_2. Additionally, there is a continuum of fundamentally informed traders indexed by the interval $[0, 1]$.[2] Each trader $f \in [0, 1]$ observes a private signal $s_f = \theta + \epsilon_f$, where $\epsilon_f \sim$ i.i.d. $N(0, \tau_\epsilon^{-1})$. There is also a continuum of uninformed but rational traders indexed by the interval $[0, \lambda_u]$. Each trader $u \in [0, \lambda_u]$ gathers neither fundamental nor non-fundamental information. Since the market is competitive, all rational agents are price takers and, therefore, (additionally) observe the market price.

For $k = n_1, n_2, f, u$, agent k's final wealth is given by $\pi_k = (\theta - P)D_k$, where D_k stands for agent k's demand for the risky asset. Without loss of generality, we normalize agents' initial wealth to zero. All rational traders are characterized by a CARA utility function, $U(\pi_k) = -\exp(-\gamma\pi_k)$. The parameter γ (> 0) measures agents' common degree of risk aversion. The random variables θ, x_1, x_2, and ϵ_f are jointly normally distributed and pairwise independent for all $f \in [0, 1]$.

4.2 Equilibrium Determination

The market price is assumed to be linear in θ, x_1, and x_2:

$$P = a_\theta\,\theta + a_1\,x_1 + a_2\,x_2, \tag{4.1}$$

for constants a_θ, a_1, and a_2. Analogous to Chapter 3, price function (4.1) and rational traders' asset demands D_k $(k = n_1, n_2, f, u; n_1 \in [0, \lambda_1], n_2 \in [0, \lambda_2], f \in [0, 1], u \in [0, \lambda_u])$ are a linear REE if D_k maximizes agent k's conditional expected utility and the asset market clears. From the results of Section 3.1, we can immediately infer that maximizing agent k's CARA utility function yields

$$D_k = \frac{\mathrm{E}(\theta\,|\,\mathcal{F}_k) - P}{\gamma \mathrm{Var}(\theta\,|\,\mathcal{F}_k)}, \tag{4.2}$$

where $\mathcal{F}_k$ stands for agent k's information set. It follows that $\mathcal{F}_{n_1} = (P, x_1)$, $\mathcal{F}_{n_2} = (P, x_2)$, $\mathcal{F}_f = (P, s_f)$, and $\mathcal{F}_u = (P)$. A non-fundamentally informed agent uses her knowledge about noise trader demand to generate a more precise signal about the fundamental asset value out of the market price. Conditional on x_i, price function (4.1) turns into the following signal about θ, which an x_i-informed trader uses to update her prior beliefs:

$$P^*_{n_i} \equiv \frac{P - a_i\,x_i}{a_\theta} = \theta + \frac{a_j}{a_\theta}\,x_j, \qquad for\;\; i, j = 1, 2,\, i \neq j. \tag{4.3}$$

Define $\beta_1 \equiv a_\theta/a_1$ and $\beta_2 \equiv a_\theta/a_2$. Then, for the x_i-informed trader, the market price is a signal about θ with precision $\beta_j^2\tau_x$ (i.e., $\mathrm{Var}^{-1}(P^*_{n_i}\,|\,\theta) = \beta_j^2\tau_x$). From (4.3), we can deduce that a rise in x_i reduces an x_i-informed trader's demand. As a noise-informed agent uses her knowledge about noise trader demand to gain a more precise signal about θ out of the market price, the signal $P^*_{n_i}$ ceteris paribus indicates a lower value of the risky fundamental asset value if x_i increases. Due to a lower expected fundamental value, a noise-informed agent decreases her demand. Thus, as in the static model of Section 3.1, a rational trader follows a contrarian strategy with respect to her information about noise.

For fundamentally informed and uninformed, rational agents, observing the price is informationally equivalent to observing

$$P^*_{f/u} \equiv \frac{P}{a_\theta} = \theta + \frac{a_1\,x_1 + a_2\,x_2}{a_\theta}. \tag{4.4}$$

Hence, $P^*_{f/u}$ is a signal about θ with precision $\tau_x/(1/\beta_1^2 + 1/\beta_2^2)$. Without non-fundamental information, the signal about θ generated by disentangling the information contained in the market price clearly has a lower precision.

Using (4.3), (4.4), and agents' private fundamental signals, the first two conditional moments of θ can be determined for all types of rational traders by using the projection theorem. Then, the price P is derived by clearing the asset market:

$$\int_0^1 D_f\,df + \int_0^{\lambda_1} D_{n_1}\,dn_1 + \int_0^{\lambda_2} D_{n_2}\,dn_2 + \int_0^{\lambda_u} D_u\,du + x_1 + x_2 = 0. \tag{4.5}$$

By plugging rational agents' demand functions from (4.2) into (4.5), we can solve for P and show that it is indeed determined by a linear function of θ, x_1, and x_2, as conjectured in (4.1). After invoking rational expectations, we obtain the coefficients of price function (4.1) in the linear REE:

Proposition 4.1. *If $\Delta_{\beta_i} < 0$ (resp., $\Delta_{\beta_i} = 0$), there exist(s) two (resp., one) linear REE, in which*

$$a_\theta = \frac{\beta_1^4\lambda_2\tau_x + \beta_2^2\left(\tau_\epsilon + \lambda_1\beta_2^2\tau_x\right) + \beta_1^2\left(\tau_\epsilon + \beta_2^2\tau_x\,\omega\right)}{\beta_1^4\lambda_2\tau_x + \beta_2^2\left(\tau_\epsilon + \lambda_1\beta_2^2\tau_x + \tau_\theta\,\omega\right) + \beta_1^2\left[\tau_\epsilon + \left(\beta_2^2\tau_x + \tau_\theta\right)\omega\right]},$$

$$a_i = (1/\beta_i)\,a_\theta,\;\; for\;\; i = 1, 2,$$

where

$$\omega \equiv 1 + \lambda_1 + \lambda_2 + \lambda_u,$$

and β_i is given by

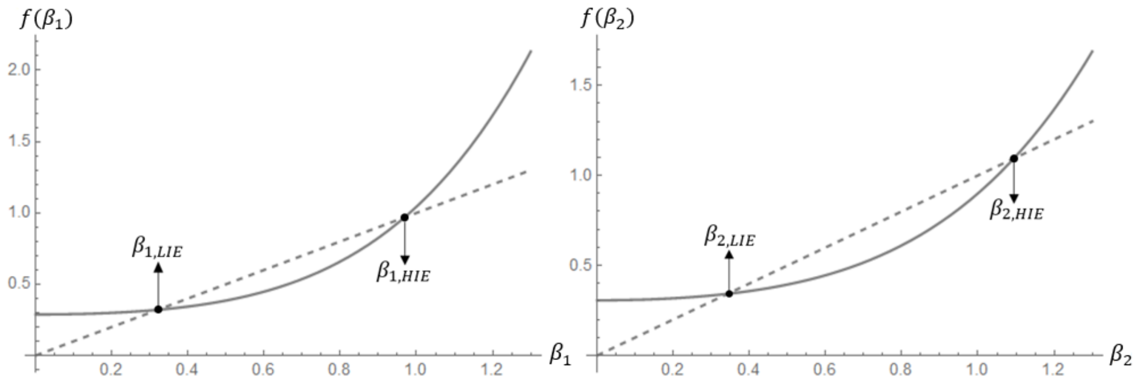
$$\beta_i = \frac{\tau_\epsilon + \lambda_i\beta_j^2\tau_x}{\gamma}, \qquad for\;\; i, j, = 1, 2,\;\; j \neq i.$$

The proof and the definition of Δ_{β_i} can be found in Appendix A. According to Proposition 4.1, there are, apart from one combination of the exogenous model parameters that yields $\Delta_{\beta_i} = 0$, two linear REE if an equilibrium exists. The number of equilibria is pinned down by the number of solutions for β_1 and β_2. The two symmetric equations in Proposition 4.1 that determine β_1 and β_2 can be further developed as follows (see the proof of Proposition 4.1 in Appendix A):

$$\beta_i = f(\beta_i) \equiv \frac{\lambda_i\lambda_j^2\tau_x^3\beta_i^4 + 2\lambda_i\lambda_j\tau_x^2\tau_\epsilon\beta_i^2 + \tau_\epsilon(\lambda_i\tau_x\tau_\epsilon + \gamma^2)}{\gamma^3},\;\; for\;\; i, j = 1, 2,\, i \neq j. \tag{4.6}$$

The equations contained in (4.6) stand for the fixed-point problems that solve for β_1 and β_2 in equilibrium. Since their solutions are hardly analytically tractable, we illustrate them by using a numerical example. Figure 4.1 numerically depicts the mapping of $f(\beta_1)$ with β_1 and $f(\beta_2)$ with β_2. The intersections of the solid curves with the dashed 45°-line represent the equilibrium values of β_1 and β_2. From

Figure 4.1: Equilibrium with two-dimensional noise



Parameters: $\gamma = 2,\ \tau_\epsilon = 0.5,\ \tau_x = 1,\ \lambda_1 = 1.2,\ \lambda_2 = 1.8$

$\beta_1 \equiv a_\theta/a_1$ and $\beta_2 \equiv a_\theta/a_2$, we can conclude that β_1 and β_1 measure how strongly the market price reacts to changes in the fundamental asset value relative to changes in the respective noise shock. Since high (resp., low) values of β_1 and β_2 imply that the market price is mainly driven by fundamentals (resp., by noise), we refer to the first intersection in the two graphs in Figure 4.1 as the low information equilibrium (LIE) and to the second intersection as the high information equilibrium (HIE). Since the numerical example assumes that $\lambda_1 < \lambda_2$, it follows that $\beta_1 < \beta_2$ in both equilibria.

4.3 Interactions at the Trading Stage

Having derived the linear REE, this section turns to the diversely noise-informed groups' interactions at the trading stage. We are particularly interested in how their trades against the two different noise trader demands are connected. Moreover, we analyze the impact of their interactions on price efficiency and assess what effects a rise in the mass of non-fundamentally informed traders exerts on the equilibria of the model.

4.3.1 Trading Intensities Against Noise

Analogous to Chapter 3, the trading intensities against noise indicate the degree of aggressiveness with which the noise-informed traders trade against the observed noise trader demand. Hence, they measure how much noise the rational, non-fundamentally informed agents actually counteract. Since there are two different groups of noise-informed traders, there are two trading intensities against noise. From a noise-informed agent's demand function (see (A17) in Appendix A), we obtain

$$I_{x_i} \equiv \int_0^{\lambda_i} \left| \frac{\partial D_{n_i}}{\partial x_i} \right| dn_i = \lambda_i \frac{\beta_j^2 \tau_x}{\gamma \beta_i}, \qquad for \;\; i, j = 1, 2, \; i \neq j. \tag{4.7}$$

Note that both trading intensities are a function of agents' conjectured values of β_1 and β_2, as rational traders use price function (4.1) to update their beliefs about θ (which then influence their demand for the risky asset). By rewriting rational traders' aggregate demand functions in a general way, we are able to find a connection between the implied values of β_1 and β_2, which follow from invoking rational expectations, and the trading intensities I_{x_1} and I_{x_2}:

$$\int_0^1 D_f \, df = I_f \, \theta + I_{P,f} \, P,$$

$$\int_0^{\lambda_i} D_{n_i} \, dn_i = I_{P,n_i} \, P - I_{x_i} \, x_i, \;\; for \;\; i = 1, 2,$$

$$\int_0^{\lambda_u} D_u \, du = I_{P,u} \, P,$$

where $I_f \equiv \int_0^1 (\partial D_f / \partial s_f) \, df$, $I_{P,f} \equiv \int_0^1 (\partial D_f / \partial P) \, df$, $I_{P,n_i} \equiv \int_0^{\lambda_i} (\partial D_{n_i} / \partial P) \, dn_i$, and $I_{P,u} \equiv \int_0^{\lambda_u} (\partial D_u / \partial P) \, du$. From (A18), it follows that $I_f = \tau_\epsilon / \gamma$. Thus, by using market-clearing condition (4.5), the implied values of the three coefficients in price function (4.1) can be written as

$$a_\theta = \frac{\tau_\epsilon}{-\gamma (I_{P,f} + I_{P,n_1} + I_{P,n_2} + I_{P,u})},$$

$$a_i = \frac{1 - I_{x_i}}{-(I_{P,f} + I_{P,n_1} + I_{P,n_2} + I_{P,u})}, \qquad for \quad i = 1, 2.$$

Hence, the implied value of β_i ($\equiv a_\theta / a_i$) is

$$\beta_i = \frac{\tau_\epsilon}{\gamma \, (1 - I_{x_i})}, \qquad for \;\; i = 1, 2. \tag{4.8}$$

By (4.8), we see that the trading intensities against noise are positively connected to the implied values of the coefficient ratios β_1 and β_2. This is intuitive, as more aggressive trading against noise makes the price react less strongly to noise relative to fundamentals, which is equivalent to a rise in β_1 and β_2. From (4.8), we can further conclude that $I_{x_i} \in [0, 1)$. This value range is deduced from the fact that β_1 and β_2 are always positive whenever a linear REE exists (see Proposition 4.1).

4.3.2 Complementarities in Trading

One central question of Section 4.3 is how the two groups of non-fundamentally informed investors interact in the financial market. That is, we are interested in how the trading intensities against noise are related. From (4.7), we already know that

$$I_{x_i} = \lambda_i \frac{\beta_j^2 \tau_x}{\gamma \beta_i}, \quad for\ \ i,j = 1,2,\ \ j \neq i.$$

Substituting for β_1 and β_2 from (4.8) and rearranging terms yields

$$I_{x_i} = \lambda_i \tau_x \frac{\left[\dfrac{\tau_\epsilon}{\gamma\left(1-I_{x_j}\right)}\right]^2}{\gamma \dfrac{\tau_\epsilon}{\gamma\left(1-I_{x_i}\right)}} = \frac{\lambda_i \tau_x \tau_\epsilon \left(1-I_{x_i}\right)}{\gamma^2 (1-I_{x_j})^2}$$

$$\Leftrightarrow I_{x_i}\left[\gamma^2(1-I_{x_j})^2 + \lambda_i \tau_x \tau_\epsilon\right] = \lambda_i \tau_x \tau_\epsilon$$

$$\Leftrightarrow I_{x_i} = \frac{\lambda_i \tau_x \tau_\epsilon}{\gamma^2(1-I_{x_j})^2 + \lambda_i \tau_x \tau_\epsilon}, \quad for\ \ i,j = 1,2,\ \ j \neq i. \tag{4.9}$$

By inspecting (4.9), the next proposition immediately follows.

Proposition 4.2. *Trading against x_i is a complement to trading against x_j (i.e., $\partial I_{x_i}/\partial I_{x_j} > 0$).*

The clear complementarity occurs due to an inference augmentation effect. A higher I_{x_j} means that more noise coming from the x_j-noise traders is offset. This benefits rational traders that do not know x_j, as they are now able to obtain a more precise signal about θ from disentangling the information contained in the market price. As a consequence, they trade more aggressively on the signal about θ generated out of the market price. Since the x_i-informed traders exclusively use their non-fundamental information to extract noise from the market price, more aggressive trading on their obtained price signal entails more aggressive trading against x_i (i.e., a higher I_{x_i}).

The identified inference augmentation effect works in the opposite direction compared to GY 2015, who deal with multidimensional *fundamental* information. In their model with two independent fundamental components, rational traders use their non-noisy information about one of the two fundamental components in two opposite ways. For one thing, they use it directly to predict fundamentals. For another, they use it together with the price to infer information about the other, unknown fundamental component. The latter function is similar to that of our model, in which agents use their *non*-fundamental information jointly with the price to infer information about fundamentals. The former function finds no counterpart in our model.

In GY 2015, as agents observe a higher value of the fundamental component they know, their demand for the risky asset is affected in two ways. On the one hand, a higher fundamental component predicts a higher fundamental value in total, thereby leading to an increase in demand. On the other hand, holding the price constant, a higher fundamental component predicts a lower value of the second, unknown fundamental component, which makes agents reduce their demand for the risky asset. In our model, by contrast, noise-informed traders' demand is affected in *one* clear way when the observed noise trader demand rises. Higher noise trader demand ceteris paribus predicts lower fundamentals, which makes noise-informed agents decrease their demand for the risky asset.

In the setup of GY 2015, as the trading intensity of one group rises, the two competing effects intensify. More aggressive trading on a fundamental component reduces the residual uncertainty about fundamentals the other group faces. This makes the other group trade more aggressively on their fundamental information too. GY 2015 call this the "uncertainty reduction effect," which favors strategic complementarity in trading. Such an effect cannot be found in our model, as non-fundamentally informed agents' trading intensities are not directly affected by the residual uncertainty about fundamentals (cf. (A17) in Appendix A and also equation (4.7)).[3] Additionally, since more aggressive trading on a fundamental component raises the informativeness of the market price about this component, the other group trades more aggressively on the signal gained from observing the market price. However, more aggressive trading on the price signal implies more aggressive trading *against* the fundamental component the other group knows. This inference augmentation effect contrasts with the uncertainty reduction effect and favors strategic substitutability in trading on different fundamentals. The resulting type of interaction is ambiguous in the setup of GY 2015 (see GY 2015, Subsection II.B).

In the present model, as I_{x_j} rises, x_i-informed traders' price signal predicts fundamentals more accurately. This makes them trade more aggressively on this signal, which entails more aggressive trading against x_i. This inference augmentation effect, in strong contrast to GY 2015, favors strategic complementarity in trading. Since this is the only effect present, trading against different types of noise is unambiguously a complement.

4.3.3 An Explanation for Equilibrium Multiplicity

The derived complementarity in trading against different types of noise is, moreover, the driving mechanism that gives rise to equilibrium multiplicity. In REE setups, equilibrium multiplicity depends crucially on rational agents' conjecture about the coefficients of the price function. If agents conjecture high or low coefficients and these different conjectures are verified in a respective equilibrium, multiple equilibria are possible. Put differently, whenever a change in agents' conjecture triggers a self-fulfilling prophecy, more than one REE can exist (see, e.g., Ganguli and Yang, 2009).

Since agents' conjecture about the values of β_1 and β_2 influences how well the market price reflects the fundamental asset value, it also affects how aggressively agents trade against noise. Thus, the conjectured values of β_1 and β_2 shape the trading intensities, as shown in (4.7). If rational agents, e.g., conjecture a high β_1, the market price becomes a precise signal about θ for the x_2-informed traders. This makes them trade aggressively against noise (i.e., I_{x_2} is high). Recalling the results contained in Proposition 4.1, I_{x_2} can be expressed in terms of the conjectured value of β_1 as follows:

$$I_{x_2} = \frac{\lambda_2 \beta_1^2 \tau_x}{\gamma \beta_2} = \frac{\lambda_2 \beta_1^2 \tau_x}{\gamma \dfrac{\tau_\epsilon + \lambda_2 \beta_1^2 \tau_x}{\gamma}} = \frac{\lambda_2 \beta_1^2 \tau_x}{\tau_\epsilon + \lambda_2 \beta_1^2 \tau_x}.$$

Thus,

$$\frac{\partial I_{x_2}}{\partial \beta_1} = \frac{(\tau_\epsilon + \lambda_2 \beta_1^2 \tau_x) 2\lambda_2 \beta_1 \tau_x - 2\lambda_2^2 \beta_1^3 \tau_x^2}{(\tau_\epsilon + \lambda_2 \beta_1^2 \tau_x)^2} = \frac{2\lambda_2 \beta_1 \tau_x \tau_\epsilon}{(\tau_\epsilon + \lambda_2 \beta_1^2 \tau_x)^2} > 0.$$

Hence, a high conjectured value of β_1 clearly translates into a high I_{x_2}. By (4.9), a high I_{x_2} leads to a high I_{x_1}, due to the explained complementarity. A high I_{x_1}, eventually, translates into a high implied value of β_1 (see (4.8)). Hence, rational agents' initial conjecture is verified in equilibrium, thereby leading to the existence of the HIE. The symmetric argument applies to the conjecture about a high β_2 and its verification. By contrast, the conjecture about low values of β_1 and β_2 and their verification justify the existence of the LIE. Without the clear complementarity in trading against different types of noise, it would be unclear whether agents' initial conjecture about high or low values of β_1 and β_2 could be verified in equilibrium. In other words, as one group of noise-informed traders vanishes, equilibrium multiplicity vanishes too. Formally, this can be seen by inspecting (4.6). If $\lambda_1 = 0$ or $\lambda_2 = 0$, the solutions for β_1 and β_2 are unique. Multidimensional non-fundamental information (i.e., $\lambda_1 > 0$, $\lambda_2 > 0$) is, thus, a necessary condition for equilibrium multiplicity.

4.3.4 Price Efficiency

Next, we analyze the impact of noise-informed traders' interactions on price efficiency. As already outlined, β_1 and β_2 can be seen as proxies for the efficiency of the market price. In equilibrium, the information that β_i conveys can be split up into two parts as follows (see Proposition 4.1):

$$\beta_i = \underbrace{\frac{\tau_\epsilon}{\gamma}}_{\substack{fundamental \\ information}} + \underbrace{\frac{\lambda_i\,\beta_j^2\,\tau_x}{\gamma}}_{\substack{non-fundamental \\ information}}, \qquad for\ \ i,j=1,2,\ i\neq j.$$

The first component of β_i represents rational traders' trading intensity on private fundamental information and indicates how much direct fundamental information β_i conveys (note also the analogy to ρ from Section 3.1). If fundamentally informed agents trade more aggressively on the private signals about fundamentals, more fundamental information is factored into the market price. This has a positive effect on β_i. In standard static REE models in the spirit of GS 1980, fundamentally informed traders' trading intensity fully determines the value of the equivalent of β_i (see, e.g., GS 1980, p. 397). In the present model, however, there is a second component that does not appear in the standard models. This component pins down how much additional information about fundamentals β_i conveys due to the existence of non-fundamental information. It also shows the crucial connection between the coefficient ratios β_1 and β_2, and, hence, the crucial connection between the two trading intensities I_{x_1} and I_{x_2}. The coefficient ratios β_1 and β_2 are clearly positively connected. Thus, the amount of information β_1 contains depends positively on the amount of information that β_2 contains and vice versa. The positive link between the coefficient ratios gives rise to the explained complementarity in trading against different types of noise.

Consequently, noise-informed agents' interaction at the trading stage benefits price efficiency. As in Chapter 3, we define price efficiency as the inverse of the variance of the fundamental asset value conditional on the market price. Using (4.4) and (4.8), we get

$$\begin{aligned}\frac{1}{\mathrm{Var}(\theta\,|\,P)} &= \tau_\theta + \frac{\tau_x}{\frac{1}{\beta_1^2}+\frac{1}{\beta_2^2}} \\ &= \tau_\theta + \frac{\tau_x\tau_\epsilon^2}{\gamma^2[(1-I_{x_1})^2+(1-I_{x_2})^2]}. \qquad (4.10)\end{aligned}$$

Hence, the total effect of a rise in I_{x_i} on price efficiency is

$$\frac{d[\mathrm{Var}^{-1}(\theta\,|\,P)]}{dI_{x_i}} = \underbrace{\frac{\partial[\mathrm{Var}^{-1}(\theta\,|\,P)]}{\partial I_{x_i}}}_{direct\ effect} + \underbrace{\frac{\partial[\mathrm{Var}^{-1}(\theta\,|\,P)]}{\partial I_{x_j}}\frac{dI_{x_j}}{dI_{x_i}}}_{\substack{complementarity \\ effect}}, \quad for\ \ i,j=1,2,\ i\neq j. \qquad (4.11)$$

By inspecting (4.9) and (4.10), one immediately sees that all derivatives in (4.11) are positive. According to (4.11), the total effect of an increase in I_{x_i} can be split up into two parts. First, as I_{x_i} rises, the x_i-informed agents counteract more noise induced by the x_i-noise traders, raising the quality of the market price as an adequate signal about the fundamental asset value. This fact is represented by the first summand in (4.11). Second, a higher I_{x_i} triggers the derived complementarity in trading against different types of noise. If more noise generated by the x_i-noise traders is counteracted, the x_j-informed traders increase their own trading intensity. A rise in I_{x_i}, thus, leads to a rise in I_{x_j}, which further improves price efficiency. This connection is described by the second summand in (4.11). Hence, a higher trading intensity against noise increases price efficiency through two channels. Due to this positive relationship, price efficiency is, of course, higher in the HIE than in the LIE.

4.3.5 Consequences of a Rise in λ_i in Equilibrium

Lastly, we examine the effects of an increase in the mass of noise-informed traders in equilibrium. On the one hand, we are interested in the influence on the trading intensities. On the other hand, we explore the impact on existence and multiplicity of equilibria in the model. The results are summarized in the next proposition (with the proof given in Appendix A):

Proposition 4.3.

(a) The total effect of a rise in λ_i on the trading intensities is given by

$$\frac{dI_{x_i}}{d\lambda_i} = \Gamma^{-1} \times \frac{\partial I_{x_i}}{\partial \lambda_i}, \tag{4.12}$$

$$\frac{dI_{x_j}}{d\lambda_i} = \Gamma^{-1} \times \frac{\partial I_{x_j}}{\partial I_{x_i}} \frac{\partial I_{x_i}}{\partial \lambda_i}, \tag{4.13}$$

where

$$\Gamma \equiv 1 - 4 I_{x_i} I_{x_j}.$$

(b) In the LIE (resp., HIE), it holds that $\Gamma > 0$ (resp., $\Gamma < 0$).

(c) If $\lambda_i = \tilde{\lambda}_i$ (see (A32) in Appendix A), then

$$\beta_i = \tilde{\beta}_i \equiv \frac{2\lambda_j \tau_\epsilon \tau_x + \sqrt{\lambda_j \tau_\epsilon \tau_x (4\lambda_j \tau_\epsilon \tau_x + 3\gamma^2)}}{3\gamma \lambda_j \tau_x}, \; \textit{for } i, j = 1, 2, \; i \neq j. \tag{4.14}$$

(d) If $\lambda_i > \tilde{\lambda}_i$, there is no equilibrium. If $\lambda_i < \tilde{\lambda}_i$, there are two equilibria.

Since all partial derivatives in (4.12) and (4.13) are clearly positive (see also the proof of Proposition 4.3 in Appendix A), part (a) in Proposition 4.3 states that a rise in λ_i decreases both trading intensities in equilibrium if $\Gamma < 0$. Part (b) shows that this always happens in the HIE, whereas the opposite effect holds true in the LIE. In other words, the direction of influence on the trading intensities is pinned down by the equilibrium rational traders coordinate on. The obtained result is in line with the existing literature on non-fundamental information and equilibrium multiplicity. Ganguli and Yang (2009) derive comparable results. In their specification, a rise in the mass of informed agents increases efficiency in one equilibrium, while decreasing it in the other equilibrium. We identify this feature in the two-dimensional noise model as well.

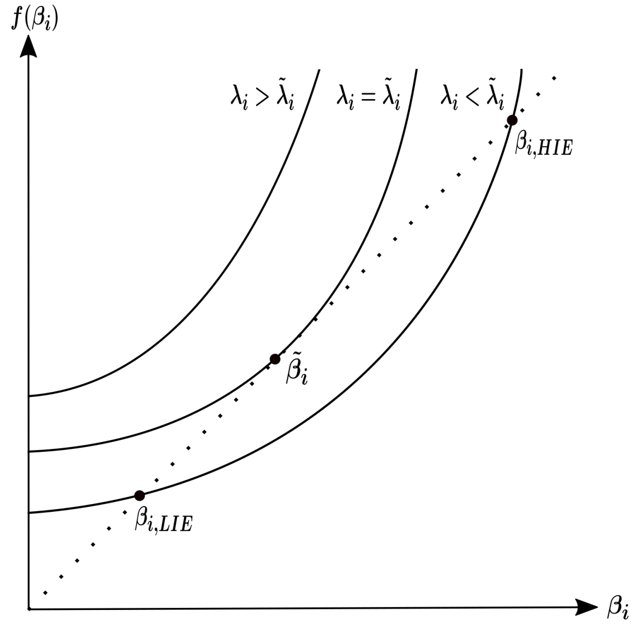
Part (c) in Proposition 4.3 shows that a critical value of λ_i exists that leads to a unique REE. In this special case, the solutions for β_1 and β_2 can be determined in closed form (see (4.14)). According to part (d), there are two equilibria (the LIE and the HIE) if the mass of noise-informed traders is sufficiently small. If the overall mass of noise-informed traders is too large, an equilibrium fails to exist.

Figure 4.2 shows the mapping of $f(\beta_i)$ from (4.6) with β_i for different values of λ_i. For $\lambda_i = \tilde{\lambda}_i$, $f(\beta_i)$ possesses a touch point with the dashed 45°-line at $\beta_i = \tilde{\beta}_i$. Consequently, $\tilde{\beta}_i$ is the unique solution of the fixed-point problem. For $\lambda_i < \tilde{\lambda}_i$, $f(\beta_i)$ has a smaller intercept and a smaller slope than for $\lambda_i = \tilde{\lambda}_i$. The LIE and the HIE arise. If $\lambda_i > \tilde{\lambda}_i$, there is no equilibrium. By carefully inspecting the expression of $f(\beta_i)$ in (4.6), we see that non-existence of equilibrium also occurs for sufficiently large values of λ_j, τ_ϵ, and τ_x and for sufficiently small values of γ.

Strong informed trading expressed by a large mass of noise-informed agents (i.e., a high λ_1 or λ_2) or precise private fundamental signals (i.e., a high τ_ϵ) exacerbates the adverse selection problem in financial markets. Aggressive trading expressed by low risk aversion (i.e., a low γ) has the same effect. The adverse selection problem in financial markets refers to the state that traders are exposed to the risk of potentially trading against other market participants that possess information superior to their own (see, e.g., Medrano and Vives, 2004). If adverse selection happens to be very intense, agents might refrain from participating in the market, thereby producing a market breakdown.

The fact that too much informed trading leads to a market breakdown due to severe adverse selection can also be found in other models related to non-fundamental information (see Ganguli and Yang, 2009; Marmora and Rytchkov, 2018). The novelty in the two-dimensional noise setup compared to Ganguli and Yang (2009) and Marmora and Rytchkov (2018) is that the precision of noise trading (i.e., τ_x) also influences the existence of an equilibrium. This result can be directly linked to the adverse selection problem in financial markets as well. Highly volatile noise

Figure 4.2: Mapping $f(\beta_i)$ with β_i - case distinction



trading (i.e., a low τ_x) alleviates adverse selection. If the impact of noise traders increases, the risk of trading against a better informed agent is mitigated. Instead, it becomes more likely to trade against an uninformed noise trader (see also Vives, 2008, Chapter 4).

4.4 Costly Signals and Information Acquisition

Thus far, observing x_1 and x_2 has not been linked to any cost. In the following, we relax this assumption and turn x_1 and x_2 into costly signals. This allows us to derive an equilibrium at the information acquisition stage with endogenous values of λ_1 and λ_2 (for a given mass of fundamentally informed traders). We additionally analyze the strategic interactions in acquiring non-fundamental information and compare our obtained results to the outcomes of the relevant literature.

4.4.1 Information Acquisition Equilibrium

Information about x_1 and x_2 can now be acquired at costs $c_1 > 0$ and $c_2 > 0$, respectively. For the sake of tractability, each fundamentally uninformed, rational trader is only able to acquire one of the two signals. Throughout the analysis, we assume, similar to GY 2015, that there are always some rational traders that decide to stay uninformed in equilibrium (i.e., $\lambda_u > 0$). This assumption allows us to omit the analysis of corner solutions in which all fundamentally uninformed, rational agents acquire information about x_1 or x_2.[4] As already pointed out by GY (2015,

p. 1740), "[t]he case of $\lambda_u > 0$ is of course empirically relevant, since in reality it is unlikely that every trader is informed." As a consequence, we are interested in the following four outcomes in equilibrium: ($\lambda_1 = \lambda_2 = 0$), ($\lambda_1 > 0$, $\lambda_2 = 0$), ($\lambda_1 = 0$, $\lambda_2 > 0$), and ($\lambda_1 > 0$, $\lambda_2 > 0$).

By comparing the ex-ante expected utility of a noise-informed trader with that of an uninformed, rational trader, we can derive the value of non-fundamental information, which is given in the next proposition (with the proof in Appendix A).

Proposition 4.4. *The value of information about noise is given by*

$$\phi_{x_i}(\beta_1, \beta_2) = \frac{1}{2\gamma} \log \left[\frac{\mathrm{Var}(\theta \,|\, P)}{\mathrm{Var}(\theta \,|\, P, x_i)} \right], \quad for \quad i = 1, 2. \tag{4.15}$$

According to (4.15), the value of non-fundamental information is determined by the ratio between the residual uncertainty about fundamentals that traders face when they only observe the market price and the residual uncertainty when they additionally know x_i. The higher the reduction in residual uncertainty compared to just observing the market price, the higher the value of information about x_i. If information about noise only marginally reduces the uncertainty traders are confronted with, its value is rather small. Note that ϕ_{x_i} depends indirectly on λ_1 and λ_2 via β_1 and β_2. From Proposition 4.1, we know that $\lambda_i = 0$ leads to $\beta_i = \tau_\epsilon/\gamma$ (for $i = 1, 2$). Therefore, if $\lambda_i > 0$, it holds that $\beta_i > \tau_\epsilon/\gamma$.

Definition (information acquisition equilibrium): Let $(\lambda_1^*, \lambda_2^*) \in \mathbb{R}_+^2$ be an *information acquisition equilibrium*. Then,

(i) $\lambda_1^* = \lambda_2^* = 0$ if $\phi_{x_1}(\tau_\epsilon/\gamma, \tau_\epsilon/\gamma) \leq c_1$, $\phi_{x_2}(\tau_\epsilon/\gamma, \tau_\epsilon/\gamma) \leq c_2$,

(ii) $\lambda_1^* > 0$, $\lambda_2^* = 0$ if $\phi_{x_1}(\beta_1, \tau_\epsilon/\gamma) = c_1$, $\phi_{x_2}(\beta_1, \tau_\epsilon/\gamma) \leq c_2$, with $\beta_1 > \tau_\epsilon/\gamma$,

(iii) $\lambda_1^* = 0$, $\lambda_2^* > 0$ if $\phi_{x_1}(\tau_\epsilon/\gamma, \beta_2) \leq c_1$, $\phi_{x_2}(\tau_\epsilon/\gamma, \beta_2) = c_2$, with $\beta_2 > \tau_\epsilon/\gamma$,

(iv) $\lambda_1^* > 0$, $\lambda_2^* > 0$ if $\phi_{x_1}(\beta_1, \beta_2) = c_1$, $\phi_{x_2}(\beta_1, \beta_2) = c_2$, with $\beta_1 > \tau_\epsilon/\gamma$, $\beta_2 > \tau_\epsilon/\gamma$.

In general, a rational agent is willing to acquire information about noise if its cost is not greater than its value. If its cost exactly equals its value, an agent is indifferent between becoming noise-informed and staying uninformed. Since we suppose that there are always some rational agents that decide to remain uninformed, the cost of observing x_i must be equal to its value whenever there are x_i-informed traders in equilibrium. An equilibrium without x_i-informed traders exists if the cost of acquiring information about x_i is equal to or exceeds its value in the situation where no rational trader in the market possesses information about x_i.

The following proposition describes the information acquisition equilibrium in dependence of the cost parameters (with the proof and the exact characterization of $g(c_1)$ given in Appendix A):

Proposition 4.5. *Let*

$$\bar{c} \equiv \frac{1}{2\gamma} \log \left[\frac{2(\gamma^2 \tau_\theta + \tau_\epsilon^2 \tau_x)}{2\gamma^2 \tau_\theta + \tau_\epsilon^2 \tau_x} \right],$$

$$f(c_1) = \frac{1}{2\gamma} \log \left\{ \frac{e^{2\gamma c_1} \left[\left(e^{2\gamma c_1} - 1 \right) \gamma^4 \tau_\theta^2 + \tau_\epsilon^4 \tau_x^2 \right]}{\left(e^{2\gamma c_1} - 1 \right) \left(\gamma^2 \tau_\theta + \tau_\epsilon^2 \tau_x \right)^2} \right\},$$

$$g(c_1) = f^{-1}(c_1) \;\; for \;\; c_1 \leq \frac{1}{2\gamma} \log \left(1 + \frac{\tau_\epsilon^2 \tau_x}{\gamma^2 \tau_\theta} \right),$$

$$i(c_1) = \frac{1}{2\gamma} \log \left(\frac{e^{2\gamma c_1}}{e^{2\gamma c_1} - 1} \right).$$

Then

(a) $\lambda_1^* = \lambda_2^* = 0$ *holds true if* $c_1 \geq \bar{c}$ *and* $c_2 \geq \bar{c}$*.*

(b) $\lambda_1^* > 0$, $\lambda_2^* = 0$ *holds true if* $c_1 < \bar{c}$ *and* $c_2 \geq f(c_1)$*.*

(c) $\lambda_1^* = 0$, $\lambda_2^* > 0$ *holds true if* $c_1 > \bar{c}$ *and* $\bar{c} > c_2 \geq g(c_1)$*.*

(d) $\lambda_1^* > 0$, $\lambda_2^* > 0$ *holds true if*

(i) $c_1 \leq \bar{c}$ *and* $i(c_1) > c_2 > f(c_1)$*;*
(ii) $c_1 > \bar{c}$ *and* $i(c_1) > c_2 > g(c_1)$*.*

(e) *There is no information acquisition equilibrium if*

(i) $c_1 \leq \bar{c}$ *and* $c_2 < f(c_1)$*;*
(ii) $c_1 > \bar{c}$ *and* $c_2 < g(c_1)$*.*

The conditions in Proposition 4.5 specify the value range of c_2 in dependence of c_1 for the respective types of equilibrium. Building on these value ranges, Figure 4.3 illustrates all possible information acquisition equilibria in the space of (c_1, c_2), where $\bar{\bar{c}}$ is the unique solution of $\bar{c} = i(c_1)$. As depicted, there are seven different areas with the following outcomes in equilibrium:

(1) $\lambda_1^* = \lambda_2^* = 0$: $c_1 \geq \bar{c}$ and $c_2 \geq \max\{\bar{c}, i(c_1)\}$,

(2) $\lambda_1^* > 0$, $\lambda_2^* = 0$: $c_1 < \bar{c}$ and $c_2 \geq i(c_1)$,

(3) $\lambda_1^* > 0$, $\lambda_2^* > 0$; $\lambda_1^* > 0$, $\lambda_2^* = 0$: $c_1 < \bar{c}$ and $i(c_1) > c_2 > f(c_1)$,[5]

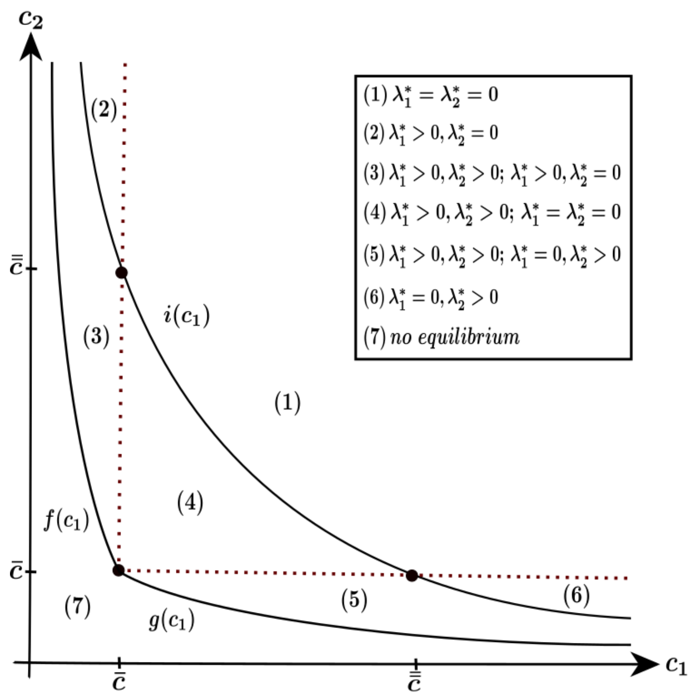
(4) $\lambda_1^* > 0$, $\lambda_2^* > 0$; $\lambda_1^* = \lambda_2^* = 0$: $c_1 = \bar{c}$ and $i(c_1) > c_2 > \bar{c}$; $\bar{\bar{c}} > c_1 > \bar{c}$ and $i(c_1) > c_2 \geq \bar{c}$,[6]

(5) $\lambda_1^* > 0$, $\lambda_2^* > 0$; $\lambda_1^* = 0$, $\lambda_2^* > 0$: $c_1 > \bar{c}$ and $\min\{\bar{c}, i(c_1)\} > c_2 > g(c_1)$,[7]

(6) $\lambda_1^* = 0$, $\lambda_2^* > 0$: $c_1 > \bar{\bar{c}}$ and $\bar{c} > c_2 \geq i(c_1)$,

(7) no equilibrium: $c_1 \leq \bar{c}$ and $c_2 < f(c_1)$; $c_1 > \bar{c}$ and $c_2 < g(c_1)$.

Figure 4.3: Information acquisition equilibrium with two-dimensional noise



In area (1), costs are too high and all agents refrain from acquiring non-fundamental information (i.e., $\lambda_1^* = \lambda_2^* = 0$). In area (2), agents only acquire information about x_1 (i.e., $\lambda_1^* > 0$, $\lambda_2^* = 0$). Areas (3), (4), and (5) define a channel that supports multiple information acquisition equilibria. In these three areas, an equilibrium with both groups of noise-informed traders (i.e., $\lambda_1^* > 0$, $\lambda_2^* > 0$) is always possible. Additionally, there is a second equilibrium, whose type depends on the exact combination of c_1 and c_2. Area (3) (resp., area (5)) also supports an equilibrium of the form $\lambda_1^* > 0$, $\lambda_2^* = 0$ (resp., $\lambda_1^* = 0$, $\lambda_2^* > 0$). In area (4), an equilibrium without non-fundamentally informed traders (i.e., $\lambda_1^* = \lambda_2^* = 0$) is possible. In area (6), equilibrium is unique and of the form $\lambda_1^* = 0$, $\lambda_2^* > 0$.

Surprisingly, an information acquisition equilibrium fails to exist for sufficiently small costs (see area (7)). Intuitively, one would expect an equilibrium with both groups of noise-informed traders in this situation. The explanation for non-existence is the following: in a potential equilibrium of the form $\lambda_1^* > 0$, $\lambda_2^* > 0$, low costs are associated with low values of information about noise. Low values of information about noise, in return, are linked to low values of β_1 and β_2. This is intuitive, as low values of β_1 and β_2 imply an uninformative price. Thus, knowing one of the two noise shocks does not significantly improve the predictive power of the market price with respect to the fundamental asset value. The price still remains a rather noisy signal about θ and the reduction in residual uncertainty about fundamentals is small. As a consequence, the value of information about noise is low. However, if β_1 and β_2 are smaller than τ_ϵ/γ, i.e., if they are smaller than fundamentally informed agents' trading intensity, an equilibrium with $\lambda_1 > 0$, $\lambda_2 > 0$ cannot exist (see also Proposition 4.1). In other words, both β_1 and β_2 can only be influenced by non-fundamental information if they convey more information than potentially contributed by the fundamentally informed traders. For sufficiently small values of c_1 and c_2 and, hence, for sufficiently low values of information about noise, this is not the case, and an information acquisition equilibrium with a positive mass of non-fundamentally informed traders fails to exist.[8]

4.4.2 Interactions at the Information Acquisition Stage

Having derived the information acquisition equilibrium, we turn to the strategic interactions in acquiring non-fundamental information. More specifically, we analyze whether acquiring information about the same noise component and about different noise components is a strategic complement or substitute. If a rise in λ_i increases (resp., decreases) ϕ_{x_i}, acquiring information about the same noise component is said to be a complement (resp., a substitute). In other words, as more traders with information about x_i enter the market, the incentive for other agents to acquire information about x_i rises (resp., shrinks), which is expressed by a higher (resp., lower) value of information about noise. Whenever a rise in λ_j increases (resp., decreases) ϕ_{x_i}, acquiring information about different noise components is a complement (resp., a substitute). That is, as more traders with information about x_j enter the market, the incentive for other traders to acquire information about x_i rises (resp., decreases).

Analogous to GY 2015, the value of information about noise, given in (4.15), can be split up as follows:

$$\begin{aligned}
\phi_{x_i} &= \frac{1}{2\gamma}\log\left[\frac{\mathrm{Var}(\theta\,|\,P)}{\mathrm{Var}(\theta\,|\,P, x_i)}\right] \\
&= \frac{1}{2\gamma}\log\left\{\left[\tau_\theta + \frac{\tau_\epsilon^2\tau_x}{\gamma^2\,(1-I_{x_j})^2}\right]\mathrm{Var}(\theta\,|\,P)\right\} \\
&= \frac{1}{2\gamma}\log\underbrace{\left[\tau_\theta + \frac{\tau_\epsilon^2\tau_x}{\gamma^2\,(1-I_{x_j})^2}\right]}_{\substack{inverse\ of \\ residual\ uncertainty}} - \frac{1}{2\gamma}\log\underbrace{\left[\frac{1}{\mathrm{Var}(\theta\,|\,P)}\right]}_{price\ efficiency}. \qquad (4.16)
\end{aligned}$$

According to (4.16), a change in λ_i or λ_j affects ϕ_{x_i} in two ways. On the one hand, it influences the residual uncertainty about fundamentals an x_i-informed trader faces.

Since an x_i-informed trader's residual uncertainty about fundamentals depends on how aggressively the x_j-informed traders trade against the observed noise trader demand, λ_i and λ_j affect this residual uncertainty by influencing I_{x_j}. A(n) decrease (resp., increase) in the residual uncertainty raises (resp., reduces) the incentive to acquire information about x_i. On the other hand, a rise in the mass of noise-informed agents affects overall price efficiency. The more efficient the market price, the lower the incentive to acquire costly non-fundamental information. Whenever an increase in λ_i or λ_j raises (resp., decreases) price efficiency, agents' incentive to free-ride on the price increases (resp., shrinks).

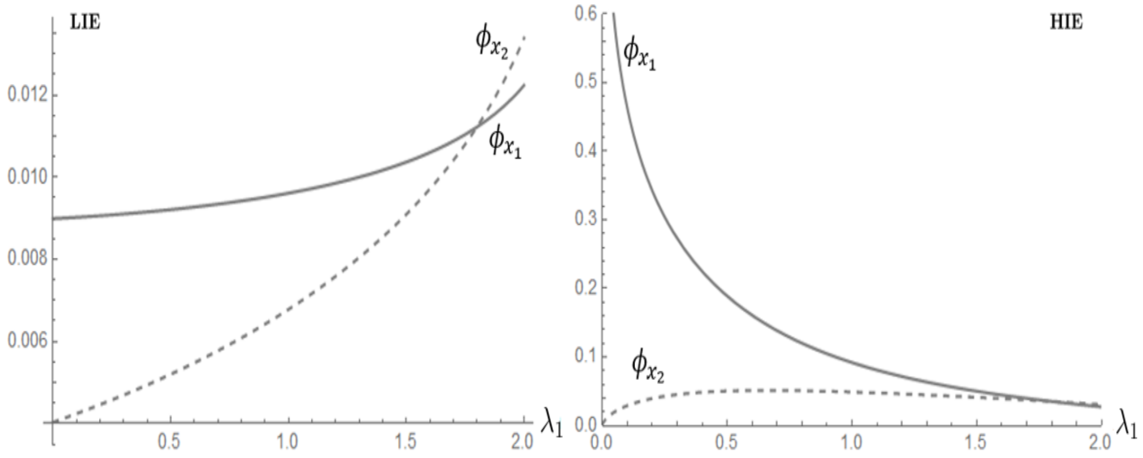
In the LIE, a rise in the mass of noise-informed traders increases both trading intensities (see Proposition 4.3). Thus, both components in (4.16) rise and, at first glance, the resulting effect on the value of information about noise is ambiguous. In the HIE, a rise in the mass of non-fundamentally informed agents leads to a fall in both trading intensities. Hence, the inverse of the residual uncertainty about fundamentals and overall price efficiency decrease, which again has opposite effects. The next proposition summarizes the results concerning agents' interactions at the information acquisition stage (with the proof delegated to Appendix A):

Proposition 4.6. *(a) If $\lambda_j = 0$, acquiring information about the same noise component is always a substitute (i.e., $d\phi_{x_i}/d\lambda_i < 0$). (b) If $\lambda_j > 0$, acquiring information about the same noise component can be a complement in the LIE and in the HIE (i.e., $d\phi_{x_i}/d\lambda_i > 0$). (c) Acquiring information about different noise components can be a complement in the LIE and in the HIE (i.e., $d\phi_{x_i}/d\lambda_j > 0$).*

The exact conditions that ensure the complementarities mentioned in parts (b) and (c) can also be found in Appendix A. Part (a) in Proposition 4.6 states that a rise in λ_i always reduces the value of information about x_i (i.e., $d\phi_{x_i}/d\lambda_i < 0$) if non-fundamental information is one-dimensional (i.e., $\lambda_j = 0$). In other words, acquiring information about the same noise component is unambiguously a substitute whenever there is only one non-fundamentally informed group present. If non-fundamental information is one-dimensional, there are no complementarities in trading against different types of noise and, thus, there is no equilibrium multiplicity. In this reduced setting, a higher λ_i always translates into a higher I_{x_i} (see the proof of Proposition 4.6 in Appendix A). If I_{x_i} rises, price efficiency increases. However, since there is no second group of noise-informed traders present, a rise in I_{x_i} does not induce any complementarity in trading, thereby leaving an x_i-informed trader's residual uncertainty about fundamentals unchanged. In total, this reduces the incentive to acquire information about x_i.

Part (b) in the proposition shows that acquiring information about the same noise component can be a complement in both equilibria if noise and non-fundamental

Figure 4.4: Interactions in information acquisition with two-dimensional noise



Parameters: $\tau_\epsilon = 0.5$, $\tau_x = 1$, $\gamma = 2$, $\tau_\theta = 1.5$, $\lambda_2 = 1.8$

information are two-dimensional. If $\lambda_j > 0$, a rise in the mass of traders with information about x_i affects not only overall price efficiency, but also an x_i-informed trader's residual uncertainty about fundamentals (through changing I_{x_j}). In the LIE (resp., in the HIE), the positive effect on the value of information about noise generated by reducing the residual uncertainty (resp., by decreasing overall price efficiency) can outweigh the negative effect induced by increasing overall price efficiency (resp., by raising the residual uncertainty). This, then, leads to complementarities in the acquisition of information about the same noise component. Part (c) shows that the same holds true for the acquisition of information about different noise components. Both in the LIE and in the HIE, acquiring information about x_i can be a complement to acquiring information about x_j.

For the sake of illustrating the analytical results, Figure 4.4 plots the value of information about x_1 and x_2 in dependence of λ_1 for a given set of parameters. In the numerical example, both acquiring information about the same noise component and about different noise components are a complement in the LIE (i.e., ϕ_{x_1} and ϕ_{x_2} are increasing in λ_1). In the HIE, acquiring information about the same noise component is a substitute, whereas acquiring information about different noise components is a complement for sufficiently small values of λ_1.

Comparison with GS 1980 and GY 2015. Table 4.1 compares the results on the strategic interactions in acquiring non-fundamental information with the relevant outcomes of GS 1980 and GY 2015, who respectively deal with one- and two-dimensional fundamental information in a setting with two-dimensional fundamentals. The third and the fourth row in Table 4.1 display the results contained in Proposition 4.6. In the seminal GS 1980 model, agents only have access to information about one of the

Table 4.1: Comparison of interactions in information acquisition

	Same component	Different components
GS 1980	substitute	/
GY 2015	substitute	substitute or complement
$\lambda_i > 0, \lambda_j = 0$	substitute	/
$\lambda_i > 0, \lambda_j > 0$	substitute or complement	substitute or complement

two risky fundamental components that jointly determine the fair value of the asset. In their setup, acquiring information about this fundamental is always a substitute. In the extension of GY 2015, which entails two groups of rational traders that possess information about one of the two fundamentals each, acquiring information about the same fundamental component is a substitute too. Hence, although fundamental information is two-dimensional, acquiring information about the same fundamental is always a substitute. In GY 2015, as the mass of fundamentally informed traders rises, the increase in overall price efficiency is always greater than the rise in the inverse of the residual uncertainty about fundamentals. Consequently, the incentive for others to acquire information about the same fundamental shrinks.

Considering non-fundamental information, in contrast to GY 2015, the change in the inverse of the residual uncertainty about fundamentals can indeed be more positive than the change in overall price efficiency. If noise and non-fundamental information are two-dimensional, acquiring information about the same noise component can be a complement in both equilibria of the model.

When noise is two-dimensional and non-fundamental information is only one-dimensional, acquiring information about the same noise component is unequivocally a substitute. This finding relates to the result on fundamental information acquisition obtained by GS 1980. GY 2015 uncover a possible complementarity in acquiring information about different fundamental components. Considering two-dimensional noise and non-fundamental information, the analogous complementarity can occur. Acquiring non-fundamental information can be characterized by cross-complementarities in both the LIE and the HIE.

Comparison with Ganguli and Yang (2009). The one-dimensional noise model proposed by Ganguli and Yang (2009) shows that acquiring private fundamental information can be a complement when traders additionally possess private non-fundamental information. Moreover, the authors prove that the *simultaneous* acquisition of private fundamental and private non-fundamental information can be a complement. Our model, by contrast, reveals complementarities in the acquisition of non-fundamental information *only*. Although not carried out, the case of acquiring information about the same noise component could also be analyzed in the model of Ganguli and Yang (2009). However, our model additionally points to the important fact that acquiring information about *different* noise components can be a complement. This kind of complementarity cannot be uncovered using the setup of Ganguli and Yang (2009).

Furthermore, our model shows that the existence of non-fundamental information does not necessarily lead to complementarities in acquiring information. Part (a) in Proposition 4.6 states that acquiring information about the same noise component is unambiguously a substitute if information about the other noise component is absent. Consequently, for complementarities in the acquisition of information about the same noise component to be possible, the dimensionality of non-fundamental information has to be equal to the dimensionality of noise. In Ganguli and Yang (2009), private information about the single noise component makes complementarities in information acquisition possible. In our setup, by contrast, private information about one of the two noise components does not generate complementarities. It is indispensable that information about *both* noise components is available. This important dependency can only be demonstrated in a multidimensional-noise setup.

To sum up, the two-dimensional noise case uncovers new kinds of complementarities in information acquisition (and also at the trading stage) that go beyond the model of Ganguli and Yang (2009). Furthermore, it sheds additional light on when non-fundamental information can generate complementarities in information acquisition and when not. The three-dimensional noise case, which will be analyzed in the next section, reveals further important differences compared to Ganguli and Yang (2009).

4.5 Three-Dimensional Noise

This section extends the model to the case of three-dimensional noise and a third group of noise-informed traders. While the three-dimensional model confirms the central results of the two-dimensional version regarding complementarities at the trading and the information acquisition stage, it additionally yields new insights and new properties that have not been recognized by the relevant literature on non-fundamental information and equilibrium multiplicity so far.

First, the three-dimensional model underscores the importance of a *sufficiently* high dimensionality of non-fundamental information in generating equilibrium multiplicity. Complementarities in trading are only strong enough to generate multiple equilibria if information about all three noise shocks is available to traders. As one of the three groups of non-fundamentally informed traders vanishes, equilibrium turns out to be unique, although complementarities in trading still exist. So, even if non-fundamental information is multidimensional, equilibrium can be unique, given that the dimensionality of noise is sufficiently high. The higher the dimensionality of noise, the higher the dimensionality of non-fundamental information needs to be in order for multiplicity of equilibria to arise. This is a new insight that cannot be gained in a setup with one-dimensional noise à la Ganguli and Yang (2009).

Second, the properties of the equilibria of the model with three-dimensional noise differ in an important way from the two-dimensional case. It can happen that an increase in λ_i leads to an increase in I_{x_i} (and β_i) in *both* the LIE and the HIE. This result also contrasts with Ganguli and Yang (2009), where a rise in the mass of informed traders unambiguously increases the equivalent of β_i in one equilibrium, while unequivocally decreasing it in the other equilibrium.

Third, the model reveals a complementarity in the acquisition of non-fundamental information that can prevail although equilibrium is unique. While acquiring information about the same noise component is always a substitute in the absence of equilibrium multiplicity, acquiring information about *different* noise components can still be a complement. This result clarifies that non-fundamental information and multiple equilibria are not inextricably linked with each other when assessing the possibility of complementarities in information acquisition.

Fourth, and perhaps most interestingly, numerical simulation shows that a market breakdown is less likely to happen in the three-dimensional noise case. If noise is three-dimensional, the adverse selection problem is less severe than in the two-dimensional case. The higher the dimensionality of noise, the smaller the informational advantage obtained from observing a single noise component. In the three-dimensional case, scenarios exist in which the market does not break down, even though the mass of noise-informed traders becomes arbitrarily large.[9] This leads to the conclusion that the dimensionality of noise and the intensity of adverse selection are negatively correlated: the higher the dimensionality of noise, the weaker adverse selection, and the lower the likelihood of a market breakdown.[10]

4.5.1 Model Assumptions and Equilibrium Determination

The model is the same as in Section 4.1 except that there is a third, independent component linked to noise trader demand, $x_3 \sim N(0, \tau_x^{-1})$, and an additional continuum of rational, noise-informed agents indexed by the interval $[0, \lambda_3]$. Each trader $n_3 \in [0, \lambda_3]$ observes x_3 and is characterized by the same CARA utility function as before. The determination of the linear REE closely follows the steps applied in Section 4.2. Traders conjecture the asset price to be linear in θ, x_1, x_2, and x_3:

$$P = a_\theta \, \theta + a_1 \, x_1 + a_2 \, x_2 + a_3 \, x_3, \tag{4.17}$$

for constants a_θ and a_i (for $i = 1, 2, 3$). A rational trader's demand function is still given by (4.2). Using her non-fundamental information, an x_i-informed agent can disentangle the information conveyed by the market price as follows:

$$P^*_{n_i} \equiv \frac{P - a_i\, x_i}{a_\theta} = \theta + \frac{a_j\, x_j + a_l\, x_l}{a_\theta}, \qquad for\;\; i, j, l = 1, 2, 3,\;\; j \neq i \neq l. \tag{4.18}$$

Define $\beta_i \equiv a_\theta / a_i$ (for $i = 1, 2, 3$). Then, according to (4.18), the market price is a signal about θ with precision $\tau_x/(1/\beta_j^2 + 1/\beta_l^2)$ for the x_i-informed trader (i.e., $\mathrm{Var}^{-1}(P^*_{n_i}|\,\theta) = \tau_x/(1/\beta_j^2 + 1/\beta_l^2)$). Equivalently, the fundamentally informed and the uninformed, rational traders observe

$$P^*_{f/u} \equiv \frac{P}{a_\theta} = \theta + \frac{a_1\, x_1 + a_2\, x_2 + a_3\, x_3}{a_\theta}. \tag{4.19}$$

Thus, the price is a signal about θ with precision $\tau_x/(1/\beta_1^2 + 1/\beta_2^2 + 1/\beta_3^2)$ for the fundamentally informed and the uninformed, rational traders (i.e., $\mathrm{Var}^{-1}(P^*_{f/u}|\,\theta) = \tau_x/(1/\beta_1^2 + 1/\beta_2^2 + 1/\beta_3^2)$). The precision of $P^*_{f/u}$ is clearly smaller than that of $P^*_{n_i}$. By using (4.18), (4.19), and fundamentally informed traders' private signals, we can compute the first two conditional moments of θ for all types of rational traders via the projection theorem. They are, then, used to compute rational agents' demand functions, which are plugged into the market-clearing condition:

$$\begin{aligned}\int_0^1 D_f\, df + \int_0^{\lambda_1} D_{n_1}\, dn_1 + \int_0^{\lambda_2} D_{n_2}\, dn_2 \\ + \int_0^{\lambda_3} D_{n_3}\, dn_3 + \int_0^{\lambda_u} D_u\, du + x_1 + x_2 + x_3 = 0.\end{aligned} \tag{4.20}$$

Solving (4.20) for P shows that it is linear in θ, x_1, x_2, and x_3, in line with (4.17). Eventually, invoking rational expectations delivers the coefficients of the price function in equilibrium (with the proof delegated to Appendix A):

Proposition 4.7. *In the linear REE, it holds that*

$$a_\theta = \frac{\tau_\epsilon + \left[\begin{array}{c}(1+\lambda_u)\mathrm{Var}^{-1}(P^*_{f/u}|\,\theta) + \lambda_1 \mathrm{Var}^{-1}(P^*_{n_1}|\,\theta) \\ + \lambda_2 \mathrm{Var}^{-1}(P^*_{n_2}|\,\theta) + \lambda_3 \mathrm{Var}^{-1}(P^*_{n_3}|\,\theta)\end{array}\right]}{\tau_\epsilon + \omega\tau_\theta + \left[\begin{array}{c}(1+\lambda_u)\mathrm{Var}^{-1}(P^*_{f/u}|\,\theta) + \lambda_1 \mathrm{Var}^{-1}(P^*_{n_1}|\,\theta) \\ + \lambda_2 \mathrm{Var}^{-1}(P^*_{n_2}|\,\theta) + \lambda_3 \mathrm{Var}^{-1}(P^*_{n_3}|\,\theta)\end{array}\right]},$$

$$a_i = (1/\beta_i)\, a_\theta,\;\; for\;\; i = 1, 2, 3,$$

where

$$\omega \equiv 1 + \lambda_u + \lambda_1 + \lambda_2 + \lambda_3,$$

and β_i *is given by*

$$\beta_i = \frac{\tau_\epsilon}{\gamma} + \frac{\lambda_i \beta_j^2 \beta_l^2 \tau_x}{\gamma(\beta_j^2 + \beta_l^2)}, \qquad for \;\; i, j, l = 1, 2, 3, \;\; j \neq i \neq l. \tag{4.21}$$

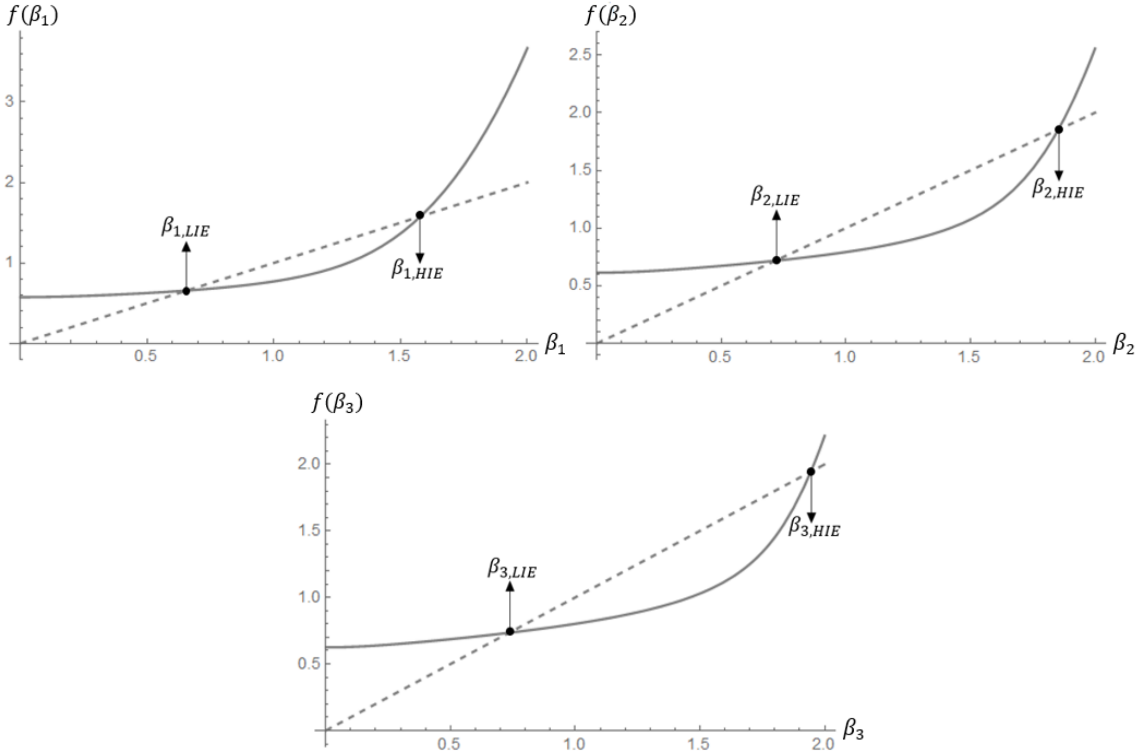
The three-equation system contained in (4.21) pins down the number of linear REE. The high non-linearity of the system, however, prevents an analytical characterization of the number of equilibria. Nevertheless, numerical analysis shows that, as in the two-dimensional noise model, equilibrium is either multiple in form of the LIE and the HIE, unique (in a special case), or non-existent. For a given set of parameter values, Figure 4.5 provides a graphical solution of the resulting fixed-point problems that determine the number of linear REE. As depicted in Figure 4.5, there are still two equilibria present, the LIE and the HIE (further numerical simulations are given in Subsection 4.5.3). Thus, increasing the dimensionality of noise does not change the possible number of equilibria. Since the numerical example assumes that $\lambda_1 < \lambda_2 < \lambda_3$, it holds that $\beta_{1,LIE} < \beta_{2,LIE} < \beta_{3,LIE}$ and $\beta_{1,HIE} < \beta_{2,HIE} < \beta_{3,HIE}$ in Figure 4.5.

In the two-dimensional noise model, multidimensionality of non-fundamental information is crucial for multiple equilibria to arise. The three-dimensional noise case points to the important fact that multidimensionality of non-fundamental information does not necessarily lead to equilibrium multiplicity. Instead, it is essential that the dimensionality of non-fundamental information is *sufficiently* high. As one group of noise-informed traders vanishes, equilibrium is unique, as described in the following proposition:

Proposition 4.8. *Let* $\lambda_i = 0$. *Then, equilibrium is unique although non-fundamental information is multidimensional (i.e.,* $\lambda_j > 0$, $\lambda_l > 0$*).*

The proof can be found in Appendix A. If noise is three-dimensional, information about *all* three components needs to be available to rational traders for multiple equilibria to show up. If not, equilibrium is unequivocally unique. This is a new insight that cannot be obtained in a setup with one-dimensional noise in the spirit of Ganguli and Yang (2009). The result in the proposition highlights that multidimensionality of non-fundamental information does *not necessarily* lead to multiple equilibria. Instead, it is crucial that the dimensionality of non-fundamental information equals the dimensionality of noise.

Figure 4.5: Equilibrium with three-dimensional noise



Parameters: $\tau_\epsilon = \tau_x = 1$, $\gamma = 2$, $\lambda_1 = 1.2$, $\lambda_2 = 1.8$, $\lambda_3 = 2$

4.5.2 Interactions at the Trading Stage

Analogous to the two-dimensional setup, by inspecting (A56), we can express x_i-informed traders' trading intensity as a function of the conjectured values of the three coefficient ratios as follows:

$$I_{x_i} \equiv \int_0^{\lambda_i} \left| \frac{\partial D_{n_i}}{\partial x_i} \right| dn_i = \lambda_i \frac{\tau_x}{\gamma \beta_i \left(\frac{1}{\beta_j^2} + \frac{1}{\beta_l^2} \right)}, \qquad for \;\; i, j, l = 1, 2, 3, \;\; j \neq i \neq l. \quad (4.22)$$

Then, by using the same argument as in Subsection 4.3.1 with three instead of two noise-informed groups, we can show that the relationship between the implied value of β_i and trading intensity I_{x_i} is still given by

$$\beta_i = \frac{\tau_\epsilon}{\gamma(1 - I_{x_i})}, \qquad for \;\; i = 1, 2, 3. \quad (4.23)$$

By plugging β_1, β_2, and β_3 from (4.23) into (4.22), we obtain

$$I_{x_i} = \frac{\lambda_i \tau_x}{\frac{\tau_\epsilon}{1 - I_{x_i}} \left\{ \frac{\gamma^2 [(1 - I_{x_j})^2 + (1 - I_{x_l})^2]}{\tau_\epsilon^2} \right\}}$$

$$= \frac{\lambda_i \tau_x \tau_\epsilon (1 - I_{x_i})}{\gamma^2 [(1 - I_{x_j})^2 + (1 - I_{x_l})^2]}$$

$$\Leftrightarrow I_{x_i} = \frac{\lambda_i \tau_x \tau_\epsilon}{\gamma^2 [(1 - I_{x_j})^2 + (1 - I_{x_l})^2] + \lambda_i \tau_x \tau_\epsilon}, \quad for \;\; i, j, l = 1, 2, 3, \;\; i \neq j \neq l. \tag{4.24}$$

By (4.24), we clearly see that trading against different types of noise is a complement in the three-dimensional model too. This is due to the same inference augmentation effect explained in Subsection 4.3.2. If $\lambda_l = 0$ and, hence, $I_{x_l} = 0$, trading against x_i is still a complement to trading against x_j (i.e., $\partial I_{x_i} / \partial I_{x_j} > 0$). Nevertheless, as stated in Proposition 4.8, equilibrium is unique in this case. Thus, complementarities in trading against different types of noise are only strong enough to generate multiple equilibria if information about all three noise components is available to traders.

4.5.3 Consequences of a Rise in λ_i in Equilibrium

Next, we are interested in the impact that a rise in λ_i exerts on the three trading intensities. Formally, we have:

Proposition 4.9. *The total effect of a rise in λ_i on the trading intensities is*

$$\frac{dI_{x_i}}{d\lambda_i} = \Gamma_1^{-1} \times (1 - \Gamma_2) \frac{\partial I_{x_i}}{\partial \lambda_i}, \tag{4.25}$$

$$\frac{dI_{x_j}}{d\lambda_i} = \Gamma_1^{-1} \times \left(\frac{\partial I_{x_j}}{\partial I_{x_i}} + \frac{\partial I_{x_j}}{\partial I_{x_l}} \frac{\partial I_{x_l}}{\partial I_{x_i}} \right) \frac{\partial I_{x_i}}{\partial \lambda_i}, \tag{4.26}$$

$$\frac{dI_{x_l}}{d\lambda_i} = \Gamma_1^{-1} \times \left(\frac{\partial I_{x_l}}{\partial I_{x_i}} + \frac{\partial I_{x_l}}{\partial I_{x_j}} \frac{\partial I_{x_j}}{\partial I_{x_i}} \right) \frac{\partial I_{x_i}}{\partial \lambda_i}, \tag{4.27}$$

where

$$\Gamma_1 \equiv 1 - \left[\Gamma_2 + \frac{\partial I_{x_i}}{\partial I_{x_j}} \left(\frac{\partial I_{x_j}}{\partial I_{x_i}} + \frac{\partial I_{x_j}}{\partial I_{x_l}} \frac{\partial I_{x_l}}{\partial I_{x_i}} \right) + \frac{\partial I_{x_i}}{\partial I_{x_l}} \left(\frac{\partial I_{x_l}}{\partial I_{x_i}} + \frac{\partial I_{x_l}}{\partial I_{x_j}} \frac{\partial I_{x_j}}{\partial I_{x_i}} \right) \right],$$

$$\Gamma_2 \equiv \frac{\partial I_{x_j}}{\partial I_{x_l}} \frac{\partial I_{x_l}}{\partial I_{x_j}}.$$

The proof can be found in Appendix A. By carefully inspecting equations (4.25) to (4.27), a crucial difference between the two- and the three-dimensional noise setup becomes visible. Recall from Proposition 4.3 that if noise is two-dimensional, an increase in λ_i unambiguously raises both trading intensities (and both coefficient ratios) in the LIE, while decreasing them in the HIE. The analogous result holds true for the setup of Ganguli and Yang (2009). In the three-dimensional case, the sign of Γ_1 pins down the sign of $dI_{x_j}/d\lambda_i$ and $dI_{x_l}/d\lambda_i$ (all partial derivatives are clearly positive, as in the two-dimensional case). However, the sign of Γ_1 does *not* alone determine the sign of $dI_{x_i}/d\lambda_i$. The sign of $dI_{x_i}/d\lambda_i$ is additionally influenced by the sign of $1-\Gamma_2$. Note that if $1-\Gamma_2<0$, it also holds that $\Gamma_1<0$. Thus, it can happen that a rise in λ_i leads to an *increase* in I_{x_i}, even though $\Gamma_1<0$. However, if $\Gamma_1>0$, we have $1-\Gamma_2>0$.

Although the exact relationship between the sign of Γ_1 and the equilibrium traders coordinate on cannot be analytically derived, the following numerical simulations indicate that $\Gamma_1>0$ (resp., $\Gamma_1<0$) is true in the LIE (resp., in the HIE), similar to the two-dimensional setup. Tables 4.2, 4.3, and 4.4 depict the equilibrium values of the coefficient ratios β_1, β_2, and β_3 in the two- and three-dimensional models for $\gamma=1$, $\tau_x=0.5$, $\lambda_2=0.5$, $\lambda_3=0.4$, $\tau_\epsilon\in\{0.3, 0.8, 1.5\}$, and $\lambda_1\in\{0.1, 1, 10, 100, 200\}$. The first number in each bracket refers to the value in the LIE. The second one refers to the respective value in the HIE.

The results of the two-dimensional case show the known pattern that an increase in the mass of noise-informed traders unequivocally raises β_1 and β_2 in the LIE and decreases them in the HIE. When looking at the three-dimensional case, in strong contrast, a rise in λ_1 can lead to an *increase* in β_1 in the HIE (e.g., when λ_1 increases from 1 to 10 in Tables 4.2 and 4.3). This is a new property linked to equilibria generated by non-fundamental information that has not been identified in the relevant literature so far.

Table 4.2: Comparison of models with two- and three-dimensional noise (1)

	Two-dimensional noise		Three-dimensional noise		
	β_1	β_2	β_1	β_2	β_3
$\lambda_1=0.1$	(0.31, 6.62)	(0.32, 11.24)	(0.30, 10.75)	(0.31, 22.22)	(0.31, 19.04)
$\lambda_1=1$	(0.35, 2.77)	(0.33, 2.22)	(0.32, 9.33)	(0.31, 6.30)	(0.31, 5.76)
$\lambda_1=10$	(−−, −−)	(−−, −−)	(0.55, 44.06)	(0.32, 4.34)	(0.32, 4.04)
$\lambda_1=100$	(−−, −−)	(−−, −−)	(2.91, 429.51)	(0.33, 4.30)	(0.32, 4.00)
$\lambda_1=200$	(−−, −−)	(−−, −−)	(5.52, 858.59)	(0.33, 4.30)	(0.32, 4.00)

Parameters: $\tau_\epsilon=0.3$, $\gamma=1$, $\tau_x=0.5$, $\lambda_2=0.5$, $\lambda_3=0.4$

Moreover, the simulations outlined in Tables 4.2 - 4.4 demonstrate that a rise in λ_1 increases (resp., decreases) β_2 and β_3 in the LIE (resp., in the HIE). By recalling (4.26) and (4.27), this leads to the conclusion that $\Gamma_1>0$ holds in the LIE and $\Gamma_1<0$ in the HIE. Since $1-\Gamma_2>0$ if $\Gamma_1>0$, a rise in λ_i cannot lead to a decrease in I_{x_i} in the LIE (as in the model with two-dimensional noise).

The numerical results contain another interesting point. It is well known from Ganguli and Yang (2009) and the two-dimensional noise model of Section 4.1 that a large mass of informed traders makes the two equilibria vanish (see part (d) in Proposition 4.3). If noise is three-dimensional, however, such a market breakdown is

Table 4.3: Comparison of models with two- and three-dimensional noise (2)

	Two-dimensional noise		Three-dimensional noise		
	β_1	β_2	β_1	β_2	β_3
$\lambda_1 = 0.1$	(0.85, 6.19)	(0.98, 10.38)	(0.82, 10.41)	(0.89, 21.26)	(0.87, 18.29)
$\lambda_1 = 1$	(−−, −−)	(−−, −−)	(1.00, 8.31)	(0.91, 5.74)	(0.89, 5.26)
$\lambda_1 = 10$	(−−, −−)	(−−, −−)	(3.35, 31.17)	(1.03, 3.61)	(0.99, 3.37)
$\lambda_1 = 100$	(−−, −−)	(−−, −−)	(28.02, 293.95)	(1.06, 3.55)	(1.03, 3.31)
$\lambda_1 = 200$	(−−, −−)	(−−, −−)	(55.30, 586.92)	(1.06, 3.55)	(1.03, 3.31)

Parameters: $\tau_\epsilon = 0.8$, $\gamma = 1$, $\tau_x = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 0.4$

Table 4.4: Comparison of models with two- and three-dimensional noise (3)

	Two-dimensional noise		Three-dimensional noise		
	β_1	β_2	β_1	β_2	β_3
$\lambda_1 = 0.1$	(1.76, 5.43)	(2.27, 8.86)	(1.58, 9.87)	(1.85, 19.78)	(1.79, 17.11)
$\lambda_1 = 1$	(−−, −−)	(−−, −−)	(2.60, 6.26)	(2.15, 4.54)	(2.05, 4.20)
$\lambda_1 = 10$	(−−, −−)	(−−, −−)	(−−, −−)	(−−, −−)	(−−, −−)
$\lambda_1 = 100$	(−−, −−)	(−−, −−)	(−−, −−)	(−−, −−)	(−−, −−)
$\lambda_1 = 200$	(−−, −−)	(−−, −−)	(−−, −−)	(−−, −−)	(−−, −−)

Parameters: $\tau_\epsilon = 1.5$, $\gamma = 1$, $\tau_x = 0.5$, $\lambda_2 = 0.5$, $\lambda_3 = 0.4$

less likely to happen than in the two-dimensional case. In Table 4.2, a market failure already occurs for a value of λ_1 between 1 and 10 in the version with two-dimensional noise. In the three-dimensional version, by contrast, a market breakdown does not take place even if $\lambda_1 = 200$. In Tables 4.3 and 4.4, τ_ϵ is increased from 0.3 to 0.8 and 1.5, respectively, raising the likelihood of a market breakdown. Although the LIE and the HIE eventually disappear in the three-dimensional case for some value of λ_1 between 1 and 10 in Table 4.4, they vanish "later" (i.e., for a higher λ_1) than in the two-dimensional case.

Exactly the same pattern can be identified when gradually increasing τ_x or gradually decreasing γ and varying λ_1. In all scenarios, a market breakdown is less likely to happen in the model with three-dimensional noise. As the mass of traders with information about the third noise component (i.e., λ_3) increases, the advantage of the three-dimensional over the two-dimensional case shrinks and a market breakdown occurs for smaller values of λ_1 than in the simulations contained in Tables 4.2, 4.3, and 4.4. Nevertheless, a market failure is still less likely to happen in the three-dimensional model, no matter which value λ_3 actually takes.

The fact that the LIE and the HIE are less likely to vanish in the three-dimensional setup can be economically motivated as follows: When noise is three-dimensional, adverse selection is less intense than in the two-dimensional setup. The higher the dimensionality of noise, the smaller the informational advantage obtained from knowing a single noise component. Thus, even if the mass of noise-informed traders becomes large, it can happen that adverse selection is not severe enough to produce a market breakdown. This leads to the conclusion that a high dimensionality of noise is beneficial for the functioning of financial markets with diversely informed traders, as it weakens adverse selection.

4.5.4 Interactions at the Information Acquisition Stage

Finally, we explore agents' strategic interactions in the acquisition of non-fundamental information.[11] The value of knowing x_i is still given by (4.15) in Proposition 4.4 (see Appendix A for the proof):

$$\phi_{x_i} = \frac{1}{2\gamma} \log \left[\frac{\mathrm{Var}(\theta \mid P)}{\mathrm{Var}(\theta \mid P, x_i)} \right], \quad for \quad i = 1, 2, 3,$$

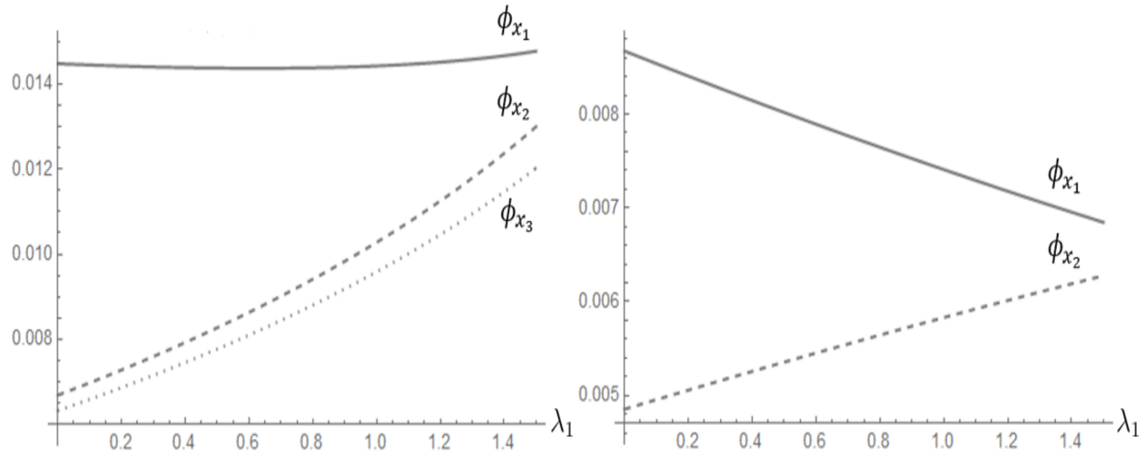
which can be written as

$$\phi_{x_i} = \frac{1}{2\gamma} \log \left[\frac{\tau_\theta + \tau_x/(\beta_j^{-2} + \beta_l^{-2})}{\tau_\theta + \tau_x/(\beta_i^{-2} + \beta_j^{-2} + \beta_l^{-2})} \right], \qquad for \;\; i, j, l = 1, 2, 3, \;\; i \neq j \neq l. \quad (4.28)$$

As in the two-dimensional noise model, the value of information about x_i is positively correlated with the inverse of an x_i-informed trader's residual uncertainty about fundamentals, represented by the numerator in (4.28). By contrast, the value of information about noise shrinks when overall price efficiency, represented by the denominator in (4.28), rises. The graph on the left-hand side in Figure 4.6 plots the LIE values of ϕ_{x_1}, ϕ_{x_2}, and ϕ_{x_3} in dependence of λ_1 for a given set of parameter values. In the numerical example, ϕ_{x_1} is increasing in λ_1 for sufficiently large values of λ_1, which confirms that acquiring information about the same noise component can be a complement in the three-dimensional model as well. The same holds true for acquiring information about different noise components, as both ϕ_{x_2} and ϕ_{x_3} are strictly increasing in λ_1 in the given example.

If $\lambda_3 = 0$, numerical simulation shows that acquiring information about the same noise component is always a substitute (i.e., ϕ_{x_i} is monotonically decreasing in λ_i, for $i = 1, 2$). More interestingly, the graph on the right-hand side in Figure 4.6 shows that ϕ_{x_2} can be increasing in λ_1, although there is no third noise-informed group (i.e., $\lambda_3 = 0$). Thus, complementarities in acquiring information about different noise components can exist even if one noise-informed group is absent. Since Ganguli and Yang (2009) and Manzano and Vives (2011), complementarities in information acquisition and equilibrium multiplicity are closely tied together

Figure 4.6: Interactions in information acquisition with three-dimensional noise



Note: parameters for the graph on the left-hand side are $\tau_\epsilon = \tau_x = 1$, $\gamma = 2$, $\tau_\theta = 1.5$, $\lambda_2 = 1.8$, $\lambda_3 = 2$. Parameters for the graph on the right-hand side are $\tau_\epsilon = \tau_x = 1$, $\gamma = 2$, $\tau_\theta = 1.5$, $\lambda_2 = 1.8$, $\lambda_3 = 0$.

when considering the effects of non-fundamental information, i.e., non-fundamental information only generates complementarities in information acquisition if it leads to multiple equilibria. This relationship is confirmed when considering the acquisition of information about the *same* noise component. Recall from Subsection 4.4.2 that equilibrium is unique and acquiring information about the same noise component is unequivocally a substitute if there is only one group of noise-informed traders present. In the three-dimensional noise setup, if one noise-informed group is absent, equilibrium is unique and acquiring information about the same noise component is a substitute too. Nevertheless, acquiring information about *different* noise components can still be a complement, as seen in the graph on the right-hand side in Figure 4.6 (i.e., ϕ_{x_2} increases with λ_1). Thus, the three-dimensional noise model uncovers a complementarity in the acquisition of non-fundamental information that can show up, although equilibrium is unique. Acquiring information about different noise components can be a complement even if non-fundamental information does not make multiple equilibria arise. This new insight clarifies that multiple equilibria are a *necessary and sufficient* condition for the possibility of complementarities in the acquisition of information about the *same* noise component. However, multiple equilibria are only a *sufficient* but not a necessary condition when considering under what circumstances acquiring information about *different* noise components can be a complement.

4.6 Implications for the Effects of PFOF

The models of Sections 4.1 and 4.5 explore diversely noise-informed traders' interactions theoretically in an environment characterized by multidimensional noise. Additionally, they investigate the resulting effects on price efficiency and the intensity of adverse selection. The model setup was motivated by the increased availability of non-fundamental information in financial markets due to the rise in PFOF. Following up on this observation, this section uses the theoretical model results to derive some implications regarding the effects of PFOF in real financial markets.

A first implication derived from the model is that the surge in PFOF is conducive to price efficiency. As shown in Subsection 4.3.4, traders' usage of their multidimensional non-fundamental information, expressed by the relevant trading intensities, benefits the efficiency of the market price. In the model, rational agents use their information about noise trading to infer information about fundamentals from the price, which makes them trade against non-fundamental information. This mitigates the influence of noise trader demand on the price relative to fundamentals. In reality, of course, it is unlikely that wholesalers engaged in PFOF use non-fundamental information to extract noise from the price and forecast fundamentals more accurately. Nevertheless, Farboodi et al. (2021, p. 16) state that this technique "is functionally equivalent to trading against dumb money, a common practice for sophisticated traders with access to retail order flow." Hence, the model property that wholesalers engaged in PFOF trade against retail trader demand closely resembles what happens in real financial markets (see also Sal Arnuk's quote on p. 73).

Moreover, as trading against different types of noise is unambiguously a complement (see Proposition 4.2), the resulting interaction of the different market participants engaged in PFOF additionally drives prices closer to fundamentals. As one wholesaler engaged in PFOF trades more aggressively against the observed retail trader demand, other wholesalers with information about other components linked to retail trading trade more aggressively too. Thus, rational traders' usage of their multidimensional non-fundamental information and the resulting complementarities in trading indicate a positive impact of PFOF on price efficiency.

Secondly, the model points to the fact that complementarities in acquiring information about different noise components can exist (see Proposition 4.6 and Subsection 4.5.4). This suggests that the incentive to acquire non-fundamental information by engaging in PFOF can increase even further as a consequence of more non-fundamental information being acquired through PFOF. Thus, if the theoretically derived complementarities manifest themselves in financial markets, the total amount of non-fundamental information obtained through PFOF should increase or at least maintain its contemporaneous level. Although this amount seems to be difficult to gauge, one can take a look at the major U.S. online brokers' PFOF-related revenue to tentatively evaluate whether complementarities in acquiring non-fundamental information manifest themselves in financial markets. The major online brokers' PFOF-related revenue can serve as a proxy for the amount of non-fundamental information in the market that is obtained through PFOF: the more non-fundamental information acquired through PFOF, the higher the major online brokers' revenue linked to PFOF (assuming a constant payment per routed order, of course).

As outlined in the Introductory Chapter, the PFOF-related revenue of four major U.S. online brokers, viz., *Robinhood*, *Charles Schwab*, *E*TRADE*, and *TD Ameritrade*, jointly rose from $900 million in 2019 to roughly $2.5 billion in 2020. In 2021, as of September 30, it already adds up to $2.4 billion (see also Figure 1.4). Of course, the majority of this immense growth in revenue is related to the recent boom in retail investing and the trend toward commission-free trading, which has forced online brokers to find new sources of revenue. Nevertheless, this sharp increase can tentatively be seen as a first hint at the manifestation of complementarities in acquiring information about different components of retail order flow. It will certainly be interesting to see how the revenue generated by PFOF will develop over the next few years. Based on the model results, one would expect the total amount of non-fundamental information in the market obtained through PFOF to increase. This should be reflected in a rise in online brokers' PFOF-related revenue or at least in a conservation of its current high level.

Finally, the model has an important implication for the intensity of adverse selection in financial markets due to the presence of PFOF. As seen in Subsection 4.5.3, a higher dimensionality of noise weakens adverse selection and makes a market breakdown less likely to occur. Hence, PFOF should not contribute significantly to exacerbating the adverse selection problem as long as the non-fundamental information obtained through PFOF is sufficiently dispersed among professional traders. The higher the dimensionality of noise and, thus, the higher the number of different market participants engaged in PFOF, the less severe adverse selection should be. According to the model results, PFOF is not or only weakly conducive to adverse selection if there are *enough different* traders engaged in PFOF. This furthermore implies that possible market concentrations in the field of PFOF should be prevented.

Notably, the SEC requires U.S. online brokers' company 606 reports, which disclose the net payments received through PFOF, to be listed by customers. Thus, these reports are a useful indicator of how many active wholesalers exist in the field of PFOF. Additionally, each wholesaler's net payments can serve as a proxy to evaluate the possessed amount of non-fundamental information, also in relation to other wholesalers. These figures should be used in the future to assess the severity of adverse selection in financial markets generated by PFOF.

5 Conclusion

The recent global surge in retail investing has shacked up the financial landscape and significantly fostered the role of non-fundamental information in financial markets, with two important developments that stand out. First, the rising mass of retail traders has contributed to a boom in the user statistics of stock message boards such as *WallStreetBets* and *StockTwits*. These forums have become a common place for private investors to share opinions and to systematically coordinate market activities, as was strikingly observed during the *Gamestop* episode in January 2021. Advances in processing the big data contained in the stock message boards enable professional traders to gauge so-called social sentiment, which they include in their trading decisions. Second, the online brokerage sector experienced an immense influx of new customers, which was accompanied by a structural change toward commission-free trading. In search of new sources of revenue, major U.S. online brokers such as *Robinhood* and *Charles Schwab* began raising more money through PFOF arrangements with wholesalers, resulting in a perhaps unprecedented availability of non-fundamental information in financial markets.

Modeling retail investors as noise traders, this thesis adds to the theoretical literature on non-fundamental information by investigating the impact of social sentiment investing and PFOF within the competitive noisy REE framework. The results of Chapter 3 indicate that social sentiment investing potentially moves prices away from fundamentals. This outcome sharply contrasts with the conventional wisdom that using non-fundamental information unambiguously raises price efficiency. In the dynamic models of Chapter 3, professional traders capitalize on social sentiment derived from stock message boards by front-running retail investors' stock market activity. Rather than trading against retail investors, professional investors ride the bubble induced by retail trading, which possibly drives the market price even further away from fundamentals.

Chapter 4 investigates the strategic interactions between different wholesalers that glean non-fundamental information through PFOF arrangements. The ensuing analysis uncovers new types of complementarities in trading and information acquisition that have been absent in the relevant literature. Perhaps most interestingly, the model reveals that a high dimensionality of noise mitigates the possibility of a market breakdown by weakening adverse selection. The theoretical results are used to derive some implications regarding the real effects of PFOF: (i) PFOF enhances price efficiency, (ii) complementarities in acquiring information about different noise components predict an increase in the amount of non-fundamental information obtained through PFOF, and (iii) non-fundamental information obtained through PFOF should be sufficiently dispersed among wholesalers to weaken adverse selection.

Although this thesis points out some important consequences of social sentiment investing and PFOF for financial markets, several open issues remain that future research should address. Notably, the outcomes of Chapters 3 and 4 yield potentially contrary results regarding the effects of social sentiment investing and PFOF on price efficiency. Thus, one could think of setting up a "unified" framework encompassing both social sentiment investing and PFOF to better assess the overall impact of these sources of non-fundamental information on price efficiency. The dynamic setups of Chapter 3 could, moreover, be extended by adding several feedback effects. On the one hand, one could introduce a feedback loop between social sentiment and noise trader demand, as not only professional investors' but also retail traders' demand is likely to be influenced by social sentiment. On the other hand, one could implement a feedback effect from the financial market to the real economy (e.g., by modeling a firm manager or a capital provider) and explore the effects of social sentiment investing on real efficiency. While the results of Chapter 3 emphasize a potentially negative impact of social sentiment investing on price efficiency, they do not consider the ensuing influence on real efficiency. However, given the static nature of existing REE models with real decision-makers, analyzing real efficiency in a dynamic framework seems to be quite challenging.

The model of Chapter 4 could also be modified in at least two reasonable ways. First, one could try to investigate the general n-dimensional noise case and evaluate whether the results derived in the two- and three-dimensional noise models are robust. However, it might also be difficult to obtain analytical results at this point. Second, Chapter 4 focuses on how professional traders with PFOF arrangements profit from non-fundamental information when trading on their own account. Nevertheless, matching and executing retail investors' orders constitute the dominant activity of wholesalers engaged in PFOF. Thus, one should also take into account the consequences of this practice when assessing the overall effects of PFOF. These and other aspects related to the special role that non-fundamental information has played in financial markets since the beginning of this decade certainly merit additional research.

A Model Proofs

Proof of Proposition 3.2. We solve for the linear dynamic REE with OLG of investors by using backward induction. That is, we first derive the equilibrium function of P_2.

Predicting θ at date 2. A date-2 agent possesses three signals to predict the fundamental asset value (i.e., P_1^*, P_2^*, and x_{2i}). Since the signals' error terms are pairwise uncorrelated, the first two conditional moments of θ are

$$\mathrm{E}(\theta \mid I_{2i}) = \frac{\tau_\epsilon \, x_{2i} + \rho_1^2(\tau_{s_1} + \tau_{\eta_1}) \, P_1^* + \rho_2^2(\tau_{s_2} + \tau_{\eta_2}) P_2^*}{\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1}) + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})},$$

$$\mathrm{Var}(\theta \mid I_{2i}) = \frac{1}{\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1}) + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})}.$$

Determining the equilibrium function of P_2. Recalling (3.25), agent i's date-2 demand for the risky asset becomes

$$\begin{aligned} D_{2i} &= \delta \, \frac{\mathrm{E}(\theta \mid I_{2i}) - P_2}{\mathrm{Var}(\theta \mid I_{2i})} \\ &= \delta \tau_\epsilon \, x_{2i} + \delta \rho_1^2(\tau_{s_1} + \tau_{\eta_1}) \, P_1^* + \delta \rho_2^2(\tau_{s_2} + \tau_{\eta_2}) P_2^* - \frac{\delta}{\mathrm{Var}(\theta \mid I_{2i})} P_2, \end{aligned}$$

which is equal to (3.29) in the main text. Further computations yield

$$\begin{aligned} &\delta \tau_\epsilon \, x_{2i} + \delta \rho_1^2(\tau_{s_1} + \tau_{\eta_1}) \, P_1^* + \delta \rho_2^2(\tau_{s_2} + \tau_{\eta_2}) P_2^* - \frac{\delta}{\mathrm{Var}(\theta \mid I_{2i})} P_2 \\ &= \delta \tau_\epsilon \, x_{2i} + \delta \rho_1^2(\tau_{s_1} + \tau_{\eta_1}) \left(\frac{P_1 + c_{11} Y_1 - c_{12} Y_2}{a_1} - \frac{1}{\rho_1} \frac{\tau_{\eta_1}}{\tau_{s_1} + \tau_{\eta_1}} Y_1 \right) \\ &\quad + \delta \rho_2^2(\tau_{s_2} + \tau_{\eta_2}) \left[\theta + \frac{1}{\rho_2} \left(s_2 - \frac{\tau_{\eta_2}}{\tau_{s_2} + \tau_{\eta_2}} Y_2 \right) \right] - \frac{\delta}{\mathrm{Var}(\theta \mid I_{2i})} P_2 \\ &= \delta \tau_\epsilon \, x_{2i} + \delta \rho_2^2(\tau_{s_2} + \tau_{\eta_2}) \, \theta + \delta \rho_2(\tau_{s_2} + \tau_{\eta_2}) \, s_2 - \frac{\delta}{\mathrm{Var}(\theta \mid I_{2i})} P_2 + \frac{\delta \rho_1^2(\tau_{s_1} + \tau_{\eta_1})}{a_1} \, P_1 \\ &\quad - \delta \left[-\rho_1^2(\tau_{s_1} + \tau_{\eta_1}) \frac{c_{11}}{a_1} + \rho_1 \tau_{\eta_1} \right] Y_1 - \delta \left[\rho_1^2(\tau_{s_1} + \tau_{\eta_1}) \frac{c_{12}}{a_1} + \rho_2 \tau_{\eta_2} \right] Y_2. \end{aligned}$$

Market clearing at date 2 implies that

$$\int_0^1 D_{2i}\, di + s_2 = 0,$$

which is equivalent to

$$\delta\tau_\epsilon \int_0^1 x_{2i}\, di + \delta\rho_2^2(\tau_{s_2} + \tau_{\eta_2})\,\theta + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})\, s_2 - \frac{\delta}{\mathrm{Var}(\theta \mid I_{2i})} P_2$$
$$+ \delta\frac{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})}{a_1}\, P_1 - \delta\left[-\rho_1^2(\tau_{s_1} + \tau_{\eta_1})\frac{c_{11}}{a_1} + \rho_1\tau_{\eta_1}\right] Y_1 \tag{A1}$$
$$- \delta\left[\rho_1^2(\tau_{s_1} + \tau_{\eta_1})\frac{c_{12}}{a_1} + \rho_2\tau_{\eta_2}\right] Y_2 + s_2 = 0.$$

Making use of the strong law of large numbers as in the static context (see Subsection 3.1.2), the error term in x_{2i} vanishes when integrating (i.e., $\int_0^1 x_{2i}\, di = \theta$). Solving (A1) for P_2 delivers

$$P_2 = \frac{\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})}{\Delta}\,\theta + \frac{1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})}{\delta\Delta}\, s_2 - \frac{\rho_1\tau_{\eta_1} - \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\frac{c_{11}}{a_1}}{\Delta}\, Y_1$$
$$- \frac{\rho_2\tau_\eta + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\frac{c_{12}}{a_1}}{\Delta}\, Y_2 + \frac{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})}{a_1\Delta}\, P_1,$$

where $\Delta \equiv \tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1}) + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})$. Invoking rational expectations immediately yields:

$$a_2 = \frac{\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})}{\Delta},$$

$$b_2 = \frac{1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})}{\delta\Delta},$$

$$c_{21} = \frac{\rho_1\tau_{\eta_1} - \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\frac{c_{11}}{a_1}}{\Delta},$$

$$c_{22} = \frac{\rho_2\tau_{\eta_2} + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\frac{c_{12}}{a_1}}{\Delta},$$

$$d_2 = \frac{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})}{a_1\Delta}.$$

Predicting θ at date 1. Recall that $I_{1i} = (P_1, x_{1i}, Y_1, Y_2)$. At date 1, an agent uses her private signal about the fundamental asset value and the information contained in P_1^* to update her prior beliefs about θ. As the error terms of the signals x_{1i} and

P_1^* are uncorrelated, we get

$$\mathrm{E}(\theta \,|\, I_{1i}) = \frac{\tau_\epsilon\, x_{1i} + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\, P_1^*}{\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})},$$

$$\mathrm{Var}(\theta \,|\, I_{1i}) = \frac{1}{\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})}.$$

Predicting s_2 at date 1. Since date-1 agents are concerned with forecasting the date-2 price, they need to predict date-2 noise trader demand too. Inspecting date-1 price function (3.23), we see that P_1 does not convey any information about s_2 that goes beyond the information already contained in Y_2. Thus, date-1 rational traders only use Y_2 to predict s_2. Using the bivariate case of the projection theorem, we get

$$\mathrm{E}(s_2 \,|\, Y_2) = \frac{\tau_{\eta_2} Y_2}{\tau_{s_2} + \tau_{\eta_2}},$$

$$\mathrm{Var}(s_2 \,|\, Y_2) = \frac{1}{\tau_{s_2} + \tau_{\eta_2}}.$$

Determining the equilibrium function of P_1. From (3.26), we have

$$D_{1i} = \delta\, \frac{\mathrm{E}(P_2 \,|\, I_{1i}) - P_1}{\mathrm{Var}(P_2 \,|\, I_{1i})}.$$

Denote $\Gamma_1 \equiv \mathrm{Var}(\theta \,|\, I_{1i})$ and $\Gamma_2 \equiv \mathrm{Var}(s_2 \,|\, I_{1i})$. Then, the first two conditional moments of P_2 are

$$\begin{aligned}\mathrm{E}(P_2 \,|\, I_{1i}) &= a_2\, \mathrm{E}(\theta \,|\, I_{1i}) + b_2\, \mathrm{E}(s_2 \,|\, I_{1i}) \; - c_{21} Y_1 - c_{22} Y_2 + d_2 P_1 \\ &= a_2 \Gamma_1 \left[\tau_\epsilon\, x_{1i} + \rho_1^2(\tau_{s_1} + \tau_{\eta_1}) P_1^*\right] + b_2 \Gamma_2 \tau_{\eta_2} Y_2 - c_{21} Y_1 - c_{22} Y_2 + d_2 P_1,\end{aligned}$$

$$\mathrm{Var}(P_2 \,|\, I_{1i}) = a_2^2 \Gamma_1 + b_2^2 \Gamma_2.$$

Thus, agent i's demand for the risky asset at date 1 can be written as

$$\begin{aligned}D_{1i} = {}& \frac{a_2 \Gamma_1}{a_2^2 \Gamma_1 + b_2^2 \Gamma_2} \left[\delta \tau_\epsilon\, x_{1i} + \delta \rho_1^2(\tau_{s_1} + \tau_{\eta_1}) P_1^*\right] \\ & + \frac{\delta}{a_2^2 \Gamma_1 + b_2^2 \Gamma_2} \left(b_2 \Gamma_2 \tau_{\eta_2} Y_2 - c_{21} Y_1 - c_{22} Y_2 + d_2 P_1\right) - \frac{\delta}{a_2^2 \Gamma_1 + b_2^2 \Gamma_2} P_1,\end{aligned}$$

which is equal to (3.30) in the main text. Market clearing at date 1 implies:

$$0 = \int_0^1 D_{1i}\, di + s_1$$

$$
\begin{aligned}
&= a_2\Gamma_1\tau_\epsilon \int_0^1 x_{1i}\,di + a_2\Gamma_1\rho_1^2(\tau_{s_1}+\tau_{\eta_1})\left(\frac{P_1+c_{11}Y_1-c_{12}Y_2}{a_1} - \frac{1}{\rho_1}\frac{\tau_{\eta_1}}{\tau_{s_1}+\tau_{\eta_1}}Y_1\right) \\
&\quad + b_2\Gamma_2\tau_{\eta_2}Y_2 - c_{21}Y_1 - c_{22}Y_2 + d_2P_1 - P_1 + \frac{a_2^2\Gamma_1+b_2^2\Gamma_2}{\delta}s_1 \\
&= a_2\Gamma_1\tau_\epsilon \int_0^1 x_{1i}\,di + \frac{a_2^2\Gamma_1+b_2^2\Gamma_2}{\delta}s_1 - \left\{a_2\Gamma_1\left[\rho_1\tau_{\eta_1} - \rho_1^2(\tau_{s_1}+\tau_{\eta_1})\frac{c_{11}}{a_1}\right] + c_{21}\right\}Y_1 \\
&\quad + \left[-a_2\Gamma_1\rho_1^2(\tau_{s_1}+\tau_{\eta_1})\frac{c_{12}}{a_1} + b_2\tau_{\eta_2}\Gamma_2 - c_{22}\right]Y_2 \\
&\quad - \left[1 - d_2 - \frac{a_2\Gamma_1\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}{a_1}\right]P_1.
\end{aligned} \tag{A2}
$$

Again, by the strong law of large numbers, we obtain $\int_0^1 x_{1i}\,di = \theta$. Solving (A2) for P_1 gives

$$
\begin{aligned}
P_1 = {} & \frac{1}{1 - d_2 - \dfrac{a_2\Gamma_1\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}{a_1}}\Bigg(a_2\Gamma_1\tau_\epsilon\,\theta + \frac{a_2^2\Gamma_1+b_2^2\Gamma_2}{\delta}s_1 \\
& - \left\{a_2\Gamma_1\left[\rho_1\tau_{\eta_1} - \rho_1^2(\tau_{s_1}+\tau_{\eta_1})\frac{c_{11}}{a_1}\right] + c_{21}\right\}Y_1 \\
& + \left[-\,a_2\Gamma_1\rho_1^2(\tau_{s_1}+\tau_{\eta_1})\frac{c_{12}}{a_1} + b_2\tau_{\eta_2}\Gamma_2 - c_{22}\right]Y_2\Bigg).
\end{aligned}
$$

By invoking rational expectations, we obtain

$$
\begin{aligned}
a_1 &= \frac{a_2\Gamma_1\tau_\epsilon}{1 - d_2 - \dfrac{a_2\Gamma_1\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}{a_1}} \\
&= \frac{a_2\Gamma_1\tau_\epsilon}{1 - \dfrac{\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}{a_1\Delta} - \dfrac{a_2\Gamma_1\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}{a_1}} \\
&= \frac{a_2\Gamma_1\tau_\epsilon}{1 - \dfrac{\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+a_2\Gamma_1\Delta)}{a_1\Delta}}.
\end{aligned}
$$

Solving for a_1 gives

$$
a_1 = a_2\Gamma_1\tau_\epsilon + \frac{\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+a_2\Gamma_1\Delta)}{\Delta}
$$

$$= \frac{a_2\Gamma_1\Delta\left[\tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\right] + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})}{\Delta}$$

$$= \frac{\dfrac{\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})}{\Delta}\dfrac{\Delta}{\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})}\left[\tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\right] + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})}{\Delta}$$

$$= \frac{[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})][\tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})] + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})[\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]}{\Delta[\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]}$$

$$= \frac{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})(\Delta + \tau_\epsilon) + \tau_\epsilon[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]}{\Delta[\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]}.$$

From the definition of ρ_1, it immediately follows that

$$b_1 = \frac{a_1}{\rho_1}.$$

Furthermore,

$$c_{11} = \frac{a_2\Gamma_1\left[\rho_1\tau_{\eta_1} - \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\frac{c_{11}}{a_1}\right] + c_{21}}{1 - d_2 - \dfrac{a_2\Gamma_1\rho_1^2(\tau_{s_1} + \tau_{\eta_1})}{a_1}}$$

$$= \frac{a_2\Gamma_1\left[\rho_1\tau_{\eta_1} - \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\frac{c_{11}}{a_1}\right] + \dfrac{\rho_1\tau_{\eta_1} - \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\frac{c_{11}}{a_1}}{\Delta}}{1 - \dfrac{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})(1 + a_2\Gamma_1\Delta)}{a_1\Delta}}$$

$$= \frac{a_2\Gamma_1 a_1\Delta\left[\rho_1\tau_{\eta_1} - \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\frac{c_{11}}{a_1}\right] + a_1\rho_1\tau_{\eta_1} - \rho_1^2(\tau_{s_1} + \tau_{\eta_1})c_{11}}{a_1\Delta - \rho_1^2(\tau_{s_1} + \tau_{\eta_1})(1 + a_2\Gamma_1\Delta)}$$

$$= \frac{a_1\rho_1\tau_{\eta_1}(1 + a_2\Gamma_1\Delta) - c_{11}\rho_1^2(\tau_{s_1} + \tau_{\eta_1})(1 + a_2\Gamma\Delta)}{a_1\Delta - \rho_1^2(\tau_{s_1} + \tau_{\eta_1})(1 + a_2\Gamma_1\Delta)}.$$

Solving for c_{11} yields

$$c_{11} = \frac{\rho_1\tau_{\eta_1}(1 + a_2\Gamma_1\Delta)}{\Delta}$$

$$= \frac{\rho_1\tau_{\eta_1}\left[1 + \dfrac{\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})}{\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})}\right]}{\Delta}$$

$$= \frac{\rho_1\tau_{\eta_1}(\Delta + \tau_\epsilon)}{\Delta[\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]}.$$

Moreover,

$$c_{12} = \frac{-a_2\Gamma_1\rho_1^2(\tau_{s_1}+\tau_{\eta_1})\frac{c_{12}}{a_1} + b_2\tau_{\eta_2}\Gamma_2 - c_{22}}{1 - d_2 - \dfrac{a_2\Gamma_1\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}{a_1}}$$

$$= \frac{-a_2\Gamma_1\rho_1^2(\tau_{s_1}+\tau_{\eta_1})\frac{c_{12}}{a_1} + b_2\tau_{\eta_2}\Gamma_2 - \dfrac{\rho_2\tau_{\eta_2}+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})\frac{c_{12}}{a_1}}{\Delta}}{1 - \dfrac{\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+a_2\Gamma_1\Delta)}{a_1\Delta}}$$

$$= \frac{-\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+a_2\Gamma_1\Delta)c_{12} + a_1\tau_{\eta_2}(b_2\Gamma_2\Delta - \rho_2)}{a_1\Delta - \rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+a_2\Gamma_1\Delta)}.$$

After solving for c_{12}, we obtain

$$c_{12} = \frac{\tau_{\eta_2}(b_2\Gamma_2\Delta - \rho_2)}{\Delta}$$

$$= \frac{\tau_{\eta_2}}{\Delta}\left\{\frac{[1+\delta\rho_2(\tau_{s_2}+\tau_{\eta_2})]\Delta}{\delta\Delta(\tau_{s_2}+\tau_{\eta_2})} - \rho_2\right\}$$

$$= \frac{\tau_{\eta_2}}{\delta\Delta(\tau_{s_2}+\tau_{\eta_2})}.$$

Determining ρ_1 and ρ_2. Recall that $\rho_2 \equiv a_2/b_2$ and $\rho_1 \equiv a_1/b_1$. This delivers

$$\rho_2 = \frac{\delta[\tau_\epsilon + \rho_2^2(\tau_{s_2}+\tau_{\eta_2})]}{1+\delta\rho_2(\tau_{s_2}+\tau_{\eta_2})}$$

$$\Leftrightarrow \rho_2[1+\delta\rho_2(\tau_{s_2}+\tau_{\eta_2})] = \delta[\tau_\epsilon + \rho_2^2(\tau_{s_2}+\tau_{\eta_2})]$$

$$\Leftrightarrow \rho_2 = \delta\tau_\epsilon,$$

which, by (3.29), is equal to $\int_0^1 (\partial D_{2i}/\partial x_{2i})\,di$. Furthermore,

$$\rho_1 = \frac{a_2\Gamma_1}{a_2^2\Gamma_1 + b_2^2\Gamma_2}\delta\tau_\epsilon,$$

which equals $\int_0^1 (\partial D_{1i}/\partial x_{1i})\,di$ (see (3.30)). Direct computations yield

$$\rho_1 = \frac{\dfrac{\tau_\epsilon + \rho_2^2(\tau_{s_2}+\tau_{\eta_2})}{\Delta[\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})]}\delta\tau_\epsilon}{\dfrac{[\tau_\epsilon + \rho_2^2(\tau_{s_2}+\tau_{\eta_2})]^2}{\Delta^2[\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})]} + \dfrac{[1+\delta\rho_2(\tau_{s_2}+\tau_{\eta_2})]^2}{\delta^2\Delta^2(\tau_{s_2}+\tau_{\eta_2})}}$$

$$= \frac{\dfrac{\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})}{\Delta[\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]}\,\delta\tau_\epsilon}{\dfrac{\delta^2(\tau_{s_2} + \tau_{\eta_2})[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]^2 + [1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})]^2[\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]}{\delta^2\Delta^2[\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})](\tau_{s_2} + \tau_{\eta_2})}}$$

$$= \frac{\delta^3\tau_\epsilon\Delta[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})](\tau_{s_2} + \tau_{\eta_2})}{\rho_2^2(\tau_{s_2} + \tau_{\eta_2})[1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})]^2 + [1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})]^2[\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]}$$

$$= \frac{\delta^3\tau_\epsilon^2\Delta[1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})](\tau_{s_2} + \tau_{\eta_2})}{\rho_2^2(\tau_{s_2} + \tau_{\eta_2})[1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})]^2 + [1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})]^2[\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]}$$

$$= \frac{\delta^3\tau_\epsilon^2(\tau_{s_2} + \tau_{\eta_2})\Delta}{[1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})][\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1}) + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]}$$

$$= \frac{\delta^3\tau_\epsilon^2\,(\tau_{s_2} + \tau_{\eta_2})}{1 + \delta^2\tau_\epsilon(\tau_{s_2} + \tau_{\eta_2})}.$$

Since ρ_1 and ρ_2 can be expressed in closed form, the derived solution for the coefficients $(a_1,\ b_1,\ c_{11},\ c_{12},\ a_2,\ b_2,\ c_{21},\ c_{22},\ d_2)$ is given in closed form too. Furthermore, the linear REE is unique, as the expressions of ρ_1 and ρ_2 are unique. □

Proof of Proposition 3.3. Denote

$$B_1 \equiv \left(\frac{1}{\rho_1} - \frac{c_{11}}{a_1}\right)^2 \frac{1}{\tau_{s_1}},$$

$$B_2 \equiv \left(\frac{c_{11}}{a_1}\right)^2 \frac{1}{\tau_{\eta_1}},$$

$$B_3 \equiv \left(\frac{c_{12}}{a_1}\right)^2 \left(\frac{1}{\tau_{s_2}} + \frac{1}{\tau_{\eta_2}}\right)$$

so that $\mathrm{Var}^{-1}(\theta \mid P_1^{**}) = \tau_\theta\ +\ (B_1 + B_2 + B_3)^{-1}$. Recalling the coefficients contained in Proposition 3.2, direct computations yield

$$B_1 = \left\{\frac{1}{\rho_1} - \frac{\rho_1\tau_{\eta_1}(\Delta + \tau_\epsilon)}{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})(\Delta + \tau_\epsilon) + \tau_\epsilon[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]}\right\}^2 \frac{1}{\tau_{s_1}}$$

$$= \left(\frac{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})(\Delta + \tau_\epsilon) + \tau_\epsilon[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})] - \rho_1^2\tau_{\eta_1}(\Delta + \tau_\epsilon)}{\rho_1\{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})(\Delta + \tau_\epsilon) + \tau_\epsilon[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]\}}\right)^2 \frac{1}{\tau_{s_1}}$$

$$= \left\{\frac{\rho_1^2\tau_{s_1}(\Delta + \tau_\epsilon) + \tau_\epsilon[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]}{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})(\Delta + \tau_\epsilon) + \tau_\epsilon[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]}\right\}^2 \frac{1}{\rho_1^2\tau_{s_1}}.$$

Moreover,

$$B_2 = \left\{\frac{\rho_1\tau_{\eta_1}(\Delta + \tau_\epsilon)}{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})(\Delta + \tau_\epsilon) + \tau_\epsilon[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]}\right\}^2 \frac{1}{\tau_{\eta_1}}$$

$$= \tau_{\eta_1} \left\{ \frac{\rho_1(\Delta + \tau_\epsilon)}{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})(\Delta + \tau_\epsilon) + \tau_\epsilon[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]} \right\}^2$$

and

$$B_3 = \left(\frac{\tau_{\eta_2}[\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]}{\delta(\tau_{s_2} + \tau_{\eta_2})\{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})(\Delta + \tau_\epsilon) + \tau_\epsilon[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]\}} \right)^2 \left(\frac{1}{\tau_{s_2}} + \frac{1}{\tau_{\eta_2}} \right)$$

$$= \frac{\tau_{\eta_2}}{\tau_{s_2}(\tau_{\eta_2} + \tau_{s_2})} \left(\frac{\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})}{\delta\{\rho_1^2(\tau_{s_1} + \tau_{\eta_1})(\Delta + \tau_\epsilon) + \tau_\epsilon[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]\}} \right)^2.$$

For $\tau_{\eta_1} = 0$, we obtain

$$B_1 = \frac{1}{\rho_1^2 \tau_{s_1}},$$

$$B_2 = 0,$$

$$B_3 = \frac{\tau_{\eta_2}}{\tau_{s_2}(\tau_{\eta_2} + \tau_{s_2})} \left(\frac{\tau_\theta + \tau_\epsilon + \rho_1^2 \tau_{s_1}}{\delta\{\rho_1^2 \tau_{s_1}(\Delta_{01} + \tau_\epsilon) + \tau_\epsilon[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]\}} \right)^2,$$

where $\Delta_{01} \equiv \tau_\theta + \tau_\epsilon + \rho_1^2 \tau_{s_1} + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})$. From Proposition 3.2 and Table 3.1,

$$\rho_{10} \equiv \rho_1|_{\tau_{\eta_2}=0} = \frac{\delta^3 \tau_\epsilon^2 \tau_{s_2}}{1 + \delta^2 \tau_\epsilon \tau_{s_2}},$$

$$\lim_{\tau_{\eta_2} \to \infty} \rho_1 = \delta \tau_\epsilon,$$

which gives

$$B_1|_{\tau_{\eta_2}=0} = \frac{1}{\rho_{10}^2 \tau_{s_1}},$$

$$\lim_{\tau_{\eta_2} \to \infty} B_1 = \frac{1}{\delta^2 \tau_\epsilon^2 \tau_{s_1}}.$$

Thus, B_1 is smaller as $\tau_{\eta_2} \to \infty$ than for $\tau_{\eta_2} = 0$, which means that the CON effect is more pronounced for $\tau_{\eta_2} = 0$. Turning to the COMSFUN effect, we obtain

$$B_3|_{\tau_{\eta_2}=0} = 0$$

and

$$\lim_{\tau_{\eta_2} \to \infty} B_3 = \lim_{\tau_{\eta_2} \to \infty} \frac{\tau_{\eta_2}}{\tau_{s_2}(\tau_{\eta_2} + \tau_{s_2})} \left(\frac{\tau_\theta + \tau_\epsilon + \rho_1^2 \tau_{s_1}}{\delta\{\rho_1^2 \tau_{s_1}(\Delta_{01} + \tau_\epsilon) + \tau_\epsilon[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]\}} \right)^2$$

$$= 0,$$

which proves part (a) in the proposition. For $\tau_{\eta_1} > 0$, direct computations yield

$$B_1|_{\tau_{\eta_2}=0} = \left[\frac{\rho_{10}^2\tau_{s_1}(\Delta_{02}+\tau_\epsilon)+\tau_\epsilon(\tau_\epsilon+\rho_2^2\tau_{s_2})}{\rho_{10}^2(\tau_{s_1}+\tau_{\eta_1})(\Delta_{02}+\tau_\epsilon)+\tau_\epsilon(\tau_\epsilon+\rho_2^2\tau_{s_2})}\right]^2 \frac{1}{\rho_{10}^2\tau_{s_1}}$$

$$= \frac{(\rho_{10}^2\tau_{s_1}+\tau_\epsilon C)^2}{\rho_{10}^2\tau_{s_1}[\rho_{10}^2(\tau_{s_1}+\tau_{\eta_1})+\tau_\epsilon C]^2}, \tag{A3}$$

where

$$\Delta_{02} \equiv \tau_\theta + \tau_\epsilon + \rho_{10}^2(\tau_{s_1}+\tau_{\eta_1}) + \rho_2^2\tau_{s_2},$$

$$C \equiv \frac{\tau_\epsilon + \rho_2^2\tau_{s_2}}{\Delta_{02}+\tau_\epsilon} < 1.$$

Moreover,

$$B_2|_{\tau_{\eta_2}=0} = \tau_{\eta_1}\left[\frac{\rho_{10}(\Delta_{02}+\tau_\epsilon)}{\rho_{10}^2(\tau_{s_1}+\tau_{\eta_1})(\Delta_{02}+\tau_\epsilon)+\tau_\epsilon[\tau_\epsilon+\rho_2^2\tau_{s_2}]}\right]^2$$

$$= \frac{\rho_{10}^2\tau_{\eta_1}}{[\rho_{10}^2(\tau_{s_1}+\tau_{\eta_1})+\tau_\epsilon C]^2}. \tag{A4}$$

Combining (A3) and (A4) delivers

$$B_1|_{\tau_{\eta_2}=0} + B_2|_{\tau_{\eta_2}=0} = \frac{(\rho_{10}^2\tau_{s_1}+\tau_\epsilon C)^2}{\rho_{10}^2\tau_{s_1}[\rho_{10}^2(\tau_{s_1}+\tau_{\eta_1})+\tau_\epsilon C]^2} + \frac{\rho_{10}^2\tau_{\eta_1}}{[\rho_{10}^2(\tau_{s_1}+\tau_{\eta_1})+\tau_\epsilon C]^2}$$

$$= \frac{(\rho_{10}^2\tau_{s_1}+\tau_\epsilon C)^2+\rho_{10}^4\tau_{s_1}\tau_{\eta_1}}{\rho_{10}^2\tau_{s_1}[\rho_{10}^2(\tau_{s_1}+\tau_{\eta_1})+\tau_\epsilon C]^2}. \tag{A5}$$

Turning to the case of $\tau_{\eta_2} \to \infty$, we obtain

$$\lim_{\tau_{\eta_2}\to\infty} B_1 = \left(\lim_{\tau_{\eta_2}\to\infty}\frac{1}{\rho_1} - \lim_{\tau_{\eta_2}\to\infty}\frac{c_{11}}{a_1}\right)^2 \frac{1}{\tau_{s_1}}.$$

By Table 3.1, we get

$$\lim_{\tau_{\eta_2}\to\infty} B_1 = \left[\frac{1}{\delta\tau_\epsilon} - \frac{\delta\tau_\epsilon\tau_{\eta_1}}{\tau_\epsilon+\delta^2\tau_\epsilon^2(\tau_{s_1}+\tau_{\eta_1})}\right]^2 \frac{1}{\tau_{s_1}} = \frac{(\tau_\epsilon+\delta^2\tau_\epsilon^2\tau_{s_1})^2}{\delta^2\tau_\epsilon^2\tau_{s_1}[\tau_\epsilon+\delta^2\tau_\epsilon^2(\tau_{s_1}+\tau_{\eta_1})]^2}. \tag{A6}$$

It is easily checked that (A6) can be greater or smaller than (A3). Furthermore,

$$\lim_{\tau_{\eta_2}\to\infty} B_2 = \left(\lim_{\tau_{\eta_2}\to\infty}\frac{c_{11}}{a_1}\right)^2 \frac{1}{\tau_{\eta_1}} = \frac{\delta^2\tau_\epsilon^2\tau_{\eta_1}}{[\tau_\epsilon+\delta^2\tau_\epsilon^2(\tau_{s_1}+\tau_{\eta_1})]^2}. \tag{A7}$$

Again, it is easily checked that (A7) can be greater or smaller than (A4). Thus,

$$\lim_{\tau_{\eta_2}\to\infty} B_1 + \lim_{\tau_{\eta_2}\to\infty} B_2 = \frac{(\tau_\epsilon + \delta^2\tau_\epsilon^2\tau_{s_1})^2}{\delta^2\tau_\epsilon^2\tau_{s_1}[\tau_\epsilon + \delta^2\tau_\epsilon^2(\tau_{s_1}+\tau_{\eta_1})]^2} + \frac{\delta^2\tau_\epsilon^2\tau_{\eta_1}}{[\tau_\epsilon + \delta^2\tau_\epsilon^2(\tau_{s_1}+\tau_{\eta_1})]^2}$$

$$= \frac{(\tau_\epsilon + \delta^2\tau_\epsilon^2\tau_{s_1})^2 + \delta^4\tau_\epsilon^4\tau_{\eta_1}\tau_{s_1}}{\delta^2\tau_\epsilon^2\tau_{s_1}[\tau_\epsilon + \delta^2\tau_\epsilon^2(\tau_{s_1}+\tau_{\eta_1})]^2}, \tag{A8}$$

which can be greater or smaller than (A5). This proves part (b) in the proposition.

As a supplement to the proof of Proposition 3.3, we show that date-1 agents can trade more or less aggressively against Y_1 for $\tau_{\eta_2} = 0$ than as $\tau_{\eta_2} \to \infty$ (as stated on p. 55 in the main text). We know that

$$\frac{c_{11}}{a_1} = \frac{|\partial D_1/\partial Y_1|}{\partial D_1/\partial\theta} \quad \Leftrightarrow \quad \left|\frac{\partial D_1}{\partial Y_1}\right| = \frac{\partial D_1}{\partial\theta}\frac{c_{11}}{a_1}.$$

For $\tau_{\eta_2} = 0$, by Proposition 3.2 and (3.34), we obtain

$$\left|\frac{\partial D_1}{\partial Y_1}\right| = \frac{\rho_{10}}{\delta\tau_\epsilon}[\delta\tau_\epsilon + \delta\rho_{10}^2(\tau_{s_1}+\tau_{\eta_1})]\frac{\rho_{10}\tau_{\eta_1}(\Delta_{02}+\tau_\epsilon)}{\rho_{10}^2(\tau_{s_1}+\tau_{\eta_1})(\Delta_{02}+\tau_\epsilon)+\tau_\epsilon(\tau_\epsilon+\rho_2^2\tau_{s_2})}$$

$$= \frac{\rho_{10}^2\tau_{\eta_1}(\Delta_{02}+\tau_\epsilon)[\tau_\epsilon+\rho_{10}^2(\tau_{s_1}+\tau_{\eta_1})]}{\tau_\epsilon[\rho_{10}^2(\tau_{s_1}+\tau_{\eta_1})(\Delta_{02}+\tau_\epsilon)+\tau_\epsilon(\tau_\epsilon+\rho_2^2\tau_{s_2})]}$$

$$= \delta^2\tau_\epsilon\tau_{\eta_1}\frac{z^2(\Delta_{02}+\tau_\epsilon)[\tau_\epsilon+\rho_{10}^2(\tau_{s_1}+\tau_{\eta_1})]}{\rho_{10}^2(\tau_{s_1}+\tau_{\eta_1})(\Delta_{02}+\tau_\epsilon)+\tau_\epsilon(\tau_\epsilon+\rho_2^2\tau_{s_2})}, \tag{A9}$$

where

$$z \equiv \frac{\delta^2\tau_\epsilon\tau_{s_2}}{1+\delta^2\tau_\epsilon\tau_{s_2}}.$$

Analogously, as $\tau_{\eta_2} \to \infty$, we get

$$\lim_{\tau_{\eta_2}\to\infty}\left|\frac{\partial D_1}{\partial Y_1}\right| = \lim_{\tau_{\eta_2}\to\infty}\frac{\partial D_1}{\partial\theta}\lim_{\tau_{\eta_2}\to\infty}\frac{c_{11}}{a_1}$$

$$= [\delta\tau_\epsilon + \delta^3\tau_\epsilon^2(\tau_{s_1}+\tau_{\eta_1})]\frac{\delta\tau_\epsilon\tau_{\eta_1}}{\tau_\epsilon+\delta^2\tau_\epsilon^2(\tau_{s_1}+\tau_{\eta_1})}$$

$$= \delta^2\tau_\epsilon\tau_{\eta_1}. \tag{A10}$$

Note that (A10) can be smaller or greater than (A9), depending on whether the fraction in (A9) is greater or smaller than unity. □

Proof of Proposition 3.4. If $\tau_{\eta_1} = 0$, we already know that

$$B_1 = \frac{1}{\rho_1^2 \tau_{s_1}},$$

$$B_2 = 0,$$

$$B_3 = \frac{\tau_{\eta_2}}{\tau_{s_2}(\tau_{\eta_2} + \tau_{s_2})} \left(\frac{\tau_\theta + \tau_\epsilon + \rho_1^2 \tau_{s_1}}{\delta\{\rho_1^2 \tau_{s_1}(\Delta_{01} + \tau_\epsilon) + \tau_\epsilon[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]\}} \right)^2.$$

Thus,

$$\frac{\partial B_1}{\partial \tau_{\eta_2}} = -\frac{2}{\rho_1^3 \tau_{s_1}} \frac{\partial \rho_1}{\partial \tau_{\eta_2}} = -\frac{2}{\rho_1^3 \tau_{s_1}} \frac{\delta^3 \tau_\epsilon^2}{[1 + \delta^2 \tau_\epsilon (\tau_{s_2} + \tau_{\eta_2})]^2}.$$

For $\tau_{\eta_2} = 0$, we obtain

$$\left.\frac{\partial B_1}{\partial \tau_{\eta_2}}\right|_{\tau_{\eta_2} = 0} = -\frac{2}{\rho_{10}^3 \tau_{s_1}} \frac{\delta^3 \tau_\epsilon^2}{(1 + \delta^2 \tau_\epsilon \tau_{s_2})^2}.$$

Furthermore, note that

$$\frac{\delta^3 \tau_\epsilon^2}{(1 + \delta^2 \tau_\epsilon \tau_{s_2})^2} = \left(\frac{\delta^3 \tau_\epsilon^2 \tau_{s_2}}{1 + \delta^2 \tau_\epsilon \tau_{s_2}} \right)^2 \frac{1}{\delta^3 \tau_\epsilon^2 \tau_{s_2}^2} = \frac{\rho_{10}^2}{\delta^3 \tau_\epsilon^2 \tau_{s_2}^2}.$$

This eventually delivers

$$\left.\frac{\partial B_1}{\partial \tau_{\eta_2}}\right|_{\tau_{\eta_2} = 0} = -\frac{2}{\delta^3 \rho_{10} \tau_\epsilon^2 \tau_{s_1} \tau_{s_2}^2}.$$

The impact on the COMSFUN effect is given by

$$\frac{\partial B_3}{\partial \tau_{\eta_2}} = \frac{1}{(\tau_{s_2} + \tau_{\eta_2})^2} \left(\frac{\tau_\theta + \tau_\epsilon + \rho_1^2 \tau_{s_1}}{\delta\{\rho_1^2 \tau_{s_1}(\Delta_{01} + \tau_\epsilon) + \tau_\epsilon[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]\}} \right)^2 + \frac{\tau_{\eta_2}}{\tau_{s_2}(\tau_{s_2} + \tau_{\eta_2})} \frac{\partial}{\partial \tau_{\eta_2}} \left[\left(\frac{\tau_\theta + \tau_\epsilon + \rho_1^2 \tau_{s_1}}{\delta\{\rho_1^2 \tau_{s_1}(\Delta_{01} + \tau_\epsilon) + \tau_\epsilon[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]\}} \right)^2 \right].$$

For $\tau_{\eta_2} = 0$, the second summand drops out (note that the denominator in the above fraction is bounded away from zero). This gives

$$\left.\frac{\partial B_3}{\partial \tau_{\eta_2}}\right|_{\tau_{\eta_2} = 0} = \frac{1}{\tau_{s_2}^2} \left\{ \frac{\tau_\theta + \tau_\epsilon + \rho_{10}^2 \tau_{s_1}}{\delta[\rho_{10}^2 \tau_{s_1}(\tau_\theta + 2\tau_\epsilon + \rho_{10}^2 \tau_{s_1} + \rho_2^2 \tau_{s_2}) + \tau_\epsilon(\tau_\epsilon + \rho_2^2 \tau_{s_2})]} \right\}^2.$$

Thus,

$$\left.\frac{\partial[\mathrm{Var}^{-1}(\theta \mid P_1^{**})]}{\partial \tau_{\eta_2}}\right|_{\tau_{\eta_2}=0} < 0 \text{ exactly if}$$

$$-\frac{2}{\delta^3 \rho_{10} \tau_\epsilon^2 \tau_{s_1} \tau_{s_2}^2} + \frac{1}{\tau_{s_2}^2}\left\{\frac{\tau_\theta + \tau_\epsilon + \rho_{10}^2 \tau_{s_1}}{\delta[\rho_{10}^2 \tau_{s_1}(\tau_\theta + 2\tau_\epsilon + \rho_{10}^2 \tau_{s_1} + \rho_2^2 \tau_{s_2}) + \tau_\epsilon(\tau_\epsilon + \rho_2^2 \tau_{s_2})]}\right\}^2 > 0,$$

which can be written as

$$\frac{2}{\delta \rho_{10} \tau_\epsilon^2 \tau_{s_1}} < \left[\frac{\tau_\theta + \tau_\epsilon + \rho_{10}^2 \tau_{s_1}}{\rho_{10}^2 \tau_{s_1}(\tau_\theta + 2\tau_\epsilon + \rho_{10}^2 \tau_{s_1} + \rho_2^2 \tau_{s_2}) + \tau_\epsilon(\tau_\epsilon + \rho_2^2 \tau_{s_2})}\right]^2. \quad \square$$

Proof of Proposition 3.5. Note that the error terms in (3.31) and (3.35) are correlated. This requires the application of the classical projection theorem, given in Appendix B.2, to determine joint price efficiency. Direct computations yield

$$\begin{aligned}
&\mathrm{Var}\,(\theta \mid P_1^{**}, P_2^{**}) \\
&= \mathrm{Var}\,(\theta) - \begin{pmatrix} \mathrm{Cov}\,(\theta, P_1^{**}) & \mathrm{Cov}\,(\theta, P_2^{**}) \end{pmatrix} \\
&\quad \times \begin{pmatrix} \mathrm{Var}\,(P_1^{**}) & \mathrm{Cov}\,(P_1^{**}, P_2^{**}) \\ \mathrm{Cov}\,(P_1^{**}, P_2^{**}) & \mathrm{Var}\,(P_2^{**}) \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{Cov}\,(\theta, P_1^{**}) \\ \mathrm{Cov}\,(\theta, P_2^{**}) \end{pmatrix} \\
&= \tau_\theta^{-1} - \frac{1}{\mathrm{Var}\,(P_1^{**})\,\mathrm{Var}\,(P_2^{**}) - [\mathrm{Cov}(P_1^{**}, P_2^{**})]^2} \begin{pmatrix} \tau_\theta^{-1} & \tau_\theta^{-1} \end{pmatrix} \\
&\quad \times \begin{pmatrix} \mathrm{Var}\,(P_2^{**}) & -\,\mathrm{Cov}\,(P_1^{**}, P_2^{**}) \\ -\,\mathrm{Cov}\,(P_1^{**}, P_2^{**}) & \mathrm{Var}\,(P_1^{**}) \end{pmatrix} \begin{pmatrix} \tau_\theta^{-1} \\ \tau_\theta^{-1} \end{pmatrix} \\
&= \tau_\theta^{-1} - \frac{\tau_\theta^{-1}}{\mathrm{Var}\,(P_1^{**})\,\mathrm{Var}\,(P_2^{**}) - [\mathrm{Cov}(P_1^{**}, P_2^{**})]^2} \\
&\quad \times \begin{pmatrix} \mathrm{Var}\,(P_2^{**}) - \mathrm{Cov}\,(P_1^{**}, P_2^{**}) & \mathrm{Var}\,(P_1^{**}) - \mathrm{Cov}\,(P_1^{**}, P_2^{**}) \end{pmatrix} \begin{pmatrix} \tau_\theta^{-1} \\ \tau_\theta^{-1} \end{pmatrix} \\
&= \tau_\theta^{-1} - \tau_\theta^{-2} \frac{\mathrm{Var}\,(P_1^{**}) + \mathrm{Var}\,(P_2^{**}) - 2\,\mathrm{Cov}\,(P_1^{**}, P_2^{**})}{\mathrm{Var}\,(P_1^{**})\,\mathrm{Var}\,(P_2^{**}) - [\mathrm{Cov}(P_1^{**}, P_2^{**})]^2}.
\end{aligned}$$

Thus,

$$\operatorname{Var}^{-1}(\theta \mid P_1^{**}, P_2^{**}) = \left\{ \tau_\theta^{-1} - \tau_\theta^{-2} \frac{\operatorname{Var}(P_1^{**}) + \operatorname{Var}(P_2^{**}) - 2\operatorname{Cov}(P_1^{**}, P_2^{**})}{\operatorname{Var}(P_1^{**})\operatorname{Var}(P_2^{**}) - [\operatorname{Cov}(P_1^{**}, P_2^{**})]^2} \right\}^{-1},$$

where

$$\operatorname{Var}(P_1^{**}) = \tau_\theta^{-1} + \left(\frac{1}{\rho_1} - \frac{c_{11}}{a_1}\right)^2 \frac{1}{\tau_{s_1}} + \left(\frac{c_{11}}{a_1}\right)^2 \frac{1}{\tau_{\eta_1}} + \left(\frac{c_{12}}{a_1}\right)^2 \left(\frac{1}{\tau_{s_2}} + \frac{1}{\tau_{\eta_2}}\right),$$

$$\operatorname{Var}(P_2^{**}) = \tau_\theta^{-1} + \left(\frac{1}{\rho_2} - \frac{c_{22}}{a_2}\right)^2 \frac{1}{\tau_{s_2}} + \left(\frac{c_{22}}{a_2}\right)^2 \frac{1}{\tau_{\eta_2}} + \left(\frac{c_{21}}{a_2}\right)^2 \left(\frac{1}{\tau_{s_1}} + \frac{1}{\tau_{\eta_1}}\right),$$

$$\operatorname{Cov}(P_1^{**}, P_2^{**}) = \tau_\theta^{-1} - \left(\frac{1}{\rho_1} - \frac{c_{11}}{a_1}\right)\frac{c_{21}}{a_2}\frac{1}{\tau_{s_1}} + \frac{c_{12}}{a_1}\left(\frac{1}{\rho_2} - \frac{c_{22}}{a_2}\right)\frac{1}{\tau_{s_2}} + \frac{c_{11}}{a_1}\frac{c_{21}}{a_2}\frac{1}{\tau_{\eta_1}} - \frac{c_{12}}{a_1}\frac{c_{22}}{a_2}\frac{1}{\tau_{\eta_2}}. \quad \square$$

Proof of Proposition 3.6. Along the proof, we will make use of three laws:

1. *Law of iterated expectations.* Let X and Z be two sets of random variables and Y a single random variable. Then, if $X \subseteq Z$, it holds that $\mathrm{E}[\mathrm{E}(Y \mid Z) \mid X] = \mathrm{E}(Y \mid X)$.

2. *Law of total conditional variance.* Let X, Y, and Z be three random variables. The law of total conditional variance states that

$\operatorname{Var}(Y \mid X) = \mathrm{E}[\operatorname{Var}(Y \mid Z, X) \mid X] + \operatorname{Var}[\mathrm{E}(Y \mid X, Z) \mid X]$.

If X, Y, and Z are normal, we have

$\operatorname{Var}(Y \mid X) = \operatorname{Var}(Y \mid Z, X) + \operatorname{Var}[\mathrm{E}(Y \mid X, Z) \mid X]$,

as $\operatorname{Var}(Y \mid Z, X)$ is non-random in this case.

3. *Law of total covariance.* Let X, Y, and Z be three random variables. The law of total covariance states that

$\operatorname{Cov}(Y, Z) = \mathrm{E}[\operatorname{Cov}(Y, Z \mid X)] + \operatorname{Cov}[\mathrm{E}(Y \mid X), \mathrm{E}(Z \mid X)]$.

If X, Y, and Z are normal, we have

$\operatorname{Cov}(Y, Z) = \operatorname{Cov}(Y, Z \mid X) + \operatorname{Cov}[\mathrm{E}(Y \mid X), \mathrm{E}(Z \mid X)]$,

as $\operatorname{Cov}(Y, Z \mid X)$ is non-random in this case.

Following Avdis (2016, Appendix B), the optimization problem of agent i can be written as follows:

$$V(\pi_i) = \max_{D_{1i}} \mathrm{E}\left[\max_{D_{2i}} \mathrm{E}\left(-e^{-\delta^{-1}\pi_i} \middle| I_{2i}\right) \middle| I_{1i}\right]$$

$$= -\min_{D_{1i}} \left[\mathrm{E} \left(e^{-\delta^{-1}(P_2 - P_1)D_{1i}} \min_{D_{2i}} \left\{ \mathrm{E} \left[e^{-\delta^{-1}(\theta - P_2)D_{2i}} \middle| I_{2i} \right] \right\} \middle| I_{1i} \right) \right]. \tag{A11}$$

Recalling the results contained in Appendix B.3.1, the innermost optimization problem in (A11) becomes

$$\begin{aligned} &\min_{D_{2i}} \left\{ \mathrm{E} \left[e^{-\delta^{-1}(\theta - P_2)D_{2i}} \middle| I_{2i} \right] \right\} \\ &= \min_{D_{2i}} \left[\exp \left(-\frac{1}{\delta} \left\{ [\mathrm{E}(\theta \mid I_{2i}) - P_2] \, D_{2i} - \frac{1}{2\delta} \mathrm{Var}(\theta \mid I_{2i}) D_{2i}^2 \right\} \right) \right]. \end{aligned} \tag{A12}$$

As before, the first-order condition of the above objective function in D_{2i} immediately gives

$$D_{2i} = \frac{\delta[\mathrm{E}(\theta \mid I_{2i}) - P_2]}{\mathrm{Var}\,(\theta \mid I_{2i})}.$$

Plugging the optimal date-2 demand function back into (A12) yields

$$\begin{aligned} &\exp \left[-\frac{1}{\delta} \left([\mathrm{E}(\theta \mid I_{2i}) - P_2] \frac{\delta[\mathrm{E}(\theta \mid I_{2i}) - P_2]}{\mathrm{Var}(\theta \mid I_{2i})} \right. \right. \\ &\left. \left. - \frac{1}{2\delta} \mathrm{Var}(\theta \mid I_{2i}) \left\{ \frac{\delta[\mathrm{E}(\theta \mid I_{2i}) - P_2]}{\mathrm{Var}(\theta \mid I_{2i})} \right\}^2 \right) \right] \\ &= \exp \left\{ -\frac{[\mathrm{E}(\theta - P_2 \mid I_{2i})]^2}{2\mathrm{Var}(\theta \mid I_{2i})} \right\}. \end{aligned} \tag{A13}$$

By plugging (A13) into (A11), the value function becomes

$$V(\pi_i) = -\min_{D_{1i}} \left[\mathrm{E} \left(\exp \left\{ -\delta^{-1}(P_2 - P_1)D_{1i} - \frac{[\mathrm{E}(\theta - P_2 | I_{2i})]^2}{2\mathrm{Var}(\theta \mid I_{2i})} \right\} \middle| I_{1i} \right) \right].$$

In Appendix B.4, we prove that

$$\mathrm{E}\left[\exp(x - y^2)\right] = \frac{\exp \left\{ \mathrm{E}(X) + \frac{1}{2}\mathrm{Var}(X) - \frac{[\mathrm{E}(Y) + \mathrm{Cov}(X, Y)]^2}{1 + 2\mathrm{Var}(Y)} \right\}}{\sqrt{1 + 2\mathrm{Var}(Y)}},$$

where X and Y are two jointly normal random variables. Setting $X = -\delta^{-1}(P_2 - P_1)D_{1i}$ and $Y = \mathrm{E}(\theta - P_2 \mid I_{2i}) / \sqrt{2\mathrm{Var}(\theta \mid I_{2i})}$, conditional on I_{1i}, we obtain

$$\mathrm{E}(X \mid I_{1i}) = -\delta^{-1}[\mathrm{E}(P_2 \mid I_{1i}) - P_1]D_{1i},$$

$$\mathrm{Var}(X \mid I_{1i}) = \delta^{-2}\,\mathrm{Var}(P_2 \mid I_{1i}) D_{1i}^2.$$

Furthermore, by the law of iterated expectations,

$$\mathrm{E}(Y \mid I_{1i}) = \frac{\mathrm{E}\left[\mathrm{E}(\theta - P_2 \mid I_{2i}) \mid I_{1i}\right]}{\sqrt{2\mathrm{Var}(\theta \mid I_{2i})}} = \frac{\mathrm{E}(\theta - P_2 \mid I_{1i})}{\sqrt{2\mathrm{Var}(\theta \mid I_{2i})}}.$$

Applying the law of total conditional variance delivers

$$\begin{aligned}
1 + 2\mathrm{Var}(Y \mid I_{1i}) &= 1 + \frac{\mathrm{Var}\left[\mathrm{E}(\theta - P_2 \mid I_{2i}) \mid I_{1i}\right]}{\mathrm{Var}\left[\theta | I_{2i}\right]} \\
&= 1 + \frac{\mathrm{Var}\left(\theta - P_2 \mid I_{1i}\right) - \mathrm{E}\left[\mathrm{Var}\left(\theta - P_2 \mid I_{2i}\right) \mid I_{1i}\right]}{\mathrm{Var}(\theta \mid I_{2i})} \\
&= 1 + \frac{\mathrm{Var}\left(\theta - P_2 \mid I_{1i}\right) - \mathrm{Var}\left(\theta - P_2 \mid I_{2i}\right)}{\mathrm{Var}(\theta \mid I_{2i})} \\
&= \frac{\mathrm{Var}\left(\theta - P_2 \mid I_{1i}\right)}{\mathrm{Var}(\theta \mid I_{2i})}.
\end{aligned}$$

Moreover,

$$\mathrm{Cov}(X, Y \mid I_{1i}) = -\frac{D_{1i}}{\delta\sqrt{2\mathrm{Var}(\theta \mid I_{2i})}}\,\mathrm{Cov}[P_2 - P_1, \mathrm{E}(\theta - P_2 \mid I_{2i}) \mid I_{1i}].$$

Note that

$$\mathrm{Cov}[P_2 - P_1, \mathrm{E}(\theta - P_2 \mid I_{2i}) \mid I_{1i}] = \mathrm{Cov}\left[\mathrm{E}(P_2 - P_1 \mid I_{2i}), \mathrm{E}(\theta - P_2 \mid I_{2i}) \mid I_{1i}\right].$$

Then, by the law of total covariance and the law of iterated expectations, we have

$$\begin{aligned}
&\mathrm{Cov}\left[\mathrm{E}(P_2 - P_1 \mid I_{2i}), \mathrm{E}(\theta - P_2 \mid I_{2i}) \mid I_{1i}\right] \\
&= \mathrm{Cov}\left[\mathrm{E}(P_2 - P_1 \mid I_{2i}), \mathrm{E}(\theta - P_2 \mid I_{2i})\right] \\
&\quad - \mathrm{Cov}\{\mathrm{E}\left[\mathrm{E}(P_2 - P_1 \mid I_{2i}) \mid I_{1i}\right], \mathrm{E}\left[\mathrm{E}(\theta - P_2 \mid I_{2i}) \mid I_{1i}\right]\} \\
&= \mathrm{Cov}\left[\mathrm{E}(P_2 - P_1 \mid I_{2i}), \mathrm{E}(\theta - P_2 \mid I_{2i})\right] - \mathrm{Cov}\left[\mathrm{E}(P_2 - P_1 \mid I_{1i}), \mathrm{E}(\theta - P_2 \mid I_{1i})\right].
\end{aligned}$$

Again applying the law of total covariance to both above terms delivers

$$\begin{aligned}
&\mathrm{Cov}\left[\mathrm{E}(P_2 - P_1 \mid I_{2i}), \mathrm{E}(\theta - P_2 \mid I_{2i})\right] - \mathrm{Cov}\left[\mathrm{E}(P_2 - P_1 \mid I_{1i}), \mathrm{E}(\theta - P_2 \mid I_{1i})\right] \\
&= \mathrm{Cov}(P_2 - P_1, \theta - P_2) - \mathrm{Cov}(P_2 - P_1, \theta - P_2 \mid I_{2i}) - \mathrm{Cov}(P_2 - P_1, \theta - P_2) \\
&\quad + \mathrm{Cov}(P_2 - P_1, \theta - P_2 \mid I_{1i})
\end{aligned}$$

$$= \mathrm{Cov}(P_2, \theta - P_2 \,|\, I_{1i}),$$

where the last equation follows from the fact that $\mathrm{Cov}(P_2 - P_1, \theta - P_2 \,|\, I_{2i}) = 0$, as $P_2 - P_1$ is non-random conditional on I_{2i}. Thus,

$$\mathrm{Cov}(X, Y \,|\, I_{1i}) = -\frac{D_{1i}}{\delta\sqrt{2\mathrm{Var}(\theta \,|\, I_{2i})}} \mathrm{Cov}(P_2, \theta - P_2 \,|\, I_{1i}).$$

Further computations yield

$$\begin{aligned}
&[\mathrm{E}(Y \,|\, I_{1i}) + \mathrm{Cov}(X, Y \,|\, I_{1i})]^2 \\
= &\left[\frac{\mathrm{E}(\theta - P_2 \,|\, I_{1i})}{\sqrt{2\mathrm{Var}(\theta \,|\, I_{2i})}} - \frac{D_{1i}}{\delta\sqrt{2\mathrm{Var}(\theta \,|\, I_{2i})}} \mathrm{Cov}(P_2, \theta - P_2 \,|\, I_{1i})\right]^2 \\
= &\frac{1}{2\mathrm{Var}(\theta \,|\, I_{2i})} \left[\mathrm{E}(\theta - P_2 \,|\, I_{1i}) - \frac{1}{\delta} D_{1i} \mathrm{Cov}(P_2, \theta - P_2 \,|\, I_{1i})\right]^2.
\end{aligned}$$

This delivers

$$\begin{aligned}
&\frac{[\mathrm{E}(Y \,|\, I_{1i}) + \mathrm{Cov}(X, Y \,|\, I_{1i})]^2}{1 + 2\mathrm{Var}(Y \,|\, I_{1i})} \\
= &\frac{1}{2\mathrm{Var}(\theta - P_2 \,|\, I_{1i})} \left[\mathrm{E}(\theta - P_2 \,|\, I_{1i}) - \frac{1}{\delta} D_{1i} \mathrm{Cov}(P_2, \theta - P_2 \,|\, I_{1i})\right]^2.
\end{aligned}$$

The value function becomes

$$\begin{aligned}
&V(\pi_i) \\
= &-\min_{D_{1i}} \left(\sqrt{\frac{\mathrm{Var}\,(\theta \,|\, I_{2i})}{\mathrm{Var}\,(\theta - P_2 \,|\, I_{1i})}} \exp\left\{ -\delta^{-1}[\mathrm{E}(P_2 \,|\, I_{1i}) - P_1] D_{1i} \right.\right. \\
&\left.\left. + \frac{1}{2\delta^2}\mathrm{Var}(P_2 \,|\, I_{1i}) D_{1i}^2 - \frac{\left[\mathrm{E}(\theta - P_2 \,|\, I_{1i}) - \frac{1}{\delta} D_{1i} \mathrm{Cov}(P_2, \theta - P_2 \,|\, I_{1i})\right]^2}{2\mathrm{Var}(\theta - P_2 \,|\, I_{1i})} \right\} \right).
\end{aligned}$$

Then, the first-order condition of the objective function in D_{1i} is

$$\begin{aligned}
&-\delta^{-1}[\mathrm{E}(P_2 \,|\, I_{1i}) - P_1] + \frac{1}{\delta^2}\mathrm{Var}(P_2 \,|\, I_{1i}) D_{1i} + \frac{\mathrm{Cov}(P_2, \theta - P_2 \,|\, I_{1i})}{\delta \mathrm{Var}(\theta - P_2 \,|\, I_{1i})} \\
&\times \left[\mathrm{E}(\theta - P_2 \,|\, I_{1i}) - \frac{1}{\delta} D_{1i} \mathrm{Cov}(P_2, \theta - P_2 \,|\, I_{1i})\right] = 0.
\end{aligned}$$

Solving for D_{1i} eventually yields the optimal demand at date 1:

$$D_{1i}\left\{\frac{1}{\delta}\mathrm{Var}(P_2\,|\,I_{1i})-\frac{[\mathrm{Cov}(P_2,\theta-P_2\,|\,I_{1i})]^2}{\delta\,\mathrm{Var}(\theta-P_2\,|\,I_{1i})}\right\}$$

$$=\ \mathrm{E}(P_2-P_1\,|\,I_{1i})-\frac{\mathrm{Cov}(P_2,\theta-P_2\,|\,I_{1i})}{\mathrm{Var}(\theta-P_2\,|\,I_{1i})}\,\mathrm{E}(\theta-P_2\,|\,I_{1i})$$

$$\Leftrightarrow D_{1i}=\delta\,\frac{\mathrm{E}[P_2-h(\theta-P_2)\,|\,I_{1i}]-P_1}{\mathrm{Var}(P_2\,|\,I_{1i})-h\,\mathrm{Cov}(P_2,\theta-P_2\,|\,I_{1i})},$$

where $h\equiv\dfrac{\mathrm{Cov}(P_2,\theta-P_2\,|\,I_{1i})}{\mathrm{Var}(\theta-P_2\,|\,I_{1i})}$.

The given demand function can be further developed as follows:

$$D_{1i}=\delta\,\frac{\mathrm{E}[P_2-\mathrm{h}(\theta-P_2)\,|\,I_{1i}]-P_1}{\mathrm{Var}(P_2\,|\,I_{1i})\left\{1-\dfrac{[\mathrm{Cov}(P_2,\theta-P_2\,|\,I_{1i})]^2}{\mathrm{Var}(P_2\,|\,I_{1i})\,\mathrm{Var}(\theta-P_2\,|\,I_{1i})}\right\}}$$

$$=\delta\,\frac{\mathrm{E}[P_2-\mathrm{h}(\theta-P_2)\,|\,I_{1i}]-P_1}{\mathrm{Var}(P_2\,|\,I_{1i})(1-Corr^2)}$$

$$=\delta\,\frac{\mathrm{E}(P_2\,|\,I_{1i})-P_1}{\mathrm{Var}(P_2\,|\,I_{1i})(1-Corr^2)}-\delta h\,\frac{\mathrm{E}(\theta-P_2\,|\,I_{1i})}{\mathrm{Var}(P_2\,|\,I_{1i})(1-Corr^2)},$$

which equals equation (3.37) in the main text. □

Proof of Proposition 3.7. As in the OLG model, we obtain the equilibrium price functions in the LLA model by using backward induction. A long-lived agent's date-2 demand function shows the same general form as that of an agent in the OLG model (cf. (3.25) and (3.36)). Furthermore, the date-2 information sets are identical in both models. They consist of one private fundamental signal, two public non-fundamental signals, and both prices. Thus, the derivation of the equilibrium function of P_2 in the LLA model follows exactly the same steps as in the OLG model. Without any further computations, we can conclude that

$$a_2=\frac{\tau_\epsilon+\rho_2^2(\tau_{s_2}+\tau_{\eta_2})}{\Delta},\ \ b_2=\frac{1+\delta\rho_2(\tau_{s_2}+\tau_{\eta_2})}{\delta\Delta},\ \ c_{21}=\frac{\rho_1\tau_{\eta_1}-\rho_1^2(\tau_{s_1}+\tau_{\eta_1})\frac{c_{11}}{a_1}}{\Delta},$$

$$c_{22}=\frac{\rho_2\tau_{\eta_2}+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})\frac{c_{12}}{a_1}}{\Delta},\ \ d_2=\frac{\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}{a_1\Delta},\ \ \rho_2\equiv\frac{a_2}{b_2}=\delta\tau_\epsilon,$$

where $\Delta\equiv\tau_\theta+\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})+\rho_2^2(\tau_{s_2}+\tau_{\eta_2})$. We know from the proof of Proposition 3.6 that a long-lived agent's date-1 demand function can be written as

$$D_{1i}=\delta\,\frac{\mathrm{E}[P_2-h(\theta-P_2)\,|\,I_{1i}]-P_1}{\mathrm{Var}(P_2\,|\,I_{1i})-h\,\mathrm{Cov}(P_2,\theta-P_2\,|\,I_{1i})},$$

where $h \equiv \dfrac{\mathrm{Cov}(P_2, \theta - P_2 \mid I_{1i})}{\mathrm{Var}(\theta - P_2 \mid I_{1i})}$.

Recall that $I_{1i} = (x_i, P_1, Y_1, Y_2)$. Then,

$$\begin{aligned}
\mathrm{Cov}(\theta - P_2, P_2 \mid I_{1i}) &= \mathrm{Cov}(\theta, P_2 \mid I_{1i}) + \mathrm{Cov}(-P_2, P_2 \mid I_{1i}) \\
&= \mathrm{Cov}(\theta, a_2\,\theta + b_2 s_2 - c_{21} Y_1 - c_{22} Y_2 + d_2 P_1 \mid I_{1i}) - \mathrm{Var}(P_2 \mid I_{1i}) \\
&= a_2 \mathrm{Var}(\theta \mid I_{1i}) + \mathrm{Cov}(\theta, s_2 \mid I_{1i}) - \mathrm{Var}(P_2 \mid I_{1i}) \\
&= a_2 \mathrm{Var}(\theta \mid I_{1i}) - \mathrm{Var}(a_2\,\theta + b_2 s_2 - c_{21} Y_1 - c_{22} Y_2 + d_2 P_1 \mid I_{1i}) \\
&\quad + \mathrm{Cov}(\theta, s_2 \mid I_{1i}) \\
&= a_2(1 - a_2)\mathrm{Var}(\theta \mid I_{1i}) - b_2^2 \mathrm{Var}(s_2 \mid I_{1i}) + \mathrm{Cov}(\theta, s_2 \mid I_{1i}).
\end{aligned}$$

By the law of total covariance, we obtain

$$\begin{aligned}
\mathrm{Cov}(\theta, s_2 \mid I_{1i}) &= \mathrm{Cov}(\theta, s_2) - \mathrm{Cov}[\mathrm{E}(\theta \mid I_{1i}), \mathrm{E}(s_2 \mid I_{1i})] \\
&= -\,\mathrm{Cov}[\mathrm{E}(\theta \mid I_{1i}), \mathrm{E}(s_2 \mid I_{1i})].
\end{aligned}$$

We know that

$$\mathrm{E}(\theta \mid I_{1i}) = \frac{\tau_\epsilon\, x_i + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\, P_1^*}{\tau_\theta + \tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})},$$

$$\mathrm{E}(s_2 \mid I_{1i}) = \frac{\tau_{\eta_2} Y_2}{\tau_{s_2} + \tau_{\eta_2}},$$

and that P_1^* is a linear function of θ, s_1, and η_1 (see Subsection 3.2.2). Thus, $\mathrm{E}(\theta \mid I_{1i})$ can be expressed as a linear function of θ, ϵ_i, s_1, and η_1 and $\mathrm{E}(s_2 \mid I_{1i})$ as a linear function of s_2 and η_2. Since the respective random variables are pairwise uncorrelated,

$$\mathrm{Cov}(\theta, s_2 \mid I_{1i}) = -\,\mathrm{Cov}[\mathrm{E}(\theta \mid I_{1i}), \mathrm{E}(s_2 \mid I_{1i})] = 0.$$

This yields:

$$\mathrm{Cov}(\theta - P_2, P_2 \mid I_{1i}) = a_2(1 - a_2)\mathrm{Var}(\theta \mid I_{1i}) - b_2^2 \mathrm{Var}(s_2 \mid I_{1i}).$$

Moreover,

$$\mathrm{Var}(\theta - P_2 \mid I_{1i}) = \mathrm{Var}(\theta - a_2\,\theta - b_2 s_2 + c_{21} Y_1 + c_{22} Y_2 - d_2 P_1 \mid I_{1i})$$

$$= \mathrm{Var}[(1-a_2)\,\theta - b_2 s_2 \,|\, I_{1i}]$$

$$= (1-a_2)^2\,\mathrm{Var}(\theta \,|\, I_{1i}) + b_2^2\,\mathrm{Var}(s_2 \,|\, I_{1i}).$$

Denote $\Gamma_1 \equiv \mathrm{Var}(\theta \,|\, I_{1i})$ and $\Gamma_2 \equiv \mathrm{Var}(s_2 \,|\, I_{1i})$. Then,

$$h = \frac{\mathrm{Cov}(P_2, \theta - P_2 \,|\, I_{1i})}{\mathrm{Var}(\theta - P_2 \,|\, I_{1i})} = \frac{a_2(1-a_2)\Gamma_1 - b_2^2\Gamma_2}{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}.$$

Furthermore,

$$\mathrm{E}[P_2 - h(\theta - P_2) \,|\, I_{1i}]$$

$$= (1+h)\mathrm{E}(a_2\,\theta + b_2 s_2 - c_{21}Y_1 - c_{22}Y_2 + d_2 P_1 \,|\, I_{1i}) - h\,\mathrm{E}(\theta \,|\, I_{1i})$$

$$= \,[(1+h)a_2 - h]\,\mathrm{E}(\theta \,|\, I_{1i}) + (1+h)\,[b_2\mathrm{E}(s_2 \,|\, I_{1i}) - c_{21}Y_1 - c_{22}Y_2 + d_2 P_1]$$

and

$$\mathrm{Var}(P_2|I_{1i}) - h\,\mathrm{Cov}(P_2, \theta - P_2 \,|\, I_{1i})$$

$$= a_2^2\Gamma_1 + b_2^2\Gamma_2 - \frac{[a_2(1-a_2)\Gamma_1 - b_2^2\Gamma_2]^2}{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}$$

$$= \frac{a_2^2(1-a_2)^2\Gamma_1^2 + b_2^4\Gamma_2 + [a_2^2 b_2^2 + b_2^2(1-a_2)^2]\Gamma_1\Gamma_2}{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}$$

$$- \frac{a_2^2(1-a_2)^2\Gamma_1^2 + b_2^4\Gamma_2^2 - 2a_2(1-a_2)b_2^2\Gamma_1\Gamma_2}{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}$$

$$= \frac{b_2^2\Gamma_1\Gamma_2[a_2^2 + (1-a_2)^2 - 2a_2(1-a_2)]}{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}$$

$$= \frac{b_2^2\Gamma_1\Gamma_2}{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}.$$

By $\mathrm{E}(\theta \,|\, I_{1i}) = \Gamma_1[\tau_\epsilon\,x_i + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\,P_1^*]$ and $\mathrm{E}(s_2 \,|\, I_{1i}) = \tau_{\eta_2}\Gamma_2 Y_2$, the date-1 demand function becomes

$$\frac{D_{1i}}{\delta} = \frac{[(1+h)a_2 - h]\Gamma_1[\tau_\epsilon\,x_i + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\,P_1^*]}{\dfrac{b_2^2\Gamma_1\Gamma_2}{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}}$$

$$+ \frac{(1+h)[b_2\tau_{\eta_2}\Gamma_2 Y_2 - c_{21}Y_1 - c_{22}Y_2 + d_2 P_1]}{\dfrac{b_2^2\Gamma_1\Gamma_2}{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}} - \frac{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}{b_2^2\Gamma_1\Gamma_2}P_1.$$

Note that

$$1+h = 1+\frac{a_2(1-a_2)\Gamma_1 - b_2^2\Gamma_2}{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}$$

$$= \frac{\Gamma_1 + a_2^2\Gamma_1 - 2a_2\Gamma_1 + a_2(1-a_2)\Gamma_1}{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}$$

$$= \frac{(1-a_2)\Gamma_1}{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}$$

and, thus,

$$(1+h)a_2 - h = \frac{a_2(1-a_2)\Gamma_1}{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2} - \frac{a_2(1-a_2)\Gamma_1 - b_2^2\Gamma_2}{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}$$

$$= \frac{b_2^2\Gamma_2}{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}.$$

With all this in hand, the date-1 demand function boils down to

$$D_{1i} = \delta\tau_\epsilon\, x_i + \delta\rho_1^2\left(\tau_{s_1}+\tau_{\eta_1}\right)P_1^* + \frac{\delta(1-a_2)}{b_2^2\Gamma_2}\left(b_2\frac{\tau_{\eta_2}}{\tau_{s_2}+\tau_{\eta_2}}Y_2 - c_{21}Y_1 - c_{22}Y_2 + d_2P_1\right)$$

$$-\,\delta\frac{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}{b_2^2\Gamma_1\Gamma_2}P_1,$$

which is equal to (3.40) in the main text. Then, market clearing at date 1 implies:

$$0 = \frac{s_1}{\delta} + \int_0^1 \frac{D_{1i}}{\delta}di$$

$$= \frac{s_1}{\delta} + \tau_\epsilon\,\theta + \rho_1^2\left(\tau_{s_1}+\tau_{\eta_1}\right)P_1^* + \frac{1-a_2}{b_2^2\Gamma_2}\left(b_2\frac{\tau_{\eta_2}}{\tau_{s_2}+\tau_{\eta_2}}Y_2 - c_{21}Y_1 - c_{22}Y_2 + d_2P_1\right)$$

$$-\,\frac{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}{b_2^2\Gamma_1\Gamma_2}P_1$$

$$= \frac{s_1}{\delta} + \tau_\epsilon\,\theta + \rho_1^2\left(\tau_{s_1}+\tau_{\eta_1}\right)\left[\frac{P_1 + c_{11}Y_1 - c_{12}Y_2}{a_1} - \frac{\tau_{\eta_1}}{\rho_1(\tau_{s_1}+\tau_{\eta_1})}Y_1\right]$$

$$+\,\frac{1-a_2}{b_2^2\Gamma_2}\left(b_2\frac{\tau_{\eta_2}}{\tau_{s_2}+\tau_{\eta_2}}Y_2 - c_{21}Y_1 - c_{22}Y_2 + d_2P_1\right) - \frac{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}{b_2^2\Gamma_1\Gamma_2}P_1$$

$$= \tau_\epsilon\,\theta + \frac{s_1}{\delta} - \left[\rho_1\tau_{\eta_1} - \rho_1^2\left(\tau_{s_1}+\tau_{\eta_1}\right)\frac{c_{11}}{a_1} + \frac{1-a_2}{b_2^2\Gamma_2}\,c_{21}\right]Y_1$$

$$+\left[\frac{1-a_2}{b_2^2\Gamma_2}\left(b_2\frac{\tau_{\eta_2}}{\tau_{s_2}+\tau_{\eta_2}} - c_{22}\right) - \rho_1^2\left(\tau_{s_1}+\tau_{\eta_1}\right)\frac{c_{12}}{a_1}\right]Y_2$$

$$- \left[\frac{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}{b_2^2\Gamma_1\Gamma_2} - \frac{1-a_2}{b_2^2\Gamma_2}\, d_2 - \frac{\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}{a_1} \right] P_1,$$

where $\int_0^1 x_i\, di = \theta$ again follows from the strong law of large numbers. Solving for P_1 gives

$$P_1 = \left[\frac{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}{b_2^2\Gamma_1\Gamma_2} - \frac{1-a_2}{b_2^2\Gamma_2}\, d_2 - \frac{\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}{a_1} \right] \Bigg\{ \tau_\epsilon\, \theta + \frac{s_1}{\delta}$$
$$- \left[\rho_1\tau_{\eta_1} - \rho_1^2\left(\tau_{s_1}+\tau_{\eta_1}\right) \frac{c_{11}}{a_1} + \frac{1-a_2}{b_2^2\Gamma_2}\, c_{21} \right] Y_1 \tag{A14}$$
$$+ \left[\frac{1-a_2}{b_2^2\Gamma_2} \left(b_2 \frac{\tau_{\eta_2}}{\tau_{s_2}+\tau_{\eta_2}} - c_{22} \right) - \rho_1^2\left(\tau_{s_1}+\tau_{\eta_1}\right) \frac{c_{12}}{a_1} \right] Y_2 \Bigg\}.$$

By invoking rational expectations, we obtain

$$a_1 = \frac{\tau_\epsilon}{\frac{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}{b_2^2\Gamma_1\Gamma_2} - \frac{1-a_2}{b_2^2\Gamma_2}\, d_2 - \frac{\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}{a_1}}$$

$$\Leftrightarrow \frac{\tau_\epsilon}{a_1} = \frac{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}{b_2^2\Gamma_1\Gamma_2} - \frac{1-a_2}{b_2^2\Gamma_2} d_2 - \frac{\rho_1^2\left(\tau_{s_1}+\tau_{\eta_1}\right)}{a_1}$$

$$= \frac{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}{b_2^2\Gamma_1\Gamma_2} - \frac{1-a_2}{b_2^2\Gamma_2} \frac{\rho_1^2\left(\tau_{s_1}+\tau_{\eta_1}\right)}{a_1\Delta} - \frac{\rho_1^2\left(\tau_{s_1}+\tau_{\eta_1}\right)}{a_1}$$

$$\Leftrightarrow a_1 = \frac{\tau_\epsilon + \frac{1-a_2}{b_2^2\Gamma_2} \frac{\rho_1^2\left(\tau_{s_1}+\tau_{\eta_1}\right)}{\Delta} + \rho_1^2\left(\tau_{s_1}+\tau_{\eta_1}\right)}{\frac{(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2}{b_2^2\Gamma_1\Gamma_2}}$$

$$= \frac{\left[\tau_\epsilon + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})\right]\Gamma_1\Gamma_2\Delta b_2^2 + (1-a_2)\Gamma_1\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}{\Delta[(1-a_2)^2\Gamma_1 + b_2^2\Gamma_2]}.$$

Furthermore,

$$\frac{c_{11}}{a_1} = \frac{\rho_1\tau_{\eta_1} - \rho_1^2\left(\tau_{s_1}+\tau_{\eta_1}\right) \frac{c_{11}}{a_1} + \frac{1-a_2}{b_2^2\Gamma_2}\, c_{21}}{\tau_\epsilon}$$

$$= \frac{\rho_1\tau_{\eta_1} - \rho_1^2\left(\tau_{s_1}+\tau_{\eta_1}\right) \frac{c_{11}}{a_1} + \frac{1-a_2}{b_2^2\Gamma_2} \frac{\rho_1\tau_{\eta_1} - \rho_1^2\left(\tau_{s_1}+\tau_{\eta_1}\right)\frac{c_{11}}{a_1}}{\Delta}}{\tau_\epsilon}$$

$$= \frac{\rho_1 \tau_{\eta_1} \left(1 + \frac{1-a_2}{\Delta b_2^2 \Gamma_2}\right) - \rho_1^2 \left(\tau_{s_1} + \tau_{\eta_1}\right) \left(1 + \frac{1-a_2}{\Delta b_2^2 \Gamma_2}\right) \frac{c_{11}}{a_1}}{\tau_\epsilon}$$

$$\Leftrightarrow \frac{c_{11}}{a_1} = \frac{\rho_1 \tau_{\eta_1} \left(1 + \frac{1-a_2}{\Delta b_2^2 \Gamma_2}\right)}{\tau_\epsilon + \rho_1^2 \left(\tau_{s_1} + \tau_{\eta_1}\right) \left(1 + \frac{1-a_2}{\Delta b_2^2 \Gamma_2}\right)}$$

$$\Leftrightarrow c_{11} = a_1 \frac{\rho_1 \tau_{\eta_1} \left(1 + \frac{1-a_2}{\Delta b_2^2 \Gamma_2}\right)}{\tau_\epsilon + \rho_1^2 \left(\tau_{s_1} + \tau_{\eta_1}\right) \left(1 + \frac{1-a_2}{\Delta b_2^2 \Gamma_2}\right)}$$

and

$$\frac{c_{12}}{a_1} = \frac{\frac{1-a_2}{b_2^2 \Gamma_2} \left(b_2 \frac{\tau_{\eta_2}}{\tau_{s_2} + \tau_{\eta_2}} - c_{22}\right) - \rho_1^2 \left(\tau_{s_1} + \tau_{\eta_1}\right) \frac{c_{12}}{a_1}}{\tau_\epsilon}$$

$$= \frac{\frac{1-a_2}{b_2^2 \Gamma_2} \left[b_2 \frac{\tau_{\eta_2}}{\tau_{s_2} + \tau_{\eta_2}} - \frac{\rho_2 \tau_{\eta_2} + \rho_1^2 \left(\tau_{s_1} + \tau_{\eta_1}\right) \frac{c_{12}}{a_1}}{\Delta}\right] - \rho_1^2 \left(\tau_{s_1} + \tau_{\eta_1}\right) \frac{c_{12}}{a_1}}{\tau_\epsilon}$$

$$= \frac{\frac{1-a_2}{b_2} \tau_{\eta_2} \left(1 - \frac{\rho_2}{\Delta b_2 \Gamma_2}\right) - \rho_1^2 \left(\tau_{s_1} + \tau_{\eta_1}\right) \left(1 + \frac{1-a_2}{\Delta b_2^2 \Gamma_2}\right) \frac{c_{12}}{a_1}}{\tau_\epsilon}$$

$$\Leftrightarrow \frac{c_{12}}{a_1} = \frac{\frac{1-a_2}{b_2} \tau_{\eta_2} \left(1 - \frac{\rho_2}{\Delta b_2 \Gamma_2}\right)}{\tau_\epsilon + \rho_1^2 \left(\tau_{s_1} + \tau_{\eta_1}\right) \left(1 + \frac{1-a_2}{\Delta b_2^2 \Gamma_2}\right)}$$

$$\Leftrightarrow c_{12} = a_1 \frac{\frac{\delta[\tau_\theta + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]}{1 + \delta \rho_2(\tau_{s_2} + \tau_{\eta_2})} \tau_{\eta_2} \left[1 - \frac{\delta \rho_2(\tau_{s_2} + \tau_{\eta_2})}{1 + \delta \rho_2(\tau_{s_2} + \tau_{\eta_2})}\right]}{\tau_\epsilon + \rho_1^2 \left(\tau_{s_1} + \tau_{\eta_1}\right) \left(1 + \frac{1-a_2}{\Delta b_2^2 \Gamma_2}\right)}$$

$$= a_1 \frac{\frac{\delta \tau_{\eta_2}[\tau_\theta + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]}{[1 + \delta \rho_2(\tau_{s_2} + \tau_{\eta_2})]^2}}{\tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1}) \left(1 + \frac{1-a_2}{b_2^2 \Gamma_2 \Delta}\right)}.$$

Eventually, by (A14), it immediately follows that

$$\rho_1 \equiv \frac{a_1}{b_1} = \delta \tau_\epsilon.$$

Since the values of ρ_1 and ρ_2 are unique and determined in closed form, the linear dynamic REE with LLA is also unique and given in closed form. □

Proof of Proposition 3.8. As in the proof of Proposition 3.3, define

$$B_1 \equiv \left(\frac{1}{\rho_1} - \frac{c_{11}}{a_1}\right)^2 \frac{1}{\tau_{s_1}},$$

$$B_2 \equiv \left(\frac{c_{11}}{a_1}\right)^2 \frac{1}{\tau_{\eta_1}},$$

$$B_3 \equiv \left(\frac{c_{12}}{a_1}\right)^2 \left(\frac{1}{\tau_{s_2}} + \frac{1}{\tau_{\eta_2}}\right)$$

so that $\mathrm{Var}^{-1}(\theta \,|\, P_1^{**}) = \tau_\theta + (B_1 + B_2 + B_3)^{-1}$. Inspecting the coefficients in Proposition 3.7, the CON effect can be written as

$$\begin{aligned} B_1 &= \left[\frac{1}{\rho_1} - \frac{\rho_1 \tau_{\eta_1}(1+\phi)}{\tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})(1+\phi)}\right]^2 \frac{1}{\tau_{s_1}} \\ &= \left\{\frac{\tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})(1+\phi) - \rho_1^2 \tau_{\eta_1}(1+\phi)}{\rho_1[\tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})(1+\phi)]}\right\}^2 \frac{1}{\tau_{s_1}} \\ &= \left\{\frac{\tau_\epsilon + \rho_1^2 \tau_{s_1}(1+\phi)}{\rho_1[\tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})(1+\phi)]}\right\}^2 \frac{1}{\tau_{s_1}}, \end{aligned}$$

where

$$\phi \equiv \frac{1 - a_2}{b_2^2 \Gamma_2 \Delta} = \frac{1 - \dfrac{[\tau_\epsilon + \rho_2^2(\tau_{s_2} + \tau_{\eta_2})]}{\Delta}}{\dfrac{[1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})]^2 \Delta}{\delta^2 \Delta^2 (\tau_{s_2} + \tau_{\eta_2})}} = \frac{\delta^2[\tau_\theta + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})](\tau_{s_2} + \tau_{\eta_2})}{[1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})]^2}.$$

The COMESCON effect is given by

$$B_2 = \tau_{\eta_1} \left[\frac{\rho_1(1+\phi)}{\tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})(1+\phi)}\right]^2.$$

Analogously, the COMSFUN effect can be expressed as

$$\begin{aligned} B_3 &= \left\{\frac{\dfrac{\delta \tau_{\eta_2}[\tau_\theta + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]}{[1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})]^2}}{\tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\,(1+\phi)}\right\}^2 \left(\frac{1}{\tau_{s_2}} + \frac{1}{\tau_{\eta_2}}\right) \\ &= \frac{\delta^2 \tau_{\eta_2}^2 [\tau_\theta + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})]^2}{[1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})]^4 [\tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})\,(1+\phi)]^2} \frac{\tau_{s_2} + \tau_{\eta_2}}{\tau_{s_2}\tau_{\eta_2}} \end{aligned}$$

$$= \tau_{\eta 2} \frac{\delta^2 [\tau_\theta + \rho_1^2 (\tau_{s_1} + \tau_{\eta 1})]^2 (\tau_{s_2} + \tau_{\eta 2})}{\tau_{s_2} [1 + \delta \rho_2 (\tau_{s_2} + \tau_{\eta 2})]^4 [\tau_\epsilon + \rho_1^2 (\tau_{s_1} + \tau_{\eta_1}) (1 + \phi)]^2}.$$

For $\tau_{\eta_1} = 0$, we obtain

$$B_1 = \frac{1}{\rho_1^2 \tau_{s_1}},$$

$$B_2 = 0,$$

$$B_3 = \tau_{\eta 2} \frac{\delta^2 (\tau_\theta + \rho_1^2 \tau_{s_1})^2 (\tau_{s_2} + \tau_{\eta 2})}{\tau_{s_2} [1 + \delta \rho_2 (\tau_{s_2} + \tau_{\eta 2})]^4 [\tau_\epsilon + \rho_1^2 \tau_{s_1} (1 + \phi_{01})]^2},$$

where $\phi_{01} \equiv \dfrac{\delta^2 (\tau_\theta + \rho_1^2 \tau_{s_1})(\tau_{s_2} + \tau_{\eta 2})}{[1 + \delta \rho_2 (\tau_{s_2} + \tau_{\eta 2})]^2}$.

Since ρ_1 is independent of τ_{η_2} (see Proposition 3.7), the CON effect is independent of τ_{η_2} for $\tau_{\eta_1} = 0$. The fact that price efficiency is maximum for $\tau_{\eta_2} = 0$ follows from the fact that $B_3 = 0$ for $\tau_{\eta_2} = 0$ and $B_3 > 0$ for $\tau_{\eta_2} > 0$.

To prove part (b) in the proposition, we first rewrite the COMSFUN effect. Denote

$$k_1 \equiv \tau_\theta + \rho_1^2 \tau_{s_1},$$

$$k_2 \equiv 1 + \delta \rho_2 (\tau_{s_2} + \tau_{\eta 2}),$$

$$k_3 \equiv k_2^2 \tau_\epsilon + \rho_1^2 \tau_{s_1} [k_2^2 + \delta^2 (\tau_{s_2} + \tau_{\eta 2}) k_1].$$

Then,

$$B_3 = \tau_{\eta 2} \frac{\delta^2 k_1^2 (\tau_{s_2} + \tau_{\eta 2})}{\tau_{s_2} k_2^4 \left\{ \tau_\epsilon + \rho_1^2 \tau_{s_1} \left[1 + \dfrac{\delta^2 k_1 (\tau_{s_2} + \tau_{\eta 2})}{k_2^2} \right] \right\}^2}$$

$$= \tau_{\eta 2} \frac{\delta^2 k_1^2 (\tau_{s_2} + \tau_{\eta 2})}{\tau_{s_2} \{ k_2^2 \tau_\epsilon + \rho_1^2 \tau_{s_1} [k_2^2 + \delta^2 (\tau_{s_2} + \tau_{\eta 2}) k_1] \}^2}$$

$$= \tau_{\eta 2} \frac{\delta^2 k_1^2 (\tau_{s_2} + \tau_{\eta 2})}{\tau_{s_2} k_3^2}.$$

Differentiating with respect to τ_{η_2} yields

$$\frac{\partial B_3}{\partial \tau_{\eta 2}} = \frac{\delta^2 k_1^2 (\tau_{s_2} + \tau_{\eta 2})}{\tau_{s_2} k_3^2} + \tau_{\eta 2} \frac{\tau_{s_2} k_3^2 \delta^2 k_1^2 - \delta^2 k_1^2 (\tau_{s_2} + \tau_{\eta 2}) \tau_{s_2} 2 k_3 (\partial k_3 / \partial \tau_{\eta 2})}{\tau_{s_2}^2 k_3^4}$$

$$= \frac{\delta^2 k_1^2 \, [k_3 (\tau_{s_2} + 2\tau_{\eta 2}) - 2(\tau_{s_2} + \tau_{\eta 2}) \tau_{\eta 2} (\partial k_3 / \partial \tau_{\eta 2})]}{\tau_{s_2} k_3^3},$$

where

$$\frac{\partial k_3}{\partial \tau_{\eta_2}} = 2k_2\tau_\epsilon\delta\rho_2 + \rho_1^2\tau_{s_1}(2k_2\delta\rho_2 + \delta^2 k_1).$$

Next, we focus on the term in square brackets in the numerator of the derivative $\partial B_3/\partial \tau_{\eta_2}$. By recalling the definition of k_3, this term can be written as

$$\begin{aligned}
&\{k_2^2\tau_\epsilon + \rho_1^2\tau_{s_1}[k_2^2 + \delta^2(\tau_{s_2} + \tau_{\eta_2})k_1]\}(\tau_{s_2} + 2\tau_{\eta_2}) \\
&\quad - 2(\tau_{s_2} + \tau_{\eta_2})\tau_{\eta_2}[2k_2\tau_\epsilon\delta\rho_2 + \rho_1^2\tau_{s_1}(2k_2\delta\rho_2 + \delta^2 k_1)] \\
&= [k_2^2(\tau_\epsilon + \rho_1^2\tau_{s_1}) + \rho_1^2\tau_{s_1}\delta^2(\tau_{s_2} + \tau_{\eta_2})k_1](\tau_{s_2} + 2\tau_{\eta_2}) \\
&\quad - 2(\tau_{s_2} + \tau_{\eta_2})\left[2\delta\rho_2 k_2(\tau_\epsilon + \rho_1^2\tau_{s_1}) + \rho_1^2\tau_{s_1}\delta^2 k_1\right] \\
&= \{[1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})]^2(\tau_\epsilon + \rho_1^2\tau_{s_1}) + \rho_1^2\tau_{s_1}\delta^2(\tau_{s_2} + \tau_{\eta_2})(\tau_\theta + \rho_1^2\tau_{s_1})\}(\tau_{s_2} + 2\tau_{\eta_2}) \\
&\quad - 2(\tau_{s_2} + \tau_{\eta_2})\left\{2\delta\rho_2[1 + \delta\rho_2(\tau_{s_2} + \tau_{\eta_2})](\tau_\epsilon + \rho_1^2\tau_{s_1}) + \rho_1^2\tau_{s_1}\delta^2(\tau_\theta + \rho_1^2\tau_{s_1})\right\} \\
&= -2\delta^2\rho_2^2(\tau_\epsilon + \rho_1^2\tau_{s_1})\,\tau_{\eta_2}^3 - 3\delta^2\rho_2^2\tau_{s_2}(\tau_\epsilon + \rho_1^2\tau_{s_1})\,\tau_{\eta_2}^2 \\
&\quad + \left[2(1 + \delta\rho_2\tau_{s_2})(\tau_\epsilon + \rho_1^2\tau_{s_1}) + \rho_1^2\tau_{s_1}\delta(\tau_\theta + \rho_1^2\tau_{s_1})\tau_{s_2}\right]\tau_{\eta_2} \\
&\quad + \tau_{s_2}[(1 + \delta\rho_2\tau_{s_2})^2(\tau_\epsilon + \rho_1^2\tau_{s_1}) + \rho_1^2\tau_{s_1}\delta(\tau_\theta + \rho_1^2\tau_{s_1})\tau_{s_2}].
\end{aligned}$$

Thus,

$$\frac{\partial B_3}{\partial \tau_{\eta_2}} = \frac{\delta^2 k_1^2\left(-b_3\tau_{\eta_2}^3 - b_2\tau_{\eta_2}^2 + b_1\tau_{\eta_2} + b_0\right)}{\tau_{s_2}k_3^3},$$

where

$$b_3 \equiv 2\delta^2\rho_2^2(\tau_\epsilon + \rho_1^2\tau_{s_1}),$$

$$b_2 \equiv 3\delta^2\rho_2^2\tau_{s_2}(\tau_\epsilon + \rho_1^2\tau_{s_1}),$$

$$b_1 \equiv 2(1 + \delta\rho_2\tau_{s_2})(\tau_\epsilon + \rho_1^2\tau_{s_1}) + \rho_1^2\tau_{s_1}\delta(\tau_\theta + \rho_1^2\tau_{s_1})\tau_{s_2},$$

$$b_0 \equiv \tau_{s_2}[(1 + \delta\rho_2\tau_{s_2})^2(\tau_\epsilon + \rho_1^2\tau_{s_1}) + \rho_1^2\tau_{s_1}\delta(\tau_\theta + \rho_1^2\tau_{s_1})\tau_{s_2}].$$

Note that the term in brackets in the numerator of the derivative is a cubic polynomial in τ_{η_2}. To determine the number of positive real roots, we use Descartes' rule of signs. This rule states that the number of positive real roots of a polynomial is either equal to the number of its sign changes or a number that is smaller by an even integer than the actual number of sign changes (see, e.g., Struik, 1986, Chapter 2).

It can be clearly seen that the cubic exhibits one sign change, which means that it possesses exactly one positive real root, $\bar{\tau}_{\eta_2}$ say. This, in return, implies that the unique extremum of B_3 lies at $\tau_{\eta_2} = \bar{\tau}_{\eta_2}$. Since $\partial B_3/\partial\tau_{\eta_2} > 0$ for $\tau_{\eta_2} = 0$ and $\partial B_3/\partial\tau_{\eta_2} < 0$ for sufficiently large values of τ_{η_2}, B_3 has a global maximum at $\tau_{\eta_2} = \bar{\tau}_{\eta_2}$. Consequently, the global minimum of $\mathrm{Var}^{-1}(\theta \mid P_1^{**})$ lies at this point. Thus, $\mathrm{Var}^{-1}(\theta \mid P_1^{**})$ is decreasing (resp., increasing) in τ_{η_2} for $\tau_{\eta_2} \lesseqgtr \bar{\tau}_{\eta_2}$. □

Proof of Proposition 3.9. Inspecting the coefficients in Proposition 3.7, for $\tau_{\eta_2} = 0$, we get

$$\frac{c_{11}}{a_1} = \frac{\rho_1\tau_{\eta_1}(1+\phi_{02})}{\tau_\epsilon + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi_{02})} = \frac{\rho_1\tau_{\eta_1}}{D\tau_\epsilon + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})}, \tag{A15}$$

where

$$\phi_{02} \equiv \delta^2\tau_{s_2}\frac{\tau_\theta + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})}{(1+\delta\rho_2\tau_{s_2})^2},$$

$$D \equiv (1+\phi_{02})^{-1} = \left[1 + \delta^2\tau_{s_2}\frac{\tau_\theta + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})}{(1+\delta\rho_2\tau_{s_2})^2}\right]^{-1} < 1.$$

Hence,

$$B_1|_{\tau_{\eta_2}=0} = \left[\frac{1}{\rho_1} - \frac{\rho_1\tau_{\eta_1}}{D\tau_\epsilon + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})}\right]^2 \frac{1}{\tau_{s_1}} = \frac{1}{\rho_1^2\tau_{s_1}}\left[\frac{D\tau_\epsilon + \rho_1^2\tau_{s_1}}{D\tau_\epsilon + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})}\right]^2,$$

$$B_2|_{\tau_{\eta_2}=0} = \frac{1}{\rho_1^2\tau_{\eta_1}}\left[\frac{\rho_1^2\tau_{\eta_1}}{D\tau_\epsilon + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})}\right]^2,$$

$$B_3|_{\tau_{\eta_2}=0} = 0.$$

By Table 3.1, we can conclude that

$$\lim_{\tau_{\eta_2}\to\infty}\frac{c_{11}}{a_1} = \frac{\rho_1\tau_{\eta_1}}{\tau_\epsilon + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})}. \tag{A16}$$

Note that (A16) is unequivocally smaller than (A15), as $D < 1$. Thus,

$$\lim_{\tau_{\eta_2}\to\infty} B_1 = \left[\frac{1}{\rho_1} - \frac{\rho_1\tau_{\eta_1}}{\tau_\epsilon + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})}\right]^2\frac{1}{\tau_{s_1}} = \frac{1}{\rho_1^2\tau_{s_1}}\left[\frac{\tau_\epsilon + \rho_1^2\tau_{s_1}}{\tau_\epsilon + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})}\right]^2,$$

$$\lim_{\tau_{\eta_2}\to\infty} B_2 = \frac{1}{\rho_1^2\tau_{\eta_1}}\left[\frac{\rho_1^2\tau_{\eta_1}}{\tau_\epsilon + \rho_1^2(\tau_{s_1}+\tau_{\eta_1})}\right]^2,$$

$$\lim_{\tau_{\eta_2}\to\infty} B_3 = 0.$$

Consequently,

$$\mathrm{Var}^{-1}(\theta \mid P_1^{**})\big|_{\tau_{\eta_2}=0} > \lim_{\tau_{\eta_2}\to\infty} \mathrm{Var}^{-1}(\theta \mid P_1^{**}) \quad \text{exactly if}$$

$$\frac{1}{\rho_1^2\tau_{s_1}}\left[\frac{D\tau_\epsilon+\rho_1^2\tau_{s_1}}{D\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}\right]^2+\frac{1}{\rho_1^2\tau_{\eta_1}}\left[\frac{\rho_1^2\tau_{\eta_1}}{D\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}\right]^2$$

$$<\frac{1}{\rho_1^2\tau_{s_1}}\left[\frac{\tau_\epsilon+\rho_1^2\tau_{s_1}}{\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}\right]^2+\frac{1}{\rho_1^2\tau_{\eta_1}}\left[\frac{\rho_1^2\tau_{\eta_1}}{\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}\right]^2.$$

Note that both sides of the above inequality would be identical if D equaled unity. Since $D < 1$, the validity of the inequality is proven if the term on the left-hand side is strictly increasing in D. Comparative-statics analysis yields

$$\frac{\partial}{\partial D}\left\{\frac{1}{\rho_1^2\tau_{s_1}}\left[\frac{D\tau_\epsilon+\rho_1^2\tau_{s_1}}{D\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}\right]^2+\frac{1}{\rho_1^2\tau_{\eta_1}}\left[\frac{\rho_1^2\tau_{\eta_1}}{D\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}\right]^2\right\}$$

$$=\frac{2}{\rho_1^2\tau_{s_1}}\frac{D\tau_\epsilon+\rho_1^2\tau_{s_1}}{D\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}\frac{\tau_\epsilon[D\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})]-\tau_\epsilon(D\tau_\epsilon+\rho_1^2\tau_{s_1})}{[D\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})]^2}$$

$$-\frac{2}{\rho_1^2\tau_{\eta_1}}\frac{\rho_1^2\tau_{\eta_1}}{D\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})}\frac{\rho_1^2\tau_{\eta_1}\tau_\epsilon}{[D\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})]^2}$$

$$=\frac{2\tau_\epsilon\rho_1^2\tau_{\eta_1}(D\tau_\epsilon+\rho_1^2\tau_{s_1})}{\rho_1^2\tau_{s_1}[D\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})]^3}-\frac{2\rho_1^4\tau_{\eta_1}^2\tau_\epsilon}{\rho_1^2\tau_{\eta_1}[D\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})]^3}$$

$$=\frac{2\tau_\epsilon^2\tau_{\eta_1}D}{\tau_{s_1}[D\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})]^3}>0.$$

This proves that price efficiency is higher for $\tau_{\eta_2} = 0$ than as $\tau_{\eta_2} \to \infty$. □

Proof of Proposition 3.10. From the proof of Proposition 3.8, we know that

$$B_1=\left\{\frac{\tau_\epsilon+\rho_1^2\tau_{s_1}(1+\phi)}{\rho_1[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)]}\right\}^2\frac{1}{\tau_{s_1}}.$$

Differentiating with respect to τ_{η_2} yields

$$\frac{\partial B_1}{\partial \tau_{\eta_2}}=\frac{2}{\tau_{s_1}}\frac{\tau_\epsilon+\rho_1^2\tau_{s_1}(1+\phi)}{\rho_1[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)]}\left\{\frac{\rho_1^3\tau_{s_1}[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)](\partial\phi/\partial\tau_{\eta_2})}{\rho_1^2[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)]^2}\right.$$

$$\left.-\frac{\rho_1^3(\tau_{s_1}+\tau_{\eta_1})[\tau_\epsilon+\rho_1^2\tau_{s_1}(1+\phi)](\partial\phi/\partial\tau_{\eta_2})}{\rho_1^2[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)]^2}\right\}$$

$$=\frac{2}{\tau_{s_1}}\frac{\tau_\epsilon+\rho_1^2\tau_{s_1}(1+\phi)}{\rho_1[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)]}\frac{-\rho_1^3\tau_{\eta_1}\tau_\epsilon(\partial\phi/\partial\tau_{\eta_2})}{\rho_1^2[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)]^2}$$

$$= -\frac{2\tau_{\eta_1}\tau_\epsilon(\partial\phi/\partial\tau_{\eta_2})[\tau_\epsilon+\rho_1^2\tau_{s_1}(1+\phi)]}{\tau_{s_1}[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)]^3}.$$

Note that $sign(\partial B_1/\partial\tau_{\eta_2}) = -sign(\partial\phi/\partial\tau_{\eta_2})$. If $\partial\phi/\partial\tau_{\eta_2} > 0$, we have $\partial(c_{11}/a_1)/\partial\tau_{\eta_2} > 0$, and the CON effect is weakened. The COMESCON effect is given by

$$B_2 = \tau_{\eta_1}\left[\frac{\rho_1(1+\phi)}{\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)}\right]^2.$$

Thus,

$$\frac{\partial B_2}{\partial\tau_{\eta_2}} = \frac{2\rho_1\tau_{\eta_1}(1+\phi)}{\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)}$$
$$\times\frac{\rho_1(\partial\phi/\partial\tau_{\eta_2})[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)]-\rho_1^3(\tau_{s_1}+\tau_{\eta_1})(1+\phi)(\partial\phi/\partial\tau_{\eta_2})}{[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)]^2}$$
$$= \frac{2\rho_1\tau_{\eta_1}(1+\phi)}{\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)}\frac{\rho_1\tau_{\eta_1}\tau_\epsilon(\partial\phi/\partial\tau_{\eta_2})}{[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)]^2}$$
$$= \frac{2\rho_2^2\tau_{\eta_1}\tau_\epsilon(1+\phi)(\partial\phi/\partial\tau_{\eta_2})}{[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)]^3}.$$

Analogously, $sign(\partial B_2/\partial\tau_{\eta_2}) = sign(\partial\phi/\partial\tau_{\eta_2})$. Combining the separate terms gives

$$\frac{\partial B_1}{\partial\tau_{\eta_2}}+\frac{\partial B_2}{\partial\tau_{\eta_2}}$$
$$= -\frac{2\tau_{\eta_1}\tau_\epsilon(\partial\phi/\partial\tau_{\eta_2})[\tau_\epsilon+\rho_1^2\tau_{s_1}(1+\phi)]}{\tau_{s_1}[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)]^3}+\frac{2\rho_2^2\tau_{\eta_1}\tau_\epsilon(1+\phi)(\partial\phi/\partial\tau_{\eta_2})}{[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)]^3}$$
$$= \frac{-2\tau_{\eta_1}\tau_\epsilon(\partial\phi/\partial\tau_{\eta_2})[\tau_\epsilon+\rho_1^2\tau_{s_1}(1+\phi)]+2\rho_2^2\tau_{s_1}\tau_{\eta_1}\tau_\epsilon(1+\phi)(\partial\phi/\partial\tau_{\eta_2})}{\tau_{s_1}[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)]^3}$$
$$= \frac{-2\tau_{\eta_1}\tau_\epsilon^2(\partial\phi/\partial\tau_{\eta_2})}{\tau_{s_1}[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})(1+\phi)]^3}.$$

As $sign(\partial B_1/\partial\tau_{\eta_2}+\partial B_2/\partial\tau_{\eta_2}) = sign(\partial B_1/\partial\tau_{\eta_2}) = -sign(\partial\phi/\partial\tau_{\eta_2})$, the impact on the CON effect dominates that on the COMESCON effect. Moreover,

$$\frac{\partial\phi}{\partial\tau_{\eta_2}} = \frac{\delta^2[1+\delta\rho_2(\tau_{s_2}+\tau_{\eta_2})]^2[\tau_\theta+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})]}{[1+\delta\rho_2(\tau_{s_2}+\tau_{\eta_2})]^4}$$
$$-\frac{-2\delta^3\rho_2(\tau_{s_2}+\tau_{\eta_2})[\tau_\theta+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})][1+\delta\rho_2(\tau_{s_2}+\tau_{\eta_2})]}{[1+\delta\rho_2(\tau_{s_2}+\tau_{\eta_2})]^4}$$
$$= \frac{[\delta^2+\delta^3\rho_2(\tau_{s_2}+\tau_{\eta_2})][\tau_\theta+\rho_1^2(\tau_{s_1}+\tau_{\eta_1})]}{[1+\delta\rho_2(\tau_{s_2}+\tau_{\eta_2})]^3}$$

$$- \frac{-2\delta^3\rho_2(\tau_{s_2}+\tau_{\eta 2})[\tau_\theta+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})]}{[1+\delta\rho_2(\tau_{s_2}+\tau_{\eta 2})]^3}$$

$$= \frac{\delta^2[1-\delta\rho_2(\tau_{s_2}+\tau_{\eta 2})][\tau_\theta+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})]}{[1+\delta\rho_2(\tau_{s_2}+\tau_{\eta 2})]^3}.$$

Consequently, ϕ and, thus, c_{11}/a_1 are increasing in $\tau_{\eta 2}$ if $1 > \delta\rho_2(\tau_{s_2}+\tau_{\eta 2})$. Further computations give

$$\frac{\partial B_1}{\partial \tau_{\eta 2}} + \frac{\partial B_2}{\partial \tau_{\eta 2}} = \frac{-2\delta^2\tau_{\eta 1}\tau_\epsilon^2[1-\delta\rho_2(\tau_{s_2}+\tau_{\eta 2})][\tau_\theta+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})]}{\tau_{s_1}[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})(1+\phi_1)]^3[1+\delta\rho_2(\tau_{s_2}+\tau_{\eta 2})]^3}.$$

For $\tau_{\eta 2} = 0$, we obtain

$$\left(\frac{\partial B_1}{\partial \tau_{\eta 2}} + \frac{\partial B_2}{\partial \tau_{\eta 2}}\right)\Bigg|_{\tau_{\eta 2}=0} = \frac{-2\delta^2\tau_{\eta 1}\tau_\epsilon^2(1-\delta\rho_2\tau_{s_2})[\tau_\theta+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})]}{\tau_{s_1}[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})(1+\phi_{02})]^3(1+\delta\rho_2\tau_{s_2})^3}.$$

Recall from the proof of Proposition 3.8 that the COMSFUN effect is given by

$$B_3 = \tau_{\eta 2}\frac{\delta^2[\tau_\theta+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})]^2(\tau_{s_2}+\tau_{\eta 2})}{\tau_{s_2}[1+\delta\rho_2(\tau_{s_2}+\tau_{\eta 2})]^4[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})\,(1+\phi)]^2}.$$

Thus,

$$\frac{\partial B_3}{\partial \tau_{\eta 2}} = \frac{\delta^2[\tau_\theta+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})]^2(\tau_{s_2}+\tau_{\eta 2})}{\tau_{s_2}[1+\delta\rho_2(\tau_{s_2}+\tau_{\eta 2})]^4[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})\,(1+\phi)]^2}$$
$$+ \tau_{\eta 2}\frac{\partial}{\partial \tau_{\eta 2}}\left\{\frac{\delta^2[\tau_\theta+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})]^2(\tau_{s_2}+\tau_{\eta 2})}{\tau_{s_2}[1+\delta\rho_2(\tau_{s_2}+\tau_{\eta 2})]^4[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})\,(1+\phi)]^2}\right\}.$$

For $\tau_{\eta 2} = 0$, the second summand in the above derivative drops out (note that the denominator of the above fraction is bounded away from zero). This gives

$$\frac{\partial B_3}{\partial \tau_{\eta 2}}\Bigg|_{\tau_{\eta 2}=0} = \frac{\delta^2[\tau_\theta+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})]^2}{(1+\delta\rho_2\tau_{s_2})^4[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})\,(1+\phi_{02})]^2}.$$

Putting all obtained results together yields:

$$\frac{\partial[\mathrm{Var}^{-1}(\theta\,|\,P_1^{**})]}{\partial \tau_{\eta 2}}\Bigg|_{\tau_{\eta 2}=0} < 0 \quad \text{exactly if}$$

$$\frac{-2\delta^2\tau_{\eta 1}\tau_\epsilon^2(1-\delta\rho_2\tau_{s_2})[\tau_\theta+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})]}{\tau_{s_1}[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})(1+\phi_{02})]^3(1+\delta\rho_2\tau_{s_2})^3}$$
$$+ \frac{\delta^2[\tau_\theta+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})]^2}{(1+\delta\rho_2\tau_{s_2})^4[\tau_\epsilon+\rho_1^2(\tau_{s_1}+\tau_{\eta 1})\,(1+\phi_{02})]^2} > 0,$$

$$\Leftrightarrow \frac{\tau_\theta + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})}{1 + \delta\rho_2\tau_{s_2}} > \frac{2\tau_{\eta_1}\tau_\epsilon^2(1 - \delta\rho_2\tau_{s_2})}{\tau_{s_1}[\tau_\epsilon + \rho_1^2(\tau_{s_1} + \tau_{\eta_1})(1 + \phi_{02})]},$$

which is equal to the inequality in the proposition. □

Proof of Proposition 4.1. Using (4.3), a non-fundamentally informed agent's conditional moments are given by

$$\mathrm{E}(\theta \mid P^*_{n_i}) = \frac{\beta_j^2 \tau_x \left(\frac{P}{a_\theta} - \frac{1}{\beta_i} x_i\right)}{\tau_\theta + \beta_j^2\,\tau_x},$$

$$\mathrm{Var}(\theta \mid P^*_{n_i}) = \frac{1}{\tau_\theta + \beta_j^2 \tau_x}, \quad for \;\; i, j = 1, 2, i \neq j.$$

Recalling (4.2), the demand function of an x_i-informed trader becomes

$$D_{n_i} = \frac{\beta_j^2 \tau_x \left(\frac{P}{a_\theta} - \frac{1}{\beta_i} x_i\right) - P\left(\tau_\theta + \beta_j^2\,\tau_x\right)}{\gamma}, \qquad for \;\; i, j = 1, 2, i \neq j. \tag{A17}$$

Concerning a fundamentally informed trader, as the error terms in s_f and $P^*_{f/u}$ are uncorrelated, we obtain

$$\mathrm{E}\left(\theta \mid s_f, P^*_{f/u}\right) = \frac{\tau_\epsilon\, s_f + \frac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2} \frac{P}{a_\theta}}{\tau_\theta + \tau_\epsilon + \frac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2}},$$

$$\mathrm{Var}\left(\theta \mid s_f, P^*_{f/u}\right) = \frac{1}{\tau_\theta + \tau_\epsilon + \frac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2}}.$$

Thus,

$$D_f = \frac{\tau_\epsilon\, s_f + \frac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2} \frac{P}{a_\theta} - P\left(\tau_\theta + \tau_\epsilon + \frac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2}\right)}{\gamma}. \tag{A18}$$

Analogously, the conditional moments of an uninformed, rational agent are

$$\mathrm{E}\left(\theta \,| P^*_{f/u}\right) = \frac{\frac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2} \frac{P}{a_\theta}}{\tau_\theta + \frac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2}},$$

$$\mathrm{Var}\left(\theta \mid P^*_{f/u}\right) = \frac{1}{\tau_\theta + \frac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2}},$$

which yields

$$D_u = \frac{\frac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2} \frac{P}{a_\theta} - P\left(\tau_\theta + \frac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2}\right)}{\gamma}. \tag{A19}$$

Using (A17), (A18), and (A19), the market-clearing condition in (4.5) can be developed as follows:

$$\frac{\tau_\epsilon \theta + \frac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2} \frac{P}{a_\theta} - P\left(\tau_\theta + \tau_\epsilon + \frac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2}\right)}{\gamma}$$

$$+ \lambda_1 \frac{\beta_2^2 \tau_x \left(\frac{P}{a_\theta} - \frac{1}{\beta_1} x_1\right) - P\left(\tau_\theta + \beta_2^2\, \tau_x\right)}{\gamma} + \lambda_2 \frac{\beta_1^2 \tau_x \left(\frac{P}{a_\theta} - \frac{1}{\beta_2} x_2\right) - P\left(\tau_\theta + \beta_1^2\, \tau_x\right)}{\gamma}$$

$$+ x_1 + x_2 \; + \lambda_u \frac{\frac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2} \frac{P}{a_\theta} - P\left(\tau_\theta + \frac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2}\right)}{\gamma} = 0.$$

As in Chapter 3, by the strong law of large numbers, the error term in s_f vanishes when integrating (i.e., $\int_0^1 s_f \, df = \theta$). Collecting terms gives

$$P\Bigg[(1+\lambda_u) \frac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2}(a_\theta^{-1} - 1) + \lambda_1 \beta_2^2 \tau_x (a_\theta^{-1} - 1) + \lambda_2 \beta_1^2 \tau_x (a_\theta^{-1} - 1)$$

$$- \tau_\theta (1 + \lambda_1 + \lambda_2 + \lambda_u) - \tau_\epsilon\Bigg] + \tau_\epsilon \theta + \gamma\, x_1 - \frac{\lambda_1 \beta_2^2 \tau_x}{\beta_1} x_1 + \gamma\, x_2 - \frac{\lambda_2 \beta_1^2 \tau_x}{\beta_2} x_2 = 0.$$

Further simplifications deliver

$$\begin{aligned} &P\left[\tau_x \left(1 - a_\theta^{-1}\right)\left(\frac{1+\lambda_u}{1/\beta_1^2 + 1/\beta_2^2} + \lambda_1 \beta_2^2 + \lambda_2 \beta_1^2\right) + \tau_\theta\left(1 + \lambda_1 + \lambda_2 + \lambda_u\right) + \tau_\epsilon\right] \\ &= \tau_\epsilon \theta + \left(\gamma - \frac{\lambda_1 \beta_2^2 \tau_x}{\beta_1}\right) x_1 + \left(\gamma - \frac{\lambda_2 \beta_1^2 \tau_x}{\beta_2}\right) x_2. \end{aligned} \tag{A20}$$

By comparing (A20) with (4.1), we obtain

$$a_\theta = \frac{\tau_\epsilon}{\tau_x \left(1 - a_\theta^{-1}\right)\left(\frac{1+\lambda_u}{\frac{1}{\beta_1^2} + \frac{1}{\beta_2^2}} + \lambda_1 \beta_2^2 + \lambda_2 \beta_1^2\right) + \tau_\theta\left(1 + \lambda_1 + \lambda_2 + \lambda_u\right) + \tau_\epsilon}$$

$$\Leftrightarrow a_\theta \left[\tau_x\left(1-a_\theta^{-1}\right)\left(\frac{1+\lambda_u}{\frac{\beta_1^2+\beta_2^2}{\beta_1^2\beta_2^2}}+\lambda_1\beta_2^2+\lambda_2\beta_1^2\right)+\tau_\theta\left(1+\lambda_1+\lambda_2+\lambda_u\right)+\tau_\epsilon\right]=\tau_\epsilon$$

$$\Leftrightarrow \tau_x\left(a_\theta-1\right)\left[\frac{(1+\lambda_u)\beta_1^2\beta_2^2}{\beta_1^2+\beta_2^2}+\lambda_1\beta_2^2+\lambda_2\beta_1^2\right]+a_\theta\tau_\theta\left(1+\lambda_1+\lambda_2+\lambda_u\right)+a_\theta\tau_\epsilon=\tau_\epsilon$$

$$\Leftrightarrow a_\theta\tau_x\left[\frac{(1+\lambda_u)\beta_1^2\beta_2^2}{\beta_1^2+\beta_2^2}+\lambda_1\beta_2^2+\lambda_2\beta_1^2\right]+a_\theta\tau_\theta\left(1+\lambda_1+\lambda_2+\lambda_u\right)+a_\theta\tau_\epsilon=\tau_\epsilon$$

$$+\tau_x\left[\frac{(1+\lambda_u)\beta_1^2\beta_2^2}{\beta_1^2+\beta_2^2}+\lambda_1\beta_2^2+\lambda_2\beta_1^2\right]$$

$$\Leftrightarrow a_\theta\left\{\tau_x\left[\frac{(1+\lambda_u)\beta_1^2\beta_2^2}{\beta_1^2+\beta_2^2}+\lambda_1\beta_2^2+\lambda_2\beta_1^2\right]+\tau_\theta\left(1+\lambda_1+\lambda_2+\lambda_u\right)+\tau_\epsilon\right\}=\tau_\epsilon$$

$$+\tau_x\left[\frac{(1+\lambda_u)\beta_1^2\beta_2^2}{\beta_1^2+\beta_2^2}+\lambda_1\beta_2^2+\lambda_2\beta_1^2\right]$$

$$\Leftrightarrow a_\theta=\frac{\tau_\epsilon+\tau_x\left[\frac{(1+\lambda_u)\beta_1^2\beta_2^2}{\beta_1^2+\beta_2^2}+\lambda_1\beta_2^2+\lambda_2\beta_1^2\right]}{\tau_x\left[\frac{(1+\lambda_u)\beta_1^2\beta_2^2}{\beta_1^2+\beta_2^2}+\lambda_1\beta_2^2+\lambda_2\beta_1^2\right]+\tau_\theta\left(1+\lambda_1+\lambda_2+\lambda_u\right)+\tau_\epsilon}.$$

After defining $\omega \equiv 1+\lambda_1+\lambda_2+\lambda_u$, we get

$$a_\theta=\frac{\frac{(\beta_1^2+\beta_2^2)(\tau_\epsilon+\lambda_1\beta_2^2\tau_x+\lambda_2\beta_1^2\tau_x)+\tau_x(1+\lambda_u)\beta_1^2\beta_2^2}{\beta_1^2+\beta_2^2}}{\frac{(\beta_1^2+\beta_2^2)(\tau_\epsilon+\tau_\theta\,\omega+\lambda_1\beta_2^2\tau_x+\lambda_2\beta_1^2\tau_x)+\tau_x(1+\lambda_u)\beta_1^2\beta_2^2}{\beta_1^2+\beta_2^2}}$$

$$=\frac{\beta_1^4\lambda_2\tau_x+\beta_2^2(\tau_\epsilon+\lambda_1\beta_2^2\tau_x)+\beta_1^2[\tau_\epsilon+\beta_2^2\tau_x(1+\lambda_1+\lambda_2+\lambda_u)]}{\beta_1^4\lambda_2\tau_x+\beta_2^2(\tau_\epsilon+\tau_\theta\,\omega+\lambda_1\beta_2^2\tau_x)+\beta_1^2[\tau_\epsilon+\tau_\theta\,\omega+\beta_2^2\tau_x(1+\lambda_1+\lambda_2+\lambda_u)]}$$

$$=\frac{\beta_1^4\lambda_2\tau_x+\beta_2^2\left(\tau_\epsilon+\lambda_1\beta_2^2\tau_x\right)+\beta_1^2\left(\tau_\epsilon+\beta_2^2\tau_x\,\omega\right)}{\beta_1^4\lambda_2\tau_x+\beta_2^2\left(\tau_\epsilon+\tau_\theta\,\omega+\lambda_1\beta_2^2\tau_x\right)+\beta_1^2[\tau_\epsilon+\left(\beta_2^2\tau_x+\tau_\theta\right)\omega]}.$$

From the definitions of β_1 and β_2, it immediately follows that

$$a_1=\frac{1}{\beta_1}a_\theta \quad \text{and} \quad a_2=\frac{1}{\beta_2}a_\theta.$$

Furthermore, by imposing rational expectations, the implied values of β_1 and β_2 are given by

$$\beta_i=\frac{\tau_\epsilon}{\gamma-\frac{\lambda_i\beta_j^2\,\tau_x}{\beta_i}}$$

$$\Leftrightarrow \beta_i \left(\gamma - \frac{\lambda_i \beta_j^2 \, \tau_x}{\beta_i} \right) = \tau_\epsilon$$

$$\Leftrightarrow \beta_i \, \gamma - \lambda_i \beta_j^2 \tau_x = \tau_\epsilon$$

$$\Leftrightarrow \beta_i = \frac{\tau_\epsilon + \lambda_i \beta_j^2 \tau_x}{\gamma}, \quad for \;\; i, j = 1, 2, \; i \neq j. \tag{A21}$$

By further developing the two-equation system contained in (A21), we can find the fixed-point equations that determine the solutions for β_1 and β_2:

$$\beta_i = \frac{\tau_\epsilon + \lambda_i \beta_j^2 \tau_x}{\gamma}$$

$$\Leftrightarrow \beta_i = \frac{\tau_\epsilon + \lambda_i \tau_x \left(\frac{\tau_\epsilon + \lambda_j \beta_i^2 \tau_x}{\gamma} \right)^2}{\gamma}$$

$$\Leftrightarrow \beta_i = f(\beta_i) \equiv \frac{\lambda_i \lambda_j^2 \beta_i^4 \tau_x^3 + 2\lambda_i \lambda_j \beta_i^2 \tau_x^2 \tau_\epsilon + \tau_\epsilon (\lambda_i \tau_x \tau_\epsilon + \gamma^2)}{\gamma^3}, \;\; for \;\; i, j = 1, 2, \; i \neq j. \tag{A22}$$

Rearranging terms in (A22) delivers

$$\lambda_i \lambda_j^2 \tau_x^3 \beta_i^4 + 2\lambda_i \lambda_j \tau_x^2 \tau_\epsilon \beta_i^2 - \gamma^3 \beta_i + \tau_\epsilon (\lambda_i \tau_x \tau_\epsilon + \gamma^2) = 0. \tag{A23}$$

The solutions for β_i are obtained by determining the roots of the quartic in (A23). To find the number of solutions, we make use of Descartes' rule of signs (see the proof of Proposition 3.8 for an explanation). From (A23), we see that the quartic incorporates two sign changes. This means that there are either two or zero positive real roots. Thus, the existence of a linear REE can be ensured if and only if the solution of (A23) delivers two positive real roots. In the present case, this occurs whenever the discriminant of (A23), Δ_{β_i} say, is non-positive. If it is negative, there are two distinct positive real roots. A discriminant equal to zero means that there are two identical positive roots (see Dickson, 1914, Chapter 4). Denote

$$b_4 \equiv \lambda_i \lambda_j^2 \tau_x^3, \quad b_2 \equiv 2\lambda_i \lambda_j \tau_x^2 \tau_\epsilon, \quad b_1 \equiv -\gamma^3, \quad b_0 \equiv \tau_\epsilon \left(\lambda_i \tau_x \tau_\epsilon + \gamma^2 \right)$$

so that the quartic in (A23) can be written as

$$b_4 \beta_i^4 + b_2 \beta_i^2 + b_1 \beta_i + b_0 = 0.$$

Following Dickson (1914, p. 41), the discriminant of a quartic function is given by

$$\Delta = -4Y^3 - 27Q^2, \tag{A24}$$

where

$$Y = b_3 b_1 - 4b_4 b_0 - \frac{1}{3}b_2^2 \quad \text{and} \quad Q = -b_3^2\, b_0 + \frac{1}{3}b_3 b_2 b_1 + \frac{8}{3}b_4 b_2 b_0 - b_4 b_1^2 - \frac{2}{27}b_2^3.$$

Note that $b_3 = 0$ in the present case. Hence,

$$\begin{aligned} Y &= -4\lambda_i \lambda_j^2 \tau_x^3 \tau_\epsilon \left(\lambda_i \tau_x \tau_\epsilon + \gamma^2\right) - \frac{1}{3}(2\lambda_j \lambda_i \tau_x^2 \tau_\epsilon)^2 \\ &= -\frac{16}{3}\lambda_j^2 \lambda_i^2 \tau_x^4 \tau_\epsilon^2 - 4\lambda_j^2 \lambda_i \tau_x^3 \tau_\epsilon \gamma^2 \\ &= -\frac{4}{3}\lambda_j^2 \lambda_i \tau_x^3 \tau_\epsilon \left(4\lambda_i \tau_x \tau_\epsilon + 3\gamma^2\right) \end{aligned} \tag{A25}$$

and

$$\begin{aligned} Q &= \frac{8}{3}\lambda_j^2 \lambda_i \tau_x^3 2\lambda_j \lambda_i \tau_x^2 \tau_\epsilon \tau_\epsilon \left(\lambda_i \tau_x \tau_\epsilon + \gamma^2\right) - \lambda_j^2 \lambda_i \tau_x^3 (-\gamma^3)^2 - \frac{2}{27}(2\lambda_j \lambda_i \tau_x^2 \tau_\epsilon)^3 \\ &= \frac{16}{3}\lambda_j^3 \lambda_i^3 \tau_x^6 \tau_\epsilon^3 + \frac{16}{3}\lambda_j^3 \lambda_i^2 \tau_x^5 \tau_\epsilon^2 \gamma^2 - \lambda_j^2 \lambda_i \tau_x^3 \gamma^6 - \frac{16}{27}\lambda_j^3 \lambda_i^3 \tau_x^6 \tau_\epsilon^3 \\ &= \frac{1}{27}\lambda_j^2 \lambda_i \tau_x^3 \left(128\lambda_j \lambda_i^2 \tau_x^3 \tau_\epsilon^3 + 144\lambda_j \lambda_i \tau_x^2 \tau_\epsilon^2 \gamma^2 - 27\gamma^6\right). \end{aligned} \tag{A26}$$

By plugging (A25) and (A26) into (A24), we eventually get the discriminant Δ_{β_i} of the quartic in (A23):

$$\begin{aligned} \Delta_{\beta_i} =& -4\left[-\frac{4}{3}\lambda_j^2 \lambda_i \tau_x^3 \tau_\epsilon \left(4\lambda_i \tau_x \tau_\epsilon + 3\gamma^2\right)\right]^3 \\ & - 27\left[\frac{1}{27}\lambda_j^2 \lambda_i \tau_x^3 \left(128\lambda_j \lambda_i^2 \tau_x^3 \tau_\epsilon^3 + 144\lambda_j \lambda_i \tau_x^2 \tau_\epsilon^2 \gamma^2 - 27\gamma^6\right)\right]^2 \\ =& \frac{256}{27}\lambda_j^6 \lambda_i^3 \tau_x^9 \tau_\epsilon^3 (4\lambda_i \tau_x \tau_\epsilon + 3\gamma^2)^3 \\ & - \frac{1}{27}\lambda_j^4 \lambda_i^2 \tau_x^6 (128\lambda_j \lambda_i^2 \tau_x^3 \tau_\epsilon^3 + 144\lambda_j \lambda_i \tau_x^2 \tau_\epsilon^2 \gamma^2 - 27\gamma^6)^2. \end{aligned}$$

Whenever $\Delta_{\beta_i} < 0$ (resp., $\Delta_{\beta_i} = 0$), there exist(s) two (resp., one) linear REE. $\square$

Proof of Proposition 4.3. By (4.9), the total effect of a rise in λ_i on I_{x_i} and I_{x_j} is given by

$$\frac{dI_{x_i}}{d\lambda_i} = \frac{\partial I_{x_i}}{\partial \lambda_i} + \frac{\partial I_{x_i}}{\partial I_{x_j}}\frac{dI_{x_j}}{d\lambda_i} \quad \text{and} \quad \frac{dI_{x_j}}{d\lambda_i} = \frac{\partial I_{x_j}}{\partial I_{x_i}}\frac{dI_{x_i}}{d\lambda_i}.$$

First, we solve for the total effect on I_{x_i}:

$$\frac{dI_{x_i}}{d\lambda_i} = \frac{\partial I_{x_i}}{\partial \lambda_i} + \frac{\partial I_{x_i}}{\partial I_{x_j}}\frac{\partial I_{x_j}}{\partial I_{x_i}}\frac{dI_{x_i}}{d\lambda_i}$$

$$\Leftrightarrow \frac{dI_{x_i}}{d\lambda_i} = \frac{\frac{\partial I_{x_i}}{\partial \lambda_i}}{1 - \frac{\partial I_{x_i}}{\partial I_{x_j}}\frac{\partial I_{x_j}}{\partial I_{x_i}}}. \tag{A27}$$

The total impact on I_{x_j} is

$$\frac{dI_{x_j}}{d\lambda_i} = \frac{\partial I_{x_j}}{\partial I_{x_i}}\left(\frac{\partial I_{x_i}}{\partial \lambda_i} + \frac{\partial I_{x_i}}{\partial I_{x_j}}\frac{dI_{x_j}}{d\lambda_i}\right)$$

$$\Leftrightarrow \frac{dI_{x_j}}{d\lambda_i} = \frac{\frac{\partial I_{x_j}}{\partial I_{x_i}}\frac{\partial I_{x_i}}{\partial \lambda_i}}{1 - \frac{\partial I_{x_i}}{\partial I_{x_j}}\frac{\partial I_{x_j}}{\partial I_{x_i}}}. \tag{A28}$$

Making use of (4.9), further computations yield

$$\frac{\partial I_{x_i}}{\partial \lambda_i} = \frac{[\gamma^2(1-I_{x_j})^2 + \lambda_i\tau_x\tau_\epsilon]\tau_x\tau_\epsilon - \lambda_i\tau_x^2\tau_\epsilon^2}{[\gamma^2(1-I_{x_j})^2 + \lambda_i\tau_x\tau_\epsilon]^2} = \frac{\gamma^2(1-I_{x_j})^2\tau_x\tau_\epsilon}{[\gamma^2(1-I_{x_j})^2 + \lambda_i\tau_x\tau_\epsilon]^2} > 0.$$

The partial $\partial I_{x_j}/\partial I_{x_i}$ in (A28) is clearly positive due to the derived complementarity in trading against different types of noise. As a consequence, the identical denominator in (A27) and (A28) pins down the sign of $dI_{x_i}/d\lambda_i$ and $dI_{x_j}/d\lambda_i$. By (4.9),

$$\frac{\partial I_{x_i}}{\partial I_{x_j}} = \frac{2\gamma^2(1-I_{x_j})\lambda_i\tau_x\tau_\epsilon}{[\gamma^2(1-I_{x_j})^2 + \lambda_i\tau_x\tau_\epsilon]^2}.$$

Next, we eliminate λ_i in the above derivative. Solving (4.9) for λ_i yields

$$\frac{\lambda_i\tau_x\tau_\epsilon}{\gamma^2(1-I_{x_j})^2 + \lambda_i\tau_x\tau_\epsilon} = I_{x_i}$$

$$\Leftrightarrow I_{x_i}\left[\gamma^2(1-I_{x_j})^2 + \lambda_i\tau_x\tau_\epsilon\right] = \lambda_i\tau_x\tau_\epsilon$$

$$\Leftrightarrow (I_{x_i}-1)\lambda_i\tau_x\tau_\epsilon = -\gamma^2 I_{x_i}(1-I_{x_j})^2$$

$$\Leftrightarrow \lambda_i = \frac{\gamma^2 I_{x_i}(1-I_{x_j})^2}{(1-I_{x_i})\tau_x\tau_\epsilon}. \tag{A29}$$

This delivers

$$\frac{\partial I_{x_i}}{\partial I_{x_j}} = \frac{\dfrac{2\gamma^4 I_{x_i}(1-I_{x_j})^3}{1-I_{x_i}}}{\left[\gamma^2(1-I_{x_j})^2 + \dfrac{\gamma^2 I_{x_i}(1-I_{x_j})^2}{1-I_{x_i}}\right]^2}$$

$$= \frac{\dfrac{2\gamma^4 I_{x_i}(1-I_{x_j})^3}{1-I_{x_i}}}{\left[\dfrac{\gamma^2(1-I_{x_j})^2}{1-I_{x_i}}\right]^2}$$

$$= \frac{2I_{x_i}(1-I_{x_i})}{1-I_{x_j}}.$$

By symmetry,

$$\frac{\partial I_{x_j}}{\partial I_{x_i}} = \frac{2I_{x_j}(1-I_{x_j})}{1-I_{x_i}}.$$

With all this in hand, we can explicitly calculate the denominator in (A27) and (A28):

$$\Gamma \equiv 1 - \frac{\partial I_{x_i}}{\partial I_{x_j}}\frac{\partial I_{x_j}}{\partial I_{x_i}} = 1 - \frac{2I_{x_i}(1-I_{x_i})}{1-I_{x_j}}\frac{2I_{x_j}(1-I_{x_j})}{1-I_{x_i}}$$

$$= 1 - 4I_{x_i}I_{x_j},$$

which proves part (a) of the proposition. Recalling (A29), further computations deliver

$$\frac{dI_{x_i}}{d\lambda_i} = \frac{\gamma^2(1-I_{x_j})^2\tau_x\tau_\epsilon}{(1-4I_{x_i}I_{x_j})[\gamma^2(1-I_{x_j})^2+\lambda_i\tau_x\tau_\epsilon]^2}$$

$$= \frac{\gamma^2(1-I_{x_j})^2\tau_x\tau_\epsilon}{(1-4I_{x_i}I_{x_j})\left\{\gamma^2(1-I_{x_j})^2 + \left[\dfrac{\gamma^2 I_{x_i}(1-I_{x_j})^2}{(1-I_{x_i})\tau_x\tau_\epsilon}\right]\tau_x\tau_\epsilon\right\}^2}$$

$$= \frac{\tau_\epsilon\tau_x(1-I_{x_i})^2}{\gamma^2(1-I_{x_j})^2(1-4I_{x_i}I_{x_j})} \tag{A30}$$

and

$$\frac{dI_{x_j}}{d\lambda_i} = \frac{1}{(1-4I_{x_i}I_{x_j})}\frac{2I_{x_j}(1-I_{x_j})}{1-I_{x_i}}\frac{\tau_\epsilon\tau_x(1-I_{x_i})^2}{\gamma^2(1-I_{x_j})^2}$$

$$= \frac{2\tau_\epsilon\tau_x(1-I_{x_i})I_{x_j}}{\gamma^2(1-I_{x_j})(1-4I_{x_i}I_{x_j})}. \tag{A31}$$

To prove part (b) in Proposition 4.3, it suffices to explore the effect of an increase in λ_i on β_i and β_j in equilibrium, as the two coefficient ratios are positively connected to the trading intensities (see (4.8)). According to (A22), in equilibrium, it must hold that $\beta_i - f(\beta_i) = 0$. Implicit differentiation with respect to λ_i delivers

$$\frac{\partial\beta_i}{\partial\beta_i}\frac{d\beta_i}{d\lambda_i} - \left[\frac{\partial f(\beta_i)}{\partial\lambda_i} + f'(\beta_i)\frac{d\beta_i}{d\lambda_i}\right] = 0$$

$$\Leftrightarrow \frac{d\beta_i}{d\lambda_i} = \frac{\frac{\partial f(\beta_i)}{\partial\lambda_i}}{1-f'(\beta_i)}.$$

Since $f(\beta_i)$ is strictly increasing in λ_i (see (A22)), it follows that $sign(d\beta_i/d\lambda_i) = sign[1-f'(\beta_i)]$. As $f(\beta_i)$ is a strictly increasing and convex function in β_i with a positive intercept (i.e., $f'(\beta_i) > 0$, $f''(\beta_i) > 0$, and $f(0) > 0$), it can be concluded that $f'(\beta_{i,LIE}) < 1$ and $f'(\beta_{i,HIE}) > 1$ (see also Figure 4.2). Hence, $d\beta_i/d\lambda_i > 0$ holds in the LIE and $d\beta_i/d\lambda_i < 0$ is true in the HIE. The analogous result holds for $\beta_{j,LIE}$ and $\beta_{j,HIE}$. The positive link between the coefficient ratios and the trading intensities proves that $\Gamma > 0$ (resp., $\Gamma < 0$) is true in the LIE (resp., HIE).

To derive part (c) in the proposition, we assume $I_{x_i} = 0.25\, I_{x_j}^{-1}$ so that $\Gamma = 0$, and examine the consequences in equilibrium. At this point, the effect of a change in λ_i on both trading intensities is undefined. Then, by (4.8), the value of β_i in equilibrium in terms of β_j is

$$\beta_i = \frac{\tau_\epsilon}{\gamma(1-I_{x_i})}$$

$$= \frac{\tau_\epsilon}{\gamma\left(1-0.25\, I_{x_j}^{-1}\right)}$$

$$= \frac{\tau_\epsilon}{\gamma\left[1-0.25\,(1-\tau_\epsilon/\gamma\beta_j)^{-1}\right]}$$

$$= \frac{\tau_\epsilon}{\gamma\left(1-\frac{\gamma\beta_j}{4\gamma\beta_j-4\tau_\epsilon}\right)}$$

$$= \frac{\tau_\epsilon}{\gamma \dfrac{3\gamma\beta_j - 4\tau_\epsilon}{4\gamma\beta_j - 4\tau_\epsilon}}$$

$$= \frac{4\tau_\epsilon(\gamma\beta_j - \tau_\epsilon)}{\gamma(3\gamma\beta_j - 4\tau_\epsilon)}.$$

Equating the above term with (A21) and rearranging terms delivers

$$\frac{\tau_\epsilon + \lambda_i\beta_j^2\tau_x}{\gamma} = \frac{4\tau_\epsilon(\gamma\beta_j - \tau_\epsilon)}{\gamma(3\gamma\beta_j - 4\tau_\epsilon)}$$

$$\Leftrightarrow (\tau_\epsilon + \lambda_i\beta_j^2\tau_x)(3\gamma\beta_j - 4\tau_\epsilon) = 4\tau_\epsilon(\gamma\beta_j - \tau_\epsilon)$$

$$\Leftrightarrow 3\gamma\lambda_i\tau_x\beta_j^3 - 4\lambda_i\tau_\epsilon\tau_x\beta_j^2 - \gamma\tau_\epsilon\beta_j = 0.$$

The above cubic polynomial has three real roots. The trivial root $\beta_{j,1} = 0$, however, violates the value range of β_j. The two other roots are given by

$$\beta_{j,2/3} = \frac{4\lambda_i\tau_\epsilon\tau_x \pm \sqrt{16\lambda_i^2\tau_\epsilon^2\tau_x^2 + 12\gamma^2\lambda_i\tau_x\tau_\epsilon}}{6\gamma\lambda_i\tau_x}.$$

Hence,

$$\beta_{j,2} = \frac{4\lambda_i\tau_\epsilon\tau_x - \sqrt{16\lambda_i^2\tau_\epsilon^2\tau_x^2 + 12\gamma^2\lambda_i\tau_x\tau_\epsilon}}{6\gamma\lambda_i\tau_x}$$

$$= \frac{2\lambda_i\tau_\epsilon\tau_x - \sqrt{4\lambda_i^2\tau_\epsilon^2\tau_x^2 + 3\gamma^2\lambda_i\tau_x\tau_\epsilon}}{3\gamma\lambda_i\tau_x}.$$

By carefully checking the above root, one sees that $\beta_{j,2} < 0$. This again contradicts the value range of β_j. Therefore, the only positive real root is given by

$$\beta_{j,3} = \frac{4\lambda_i\tau_\epsilon\tau_x + \sqrt{16\lambda_i^2\tau_\epsilon^2\tau_x^2 + 12\gamma^2\lambda_i\tau_x\tau_\epsilon}}{6\gamma\lambda_i\tau_x}$$

$$= \frac{2\lambda_i\tau_\epsilon\tau_x + \sqrt{\lambda_i\tau_\epsilon\tau_x(4\lambda_i\tau_\epsilon\tau_x + 3\gamma^2)}}{3\gamma\lambda_i\tau_x}.$$

By symmetry,

$$\beta_i = \tilde{\beta}_i \equiv \frac{2\lambda_j\tau_\epsilon\tau_x + \sqrt{\lambda_j\tau_\epsilon\tau_x(4\lambda_j\tau_\epsilon\tau_x + 3\gamma^2)}}{3\gamma\lambda_j\tau_x}.$$

Whenever $\Gamma = 0$, the equilibrium values of β_1 and β_2 are given by $\tilde{\beta}_1$ and $\tilde{\beta}_2$, which are unique and given in closed form. From Proposition 4.1, we know that the solutions for β_1 and β_2 are unique if and only if the discriminant Δ_{β_i} belonging to (A23) equals zero. Thus, $\Gamma = 0$ is associated with the special case where exactly one linear REE exists.

By equating $\tilde{\beta}_i$ with (A21) and substituting for β_j, we can derive the critical value of λ_i that is linked to the existence of a unique linear REE:

$$
\begin{aligned}
\tilde{\beta}_i &= \frac{\tau_\epsilon + \lambda_i \beta_j^2 \tau_x}{\gamma} \\
&= \frac{\tau_\epsilon + \lambda_i \tau_x \dfrac{(\tau_\epsilon + \lambda_j \tilde{\beta}_i^2\, \tau_x)^2}{\gamma^2}}{\gamma} \\
\Leftrightarrow \lambda_i &= \frac{\gamma^3 \left(\tilde{\beta}_i - \dfrac{\tau_\epsilon}{\gamma}\right)}{\tau_x(\tau_\epsilon + \lambda_j \tilde{\beta}_i^2\, \tau_x)^2} \\
&= \frac{\gamma^3 \left[\dfrac{2\lambda_j\tau_\epsilon\tau_x + \sqrt{\lambda_j\tau_\epsilon\tau_x(4\lambda_j\tau_\epsilon\tau_x + 3\gamma^2)}}{3\gamma\lambda_j\tau_x} - \dfrac{\tau_\epsilon}{\gamma}\right]}{\tau_x(\tau_\epsilon + \lambda_j \tilde{\beta}_i^2\, \tau_x)^2} \\
&= \frac{\gamma^2[\sqrt{\lambda_j\tau_\epsilon\tau_x(4\lambda_j\tau_\epsilon\tau_x + 3\gamma^2)} - \lambda_j\tau_\epsilon\tau_x]}{3\lambda_j\tau_x^2(\tau_\epsilon + \lambda_j \tilde{\beta}_i^2\, \tau_x)^2} \\
&= \frac{\gamma^2[\sqrt{\lambda_j\tau_\epsilon\tau_x(4\lambda_j\tau_\epsilon\tau_x + 3\gamma^2)} - \lambda_j\tau_\epsilon\tau_x]}{3\lambda_j\tau_x^2 \left\{\tau_\epsilon + \lambda_j \dfrac{[2\lambda_j\tau_\epsilon\tau_x + \sqrt{\lambda_j\tau_\epsilon\tau_x(4\lambda_j\tau_\epsilon\tau_x + 3\gamma^2)}]^2}{9\gamma^2\lambda_j^2\tau_x^2}\, \tau_x\right\}^2} \\
&= \frac{27\gamma^6\lambda_j[\sqrt{\lambda_j\tau_\epsilon\tau_x(4\lambda_j\tau_\epsilon\tau_x + 3\gamma^2)} - \lambda_j\tau_\epsilon\tau_x]}{\{9\gamma^2\lambda_j\tau_\epsilon\tau_x + [2\lambda_j\tau_\epsilon\tau_x + \sqrt{\lambda_j\tau_\epsilon\tau_x(4\lambda_j\tau_\epsilon\tau_x + 3\gamma^2)}]^2\}^2} \equiv \tilde{\lambda}_i > 0. \qquad \text{(A32)}
\end{aligned}
$$

Since $f(\beta_i)$ is increasing in λ_i (see (A22)), we can further conclude that $\lambda_i < \tilde{\lambda}_i$ is a necessary and sufficient condition for the existence of the LIE and the HIE. If $\lambda_i > \tilde{\lambda}_i$, there is no solution to the underlying fixed-point equation in (A22) and a linear REE fails to exist. This proves part (d) in the proposition. □

Proof of Proposition 4.4. By turning x_1 and x_2 into costly signals, the wealth function of a noise-informed trader changes to $\pi_{n_i} = (\theta - P)D_{n_i} - c_i$, for $i = 1, 2$. Since c_i is a constant, the wealth function still follows a normal distribution. By recalling the results from Section 3.1, conditional expected utility becomes

$$
\mathrm{E}[U(\pi_{n_i}) \,|\, x_i, P] = -\exp\left\{-\gamma\left[\mathrm{E}(\pi_{n_i} \,|\, x_i, P) - \frac{\gamma}{2}\mathrm{Var}(\pi_{n_i} \,|\, x_i, P)\right]\right\}.
$$

Inspecting the wealth function delivers

$$\mathrm{E}[U(\pi_{n_i}) \,|\, x_i, P] = -\exp\left(-\gamma\left\{[\mathrm{E}(\theta \,|\, x_i, P) - P]D_{n_i} - c_i - \frac{\gamma}{2}\mathrm{Var}(\theta \,|\, x_i, P)D_{n_i}^2\right\}\right).$$

Since agents are characterized by constant absolute risk aversion, their demand for the risky asset does not depend on their initial wealth (i.e., the cost of acquiring non-fundamental information does not change their optimal demand). Plugging the optimal demand for the risky asset from (4.2) into the expression of the conditional expected utility yields

$$\begin{aligned}\mathrm{E}[U(\pi_{n_i}) \,|\, x_i, P] &= -\exp\left(-\gamma\left\{\frac{[\mathrm{E}(\theta \,|\, x_i, P) - P]^2}{\gamma\mathrm{Var}(\theta \,|\, x_i, P)} - c_i - \frac{\gamma}{2}\frac{[\mathrm{E}(\theta \,|\, x_i, P) - P]^2}{\gamma^2\mathrm{Var}(\theta \,|\, x_i, P)}\right\}\right)\\ &= -\exp(\gamma c_i)\exp\left\{-\frac{[\mathrm{E}(\theta \,|\, x_i, P) - P]^2}{2\mathrm{Var}(\theta \,|\, x_i, P)}\right\}.\end{aligned}$$

Taking expectations conditional on P gives

$$\begin{aligned}&\mathrm{E}\left\{\mathrm{E}\left[U(\pi_{n_i}) \,|\, x_i, P\right] \,|\, P\right\}\\ =& -\exp(\gamma c_i)\ \mathrm{E}\left(\exp\left\{-\frac{[\mathrm{E}(\theta \,|\, x_i, P) - P]^2}{2\,\mathrm{Var}(\theta \,|\, x_i, P)}\right\}\middle| P\right)\\ =& -\exp(\gamma c_i)\ \mathrm{E}\left(\exp\left\{-\frac{\mathrm{Var}\left[\mathrm{E}\left(\theta \,|\, x_i, P\right) \,|\, P\right]}{2\,\mathrm{Var}(\theta \,|\, x_i, P)}\frac{[\mathrm{E}(\theta \,|\, x_i, P) - P]^2}{\mathrm{Var}[\mathrm{E}(\theta \,|\, x_i, P)|P]}\right\}\middle| P\right)\\ =& -\exp(\gamma c_i)\ \mathrm{E}\left(\exp\left\{-\frac{\mathrm{Var}[\mathrm{E}(\theta \,|\, x_i, P)|P]}{2\,\mathrm{Var}(\theta \,|\, x_i, P)}\, z^2\right\}\middle| P\right),\end{aligned}$$

where $z \equiv \dfrac{\mathrm{E}(\theta \,|\, x_i, P) - P}{\sqrt{\mathrm{Var}[\mathrm{E}(\theta \,|\, x_i, P) \,|\, P]}}$.

As z is a sum of linear transformations of normal random variables, it is normally distributed too (note that $\mathrm{Var}[\mathrm{E}(\theta \,|\, x_i, P) \,|\, P]$ is non-random). Conditional on P, z still follows a normal distribution with mean

$$\begin{aligned}\mathrm{E}\left\{\frac{\mathrm{E}(\theta \,|\, x_i, P) - P}{\sqrt{\mathrm{Var}[\mathrm{E}(\theta \,|\, x_i, P)|P]}}\middle| P\right\} &= \frac{\mathrm{E}\left[\mathrm{E}(\theta \,|\, x_i, P) - P \,\middle|\, P\right]}{\sqrt{\mathrm{Var}[\mathrm{E}(\theta \,|\, x_i, P)|P]}}\\ &= \frac{\mathrm{E}(\theta \,|\, P) - P}{\sqrt{\mathrm{Var}[\mathrm{E}(\theta \,|\, x_i, P)|P]}},\end{aligned}$$

where the last equation follows from the law of iterated expectations. The variance of z conditional on P is

$$\operatorname{Var}\left\{ \frac{\mathrm{E}(\theta \,|\, x_i, P) - P}{\sqrt{\operatorname{Var}[\mathrm{E}(\theta \,|\, x_i, P) \,|\, P]}} \middle| P \right\} = \frac{\operatorname{Var}[\mathrm{E}(\theta \,|\, x_i, P) - P|P]}{\operatorname{Var}[\mathrm{E}(\theta \,|\, x_i, P)|P]}$$

$$= \frac{\operatorname{Var}[\mathrm{E}(\theta \,|\, x_i, P)|P]}{\operatorname{Var}[\mathrm{E}(\theta \,|\, x_i, P)|P]}$$

$$= 1.$$

Since the variance of z conditional on P equals unity, z^2 follows a noncentral chi-square distribution conditional on P. In Appendix B.3.2, we prove that

$$\mathrm{E}[\exp(tz^2) \,|\, P] = \frac{1}{\sqrt{1-2t}} \exp\left\{ \frac{t\,[\mathrm{E}(z \,|\, P)]^2}{1-2t} \right\}, \quad for \;\; t < 0.5.$$

By setting $t = -\dfrac{\operatorname{Var}[\mathrm{E}(\theta \,|\, x_i, P) \,|\, P]}{2\operatorname{Var}(\theta \,|\, x_i, P)}$, we get

$$\frac{1}{\sqrt{1-2t}} = \frac{1}{\sqrt{1 + \dfrac{\operatorname{Var}[\mathrm{E}(\theta \,|\, x_i, P) \,|\, P]}{\operatorname{Var}(\theta \,|\, x_i, P)}}}.$$

Using the law of total conditional variance yields

$$\operatorname{Var}(\theta \,|\, P) = \mathrm{E}[\operatorname{Var}(\theta \,|\, x_i, P) \,|\, P] + \operatorname{Var}[\mathrm{E}(\theta \,|\, x_i, P) \,|\, P]$$

$$\Leftrightarrow \operatorname{Var}[\mathrm{E}(\theta \,|\, x_i, P) \,|\, P] = \operatorname{Var}(\theta \,|\, P) - \mathrm{E}[\operatorname{Var}(\theta \,|\, x_i, P) \,|\, P]$$

$$= \operatorname{Var}(\theta \,|\, P) - \operatorname{Var}(\theta \,|\, x_i, P),$$

where the last step follows from the fact that $\operatorname{Var}(\theta \,|\, x_i, P)$ is non-random. Thus,

$$\frac{1}{\sqrt{1-2t}} = \frac{1}{\sqrt{1 + \dfrac{\operatorname{Var}[\mathrm{E}(\theta \,|\, x_i, P) \,|\, P]}{\operatorname{Var}(\theta \,|\, x_i, P)}}}$$

$$= \frac{1}{\sqrt{1 + \dfrac{\operatorname{Var}(\theta \,|\, P) - \operatorname{Var}(\theta \,|\, x_i, P)}{\operatorname{Var}(\theta \,|\, x_i, P)}}}$$

$$= \sqrt{\frac{\operatorname{Var}(\theta \,|\, x_i, P)}{\operatorname{Var}(\theta \,|\, P)}}$$

and

$$\exp\left\{\frac{t\,[\mathrm{E}(z\,|\,P)]^2}{1-2t}\right\} = \exp\left(\frac{-\frac{\mathrm{Var}\,[\mathrm{E}\,(\theta|x_i,P)|P]}{2\,\mathrm{Var}\,(\theta|x_i,P)}\left\{\frac{\mathrm{E}\,(\theta|P)-P}{\sqrt{\mathrm{Var}\,[\mathrm{E}\,(\theta|x_i,P)|P]}}\right\}^2}{\frac{\mathrm{Var}\,(\theta|P)}{\mathrm{Var}\,(\theta|x_i,P)}}\right)$$

$$= \exp\left\{-\frac{[\mathrm{E}\,(\theta|P)-P]^2}{2\,\mathrm{Var}\,(\theta|P)}\right\}.$$

Putting the obtained results together delivers

$$\mathrm{E}\left(\exp\left\{-\frac{\mathrm{Var}[\mathrm{E}(\theta\,|\,x_i,P)\,|\,P]}{2\,\mathrm{Var}(\theta\,|\,x_i,P)}\,z^2\right\}\Bigg|P\right)$$

$$= \sqrt{\frac{\mathrm{Var}(\theta\,|\,x_i,P)}{\mathrm{Var}(\theta\,|\,P)}}\,\exp\left\{-\frac{[\mathrm{E}(\theta\,|\,P)-P]^2}{2\mathrm{Var}(\theta\,|\,P)}\right\}.$$

Again, making use of the law of iterated expectations, we get

$$\mathrm{E}\left\{\mathrm{E}[U(\pi_{n_i})\,|\,x_i,P]\,|\,P\right\} = -\exp\left(\gamma c_i\right)\sqrt{\frac{\mathrm{Var}(\theta\,|\,x_i,P)}{\mathrm{Var}(\theta\,|\,P)}}\,\exp\left\{-\frac{[\mathrm{E}(\theta\,|\,P)-P]^2}{2\,\mathrm{Var}(\theta\,|\,P)}\right\}$$

$$\Leftrightarrow \mathrm{E}[U(\pi_{n_i})\,|\,P] = -\exp\left(\gamma c_i\right)\sqrt{\frac{\mathrm{Var}(\theta\,|\,x_i,P)}{\mathrm{Var}(\theta\,|\,P)}}\,\exp\left\{-\frac{[\mathrm{E}(\theta\,|\,P)-P]^2}{2\,\mathrm{Var}(\theta\,|\,P)}\right\}.$$

Taking unconditional expectations and using the law of iterated expectations finally yields

$$\mathrm{E}\{\mathrm{E}[U(\pi_{n_i})\,|\,P]\} = -\exp\left(\gamma c_i\right)\,\sqrt{\frac{\mathrm{Var}(\theta\,|\,x_i,P)}{\mathrm{Var}(\theta\,|\,P)}}\,\mathrm{E}\left(\exp\left\{-\frac{[\mathrm{E}(\theta\,|\,P)-P]^2}{2\,\mathrm{Var}(\theta\,|\,P)}\right\}\right)$$

$$\Leftrightarrow \mathrm{E}[U(\pi_{n_i})] = -\exp\left(\gamma c_i\right)\,\sqrt{\frac{\mathrm{Var}(\theta\,|\,x_i,P)}{\mathrm{Var}(\theta\,|\,P)}}\,\mathrm{E}\left(\exp\left\{-\frac{[\mathrm{E}(\theta\,|\,P)-P]^2}{2\,\mathrm{Var}(\theta\,|\,P)}\right\}\right). \tag{A33}$$

Analogously, the conditional expected utility of an uninformed, rational trader is given by

$$\mathrm{E}[U(\pi_u)\,|\,P] = -\,\exp\left\{-\frac{[\mathrm{E}(\theta\,|\,P)-P]^2}{2\,\mathrm{Var}(\theta\,|\,P)}\right\}.$$

Taking unconditional expectations and using the law of iterated expectations delivers

$$\mathrm{E}\left[U(\pi_u)\right] = -\,\mathrm{E}\left(\exp\left\{-\frac{[\mathrm{E}(\theta\,|\,P)-P]^2}{2\,\mathrm{Var}(\theta\,|\,P)}\right\}\right). \tag{A34}$$

By comparing the ex-ante expected utility of a noise-informed trader in (A33) with that of an uninformed, rational trader in (A34), we can derive the value of non-fundamental information:

$$
\begin{aligned}
& \mathrm{E}\left[U(\pi_{n_i})\right] \gtreqless \mathrm{E}\left[U(\pi_u)\right] \\
\Leftrightarrow\ & \frac{\mathrm{E}\left[U(\pi_{n_i})\right]}{\mathrm{E}\left[U(\pi_u)\right]} \lesseqgtr 1 \\
\Leftrightarrow\ & \exp\left(\gamma c_i\right) \sqrt{\frac{\mathrm{Var}(\theta \,|\, x_i, P)}{\mathrm{Var}(\theta \,|\, P)}} \lesseqgtr 1 \\
\Leftrightarrow\ & \exp\left(\gamma c_i\right) \lesseqgtr \sqrt{\frac{\mathrm{Var}(\theta \,|\, P)}{\mathrm{Var}(\theta \,|\, x_i, P)}} \\
\Leftrightarrow\ & c_i \lesseqgtr \frac{1}{2\gamma} \log\left[\frac{\mathrm{Var}(\theta \,|\, P)}{\mathrm{Var}(\theta \,|\, x_i, P)}\right]. \qquad \text{(A35)}
\end{aligned}
$$

The left-hand (resp., right-hand) side in (A35) represents the cost (resp., the value) of non-fundamental information. If its cost is inferior to (resp., exceeds) its value, the ex-ante expected utility of a noise-informed agent exceeds (resp., falls short of) that of an uninformed, rational agent. Whenever the cost of information about noise exactly equals its value, both expected utilities are the same. □

Proof of Proposition 4.5. Direct computations yield

$$
\phi_{x_1}(\beta_1, \beta_2) = \frac{1}{2\gamma} \log\left(\frac{\tau_\theta + \beta_2^2 \tau_x}{\tau_\theta + \dfrac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2}}\right) = \frac{1}{2\gamma} \log\left[\frac{(\beta_1^2 + \beta_2^2)(\tau_\theta + \beta_2^2 \tau_x)}{\beta_2^2 \tau_\theta + \beta_1^2(\tau_\theta + \beta_2^2 \tau_x)}\right], \quad \text{(A36)}
$$

$$
\phi_{x_2}(\beta_1, \beta_2) = \frac{1}{2\gamma} \log\left(\frac{\tau_\theta + \beta_1^2 \tau_x}{\tau_\theta + \dfrac{\tau_x}{1/\beta_1^2 + 1/\beta_2^2}}\right) = \frac{1}{2\gamma} \log\left[\frac{(\beta_1^2 + \beta_2^2)(\tau_\theta + \beta_1^2 \tau_x)}{\beta_2^2 \tau_\theta + \beta_1^2(\tau_\theta + \beta_2^2 \tau_x)}\right]. \quad \text{(A37)}
$$

Case 1. First, we look at the case where no one acquires information about noise (i.e., $\lambda_1^* = \lambda_2^* = 0$). In this situation, no agent finds it beneficial to acquire information about noise, given that there is no single trader in the market possessing non-fundamental information. In equilibrium, it follows that

$$
c_1 \geq \phi_{x_1}\left(\tau_\epsilon/\gamma, \tau_\epsilon/\gamma\right) \text{ and } c_2 \geq \phi_{x_2}\left(\tau_\epsilon/\gamma, \tau_\epsilon/\gamma\right).
$$

Due to symmetry of (A36) and (A37), we obtain

$$\phi_{x_1}(\tau_\epsilon/\gamma, \tau_\epsilon/\gamma) = \phi_{x_2}(\tau_\epsilon/\gamma, \tau_\epsilon/\gamma) = \frac{1}{2\gamma}\log\left(\frac{\tau_\theta + \dfrac{\tau_\epsilon^2\tau_x}{\gamma^2}}{\tau_\theta + \dfrac{\tau_\epsilon^2\tau_x}{2\gamma^2}}\right)$$

$$= \frac{1}{2\gamma}\log\left(\frac{\dfrac{\gamma^2\tau_\theta + \tau_\epsilon^2\tau_x}{\gamma^2}}{\dfrac{2\gamma^2\tau_\theta + \tau_\epsilon^2\tau_x}{2\gamma^2}}\right)$$

$$= \frac{1}{2\gamma}\log\left[\frac{2(\gamma^2\tau_\theta + \tau_\epsilon^2\tau_x)}{2\gamma^2\tau_\theta + \tau_\epsilon^2\tau_x}\right] \equiv \bar{c}.$$

Therefore, in an information acquisition equilibrium of the form $\lambda_1^* = \lambda_2^* = 0$, it holds that

$$c_1 \geq \bar{c} \quad \text{and} \quad c_2 \geq \bar{c}. \tag{A38}$$

Case 2. In the second case, we turn to the situation where agents acquire information about x_1 only (i.e., $\lambda_1^* > 0$, $\lambda_2^* = 0$). Thus, in equilibrium,

$$\phi_{x_1}(\beta_1, \tau_\epsilon/\gamma) = c_1 \quad \text{and} \quad \phi_{x_2}(\beta_1, \tau_\epsilon/\gamma) \leq c_2,$$

with $\beta_1 > \tau_\epsilon/\gamma$ (see also (A21)). The value of β_1 in equilibrium in terms of c_1 is, then, given by

$$c_1 = \frac{1}{2\gamma}\log\left(\frac{\tau_\theta + \dfrac{\tau_\epsilon^2\tau_x}{\gamma^2}}{\tau_\theta + \dfrac{\tau_x}{\dfrac{1}{\beta_1^2} + \dfrac{\gamma^2}{\tau_\epsilon^2}}}\right)$$

$$\Leftrightarrow e^{2\gamma c_1} = \frac{\dfrac{\gamma^2\tau_\theta + \tau_\epsilon^2\tau_x}{\gamma^2}}{\dfrac{\beta_1^2(\tau_\epsilon^2\tau_x + \gamma^2\tau_\theta) + \tau_\theta\tau_\epsilon^2}{\beta_1^2\gamma^2 + \tau_\epsilon^2}}$$

$$\Leftrightarrow e^{2\gamma c_1} = \frac{(\beta_1^2\gamma^2 + \tau_\epsilon^2)(\gamma^2\tau_\theta + \tau_\epsilon^2\tau_x)}{\gamma^2[\beta_1^2(\tau_\epsilon^2\tau_x + \gamma^2\tau_\theta) + \tau_\theta\tau_\epsilon^2]}$$

$$\Leftrightarrow \beta_1^2\left[\gamma^2(e^{2\gamma c_1} - 1)(\tau_\epsilon^2\tau_x + \gamma^2\tau_\theta)\right] = \tau_\epsilon^2\left[\tau_\epsilon^2\tau_x - (e^{2\gamma c_1} - 1)\gamma^2\tau_\theta\right]$$

$$\Leftrightarrow \beta_1 = \frac{\tau_\epsilon \sqrt{\tau_\epsilon^2 \tau_x - (e^{2\gamma c_1} - 1)\gamma^2 \tau_\theta}}{\gamma \sqrt{(e^{2\gamma c_1} - 1)(\tau_\epsilon^2 \tau_x + \gamma^2 \tau_\theta)}}.$$

For $\lambda_1^* > 0$ to be true in equilibrium, it must hold that $\beta_1 > \tau_\epsilon/\gamma$, which is equivalent to

$$\frac{\tau_\epsilon \sqrt{\tau_\epsilon^2 \tau_x - (e^{2\gamma c_1} - 1)\gamma^2 \tau_\theta}}{\gamma \sqrt{(e^{2\gamma c_1} - 1)(\tau_\epsilon^2 \tau_x + \gamma^2 \tau_\theta)}} - \frac{\tau_\epsilon}{\gamma} > 0$$

$$\Leftrightarrow \frac{\tau_\epsilon^2 \tau_x - (e^{2\gamma c_1} - 1)\gamma^2 \tau_\theta}{(e^{2\gamma c_1} - 1)(\tau_\epsilon^2 \tau_x + \gamma^2 \tau_\theta)} > 1$$

$$\Leftrightarrow 2(\gamma^2 \tau_\theta + \tau_\epsilon^2 \tau_x) > e^{2\gamma c_1}(\tau_\epsilon^2 \tau_x + 2\gamma^2 \tau_\theta)$$

$$\Leftrightarrow c_1 < \frac{1}{2\gamma} \log \left[\frac{2\left(\gamma^2 \tau_\theta + \tau_\epsilon^2 \tau_x\right)}{2\gamma^2 \tau_\theta + \tau_\epsilon^2 \tau_x} \right] = \bar{c}.$$

Since β_1 is decreasing in c_1 and $\beta_1 = \tau_\epsilon/\gamma$ for $c_1 = \bar{c}$, $\beta_1 \in \mathbb{R}_{++}$ holds for sure for all $c_1 < \bar{c}$. Furthermore, we can express the value of information about x_2 in terms of c_1 as

$$\phi_{x_2}\left(\beta_1, \tau_\epsilon/\gamma\right) = \frac{1}{2\gamma} \log \left\{ \frac{\tau_\theta + \dfrac{\tau_\epsilon^2[\tau_\epsilon^2 \tau_x - (e^{2\gamma c_1} - 1)\gamma^2 \tau_\theta]}{(e^{2\gamma c_1} - 1)\gamma^2(\tau_\epsilon^2 \tau_x + \gamma^2 \tau_\theta)} \tau_x}{\tau_\theta + \dfrac{\tau_x}{\dfrac{(e^{2\gamma c_1} - 1)\gamma^2(\tau_\epsilon^2 \tau_x + \gamma^2 \tau_\theta)}{\tau_\epsilon^2[\tau_\epsilon^2 \tau_x - (e^{2\gamma c_1} - 1)\gamma^2 \tau_\theta]} + \dfrac{\gamma^2}{\tau_\epsilon^2}}} \right\}$$

$$= \frac{1}{2\gamma} \log \left\{ \frac{\tau_\theta + \dfrac{\tau_\epsilon^2[\tau_\epsilon^2 \tau_x - (e^{2\gamma c_1} - 1)\gamma^2 \tau_\theta]}{(e^{2\gamma c_1} - 1)\gamma^2(\tau_\epsilon^2 \tau_x + \gamma^2 \tau_\theta)} \tau_x}{\tau_\theta + \dfrac{\tau_x}{\dfrac{e^{2\gamma c_1}\gamma^2 \tau_x}{\tau_\epsilon^2 \tau_x - (e^{2\gamma c_1} - 1)\gamma^2 \tau_\theta}}} \right\}$$

$$= \frac{1}{2\gamma} \log \left[\frac{\dfrac{(e^{2\gamma c_1} - 1)\gamma^4 \tau_\theta^2 + \tau_\epsilon^4 \tau_x^2}{(e^{2\gamma c_1} - 1)\gamma^2(\gamma^2 \tau_\theta + \tau_\epsilon^2 \tau_x)}}{\dfrac{\gamma^2 \tau_\theta + \tau_\epsilon^2 \tau_x}{e^{2\gamma c_1}\gamma^2}} \right]$$

$$= \frac{1}{2\gamma} \log \left\{ \frac{e^{2\gamma c_1}\left[(e^{2\gamma c_1} - 1)\ \gamma^4 \tau_\theta^2 + \tau_\epsilon^4 \tau_x^2\right]}{(e^{2\gamma c_1} - 1)\left(\gamma^2 \tau_\theta + \tau_\epsilon^2 \tau_x\right)^2} \right\} \equiv f(c_1).$$

Therefore, in an equilibrium of the form $\lambda_1^* > 0$, $\lambda_2^* = 0$, it must hold that

$$c_1 \in (0\,, \bar{c}) \quad \text{and} \quad c_2 \geq f(c_1). \tag{A39}$$

By (A21), we can compute a unique $\lambda_1^* > 0$ by using the unique value of β_1 and $\beta_2 = \tau_\epsilon/\gamma$.

Case 3. The third case deals with the situation where no one possesses information about x_1 and some agents acquire information about x_2 (i.e., $\lambda_1^* = 0$, $\lambda_2^* > 0$). This case is symmetric to the second one. Hence, it can be concluded without any further calculations that such an equilibrium requires

$$c_1 \geq f(c_2) \quad \text{and} \quad c_2 \in (0,\, \bar{c}), \tag{A40}$$

where

$$f(c_2) \equiv \frac{1}{2\gamma} \log \left\{ \frac{e^{2\gamma c_2}\left[\left(e^{2\gamma c_2}-1\right)\gamma^4\tau_\theta^2 + \tau_\epsilon^4\tau_x^2\right]}{\left(e^{2\gamma c_2}-1\right)\left(\gamma^2\tau_\theta + \tau_\epsilon^2\tau_x\right)^2} \right\}.$$

Analogously, by (A21), we can calculate a unique $\lambda_2^* > 0$ by using the unique value of β_2 and $\beta_1 = \tau_\epsilon/\gamma$.

Notably, the condition in (A40) expresses the value range of c_1 as a function of c_2. The condition in (A39), by contrast, indicates the value range of c_2 in terms of c_1. To make both conditions better comparable, we rewrite the condition in (A40) in such a way that it expresses the value range of c_2 in terms of c_1, as the condition in (A39) already does. To get there, we first analyze the monotonicity of $f(c_2)$:

$$\begin{aligned}
&f'(c_2) \\
&= \frac{2\gamma e^{2\gamma c_2}}{2\gamma} \frac{\left(e^{2\gamma c_2}-1\right)\left(\gamma^2\tau_\theta + \tau_\epsilon^2\tau_x\right)^2}{e^{2\gamma c_2}\left[\left(e^{2\gamma c_2}-1\right)\gamma^4\tau_\theta^2 + \tau_\epsilon^4\tau_x^2\right]} \\
&\quad \times \frac{(e^{2\gamma c_2}-1)[(e^{2\gamma c_2}-1)\gamma^4\tau_\theta^2 + \tau_\epsilon^4\tau_x^2 + \gamma^4\tau_\theta^2\, e^{2\gamma c_2}] - e^{2\gamma c_2}[(e^{2\gamma c_2}-1)\gamma^4\tau_\theta^2 + \tau_\epsilon^4\tau_x^2]}{(e^{2\gamma c_2}-1)^2(\gamma^2\tau_\theta + \tau_\epsilon^2\tau_x)^2} \\
&= \frac{\left(e^{2\gamma c_2}-1\right)\left(\gamma^2\tau_\theta + \tau_\epsilon^2\tau_x\right)^2}{\left(e^{2\gamma c_2}-1\right)\gamma^4\tau_\theta^2 + \tau_\epsilon^4\tau_x^2} \frac{(e^{2\gamma c_2}-1)^2\gamma^4\tau_\theta^2 - \tau_\epsilon^4\tau_x^2}{(e^{2\gamma c_2}-1)^2(\gamma^2\tau_\theta + \tau_\epsilon^2\tau_x)^2} \\
&= \frac{(e^{2\gamma c_2}-1)^2\gamma^4\tau_\theta^2 - \tau_\epsilon^4\tau_x^2}{(e^{2\gamma c_2}-1)[(e^{2\gamma c_2}-1)\gamma^4\tau_\theta^2 + \tau_\epsilon^4\tau_x^2]}.
\end{aligned}$$

Hence,

$$\begin{aligned}
f'(c_2) \gtreqless 0 &\Leftrightarrow \frac{(e^{2\gamma c_2}-1)^2\gamma^4\tau_\theta^2 - \tau_\epsilon^4\tau_x^2}{(e^{2\gamma c_2}-1)[(e^{2\gamma c_2}-1)\gamma^4\tau_\theta^2 + \tau_\epsilon^4\tau_x^2]} \gtreqless 0 \\
&\Leftrightarrow c_2 \gtreqless \frac{1}{2\gamma} \log\left(1 + \frac{\tau_\epsilon^2\tau_x}{\gamma^2\tau_\theta}\right) \equiv \tilde{c} > \bar{c}.
\end{aligned}$$

Furthermore,

$$f(\tilde{c}) = \frac{1}{2\gamma}\log\left[\frac{\left(1+\dfrac{\tau_\epsilon^2\tau_x}{\gamma^2\tau_\theta}\right)\left(\dfrac{\tau_\epsilon^2\tau_x}{\gamma^2\tau_\theta}\,\gamma^4\tau_\theta^2+\tau_\epsilon^4\tau_x^2\right)}{\dfrac{\tau_\epsilon^2\tau_x}{\gamma^2\tau_\theta}\left(\gamma^2\tau_\theta+\tau_\epsilon^2\tau_x\right)^2}\right]$$

$$= \frac{1}{2\gamma}\log\left[\frac{\dfrac{\gamma^2\tau_\theta+\tau_\epsilon^2\tau_x}{\gamma^2\tau_\theta}\left(\tau_\epsilon^2\tau_x\gamma^2\tau_\theta+\tau_\epsilon^4\tau_x^2\right)}{\dfrac{\tau_\epsilon^2\tau_x}{\gamma^2\tau_\theta}\left(\gamma^2\tau_\theta+\tau_\epsilon^2\tau_x\right)^2}\right]$$

$$= \frac{1}{2\gamma}\log\left[\frac{\tau_\epsilon^2\tau_x(\gamma^2\tau_\theta+\tau_\epsilon^2\tau_x)^2}{\tau_\epsilon^2\tau_x\left(\gamma^2\tau_\theta+\tau_\epsilon^2\tau_x\right)^2}\right]$$

$$= 0.$$

Consequently, the point $(\tilde{c},\,0)$ represents the global minimum of $f(c_2)$ (and $f(c_1)$). Moreover, since $f(c_2)$ is a quadratic function of c_2, solving $c_1 = f(c_2)$ for c_2 delivers two solutions. The first one, $g(c_1)$ say, is characterized by $0 < g(c_1) < \tilde{c}$ for $c_1 \in \mathbb{R}_{++}$. The second one, $h(c_1)$ say, is characterized by $h(c_1) > \tilde{c}$ for $c_1 \in \mathbb{R}_{++}$. Direct computations yield

$$c_1 = f(c_2) \equiv \frac{1}{2\gamma}\log\left\{\frac{e^{2\gamma c_2}\left[(e^{2\gamma c_2}-1)\;\gamma^4\tau_\theta^2+\tau_\epsilon^4\tau_x^2\right]}{(\gamma^2\tau_\theta+\tau_\epsilon^2\tau_x)^2\,(e^{2\gamma c_2}-1)}\right\}$$

$$\Leftrightarrow e^{2\gamma c_1}(e^{2\gamma c_2}-1)\left(\gamma^2\tau_\theta+\tau_\epsilon^2\tau_x\right)^2 = e^{2\gamma c_2}\left[\left(e^{2\gamma c_2}-1\right)\;\gamma^4\tau_\theta^2+\tau_\epsilon^4\tau_x^2\right]$$

$$\Leftrightarrow \gamma^4\tau_\theta^2\,e^{4\gamma c_2}+\left[\tau_\epsilon^4\tau_x^2-\gamma^4\tau_\theta^2-(\gamma^2\tau_\theta+\tau_\epsilon^2\tau_x)^2e^{2\gamma c_1}\right]e^{2\gamma c_2}+(\gamma^2\tau_\theta+\tau_\epsilon^2\tau_x)^2\,e^{2\gamma c_1}=0.$$

Hence,

$$g(c_1) = \frac{1}{2\gamma}\log\left[\frac{\psi_1(c_1)-\sqrt{\psi_2(c_1)}}{2\gamma^4\tau_\theta^2}\right], \tag{A41}$$

$$h(c_1) = \frac{1}{2\gamma}\log\left[\frac{\psi_1(c_1)+\sqrt{\psi_2(c_1)}}{2\gamma^4\tau_\theta^2}\right], \tag{A42}$$

where

$$\psi_1(c_1) = -\tau_\epsilon^4\tau_x^2+\gamma^4\tau_\theta^2+(\gamma^2\tau_\theta+\tau_\epsilon^2\tau_x)^2e^{2\gamma c_1},$$

$$\psi_2(c_1) = [\tau_\epsilon^4\tau_x^2-\gamma^4\tau_\theta^2-(\gamma^2\tau_\theta+\tau_\epsilon^2\tau_x)^2e^{2\gamma c_1}]^2-4\gamma^4\tau_\theta^2(\gamma^2\tau_\theta+\tau_\epsilon^2\tau_x)^2\,e^{2\gamma c_1}.$$

Since $f(c_2)$ is decreasing in c_2 for $c_2 < \tilde{c}$, $g(c_1)$ is decreasing in c_1 for $c_1 \in \mathbb{R}_{++}$.

Analogously, as $f(c_2)$ is increasing in c_2 for $c_2 > \tilde{c}$, $h(c_1)$ is increasing in c_1 for $c_1 \in \mathbb{R}_{++}$. Thus, $c_1 \geq f(c_2)$ is equivalent to

$$h(c_1) \geq c_2 \geq g(c_1).$$

Recall from (A40) that an equilibrium of the form $\lambda_1^* = 0$, $\lambda_2^* > 0$ requires $c_2 < \bar{c}$. Since $h(c_1) > \tilde{c} > \bar{c}$, the value range of c_2 in terms of c_1 is given by

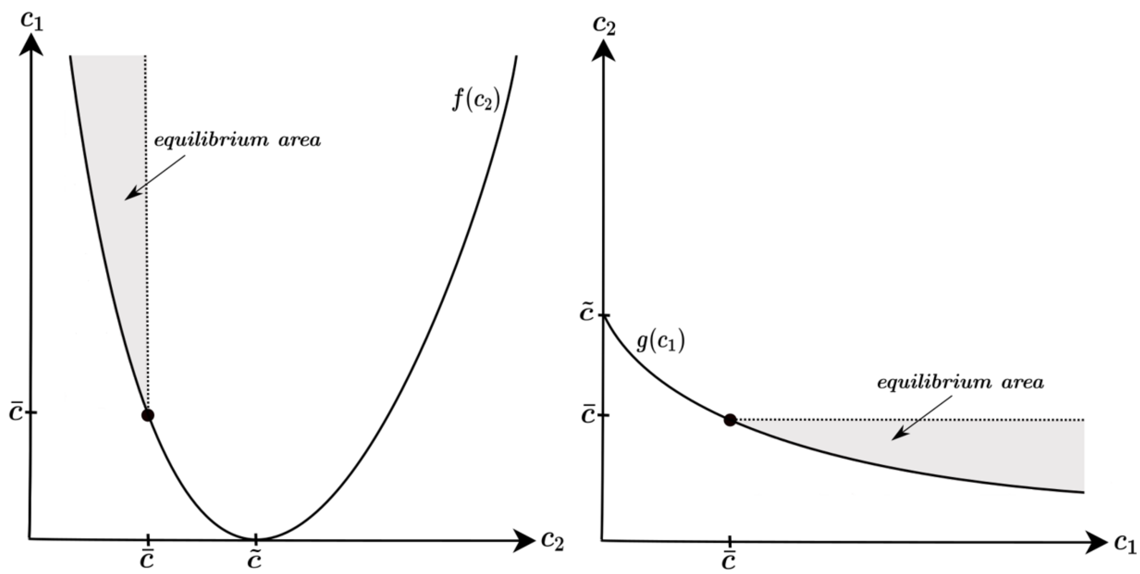
$$\bar{c} > c_2 \geq g(c_1).$$

Furthermore, due to symmetry of cases 2 and 3, we know that $\beta_2 = \tau_\epsilon/\gamma$ for $c_2 = \bar{c}$. Since $\phi_{x_1}(\tau_\epsilon/\gamma, \beta_2)(\equiv f(c_2))$ is decreasing in c_2 for $c_2 < \tilde{c}$ and $\phi_{x_1}(\tau_\epsilon/\gamma, \tau_\epsilon/\gamma) = \bar{c}$, c_1 reaches its infimum at $\bar{c}$. Therefore, the condition that supports an information acquisition equilibrium of the form $\lambda_1^* = 0$, $\lambda_2^* > 0$, given in (A40), can be written as

$$c_1 > \bar{c} \quad \text{and} \quad \bar{c} > c_2 \geq g(c_1). \tag{A43}$$

From $f(\bar{c}) = \bar{c}$, it follows that $g(\bar{c}) = \bar{c}$. Since $g(c_1)$ is decreasing in c_1, we can conclude that $\bar{c} > g(c_1)$ holds for all $c_1 > \bar{c}$. Figure A.1 illustrates the two possible ways of expressing the value range of an equilibrium of the form $\lambda_1^* = 0$, $\lambda_2^* > 0$. The graph on the left-hand side in Figure A.1 corresponds to the condition in (A40), the one on the right-hand side to the condition in (A43).

Figure A.1: Two manners of depicting the equilibrium area of $\lambda_1^* = 0$, $\lambda_2^* > 0$



Case 4. The fourth and last case implies that information about x_1 and x_2 is acquired in equilibrium (i.e., $\lambda_1^* > 0$, $\lambda_2^* > 0$). Thus,

$$\phi_{x_1}(\beta_1, \beta_2) = c_1 \quad \text{and} \quad \phi_{x_2}(\beta_1, \beta_2) = c_2, \tag{A44}$$

with $\beta_1 > \tau_\epsilon/\gamma$, $\beta_2 > \tau_\epsilon/\gamma$. We first derive how a change in c_1 affects the equilibrium values of β_1 and β_2. Implicit differentiation of the system in (A44) with respect to c_1 yields

$$\begin{cases} \dfrac{\partial \phi_{x_1}}{\partial \beta_1}\dfrac{d\beta_1}{dc_1} + \dfrac{\partial \phi_{x_1}}{\partial \beta_2}\dfrac{d\beta_2}{dc_1} = 1, \\[2ex] \dfrac{\partial \phi_{x_2}}{\partial \beta_1}\dfrac{d\beta_1}{dc_1} + \dfrac{\partial \phi_{x_2}}{\partial \beta_2}\dfrac{d\beta_2}{dc_1} = 0. \end{cases}$$

Thus,

$$\frac{d\beta_2}{dc_1} = -\frac{\partial \phi_{x_2}/\partial \beta_1}{\partial \phi_{x_2}/\partial \beta_2}\frac{d\beta_1}{dc_1}.$$

This delivers

$$\begin{aligned} &\frac{\partial \phi_{x_1}}{\partial \beta_1}\frac{d\beta_1}{dc_1} - \frac{\partial \phi_{x_1}}{\partial \beta_2}\frac{\partial \phi_{x_2}/\partial \beta_1}{\partial \phi_{x_2}/\partial \beta_2}\frac{d\beta_1}{dc_1} = 1 \\ \Leftrightarrow &\frac{d\beta_1}{dc_1} = \frac{\partial \phi_{x_2}/\partial \beta_2}{\dfrac{\partial \phi_{x_1}}{\partial \beta_1}\dfrac{\partial \phi_{x_2}}{\partial \beta_2} - \dfrac{\partial \phi_{x_1}}{\partial \beta_2}\dfrac{\partial \phi_{x_2}}{\partial \beta_1}} \end{aligned}$$

and, hence,

$$\frac{d\beta_2}{dc_1} = -\frac{\partial \phi_{x_2}/\partial \beta_1}{\dfrac{\partial \phi_{x_1}}{\partial \beta_1}\dfrac{\partial \phi_{x_2}}{\partial \beta_2} - \dfrac{\partial \phi_{x_1}}{\partial \beta_2}\dfrac{\partial \phi_{x_2}}{\partial \beta_1}}.$$

Using (A36), we obtain

$$\begin{aligned} \frac{\partial \phi_{x_1}}{\partial \beta_1} &= \frac{1}{2\gamma}\frac{\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta + \beta_2^2\tau_x)}{(\beta_1^2+\beta_2^2)(\tau_\theta + \beta_2^2\tau_x)}\frac{\begin{Bmatrix} 2\beta_1[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta + \beta_2^2\tau_x)](\tau_\theta + \beta_2^2\tau_x) \\ -2\beta_1(\tau_\theta + \beta_2^2\tau_x)(\beta_1^2+\beta_2^2)(\tau_\theta + \beta_2^2\tau_x) \end{Bmatrix}}{[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta + \beta_2^2\tau_x)]^2} \\ &= -\frac{1}{2\gamma}\frac{\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta + \beta_2^2\tau_x)}{(\beta_1^2+\beta_2^2)(\tau_\theta + \beta_2^2\tau_x)}\frac{2\beta_1\beta_2^4\tau_x(\tau_\theta + \beta_2^2\tau_x)}{[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta + \beta_2^2\tau_x)]^2} \\ &= -\frac{\beta_1\beta_2^4\tau_x}{\gamma(\beta_1^2+\beta_2^2)[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta + \beta_2^2\tau_x)]} \end{aligned}$$

and

$$\frac{\partial \phi_{x_1}}{\partial \beta_2} = \frac{1}{2\gamma}\frac{\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta + \beta_2^2\tau_x)}{(\beta_1^2+\beta_2^2)(\tau_\theta+\beta_2^2\tau_x)}\frac{\left\{\begin{array}{c} 2\beta_2[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta+\beta_2^2\tau_x)][\tau_\theta + (\beta_1^2+2\beta_2^2)\tau_x] \\ -2\beta_2(\tau_\theta+\beta_1^2\tau_x)(\beta_1^2+\beta_2^2)(\tau_\theta+\beta_2^2\tau_x)\end{array}\right\}}{[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta+\beta_2^2\tau_x)]^2}$$

$$= \frac{1}{2\gamma}\frac{\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta + \beta_2^2\tau_x)}{(\beta_1^2+\beta_2^2)(\tau_\theta+\beta_2^2\tau_x)}\frac{2\beta_2^3\tau_x[\beta_2^2\tau_\theta + \beta_1^2(2\tau_\theta+\beta_2^2\tau_x)]}{[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta+\beta_2^2\tau_x)]^2}$$

$$= \frac{\beta_2^3\tau_x[\beta_2^2\tau_\theta + \beta_1^2(2\tau_\theta+\beta_2^2\tau_x)]}{\gamma(\beta_1^2+\beta_2^2)(\tau_\theta+\beta_2^2\tau_x)[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta+\beta_2^2\tau_x)]}.$$

Symmetry immediately delivers

$$\frac{\partial \phi_{x_2}}{\partial \beta_1} = \frac{\beta_1^3\tau_x[\beta_1^2\tau_\theta + \beta_2^2(2\tau_\theta+\beta_1^2\tau_x)]}{\gamma(\beta_1^2+\beta_2^2)(\tau_\theta+\beta_1^2\tau_x)[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta+\beta_2^2\tau_x)]}$$

and

$$\frac{\partial \phi_{x_2}}{\partial \beta_2} = -\frac{\beta_1^4\beta_2\tau_x}{\gamma(\beta_1^2+\beta_2^2)[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta+\beta_2^2\tau_x)]}.$$

Thus,

$$\begin{aligned}
&\frac{\partial \phi_{x_1}}{\partial \beta_1}\frac{\partial \phi_{x_2}}{\partial \beta_2} - \frac{\partial \phi_{x_1}}{\partial \beta_2}\frac{\partial \phi_{x_2}}{\partial \beta_1} \\
&= \frac{\beta_1^5\beta_2^5\tau_x^2}{\gamma^2(\beta_1^2+\beta_2^2)^2(\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta+\beta_2^2\tau_x))^2} \\
&\quad - \frac{\beta_1^3\beta_2^3\tau_x^2(\beta_1^2\tau_\theta + \beta_2^2[2\tau_\theta+\beta_1^2\tau_x)][\beta_2^2\tau_\theta + \beta_1^2(2\tau_\theta+\beta_2^2\tau_x)]}{\gamma^2(\beta_1^2+\beta_2^2)^2(\tau_\theta+\beta_1^2\tau_x)(\tau_\theta+\beta_2^2\tau_x)[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta+\beta_2^2\tau_x)]^2} \\
&= \frac{\beta_1^5\beta_2^5\tau_x^2(\tau_\theta+\beta_1^2\tau_x)(\tau_\theta+\beta_2^2\tau_x)}{\gamma^2(\beta_1^2+\beta_2^2)^2(\tau_\theta+\beta_1^2\tau_x)(\tau_\theta+\beta_2^2\tau_x)[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta+\beta_2^2\tau_x)]^2} \\
&\quad - \frac{\beta_1^5\beta_2^5\tau_x^2(\tau_\theta+\beta_1^2\tau_x)(\tau_\theta+\beta_2^2\tau_x) + 2\beta_1^3\beta_2^3\tau_\theta\tau_x(\beta_1^2+\beta_2^2)[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta+\beta_2^2\tau_x)]}{\gamma^2(\beta_1^2+\beta_2^2)^2(\tau_\theta+\beta_1^2\tau_x)(\tau_\theta+\beta_2^2\tau_x)[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta+\beta_2^2\tau_x)]^2} \\
&= -\frac{2\beta_1^3\beta_2^3\tau_\theta\tau_x^2}{\gamma^2(\beta_1^2+\beta_2^2)(\tau_\theta+\beta_1^2\tau_x)(\tau_\theta+\beta_2^2\tau_x)[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta+\beta_2^2\tau_x)]}.
\end{aligned}$$

Eventually,

$$\frac{d\beta_1}{dc_1} = \frac{\dfrac{\beta_1^4\beta_2\tau_x}{\gamma(\beta_1^2+\beta_2^2)[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta+\beta_2^2\tau_x)]}}{\dfrac{2\beta_1^3\beta_2^3\tau_\theta\tau_x^2}{\gamma^2(\beta_1^2+\beta_2^2)(\tau_\theta+\beta_1^2\tau_x)(\tau_\theta+\beta_2^2\tau_x)[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta+\beta_2^2\tau_x)]}}$$

$$= \frac{\gamma\beta_1(\tau_\theta + \beta_1^2\tau_x)(\tau_\theta + \beta_2^2\tau_x)}{2\beta_2^2\tau_\theta\tau_x} > 0$$

and

$$\frac{d\beta_2}{dc_1} = \frac{\dfrac{\beta_1^3\tau_x[\beta_1^2\tau_\theta + \beta_2^2(2\tau_\theta + \beta_1^2\tau_x)]}{\gamma(\beta_1^2+\beta_2^2)(\tau_\theta + \beta_1^2\tau_x)[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta + \beta_2^2\tau_x)]}}{\dfrac{2\beta_1^3\beta_2^3\tau_\theta\tau_x^2}{\gamma^2(\beta_1^2+\beta_2^2)(\tau_\theta + \beta_1^2\tau_x)(\tau_\theta + \beta_2^2\tau_x)[\beta_2^2\tau_\theta + \beta_1^2(\tau_\theta + \beta_2^2\tau_x)]}}$$

$$= \frac{\gamma(\tau_\theta + \beta_2^2\tau_x)[\beta_1^2\tau_\theta + \beta_2^2(2\tau_\theta + \beta_1^2\tau_x)]}{2\beta_2^3\tau_\theta\tau_x} > 0.$$

Turning to the case of c_2, one can immediately conclude due to symmetry that

$$\frac{d\beta_1}{dc_2} = \frac{\gamma(\tau_\theta + \beta_1^2\tau_x)[\beta_2^2\tau_\theta + \beta_1^2(2\tau_\theta + \beta_2^2\tau_x)]}{2\beta_1^3\tau_\theta\tau_x} > 0,$$

$$\frac{d\beta_2}{dc_2} = \frac{\gamma\beta_2(\tau_\theta + \beta_1^2\tau_x)(\tau_\theta + \beta_2^2\tau_x)}{2\beta_1^2\tau_\theta\tau_x} > 0.$$

Thus, both β_1 and β_2 are increasing in both cost parameters in equilibrium.

Next, we derive the explicit expressions of β_1 and β_2 in the information acquisition equilibrium. Recalling the system in (A44), the values of β_1 and β_2 are obtained by simultaneously solving

$$\frac{1}{2\gamma}\log\left(\frac{\tau_\theta + \beta_2^2\tau_x}{\tau_\theta + \dfrac{\tau_x}{\dfrac{1}{\beta_1^2} + \dfrac{1}{\beta_2^2}}}\right) = c_1 \quad \text{and} \quad \frac{1}{2\gamma}\log\left(\frac{\tau_\theta + \beta_1^2\tau_x}{\tau_\theta + \dfrac{\tau_x}{\dfrac{1}{\beta_1^2} + \dfrac{1}{\beta_2^2}}}\right) = c_2.$$

Solving the first above equation for β_1 yields

$$c_1 = \frac{1}{2\gamma}\log\left(\frac{\tau_\theta + \beta_2^2\tau_x}{\tau_\theta + \dfrac{\tau_x}{\dfrac{1}{\beta_1^2} + \dfrac{1}{\beta_2^2}}}\right)$$

$$\Leftrightarrow e^{2\gamma c_1} = \frac{\tau_\theta + \beta_2^2\tau_x}{\tau_\theta + \dfrac{\tau_x}{\dfrac{\beta_1^2+\beta_2^2}{\beta_1^2\beta_2^2}}}$$

$$\Leftrightarrow e^{2\gamma c_1} = \frac{\tau_\theta + \beta_2^2 \tau_x}{\frac{\tau_\theta(\beta_1^2+\beta_2^2) + \tau_x\, \beta_1^2\beta_2^2}{\beta_1^2+\beta_2^2}}$$

$$\Leftrightarrow e^{2\gamma c_1}[\tau_\theta(\beta_1^2+\beta_2^2) + \tau_x\, \beta_1^2\beta_2^2] - (\tau_\theta + \beta_2^2\tau_x)(\beta_1^2+\beta_2^2) = 0$$

$$\Leftrightarrow [(e^{2\gamma c_1}-1)\tau_\theta + (e^{2\gamma c_1}-1)\tau_x\beta_2^2]\beta_1^2 - \tau_x\beta_2^4 + (e^{2\gamma c_1}-1)\tau_\theta\beta_2^2 = 0.$$

As β_1 can only take positive values, it follows that

$$\beta_1 = \sqrt{\frac{\beta_2^2[\beta_2^2\tau_x - (e^{2\gamma c_1}-1)\tau_\theta]}{(e^{2\gamma c_1}-1)(\tau_\theta + \tau_x\beta_2^2)}}.$$

Symmetry immediately yields

$$\beta_2 = \sqrt{\frac{\beta_1^2[\beta_1^2\tau_x - (e^{2\gamma c_2}-1)\tau_\theta]}{(e^{2\gamma c_2}-1)(\tau_\theta + \tau_x\beta_1^2)}}.$$

Denote $\xi_1 \equiv e^{2\gamma c_1} - 1$ and $\xi_2 = e^{2\gamma c_2} - 1$. Then, we can further solve for β_1:

$$\beta_1\sqrt{\xi_1(\tau_\theta + \tau_x\beta_2^2)} = \sqrt{\beta_2^2(\beta_2^2\tau_x - \xi_1\tau_\theta)}$$

$$\Leftrightarrow \beta_1^2\xi_1(\tau_\theta + \tau_x\beta_2^2) = \beta_2^2(\beta_2^2\tau_x - \xi_1\tau_\theta)$$

$$\Leftrightarrow \tau_\theta\xi_1\beta_1^2 + [(\tau_x\beta_1^2 + \tau_\theta)\xi_1 - \beta_2^2\tau_x]\beta_2^2 = 0$$

$$\Leftrightarrow \tau_\theta\xi_1\beta_1^2 + \left[(\tau_x\beta_1^2+\tau_\theta)\xi_1 - \frac{\tau_x\beta_1^2(\beta_1^2\tau_x - \xi_2\tau_\theta)}{\xi_2(\tau_\theta+\tau_x\beta_1^2)}\right]\frac{\beta_1^2(\beta_1^2\tau_x - \xi_2\tau_\theta)}{\xi_2(\tau_\theta+\tau_x\beta_1^2)} = 0$$

$$\Leftrightarrow \tau_\theta\xi_1\beta_1^2\xi_2^2(\tau_\theta+\tau_x\beta_1^2)^2$$
$$+ [(\tau_x\beta_1^2+\tau_\theta)^2\xi_1\xi_2 - \tau_x\beta_1^2(\tau_x\beta_1^2 - \xi_2\tau_\theta)]\beta_1^2(\tau_x\beta_1^2 - \xi_2\tau_\theta) = 0$$

$$\Leftrightarrow (\tau_x\beta_1^2+\tau_\theta)^2\tau_x\xi_1\xi_2\beta_1^4 - \tau_x\beta_1^4(\tau_x\beta_1^2 - \xi_2\tau_\theta)^2 = 0.$$

Expanding yields

$$\tau_x^3\xi_1\xi_2\beta_1^8 + \tau_\theta^2\tau_x\xi_1\xi_2\beta_1^4 + 2\tau_\theta\tau_x^2\xi_1\xi_2\beta_1^6 - \tau_x^3\beta_1^8 - \tau_\theta^2\tau_x\xi_2^2\beta_1^4 + 2\tau_\theta\tau_x^2\xi_2\beta_1^6 = 0.$$

Collecting terms and simplifying gives

$$\tau_x^3(\xi_1\xi_2 - 1)\beta_1^4 + 2\tau_\theta\tau_x^2\xi_2(\xi_1+1)\beta_1^2 + \tau_\theta^2\tau_x\xi_2(\xi_1 - \xi_2) = 0. \tag{A45}$$

Symmetry delivers

$$\tau_x^3(\xi_1\xi_2 - 1)\beta_2^4 + 2\tau_\theta\tau_x^2\xi_1(\xi_2 + 1)\beta_2^2 + \tau_\theta^2\tau_x\xi_1(\xi_2 - \xi_1) = 0. \tag{A46}$$

To determine the roots of the biquadratic equation in (A45), we introduce the auxiliary variable $\mathrm{z} \equiv \beta_1^2$. Hence, (A45) becomes

$$\tau_x^3(\xi_1\xi_2 - 1)\,\mathrm{z}^2 + 2\tau_\theta\tau_x^2\xi_2(\xi_1 + 1)\,\mathrm{z} + \tau_\theta^2\tau_x\xi_2(\xi_1 - \xi_2) = 0.$$

Consequently,

$$\mathrm{z}_{1/2} = \frac{-2\tau_\theta\tau_x^2\xi_2(\xi_1 + 1) \pm \sqrt{4\tau_\theta^2\tau_x^4\xi_2^2(\xi_1 + 1)^2 - 4\tau_\theta^2\tau_x^4\xi_2(\xi_1\xi_2 - 1)(\xi_1 - \xi_2)}}{2\tau_x^3(\xi_1\xi_2 - 1)}.$$

Simplifying yields

$$\mathrm{z}_1 = -\frac{[\xi_2(\xi_1 + 1) - \sqrt{\xi_1\xi_2}(\xi_2 + 1)]\tau_\theta}{\tau_x(\xi_1\xi_2 - 1)}$$

and

$$\mathrm{z}_2 = -\frac{[\xi_2(\xi_1 + 1) + \sqrt{\xi_1\xi_2}(\xi_2 + 1)]\tau_\theta}{\tau_x(\xi_1\xi_2 - 1)}.$$

Since β_1 needs to be positive, the only possible roots of the biquadratic in (A45) are $\sqrt{\mathrm{z}_1}$ and $\sqrt{\mathrm{z}_2}$. Using Descartes' rule of signs, we see that (A45) possesses two positive real roots if $\xi_1\xi_2 < 1$ and $\xi_1 < \xi_2$. In this case, (A46) has one positive real root. Inversely, if $\xi_1\xi_2 < 1$ and $\xi_1 > \xi_2$, (A45) has one positive real root and (A46) two. If $\xi_1\xi_2 > 1$ and $\xi_1 \neq \xi_2$, either (A45) or (A46) exhibits one sign change and the other polynomial none. If $\xi_1\xi_2 > 1$ and $\xi_1 = \xi_2$, neither (A45) nor (A46) shows a single sign change. Thus, $\xi_1\xi_2 < 1$ is a necessary and sufficient condition for a unique pair $(\beta_1, \beta_2) \in \mathbb{R}_{++}^2$.

Consequently, we need to identify the unique root that is always positive whenever $\xi_1\xi_2 < 1$. Inspecting z_1 and z_2 shows that z_2 exclusively fulfills this condition. Since $\mathrm{z} \equiv \beta_1^2$, we can conclude that $\sqrt{\mathrm{z}_2}$ is the unique root of the biquadratic in (A45) that is consistent with $(\beta_1, \beta_2) \in \mathbb{R}_{++}^2$. Eventually,

$$\beta_1 = \sqrt{-\tau_\theta\frac{\xi_2(\xi_1 + 1) + \sqrt{\xi_1\xi_2}(\xi_2 + 1)}{\tau_x(\xi_1\xi_2 - 1)}}.$$

Recalling the definitions of ξ_1 and ξ_2, we obtain

$$\xi_1\xi_2 - 1 \equiv (e^{2\gamma c_1} - 1)(e^{2\gamma c_2} - 1) - 1 = e^{2\gamma(c_1 + c_2)} - (e^{2\gamma c_1} + e^{2\gamma c_2})$$

and

$$\xi_2(\xi_1+1) \equiv (e^{2\gamma c_2}-1)[(e^{2\gamma c_1}-1)+1] = e^{2\gamma(c_1+c_2)} - e^{2\gamma c_1}.$$

Thus,

$$\beta_1 = \sqrt{-\tau_\theta \frac{e^{2\gamma(c_1+c_2)} - e^{2\gamma c_1} + \sqrt{(e^{2\gamma c_1}-1)(e^{2\gamma c_2}-1)}\, e^{2\gamma c_2}}{\tau_x[e^{2\gamma(c_1+c_2)} - (e^{2\gamma c_1}+e^{2\gamma c_2})]}} \tag{A47}$$

and by symmetry

$$\beta_2 = \sqrt{-\tau_\theta \frac{e^{2\gamma(c_1+c_2)} - e^{2\gamma c_2} + \sqrt{(e^{2\gamma c_1}-1)(e^{2\gamma c_2}-1)}\, e^{2\gamma c_1}}{\tau_x[e^{2\gamma(c_1+c_2)} - (e^{2\gamma c_1}+e^{2\gamma c_2})]}}. \tag{A48}$$

By inspecting (A47) and (A48), we immediately see that $(\beta_1, \beta_2) \in \mathbb{R}^2_{++}$ requires

$$e^{2\gamma(c_1+c_2)} - (e^{2\gamma c_1}+e^{2\gamma c_2}) < 0$$

$$\Leftrightarrow e^{2\gamma c_2}(e^{2\gamma c_1}-1) < e^{2\gamma c_1}$$

$$\Leftrightarrow c_2 < \frac{1}{2\gamma}\log\left(\frac{e^{2\gamma c_1}}{e^{2\gamma c_1}-1}\right) \equiv i(c_1).$$

Note that the above condition can also be expressed as

$$e^{2\gamma(c_1+c_2)} - (e^{2\gamma c_1}+e^{2\gamma c_2}) < 0$$

$$\Leftrightarrow e^{2\gamma c_1}(e^{2\gamma c_2}-1) < e^{2\gamma c_2}$$

$$\Leftrightarrow c_1 < \frac{1}{2\gamma}\log\left(\frac{e^{2\gamma c_2}}{e^{2\gamma c_2}-1}\right) \equiv i(c_2).$$

An information acquisition equilibrium of the form $\lambda_1^* > 0$, $\lambda_2^* > 0$ further requires $\gamma\beta_1 > \tau_\epsilon$ and $\gamma\beta_2 > \tau_\epsilon$ (see also (A21)). In what follows, we investigate the case $\lambda_1^* > 0$ and derive the respective results for $\lambda_2^* > 0$ by symmetry. Note from (A47) that for $c_1 = 0$,

$$\beta_1 = \sqrt{\frac{(e^{2\gamma c_2}-1)\tau_\theta}{\tau_x}}.$$

Since β_1 is increasing in c_1, $\lambda_1^* > 0$ holds for all $c_1 \in (0,\, i(c_2))$ if

$$\gamma\sqrt{\frac{(e^{2\gamma c_2}-1)\tau_\theta}{\tau_x}} - \tau_\epsilon \geq 0$$

$$\Leftrightarrow e^{2\gamma c_2} - 1 \geq \frac{\tau_\epsilon^2 \tau_x}{\gamma^2 \tau_\theta}$$

$$\Leftrightarrow c_2 \geq \frac{1}{2\gamma} \log\left(1 + \frac{\tau_\epsilon^2 \tau_x}{\gamma^2 \tau_\theta}\right) = \tilde{c}.$$

Whenever $c_2 < \tilde{c}$, using (A47), $\lambda_1^* > 0$ holds exactly if

$$\gamma \sqrt{-\tau_\theta \frac{e^{2\gamma(c_1+c_2)} - e^{2\gamma c_1} + \sqrt{(e^{2\gamma c_1} - 1)(e^{2\gamma c_2} - 1)}\, e^{2\gamma c_2}}{\tau_x [e^{2\gamma(c_1+c_2)} - (e^{2\gamma c_1} + e^{2\gamma c_2})]}} - \tau_\epsilon > 0$$

$$\Leftrightarrow -\frac{e^{2\gamma(c_1+c_2)} - e^{2\gamma c_1} + \sqrt{(e^{2\gamma c_1} - 1)(e^{2\gamma c_2} - 1)}\, e^{2\gamma c_2}}{e^{2\gamma(c_1+c_2)} - (e^{2\gamma c_1} + e^{2\gamma c_2})} > E$$

$$\Leftrightarrow \sqrt{(e^{2\gamma c_1} - 1)(e^{2\gamma c_2} - 1)}\, e^{2\gamma c_2} > -E[e^{2\gamma(c_1+c_2)} - (e^{2\gamma c_1} + e^{2\gamma c_2})] - e^{2\gamma(c_1+c_2)} + e^{2\gamma c_1}$$

$$\Leftrightarrow (e^{2\gamma c_1} - 1)(e^{2\gamma c_2} - 1)\, e^{4\gamma c_2} > \left[-e^{2\gamma c_1} e^{2\gamma c_2}\,(1+E) + e^{2\gamma c_1}\,(1+E) + E e^{2\gamma c_2}\right]^2$$

$$\Leftrightarrow (e^{2\gamma c_1} - 1)(e^{2\gamma c_2} - 1)\, e^{4\gamma c_2} > \left[-e^{2\gamma c_1}(1+E)(e^{2\gamma c_2} - 1) + E e^{2\gamma c_2}\right]^2$$

$$\Leftrightarrow e^{2\gamma c_1}(e^{2\gamma c_2} - 1)\, e^{4\gamma c_2} - (e^{2\gamma c_2} - 1)\, e^{4\gamma c_2} > \left[-e^{2\gamma c_1}(1+E)(e^{2\gamma c_2} - 1) + E e^{2\gamma c_2}\right]^2$$

$$\Leftrightarrow a\, e^{4\gamma c_1} + b\, e^{2\gamma c_1} + c > 0,$$

where

$$E \equiv \tau_\epsilon^2 \tau_x / \gamma^2 \tau_\theta,$$

$$a \equiv -(E+1)^2 (e^{2\gamma c_2} - 1)^2,$$

$$b \equiv e^{2\gamma c_2}(e^{2\gamma c_2} - 1)[e^{2\gamma c_2} + 2E(E+1)],$$

$$c \equiv -e^{4\gamma c_2}(e^{2\gamma c_2} - 1 + E^2).$$

The solutions of the respective quadratic equation are given by

$$c_{1,1/2} = \frac{1}{2\gamma} \log\left[\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right],$$

with

$$\begin{aligned} b^2 - 4ac &= e^{4\gamma c_2}(e^{2\gamma c_2} - 1)^2 [e^{2\gamma c_2} + 2E(E+1)]^2 \\ &\quad - 4(E+1)^2 (e^{2\gamma c_2} - 1)^2 e^{4\gamma c_2} (e^{2\gamma c_2} - 1 + E^2) \\ &= e^{4\gamma c_2}(e^{2\gamma c_2} - 1)^2 (e^{2\gamma c_2} - 2E - 2)^2. \end{aligned}$$

Thus,

$$c_{1,1} = \frac{1}{2\gamma}\log\left(\frac{\left\{\begin{matrix} -e^{2\gamma c_2}(e^{2\gamma c_2}-1)[e^{2\gamma c_2}+2E(E+1)] \\ -e^{2\gamma c_2}(e^{2\gamma c_2}-1)(e^{2\gamma c_2}-2E-2) \end{matrix}\right\}}{-2(E+1)^2(e^{2\gamma c_2}-1)^2}\right)$$

$$= \frac{1}{2\gamma}\log\left[\frac{2e^{2\gamma c_2}(e^{2\gamma c_2}-1)(e^{2\gamma c_2}+E^2-1)}{2(E+1)^2(e^{2\gamma c_2}-1)^2}\right]$$

$$= \frac{1}{2\gamma}\log\left[\frac{e^{2\gamma c_2}\left(e^{2\gamma c_2}+\frac{\tau_\epsilon^4\tau_x^2}{\gamma^4\tau_\theta^2}-1\right)}{\left(\frac{\tau_\epsilon^2\tau_x}{\gamma^2\tau_\theta}+1\right)^2(e^{2\gamma c_2}-1)}\right]$$

$$= \frac{1}{2\gamma}\log\left\{\frac{e^{2\gamma c_2}\left[(e^{2\gamma c_2}-1)\,\gamma^4\tau_\theta^2+\tau_\epsilon^4\tau_x^2\right]}{(e^{2\gamma c_2}-1)\left(\gamma^2\tau_\theta+\tau_\epsilon^2\tau_x\right)^2}\right\} = f(c_2)$$

and

$$c_{1,2} = \frac{1}{2\gamma}\log\left(\frac{\left\{\begin{matrix} -e^{2\gamma c_2}(e^{2\gamma c_2}-1)[e^{2\gamma c_2}+2E(E+1)] \\ +e^{2\gamma c_2}(e^{2\gamma c_2}-1)(e^{2\gamma c_2}-2E-2) \end{matrix}\right\}}{-2(E+1)^2(e^{2\gamma c_2}-1)^2}\right)$$

$$= \frac{1}{2\gamma}\log\left[\frac{2e^{2\gamma c_2}(e^{2\gamma c_2}-1)(1+E)^2}{2(E+1)^2(e^{2\gamma c_2}-1)^2}\right]$$

$$= \frac{1}{2\gamma}\log\left(\frac{e^{2\gamma c_2}}{e^{2\gamma c_2}-1}\right) = i(c_2).$$

Comparing $f(c_2)$ and $i(c_2)$ yields

$$\frac{1}{2\gamma}\log\left(\frac{e^{2\gamma c_2}}{e^{2\gamma c_2}-1}\right) \gtreqless \frac{1}{2\gamma}\log\left\{\frac{e^{2\gamma c_2}\left[(e^{2\gamma c_2}-1)\,\gamma^4\tau_\theta^2+\tau_\epsilon^4\tau_x^2\right]}{(e^{2\gamma c_2}-1)\left(\gamma^2\tau_\theta+\tau_\epsilon^2\tau_x\right)^2}\right\}$$

$$\Leftrightarrow 1 \gtreqless \frac{(e^{2\gamma c_2}-1)\,\gamma^4\tau_\theta^2+\tau_\epsilon^4\tau_x^2}{\left(\gamma^2\tau_\theta+\tau_\epsilon^2\tau_x\right)^2}$$

$$\Leftrightarrow e^{2\gamma c_2}\gamma^4\tau_\theta^2 \lesseqgtr 2\gamma^4\tau_\theta^2+2\gamma^2\tau_\theta\tau_\epsilon^2\tau_x$$

$$\Leftrightarrow c_2 \lesseqgtr \frac{1}{2\gamma}\log\left(2+\frac{2\tau_\epsilon^2\tau_x}{\gamma^2\tau_\theta}\right) \equiv \hat{c} > \tilde{c}.$$

Thus, we can conclude that $i(c_2) > f(c_2)$ for all $c_2 < \tilde{c}$. Furthermore, we already know that $\beta_1 \notin \mathbb{R}$ if $c_1 \geq i(c_2)$. Hence, given $c_2 < \tilde{c}$, $f(c_2)$ is the unique value of c_1

that leads to $\beta_1 = \tau_\epsilon/\gamma$. Since β_1 is increasing in c_1, $\lambda_1^* > 0$ holds for

$$\begin{cases} i(c_2) > c_1 > f(c_2) \text{ if } c_2 < \tilde{c}, \\ c_1 \in (0,\, i(c_2)) \text{ if } c_2 \geq \tilde{c}. \end{cases} \tag{A49}$$

Recall that β_2 is increasing in c_2. Then, by symmetry, we can immediately conclude that $\lambda_2^* > 0$ holds for

$$\begin{cases} i(c_1) > c_2 > f(c_1) \text{ if } c_1 < \tilde{c}, \\ c_2 \in (0,\, i(c_1)) \text{ if } c_1 \geq \tilde{c}. \end{cases} \tag{A50}$$

Next, we unite the conditions in (A49) and (A50) by deducing the value range of c_2 in dependence of c_1 that simultaneously ensures $\lambda_1^* > 0$ and $\lambda_2^* > 0$. This will also make the conditions in (A49) and (A50) better comparable to those in (A39) and (A43). We already know that $c_2 < \tilde{c}$ and $c_1 = f(c_2)$ would lead to $\beta_1 = \tau_\epsilon/\gamma$. Furthermore, we have shown that solving $c_1 = f(c_2)$ for c_2 delivers two solutions, viz., $g(c_1)$ and $h(c_1)$ (see (A41) and (A42)). Since $h(c_1) > \tilde{c}$ for $c_1 \in \mathbb{R}_{++}$, $c_2 = h(c_1)$ cannot yield $\beta_1 = \tau_\epsilon/\gamma$. Thus, as $g(c_1) < \tilde{c}$ for $c_1 \in \mathbb{R}_{++}$, $g(c_1)$ is the unique value of c_2 that leads to $\beta_1 = \tau_\epsilon/\gamma$. Given that β_1 is increasing in c_2, $c_2 > g(c_1)$ is a necessary condition for $\lambda_1^* > 0$ to hold. As already derived, the supremum of c_2 in terms of c_1 is $i(c_1)$. Since $g(c_1) \in (0,\, \tilde{c})$ and $\tilde{c} < \hat{c}$, $i(c_1) > g(c_1)$ for all $c_1 \in \mathbb{R}_{++}$. Hence, the condition in (A49) can be written as

$$i(c_1) > c_2 > g(c_1).$$

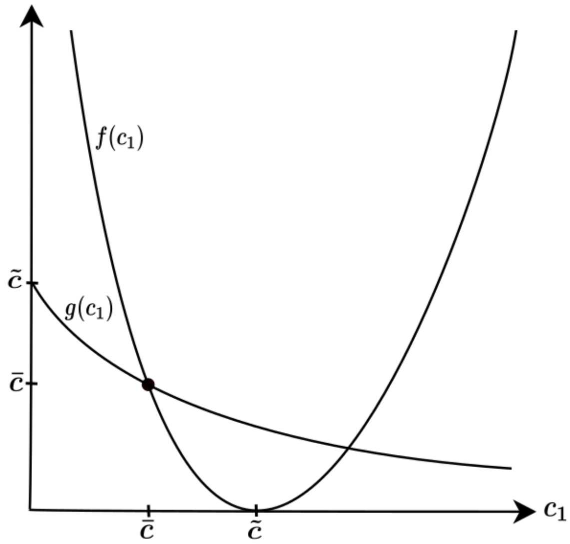
Furthermore, note that $g(c_1)$ is the inverse function of $f(c_1)$ for $c_1 \leq \tilde{c}$. The relationship between the two functions is depicted in Figure A.2. As $f(\bar{c}) = \bar{c}$, $\lim_{c_1 \to 0} f(c_1) = \infty$, and $\lim_{c_1 \to 0} g(c_1) = \tilde{c}$, $f(c_1) > g(c_1)$ holds for $c_1 \in (0,\, \bar{c})$ and $g(c_1) > f(c_1)$ holds for sure for $c_1 \in (\bar{c},\, \tilde{c})$. If $c_1 \geq \tilde{c}$, $\lambda_2^* > 0$ holds for all $c_2 \in (0,\, i(c_1))$ and the condition $c_2 > f(c_1)$ in (A50) becomes irrelevant. For $\lambda_1^* > 0$ to be true in this case, $i(c_1) > c_2 > g(c_1)$ still needs to be valid.

Eventually, we can conclude that an information acquisition equilibrium of the form $\lambda_1^* > 0$, $\lambda_2^* > 0$ requires

$$\begin{cases} i(c_1) > c_2 > f(c_1) \text{ if } c_1 \in (0,\, \bar{c}], \\ i(c_1) > c_2 > g(c_1) \text{ if } c_1 > \bar{c}. \end{cases} \tag{A51}$$

Using (A21), we can determine a unique pair $(\lambda_1^*,\, \lambda_2^*) \in \mathbb{R}_{++}^2$ through the unique values of β_1 and β_2 from (A47) and (A48). Note also that the conditions in (A38), (A39), (A43), and (A51) together yield the conditions in the proposition. $\square$

Figure A.2: The relationship between $f(c_1)$ and $g(c_1)$



Proof of Proposition 4.6. By (4.9), if $\lambda_j = 0$, it holds that $I_{x_j} = 0$. In this scenario, a rise in λ_i does not affect the first term in (4.16). Thus,

$$\frac{d\phi_{x_i}}{d\lambda_i} = -\frac{1}{2\gamma}\mathrm{Var}(\theta \,|\, P)\left\{\frac{\partial[\mathrm{Var}^{-1}(\theta \,|\, P)]}{\partial I_{x_i}}\frac{dI_{x_i}}{d\lambda_i}\right\}.$$

Building on (4.10), we can immediately derive that

$$\frac{\partial \mathrm{Var}^{-1}(\theta \,|\, P)}{\partial I_{x_i}} = \frac{2\tau_\epsilon^2\tau_x(1 - I_{x_i})}{\gamma^2\left[(1 - I_{x_i})^2 + (1 - I_{x_j})^2\right]^2} \tag{A52}$$

and by symmetry

$$\frac{\partial \mathrm{Var}^{-1}(\theta \, P)}{\partial I_{x_j}} = \frac{2\tau_\epsilon^2\tau_x(1 - I_{x_j})}{\gamma^2\left[(1 - I_{x_i})^2 + (1 - I_{x_j})^2\right]^2}. \tag{A53}$$

If $I_{x_j} = 0$, from (4.10),

$$\mathrm{Var}(\theta \,|\, P) = \frac{1}{\tau_\theta + \dfrac{\tau_\epsilon^2\tau_x}{\gamma^2[(1 - I_{x_i})^2 + 1]}},$$

and from (A52),

$$\frac{\partial[\mathrm{Var}^{-1}(\theta \,|\, P)]}{\partial I_{x_i}} = \frac{2\tau_\epsilon^2\tau_x(1 - I_{x_i})}{\gamma^2[(1 - I_{x_i})^2 + 1]^2}.$$

By (4.9), we see that

$$I_{x_i} = \frac{\lambda_i \tau_x \tau_\epsilon}{\gamma^2 + \lambda_i \tau_x \tau_\epsilon}.$$

Hence,

$$\frac{dI_{x_i}}{d\lambda_i} = \frac{(\gamma^2 + \lambda_i \tau_x \tau_\epsilon)\tau_x \tau_\epsilon - \lambda_i \tau_x^2 \tau_\epsilon^2}{(\gamma^2 + \lambda_i \tau_x \tau_\epsilon)^2} = \frac{\gamma^2 \tau_x \tau_\epsilon}{(\gamma^2 + \lambda_i \tau_x \tau_\epsilon)^2} \quad > 0.$$

From (A29), we know that

$$\lambda_i = \frac{\gamma^2 I_{x_i}}{(1 - I_{x_i})\tau_x \tau_\epsilon}.$$

Thus, we can further simplify:

$$\frac{dI_{x_i}}{d\lambda_i} = \frac{\gamma^2 \tau_x \tau_\epsilon}{\left[\gamma^2 + \dfrac{\gamma^2 I_{x_i}}{(1 - I_{x_i})}\right]^2} = \frac{(1 - I_{x_i})^2 \tau_x \tau_\epsilon}{\gamma^2}.$$

Aggregating all those results delivers

$$\begin{aligned}
\frac{d\phi_{x_i}}{d\lambda_i} &= - \frac{1}{2\gamma} \frac{1}{\tau_\theta + \dfrac{\tau_\epsilon^2 \tau_x}{\gamma^2[(1 - I_{x_i})^2 + 1]}} \left\{ \frac{2\tau_\epsilon^2 \tau_x (1 - I_{x_i})}{\gamma^2[(1 - I_{x_i})^2 + 1]^2} \frac{(1 - I_{x_i})^2 \tau_x \tau_\epsilon}{\gamma^2} \right\} \\
&= - \frac{1}{2\gamma} \frac{1}{\dfrac{\gamma^2 \tau_\theta [(1 - I_{x_i})^2 + 1] + \tau_\epsilon^2 \tau_x}{\gamma^2[(1 - I_{x_i})^2 + 1]}} \frac{2\tau_\epsilon^3 \tau_x^2 (1 - I_{x_i})^3}{\gamma^4[(1 - I_{x_i})^2 + 1]^2} \\
&= - \frac{\tau_\epsilon^3 \tau_x^2 (1 - I_{x_i})^3}{\gamma^3[(1 - I_{x_i})^2 + 1]\{\gamma^2 \tau_\theta [(1 - I_{x_i})^2 + 1] + \tau_\epsilon^2 \tau_x\}} \quad < 0,
\end{aligned}$$

which proves part (a) of the proposition. If $\lambda_j > 0$, both terms in (4.16) are affected by a rise in λ_i. Comparative-statics analysis of the first term in (4.16) yields

$$\begin{aligned}
\frac{d\left\{\log\left[\tau_\theta + \dfrac{\tau_\epsilon^2 \tau_x}{\gamma^2 \, (1 - I_{x_j})^2}\right]\right\}}{d\lambda_i} &= \frac{1}{\tau_\theta + \dfrac{\tau_\epsilon^2 \tau_x}{\gamma^2 \, (1 - I_{x_j})^2}} \frac{2\tau_\epsilon^2 \tau_x}{\gamma^2 (1 - I_{x_j})^3} \frac{dI_{x_j}}{d\lambda_i} \\
&= \frac{1}{\dfrac{\tau_\theta \, \gamma^2 \, (1 - I_{x_j})^2 + \tau_\epsilon^2 \tau_x}{\gamma^2 \, (1 - I_{x_j})^2}} \frac{2\tau_\epsilon^2 \tau_x}{\gamma^2 (1 - I_{x_j})^3} \frac{dI_{x_j}}{d\lambda_i} \\
&= \frac{2\tau_\epsilon^2 \tau_x}{(1 - I_{x_j})[\gamma^2 \, (1 - I_{x_j})^2 \, \tau_\theta + \tau_\epsilon^2 \tau_x]} \frac{dI_{x_j}}{d\lambda_i}.
\end{aligned}$$

Recalling (A31), we obtain

$$\frac{d\left\{\log\left[\tau_\theta + \frac{\tau_\epsilon^2\tau_x}{\gamma^2\,(1-I_{x_j})^2}\right]\right\}}{d\lambda_i} = \frac{2\tau_\epsilon^2\tau_x}{(1-I_{x_j})[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta + \tau_\epsilon^2\tau_x]} \times \frac{2\tau_\epsilon\tau_x(1-I_{x_i})I_{x_j}}{\gamma^2(1-I_{x_j})(1-4I_{x_i}I_{x_j})}$$

$$= \frac{4\tau_\epsilon^3\tau_x^2(1-I_{x_i})I_{x_j}}{\gamma^2(1-4I_{x_i}I_{x_j})(1-I_{x_j})^2[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta + \tau_\epsilon^2\tau_x]}.$$

By carefully inspecting the above term, we see that the effect of a rise in λ_i depends crucially on the sign of $1-4I_{x_i}I_{x_j}$. In other words, it depends on the equilibrium rational traders coordinate on. In the LIE, a rise in λ_i translates into a higher I_{x_j}. Hence, the inverse of an x_i-informed agent's residual uncertainty about fundamentals rises, thereby increasing the incentive to acquire information about x_i. By contrast, in the HIE, it holds that $1-4I_{x_i}I_{x_j} < 0$. Therefore, an increase in λ_i triggers a smaller I_{x_j}, reducing the incentive to acquire information about x_i.

Comparative-statics analysis of the second term in (4.16) gives

$$\frac{d\left\{\log\left[\mathrm{Var}^{-1}(\theta\,|\,P)\right]\right\}}{d\lambda_i} = \mathrm{Var}(\theta\,|\,P)\left\{\frac{\partial[\mathrm{Var}^{-1}(\theta\,|\,P)]}{\partial I_{x_i}}\frac{dI_{x_i}}{d\lambda_i} + \frac{\partial[\mathrm{Var}^{-1}(\theta\,|\,P)]}{\partial I_{x_j}}\frac{dI_{x_j}}{d\lambda_i}\right\}.$$

Making use of (A30), (A31), (A52), and (A53) delivers

$$\frac{\partial[\mathrm{Var}^{-1}(\theta\,|\,P)]}{\partial I_{x_i}}\frac{dI_{x_i}}{d\lambda_i} = \frac{2\tau_\epsilon^2\tau_x(1-I_{x_i})}{\gamma^2[(1-I_{x_i})^2+(1-I_{x_j})^2]^2}\frac{\tau_\epsilon\tau_x(1-I_{x_i})^2}{\gamma^2(1-I_{x_j})^2(1-4I_{x_i}I_{x_j})}$$

$$= \frac{2\tau_\epsilon^3\tau_x^2(1-I_{x_i})^3}{\gamma^4(1-I_{x_j})^2(1-4I_{x_i}I_{x_j})[(1-I_{x_i})^2+(1-I_{x_j})^2]^2}$$

and

$$\frac{\partial[\mathrm{Var}^{-1}(\theta\,|\,P)]}{\partial I_{x_j}}\frac{dI_{x_j}}{d\lambda_i} = \frac{2\tau_\epsilon^2\tau_x(1-I_{x_j})}{\gamma^2[(1-I_{x_i})^2+(1-I_{x_j})^2]^2}\frac{2\tau_\epsilon\tau_x(1-I_{x_i})I_{x_j}}{\gamma^2(1-I_{x_j})(1-4I_{x_i}I_{x_j})}$$

$$= \frac{4\tau_\epsilon^3\tau_x^2(1-I_{x_i})(1-I_{x_j})I_{x_j}}{\gamma^4(1-I_{x_j})(1-4I_{x_i}I_{x_j})[(1-I_{x_i})^2+(1-I_{x_j})^2]^2}.$$

Therefore,

$$\frac{d\left\{\log\left[\mathrm{Var}^{-1}(\theta\,|\,P)\right]\right\}}{d\lambda_i} = \mathrm{Var}(\theta\,|\,P)\,\frac{2\tau_\epsilon^3\tau_x^2(1-I_{x_i})[(1-I_{x_i})^2+2\,(1-I_{x_j})^2I_{x_j}]}{\gamma^4(1-4I_{x_i}I_{x_j})(1-I_{x_j})^2[(1-I_{x_i})^2+(1-I_{x_j})^2]^2}$$

$$= \frac{1}{\tau_\theta + \frac{\tau_\epsilon^2 \tau_x}{\gamma^2[(1-I_{x_i})^2 + (1-I_{x_j})^2]}}$$

$$\times \frac{2\tau_\epsilon^3 \tau_x^2 (1-I_{x_i})[(1-I_{x_i})^2 + 2\,(1-I_{x_j})^2 I_{x_j}]}{\gamma^4(1-4I_{x_i}I_{x_j})(1-I_{x_j})^2[(1-I_{x_i})^2 + (1-I_{x_j})^2]^2}$$

$$= \frac{\gamma^2[(1-I_{x_i})^2 + (1-I_{x_j})^2]}{\tau_\theta\gamma^2[(1-I_{x_i})^2 + (1-I_{x_j})^2] + \tau_\epsilon^2\tau_x}$$

$$\times \frac{2\tau_\epsilon^3 \tau_x^2 (1-I_{x_i})[(1-I_{x_i})^2 + 2\,(1-I_{x_j})^2 I_{x_j}]}{\gamma^4(1-4I_{x_i}I_{x_j})(1-I_{x_j})^2[(1-I_{x_i})^2 + (1-I_{x_j})^2]^2}$$

$$= \frac{2\tau_\epsilon^3 \tau_x^2 (1-I_{x_i})[(1-I_{x_i})^2 + 2\,(1-I_{x_j})^2 I_{x_j}]}{\gamma^2\chi(1-4I_{x_i}I_{x_j})(1-I_{x_j})^2(\tau_\theta\gamma^2\chi + \tau_\epsilon^2\tau_x)},$$

where $\chi \equiv (1-I_{x_i})^2 + (1-I_{x_j})^2$. The sign of the above term again depends on the sign of $1 - 4I_{x_i}I_{x_j}$ and, thus, on the equilibrium traders coordinate on. In the LIE, a rise in λ_i raises both I_{x_i} and I_{x_j}. Consequently, price efficiency increases, which reduces the incentive to acquire information about x_i. In the HIE, I_{x_i} and I_{x_j} decrease in response to a higher λ_i. This raises the incentive to acquire information about x_i.

Thus, a change in λ_i triggers a change in both terms in (4.16) in the same direction. The exact direction depends on the equilibrium. In the LIE, both terms rise. In the HIE, the opposite happens. Putting the separately derived results together yields

$$\frac{d\phi_{x_i}}{d\lambda_i} = \frac{1}{2\gamma}\Bigg\{\frac{4\tau_\epsilon^3\tau_x^2(1-I_{x_i})I_{x_j}}{\gamma^2(1-4I_{x_i}I_{x_j})(1-I_{x_j})^2[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta + \tau_\epsilon^2\tau_x]}$$

$$- \frac{2\tau_\epsilon^3 \tau_x^2 (1-I_{x_i})[(1-I_{x_i})^2 + 2\,(1-I_{x_j})^2 I_{x_j}]}{\gamma^2\chi(1-4I_{x_i}I_{x_j})(1-I_{x_j})^2(\tau_\theta\gamma^2\chi + \tau_\epsilon^2\tau_x)}\Bigg\}$$

$$= \frac{1}{2\gamma}\Bigg\{\frac{4\tau_\epsilon^3\tau_x^2(1-I_{x_i})I_{x_j}\chi(\tau_\theta\gamma^2\chi + \tau_\epsilon^2\tau_x)}{\gamma^2(1-4I_{x_i}I_{x_j})(1-I_{x_j})^2\chi[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta + \tau_\epsilon^2\tau_x](\tau_\theta\gamma^2\chi + \tau_\epsilon^2\tau_x)}$$

$$- \frac{2\tau_\epsilon^3 \tau_x^2 (1-I_{x_i})[(1-I_{x_i})^2 + 2\,(1-I_{x_j})^2 I_{x_j}][\gamma^2(1-I_{x_j})^2\tau_\theta + \tau_\epsilon^2\tau_x]}{\gamma^2(1-4I_{x_i}I_{x_j})(1-I_{x_j})^2\chi[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta + \tau_\epsilon^2\tau_x](\tau_\theta\gamma^2\chi + \tau_\epsilon^2\tau_x)}\Bigg\}$$

$$= \frac{2\tau_\epsilon^3\tau_x[(1-I_{x_i})I_{x_j}\chi^2\tau_\theta\gamma^2 + (1-I_{x_i})I_{x_j}\chi\tau_\epsilon^2\tau_x]}{\gamma^3(1-4I_{x_i}I_{x_j})(1-I_{x_j})^2\chi[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta + \tau_\epsilon^2\tau_x](\tau_\theta\gamma^2\chi + \tau_\epsilon^2\tau_x)}$$

$$- \frac{\tau_\epsilon^3\tau_x[(1-I_{x_i})^3 + 2(1-I_{x_j})^2(1-I_{x_i})I_{x_j}](\tau_\theta\gamma^2(1-I_{x_j})^2 + \tau_\epsilon^2\tau_x)}{\gamma^3(1-4I_{x_i}I_{x_j})(1-I_{x_j})^2\chi[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta + \tau_\epsilon^2\tau_x](\tau_\theta\gamma^2\chi + \tau_\epsilon^2\tau_x)}$$

$$= \frac{\tau_\epsilon^3\tau_x^2[2(1-I_{x_i})^5 I_{x_j} + 4(1-I_{x_i})^3(1-I_{x_j})^2 I_{x_j} - (1-I_{x_i})^3(1-I_{x_j})^2]\tau_\theta\gamma^2}{\gamma^3(1-4I_{x_i}I_{x_j})(1-I_{x_j})^2\chi[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta + \tau_\epsilon^2\tau_x](\tau_\theta\gamma^2\chi + \tau_\epsilon^2\tau_x)}$$

$$+\frac{\tau_\epsilon^3\tau_x^2[2(1-I_{x_i})^3 I_{x_j}-(1-I_{x_i})^3]\tau_\epsilon^2\tau_x}{\gamma^3(1-4I_{x_i}I_{x_j})(1-I_{x_j})^2\chi[\gamma^2(1-I_{x_j})^2\tau_\theta+\tau_\epsilon^2\tau_x](\tau_\theta\gamma^2\chi+\tau_\epsilon^2\tau_x)}$$

$$=\frac{(1-I_{x_i})^3\tau_\epsilon^3\tau_x^2\left\{\tau_\theta\gamma^2[2(1-I_{x_i})^2 I_{x_j}+(1-I_{x_j})^2(4I_{x_j}-1)]-\tau_\epsilon^2\tau_x(1-2I_{x_j})\right\}}{\gamma^3(1-4I_{x_i}I_{x_j})(1-I_{x_j})^2\chi[\gamma^2(1-I_{x_j})^2\tau_\theta+\tau_\epsilon^2\tau_x](\tau_\theta\gamma^2\chi+\tau_\epsilon^2\tau_x)}. \quad \text{(A54)}$$

The following analysis differentiates between the two possible equilibria. We begin with the LIE, in which it holds that $1-4I_{x_i}I_{x_j}>0$. This leads to a strictly positive denominator in (A54). Thus, acquiring information about the same noise component is a complement (resp., a substitute) if the numerator in (A54) is positive (resp., negative). This yields

$$\frac{d\phi_{x_i}^{LIE}}{d\lambda_i}\gtrless 0 \;\;\Leftrightarrow\;\; \tau_\theta\gamma^2[2(1-I_{x_i})^2 I_{x_j}+(1-I_{x_j})^2(4I_{x_j}-1)]\gtrless\tau_\epsilon^2\tau_x(1-2I_{x_j}).$$

Note that if $I_{x_j}\geq 0.5$, acquiring information about the same noise component is unambiguously a complement. Whenever $I_{x_j}<0.5$, it can be either a complement or a substitute. Recall that the endogenous values of both I_{x_i} and I_{x_j} do not vary with τ_θ. Thus, if $I_{x_j}<0.5$, both scenarios are plausible (given that the term in square brackets on the left-hand side in the above inequality is positive).

Next, we consider the HIE. In the HIE, it holds that $1-4I_{x_i}I_{x_j}<0$. This leads to a strictly negative denominator in (A54). Thus, acquiring information about the same noise component is a complement (resp., a substitute) if the numerator in (A54) is negative (resp., positive). Hence,

$$\frac{d\phi_{x_i}^{HIE}}{d\lambda_i}\lessgtr 0 \;\;\Leftrightarrow\;\; \tau_\theta\gamma^2[2(1-I_{x_i})^2 I_{x_j}+(1-I_{x_j})^2(4I_{x_j}-1)]\gtrless\tau_\epsilon^2\tau_x(1-2I_{x_j}).$$

Note that in the HIE, it must hold that $4I_{x_j}-1>0$. From $I_{x_i}>0.25I_{x_j}^{-1}$ (i.e., $\Gamma<0$) and $I_{x_i}\in[0,1)$, it follows that $I_{x_j}>0.25$. Otherwise, the existence of the HIE is not possible. We further conclude that acquiring information about the same noise component is always a substitute whenever $I_{x_j}\geq 0.5$. If $0.25<I_{x_j}<0.5$, it can be either a complement or a substitute.

Lastly, to prove part (c) of the proposition, we investigate the influence of a change in λ_j on ϕ_{x_i}. As in the analysis before, we consider the two terms in (4.16) separately. Direct computations yield

$$\frac{d\left\{\log\left[\tau_\theta+\dfrac{\tau_\epsilon^2\tau_x}{\gamma^2(1-I_{x_j})^2}\right]\right\}}{d\lambda_j}=\frac{1}{\tau_\theta+\dfrac{\tau_\epsilon^2\tau_x}{\gamma^2(1-I_{x_j})^2}}\frac{2\tau_\epsilon^2\tau_x}{\gamma^2(1-I_{x_j})^3}\frac{dI_{x_j}}{d\lambda_j}$$

$$= \frac{1}{\frac{\tau_\theta\,\gamma^2\,(1-I_{x_j})^2+\tau_\epsilon^2\tau_x}{\gamma^2\,(1-I_{x_j})^2}} \frac{2\tau_\epsilon^2\tau_x}{\gamma^2(1-I_{x_j})^3}\frac{dI_{x_j}}{d\lambda_j}$$

$$= \frac{2\tau_\epsilon^2\tau_x}{(1-I_{x_j})[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta+\tau_\epsilon^2\tau_x]}\frac{dI_{x_j}}{d\lambda_j}.$$

Using the symmetric counterpart of (A30) delivers

$$\frac{d\left\{\log\left[\tau_\theta+\frac{\tau_\epsilon^2\tau_x}{\gamma^2\,(1-I_{x_j})^2}\right]\right\}}{d\lambda_j} = \frac{2\tau_\epsilon^2\tau_x}{(1-I_{x_j})(\gamma^2\,(1-I_{x_j})^2\,\tau_\theta+\tau_\epsilon^2\tau_x)}$$

$$\times\frac{\tau_\epsilon\tau_x(1-I_{x_j})^2}{\gamma^2(1-I_{x_i})^2(1-4I_{x_i}I_{x_j})}$$

$$= \frac{2\tau_\epsilon^3\tau_x^2(1-I_{x_j})}{\gamma^2(1-4I_{x_i}I_{x_j})(1-I_{x_i})^2[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta+\tau_\epsilon^2\tau_x]}.$$

Similar to the previous case, the effect of a rise in λ_j depends on the equilibrium agents coordinate on. In the LIE, a rise in λ_j leads to a higher I_{x_j}, increasing the incentive to acquire information about x_i. In the HIE, it holds that $1-4I_{x_i}I_{x_j}<0$. An increase in λ_j leads to a smaller I_{x_j}, reducing the incentive to acquire information about x_i.

Comparative-statics analysis of the second term in (4.16) yields

$$\frac{d\left\{\log\left[\text{Var}^{-1}(\theta\,|\,P)\right]\right\}}{d\lambda_j} = \text{Var}(\theta\,|\,P)\left\{\frac{\partial\left[\text{Var}^{-1}(\theta\,|\,P)\right]}{\partial I_{x_i}}\frac{dI_{x_i}}{d\lambda_j}+\frac{\partial\left[\text{Var}^{-1}(\theta\,|\,P)\right]}{\partial I_{x_j}}\frac{dI_{x_j}}{d\lambda_j}\right\}.$$

Using the symmetric counterparts of (A30) and (A31), (A52), and (A53) yields

$$\frac{\partial[\text{Var}^{-1}(\theta\,|\,P)]}{\partial I_{x_i}}\frac{dI_{x_i}}{d\lambda_j} = \frac{2\tau_\epsilon^2\tau_x(1-I_{x_i})}{\gamma^2[(1-I_{x_i})^2+(1-I_{x_j})^2]^2}\frac{2\tau_\epsilon\tau_x(1-I_{x_j})I_{x_i}}{\gamma^2(1-I_{x_i})(1-4I_{x_i}I_{x_j})}$$

$$= \frac{4\tau_\epsilon^3\tau_x^2(1-I_{x_i})(1-I_{x_j})I_{x_i}}{\gamma^4(1-I_{x_i})(1-4I_{x_i}I_{x_j})[(1-I_{x_i})^2+(1-I_{x_j})^2]^2}$$

and

$$\frac{\partial[\text{Var}^{-1}(\theta\,|\,P)]}{\partial I_{x_j}}\frac{dI_{x_j}}{d\lambda_j} = \frac{2\tau_\epsilon^2\tau_x(1-I_{x_j})}{\gamma^2[(1-I_{x_i})^2+(1-I_{x_j})^2]^2}\frac{\tau_\epsilon\tau_x(1-I_{x_j})^2}{\gamma^2(1-I_{x_i})^2(1-4I_{x_i}I_{x_j})}$$

$$= \frac{2\tau_\epsilon^3\tau_x^2(1-I_{x_j})^3}{\gamma^4(1-I_{x_i})^2(1-4I_{x_i}I_{x_j})[(1-I_{x_i})^2+(1-I_{x_j})^2]^2}.$$

Thus,

$$\frac{d\left\{\log\left[\mathrm{Var}^{-1}(\theta\,|\,P)\right]\right\}}{d\lambda_j} = \mathrm{Var}\,(\theta|P)\,\frac{2\tau_\epsilon^3\tau_x^2(1-I_{x_j})[(1-I_{x_j})^2+2\,(1-I_{x_i})^2I_{x_i}]}{\gamma^4(1-4I_{x_i}I_{x_j})(1-I_{x_i})^2[(1-I_{x_i})^2+(1-I_{x_j})^2]^2}$$

$$= \frac{1}{\tau_\theta + \dfrac{\tau_\epsilon^2\tau_x}{\gamma^2[(1-I_{x_i})^2+(1-I_{x_j})^2]}}$$

$$\times\,\frac{2\tau_\epsilon^3\tau_x^2(1-I_{x_j})[(1-I_{x_j})^2+2\,(1-I_{x_i})^2I_{x_i}]}{\gamma^4(1-4I_{x_i}I_{x_j})(1-I_{x_i})^2[(1-I_{x_i})^2+(1-I_{x_j})^2]^2}$$

$$= \frac{\gamma^2[(1-I_{x_i})^2+(1-I_{x_j})^2]}{\tau_\theta\gamma^2((1-I_{x_i})^2+(1-I_{x_j})^2)+\tau_\epsilon^2\tau_x}$$

$$\times\,\frac{2\tau_\epsilon^3\tau_x^2(1-I_{x_j})[(1-I_{x_j})^2+2\,(1-I_{x_i})^2I_{x_i}]}{\gamma^4(1-4I_{x_i}I_{x_j})(1-I_{x_i})^2[(1-I_{x_i})^2+(1-I_{x_j})^2]^2}$$

$$= \frac{2\tau_\epsilon^3\tau_x^2(1-I_{x_j})[(1-I_{x_j})^2+2\,(1-I_{x_i})^2I_{x_i}]}{\gamma^2\chi(1-4I_{x_i}I_{x_j})(1-I_{x_i})^2(\tau_\theta\gamma^2\chi+\tau_\epsilon^2\tau_x)},$$

where $\chi \equiv (1-I_{x_i})^2+(1-I_{x_j})^2$. The sign of the above term again depends on the equilibrium traders coordinate on. In the LIE, a rise in λ_j raises both I_{x_i} and I_{x_j}. Consequently, price efficiency increases. This reduces the incentive to acquire information about x_i. In the HIE, by contrast, I_{x_i} and I_{x_j} shrink in response to a higher λ_j. This raises the incentive to acquire information about x_i.

Analogous to the case before, a change in λ_j triggers a change in both terms in (4.16) in the same direction. The exact direction is determined by the respective equilibrium. Further calculations yield

$$\frac{d\phi_{x_i}}{d\lambda_j} = \frac{1}{2\gamma}\left\{\frac{2\tau_\epsilon^3\tau_x^2(1-I_{x_j})}{\gamma^2(1-4I_{x_i}I_{x_j})(1-I_{x_i})^2[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta+\tau_\epsilon^2\tau_x]}\right.$$

$$\left.-\,\frac{2\tau_\epsilon^3\tau_x^2(1-I_{x_j})[(1-I_{x_j})^2+2\,(1-I_{x_i})^2I_{x_i}]}{\gamma^2\chi(1-4I_{x_i}I_{x_j})(1-I_{x_i})^2(\tau_\theta\gamma^2\chi+\tau_\epsilon^2\tau_x)}\right\}$$

$$= \frac{1}{2\gamma}\left\{\frac{2\tau_\epsilon^3\tau_x^2(1-I_{x_j})\chi(\tau_\theta\gamma^2\chi+\tau_\epsilon^2\tau_x)}{\gamma^2(1-4I_{x_i}I_{x_j})(1-I_{x_i})^2\chi[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta+\tau_\epsilon^2\tau_x](\tau_\theta\gamma^2\chi+\tau_\epsilon^2\tau_x)}\right.$$

$$\left.-\,\frac{2\tau_\epsilon^3\tau_x^2(1-I_{x_j})[(1-I_{x_j})^2+2\,(1-I_{x_i})^2I_{x_i}][\gamma^2(1-I_{x_j})^2\tau_\theta+\tau_\epsilon^2\tau_x]}{\gamma^2(1-4I_{x_i}I_{x_j})(1-I_{x_i})^2\chi[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta+\tau_\epsilon^2\tau_x](\tau_\theta\gamma^2\chi+\tau_\epsilon^2\tau_x)}\right\}$$

$$= \frac{\tau_\epsilon^3\tau_x^2[(1-I_{x_j})\chi^2\tau_\theta\gamma^2+(1-I_{x_j})\chi\tau_\epsilon^2\tau_x]}{\gamma^3(1-4I_{x_i}I_{x_j})(1-I_{x_i})^2\chi(\gamma^2\,(1-I_{x_j})^2\,\tau_\theta+\tau_\epsilon^2\tau_x)(\tau_\theta\gamma^2\chi+\tau_\epsilon^2\tau_x)}$$

$$
\begin{aligned}
&-\frac{\tau_\epsilon^3\tau_x^2[(1-I_{x_j})^3+2(1-I_{x_i})^2(1-I_{x_j})I_{x_i}][\tau_\theta\gamma^2(1-I_{x_j})^2+\tau_\epsilon^2\tau_x]}{\gamma^3(1-4I_{x_i}I_{x_j})(1-I_{x_i})^2\chi[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta+\tau_\epsilon^2\tau_x](\tau_\theta\gamma^2\chi+\tau_\epsilon^2\tau_x)}\\
&=\frac{\tau_\epsilon^3\tau_x^2(1-I_{x_i})^2(1-I_{x_j})[(1-I_{x_i})^2+2(1-I_{x_j})^2-2(1-I_{x_j})^2I_{x_i}]\tau_\theta\gamma^2}{\gamma^3(1-4I_{x_i}I_{x_j})(1-I_{x_i})^2\chi[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta+\tau_\epsilon^2\tau_x](\tau_\theta\gamma^2\chi+\tau_\epsilon^2\tau_x)}\\
&\quad+\frac{\tau_\epsilon^3\tau_x^2(1-2I_{x_i})(1-I_{x_i})^2(1-I_{x_j})\tau_\epsilon^2\tau_x}{\gamma^3(1-4I_{x_i}I_{x_j})(1-I_{x_i})^2\chi[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta+\tau_\epsilon^2\tau_x](\tau_\theta\gamma^2\chi+\tau_\epsilon^2\tau_x)}\\
&=\frac{\tau_\epsilon^3\tau_x^2(1-I_{x_j})\left\{\tau_\theta\gamma^2[(1-I_{x_i})^2+2(1-I_{x_j})^2(1-I_{x_i})]+\tau_\epsilon^2\tau_x(1-2I_{x_i})\right\}}{\gamma^3(1-4I_{x_i}I_{x_j})\chi[\gamma^2\,(1-I_{x_j})^2\,\tau_\theta+\tau_\epsilon^2\tau_x](\tau_\theta\gamma^2\chi+\tau_\epsilon^2\tau_x)}.
\end{aligned}
\tag{A55}
$$

In the LIE, the denominator in (A55) is positive, since $1-4I_{x_i}I_{x_j}>0$. Consequently, acquiring information about different noise components is a complement (resp., a substitute) if the numerator in (A55) is positive (resp., negative). This gives

$$
\frac{d\phi_{x_i}^{LIE}}{d\lambda_j}\gtreqless 0 \;\Leftrightarrow\; \tau_\theta\gamma^2[(1-I_{x_i})^2+2(1-I_{x_j})^2(1-I_{x_i})]+\tau_\epsilon^2\tau_x(1-2I_{x_i})\gtreqless 0.
$$

If $I_{x_i}\leq 0.5$, acquiring information about different noise components is unequivocally a complement. Whenever $I_{x_i}>0.5$, it can be either a complement or a substitute. In the HIE, the denominator in (A55) is negative. Thus, acquiring information about different noise components is a complement (resp., a substitute) if the numerator in (A55) is negative (resp., positive). Thus,

$$
\frac{d\phi_{x_i}^{HIE}}{d\lambda_j}\lesseqgtr 0 \;\Leftrightarrow\; \tau_\theta\gamma^2[(1-I_{x_i})^2+2(1-I_{x_j})^2(1-I_{x_i})]+\tau_\epsilon^2\tau_x(1-2I_{x_i})\gtreqless 0.
$$

In the HIE, contrary to the LIE, if $I_{x_i}\leq 0.5$, acquiring information about different noise components is a substitute. Whenever $I_{x_i}>0.5$, both scenarios are possible, since the endogenous values of I_{x_i} and I_{x_j} are independent of τ_θ. □

Proof of Proposition 4.7. Recalling (4.18), a noise-informed agent's conditional moments are

$$
\mathrm{E}(\theta\,|\,P_{n_i}^*)=\frac{\dfrac{\tau_x}{\beta_j^{-2}+\beta_l^{-2}}\left(\dfrac{P}{a_\theta}-\dfrac{1}{\beta_i}x_i\right)}{\tau_\theta+\tau_x/(\beta_j^{-2}+\beta_l^{-2})},
$$

$$
\mathrm{Var}(\theta\,|\,P_{n_i}^*)=\frac{1}{\tau_\theta+\tau_x/(\beta_j^{-2}+\beta_l^{-2})}.
$$

Then, by (4.2), the demand function of an x_i-informed trader becomes

$$D_{n_i} = \frac{\dfrac{\tau_x}{\beta_j^{-2}+\beta_l^{-2}}\left(\dfrac{P}{a_\theta} - \dfrac{1}{\beta_i}x_i\right) - P\left[\tau_\theta + \tau_x/(\beta_j^{-2}+\beta_l^{-2})\right]}{\gamma}, \tag{A56}$$

for $i, j, l = 1, 2, 3, i \neq j \neq l$. For a fundamentally informed trader, we obtain

$$\mathrm{E}\left(\theta \,|\, s_f, P^*_{f/u}\right) = \frac{\tau_\epsilon\, s_f + \dfrac{\tau_x}{\beta_1^{-2}+\beta_2^{-2}+\beta_3^{-2}}\dfrac{P}{a_\theta}}{\tau_\theta + \tau_\epsilon + \tau_x/(\beta_1^{-2}+\beta_2^{-2}+\beta_3^{-2})},$$

$$\mathrm{Var}\left(\theta \,|\, s_f, P^*_{f/u}\right) = \frac{1}{\tau_\theta + \tau_\epsilon + \tau_x/(\beta_1^{-2}+\beta_2^{-2}+\beta_3^{-2})}.$$

Thus,

$$D_f = \frac{\tau_\epsilon\, s_f + \dfrac{\tau_x}{\beta_1^{-2}+\beta_2^{-2}+\beta_3^{-2}}\dfrac{P}{a_\theta} - P\left(\tau_\theta + \tau_\epsilon + \dfrac{\tau_x}{\beta_1^{-2}+\beta_2^{-2}+\beta_3^{-2}}\right)}{\gamma}. \tag{A57}$$

Analogously, the conditional moments of an uninformed, rational agent are

$$\mathrm{E}\left(\theta \,|P^*_{f/u}\right) = \frac{\dfrac{\tau_x}{\beta_1^{-2}+\beta_2^{-2}+\beta_3^{-2}}\dfrac{P}{a_\theta}}{\tau_\theta + \tau_x/(\beta_1^{-2}+\beta_2^{-2}+\beta_3^{-2})},$$

$$\mathrm{Var}\left(\theta \,|\, P^*_{f/u}\right) = \frac{1}{\tau_\theta + \tau_x/(\beta_1^{-2}+\beta_2^{-2}+\beta_3^{-2})},$$

which gives

$$D_u = \frac{\dfrac{\tau_x}{\beta_1^{-2}+\beta_2^{-2}+\beta_3^{-2}}\dfrac{P}{a_\theta} - P\left(\tau_\theta + \dfrac{\tau_x}{\beta_1^{-2}+\beta_2^{-2}+\beta_3^{-2}}\right)}{\gamma}. \tag{A58}$$

Using $\int_0^1 s_f\, df = \theta$ (which again follows from the strong law of large numbers), (A56), (A57), and (A58), the market-clearing condition in (4.20) can be written as

$$\frac{\tau_\epsilon\, \theta + \dfrac{\tau_x}{\beta_1^{-2}+\beta_2^{-2}+\beta_3^{-2}}\dfrac{P}{a_\theta} - P\left(\tau_\theta + \tau_\epsilon + \dfrac{\tau_x}{\beta_1^{-2}+\beta_2^{-2}+\beta_3^{-2}}\right)}{\gamma}$$

$$+ \lambda_1 \frac{\dfrac{\tau_x}{\beta_2^{-2}+\beta_3^{-2}}\left(\dfrac{P}{a_\theta} - \dfrac{1}{\beta_1}x_1\right) - P\left(\tau_\theta + \dfrac{\tau_x}{\beta_2^{-2}+\beta_3^{-2}}\right)}{\gamma}$$

$$+ \lambda_2 \frac{\frac{\tau_x}{\beta_1^{-2}+\beta_3^{-2}}\left(\frac{P}{a_\theta} - \frac{1}{\beta_2}x_1\right) - P\left(\tau_\theta + \frac{\tau_x}{\beta_1^{-2}+\beta_3^{-2}}\right)}{\gamma}$$

$$+ \lambda_3 \frac{\frac{\tau_x}{\beta_1^{-2}+\beta_2^{-2}}\left(\frac{P}{a_\theta} - \frac{1}{\beta_3}x_3\right) - P\left(\tau_\theta + \frac{\tau_x}{\beta_1^{-2}+\beta_2^{-2}}\right)}{\gamma}$$

$$+ x_1 + x_2 + x_3 + \lambda_u \frac{\frac{\tau_x}{\beta_1^{-2}+\beta_2^{-2}+\beta_3^{-2}}\frac{P}{a_\theta} - P\left(\tau_\theta + \frac{\tau_x}{\beta_1^{-2}+\beta_2^{-2}+\beta_3^{-2}}\right)}{\gamma} = 0.$$

Recall that

$$\frac{\tau_x}{\beta_1^{-2}+\beta_2^{-2}+\beta_3^{-2}} = \mathrm{Var}^{-1}\left(P^*_{f/u}|\,\theta\right),$$

$$\frac{\tau_x}{\beta_2^{-2}+\beta_3^{-2}} = \mathrm{Var}^{-1}\left(P^*_{n_1}|\,\theta\right),$$

$$\frac{\tau_x}{\beta_1^{-2}+\beta_3^{-2}} = \mathrm{Var}^{-1}\left(P^*_{n_2}|\,\theta\right),$$

$$\frac{\tau_x}{\beta_1^{-2}+\beta_2^{-2}} = \mathrm{Var}^{-1}\left(P^*_{n_3}|\,\theta\right).$$

Thus, the market-clearing condition can be rearranged as follows:

$$\begin{aligned} &[(1+\lambda_u+\lambda_1+\lambda_2+\lambda_3)\tau_\theta + \tau_\epsilon]\ P \\ &+ (1-a_\theta^{-1})\begin{bmatrix}(1+\lambda_u)\mathrm{Var}^{-1}(P^*_{f/u}|\,\theta) + \lambda_1\mathrm{Var}^{-1}(P^*_{n_1}|\,\theta) \\ +\lambda_2\mathrm{Var}^{-1}(P^*_{n_2}|\,\theta) + \lambda_3\mathrm{Var}^{-1}(P^*_{n_3}|\,\theta)\end{bmatrix} P \\ = &\ \tau_\epsilon\,\theta + \left[\gamma - \lambda_1\frac{\tau_x}{(\beta_2^{-2}+\beta_3^{-2})\beta_1}\right] x_1 + \left[\gamma - \lambda_2\frac{\tau_x}{(\beta_1^{-2}+\beta_3^{-2})\beta_2}\right] x_2 \\ &+ \left[\gamma - \lambda_3\frac{\tau_x}{(\beta_1^{-2}+\beta_2^{-2})\beta_3}\right] x_3. \end{aligned} \tag{A59}$$

Denote $\omega \equiv 1 + \lambda_u + \lambda_1 + \lambda_2 + \lambda_3$. Then, by comparing (A59) with (4.17) and invoking rational expectations, we get

$$\frac{\tau_\epsilon}{a_\theta} = \omega\tau_\theta + \tau_\epsilon + (1-a_\theta^{-1})\begin{bmatrix}(1+\lambda_u)\mathrm{Var}^{-1}(P^*_{f/u}|\,\theta) + \lambda_1\mathrm{Var}^{-1}(P^*_{n_1}|\,\theta) \\ +\lambda_2\mathrm{Var}^{-1}(P^*_{n_2}|\,\theta) + \lambda_3\mathrm{Var}^{-1}(P^*_{n_3}|\,\theta)\end{bmatrix}$$

$$\Leftrightarrow \tau_\epsilon = (\omega\tau_\theta + \tau_\epsilon)\,a_\theta + (a_\theta - 1)\begin{bmatrix}(1+\lambda_u)\mathrm{Var}^{-1}(P^*_{f/u}|\,\theta) + \lambda_1\mathrm{Var}^{-1}(P^*_{n_1}|\,\theta) \\ +\lambda_2\mathrm{Var}^{-1}(P^*_{n_2}|\,\theta) + \lambda_3\mathrm{Var}^{-1}(P^*_{n_3}|\,\theta)\end{bmatrix}$$

$$\Leftrightarrow a_\theta = \frac{\tau_\epsilon + \begin{bmatrix} (1+\lambda_u)\text{Var}^{-1}(P^*_{f/u}|\,\theta) + \lambda_1 \text{Var}^{-1}(P^*_{n_1}|\,\theta) \\ +\lambda_2 \text{Var}^{-1}(P^*_{n_2}|\,\theta) + \lambda_3 \text{Var}^{-1}(P^*_{n_3}|\,\theta) \end{bmatrix}}{\tau_\epsilon + \omega\tau_\theta + \begin{bmatrix} (1+\lambda_u)\text{Var}^{-1}(P^*_{f/u}|\,\theta) + \lambda_1 \text{Var}^{-1}(P^*_{n_1}|\,\theta) \\ +\lambda_2 \text{Var}^{-1}(P^*_{n_2}|\,\theta) + \lambda_3 \text{Var}^{-1}(P^*_{n_3}|\,\theta) \end{bmatrix}},$$

which is positive. By the definition of β_i, it immediately follows that

$$a_i = \frac{a_\theta}{\beta_i}, \quad for \;\; i = 1, 2, 3.$$

Furthermore, by (A59), the implied value of β_i is

$$\beta_i = \frac{\tau_\epsilon}{\gamma - \dfrac{\lambda_i \tau_x}{(\beta_j^{-2} + \beta_l^{-2})\beta_i}}$$

$$\Leftrightarrow \beta_i \gamma - \frac{\lambda_i \tau_x}{\beta_j^{-2} + \beta_l^{-2}} = \tau_\epsilon$$

$$\Leftrightarrow \beta_i = \frac{\tau_\epsilon}{\gamma} + \frac{\lambda_i \tau_x}{\gamma(\beta_j^{-2} + \beta_l^{-2})}, \quad for \;\; i, j, l = 1, 2, 3, \; i \neq j \neq l,$$

which is equal to (4.21) in the proposition. □

Proof of Proposition 4.8. By (4.21), $\beta_i = \tau_\epsilon/\gamma$ if $\lambda_i = 0$. Then, the equation that determines β_j in equilibrium becomes

$$\begin{aligned} \beta_j &= \frac{\tau_\epsilon}{\gamma} + \frac{\lambda_j \tau_x}{\gamma\left(\dfrac{\gamma^2}{\tau_\epsilon^2} + \dfrac{1}{\beta_l^2}\right)} \\ &= \frac{\tau_\epsilon}{\gamma} + \frac{\lambda_j \tau_x \tau_\epsilon^2 \beta_l^2}{\gamma\,(\gamma^2 \beta_l^2 + \tau_\epsilon^2)} \\ &= \frac{\tau_\epsilon^3 + \beta_l^2 \tau_\epsilon(\gamma^2 + \lambda_j \tau_\epsilon \tau_x)}{\gamma(\tau_\epsilon^2 + \beta_l^2 \gamma^2)}. \end{aligned} \tag{A60}$$

Next, we plug the solution for β_j contained in (A60) into the respective equation that determines β_l in equilibrium:

$$\begin{aligned} \beta_l &= \frac{\tau_\epsilon}{\gamma} + \frac{\lambda_l \tau_x}{\gamma\left\{\dfrac{\gamma^2}{\tau_\epsilon^2} + \dfrac{\gamma^2(\tau_\epsilon^2 + \beta_l^2\gamma^2)^2}{\left[\tau_\epsilon^3 + \beta_l^2 \tau_\epsilon(\gamma^2 + \lambda_j \tau_\epsilon \tau_x)\right]^2}\right\}} \\ &= \frac{\tau_\epsilon}{\gamma} + \frac{\lambda_l \tau_x \tau_\epsilon^2 \left[\tau_\epsilon^2 + \beta_l^2(\gamma^2 + \lambda_j \tau_\epsilon \tau_x)\right]^2}{\gamma^3\left\{\left[\tau_\epsilon^2 + \beta_l^2(\gamma^2 + \lambda_j \tau_\epsilon \tau_x)\right]^2 + (\tau_\epsilon^2 + \beta_l^2\gamma^2)^2\right\}} \equiv f(\beta_l). \end{aligned} \tag{A61}$$

The number of solutions of the fixed-point equation in (A61) is equal to the number of equilibria in the model. Denote

$$g(\beta_l) \equiv \tau_\epsilon^2 + \beta_l^2(\gamma^2 + \lambda_j \tau_\epsilon \tau_x),$$

$$h(\beta_l) \equiv \tau_\epsilon^2 + \beta_l^2 \gamma^2.$$

Then, differentiating yields

$$
\begin{aligned}
& f'(\beta_l) \\
&= \frac{\lambda_l \tau_x \tau_\epsilon^2}{\gamma^3} \frac{\{[g(\beta_l)]^2 + [h(\beta_l)]^2\} 2g(\beta_l)g'(\beta_l) - [g(\beta_l)]^2[2g(\beta_l)g'(\beta_l) + 2h(\beta_l)h'(\beta_l)]}{\{[g(\beta_l)]^2 + [h(\beta_l)]^2\}^2} \\
&= \frac{\lambda_l \tau_x \tau_\epsilon^2}{\gamma^3} \frac{2[h(\beta_l)]^2 g(\beta_l)g'(\beta_l) - 2[g(\beta_l)]^2 h(\beta_l)h'(\beta_l)}{[g(\beta_l) + h(\beta_l)]^2} \\
&= \frac{\lambda_l \tau_x \tau_\epsilon^2}{\gamma^3} \\
&\quad \times \frac{\left\{ \begin{array}{c} 4(\tau_\epsilon^2 + \beta_l^2\gamma^2)^2 \, [\tau_\epsilon^2 + \beta_l^2(\gamma^2 + \lambda_j \tau_\epsilon \tau_x)] \\ \times (\gamma^2 + \lambda_j \tau_\epsilon \tau_x)\beta_l \end{array} \right\} - \left\{ \begin{array}{c} [\tau_\epsilon^2 + \beta_l^2(\gamma^2 + \lambda_j \tau_\epsilon \tau_x)]^2 \\ \times 4\gamma^2 \beta_l (\tau_\epsilon^2 + \beta_l^2 \gamma^2) \end{array} \right\}}{\{[g(\beta_l)]^2 + [h(\beta_l)]^2\}^2} \\
&= \frac{4\lambda_l \tau_x \tau_\epsilon^2 \beta_l [\tau_\epsilon^2 + \beta_l^2(\gamma^2 + \lambda_j \tau_\epsilon \tau_x)](\tau_\epsilon^2 + \beta_l^2\gamma^2)}{\gamma^3} \\
&\quad \times \frac{(\tau_\epsilon^2 + \beta_l^2\gamma^2)(\gamma^2 + \lambda_j \tau_\epsilon \tau_x) - \gamma^2[\tau_\epsilon^2 + \beta_l^2(\gamma^2 + \lambda_j \tau_\epsilon \tau_x)]}{\{[g(\beta_l)]^2 + [h(\beta_l)]^2\}^2} \\
&= \frac{4\lambda_j \lambda_l \tau_x^2 \tau_\epsilon^5 \beta_l [\tau_\epsilon^2 + \beta_l^2(\gamma^2 + \lambda_j \tau_\epsilon \tau_x)](\tau_\epsilon^2 + \beta_l^2\gamma^2)}{\gamma^3 \{[g(\beta_l)]^2 + [h(\beta_l)]^2\}^2} \\
&= \frac{4\lambda_j \lambda_l \tau_x^2 \tau_\epsilon^5 \beta_l \, g(\beta_l) h(\beta_l)}{\gamma^3 \{[g(\beta_l)]^2 + [h(\beta_l)]^2\}^2} > 0.
\end{aligned}
$$

Thus, $f(\beta_l)$ is a monotonically increasing function in β_l. Differentiating a second time delivers

$$
\begin{aligned}
f''(\beta_l) = \frac{4\lambda_j \lambda_l \tau_x^2 \tau_\epsilon^5}{\gamma^3} \Bigg(& \frac{g(\beta_l)h(\beta_l)}{\{[g(\beta_l)]^2 + [h(\beta_l)]^2\}^2} \\
& + \beta_l \frac{\begin{array}{c} \{[g(\beta_l)]^2 + [h(\beta_l)]^2\}^2 [g(\beta_l)h'(\beta_l) + h(\beta_l)g(\beta_l)] \\ - 4\{[g(\beta_l)]^2 + [h(\beta_l)]^2\}[g(\beta_l)g'(\beta_l) + h(\beta_l)h'(\beta_l)]g(\beta_l)h(\beta_l) \end{array}}{\{[g(\beta_l)]^2 + [h(\beta_l)]^2\}^4} \Bigg)
\end{aligned}
$$

$$= \frac{4\lambda_j \lambda_l \tau_x^2 \tau_\epsilon^5}{\gamma^3} \frac{\begin{array}{c} g(\beta_l)h(\beta_l)\{[g(\beta_l)]^2 + [h(\beta_l)]^2 - 4\beta_l[g(\beta_l)g'(\beta_l) + h(\beta_l)h'(\beta_l)]\} \\ + \beta_l\{[g(\beta_l)]^2 + [h(\beta_l)]^2\}[g(\beta_l)h'(\beta_l) + h(\beta_l)g'(\beta_l)] \end{array}}{\{[g(\beta_l)]^2 + [h(\beta_l)]^2\}^3}.$$

Direct computations yield

$$\begin{aligned}
& g(\beta_l)h(\beta_l)\{[g(\beta_l)]^2 + [h(\beta_l)]^2 - 4\beta_l[g(\beta_l)g'(\beta_l) + h(\beta_l)h'(\beta_l)]\} \\
&= g(\beta_l)h(\beta_l)\big\{[\tau_\epsilon^2 + \beta_l^2(\gamma^2 + \lambda_j\tau_\epsilon\tau_x)]^2 + (\tau_\epsilon^2 + \beta_l^2\gamma^2)^2 \\
&\quad - 8\beta_l^2(\gamma^2 + \lambda_j\tau_x\tau_\epsilon)[\tau_\epsilon^2 + \beta_l^2(\gamma^2 + \lambda_j\tau_\epsilon\tau_x)] - 8\gamma^2\beta_l^2(\tau_\epsilon^2 + \beta_l^2\gamma^2)\big\} \\
&= [\tau_\epsilon^2 + \beta_l^2(\gamma^2 + \lambda_j\tau_\epsilon\tau_x)](\tau_\epsilon^2 + \beta_l^2\gamma^2) \\
&\quad \times \{-7[\gamma^4 + (\gamma^2 + \lambda_j\tau_x\tau_\epsilon)^2]\beta_j^4 - 6\tau_\epsilon^2(2\gamma^2 + \lambda_j\tau_x\tau_\epsilon)\beta_l^2 + 2\tau_\epsilon^4\} \\
&= [\gamma^2(\gamma^2 + \lambda_j\tau_x\tau_\epsilon)\beta_l^4 + \tau_\epsilon^2(2\gamma^2 + \lambda_j\tau_x\tau_\epsilon)\beta_l^2 + \tau_\epsilon^4] \\
&\quad \times \{-7[\gamma^4 + (\gamma^2 + \lambda_j\tau_x\tau_\epsilon)^2]\beta_j^4 - 6\tau_\epsilon^2(2\gamma^2 + \lambda_j\tau_x\tau_\epsilon)\beta_l^2 + 2\tau_\epsilon^4\} \\
&= -7\gamma^2(\gamma^2 + \lambda_j\tau_x\tau_\epsilon)[\gamma^4 + (\gamma^2 + \lambda_j\tau_x\tau_\epsilon)^2]\beta_l^8 \\
&\quad - \tau_\epsilon^2(2\gamma^2 + \lambda_j\tau_x\tau_\epsilon)\{6\gamma^2(\gamma^2 + \lambda_j\tau_x\tau_\epsilon) + 7[\gamma^4 + (\gamma^2 + \lambda_j\tau_x\tau_\epsilon)^2]\}\beta_l^6 \\
&\quad - \tau_\epsilon^4\{-2\gamma^2(\gamma^2 + \lambda_j\tau_x\tau_\epsilon) + 6(2\gamma^2 + \lambda_j\tau_x\tau_\epsilon)^2 + 7[\gamma^4 + (\gamma^2 + \lambda_j\tau_x\tau_\epsilon)^2]\}\beta_l^4 \\
&\quad - 4\tau_\epsilon^6(2\gamma^2 + \lambda_j\tau_x\tau_\epsilon)\beta_l^2 + 2\tau_\epsilon^8
\end{aligned} \tag{A62}$$

and

$$\begin{aligned}
& \beta_l\{[g(\beta_l)]^2 + [h(\beta_l)]^2\}[g(\beta_l)h'(\beta_l) + h(\beta_l)g'(\beta_l)] \\
&= \beta_l\{[\tau_\epsilon^2 + \beta_l^2(\gamma^2 + \lambda_j\tau_\epsilon\tau_x)]^2 + (\tau_\epsilon^2 + \beta_l^2\gamma^2)^2\} \\
&\quad \times \{2\gamma^2\beta_l[\tau_\epsilon^2 + \beta_l^2(\gamma^2 + \lambda_j\tau_\epsilon\tau_x)] + 2(\gamma^2 + \lambda_j\tau_x\tau_\epsilon)(\tau_\epsilon^2 + \beta_l^2\gamma^2)\beta_l\} \\
&= \{[\gamma^4 + (\gamma^2 + \lambda_j\tau_x\tau_\epsilon)^2]\beta_l^5 + 2\tau_\epsilon^2(2\gamma^2 + \lambda_j\tau_x\tau_\epsilon)\beta_l^3 + 2\tau_\epsilon^4\beta_l\} \\
&\quad \times [2\tau_\epsilon^2(2\gamma^2 + \lambda_j\tau_x\tau_\epsilon)\beta_l + 4\gamma^2(\gamma^2 + \lambda_j\tau_x\tau_\epsilon)\beta_l^3] \\
&= 4\gamma^2(\gamma^2 + \lambda_j\tau_x\tau_\epsilon)[\gamma^4 + (\gamma^2 + \lambda_j\tau_x\tau_\epsilon)^2]\beta_l^8 \\
&\quad + 2\tau_\epsilon^2(2\gamma^2 + \lambda_j\tau_x\tau_\epsilon)[\gamma^4 + (\gamma^2 + \lambda_j\tau_x\tau_\epsilon)^2 + 4\gamma^2(\gamma^2 + \lambda_j\tau_x\tau_\epsilon)]\beta_l^6 \\
&\quad + 4\tau_\epsilon^4[(2\gamma^2 + \lambda_j\tau_x\tau_\epsilon)^2 + 2\gamma^2(\gamma^2 + \lambda_j\tau_x\tau_\epsilon)]\beta_l^4 + 4\tau_\epsilon^6(2\gamma^2 + \lambda_j\tau_x\tau_\epsilon)\beta_l^2.
\end{aligned} \tag{A63}$$

Next, we pairwise compare the terms linked to β_l^8, β_l^6, β_l^4, and β_l^2 in (A62) and (A63):

$$
\begin{aligned}
& -7\gamma^2(\gamma^2+\lambda_j\tau_x\tau_\epsilon)[\gamma^4+(\gamma^2+\lambda_j\tau_x\tau_\epsilon)^2]\beta_l^8 \\
& +4\gamma^2(\gamma^2+\lambda_j\tau_x\tau_\epsilon)[\gamma^4+(\gamma^2+\lambda_j\tau_x\tau_\epsilon)^2]\beta_l^8 \\
= & -3\gamma^2(\gamma^2+\lambda_j\tau_x\tau_\epsilon)[\gamma^4+(\gamma^2+\lambda_j\tau_x\tau_\epsilon)^2]\beta_l^8
\end{aligned}
$$

and

$$
\begin{aligned}
& -\tau_\epsilon^2(2\gamma^2+\lambda_j\tau_x\tau_\epsilon)\{6\gamma^2(\gamma^2+\lambda_j\tau_x\tau_\epsilon)+7[\gamma^4+(\gamma^2+\lambda_j\tau_x\tau_\epsilon)^2]\}\beta_l^6 \\
& +2\tau_\epsilon^2(2\gamma^2+\lambda_j\tau_x\tau_\epsilon)[\gamma^4+(\gamma^2+\lambda_j\tau_x\tau_\epsilon)^2+4\gamma^2(\gamma^2+\lambda_j\tau_x\tau_\epsilon)]\beta_l^6 \\
= & \tau_\epsilon^2(2\gamma^2+\lambda_j\tau_x\tau_\epsilon)\{-5[\gamma^4+(\gamma^2+\lambda_j\tau_x\tau_\epsilon)^2]+2\gamma^2(\gamma^2+\lambda_j\tau_x\tau_\epsilon)\}\beta_l^6 \\
= & -\tau_\epsilon^2(2\gamma^2+\lambda_j\tau_x\tau_\epsilon)[8\gamma^2(\gamma^2+\lambda_j\tau_x\tau_\epsilon)+5(\lambda_j\tau_x\tau_\epsilon)^2]\beta_l^6
\end{aligned}
$$

and

$$
\begin{aligned}
& -\tau_\epsilon^4\{-2\gamma^2(\gamma^2+\lambda_j\tau_x\tau_\epsilon)+6(2\gamma^2+\lambda_j\tau_x\tau_\epsilon)^2+7[\gamma^4+(\gamma^2+\lambda_j\tau_x\tau_\epsilon)^2]\}\beta_l^4 \\
& +4\tau_\epsilon^4[(2\gamma^2+\lambda_j\tau_x\tau_\epsilon)^2+2\gamma^2(\gamma^2+\lambda_j\tau_x\tau_\epsilon)]\beta_l^4 \\
= & \tau_\epsilon^4\{10\gamma^2(\gamma^2+\lambda_j\tau_x\tau_\epsilon)-2(2\gamma^2+\lambda_j\tau_x\tau_\epsilon)^2-7[\gamma^4+(\gamma^2+\lambda_j\tau_x\tau_\epsilon)^2]\}\beta_l^4 \\
= & -3\tau_\epsilon^4[4\gamma^2(\gamma^2+\lambda_j\tau_x\tau_\epsilon)+3(\lambda_j\tau_x\tau_\epsilon)^2]\beta_l^4
\end{aligned}
$$

and

$$
-4\tau_\epsilon^6(2\gamma^2+\lambda_j\tau_x\tau_\epsilon)\beta_l^2+4\tau_\epsilon^6(2\gamma^2+\lambda_j\tau_x\tau_\epsilon)\beta_l^2=0.
$$

Putting all obtained results together, we finally get

$$
f''(\beta_l)=\frac{4\lambda_j\lambda_l\tau_x^2\tau_\epsilon^5}{\gamma^3}\frac{-b_8\beta_l^8-b_6\beta_l^6-b_4\beta_l^4+2\tau_\epsilon^8}{\{[\tau_\epsilon^2+\beta_l^2(\gamma^2+\lambda_j\tau_\epsilon\tau_x)]^2+(\tau_\epsilon^2+\beta_l^2\gamma^2)^2\}^3}, \tag{A64}
$$

where

$$
\begin{aligned}
& b_8\equiv 3\gamma^2(\gamma^2+\lambda_j\tau_x\tau_\epsilon)[\gamma^4+(\gamma^2+\lambda_j\tau_x\tau_\epsilon)^2], \\
& b_6\equiv \tau_\epsilon^2(2\gamma^2+\lambda_j\tau_x\tau_\epsilon)[8\gamma^2(\gamma^2+\lambda_j\tau_x\tau_\epsilon)+5(\lambda_j\tau_x\tau_\epsilon)^2], \\
& b_4\equiv 3\tau_\epsilon^4[4\gamma^2(\gamma^2+\lambda_j\tau_x\tau_\epsilon)+3(\lambda_j\tau_x\tau_\epsilon)^2].
\end{aligned}
$$

Note that the numerator in (A64) is an octic polynomial in β_l. Its terms are all negative except for the constant (i.e., $2\tau_\epsilon^8$), which is positive. Thus, by Descartes' rule of signs, we can conclude that the octic has exactly one positive real root, since it exhibits exactly one sign change. We denote this positive real root $\bar{\beta}_l$. As $f''(0) > 0$, $f(\beta_l)$ is strictly convex in β_l for $\beta_l < \bar{\beta}_l$ and strictly concave in β_l for $\beta_l > \bar{\beta}_l$. By (4.21), any equilibrium solution for β_l must be greater than τ_ϵ/γ if $\lambda_l > 0$. Since

$$f(0) = \frac{\tau_\epsilon}{\gamma} + \frac{\lambda_l \tau_x \tau_\epsilon^6}{2\gamma^3 \tau_\epsilon^4} = \frac{\tau_\epsilon(2\gamma^2 + \lambda_l \tau_x \tau_\epsilon)}{2\gamma^3} > 0,$$

$f(\beta_l) > \beta_l$ holds for sure for all $\beta_l \leq \tau_\epsilon/\gamma$. Next, we investigate the curvature of $f(\beta_l)$ at $\beta_l = \tau_\epsilon/\gamma$:

$$\begin{aligned}
f''(\tau_\epsilon/\gamma) &= \frac{4\lambda_j \lambda_l \tau_x^2 \tau_\epsilon^5}{\gamma^3} \frac{-\, b_8 \tau_\epsilon^8 \gamma^{-8} - b_6 \tau_\epsilon^6 \gamma^{-6} - b_4 \tau_\epsilon^4 \gamma^{-4} + 2\tau_\epsilon^8}{[(\tau_\epsilon^2 + \tau_\epsilon^2 + \lambda_j \tau_\epsilon^3 \tau_x \gamma^{-2})^2 + (\tau_\epsilon^2 + \tau_\epsilon^2)^2]^3} \\
&= \frac{4\lambda_j \lambda_l \tau_x^2 \tau_\epsilon^5}{\gamma^3} \frac{\gamma^{-8} \tau_\epsilon^4 (-\, b_8 \tau_\epsilon^4 - b_6 \tau_\epsilon^2 \gamma^2 - b_4 \gamma^4 + 2\tau_\epsilon^4 \gamma^8)}{\gamma^{-12} \tau_\epsilon^{12} [(2\gamma^2 + \lambda_j \tau_\epsilon \tau_x)^2 + 4\gamma^4]^3} \\
&= \frac{4\lambda_j \lambda_l \tau_x^2 \gamma}{\tau_\epsilon^3} \frac{-\, b_8 \tau_\epsilon^4 - b_6 \tau_\epsilon^2 \gamma^2 - b_4 \gamma^4 + 2\tau_\epsilon^4 \gamma^8}{[(2\gamma^2 + \lambda_j \tau_\epsilon \tau_x)^2 + 4\gamma^4]^3}.
\end{aligned}$$

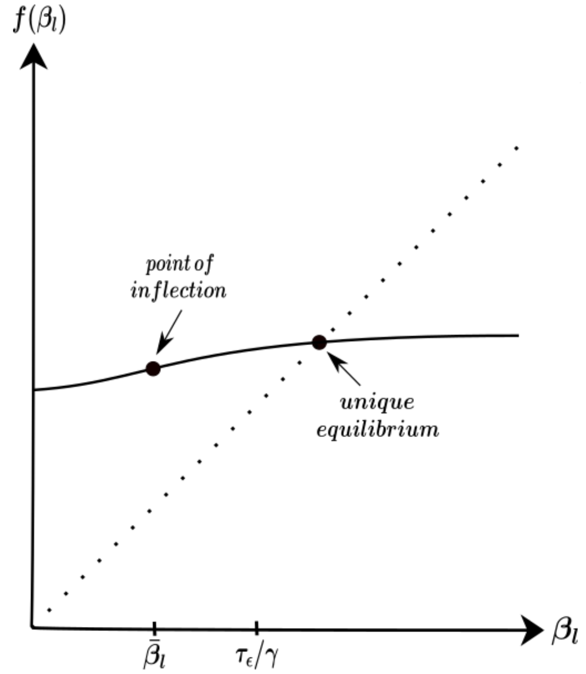
Direct computations yield

$$\begin{aligned}
&-\, b_8 \tau_\epsilon^4 - b_6 \tau_\epsilon^2 \gamma^2 \\
=\; &-\, 3\tau_\epsilon^4 \gamma^2 (\gamma^2 + \lambda_j \tau_x \tau_\epsilon)[\gamma^4 + (\gamma^2 + \lambda_j \tau_x \tau_\epsilon)^2] \\
&-\, \tau_\epsilon^4 \gamma^2 (2\gamma^2 + \lambda_j \tau_x \tau_\epsilon)[8\gamma^2(\gamma^2 + \lambda_j \tau_x \tau_\epsilon) + 5(\lambda_j \tau_x \tau_\epsilon)^2] \\
=\; &-\, 3\tau_\epsilon^4 \gamma^2 (2\gamma^6 + 2\gamma^4 \lambda_j \tau_x \tau_\epsilon + \gamma^2 \lambda_j^2 \tau_x^2 \tau_\epsilon^2 + 2\gamma^4 \lambda_j \tau_x \tau_\epsilon + 2\gamma^2 \lambda_j^2 \tau_x^2 \tau_\epsilon^2 + \lambda_j^3 \tau_x^3 \tau_\epsilon^3) \\
&-\, \tau_\epsilon^4 \gamma^2 (16\gamma^6 + 16\gamma^4 \lambda_j \tau_x \tau_\epsilon + 10\gamma^2 \lambda_j^2 \tau_x^2 \tau_\epsilon^2 + 8\gamma^4 \lambda_j \tau_x \tau_\epsilon + 8\gamma^2 \lambda_j^2 \tau_x^2 \tau_\epsilon^2 + 5\lambda_j^3 \tau_x^3 \tau_\epsilon^3) \\
=\; &-\, \tau_\epsilon^4 \gamma^2 (22\gamma^6 + 36\gamma^4 \lambda_j \tau_x \tau_\epsilon + 27\gamma^2 \lambda_j^2 \tau_x^2 \tau_\epsilon^2 + 8\lambda_j^3 \tau_x^3 \tau_\epsilon^3)
\end{aligned}$$

as well as

$$\begin{aligned}
-b_4 \gamma^4 + 2\tau_\epsilon^4 \gamma^8 &= -3\tau_\epsilon^4 \gamma^4 [4\gamma^2(\gamma^2 + \lambda_j \tau_x \tau_\epsilon) + 3(\lambda_j \tau_x \tau_\epsilon)^2] + 2\tau_\epsilon^4 \gamma^8 \\
&= -12\tau_\epsilon^4 \gamma^8 - 3\tau_\epsilon^4 \gamma^2 (4\gamma^4 \lambda_j \tau_x \tau_\epsilon + 3\gamma^2 \lambda_j^2 \tau_x^2 \tau_\epsilon^2) + 2\tau_\epsilon^4 \gamma^8 \\
&= -\tau_\epsilon^4 \gamma^2 (10\gamma^6 + 12\gamma^4 \lambda_j \tau_x \tau_\epsilon + 9\gamma^2 \lambda_j^2 \tau_x^2 \tau_\epsilon^2).
\end{aligned}$$

Figure A.3: Mapping $f(\beta_l)$ with β_l for $\lambda_i = 0$



Thus,

$$
\begin{aligned}
& - b_8\tau_\epsilon^4 - b_6\tau_\epsilon^2\gamma^2 - b_4\gamma^4 + 2\tau_\epsilon^4\gamma^8 \\
= & - \tau_\epsilon^4\gamma^2(22\gamma^6 + 36\gamma^4\lambda_j\tau_x\tau_\epsilon + 27\gamma^2\lambda_j^2\tau_x^2\tau_\epsilon^2 + 8\lambda_j^3\tau_x^3\tau_\epsilon^3) \\
& - \tau_\epsilon^4\gamma^2(10\gamma^6 + 12\gamma^4\lambda_j\tau_x\tau_\epsilon + 9\gamma^2\lambda_j^2\tau_x^2\tau_\epsilon^2) \\
= & - 4\tau_\epsilon^4\gamma^2(8\gamma^6 + 12\gamma^4\lambda_j\tau_x\tau_\epsilon + 9\gamma^2\lambda_j^2\tau_x^2\tau_\epsilon^2 + 2\lambda^3\tau_x^3\tau_\epsilon^3).
\end{aligned}
$$

Eventually,

$$
\begin{aligned}
f''(\tau_\epsilon/\gamma) &= -\frac{4\lambda_j\lambda_l\tau_x^2\gamma}{\tau_\epsilon^3}\frac{4\tau_\epsilon^4\gamma^2(8\gamma^6 + 12\gamma^4\lambda_j\tau_x\tau_\epsilon + 9\gamma^2\lambda_j^2\tau_x^2\tau_\epsilon^2 + 2\lambda^3\tau_x^3\tau_\epsilon^3)}{[(2\gamma^2 + \lambda_j\tau_\epsilon\tau_x)^2 + 4\gamma^4]^3} \\
&= -\frac{16\lambda_j\lambda_l\tau_x^2\tau_\epsilon\gamma^3(8\gamma^6 + 12\gamma^4\lambda_j\tau_x\tau_\epsilon + 9\gamma^2\lambda_j^2\tau_x^2\tau_\epsilon^2 + 2\lambda^3\tau_x^3\tau_\epsilon^3)}{[(2\gamma^2 + \lambda_j\tau_\epsilon\tau_x)^2 + 4\gamma^4]^3} < 0.
\end{aligned}
$$

This implies that $f(\beta_l)$ is for sure concave for all $\beta_l \geq \tau_\epsilon/\gamma$, which furthermore means that $\bar{\beta}_l < \tau_\epsilon/\gamma$. Thus, any potential solution for β_l lies in the region where $f(\beta_l)$ is strictly concave. Given that $f(\beta_l) > \beta_l$ for $\beta_l = \tau_\epsilon/\gamma$, $f(\beta_l)$ unequivocally intersects with the 45°-line exactly once (see also Figure A.3). This proves that equilibrium is unique. □

Proof of Proposition 4.9. Formally, by (4.24), we obtain

$$\frac{dI_{x_i}}{d\lambda_i} = \frac{\partial I_{x_i}}{\partial\lambda_i} + \frac{\partial I_{x_i}}{\partial I_{x_j}}\frac{dI_{x_j}}{d\lambda_i} + \frac{\partial I_{x_i}}{\partial I_{x_l}}\frac{dI_{x_l}}{d\lambda_i}, \tag{A65}$$

$$\frac{dI_{x_j}}{d\lambda_i} = \frac{\partial I_{x_j}}{\partial I_{x_i}}\frac{dI_{x_i}}{d\lambda_i} + \frac{\partial I_{x_j}}{\partial I_{x_l}}\frac{dI_{x_l}}{d\lambda_i}, \tag{A66}$$

$$\frac{dI_{x_l}}{d\lambda_i} = \frac{\partial I_{x_l}}{\partial I_{x_i}}\frac{dI_{x_i}}{d\lambda_i} + \frac{\partial I_{x_l}}{\partial I_{x_j}}\frac{dI_{x_j}}{d\lambda_i}. \tag{A67}$$

Plugging (A67) into (A66) and rearranging terms delivers

$$\frac{dI_{x_j}}{d\lambda_i} = \frac{\partial I_{x_j}}{\partial I_{x_i}}\frac{dI_{x_i}}{d\lambda_i} + \frac{\partial I_{x_j}}{\partial I_{x_l}}\left(\frac{\partial I_{x_l}}{\partial I_{x_i}}\frac{dI_{x_i}}{d\lambda_i} + \frac{\partial I_{x_l}}{\partial I_{x_j}}\frac{dI_{x_j}}{d\lambda_i}\right)$$

$$\Leftrightarrow \frac{dI_{x_j}}{d\lambda_i} = \frac{\frac{\partial I_{x_j}}{\partial I_{x_i}} + \frac{\partial I_{x_j}}{\partial I_{x_l}}\frac{\partial I_{x_l}}{\partial I_{x_i}}}{1 - \frac{\partial I_{x_j}}{\partial I_{x_l}}\frac{\partial I_{x_l}}{\partial I_{x_j}}}\frac{dI_{x_i}}{d\lambda_i}. \tag{A68}$$

By (A68), (A67) can be written as

$$\begin{aligned}
\frac{dI_{x_l}}{d\lambda_i} &= \frac{\partial I_{x_l}}{\partial I_{x_i}}\frac{dI_{x_i}}{d\lambda_i} + \frac{\partial I_{x_l}}{\partial I_{x_j}}\left[\frac{\left(\frac{\partial I_{x_j}}{\partial I_{x_i}} + \frac{\partial I_{x_j}}{\partial I_{x_l}}\frac{\partial I_{x_l}}{\partial I_{x_i}}\right)\frac{dI_{x_i}}{d\lambda_i}}{1 - \frac{\partial I_{x_j}}{\partial I_{x_l}}\frac{\partial I_{x_l}}{\partial I_{x_j}}}\right] \\
&= \left[\frac{\partial I_{x_l}}{\partial I_{x_i}} + \frac{\frac{\partial I_{x_l}}{\partial I_{x_j}}\left(\frac{\partial I_{x_j}}{\partial I_{x_i}} + \frac{\partial I_{x_j}}{\partial I_{x_l}}\frac{\partial I_{x_l}}{\partial I_{x_i}}\right)}{1 - \frac{\partial I_{x_j}}{\partial I_{x_l}}\frac{\partial I_{x_l}}{\partial I_{x_j}}}\right]\frac{dI_{x_i}}{d\lambda_i} \\
&= \frac{\frac{\partial I_{x_l}}{\partial I_{x_i}} + \frac{\partial I_{x_l}}{\partial I_{x_j}}\frac{\partial I_{x_j}}{\partial I_{x_i}}}{1 - \frac{\partial I_{x_j}}{\partial I_{x_l}}\frac{\partial I_{x_l}}{\partial I_{x_j}}}\frac{dI_{x_i}}{d\lambda_i}.
\end{aligned} \tag{A69}$$

Plugging (A68) and (A69) into (A65) gives

$$\frac{dI_{x_i}}{d\lambda_i} = \frac{\partial I_{x_i}}{\partial\lambda_i} + \frac{\partial I_{x_i}}{\partial I_{x_j}}\frac{\frac{\partial I_{x_j}}{\partial I_{x_i}} + \frac{\partial I_{x_j}}{\partial I_{x_l}}\frac{\partial I_{x_l}}{\partial I_{x_i}}}{1 - \frac{\partial I_{x_j}}{\partial I_{x_l}}\frac{\partial I_{x_l}}{\partial I_{x_j}}}\frac{dI_{x_i}}{d\lambda_i}$$

$$+\frac{\partial I_{x_i}}{\partial I_{x_l}}\frac{\dfrac{\partial I_{x_l}}{\partial I_{x_i}}+\dfrac{\partial I_{x_l}}{\partial I_{x_j}}\dfrac{\partial I_{x_j}}{\partial I_{x_i}}}{1-\dfrac{\partial I_{x_j}}{\partial I_{x_l}}\dfrac{\partial I_{x_l}}{\partial I_{x_j}}}\frac{dI_{x_i}}{d\lambda_i}$$

$$\Leftrightarrow\left[1-\frac{\dfrac{\partial I_{x_i}}{\partial I_{x_j}}\left(\dfrac{\partial I_{x_j}}{\partial I_{x_i}}+\dfrac{\partial I_{x_j}}{\partial I_{x_l}}\dfrac{\partial I_{x_l}}{\partial I_{x_i}}\right)}{1-\dfrac{\partial I_{x_j}}{\partial I_{x_l}}\dfrac{\partial I_{x_l}}{\partial I_{x_j}}}-\frac{\dfrac{\partial I_{x_i}}{\partial I_{x_l}}\left(\dfrac{\partial I_{x_l}}{\partial I_{x_i}}+\dfrac{\partial I_{x_l}}{\partial I_{x_j}}\dfrac{\partial I_{x_j}}{\partial I_{x_i}}\right)}{1-\dfrac{\partial I_{x_j}}{\partial I_{x_l}}\dfrac{\partial I_{x_l}}{\partial I_{x_j}}}\right]\frac{dI_{x_i}}{d\lambda_i}=\frac{\partial I_{x_i}}{\partial \lambda_i}$$

$$\Leftrightarrow\frac{dI_{x_i}}{d\lambda_i}=\Gamma_1^{-1}\left(1-\frac{\partial I_{x_j}}{\partial I_{x_l}}\frac{\partial I_{x_l}}{\partial I_{x_j}}\right)\frac{\partial I_{x_i}}{\partial \lambda_i}, \tag{A70}$$

where

$$\Gamma_1\equiv 1-\left[\frac{\partial I_{x_j}}{\partial I_{x_l}}\frac{\partial I_{x_l}}{\partial I_{x_j}}+\frac{\partial I_{x_i}}{\partial I_{x_j}}\left(\frac{\partial I_{x_j}}{\partial I_{x_i}}+\frac{\partial I_{x_j}}{\partial I_{x_l}}\frac{\partial I_{x_l}}{\partial I_{x_i}}\right)+\frac{\partial I_{x_i}}{\partial I_{x_l}}\left(\frac{\partial I_{x_l}}{\partial I_{x_i}}+\frac{\partial I_{x_l}}{\partial I_{x_j}}\frac{\partial I_{x_j}}{\partial I_{x_i}}\right)\right].$$

Finally, plugging (A70) into (A68) and (A69) yields

$$\frac{dI_{x_j}}{d\lambda_i}=\frac{\dfrac{\partial I_{x_j}}{\partial I_{x_i}}+\dfrac{\partial I_{x_j}}{\partial I_{x_l}}\dfrac{\partial I_{x_l}}{\partial I_{x_i}}}{1-\dfrac{\partial I_{x_j}}{\partial I_{x_l}}\dfrac{\partial I_{x_l}}{\partial I_{x_j}}}\times\Gamma_1^{-1}\left(1-\frac{\partial I_{x_j}}{\partial I_{x_l}}\frac{\partial I_{x_l}}{\partial I_{x_j}}\right)\frac{\partial I_{x_i}}{\partial \lambda_i}$$

$$=\Gamma_1^{-1}\left(\frac{\partial I_{x_j}}{\partial I_{x_i}}+\frac{\partial I_{x_j}}{\partial I_{x_l}}\frac{\partial I_{x_l}}{\partial I_{x_i}}\right)\frac{\partial I_{x_i}}{\partial \lambda_i}$$

and

$$\frac{dI_{x_l}}{d\lambda_i}=\frac{\dfrac{\partial I_{x_l}}{\partial I_{x_i}}+\dfrac{\partial I_{x_l}}{\partial I_{x_j}}\dfrac{\partial I_{x_j}}{\partial I_{x_i}}}{1-\dfrac{\partial I_{x_j}}{\partial I_{x_l}}\dfrac{\partial I_{x_l}}{\partial I_{x_j}}}\times\Gamma_1^{-1}\left(1-\frac{\partial I_{x_j}}{\partial I_{x_l}}\frac{\partial I_{x_l}}{\partial I_{x_j}}\right)\frac{\partial I_{x_i}}{\partial \lambda_i}$$

$$=\Gamma_1^{-1}\left(\frac{\partial I_{x_l}}{\partial I_{x_i}}+\frac{\partial I_{x_l}}{\partial I_{x_j}}\frac{\partial I_{x_j}}{\partial I_{x_i}}\right)\frac{\partial I_{x_i}}{\partial \lambda_i}.\quad\square$$

B Technical Appendix

This Technical Appendix aims to provide the necessary mathematical and statistical background knowledge that is needed to understand some of the results derived in the main text and in Appendix A.

B.1 Important Properties of Normal Random Variables

The used theoretical framework relies on the assumption that all introduced random variables are (jointly) normally distributed. Additionally, the framework works with linear transformations of normal random variables and sums of and differences between independent normal random variables. The aim of this section of Appendix B is to prove that linear transformations of normal random variables and sums of and differences between independent normal variables are again normally distributed.

B.1.1 Linear Transformations

Consider an arbitrary continuous random variable $X \sim N(\mu, \sigma^2)$ and a linear transformation of this variable $Y = a + bX$, for constants a and b. It holds that $Y \sim N(a + b\mu, b^2\sigma^2)$.

Proof. The probability density function (PDF) of the normal random variable X evaluated at value x is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\,\sigma^2}\right].$$

The PDF can be used to determine the cumulative distribution function (CDF). The CDF indicates the probability that a random variable takes on a realisation that is smaller than or equal to a specific value. For example, the probability that the normal random variable X takes on a realisation that is smaller than or equal to some value m is

$$F_X(m) = P(X \leq m) = \int_{x=-\infty}^{x=m} f_X(x)\, dx.$$

To determine the distribution of the random variable Y, we need to derive its PDF, which can be done by computing its CDF. More specifically, we are interested in determining

$$F_Y(m) = P(Y \leq m) = \int_{y=-\infty}^{y=m} f_Y(y)\, dy.$$

Recalling the relationship between Y and X, we obtain

$$P(Y \leq m) = P(a + bX \leq m) = P\left(X \leq \frac{m-a}{b}\right) = F_X\left(\frac{m-a}{b}\right).$$

The probability that X takes on a value that is smaller than or equal to $(m-a)/b$ is given by

$$\begin{aligned} F_X\left(\frac{m-a}{b}\right) &= \int_{x=-\infty}^{x=(m-a)/b} f_X(x)\, dx \\ &= \int_{x=-\infty}^{x=(m-a)/b} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\,\sigma^2}\right] dx. \end{aligned}$$

Thus, we can conclude that

$$F_Y(m) = P(Y \leq m) = \int_{x=-\infty}^{x=(m-a)/b} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\,\sigma^2}\right] dx.$$

Next, we apply a change of variables to the above CDF, which allows us to express it in terms of y rather than x. We have assumed that $y = a + bx$. Consequently, the boundaries of the integral become

$$\begin{aligned} x = \frac{m-a}{b} \quad &\Leftrightarrow \quad y = a + b\,\frac{m-a}{b} = m, \\ x = -\infty \quad &\Leftrightarrow \quad y = -\infty. \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{dy}{dx} = b \quad &\Leftrightarrow \quad dx = \frac{dy}{b}, \\ y = a + bx \quad &\Leftrightarrow \quad x = \frac{y-a}{b}. \end{aligned}$$

Hence, the CDF can be written as

$$F_Y(m) = P(Y \leq m) = \int_{y=-\infty}^{y=m} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{[(y-a)/b-\mu]^2}{2\,\sigma^2}\right\} \frac{dy}{b}$$

$$= \int_{y=-\infty}^{y=m} \frac{1}{\sqrt{2\pi b^2\sigma^2}} \exp\left[-\frac{(1/b^2)\,(y-a-b\mu)^2}{2\,\sigma^2}\right] dy$$

$$= \int_{y=-\infty}^{y=m} \underbrace{\frac{1}{\sqrt{2\pi b^2\sigma^2}} \exp\left\{-\frac{[y-(a+b\mu)]^2}{2b^2\sigma^2}\right\}}_{=f_Y(y)} dy.$$

The term under the integral represents the PDF of the random variable Y. According to the PDF, Y follows a normal distribution with mean $a + b\mu$ and variance $b^2\sigma^2$. □

B.1.2 Sums and Differences

Consider two continuous, independent random variables $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$. Then, $X + Y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$ and $X - Y \sim N(\mu_x - \mu_y, \sigma_x^2 + \sigma_y^2)$.

Proof. The aim is to determine the PDF of $X + Y$, which indicates the distribution of the sum of the two normal random variables. The PDF of $X + Y$ can then be used to determine the distribution of the difference between the two normal random variables. Since X and Y are independent, the PDF of $X+Y$ equals the convolution of the PDFs of X and Y (see, e.g., Grinstead and Snell, 1998, Section 7.2). This implies

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(t) f_Y(z-t)\,dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left[-\frac{(t-\mu_x)^2}{2\,\sigma_x^2}\right] \frac{1}{\sqrt{2\pi\sigma_y^2}} \exp\left[-\frac{(z-t-\mu_y)^2}{2\,\sigma_y^2}\right] dt$$

$$= \frac{1}{2\pi\sigma_x\sigma_y} \int_{-\infty}^{\infty} \exp\left\{-\left[\frac{(t-\mu_x)^2}{2\,\sigma_x^2} + \frac{(z-t-\mu_y)^2}{2\,\sigma_y^2}\right]\right\} dt.$$

The term in square brackets can be developed as follows:

$$\frac{(t-\mu_x)^2}{2\,\sigma_x^2} + \frac{(z-t-\mu_y)^2}{2\,\sigma_y^2}$$

$$= \frac{t^2 + \mu_x^2 - 2t\mu_x}{2\,\sigma_x^2} + \frac{(z-\mu_y)^2 - 2t(z-\mu_y) + t^2}{2\,\sigma_y^2}$$

$$= \frac{t^2 + \mu_x^2 - 2t\mu_x}{2\,\sigma_x^2} + \frac{(z-\mu_y)^2 - 2t(z-\mu_y) + t^2}{2\,\sigma_y^2}$$

$$= t^2\left(\frac{1}{2\sigma_x^2} + \frac{1}{2\sigma_y^2}\right) - 2t\left(\frac{\mu_x}{2\sigma_x^2} + \frac{z-\mu_y}{2\sigma_y^2}\right) + \frac{\mu_x^2}{2\sigma_x^2} + \frac{(z-\mu_y)^2}{2\sigma_y^2}$$

$$= \frac{1}{2}\frac{\sigma_x^2+\sigma_y^2}{\sigma_x^2\sigma_y^2}\left[t^2 - 2t\,\frac{\mu_x\sigma_y^2 + (z-\mu_y)\sigma_x^2}{\sigma_x^2+\sigma_y^2} + \frac{\mu_x^2\sigma_y^2 + (z-\mu_y)^2\sigma_x^2}{\sigma_x^2+\sigma_y^2}\right].$$

Completing the square delivers

$$\frac{(t-\mu_x)^2}{2\,\sigma_x^2} + \frac{(z-t-\mu_y)^2}{2\,\sigma_y^2}$$

$$= \frac{1}{2}\frac{\sigma_x^2+\sigma_y^2}{\sigma_x^2\sigma_y^2}\Bigg\{\left[t - \frac{\mu_x\sigma_y^2 + (z-\mu_y)\sigma_x^2}{\sigma_x^2+\sigma_y^2}\right]^2 + \frac{\mu_x^2\sigma_y^2 + (z-\mu_y)^2\sigma_x^2}{\sigma_x^2+\sigma_y^2}$$

$$- \left[\frac{\mu_x\sigma_y^2 + (z-\mu_y)\sigma_x^2}{\sigma_x^2+\sigma_y^2}\right]^2\Bigg\}.$$

Note that

$$\frac{\mu_x^2\sigma_y^2 + (z-\mu_y)^2\sigma_x^2}{\sigma_x^2+\sigma_y^2} - \left[\frac{\mu_x\sigma_y^2 + (z-\mu_y)\sigma_x^2}{\sigma_x^2+\sigma_y^2}\right]^2$$

$$= \frac{\left[\mu_x^2\sigma_y^2 + (z-\mu_y)^2\sigma_x^2\right](\sigma_x^2+\sigma_y^2) - \mu_x^2\sigma_y^4 - (z-\mu_y)^2\sigma_x^4 - 2\mu_x(z-\mu_y)\sigma_x^2\sigma_y^2}{(\sigma_x^2+\sigma_x^2)^2}$$

$$= \frac{\sigma_x^2\sigma_y^2\left[\mu_x^2 + (z-\mu_y)^2 - 2\mu_x(z-\mu_y)\right]}{(\sigma_x^2+\sigma_y^2)^2}$$

$$= \frac{\sigma_x^2\sigma_y^2\left[\mu_x^2 - (z-\mu_y)\right]^2}{(\sigma_x^2+\sigma_y^2)^2}$$

$$= \frac{\sigma_x^2\sigma_y^2\left[z - (\mu_x+\mu_y)\right]^2}{(\sigma_x^2+\sigma_y^2)^2}.$$

Thus, we obtain

$$\frac{(t-\mu_x)^2}{2\,\sigma_x^2} + \frac{(z-t-\mu_y)^2}{2\,\sigma_y^2}$$

$$= \frac{\sigma_x^2+\sigma_y^2}{2\sigma_x^2\sigma_y^2}\left[t - \frac{\mu_x\sigma_y^2 + (z-\mu_y)\sigma_x^2}{\sigma_x^2+\sigma_y^2}\right]^2 + \frac{\left[z-(\mu_x+\mu_y)\right]^2}{2(\sigma_x^2+\sigma_y^2)}.$$

The PDF of $X + Y$ becomes

$$f_{X+Y}(z) = \frac{1}{2\pi\sigma_x\sigma_y} \int_{-\infty}^{\infty} \exp\left\{ -\frac{\sigma_x^2+\sigma_y^2}{2\sigma_x^2\sigma_y^2}\left[t - \frac{\mu_x\sigma_y^2 + (z-\mu_y)\sigma_x^2}{\sigma_x^2+\sigma_y^2}\right]^2 - \frac{[z-(\mu_x+\mu_y)]^2}{2(\sigma_x^2+\sigma_y^2)}\right\} dt$$

$$= \frac{\exp\left\{-\frac{[z-(\mu_x+\mu_y)]^2}{2(\sigma_x^2+\sigma_y^2)}\right\}}{2\pi\sigma_x\sigma_y} \times \int_{-\infty}^{\infty} \exp\left\{-\frac{\sigma_x^2+\sigma_y^2}{2\sigma_x^2\sigma_y^2}\left[t - \frac{\mu_x\sigma_y^2 + (z-\mu_y)\sigma_x^2}{\sigma_x^2+\sigma_y^2}\right]^2\right\} dt.$$

Recall from Appendix B.1.1 that the CDF of a random variable evaluated at a particular value indicates the probability that the random variable takes on a value that is smaller than or equal to that particular value. The probability that a random variable takes on a value that is smaller than or "equal to" positive infinity is, of course, one. Thus, in general form, we obtain for any normal random variable that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right] dt = 1,$$

where the integrand stands for the PDF of the normal random variable. Note that the integral in $f_{X+Y}(z)$ can be rewritten as follows:

$$\int_{-\infty}^{\infty} \exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right] dt,$$

where $\mu = \frac{\mu_x\sigma_y^2 + (z-\mu_y)\sigma_x^2}{\sigma_x^2+\sigma_y^2}$ and $\sigma^2 = \frac{\sigma_x^2\sigma_y^2}{\sigma_x^2+\sigma_x^2}$.

Since

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right] dt = 1,$$

we can immediately conclude that

$$\int_{-\infty}^{\infty} \exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right] dt = \sqrt{2\pi\sigma^2} = \sqrt{2\pi}\frac{\sigma_x\sigma_y}{\sqrt{\sigma_x^2+\sigma_y^2}}.$$

Eventually, the PDF of $X+Y$ becomes

$$f_{X+Y}(z) = \frac{\exp\left\{-\frac{[z-(\mu_x+\mu_y)]^2}{2(\sigma_x^2+\sigma_y^2)}\right\}}{2\pi\sigma_x\sigma_y} \times \sqrt{2\pi}\frac{\sigma_x\sigma_y}{\sqrt{\sigma_x^2+\sigma_y^2}}$$

$$= \frac{\exp\left\{-\frac{[z-(\mu_x+\mu_y)]^2}{2\left(\sigma_x^2+\sigma_y^2\right)}\right\}}{\sqrt{2\pi\left(\sigma_x^2+\sigma_y^2\right)}}.$$

It can be clearly seen that the PDF of $X+Y$ shows the form of the PDF of a normal random variable with mean $\mu_x+\mu_y$ and variance $\sigma_x^2+\sigma_y^2$.

Having derived the distribution of the sum of two independent normal random variables, we can also determine the distribution of the difference between two independent normal variables (i.e., $X-Y$). Recall from Appendix B.1.1 that $-Y$ is just a linear transformation of the normal random variable Y, which is again normally distributed with mean $-\mu_y$ and variance σ_y^2. Thus, the difference between X and Y can be interpreted as the sum of X and the linear transformation $-Y$. This immediately gives that $X-Y$ is normally distributed with mean $\mu_x-\mu_y$ and variance $\sigma_x^2+\sigma_y^2$. □

B.2 Projection Theorem

Consider an n-dimensional random vector $X=(X_1\ X_2)^T$ that is characterized by the two subvectors X_1 and X_2 of arbitrary dimensions $l\times 1$ and $k\times 1$ with $l+k=n$. Each of the n random variables is assumed to follow a normal distribution. Following Rao (1973, Chapter 8), it holds that $X\sim N_n(\mu,\Sigma)$, i.e., X follows an n-variate normal distribution with

$$\mu = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}.$$

μ is an n-dimensional vector that can be partitioned into μ_1 of dimension $l\times 1$ and μ_2 of dimension $k\times 1$, thereby representing the mean vectors of X_1 and X_2, respectively. Σ_{11} of dimension $l\times l$ stands for the variance-covariance matrix of X_1, Σ_{12} of dimension $l\times k$ and Σ_{21} of dimension $k\times l$ for the covariance matrix of X_1 and X_2, respectively, and Σ_{22} of dimension $k\times k$ for the variance-covariance matrix of X_2. Note that $\Sigma_{21}=\Sigma_{12}^T$.

The projection theorem (see, e.g., Brunnermeier, 2001, p. 12) states that

$$(X_1|X_2 = x_2) \sim N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}).$$

B.2.1 Derivation

Since the projection theorem constitutes a central theorem in this thesis, we expound a derivation below. We focus on the case where X_1 is one-dimensional, as the projection theorem is exclusively used in this variant in the main text. If X_1 is one-dimensional, Σ_{11} is one-dimensional too. Furthermore, Σ_{12} is a row vector of dimension $1 \times n-1$, Σ_{21} a column vector of dimension $n-1 \times 1$, and Σ_{22} a matrix of dimension $n-1 \times n-1$. The following proof is an extensive and adjusted version of Wang (2006):

Matrix properties and theorems

Before expounding the actual proof, some important properties of matrix calculations (see, e.g., Gentle, 2017) need to be stated:

- Matrix multiplication is associative (i.e., $(A \times B)C = A(B \times C)$).
- Matrix multiplication is generally not commutative (i.e., $AB \neq BA$).
- Matrix multiplication is distributive (i.e., $(A + B)C = AC + BC$).
- The inverse of a symmetric matrix is symmetric too.
- The product of a matrix and its inverse yields the identity matrix (i.e., $A \times A^{-1} = I$).
- It holds that $A^T = A$ if A is symmetric.
- It generally holds that $(AB)^T = B^T A^T$.
- The determinant of a matrix product is equal to the product of the respective determinants (i.e., $|AB| = |A||B|$).

Furthermore, we need three important theorems:

Theorem 1. Let A, B, C, and D be four matrices of dimension $k \times k$, $k \times l$, $l \times l$, and $l \times k$, respectively. Then,

$$(A - BC^{-1}D)^{-1} = A^{-1} + A^{-1}B(C - DA^{-1}B)^{-1}DA^{-1}.$$

This theorem is a variant of the Sherman-Morrison-Woodbury formula (see, e.g., Golub and Van Loan, 2013, Chapter 2).

Proof. The theorem is proven by showing that

$$(A - BC^{-1}D)[A^{-1} + A^{-1}B(C - DA^{-1}B)^{-1}DA^{-1}] = I.$$

Direct computations yield

$$\begin{aligned}
&(A - BC^{-1}D)[A^{-1} + A^{-1}B(C - DA^{-1}B)^{-1}DA^{-1}] \\
&= (A - BC^{-1}D)A^{-1} + (A - BC^{-1}D)A^{-1}B(C - DA^{-1}B)^{-1}DA^{-1} \\
&= I - BC^{-1}DA^{-1} + (B - BC^{-1}DA^{-1}B)(C - DA^{-1}B)^{-1}DA^{-1} \\
&= I - BC^{-1}DA^{-1} + BC^{-1}(C - DA^{-1}B)(C - DA^{-1}B)^{-1}DA^{-1} \\
&= I - BC^{-1}DA^{-1} + BC^{-1}DA^{-1} \\
&= I. \quad \square
\end{aligned}$$

Theorem 2. Consider a square, symmetric matrix M that can be divided into four blocks, each of which represents an own matrix, i.e.,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The matrix M is called a partitioned matrix or block matrix. The respective inverse matrix N can be divided in the same manner:

$$N = M^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix},$$

where the following holds:

- A and E are assumed to be of dimension $n \times n$.
- D and H are assumed to be of dimension $m \times m$.
- B and F are assumed to be of dimension $n \times m$.
- C and G are assumed to be of dimension $m \times n$.

Then,

$$E = (A - BD^{-1}C)^{-1},$$

$F = -(A - BD^{-1}C)^{-1}BD^{-1}$,

$G = -(D - CA^{-1}B)^{-1}CA^{-1}$,

$H = (D - CA^{-1}B)^{-1}$.

Proof. Since N is the inverse matrix of M, it must hold that $MN = I$. This delivers

$$MN = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Thus,

$$AE + BG = I \;\Leftrightarrow\; A^{-1}AE + A^{-1}BG = A^{-1} \;\Leftrightarrow\; E = A^{-1} - A^{-1}BG,$$

$$AF + BH = 0 \;\Leftrightarrow\; A^{-1}AF + A^{-1}BH = 0 \;\Leftrightarrow\; F = -A^{-1}BH,$$

$$CE + DG = 0 \;\Leftrightarrow\; D^{-1}CE + D^{-1}DG = 0 \;\Leftrightarrow\; G = -D^{-1}CE,$$

$$CF + DH = I \;\Leftrightarrow\; D^{-1}CF + D^{-1}DH = D^{-1} \;\Leftrightarrow\; H = D^{-1} - D^{-1}CF.$$

Putting these four results together yields

$$\begin{aligned}
& E = A^{-1} - A^{-1}BG \\
\Leftrightarrow\; & E = A^{-1} - A^{-1}B(-D^{-1}CE) \\
\Leftrightarrow\; & (I - A^{-1}BD^{-1}C)E = A^{-1} \\
\Leftrightarrow\; & (A - BD^{-1}C)E = I \\
\Leftrightarrow\; & E = (A - BD^{-1}C)^{-1}, \\
& F = -A^{-1}BH \\
\Leftrightarrow\; & F = -A^{-1}B(D^{-1} - D^{-1}CF) \\
\Leftrightarrow\; & (I - A^{-1}BD^{-1}C)F = -A^{-1}BD^{-1} \\
\Leftrightarrow\; & (A - BD^{-1}C)F = -BD^{-1} \\
\Leftrightarrow\; & F = -(A - BD^{-1}C)BD^{-1}, \\
& G = -D^{-1}CE
\end{aligned}$$

$$\Leftrightarrow G = -D^{-1}C(A^{-1} - A^{-1}BG)$$

$$\Leftrightarrow (I - D^{-1}CA^{-1}B)G = -D^{-1}CA^{-1}$$

$$\Leftrightarrow (D - CA^{-1}B)G = -CA^{-1}$$

$$\Leftrightarrow G = -(D - CA^{-1}B)^{-1}CA^{-1},$$

$$H = D^{-1} - D^{-1}CF$$

$$\Leftrightarrow H = D^{-1} - D^{-1}C(-A^{-1}BH)$$

$$\Leftrightarrow (I - D^{-1}CA^{-1}B)H = D^{-1}$$

$$\Leftrightarrow (D - CA^{-1}B)H = I$$

$$\Leftrightarrow H = (D - CA^{-1}B)^{-1}. \quad \square$$

Theorem 3. The determinant of a partitioned, symmetric matrix can be written as

$$|M| = \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right| = |D||A - BD^{-1}C|.$$

Proof. The matrix M can be decomposed as follows:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & B \\ 0 & D \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I \end{pmatrix}.$$

Thus,

$$|M| = \left| \begin{pmatrix} I & B \\ 0 & D \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} I & B \\ 0 & D \end{pmatrix} \right| \left| \begin{pmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I \end{pmatrix} \right|.$$

Following the results on determinants of partitioned matrices (see, e.g., Silvester, 2000), we obtain

$$|M| = \left|D \times I - B \times 0\right| \left|(A - BD^{-1}C) \times I - D^{-1}C \times 0\right|$$

$$= |D||A - BD^{-1}C|. \quad \square$$

Conditional density function

To derive the projection theorem for the case where X_1 is one-dimensional, we need to determine the conditional distribution of X_1 given X_2. According to Lindgren et al. (2013, Appendix A), the conditional density function of X_1 given X_2 can be computed as follows:

$$f_{X_1|X_2}(x_1, x_2) = \frac{f_{(X_1,X_2)}(x_1, x_2)}{f_{X_2}(x_2)}.$$

The joint density function of X_1 and X_2 is given by

$$f_{(X_1,X_2)}(x_1, x_2) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}$$
$$\times \exp\left\{-\frac{1}{2}\left[(x_1-\mu_1)^T, (x_2-\mu_2)^T\right]\Sigma^{-1}\big(x_1-\mu_1, x_2-\mu_2\big)\right\}.$$

As Σ can be displayed as a 2×2 block matrix, we can write

$$\Sigma^{-1} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{pmatrix}.$$

According to Theorems 1 and 2,

$$\Sigma^{11} = (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1},$$

$$\Sigma^{12} = -(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}\Sigma_{12}\Sigma_{22}^{-1},$$

$$\Sigma^{21} = -(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})^{-1}\Sigma_{21}\Sigma_{11}^{-1},$$

$$\Sigma^{22} = (\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})^{-1} = \Sigma_{22}^{-1} + \Sigma_{22}^{-1}\Sigma_{21}(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}\Sigma_{12}\Sigma_{22}^{-1}.$$

Since X_1 is assumed to be one-dimensional, Σ^{11} is one-dimensional, Σ^{12} a row vector, Σ^{21} a column vector, and Σ^{22} an $n-1 \times n-1$ matrix, equivalent to the entries of Σ. Furthermore,

$$F(x_1, x_2) \equiv \left[(x_1-\mu_1)^T, (x_2-\mu_2)^T\right]\Sigma^{-1}\big(x_1-\mu_1, x_2-\mu_2\big)$$
$$= \Big[(x_1-\mu_1), (x_2-\mu_2)^T\Big]\begin{pmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{pmatrix}\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}$$

$$
\begin{aligned}
&= \Big[(x_1-\mu_1)\Sigma^{11} + (x_2-\mu_2)^T\Sigma^{21}, (x_1-\mu_1)\Sigma^{12} + (x_2-\mu_2)^T\Sigma^{22}\Big] \\
&\quad \times \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix} \\
&= (x_1-\mu_1)^2\Sigma^{11} + (x_2-\mu_2)^T\Sigma^{21}(x_1-\mu_1) + (x_1-\mu_1)\Sigma^{12}(x_2-\mu_2) \\
&\quad + (x_2-\mu_2)^T\Sigma^{22}(x_2-\mu_2) \\
&= (x_1-\mu_1)^2\Sigma^{11} + [\Sigma^{12}(x_2-\mu_2)]^T(x_1-\mu_1) + (x_1-\mu_1)\Sigma^{12}(x_2-\mu_2) \\
&\quad + (x_2-\mu_2)^T\Sigma^{22}(x_2-\mu_2) \\
&= (x_1-\mu_1)^2\Sigma^{11} + 2\Sigma^{12}(x_2-\mu_2)(x_1-\mu_1) + (x_2-\mu_2)^T\Sigma^{22}(x_2-\mu_2).
\end{aligned}
$$

The last step follows from the fact that $\Sigma^{12}(x_2-\mu_2)$ is one-dimensional, since Σ^{12} is a row vector and $(x_2-\mu_2)$ a column vector. Plugging the results for the entries of Σ^{-1} into the above equation yields

$$
\begin{aligned}
F(x_1,x_2) &= (x_1-\mu_1)^2(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} \\
&\quad - 2\,(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)(x_1-\mu_1) \\
&\quad + (x_2-\mu_2)^T\left[\Sigma_{22}^{-1} + \Sigma_{22}^{-1}\Sigma_{21}(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}\Sigma_{12}\Sigma_{22}^{-1}\right](x_2-\mu_2) \\
&= (x_1-\mu_1)^2(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} \\
&\quad - 2\,(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)(x_1-\mu_1) \\
&\quad - (x_2-\mu_2)^T\Sigma_{22}^{-1}(x_2-\mu_2) \\
&\quad - (x_2-\mu_2)^T\Sigma_{22}^{-1}\Sigma_{21}(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2) \\
&= (x_1-\mu_1)^2(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} \\
&\quad - 2\,(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)(x_1-\mu_1) \\
&\quad + (x_2-\mu_2)^T\Sigma_{22}^{-1}(x_2-\mu_2) \\
&\quad + [\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)]^T(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2) \\
&= (x_2-\mu_2)^T\Sigma_{22}^{-1}(x_2-\mu_2) \\
&\quad + [(x_1-\mu_1) - \Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)]^2(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1},
\end{aligned}
$$

where the last equation follows from the fact that $\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)$ is one-dimensional. Thus,

$$f_{(X_1,X_2)}(x_1,x_2) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(x_2-\mu_2)^T\Sigma_{22}^{-1}(x_2-\mu_2)\right]$$
$$\exp\left\{-\frac{1}{2}\left[(x_1-\mu_1)-\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)\right]^2\left(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)^{-1}\right\}.$$

Following Lindgren et al. (2013, Appendix A), the density function of the normal random vector X_2 of dimension $n-1\times 1$ is given by

$$f_{X_2}(x_2) = \frac{1}{(2\pi)^{(n-1)/2}|\Sigma_{22}|^{1/2}} \exp\left[-\frac{1}{2}(x_2-\mu_2)^T\Sigma_{22}^{-1}(x_2-\mu_2)\right].$$

Hence,

$$f_{X_1|X_2}(x_1,x_2) = \frac{f_{(X_1,X_2)}(x_1,x_2)}{f_{X_2}(x_2)}$$
$$= \frac{(2\pi)^{(n-1)/2}|\Sigma_{22}|^{1/2}}{(2\pi)^{n/2}|\Sigma|^{1/2}}$$
$$\exp\left\{-\frac{1}{2}\left[(x_1-\mu_1)-\Sigma_{12}\Sigma_{22}^{-1}(X_2-\mu_2)\right]^2\left(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)^{-1}\right\}.$$

According to Theorem 3,

$$|\Sigma| = |\Sigma_{22}||\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}|.$$

Thus,

$$f_{X_1|X_2}(x_1,x_2) = \frac{(2\pi)^{(n-1)/2}|\Sigma_{22}|^{1/2}}{(2\pi)^{n/2}|\Sigma_{22}|^{1/2}|\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}|^{1/2}}$$
$$\exp\left\{-\frac{1}{2}\left[(x_1-\mu_1)-\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)\right]^2\left(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)^{-1}\right\}$$
$$= \frac{1}{(2\pi)^{1/2}|\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}|^{1/2}}$$
$$\exp\left(-\frac{1}{2}\left\{x_1-\left[\mu_1+\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)\right]\right\}^2\left(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)^{-1}\right).$$

The above function stands for the density function of the one-dimensional normal random variable X_1 with mean $\mu_1+\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2)$ and variance $\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. In other words, conditional on the normal random vector X_2, X_1 is normally distributed with mean $\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$ and variance $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. □

B.2.2 Important Corollaries

Corollary 1. Consider the bivariate normal random variables $Z = X + \epsilon$ and $Y = \epsilon + \eta$ with $X \sim N(\mu_x, \sigma_x^2)$, $\epsilon \sim N(0, \sigma_\epsilon^2)$, and $\eta \sim N(0, \sigma_\eta^2)$. The random variables X, ϵ, and η are assumed to be pairwise uncorrelated. It holds that $\mathrm{E}(X \mid Z, Y) = \mathrm{E}(X \mid \hat{Z})$ and $\mathrm{Var}(X \mid Z, Y) = \mathrm{Var}(X \mid \hat{Z})$ with $\hat{Z} = Z - \mathrm{E}(\epsilon \mid Y)$.

Proof. Using the projection theorem yields

$$\begin{aligned}
\mathrm{E}(X \mid Z, Y) &= \mu_x + \begin{pmatrix} \sigma_x^2 & 0 \end{pmatrix} \begin{pmatrix} \sigma_z^2 & \sigma_\epsilon^2 \\ \sigma_\epsilon^2 & \sigma_y^2 \end{pmatrix}^{-1} \begin{pmatrix} Z - \mu_x \\ Y \end{pmatrix} \\
&= \mu_x + \begin{pmatrix} \sigma_x^2 & 0 \end{pmatrix} \frac{1}{\sigma_z^2\sigma_y^2 - \sigma_\epsilon^4} \begin{pmatrix} \sigma_y^2 & -\sigma_\epsilon^2 \\ -\sigma_\epsilon^2 & \sigma_z^2 \end{pmatrix} \begin{pmatrix} Z - \mu_x \\ Y \end{pmatrix} \\
&= \mu_x + \frac{1}{\sigma_z^2\sigma_y^2 - \sigma_\epsilon^4} \begin{pmatrix} \sigma_x^2\sigma_y^2 & -\sigma_x^2\sigma_\epsilon^2 \end{pmatrix} \begin{pmatrix} Z - \mu_x \\ Y \end{pmatrix} \\
&= \mu_x + \frac{1}{\sigma_z^2\sigma_y^2 - \sigma_\epsilon^4} \left[\sigma_x^2\sigma_y^2(Z - \mu_x) - \sigma_x^2\sigma_\epsilon^2 Y\right] \\
&= \mu_x + \frac{1}{\sigma_x^2(\sigma_\epsilon^2 + \sigma_\eta^2) + \sigma_\epsilon^2\sigma_\eta^2} \left[\sigma_x^2(\sigma_\epsilon^2 + \sigma_\eta^2)(Z - \mu_x) - \sigma_x^2\sigma_\epsilon^2 Y\right]
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}(X \mid Z, Y) &= \sigma_x^2 - \begin{pmatrix} \sigma_x^2 & 0 \end{pmatrix} \begin{pmatrix} \sigma_z^2 & \sigma_\epsilon^2 \\ \sigma_\epsilon^2 & \sigma_y^2 \end{pmatrix}^{-1} \begin{pmatrix} \sigma_x^2 \\ 0 \end{pmatrix} \\
&= \sigma_x^2 - \begin{pmatrix} \sigma_x^2 & 0 \end{pmatrix} \frac{1}{\sigma_z^2\sigma_y^2 - \sigma_\epsilon^4} \begin{pmatrix} \sigma_y^2 & -\sigma_\epsilon^2 \\ -\sigma_\epsilon^2 & \sigma_z^2 \end{pmatrix} \begin{pmatrix} \sigma_x^2 \\ 0 \end{pmatrix} \\
&= \sigma_x^2 - \frac{1}{\sigma_z^2\sigma_y^2 - \sigma_\epsilon^4} \begin{pmatrix} \sigma_x^2\sigma_y^2 & -\sigma_x^2\sigma_\epsilon^2 \end{pmatrix} \begin{pmatrix} \sigma_x^2 \\ 0 \end{pmatrix}
\end{aligned}$$

$$= \sigma_x^2 - \frac{\sigma_x^4 \sigma_y^2}{\sigma_z^2 \sigma_y^2 - \sigma_\epsilon^4}$$

$$= \sigma_x^2 - \frac{\sigma_x^4 (\sigma_\epsilon^2 + \sigma_\eta^2)}{\sigma_x^2 (\sigma_\epsilon^2 + \sigma_\eta^2) + \sigma_\epsilon^2 \sigma_\eta^2}$$

$$= \frac{\sigma_x^2 \sigma_\epsilon^2 \sigma_\eta^2}{\sigma_x^2 (\sigma_\epsilon^2 + \sigma_\eta^2) + \sigma_\epsilon^2 \sigma_\eta^2}.$$

The corollary is proved if the respective calculations with the combined signal $\hat{Z}$ deliver identical results. According to the bivariate case of the projection theorem,

$$\mathrm{E}(X \,|\, \hat{Z}) = \mathrm{E}(X) + \frac{\mathrm{Cov}(X, \hat{Z})}{\mathrm{Var}(\hat{Z})} \left[\hat{Z} - \mathrm{E}(X) \right].$$

Note that $\hat{Z} = Z - \mathrm{E}(\epsilon \,|\, Y) = Z - \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_\eta^2} Y$.
This gives

$$\mathrm{Var}(\hat{Z}) = \mathrm{Var}(Z) + \mathrm{Var}\left(\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_\eta^2} Y \right) - 2\,\mathrm{Cov}\left(Z, \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_\eta^2} Y \right)$$

$$= \sigma_z^2 + \frac{\sigma_\epsilon^4}{(\sigma_\epsilon^2 + \sigma_\eta^2)^2} (\sigma_\epsilon^2 + \sigma_\eta^2) - 2 \frac{\sigma_\epsilon^4}{\sigma_\epsilon^2 + \sigma_\eta^2}$$

$$= \sigma_z^2 - \frac{\sigma_\epsilon^4}{\sigma_\epsilon^2 + \sigma_\eta^2}.$$

Hence,

$$\mathrm{E}(X \,|\, \hat{Z}) = \mu_x + \frac{\sigma_x^2}{\sigma_z^2 - \dfrac{\sigma_\epsilon^4}{\sigma_\epsilon^2 + \sigma_\eta^2}} \left(Z - \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + \sigma_\eta^2} Y - \mu_x \right)$$

$$= \mu_x + \frac{\sigma_x^2 \sigma_y^2}{\sigma_z^2 \sigma_y^2 - \sigma_\epsilon^4} \left(Z - \mu_x - \frac{\sigma_\epsilon^2}{\sigma_y^2} Y \right)$$

$$= \mu_x + \frac{1}{\sigma_z^2 \sigma_y^2 - \sigma_\epsilon^4} \left[\sigma_x^2 \sigma_y^2 (Z - \mu_x) - \sigma_x^2 \sigma_\epsilon^2 Y \right]$$

$$= \mu_x + \frac{1}{\sigma_x^2 (\sigma_\epsilon^2 + \sigma_\eta^2) + \sigma_\epsilon^2 \sigma_\eta^2} \left[\sigma_x^2 (\sigma_\epsilon^2 + \sigma_\eta^2)(Z - \mu_x) - \sigma_x^2 \sigma_\epsilon^2 Y \right].$$

Moreover, using the projection theorem,

$$\mathrm{Var}(X \,|\, \hat{Z}) = \mathrm{Var}(X) - \frac{[\mathrm{Cov}(X, \hat{Z})]^2}{\mathrm{Var}(\hat{Z})}$$

$$= \sigma_x^2 - \frac{\sigma_x^4}{\sigma_z^2 - \dfrac{\sigma_\epsilon^4}{\sigma_\epsilon^2 + \sigma_\eta^2}}$$

$$= \sigma_x^2 - \frac{\sigma_x^4 \sigma_y^2}{\sigma_z^2 \sigma_y^2 - \sigma_\epsilon^4}$$

$$= \frac{\sigma_x^2 \sigma_\epsilon^2 \sigma_\eta^2}{\sigma_x^2(\sigma_\epsilon^2 + \sigma_\eta^2) + \sigma_\epsilon^2 \sigma_\eta^2}.$$

By comparing the relevant results, it can be seen that they are pairwise identical. □

Using the analogous proof, one can show that the corollary still holds if additional signals about X with independent error terms are added. With the multivariate normal variables $Z_1 = X + \epsilon_1$, $Z_2 = X + \epsilon_2$, and $Y_1 = \epsilon_1 + \eta$, it holds that $\mathrm{E}(X \mid Z_1, Z_2, Y_1) = \mathrm{E}(X \mid \hat{Z}_1, Z_2)$ and $\mathrm{Var}(X \mid Z_1, Z_2, Y_1) = \mathrm{Var}(X \mid \hat{Z}_1, Z_2)$ given that $X, \epsilon_1, \epsilon_2$, and η are pairwise uncorrelated.

Furthermore, it can be shown that the corollary can be applied to cases with more than one combined signal. With the multivariate normal variables $Z_1 = X + \epsilon_1$, $Z_2 = X + \epsilon_2$, $Y_1 = \epsilon_1 + \eta_1$, and $Y_2 = \epsilon_2 + \eta_2$, it follows that $\mathrm{E}(X \mid Z_1, Z_2, Y_1, Y_2) = \mathrm{E}(X \mid \hat{Z}_1, \hat{Z}_2)$ and $\mathrm{Var}(X \mid Z_1, Z_2, Y_1, Y_2) = \mathrm{Var}(X \mid \hat{Z}_1, \hat{Z}_2)$, where $\hat{Z}_1 = Z_1 - \mathrm{E}(\epsilon_1 \mid Y_1)$ and $\hat{Z}_2 = Z_2 - \mathrm{E}(\epsilon_2 \mid Y_2)$. This holds true as long as $X, \epsilon_1, \epsilon_2, \eta_1$, and η_2 are pairwise uncorrelated.

Corollary 2. Consider an arbitrary continuous random variable $X \sim N(\mu_x, \tau_x^{-1})$ and K multivariate normal signals of the form $s_k = X + \epsilon_k$ with $\epsilon_k \sim$ i.i.d. $N(0, \tau_{\epsilon_k}^{-1})$. The noise terms ϵ_k are assumed to be independent of X. It holds that

$$\mathrm{E}(X \mid s_1, s_2, ..., s_K) = \mu_x + \frac{1}{\tau_x + \sum_{k=1}^K \tau_{\epsilon_k}} \sum_{k=1}^K \tau_{\epsilon_k}(s_k - \mu_x),$$

$$\mathrm{Var}(X \mid s_1, s_2, ..., s_K) = \frac{1}{\tau_x + \sum_{k=1}^K \tau_{\epsilon_k}}.$$

The parameter τ stands for the precision of a normal random variable, which is the inverse of its variance (e.g., $\tau_x = 1/\mathrm{Var}(X)$ for the normal random variable X).

Proof. Using the projection theorem, we obtain

$$\mathrm{E}(X \mid s_1, s_2, ..., s_K) = \mu_x + \begin{pmatrix} \tau_x^{-1} & \tau_x^{-1} & \ldots & \tau_x^{-1} \end{pmatrix}$$

$$\times \begin{pmatrix} \tau_x^{-1} + \tau_{\epsilon_1}^{-1} & \tau_x^{-1} & ... & \tau_x^{-1} \\ \tau_x^{-1} & \tau_x^{-1} + \tau_{\epsilon_2}^{-1} & ... & \tau_x^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ \tau_x^{-1} & \tau_x^{-1} & \cdots & \tau_x^{-1} + \tau_{\epsilon_K}^{-1} \end{pmatrix}^{-1} \begin{pmatrix} s_1 - \mu_x \\ s_2 - \mu_x \\ \vdots \\ s_K - \mu_x \end{pmatrix}.$$

Since the elements on the off-diagonals of the signals' variance-covariance matrix are identical, we can decompose the matrix as follows:

$$\begin{pmatrix} \tau_x^{-1} + \tau_{\epsilon_1}^{-1} & \tau_x^{-1} & ... & \tau_x^{-1} \\ \tau_x^{-1} & \tau_x^{-1} + \tau_{\epsilon_2}^{-1} & ... & \tau_x^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ \tau_x^{-1} & \tau_x^{-1} & \cdots & \tau_x^{-1} + \tau_{\epsilon_K}^{-1} \end{pmatrix} = \begin{pmatrix} \tau_{\epsilon_1}^{-1} & 0 & ... & 0 \\ 0 & \tau_{\epsilon_2}^{-1} & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tau_{\epsilon_K}^{-1} \end{pmatrix} + \begin{pmatrix} \tau_x^{-1} \\ \tau_x^{-1} \\ \vdots \\ \tau_x^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}.$$

By defining

$$A \equiv \begin{pmatrix} \tau_{\epsilon_1}^{-1} & 0 & ... & 0 \\ 0 & \tau_{\epsilon_2}^{-1} & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tau_{\epsilon_K}^{-1} \end{pmatrix}, \; u \equiv \begin{pmatrix} \tau_x^{-1} \\ \tau_x^{-1} \\ \vdots \\ \tau_x^{-1} \end{pmatrix}, \; \text{ and } \; v^T \equiv \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix},$$

we can make use of the Sherman-Morrison formula (see, e.g., Bartlett, 1951), which states that

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}.$$

Since the matrix A is diagonal, we can determine its inverse by simply inverting the elements on the main diagonal. Thus,

$$A^{-1}uv^TA^{-1} = \begin{pmatrix} \tau_{\epsilon_1} & 0 & ... & 0 \\ 0 & \tau_{\epsilon_2} & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tau_{\epsilon_K} \end{pmatrix} \begin{pmatrix} \tau_x^{-1} \\ \tau_x^{-1} \\ \vdots \\ \tau_x^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \tau_{\epsilon_1} & 0 & ... & 0 \\ 0 & \tau_{\epsilon_2} & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tau_{\epsilon_K} \end{pmatrix}$$

$$= \begin{pmatrix} \tau_x^{-1}\tau_{\epsilon_1} \\ \tau_x^{-1}\tau_{\epsilon_2} \\ \vdots \\ \tau_x^{-1}\tau_{\epsilon_K} \end{pmatrix} \begin{pmatrix} \tau_{\epsilon_1} & \tau_{\epsilon_2} & \cdots & \tau_{\epsilon_K} \end{pmatrix}$$

$$= \tau_x^{-1} \begin{pmatrix} \tau_{\epsilon_1}^2 & \tau_{\epsilon_1}\tau_{\epsilon_2} & ... & \tau_{\epsilon_1}\tau_{\epsilon_K} \\ \tau_{\epsilon_2}\tau_{\epsilon_1} & \tau_{\epsilon_2}^2 & \cdots & \tau_{\epsilon_2}\tau_{\epsilon_K} \\ \vdots & \vdots & \vdots & \vdots \\ \tau_{\epsilon_K}\tau_{\epsilon_1} & \tau_{\epsilon_K}\tau_{\epsilon_2} & \cdots & \tau_{\epsilon_K}^2 \end{pmatrix}$$

and

$$1 + v^TA^{-1}u = 1 + \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \tau_{\epsilon_1} & 0 & ... & 0 \\ 0 & \tau_{\epsilon_2} & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tau_{\epsilon_K} \end{pmatrix} \begin{pmatrix} \tau_x^{-1} \\ \tau_x^{-1} \\ \vdots \\ \tau_x^{-1} \end{pmatrix}$$

$$= 1 + \begin{pmatrix} \tau_{\epsilon_1} & \tau_{\epsilon_2} & \cdots & \tau_{\epsilon_K} \end{pmatrix} \begin{pmatrix} \tau_x^{-1} \\ \tau_x^{-1} \\ \vdots \\ \tau_x^{-1} \end{pmatrix}$$

$$= 1 + \tau_x^{-1} \sum_{k=1}^{K} \tau_{\epsilon_k}.$$

Denote $\nu \equiv \frac{\tau_x^{-1}}{1+\tau_x^{-1}\sum_{k=1}^{K}\tau_{\epsilon_k}} = \frac{1}{\tau_x+\sum_{k=1}^{K}\tau_{\epsilon_k}}$. Then,

$$(A+uv^T)^{-1} = \begin{pmatrix} \tau_{\epsilon_1} & 0 & ... & 0 \\ 0 & \tau_{\epsilon_2} & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \tau_{\epsilon_K} \end{pmatrix} - \nu \begin{pmatrix} \tau_{\epsilon_1}^2 & \tau_{\epsilon_1}\tau_{\epsilon_2} & ... & \tau_{\epsilon_1}\tau_{\epsilon_K} \\ \tau_{\epsilon_2}\tau_{\epsilon_1} & \tau_{\epsilon_2}^2 & ... & \tau_{\epsilon_2}\tau_{\epsilon_K} \\ \vdots & \vdots & \vdots & \vdots \\ \tau_{\epsilon_K}\tau_{\epsilon_1} & \tau_{\epsilon_K}\tau_{\epsilon_2} & \cdots & \tau_{\epsilon_K}^2 \end{pmatrix}$$

$$= \begin{pmatrix} \tau_{\epsilon_1}-\tau_{\epsilon_1}^2\nu & -\tau_{\epsilon_1}\tau_{\epsilon_2}\nu & ... & -\tau_{\epsilon_1}\tau_{\epsilon_K}\nu \\ -\tau_{\epsilon_2}\tau_{\epsilon_1}\nu & \tau_{\epsilon_2}-\tau_{\epsilon_2}^2\nu & ... & -\tau_{\epsilon_2}\tau_{\epsilon_K}\nu \\ \vdots & \vdots & \vdots & \vdots \\ -\tau_{\epsilon_K}\tau_{\epsilon_1}\nu & -\tau_{\epsilon_K}\tau_{\epsilon_2}\nu & \cdots & \tau_{\epsilon_K}-\tau_{\epsilon_K}^2\nu \end{pmatrix}.$$

Hence,

$$\mathrm{E}(X \mid s_1, s_2, ..., s_K) = \mu_x + \begin{pmatrix} \tau_x^{-1} & \tau_x^{-1} & ... & \tau_x^{-1} \end{pmatrix}$$
$$\times \begin{pmatrix} \tau_{\epsilon_1}-\tau_{\epsilon_1}^2\nu & -\tau_{\epsilon_1}\tau_{\epsilon_2}\nu & ... & -\tau_{\epsilon_1}\tau_{\epsilon_K}\nu \\ -\tau_{\epsilon_2}\tau_{\epsilon_1}\nu & \tau_{\epsilon_2}-\tau_{\epsilon_2}^2\nu & ... & \tau_{\epsilon_2}\tau_{\epsilon_K}\nu \\ \vdots & \vdots & \vdots & \vdots \\ -\tau_{\epsilon_K}\tau_{\epsilon_1}\nu & -\tau_{\epsilon_K}\tau_{\epsilon_2}\nu & \cdots & \tau_{\epsilon_K}-\tau_{\epsilon_K}^2\nu \end{pmatrix} \begin{pmatrix} s_1-\mu_x \\ s_2-\mu_x \\ \vdots \\ s_K-\mu_x \end{pmatrix}.$$

Note that

$$\tau_x^{-1}\left(\tau_{\epsilon_1}-\tau_{\epsilon_1}^2\nu-\tau_{\epsilon_1}\tau_{\epsilon_2}\nu-...-\tau_{\epsilon_1}\tau_{\epsilon_K}\nu\right)$$
$$= \tau_x^{-1}\left[\tau_{\epsilon_1}-\tau_{\epsilon_1}\left(\frac{\tau_{\epsilon_1}}{\tau_x+\sum_{k=1}^{K}\tau_{\epsilon_k}}+\frac{\tau_{\epsilon_2}}{\tau_x+\sum_{k=1}^{K}\tau_{\epsilon_k}}+...+\frac{\tau_{\epsilon_K}}{\tau_x+\sum_{k=1}^{K}\tau_{\epsilon_k}}\right)\right]$$
$$= \tau_x^{-1}\left(\tau_{\epsilon_1}-\tau_{\epsilon_1}\frac{\sum_{k=1}^{K}\tau_{\epsilon_k}}{\tau_x+\sum_{k=1}^{K}\tau_{\epsilon_k}}\right)$$
$$= \frac{\tau_{\epsilon_1}}{\tau_x+\sum_{k=1}^{K}\tau_{\epsilon_k}}.$$

This eventually gives

$$\mathrm{E}(X \mid s_1, s_2, ..., s_K) = \mu_x + \begin{pmatrix} \frac{\tau_{\epsilon_1}}{\tau_x + \sum_{k=1}^K \tau_{\epsilon_k}} & \frac{\tau_{\epsilon_2}}{\tau_x + \sum_{k=1}^K \tau_{\epsilon_k}} & ... & \frac{\tau_{\epsilon_K}}{\tau_x + \sum_{k=1}^K \tau_{\epsilon_k}} \end{pmatrix}$$

$$\times \begin{pmatrix} s_1 - \mu_x & s_2 - \mu_x & \cdots & s_K - \mu_x \end{pmatrix}^T$$

$$= \mu_x + \frac{\tau_{\epsilon_1}}{\tau_x + \sum_{k=1}^K \tau_{\epsilon_k}}(s_1 - \mu_x) + \frac{\tau_{\epsilon_2}}{\tau_x + \sum_{k=1}^K \tau_{\epsilon_k}}(s_2 - \mu_x) + ...$$

$$+ \frac{\tau_{\epsilon_K}}{\tau_x + \sum_{k=1}^K \tau_{\epsilon_k}}(s_K - \mu_x)$$

$$= \mu_x + \frac{1}{\tau_x + \sum_{k=1}^K \tau_{\epsilon_k}} \sum_{k=1}^K \tau_{\epsilon_k}(s_k - \mu_x).$$

Furthermore,

$$\mathrm{Var}(X \mid s_1, s_2, ..., s_K) = \tau_x^{-1} - \begin{pmatrix} \tau_x^{-1} & \tau_x^{-1} & ... & \tau_x^{-1} \end{pmatrix}$$

$$\times \begin{pmatrix} \tau_x^{-1} + \tau_{\epsilon_1}^{-1} & \tau_x^{-1} & ... & \tau_x^{-1} \\ \tau_x^{-1} & \tau_x^{-1} + \tau_{\epsilon_2}^{-1} & ... & \tau_x^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ \tau_x^{-1} & \tau_x^{-1} & \cdots & \tau_x^{-1} + \tau_{\epsilon_K}^{-1} \end{pmatrix}^{-1} \begin{pmatrix} \tau_x^{-1} \\ \tau_x^{-1} \\ \vdots \\ \tau_x^{-1} \end{pmatrix}$$

$$= \tau_x^{-1} - \begin{pmatrix} \tau_x^{-1} & \tau_x^{-1} & ... & \tau_x^{-1} \end{pmatrix}$$

$$\times \begin{pmatrix} \tau_{\epsilon_1} - \tau_{\epsilon_1}^2 \nu & -\tau_{\epsilon_1}\tau_{\epsilon_2}\nu & ... & -\tau_{\epsilon_1}\tau_{\epsilon_K}\nu \\ -\tau_{\epsilon_2}\tau_{\epsilon_1}\nu & \tau_{\epsilon_2} - \tau_{\epsilon_2}^2 \nu & ... & \tau_{\epsilon_2}\tau_{\epsilon_K}\nu \\ \vdots & \vdots & \vdots & \vdots \\ -\tau_{\epsilon_K}\tau_{\epsilon_1}\nu & -\tau_{\epsilon_K}\tau_{\epsilon_2}\nu & \cdots & \tau_{\epsilon_K} - \tau_{\epsilon_K}^2 \nu \end{pmatrix} \begin{pmatrix} \tau_x^{-1} \\ \tau_x^{-1} \\ \vdots \\ \tau_x^{-1} \end{pmatrix}$$

$$= \tau_x^{-1} - \begin{pmatrix} \frac{\tau_{\epsilon_1}}{\tau_x + \sum_{k=1}^K \tau_{\epsilon_k}} & \frac{\tau_{\epsilon_2}}{\tau_x + \sum_{k=1}^K \tau_{\epsilon_k}} & ... & \frac{\tau_{\epsilon_K}}{\tau_x + \sum_{k=1}^K \tau_{\epsilon_k}} \end{pmatrix}$$

$$\times \begin{pmatrix} \tau_x^{-1} & \tau_x^{-1} & \cdots & \tau_x^{-1} \end{pmatrix}^T$$

$$= \tau_x^{-1} - \tau_x^{-1} \frac{\sum_{k=1}^{K} \tau_{\epsilon_k}}{\tau_x + \sum_{k=1}^{K} \tau_{\epsilon_k}}$$

$$= \frac{1}{\tau_x + \sum_{k=1}^{K} \tau_{\epsilon_k}}. \quad \square$$

B.3 Moment-Generating Functions

In general, the moment-generating function of an arbitrary random variable V is

$$M_V(t) = \mathrm{E}[\exp(tv)], \quad t \in \mathbb{R}.$$

By differentiating this function n-times with respect to t and evaluating it at $t = 0$, the n-th moment of the random variable V can be found. That is why the function is called the moment-generating function (see, e.g., Grimmett and Welsh, 2014, Chapter 7). Since the models in the main text assume an exponential utility function, we are confronted with moment-generating functions when considering expected utility. As this thesis deals with random variables that follow a normal or a noncentral chi-square distribution, we expound a derivation of their moment-generating functions below.

B.3.1 Normally Distributed Variable

Consider an arbitrary continuous random variable $W \sim N(\mu, \sigma^2)$. Its moment-generating function is

$$M_W(t) = \mathrm{E}[\exp(tw)] = \exp\left[t\left(\mu + \frac{t}{2}\sigma^2\right)\right]. \tag{B1}$$

Proof. In order to compute the expected value of a function of a random variable, it suffices to know the density function of the respective random variable. In this case, no further information about the density of the actual function is required. By the law of the unconscious statistician (see, e.g., Allen, 2006, Chapter 1),

$$\mathrm{E}[\exp(tw)] = \int_{-\infty}^{\infty} \exp(tw) f_W(w)\, dw.$$

Recalling the results of Appendix B.1.1, we get

$$\mathrm{E}[\exp(tw)] = \int_{-\infty}^{\infty} \exp(tw) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(w-\mu)^2}{2\,\sigma^2}\right] dw$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left[t\,w - \frac{(w-\mu)^2}{2\,\sigma^2}\right] dw$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(\frac{2tw\sigma^2 - w^2 - \mu^2 + 2w\mu}{2\,\sigma^2}\right) dw.$$

Now, we manipulate the above function in such a way that the integrand stands for the PDF of an arbitrary normal random variable W. Note that

$$\begin{aligned} \mathrm{E}[\exp(tw)] &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left[\frac{-w^2 + 2w(\mu + t\sigma^2) - \mu^2}{2\sigma^2}\right] \\ &\quad \times \exp\left(\frac{-2t\mu\sigma^2 - t^2\sigma^4 + 2t\mu\sigma^2 + t^2\sigma^4}{2\sigma^2}\right) dw. \end{aligned}$$

Further simplifications yield

$$\begin{aligned} \mathrm{E}[\exp(tw)] &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left[\frac{-w^2 + 2w\left(\mu + t\sigma^2\right) - \mu^2 - 2t\mu\sigma^2 - t^2\sigma^4}{2\sigma^2}\right] dw \\ &\quad \times \exp\left(\frac{2t\mu\sigma^2 + t^2\sigma^4}{2\sigma^2}\right) \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{[w - (\mu + t\sigma^2)]^2}{2\,\sigma^2}\right\} dw \exp\left(t\mu + \frac{t^2\sigma^2}{2}\right). \end{aligned}$$

In fact, the integrand stands for the PDF of the normal random variable W with mean $\mu + t\sigma^2$ and variance σ^2. Together with the integral sign, it represents the CDF of the random variable W evaluated at positive infinity. From Appendix B.1.2, we know that its value equals unity. Eventually,

$$\begin{aligned} \mathrm{E}[\exp(tw)] &= \exp\left(t\mu + \frac{t^2\sigma^2}{2}\right) \\ &= \exp\left[t\left(\mu + \frac{t}{2}\sigma^2\right)\right]. \quad \square \end{aligned}$$

Note that the analogous result holds for taking conditional expectations. Let U be a vector of jointly normal random variables. Then,

$$\mathrm{E}[\exp(tw)|u] = \int_{-\infty}^{\infty} \exp(tw) f_{W|U}(w, u)\, dw.$$

From Appendix B.2.1, we know that conditional on U, the random variable W is still normally distributed. This delivers

$$\mathrm{E}[\exp(tw)|u] = \int_{-\infty}^{\infty} \exp(tw) \frac{1}{\sqrt{2\pi\sigma^2_{w|u}}} \exp\left[-\frac{\left(w - \mu_{w|u}\right)^2}{2\,\sigma^2_{w|u}}\right] dw,$$

where $\mu_{w|u} \equiv \mathrm{E}(W \,|\, U)$ and $\sigma^2_{w|u} \equiv \mathrm{Var}(W \,|\, U)$. After carrying out the analogous calculations, we end up with

$$\mathrm{E}[\exp(tw)|u] = \exp\left[t\left(\mu_{w|u} + \frac{t}{2}\sigma^2_{w|u}\right)\right].$$

B.3.2 Noncentral Chi-Square Distributed Variable

Consider an arbitrary continuous random variable $Z \sim N(\mu, 1)$. Then, $Z^2 \sim \chi^2_{1,\lambda}$, i.e., the random variable Z^2 follows a noncentral chi-square distribution with one degree of freedom and noncentrality parameter $\lambda = \mu^2$. For $t < 0.5$, its moment-generating function is

$$M_Z(t) = \mathrm{E}[\exp(tz^2)] = \frac{1}{\sqrt{1-2t}} \exp\left(\frac{t\mu^2}{1-2t}\right). \tag{B2}$$

Proof. Applying the law of the unconscious statistician delivers

$$\begin{aligned}
\mathrm{E}\left[\exp(tz^2)\right] &= \int_{-\infty}^{\infty} \exp(tz^2) f_Z(z)\, dz \\
&= \int_{-\infty}^{\infty} \exp(tz^2) \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(z-\mu)^2}{2}\right] dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[tz^2 - \frac{(z-\mu)^2}{2}\right] dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{(1-2t)z^2 + \mu^2 - 2z\mu}{2}\right] dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{z^2 + \mu^2(1-2t)^{-1} - 2z\mu(1-2t)^{-1}}{2(1-2t)^{-1}}\right] dz.
\end{aligned}$$

Similar to the method applied in Appendix B.3.1, we manipulate the above equation in such a way that the integrand represents the PDF of an arbitrary normal random variable Z. Note that

$$\begin{aligned}
&\exp\left[-\frac{z^2 + \mu^2(1-2t)^{-1} - 2z\mu(1-2t)^{-1}}{2(1-2t)^{-1}}\right] \\
&= \exp\left[-\frac{z^2 + \mu^2(1-2t)^{-1} - 2z\mu(1-2t)^{-1} + \mu^2(1-2t)^{-2} - \mu^2(1-2t)^{-2}}{2(1-2t)^{-1}}\right] \\
&= \exp\left\{-\frac{[z - \mu(1-2t)^{-1}]^2}{2(1-2t)^{-1}}\right\} \exp\left[\frac{\mu^2(1-2t)^{-2} - \mu^2(1-2t)^{-1}}{2(1-2t)^{-1}}\right]
\end{aligned}$$

$$= \exp\left\{-\frac{[z-\mu(1-2t)^{-1}]^2}{2(1-2t)^{-1}}\right\}\exp\left(\frac{t\mu^2}{1-2t}\right).$$

Hence,

$$\begin{aligned}
\mathrm{E}[\exp(tz^2)] &= \exp\left(\frac{t\mu^2}{1-2t}\right)\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{[z-\mu(1-2t)^{-1}]^2}{2(1-2t)^{-1}}\right\}dz \\
&= \exp\left(\frac{t\mu^2}{1-2t}\right)\int_{-\infty}^{\infty}\frac{(1-2t)^{-1/2}}{\sqrt{2\pi}(1-2t)^{-1/2}}\exp\left\{-\frac{[z-\mu(1-2t)^{-1}]^2}{2(1-2t)^{-1}}\right\}dz \\
&= \frac{\exp\left(\frac{t\mu^2}{1-2t}\right)}{\sqrt{1-2t}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}(1-2t)^{-1/2}}\exp\left\{-\frac{[z-\mu(1-2t)^{-1}]^2}{2(1-2t)^{-1}}\right\}dz.
\end{aligned}$$

As a matter of fact, the integrand represents the PDF of the normal random variable Z with mean $\mu/(1-2t)$ and variance $(1-2t)^{-1}$. Thus, the value of the integral, which stands for the CDF of Z evaluated at positive infinity, is unity. Eventually,

$$\mathrm{E}[\exp(tz^2)] = \frac{1}{\sqrt{1-2t}}\exp\left(\frac{t\mu^2}{1-2t}\right). \quad \square$$

As before, the analogous result holds for taking expectations conditional on jointly normal random variables. Let U be a vector of jointly normal random variables and suppose that $Z|U \sim \mathrm{N}(\mu_{z|u}, 1)$, i.e., conditional on U, the random variable Z^2 follows a noncentral chi-square distribution. Then,

$$\mathrm{E}[\exp(tz^2)|u] = \frac{1}{\sqrt{1-2t}}\exp\left(\frac{t\mu_{z|u}^2}{1-2t}\right).$$

B.4 A Further Property of Normal Variables and CARA Utility

Consider two continuous, bivariate normal random variables $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$ with $\mathrm{Cov}(X,Y) \equiv \sigma_{xy}$. It holds that

$$\mathrm{E}\left[\exp\left(x-y^2\right)\right] = \frac{\exp\left[\mu_x + \frac{1}{2}\sigma_x^2 - \frac{(\mu_y+\sigma_{xy})^2}{1+2\sigma_y^2}\right]}{\sqrt{1+2\sigma_y^2}}. \tag{B3}$$

This formula can be found, e.g., in Demange and Laroque (1995, p. 252).

Proof. Analogous to the univariate case, it suffices to know the joint density function of the bivariate normal variables to calculate the expected value of the underlying function. According to the bivariate case of the law of the unconscious statistician,

$$\mathrm{E}\left[\exp\left(x-y^2\right)\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(x-y^2\right)f_{XY}(x,y)\,dx\,dy,$$

where

$$f_{XY}(x,y)=\frac{\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}-2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right\}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}$$

stands for the joint density function of X and Y, and $\rho=\sigma_{xy}/\sigma_x\sigma_y$ for the correlation between X and Y. Further computations deliver

$$\begin{aligned}\mathrm{E}\left[\exp\left(x-y^2\right)\right]=&\int_{-\infty}^{\infty}\exp\left[-y^2-\frac{(y-\mu_y)^2}{2(1-\rho^2)\sigma_y^2}\right]\\&\times\int_{-\infty}^{\infty}\frac{\exp\left[x-\frac{(x-\mu_x)^2}{2(1-\rho^2)\sigma_x^2}+\frac{2\rho(x-\mu_x)(y-\mu_y)}{2(1-\rho^2)\sigma_x\sigma_y}\right]}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}dx\,dy.\end{aligned}$$

In the next step, we focus on the exponential of the innermost integrand, which can be developed as follows:

$$\begin{aligned}&\exp\left[x-\frac{(x-\mu_x)^2}{2(1-\rho^2)\sigma_x^2}+\frac{2\rho(x-\mu_x)(y-\mu_y)}{2(1-\rho^2)\sigma_x\sigma_y}\right]\\
=&\exp\left\{\frac{x[2(1-\rho^2)\sigma_x^2\sigma_y]-\sigma_y(x-\mu_x)^2+2\rho\sigma_x(x-\mu_x)(y-\mu_y)}{2(1-\rho^2)\sigma_x^2\sigma_y}\right\}\\
=&\exp\left\{\frac{-\sigma_y\,x^2+2x[(1-\rho^2)\sigma_x^2\sigma_y+\mu_x\sigma_y+\rho\sigma_x(y-\mu_y)]-2\rho\sigma_x\mu_x(y-\mu_y)-\mu_x^2\sigma_y}{2(1-\rho^2)\sigma_x^2\sigma_y}\right\}\\
=&\exp\left\{\frac{-x^2+2x[(1-\rho^2)\sigma_x^2+\mu_x+\rho\sigma_x\sigma_y^{-1}(y-\mu_y)]}{2(1-\rho^2)\sigma_x^2}\right\}\\
&\times\exp\left[\frac{-2\rho\sigma_x\sigma_y^{-1}\mu_x(y-\mu_y)-\mu_x^2}{2(1-\rho^2)\sigma_x^2}\right]\\
=&\exp\left(-\frac{\left\{x-\left[(1-\rho^2)\sigma_x^2+\mu_x+\rho\sigma_x\sigma_y^{-1}(y-\mu_y)\right]\right\}^2}{2(1-\rho^2)\sigma_x^2}\right)\\
&\underbrace{\times\exp\left\{\frac{\left[(1-\rho^2)\sigma_x^2+\mu_x+\rho\sigma_x\sigma_y^{-1}(y-\mu_y)\right]^2-2\rho\sigma_x\sigma_y^{-1}\mu_x(y-\mu_y)-\mu_x^2}{2(1-\rho^2)\sigma_x^2}\right\}}_{\equiv\Phi(y)}.\end{aligned}$$

Hence,

$$\mathrm{E}\left[\exp\left(x-y^2\right)\right]=\int_{-\infty}^{\infty}\Phi(y)\exp\left[-y^2-\frac{(y-\mu_y)^2}{2(1-\rho^2)\sigma_y^2}\right]$$

$$\int_{-\infty}^{\infty}\frac{\exp\left\{-\frac{\left\{x-\left[(1-\rho^2)\sigma_x^2+\mu_x+\rho\sigma_x\sigma_y^{-1}(y-\mu_y)\right]\right\}^2}{2(1-\rho^2)\sigma_x^2}\right\}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}dx\,dy$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_y}\Phi(y)\exp\left[-y^2-\frac{(y-\mu_y)^2}{2(1-\rho^2)\sigma_y^2}\right]$$

$$\int_{-\infty}^{\infty}\frac{\exp\left\{-\frac{\left\{x-\left[(1-\rho^2)\sigma_x^2+\mu_x+\rho\sigma_x\sigma_y^{-1}(y-\mu_y)\right]\right\}^2}{2(1-\rho^2)\sigma_x^2}\right\}}{\sqrt{2\pi}\sigma_x\sqrt{1-\rho^2}}dx\,dy.$$

Note that the term under the innermost integral sign stands for the PDF of a normal random variable X. Thus, solving the innermost integral yields unity. We are left with

$$\mathrm{E}\left[\exp\left(x-y^2\right)\right]=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_y}\Phi(y)\exp\left[-y^2-\frac{(y-\mu_y)^2}{2(1-\rho^2)\sigma_y^2}\right]dy.$$

Further computations yield

$$\Phi(y)=\exp\left\{\frac{\left[(1-\rho^2)\sigma_x^2+\mu_x+\rho\sigma_x\sigma_y^{-1}(y-\mu_y)\right]^2-2\rho\sigma_x\sigma_y^{-1}\mu_x(y-\mu_y)-\mu_x^2}{2(1-\rho^2)\sigma_x^2}\right\}$$

$$=\exp\left[\frac{(1-\rho^2)^2\sigma_x^4+2(1-\rho^2)\sigma_x^2\mu_x}{2(1-\rho^2)\sigma_x^2}\right]$$

$$\times\exp\left[\frac{\rho^2\sigma_x^2\sigma_y^{-2}(y-\mu_y)^2+2\rho(1-\rho^2)\sigma_x^3\sigma_y^{-1}(y-\mu_y)}{2(1-\rho^2)\sigma_x^2}\right]$$

$$=\exp\left[\mu_x+\frac{1}{2}(1-\rho^2)\sigma_x^2\right]$$

$$\times\exp\left[\frac{\rho^2\sigma_y^{-2}(y-\mu_y)^2+2\rho(1-\rho^2)\sigma_x\sigma_y^{-1}(y-\mu_y)}{2(1-\rho^2)}\right].$$

This delivers

$$\mathrm{E}\left[\exp\left(x-y^2\right)\right]=\frac{\exp\left[\mu_x+\frac{1}{2}(1-\rho^2)\sigma_x^2\right]}{\sqrt{2\pi}\sigma_y}\int_{-\infty}^{\infty}\exp\left[-y^2-\frac{(y-\mu_y)^2}{2(1-\rho^2)\sigma_y^2}\right]$$

$$\times \exp\left[\frac{\rho^2\sigma_y^{-2}(y-\mu_y)^2 + 2\rho(1-\rho^2)\sigma_x\sigma_y^{-1}(y-\mu_y)}{2(1-\rho^2)}\right] dy.$$

The integrand becomes

$$\begin{aligned}
&\exp\left[-y^2 - \frac{(y-\mu_y)^2}{2(1-\rho^2)\sigma_y^2} + \frac{\rho^2\sigma_y^{-2}(y-\mu_y)^2 + 2\rho(1-\rho^2)\sigma_x\sigma_y^{-1}(y-\mu_y)}{2(1-\rho^2)}\right] \\
&= \exp\left[\frac{-2(1-\rho^2)\sigma_y^2 y^2 - (y-\mu_y)^2 + \rho^2(y-\mu_y)^2 + 2\rho(1-\rho^2)\sigma_x\sigma_y(y-\mu_y)}{2\sigma_y^2(1-\rho^2)}\right] \\
&= \exp\left\{\frac{y^2\left[-2(1-\rho^2)\sigma_y^2 - 1 + \rho^2\right] + 2y\left[\mu_2 - \rho^2\mu_2 + \rho(1-\rho^2)\sigma_x\sigma_y\right]}{2\sigma_y^2(1-\rho^2)}\right\} \\
&\quad \times \exp\left[\frac{-\mu_y^2 + \rho^2\mu_y^2 - 2\rho(1-\rho^2)\sigma_x\sigma_y\mu_2}{2\sigma_y^2(1-\rho^2)}\right] \\
&= \exp\left[\frac{-y^2\left(1+2\sigma_y^2\right) + 2y\left(\mu_2 + \rho\sigma_x\sigma_y\right)}{2\sigma_y^2}\right] \exp\left[-\frac{\mu_y\left(\mu_y + 2\rho\sigma_x\sigma_y\right)}{2\sigma_y^2}\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{E}\left[\exp\left(x - y^2\right)\right] =& \frac{\exp\left[\mu_x + \frac{1}{2}(1-\rho^2)\sigma_x^2 - \frac{\mu_y\left(\mu_y + 2\rho\sigma_x\sigma_y\right)}{2\sigma_y^2}\right]}{\sqrt{2\pi}\sigma_y} \\
&\times \int_{-\infty}^{\infty} \exp\left[\frac{-y^2\left(1+2\sigma_y^2\right) + 2y\left(\mu_y + \rho\sigma_x\sigma_y\right)}{2\sigma_y^2}\right] dy.
\end{aligned}$$

Further manipulations yield

$$\begin{aligned}
&\exp\left[\frac{-y^2\left(1+2\sigma_y^2\right) + 2y\left(\mu_y + \rho\sigma_x\sigma_y\right)}{2\sigma_y^2}\right] \\
&= \exp\left[\frac{-y^2 + 2y\dfrac{\mu_y + \rho\sigma_x\sigma_y}{1+2\sigma_y^2}}{\dfrac{2\sigma_y^2}{1+2\sigma_y^2}} - \frac{\dfrac{(\mu_y + \rho\sigma_x\sigma_y)^2}{(1+2\sigma_y^2)^2} - \dfrac{(\mu_y + \rho\sigma_x\sigma_y)^2}{(1+2\sigma_y^2)^2}}{\dfrac{2\sigma_y^2}{1+2\sigma_y^2}}\right] \\
&= \exp\left[-\frac{\left(y - \dfrac{\mu_y + \rho\sigma_x\sigma_y}{1+2\sigma_y^2}\right)^2}{\dfrac{2\sigma_y^2}{1+2\sigma_y^2}}\right] \exp\left[\frac{(\mu_y + \rho\sigma_x\sigma_y)^2}{2\sigma_y^2(1+2\sigma_y^2)}\right].
\end{aligned}$$

Hence,

$$\mathrm{E}\left[\exp\left(x-y^2\right)\right]=\frac{\exp\left[\mu_x+\frac{1}{2}(1-\rho^2)\sigma_x^2-\frac{\mu_y\left(\mu_y+2\rho\sigma_x\sigma_y\right)}{2\sigma_y^2}+\frac{(\mu_y+\rho\sigma_x\sigma_y)^2}{2\sigma_y^2(1+2\sigma_y^2)}\right]}{\sqrt{2\pi}\sigma_y}$$

$$\times\int_{-\infty}^{\infty}\exp\left[-\frac{\left(y-\frac{\mu_y+\rho\sigma_x\sigma_y}{1+2\sigma_y^2}\right)^2}{\frac{2\sigma_y^2}{1+2\sigma_y^2}}\right]dy$$

$$=\frac{\exp\left[\mu_x+\frac{1}{2}(1-\rho^2)\sigma_x^2-\frac{\mu_y\left(\mu_y+2\rho\sigma_x\sigma_y\right)}{2\sigma_y^2}+\frac{(\mu_y+\rho\sigma_x\sigma_y)^2}{2\sigma_y^2(1+2\sigma_y^2)}\right]}{\sqrt{1+2\sigma_y^2}}$$

$$\times\int_{-\infty}^{\infty}\frac{1}{\sqrt{\frac{2\pi\sigma_y^2}{1+2\sigma_y^2}}}\exp\left[-\frac{\left(y-\frac{\mu_y+\rho\sigma_x\sigma_y}{1+2\sigma_y^2}\right)^2}{\frac{2\sigma_y^2}{1+2\sigma_y^2}}\right]dy.$$

Again, the integrand stands for the PDF of a normal random variable. Consequently, the value of the integral equals unity. Finally, we obtain

$$\mathrm{E}\left[\exp\left(x-y^2\right)\right]=\frac{\exp\left[\mu_x+\frac{1}{2}(1-\rho^2)\sigma_x^2-\frac{\mu_y\left(\mu_y+2\rho\sigma_x\sigma_y\right)}{2\sigma_y^2}+\frac{(\mu_y+\rho\sigma_x\sigma_y)^2}{2\sigma_y^2(1+2\sigma_y^2)}\right]}{\sqrt{1+2\sigma_y^2}}$$

$$=\frac{\exp\left[\mu_x+\frac{1}{2}\sigma_x^2-\frac{\sigma_{xy}^2+\mu_y\left(\mu_y+2\sigma_{xy}\right)}{2\sigma_y^2}+\frac{(\mu_y+\sigma_{xy})^2}{2\sigma_y^2(1+2\sigma_y^2)}\right]}{\sqrt{1+2\sigma_y^2}}$$

$$=\frac{\exp\left[\mu_x+\frac{1}{2}\sigma_x^2-\frac{(\mu_y+\sigma_{xy})^2(1+2\sigma_y^2)}{2\sigma_y^2(1+2\sigma_y^2)}+\frac{(\mu_y+\sigma_{xy})^2}{2\sigma_y^2(1+2\sigma_y^2)}\right]}{\sqrt{1+2\sigma_y^2}}$$

$$=\frac{\exp\left[\mu_x+\frac{1}{2}\sigma_x^2-\frac{(\mu_y+\sigma_{xy})^2}{1+2\sigma_y^2}\right]}{\sqrt{1+2\sigma_y^2}}.\quad\square$$

As before, the analogous result holds for taking expectations conditional on jointly normally distributed variables by substituting the unconditional moments by the respective conditional equivalents.

Furthermore, note that (B3) nests (B1) from Appendix B.3.1. If we set $x = tw$

and $y = 0$ (and, thus, $\mu_y = \sigma_y^2 = \sigma_{xy} = 0$) in (B3), we end up with

$$\mathrm{E}\left[\exp\left(x\right)\right] = \exp\left(\mu_x + \frac{1}{2}\sigma_x^2\right)$$

$$\Leftrightarrow \ \mathrm{E}\left[\exp\left(tw\right)\right] = \exp\left(t\mu_w + \frac{t^2}{2}\sigma_w^2\right),$$

which is equal to (B1). In this case, the random variable X is just a linear transformation of the normal variable W. Thus, X is normally distributed too.

However, (B3) does not nest (B2) from Appendix B.3.2. Instead, (B3) with $x = 0$ can be seen as a special case of (B2) with $z = y/\sigma_y$ and $t = -\sigma_y^2$. That is, $\mathrm{E}\left[\exp\left(-y^2\right)\right]$ (resp., $\mathrm{E}\left[\exp\left(-y^2\right)\middle|\, u\right]$) can be calculated in two ways. First, (B3) with $x = 0$ (and, thus, $\mu_x = \sigma_x^2 = \sigma_{xy} = 0$) can be used. Second, the original term can be transformed into the moment-generating function of a noncentral chi-square distributed variable by setting $z = y/\sigma_y$ (resp., $z = y/\sigma_{y|u}$) and $t = -\sigma_y^2$ (resp., $t = -\sigma_{y|u}^2$) so that $\sigma_z^2 = 1$ (resp., $\sigma_{z|u}^2 = 1$) and $tz^2 = -y^2$. Then, (B2) can be used (as done in Appendix A).

Notes

1. Introduction

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2. Literature Review

1. An overview of the work of Kahneman and Tversky and of other related studies can be found, e.g., in Shefrin (2000, Chapter 2).

2. A different strand of the theoretical asset-pricing literature labels positive feedback traders as "chartists." The corresponding (non-microfounded) chartist-fundamentalist approach pioneered by Zeeman (1974) is, however, less relevant for this thesis. Hommes (2006) and Westerhoff (2009), e.g., provide an introduction to the field.

3. Bollen et al. (2011), Zhang et al. (2011), Sprenger et al. (2014), and Agrawal et al. (2018) are further examples that relate twitter sentiment to future prices and stock market returns.

4. GS 1980 state that their model is based on the work of Lucas (1972), Green (1973), Grossman (1975, 1976, 1978), and Kihlstrom and Mirman (1975). Nevertheless, due to its remarkable influence, the contribution of GS 1980 is considered to be the origin of the competitive noisy REE framework.

5. Laffont (1985) provides another early example of a welfare analysis.

6. A different strand of the literature avoids the need for random asset supply and noise traders by introducing risk-averse, rationally behaving hedgers (see, e.g., Medrano and Vives, 2004, and Bond and García, 2020). This way of modeling, however, would miss the basic spirit of this thesis.

7. It is worth noting that a second strand of the theoretical literature exists that explores the consequences of non-fundamental information, using the setup developed by Kyle (1985). This setup, however, is characterized by strategic trading and risk-neutral market participants, making it less relevant for this thesis. Important contributions include Madrigal (1996), Yu (1999), Foucault and Lescourret (2003), Bernhardt and Taub (2008), Cheynel and Levine (2012), Demarquette (2016), Yang and Zhu (2017), and Sadzik and Woolnough (2021).

8. In Diamond and Verrecchia (1981) and Verrecchia (1982), rational traders are also endowed with an uncertain amount of the risky asset. Contrary to Ganguli and Yang (2009), agents' endowments do not share a common component, preventing the existence of multiple equilibria in their setup.

3. Social Sentiment Investing and Price Efficiency

1. The story behind the KBC dates back to Keynes' (1936) classical work. In a fictive beauty contest, participants have to select the six individuals among a hundred photographs who they think are considered the most attractive by the other participants. Thus, rather than naively picking the

six they personally consider the most attractive, the participants should form expectations about which photographs the others will pick. Since the other participants also form expectations about which photographs the rest will pick, forecasting others' choices entails forming expectations about others' expectations (i.e., higher-order expectations).

2. See, e.g., Admati (1985), Judd (1985), and Uhlig (1996) for more detailed discussions on adopting the strong law of large numbers for a continuum of independent random variables.

3. The case of persistent noise trading in dynamic REE models is studied by Cespa and Vives (2012, 2015) and Avdis (2016). More details on these contributions can be found in Section 2.2.

4. Payment for Order Flow and Multidimensional Noise

1. One could also interpret each component of noise trader demand as the sum of a fixed number of single noise trader demands.

2. Any mass different from unity would leave all derived results unchanged.

3. This difference becomes even more visible when inspecting equations (11) and (12) on p. 1731 in GY 2015 and comparing them to the relevant equations in this model.

4. Imposing a restriction on the overall mass of fundamentally uninformed, rational traders would significantly complicate the derivation of an equilibrium at the information acquisition stage, however leaving all derived results qualitatively unchanged.

5. Note that in the special case $c_1 < \bar{c}$ and $c_2 = f(c_1)$, $\lambda_1^* > 0, \lambda_2^* = 0$ is the unique equilibrium (see also Proposition 4.5).

6. Note that in the special case $c_1 = c_2 = \bar{c}$, $\lambda_1^* = \lambda_2^* = 0$ is the unique equilibrium (see also Proposition 4.5).

7. Note that in the special case $c_1 > \bar{c}$ and $c_2 = g(c_1)$, $\lambda_1^* = 0, \lambda_2^* > 0$ is the unique equilibrium (see also Proposition 4.5).

8. The obtained result would not change qualitatively if we allowed for the excluded corner solutions. There would still exist an area near the origin in Figure 4.3 where an equilibrium at the information acquisition stage would fail to exist.

9. If one allows for a group of rational traders that observes two of the three noise shocks, the two equilibria again exhibit the classical properties and a large mass of noise-informed traders always leads to a market breakdown.

10. It is known that the adverse selection problem vanishes if the error terms in rational traders' private fundamental signals are correlated (see Manzano and Vives, 2011). However, Section 4.5 shows that adverse selection is significantly weakened if noise is three-dimensional and non-fundamental information is sufficiently dispersed.

11. We refrain from deriving an equilibrium at the information acquisition stage with endogenous values of λ_1, λ_2, and λ_3, since the complexity of the three-dimensional model prevents a full

analytical characterization of such an equilibrium. However, this is not problematic, as all relevant results on interactions in information acquisition can be gained without deriving an information acquisition equilibrium with exogenous cost parameters. Another example of analyzing interactions in information acquisition without deriving an information acquisition equilibrium beforehand can be found, e.g., in Manzano and Vives (2011, Section 4.2).

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