

A semantic approach to nonmonotonic reasoning: Inference operations and choice

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Abstract

This paper presents a uniform semantic treatment of nonmonotonic inference operations that allow for inferences from *infinite* sets of premisses. The semantics is formulated in terms of *selection functions* and is a generalisation of the *preferential semantics* of Shoham, Kraus et al., and Makinson. A selection function picks out from a given set of possible states (worlds, situations, models) a subset consisting of those states that are, in some sense, the most preferred ones. A proposition α is a *nonmonotonic consequence* of a set of propositions Γ iff α holds in all the most preferred Γ -states. In the literature on revealed preference theory, there are a number of well-known theorems concerning the representability of selection functions, satisfying certain properties, in terms of underlying preference relations. Such theorems are utilised here to give corresponding representation theorems for nonmonotonic inference operations. At the end of the paper, the connection between nonmonotonic inference and belief revision, in the sense of Alchourrón, Gärdenfors, and Makinson, is explored. In this connection, *infinitary belief revision operations*, that allow for the revision of a theory with a possibly infinite set of propositions, are introduced and characterised axiomatically. Several semantic representation theorems are proved for operations of this kind.

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1 | INTRODUCTION

In standard deductive logic, a proposition α is a logical consequence of a set of propositions Γ (in symbols, $\Gamma \vdash \alpha$) just in case α holds (or is true) in every possible state (situation, world) in which all the propositions in Γ hold. In other words, we have the following semantic characterisation of logical consequence:

$$\Gamma \vdash \alpha \text{ iff } \llbracket \Gamma \rrbracket \subseteq \llbracket \alpha \rrbracket,$$

where $\llbracket \alpha \rrbracket$ and $\llbracket \Gamma \rrbracket$ are the sets of all possible states in which, respectively, α and the set of all propositions in Γ hold. If $\Gamma \subseteq \Delta$, then, of course, $\llbracket \Delta \rrbracket \subseteq \llbracket \Gamma \rrbracket$. It follows, that standard deductive logic is monotonic, that is:

$$\text{if } \Gamma \vdash \alpha \text{ and } \Gamma \subseteq \Delta, \text{ then } \Delta \vdash \alpha. \quad (\textit{Monotonicity})$$

Notions of *plausible inference* or *default reasoning* do not in general satisfy monotonicity. From the information that x is a Quaker, we may plausibly infer that x is a pacifist. However, from the information that x is a Quaker and a Republican, it is not a plausible inference to conclude that x is a pacifist. Of course, the phenomenon of *nonmonotonicity* is familiar also from probabilistic contexts: from α being highly probable given β , we may not conclude in general that α is highly probable given $\beta \wedge \gamma$.

A common idea in the literature on nonmonotonic reasoning is the following: α is a nonmonotonic consequence of Γ (in symbols, $\Gamma \sim \alpha$) just in case α holds in all those Γ -states that are maximally plausible (from the viewpoint of some agent). Or more abstractly, $\Gamma \sim \alpha$ obtains if α holds in all the *best preferred* Γ -states, namely in those Γ -states to which no other Γ -state is strictly preferred (or better).

Formally we represent this idea by introducing a *selection function* S which, given a set X of possible states, picks out the set $S(X)$ of all the “best” elements in X . The relation \sim of nonmonotonic consequence (or plausible inference) is then defined in terms of S in the following way:

$$\Gamma \sim \alpha \text{ iff } S(\llbracket \Gamma \rrbracket) \subseteq \llbracket \alpha \rrbracket.$$

This definition will in general lead to \sim being nonmonotonic, since there is no guarantee that $S(\llbracket \Gamma \rrbracket) \subseteq \llbracket \alpha \rrbracket$ will imply that $S(\llbracket \Gamma \cup \Delta \rrbracket) \subseteq \llbracket \alpha \rrbracket$ (see Figure 1). Clearly, one of the best preferred $\Gamma \cup \Delta$ -states may fail to be a best preferred member of the more inclusive class of Γ -states. Therefore, it need not be the case that $S(\llbracket \Gamma \cup \Delta \rrbracket) \subseteq S(\llbracket \Gamma \rrbracket)$. Neither does it follow that $S(\llbracket \Gamma \cup \Delta \rrbracket) \subseteq \llbracket \alpha \rrbracket$.

Different choices of underlying language, different conceptions of possible states, and different formal requirements on the selection function will give rise to different nonmonotonic logics. In this paper we shall explore some of the possibilities that ensue. In particular, we are going to study correspondences between various conditions on the selection function S —many of which are well-known from the literature on preference and choice—and conditions on the inference relation \sim . In this connection it is often more natural to look at nonmonotonic inference as a Tarski-style inference *operation* C on sets of propositions rather than as an inference *relation* \sim . The two notions are simply related by the equation: $C(\Gamma) = \{\alpha : \Gamma \sim \alpha\}$.

The essential idea behind our semantic modelling of nonmonotonic inference goes back to McCarthy’s classical paper (McCarthy, 1980) on circumscription. McCarthy presents circumscription as a formalised rule of nonmonotonic inference (he calls it a *rule of conjecture*) which is used in conjunction with the rules of standard logic. There are many versions of

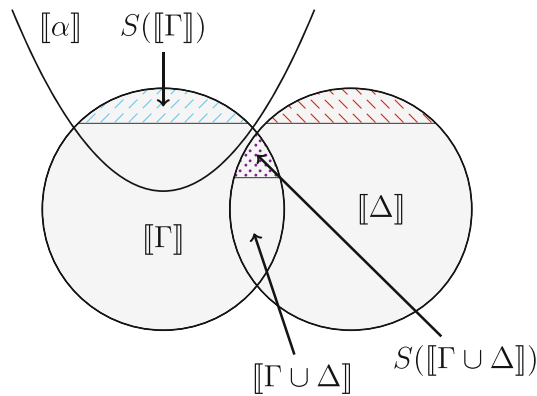


FIGURE 1 Selection functions lead to nonmonotonic consequences

circumscription, but the essential model-theoretic idea is the same: among all the models of a formula α , some are singled out as being *minimal*. Minimality here can mean various things, for instance: (i) *Domain Circumscription*: the minimal models of α are those that have no proper submodels that are also models of α , (ii) *Predicate Circumscription*: the extensions of some designated predicates are minimised, while the domain together with the extensions of all other predicates are kept fixed, (iii) *Parameterised Predicate Circumscription*: this case is like (ii), except that the extensions of some predicates (the parameters) are allowed to vary freely, and (iv) *Prioritised Circumscription*: there is a priority ordering of the predicates to be minimised: minimising a predicate with higher priority is always preferred to minimizing a predicate with lower priority.

Given some notion of a minimal model, one can define a corresponding notion of *minimal entailment*: α minimally entails β iff all minimal α -models are β -models.¹ In order to single out the minimal models of α , and thereby the sentences that are minimally entailed by α , a new sentence, called the *circumscription* of α , is associated with α . This new sentence has as its models just the minimal models of α . Thus, α minimally entails β just in case β is a logical consequence of the circumscription of α . It should be noted, however, that the circumscription of α is in general a sentence of second-order logic.

To make all this a little more concrete, let us look at a special case: predicate circumscription (McCarthy, 1980). Let $\alpha(P)$ be a sentence involving the predicate P (for simplicity, we let P be unary). The (*Predicate*) *Circumscription* of α with P is the second-order sentence $Circum(\alpha, P)$ defined as:²

$$\alpha(P) \wedge \forall Q(Q < P \rightarrow \neg\alpha(Q)),$$

where $Q < P$ is an abbreviation for the sentence $\forall x(Q(x) \rightarrow P(x)) \wedge \exists x(P(x) \wedge \neg Q(x))$. Next, we introduce a strict partial ordering \sqsubset_P on models of the language under consideration: $M \sqsubset_P N$ iff M and N have the same domain, all predicate symbols in the language besides

¹Cf. McCarthy (1980).

²In McCarthy (1980), the circumscription of a sentence is a first-order schema instead of a second-order sentence (cf. the difference between the induction schema of first-order Peano arithmetic and the induction axiom of second-order Peano arithmetic). In later work by John McCarthy and Vladimir Lifschitz (cf. Lifschitz, 1985, 1987; McCarthy, 1986), the circumscription of a sentence is defined as a single second-order sentence. In general, a first-order schema is weaker than the corresponding second-order sentence and is not strong enough to characterise the minimal models (example: the existence of *non-standard models* of first-order Peano arithmetic). In certain cases, though, the second-order formulation may be equivalent to a first-order sentence or a set of first-order sentences.

P have the same extension in M and N but the extension of P in M is a proper subset of its extension in N . We say that a model M is a P -minimal model of α , if $M \models \alpha$ (M is a model of α) and there is no model N such that $N \sqsubset_P M$ and $N \models \alpha$. Now, the models of $Circum(\alpha, P)$ are exactly the P -minimal models of α . Following McCarthy (1980), we say that α *minimally entails* β with respect to P (in symbols, $\alpha \vdash_P \beta$) if all P -minimal models of α are models of β . Thus, $\alpha \vdash_P \beta$ holds just in case β is a logical consequence of $Circum(\alpha, P)$. Since a P -minimal model of $\alpha \wedge \beta$ may not be a P -minimal model of α , minimal entailment with respect to P is nonmonotonic.

Example 1. Let α be the conjunction of the sentences:³

- (1) $bird(tweety)$;
- (2) $\forall x(bird(x) \wedge \neg ab_1(x) \rightarrow canfly(x))$;
- (3) $\forall x(penguin(x) \rightarrow bird(x))$;
- (4) $\forall x(penguin(x) \rightarrow ab_1(x))$;
- (5) $\forall x(penguin(x) \wedge \neg ab_2(x) \rightarrow \neg canfly(x))$.

saying that (1) Tweety is a bird, (2) birds that are not abnormal₁ can fly, (3) all penguins are birds, (4) penguins are abnormal₁, and (5) all penguins, except those that are abnormal₂ cannot fly. Applying predicate circumscription to the abnormality predicates ab_1 and ab_2 (that is, minimising the extension of these two predicates while keeping the extensions of the other predicates fixed) we infer from α that Tweety can fly. We cannot make this inference from $\alpha \wedge penguin(tweety)$. Since Tweety is a penguin, she is abnormal₁. Hence, (2) cannot be used to infer that she can fly. On the other hand, it follows by minimality that Tweety is not abnormal₂. Therefore, it follows by (5) that she cannot fly.

Example 2. Prioritised Circumscription. Let β be the conjunction of

- (1) $bird(tweety)$;
- (2) $\forall x(bird(x) \wedge \neg ab_1(x) \rightarrow canfly(x))$;
- (3) $\forall x(penguin(x) \rightarrow bird(x))$;
- (5) $\forall x(penguin(x) \wedge \neg ab_2(x) \rightarrow \neg canfly(x))$.

That is, β is like α , except for not containing the so-called cancellation of inheritance axiom (4). Using ordinary predicate circumscription, we can only infer from $\beta \wedge penguin(tweety)$ that one of the following cases obtains:

- (i) Tweety is an abnormal₁ bird that cannot fly;
- (ii) Tweety is an abnormal₂ penguin that can fly.

Nothing follows concerning Tweety's ability to fly. Intuitively, however, it seems reasonable to conjecture from $\beta \wedge penguin(tweety)$ that Tweety cannot fly. The cases (i) and (ii) are not symmetrical: the information that Tweety is a penguin is *more specific* than the information that she is a bird. It seems reasonable to give higher priority to minimising abnormality with respect to the more specific predicate. In the choice between minimising abnormality₁ and abnormality₂, we choose the latter. Hence, we conclude that Tweety is not abnormal₂. Then, it follows by (5) that she cannot fly.

Shoham (1987, 1988) generalised the concept of circumscription, or minimal entailment, to a more abstract notion: *preferential entailment*. Shoham's idea was to start from

³This is a shortened version of an example in McCarthy (1986).

any ordinary model-theoretic semantics for a formal language \mathcal{L} and add a new primitive notion to it: a strict partial ordering \sqsubset of all the models of \mathcal{L} . Intuitively, $M \sqsubset N$ means that the model M is *preferred over* the model N . Then, M is defined to be a *preferred model* of α iff (i) $M \models \alpha$, and (ii) there is no model N such that $N \models \alpha$ and $N \sqsubset M$. Finally, α is said to *preferentially entail* β (in symbols, $\alpha \sim_{\sqsubset} \beta$) just in case every preferred model of α is a model of β .

Shoham (1988) emphasises three ways in which his own approach generalises that of McCarthy's: (i) Preferential entailment can be defined relative to any logic having a model-theoretic semantics, not just to standard first-order logic—starting, for instance, with a modal logic and a preference relation over its Kripke-models, one can define the corresponding nonmonotonic modal logic; (ii) a notion of preferential entailment can be defined in terms of any partial ordering of models—that is, one is not limited to those orderings that correspond to circumscription axioms; and (iii) there is a shift of emphasis from syntax—circumscription axioms—to semantics—partial orderings of models.

In the work of Kraus et al. (1990), Shoham's approach is generalised further: a new primitive is introduced into the semantics—the notion of a *state*. Each state is labelled by a *set* of models of the underlying nonmonotonic logic and the states, not the models, are ordered by a binary relation \sqsubset . In general, it is not assumed that \sqsubset satisfies any of the usual properties like irreflexivity or transitivity. A formula α *holds* in a state u (u is an α -state) iff α is true at every model that is labelled by the state. A state u is a *preferred α -state* iff (i) u is an α -state and there is no α -state v such that $v \sqsubset u$. α *preferentially entails* β , in symbols, $\alpha \sim \beta$, if all preferred α -states are β -states. The main objective of Kraus et al. (1990) is to study nonmonotonic inference relations \sim both in terms of abstract proof-theoretic properties and semantically in terms of preferential models. Several important classes of inference relations are characterised semantically by means of representation theorems.

The study of abstract non-monotonic inference relations was initiated by Gabbay (1985) who took \sim to be a relation between a *finite* set Γ of premises and a single conclusion α . Gabbay (1985) defined a *nonmonotonic logic* as a relation of the described sort satisfying the following conditions:

- | | |
|--|-------------------------------------|
| if $\alpha \in \Gamma$, then $\Gamma \sim \alpha$; | <i>(Reflexivity)</i> |
| if $\Gamma \sim \alpha$ and $\Gamma, \alpha \sim \beta$, then $\Gamma \sim \beta$; | <i>(Finitary Cut)</i> |
| if $\Gamma \sim \alpha$ and $\Gamma \sim \beta$, then $\Gamma, \alpha \sim \beta$. | <i>(Finitary Cautious Monotony)</i> |

He argued that these requirements should be satisfied by any reasonable inference relation. As we have seen, Shoham (1987, 1988) and Kraus et al. (1990) define \sim as a relation taking only single propositions as premises. In the presence of conjunction in the object language, this is essentially equivalent to allowing finite sets of propositions as premises. A more general treatment is proposed in Makinson (1989), where \sim is allowed to take *infinite* sets of premises. This generalisation makes it possible for Makinson to redefine nonmonotonic consequence as a Tarski-style operation C on arbitrary sets of sentences. Generalising Gabbay's conditions to the infinitary case and expressing them in terms of C rather than \sim , Makinson (1989) obtains the following conditions:

- | | |
|---|----------------------------|
| $\Gamma \subseteq C(\Gamma)$; | <i>(Inclusion)</i> |
| $\Gamma \subseteq \Delta \subseteq C(\Gamma)$ implies $C(\Delta) \subseteq C(\Gamma)$; | <i>(Infinitary Cut)</i> |
| $\Gamma \subseteq \Delta \subseteq C(\Gamma)$ implies $C(\Gamma) \subseteq C(\Delta)$. | <i>(Cautious Monotony)</i> |

An operation on sets of sentences satisfying these conditions is called by Makinson a *cumulative inference operation*. Makinson (1994) is a comprehensive survey—from an abstract logical point of view—of systems of nonmonotonic logic: its focus is on properties of the inference relations (or operations) that are associated with the various systems.

In the present paper, we follow Makinson—and differ from Kraus et al.—in viewing nonmonotonic consequence as an operation C on arbitrary sets of sentences (or equivalently, as a relation $\Gamma \sim \alpha$, where Γ is allowed to be infinite). In addition, we modify the preferential semantics of Shoham and Kraus et al. by defining C , in the way previously described, in terms of a *selection function* S on sets of states rather than in terms of a *preference relation* on states. This treatment is more general, since a given selection function may not be definable in terms of any preference relation.

Utilising various well-known results from preference theory on the rationalisability of a selection function by an underlying preference ordering (cf., Moulin, 1985), we are able to prove a series of *representation theorems* for nonmonotonic inference. The general strategy in proving these results is the following: First, it is shown that any inference operation C that satisfies some set X of conditions may be defined in terms of a selection function S on sets of states satisfying a corresponding set of conditions X^* . Next, it is shown that if S satisfies the conditions X^* , then S is based on a preference relation P between states (read: xPy as state x is preferred over state y) satisfying some suitable conditions like asymmetry, transitivity, etc. Finally, the two steps are combined to yield a representation theorem for the inference operation C in terms of the preference relation P . The connection between C and P is given by:

$$\alpha \in C(\Gamma) \text{ iff } \forall x[(x \in \llbracket \Gamma \rrbracket \wedge \forall y(y \in \llbracket \Gamma \rrbracket \rightarrow \neg yPx) \rightarrow x \in \llbracket \alpha \rrbracket],$$

that is, α is a nonmonotonic consequence of Γ iff every P -maximal member of $\llbracket \Gamma \rrbracket$ is also a member of $\llbracket \alpha \rrbracket$.

At the end of the paper, we shall also briefly consider *dyadic inference operations* C , where $C_{\Delta}(\Gamma)$ is the set of all nonmonotonic consequences of the set of *premises* Γ relative to the *background assumptions* Δ . Dyadic inference operations may be defined from *dyadic selection functions* on sets of states:⁴

$$\alpha \in C_{\Delta}(\Gamma) \text{ iff } S(\llbracket \Delta \rrbracket, \llbracket \Gamma \rrbracket) \subseteq \llbracket \alpha \rrbracket.$$

The notion of a dyadic nonmonotonic inference operation is, of course, closely related to Gärdenfors' concept of *theory revision*. If K_{α}^* is the revision of a theory K with the proposition α , then we have the following natural connection:

$$\beta \in K_{\alpha}^* \text{ iff } \beta \in C_K(\{\alpha\}).$$

or more briefly:

$$K_{\alpha}^* = C_K(\{\alpha\}).$$

That is, the revision of K with α is identified with the theory consisting of all the nonmonotonic consequences of α relative to the background theory K .⁵ Conversely, a dyadic nonmonotonic inference relation may be viewed as a generalisation of ordinary theory revision: $C_{\Delta}(\Gamma)$ may be thought of as the result of revising Δ with the *set* Γ .

⁴Binary selection functions were studied by Kanger (2001). However Kanger's interpretation of $S(V, X)$, where X, V are subsets of some grand domain U , differs from the one employed here. Kanger took $S(V, X)$ to be "the set of those alternatives of $V \cap X$ which, compared with alternatives of V , are regarded as not being worse than any alternative of $V \cap X$ " (Kanger, 2001, p. 216). Here, on the other hand, $S(V, X)$ is interpreted as the set of all those alternatives of X which are not farther removed from the set V than any alternatives in X . Thus, we think of the elements of V as the "ideal" alternatives; and the elements of $S(V, X)$ are the elements of X that are as close to being ideal as possible.

⁵This method of translating back and forth between theories of belief revision and nonmonotonic inference (with single propositions as premises) was suggested by Makinson and Gärdenfors (1991).

2 | DEDUCTIVE LOGICS

This section consists essentially of a review of selected, but well-known, material about consequence relations and consequence operations, some of it going back to the work of Tarski in the 1920s and 1930s. The concepts introduced here are basic to the development of nonmonotonic logic in the rest of the paper.

We assume that a fixed object language \mathcal{L} is given. The details of \mathcal{L} are left open, except that we assume \mathcal{L} to contain the standard connectives: \perp (falsity), \rightarrow (the material conditional), \wedge (conjunction) and \vee (disjunction). Hence, the set Φ of *sentences* of \mathcal{L} is closed under the rules: (i) $\perp \in \Phi$, and (ii) if $\alpha, \beta \in \Phi$, then $(\alpha \rightarrow \beta), (\alpha \wedge \beta), (\alpha \vee \beta) \in \Phi$. $\neg\alpha$ is taken as a metalinguistic abbreviation of $(\alpha \rightarrow \perp)$.

If Γ is a set of sentences in \mathcal{L} and α is a sentence in \mathcal{L} , then we write $\Gamma \vdash_0 \alpha$ just in case α is a *tautological consequence* of Γ (that is, if α follows from Γ in classical propositional logic). We also write $Cn_0(\Gamma) = \{\alpha : \Gamma \vdash_0 \alpha\}$, that is, $Cn_0(\Gamma)$ is the *closure* of Γ under tautological consequence.

By a *consequence relation* we shall understand a binary relation \vdash which takes sets of sentences (in \mathcal{L}) as its first argument and single sentences (in \mathcal{L}) as its second and which satisfies the following conditions:

- (\vdash 1) if $\alpha \in \Gamma$, then $\Gamma \vdash \alpha$; (*Reflexivity*)
- (\vdash 2) if $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \alpha$; (*Monotonicity*)
- (\vdash 3) if $\Gamma \cup \Delta \vdash \beta$ and for each $\alpha \in \Delta$, $\Gamma \vdash \alpha$, then $\Gamma \vdash \beta$. (*Cut*)

Here, Γ and Δ are any sets of sentences and α, β are any sentences.

By a *deductive logic* L we shall understand a finitary consequence relation, that is, a consequence relation \vdash_L that satisfies:

- (\vdash 4) if $\Gamma \vdash_L \alpha$, then for some *finite* $\Delta \subseteq \Gamma$, $\Delta \vdash_L \alpha$. (*Finiteness*)

We say that a deductive logic L is $\{\wedge, \vee\}$ -*normal* if it satisfies the standard natural deduction rules for conjunction and disjunction, that is,

- (\wedge I) $\Gamma, \alpha, \beta \vdash_L \alpha \wedge \beta$;
- (\wedge E) $\Gamma, \alpha \wedge \beta \vdash_L \alpha$; and $\Gamma, \alpha \wedge \beta \vdash_L \beta$;
- (\vee I) $\Gamma, \alpha \vdash_L \alpha \vee \beta$; and $\Gamma, \beta \vdash_L \alpha \vee \beta$;
- (\vee E) if $\Gamma, \alpha \vdash_L \gamma$ and $\Gamma, \beta \vdash_L \gamma$, then $\Gamma, \alpha \vee \beta \vdash_L \gamma$.

By a *classical logic* we understand a deductive logic that satisfies the following two conditions:

- (\vdash 5) if $\Gamma \vdash_0 \alpha$, then $\Gamma \vdash_L \alpha$; (*Supraclassicality*)
- (\vdash 6) if $\Gamma, \alpha \vdash_L \beta$, then $\Gamma \vdash_L \alpha \rightarrow \beta$. (*Deduction Theorem*)

That is, a classical logic is a deductive logic which extends the classical propositional calculus and satisfies the deduction theorem. Every classical logic is, of course, $\{\wedge, \vee\}$ -*normal*.

A deductive logic L can equivalently be presented as a *finitary consequence operation* Cn_L , that is, an operation that takes sets of sentences in \mathcal{L} into sets of sentences in \mathcal{L} and satisfies the following conditions:

- (Cn1) $\Gamma \subseteq Cn_L(\Gamma)$; (*Inclusion*)
- (Cn2) if $\Gamma \subseteq \Delta$, then $Cn_L(\Gamma) \subseteq Cn_L(\Delta)$; (*Monotonicity*)

- (Cn3) $Cn_L(Cn_L(\Gamma)) \subseteq Cn_L(\Gamma)$; (Iteration)
 (Cn4) $Cn_L(\Gamma) \subseteq \cup \{Cn_L(\Delta) : \Delta \subseteq \Gamma \text{ and } \Delta \text{ is finite}\}$. (Finiteness)

In the presence of (Cn1) and (Cn2), (Cn3) is equivalent to the cut rule:

- (Cn3') If $\Delta \subseteq Cn_L(\Gamma)$, then $Cn_L(\Gamma \cup \Delta) \subseteq Cn_L(\Gamma)$. (Cut)

Lemma 2.1. *If Cn is a consequence operation, that is, satisfies (Cn1) – (Cn3), then it also satisfies:*

- (i) $\Gamma \subseteq Cn(\Delta)$ iff $Cn(\Gamma) \subseteq Cn(\Delta)$;
 (ii) $Cn(\Gamma \cup \Delta) = Cn(\Gamma \cup Cn(\Delta)) = Cn(Cn(\Gamma) \cup Cn(\Delta))$.

This lemma, like several of the theorems and lemmas below, is proved in the Appendix.

Of course, L is a classical logic if, in addition to (Cn1) – (Cn4), it satisfies the following two conditions:

- (Cn5) $Cn_0(\Gamma) \subseteq Cn_L(\Gamma)$; (Supraclassicality)
 (Cn6) If $\beta \in Cn_L(\Gamma \cup \{\alpha\})$, then $\alpha \rightarrow \beta \in Cn_L(\Gamma)$. (Deduction Theorem)

The two presentations of a deductive logic L are related by the following conditions:

$$Cn_L(\Gamma) = \{\alpha : \Gamma \vdash_L \alpha\};$$

and

$$\Gamma \vdash_L \alpha \text{ iff } \alpha \in Cn_L(\Gamma).$$

If $\alpha \in Cn_L(\Gamma)$, we say that α is an L -consequence of Γ . We say that α is an L -theorem, if $\alpha \in Cn_L(\emptyset)$.

Lemma 2.2. *If L is a classical logic, then it satisfies the following conditions:*

- (\vdash 7) for all α , $\perp \vdash_L \alpha$; (Falsity)
 (\vdash 8) if $\Gamma \vdash_L \alpha \rightarrow \beta$ and $\Gamma \vdash_L \alpha$, then $\Gamma \vdash_L \beta$; (Modus Ponens)
 (\vdash 9) if Γ , $\alpha \rightarrow \perp \vdash_L \perp$, then $\Gamma \vdash_L \alpha$. (Reductio Ad Absurdum)

We omit the straightforward proof of Lemma 2.2.

Let L be a deductive logic. L is (absolutely) *inconsistent* if $\emptyset \vdash_L \perp$ and *consistent*, otherwise. A set Γ of sentences is said to be *L -inconsistent* if $\Gamma \vdash_L \perp$. Γ is *L -consistent* if it is not L -inconsistent. A sentence α is said to be *L -consistent* if $\{\alpha\}$ is L -consistent. Γ is an *L -theory* iff $\Gamma = Cn_L(\Gamma)$. A set Γ is *L -maximal* iff Γ is L -consistent and for every Δ , if $\Gamma \subseteq \Delta$ and Δ is L -consistent, then $\Gamma = \Delta$.

In view of the next lemma, we may speak of L -maximal sets as *L -maximal theories*. Observe the use of Iteration (i.e., Cut) in the proof of the lemma.

Lemma 2.3. *Let L be a deductive logic. Then every L -maximal set is an L -theory.*

The proof of the following lemma uses Inclusion, Cut, Monotonicity, Finiteness and the Axiom of Choice in the form of Zorn's Lemma.

Lemma 2.4. (Lindenbaum's Lemma) *Let L be a deductive logic.*

- (a) *Every L -consistent set is included in an L -maximal theory.*
- (b) *If $\alpha \notin \text{Cn}_L(\Gamma)$, then there exists an L -maximal theory m such that $\text{Cn}_L(\Gamma) \subseteq m$ and $\alpha \notin m$.*

If L is a deductive logic, then we write $\mathcal{M}_L, \mathcal{T}_L$ for the set of all L -maximal theories and the set of all L -theories, respectively. m, m', m'', \dots are variables ranging over L -maximal theories and G, H, K, T, T', \dots range over L -theories. We also introduce the following notation:

$$\begin{aligned} \text{for any } \Gamma \subseteq \Phi, |\Gamma|_L &= \{m \in \mathcal{M}_L : \Gamma \subseteq m\}; \\ \text{for } \alpha \in \Phi, |\alpha|_L &= \{|\alpha|\}_L = \{m \in \mathcal{M}_L : \alpha \in m\}. \end{aligned}$$

In what follows, we shall often suppress the subscript L in contexts where the logic is assumed to be fixed.

Lemma 2.5. *Let L be a deductive logic. Then, $\text{Cn}_L(\Gamma) = \bigcap(|\Gamma|_L)$. That is, α is an L -consequence of Γ iff α belongs to every L -maximal extension of Γ .*

Proof. (\Rightarrow) Suppose that $\alpha \in \text{Cn}_L(\Gamma)$ and that m is an L -maximal set such that $\Gamma \subseteq m$. It follows by Monotonicity that $\alpha \in \text{Cn}_L(m)$. Since $m = \text{Cn}_L(m)$ (Lemma 2.1), $\alpha \in m$.

(\Leftarrow) Suppose that $\alpha \notin \text{Cn}_L(\Gamma)$. By Lemma 2.4, there exists an L -maximal theory m such that $\text{Cn}_L(\Gamma) \subseteq m$ and $\alpha \notin m$. □

Lemma 2.6. *Let L be a $\{\wedge, \vee\}$ -normal deductive logic. Then, every L -maximal set m satisfies the conditions:*

- (i) $\alpha \wedge \beta \in m$ iff $\alpha \in m$ and $\beta \in m$;
- (ii) $\alpha \vee \beta \in m$ iff $\alpha \in m$ or $\beta \in m$.

Lemma 2.7. *Let L be a classical logic. Then,*

- (a) Γ is an L -theory iff
 - (i) $\text{Cn}_L(\emptyset) \subseteq \Gamma$ (i.e., all L -theorems are in Γ);
 - (ii) if $\alpha \in \Gamma$ and $\alpha \rightarrow \beta \in \Gamma$, then $\beta \in \Gamma$ (i.e., Γ is closed under modus ponens).
- (b) Γ is an L -maximal set iff it satisfies the conditions:
 - (i) $\text{Cn}_L(\emptyset) \subseteq \Gamma$;
 - (ii) $\perp \notin \Gamma$;
 - (iii) $\alpha \rightarrow \beta \notin \Gamma$ iff $\alpha \in \Gamma$ and $\beta \notin \Gamma$.

3 | NONMONOTONIC INFERENCE

We assume that a fixed consistent deductive logic L is given. We are next going to introduce the notion of an *inference relation* based on the underlying deductive logic L . We shall assume that all relations of nonmonotonic inference that we are going to study are inference relations in the sense defined below. In addition, we introduce the notion of an *inference operation* which is just a notational variant of that of an inference relation.

Definition 3.1. (a) By an inference relation based on L we understand a relation $\vdash \subseteq \wp(\Phi) \times \Phi$ satisfying the following conditions for all sets Γ and Δ of sentences and sentences α, β :⁶

- (\vdash 1) if $\alpha \in \Gamma$, then $\Gamma \vdash \alpha$; (*Reflexivity*)
- (\vdash 2) if $\Gamma \vdash \alpha$, for all $\alpha \in \Delta$, and $\Delta \vdash_L \beta$, then $\Gamma \vdash \beta$; (*Closure*)
- (\vdash 3) if for all α , $\Gamma \vdash_L \alpha$ iff $\Delta \vdash_L \alpha$, then $\Gamma \vdash \beta$ iff $\Delta \vdash \beta$. (*Congruence*)

According to (\vdash 1), an element of Γ is a nonmonotonic consequence of Γ . (\vdash 2) says that if β is an L -consequence of a set of nonmonotonic consequences of Γ , then β is itself a nonmonotonic consequence of Γ . In other words, the set of nonmonotonic consequences of Γ is closed under L -consequence. According to (\vdash 3), L -equivalent sets of sentences have the same nonmonotonic consequences.

(b) An inference operation based on L is an operation $C : \wp(\Phi) \rightarrow \wp(\Phi)$ satisfying the following conditions:

- (C1) $\Gamma \subseteq C(\Gamma)$; (*Inclusion*)
- (C2) $Cn_L(C(\Gamma)) \subseteq C(\Gamma)$; (*Closure*)
- (C3) if $Cn_L(\Gamma) = Cn_L(\Delta)$, then $C(\Gamma) = C(\Delta)$. (*Congruence*)

Of course, there is a one-to-one correspondence between inference relations based on L and inference operations based on L . That is, we define the inference operation corresponding to \vdash by:

$$C(\Gamma) = \{\alpha : \Gamma \vdash \alpha\}.$$

Conversely, given C , we define \vdash by:

$$\Gamma \vdash \alpha \text{ iff } \alpha \in C(\Gamma).$$

Lemma 3.2. *If C is an inference operation based on L , then for all Γ :*

$$C(Cn_L(\Gamma)) = Cn_L(C(\Gamma)) = C(\Gamma).$$

Proof. Since, $Cn_L(\Gamma) = Cn_L(Cn_L(\Gamma))$, we have by (C3), $C(\Gamma) = C(Cn_L(\Gamma))$. But, $C(\Gamma) = Cn_L(C(\Gamma))$, by (Cn1) and (C2). □

Lemma 3.3. *If C is an inference operation based on L , then C satisfies the condition:*

$$Cn_L(\Gamma) \subseteq C(\Gamma).$$

In other words,

$$\text{if } \Gamma \vdash_L \alpha, \text{ then } \Gamma \vdash \alpha.$$

Proof. $\Gamma \subseteq C(\Gamma)$, by (C1). Hence, $Cn_L(\Gamma) \subseteq Cn_L(C(\Gamma))$, by (Cn2). But $Cn_L(C(\Gamma)) = C(\Gamma)$. It follows that $Cn_L(\Gamma) \subseteq C(\Gamma)$.

⁶For any set A , we let $\wp(A)$ be the power set of A .

Lemma 3.4. *Suppose that L is a classical logic. Then, \vdash is an inference relation based on L iff it satisfies the following conditions:*

- (\vdash 1) *if $\alpha \in \Gamma$, then $\Gamma \vdash \alpha$;*
- (\vdash 3) *if $Cn_L(\Gamma) = Cn_L(\Delta)$, then $\Gamma \vdash \alpha$ iff $\Delta \vdash \alpha$;*
- (\vdash 4) *if $Cn_L(\{\alpha\}) = Cn_L(\{\beta\})$, then $\Gamma \vdash \alpha$ iff $\Gamma \vdash \beta$;*
- (\vdash 5) *if $\Gamma \vdash \alpha \wedge \beta$, then $\Gamma \vdash \alpha$ and $\Gamma \vdash \beta$;*
- (\vdash 6) *if $\Gamma \vdash \alpha$ and $\Gamma \vdash \beta$, then $\Gamma \vdash \alpha \wedge \beta$;*
- (\vdash 7) $\Gamma \vdash \top$.

It is easy to verify that conditions (\vdash 4) and (\vdash 5) may be replaced in Lemma 3.4 by the single condition:⁷

- (\vdash 8) If $\Gamma \vdash \alpha$ and $\alpha \vdash_L \beta$, then $\Gamma \vdash \beta$. (*Right Weakening*)

In the next definition, we introduce the notion of an L -maximal theory being Γ -optimal with respect to an inference relation \vdash . The L -maximal theories may be thought of as (descriptions of) those *possible worlds* that are allowed by the underlying logic L . We may think of $\Gamma \vdash \alpha$ as expressing a (conditional) disposition on the part of an agent to *expect* α to be true, if she were to be given Γ as her total new information. The set $C(\Gamma) = \{\alpha : \Gamma \vdash \alpha\}$, then, consists of all the agents Γ -expectations.⁸ A possible world is Γ -optimal if all the Γ -expectations of the agent are true in it. In other words, after having received the total information Γ , the agent would not be *surprised* at all if any of the Γ -optimal worlds turned out to be the actual one.

Definition 3.5. Let L be a deductive logic and \vdash an inference relation based on L . Let Γ be any set of sentences and m any L -maximal theory. We say that m is Γ -optimal (with respect to \vdash) if for all α , if $\Gamma \vdash \alpha$, then $\alpha \in m$. In other words, m is Γ -optimal iff $C(\Gamma) \subseteq m$.

Lemma 3.6. *Let L be a deductive logic and \vdash an inference relation based on L . Then, $\Gamma \vdash \alpha$ iff for every Γ -optimal m , $\alpha \in m$.*

Proof. (\Rightarrow) This direction follows immediately from the definition of Γ -optimality.

(\Leftarrow) Suppose that $\alpha \notin C(\Gamma)$. By Lemma 3.2, $C(\Gamma) = Cn_L(C(\Gamma))$. Hence, $\alpha \notin Cn_L(C(\Gamma))$. Then, by Lemma 2.4, there exists an L -maximal theory m such that $Cn_L(C(\Gamma)) \subseteq m$ and $\alpha \notin m$. Hence, $C(\Gamma) \subseteq m$ and $\alpha \notin m$. \square

4 | SEMANTICS: MODELS USING SET-VALUED SELECTION FUNCTIONS

In the following we let L be a deductive logic which we assume to be $\{\wedge, \vee\}$ -normal. The notion of a model based on L will be introduced in two steps. First, we define the notion of structure based on L . After having defined the requisite concepts, a model will be defined as a structure of a special kind.

⁷Cf. Gärdenfors and Makinson (1994).

⁸This way of talking about an agent's expectations is inspired by Gärdenfors and Makinson (1994).

Definition 4.1. A *structure* based on L is a 4-tuple $\mathcal{M} = \langle U, \mathcal{V}, \mathfrak{l}, S \rangle$, where

- (i) U is a non-empty set, the elements of which are called *states* (these might be thought of as representing the possible belief states of an agent). We use the lower case letters x, y, z, u as variables ranging over U . The letters X, Y, Z will be variables ranging over $\wp(U)$
- (ii) \mathcal{V} is a non-empty family of subsets of U .
- (iii) \mathfrak{l} (the labeling function) is a function that assigns to every state $u \in U$ a non-empty set $\mathfrak{l}(u)$ of L -maximal theories. We may think of the members of $\mathfrak{l}(u)$ as representing those possible worlds that are compatible with the agent's beliefs in state u (the agent's doxastically possible worlds in state u).
- (iv) S is a function from \mathcal{V} to \mathcal{V} such that for every $X \in \mathcal{V}$, $S(X) \subseteq X$. Such a function we call a selection function on \mathcal{V} .

Let $\mathcal{M} = \langle U, \mathcal{V}, \mathfrak{l}, S \rangle$ be a structure based on L . We say that a sentence α *holds* (or *is accepted*) in the state $u \in U$ (relative to \mathcal{M}) and write $\mathcal{M} \Vdash_u \alpha$ iff for every $m \in \mathfrak{l}(u)$, $\alpha \in m$. That is, $\mathcal{M} \Vdash_u \alpha$ obtains just in case $\mathfrak{l}(u) \subseteq |\alpha|_L$. Intuitively, a sentence α is accepted in a state u just in case α is true in all possible worlds that are compatible with the agent's beliefs in the state u .

The set of all states in which α holds will be written $\llbracket \alpha \rrbracket^{\mathcal{M}}$ (or just $\llbracket \alpha \rrbracket$). Thus,

$$\llbracket \alpha \rrbracket = \{u \in U : \mathcal{M} \Vdash_u \alpha\}.$$

For a set of sentences Γ , we write:

$$\llbracket \Gamma \rrbracket = \bigcap \{ \llbracket \alpha \rrbracket : \alpha \in \Gamma \},$$

that is, $\llbracket \Gamma \rrbracket$ is the set of all states in which all sentences in Γ are accepted.

Given any set X of states in \mathcal{M} , we may also define the set $t(X)$ of sentences that are accepted in all the states in X , that is,

$$t(X) = \{ \alpha : X \subseteq \llbracket \alpha \rrbracket \}.$$

Notice, that the pair of mappings $\llbracket \dots \rrbracket$ and t together form a *Galois connection* between $\wp(\Phi)$ and $\wp(U)$, that is, they satisfy:⁹

- (i) if $\Gamma \subseteq \Delta$, then $\llbracket \Delta \rrbracket \subseteq \llbracket \Gamma \rrbracket$;
- (ii) if $X \subseteq Y$, then $t(Y) \subseteq t(X)$;
- (iii) $\Gamma \subseteq t(\llbracket \Gamma \rrbracket)$;
- (iv) $X \subseteq \llbracket t(X) \rrbracket$.

It follows from (i)–(iv) that these mappings also satisfy:

- (v) $\llbracket \Gamma \rrbracket = \llbracket t(\llbracket \Gamma \rrbracket) \rrbracket$;
- (vi) $t(X) = t(\llbracket t(X) \rrbracket)$.

For every set $X \subseteq U$, we define the *closure* of X , $Cl(X)$, as the set

$$(*) \quad \llbracket t(X) \rrbracket = \bigcap \{ \llbracket \alpha \rrbracket : X \subseteq \llbracket \alpha \rrbracket \}.$$

⁹The notion of a Galois connection and its use in model theory is discussed, for example, in Cohn (1965).

A set X of states in \mathcal{M} is said to be *closed* if $X = Cl(X)$. The closure of X is the intersection of all closed subsets of U that include X .¹⁰

Lemma 4.2. *Let $\mathcal{M} = \langle U, V, \iota, S \rangle$ be a structure based on L . Then, the operator $Cl: \wp(U) \rightarrow \wp(U)$, defined by the equation $(*)$ above, satisfies the following conditions. For all $X, Y \subseteq U$,*

- (Cl 1) *If $X \subseteq Y$, then $Cl(X) \subseteq Cl(Y)$;*
- (Cl 2) *$X \subseteq Cl(X)$;*
- (Cl 3) *$Cl(Cl(X)) = Cl(X)$;*
- (Cl 4) *$Cl(\emptyset) = \emptyset$.*

We are now ready to define the notion of a model based on L .

Definition 4.3. Let $\mathcal{M} = \langle U, V, \iota, S \rangle$ be a structure based on L . We say that \mathcal{M} is a *model* (based on L) if the family V satisfies the following conditions:

- (i) for every $X \subseteq U$, $Cl(X) \in V$;
- (ii) if $X, Y \in V$, then $X \cup Y \in V$;
- (iii) for any non-empty family F of members of V , $\bigcap_{X \in F} X \in V$.

That is, a model is a structure in which the domain V of the selection function contains all closed subsets of U and is closed under finite unions and arbitrary intersections.

For any model $\mathcal{M} = \langle U, V, \iota, S \rangle$ based on L , we define two corresponding relations $\vDash_{\mathcal{M}}$ and $\vdash_{\mathcal{M}}$ between sets of sentences and single sentences:

$$\begin{aligned} \Gamma \vDash_{\mathcal{M}} \alpha & \text{ iff } \llbracket \Gamma \rrbracket \subseteq \llbracket \alpha \rrbracket; \text{ and} \\ \Gamma \vdash_{\mathcal{M}} \alpha & \text{ iff } S(\llbracket \Gamma \rrbracket) \subseteq \llbracket \alpha \rrbracket. \end{aligned}$$

That is, $\Gamma \vDash_{\mathcal{M}} \alpha$ obtains just in case α is accepted in all the Γ -states (i.e., in all the states in which all sentences in Γ are accepted). And, $\Gamma \vdash_{\mathcal{M}} \alpha$ obtains just in case α is accepted in all the *most preferred* Γ -states.

Lemma 4.4. *If $\mathcal{M} = \langle U, V, \iota, S \rangle$ is a model based on L , then $\vDash_{\mathcal{M}}$ is a consequence relation which extends L , that is, such that:*

$$\vdash_L \subseteq \vDash_{\mathcal{M}}$$

and $\vdash_{\mathcal{M}}$ is an inference relation based on L .

Proof. The easy verification that $\vDash_{\mathcal{M}}$ is a consequence relation extending L is omitted. We prove that $\vdash_{\mathcal{M}}$ is an inference relation based on L .

Reflexivity: Suppose $\alpha \in \Gamma$. Then clearly, $\llbracket \Gamma \rrbracket \subseteq \llbracket \alpha \rrbracket$. However, $S(\llbracket \Gamma \rrbracket) \subseteq \llbracket \Gamma \rrbracket$. Hence, $S(\llbracket \Gamma \rrbracket) \subseteq \llbracket \alpha \rrbracket$. That is, $\Gamma \vdash_{\mathcal{M}} \alpha$.

Closure: Suppose $\Gamma \vdash_{\mathcal{M}} \alpha$, for all $\alpha \in \Delta$, and $\Delta \vdash_L \beta$. Then, $S(\llbracket \Gamma \rrbracket) \subseteq \bigcap \{ \llbracket \alpha \rrbracket : \alpha \in \Delta \}$, that is, $S(\llbracket \Gamma \rrbracket) \subseteq \llbracket \Delta \rrbracket$. Furthermore, since $\Delta \vdash_L \beta$, $\llbracket \Delta \rrbracket \subseteq \llbracket \beta \rrbracket$. It follows that $S(\llbracket \Gamma \rrbracket) \subseteq \llbracket \beta \rrbracket$, that is, $\Gamma \vdash_{\mathcal{M}} \beta$.

¹⁰In order to be fully explicit, we should write $Cl_L(X)$ rather than $Cl(X)$ and speak of it as the L -closure of X . Similarly, we should say that X is L -closed, if $X = Cl_L(X)$. In most cases, however, the reference to L can be left implicit.

Congruence: Suppose that for all α , $\Gamma \vdash_L \alpha$ iff $\Delta \vdash_L \alpha$. Then, $|\Gamma|_L = |\Delta|_L$. It follows that also $\llbracket \Gamma \rrbracket = \llbracket \Delta \rrbracket$. Hence, $S(\llbracket \Gamma \rrbracket) = S(\llbracket \Delta \rrbracket)$. Finally, we have that $\Gamma \vdash_{\mathcal{M}} \beta$ iff $\Delta \vdash_{\mathcal{M}} \beta$. \square

We speak of $\vDash_{\mathcal{M}}$ and $\vdash_{\mathcal{M}}$ as the consequence relation and the inference relation, respectively, *determined by* \mathcal{M} .

We say that a model $\mathcal{M} = \langle U, V, \iota, S \rangle$ is a *world model* if $\iota(u)$ is a unit set for each $u \in U$. In a world model, the set of sentences accepted in a state is always L -maximal. Since L is assumed to be $\{\wedge, \vee\}$ -normal, we have for any model \mathcal{M} and all sentences α and β :

- (i) $\llbracket \alpha \wedge \beta \rrbracket^{\mathcal{M}} = \llbracket \alpha \rrbracket^{\mathcal{M}} \cap \llbracket \beta \rrbracket^{\mathcal{M}}$;
- (ii) $\llbracket \alpha \rrbracket^{\mathcal{M}} \cup \llbracket \beta \rrbracket^{\mathcal{M}} \subseteq \llbracket \alpha \vee \beta \rrbracket^{\mathcal{M}}$;
- (iii) if \mathcal{M} is a world model, $\llbracket \alpha \vee \beta \rrbracket^{\mathcal{M}} \subseteq \llbracket \alpha \rrbracket^{\mathcal{M}} \cup \llbracket \beta \rrbracket^{\mathcal{M}}$;
- (iv) if L is classical, $\llbracket \alpha \rrbracket^{\mathcal{M}} \cap \llbracket \neg \alpha \rrbracket^{\mathcal{M}} = \emptyset$;
- (v) if L is classical and \mathcal{M} is a world model, $\llbracket \alpha \rrbracket^{\mathcal{M}} \cup \llbracket \neg \alpha \rrbracket^{\mathcal{M}} = U$.

A model $\mathcal{M} = \langle U, V, \iota, S \rangle$ is said to be *full* if $V = \wp(U)$.

Theorem 4.5. *Let L be a $\{\wedge, \vee\}$ -normal deductive logic and let \vdash be an inference relation based on L . Then, there exists a world model $\mathcal{M} = \langle U, V, \iota, S \rangle$ (based on L) such that:*

$\vdash_L = \vDash_{\mathcal{M}}$ and $\vdash = \vdash_{\mathcal{M}}$,
that is, \vdash_L and \vdash are, respectively, the consequence relation and inference relation determined by \mathcal{M} .

Proof. We define a structure $\mathcal{M} = \langle U, V, \iota, S \rangle$, which we shall call the *canonical model* for \vdash_L and \vdash , as follows:

- (i) $U = \mathcal{M}_L$, that is, U is the set of all L -maximal theories;
- (ii) $V = \{|\Gamma|_L : \Gamma \text{ is a set of sentences in } \mathcal{L}\}$. That is, V consists of all closed subsets of U ;
- (iii) for each $u \in U$, $\iota(u) = \{u\}$;
- (iv) We define S as follows: For any set $X \in V$, consider *the theory* $t(X)$ *determined by* X , namely:

$$t(X) = \{\alpha : \forall m \in X, \alpha \in m\} = \bigcap X.$$

Now, define:

$$S(X) = |C(t(X))|_L = \{m \in U : \text{for all } \alpha, \text{ if } t(X) \vdash \alpha, \text{ then } \alpha \in m\}.$$

That is, $S(X)$ is the set of $t(X)$ -optimal L -maximal theories.

Let $X \in V$. Then, $X = |t(X)|_L$. Since $t(X) \subseteq C(t(X))$, we get that $|C(t(X))|_L \subseteq |t(X)|_L$. That is, $S(X) \subseteq X$. Thus, the canonical model for \vdash_L and \vdash is a structure based on L . In this structure, we have $\llbracket \Gamma \rrbracket = |\Gamma|_L$, for all Γ . Hence, $\vdash_L = \vDash_{\mathcal{M}}$.

We now claim that:

$$(*) \quad \Gamma \vdash \alpha \text{ iff } S(|\Gamma|) \subseteq |\alpha|.$$

Proof of ():* $S(|\Gamma|)$ is the set of all $\bigcap(|\Gamma|)$ -optimal L -maximal sets. But $\bigcap(|\Gamma|) = Cn_L(\Gamma)$, so $S(|\Gamma|)$ is the set of all L -maximal sets that are $Cn_L(\Gamma)$ -optimal. However, m is $Cn_L(\Gamma)$ -optimal iff m is Γ -optimal (since $C(Cn(\Gamma)) = C(\Gamma)$).

Hence, $S(|\Gamma|)$ is the set of all L -maximal theories that are Γ -optimal. It follows by lemma 3.6 that $\Gamma \vdash \alpha$ iff for all $m \in S(|\Gamma|)$, $\alpha \in m$. Q.E.D.

By definition, we have:

$$(**) \Gamma \vdash_{\mathcal{M}} \alpha \text{ iff } S(|\Gamma|) \subseteq |\alpha|.$$

From (*) and (**) and the fact that $|\Gamma| = |\Gamma|_L$, we get that $\Gamma \vdash \alpha$ iff $\Gamma \vdash_{\mathcal{M}} \alpha$.

It only remains to show that the canonical model $\mathcal{M} = \langle U, V, \iota, S \rangle$ for \vdash_L and \vdash is indeed a model, that is, satisfies conditions (i)–(iii) of Definition 4.3:

Condition (i) is immediate from the definition of V .

Condition (ii): We first prove that the closure operation of the canonical model satisfies:

$$(*) \quad Cl(X \cup Y) = Cl(X) \cup Cl(Y).$$

First of all, $X \subseteq X \cup Y$. Hence, $t(X \cup Y) \subseteq t(X)$, and, therefore, $|\!|t(X)|\!| \subseteq |\!|t(X \cup Y)|\!|$. That is, $Cl(X) \subseteq Cl(X \cup Y)$. In the same way, we get $Cl(Y) \subseteq Cl(X \cup Y)$. Thus, $Cl(X) \cup Cl(Y) \subseteq Cl(X \cup Y)$.

In order to prove the other direction, assume that $m \in Cl(X \cup Y)$, that is, $m \in |\!|t(X \cup Y)|\!|$. Then, we have: $t(X \cup Y) \subseteq m$. But, $t(X \cup Y) = t(X) \cap t(Y)$. Hence, $t(X) \cap t(Y) \subseteq m$. Suppose now, that $m \notin Cl(X) \cup Cl(Y)$. Then, $m \notin Cl(X)$ and $m \notin Cl(Y)$. Hence, there must exist sentences α, β such that $\alpha \in t(X)$, $\alpha \notin m$; $\beta \in t(Y)$ and $\beta \notin m$. Consider now the sentence $\alpha \vee \beta$. Since, $\alpha \notin m$ and $\beta \notin m$, it follows by the $\{\wedge, \vee\}$ -normality of L that $\alpha \vee \beta \notin m$. But, on the other hand, $\alpha \vee \beta \in t(X) \cap t(Y)$. By contradiction, we conclude that $m \in Cl(X) \cup Cl(Y)$. Thus, $Cl(X \cup Y) \subseteq Cl(X) \cup Cl(Y)$.

Suppose now that $X, Y \in V$. By the definition of V , $X = Cl(X)$ and $Y = Cl(Y)$. Hence, $X \cup Y = Cl(X) \cup Cl(Y) = (\text{by } (*)) \quad Cl(X \cup Y)$. But, by the definition of V , $Cl(X \cup Y) \in V$. Thus, $X \cup Y \in V$.

Condition (iii). Let F be a non-empty family of elements in V . By (Cl 2) of Lemma 4.2, $\bigcap F \subseteq Cl(\bigcap F)$. On the other hand, $\bigcap F \subseteq X$, for each $X \in F$. Hence, by (Cl 1), $Cl(\bigcap F) \subseteq Cl(X) = X$, for each $X \in F$. Thus, $Cl(\bigcap F) \subseteq \bigcap F$. We have shown that $\bigcap F = Cl(\bigcap F)$. So by the definition of V , $\bigcap F \in V$. \square

Remark 4.6. Let \vdash be an inference relation based on L . Let $\mathcal{M} = \langle U, V, \iota, S \rangle$ be the corresponding canonical model. Then, we have for all $\Gamma \subseteq \Phi$ and $X \in V$:

- (i) $Cn_L(\Gamma) = t(|\Gamma|)$;
- (ii) $X \neq \emptyset$ iff $t(X)$ is L -consistent;
- (iii) $C(\Gamma) = t(S(|\Gamma|))$; and
- (iv) $S(X) = |\!|C(t(X))|\!|$.

It follows that for all sets of sentences Γ and all $X \in V$,

- (v) $C(t(X)) = t(S(X))$; and
- (vi) $S(|\Gamma|) = |\!|C(\Gamma)|\!|$,

that is, the two diagrams in Figure 2 commute.

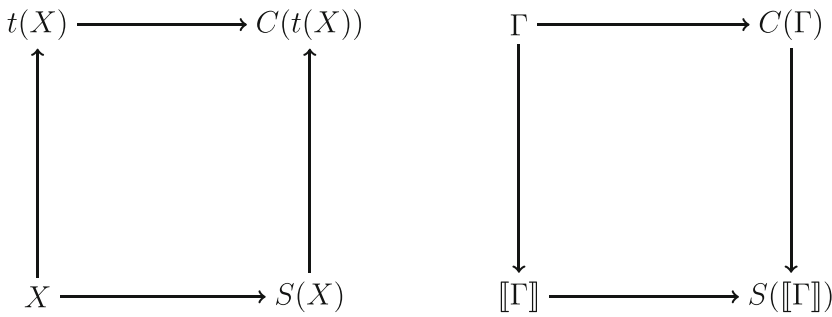


FIGURE 2 Two commuting diagrams

By a *canonical model* for L we understand a model $\mathcal{M} = \langle U, V, \imath, S \rangle$ such that:

- (i) U is the set of all maximal L -theories;
- (ii) V is the set of all closed subsets of U , that is,

$$V = \{ \llbracket t(X) \rrbracket : X \subseteq U \} = \{ Cl(X) : X \subseteq U \};$$

- (iii) for each $u \in U$, $\imath(u) = \{u\}$.

It is easy to see that a canonical model for L is the canonical model *for* L and the inference operation $C_{\mathcal{M}}$ defined by: $C_{\mathcal{M}}(\Gamma) = t(S(\llbracket \Gamma \rrbracket))$. That is, we also have: $S(X) = \llbracket C_{\mathcal{M}}(t(X)) \rrbracket$.

The next lemma states that the set V of a canonical model has certain important closure properties: V is closed under finite unions and arbitrary intersections and contains all singleton sets. It follows that V contains all *finite* subsets of U .

Lemma 4.7. *Let $\mathcal{M} = \langle U, V, \imath, S \rangle$ be a canonical model for L . Then, for all $X, Y \in V$, and $m \in U$,*

- (i) $Cl(X \cup Y) = Cl(X) \cup Cl(Y)$;¹¹
- (ii) $Cl(\{m\}) = \{m\}$, that is, all singleton sets are closed;
- (iii) all finite subsets of U are members of V .

Proof. We have already proved

- (i) in the course of proving Theorem 4.5. Observe that in the proof of (i), we used the fact that L is closed under the standard natural deduction rules for \vee (see the Appendix).
- (ii) $Cl(\{m\}) = \llbracket t(m) \rrbracket = \llbracket m \rrbracket$, since $t(m) = m$. But $\llbracket m \rrbracket = \{n \in U : m \subseteq n\} = \{m\}$, since U is the set of all L -maximal theories. Hence, $Cl(\{m\}) = \{m\}$.
- (iii) follows immediately from (i) and (ii). □

We shall now consider some natural conditions that we might want to impose on the selection function in a model. Most of these are taken from the literature on choice functions and

¹¹Together with (Cl 1)–(Cl 4), this condition implies that the closure operation of a canonical model is a topological closure operation in the sense of Kuratowski (see, for instance, Kelley (1955), p. 43).

preference relations.¹² Some, however, borrow their names from the corresponding conditions on inference operations: for any $X, Y \in \mathcal{V}$,

- (cp) If $X \neq \emptyset$, then $S(X) \neq \emptyset$; (*Consistency Preservation*)
- (it) $S(S(X)) = S(X)$; (*Iteration*)
- (c) if $S(X) \subseteq Y \subseteq X$, then $S(X) \subseteq S(Y)$; (*Cut*)
- (d) $S(X \cup Y) \subseteq S(X) \cup S(Y)$; (*Distributivity*)
- (ch) $S(X) \cap Y \subseteq S(X \cap Y)$; (*Chernoff*)
- (aiz) if $S(X) \subseteq Y \subseteq X$, then $S(Y) \subseteq S(X)$; (*Aizerman*)
- (pi) $S(S(X) \cup S(Y)) = S(X \cup Y)$; (*Path Independence*)
- (g) Let $\emptyset \neq F \subseteq \mathcal{V}$ such that $\bigcup_{X \in F} X \in \mathcal{V}$. Then,
 $\bigcap_{X \in F} S(X) \subseteq S(\bigcup_{X \in F} X)$; (*Gamma*)
- (s) if $S(X) \cap S(Y) \neq \emptyset$, then $S(X \cap Y) \subseteq S(X) \cap S(Y)$; (*Sen*)
- (iia) if $S(X) \cap Y \neq \emptyset$, then $S(X \cap Y) = S(X) \cap Y$. (*Arrow*)

The condition (ch)—originally introduced by Chernoff (1954)—is identical to Sen's (1971) *Property α* .¹³ The following formulation is easily seen to be equivalent to the one above:

$$(\alpha) \text{ If } Y \subseteq X, \text{ then } S(X) \cap Y \subseteq S(Y).$$

Intuitively, if x is a best choice in the set X , then x is still a best choice in any subset of X to which x belongs.

The conditions Aizerman and Cut together say that:

$$\text{if } S(X) \subseteq Y \subseteq X, \text{ then } S(X) = S(Y).$$

That is, deleting from a set only such members that are not among its best members does not affect which members are best in the set.

The condition (g) is called *Property γ* by Sen (1971). It says that if x is a best choice in every set X in a family of sets, then x is also a best choice in their union. It has the following finitary consequence:

$$S(X) \cap S(Y) \subseteq S(X \cup Y).$$

The condition Sen may also be formulated as:

$$\text{if } X \subseteq Y \text{ and } S(X) \cap S(Y) \neq \emptyset, \text{ then } S(X) \subseteq S(Y).$$

It is called *Property β* by Sen (1971).

Arrow's condition (iia) of *Independence of Irrelevant Alternatives* can be rewritten as:

$$\text{If } Y \subseteq X \text{ and } S(X) \cap Y \neq \emptyset, \text{ then } S(X) \cap Y = S(Y).$$

That is, if Y is a subset of X that contains some of the best members of X , then the best members of Y are precisely the best member of X that belong to Y .

The results of the next lemma are either well known (see Moulin (1985)) or obvious. Their proofs are included in the Appendix for the sake of completeness and easy reference.

¹²Cf. for example Moulin (1985).

¹³See Moulin (1985) for additional references and details about the origin of some of the conditions.

Lemma 4.8.

- (i) (c) implies (it) ;
- (ii) (ch) implies (c) ;
- (iii) (ch) implies (d) ;
- (iv) (c) , (aiz) and (d) together imply (pi) . If the model is full, that is, if $V = \wp(U)$, then (pi) is equivalent to (ch) and (aiz) ;¹⁴
- (v) (cp) and (ia) together imply (ch) , (aiz) and (g) ;
- (vi) (ia) is equivalent to (ch) and (s) .

Next we turn to conditions on the inference operation C :

- (CP) if $\perp \notin Cn(\Gamma)$, then $\perp \notin C(\Gamma)$; (*Consistency Preservation*)
- (It) $C(C(\Gamma)) = C(\Gamma)$; (*Iteration*)
- (C) if $\Gamma \subseteq \Delta \subseteq C(\Gamma)$, then $C(\Delta) \subseteq C(\Gamma)$; (*Cut*)
- (D) $C(\Gamma) \cap C(\Delta) \subseteq C(Cn(\Gamma) \cap Cn(\Delta))$; (*Distributivity*)
- (Ch) $C(\Gamma \cup \Delta) \subseteq Cn(C(\Gamma) \cup \Delta)$; (*Chernoff*)
- (Aiz) if $\Gamma \subseteq \Delta \subseteq C(\Gamma)$, then $C(\Gamma) \subseteq C(\Delta)$; (*Aizerman*)
- (PI) $C(C(\Gamma) \cap C(\Delta)) = C(Cn(\Gamma) \cap Cn(\Delta))$; (*Path Independence*)
- (G) $C(\bigcap_{\Gamma \in F} Cn(\Gamma)) \subseteq Cn(\bigcup_{\Gamma \in F} C(\Gamma))$, where F is any non-empty family of sets of sentences; (*Gamma*)
- (S) If $C(\Gamma) \cup C(\Delta)$ is L -consistent, then $C(\Gamma) \cup C(\Delta) \subseteq C(\Gamma \cup \Delta)$; (*Sen*)
- (IIA) if $C(\Gamma) \cup \Delta$ is L -consistent, then $C(\Gamma \cup \Delta) = Cn(C(\Gamma) \cup \Delta)$. (*Arrow*)

Notice that Aizerman is the condition that Makinson (1989) calls *Cautious Monotony*. It is also worth mentioning that Arrow implies the following generalisation of the so-called condition of *Rational Monotony*:¹⁵

If $C(\Gamma) \cup \Delta$ is L -consistent, then $C(\Gamma) \subseteq C(\Gamma \cup \Delta)$.

In the presence of (CP), this principle implies Sen, but is not implied by Sen. Rational monotony is the special case of the principle for which Δ is a unit set.

Theorem 4.9. *Let L be a $\{\wedge, \vee\}$ -normal deductive logic and $\mathcal{M} = \langle U, V, \iota, S \rangle$ a canonical model based on L . Let $C = C_{\mathcal{M}}$ be the inference operation that is determined by \mathcal{M} . If (x) is any of the conditions (cp) – (ia) , then S satisfies (x) iff the inference operation C satisfies the corresponding condition (X) among (CP) – (IIA) .*

In view of Lemma 4.8 and Theorem 4.9, we have the following result connecting the various conditions on C .

Lemma 4.10.

- (i) (C) implies (It) ;
- (ii) (Ch) implies (C) ;
- (iii) (Ch) implies (D) ;
- (iv) (C) , (Aiz) and (D) together imply (PI) ;
- (v) (CP) and (IIA) together imply (Ch) , (Aiz) and (G) ;
- (vi) (IIA) is equivalent to (Ch) and (S) .

¹⁴David Makinson (personal communication) has proved that (pi) does not in general imply (ch) .

¹⁵Cf. Makinson (1989) and Kraus et al. (1990).

We conclude this section by discussing some consequences of the conditions above in the context of L being classical. First of all, L being classical implies that a finite set of premises may be treated as a conjunction, that is,

$$\alpha \wedge \beta \vdash \gamma \text{ iff } \{\alpha, \beta\} \vdash \gamma.$$

Hence, the infinitary conditions (C) and (Aiz) above have—in the classical case—the following finitary consequences:

$$\begin{aligned} \text{if } \alpha \vdash \beta \text{ and } \alpha \wedge \beta \vdash \gamma, \text{ then } \alpha \vdash \gamma; & \quad (\text{Cut}) \\ \text{if } \alpha \vdash \beta \text{ and } \alpha \vdash \gamma, \text{ then } \alpha \wedge \beta \vdash \gamma. & \quad (\text{Cautious Monotony}) \end{aligned}$$

The next Lemma is due to Makinson (1994).

Lemma 4.11. (Makinson) *Suppose that C is an inference operation based on a classical logic L . If C satisfies Distribution, then the following conditions are also satisfied:*

- (i) *if $\Gamma, \alpha \vdash \gamma$ and $\Gamma, \beta \vdash \gamma$, then $\Gamma, \alpha \vee \beta \vdash \gamma$;* (Disjunction in the Antecedent)
- (ii) *if $\Gamma, \alpha \vdash \gamma$ and $\Gamma, \neg \alpha \vdash \gamma$, then $\Gamma \vdash \gamma$;* (Proof by Cases)
- (iii) *if $\Gamma, \alpha \vdash \beta$, then $\Gamma \vdash \alpha \rightarrow \beta$.* (Conditionalisation)
- (iv) *if $\Gamma \vdash \alpha$ and $\Gamma, \beta \vdash \neg \alpha$, then $\Gamma \vdash \neg \beta$.*

Since Chernoff implies Distribution, the assumption of Chernoff yields, in the context of classical logic, Conditions (i)–(iv).

Condition (iv) would license inferences of the kind:

- (1) If Squeaky is a mammal, then it is expected that Squeaky cannot fly.
- (2) If Squeaky is a mammal and a bat, then it is expected that Squeaky can fly.
- (3) Hence: if Squeaky is a mammal, then it is expected that Squeaky is not a bat.

If L is classical and C satisfies Arrow, then we also have:

$$\text{If } \Gamma \vdash \gamma \text{ and } \Gamma \not\vdash \neg \beta, \text{ then } \Gamma, \beta \vdash \gamma. \quad (\text{Rational Monotony})$$

This principle yields inferences of the kind:

- (1) If Squeaky is a mammal, then it is expected that Squeaky cannot fly.
- (2) If Squeaky is a mammal, then it is not expected that Squeaky is not a dog.
- (3) Hence: if Squeaky is a mammal and a dog, then it is expected that Squeaky cannot fly.

5 | REPRESENTATION THEOREMS

In the last section, we proved a series of results connecting properties of the inference operation C with properties of the selection function S in the canonical model corresponding to C . In this section we wish to explore under what conditions a given selection function can be defined in terms of an underlying preference relation P on the set U of all states. In order to make this question precise, we introduce the notion of a choice structure:¹⁶

¹⁶The term “choice structure” is borrowed from Hansson (1968), although his notion of a choice structure is not exactly the one defined here: Hansson’s choice structures satisfy weaker structural conditions on the set V , but stronger conditions on the selection function S .

Definition 5.1. A choice structure is an ordered triple $\mathcal{S} = \langle U, V, S \rangle$, where U is a non-empty set, V is a non-empty family of subsets of U , S is a function from V to V , such that:

- (i) for each $x \in U$, $\{x\} \in V$;
- (ii) $\emptyset \in V$;
- (iii) if $X, Y \in V$, then $X \cup Y \in V$;
- (iv) for any non-empty family F of members of V , $\bigcap_{X \in F} X \in V$;
- (v) for each $X \in V$, $S(X) \subseteq X$.

U is the *domain* of \mathcal{S} and the elements of U are here called *states* (or *points*). Axioms (ii)–(iv) say that the elements of V form the closed sets of a topological space over U . Hence, it is appropriate to refer to the elements of V as the *closed sets* of \mathcal{S} . For any $X \subseteq U$, we write $Cl(X)$ for the *closure* of X , that is, the intersection of all closed sets that include X . Cl , of course, satisfies the axioms (Cl 1)–(Cl 4) of a topological closure operation. A topological space satisfying condition (i), that all singleton sets are closed, is called a T_1 -space. It follows from (i) and (iii) that all finite sets are members of V . S is the *selection function* (or the *choice function*) of the structure \mathcal{S} . According to (v), S selects a subset of elements from any closed subset X of U . Since S is an operation on V , $S(X)$ is always a closed set.

The principal case we are interested in is the following: A $\{\wedge, \vee\}$ -normal deductive logic L and an inference operation C based on L are given. $\mathcal{S} = \langle U, V, S \rangle$ is defined in terms of L and C as follows: (i) U is the set of all L -maximal theories; (ii) $V = \{X \subseteq U : X = \llbracket t(X) \rrbracket\}$; (iii) for each $X \in V$, $S(X) = \llbracket C(t(X)) \rrbracket$. In this case, $\mathcal{S} = \langle U, V, S \rangle$ is essentially identical to the *canonical model* for L and C .

In this section, we shall think of the set U of states as being provided with a preference relation $P \subseteq U \times U$ (we read xPy as: x is *better than* y). In terms of such a relation, we can define the selection function $S : V \rightarrow V$ as follows: for all $X \in V$:

$$(*) \quad S(X) = \{x : x \in X \ \& \ (\forall y)(y \in X \rightarrow \neg(yPx))\}.$$

That is, $S(X)$ is the set of all P -maximal elements of X . We say that S is *based on* the relation P —and that P *rationalises* S —if S is defined from P by means of the equation (*). S is said to be *rationalisable* if there is a relation that rationalises it.

We use the following terminology for preference relations: We use xRy as an abbreviation for $\neg(yPx)$. P is said to be:

- (i) a *strict partial ordering* iff P is asymmetric and transitive;
- (ii) a *strict weak ordering* iff P is asymmetric and $\forall xyz(xRy \wedge yPz \rightarrow xPz)$;
- (iii) a *strict linear order* iff P is a strict partial ordering and $\forall xy(xRy \wedge x \neq y \rightarrow xPy)$;
- (iv) *neat* iff every non-empty element X of V contains a P -maximal element, that is, an x such that $\forall y(y \in X \rightarrow \neg(yPx))$.¹⁷

Lemma 5.2. *Let $S : V \rightarrow V$ be a selection function and P a relation that rationalises S . Then,*

- (a) P is *irreflexive* iff S satisfies the condition:

$$(ir) \quad S(\{x\}) \neq \emptyset, \text{ for each } x \in U.$$

(Irreflexivity)

¹⁷Our terminology here, differs from Kanger (2001) who uses “neat” to refer to the stronger property of P^{-1} (the converse of P) being well-founded. Thus, P is neat in Kanger’s sense, just in case every non-empty subset of U has a P -maximal element, that is, iff there are no infinitely ascending P -chains in U . Neatness in our sense only requires every non-empty closed subset of U to contain a P -maximal element. Neatness is analogous to the *limit assumption* of Lewis (1973).

- (b) P is neat iff S satisfies Consistency Preservation.
 (c) If P is irreflexive, then P is unique and for all $x, y \in U$,
 xPy iff $y \notin S(\{x, y\})$.

Proof. (a) and (b) are trivial. We prove (c). By (*), we have:

$$y \in S(\{x, y\}) \text{ iff } \neg yPy \text{ and } \neg xPy.$$

The irreflexivity of P then yields:

$$xPy \text{ iff } y \notin S(\{x, y\}). \quad \square$$

Lemma 5.3. *A selection function S is rationalisable iff it satisfies the condition:*

$$\text{for all } X \in V, S(X) = \{x \in X : (\forall y)(y \in X \rightarrow x \in S(\{x, y\}))\}.$$

Proof. Suppose that P rationalises S . Then, we have for all $Z \in V$ and all $z \in Z$,

$$(1) \quad z \in S(Z) \leftrightarrow (\forall y)(y \in Z \rightarrow \neg(yPz)).$$

Let $X \in V$ and $x \in X$. We want to show that:

$$(2) \quad x \in S(X) \leftrightarrow (\forall y)(y \in X \rightarrow x \in S(\{x, y\})).$$

Suppose that $x \in S(X)$ and that $y \in X$. By (1), we then have: $\neg(yPx)$ and $\neg(xPy)$. Hence, by (1) again, we get: $x \in S(\{x, y\})$. To prove the other direction of (2), assume that $(\forall y)(y \in X \rightarrow x \in S(\{x, y\}))$. Suppose also that $x \notin S(X)$. Then, by (1), there is a $y \in X$ such that yPx . Applying (1) again, we get $x \notin S(\{x, y\})$, that is, a contradiction.

To prove the other direction of the lemma, suppose that for all $X \in V$,

$$(3) \quad S(X) = \{x \in X : (\forall y)(y \in X \rightarrow x \in S(\{x, y\}))\}.$$

Define $P \subseteq U \times U$ by the condition:

$$(4) \quad xPy \text{ iff } y \notin S(\{x, y\}).$$

(3) and (4) then yield:

$$S(X) = \{x \in X : (\forall y)(y \in X \rightarrow \neg(yPx))\}.$$

that is, P rationalises S . □

In the theory of preference and choice, there are many well-known theorems relating conditions on the selection function S , like Chernoff, Aizerman, Gamma, etc., to the existence of an underlying preference relation P .¹⁸ The following theorem is a slight strengthening of a result by Sen (1971).¹⁹

¹⁸Cf. Hansson (1968), Sen (1971), Moulin (1985), Kanger (2001).

¹⁹Sen proved that a selection function that satisfies Consistency Preservation is rationalisable iff it satisfies Chernoff and Gamma. Theorem 2 of Moulin (1985) is Sen's theorem for the case when U is finite.

Theorem 5.4. *A selection function S is rationalisable iff it satisfies Chernoff and Gamma (i.e., Sen's Properties α and γ):*

- (ch) $S(X) \cap Y \subseteq S(X \cap Y)$;
 (g) Let $\emptyset \neq F \subseteq V$ be such that $\bigcup_{X \in F} X \in V$. Then, $\bigcap_{X \in F} S(X) \subseteq S(\bigcup_{X \in F} X)$.

Proof. We omit the straightforward verification that every rationalisable selection function satisfies Chernoff and Gamma.

For the other direction, suppose that S satisfies Chernoff and Gamma. By Lemma 5.3, it is sufficient to prove that for all $X \in V$ and $x \in X$, $x \in S(X)$ iff $(\forall y)(y \in X \rightarrow x \in S(\{x, y\}))$.

Assume first that $x \in S(X)$. Consider any $y \in X$. Chernoff then yields that $S(X) \cap \{x, y\} \subseteq S(\{x, y\})$, which implies that $x \in S(\{x, y\})$.

Next, we assume that x is such that $(\forall y)(y \in X \rightarrow x \in S(\{x, y\}))$. This means that: $x \in \bigcap_{y \in X} S(\{x, y\})$. But $X = \bigcup_{y \in X} \{x, y\}$. Gamma then yields $\bigcap_{y \in X} S(\{x, y\}) \subseteq S(\bigcup_{y \in X} \{x, y\}) = S(X)$. Hence, $x \in S(X)$. \square

Lemma 5.5.²⁰ *Suppose that S is based on P .*

- (a) *If S is neat and satisfies Aizerman:*
 (aiz) *if $S(X) \subseteq Y \subseteq X$, then $S(Y) \subseteq S(X)$,*

then P is transitive.

- (b) *If P is transitive and P^{-1} is well-founded (that is, there are no infinitely ascending P -chains in U), then S satisfies Aizerman.*

Lemma 5.6. *Suppose that S is based on a neat and transitive relation P . Then, S satisfies Sen:*

- (s) *if $S(X) \cap S(Y) \neq \emptyset$, then $S(X \cap Y) \subseteq S(X) \cap S(Y)$.*

iff P satisfies the condition:

$$\forall xyz(xRy \wedge yPz \rightarrow xPz),$$

(that is, iff P is a neat strict weak ordering).

Proof. Cf. Kanger (2001), Theorem 8.1.

Theorem 5.7.

- (a) *S is based on a neat relation P iff S satisfies (cp), Chernoff and Gamma.*
 (b) *If S satisfies (cp), Chernoff, Gamma and Aizerman, then S is based on a neat and transitive relation (that is, a neat strict partial ordering).*
 (c) *If any of the following equivalent conditions is satisfied:*

- (i) *S satisfies (cp), Chernoff, Gamma, Aizerman and Sen;*
 (ii) *S satisfies (cp), Chernoff and Sen;*
 (iii) *S satisfies (cp) and (ia),*

then S is based on a neat strict weak ordering.

²⁰Moulin (1985) proves a finitary version of this lemma which may be formulated as follows: Suppose that S is based on P and that the set U of all alternatives is *finite*. Then, S satisfies Aizerman iff P is transitive. This is, of course, a consequence of Lemma 5.5.

Proof. By Lemma 4.8, Lemma 5.2 (b), Theorem 5.4, Lemma 5.5 and Lemma 5.6. \square

Theorems 4.9, 5.4 and 5.7 together give us:

Theorem 5.8. (*Representation Theorem I*) *Let C be an inference relation based on the deductive logic L . Let $\mathcal{M} = \langle U, V, \iota, S \rangle$ be the canonical model for L and C . Then, $L = L_{\mathcal{M}}$ and $C = C_{\mathcal{M}}$ and:*

- (i) *C satisfies Chernoff and Gamma iff S is based on a relation $P \subseteq U \times U$.*
- (ii) *If C satisfies (CP), Chernoff, Gamma and Aizerman, then S is based on a neat strict partial ordering $P \subseteq U \times U$.*
- (iii) *If C satisfies (CP) and Arrow, then S is based on a neat strict weak ordering $P \subseteq U \times U$.*

6 | DYADIC INFERENCE OPERATIONS AND INFINITARY BELIEF REVISION

In this section we shall explore the connection between nonmonotonic inference and belief revision in the sense of Alchourrón et al.²¹ In doing so, we generalise the notion of belief revision to allow for the revision of a set of beliefs with a, possibly infinite, *set* of propositions representing the new information.²² A representation theorem is proved for the generalised notion of belief revision in terms of systems of spheres of the kind introduced by Grove (1988).

In Makinson and Gärdenfors (1991) a method is described for translating postulates for belief revision into postulates for nonmonotonic inference, and vice versa.²³ The basic idea here is to interpret $\beta \in K_{\alpha}^*$ as a claim that β is a nonmonotonic consequence of α , relative to the *background* (or *default*) theory K . That is, $\beta \in K_{\alpha}^*$ is translated as $\alpha \vdash_K \beta$, where \vdash_K is a nonmonotonic inference relation associated with the theory K . Expressing this equivalence, in terms of an inference *operation* C_K instead, we get, for a fixed K , the identity:

$$K_{\alpha}^* = C_K(\{\alpha\}).$$

This idea can be generalised: thinking of C as a *binary* operation and allowing K to be replaced by an arbitrary *set* of sentences Δ , we get:

$$\Delta_{\alpha}^* = C(\Delta, \{\alpha\}).$$

In other words, for any Δ and α , the revision of Δ with α is identified with the set of nonmonotonic consequences of α , relative to the default assumptions Δ . Now, in order to get complete *interdefinability* between the notions of belief revision and nonmonotonic inference, we need just another step: we must allow for the possibility of a set Δ being revised with a possibly infinite *set* of propositions Γ . Then, for all Γ and Δ , we obtain:

$$\Delta_{\Gamma}^* = C(\Delta, \Gamma).$$

²¹Cf. Alchourrón et al. (1985) and Gärdenfors (1988).

²²Infinitary belief revision has been studied before in the literature. Fuhrmann (1988) considers both infinitary belief contraction operations and infinitary belief revision operations. He refers to these kinds of operations as multiple contraction and multiple revision, respectively. Via a generalisation of the so-called Levi identity, Fuhrmann defines infinitary belief revision in terms of infinitary belief contraction. He also formulates a set of postulates for infinitary belief revision which is equivalent to (BC1)–(BC4) together with (BC9). (Fuhrmann, 1988, p. 159). S. O. Hansson (1989) contains a theory of infinitary belief contraction.

²³See also the discussion in Gärdenfors (1990).

Conversely, given a notion of belief revision Δ_Γ^* , we may, of course, define the corresponding notion of nonmonotonic inference via the same equality.

In our formal treatment, however, we shall not make a complete identification between the notions of belief revision and nonmonotonic inference. Instead, we take the former as a special case of the latter—in the sense of being characterised by stronger axioms. The axioms for belief revision presented here are straightforward generalisations to the infinitary case of Gärdenfors' (1988) basic axioms (K*1) – (K*6) for finitary belief revision.²⁴

Definition 6.1. Let L be a consistent deductive logic.

(a) A dyadic inference operation based on L is an operation $C: \wp(\Phi) \times \wp(\Phi) \rightarrow \wp(\Phi)$ satisfying the following conditions. For easy readability, we shall write $C_\Delta(\Gamma)$ instead of $C(\Delta, \Gamma)$. We also write $\Gamma + \Delta$ as an abbreviation of $Cn_L(\Gamma \cup \Delta)$. We speak of $\Gamma + \Delta$ as the *expansion* of Γ with Δ .

- (BC1) $C_\Delta(\Gamma)$ is an L -theory; (Closure)
 (BC2) $\Gamma \subseteq C_\Delta(\Gamma)$; (Success)
 (BC3) if $Cn_L(\Gamma) = Cn_L(\Delta)$ and $Cn_L(\Sigma) = Cn_L(\Pi)$, then $C_\Sigma(\Gamma) = C_\Pi(\Delta)$. (Congruence)

(b) An (infinitary) belief revision operation based on L is a dyadic inference operation C satisfying—in addition to (BC1)–(BC3)—the following conditions:

- (BC4) if $\perp \notin \Delta + \Gamma$, then $C_\Delta(\Gamma) = \Delta + \Gamma$; (Expansion)
 (BC5) if $\perp \notin Cn_L(\Gamma)$, then $\perp \notin C_\Delta(\Gamma)$. (Consistency Preservation)

Here, the preferred reading of $C_\Delta(\Gamma)$ is: 'the result of revising the set Δ with the new information Γ '.

Notice that (BC1)–(BC3) say no more than that, for any fixed Δ , $C_\Delta(\dots)$ is an inference operation in the sense of Definition 3.1 (b). The axioms (BC1)–(BC5) should be compared with the corresponding axioms in Gärdenfors (1988), namely:

- (K*1) K_α^* is an L -theory; (Closure)
 (K*2) $\alpha \in K_\alpha^*$; (Success)
 (K*6) if $Cn_L(\{\alpha\}) = Cn_L(\{\beta\})$, then $K_\alpha^* = K_\beta^*$. (Congruence)

The following two axioms correspond to *Expansion*:

- (K*3) $K_\alpha^* \subseteq K + \{\alpha\}$;
 (K*4) if $\neg\alpha \notin K$, then $K + \{\alpha\} \subseteq K_\alpha^*$.

Finally, there is:

- (K*5) if α is L -consistent, then K_α^* is L -consistent. (Consistency Preservation)

To the basic axioms (BC1)–(BC5) for belief revision, we might want to add some of the following supplementary axioms:

²⁴The reader should perhaps also be reminded that we make weaker assumptions concerning the underlying logic than Gärdenfors does. We assume only that it is a deductive logic, that is, a finitary Tarski-style consequence relation. He assumes, in addition, that it is classical, that is, is closed under the axioms and rules (including the deduction theorem) of classical propositional logic.

- (BC6) $C_{\Pi}(\Gamma \cup \Delta) \subseteq C_{\Pi}(\Gamma) + \Delta$; (Chernoff)
 (BC7) $C_{\Pi}(\bigcap_{\Gamma \in F} Cn(\Gamma)) \subseteq Cn(\bigcup_{\Gamma \in F} C_{\Pi}(\Gamma))$, where F is any
 non-empty family of sets of sentences; (Gamma)
 (BC8) if $\Gamma \subseteq \Delta \subseteq C_{\Pi}(\Gamma)$, then $C_{\Pi}(\Gamma) \subseteq C_{\Pi}(\Delta)$; (Aizerman)
 (BC9) if $\perp \notin C_{\Pi}(\Gamma) + \Delta$, then $C_{\Pi}(\Gamma \cup \Delta) = C_{\Pi}(\Gamma) + \Delta$. (Arrow)

Provided that $Cn_L(\{\alpha \wedge \beta\}) = Cn_L(\{\alpha, \beta\})$, (BC6) yields the following supplementary axiom of Gärdenfors:

$$(K^*7) \quad K_{\alpha \wedge \beta}^* \subseteq K_{\alpha}^* + \beta.$$

Under the same provision, (BC9) yields *Revision by Conjunction*:

$$\text{if } K_{\alpha}^* + \beta \text{ is } L\text{-consistent, then } K_{\alpha \wedge \beta}^* = K_{\alpha}^* + \beta,$$

which is equivalent to (K*7) together with the other of Gärdenfors' supplementary axiom:

$$(K^*8) \quad \text{if } \neg\beta \notin K_{\alpha}^*, \text{ then } K_{\alpha}^* + \beta \subseteq K_{\alpha \wedge \beta}^*.$$

It is straightforward to modify the notion of a model $\mathcal{M} = \langle U, V, \mathfrak{I}, S \rangle$ based on L , that was introduced in Section 4, in such a way as to get a semantics for dyadic inference operations. The only difference occurs in clause (iii) of Definition 4.1, which has to be changed to:

(iii') S is a function from $V \times V$ to V such that for all $X, Y \in V$, $S(X, Y) \subseteq Y$. Such a function we call a *dyadic selection function* on V .

Each model $\mathcal{M} = \langle U, V, \mathfrak{I}, S \rangle$, of the new kind, determines two operations:

$$Cn^{\mathcal{M}}(\Gamma) = t(\llbracket \Gamma \rrbracket) \text{ and } C_{\Delta}^{\mathcal{M}}(\Gamma) = t(S(\llbracket \Delta \rrbracket, \llbracket \Gamma \rrbracket)),$$

where the first operation is a consequence operation that extends L and the second is a dyadic inference operation based on L (cf. Lemma 4.4). Now, for any $\{\wedge, \vee\}$ -normal deductive logic L and any dyadic inference operation C , we may define the corresponding *canonical model* $\mathcal{M} = \langle U, V, \mathfrak{I}, S \rangle$, where:

- (i) $U = \mathcal{M}_L$, that is, U is the set of all L -maximal theories;
- (ii) V is the set of all closed subsets of U ;
- (iii) for each $u \in U$, $\mathfrak{I}(u) = \{u\}$;
- (iv) for any sets $X, Y \in V$,
 $S(X, Y) = |C(t(X), t(Y))|_L = \{m \in U : C(t(X), t(Y)) \subseteq m\}$.

The proof of Theorem 4.5 carries over unchanged, so we have that $Cn_L = Cn^{\mathcal{M}}$ and for all Γ, Δ , $C_{\Delta}(\Gamma) = C_{\Delta}^{\mathcal{M}}(\Gamma)$. That is, Cn_L and C are, respectively, the consequence operation and the dyadic inference operation that are determined by the canonical model \mathcal{M} .

In addition to letting the selection functions take an extra argument, we apply the same procedure to the preference relations. That is, we write xP_Xy and read it as: x is preferred over (or better than) y , relative to X . Intuitively, xP_Xy means that x is closer to the optimal alternatives in X than y . We shall refer to ternary relations $P \subseteq U \times \wp(U) \times U$ as (*relativised preference relations*). Properties of relations like reflexivity, transitivity, being a weak ordering, etc., carry over to relativised preference relations as follows: a given property is said to

apply to P iff for each X , P_X has the property in question. A dyadic selection function $S: V \times V \rightarrow V$ is said to be *based on* a given preference relation if for all $X, Y \in V$,

$$S(X, Y) = \{x : x \in Y \ \& \ (\forall y)(y \in Y \rightarrow \neg(yP_Xx))\}.$$

We have now introduced the concepts that are required in order to state the following representation theorem.

Theorem 6.2. (*Representation Theorem II*) *Let C be a belief revision operation based on the deductive logic L . Let $\mathcal{M} = \langle U, V, \mathfrak{I}, S \rangle$ be the canonical model for L and C . Then, $Cn_L = Cn^{\mathcal{M}}$, for all Γ, Δ , $C_{\Delta}(\Gamma) = C_{\Delta}^{\mathcal{M}}(\Gamma)$ and for all $X, Y \in V$:*

- (i) if $Y \neq \emptyset$, then $S(X, Y) \neq \emptyset$; (*Consistency Preservation*)
- (ii) if $X \cap Y \neq \emptyset$, then $S(X, Y) = X \cap Y$. (*Expansion*)

Moreover,

- (a) If C , in addition to the basic axioms (BC1)–(BC5), also satisfies axioms (BC6) (*Chernoff*) and (BC7) (*Gamma*), then there exists a (relativised) preference relation $P \subseteq U \times \wp(U) \times U$ such that P is neat and S is based on P .
- (b) If C satisfies the conditions (BC1)–(BC8), then there exists a relation $P \subseteq U \times \wp(U) \times U$ such that P is neat and transitive and S is based on P .
- (c) If C satisfies (BC1)–(BC5) together with (BC9) (*Arrow*), then there exists a relation $P \subseteq U \times \wp(U) \times U$ such that P is neat strict weak ordering and S is based on P .

Proof. (to be written)

We are next going to prove that any (infinitary) belief revision operation C that satisfies (BC1)–(BC5) together with (BC9) can be defined in terms of “systems of spheres” of the kind defined in Grove (1988). Theorems 6.4 and 6.5 below for infinitary belief revision operations should be compared with Grove’s (1988) Theorems 1 and 2 for finitary belief revision operations. In the following, we let L be a fixed deductive logic and U the set \mathcal{M}_L of all L -maximal theories. V is the set of all closed subsets of U .

Definition 6.3.

(a) A *family of spheres* centred on $X \in V$ is a collection $\$X$ of elements in V satisfying the conditions:²⁵

- (\\$1) $\$X$ is totally ordered by \subseteq , that is, if $Y, Z \in \$X$, then $Y \subseteq Z$ or $Z \subseteq Y$;
- (\\$2) X is the \subseteq -minimum of $\$X$, that is $X \in \$X$ and for all $Y \in \$X$, $X \subseteq Y$;
- (\\$3) $U \in \$X$;
- (\\$4) if $Y \in V$, then there exists an element Z of $\$X$, such that $Y \cap Z \neq \emptyset$ and for all $Z' \in \$X$, if $Y \cap Z' \neq \emptyset$, then $Z \subseteq Z'$. In other words, for every closed set $Y \in V$, there exists a smallest sphere in $\$X$ intersecting Y .

(b) A *system of spheres* is a function $\$$ that associates a family of spheres with any set $X \in V$.

²⁵(\\$4) is a strengthening of Grove’s (1988) limit assumption. For a discussion of the limit assumption in the context of possible worlds semantics for counterfactuals, see Lewis (1973).

Let $P \subseteq U \times U$ be a neat strict weak ordering of U . As usual xRy is defined as $\neg yPx$. We say that X is a P -sphere iff $X \in V$ and $(\forall x, y)(x \in X \text{ and } xRy \rightarrow y \in X)$. The set $\$$ of all P -spheres is then a family of spheres centred around $\bigcap \$$.

With any system of spheres $\$X$, we may associate a dyadic selection function $S: V \times V \rightarrow V$ in the following way: for all $X, Y \in V$, (i) if $Y \neq \emptyset$, we let $S(X, Y) = Z_0 \cap Y$, where Z_0 is the smallest sphere in $\$X$ that intersects Y ; and (ii) $S(X, \emptyset) = \emptyset$.

Theorem 6.4. *Let $\$$ be any system of spheres and let S be the associated dyadic selection function. For all Γ and Δ , let $C_\Delta(\Gamma) = t(S(\llbracket \Delta \rrbracket, \llbracket \Gamma \rrbracket))$. Then C is a belief revision operation, satisfying axioms $(BC1)–(BC5)$, $(BC9)$.*

Theorem 6.5. *(Representation Theorem III) Let C be any belief revision operation satisfying axioms $(BC1)–(BC5)$ and $(BC9)$. Then, there exists a system of spheres $\$$ such that for all Γ, Δ ,*

$$C_\Delta(\Gamma) = t(S(\llbracket \Delta \rrbracket, \llbracket \Gamma \rrbracket)),$$

where S is the dyadic selection function associated with $\$$.

(Proofs will be added)

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APPENDIX: PROOFS OF LEMMAS AND THEOREMS

Proof of Lemma 2.1. We omit the easy verification of (i) and proceed to prove (ii). By (Cn1), $\Gamma \cup \Delta \subseteq \Gamma \cup Cn(\Delta)$. (Cn2) then yields:

$$Cn(\Gamma \cup \Delta) \subseteq Cn(\Gamma \cup Cn(\Delta)).$$

We also have, by (Cn1) and (Cn2), that:

$$Cn(\Gamma \cup Cn(\Delta)) \subseteq Cn(Cn(\Gamma) \cup Cn(\Delta)).$$

It remains to prove that $Cn(Cn(\Gamma) \cup Cn(\Delta)) \subseteq Cn(\Gamma \cup \Delta)$. By Monotonicity we get:

$$\Gamma \subseteq Cn(\Gamma \cup \Delta) \text{ and } \Delta \subseteq Cn(\Gamma \cup \Delta).$$

(i) then yields:

$$Cn(\Gamma) \subseteq Cn(\Gamma \cup \Delta) \text{ and } Cn(\Delta) \subseteq Cn(\Gamma \cup \Delta).$$

Hence,

$$Cn(\Gamma) \cup Cn(\Delta) \subseteq Cn(\Gamma \cup \Delta).$$

Monotonicity then yields:

$$Cn(Cn(\Gamma) \cup Cn(\Delta)) \subseteq Cn(Cn(\Gamma \cup \Delta)).$$

Finally, using (Cn3), we get:

$$Cn(Cn(\Gamma) \cup Cn(\Delta)) \subseteq Cn(\Gamma \cup \Delta). \quad \square$$

Proof of Lemma 2.3. Suppose Γ is L -maximal. We prove that $Cn_L(\Gamma) \subseteq \Gamma$. By (Cn1), $\Gamma \subseteq Cn_L(\Gamma)$. Suppose that $Cn_L(\Gamma)$ is not L -consistent. Then, $Cn_L(\Gamma) \vdash_L \perp$, that is, $\perp \in Cn_L(Cn_L(\Gamma))$. By (Cn3), $\perp \in Cn_L(\Gamma)$, which is impossible. Hence, $Cn_L(\Gamma)$ is an L -consistent superset of Γ . It follows by the L -maximality of Γ that $\Gamma = Cn_L(\Gamma)$. \square

Proof of Lemma 2.4. First we notice that (a) follows from (b). Substituting \perp for α in (a) we get: If Γ is L -consistent, then $Cn_L(\Gamma)$ is included in an L -maximal theory. (a) then follows, by Inclusion.

Next, we prove (b). Suppose $\alpha \notin Cn_L(\Gamma)$. Let

$$X = \{\Delta : \Delta \text{ is } L\text{-consistent, } Cn_L(\Gamma) \subseteq \Delta \text{ and } \alpha \notin \Delta\}.$$

X is non-empty, since $Cn_L(\Gamma) \in X$ (the claim that $Cn_L(\Gamma)$ is L -consistent presupposes Iteration (i.e., Cut)). Let Y be any non-empty chain in X . Consider $\Sigma = \bigcup Y$. Clearly $Cn_L(\Gamma) \subseteq \Sigma$. We claim that Σ is L -consistent. Indirect proof: Suppose not. Then $\Sigma \vdash_L \perp$. By Finiteness, there is a finite $\Sigma' \subseteq \Sigma$ such that $\Sigma' \vdash_L \perp$. But then, since Σ is simply ordered by inclusion and Σ' is finite, there must exist some $\Delta \in Y$ such that $\Sigma' \subseteq \Delta$. It follows, by Monotonicity, that Δ is L -inconsistent. Contradiction. It remains to prove that $\alpha \notin \Sigma$. But this is clear, since otherwise $\alpha \in \Delta$, for some $\Delta \in Y$, which is impossible. Thus, Σ satisfies the necessary conditions for being a member of X . It follows by Zorn's Lemma that X has a maximal element. \square

Proof of Lemma 3.4. (\Rightarrow) Suppose that L is a classical logic and that \vdash is an inference relation based on L . We verify that \vdash satisfies conditions ($\vdash 4$)–($\vdash 7$).

($\vdash 4$) Suppose that $Cn_L(\{\alpha\}) = Cn_L(\{\beta\})$. Then, $\alpha \vdash_L \beta$ and $\beta \vdash_L \alpha$. $\Gamma \vdash \alpha$ and $\alpha \vdash_L \beta$ yield via ($\vdash 2$) that $\Gamma \vdash \beta$. Similarly, from $\Gamma \vdash \beta$ and $\beta \vdash_L \alpha$, one obtains $\Gamma \vdash \alpha$. Thus, $\Gamma \vdash \alpha$ iff $\Gamma \vdash \beta$.

($\vdash 5$) Suppose that $\Gamma \vdash \alpha \wedge \beta$. Since L is classical, $\alpha \wedge \beta \vdash_L \alpha$ and $\alpha \wedge \beta \vdash_L \beta$. Hence, by ($\vdash 2$), $\Gamma \vdash \alpha$ and $\Gamma \vdash \beta$.

($\vdash 6$) Suppose that $\Gamma \vdash \alpha$ and $\Gamma \vdash \beta$. L being classical yields $\alpha, \beta \vdash_L \alpha \wedge \beta$. Hence, by ($\vdash 2$), $\Gamma \vdash \alpha \wedge \beta$.

($\vdash 7$) Letting $\Delta = \emptyset$ and $\beta = \top$ in ($\vdash 2$), we get:

if $\Gamma \vdash \alpha$, for all $\alpha \in \emptyset$, and $\emptyset \vdash_L \top$, then $\Gamma \vdash \top$.

$\Gamma \vdash \alpha$, for all $\alpha \in \emptyset$ holds vacuously and $\emptyset \vdash_L \top$ holds because L is classical. Hence, $\Gamma \vdash \top$.

(\Leftarrow) Suppose that L is a classical logic and that \vdash satisfies ($\vdash 1$), ($\vdash 3$)–($\vdash 7$). We must prove that \vdash then also satisfies ($\vdash 2$).

Suppose that $\Gamma \vdash \alpha$, for all $\alpha \in \Delta$, and $\Delta \vdash_L \beta$. We wish to prove that $\Gamma \vdash_L \beta$.

We first consider the case when $\Delta \neq \emptyset$. Then there are $\beta_1, \dots, \beta_n \in \Delta$ ($n \geq 0$) such that $\Gamma \vdash \beta_1, \dots, \Gamma \vdash \beta_n$ and $\{\beta_1, \dots, \beta_n\} \vdash_L \beta$. Using ($\vdash 4$) and ($\vdash 6$), we get that $\Gamma \vdash \beta_1 \wedge \dots \wedge \beta_n$. Furthermore, $\beta_1 \wedge \dots \wedge \beta_n \vdash_L \beta$. This implies that $\emptyset \vdash_L ((\beta_1 \wedge \dots \wedge \beta_n) \wedge \beta) \leftrightarrow \beta_1 \wedge \dots \wedge \beta_n$. Using ($\vdash 4$), we get $\Gamma \vdash_L (\beta_1 \wedge \dots \wedge \beta_n) \wedge \beta$. This yields by ($\vdash 5$) that $\Gamma \vdash \beta$.

We now consider the case when $\Delta = \emptyset$. We need to prove that if $\emptyset \vdash_L \beta$, then $\Gamma \vdash \beta$. But if $\emptyset \vdash_L \beta$, then $\emptyset \vdash_L (\beta \leftrightarrow \top)$. We have $\Gamma \vdash \top$, by ($\vdash 7$). Hence, $\Gamma \vdash \beta$, by ($\vdash 4$). \square

Proof of Lemma 4.8.

(i) Suppose that S satisfies condition (c). By the definition of a model $S(S(X)) \subseteq S(X) \subseteq X$. Hence, by condition (c), $S(X) \subseteq S(S(X))$. Thus, $S(S(X)) = S(X)$.

(ii) Assume that $S(X) \subseteq Y \subseteq X$. By Chernoff:

$$S(X) \cap Y \subseteq S(X \cap Y).$$

By the assumption: $S(X) \cap Y = S(X)$ and $X \cap Y = Y$. Hence, $S(X) \subseteq S(Y)$.

(iii) We assume that Chernoff holds. It follows that:

$$S(X \cup Y) \cap X \subseteq S((X \cup Y) \cap X); \text{ and}$$

$$S(X \cup Y) \cap Y \subseteq S((X \cup Y) \cap Y).$$

However, $(X \cup Y) \cap X = X$ and $(X \cup Y) \cap Y = Y$. So,

$$S(X \cup Y) \cap X \subseteq S(X); \text{ and}$$

$$S(X \cup Y) \cap Y \subseteq S(Y).$$

Since $S(X \cup Y) \subseteq X \cup Y$, we get:

$$S(X \cup Y) \subseteq S(X) \cup S(Y).$$

(iv) Suppose that S satisfies Cut, Aizerman and Distributivity. By Distribution and the definition of a selection function:

$$S(X \cup Y) \subseteq S(X) \cup S(Y) \subseteq X \cup Y.$$

From this we get, using Cut and Aizerman:

$$S(S(X) \cup S(Y)) = S(X \cup Y)$$

which is Path Independence.

We next prove that for full models, Path Independence is equivalent to Chernoff and Aizerman. Since Chernoff implies Cut and Distribution, we have in general that Chernoff and Aizerman imply Path Independence.

In order to prove the other direction, suppose that S is the selection function of a full model which satisfies Path Independence. First, we prove Chernoff in the formulation:

(α) if $Y \subseteq X$, then $S(X) \cap Y \subseteq S(Y)$.

Suppose $Y \subseteq X$. Applying Path Independence to the sets Y and $X - Y$, we get:

$$(1) \quad S(X) = S(Y \cup (X - Y)) = S(S(Y) \cup S(X - Y)).$$

Here, we need the assumption of fullness in order to be sure that $X - Y \in \mathcal{V}$. But,

$$(2) \quad S(S(Y) \cup S(X - Y)) \subseteq S(Y) \cup S(X - Y) \subseteq S(Y) \cup (X - Y).$$

(1) and (2) yield:

$$S(X) \subseteq S(Y) \cup (X - Y).$$

Hence,

$$S(X) \cap Y \subseteq (S(Y) \cup (X - Y)) \cap Y.$$

That is,

$$S(X) \cap Y \subseteq S(Y).$$

To prove that Path Independence, in the presence of fullness, implies Aizerman, suppose that $S(X) \subseteq Y \subseteq X$. Cut (which follows from Chernoff) then yields $S(X) \subseteq S(Y)$. Path Independence yields:

$$S(X) = S(X \cup Y) = S(S(X) \cup S(Y)) = S(S(Y)).$$

Since Chernoff yields Iteration, we also have $S(S(Y)) = S(Y)$. Thus, $S(X) = S(Y)$. We have shown:

(aiz) if $S(X) \subseteq Y \subseteq X$, then $S(Y) \subseteq S(X)$.

(v) Suppose that S satisfies Consistency Preservation and Arrow. Suppose $Y \subseteq X$. If $S(X) \cap Y = \emptyset$, then clearly $S(X) \cap Y \subseteq S(Y)$. However, if $S(X) \cap Y \neq \emptyset$, we have $S(X) \cap Y \subseteq S(Y)$, by Arrow. Hence, we have derived (α) which is equivalent to Chernoff. (Notice, that we did not use (cp) in this derivation.)

In order to prove Aizerman, assume that $S(X) \subseteq Y \subseteq X$. If $S(X) \cap Y = \emptyset$, then $S(X) = \emptyset$. Consistency Preservation then yields $X = \emptyset$ and $Y = \emptyset$. Hence, $\emptyset = S(Y) \subseteq S(X)$, in this case. Hence, we may assume that $S(X) \cap Y \neq \emptyset$. Arrow then yields:

$$S(X) \cap Y = S(Y).$$

But since $S(X) \subseteq Y$, we get $S(X) = S(Y)$.

Finally, we prove Gamma. Let $\emptyset \neq F \subseteq V$ such that $\bigcup_{X \in F} X \in V$. If $X = \emptyset$, for all $X \in F$, then Gamma holds trivially. We therefore suppose that for some $X \in F$, $X \neq \emptyset$. Then, $\bigcup_{X \in F} X \neq \emptyset$. By Consistency Preservation $S(\bigcup_{X \in F} X) \neq \emptyset$. Hence, for some $Y \in F$, $S(\bigcup_{X \in F} X) \cap Y \neq \emptyset$. Arrow then yields:

If $Y \subseteq \bigcup_{X \in F} X$ and $S(\bigcup_{X \in F} X) \cap Y \neq \emptyset$, then $S(\bigcup_{X \in F} X) \cap Y = S(Y)$.

Thus, $S(\bigcup_{X \in F} X) \cap Y = S(Y)$. It follows that $S(Y) \subseteq S(\bigcup_{X \in F} X)$. Since $\bigcap_{X \in F} S(X) \subseteq S(Y)$, we obtain:

$$\bigcap_{X \in F} S(X) \subseteq S(\bigcup_{X \in F} X).$$

(vi) We first assume that S satisfies Arrow. As we have already shown in (v) above, S then satisfies Chernoff. To prove Sen, assume that $X \subseteq Y$ and $S(X) \cap S(Y) \neq \emptyset$. By Arrow, we then get:

$$S(Y \cap S(X)) = S(Y) \cap S(X).$$

But, $S(X) \subseteq X \subseteq Y$. Hence, $Y \cap S(X) = S(X)$. We get:

$$S(S(X)) = S(Y) \cap S(X).$$

But Arrow implies Chernoff which in turn implies iteration: $S(S(X)) = S(X)$. Hence,

$$S(X) = S(Y) \cap S(X),$$

that is, $S(X) \subseteq S(Y)$.

For the other direction, assume that S satisfies Chernoff and Sen. We prove Arrow in the form: If $Y \subseteq X$ and $S(X) \cap Y \neq \emptyset$, then $S(X) \cap Y = S(Y)$. Thus, assume that $Y \subseteq X$ and $S(X) \cap Y \neq \emptyset$. By Chernoff, we have: $S(X) \cap Y \subseteq S(X \cap Y) = S(Y)$. It follows that $S(X) \cap Y \subseteq S(X) \cap S(Y)$ and that $S(X) \cap S(Y) \neq \emptyset$. Sen then yields $S(Y) \subseteq S(X)$. Hence, $S(Y) \subseteq S(X) \cap Y$. But we already have $S(X) \cap Y \subseteq S(Y)$. Thus, $S(X) \cap Y = S(Y)$. \square

Proof of Theorem 4.9. Let $\mathcal{M} = \langle U, V, I, S \rangle$ be a canonical model based on the $\{\wedge, \vee\}$ normal deductive logic L and let $C = C_{\mathcal{M}}$. Then, we have: $C(\Gamma) = t(S(\llbracket \Gamma \rrbracket))$ and $S(X) = \llbracket C(t(X)) \rrbracket$.

Consistency Preservation: We assume (cp) and prove (CP). Suppose that $\perp \notin Cn(\Gamma)$. Then $\llbracket \Gamma \rrbracket \neq \emptyset$, so by (cp), $S(\llbracket \Gamma \rrbracket) \neq \emptyset$. But $S(\llbracket \Gamma \rrbracket) = \llbracket t(S(\llbracket \Gamma \rrbracket)) \rrbracket = \llbracket C(\Gamma) \rrbracket$. Thus, $\llbracket C(\Gamma) \rrbracket \neq \emptyset$. It follows that $C(\Gamma)$ is L -consistent.

For the other direction, assume that C satisfies (CP). Also assume that $X \neq \emptyset$. Then, $t(X)$ is L -consistent, so it follows by (CP) that $C(t(X))$ is L -consistent. Hence, $\llbracket C(t(X)) \rrbracket \neq \emptyset$, that is, $S(X) \neq \emptyset$.

Iteration: We assume (it) and prove (It). $C(\Gamma) = t(S(\llbracket \Gamma \rrbracket)) =$ by (it) $t(S(S(\llbracket \Gamma \rrbracket))) = t(S(\llbracket t(S(\llbracket \Gamma \rrbracket)) \rrbracket)) = C(t(S(\llbracket \Gamma \rrbracket))) = C(C(\Gamma))$.

Next, we assume (It). Then, $S(X) = \llbracket C(t(X)) \rrbracket = \llbracket C(C(t(X))) \rrbracket = \llbracket C(Cn(C(t(X)))) \rrbracket = \llbracket C(t(\llbracket C(t(X)) \rrbracket)) \rrbracket = S(\llbracket C(t(X)) \rrbracket) = S(S(X))$.

Cut: First, we assume (c) and prove (C). Suppose $\Gamma \subseteq \Delta \subseteq C(\Gamma)$. It follows that: $\llbracket C(\Gamma) \rrbracket \subseteq \llbracket \Delta \rrbracket \subseteq \llbracket \Gamma \rrbracket$. Hence, $\llbracket t(S(\llbracket \Gamma \rrbracket)) \rrbracket \subseteq \llbracket \Delta \rrbracket \subseteq \llbracket \Gamma \rrbracket$. But $S(\llbracket \Gamma \rrbracket) = \llbracket t(S(\llbracket \Gamma \rrbracket)) \rrbracket$, so $S(\llbracket \Gamma \rrbracket) \subseteq \llbracket \Delta \rrbracket \subseteq \llbracket \Gamma \rrbracket$. Condition (c) now yields: $S(\llbracket \Gamma \rrbracket) \subseteq S(\llbracket \Delta \rrbracket)$. Hence, $t(S(\llbracket \Delta \rrbracket)) \subseteq t(S(\llbracket \Gamma \rrbracket))$. Finally, we get $C(\Delta) \subseteq C(\Gamma)$.

In order to prove the converse, assume (C) and that $S(X) \subseteq Y \subseteq X$. Then we get: $t(X) \subseteq t(Y) \subseteq t(S(X))$. But, $S(X) = S(\llbracket t(X) \rrbracket)$, so it follows that $t(X) \subseteq t(Y) \subseteq t(S(\llbracket t(X) \rrbracket))$. $t(S(\llbracket t(X) \rrbracket)) = C(t(X))$. Hence, $t(X) \subseteq t(Y) \subseteq C(t(X))$. Condition (C) now yields, $C(t(Y)) \subseteq C(t(X))$. Thus, $\llbracket C(t(X)) \rrbracket \subseteq \llbracket C(t(Y)) \rrbracket$. Finally, we obtain $S(X) \subseteq S(Y)$.

Distributivity: First, we prove (D) from (d). By (d), we get:

$$S(\llbracket \Gamma \rrbracket \cup \llbracket \Delta \rrbracket) \subseteq S(\llbracket \Gamma \rrbracket) \cup S(\llbracket \Delta \rrbracket).$$

However, $t(\llbracket \Gamma \rrbracket \cup \llbracket \Delta \rrbracket) = Cn(\Gamma) \cap Cn(\Delta)$. Hence,

$$\llbracket t(\llbracket \Gamma \rrbracket \cup \llbracket \Delta \rrbracket) \rrbracket = \llbracket Cn(\Gamma) \cap Cn(\Delta) \rrbracket.$$

It follows that:

$$S(\llbracket \Gamma \rrbracket \cup \llbracket \Delta \rrbracket) = S(\llbracket t(\llbracket \Gamma \rrbracket \cup \llbracket \Delta \rrbracket) \rrbracket) = S(\llbracket Cn(\Gamma) \cap Cn(\Delta) \rrbracket).$$

Hence, we have:

$$S(\llbracket Cn(\Gamma) \cap Cn(\Delta) \rrbracket) \subseteq S(\llbracket \Gamma \rrbracket) \cup S(\llbracket \Delta \rrbracket).$$

However, this in turn implies:

$$\llbracket C(t(\llbracket Cn(\Gamma) \cap Cn(\Delta) \rrbracket)) \rrbracket \subseteq \llbracket C(t(\llbracket \Gamma \rrbracket)) \rrbracket \cup \llbracket C(t(\llbracket \Delta \rrbracket)) \rrbracket.$$

That is,

$$\llbracket C(Cn(\Gamma) \cap Cn(\Delta)) \rrbracket \subseteq \llbracket C(\Gamma) \rrbracket \cup \llbracket C(\Delta) \rrbracket.$$

But, $\llbracket C(\Gamma) \rrbracket \cup \llbracket C(\Delta) \rrbracket \subseteq \llbracket C(\Gamma) \cap C(\Delta) \rrbracket$, so

$$\llbracket C(Cn(\Gamma) \cap Cn(\Delta)) \rrbracket \subseteq \llbracket C(\Gamma) \cap C(\Delta) \rrbracket.$$

It follows that:

$$t(\llbracket C(\Gamma) \cap C(\Delta) \rrbracket) \subseteq t(\llbracket C(Cn(\Gamma) \cap Cn(\Delta)) \rrbracket).$$

That is:

$$C(\Gamma) \cap C(\Delta) \subseteq C(Cn(\Gamma) \cap Cn(\Delta)).$$

In order to prove the converse, assume (D). Then, we have:

$$C(t(X)) \cap C(t(Y)) \subseteq C(t(\llbracket t(X) \rrbracket)) \cap t(\llbracket t(Y) \rrbracket).$$

That is,

$$C(t(X)) \cap C(t(Y)) \subseteq C(t(X) \cap t(Y)).$$

But, $t(X) \cap t(Y) = t(X \cup Y)$, so:

$$C(t(X)) \cap C(t(Y)) \subseteq C(t(X \cup Y)).$$

In other words,

$$t(S(X)) \cap t(S(Y)) \subseteq t(S(X \cup Y)).$$

Since $t(S(X)) \cap t(S(Y)) = t(S(X) \cup S(Y))$,

$$t(S(X) \cup S(Y)) \subseteq t(S(X \cup Y)).$$

This in turn yields:

$$Cl(S(X \cup Y)) \subseteq Cl(S(X) \cup S(Y)).$$

But according to Lemma 4.2, $Cl(S(X) \cup S(Y)) = Cl(S(X)) \cup Cl(S(Y)) = S(X) \cup S(Y)$. Thus, finally, we get:

$$S(X \cup Y) \subseteq S(X) \cup S(Y).$$

Chernoff: First, we prove (Ch) from (ch). By (ch), we have:

$$S(\llbracket \Gamma \rrbracket) \cap \llbracket \Delta \rrbracket \subseteq S(\llbracket \Gamma \rrbracket \cap \llbracket \Delta \rrbracket).$$

That is:

$$\llbracket C(t(\llbracket \Gamma \rrbracket)) \rrbracket \cap \llbracket \Delta \rrbracket \subseteq \llbracket C(t(\llbracket \Gamma \rrbracket \cap \llbracket \Delta \rrbracket)) \rrbracket.$$

Hence,

$$t(\llbracket C(t(\llbracket \Gamma \rrbracket \cap \llbracket \Delta \rrbracket)) \rrbracket) \subseteq t(\llbracket C(t(\llbracket \Gamma \rrbracket)) \rrbracket) \cap \llbracket \Delta \rrbracket.$$

But, $\llbracket C(t(\llbracket \Gamma \rrbracket)) \rrbracket \cap \llbracket \Delta \rrbracket = \llbracket C(t(\llbracket \Gamma \rrbracket)) \cup \Delta \rrbracket$, so:

$$t(\llbracket C(t(\llbracket \Gamma \rrbracket \cap \llbracket \Delta \rrbracket)) \rrbracket) \subseteq t(\llbracket C(t(\llbracket \Gamma \rrbracket)) \cup \Delta \rrbracket).$$

That is,

$$C(t(\llbracket \Gamma \rrbracket \cap \llbracket \Delta \rrbracket)) \subseteq Cn(C(\Gamma) \cup \Delta).$$

$t(\llbracket \Gamma \rrbracket \cap \llbracket \Delta \rrbracket) = Cn(\Gamma \cup \Delta)$, and $C(Cn(\Gamma \cup \Delta)) = C(\Gamma \cup \Delta)$, so

$$C(\Gamma \cup \Delta) \subseteq Cn(C(\Gamma) \cup \Delta).$$

For the other direction, assume (Ch). Then, we have for any $X, Y \in V$:

$$C(t(X) \cup t(Y)) \subseteq Cn(C(t(X)) \cup t(Y)).$$

Hence,

$$C(t(X \cap Y)) \subseteq Cn(t(S(X)) \cup t(Y)).$$

That is,

$$t(S(X \cap Y)) \subseteq t(S(X) \cap Y).$$

Hence,

$$\llbracket t(S(X) \cap Y) \rrbracket \subseteq \llbracket t(S(X \cap Y)) \rrbracket.$$

But, $S(X) \cap Y \subseteq \llbracket t(S(X) \cap Y) \rrbracket$ and $\llbracket t(S(X \cap Y)) \rrbracket = S(X \cap Y)$. Thus, it follows that:

$$S(X) \cap Y \subseteq S(X \cap Y).$$

Aizerman: First, we assume (aiz) and prove (Aiz). Suppose that $\Gamma \subseteq \Delta \subseteq C(\Gamma)$. It follows that $\llbracket C(\Gamma) \rrbracket \subseteq \llbracket \Delta \rrbracket \subseteq \llbracket \Gamma \rrbracket$. Hence, $S(\llbracket \Gamma \rrbracket) \subseteq \llbracket \Delta \rrbracket \subseteq \llbracket \Gamma \rrbracket$. Condition (aiz) then yields $S(\llbracket \Delta \rrbracket) \subseteq S(\llbracket \Gamma \rrbracket)$. This in turn implies: $t(S(\llbracket \Gamma \rrbracket)) \subseteq t(S(\llbracket \Delta \rrbracket))$, that is, $C(\Gamma) \subseteq C(\Delta)$.

In order to prove the converse, assume (Aiz) and $S(X) \subseteq Y \subseteq X$. Then we get: $t(X) \subseteq t(Y) \subseteq t(S(X))$, that is, $t(X) \subseteq t(Y) \subseteq C(t(X))$. (Aiz) then yields: $C(t(X)) \subseteq C(t(Y))$. This, in turn, implies: $t(S(X)) \subseteq t(S(Y))$. Hence, $\llbracket t(S(Y)) \rrbracket \subseteq \llbracket t(S(X)) \rrbracket$, that is, $S(Y) \subseteq S(X)$.

Gamma: First, we assume (g) and prove (G). Condition (g) yields:

$$\bigcap_{\Gamma \in F} S(\llbracket \Gamma \rrbracket) \subseteq S(\bigcup_{\Gamma \in F} \llbracket \Gamma \rrbracket),$$

where F is a non-empty family of sets of sentences. Now, for each Γ , $S(\llbracket \Gamma \rrbracket) = \llbracket C(\Gamma) \rrbracket$. We also have: $S(\bigcup_{\Gamma \in F} \llbracket \Gamma \rrbracket) = S(Cl(\bigcup_{\Gamma \in F} \llbracket \Gamma \rrbracket))$. Thus,

$$(1) \bigcap_{\Gamma \in F} \llbracket C(\Gamma) \rrbracket \subseteq S(Cl(\bigcup_{\Gamma \in F} \llbracket \Gamma \rrbracket)).$$

However, $t(\bigcup_{\Gamma \in F} \llbracket \Gamma \rrbracket) = \bigcap_{\Gamma \in F} Cn(\Gamma)$, so $Cl(\bigcup_{\Gamma \in F} \llbracket \Gamma \rrbracket) = \llbracket \bigcap_{\Gamma \in F} Cn(\Gamma) \rrbracket$. (1) then yields:

$$\bigcap_{\Gamma \in F} \llbracket C(\Gamma) \rrbracket \subseteq S(\llbracket \bigcap_{\Gamma \in F} Cn(\Gamma) \rrbracket).$$

This, in turn, yields:

$$\bigcap_{\Gamma \in F} \llbracket C(\Gamma) \rrbracket \subseteq \llbracket C(\bigcap_{\Gamma \in F} Cn(\Gamma)) \rrbracket.$$

Hence,

$$t(\llbracket C(\bigcap_{\Gamma \in F} Cn(\Gamma)) \rrbracket) \subseteq t(\bigcap_{\Gamma \in F} \llbracket C(\Gamma) \rrbracket).$$

But, $t(\llbracket C(\bigcap_{\Gamma \in F} Cn(\Gamma)) \rrbracket) = C(\bigcap_{\Gamma \in F} Cn(\Gamma))$ and $t(\bigcap_{\Gamma \in F} \llbracket C(\Gamma) \rrbracket) = Cn(\bigcup_{\Gamma \in F} C(\Gamma))$. Hence, we get:

$$C(\bigcap_{\Gamma \in F} Cn(\Gamma)) \subseteq Cn(\bigcup_{\Gamma \in F} C(\Gamma)).$$

Next, we assume (G) and prove (g). By (G), we have:

$$C(\bigcap_{X \in F} Cn(t(X))) \subseteq Cn(\bigcup_{X \in F} C(t(X))),$$

where F is any non-empty family of elements in V such that $\bigcup_{X \in F} X \in V$. Simplifying and using: $C(t(X)) = t(S(X))$, we get:

$$C(\bigcap_{X \in F} t(X)) \subseteq Cn(\bigcup_{X \in F} t(S(X))).$$

But, $C(\bigcap_{X \in F} t(X)) = t(S(\llbracket \bigcap_{X \in F} t(X) \rrbracket))$, so:

$$t(S(\llbracket \bigcap_{X \in F} t(X) \rrbracket)) \subseteq Cn(\bigcup_{X \in F} t(S(X))).$$

Hence,

$$\llbracket Cn(\bigcup_{X \in F} t(S(X))) \rrbracket \subseteq \llbracket t(S(\llbracket \bigcap_{X \in F} t(X) \rrbracket)) \rrbracket.$$

It follows that:

$$\llbracket \bigcup_{X \in F} t(S(X)) \rrbracket \subseteq S(\llbracket \bigcap_{X \in F} t(X) \rrbracket).$$

But, $\llbracket \bigcup_{X \in F} t(S(X)) \rrbracket = \bigcap_{X \in F} \llbracket t(S(X)) \rrbracket$, so we get:

$$\bigcap_{X \in F} \llbracket t(S(X)) \rrbracket \subseteq S(\llbracket \bigcap_{X \in F} t(X) \rrbracket).$$

However, $\bigcap_{X \in F} t(X) = t(\bigcup_{X \in F} X)$. Hence, $\llbracket \bigcap_{X \in F} t(X) \rrbracket = Cl(\bigcup_{X \in F} X)$. We get:

$$\bigcap_{X \in F} \llbracket t(S(X)) \rrbracket \subseteq S(Cl(\bigcup_{X \in F} X)).$$

But, $\llbracket t(S(X)) \rrbracket = S(X)$ and $S(Cl(\bigcup_{X \in F} X)) = S(\bigcup_{X \in F} X)$, so we finally get:

$$\bigcap_{X \in F} S(X) \subseteq S(\bigcup_{X \in F} X).$$

Sen: We first assume (s) and prove (S). Suppose that $C(\Gamma) \cup C(\Delta) = t(S(\llbracket \Gamma \rrbracket)) \cup t(S(\llbracket \Delta \rrbracket))$ is L -consistent. It follows that $S(\llbracket \Gamma \rrbracket) \cap S(\llbracket \Delta \rrbracket) \neq \emptyset$. Hence, by (s),

$$S(\llbracket \Gamma \rrbracket \cap \llbracket \Delta \rrbracket) \subseteq S(\llbracket \Gamma \rrbracket) \cap S(\llbracket \Delta \rrbracket).$$

It follows that,

$$t(S(\llbracket \Gamma \rrbracket) \cap S(\llbracket \Delta \rrbracket)) \subseteq t(S(\llbracket \Gamma \rrbracket) \cap \llbracket \Delta \rrbracket)).$$

That is,

$$Cn(t(S(\llbracket \Gamma \rrbracket)) \cup t(S(\llbracket \Delta \rrbracket))) \subseteq t(S(\llbracket \Gamma \cup \Delta \rrbracket)).$$

In other words:

$$Cn(C(\Gamma) \cup C(\Delta)) \subseteq C(\Gamma \cup \Delta).$$

But $C(\Gamma) \cup C(\Delta) \subseteq Cn(C(\Gamma) \cup C(\Delta))$, so we get:

$$C(\Gamma) \cup C(\Delta) \subseteq C(\Gamma \cup \Delta).$$

In order to prove the other direction, we assume that $S(X) \cap S(Y) \neq \emptyset$. Hence $t(S(X) \cap S(Y))$ is L -consistent. But, $t(S(X) \cap S(Y)) = Cn(t(S(X)) \cup t(S(Y))) = Cn(C(t(X)) \cup C(t(Y)))$. Thus, $C(t(X)) \cup C(t(Y))$ is L -consistent. It follows, by (S), that:

$$C(t(X)) \cup C(t(Y)) \subseteq C(t(X) \cup t(Y)).$$

But, $t(X) \cup t(Y) = t(X \cap Y)$. Hence,

$$C(t(X)) \cup C(t(Y)) \subseteq C(t(X \cap Y)).$$

This, however, means that:

$$t(S(X)) \cup t(S(Y)) \subseteq t(S(X \cap Y)).$$

But, $t(S(X)) \cup t(S(Y)) = t(S(X) \cap S(Y))$, so:

$$t(S(X) \cap S(Y)) \subseteq t(S(X \cap Y)).$$

Hence,

$$Cl(S(X \cap Y)) \subseteq Cl(S(X) \cap S(Y)).$$

But, $Cl(S(X \cap Y)) = S(X \cap Y)$ and the intersection of two closed sets is closed, so $Cl(S(X) \cap S(Y)) = S(X) \cap S(Y)$. Hence,

$$S(X \cap Y) \subseteq S(X) \cap S(Y).$$

Arrow: We first assume (IIA) and prove (iia). Suppose that $S(X) \cap Y \neq \emptyset$. That is, $\llbracket C(t(X)) \rrbracket \cap Y \neq \emptyset$, which means that $C(t(X)) \cup t(Y)$ is L -consistent. It follows by (IIA) that:

$$C(t(X) \cup t(Y)) = Cn(C(t(X)) \cup t(Y)).$$

This implies:

$$\llbracket C(t(X) \cup t(Y)) \rrbracket = \llbracket C(t(X)) \cup t(Y) \rrbracket = \llbracket C(t(X)) \rrbracket \cap \llbracket t(Y) \rrbracket.$$

But $C(t(X) \cup t(Y)) = C(Cn(t(X) \cup t(Y))) = C(t(X \cap Y))$. Hence,

$$\llbracket C(t(X \cap Y)) \rrbracket = \llbracket C(t(X)) \rrbracket \cap \llbracket t(Y) \rrbracket.$$

This means that:

$$S(X \cap Y) = S(X) \cap Cl(Y).$$

But $Y \in \mathcal{V}$, so $Cl(Y) = Y$. Hence,

$$S(X \cap Y) = S(X) \cap Y.$$

(Add proof of the other direction)

Proof of Lemma 5.5.

(a) Suppose that S is based on P and that P is neat. Then, we have that:

$$xPy \text{ iff } S(\{x, y\}) = \{x\}.$$

Now, assume that S satisfies Aizerman and that xPy and yPz . We want to show that xPz . Since, S is rationalisable, it satisfies Cut, so we have:

$$\text{if } S(X) \subseteq Y \subseteq X, \text{ then } S(Y) = S(X).$$

Now, let $X = \{x, y, z\}$ and $Y = \{x, z\}$. Since xPy and yPz , we have that $y \notin S(X)$ and $z \notin S(X)$. Since P is neat, S satisfies (cp). Thus, $S(X) \neq \emptyset$. It follows that $S(X) = \{x\}$. Thus, we have $S(X) \subseteq Y \subseteq X$. So by Aizerman (and Cut), $S(Y) = S(X)$, that is, $S(\{x, z\}) = \{x\}$. We conclude that xPz .

(b) Assume that P^{-1} is well-founded and that P is transitive. In order to prove Aizerman, assume that $S(X) \subseteq Y \subseteq X$. Let $x_0 \in S(Y)$. In order to derive a contradiction, we assume that

$x_0 \notin S(X)$. It follows that there exists some $x_1 \in X$ such that $x_1 P x_0$. If $x_1 \notin S(X)$, then there exists some $x_2 \in X$ such that $x_2 P x_1$, and so on. For any n , if $x_n \notin S(X)$, we choose x_{n+1} in such a way that $x_{n+1} P x_n$ and $x_{n+1} \in X$; and if $x_n \in S(X)$, we terminate the process. Since P^{-1} is well-founded, this process must terminate after a finite number of steps. Thus, we get a finite sequence (with at least two terms) x_0, x_1, \dots, x_n of elements in X such that $x_0 \notin S(X)$ and $x_n \in S(X)$ and $x_n P x_{n-1} P \dots P x_1 P x_0$. By the transitivity of P , $x_n P x_0$. Since $S(X) \subseteq Y$, we get that $x_n \in Y$. Thus, we have: $x_0 \in S(Y)$ (by assumption), $x_n P x_0$ and $x_n \in Y$, that is, a contradiction. Hence, we have proved that $S(Y) \subseteq S(X)$. \square