Canonical measures of subvarieties of abelian varieties



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Stefan Stadlöder

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Die Arbeit wurde angeleitet von:	Prof. Dr. Walter Gubler

Prüfungsausschuss:

Vorsitzende:	Prof. Dr. Clara Löh
Erst-Gutachter:	Prof. Dr. Walter Gubler
Zweit-Gutachter:	Prof. Dr. Klaus Künnemann
weiterer Prüfer:	Prof. Dr. Moritz Kerz
Ersatzprüfer:	Prof. Dr. Guido Kings

Abstract

Let $(\mathbb{K}, |.|)$ be an algebraically closed non-trivially valued non-archimedean field, Γ its value group, A an abelian $\mathbb{K}$-variety, L an ample line bundle on A, and X an irreducible closed $\mathbb{K}$-subvariety of A.

We will show that the support Σ_X of a canonical measure on X with respect to L is a Γ-rational piecewise linear space. This is a generalization of [Gub10, Theorem 6.12], where it is also assumed that the absolute value $|.|$ on $\mathbb{K}$ is induced by a discrete valuation.

Furthermore, we will introduce so-called toric metrics on L, which are defined as the invariant continuous metrics on L under the natural torus action on A provided by Raynaud's uniformization theory. We will then generalize [Gub10, Theorem 6.7], which gives an explicit formula for the non-archimedean Monge-Ampère measure with respect to the pull-back of a canonical metric along a certain strictly semi-stable alteration. This will be done with regards to three aspects: Firstly, we will prove the result more generally for semi-positive toric metrics instead of canonical metrics. Secondly, we will drop the additional assumption that the absolute value $|.|$ on $\mathbb{K}$ is induced by a discrete valuation. And thirdly, we will replace strictly semi-stable alterations by strictly poly-stable alterations.

This implies that a polytopal decomposition of Σ_X exists such that on its polytopes the non-archimedean Monge-Ampère measure with respect to a semi-positive toric metric $||.||$ on L equals a non-negative rational multiple of the real Monge-Ampère measure with respect to a certain convex function induced by $||.||$.

Contents

1 Introduction **7**

2 General basics **19**
2.1 Valuations 19
2.2 Category theory 20
2.3 Sheaves 23
2.4 Convex geometry 25
2.5 Abelian varieties and line bundles 30
2.6 Quadratic functions 32

3 Analytic and formal geometry **35**
3.1 The Berkovich spectrum 35
3.2 Affinoid algebras and affinoid spaces 37
3.3 Analytic spaces 40
3.4 Formal geometry 44

4 Measures **51**
4.1 Metrics 51
4.2 The non-archimedean Monge-Ampère measure 55
4.3 The real Monge-Ampère measure 57

5 Construction of a Mumford model **61**
5.1 Raynaud's uniformization theory 61
5.2 Mumford models 68
5.3 Line bundles on Mumford models 72
5.4 Metric-function correspondence and toric metrics 81
5.5 Generic and transversal polytopal decompositions 86

6 Strictly poly-stable formal schemes **105**
6.1 Strata and poly-simplices 105
6.2 Skeletons 107
6.3 Subdivisions of the skeleton 116
6.4 Application to a Mumford model 119

7 Main results **127**
7.1 Monge-Ampère measures of semi-positive toric metrics . . . 127
7.2 The canonical subset 134

Glossary of Notation **143**

References **151**

1 Introduction

Arithmetic geometry is the field in mathematics which applies techniques from algebraic geometry (for example schemes of finite type over the ring of integers $\mathcal{O}_K$ for a number field K) to problems in number theory (especially integer solutions of polynomial equations, where one can equivalently ask for rational points of curves). Central tools when looking for rational points of varieties are so-called heights [BG07]. Typically, they attach to any point of an irreducible projective variety over a number field or function field a non-negative real number, which measures the arithmetic complexity of the point. Of special interest are small points, which means points with a height smaller than a given constant $\epsilon > 0$.

The Bogomolov conjecture, which exists in different versions, claims that a non-special closed subvariety X of an abelian variety A cannot have dense small points. Its original version for number fields was proved by Ullmo [Ull98] for a curve inside its Jacobian and by Zhang [Zha98] in general. By assuming an archimedean valuation and a strictly positive curvature of the ample line bundle, they showed that for an abelian variety A over a number field the small points in $X(\overline{\mathbb{Q}})$ satisfy some equidistribution property [Zha98, Theorem 1.1]. Chambert-Loir [Cha06] introduced non-archimedean Monge-Ampère measures in the framework of Berkovich's analytic spaces [Ber90; Ber93] and proved an analogous equidistribution property in the case of an ample model metric. Later, Yuan [Yua08] generalized the equidistribution property both in the archimedean [SUZ97] and in the non-archimedean case. The proof required that the metric is only semi-positive.

Studying this non-archimedean Monge-Ampère measure for a closed subvariety X of A is the main goal of this thesis. This dissertation is based on the previous work of Gubler [Gub07a; Gub10]. The first result of this kind is [Gub07a, Theorem 1.3]. The latter assumes that A is totally degenerate over $\mathbb{K}$, where $(\mathbb{K}, |.|)$ is defined as the completion of the algebraic closure

of the completion of a discretely and non-trivially valued non-archimedean field $(K, |.|)$. Let X be a closed K-subvariety of A of pure dimension $d \in \mathbb{N}$. Then [Gub07a, Theorem 1.3] says that there exists a polytopal decomposition of the tropical variety of $X \times_K \mathbb{K}$ such that on any of its d-dimensional polytopes the push-forward of the canonical measure (which means the non-archimedean Monge-Ampère measure corresponding to the canonical metrics) with respect to the tropicalization map is given by a positive real multiple of the Lebesgue measure. Gubler used this result to prove the geometric Bogomolov conjecture in the case of an abelian variety which is totally degenerate at some place, stated in [Gub07b, Theorem 1.1]. In [Gub10, Theorem 1.1], Gubler could remove the assumption that the abelian variety is totally degenerate at some place by using Raynaud's uniformization theory and Mumford models, which we will see below. The latter result was central for proving Yamaki's reduction theorem [Yam17, Theorem 3.10], which again was an important part for the general proof of the geometric Bogomolov conjecture by Xie and Yuan [XY21, Theorem 1.1].

Let $(\mathbb{K}, |.|)$ be an algebraically closed non-trivially valued non-archimedean field with value group Γ. Unlike before, we do not assume that it is induced by a discrete valuation. Raynaud [Ray70] (proofs were given by Bosch and Lütkebohmert [BL91, §1], adapted to Berkovich's analytic spaces in [Ber90, §6.5]) stated that for any abelian $\mathbb{K}$-variety A there exists a unique formal $\mathbb{K}^\circ$-group scheme $\mathscr{A}_0$ (denoted by N in [Ber90, §6.5]) and a unique monomorphism of $\mathbb{K}$-analytic groups $a\colon \mathscr{A}_0^{\mathrm{an}} \to A^{\mathrm{an}}$ inducing an isomorphism onto a $\mathbb{K}$-analytic domain on A^{an} such that $\mathscr{A}_0$ has semi-abelian reduction (see also [Gub10, 4.1]). This implies that there exists a number $n \in \mathbb{N}$, an abelian $\mathbb{K}$-variety B of good reduction, a $\mathbb{K}$-algebraic group E, a $\mathbb{K}$-analytic subgroup M' of E^{an} together with two short exact sequences, forming the diagram of $\mathbb{K}$-analytic groups

$$\begin{array}{ccccc} & & M' & & \\ & & \big\downarrow & & \\ \mathbb{T}^{\mathrm{an}} & \hookrightarrow & E^{\mathrm{an}} & \overset{q}{\twoheadrightarrow} & B^{\mathrm{an}} \\ & & \big\downarrow{\scriptstyle p} & & \\ & & A^{\mathrm{an}}, & & \end{array}$$

which is called Raynaud's uniformization cross and where $\mathbb{T}$ denotes the $\mathbb{K}$-torus of rank n. Moreover, there exists an epimorphism of topological spaces $\mathrm{trop}\colon E^{\mathrm{an}} \twoheadrightarrow N_{\mathbb{R}} := \hom_{\mathbb{R}-\mathbf{Vec}}(M, \mathbb{R})$, where M is the character group (free abelian of rank n) of the $\mathbb{K}$-torus $\mathbb{T}$, extending the canonical tropicalization map $\mathrm{trop}\colon \mathbb{T}^{\mathrm{an}} \twoheadrightarrow N_{\mathbb{R}}, ||.|| \mapsto (m \mapsto -\log||\chi^m||)$. By defining $\Lambda := \mathrm{trop}(M')$, which forms a lattice of full rank in the n-dimensional $\mathbb{R}$-vector space $N_{\mathbb{R}}$, the canonical tropicalization map reduces to an epimorphism of topological spaces $\overline{\mathrm{trop}}\colon A^{\mathrm{an}} \twoheadrightarrow N_{\mathbb{R}}/\Lambda$. This epimorphism was used by Mumford [Mum72] to construct a formal $\mathbb{K}^\circ$-model $\mathscr{A}_{\overline{\mathscr{C}}}$ of A for any Γ-rational polytopal decomposition $\overline{\mathscr{C}}$ of $N_{\mathbb{R}}/\Lambda$.

In Section 5.3 we will see that any line bundle L on A induces a so-called Λ-cocycle which is defined as a family of continuous functions $(z_\lambda\colon N_{\mathbb{R}} \to \mathbb{R})_{\lambda\in\Lambda}$ such that $z_{\lambda+\mu}(u) = z_\lambda(\mu+u) + z_\lambda(u)$ for all $\lambda, \mu \in \Lambda$ and $u \in N_{\mathbb{R}}$, and such that $z\colon \Lambda \to \mathbb{R}, \lambda \mapsto z_\lambda(0)$ is a $\mathbb{Z}$-quadratic function. The latter extends uniquely to an $\mathbb{R}$-quadratic function $z\colon N_{\mathbb{R}} \to \mathbb{R}$, which is the canonical function (see Remark 5.4.10) fulfilling the cocycle rule: We say that a function $f\colon N_{\mathbb{R}} \to \mathbb{R}$ fulfils the cocycle rule if for any $\lambda \in \Lambda$, $u \in N_{\mathbb{R}}$ the equality $f(\lambda+u) = z_\lambda(u)+f(u)$ holds true. Furthermore, we say that a function $f\colon N_{\mathbb{R}} \to \mathbb{R}$ is $\mathscr{C}$-polyhedral, where $\mathscr{C}$ is a polytopal decomposition of $N_{\mathbb{R}}$, if the restriction $f|_\Delta$ is $\mathbb{R}$-affine for any polytope $\Delta \in \mathscr{C}$. The function f is moreover called Γ-rational if all restrictions to n-dimensional polytopes $\Delta \in \mathscr{C}$ are Γ-rational. The latter property is also known as $(\mathbb{Z}, \Gamma)$-piecewise linear [GJR21, §2.5].

We want to construct a formal $\mathbb{K}^\circ$-model $\mathscr{L}$ of a rigidified line bundle L because, due to [Gub07a, Proposition 3.11], the non-archimedean Monge-Ampère measure of formal metrics has the very convenient form of a discrete measure. If the abelian $\mathbb{K}$-variety A has good reduction, one receives a canonical formal $\mathbb{K}^\circ$-model $\mathscr{L}$ of L. In this case, the canonical metric $||.||_{L,\mathrm{can}}$ on L coincides with the associated formal metric $||.||_{\mathscr{L}}$ on L (see [Gub98, 7.3]). Since in other cases we do not have canonical formal $\mathbb{K}^\circ$-models of L, we have to take advantage of Raynaud's uniformization cross: Due to [BL91, §4], there exists a rigidified line bundle H on B such that $p^*L^{\mathrm{an}} = q^*H^{\mathrm{an}}$ as rigidified line bundles on E^{an} (in [BL91, §4]: as M'-linearized cubical line bundles). Since B has good reduction, we have a canonical formal $\mathbb{K}^\circ$-model $\mathscr{H}$ of H and the canonical metric $||.||_{H,\mathrm{can}}$ and the associated formal metric $||.||_{\mathscr{H}}$ coincide.

Gubler established a metric-function correspondence [Gub10, Proposition 4.11], which yields a bijection between the set of Γ-rational $\mathscr{C}$-polyhedral functions $f_{\mathscr{L}}\colon N_{\mathbb{R}} \to \mathbb{R}$ fulfilling the cocycle rule, and the set of isomorphism classes of formal $\mathbb{K}^\circ$-models $\mathscr{L}$ of L which are trivial over the formal open subsets of the Mumford model $\mathscr{A}_{\overline{\mathscr{C}}}$. It is determined by the equation $f_{\mathscr{L}} \circ \mathrm{trop} = -\log \frac{p^*||.||_{\mathscr{L}}}{q^*||.||_{\mathscr{H}}}$.

Since in our situation the considered metric $||.||$ on L will in general not be formal with respect to a Mumford model, we need a uniform approximation process of the metric $||.||$ by formal metrics (with respect to Mumford models). To handle a larger class than canonical metrics, we introduce so-called toric metrics in Section 5.4: A continuous metric $||.||$ on L is called *toric* if there exists a morphism of topological spaces $\psi\colon A^{\mathrm{an}} \to \mathbb{R}$ such that ψ is $\mathscr{A}_0^{\mathrm{an}}$-invariant (with respect to the monomorphism $a\colon \mathscr{A}_0^{\mathrm{an}} \hookrightarrow A^{\mathrm{an}}$ of $\mathbb{K}$-analytic groups) and $||.|| = e^{-\psi}||.||_{L,\mathrm{can}}$. In this case, we will denote the toric metric by $||.||_\psi$. This definition leads to the following generalization of the metric-function correspondence:

Proposition A (= Proposition 5.4.9)**.** *There exists a bijection between the set of toric metrics $||.||_\psi$ on L and the set of morphisms of topological spaces $f_\psi\colon N_{\mathbb{R}} \to \mathbb{R}$ fulfilling the cocycle rule which is an extension of the bijection in [Gub10, Proposition 4.11] and becomes an isomorphism of metric spaces with respect to uniform convergence. The bijection is determined by the equation*

$$f_\psi \circ \mathrm{trop} = -\log \frac{p^*||.||_\psi}{q^*||.||_{H,\mathrm{can}}}.$$

We denote the corresponding function to the toric metric $||.||_\psi$ by f_ψ. The canonical function z corresponds to the canonical metric $||.||_{L,\mathrm{can}}$. It is strictly convex if L is ample. We assume for the rest of the introduction that L is ample. For this case in [GS22, Theorem 4.10] it is shown that the metric $||.||_\psi$ is semi-positive if and only if the function f_ψ is convex. Thanks to this metric-function correspondence, we can study all toric metrics combinatorially. In this light, the following construction was done:

The proof strategy of [Gub10, Theorem 6.7], where only canonical metrics are considered and the absolute value $|.|$ on $\mathbb{K}$ is induced by a discrete valuation (that means $\Gamma = \mathbb{Q}$), is to approximate the $\mathbb{R}$-quadratic function $z\colon N_{\mathbb{R}} \to \mathbb{R}$ uniformly by strongly polyhedral convex functions $(f_m)_{m\in\mathbb{N}\setminus\{0\}}$

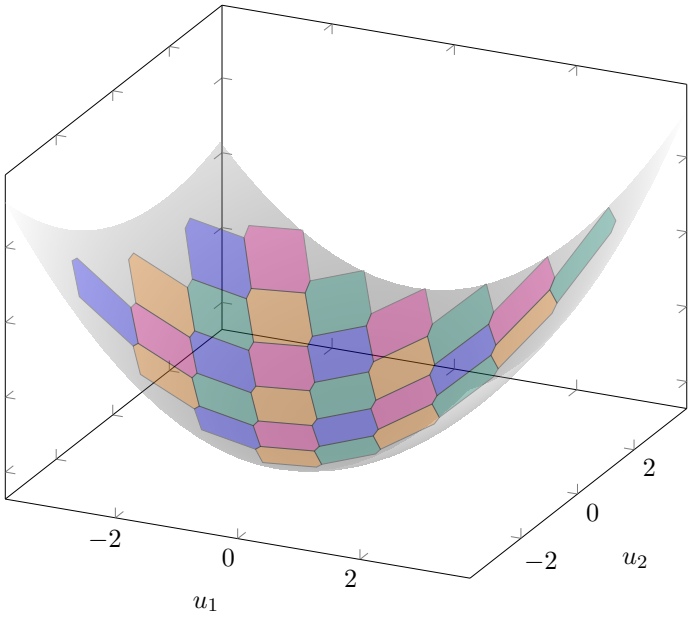


Figure 1.1: Visualization for $n = 2$: The $\mathbb{R}$-quadratic function z (black) is approximated by strongly polyhedral convex functions fulfilling the cocycle rule (coloured).

and to find a sequence $(N_m)_{m\in\mathbb{N}\setminus\{0\}}$ in $\mathbb{N}\setminus\{0\}$ such that $N_m f_m$ is Γ-rational and fulfils the cocycle rule. On top of that in [Gub07a, Lemma 8.4] it is shown that there exists a Γ'-rational polytopal decomposition $\overline{\mathscr{C}}$ of $N_{\mathbb{R}}/\Lambda$, where Γ' is an arbitrary infinite-dimensional $\mathbb{Q}$-vector subspace of $\mathbb{R}$ containing Γ such that $\overline{m^{-1}\mathscr{C}}$ is $\overline{\Sigma}$-generic for all $m \in \mathbb{N}\setminus\{0\}$, where $\overline{\Sigma}$ is a finite set of Γ-rational polytopes in $N_{\mathbb{R}}/\Lambda$. A polytopal decomposition $\overline{\mathscr{C}}$ of $N_{\mathbb{R}}/\Lambda$ is called $\overline{\Sigma}$-generic if for all $\overline{\sigma} \in \overline{\Sigma}$, $\overline{\Delta} \in \overline{\mathscr{C}}$ and $D := \dim_{\mathbb{R}}(\sigma)+\dim_{\mathbb{R}}(\Delta)-n$ we have $\dim_{\mathbb{R}}(\mathbb{A}_\sigma \cap \mathbb{A}_\Delta) = D$ in the case $D \geq 0$, and $\mathbb{A}_\sigma \cap \mathbb{A}_\Delta = \varnothing$ otherwise, where $\mathbb{A}_\sigma$ respectively $\mathbb{A}_\Delta$ denotes the smallest $\mathbb{R}$-affine subspace of $N_{\mathbb{R}}$ containing the polytope σ respectively Δ (see Section 5.5). Approximating the canonical metric $||.||_{L,\mathrm{can}}$ uniformly by the metrics $||.||_{\mathscr{L}_m}^{\otimes 1/N_m}$ ($||.||_{\mathscr{L}_m}$ is given by the metric-function correspondence for $N_m f_m$), the corresponding non-archimedean Monge-Ampère measure will be approximated weakly by discrete measures. Since the polytopal decompositions $\overline{m^{-1}\mathscr{C}}$ are

$\overline{\Sigma}$-generic, its multiplicities can be calculated by a formula given in [Gub10, Proposition 5.18].

The greatest part of this thesis is dedicated to deducing Proposition B from [Gub07a, Lemma 8.4] (for A totally degenerate) respectively [Gub10, Lemma 6.5] (for any abelian $\mathbb{K}$-variety A) by generalizing the following four aspects: (1) We will not assume that the absolute value $|.|$ on $\mathbb{K}$ is induced by a discrete valuation. (2) We will not assume that the polytopes in $\overline{\Sigma}$ are Γ-rational. (3) We will replace Γ' by Γ. (4) We will consider any semi-positive toric metric $||.||_\psi$ instead of just the canonical metric.

Proposition B (= Proposition 5.5.5 + Lemma 5.5.6)**.** *Let the line bundle L on A be ample. Then, for any convex function f_ψ fulfilling the cocycle rule, there exists a sequence $(\overline{\mathscr{C}_k})_{k\in\mathbb{N}}$ of Γ-rational polytopal decompositions of $N_\mathbb{R}/\Lambda$, a sequence $(N_k)_{k\in\mathbb{N}}$ in $\mathbb{N}\backslash\{0\}$, a sequence $(f_k)_{k\in\mathbb{N}}$ of strongly $\mathscr{C}_k$-polyhedral convex functions fulfilling the cocycle rule and converging uniformly to f_ψ, and a sequence $(\mathscr{L}_k)_{k\in\mathbb{N}}$ of formal $\mathbb{K}^\circ$-models of $L^{\otimes N_k}$ determined on the Mumford model $\mathscr{A}_{\overline{\mathscr{C}_k}}$ such that for all $k \in \mathbb{N}$ the following holds:*

(i) The strongly $\mathscr{C}_k$-polyhedral convex function $N_k f_k$ is Γ-rational.

(ii) For all $m \in \mathbb{Z}\backslash\{0\}$ the polytopal decomposition $\overline{m^{-1}\mathscr{C}_k}$ of $N_\mathbb{R}/\Lambda$ is $\overline{\Sigma}$-generic.

(iii) The metric $||.||_{\mathscr{L}_k}^{\otimes 1/N_k}$ is semi-positive and converges for $k \to +\infty$ uniformly to the toric metric $||.||_\psi$.

The first two properties could be proven purely combinatorially (see Proposition 5.5.5). Hence Proposition B also improves [BGJK21, Proposition 8.2.6], which states a Γ-rational polyhedral convex approximation result for polyhedral convex functions fulfilling the cocycle rule, by possibly assuming the approximating functions to be Γ-rational *strongly* polyhedral convex.

Since the function f_ψ is in general not $\mathbb{R}$-quadratic anymore, we could not construct a single polytopal decomposition $\overline{\mathscr{C}}$ of $N_\mathbb{R}/\Lambda$ such that the sequence of $\overline{\Sigma}$-generic Γ-rational polytopal decompositions of $N_\mathbb{R}/\Lambda$ in the approximation process has the form $(\overline{m^{-1}\mathscr{C}})_{m\in\mathbb{N}\backslash\{0\}}$. Instead, the elements of the sequence $(\overline{\mathscr{C}_k})_{k\in\mathbb{N}}$ are independent from each other. Due to this change, in property (ii) the case $m \neq 1$ becomes obsolete for the proof of

Theorem C later. But as we have found a nice proof for all $m \in \mathbb{Z}\backslash\{0\}$, we have kept the more general case.

Let X be an irreducible closed $\mathbb{K}$-subvariety of A, and define $d := \dim(X) \in \mathbb{N}$. Let $\overline{\mathscr{C}}_{\mathrm{alt}}$ be a Γ-rational polytopal decomposition of $N_{\mathbb{R}}/\Lambda$ with associated Mumford model $\mathscr{A}_{\mathrm{alt}}$ of A, and denote the closure of X^{an} in $\mathscr{A}_{\mathrm{alt}}$ by $\mathscr{X}_{\mathrm{alt}}$. In [Gub07a; Gub10] Gubler has chosen a strictly semi-stable alteration of X respectively $\mathscr{X}_{\mathrm{alt}}$, which is a pair $(\mathscr{X}', \varphi_{\mathrm{alt}})$ consisting of a strictly semi-stable admissible formal $\mathbb{K}^\circ$-scheme $\mathscr{X}'$ [Jon96; Ber99] and an epimorphism of admissible formal $\mathbb{K}^\circ$-schemes $\varphi_{\mathrm{alt}} \colon \mathscr{X}' \twoheadrightarrow \mathscr{X}_{\mathrm{alt}}$ such that there exists an alteration (X', f) of X with $(\mathscr{X}')^{\mathrm{an}} = (X')^{\mathrm{an}}$ and $\varphi_{\mathrm{alt}}^{\mathrm{an}} = f^{\mathrm{an}}$ in the category of $\mathbb{K}$-analytic spaces. An alteration (X', f) of X consists of an irreducible smooth $\mathbb{K}$-variety X' and a proper generically finite epimorphism of $\mathbb{K}$-varieties $f \colon X' \twoheadrightarrow X$ (see Section 7.1).

If the absolute value $|.|$ on $\mathbb{K}$ is induced by a discrete valuation, such a strictly semi-stable alteration of X always exists by [Jon96, Theorem 6.5]. As one of the main goals of this thesis is to remove this assumption, we cannot apply this theorem anymore. Fortunately, Adiprasito, Liu, Pak and Temkin have proved in [ALPT19, Theorem 5.2.19] that in this case there exists at least a strictly poly-stable alteration of X, which is defined analogously. A strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme $\mathscr{X}'$ is locally étale a finite product of strictly semi-stable ones.

The striking advantage of these alterations is the fact that for a strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme $\mathscr{X}'$, Berkovich constructed a so-called skeleton $S(\mathscr{X}')$, which is a closed subset of $(X')^{\mathrm{an}}$ and a union of poly-simplices Δ_S (inclusions and intersections possible) where S runs over the set of strata $\mathrm{str}(\tilde{\mathscr{X}}')$ of the special fibre of $\mathscr{X}'$ (see Sections 6.1 and 6.2). A poly-simplex is defined as a finite product of simplices. In Section 6.4 we will see that there exists a unique continuous function $\overline{f}_{\mathrm{aff}} \colon S(\mathscr{X}') \to N_{\mathbb{R}}/\Lambda$ making the diagram

$$\begin{array}{ccc} (X')^{\mathrm{an}} & \xrightarrow{f^{\mathrm{an}}} & A^{\mathrm{an}} \\ \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle \overline{\mathrm{trop}}} \\ S(\mathscr{X}') & \xrightarrow{\overline{f}_{\mathrm{aff}}} & N_{\mathbb{R}}/\Lambda \end{array}$$

commutative in the category of topological spaces, where τ denotes the retraction map [Ber99, §5]. For every stratum $S \in \mathrm{str}(\tilde{\mathscr{X}}')$, the restriction

of the function $\overline{f}_{\mathrm{aff}}$ to the poly-simplex Δ_S has a lift $f_{S,\mathrm{aff}}\colon \Delta_S \to N_{\mathbb{R}}$, which extends to a Γ-rational $\mathbb{R}$-affine function $f_{S,\mathrm{aff}}\colon \mathbb{R}^{r_S} \to N_{\mathbb{R}}$ for $r_S := \dim_{\mathbb{R}}(\Delta_S)$. A poly-simplex Δ_S is called non-degenerate with respect to f^{an} if and only if the function $f_{S,\mathrm{aff}}$ is injective and $\deg_{\mathscr{H}}(\overline{S}) \neq 0$ is fulfilled. The latter denotes the degree of the irreducible component $\overline{S}$ of $\tilde{\mathscr{X}}'$ with respect to the pull-back of the line bundle $q^*\tilde{\mathscr{H}}$ along a lift $\tilde{\Phi}_{\mathrm{alt}}\colon \overline{S} \to \tilde{\mathscr{E}}_{\mathrm{alt}}$ of the restricted morphism of $\tilde{\mathbb{K}}$-varieties $\tilde{\varphi}_{\mathrm{alt}}\colon \overline{S} \to \tilde{\mathscr{A}}_{\mathrm{alt}}$.

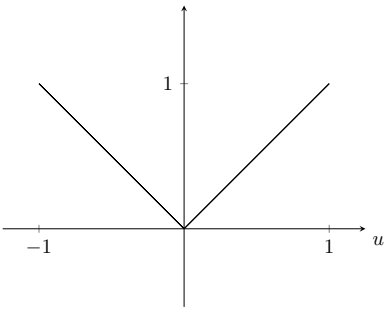


Figure 1.2: The real Monge-Ampère measure for the depicted function g is given as the discrete measure $\mathrm{MA}(g) = 2\delta_0$ where δ_0 denotes the Dirac measure at $u = 0$.

We will see that the non-archimedean Monge-Ampère measure with respect to toric metrics on $(X')^{\mathrm{an}}$ is at most supported on the poly-simplices Δ_S which are non-degenerate with respect to f^{an}, and that on the relative interior of the latter the non-archimedean Monge-Ampère measure equals some positive multiple of the real Monge-Ampère measure. The real Monge-Ampère measure with respect to a strongly polyhedral convex function g on $\mathbb{R}^{r_S}$ is a discrete measure with support in the vertices of the corresponding polyhedral decomposition of $\mathbb{R}^{r_S}$ and its multiplicities are given by the Lebesgue measure of the gradient image of these vertices under g. This gradient image of a vertex ξ is given by the subset of all elements m of the dual space of $\mathbb{R}^{r_S}$ such that for all $u \in \mathbb{R}^{r_S}$ the inequality $\langle m, u - \xi\rangle \leq g(u) - g(\xi)$ holds true. See Figure 1.2 for an example.

Using uniform convergence, the real Monge-Ampère measure can be extended from strongly polyhedral convex functions to convex functions on $\mathbb{R}^{r_S}$. Doing this uniform approximation for the other side of the metric-function correspondence as well yields the following result:

Theorem C (= Theorem 7.1.7)**.** *Let the line bundle L on A be ample, let $(\mathscr{X}', \varphi_{\mathrm{alt}})$ be a strictly poly-stable alteration of $\mathscr{X}_{\mathrm{alt}}$, let $||.||_\psi$ be a semi-positive toric metric on L, and take the non-archimedean Monge-Ampère measure*

$$\mu'_\psi := c_1((f^{\mathrm{an}})^*(L, ||.||_\psi))^{\wedge d}$$

on the measurable space $(X')^{\mathrm{an}}$. We then get the following results:

(i) For the support of the regular Borel measure μ'_ψ on $(X')^{\mathrm{an}}$ we have

$$\mathrm{Supp}(\mu'_\psi) \subseteq \bigcup_{\substack{S \in \mathrm{str}(\tilde{\mathscr{X}}') \\ \Delta_S \text{non-deg.} \\ \text{w.r.t. } f^{\mathrm{an}}}} \Delta_S.$$

(ii) Let Ω be a Borel-measurable subset of the relative interior of Δ_S for some stratum S of $\tilde{\mathscr{X}}'$. We then get the equality

$$\mu'_\psi(\Omega) = \frac{d!}{(d - \dim_{\mathbb{R}}(\Delta_S))!} \cdot \deg_{\mathscr{H}}(\overline{S}) \cdot \mathrm{MA}(f_\psi \circ f_{S,\mathrm{aff}})(\Omega),$$

where f_ψ is the function corresponding to $||.||_\psi$ according to Proposition A.

In the case of canonical metrics, we get the following result, which is a generalization of [Gub10, Theorem 6.7] by dropping the assumption that the absolute value $|.|$ on $\mathbb{K}$ is induced by a discrete valuation:

Corollary D (= Corollary 7.1.9)**.** *If in Theorem C the metric is canonical, we get equality in (i) and the formula in (ii) transforms to*

$$\mu'_{\mathrm{can}}(\Omega) = \frac{d!}{(d - \dim_{\mathbb{R}}(\Delta_S))!} \cdot \deg_{\mathscr{H}}(\overline{S}) \cdot \det(\boldsymbol{Q}_S) \cdot \mathrm{vol}_{r_S}(\Omega)$$

where Ω is a Borel-measurable subset of $\mathrm{relint}(\Delta_S)$ *for some $S \in \mathrm{str}(\tilde{\mathscr{X}}')$, and $\boldsymbol{Q}_S$ is a not necessarily unique symmetric $r_S \times r_S$-matrix given by $z \circ f_{S,\mathrm{aff}} \colon \Delta_S \to \mathbb{R}$, whose determinant is unique and which is invertible if and only if the poly-simplex Δ_S is non-degenerate with respect to f^{an}.*

So far we have considered the canonical measure μ'_{can} only on the measurable space $(X')^{\mathrm{an}}$. We call the support of the canonical measure $\mu_{\mathrm{can}} := c_1((L, ||.||_{L,\mathrm{can}})|_{X^{\mathrm{an}}})^{\wedge d}$ on X^{an} the canonical subset of X^{an} and denote it by Σ_X. It is well-defined and independent of the choice of L. The referee of the paper [Gub10] suggested that this canonical subset is a Γ-rational piecewise linear space [Ber04, §1], which was afterwards proved in [Gub10, Theorem 6.12]. In the meantime Ducros developed in [Duc12] criteria to check whether a subset of a $\mathbb{K}$-analytic space is a so-called Γ-rational skeleton. Since every Γ-rational skeleton is a Γ-rational piecewise linear space, the following result generalizes [Gub10, Theorem 6.12] from strictly semi-stable alterations to poly-stable ones and by dropping the assumption that the absolute value $|.|$ on $\mathbb{K}$ is induced by a discrete valuation:

Theorem E (= Theorem 7.2.13)**.** *Let the line bundle L on A be ample. The canonical subset Σ_X is a Γ-rational skeleton of X^{an}. Moreover, for any choice of the strictly poly-stable alteration $(\mathscr{X}', \varphi_{\mathrm{alt}})$ of $\mathscr{X}_{\mathrm{alt}}$ and any $S \in \mathrm{str}(\tilde{\mathscr{X}'})$ with Δ_S being non-degenerate with respect to f^{an}, the induced morphism of topological spaces $f^{\mathrm{an}}\colon \Delta_S \to f^{\mathrm{an}}(\Delta_S)$ yields an isomorphism of Γ-rational piecewise linear spaces.*

[Gub10, Theorem 6.12] was important for proving Yamaki's reduction theorem [Yam17, Theorem 3.10].

Organization of this thesis. Chapter 2 is a collection of various basics, which are quite independent from each other. Chapter 3 recollects basics on analytic and formal geometry and Chapter 4 introduces metrics, the non-archimedean and real Monge-Ampère measure. All these three chapters are completely based on references and do not contain new results. Chapter 5 is guided very narrowly by [Gub10, §4] and introduces Raynaud's uniformization theory (see Section 5.1) and Mumford models (see Section 5.2), which are also completely based on references.

The most important new ideas of this thesis are contained in the remaining three sections of Chapter 5: In Section 5.3 we find an $\mathbb{R}$-quadratic function $z\colon N_{\mathbb{R}} \to \mathbb{R}$, which yields the "tropical version" of the canonical metric $||.||_{L,\mathrm{can}}$. Afterwards we define what it means for a function $N_{\mathbb{R}} \to \mathbb{R}$ to fulfil the cocycle rule and give a purely combinatorial approximation result, which is important for the proof of Proposition B.

Section 5.4 newly defines toric metrics and generalizes in Proposition A the metric-function correspondence [Gub10, Proposition 4.11] from formal (associated to Mumford models) to toric metrics, which attaches to every toric metric its "tropical version", namely a continuous function $N_{\mathbb{R}} \to \mathbb{R}$ fulfilling the cocycle rule, as explained in the previous paragraph. In Section 5.5 we construct Mumford models having a very favourable form by deducing Proposition B from [Gub07a, Lemma 8.4] respectively [Gub10, Lemma 6.5]. Compared to the latter two results, in Proposition B we replace canonical metrics with semi-positive toric metrics and drop the assumption that the absolute value $|.|$ on $\mathbb{K}$ is induced by a discrete valuation.

Chapter 6 is guided very narrowly by [Gub10, §5] and collects the necessary basics for strictly poly-stable alterations. Section 6.1 and 6.2 recapitulate the definitions of a strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme $\mathscr{X}'$ (formal $\mathbb{K}^\circ$-model of X') and Berkovich's skeleton on it [Ber99]. In Section 6.3 we will explain how a subdivision of the skeleton affects $\mathscr{X}'$ and in Section 6.4 we will apply the collected theory to a Mumford model $\mathscr{A}_{\mathrm{alt}}$. In particular we will formulate [Gub10, Proposition 5.11] for the poly-stable case, which shows that the "tropical version" of the strictly poly-stable alteration is composed by Γ-rational $\mathbb{R}$-affine functions. Moreover, we reformulate [Gub10, Proposition 5.18], which computes the degree of an irreducible component of the special fibre of $\mathscr{X}'$ under some assumptions.

Finally in Chapter 7 we prove the two main results Theorem C (including Corollary D) and Theorem E of this thesis, which are a generalization of [Gub10, Theorem 6.7] respectively [Gub10, Theorem 6.12]. The glossary of notation at the end supports the reader in the case of uncertainties regarding notations. The list of references concludes this thesis.

Terminology. The set of natural numbers $\mathbb{N}$ includes 0. The usual archimedean absolute value on $\mathbb{R}$ is written as $|.|_{\mathbb{R}}$. By a ring we mean a commutative ring with 1. We call a morphism monic respectively epic if it is a monomorphism respectively an epimorphism and we will draw them as $\hookrightarrow$ respectively $\twoheadrightarrow$. Isomorphisms will be drawn as $\xrightarrow{\sim}$ and canonical isomorphisms as $=$. Eventually, we will define the property Λ-periodic for a polytopal complex differently to the literature (see Remark 2.4.7).

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2 General basics

In this chapter we will collect basics from various areas in order to have a sound basis for the rest of this thesis. In particular we will consider valuations, category theory (especially group objects), sheaves, convex geometry, abelian varieties together with line bundles on them, and quadratic functions.

2.1 Valuations

The goal of this section is to give some basics on valuations of fields. Use the beginning of [EP05] as a reference.

Definition 2.1.1. Let K be a field. An *absolute value* on K is a function $|.|\colon K \to \mathbb{R}_{\geq 0}$ satisfying for all $x, y \in K$ the following properties

(i) $|x| \geq 0$, and $|x| = 0$ if and only if $x = 0$,

(ii) $|x \cdot y| = |x| \cdot |y|$, and

(iii) $|x + y| \leq |x| + |y|$.

It is called *non-archimedean* if it satisfies the strong triangle inequality

(iii)' $|x + y| \leq \max(|x|, |y|)$.

Definition 2.1.2. A *valued field* is a pair $(K, |.|)$ consisting of a field K and an absolute value $|.|$ on K. A valued field $(\mathbb{K}, |.|)$ is called *algebraically closed* if the field $\mathbb{K}$ is algebraically closed.

Example 2.1.3. Let K be a field. The function

$$|.|\colon K \to \mathbb{R}_{\geq 0},$$
$$x \mapsto \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

is an absolute value on K and is called the *trivial* absolute value on K. An absolute value on K that is different from this one is called *non-trivial*. A *non-trivially valued field* is a valued field $(K, |.|)$ such that $|.|$ is non-trivial.

Definition 2.1.4. A metric space (M, d) is called *complete* if every Cauchy sequence converges in M. Every valued field $(K, |.|)$ yields a metric space by taking the underlying set of K and defining the metric $d(x, y) := |x - y|$ for $x, y \in K$. A valued field $(K, |.|)$ is called *complete* if this metric space is complete. A *non-archimedean field* is a complete valued field $(K, |.|)$ such that the absolute value $|.|$ is non-archimedean. For a non-archimedean field $(K, |.|)$ we define the *valuation ring* $K^\circ := \{x \in K \mid |x| \leq 1\}$ with its maximal ideal $K^{\circ\circ} := \{x \in K \mid |x| < 1\}$ and the corresponding *residue field* $\tilde{K} := K^\circ / K^{\circ\circ}$.

Definition 2.1.5. The *value group* Γ of a valued field $(K, |.|)$ is defined to be the image of the morphism of groups $-\log|.|\colon (K\backslash\{0\}, \cdot) \to (\mathbb{R}, +)$.

Lemma 2.1.6. *Let* $(\mathbb{K}, |.|)$ *be an algebraically closed valued field. Then the value group* Γ *of* $(\mathbb{K}, |.|)$ *is a* $\mathbb{Q}$*-vector subspace of* $\mathbb{R}$*.*

Proof. Let $n \in \mathbb{N}\backslash\{0\}$ and $x \in \mathbb{K}$. Then the roots $y_1, ..., y_n$ of the polynomial $T^n - x \in \mathbb{K}[T]$ are also in $\mathbb{K}$ and thus $n \cdot (-\log|y_i|) = -\log|x|$ in Γ meaning that the value group Γ is divisible. A group G is called divisible if for every $n \in \mathbb{N}\backslash\{0\}$ and $g \in G$ there exists a $h \in G$ such that $h^n = g$. Since Γ is an abelian group (as the group $(K\backslash\{0\}, \cdot)$ is abelian), it is also a $\mathbb{Z}$-module. Since we have shown that Γ is divisible, it becomes a $\mathbb{Q}$-vector space. □

2.2 Category theory

The following section is based on [Lan78] and provides some basic categorical language for the whole thesis.

Definition 2.2.1. We call a morphism *monic* respectively *epic* if it is a monomorphism respectively an epimorphism.

Definition 2.2.2. Let $\mathcal{C}$ be a category. The *(co)equalizer* of two morphisms $X \rightrightarrows Y$ is the (co)limit of that diagram.

Definition 2.2.3. Let $\mathcal{C}$ be a category. A *subobject* of an object A is an equivalence class of monomorphisms with codomain A, where we call two monomorphisms $m_1 \colon B_1 \hookrightarrow A$ and $m_2 \colon B_2 \hookrightarrow A$ equivalent if there exists an isomorphism $f \colon B_1 \xrightarrow{\sim} B_2$ such that $m_2 \circ f = m_1$.

Definition 2.2.4. Let $\mathcal{C}$ be a category with zero morphisms. We define the *(co)kernel* of a morphism $f \colon X \to Y$ to be the (co)equalizer of f and the zero morphism $0 \colon X \to Y$. The *image* of a morphism $f \colon X \to Y$ is the kernel of the cokernel of f. The image of a subobject $g \colon U \hookrightarrow X$ under a morphism $f \colon X \to Y$ is the image of the morphism $f \circ g$.

Definition 2.2.5. Let $\mathcal{C}$ be a category with finite products and a terminal object 0. A *group object* in $\mathcal{C}$ is an object G together with morphisms $m \colon G \times G \to G$, $e \colon 0 \to G$, and $i \colon G \to G$, such that the following axioms from [Lan78, III, §6] hold:

(i) (Associativity) The diagram

$$\begin{array}{ccccc} G \times (G \times G) & & \xrightarrow{\quad\alpha\quad} & & (G \times G) \times G \\ {\scriptstyle e\times m}\downarrow & & & & \downarrow{\scriptstyle m\times e} \\ G \times G & & & & G \times G \\ & {\scriptstyle m}\searrow & & \swarrow{\scriptstyle m} & \\ & & G & & \end{array}$$

commutes; α is the associativity isomorphism given by [Lan78, III, §5, Proposition 1, (1)].

(ii) (Identity) The diagram

$$\begin{array}{ccccc} 0 \times G & \xrightarrow{e\times \mathrm{id}} & G \times G & \xleftarrow{\mathrm{id}\times e} & G \times 0 \\ & {\scriptstyle \lambda}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle \sigma} & \\ & & G & & \end{array}$$

commutes; λ, σ are the isomorphisms given by [Lan78, III, §5, Proposition 1, (2)].

(iii) (Inverse) The diagram

$$\begin{array}{ccccc} G & \xrightarrow{\delta_G} & G \times G & \xrightarrow{\mathrm{id} \times i} & G \times G \\ {\scriptstyle 0}\downarrow & & & & \downarrow{\scriptstyle m} \\ 0 & & \xrightarrow{\qquad e \qquad} & & G \end{array}$$

commutes; δ_G is the diagonal morphism defined before [Lan78, III, §5, Proposition 1].

A morphism of group objects in $\mathcal{C}$ is a morphism $\varphi\colon G_1 \to G_2$ with the obvious conditions. The group objects of $\mathcal{C}$ form a category denoted by $\mathbf{Grp}(\mathcal{C})$. Furthermore, a group object is called *abelian* if

(iv) (Commutative) the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\gamma_G} & G \times G \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & G & \end{array}$$

commutes; γ_G is the morphism which switches the two factors of the product.

The abelian group objects of $\mathcal{C}$ form a subcategory of $\mathbf{Grp}(\mathcal{C})$ denoted by $\mathbf{Grp}_{\mathrm{Ab}}(\mathcal{C})$.

Definition 2.2.6. Let $\mathcal{C}$ be a category with finite products and a terminal object 0. A *group action* of a group object G on an object X is a morphism $\triangleleft\colon G \times X \to X$ such that

(i) the composition

$$X \xrightarrow{\lambda^{-1}} 0 \times X \xrightarrow{e \times \mathrm{id}} G \times X \xrightarrow{\triangleleft} X$$

is the identity morphism, and

(ii) the diagram

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times e} & G \times X \\ {\scriptstyle e \times \triangleleft}\downarrow & & \downarrow{\scriptstyle \triangleleft} \\ G \times X & \xrightarrow{\triangleleft} & X \end{array}$$

commutes.

Lemma 2.2.7. *Let $\mathcal{C}$ be a category with finite products and a terminal object 0. Let $f\colon H \to G$ be a morphism of the category $\mathbf{Grp}(\mathcal{C})$. Then the composition of the morphisms*

$$H \times G \xrightarrow{f\times \mathrm{id}} G \times G \xrightarrow{m} G$$

in the category $\mathbf{Grp}(\mathcal{C})$ is a group action.

Proof. We check the properties of Definition 2.2.6:

(i) Follows by the left triangular of the commutative diagram in property (ii) in Definition 2.2.5.

(ii) Follows by property (i) in Definition 2.2.5 and the fact that f is a morphism of the category $\mathbf{Grp}(\mathcal{C})$. □

Definition 2.2.8. A category $\mathcal{C}$ *has zero morphisms* if for any two objects A, B there exists a morphism $0\colon A \to B$ such that for all objects X, Y, Z and morphisms $f\colon Y \to Z$, $g\colon X \to Y$ the diagram

$$\begin{array}{ccc} X & \xrightarrow{0} & Y \\ \downarrow{\scriptstyle g} & \searrow^{0} & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{0} & Z \end{array}$$

commutes.

Lemma 2.2.9. *Let $\mathcal{C}$ be a category with finite products and a terminal object 0. Then the category $\mathbf{Grp}(\mathcal{C})$ has zero morphisms.*

Proof. Define $0\colon G \to 0 \xrightarrow{e} H$. □

2.3 Sheaves

The goal of this section is to define sheaves in a general way. It is based on [Stacks, Chapter 00UZ], which is in turn based on [SGA4, Exposé II]. Note that the site here is defined differently from [SGA4].

Definition 2.3.1 ([SGA4, Exposé II, Definition 1.3])**.** Let $\mathcal{C}$ be a category. A *Grothendieck pretopology* on $\mathcal{C}$ is a set $\mathrm{Cov}(\mathcal{C})$ of families $(U_i \to U)_{i\in I}$ of morphisms in $\mathcal{C}$, called *covering families*, such that the following conditions hold true:

(i) If $\varphi \in \hom_{\mathcal{C}}(U, V)$, then $(\varphi) \in \mathrm{Cov}(\mathcal{C})$.

(ii) If $(U_i \to U)_{i\in I}$ and $(U_{ij} \to U_i)_{j\in J_i}$ are in $\mathrm{Cov}(\mathcal{C})$, then the composition $(U_{ij} \to U_i \to U)_{i\in I, j\in J_i}$ is also in $\mathrm{Cov}(\mathcal{C})$.

(iii) If $(U_i \to U)_{i\in I}$ is in $\mathrm{Cov}(\mathcal{C})$ and if $\varphi \in \hom_{\mathcal{C}}(V, U)$ for any $V \in \mathrm{ob}(\mathcal{C})$, then the fibred products $U_i \times_U V$ exist in $\mathcal{C}$ and $(U_i \times_U V \to V)_{i\in I}$ is in $\mathrm{Cov}(\mathcal{C})$.

Definition 2.3.2 ([Stacks, Definition 00VH])**.** A *site* is a pair $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$ consisting of a category $\mathcal{C}$ and a Grothendieck pretopology $\mathrm{Cov}(\mathcal{C})$ on $\mathcal{C}$.

Example 2.3.3. Let X be a topological space. Then the open subsets of X with inclusions $U \hookrightarrow V$ as morphisms form a category denoted by $\mathbf{Top}(X)$. Taking $\mathrm{Cov}(\mathbf{Top}(X))$ to be the maximal set of covering families, i.e. all possible families $(U_i \to U)_{i\in I}$ of morphisms in $\mathbf{Top}(X)$, yields a Grothendieck pretopology on $\mathbf{Top}(X)$. We call the corresponding site the *canonical site* of X and denote it also by X.

Definition 2.3.4 ([Stacks, Definition 00V3])**.** Let $\mathcal{C}$, $\mathcal{D}$ be categories. A *presheaf* $\mathscr{F}$ of $\mathcal{D}$ on $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$. A morphism of presheaves $\mathscr{F} \to \mathscr{G}$ of $\mathcal{D}$ on $\mathcal{C}$ is a natural transformation from $\mathscr{F}$ to $\mathscr{G}$.

Definition 2.3.5 ([Stacks, Definition 00VM], [Stacks, Definition 00XA])**.** Let $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$ be a site, and $\mathcal{D}$ be a category which admits all limits. A *sheaf* $\mathscr{F}$ of $\mathcal{D}$ on $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$ is a presheaf of $\mathcal{D}$ on $\mathcal{C}$ such that the diagram

$$\mathscr{F}(U) \to \prod_{i\in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathscr{F}(U_i \times_U U_j)$$

is an equalizer in $\mathcal{D}$ for all covering families $(U_i \to U)_{i\in I}$. A morphism of sheaves of $\mathcal{D}$ on $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$ is a morphism of the underlying presheaves. This forms a category of sheaves of $\mathcal{D}$ on $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$, which will be denoted by $\mathbf{Sh}((\mathcal{C}, \mathrm{Cov}(\mathcal{C})), \mathcal{D})$. It is also called a *topos*.

Example 2.3.6. Let X be a topological space. The topos $\mathbf{Sh}(X, \mathbf{Rng})$, where we denote the canonical site on X also by X, is equivalent to the category of sheaves of rings defined in [Har77, §II.1].

Definition 2.3.7 ([Stacks, Definition 00WV])**.** A *continuous functor* $\psi\colon (\mathcal{C}, \mathrm{Cov}(\mathcal{C})) \to (\mathcal{D}, \mathrm{Cov}(\mathcal{D}))$ is a functor $\psi\colon \mathcal{C} \to \mathcal{D}$ such that for every covering family $(U_i \to U)_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ we have

(i) $(\psi(U_i) \to \psi(U))_{i\in I} \in \mathrm{Cov}(\mathcal{D})$, and

(ii) for any $\alpha \in \hom_{\mathcal{C}}(V, U)$, the morphism $\psi(U_i \times_U V) \to \psi(U_i) \times_{\psi(U)} \psi(V)$ is an isomorphism in $\mathcal{D}$.

Lemma 2.3.8. *Let $\psi\colon (\mathcal{C}, \mathrm{Cov}(\mathcal{C})) \to (\mathcal{D}, \mathrm{Cov}(\mathcal{D}))$ be a continuous functor, and $\mathscr{F} \in \mathrm{ob}(\mathbf{Sh}((\mathcal{D}, \mathrm{Cov}(\mathcal{D})), \mathcal{E}))$, where $\mathcal{E}$ is a category which admits all limits. Then ψ induces a functor*

$$\psi^s\colon \mathbf{Sh}((\mathcal{D}, \mathrm{Cov}(\mathcal{D})), \mathcal{E}) \to \mathbf{Sh}((\mathcal{C}, \mathrm{Cov}(\mathcal{C})), \mathcal{E})$$

such that $\psi^s\mathscr{F}(U) = \mathscr{F}(\psi(U))$ in $\mathcal{E}$ for any $U \in \mathrm{ob}(\mathcal{C})$. Furthermore, there exists a left adjoint functor

$$\psi_s\colon \mathbf{Sh}((\mathcal{C}, \mathrm{Cov}(\mathcal{C})), \mathcal{E}) \to \mathbf{Sh}((\mathcal{D}, \mathrm{Cov}(\mathcal{D})), \mathcal{E}).$$

Proof. [SGA4, Exposé III, Proposition 1.2] or [Stacks, Section 00WU] □

Definition 2.3.9 ([Stacks, Definition 00X1])**.** A *morphism of sites* $\varphi\colon (\mathcal{D}, \mathrm{Cov}(\mathcal{D})) \to (\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$ is a continuous functor $\psi\colon (\mathcal{D}, \mathrm{Cov}(\mathcal{D})) \to (\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$ such that the functor ψ_s is exact. The sites form a category. The *push-forward functor* of φ is defined to be ψ^s and is denoted by φ_*. The *pull-back functor* of φ is defined to be ψ_s and is denoted by φ^*.

2.4 Convex geometry

Let $n \in \mathbb{N}$ and M be a free abelian group of rank n. The following section is based on [Gub07a, Appendix A]. See also [Gub13, Appendix A] or [GK17, Appendix A], who all have used [Roc70] and [McM03] as references. The goal is to provide the necessary tools for Chapter 5.

Definition 2.4.1. We define the $\mathbb{Z}$-module $N := \hom_{\mathbb{Z}-\mathbf{Mod}}(M, \mathbb{Z})$, which is again a free abelian group of rank n. The definitions of the n-dimensional $\mathbb{Q}$-vector spaces $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$, $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ and the n-dimensional $\mathbb{R}$-vector spaces $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ give a canonical way to consider M respectively N as a subobject of $M_{\mathbb{Q}}$,$M_{\mathbb{R}}$ respectively $N_{\mathbb{Q}}$,$N_{\mathbb{R}}$ in the category of $\mathbb{Z}$-modules. In the same way we may consider $M_{\mathbb{Q}}$ respectively

$N_{\mathbb{Q}}$ as a subobject of $M_{\mathbb{R}}$ respectively $N_{\mathbb{R}}$ in the category of $\mathbb{Q}$-vector spaces. Moreover we define the *dual pairing* to be the morphism of $\mathbb{Z}$-modules

$$\begin{aligned}\langle .,. \rangle : M \times N &\to \mathbb{Z},\\ (m,n) &\mapsto n(m),\end{aligned}$$

which extends uniquely to $M_{\mathbb{Q}} \times N_{\mathbb{Q}}$ and $M_{\mathbb{R}} \times N_{\mathbb{R}}$ by tensoring with $\mathbb{Q}$ and $\mathbb{R}$.

Definition 2.4.2. A *closed half space/open half space/hyperplane* in $N_{\mathbb{R}}$ is a subset of $N_{\mathbb{R}}$ of the form $H_{\square}(m,c) := \{u \in N_{\mathbb{R}} \mid \langle m,u\rangle \square\, c\}$ where $m \in M_{\mathbb{R}}$, $c \in \mathbb{R}$, and $\square$ has to be replaced by $\geq/>/=$. It is called Γ-*rational* for a subgroup Γ of $(\mathbb{R},+)$ if there exist choices $m \in M$ and $c \in \Gamma$. A *polyhedron* Δ in $N_{\mathbb{R}}$ is a subset $\Delta \subseteq N_{\mathbb{R}}$ which is equal to the intersection of finitely many closed half spaces in $N_{\mathbb{R}}$. We call a polyhedron $\Delta \subseteq N_{\mathbb{R}}$ Γ-*rational* if all the closed half spaces can be chosen to be Γ-rational. A *closed face* of a polyhedron Δ is a subset of $N_{\mathbb{R}}$ which is either equal to Δ, to $\varnothing$, or to $H_{=}(m,c) \cap \Delta$ where it must hold $\Delta \subseteq H_{\geq}(m,c)$ for $m \in M_{\mathbb{R}}$, $c \in \mathbb{R}$. A *vertex* of a polyhedron Δ is an element $\xi \in \Delta$ such that the subset $\{\xi\}$ is a closed face of Δ. An *open face* of a polyhedron Δ is a closed face of Δ without all its properly contained closed faces. We denote by $\operatorname{relint}(\Delta)$ the unique open face of a polyhedron Δ which is dense in Δ. We define the $\mathbb{R}$-affine subspace $\mathbb{A}_{\Delta}$ of $N_{\mathbb{R}}$ associated to a polyhedron Δ in $N_{\mathbb{R}}$ to be the smallest $\mathbb{R}$-affine subspace of $N_{\mathbb{R}}$ with $\Delta \subseteq \mathbb{A}_{\Delta}$. Denote by $\mathbb{V}_{\Delta}$ the $\mathbb{R}$-vector subspace of $N_{\mathbb{R}}$ corresponding to $\mathbb{A}_{\Delta}$. The dimension $\dim_{\mathbb{R}}(\Delta)$ of a polyhedron is defined to be the dimension of the $\mathbb{R}$-vector space $\mathbb{V}_{\Delta}$. A (Γ-rational) *polytope* is a bounded (Γ-rational) polyhedron. A (Γ-rational) *polytopal set* S in $N_{\mathbb{R}}$ is a subset $S \subseteq N_{\mathbb{R}}$ equal to a finite union of (Γ-rational) polytopes in $N_{\mathbb{R}}$.
A (Γ-rational) *polytopal complex* $\mathscr{C}$ in $N_{\mathbb{R}}$ is a locally finite set of (Γ-rational) polytopes in $N_{\mathbb{R}}$ such that we have:

(i) If $\Delta \in \mathscr{C}$, then all closed faces of Δ are in $\mathscr{C}$.

(ii) If $\Delta_1, \Delta_2 \in \mathscr{C}$, then $\Delta_1 \cap \Delta_2$ is either empty or a closed face of Δ_1 and Δ_2.

A *vertex* of a polytopal complex $\mathscr{C}$ in $N_{\mathbb{R}}$ is an element $\xi \in N_{\mathbb{R}}$ such that $\{\xi\} \in \mathscr{C}$. The *support* of a polytopal complex $\mathscr{C}$ in $N_{\mathbb{R}}$ is defined to be the

subset $\mathrm{Supp}(\mathscr{C}) := \bigcup_{\Delta\in\mathscr{C}} \Delta$ of $N_\mathbb{R}$. A *(Γ-rational) polytopal decomposition* of a (Γ-rational) polytopal set S in $N_\mathbb{R}$ is a (Γ-rational) polytopal complex $\mathscr{C}$ in $\mathbb{R}^n$ such that the support of $\mathscr{C}$ is equal to S. A (Γ-rational) *polytopal subdivision* $\mathscr{D}$ of a (Γ-rational) polytopal complex $\mathscr{C}$ in $N_\mathbb{R}$ is a (Γ-rational) polytopal complex $\mathscr{D}$ in $N_\mathbb{R}$ such that every polytope $\Delta \in \mathscr{C}$ has a (Γ-rational) polytopal decomposition $\mathscr{D}_\Delta \subseteq \mathscr{D}$. Let $\mathscr{C}_1$, $\mathscr{C}_2$ be (Γ-rational) polytopal complexes in $N_\mathbb{R}$. We then define $\mathscr{C}_1 \cap \mathscr{C}_2$ to be the (Γ-rational) polytopal complex in $N_\mathbb{R}$ consisting of all subsets of the form $\Delta_1 \cap \Delta_2 \subseteq N_\mathbb{R}$ for $\Delta_1 \in \mathscr{C}_1$ and $\Delta_2 \in \mathscr{C}_2$.

Lemma 2.4.3. *Let $\mathscr{C}_1$, $\mathscr{C}_2$ be (Γ-rational) polytopal complexes in $N_\mathbb{R}$. Then $\mathscr{C}_1 \cap \mathscr{C}_2$ is indeed a (Γ-rational) polytopal complex in $N_\mathbb{R}$.*

Proof. At first it follows directly by the definition of a polytope that the intersection of finitely many polytopes is again a polytope. We check the conditions of a polytopal complex in Definition 2.4.2:

(i) Let $\Delta_1 \in \mathscr{C}_1$, $\Delta_2 \in \mathscr{C}_2$. Let τ be a closed face of $\Delta_1 \cap \Delta_2$. The cases $\tau = \Delta_1 \cap \Delta_2$ and $\tau = \varnothing$ are clear. Let $m \in M_\mathbb{R}$, $c \in \mathbb{R}$ such that $\Delta_1 \cap \Delta_2 \subseteq H_\geq(m,c)$ and $\tau = H_=(m,c) \cap (\Delta_1 \cap \Delta_2)$. Since $\mathscr{C}_1$, $\mathscr{C}_2$ are already polytopal complexes in $N_\mathbb{R}$, we get $H_=(m,c) \cap \Delta_1 \in \mathscr{C}_1$ and $H_=(m,c) \cap \Delta_2 \in \mathscr{C}_2$, which implies $\tau \in \mathscr{C}_1 \cap \mathscr{C}_2$.

(ii) Let $\Delta_1, \Delta_1' \in \mathscr{C}_1$, $\Delta_2, \Delta_2' \in \mathscr{C}_2$. Consider just the non-trivial cases, meaning $\Delta_1 \cap \Delta_1' = \Delta_1 \cap H_=(m_1,c_1)$ and $\Delta_2 \cap \Delta_2' = \Delta_2 \cap H_=(m_2,c_2)$, and $\Delta_1 \subseteq H_\geq(m_1,c_1)$ and $\Delta_2 \subseteq H_\geq(m_2,c_2)$ for $m_1, m_2 \in M_\mathbb{R}$, $c_1, c_2 \in \mathbb{R}$. Finally for $\Delta := (\Delta_1 \cap \Delta_2) \cap (\Delta_1' \cap \Delta_2')$ we get $\Delta = \Delta_1 \cap \Delta_2 \cap H_=(m_1,c_1) \cap H_=(m_2,c_2)$, and $\Delta_1 \cap \Delta_2 \subseteq H_\geq(m_1,c_1)$ and $\Delta_1 \cap \Delta_2 \subseteq H_\geq(m_2,c_2)$. These properties imply that there exist $m \in M_\mathbb{R}$, $c \in \mathbb{R}$ such that $\Delta = \Delta_1 \cap \Delta_2 \cap H_=(m,c)$ and $\Delta_1 \cap \Delta_2 \subseteq H_\geq(m,c)$, which means that Δ is a closed face of $\Delta_1 \cap \Delta_2$.

The Γ-rational case follows analogously. □

Definition 2.4.4. A *lattice* in the n-dimensional $\mathbb{R}$-vector space $N_\mathbb{R}$ is a $\mathbb{Z}$-submodule Λ of $N_\mathbb{R}$ which is generated by $\mathbb{R}$-linearly independent vectors. It is called *complete* (often also called *full*) if we have $\Lambda \otimes_\mathbb{Z} \mathbb{R} = N_\mathbb{R}$.

Definition 2.4.5. Let Λ be a complete lattice in the n-dimensional $\mathbb{R}$-vector space $N_\mathbb{R}$, and let $\lambda\colon N_\mathbb{R} \twoheadrightarrow N_\mathbb{R}/\Lambda$ be the quotient function. A

(Γ-rational) *polytope* $\overline{\Delta}$ in $N_{\mathbb{R}}/\Lambda$ is a subset $\overline{\Delta} \subseteq N_{\mathbb{R}}/\Lambda$ such that there exists a (Γ-rational) polytope Δ in $N_{\mathbb{R}}$ such that $\lambda(\Delta) = \overline{\Delta}$, and for any $y \in \overline{\Delta}$ there is exactly one $x \in \Delta$ with $\lambda(x) = y$. We call such a (Γ-rational) polytope Δ in $N_{\mathbb{R}}$ *induced* by $\overline{\Delta}$. A *closed face* of a polytope $\overline{\Delta}$ in $N_{\mathbb{R}}/\Lambda$ is a polytope $\overline{\sigma}$ in $N_{\mathbb{R}}/\Lambda$ such that there exist induced polytopes Δ and σ in $N_{\mathbb{R}}$ where σ is a closed face of Δ. A (Γ-rational) *polytopal set* $\overline{S}$ in $N_{\mathbb{R}}/\Lambda$ is a subset $\overline{S} \subseteq N_{\mathbb{R}}/\Lambda$ equal to a finite union of (Γ-rational) polytopes in $N_{\mathbb{R}}/\Lambda$.
A *(Γ-rational) polytopal complex* $\overline{\mathscr{C}}$ in $N_{\mathbb{R}}/\Lambda$ is a locally finite set of (Γ-rational) polytopes in $N_{\mathbb{R}}/\Lambda$ such that we have:

(i) If $\overline{\Delta} \in \overline{\mathscr{C}}$, then all closed faces of $\overline{\Delta}$ are in $\overline{\mathscr{C}}$.

(ii) If $\overline{\Delta}_1, \overline{\Delta}_2 \in \overline{\mathscr{C}}$, then $\overline{\Delta}_1 \cap \overline{\Delta}_2$ is either empty or a closed face of $\overline{\Delta}_1$ and $\overline{\Delta}_2$.

The *support* of a polytopal complex $\overline{\mathscr{C}}$ in $N_{\mathbb{R}}/\Lambda$ is defined to be the subset $\mathrm{Supp}(\overline{\mathscr{C}}) := \bigcup_{\overline{\Delta} \in \overline{\mathscr{C}}} \overline{\Delta}$ of $N_{\mathbb{R}}/\Lambda$. A (Γ-rational) *polytopal decomposition* of a (Γ-rational) polytopal set $\overline{S}$ in $N_{\mathbb{R}}/\Lambda$ is a (Γ-rational) polytopal complex $\overline{\mathscr{C}}$ in $N_{\mathbb{R}}/\Lambda$ such that the support of $\overline{\mathscr{C}}$ is equal to $\overline{S}$. A (Γ-rational) *polytopal subdivision* $\overline{\mathscr{D}}$ of a (Γ-rational) polytopal complex $\overline{\mathscr{C}}$ in $N_{\mathbb{R}}/\Lambda$ is a (Γ-rational) polytopal complex $\overline{\mathscr{D}}$ in $N_{\mathbb{R}}/\Lambda$ such that every polytope $\overline{\Delta} \in \overline{\mathscr{C}}$ has a (Γ-rational) polytopal decomposition $\overline{\mathscr{D}}_{\overline{\Delta}} \subseteq \overline{\mathscr{D}}$. For any (Γ-rational) polytopal complex $\overline{\mathscr{C}}$ in $N_{\mathbb{R}}/\Lambda$, we define $\mathscr{C}$ to be the (Γ-rational) polytopal complex in $N_{\mathbb{R}}$ consisting of all (Γ-rational) polytopes $\Delta \subseteq N_{\mathbb{R}}$ such that $\overline{\Delta} \in \overline{\mathscr{C}}$.

Definition 2.4.6. Let Λ be a complete lattice in the n-dimensional $\mathbb{R}$-vector space $N_{\mathbb{R}}$. A polytopal complex $\mathscr{C}$ in $N_{\mathbb{R}}$ is called Λ-*periodic* if for all elements $\Delta \in \mathscr{C}$, $\lambda \in \Lambda$ we have $\Delta + \lambda \in \mathscr{C}$.

Remark 2.4.7. Definition 2.4.6 is weaker than the definition in [BGJK21, Remark 8.2.5]. Here we have that for every polytopal complex $\overline{\mathscr{C}}$ in $N_{\mathbb{R}}/\Lambda$ the induced polytopal complex $\mathscr{C}$ in $N_{\mathbb{R}}$ is Λ-periodic but not vice versa. In our definition the polytopes Δ and $\Delta + \lambda$ could intersect for $\lambda \neq 0$, which is not allowed for $\overline{\mathscr{C}}$. For the definition in [BGJK21, Remark 8.2.5] the above implication becomes an equivalence because it avoids this case.

Definition 2.4.8. Let $\overline{\mathscr{C}_1}, \overline{\mathscr{C}_2}$ be (Γ-rational) polytopal complexes in $N_\mathbb{R}/\Lambda$. We then define $\overline{\mathscr{C}_1} \cap \overline{\mathscr{C}_2}$ to be the (Γ-rational) polytopal complex $\overline{\mathscr{C}_1 \cap \mathscr{C}_2}$ in $N_\mathbb{R}/\Lambda$ inducing $\mathscr{C}_1 \cap \mathscr{C}_2$ in $N_\mathbb{R}$.

Lemma 2.4.9. *The polytopal complex $\overline{\mathscr{C}_1 \cap \mathscr{C}_2}$ in $N_\mathbb{R}/\Lambda$ is well-defined.*

Proof. The polytopal complex $\mathscr{C}_1 \cap \mathscr{C}_2$ is Λ-periodic and avoids the case described in Remark 2.4.7 because $\mathscr{C}_1$ and $\mathscr{C}_2$ already avoid this case. □

Definition 2.4.10. A function $f\colon N_\mathbb{R} \to \mathbb{R}$ is called *convex* if for all $u_1, u_2 \in N_\mathbb{R}$ with $u_1 \neq u_2$ and $t \in [0,1]$ we have $f(tu_1 + (1-t)u_2) \leq tf(u_1) + (1-t)f(u_2)$. It is called *strictly convex* if we may replace $\leq$ by $<$.

Definition 2.4.11. Let Λ be a complete lattice in the n-dimensional $\mathbb{R}$-vector space $N_\mathbb{R}$. A function $f\colon N_\mathbb{R} \to \mathbb{R}$ is called Λ-*periodic* if for all $u \in N_\mathbb{R}$, $\lambda \in \Lambda$ we have $f(u) = f(u+\lambda)$.

Definition 2.4.12. An $\mathbb{R}$-*affine function* is a function between $\mathbb{R}$-vector spaces $f\colon V_1 \to V_2$ such that there exists an element $c \in V_2$ (called *intercept*) providing that $f^{(l)} := f - c\colon V_1 \to V_2$ (called *linearization*) is a morphism of $\mathbb{R}$-vector spaces. If V_2 is one-dimensional, it can also be called *slope* instead of linearization. Let U be a subset of V_1. A function $g\colon U \to V_2$ is called $\mathbb{R}$-*affine* if there exists an $\mathbb{R}$-affine function $f\colon V_1 \to V_2$ such that its restriction to U is equal to g.
Let Γ be a $\mathbb{Z}$-submodule of $\mathbb{R}$, and let M_1, M_2 be free $\mathbb{Z}$-modules. An $\mathbb{R}$-affine function $f\colon M_1 \otimes_\mathbb{Z} \mathbb{R} \to M_2 \otimes_\mathbb{Z} \mathbb{R}$ is called Γ-*rational* if the linearization of f is induced by a morphism of $\mathbb{Z}$-modules $f^{(l)}\colon M_1 \to M_2$, and the intercept of f lies in $M_2 \otimes_\mathbb{Z} \Gamma$. This definition extends to $\mathbb{R}$-affine functions $g\colon U \to M_2 \otimes_\mathbb{Z} \mathbb{R}$ if the smallest $\mathbb{R}$-affine subspace of $M_1 \otimes_\mathbb{Z} \mathbb{R}$ containing the subset U is equal to $M_1 \otimes_\mathbb{Z} \mathbb{R}$.

Definition 2.4.13. Let $\mathscr{C}$ be a polyhedral decomposition of $N_\mathbb{R}$. A function $f\colon N_\mathbb{R} \to \mathbb{R}$ is called $\mathscr{C}$-*polyhedral* if its restriction to any polyhedron $\Delta \in \mathscr{C}$ is $\mathbb{R}$-affine. Let $\mathscr{N} \subseteq \mathscr{C}$ be the subset of n-dimensional polyhedrons, and Γ be a $\mathbb{Z}$-submodule of $\mathbb{R}$. Furthermore, we denote the slope of f on the polyhedron $\eta \in \mathscr{N}$ by $m_\eta \in M_\mathbb{R}$ and its intercept by $c_\eta \in \mathbb{R}$. A $\mathscr{C}$-polyhedral function $f\colon N_\mathbb{R} \to \mathbb{R}$ is called Γ-*rational* if the $\mathbb{R}$-affine functions $f|_\eta$ on all $\eta \in \mathscr{N}$ are Γ-rational, which means $m_\eta \in M$ and $c_\eta \in \Gamma$.

A function $f\colon N_{\mathbb{R}} \to \mathbb{R}$ is called *strongly $\mathscr{C}$-polyhedral at $u \in N_{\mathbb{R}}$* if it is $\mathscr{C}$-polyhedral and for any $\eta_1, \eta_2 \in \mathscr{N}$ with $\eta_1 \neq \eta_2$ and $u \in \eta_1 \cap \eta_2$ we have $m_{\eta_1} \neq m_{\eta_2}$. A function $f\colon N_{\mathbb{R}} \to \mathbb{R}$ is called *strongly $\mathscr{C}$-polyhedral* if it is strongly $\mathscr{C}$-polyhedral at every $u \in N_{\mathbb{R}}$.

Remark 2.4.14. A function $f\colon N_{\mathbb{R}} \to \mathbb{R}$ is strongly $\mathscr{C}$-polyhedral convex in the sense of [Gub07a, Definition A.3] if and only if it is strongly $\mathscr{C}$-polyhedral and convex.

Definition 2.4.15. The *distance of uniform convergence* of two functions $f, g\colon N_{\mathbb{R}} \to \mathbb{R}$ is defined to be the element

$$d(f,g) := \sup_{u \in N_{\mathbb{R}}} |f(u) - g(u)|_{\mathbb{R}} \in \mathbb{R} \cup \{+\infty\}.$$

We say a sequence $(f_k)_{k\in\mathbb{N}}$ of functions $f_k\colon N_{\mathbb{R}} \to \mathbb{R}$ *converges uniformly* to a function $f\colon N_{\mathbb{R}} \to \mathbb{R}$ if the sequence $(d(f_k, f))_{k\in\mathbb{N}}$ in $\mathbb{R}\cup\{+\infty\}$ converges to zero.

2.5 Abelian varieties and line bundles

Let K be a field. The following section is based on [BG07, Chapter 8].

Definition 2.5.1 ([Har77, §II.4 p. 105])**.** A *K-variety* is a reduced separated scheme of finite type over K.

Definition 2.5.2. A scheme X of finite type over K is called *geometrically irreducible/reduced* if the scheme $X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$ is irreducible/reduced.

Definition 2.5.3. A K-variety X is called *complete* if for all K-varieties Y the morphism of schemes $p_2\colon X \times_{\mathrm{Spec}(K)} Y \to Y$ is closed. A morphism of schemes is called closed if it maps closed subsets to closed subsets.

Definition 2.5.4. An *abelian K-variety* is a group object in the category of geometrically irreducible and geometrically reduced complete K-varieties.

Remark 2.5.5. Definition 2.5.4 is equivalent to [BG07, Definition 8.2.3].

Lemma 2.5.6. *Any abelian K-variety is abelian as a group object.*

Proof. [BG07, Corollary 8.2.10] □

Definition 2.5.7 ([BG07, 9.5.6])**.** Let A be an abelian K-variety. A *rigidified line bundle* on A is a pair (L, ρ) consisting of a line bundle L on A and an element $\rho \in L_0(K)\backslash\{0\}$. The latter element is called a *rigidification* of L. A morphism of rigidified line bundles $\varphi\colon (L, \rho) \to (L', \rho')$ is a morphism of line bundles $\varphi\colon L \to L'$ such that ρ is mapped to ρ' under the induced function $\varphi\colon L_0(K) \to L_0'(K)$. Remember that L_0 is the pull-back of the morphisms $L \to A$ and $0\colon \mathrm{Spec}(K) \to A$ in the category of K-varieties. The tensor product of two rigidified line bundles is defined by $(L_1, \rho_1) \otimes (L_2, \rho_2) := (L_1 \otimes L_2, \rho_1 \otimes \rho_2)$.

Lemma 2.5.8. *Let A be an abelian K-variety. The definition*

$$\begin{aligned}[m]\colon A &\to A,\\ a &\mapsto \underbrace{a + ... + a}_{m\text{-}times}\end{aligned}$$

yields for any $m \in \mathbb{Z}$ an endomorphism of abelian K-varieties.

Proof. [BG07, 8.2.11] □

Definition 2.5.9 ([BG07, Example 8.6.3])**.** Let A be an abelian K-variety. A line bundle L on A is called *odd* if $[-1]^*L \cong L^{-1}$, and *even* if $[-1]^*L \cong L$ in the category of line bundles on A. A rigidified line bundle is called odd/even if the underlying line bundle is odd/even.

Lemma 2.5.10. *Let A be an abelian K-variety, and (L, ρ) be a rigidified line bundle on A. Then there exists an odd rigidified line bundle (L_-, ρ_-) and an even rigidified line bundle (L_+, ρ_+) such that*

$$(L, \rho) \cong (L_-, \rho_-) \otimes (L_+, \rho_+).$$

Proof. [BG07, Corollary 8.8.2] □

Lemma 2.5.11. *Let A be an abelian K-variety. Then for any $m \in \mathbb{Z}$ we have*

(i) $[m]^*L_- \cong L_-^{\otimes m}$ *if L_- is an odd line bundle on A, and*

(ii) $[m]^*L_+ \cong L_+^{\otimes m^2}$ *if L_+ is an even line bundle on A.*

Proof. [BG07, Proposition 8.7.1] □

Definition 2.5.12 ([BG07, A.6.10])**.** Let X be a projective K-variety. Then a line bundle L on X is called *very ample* if there is a closed embedding $i\colon X \to \mathbb{P}^n_K$ such that L is isomorphic to $i^*\mathcal{O}_{\mathbb{P}^n_K}(1)$ as line bundles on X, where $i\colon X \to \mathbb{P}^n_K$ being a closed embedding means that it is a morphism of K-varieties such that it induces an isomorphism onto a closed K-subvariety of X. A line bundle L is called *ample* if there exists an $m \in \mathbb{N}$ such that $L^{\otimes m}$ is very ample.

Remark 2.5.13. Since by [BG07, Corollary 8.5.7] any abelian K-variety is projective, Definition 2.5.12 applies also to abelian K-varieties.

2.6 Quadratic functions

Let R be a non-trivial ring. The following is a generalization of [BG07, 8.6.5] from abelian groups to R-modules and provides some useful terms for Chapter 5.

Definition 2.6.1. Let M_1, M_2, N be R-modules. An *R-bilinear form* is a morphism of R-modules $M_1 \times M_2 \to N$. It is called *symmetric* if $M_1 = M_2$ and if its composition with the commutator isomorphism $M_2 \times M_1 \xrightarrow{\sim} M_1 \times M_2$ does not affect the R-bilinear form.

Definition 2.6.2. Let M, N be R-modules, and $\varphi\colon M \to N$ be a morphism of the underlying sets. Then φ is called an *R-quadratic function* if the function

$$\begin{aligned} b\colon M \times M &\to N, \\ (m_1, m_2) &\mapsto \varphi(m_1 + m_2) - \varphi(m_1) - \varphi(m_2) \end{aligned}$$

is an R-bilinear form. In this case b is called the *associated R-bilinear form*, which is always symmetric. An *R-quadratic form* is an R-quadratic function $\varphi\colon M \to N$ such that for any $r \in R$ we have $\varphi(rm) = r^2\varphi(m)$.

Lemma 2.6.3. *Let M, N be R-modules, and $\varphi\colon M \to N$ be an R-quadratic function. Then*

$$\begin{aligned} q\colon M &\to N, \\ m &\mapsto \varphi(m) + \varphi(-m) \end{aligned}$$

is an R-quadratic form and

$$\begin{aligned} l\colon M &\to N, \\ m &\mapsto \varphi(m) - \varphi(-m) \end{aligned}$$

is a morphism of R-modules and we get $2\varphi = q + l$ in the category of sets. Furthermore, we have $q(m) = b(m, m)$ for all $m \in M$.

Proof. [BG07, 8.6.5] (for abelian groups) generalizes to R-modules. □

Definition 2.6.4. Let M, N be R-modules, and $\varphi\colon M \to N$ be an R-quadratic function. We then call q from Lemma 2.6.3 the *associated R-quadratic form* of φ, and l the *associated R-linear form* of φ.

3 Analytic and formal geometry

Working with non-archimedean fields has the great disadvantage that the valuation topology on rational points induced by balls is totally disconnected, which means that their only connected components are the singletons and the empty set. One can circumvent this problem by considering their analytifications, developed by Berkovich [Ber90; Ber93]. For this purpose, we will present in the following three sections the necessary parts of Berkovich's theory. In the fourth section we will collect basics from formal geometry, in particular formal models, which will be needed for the rest of this thesis.

Let $(K, |.|)$ be a non-trivially valued non-archimedean field, and $(\mathbb{K}, |.|)$ be an algebraically closed non-trivially valued non-archimedean field.

3.1 The Berkovich spectrum

The following section is based on [Ber90, §1].

Definition 3.1.1. Let G be an abelian group. A *semi-norm* on G is a function $||.||\colon G \to \mathbb{R}_{\geq 0}$ such that

(i) $||0|| = 0$, and

(ii) $||f - g|| \leq ||f|| + ||g||$

for all $f, g \in G$. Two semi-norms $||.||_1$ and $||.||_2$ on G are called *equivalent* if there exist $c, c' > 0$ such that we have $c||g|| \leq ||g||' \leq c'||g||$ for all $g \in G$.

Definition 3.1.2. A *semi-normed group* is a pair $(G, ||.||)$ consisting of an abelian group G and a semi-norm $||.||$ on G. A morphism of semi-normed groups $\varphi\colon (G, ||.||_G) \to (H, ||.||_H)$ is a morphism of the underlying groups. The semi-normed groups form a category.

Definition 3.1.3. Let $(G, ||.||_G)$ be a semi-normed group, and H be a subgroup of G. Then the *residue semi-norm* of the quotient group G/H is defined as the function

$$||.||_{G/H} \colon G/H \to \mathbb{R}_{\geq 0}, \quad \overline{g} \mapsto \inf_{g' \in \pi^{-1}(\overline{g})} ||g'||_G$$

where $\pi \colon G \to G/H$ is the quotient morphism of groups.

Definition 3.1.4. A morphism of semi-normed groups $\varphi \colon (G, ||.||_G) \to (H, ||.||_H)$ is called *admissible* if the residue semi-norm $||.||_{G/\ker(\varphi)}$ of the quotient group $G/\ker(\varphi)$ is equivalent to the restriction of the semi-norm $||.||_H$ to $\mathrm{Im}(\varphi)$.

Definition 3.1.5. Let R be a ring. We define a *semi-norm* on R as a function $||.|| \colon R \to \mathbb{R}_{\geq 0}$ such that

(i) $||0|| = 0$, $||1|| = 1$,

(ii) $||f - g|| \leq ||f|| + ||g||$, and

(iii) $||fg|| \leq ||f|| \cdot ||g||$

for all $f, g \in R$. A semi-norm is called *multiplicative* if we have $||fg|| = ||f|| \cdot ||g||$ instead of condition (iii). A *norm* on R is a semi-norm $||.||$ on R such that $||f|| = 0$ if and only if $f = 0$.

Definition 3.1.6. A *Banach ring* is a pair $(R, ||.||_R)$ consisting of a ring R and a norm $||.||_R$ on the ring R such that R is complete with respect to $||.||_R$ when we consider $(R, ||.||_R)$ as a metric space.

Definition 3.1.7 ([Ber90, §1.2])**.** Let $(R, ||.||_R)$ be a Banach ring. A semi-norm $||.||$ on $(R, ||.||_R)$ is called *bounded* if there exists a constant $c > 0$ such that $||f|| \leq c||f||_R$ for all $f \in R$. The *Berkovich spectrum* $\mathscr{M}(R)$ is the topological space consisting of the set of all bounded multiplicative semi-norms and the weakest topology such that for all $f \in R$ the function $\mathscr{M}(R) \to \mathbb{R}, ||.|| \mapsto ||f||$ is continuous. We call this topology the *Berkovich topology*.

Definition 3.1.8 ([Ber90, Remark 1.2.2(i)])**.** Let $(R, ||.||_R)$ be a Banach ring, and $x = ||.||_x \in \mathscr{M}(R)$. The bounded multiplicative semi-norm $||.||_x \colon R \to \mathbb{R}_{\geq 0}$ induces an absolute value $|.|_x \colon \mathrm{Frac}(R/\ker(||.||_x)) \to \mathbb{R}_{\geq 0}$. Define $\mathscr{H}(x)$ to be the completion of $\mathrm{Frac}(R/\ker(||.||_x))$ with respect to the absolute value $|.|_x$. Furthermore, we get a canonical morphism of rings $\chi_x \colon R \to \mathscr{H}(x)$ and we define the element $f(x) := \chi_x(f)$ for $f \in R$.

Definition 3.1.9 ([Ber90, Remark 1.2.5(i)])**.** Let $(R, ||.||_R)$ be a Banach ring. Then the *kernel map* of $(R, ||.||_R)$ is defined to be the epimorphism of topological spaces

$$\begin{aligned} \pi \colon \mathscr{M}(R) &\twoheadrightarrow \mathrm{Spec}(R), \\ ||.||_x &\mapsto \ker(||.||_x)\,. \end{aligned}$$

Definition 3.1.10 ([Ber90, §1.3])**.** Let $(R, ||.||_R)$ be a Banach ring. We then define the *spectral radius* to be the bounded semi-norm

$$\begin{aligned} \rho \colon R &\to \mathbb{R}_{\geq 0}, \\ f &\mapsto \lim_{n \to +\infty} \sqrt[n]{||f^n||_R}. \end{aligned}$$

3.2 Affinoid algebras and affinoid spaces

This section is based on [Ber90, §2]. Let $n \in \mathbb{N}$ and $T = (T_1, ..., T_n)$.

Definition 3.2.1. A *Banach K-algebra* $\mathscr{A}$ is a K-vector space $(\mathscr{A}, +, \cdot)$ together with a K-bilinear, associative, commutative operation $\cdot \colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ and a norm $||.||_{\mathscr{A}}$ on the K-vector space $\mathscr{A}$ such that $(\mathscr{A}, ||.||_{\mathscr{A}})$ is complete when considered as a metric space. A morphism of Banach K-algebras $(\mathscr{A}, +, \cdot, ||.||_{\mathscr{A}}) \to (\mathscr{B}, +, \cdot, ||.||_{\mathscr{B}})$ is a morphism of the underlying K-algebras. The Banach K-algebras form a category.

Example 3.2.2. The set

$$K\{T\} := \Big\{ \sum_{i \in \mathbb{N}^n} a_i T^i \in K[[T]] \;\Big|\; |a_i| \to 0 \text{ as } ||i|| \to +\infty \Big\}$$

where $||i|| := i_1 + ... + i_n$ together with the Gauß norm

$$\Big|\Big| \sum_{i \in \mathbb{N}^n} a_i T^i \Big|\Big|_{K\{T\}} := \max_{i \in \mathbb{N}^n} |a_i|$$

yields a Banach K-algebra.

Proof. [Ber90, §2.1] □

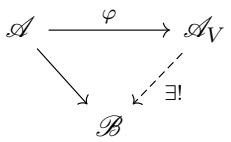
Definition 3.2.3. A *K-affinoid algebra* is a Banach K-algebra $\mathscr{A}$ such that there exists an epimorphism of Banach K-algebras

$$\left(K\{T\}, ||.||_{K\{T\}}\right) \twoheadrightarrow (\mathscr{A}, ||.||_{\mathscr{A}})$$

which is admissible in the underlying category of semi-normed groups. A morphism of K-affinoid algebras is a morphism of the underlying Banach K-algebras. The K-affinoid algebras form a category $K-\mathbf{affAlg}$.

Remark 3.2.4. The strictly K-affinoid algebra in [Ber90, Definition 2.1.1] is the same as the affinoid K-algebra defined in [Bos14, §3.1, Definition 1] and the K-affinoid algebra defined in [BGR84, §6.1, Definition 1] and in Definition 3.2.3 above.

Definition 3.2.5 ([Ber90, Definition 2.2.1])**.** Let $\mathscr{A}$ be a K-affinoid algebra. A *K-affinoid domain* on the topological space $\mathscr{M}(\mathscr{A})$ is a closed subset $V \subseteq \mathscr{M}(\mathscr{A})$ such that there exists a morphism of K-affinoid algebras $\varphi\colon \mathscr{A} \to \mathscr{A}_V$ satisfying the following universal property: Given a morphism of K-affinoid algebras $\mathscr{A} \to \mathscr{B}$ such that the image of $\mathscr{M}(\mathscr{B})$ in $\mathscr{M}(\mathscr{A})$ lies in V, there exists a unique morphism of K-affinoid algebras $\mathscr{A}_V \to \mathscr{B}$ making the diagram in the category of K-affinoid algebras



commutative.

Remark 3.2.6. Due to [Ber90, Remark 2.1.13] and Remark 3.2.4, morphisms of K-affinoid algebras in our sense are always bounded, which allows us to skip this condition for the morphisms in Definition 3.2.5.

Lemma 3.2.7. *Let $\mathscr{A}$ be a K-affinoid algebra. For any K-affinoid domain V on the topological space $\mathscr{M}(\mathscr{A})$, the topological spaces V and $\mathscr{M}(\mathscr{A}_V)$ are isomorphic.*

Proof. [Ber90, Proposition 2.2.4(i)] □

Example 3.2.8 ([Ber90, Remark 2.2.2(i)])**.** Define the multiplicative subgroup

$$\sqrt{|K^\times|} := \{r \in \mathbb{R}_{>0} \mid \exists \alpha \in K\backslash\{0\}, k \in \mathbb{N}\backslash\{0\} \text{ such that } |\alpha| = r^k\}$$

of $\mathbb{R}_{>0}$. For $n, m \in \mathbb{N}$, $p \in (\sqrt{|K^\times|})^n$, $q \in (\sqrt{|K^\times|})^m$, $f \in \mathscr{A}^n$, $g \in \mathscr{A}^m$ the subset

$$\mathscr{M}(\mathscr{A})\left(\frac{p^{-1}f}{q^{-1}g}\right) := \left\{ x \in \mathscr{M}(\mathscr{A}) \;\middle|\; \begin{array}{l} |f_i(x)|_{\overline{x}} \leq p_i, i \in \{1, ..., n\}, \\ |g_j(x)|_{\overline{x}} \geq p_j, j \in \{1, ..., m\} \end{array} \right\}$$

is a K-affinoid domain on $\mathscr{M}(\mathscr{A})$. It is called a *Laurent domain* in $\mathscr{M}(\mathscr{A})$. If $m = 0$, it is called a *Weierstrass domain* in $\mathscr{M}(\mathscr{A})$. The corresponding K-affinoid algebra is denoted by $\mathscr{A}\left\langle \frac{p^{-1}f}{q^{-1}g} \right\rangle$.

Definition 3.2.9. Let $\mathscr{A}$ be a K-affinoid algebra. A *special subset* of the topological space $\mathscr{M}(\mathscr{A})$ is a closed subset $V \subseteq \mathscr{M}(\mathscr{A})$ that is a finite union of K-affinoid domains on $\mathscr{M}(\mathscr{A})$.

Definition 3.2.10 ([Ber90, §2.3])**.** A *K-affinoid space* is defined to be a locally ringed space $(\mathscr{M}(\mathscr{A}), \mathcal{O}_{\mathscr{M}(\mathscr{A})})$ for a K-affinoid algebra $\mathscr{A}$, where we take the Berkovich topology and define the sheaf

$$\begin{aligned} \mathcal{O}_{\mathscr{M}(\mathscr{A})} \colon \mathbf{Top}(\mathscr{M}(\mathscr{A})) &\to \mathbf{Rng}, \\ U &\mapsto \varprojlim_{V \subseteq U \text{ special subset}} \mathscr{A}_V. \end{aligned}$$

A morphism of K-affinoid spaces $\varphi \colon \mathscr{M}(\mathscr{A}) \to \mathscr{M}(\mathscr{B})$ is a morphism of locally ringed spaces induced by a morphism of K-affinoid algebras $\mathscr{B} \to \mathscr{A}$. The K-affinoid spaces form a category denoted by $K-\mathbf{affSpc}$.

Lemma 3.2.11. *The definition*

$$\begin{aligned} \Upsilon \colon K-\mathbf{affAlg} &\to K-\mathbf{affSpc}, \\ \mathscr{A} &\mapsto \mathscr{M}(\mathscr{A}) \end{aligned}$$

yields a contravariant functor and the category $K-\mathbf{affSpc}$ is dual to $K-\mathbf{affAlg}$.

Proof. The beginning of [Ber90, §2.3] □

Definition 3.2.12 ([Ber90, §2.4])**.** Let $\mathscr{A}$ be a K-affinoid algebra. Then the definition of the subset $\mathscr{A}^\circ := \{f \in \mathscr{A} \mid \rho(f) \leq 1\}$ of $\mathscr{A}$ (see Definition 3.1.10 for the definition of ρ) yields a subring of the ring $\mathscr{A}$ and the subset $\mathscr{A}^{\circ\circ} := \{f \in \mathscr{A} \mid \rho(f) < 1\}$ of $\mathscr{A}^\circ$ forms an ideal in the ring $\mathscr{A}^\circ$. This allows us to define the quotient ring $\tilde{\mathscr{A}} := \mathscr{A}^\circ/\mathscr{A}^{\circ\circ}$. Finally the *reduction* of the K-affinoid space $\mathscr{M}(\mathscr{A})$ is defined to be the scheme $\mathrm{Spec}(\tilde{\mathscr{A}})$.

Lemma 3.2.13. *Let $\mathscr{A}$ be a K-affinoid algebra, and $x = ||.||_x \in \mathscr{M}(\mathscr{A})$. Then the function $\widetilde{\chi_x}\colon \tilde{\mathscr{A}} \to \widetilde{\mathscr{H}(x)}$ (see Definition 3.1.8) is well-defined and forms a morphism of rings.*

Proof. Beginning of [Ber90, §2.4] □

Definition 3.2.14 ([Ber90, §2.4])**.** Let $\mathscr{A}$ be a K-affinoid algebra. Then the *reduction map* of the K-affinoid space $\mathscr{M}(\mathscr{A})$ is defined to be the epimorphism of topological spaces

$$\begin{aligned} \mathrm{red}\colon \mathscr{M}(\mathscr{A}) &\twoheadrightarrow \mathrm{Spec}(\tilde{\mathscr{A}}), \\ x &\mapsto \ker(\widetilde{\chi_x})\,. \end{aligned}$$

3.3 Analytic spaces

The following section is mainly based on [Ber90, §3] and [Ber93, §1].

Definition 3.3.1 ([Ber93, §1.1])**.** Let X be a topological space. A *quasi-net* τ on X is a collection τ of subsets of X such that for each $x \in X$ there exist $V_1, ..., V_n \in \tau$ such that $x \in V_1 \cap ... \cap V_n$ and $V_1 \cup ... \cup V_n$ is a neighbourhood of x. Let τ be a collection of subsets of X. For a subset $Y \subseteq X$ we define the collection $\tau|_Y := \{V \in \tau \mid V \subseteq Y\}$. A *net* τ on X is a quasi-net τ on X such that for any $U, V \in \tau$ the collection $\tau|_{U\cap V}$ is a quasi-net on the topological space $U \cap V$ with the induced topology from X.

Definition 3.3.2 ([Ber93, §1.2])**.** Let X be a locally Hausdorff topological space, and τ a net of compact subsets of X. A *K-affinoid atlas $\mathscr{A}$ on X with respect to τ* is a map which assigns to each $V \in \tau$ a K-affinoid algebra $\mathscr{A}_V$ and an isomorphism of topological spaces $V \to \mathscr{M}(\mathscr{A}_V)$, and to each pair $U, V \in \tau$ with $U \subseteq V$ a bounded morphism of K-affinoid

algebras $\alpha_{V/U}\colon \mathscr{A}_V \to \mathscr{A}_U$ that identifies $(U, \mathscr{A}_U)$ with a K-affinoid domain in $(V, \mathscr{A}_V)$.
A *K-analytic space* is a triple $(X, \mathscr{A}, \tau)$ consisting of a locally Hausdorff topological space X, a K-affinoid atlas $\mathscr{A}$ on X with respect to τ, and a net τ of compact subsets of X. A K-analytic space $(X, \mathscr{A}, \tau)$ is called *good* if any $x \in X$ has a K-affinoid neighbourhood. A K-affinoid (Weierstrass, Laurent) domain on a K-analytic space $(X, \mathscr{A}, \tau)$ is a K-affinoid (Weierstrass, Laurent) domain on some $V \in \tau$ (see Definition 3.2.5).

Remark 3.3.3. For the definition of a morphism between K-analytic spaces we refer to [Ber93, §1.2]. The work there yields a category of K-analytic spaces denoted by $K-\mathbf{An}$ and its subcategory of good K-analytic spaces, which is equivalent to the category of K-analytic spaces defined in [Ber90] (see [Ber93, §1.5]).

Definition 3.3.4. Let X be a K-analytic space. A *K-rational point* of X is a section of the morphism of K-analytic spaces $X \to \mathscr{M}(K)$. We denote the set of all K-rational points of X by $X(K)$. Furthermore we equip the set $X(K)$ with the coarsest topology such that the injective function

$$\begin{aligned} X(K) &\hookrightarrow X, \\ \{P\colon \mathscr{M}(K) = \{|.|_K\} \to X\} &\mapsto P(|.|_K) \end{aligned}$$

becomes continuous.

Definition 3.3.5 ([Ber93, Definition 1.3.1])**.** Let X be a K-analytic space. A *K-analytic domain* on X is a subset $Y \subseteq X$ such that for any point $y \in Y$ there exist K-affinoid domains $V_1, ..., V_n \subseteq Y$ and such that $y \in V_1 \cap ... \cap V_n$ and the set $V_1 \cup ... \cup V_n$ is a neighbourhood of y in the topological space Y. The category of K-analytic domains on X denoted by $\mathbf{Dom}(X)$ has the K-analytic domains on X as objects and inclusions as morphisms.
For any $Y \in \mathrm{ob}(\mathbf{Dom}(X))$ we define $\mathrm{Cov}(\mathbf{Dom}(X))_Y$ to be the set of all families $(Y_i \to Y)_{i\in I}$ of morphisms in $\mathbf{Dom}(X)$, such that the families $(Y_i)_{i\in I}$, when considered in the category of sets, form quasi-nets (see Definition 3.3.1) on the topological space Y. Finally we define the set

$$\mathrm{Cov}(\mathbf{Dom}(X)) := \bigcup_{Y\in \mathrm{ob}(\mathbf{Dom}(X))} \mathrm{Cov}(\mathbf{Dom}(X))_Y.$$

Lemma 3.3.6. *The set* $\mathrm{Cov}(\mathbf{Dom}(X))$ *forms a Grothendieck pretopology on* $\mathbf{Dom}(X)$ *(see Definition 2.3.1).*

Proof. [Ber93, §1.3, p. 25] □

Definition 3.3.7. Let X be a K-analytic space. We then call the canonical site (see Example 2.3.3) of the underlying topological space X the *Berkovich site* and denote it also by X. Furthermore we call the site $X_G := (\mathbf{Dom}(X), \mathrm{Cov}(\mathbf{Dom}(X)))$ the *G-site*.

Lemma 3.3.8. *Let X be a K-analytic space. There is a canonical morphism of sites $\pi_G \colon X_G \to X$ from the G-site to the Berkovich site. Furthermore the push-forward functor $(\pi_G)_* : \mathbf{Sh}(X_G, \mathbf{Rng}) \to \mathbf{Sh}(X, \mathbf{Rng})$ (see Definition 2.3.9) is simply the restriction functor.*

Proof. If the valuation is non-trivial, all open subsets of X are K-analytic domains on X. Thus we get a canonical functor $\pi_G \colon \mathbf{Top}(X) \to \mathbf{Dom}(X)$, which yields by [Ber93, §1.3, pp. 25/26] a morphism of sites (see Definition 2.3.9) and the equality of rings $(\pi_G)_* \mathscr{F}(U) = \mathscr{F}(\pi_G(U))$ for any $\mathscr{F} \in \mathrm{ob}(\mathbf{Sh}(X_G, \mathbf{Rng}))$ and $U \in \mathrm{ob}(\mathbf{Dom}(X))$, which is simply the restriction of the sheaf $\mathscr{F}$ to open subsets. □

Lemma 3.3.9. *Let X be a K-analytic space. The functor*

$$\begin{aligned} \mathcal{O}_{X_G} \colon \mathbf{Dom}(X)^{\mathrm{op}} &\to \mathbf{Rng}, \\ Y &\mapsto \hom_{K-\mathbf{An}}(Y, \mathbb{A}_K^{1,\mathrm{an}}) \end{aligned}$$

is an object of the topos $\mathbf{Sh}(X_G, \mathbf{Rng})$ (see Definition 2.3.5).

Proof. [Ber93, Remark on p. 25] □

Example 3.3.10. For any K-affinoid algebra $\mathscr{A}$ we have the equality of rings $\hom_{K-\mathbf{An}}(\mathscr{M}(\mathscr{A}), \mathbb{A}_K^{1,\mathrm{an}}) = \mathscr{A}$.

Proof. The morphism $\mathscr{M}(\mathscr{A}) \to \mathscr{M}(K[T_1])$ of K-affinoid spaces is equivalent to a morphism $K[T_1] \to \mathscr{A}$ of K-affinoid algebras, which is uniquely determined by the image of T_1. □

Definition 3.3.11. Let X be a K-analytic space. We call $\mathcal{O}_{X_G}$ the *structure sheaf of* X_G. Furthermore we define $\mathcal{O}_X := (\pi_G)_* \mathcal{O}_{X_G}$ to be the *structure sheaf of* X.

Example 3.3.12. If X is a K-affinoid space, the structure sheaf $\mathcal{O}_X$ of X coincides with the sheaf of rings $\mathcal{O}_X$ given by Definition 3.2.10.

Proof. [Ber93, Beginning of p. 26] □

Definition 3.3.13 ([Ber93, §1.3])**.** Let X be a K-analytic space. The category of sheaves of $\mathcal{O}_{X_G}$- respectively $\mathcal{O}_X$-modules is denoted by $\mathbf{Mod}(X_G)$ respectively $\mathbf{Mod}(X)$. Furthermore, the objects of these categories together with the tensor product yield monoids with $\otimes$ as binary operation. A *line bundle* on X is an invertible element of $(\mathbf{Mod}(X_G), \otimes)$, which means a locally free sheaf of $\mathcal{O}_{X_G}$-modules of rank 1.

Definition 3.3.14. Let X be a K-analytic space, L be a line bundle on X, and $x \in X$. The fibre of L in x is defined to be the one-dimensional $\mathscr{H}(x)$-vector space

$$L(x) := L_x \otimes_{\mathcal{O}_{X_G,x}} \mathscr{H}(x),$$

where $\mathcal{O}_{X_G,x}$ respectively L_x is the stalk of $\mathcal{O}_{X_G}$ respectively L in x.

Remark 3.3.15. In [Ber93, §2.6] (see also [Ber90, Theorem 3.4.1]) it is described that there exists a functor that associates to any scheme X locally of finite type over K a good K-analytic space denoted by X^{an}. On top of that we get an epimorphism of locally ringed spaces $\pi\colon X^{\mathrm{an}} \to X$. It associates to any sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ a sheaf of $\mathcal{O}_{X_G^{\mathrm{an}}}$-modules $\mathscr{F}^{\mathrm{an}}$. In particular we get for any line bundle L on X a line bundle L^{an} on X^{an}. If X is given by $\mathrm{Spec}(\mathscr{A})$ for a K-affinoid algebra $\mathscr{A}$, then X^{an} is given by the corresponding K-affinoid space $\mathscr{M}(\mathscr{A})$ and the underlying epimorphism of topological spaces of the epimorphism of locally ringed spaces $\pi\colon X^{\mathrm{an}} \to X$ is given by the kernel map (see Definition 3.1.9 and [Ber93, Proposition 2.1.1]).

Definition 3.3.16 ([Ber90, §2.3 on p. 34], [Ber93, §1.2 on p. 23], [Duc18, 1.4.5])**.** The *local dimension* $\dim_x(X)$ of a K-analytic space X at $x \in X$ is defined to be the infimum of the Krull dimension of the K-affinoid algebras

$\mathscr{A}_V$, where V runs over all K-affinoid domains of X (see Definition 3.2.5) containing x. The *dimension* $\dim(X)$ of a K-analytic space X is defined to be the supremum of the local dimensions $\dim_x(X)$ over all $x \in X$.

Definition 3.3.17 ([Tem04, §1], [Duc18, A.4.7])**.** The *graded field* of a valued field $(K, |.|)$ is defined to be the graded ring

$$\bigoplus_{r \in \mathbb{R}_{>0}} \{\alpha \in K \mid |\alpha| \leq r\}/\{\alpha \in K \mid |\alpha| < r\}$$

and denoted by K_{grad}. Its homogeneous elements are invertible.

Definition 3.3.18 ([Duc18, 1.4.10, 1.4.6, 1.2.4])**.** An *Abhyankar point* of a K-analytic space X is an element $x \in X$ such that $d_K(x) = \dim_x(X)$ where we denote by $d_K(x)$ the transcendence degree of the extension of the graded fields $\mathscr{H}(x)_{\text{grad}}/K_{\text{grad}}$. The inequality $d_K(x) \leq \dim_x(X)$ holds always true and is called Abhyankar's inequality.

3.4 Formal geometry

In the following section, which is based on [Gub98, §1], we will define two categories and then show the equivalence of two subcategories:

Definition 3.4.1 ([Bos77, Definition 1.4], [Gub98, 1.8])**.** Let $\mathscr{A}$ be a K-affinoid algebra. The *formal topology* on the set $\mathscr{M}(\mathscr{A})$ is defined to be the coarsest topology such that the reduction map $\text{red}\colon \mathscr{M}(\mathscr{A}) \to \text{Spec}(\tilde{\mathscr{A}})$ becomes continuous, where we consider $\text{Spec}(\tilde{\mathscr{A}})$ as a topological space with the Zariski topology. The topological space consisting of the set $\mathscr{M}(\mathscr{A})$ and its formal topology is denoted by $\text{Spf}(\mathscr{A})$.
The open subsets $\text{Spec}(\tilde{\mathscr{A}}_{\tilde{f}})$ with $f \in \mathscr{A}^\circ$ form a basis of the topology on $\text{Spec}(\tilde{\mathscr{A}})$. The preimage of the open subsets $\text{Spec}(\tilde{\mathscr{A}}_{\tilde{f}})$ under the reduction map red are the formal-open subsets $\mathscr{M}(\mathscr{A})(\frac{f}{1}) \subseteq \text{Spf}(\mathscr{A})$, which are a Weierstrass domain on the K-affinoid space $\mathscr{M}(\mathscr{A})$. The corresponding K-affinoid algebras $\mathscr{A}\left\langle \frac{f}{1} \right\rangle$ yield a sheaf $\mathcal{O}_{\text{Spf}(\mathscr{A})} \in \text{ob}(\mathbf{Sh}(\text{Spf}(\mathscr{A}), \mathbf{Alg}_K))$, where we denote the canonical site on the topological space $\text{Spf}(\mathscr{A})$ also by $\text{Spf}(\mathscr{A})$. The pair $\left(\text{Spf}(\mathscr{A}), \mathcal{O}_{\text{Spf}(\mathscr{A})}\right)$ is called *formal K-affinoid variety.*
A *formal K-analytic variety* $\mathfrak{X}$ is a triple $(\mathfrak{X}, (\mathfrak{U}_i)_{i \in I}, \mathcal{O}_{\mathfrak{X}})$ consisting of a topological space $\mathfrak{X}$, a locally finite collection of open subsets $(\mathfrak{U}_i)_{i \in I}$

that cover $\mathfrak{X}$ together with a sheaf $\mathcal{O}_{\mathfrak{X}} \in \mathrm{ob}(\mathbf{Sh}(\mathfrak{X}, \mathbf{Alg}_K))$, where we denote the canonical site on the topological space $\mathfrak{X}$ also by $\mathfrak{X}$, such that for any element $i \in I$ the pair $(\mathfrak{U}_i, \mathcal{O}_{\mathfrak{X}}|_{\mathfrak{U}_i})$ is a formal K-affinoid variety $\big(\mathrm{Spf}(\mathscr{A}_i), \mathcal{O}_{\mathrm{Spf}(\mathscr{A}_i)}\big)$. A morphism of formal K-analytic varieties is a pair $(\varphi, \varphi^{\#})$ consisting of a morphism of topological spaces $\varphi\colon \mathfrak{X} \to \mathfrak{Y}$ and a morphism of sheaves $\varphi^{\#}\colon \mathcal{O}_{\mathfrak{Y}} \to \varphi_*\mathcal{O}_{\mathfrak{X}}$ (in the category $\mathbf{Sh}(\mathfrak{Y}, \mathbf{Alg}_K)$) which is locally given by a morphism of formal K-affinoid varieties. The formal K-analytic varieties form a category.

Definition 3.4.2. Let $\mathfrak{X}$ be a formal K-analytic variety. The *generic fibre* $\mathfrak{X}^{\mathrm{an}}$ of $\mathfrak{X}$ is defined to be the good K-analytic space locally given by $\mathscr{M}(\mathscr{A}_i)$. The *special fibre* $\tilde{\mathfrak{X}}$ of $\mathfrak{X}$ is defined to be the $\tilde{K}$-scheme locally given by $\mathrm{Spec}(\tilde{\mathscr{A}_i})$.

For the second category we need the theory of adic rings, for which we refer to [Bos14, §7.1]. In particular any K°-algebra is an adic ring:

Definition 3.4.3 ([EGAI, 10.1.1], [Bos14, §7.2])**.** Let A be a K°-algebra. A prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$ is called *open* if there exist $m \in \mathbb{N}\backslash\{0\}$ and $\pi \in K^{\circ\circ}\backslash\{0\}$ such that $\pi^m A \subseteq \mathfrak{p}$. We define $\mathrm{Spf}(A) \subseteq \mathrm{Spec}(A)$ to be the closed topological subspace of open prime ideals.

Definition 3.4.4 ([EGAI, Definition 10.1.2], [Bos14, §7.2])**.** A *formal affine K°-scheme* is a pair $\big(\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)}\big)$ for a K°-algebra A, consisting of the topological space $\mathrm{Spf}(A)$ and the sheaf

$$\mathcal{O}_{\mathrm{Spf}(A)} := \varprojlim_{m \in \mathbb{N}} \mathcal{O}_{\mathrm{Spec}(A/\pi^m A)} \in \mathrm{ob}(\mathbf{Sh}(\mathrm{Spf}(A), \mathbf{TopAlg}_{K^\circ}))$$

for an arbitrary choice $\pi \in K^{\circ\circ}\backslash\{0\}$.

Lemma 3.4.5. *A formal affine K°-scheme is well-defined and independent of the choice of π.*

Proof. At first, by [Bos14, §7.1, Remark 7], the valuation ring K° is an adic ring with ideal of definition $(\pi) \subseteq K^\circ$ (see [Bos14, §7.1] for the definitions) for any $\pi \in K^{\circ\circ}\backslash\{0\}$. Secondly, the (π)-adic topology transfers from K° to A and it does not depend on the choice of π according to [Bos14, §7.2]. □

Definition 3.4.6 ([EGAI, §10])**.** A *formal K°-scheme* $\mathscr{X}$ is a triple $(\mathscr{X}, (\mathscr{U}_i)_{i\in I}, \mathcal{O}_{\mathscr{X}})$ consisting of a topological space $\mathscr{X}$, a locally finite collection of open subsets $(\mathscr{U}_i)_{i\in I}$ that cover $\mathscr{X}$, and a sheaf $\mathcal{O}_{\mathscr{X}} \in \mathrm{ob}(\mathbf{Sh}(\mathscr{X}, \mathbf{TopAlg}_{K^\circ}))$, where we denote the canonical site on the topological space $\mathscr{X}$ also by $\mathscr{X}$, such that for any element $i \in I$ the pair $(\mathscr{U}_i, \mathcal{O}_{\mathscr{X}}|_{\mathscr{U}_i})$ is a formal affine K°-scheme $\big(\mathrm{Spf}(A_i), \mathcal{O}_{\mathrm{Spf}(A_i)}\big)$. The formal K°-schemes form a category.

Definition 3.4.7 ([Ber99, §1], [Gub07a, 2.8])**.** A morphism of formal K°-schemes $\varphi\colon \mathscr{X} \to \mathscr{Y}$ is called *étale* if for each ideal of definition $\mathscr{J} \subseteq \mathcal{O}_{\mathscr{Y}}$, the induced morphism of schemes $(\tilde{\mathscr{X}}, \mathcal{O}_{\mathscr{X}}/\mathscr{J}\,\mathcal{O}_{\mathscr{X}}) \to (\tilde{\mathscr{Y}}, \mathcal{O}_{\mathscr{Y}}/\mathscr{J})$ is étale.

Definition 3.4.8 ([Gub98, 1.10])**.** A K°-algebra A is called *admissible* if it has no K°-torsion and if there exists an epimorphism of K°-algebras $\zeta_A\colon K^\circ\{T\} \twoheadrightarrow A$, where the set

$$K^\circ\{T\} := \Big\{ \sum_{i\in\mathbb{N}^n} a_i T^i \in K^\circ[[T]] \;\Big|\; |a_i| \to 0 \text{ as } ||i|| \to +\infty \Big\}$$

for $||i|| := i_1 + ... + i_n$ defines a K°-algebra.

Remark 3.4.9. By [BL93, Proposition 1.1], the kernel $\ker(\zeta_A)$ is a finitely generated ideal in the ring $K^\circ\{T\}$. Therefore, the admissible K°-algebra in Definition 3.4.8 is the same as the admissible K°-algebra defined in [BL93, §1] and [Bos14, §7.3], because we can take the finitely generated ideal $\ker(\zeta_A) \subseteq K^\circ\{T\}$ to be $\mathfrak{a}$ and because the (π)-adic topology transfers from K° to A.

Definition 3.4.10 ([Gub98, 1.10])**.** We call a formal affine K°-scheme $\big(\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)}\big)$ *admissible* if the K°-algebra A is admissible. A formal K°-scheme $\mathscr{X}$ is called *admissible* if the formal affine K°-schemes in Definition 3.4.6 are admissible. The admissible formal K°-schemes form a subcategory of the category of formal K°-schemes.

Definition 3.4.11. Let $\mathscr{X}$ be a formal K°-scheme. The *special fibre* $\tilde{\mathscr{X}}$ of $\mathscr{X}$ is defined to be the $\tilde{K}$-scheme locally given by $\mathrm{Spec}(\tilde{A}_i)$ where $\tilde{A}_i$ is the $\tilde{K}$-algebra given by the quotient $\tilde{K}$-algebra $A_i/K^{\circ\circ}A_i$.

Definition 3.4.12. Let $\mathscr{X}$ be an admissible formal K°-scheme with reduced special fibre. Then the *reduction map* of $\mathscr{X}$ is defined to be the epimorphism of sets $\mathrm{red}\colon \mathscr{X}^{\mathrm{an}} \to \tilde{\mathscr{X}}$ locally given by the reduction map $\mathrm{red}\colon \mathscr{M}(\mathscr{A}) \to \mathrm{Spec}(\tilde{\mathscr{A}})$ (see Definition 3.2.14).

Lemma 3.4.13. *Let $\mathscr{X}$ be an admissible formal K°-scheme with reduced special fibre. Then for any irreducible component Y of $\tilde{\mathscr{X}}$, there exists a unique element $\xi_Y \in \mathscr{X}^{\mathrm{an}}$ (called Shilov point) with $\mathrm{red}(\xi_Y) = \eta_Y$, where $\eta_Y \in \tilde{\mathscr{X}}$ is the generic point of Y.*

Proof. Follows locally by [Ber90, Proposition 2.4.4(ii)] □

Now we will compare these two categories:

Definition 3.4.14. Let $\mathfrak{X}$ be a reduced formal $\mathbb{K}$-analytic variety. The admissible formal $\mathbb{K}^\circ$-scheme $\mathfrak{X}^{\text{f-sch}}$ with reduced special fibre is locally given by $\mathrm{Spf}(A_i)$ with the admissible $\mathbb{K}^\circ$-algebra $A_i := \mathscr{A}_i^\circ$.

Remark 3.4.15. If the field $\mathbb{K}$ was not algebraically closed, the $\mathbb{K}^\circ$-algebra $\mathscr{A}_i^\circ$ would not be necessarily admissible. See the last paragraph of [BGR84, Chapter 6] for a counterexample.

Definition 3.4.16. Let $\mathscr{X}$ be an admissible formal K°-scheme. The formal K-analytic variety $\mathscr{X}^{\text{f-an}}$ is locally given by $\mathrm{Spf}(\mathscr{A}_i)$ with the K-affinoid algebra $\mathscr{A}_i := A_i \otimes_{K^\circ} K$.

Proposition 3.4.17. *The functor from the subcategory of reduced formal $\mathbb{K}$-analytic varieties of the category of formal $\mathbb{K}$-analytic varieties to the subcategory of admissible formal $\mathbb{K}^\circ$-schemes with reduced special fibre of the category of admissible formal $\mathbb{K}^\circ$-schemes given by*

$$\begin{aligned} \mathfrak{X} &\mapsto \mathfrak{X}^{\text{f-sch}} \\ \mathscr{X}^{\text{f-an}} &\mapsfrom \mathscr{X} \end{aligned}$$

induces an equivalence of categories.

Proof. [BL86, Lemma 1.1] in the case of a discrete complete valuation, [Gub98, Proposition 1.11] in general □

Definition 3.4.18. Let $\mathscr{X}$ be an admissible formal K°-scheme with reduced special fibre. The *generic fibre* $\mathscr{X}^{\mathrm{an}}$ of $\mathscr{X}$ is defined to be the K-analytic space $\left(\mathscr{X}^{\text{f-an}}\right)^{\mathrm{an}}$ (the generic fibre of $\mathscr{X}^{\text{f-an}}$).

Definition 3.4.19 ([Gub07a, 3.1])**.** Let X be a proper K-scheme. A *formal K°-model* of X is an admissible formal K°-scheme $\mathscr{X}$ such that the generic fibre $\mathscr{X}^{\mathrm{an}}$ of $\mathscr{X}$ is isomorphic to X^{an} in the category of K-analytic spaces.

Definition 3.4.20. Let X, Y be good K-analytic spaces, $\varphi\colon Y \hookrightarrow X$ be a closed immersion, and $\mathscr{X}$ be a formal K°-model of X. Then the *closure* of Y in $\mathscr{X}$ is a formal K°-model $\mathscr{Y}$ of Y together with a closed immersion of admissible formal K°-schemes $\mathscr{Y} \hookrightarrow \mathscr{X}$.

Lemma 3.4.21. *The closure of Y in $\mathscr{X}$ exists.*

Proof. [Gub98, Proposition 3.3] □

Definition 3.4.22. A *line bundle* on a formal K°-scheme $\mathscr{X}$ is an invertible sheaf of $\mathcal{O}_{\mathscr{X}}$-modules.

Definition 3.4.23 ([Gub98, Definition 7.2])**.** Let X be a proper K-scheme, L be a line bundle on X, and $\mathscr{X}$ be a formal K°-model of X. A *formal K°-model of L determined on $\mathscr{X}$* is a line bundle $\mathscr{L}$ on $\mathscr{X}$ such that its generic fibre $\mathscr{L}^{\mathrm{an}}$ is isomorphic to L^{an} in the category of line bundles on X^{an}.

Definition 3.4.24. Let X be a proper K-scheme, L be a line bundle on X, let $\mathscr{X}$ be a formal K°-model of X, and $\mathscr{L}$ be a formal K°-model of L on $\mathscr{X}$. A *formal trivialization* of $\mathscr{L}$ is a pair $(\mathscr{U}, \varphi)$ consisting of an open subset $\mathscr{U} \subseteq \mathscr{X}$ and an isomorphism $\varphi\colon \mathscr{L}|_{\mathscr{U}} \to \mathcal{O}_{\mathscr{X}}|_{\mathscr{U}}$ of invertible sheaves of $\mathcal{O}_{\mathscr{X}}|_{\mathscr{U}}$-modules.

Definition 3.4.25. Let $\mathscr{X}$ be a formal K°-scheme. Then a line bundle $\mathscr{L}$ on $\mathscr{X}$ is called *very ample* if there is a closed embedding $i\colon \mathscr{X} \to \mathbb{P}^n_{K^\circ}$ such that $\mathscr{L}$ is isomorphic to $i^*\mathcal{O}_{\mathbb{P}^n_{K^\circ}}(1)$ as line bundles on $\mathscr{X}$, where $i\colon \mathscr{X} \to \mathbb{P}^n_{K^\circ}$ being a closed embedding means that it is a morphism of formal K°-schemes such that it induces an isomorphism onto a closed formal K°-subscheme of $\mathscr{X}$. A line bundle $\mathscr{L}$ is called *ample* if there exists an $m \in \mathbb{N}$ such that $\mathscr{L}^{\otimes m}$ is very ample.

Lemma 3.4.26. *If a line bundle on a formal K°-scheme is ample, then its generic and special fibre are also ample.*

Proof. The definitions of being ample are analogous in the different categories. □

4 Measures

The goal of this chapter is to introduce the non-archimedean and real Monge-Ampère measure, which will be compared in Theorem C. Since the first one depends on the choice of a metric, we will start by introducing metrics in detail.

Let $(\mathbb{K}, |.|)$ be an algebraically closed non-trivially valued non-archimedean field.

4.1 Metrics

The following section is based on [Gub98, §7] and defines metrics which will be used afterwards to define the non-archimedean Monge-Ampère measure. Let X be a proper $\mathbb{K}$-scheme, and L be a line bundle on X.

Definition 4.1.1 ([Gub98, 7.1], [Cha06, 2.1])**.** A *metric* $||.||$ on L is a family $(||.||_x)_{x\in X^{\mathrm{an}}}$ of norms on the fibres $L^{\mathrm{an}}(x)$ (see Definition 3.3.14) which means functions

$$||.||_x \colon L^{\mathrm{an}}(x) \to \mathbb{R}_{\geq 0}$$

not identically zero such that $||\lambda v||_x = |\lambda|_x \cdot ||v||_x$ for all $\lambda \in \mathscr{H}(x)$ (see Definition 3.1.8) and $v \in L^{\mathrm{an}}(x)$.
A metric $||.||$ on L is called *continuous* if for any $\mathbb{K}$-analytic domain Y on X^{an} and for any $s \in L^{\mathrm{an}}(Y)$ the function $Y \to \mathbb{R}_{\geq 0}, x \mapsto ||s(x)||_x$ is continuous. We denote the set of continuous metrics on L by $\mathcal{C}(L)$. As usual, we define tensor metrics and pull-back metrics, which are again continuous if the considered metrics are continuous.

Example 4.1.2. Let g be a continuous function on the topological space X^{an}. Then for any $x \in X^{\mathrm{an}}$ the definition $||1||_{g,x} := e^{-g(x)}$ gives a continuous metric $||.||_g$ on $\mathcal{O}_X$.

Proof. [Cha06, 2.1] □

Definition 4.1.3 ([Gub98, Definition 7.11], [Gub07a, 3.2], [Gub10, 3.1])**.** The *distance of uniform convergence* of two continuous metrics $||.||$ and $||.||'$ on L is defined to be the real number

$$d\big(||.||, ||.||'\big) := \sup_{x \in X^{\mathrm{an}}} \left| \log \frac{||v||_x}{||v||'_x} \right|_{\mathbb{R}}$$

for some $v \in L^{\mathrm{an}}(x) \backslash \{0\}$. We say that a sequence $(||.||_k)_{k \in \mathbb{N}}$ of continuous metrics on L *converges uniformly* to a continuous metric $||.||$ on L if the sequence $(d(||.||_k, ||.||))_{k \in \mathbb{N}}$ in $\mathbb{R}$ converges to zero.

Lemma 4.1.4. *The distance of uniform convergence is independent of the choice of v. Moreover, $(\mathcal{C}(L), d)$ forms a metric space.*

Proof. Another choice of v would just yield a factor $|\lambda|_x$ in the nominator and denominator of $\frac{||v||_x}{||v||'_x}$. The axioms for d being a metric follow mainly by the axioms of the absolute value $|.|_{\mathbb{R}}$. □

Definition 4.1.5 ([Gub98, 7.3], [Gub07a, 3.2])**.** Let $\mathscr{X}$ be a formal $\mathbb{K}^\circ$-model of X, and $\mathscr{L}$ be a formal $\mathbb{K}^\circ$-model of L on $\mathscr{X}$. Then the *formal metric* associated to $\mathscr{L}$ is defined to be the metric $||.||_{\mathscr{L}}$ on L with the following property: For any formal trivialization $(\mathscr{U}, \varphi)$ of $\mathscr{L}$ (see Definition 3.4.24), we get for all $x \in \mathscr{U}^{\mathrm{an}}$ and $s \in L^{\mathrm{an}}(\mathscr{U}^{\mathrm{an}})$ the equality $||s(x)||_{\mathscr{L},x} = |\gamma(x)|_x$, where it is defined $\gamma := \varphi(s) \in \mathcal{O}_{X^{\mathrm{an}}_G}(\mathscr{U}^{\mathrm{an}})$ and the absolute value $|.|_x$ is taken from the image of $\gamma(x)$ under the canonical morphism of rings $\mathcal{O}_{X^{\mathrm{an}}_G, x} \to \mathscr{H}(x)$.

Lemma 4.1.6. *Let $\mathscr{X}$ be a formal $\mathbb{K}^\circ$-model of X, and $\mathscr{L}$ be a formal $\mathbb{K}^\circ$-model of L on $\mathscr{X}$. The formal metric associated to $\mathscr{L}$ exists and is unique. Furthermore it is continuous.*

Proof. [Gub98, Lemma 7.4]. Continuity follows directly by Definition 4.1.5. □

Definition 4.1.7 ([Gub07a, 3.2])**.** A *formal metric* is a metric $||.||$ on L such that there exists a formal $\mathbb{K}^\circ$-model $\mathscr{X}$ of X, and a formal $\mathbb{K}^\circ$-model $\mathscr{L}$ of L on $\mathscr{X}$ such that $||.||$ is equal to the associated formal metric $||.||_{\mathscr{L}}$ on L.

A *model metric* on L (also called *root of a formal metric* in [Gub07a, 3.2]) is a metric $||.||$ on L such that there exists a formal $\mathbb{K}^\circ$-model $\mathscr{X}$ of X, an $N \in \mathbb{N}\backslash\{0\}$, and a formal $\mathbb{K}^\circ$-model $\mathscr{L}_N$ of $L^{\otimes N}$ on $\mathscr{X}$ such that $||.||^{\otimes N} = ||.||_{\mathscr{L}_N}$.

Definition 4.1.8 ([Kle66, Ch. I, §4], [Laz04, Definition 1.4.1])**.** Let K be a field. A line bundle L on a proper K-scheme X is called *nef* if for all closed integral K-curves $C \hookrightarrow X$ we have $\deg(L|_C) \geq 0$.

Lemma 4.1.9. *Let K be a field. Every ample line bundle L on a proper K-scheme X is nef.*

Proof. Paragraph after [Laz04, Definition 1.4.1] □

Definition 4.1.10 ([GM19, 3.1])**.** A model metric $||.||$ is called *semi-positive* if the formal $\mathbb{K}^\circ$-models in Definition 4.1.7 can be chosen such that the line bundle $\tilde{\mathscr{L}}_N$ on $\tilde{\mathscr{X}}$ is nef (see Definition 4.1.8).

Lemma 4.1.11. *Any model metric on L is continuous, and the subset of model metrics on L is dense in the metric space $(\mathcal{C}(L), d)$.*

Proof. The first statement follows by Lemma 4.1.6 and the fact that root functions are continuous. For the second statement see [Gub07a, Proposition 3.3]. □

Definition 4.1.12 ([Zha95, (1.3)], [Cha06, 2.2])**.** A continuous metric on L is called *semi-positive* if it is the limit of a sequence of semi-positive model metrics in the metric space $(\mathcal{C}(L), d)$.

Remark 4.1.13. According to [GM19, §1], originally Zhang [Zha95] introduced semi-positive adelic metrics, which where generalized by Gubler [Gub98]. It turned out that these metrics are continuous, which justifies our above definition. The more general Definition 4.1.12 was taken from [Cha06, 2.2].

Lemma 4.1.14. *A model metric is semi-positive in the sense of Definition 4.1.10 if and only if it is semi-positive in the sense of Definition 4.1.12.*

Proof. [GM19, Corollary 1.4] □

Lemma 4.1.15. *Let A be an abelian $\mathbb{K}$-variety, (L, ρ) be a rigidified line bundle on A. Then there exists a unique continuous metric $||.||_\rho$ on L such that for any odd rigidified line bundle (L_-, ρ_-) and even rigidified line bundle (L_+, ρ_+) on A with $(L, \rho) = (L_-, \rho_-) \otimes (L_+, \rho_+)$ as rigidified line bundles on A, we have:*

(i) There exists a unique continuous metric $||.||_{\rho_-}$ on L_- such that for all $m \in \mathbb{Z}$, we have $[m]^*||.||_{\rho_-} = ||.||_{\rho_-}^{\otimes m}$. In particular, for any continuous metric $||.||$ on L_- we have

$$||.||_{\rho_-} = \lim_{m \to +\infty} [m]^* ||.||^{\otimes 1/m}$$

in the metric space $(\mathcal{C}(L_-), d)$.

(ii) There exists a unique continuous metric $||.||_{\rho_+}$ on L_+ such that for all $m \in \mathbb{Z}$, we have $[m]^*||.||_{\rho_+} = ||.||_{\rho_+}^{\otimes m^2}$. In particular, for any continuous metric $||.||$ on L_+ we have

$$||.||_{\rho_+} = \lim_{m \to +\infty} [m]^* ||.||^{\otimes 1/m^2}$$

in the metric space $(\mathcal{C}(L_+), d)$.

(iii) We have the equality $||.||_\rho = ||.||_{\rho_-} \otimes ||.||_{\rho_+}$ of metrics on L.

Proof. [BG07, Theorem 9.5.7] where we have used that any continuous metric is locally bounded. □

Definition 4.1.16. Let A be an abelian $\mathbb{K}$-variety, and L be a line bundle on A. A metric $||.||$ on L is called *canonical* if there exists a rigidification ρ on L such that the metrics $||.||$ and $||.||_\rho$ coincide. We will denote the canonical metric on L with respect to a chosen rigidification ρ by $||.||_{L,\mathrm{can}}$.

Lemma 4.1.17. *Let A be an abelian $\mathbb{K}$-variety, L be a line bundle on A, and ρ_1, ρ_2 be two rigidifications on L. Then there exists an element $a \in \mathbb{K}\backslash\{0\}$ such that we get the equality $||.||_{\rho_1} = |a| \cdot ||.||_{\rho_2}$ of metrics on L.*

Proof. [BG07, Remark 9.5.9] □

4.2 The non-archimedean Monge-Ampère measure

Chambert-Loir constructed in [Cha06] some measure (see Definition 4.2.3) when searching for a non-archimedean analogue of Zhang's equidistribution theorem [Zha98, Theorem 1.1]. Even though in [Gub07a; Gub10], where in the first paper this measure was further developed, it is called Chambert-Loir's measure, in the following we will call it the non-archimedean Monge-Ampère measure due to [CD12, §6.9] and [GK17, §10].
Let X be a proper $\mathbb{K}$-scheme, and define $d := \dim(X) \in \mathbb{N}$. We refer to [Tao10, §1] for basics on measures.

Definition 4.2.1 ([Vil20, Definition 4.9])**.** Let Y be a Hausdorff topological space. A *positive Radon measure* on Y is a measure μ on the σ-algebra $\mathcal{B}(Y)$ of Borel sets of Y such that

(i) (locally finite) for every $y \in Y$ there exists a neighbourhood U such that $\mu(U) \in \mathbb{R}_{\geq 0}$,

(ii) (inner regular) for every open subset U of Y, the element $\mu(U)$ is the supremum of $\mu(C)$ on $\mathbb{R}_{\geq 0} \cup \{+\infty\}$ where C runs over all compact subsets of U, and

(iii) (outer regular) for every Borel set B of XY, the element $\mu(B)$ is the infimum of $\mu(V)$ on $\mathbb{R}_{\geq 0} \cup \{+\infty\}$ where V runs over all open subsets of Y containing B.

Example 4.2.2. For any $r \in \mathbb{N}$ the Lebesgue measure on $\mathbb{R}^r$ is a positive Radon measure.

Definition 4.2.3. For every $j \in \{1, ..., d\}$ let L_j be a line bundle on X, and $||.||_j$ be a semi-positive continuous metric on L_j. These choices determine a positive Radon measure $c_1(L_1, ||.||_1) \wedge ... \wedge c_1(L_d, ||.||_d)$ on X^{an}. See [Gub07a, 3.8] for the precise definition. It is called the *non-archimedean Monge-Ampère measure.*

Lemma 4.2.4. *The non-archimedean Monge-Ampère measure is multilinear and symmetric in* $(L_1, ||.||_1), ..., (L_d, ||.||_d)$.

Proof. [Gub07a, Corollary 3.9(a)] □

Lemma 4.2.5. *Let $(L_j)_{j\in\{1,...,d\}}$ be line bundles on X, let $\mathscr{X}$ be a formal $\mathbb{K}^\circ$-model of X, and let $\mathscr{L}_j$ be a formal $\mathbb{K}^\circ$-model of L_j on $\mathscr{X}$ for every $j \in \{1, ..., d\}$. We then get the equality*

$$c_1(L_1, ||.||_{\mathscr{L}_1}) \wedge ... \wedge c_1(L_d, ||.||_{\mathscr{L}_d}) = \sum_Y \deg_{\tilde{\mathscr{L}}_1,...,\tilde{\mathscr{L}}_d}(Y)\delta_{\xi_Y}$$

of measures on X^{an}, where Y runs over the irreducible components of $\tilde{\mathscr{X}}$ and δ_{ξ_Y} is the Dirac measure in $\xi_Y \in X^{\mathrm{an}}$, which is the unique point mapped to the generic point η_Y by the reduction (see Lemma 3.4.13).

Proof. [Gub07a, Proposition 3.11] or [Gub10, Proposition 3.8(d)] □

Definition 4.2.6. Let S be a topological space. A sequence of Borel measures $(\mu_n)_{n\in\mathbb{N}}$ *converges weakly* to a Borel measure μ on S if for all bounded continuous functions $f\colon S \to \mathbb{R}$ we have the equality

$$\lim_{n\to+\infty} \int_S f \,\mathrm{d}\mu_n = \int_S f \,\mathrm{d}\mu.$$

Lemma 4.2.7. *Let $(L_j)_{j\in\{1,...,d\}}$ be line bundles on X, and $(||.||_j)_{j\in\{1,...,d\}}$ be semi-positive continuous metrics on L_j. If for any $j \in \{1, ..., d\}$ a sequence $(||.||_{j,k})_{k\in\mathbb{N}}$ of metrics on L_j converges uniformly to $||.||_j$, then the sequence $(c_1(L_1, ||.||_{1,k}) \wedge ... \wedge c_1(L_d, ||.||_{d,k}))_{k\in\mathbb{N}}$ of non-archimedean Monge-Ampère measures converges weakly to the non-archimedean Monge-Ampère measure $c_1(L_1, ||.||_1) \wedge ... \wedge c_1(L_d, ||.||_d)$.*

Proof. [Gub07a, Proposition 3.12] or [Gub10, Proposition 3.8(f)] □

Definition 4.2.8 ([Gub10, 3.9])**.** Let A be an abelian K-variety, let X be a geometrically integral proper K-subvariety of A with corresponding closed immersion $i\colon X \hookrightarrow A$, define $d := \dim(X) \in \mathbb{N}$, and let $((L_j, \rho_j))_{j\in\{1,...,d\}}$ be rigidified line bundles on A. Then the *canonical measure* on X^{an} is defined to be the non-archimedean Monge-Ampère measure

$$\mu_{L,\mathrm{can}} := c_1(i^*L_1, i^*||.||_{\rho_1}) \wedge ... \wedge c_1(i^*L_d, i^*||.||_{\rho_d}).$$

4.3 The real Monge-Ampère measure

In the following we will treat the finite-dimensional version of the real Monge-Ampère measure, as defined by Rauch and Taylor [RT77]. It was motivated by the Monge-Ampère equations from differential geometry, first studied by Monge (1784) and later by Ampère (1819). Let $r \in \mathbb{N}$.

Definition 4.3.1 ([RT77, Definition 2.2 and 2.6])**.** Let $g\colon \mathbb{R}^r \to \mathbb{R}$ be a convex function, and let $u_0 \in \mathbb{R}^r$. We then define the *gradient image* of u_0 under g as the subset

$$\nabla g(u_0) := \{m \in (\mathbb{R}^r)^* \mid \forall u \in \mathbb{R}^r\colon \langle m, u - u_0\rangle \leq g(u) - g(u_0)\}$$

of the dual r-dimensional $\mathbb{R}$-vector space $(\mathbb{R}^r)^* := \hom_{\mathbb{R}-\mathbf{Vec}}(\mathbb{R}^r, \mathbb{R})$, where $\langle .,. \rangle$ is the dual pairing of $\mathbb{R}^r$ and $(\mathbb{R}^r)^*$. Furthermore, for any $U \subseteq \mathbb{R}^r$ we define the subset

$$\nabla g(U) := \bigcup_{u_0 \in U} \nabla g(u_0)$$

of $(\mathbb{R}^r)^*$. Finally we define the *real Monge-Ampère measure* associated to g as the Borel measure

$$\begin{aligned} \mathrm{MA}(g)\colon \mathcal{B}((\mathbb{R}^r)^*) &\to \mathbb{R}_{\geq 0}, \\ U &\mapsto \mathrm{vol}_r(\nabla g(U)), \end{aligned}$$

where vol_r is the Lebesgue measure on $(\mathbb{R}^r)^*$. Using the canonical isomorphism of $\mathbb{R}$-vector spaces $(\mathbb{R}^r)^* = \mathbb{R}^r$ the real Monge-Ampère measure $\mathrm{MA}(g)$ is also defined on $\mathbb{R}^r$.

Lemma 4.3.2. *The real Monge-Ampère measure is well-defined.*

Proof. [RT77, Section 2] □

Lemma 4.3.3. *Let $\mathscr{D}$ be a polyhedral decomposition of $\mathbb{R}^r$. For a strongly polyhedral convex function $g\colon \mathbb{R}^r \to \mathbb{R}$ with respect to $\mathscr{D}$, we get the equality*

$$\mathrm{MA}(g) = \sum_{\xi} \mathrm{vol}_r(\nabla g(\xi))\delta_\xi,$$

where ξ runs over all vertices of the polyhedral decomposition $\mathscr{D}$, and δ_ξ is the Dirac measure in ξ.

Proof. The r-dimensional volume of $\nabla g(u_0)$ is zero if $u_0 \in \mathbb{R}^r$ is not a vertex of $\mathscr{D}$. □

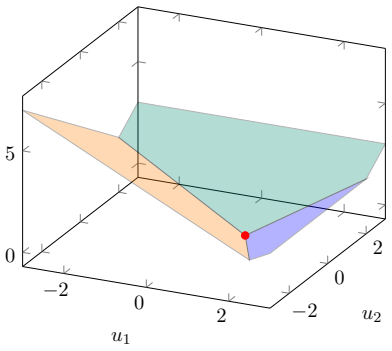
Example 4.3.4. Let $r = 2$, let

$$m_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, m_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, m_3 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

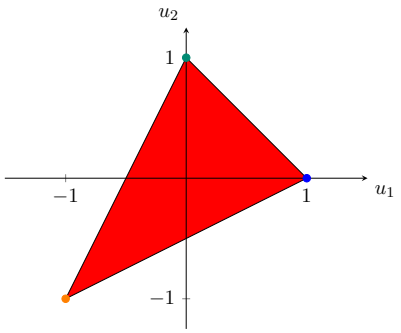
in $(\mathbb{R}^2)^*$, and let $c_1 = 1$, $c_2 = 0$, $c_3 = -1$ in $\mathbb{R}$. For any $i \in \{1, 2, 3\}$ define

$$\begin{aligned} g_i \colon \mathbb{R}^2 &\to \mathbb{R}, \\ u &\mapsto \langle m_i, u \rangle - c_i \end{aligned}$$

and $g := \max_{i \in \{1,2,3\}} g_i$. Finally, the subsets $\Delta_i := \{u \in \mathbb{R}^2 \mid g(u) = g_i(u)\}$ generate a polyhedral decomposition $\mathscr{D}$ of $\mathbb{R}^2$. See Figure 4.1 for a visualization.



(a) The function g on $\mathbb{R}^2$



(b) The gradient image of $(1, 0)$ under g is a polytope in the dual $\mathbb{R}$-vector space $(\mathbb{R}^2)^*$.

Figure 4.1: Every j-dimensional polyhedron becomes a $(2 - j)$-dimensional polyhedron in the dual space if g is strongly polyhedral convex.

Lemma 4.3.5. *If the function $g \colon \mathbb{R}^r \to \mathbb{R}$ is an $\mathbb{R}$-quadratic function (see Definition 2.6.2) and $q \colon \mathbb{R}^r \to \mathbb{R}, u \mapsto u^T \boldsymbol{Q} u$ is the associated $\mathbb{R}$-quadratic form (see Definition 2.6.4), we get the formula*

$$\mathrm{MA}(g) = \det(\boldsymbol{Q}) \cdot \mathrm{vol}_r \,.$$

Proof. If g is given by $g(u) = u^T \boldsymbol{A} u + bu + c$, then by Definition 2.6.4 we have $q(u) = 2u^T \boldsymbol{A} u$ and hence $\nabla^2 g = 2\boldsymbol{A} = \boldsymbol{Q}$. See also the paragraph after [RT77, Definition 2.6] for the more general case of two times continuously differentiable functions. □

Lemma 4.3.6. *Let $(g_k)_{k\in\mathbb{N}}$ be a sequence of convex functions $g_k \colon \mathbb{R}^r \to \mathbb{R}$ converging pointwise to a convex function g, then the sequence of Borel measures $(\mathrm{MA}(g_k))_{k\in\mathbb{N}}$ converges weakly to the Borel measure $\mathrm{MA}(g)$.*

Proof. [Vil20, 4.1] □

5 Construction of a Mumford model

The core of our proofs of the main theorems will be combinatorial. The goal of this chapter is to provide all necessary combinatorial preliminary results. We will define in Section 5.4 what we mean by a toric metric and will afterwards show that every such metric has a unique pendant given by a continuous function on a finite-dimensional $\mathbb{R}$-vector space. As we have seen in Chapter 4 that both Monge-Ampère measures behave well with respect to uniform convergence, we may approximate this function uniformly by polyhedral functions. For these polyhedral functions and their corresponding metrics both Monge-Ampère measures become discrete, which makes it easier to compare them. Note that we will just consider convex functions.

Let $(\mathbb{K}, |.|)$ be an algebraically closed non-trivially valued non-archimedean field, let Γ be its value group, and let A be an abelian $\mathbb{K}$-variety.

5.1 Raynaud's uniformization theory

In the following, we will present the results of Bosch and Lütkebohmert about Raynaud's uniformization theory, which was stated by Raynaud without giving proofs. We will formulate the results using the analytic spaces of Berkovich instead of the rigid spaces of Tate, which is guaranteed by [Ber93, Theorem 1.6.1].

Just until Definition 5.1.11, instead of assuming an abelian $\mathbb{K}$-variety A, we let $n \in \mathbb{N}$, and M be a free abelian group of rank n.

Definition 5.1.1. The category of *(abelian)* $\mathbb{K}$*-algebraic groups* is defined to be the category of (abelian) group objects of the category of $\mathbb{K}$-varieties. The category of *(abelian)* $\mathbb{K}$*-analytic groups* is defined to be the category of (abelian) group objects of the category of $\mathbb{K}$-analytic spaces. The category of *(abelian) formal group* $\mathbb{K}^\circ$*-schemes* is defined to be the category

of (abelian) group objects of the category of admissible formal $\mathbb{K}^\circ$-schemes. The category of *(abelian) formal* $\mathbb{K}$*-analytic groups* is defined to be the category of (abelian) group objects of the category of formal $\mathbb{K}$-analytic varieties. Note that all these categories are well-defined because they admit finite products and have terminal object $\mathrm{Spec}(\mathbb{K})$, $\mathscr{M}(\mathbb{K})$, $\mathrm{Spf}(\mathbb{K}^\circ)$ respectively $\mathrm{Spf}(\mathbb{K})$ (see Definition 2.2.5).

Definition 5.1.2. Define $(\chi^M, \cdot)$ to be a group equal to $(M, +)$ in the category of groups and denote the element in χ^M corresponding to $m \in M$ by χ^m. Furthermore, by using the multiplication on the group χ^M, we define the $\mathbb{K}$-algebra

$$\mathbb{K}[[M]] := \left\{ \sum_{m \in M} a_m \chi^m \;\middle|\; a_m \in \mathbb{K} \right\}.$$

The *support* of a formal sum $f \in \mathbb{K}[[M]]$ is defined to be the subset

$$\mathrm{Supp}(f) := \{m \in M \mid a_m \neq 0\}$$

of M. We define $\mathbb{K}[M]$ to be the $\mathbb{K}$-subalgebra of $\mathbb{K}[[M]]$ consisting of all elements with finite support.

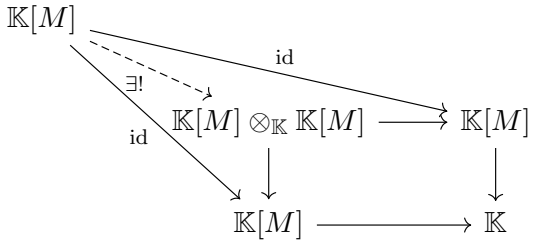
Definition 5.1.3. A $\mathbb{K}$*-torus* of rank n is the $\mathbb{K}$-variety

$$\mathbb{T} := \mathrm{Spec}(\mathbb{K}[M]).$$

Remark 5.1.4. In the literature one considers the element $\chi^m \in \chi^M$ as a character, which means a morphism of $\mathbb{K}$-varieties $\chi^m \colon \mathbb{T} \to \mathrm{Spec}(\mathbb{K}[T_1^{\pm 1}])$ given by the morphism of rings $\mathbb{K}[T_1^{\pm 1}] \to \mathbb{K}[M], T_1 \mapsto \chi^m$.

Lemma 5.1.5. *$\mathbb{T}$ is an abelian $\mathbb{K}$-algebraic group and $\mathbb{T}^{\mathrm{an}}$ is an abelian $\mathbb{K}$-analytic group.*

Proof. The morphism $m \colon \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ is given by the unique morphism of $\mathbb{K}$-algebras



given by the pull-back in the category of $\mathbb{K}$-algebras. The morphisms $e\colon 0 \to \mathbb{T}$ and $i\colon \mathbb{T} \to \mathbb{T}$ are clear and the analogous holds for the $\mathbb{K}$-analytic space $\mathbb{T}^{\mathrm{an}}$. The group objects are abelian because we use the identity morphism on the $\mathbb{K}$-algebra $\mathbb{K}[M]$ for the above pull-back both times. □

Definition 5.1.6 ([Gub13, §3.1])**.** The *canonical tropicalization map* is defined to be the function

$$\begin{aligned} \mathrm{trop}\colon \mathbb{T}^{\mathrm{an}} &\to N_{\mathbb{R}}, \\ ||.|| &\mapsto (m \mapsto -\log ||\chi^m||) \end{aligned}$$

where we use the fact that the $\mathbb{K}$-analytic group $\mathbb{T}^{\mathrm{an}}$ consists as a set of the multiplicative seminorms on $\mathbb{K}[M]$ extending the absolute value $|.|$ on $\mathbb{K}$.

Lemma 5.1.7. *The canonical tropicalization map* $\mathrm{trop}\colon \mathbb{T}^{\mathrm{an}} \to N_{\mathbb{R}}$ *is an epimorphism of topological spaces.*

Proof. [Gub13, §3.1] □

Lemma 5.1.8. *Let* Δ *be a* Γ*-rational polytope in* $N_{\mathbb{R}}$*. Then the set*

$$U_\Delta := \mathrm{trop}^{-1}(\Delta)$$

forms a Weierstrass domain in the $\mathbb{K}$*-analytic group* $\mathbb{T}^{\mathrm{an}}$*. Denote the corresponding* $\mathbb{K}$*-affinoid algebra by* $\mathbb{K}\langle U_\Delta \rangle$*.*

Proof. [Gub07a, Proposition 4.1] □

Definition 5.1.9. We define the *formal torus* to be the formal group $\mathbb{K}^\circ$-scheme

$$\mathbb{T}_0 := \mathrm{Spf}\big(\mathbb{K}\langle U_{\{0\}} \rangle^\circ\big)\,.$$

Lemma 5.1.10. *The monomorphism of topological spaces* $\{0\} \hookrightarrow N_{\mathbb{R}}$ *induces a monomorphism of* $\mathbb{K}$*-analytic groups* $t\colon \mathbb{T}_0^{\mathrm{an}} \hookrightarrow \mathbb{T}^{\mathrm{an}}$*.*

Proof. Lemma 5.1.8 for $\Delta = \{0\}$ □

Definition 5.1.11. A *semi-abelian* $\mathbb{K}$*-variety* is a $\mathbb{K}$-algebraic group G such that there exists an abelian $\mathbb{K}$-variety A and a short exact sequence

$$0 \to \mathbb{T} \to G \to A \to 0$$

in the category of $\mathbb{K}$-algebraic groups.

From now on let A be an abelian $\mathbb{K}$-variety, as explained in the beginning of this chapter.

Proposition 5.1.12 (Bosch-Lütkebohmert)**.** *There exists a unique formal group $\mathbb{K}^\circ$-scheme $\mathscr{A}_0$ (with reduced special fibre) and a unique monomorphism of $\mathbb{K}$-analytic groups $a\colon \mathscr{A}_0^{\mathrm{an}} \hookrightarrow A^{\mathrm{an}}$ inducing an isomorphism of topological sets onto a compact $\mathbb{K}$-analytic domain on A^{an} such that $\tilde{\mathscr{A}_0}$ is a semi-abelian $\tilde{\mathbb{K}}$-variety, i.e. there exists an abelian $\tilde{\mathbb{K}}$-variety $B_{\tilde{\mathbb{K}}}$ and a short exact sequence*

$$0 \to \tilde{\mathbb{T}}_0 \to \tilde{\mathscr{A}_0} \to B_{\tilde{\mathbb{K}}} \to 0 \tag{5.1}$$

in the category of $\tilde{\mathbb{K}}$-algebraic groups. Furthermore, there exists a unique abelian formal group $\mathbb{K}^\circ$-scheme $\mathscr{B}$ with special fibre $\tilde{\mathscr{B}} = B_{\tilde{\mathbb{K}}}$ in the category of $\tilde{\mathbb{K}}$-algebraic groups, such that we have a short exact sequence

$$0 \to \mathbb{T}_0 \xrightarrow{r_0} \mathscr{A}_0 \xrightarrow{b_0} \mathscr{B} \to 0$$

in the category of formal group $\mathbb{K}^\circ$-schemes, which is the lifting of (5.1). Finally we also get a short exact sequence of the generic fibres

$$0 \to \mathbb{T}_0^{\mathrm{an}} \xrightarrow{r} \mathscr{A}_0^{\mathrm{an}} \xrightarrow{b} \mathscr{B}^{\mathrm{an}} \to 0 \tag{5.2}$$

in the category of $\mathbb{K}$-analytic groups.

Proof. [BL84, Theorem 8.2]; [BL91, Theorem 1.1]; See also: [BL86, Proposition 2.1]; [Gub10, §4.1]; [Ber90, §6.5] □

Definition 5.1.13. We define the *Raynaud uniformization* of A to be the pushout of the monomorphisms $r\colon \mathbb{T}_0^{\mathrm{an}} \hookrightarrow \mathscr{A}_0^{\mathrm{an}}$ and $t\colon \mathbb{T}_0^{\mathrm{an}} \hookrightarrow \mathbb{T}^{\mathrm{an}}$ in the category of $\mathbb{K}$-analytic groups, and denote it by E_{an}.

Lemma 5.1.14. *The Raynaud uniformization is well-defined and the two arising morphisms of the pushout square*

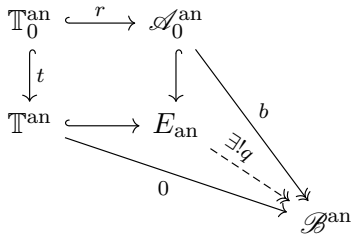
$$\begin{array}{ccc} \mathbb{T}_0^{\mathrm{an}} & \xrightarrow{r} & \mathscr{A}_0^{\mathrm{an}} \\ \downarrow{\scriptstyle t} & & \downarrow \\ \mathbb{T}^{\mathrm{an}} & \longrightarrow & E_{\mathrm{an}} \end{array}$$

are monic. Furthermore it is also a pushout square in the underlying category of topological spaces.

Proof. [Ber90, §6.5] □

Proposition 5.1.15 (Bosch-Lütkebohmert)**.** *Consider in the following the category of $\mathbb{K}$-analytic groups unless otherwise stated:*

(i) There exists a unique epimorphism $q\colon E_{\mathrm{an}} \twoheadrightarrow \mathscr{B}^{\mathrm{an}}$ such that the following diagram



(5.3)

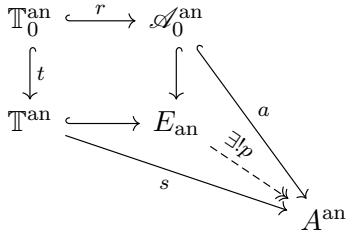
commutes.

(ii) There exists a unique morphism $s\colon \mathbb{T}^{\mathrm{an}} \to A^{\mathrm{an}}$ such that the diagram

$$\begin{array}{ccc} \mathbb{T}_0^{\mathrm{an}} & \xrightarrow{r} & \mathscr{A}_0^{\mathrm{an}} \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle a} \\ \mathbb{T}^{\mathrm{an}} & \xrightarrow{\exists! s} & A^{\mathrm{an}} \end{array}$$

commutes.

(iii) There exists a unique epimorphism $p\colon E_{\mathrm{an}} \twoheadrightarrow A^{\mathrm{an}}$ *such that the diagram*



commutes.

(iv) There exist a $\mathbb{K}$*-algebraic group* E *and an abelian* $\mathbb{K}$*-variety* B *such that* $E^{\mathrm{an}} = E_{\mathrm{an}}$ *and* $B^{\mathrm{an}} = \mathscr{B}^{\mathrm{an}}$ *in the category of* $\mathbb{K}$*-analytic groups. Furthermore, there exists a short exact sequence*

$$0 \longrightarrow \mathbb{T} \longrightarrow E \xrightarrow{q} B \longrightarrow 0$$

in the category of $\mathbb{K}$*-algebraic groups inducing the bottom short exact sequence of (5.3) on their analytifications.*

(v) We define the object M' *by the short exact sequence*

$$0 \longrightarrow M' \longrightarrow E^{\mathrm{an}} \xrightarrow{p} A^{\mathrm{an}} \longrightarrow 0 \tag{5.4}$$

in the category of $\mathbb{K}$*-analytic groups, that means as the kernel of* p*. Then* M' *is a discrete subgroup of the topological group* $E^{\mathrm{an}}(\mathbb{K})$ *and it is isomorphic to a free abelian group of rank* n *in the category of groups.*

Proof. (i) The pushout property and the short exact sequence (5.2). Alternatively see [BL91, p. 655]

(ii) [BL91, Theorem 1.2 and §2]

(iii) Apart from the epimorphism property, this follows by the pushout property as well. For p being epic see also [BL91, Theorem 1.2 and §2].

(iv) [BL91, Theorem 6.13] with GAGA principle; [Gub10, §4.1]

(v) [BL91, Theorem 1.2(b)]; [Gub10, §4.1] □

Remark 5.1.16. The (abelian) $\mathbb{K}$-analytic group $M' = (M', \mathscr{A}, \tau)$ has the following explicit description: The locally Hausdorff topological space M' and the group structure are given by a free abelian group of rank n with the discrete topology. Therefore the choice of A includes the choice of a free abelian group M of rank n (isomorphic to M' as a group). The net τ is isomorphic to M' as a set, and for any $V \in \tau$ we have the $\mathbb{K}$-affinoid algebra $\mathscr{A}_V = \mathbb{K}$. This locally yields for any $\gamma \in M'$ a morphism $\gamma\colon \mathscr{M}(\mathbb{K}) \to E^{\mathrm{an}}$ in the category of $\mathbb{K}$-analytic spaces, which is a $\mathbb{K}$-rational point of E^{an}. Eventually, these morphisms form a subset M' of $E^{\mathrm{an}}(\mathbb{K})$, which is also a discrete subgroup of the topological group $E^{\mathrm{an}}(\mathbb{K})$.

Remark 5.1.17. We denote the terminal object always by 0, even in the context of tori. For this reason we will, for instance, write $0 \hookrightarrow \mathbb{T}_0^{\mathrm{an}}$ instead of $1 \hookrightarrow \mathbb{T}_0^{\mathrm{an}}$.

Definition 5.1.18. The diagram in the category of $\mathbb{K}$-analytic groups consisting of the short exact sequences (5.3) and (5.4)

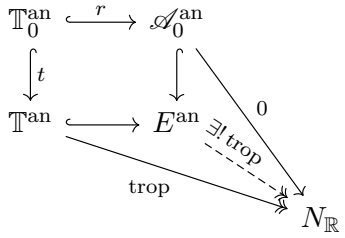
$$\begin{array}{ccccc} & & M' & & \\ & & \big\downarrow & & \\ \mathbb{T}^{\mathrm{an}} & \hookrightarrow & E^{\mathrm{an}} & \overset{q}{\twoheadrightarrow} & B^{\mathrm{an}} \\ & & \big\downarrow p & & \\ & & A^{\mathrm{an}} & & \end{array}$$

is called *Raynaud's uniformization cross.*

Definition 5.1.19. There are two extreme cases: If $B = 0$, we say the abelian $\mathbb{K}$-variety A is *totally degenerate.* If $\mathbb{T} = 0$, we say the abelian $\mathbb{K}$-variety A has *good reduction.* The abelian $\mathbb{K}$-variety B always has good reduction.

Lemma 5.1.20. *We can do the following constructions:*

(i) We may extend the canonical tropicalization map by the pushout diagram



in the category of topological spaces.

(ii) Defining Λ to be the image (see Definition 2.2.4) of M' under the epimorphism $\mathrm{trop}\colon E^{\mathrm{an}} \twoheadrightarrow N_{\mathbb{R}}$*, we get the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & E^{\mathrm{an}} & \longrightarrow & A^{\mathrm{an}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{trop}} & & \downarrow{\scriptstyle \mathrm{trop}} & & \vdots{\scriptstyle \exists!\overline{\mathrm{trop}}} & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & N_{\mathbb{R}} & \longrightarrow & N_{\mathbb{R}}/\Lambda & \longrightarrow & 0 \end{array}$$

in the category of topological spaces, where the horizontal lines are two short exact sequences in the category of $\mathbb{K}$-analytic groups respectively in the category of groups. The subobject Λ is a complete lattice in $N_{\mathbb{R}}$.

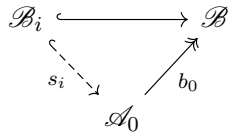
Proof. (i) By Lemma 5.1.10, the outer square commutes, and therefore the existence and uniqueness of the morphism of topological spaces $\mathrm{trop}\colon E^{\mathrm{an}} \to N_{\mathbb{R}}$ follows by the pushout property of the Raynaud extension. It is furthermore epic, because the morphism of topological spaces $\mathrm{trop}\colon \mathbb{T}^{\mathrm{an}} \twoheadrightarrow N_{\mathbb{R}}$ is already epic.

(ii) [BL91, Theorem 1.2(b)] □

5.2 Mumford models

As mentioned in [Gub10, 4.7], the following is a construction of Mumford [Mum72], which is useful for compactifying moduli spaces of abelian varieties (see [CF91]). This specific kind of models will help us later considering the problem combinatorially.

Lemma 5.2.1. *For the formal group $\mathbb{K}^\circ$-scheme $\mathscr{B}$, there exists a choice of the atlas $\mathscr{B} = (\mathscr{B}, (\mathscr{B}_i = \mathrm{Spf}(A_i))_{i\in I}, \mathcal{O}_{\mathscr{B}})$ and a family $(s_i \colon \mathscr{B}_i \hookrightarrow \mathscr{A}_0)_{i\in I}$ of monomorphisms of admissible formal $\mathbb{K}^\circ$-schemes such that for all $i \in I$ the diagram*



is commutative and the composition

$$\mathscr{B}_i \times \mathbb{T}_0 \xhookrightarrow{s_i \times r_0} \mathscr{A}_0 \times \mathscr{A}_0 \xrightarrow{m} \mathscr{A}_0$$

is an open immersion of admissible formal $\mathbb{K}^\circ$-schemes.

Proof. [BL91, penultimate paragraph of p. 655] or [Gub10, §4.2]. It can be translated to the category of admissible formal $\mathbb{K}^\circ$-schemes via Proposition 3.4.17. □

Definition 5.2.2 ([Gub10, §4.2])**.** The open immersions of admissible formal $\mathbb{K}^\circ$-schemes $(s_i \colon \mathscr{B}_i \hookrightarrow \mathscr{A}_0)_{i\in I}$ are called *local trivializations* of $\mathscr{B}$. Then, for any $i \in I$, we define the epimorphism of topological spaces

$$\mathrm{trop}_i \colon \mathscr{B}_i^{\mathrm{an}} \times \mathbb{T}^{\mathrm{an}} \xrightarrow{p_2} \mathbb{T}^{\mathrm{an}} \xrightarrow{\mathrm{trop}} N_{\mathbb{R}}.$$

Remark 5.2.3. The epimorphisms of topological spaces $\mathrm{trop} \colon E^{\mathrm{an}} \twoheadrightarrow N_{\mathbb{R}}$ and $\overline{\mathrm{trop}} \colon A^{\mathrm{an}} \twoheadrightarrow N_{\mathbb{R}}/\Lambda$ of Lemma 5.1.20 coincide with the ones defined in [Gub10, §4.2] via local trivializations, here written as $\mathrm{trop}_i \colon \mathscr{B}_i^{\mathrm{an}} \times \mathbb{T}^{\mathrm{an}} \twoheadrightarrow N_{\mathbb{R}}$. The reason is that the factor $\mathscr{A}_0^{\mathrm{an}}$ is sent to zero by $\mathrm{trop} \colon E^{\mathrm{an}} \twoheadrightarrow N_{\mathbb{R}}$ due to Lemma 5.1.20(i).

Lemma 5.2.4. *Let Δ be a Γ-rational polytope in $N_{\mathbb{R}}$. Then the set*

$$U_{i,\Delta} := \mathrm{trop}_i^{-1}(\Delta) = \mathscr{B}_i^{\mathrm{an}} \times U_\Delta$$

forms a Weierstrass domain on the $\mathbb{K}$-analytic space $\mathscr{B}_i^{\mathrm{an}} \times \mathbb{T}^{\mathrm{an}}$. We denote the corresponding $\mathbb{K}$-affinoid algebra by $\mathbb{K}\langle U_{i,\Delta}\rangle$.

Proof. [Gub10, Lemma 4.6] □

Definition 5.2.5 ([Gub10, 4.7])**.** Let Δ be a Γ-rational polytope in $N_\mathbb{R}$, and let $i \in I$. We call $U_{i,\Delta}$ the *polytopal domain* associated to (i, Δ), and we define the admissible formal affine $\mathbb{K}^\circ$-scheme $\mathscr{U}_{i,\Delta} := \mathrm{Spf}(\mathbb{K}\langle U_{i,\Delta}\rangle^\circ)$.

Lemma 5.2.6. *Let $\overline{\mathscr{C}}$ be a Γ-rational polytopal decomposition of $N_\mathbb{R}/\Lambda$, and $\mathscr{C}$ be the induced Γ-rational polytopal decomposition of $N_\mathbb{R}$ (see Definition 2.4.5). Then the definitions $I' := I \times \mathscr{C}$ (as sets),*

$$\mathscr{U}_{(i,\Delta_1)(j,\Delta_2)} := \begin{cases} \mathscr{U}_{k,\Delta_1\cap\Delta_2} & \text{if } \Delta_1 \cap \Delta_2 \neq \varnothing \\ \varnothing & \text{if } \Delta_1 \cap \Delta_2 = \varnothing, \end{cases}$$

where $k \in I$ such that $\mathscr{B}_i \cap \mathscr{B}_j = \mathscr{B}_k$, together with $\varphi_{(i,\Delta_1)(j,\Delta_2)} := \mathrm{id}$ yield a gluing datum in the category of admissible formal $\mathbb{K}^\circ$-schemes with reduced special fibre (analogous to [GW10, Definition 3.9] in the category of schemes).

Proof. Since the intersection of two affine opens is again affine open in the category of admissible formal $\mathbb{K}^\circ$-schemes, one can assume $\mathscr{B}$ to have the following property: Let $i, j \in I$. Then there exists a $k \in I$ such that $\mathscr{B}_i \cap \mathscr{B}_j = \mathscr{B}_k$.
Let $\Delta_1 \subseteq \Delta_2$ be Γ-rational polytopes in $N_\mathbb{R}$ such that Δ_1 is a closed face of Δ_2, and let $i, j \in I$ such that $\mathscr{B}_i \subseteq \mathscr{B}_j$. The inclusion $\Delta_1 \subseteq \Delta_2$ implies a monomorphism of $\mathbb{K}$-affinoid spaces $U_{\Delta_1} \hookrightarrow U_{\Delta_2}$ of the associated Weierstrass domains, which corresponds together with $\mathscr{B}_i \subseteq \mathscr{B}_j$ to an epimorphism of admissible $\mathbb{K}^\circ$-algebras $\mathbb{K}\langle U_{j,\Delta_2}\rangle^\circ \twoheadrightarrow \mathbb{K}\langle U_{i,\Delta_1}\rangle^\circ$. Finally, this induces a monomorphism of admissible formal affine $\mathbb{K}^\circ$-schemes $\mathscr{U}_{i,\Delta_1} \hookrightarrow \mathscr{U}_{j,\Delta_2}$ since Δ_1 is a closed face of Δ_2, and $\mathscr{B}_i \subseteq \mathscr{B}_j$.
By the definition of a polytopal complex (see Definition 2.4.2 (ii)), $\Delta_1 \cap \Delta_2$ is either empty or a closed face of Δ_1 and Δ_2. In the second case we get a monomorphism of admissible formal affine $\mathbb{K}^\circ$-schemes $\mathscr{U}_{k,\Delta_1\cap\Delta_2} \hookrightarrow \mathscr{U}_{i,\Delta_1}$. These monomorphisms together with the isomorphisms $\varphi_{(i,\Delta_1)(j,\Delta_2)} = \mathrm{id}$ fulfil the analogous conditions of [GW10, Definition 3.9]. Finally, we can take the considered subcategory (with reduced special fibre), because the reduction $\widetilde{\mathscr{U}_{i,\Delta}} = \mathrm{Spec}(\widetilde{\mathbb{K}\langle U_{i,\Delta}\rangle})$ is an affine $\tilde{\mathbb{K}}$-variety. □

Definition 5.2.7 ([Gub10, 4.7])**.** Let $\overline{\mathscr{C}}$ be a Γ-rational polytopal decomposition of $N_\mathbb{R}/\Lambda$. We then define the admissible formal $\mathbb{K}^\circ$-scheme $\mathscr{E}_{\mathscr{C}}$

associated to $\mathscr{C}$ to be the gluing object of the gluing datum from Lemma 5.2.6 in the category of admissible formal $\mathbb{K}^\circ$-schemes with reduced special fibre, whose existence can be shown analogously to [GW10, Proposition 3.10].

Lemma 5.2.8. *For the formal group $\mathbb{K}^\circ$-scheme $\mathscr{B}$ there exists a choice of the atlas $\mathscr{B} = (\mathscr{B}, (\mathscr{B}_i = \mathrm{Spf}(A_i))_{i\in I}, \mathcal{O}_{\mathscr{B}})$ fulfilling the properties in Lemma 5.2.1 and additionally the property that for all $x \in q(M') \subseteq \mathscr{B}^{\mathrm{an}}$ and $i \in I$ there exists a $j \in I$ such that $x + \mathscr{B}_i = \mathscr{B}_j$ as subobjects of the topological space $\mathscr{B}^{\mathrm{an}}$.*

Proof. [Gub10, 4.7] □

Definition 5.2.9 ([Gub10, 4.7])**.** Let $\overline{\mathscr{C}}$ be a Γ-rational polytopal decomposition of $N_\mathbb{R}/\Lambda$. We may form the quotient of $\mathscr{E}_{\mathscr{C}}^{\text{f-an}}$ by M', which yields a reduced formal $\mathbb{K}$-analytic variety $\mathfrak{A}_{\overline{\mathscr{C}}}$ (by Lemma 5.2.8, see [Gub10, 4.7]). The admissible formal $\mathbb{K}^\circ$-scheme $\mathscr{A}_{\overline{\mathscr{C}}} := \mathfrak{A}_{\overline{\mathscr{C}}}^{\text{f-sch}}$ with reduced special fibre is called the *Mumford model* of A associated to $\overline{\mathscr{C}}$.

Lemma 5.2.10. *Let $\overline{\mathscr{C}}$ be a Γ-rational polytopal decomposition of $N_\mathbb{R}/\Lambda$. The Mumford model $\mathscr{A}_{\overline{\mathscr{C}}}$ of A associated to $\overline{\mathscr{C}}$ is well-defined and indeed a formal $\mathbb{K}^\circ$-model of A. Furthermore we get morphisms of admissible formal $\mathbb{K}^\circ$-schemes $p\colon \mathscr{E}_{\mathscr{C}} \to \mathscr{A}_{\overline{\mathscr{C}}}$ and $q\colon \mathscr{E}_{\mathscr{C}} \to \mathscr{B}$ extending the morphisms on the generic fibres.*

Proof. [Gub10, 4.7] □

Lemma 5.2.11. *Let $\overline{\mathscr{C}}$ be a Γ-rational polytopal decomposition of $N_\mathbb{R}/\Lambda$, let $\mathscr{A}_1$ be the Mumford model of A associated to $\overline{\mathscr{C}}$, and let $m \in \mathbb{Z}\backslash\{0\}$. Then $\overline{m^{-1}\mathscr{C}} := \left\{\overline{m^{-1}\Delta} \mid \Delta \in \mathscr{C}\right\}$ is also a Γ-rational polytopal decomposition of $N_\mathbb{R}/\Lambda$. Denote the corresponding Mumford model of A by $\mathscr{A}_m$. Then the endomorphism of abelian $\mathbb{K}$-varieties $[m]\colon A \to A$ has a unique extension to a morphism $[m]\colon \mathscr{A}_m \to \mathscr{A}_1$ of admissible formal $\mathbb{K}^\circ$-schemes.*

Proof. [Gub07a, Proposition 6.4(a)] □

5.3 Line bundles on Mumford models

The goal of this section is to describe formal $\mathbb{K}^\circ$-models of L determined on a Mumford model of A (see Definition 3.4.23). We will start with a completely combinatorial part. Until Lemma 5.3.6 we may assume that $N_\mathbb{R}$ is just an arbitrary finite-dimensional $\mathbb{R}$-vector space and that Λ is an arbitrary complete lattice in $N_\mathbb{R}$. Afterwards, we will assume that both are induced by an abelian $\mathbb{K}$-variety A, and we let L be a line bundle on A.

Definition 5.3.1. A Λ-*cocycle* is a family of morphisms of topological spaces $(z_\lambda \colon N_\mathbb{R} \to \mathbb{R})_{\lambda\in\Lambda}$ such that

(i) for any $\lambda, \mu \in \Lambda$ and $u \in N_\mathbb{R}$ the equality

$$z_{\lambda+\mu}(u) = z_\lambda(\mu + u) + z_\lambda(u)$$

holds true, and

(ii) the definition

$$\begin{aligned} z \colon \Lambda &\to \mathbb{R}, \\ \lambda &\mapsto z_\lambda(0) \end{aligned}$$

yields a $\mathbb{Z}$-quadratic function (see Definition 2.6.2).

Lemma 5.3.2. *Let $(z_\lambda \colon N_\mathbb{R} \to \mathbb{R})_{\lambda\in\Lambda}$ be a Λ-cocycle. Then*

(i) the $\mathbb{Z}$-quadratic function $z \colon \Lambda \to \mathbb{R}$ extends uniquely to an $\mathbb{R}$-quadratic function $z \colon N_\mathbb{R} \to \mathbb{R}$,

(ii) for $\lambda \in \Lambda$, $u \in N_\mathbb{R}$ we have $z_\lambda(u) - z(\lambda) = b(\lambda, u)$ where b is the associated $\mathbb{R}$-bilinear form of the $\mathbb{R}$-quadratic function $z \colon N_\mathbb{R} \to \mathbb{R}$,

(iii) and for $\lambda \in \Lambda$, $u \in N_\mathbb{R}$ we get the equality $z(\lambda + u) - z(u) = z_\lambda(u)$.

Proof. (i) By Definition 5.3.1(ii) we have for the associated $\mathbb{Z}$-bilinear form $b \colon \Lambda \times \Lambda \to \mathbb{R}$ and for the associated $\mathbb{Z}$-linear form $l \colon \Lambda \to \mathbb{R}$ the equality $2z(\lambda) = b(\lambda, \lambda) + l(\lambda)$ for all $\lambda \in \Lambda$. They can be uniquely extended to an $\mathbb{R}$-bilinear form $b \colon N_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$ and a morphism of $\mathbb{R}$-vector spaces $l \colon N_\mathbb{R} \to \mathbb{R}$ and finally the definition $z(u) := \frac{b(u,u)+l(u)}{2}$ for all $u \in N_\mathbb{R}$ yields the desired $\mathbb{R}$-quadratic function. Uniqueness follows by the uniqueness of the extensions of b and l.

(ii) Definition 5.3.1(i) for $u = 0$ yields the equality $z_{\lambda+\mu}(0) = z_\lambda(\mu) + z_\mu(0)$, which can be equivalently written as $z(\lambda+\mu) = z_\lambda(\mu) + z(\mu)$. Combining this with Definition 5.3.1(ii), we get for all $\lambda, \mu \in \Lambda$ the equality

$$b(\lambda,\mu) = \underbrace{z(\lambda+\mu) - z(\mu)}_{=z_\lambda(\mu)} - z(\lambda)$$

which linearly extends to the required formula.

(iii) Using (i) we have

$$\begin{aligned} z(\lambda+u) - z(u) &= \tfrac{1}{2}(b(\lambda+u,\lambda+u) + l(\lambda+u) - b(u,u) - l(u)) \\ &= \tfrac{1}{2}(b(\lambda,\lambda) + l(\lambda)) + b(\lambda,u) \\ &= z(\lambda) + b(\lambda,u) \\ &= z_\lambda(u) \end{aligned}$$

where the last equality follows by (ii). □

Definition 5.3.3. Let $(z_\lambda\colon N_\mathbb{R} \to \mathbb{R})_{\lambda\in\Lambda}$ be a Λ-cocycle. A function $f\colon N_\mathbb{R} \to \mathbb{R}$ fulfils the *cocycle rule* if for any $\lambda \in \Lambda$, $u \in N_\mathbb{R}$ we have the equality $f(\lambda+u) = z_\lambda(u) + f(u)$.

Lemma 5.3.4. *Let $(z_\lambda\colon N_\mathbb{R} \to \mathbb{R})_{\lambda\in\Lambda}$ be a Λ-cocycle. A function $f\colon N_\mathbb{R} \to \mathbb{R}$ fulfils the cocycle rule if and only if the function $f - z\colon N_\mathbb{R} \to \mathbb{R}$ is Λ-periodic (see Definition 2.4.11), where $z\colon N_\mathbb{R} \to \mathbb{R}$ is the $\mathbb{R}$-quadratic function from Lemma 5.3.2(i).*

Proof. We have $z(\lambda+u) - z(u) = z_\lambda(u) = f(\lambda+u) - f(u)$, where the first equality follows by Lemma 5.3.2(iii), and the second equality is true for all $\lambda \in \Lambda$, $u \in N_\mathbb{R}$ if and only if the function $f\colon N_\mathbb{R} \to \mathbb{R}$ fulfils the cocycle rule. □

Lemma 5.3.5. *Let $(z_\lambda\colon N_\mathbb{R} \to \mathbb{R})_{\lambda\in\Lambda}$ be the Λ-cocycle, let $z\colon N_\mathbb{R} \to \mathbb{R}$ be the $\mathbb{R}$-quadratic function from Lemma 5.3.2(i), let b respectively l be the associated $\mathbb{R}$-bilinear respectively $\mathbb{R}$-linear form (see Definition 2.6.2, Definition 2.6.4), let $f\colon N_\mathbb{R} \to \mathbb{R}$ be an arbitrary morphism of topological spaces fulfilling the cocycle rule, and for any $m \in \mathbb{Z}\backslash\{0\}$ we will write $f(m\cdot.)\colon N_\mathbb{R} \to \mathbb{R}, u \mapsto f(mu)$.*

(i) If $l = 0$, then $\frac{1}{m^2} f(m \cdot .) \xrightarrow[m \to +\infty]{} z$ uniformly.

(ii) If $b = 0$, then $\frac{1}{m} f(m \cdot .) \xrightarrow[m \to +\infty]{} z$ uniformly.

Proof. We will just prove (i) because (ii) works analogously. Since f fulfils the cocycle rule, by Lemma 5.3.4 the morphism of topological spaces $f - z$ is Λ-periodic. As the fundamental domain of the complete lattice Λ in $N_{\mathbb{R}}$ is compact and morphisms of topological spaces on compact subsets are bounded, the morphism of topological spaces $f - z$ is therefore bounded. Hence the morphism of topological spaces

$$\tfrac{1}{m^2} f(m \cdot .) - \underbrace{\tfrac{1}{m^2} z(m \cdot .)}_{=z} \colon N_{\mathbb{R}} \to \mathbb{R}$$

converges uniformly to 0 for $m \to +\infty$ as $\frac{1}{m^2}$ converges to zero. □

Lemma 5.3.6. *Let $(z_\lambda \colon N_{\mathbb{R}} \to \mathbb{R})_{\lambda \in \Lambda}$ be a Λ-cocycle, assume that the $\mathbb{R}$-quadratic function $z \colon N_{\mathbb{R}} \to \mathbb{R}$ is strictly convex, let $\mathscr{C}$ be a Λ-periodic (see Definition 2.4.6) polyhedral decomposition of $N_{\mathbb{R}}$, and let f be a $\mathscr{C}$-polyhedral convex function fulfilling the cocycle rule. Then there exists a sequence $(\overline{\mathscr{C}_k})_{k \in \mathbb{N}}$ of polytopal decompositions of $N_{\mathbb{R}}/\Lambda$ and a sequence $(f_k)_{k \in \mathbb{N}}$ of strongly $\mathscr{C}_k$-polyhedral convex functions fulfilling the cocycle rule, such that $(f_k)_{k \in \mathbb{N}}$ converges uniformly to f.*

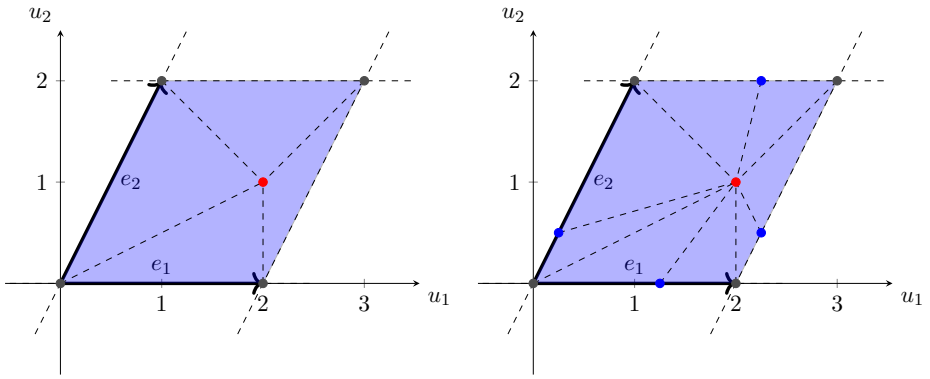
Proof. Step 1: We may assume f strongly $\mathscr{C}$-polyhedral convex and $\mathscr{C}$ being a Λ-periodic (see Definition 2.4.6) polytopal decomposition. Furthermore, we may assume that for any $\Delta \in \mathscr{C}$ and $u \in N_{\mathbb{R}}$ there exists at most one $\lambda \in \Lambda$ such that $u + \lambda \in \operatorname{relint}(\Delta)$.

By unifying all n-dimensional polyhedrons of $\mathscr{C}$ with the same slopes (and adjusting their closed faces) we may assume that f is strongly $\mathscr{C}$-polyhedral convex, $\mathscr{C}$ remains Λ-periodic (see Definition 2.4.6) and still fulfils the cocycle rule. As by Lemma 5.3.4 the function $f - z$ is Λ-periodic and by assumption the $\mathbb{R}$-quadratic function z is strictly convex, all polyhedrons Δ of $\mathscr{C}$ are bounded.
Assume for a contradiction that there exist $\Delta \in \mathscr{C}$, $u \in N_{\mathbb{R}}$, and $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \neq \lambda_2$ such that $u + \lambda_1, u + \lambda_2 \in \operatorname{relint}(\Delta)$. By the convexity we get $[u + \lambda_1, u + \lambda_2] \subseteq \operatorname{relint}(\Delta)$ and by the Λ-periodicity of $\mathscr{C}$ we get

$[u+\lambda_1+(\lambda_2-\lambda_1), u+\lambda_2+(\lambda_2-\lambda_1)] \subseteq \mathrm{relint}(\Delta+(\lambda_2-\lambda_1))$. Since for any element of $N_\mathbb{R}$ and particularly for the point $u+\lambda_1+(\lambda_2-\lambda_1) = u+\lambda_2$ there exists exactly one element of $\mathscr{C}$ such that this point lies in its relative interior, this implies the equality $\Delta = \Delta+(\lambda_2-\lambda_1)$ of subsets of $N_\mathbb{R}$. Proceeding in this way, we get the inclusion $u+\lambda_1+\mathbb{R}(\lambda_2-\lambda_1) \subseteq \mathrm{relint}(\Delta)$, which is a contradiction to $\Delta \in \mathscr{C}$ being bounded.

Step 2: f_k strongly $\mathscr{C}_k$-polyhedral convex uniformly converging to f

For any $\Delta \in \mathscr{C}$ we may choose an element $\xi_\Delta \in \mathrm{relint}(\Delta)$ such that for any $\lambda \in \Lambda$ we have $\xi_{\Delta+\lambda} = \xi_\Delta + \lambda$, which is possible by Step 1. Choose a finite subset $\mathscr{C}_0 \subseteq \mathscr{C}$ such that for any $\Delta \in \mathscr{C}$ there exist unique elements $\Delta_0 \in \mathscr{C}_0$ and $\lambda \in \Lambda$ with $\Delta = \Delta_0 + \lambda$. Define for all $j \in \{0,...,n\}$ the finite subsets $V_j := \{\xi_{\Delta_0} \mid \Delta_0 \in \mathscr{C}_0 \text{ is } j\text{-dimensional}\}$ and for $j \in \{0,...,n\}$ the sets $V_{\geq j} := \bigcup_{n \geq i \geq j} V_i$ and $V_{\leq j} := \bigcup_{0 \leq i \leq j} V_i$.



(a) $\mathscr{C}_k^{(j)}$ for $j = 2$: Every 2-dimensional polytope is the convex hull of an at most 1-dimensional polytope of $\mathscr{C}$ and exactly one vertex of $V_2 + \Lambda$.

(b) $\mathscr{C}_k^{(j)}$ for $j = 1$: Every 2-dimensional polytope is the convex hull of exactly one vertex of $V_i + \Lambda$ for every $i \in \{0,1,2\}$.

Figure 5.1: Example for $n = 2$ and $\Lambda = \mathbb{Z}e_1 + \mathbb{Z}e_2$: Let $\mathscr{C}$ be the polytopal decomposition of $N_\mathbb{R}$ having all Λ-translates of the closure of the fundamental domain as n-dimensional polytopes. The elements $\xi_\Delta \in \mathrm{relint}(\Delta)$ "break" the polytopes in $\mathscr{C}$. The red point lies in $V_2 + \Lambda$, the blue ones lie in $V_1 + \Lambda$, and the gray ones lie in $V_0 + \Lambda$.

For any $j \in \{1, ..., n+1\}$ and $\Delta \in \mathscr{C}$ we recursively define the set

$$F^{(j)}(\Delta) := \begin{cases} \{\Delta\} & \text{if } \dim_{\mathbb{R}}(\Delta) < j \\ \bigcup_{\sigma} \{C(\tau, \xi_\Delta) \mid \tau \in F^{(j)}(\sigma)\} & \text{if } \dim_{\mathbb{R}}(\Delta) \geq j \end{cases}$$

where σ runs over all $(\dim_{\mathbb{R}}(\Delta)-1)$-dimensional closed faces of Δ in $\mathscr{C}$ and $C(\tau, \xi_\Delta)$ is the convex hull of τ and ξ_Δ in $N_{\mathbb{R}}$, which is a polytope (provided by [Zie95, Theorem 1.1], which proves that a polytope is the convex hull of a finite set of points). Then every polytope $\Delta' \in F^{(j)}(\Delta)$ is the convex hull of an at most $(j-1)$-dimensional polytope in $\mathscr{C}$ (has only vertices in $V_0+\Lambda$) and exactly one vertex of $V_i + \Lambda$ for every $i \in \{j, ..., n\}$. Furthermore, the dimension of Δ' is equal to the dimension of Δ because the vertex ξ_Δ lies in the relative interior of Δ.

Let $\mathscr{N} \subseteq \mathscr{C}$ be the subset of n-dimensional polytopes. Then the set $\mathscr{N}_k^{(j)} := \bigcup_{\Delta \in \mathscr{N}} F^{(j)}(\Delta)$ of n-dimensional polytopes generates (by taking intersections) a Λ-periodic (see Definition 2.4.6) polytopal decomposition $\mathscr{C}_k^{(j)}$ of $N_{\mathbb{R}}$ because for every $\lambda \in \Lambda$ the equality $F^{(j)}(\Delta+\lambda) = F^{(j)}(\Delta)+\lambda$ holds true (see Figure 5.1 for a visualization). We have $\mathscr{C} = \mathscr{C}_k^{(n+1)}$ and we will define $\mathscr{C}_k := \mathscr{C}_k^{(1)}$. In the following, k will be an element of $\mathbb{N}$ but all considered polytopal decompositions are independent of the choice of k.

In order to construct a strongly $\mathscr{C}_k$-polyhedral convex function f_k, we will do an induction over $j \in \{1, ..., n+1\}$:

Induction claim: Let $j \in \{1, ..., n+1\}$. Then there exists a strongly $\mathscr{C}_k^{(j)}$-polyhedral convex function $f_k^{(j)}$ such that for all vertices ξ of $\mathscr{C}_k^{(j)}$ we have $|f(\xi) - f_k^{(j)}(\xi)|_{\mathbb{R}} < \frac{1}{k+1}$, and for all $\xi \in V_{\leq (j-1)} + \Lambda$ we have $f(\xi) = f_k^{(j)}(\xi)$.

Induction start: Let $j = n+1$ and define $f_k^{(n+1)} := f$, which fulfils the induction claim because f is strongly $\mathscr{C}$-polyhedral convex due to Step 1 and we have $\mathscr{C} = \mathscr{C}_k^{(n+1)}$.

Induction step: Let $j \in \{1, ..., n\}$, assume that the induction claim holds true for $j+1$. By this we get that the function $f_k^{(j)} := f_k^{(j+1)}$ is ~~strongly~~ $\mathscr{C}_k^{(j)}$-polyhedral convex and for all vertices ξ of $\mathscr{C}_k^{(j+1)}$ we have $|f(\xi) - f_k^{(j)}(\xi)|_{\mathbb{R}} < \frac{1}{k+1}$. Because of the last part of the induction claim, the inequality $|f(\xi) - f_k^{(j)}(\xi)|_{\mathbb{R}} < \frac{1}{k+1}$ holds over and above that for all vertices ξ of $\mathscr{C}_k^{(j)}$. Since

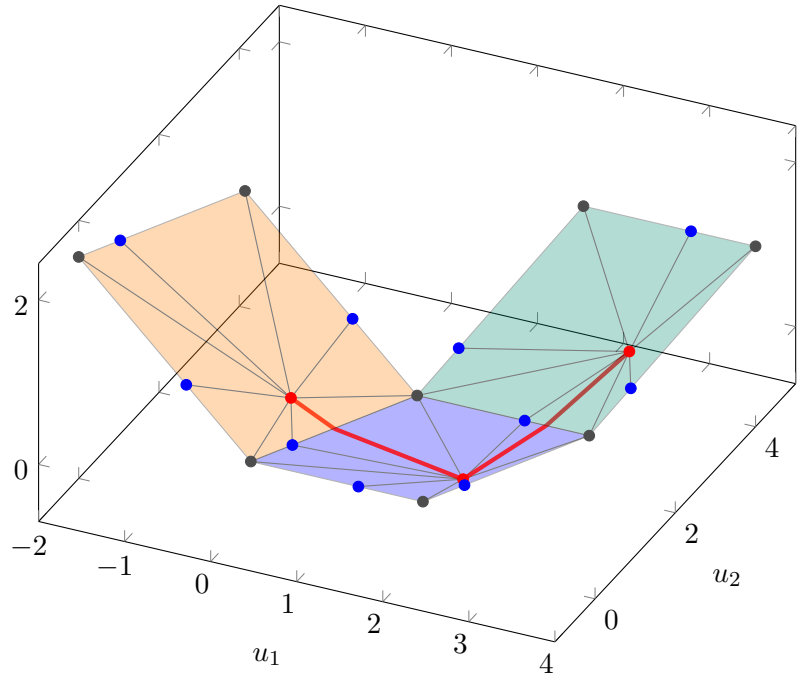


Figure 5.2: The function $f_k^{(2)}$ is strongly $\mathscr{C}_k^{(2)}$-polyhedral convex but not strongly $\mathscr{C}_k^{(1)}$-polyhedral convex.

every n-dimensional polytope of $\mathscr{C}_k^{(j)}$ has exactly one vertex in $V_j + \Lambda$, for any $\xi \in V_j$ there exists an element $\varepsilon(\xi) \in \mathbb{R}_{>0}$ such that for any re-definition

$$f(\xi) > f_k^{(j)}(\xi) > f(\xi) - \max\Big(\varepsilon(\xi), \tfrac{1}{k+1}\Big)$$

of the function $f_k^{(j)}$ at $\xi \in V_j$ (and hence all vertices in $V_j + \Lambda$) the new function $f_k^{(j)}$ satisfies the induction claim for j. Note that this is well-defined because every n-dimensional polytope of $\mathscr{C}_k^{(j)}$ is given as a convex hull of an at most $(j-1)$-dimensional polytope in $\mathscr{C}$ (only vertices in $V_0+\Lambda$) and exactly one vertex of $V_i + \Lambda$ for every $i \geq j$. Omitting the vertex in $V_j + \Lambda$ would yield an $(n-1)$-dimensional polytope. This completes the induction and the induction claim for $j = 1$ proves the statement of Step 2 (define $f_k := f_k^{(1)}$). See Figure 5.2 for a visualization.

Step 3: $\mathscr{C}_k$ induces a polytopal decomposition $\overline{\mathscr{C}_k}$ of $N_{\mathbb{R}}/\Lambda$.

Assume for a contradiction that there exists a $\sigma \in \mathscr{C}_k$ and $\lambda \in \Lambda\backslash\{0\}$ such that $\tau := \sigma \cap (\sigma + \lambda) \neq \varnothing$. Then τ and $\tau + \lambda$ are closed faces of $\sigma + \lambda \in \mathscr{C}_k$, and for every vertex ξ of τ, both ξ and $\xi + \lambda$ are vertices of $\sigma + \lambda$.
In Step 2 we have seen that every polytope $\Delta \in \mathscr{N}_k^{(1)}$ has exactly one vertex in $V_i + \Lambda$ for all $i \in \{0, ..., n\}$ and thus every polytope $\Delta \in \mathscr{C}_k^{(1)} = \mathscr{C}_k$ has at most one vertex in $V_i + \Lambda$ for all $i \in \{0, ..., n\}$. This yields a contradiction for $\Delta = \sigma + \lambda$, because there exists a $j \in \{0, ..., n\}$ such that the two different vertices ξ and $\xi + \lambda$ both lie in $V_j + \Lambda$. Therefore we get a polytopal decomposition $\overline{\mathscr{C}_k}$ of $N_{\mathbb{R}}/\Lambda$, which is independent of the choice of $k \in \mathbb{N}$. □

Lemma 5.3.7. *Let L be a line bundle on A. For any rigidification ρ_L on L there exists a rigidified line bundle (H, ρ_H) on the $\mathbb{K}$-algebraic group B and a line bundle $\mathscr{H}$ on the formal group K°-scheme $\mathscr{B}$ such that $H^{\mathrm{an}} = \mathscr{H}^{\mathrm{an}}$ as line bundles on $\mathbb{K}$-analytic spaces, such that $||.||_{\mathscr{H}} = ||.||_{H,\mathrm{can}}$ as metrics on B, such that $p^*L^{\mathrm{an}} = q^*H^{\mathrm{an}}$ as line bundles on E^{an}, and such that $p^*\rho_L^{\mathrm{an}} = q^*\rho_H^{\mathrm{an}}$ in the category of $\mathbb{K}$-analytic spaces.*

Proof. [BL91, §4, particularly Proposition 4.4] □

Remark 5.3.8. Let L be a line bundle on A. As described in [Gub10, 4.3], the group action $M' \times E^{\mathrm{an}} \to E^{\mathrm{an}}$ in the category of $\mathbb{K}$-analytic groups, which is induced by the monomorphism of $\mathbb{K}$-analytic groups $M' \hookrightarrow E^{\mathrm{an}}$ (see Lemma 2.2.7), induces for any $\gamma \in M'$ an isomorphism of fields $\mathscr{H}(x) \xrightarrow{\sim} \mathscr{H}(\gamma \cdot x)$ and an isomorphism of $\mathscr{H}(x)$-vector spaces $\alpha_\gamma \colon p^*L^{\mathrm{an}}(x) \xrightarrow{\sim} p^*L^{\mathrm{an}}(\gamma \cdot x)$. It was used that $M' \subseteq E^{\mathrm{an}}(\mathbb{K})$ (see Proposition 5.1.15(v)).

Lemma 5.3.9. *Let L be a line bundle on A. For any $\gamma \in M'$, we define the function*

$$\begin{aligned} Z_\gamma \colon E^{\mathrm{an}} &\to \mathbb{R}_{>0}, \\ x &\mapsto \frac{q^*||w||_{\mathscr{H},x}}{q^*||\alpha_\gamma(w)||_{\mathscr{H},\gamma\cdot x}} \end{aligned}$$

for $w \in p^*L^{\mathrm{an}}(x)$ with $q^*||w||_{\mathscr{H},x} \neq 0$, where $\mathscr{H}$ is a line bundle on the formal group $\mathbb{K}^\circ$-scheme $\mathscr{B}$ given by Lemma 5.3.7 for some choice of the rigidification ρ_L on L, and where $||.||_{\mathscr{H}}$ denotes the formal metric associated

to $\mathscr{H}$ (see Definition 4.1.5) and $q^*||.||_{\mathscr{H}}$ denotes its pull-back along q (see Definition 4.1.1). We then get the following properties:

(i) For any $\gamma \in M'$ the function Z_γ is well-defined and continuous.

(ii) For any $\beta, \gamma \in M'$ and $x \in E^{\mathrm{an}}$ we get $Z_{\beta\gamma}(x) = Z_\beta(\gamma \cdot x) \cdot Z_\gamma(x)$.

Proof. (i) Another choice of $w \in p^*L^{\mathrm{an}}(x)$ or of ρ_L would multiply the nominator and denominator with the same factor. Hence the function Z_γ is well-defined. Its continuity follows by Lemma 4.1.6. See also [Gub10, 4.3].

(ii) We have

$$\frac{q^*||w||_{\mathscr{H},x}}{q^*||\alpha_{\beta\gamma}(w)||_{\mathscr{H},\beta\gamma\cdot x}} = \frac{q^*||\alpha_\gamma(w)||_{\mathscr{H},\gamma\cdot x}}{q^*||\alpha_\beta(\alpha_\gamma(w))||_{\mathscr{H},\beta\gamma\cdot x}} \cdot \frac{q^*||w||_{\mathscr{H},x}}{q^*||\alpha_\gamma(w)||_{\mathscr{H},\gamma\cdot x}}$$

for all $\beta, \gamma \in M'$, $x \in E^{\mathrm{an}}$, and $w \in p^*L^{\mathrm{an}}(x)$ with $q^*||w||_{\mathscr{H},x} \neq 0$. □

Lemma 5.3.10. *Let L be a line bundle on A. For any $\lambda \in \Lambda$ there exists a unique morphism $z_\lambda \colon N_{\mathbb{R}} \to \mathbb{R}$ making the diagram*

$$\begin{array}{ccc} E^{\mathrm{an}} & \xrightarrow{\mathrm{trop}} & N_{\mathbb{R}} \\ \downarrow{\scriptstyle Z_\gamma} & & \vdots{\scriptstyle \exists! z_\lambda} \\ \mathbb{R}_{>0} & \xrightarrow{-\log} & \mathbb{R} \end{array}$$

in the category of topological spaces commutative for any $\gamma \in M'$ with $\mathrm{trop}(\gamma) = \lambda$ and forming a Λ-cocycle.

Proof. By the description of the group action α in [BL91, Proposition 4.9], the value $Z_\gamma(x)$ depends only on $\mathrm{trop}(x)$. Lemma 5.3.9(ii) and the above commutative diagram check condition (i) in Definition 5.3.1. For condition (ii) in Definition 5.3.1, one has to show that

$$\begin{aligned} b \colon \Lambda \times \Lambda &\to \mathbb{R}, \\ (\mu, \lambda) &\mapsto z_{\mu+\lambda}(0) - z_\mu(0) - z_\lambda(0) \end{aligned}$$

is a $\mathbb{Z}$-bilinear form. This requires the theory of cubical line bundles and is a non-trivial proof which is based on [BL91, §4]. For the statement we refer to [Gub10, 4.3]. □

Lemma 5.3.11. *Let L be a line bundle on A, let $(z_\lambda \colon N_\mathbb{R} \to \mathbb{R})_{\lambda\in\Lambda}$ be the Λ-cocycle induced by L (see Lemma 5.3.10), let $z \colon N_\mathbb{R} \to \mathbb{R}$ be the $\mathbb{R}$-quadratic function from Lemma 5.3.2(i), and let b respectively l be the associated $\mathbb{R}$-bilinear respectively $\mathbb{R}$-linear form (see Definition 2.6.2, Definition 2.6.4). Then*

(i) the image of the subset $\Lambda \subseteq N_\mathbb{R}$ with respect to z is contained in Γ,

(ii) the line bundle L is even if and only if $l = 0$, and L is odd if and only if $b = 0$, and

(iii) the line bundle L is ample if and only if the line bundle H is ample and b is positive definite. In this case, the line bundle $\mathscr{H}$ is also ample (see Definition 3.4.25) and the $\mathbb{R}$-quadratic function $z \colon N_\mathbb{R} \to \mathbb{R}$ is strictly convex.

Proof. (i) By the definition of the function Z_γ (see Lemma 5.3.9) in combination with the definition of the formal metric $||.||_{\mathscr{H}}$ (see Definition 4.1.5), we get that for any $x \in E^{\mathrm{an}}$ there exists an element $a \in \mathbb{K}\backslash\{0\}$ such that $Z_\gamma(x) = |a|$. Then applying the commutative diagram in Lemma 5.3.10 yields $z_\lambda(\mathrm{trop}(x)) \in \Gamma$ which implies the statement because the function $\mathrm{trop} \colon E^{\mathrm{an}} \twoheadrightarrow N_\mathbb{R}$ is surjective.

(ii) We refer to the last sentence of [Gub10, 4.3] for the statement.

(iii) For the equivalence we refer to [Gub10, 4.3] and [BL91, Theorem 6.13]. Since b is positive definite, the function z is strictly convex. It remains to show that the line bundle $\mathscr{H}$ is ample. At first we will show that the even line bundle $\mathscr{H}_0 := \mathscr{H}\hat{\otimes}[-1]^*\mathscr{H}$ is ample. The latter follows by [Ray70, Theorem XI.1.13] for $S = \mathrm{Spf}(\mathbb{K}^\circ)$ (see [DGH21, Remark 3.1] for an explanation) in combination with the fact that the formal $\mathbb{K}^\circ$-model $\mathscr{H}_0$ of the even line bundle $H_0 := H \otimes [-1]^*H$ is unique and by using the noetherian approximation (see [EGAIV.3, Lemme 8.10.5.2]) for fixing the leak that $S = \mathrm{Spf}(\mathbb{K}^\circ)$ is not necessarily noetherian. Since we have

$$\mathscr{H}^{\otimes 2} = \underbrace{\mathscr{H}_0}_{\text{even}} \hat{\otimes} (\underbrace{\mathscr{H}\hat{\otimes}([-1]^*\mathscr{H})^{-1}}_{\text{odd}}),$$

the Nakai-Moishezon Criterion (see [Har77, Theorem A.5.1]) shows that the special fibre of the line bundle $\mathscr{H}^{\otimes 2}$ is ample because $\mathscr{H}_0$ is ample and all intersection numbers with an odd line bundle are zero. Hence the special fibre $\tilde{\mathscr{H}}$ of $\mathscr{H}$ is ample. Finally by [EGAIII.1, Théorème 4.7.1] in combination with the noetherian approximation (see [EGAIV.3, Lemme 8.10.5.2]), the fact $\tilde{\mathscr{H}}$ being ample implies that the line bundle $\mathscr{H}$ itself is ample. Note that in the application of [EGAIII.1, Théorème 4.7.1] we have $U = \mathrm{Spf}(\mathbb{K}^\circ)$ because the closure of the trivial semi-norm on $\mathbb{K}^\circ$ with respect to the formal topology is the whole topological space $\mathrm{Spf}(\mathbb{K}^\circ)$. The noetherian approximation (see [EGAIV.3, Lemme 8.10.5.2]) is used for fixing the gap that $S = \mathrm{Spf}(\mathbb{K}^\circ)$ is not necessarily noetherian. See also [BGJK21, Remark 8.2.5] for the statement $\mathscr{H}$ being ample. □

5.4 Metric-function correspondence and toric metrics

Let L be a line bundle on A. The goal of this section is to establish a bijection between a certain subset of continuous metrics on L and the set of continuous functions $N_\mathbb{R} \to \mathbb{R}$ fulfilling the cocycle rule. We will name these metrics toric.

Proposition 5.4.1 (Metric-function correspondence for formal metrics)**.** *Let $\overline{\mathscr{C}}$ be a Γ-rational polytopal decomposition of $N_\mathbb{R}/\Lambda$. Then there is a bijective correspondence between the isomorphism classes of formal $\mathbb{K}^\circ$-models $\mathscr{L}$ of L determined on $\mathscr{A}_{\overline{\mathscr{C}}}$, which are trivial on the admissible formal affine $\mathbb{K}^\circ$-schemes $\mathscr{U}_{i,\overline{\Delta}}$ for each $i \in I$, $\overline{\Delta} \in \overline{\mathscr{C}}$, and morphisms of topological spaces $f_\mathscr{L} \colon N_\mathbb{R} \to \mathbb{R}$ being Γ-rational $\mathscr{C}$-polyhedral and fulfilling the cocycle rule.*
Let $||.||_\mathscr{L}$ be the associated formal metric on L. Any choice of a rigidification ρ_L on L yields a canonical bijection, in the sense that the morphism of topological spaces $f_\mathscr{L} \colon N_\mathbb{R} \to \mathbb{R}$ is uniquely determined by making the diagram in the category of topological spaces

$$\begin{array}{ccc} E^{\mathrm{an}} & \xrightarrow{\mathrm{trop}} & N_\mathbb{R} \\ {\scriptstyle \frac{p^*||.||_\mathscr{L}}{q^*||.||_\mathscr{H}}}\downarrow & & \vdots\, {\scriptstyle f_\mathscr{L}} \\ \mathbb{R}_{>0} & \xrightarrow[\sim]{-\log} & \mathbb{R} \end{array}$$

commutative. The quotient $\frac{p^*||.||_{\mathscr{L}}}{q^*||.||_{\mathscr{H}}}$ *is evaluated at any non-zero local section.*

Proof. [Gub10, Proposition 4.11] □

Definition 5.4.2. A continuous metric $||.||$ on L is called *toric* if there exists a morphism of topological spaces $\psi\colon A^{\mathrm{an}} \to \mathbb{R}$ such that

(i) ψ is $\mathscr{A}_0^{\mathrm{an}}$-invariant, that means for the group action $\triangleleft_a\colon \mathscr{A}_0^{\mathrm{an}} \times A^{\mathrm{an}} \to A^{\mathrm{an}}$ given by the monomorphism $a\colon \mathscr{A}_0^{\mathrm{an}} \hookrightarrow A^{\mathrm{an}}$ of $\mathbb{K}$-analytic groups (see Proposition 5.1.12 and Lemma 2.2.7), the diagram

$$\begin{array}{ccc} \mathscr{A}_0^{\mathrm{an}} \times A^{\mathrm{an}} & \xrightarrow{\triangleleft_a} & A^{\mathrm{an}} \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \psi} \\ A^{\mathrm{an}} & \xrightarrow{\psi} & \mathbb{R} \end{array}$$

commutes in the category of topological spaces;

(ii) and $||.|| = e^{-\psi}||.||_{L,\mathrm{can}}$ in the set $\mathcal{C}(L)$, where $||.||_{L,\mathrm{can}}$ denotes a canonical metric on L for some rigidification ρ_L on L.

In this case, and if we have fixed the choice of a rigidification ρ_L on L, we will denote the toric metric by $||.||_\psi$.

Lemma 5.4.3. *If condition (ii) in Definition 5.4.2 is fulfilled for one rigidification on L, then it is fulfilled for all rigidifications on L.*

Proof. By Lemma 4.1.17, another choice of the rigidification would just lead to a multiplication of ψ with a non-zero constant. □

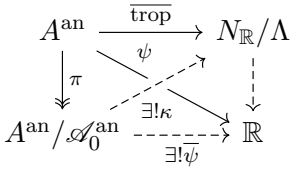
Lemma 5.4.4. *A metric $||.||$ on L is toric if and only if there exists a morphism of topological spaces $\overline{\psi}\colon N_{\mathbb{R}}/\Lambda \to \mathbb{R}$ such that we get the equality $||.|| = e^{-\overline{\psi}\circ\overline{\mathrm{trop}}}||.||_{L,\mathrm{can}}$ of metrics on L, where $||.||_{L,\mathrm{can}}$ denotes the canonical metric on L for some rigidification ρ_L on L.*

Proof. "$\Longrightarrow$": For any $x \in A^{\mathrm{an}}$ we define the subset

$$\mathscr{A}_0^{\mathrm{an}}x := \{\triangleleft_a(w) \mid w \in \mathscr{A}_0^{\mathrm{an}} \times A^{\mathrm{an}}, p_2(w) = x\}$$

of A^{an}. For different elements $x, x' \in A^{\mathrm{an}}$, the subsets $\mathscr{A}_0^{\mathrm{an}}x$ and $\mathscr{A}_0^{\mathrm{an}}x'$ are either equal or disjoint. In [Ber90, §5.1], Berkovich defined a topological

space $A^{\mathrm{an}}/\mathscr{A}_0^{\mathrm{an}}$ having these subsets as elements and having the coarsest topology such that the function $\pi\colon A^{\mathrm{an}} \twoheadrightarrow A^{\mathrm{an}}/\mathscr{A}_0^{\mathrm{an}}, x \mapsto \mathscr{A}_0^{\mathrm{an}}x$ is continuous (called quotient topology). In [Ber90, §6.5], it is proved that there exists a unique isomorphism of topological spaces $\kappa\colon A^{\mathrm{an}}/\mathscr{A}_0^{\mathrm{an}} \xrightarrow{\sim} N_{\mathbb{R}}/\Lambda$ such that the upper left triangular of the diagram in the category of topological spaces



commutes. More precisely, Berkovich showed the analogous case for the group action $\triangleleft_t\colon \mathbb{T}_0^{\mathrm{an}} \times \mathbb{T}^{\mathrm{an}} \to \mathbb{T}^{\mathrm{an}}$ induced by $t\colon \mathbb{T}_0^{\mathrm{an}} \hookrightarrow \mathbb{T}^{\mathrm{an}}$ (see Lemma 2.2.7), but this result can be transferred to the case here. The function

$$\begin{aligned}\overline{\psi}\colon A^{\mathrm{an}}/\mathscr{A}_0^{\mathrm{an}} &\to \mathbb{R},\\ \mathscr{A}_0^{\mathrm{an}}x &\mapsto \psi(x)\end{aligned}$$

is well-defined and continuous, because the function ψ is $\mathscr{A}_0^{\mathrm{an}}$-invariant and continuous, and it is the unique morphism of topological spaces making the lower left triangular of the above diagram commutative. Finally, redefining $\overline{\psi} := \overline{\psi} \circ \kappa^{-1}\colon N_{\mathbb{R}}/\Lambda \to \mathbb{R}$ yields the desired morphism of topological spaces.

"$\Longleftarrow$": Define $\psi := \overline{\psi} \circ \overline{\mathrm{trop}}$. Then the above considerations show that ψ factorizes through $\pi\colon A^{\mathrm{an}} \twoheadrightarrow A^{\mathrm{an}}/\mathscr{A}_0^{\mathrm{an}}$, which means that ψ is $\mathscr{A}_0^{\mathrm{an}}$-invariant. □

Lemma 5.4.5. *Any toric metric is continuous.*

Proof. The canonical metric is continuous and the function $\overline{\psi} \circ \overline{\mathrm{trop}}$ is continuous. □

Example 5.4.6. Any canonical metric on L is toric.

Proof. Take $\psi = 0$ or another constant function depending on the choice of the rigidification ρ_L. □

Definition 5.4.7. Choose a rigidification ρ_L on L. For any toric metric $||.||_\psi$ we define the morphism of topological spaces

$$f_\psi := z + \overline{\psi} \circ \lambda \colon N_\mathbb{R} \to \mathbb{R},$$

where $z \colon N_\mathbb{R} \to \mathbb{R}$ is the $\mathbb{R}$-quadratic function from Lemma 5.3.2(i), $\overline{\psi} \colon N_\mathbb{R}/\Lambda \to \mathbb{R}$ is the morphism of topological spaces from Lemma 5.4.4, and $\lambda \colon N_\mathbb{R} \twoheadrightarrow N_\mathbb{R}/\Lambda$ is the quotient function from Definition 2.4.5.

Lemma 5.4.8. *The function f_ψ depends on the choice of a rigidification ρ_L on L. Another choice affects f_ψ by the addition of a constant in Γ. By contrast, the $\mathbb{R}$-quadratic function z is independent of the choice of ρ_L.*

Proof. The morphism of topological spaces ψ depends on the choice of ρ_L and by Lemma 4.1.17 another choice leads to the addition of an element of Γ. The uniqueness of z follows because Lemma 5.3.10 is independent of the choice of ρ_L. □

Proposition 5.4.9 (Metric-function correspondence for toric metrics)**.** *Choose a rigidification ρ_L on L. The map $||.||_\psi \mapsto f_\psi$ gives a bijective function between the set of toric metrics on L and the set of morphisms of topological spaces $N_\mathbb{R} \to \mathbb{R}$ fulfilling the cocycle rule, such that the diagram*

$$\begin{array}{ccc} E^{\mathrm{an}} & \xrightarrow{\mathrm{trop}} & N_\mathbb{R} \\ {\scriptstyle \frac{p^*||.||_\psi}{q^*||.||_{H,\mathrm{can}}}}\downarrow & & \downarrow{\scriptstyle f_\psi} \\ \mathbb{R}_{>0} & \xrightarrow[\sim]{-\log} & \mathbb{R} \end{array}$$

commutes in the category of topological spaces. Furthermore, this bijection is an extension of the bijection in Proposition 5.4.1, and finally it becomes an isomorphism of metric spaces with respect to the distances of uniform convergence in Definition 4.1.3 and Definition 2.4.15.

Proof. By Lemma 5.4.4 we have a bijective function $||.||_\psi \mapsto \overline{\psi}$ between the set of toric metrics on L and the set of morphism of topological spaces $\overline{\psi} \colon N_\mathbb{R}/\Lambda \to \mathbb{R}$. By Lemma 5.3.4 we have a bijective function $f_\psi \mapsto \overline{\psi}$ between the set of morphisms of topological spaces $f_\psi \colon N_\mathbb{R}/\Lambda \to \mathbb{R}$ fulfilling the cocycle rule and the set of morphism of topological spaces $\overline{\psi} \colon N_\mathbb{R}/\Lambda \to \mathbb{R}$. Therefore, the function $||.||_\psi \mapsto f_\psi$ is also bijective.

Our strategy is to prove at first the case of a canonical metric. For this purpose, we will do an approximation by formal metrics associated to a line bundle on a Mumford model (a variation of Tate's limit argument, see [Gub10, Example 3.7]). This allows us to apply the metric-function correspondence for formal metrics (see Proposition 5.4.1). Consider first the case L even. Let $\overline{\mathscr{C}}$ be a polytopal decomposition of $N_{\mathbb{R}}/\Lambda$ and $\mathscr{L}$ be a formal $\mathbb{K}^{\circ}$-model of L determined on $\mathscr{A}_{\overline{\mathscr{C}}}$ which is trivial on the admissible formal affine $\mathbb{K}^{\circ}$-schemes $\mathscr{U}_{i,\overline{\Delta}}$ for each $i \in I$, $\overline{\Delta} \in \overline{\mathscr{C}}$. Then, for any $m \in \mathbb{Z}\backslash\{0\}$, we get by Proposition 5.4.1 and due to the equality $f_{[m]^*\mathscr{L}}(.) = f_{\mathscr{L}}(m\,\cdot .)$ on $N_{\mathbb{R}}$ the commutative diagram

$$\begin{array}{ccc} E^{\mathrm{an}} & \xrightarrow{\mathrm{trop}} & N_{\mathbb{R}} \\ {\scriptstyle \frac{p^*||.||_{[m]^*\mathscr{L}}^{\otimes 1/m^2}}{q^*||.||_{[m]^*\mathscr{H}}^{\otimes 1/m^2}}}\downarrow & & \downarrow{\scriptstyle \frac{1}{m^2} f_{\mathscr{L}}(m\cdot .)} \\ \mathbb{R}_{>0} & \xrightarrow[\sim]{-\log} & \mathbb{R} \end{array}$$

in the category of topological spaces, where we refer to Lemma 5.2.11 for the definition of the morphism of admissible formal $\mathbb{K}^{\circ}$-schemes $[m]\colon \mathscr{A}_{\overline{m^{-1}\mathscr{C}}} \to \mathscr{A}_{\overline{\mathscr{C}}}$. The morphism of formal group $\mathbb{K}^{\circ}$-schemes $[m]\colon \mathscr{B} \to \mathscr{B}$ is canonical. By Lemma 4.1.15 for the left-hand side and by Lemma 5.3.5 in combination with Lemma 5.3.11(ii) for the right-hand side, taking the uniform limit $m \to +\infty$ yields the commutative diagram

$$\begin{array}{ccc} E^{\mathrm{an}} & \xrightarrow{\mathrm{trop}} & N_{\mathbb{R}} \\ {\scriptstyle \frac{p^*||.||_{L,\mathrm{can}}}{q^*||.||_{H,\mathrm{can}}}}\downarrow & & \downarrow{\scriptstyle z} \\ \mathbb{R}_{>0} & \xrightarrow[\sim]{-\log} & \mathbb{R} \end{array}$$

in the category of topological spaces. After replacing m^2 by m, the same holds for L odd. Combining both cases, we get the last commutative diagram for any line bundle L. We can modify this diagram for any morphism of topological spaces $\overline{\psi}\colon N_{\mathbb{R}}/\Lambda \to \mathbb{R}$, to the commutative diagram in the category of topological spaces

$$\begin{array}{ccc} E^{\mathrm{an}} & \xrightarrow{\mathrm{trop}} & N_{\mathbb{R}} \\ {\scriptstyle e^{-\psi\circ\mathrm{trop}}\frac{p^*||.||_{L,\mathrm{can}}}{q^*||.||_{H,\mathrm{can}}}}\downarrow & & \downarrow{\scriptstyle f_{\psi}} \\ \mathbb{R}_{>0} & \xrightarrow[\sim]{-\log} & \mathbb{R}. \end{array}$$

Finally this bijection is an extension of Proposition 5.4.1 because of the equality $||.||_{H,\mathrm{can}} = ||.||_{\mathscr{H}}$ of metrics on H, and it is furthermore an isomorphism of metric spaces, which follows by comparing Definition 4.1.3 with Definition 2.4.15 when using the above commutative diagram. □

Remark 5.4.10. According to Lemma 5.3.5, Lemma 5.4.8, and Proposition 5.4.9, the $\mathbb{R}$-quadratic function z has a distinguished role in the set of continuous functions fulfilling the cocycle role analogous to the canonical metric. One might call the function therefore canonical.

Theorem 5.4.11. *Choose a rigidification ρ_L on L. The metric $||.||_\psi$ is semi-positive if the corresponding function f_ψ (see Proposition 5.4.9) is convex. If L is ample, the converse is also true.*

Proof. [GS22, Theorem 4.10] □

5.5 Generic and transversal polytopal decompositions

The main part in the proof of [Gub10, Theorem 6.7], which we will generalize in Theorem C, is the degree formula in [Gub10, Proposition 5.18], which requires a certain transversality assumption. The goal of this section is to construct a Γ-rational polytopal decomposition $\overline{\mathscr{C}}$ of $N_{\mathbb{R}}/\Lambda$ which fulfils this transversality assumption.

Again, we will start with a completely combinatorial part. Until Proposition 5.5.5 we may assume that M is just an arbitrary free abelian group of rank $n \in \mathbb{N}$, that Λ is an arbitrary complete lattice in $N_{\mathbb{R}} := \hom_{\mathbb{R}-\mathbf{Vec}}(M, \mathbb{R})$, and that $\Gamma \subseteq \mathbb{R}$ is an arbitrary $\mathbb{Q}$-vector subspace. Afterwards we will assume that the first two are induced by an abelian $\mathbb{K}$-variety A, the latter is the value group of an algebraically closed non-trivially valued non-archimedean field, and we let L be a line bundle on A.

Let $\overline{\Sigma}$ be a finite set of polytopes in $N_{\mathbb{R}}/\Lambda$.

Definition 5.5.1. Let Δ_1, Δ_2 be two polyhedrons in $N_{\mathbb{R}}$. Then the *expected dimension* of Δ_1 and Δ_2 is defined to be the integer

$$D(\Delta_1, \Delta_2) := \dim_{\mathbb{R}}(\Delta_1) + \dim_{\mathbb{R}}(\Delta_2) - n.$$

Definition 5.5.2 ([Gub07a, 8.1])**.** A polytopal decomposition $\overline{\mathscr{C}}$ of $N_{\mathbb{R}}/\Lambda$ is called $\overline{\Sigma}$-*generic* if the following conditions hold for every $\overline{\sigma} \in \overline{\Sigma}$, $\overline{\Delta} \in \overline{\mathscr{C}}$ and all corresponding polytopes $\sigma \subseteq N_{\mathbb{R}}$, $\Delta \subseteq N_{\mathbb{R}}$ (see Definition 2.4.5):

(i) If $D(\sigma, \Delta) \geq 0$, then $\dim_{\mathbb{R}}(\mathbb{A}_\sigma \cap \mathbb{A}_\Delta) = D(\sigma, \Delta)$.

(ii) If $D(\sigma, \Delta) < 0$, then $\mathbb{A}_\sigma \cap \mathbb{A}_\Delta = \varnothing$.

Definition 5.5.3 ([Gub07a, 8.1])**.** A polytopal decomposition $\overline{\mathscr{C}}$ of $N_{\mathbb{R}}/\Lambda$ is called $\overline{\Sigma}$-*transversal* if one of the following two conditions hold for every $\overline{\sigma} \in \overline{\Sigma}$, $\overline{\Delta} \in \overline{\mathscr{C}}$ and all corresponding polytopes $\sigma \subseteq N_{\mathbb{R}}$, $\Delta \subseteq N_{\mathbb{R}}$ (see Definition 2.4.5):

(i) $\dim_{\mathbb{R}}(\sigma \cap \Delta) = D(\sigma, \Delta)$

(ii) $\sigma \cap \Delta = \varnothing$

Lemma 5.5.4. *If for every $\overline{\sigma} \in \overline{\Sigma}$ all of its closed faces are also in $\overline{\Sigma}$, then every $\overline{\Sigma}$-generic polytopal decomposition $\overline{\mathscr{C}}$ of $N_{\mathbb{R}}/\Lambda$ is $\overline{\Sigma}$-transversal.*

Proof. [Gub07a, Proposition 8.2] □

Proposition 5.5.5. *Let Γ be a $\mathbb{Q}$-vector subspace of $\mathbb{R}$, and $(z_\lambda \colon N_{\mathbb{R}} \to \mathbb{R})_{\lambda \in \Lambda}$ be a Λ-cocycle such that the $\mathbb{R}$-quadratic function $z \colon N_{\mathbb{R}} \to \mathbb{R}$ is strictly convex. Let b respectively l be the associated $\mathbb{R}$-bilinear respectively $\mathbb{R}$-linear form of z (see Definition 2.6.2, Definition 2.6.4), and assume that $b(., \lambda) \in M$, $b(\lambda, \lambda) \in \Gamma$, $l(\lambda) \in \Gamma$ for all $\lambda \in \Lambda$. Then, for any convex function f_ψ fulfilling the cocycle rule, there exists a sequence $(\overline{\mathscr{C}_k})_{k \in \mathbb{N}}$ of Γ-rational polytopal decompositions of $N_{\mathbb{R}}/\Lambda$, a sequence $(N_k)_{k \in \mathbb{N}}$ in $\mathbb{N}\backslash\{0\}$, and a sequence $(f_k)_{k \in \mathbb{N}}$ of strongly $\mathscr{C}_k$-polyhedral convex functions fulfilling the cocycle rule and converging uniformly to f_ψ, such that for all $k \in \mathbb{N}$ the following holds:*

(i) The strongly $\mathscr{C}_k$-polyhedral convex function $N_k f_k$ is Γ-rational.

(ii) For all $m \in \mathbb{Z}\backslash\{0\}$ the polytopal decomposition $\overline{m^{-1}\mathscr{C}_k}$ of $N_{\mathbb{R}}/\Lambda$ is $\overline{\Sigma}$-generic.

Proof. **Step I:** We may assume f_ψ strongly polyhedral convex.

Since the necessary assumptions for the purely combinatorial proof of [BGJK21, Proposition 8.2.6] are also provided here, the result is applicable in our situation. Consequently, the function f_ψ is the uniform limit of a sequence $(f_k)_{k\in\mathbb{N}}$ of $\mathscr{C}_k$-polyhedral convex functions fulfilling the cocycle rule, where $\mathscr{C}_k$ is a Λ-periodic (see Definition 2.4.6) polytopal decomposition of $N_\mathbb{R}$. Combining this with Lemma 5.3.6, we get that the function f_ψ is the uniform limit of a sequence $(f_k)_{k\in\mathbb{N}}$ of strongly $\mathscr{C}_k$-polyhedral convex functions fulfilling the cocycle rule, where $\overline{\mathscr{C}_k}$ is a polytopal decomposition of $N_\mathbb{R}/\Lambda$. We may therefore assume that there exists a polytopal decomposition $\overline{\mathscr{C}}$ of $N_\mathbb{R}/\Lambda$ such that f_ψ is strongly $\mathscr{C}$-polyhedral convex.

Step II: Construction of $(\overline{\mathscr{C}_k})_{k\in\mathbb{N}}$, $(N_k)_{k\in\mathbb{N}}$, and $(f_k)_{k\in\mathbb{N}}$ + Property (i): $N_k f_k$ is Γ-rational.

Step II.1: f_ψ as the maximum of $\mathbb{R}$-affine functions

Denote by $\mathscr{N}$ the subset of n-dimensional polytopes of $\mathscr{C}$, and its elements by η instead of Δ like in Definition 2.4.13. They generate the polytopal complex $\mathscr{C}$ by taking intersections. Since f_ψ is $\mathscr{C}$-polyhedral, for any $\eta \in \mathscr{N}$ there exist $m_\eta \in M_\mathbb{R}$ and $c_\eta \in \mathbb{R}$ such that for the definition

$$\begin{aligned} A_\eta \colon N_\mathbb{R} &\to \mathbb{R}, \\ u &\mapsto \langle m_\eta, u\rangle - c_\eta \end{aligned}$$

the functions f_ψ and A_η coincide on $\eta \subseteq N_\mathbb{R}$. As f_ψ is $\mathscr{C}$-polyhedral and convex, we get the equality $f_\psi = \max_{\eta\in\mathscr{N}} A_\eta$ of functions.

Step II.2: Choice of representatives of the set $\mathscr{N}$

Since $\overline{\mathscr{C}}$ is a polytopal decomposition of $N_\mathbb{R}/\Lambda$, there exists a finite subset $\mathscr{N}_0 \subseteq \mathscr{N}$ such that for any $\eta \in \mathscr{N}$ there exists a unique $\eta_0 \in \mathscr{N}_0$ and a unique $\lambda \in \Lambda$ such that $\eta = \eta_0 + \lambda$ as subsets of $N_\mathbb{R}$. This defines a function

$$\begin{aligned} \omega \colon \mathscr{N} &\to \mathscr{N}_0 \times \Lambda, \\ \eta &\mapsto (\eta_0, \lambda) \end{aligned} \tag{5.5}$$

and we denote the projections onto the factors in the category of sets by ω_1 and ω_2.

Step II.3: The definition of the function f_ψ on all representatives $\eta_0 \in \mathscr{N}_0$ determines the whole function f_ψ.

Let b be the associated $\mathbb{R}$-bilinear form and l be the associated $\mathbb{R}$-linear form of the $\mathbb{R}$-quadratic function z (see Definition 2.6.2, Definition 2.6.4, Lemma 5.3.2(i)). The $\mathbb{R}$-bilinear form b is positive definite because the $\mathbb{R}$-quadratic function z is strictly convex. Since the function f_ψ fulfils the cocycle rule, we get for all $\eta \in \mathscr{N}$ and $u \in \eta \subseteq N_\mathbb{R}$ the equality

$$A_{\eta+\lambda}(\lambda + u) = z_\lambda(u) + A_\eta(u), \tag{5.6}$$

which is equivalent to the equality

$$\underbrace{\langle m_{\eta+\lambda}, \lambda + u\rangle - \langle m_\eta, u\rangle}_{=\langle m_{\eta+\lambda} - m_\eta, u\rangle + \langle m_{\eta+\lambda}, \lambda\rangle} - c_{\eta+\lambda} + c_\eta = \underbrace{z_\lambda(u)}_{=z(\lambda)+b(u,\lambda) \text{ by Lemma 5.3.2(ii)}}. \tag{5.7}$$

Note that these equalities are true for $u \in \eta \subseteq N_\mathbb{R}$ if and only if they are true for $u \in N_\mathbb{R}$, because the functions are $\mathbb{R}$-affine and the polytope η is n-dimensional. Using the equality $z(\lambda) = \frac{1}{2}(b(\lambda, \lambda) + l(\lambda))$ (see Lemma 2.6.3), the equation (5.7) is again equivalent to the equalities

$$m_{\eta+\lambda} = m_\eta + \underbrace{b(., \lambda)}_{\in M} \quad \text{and} \quad c_{\eta+\lambda} = c_\eta + \underbrace{\langle m_\eta, \lambda\rangle + \tfrac{1}{2}b(\lambda, \lambda) - \tfrac{1}{2}l(\lambda)}_{\in \Gamma \text{ if } m_\eta \in M_\mathbb{Q}}, \tag{5.8}$$

where we have $b(., \lambda) \in M$ by assumption and where on top of that we have used for $\frac{1}{2}b(\lambda, \lambda), \frac{1}{2}l(\lambda) \in \Gamma$ that the subset $\Gamma \subseteq \mathbb{R}$ is a $\mathbb{Q}$-vector subspace. Note that $b(., \lambda) \neq 0$ for $\lambda \neq 0$ because the $\mathbb{R}$-bilinear form b is positive definite. By (5.8) the function f_ψ is already determined by all elements $m_{\eta_0} \in M_\mathbb{R}$ and $c_{\eta_0} \in \mathbb{R}$ with $\eta_0 \in \mathscr{N}_0$.

Step II.4: Rational approximation of f_ψ by strongly $\mathscr{C}_k$-polyhedral convex functions f_k fulfilling the cocycle rule, such that $\mathscr{C}_k$ induces a Γ-rational polytopal decomposition $\overline{\mathscr{C}_k}$ of $N_\mathbb{R}/\Lambda$

Take arbitrary sequences

$$\underbrace{m_{\eta_0,k}}_{\in M_\mathbb{Q}} \xrightarrow[k\to+\infty]{k\in\mathbb{N}} m_{\eta_0} \quad \text{and} \quad \underbrace{c_{\eta_0,k}}_{\in\Gamma} \xrightarrow[k\to+\infty]{k\in\mathbb{N}} c_{\eta_0} \tag{5.9}$$

for all $\eta_0 \in \mathscr{N}_0$, which is possible because $\mathbb{Q}$ is dense in $\mathbb{R}$ in combination with the fact that the subset $\Gamma \subseteq \mathbb{R}$ is a $\mathbb{Q}$-vector subspace. Define for all

$k \in \mathbb{N}$, $\eta \in \mathscr{N} \setminus \mathscr{N}_0$ elements $m_{\eta,k} \in M_{\mathbb{Q}}$ and $c_{\eta,k} \in \Gamma$ by using the equalities (5.8), and define the functions

$$A_{\eta,k} \colon N_{\mathbb{R}} \to \mathbb{R}, \quad u \mapsto \langle m_{\eta,k}, u \rangle - c_{\eta,k} \qquad \text{and} \qquad f_k := \sup_{\eta \in \mathscr{N}} A_{\eta,k}.$$

In Step II.3, the equalities (5.8) are equivalent to the equality (5.6). The analogue holds for $A_{\eta,k}$ instead of A_η. Consequently, for any $k \in \mathbb{N}$ the function $f_k \colon N_{\mathbb{R}} \to \mathbb{R}$ fulfils the cocycle rule.
For f_k being $\mathscr{C}_k$-polyhedral, we have to prove that in the definition of f_k we may replace the supremum by the maximum. It suffices to check the case $|\mathscr{N}_0| = 1$ because the set $\mathscr{N}_0$ is finite and for finitely many functions its maximum is well-defined. Let $u \in N_{\mathbb{R}}$ and let η_0 be the only element of the set $\mathscr{N}_0$. Then

$$\begin{aligned} f_k(u) &= \sup_{\lambda \in \Lambda} A_{\eta_0+\lambda,k}(u) \\ &= \sup_{\lambda \in \Lambda} \left(\langle m_{\eta_0+\lambda,k}, u \rangle - c_{\eta_0+\lambda,k} \right) \\ &= \sup_{\lambda \in \Lambda} \left(\langle m_{\eta_0,k}, u \rangle + b(u,\lambda) - c_{\eta_0,k} - \langle m_{\eta_0,k}, \lambda \rangle - \tfrac{1}{2} b(\lambda,\lambda) + \tfrac{1}{2} l(\lambda) \right). \end{aligned}$$

Since the $\mathbb{R}$-bilinear form b is positive definite, since its coefficient $-\frac{1}{2}$ in the term is negative, since all the other terms are at most linear in the variable λ, and since the subset $\Lambda \subseteq N_{\mathbb{R}}$ is discrete, we can replace the supremum with the maximum and get consequently $f_k = \max_{\eta \in \mathscr{N}} A_{\eta,k}$.
Furthermore, by the same arguments we get $f_k(u) \xrightarrow{k \to +\infty} f_\psi(u)$ for all $u \in N_{\mathbb{R}}$. This pointwise convergence together with the fact that the functions f_k are convex (given by a maximum of $\mathbb{R}$-affine functions) provide uniform convergence.
For all $k \in \mathbb{N}$ and $\eta \in \mathscr{N}$ the subsets

$$\begin{aligned} \Delta_{\eta,k} &:= \{ u \in N_{\mathbb{R}} \mid A_{\eta,k}(u) = f_k(u) \} \\ &= \{ u \in N_{\mathbb{R}} \mid A_{\eta,k}(u) \geq A_{\eta',k}(u) \ \forall \eta' \in \mathscr{N} \} \\ &= \{ u \in N_{\mathbb{R}} \mid \langle m_{\eta,k} - m_{\eta',k}, u \rangle - c_{\eta,k} + c_{\eta',k} \geq 0 \ \forall \eta' \in \mathscr{N} \} \end{aligned}$$

of $N_{\mathbb{R}}$ are Γ-rational polyhedrons (the empty set is also possible), because of the equality $f_k = \max_{\eta \in \mathscr{N}} A_{\eta,k}$. Since f_ψ is strongly $\mathscr{C}$-polyhedral convex by Step I, there exists a $k' \in \mathbb{N}$ such that for all $k \geq k'$ the sets $\Delta_{\eta,k}$

are n-dimensional Γ-rational polytopes, generate a Γ-rational polytopal decomposition $\mathscr{C}_k$ of $N_{\mathbb{R}}$ and f_k is strongly $\mathscr{C}_k$-polyhedral convex. We may assume $k' = 0$.
Since $A_{\eta,k}$ fulfils the analogue of equality (5.6) and since f_k fulfils the cocycle rule, for any $u \in N_{\mathbb{R}}$, $\lambda \in \Lambda$ the equality $f_k(u+\lambda) = A_{\eta+\lambda,k}(u+\lambda)$ is equivalent to the equality $f_k(u) = A_{\eta,k}(u)$. This shows for all $\lambda \in \Lambda$ the equality $\Delta_{\eta,k} + \lambda = \Delta_{\eta+\lambda,k}$ of subsets of $N_{\mathbb{R}}$, which means that $\mathscr{C}_k$ is Λ-periodic. Since $\mathscr{C}$ induces $\overline{\mathscr{C}}$, the same holds for $\mathscr{C}_k$ because of the limit $\Delta_{\eta,k} \xrightarrow[k\to+\infty]{k\in\mathbb{N}} \eta$ with respect to the Hausdorff distance, provided by the convergences (5.9) and f_ψ being strongly $\mathscr{C}$-polyhedral convex (see Step I).

Step II.5: Property (i): There exists $N_k \in \mathbb{N}\backslash\{0\}$ such that $N_k f_k$ is additionally Γ-rational.

For any $k \in \mathbb{N}$ let $N_k \in \mathbb{N}\backslash\{0\}$ be the minimal element such that for all $\eta_0 \in \mathscr{N}_0$ we have $N_k m_{\eta_0,k} \in M$, which exists because the set $\mathscr{N}_0$ is finite. Since $b(.,\lambda) \in M$ (see Step II.3), we have $N_k m_{\eta,k} \in M$ for all $\eta \in \mathscr{N}$.

Step III: Property (ii): $\overline{m^{-1}\mathscr{C}_k}$ may be assumed to be $\overline{\Sigma}$-generic.

Step III.1: There exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that the following holds: For any convergent sequences $(m_{\eta,k})_{k\in\mathbb{N}}$ and $(c_{\eta,k})_{k\in\mathbb{N}}$ (5.9) we may assume that for any $k \in \mathbb{N}$, $\eta, \check{\eta} \in \mathscr{N}$ the property $\Delta_{\eta,k} \cap \Delta_{\check{\eta},k} \neq \varnothing$ implies

$$\omega_2(\check{\eta}) - \omega_2(\eta) \in \Lambda_0, \tag{5.10}$$

where we have used the second projection ω_2 of the function (5.5).

Proof of Step III.1: Step III.1.a: The statement of Step III.1 holds true for $\overline{\mathscr{C}}$ from Step I instead of $\overline{\mathscr{C}_k}$.

Proof of Step III.1.a: Let $\eta_0 \in \mathscr{N}_0$, which implies $\omega_2(\eta_0) = 0$. Since η_0 is a polytope, there are just finitely many other polytopes $\check{\eta}_0 \in \mathscr{N}$ having non-empty intersection with η_0. By adding for all these polytopes the elements $\omega_2(\check{\eta}_0)$ to the set Λ_0, we provide the desired property for the choice of η_0 because of $\omega_2(\eta_0) = 0$. After doing this process for all possible choices of $\eta_0 \in \mathscr{N}_0$, the set Λ_0 is still finite because the set $\mathscr{N}_0$ is finite. The statement extends to all $\eta \in \mathscr{N}$ by the Λ-periodicity of $\mathscr{C}$.

Step III.1.b: The statement of Step III.1 holds also true for sufficiently good approximations $\mathscr{C}_k$ of $\mathscr{C}$.

Proof of Step III.1.b: As we have seen in Step II.4, the sequence of polytopes $(\Delta_{\eta,k})_{k\in\mathbb{N}}$ of $\mathscr{C}_k$ converges to the polytope $\eta \in \mathscr{N}$ with respect to the Hausdorff distance. Let $\eta_0 \in \mathscr{N}_0$, and $\check{\eta} \in \mathscr{N}$ such that $\eta_0 \cap \check{\eta} = \varnothing$. Since both subsets of $N_\mathbb{R}$ are compact, there exists a $k'(\eta_0, \check{\eta}) \in \mathbb{N}$ such that for all $k \geq k'(\eta_0, \check{\eta})$ we have $\Delta_{\eta_0,k} \cap \Delta_{\check{\eta},k} = \varnothing$. Define $k' := \max_{\eta_0\in\mathscr{N}_0} \sup_{\check{\eta}\in\mathscr{N}} k'(\eta_0, \check{\eta})$. By the compactness of the polytopes and the Λ-periodicity of the $\mathscr{C}_k$, the supremum can be replaced by a maximum and hence $k' \in \mathbb{N}$. Again, we may assume $k' = 0$. Furthermore, the Λ-periodicity of the $\mathscr{C}_k$ extends the implication from all $\eta_0 \in \mathscr{N}_0$ to all $\eta \in \mathscr{N}$. Contraposition of this implication shows that the property $\Delta_{\eta,k} \cap \Delta_{\check{\eta},k} \neq \varnothing$ implies $\eta \cap \check{\eta} \neq \varnothing$, which again by Step III.1.a implies $\omega_2(\check{\eta}) - \omega_2(\eta) \in \Lambda_0$ (5.10).

Step III.2: There exist a finite subset $\Sigma_0 \subseteq \Sigma$, finite sets $(I_{\sigma_0})_{\sigma_0\in\Sigma_0}$, elements $(m_{\sigma_0,i})_{\sigma_0\in\Sigma_0,i\in I_{\sigma_0}}$ in $M_\mathbb{R}$, and elements $(c_{\sigma_0,i})_{\sigma_0\in\Sigma_0,i\in I_{\sigma_0}}$ in $\mathbb{R}$ such that every element $\sigma \in \Sigma$ has a unique representation $\sigma = \sigma_0 + \lambda_\sigma$ with $\sigma_0 \in \Sigma_0$, $\lambda_\sigma \in \Lambda$, and we get

$$\mathbb{A}_\sigma = \bigcap_{i\in I_{\sigma_0}} H_=(m_{\sigma_0,i}, c_{\sigma_0,i} + \langle m_{\sigma_0,i}, \lambda_\sigma\rangle) \tag{5.11}$$

and $\dim_\mathbb{R}(\mathbb{A}_\sigma) = n - |I_{\sigma_0}|$.

Proof of Step III.2: Step III.2.a: There exists a finite subset $\Sigma_0 \subseteq \Sigma$ serving as a representation system.

Proof of Step III.2.a: Since the set $\overline{\Sigma}$ is finite, we can choose a finite subset $\Sigma_0 \subseteq \Sigma$ such that for all $\overline{\sigma} \in \overline{\Sigma}$ there exists exactly one $\sigma_0 \in \Sigma_0$ such that $\lambda(\sigma_0) = \overline{\sigma}$ where $\lambda\colon N_\mathbb{R} \twoheadrightarrow N_\mathbb{R}/\Lambda$ is the quotient function.

Step III.2.b: The case $\lambda_\sigma = 0$

Proof of Step III.2.b: By definition, for any polytope $\sigma_0 \in \Sigma_0$ there exists a finite set J_{σ_0} such that for all $i \in J_{\sigma_0}$ there exist elements $m_{\sigma_0,i} \in M_\mathbb{R}$, $c_{\sigma_0,i} \in \mathbb{R}$ with $\sigma_0 = \bigcap_{i\in J_{\sigma_0}} H_\geq(m_{\sigma_0,i}, c_{\sigma_0,i})$. Since $\mathbb{A}_{\sigma_0}$ is the smallest $\mathbb{R}$-affine subspace of $N_\mathbb{R}$ which contains the polytope σ_0, there exists a subset

$I_{\sigma_0} \subseteq J_{\sigma_0}$ such that we get $\mathbb{A}_{\sigma_0} = \bigcap_{i \in I_{\sigma_0}} H_=(m_{\sigma_0,i}, c_{\sigma_0,i})$ and $\dim_{\mathbb{R}}(\mathbb{A}_{\sigma_0}) = n - |I_{\sigma_0}|$.

Step III.2.c: The case $\lambda_\sigma \neq 0$

Proof of Step III.2.c: Let $\sigma \in \Sigma$ with its unique representation $\sigma = \sigma_0 + \lambda_\sigma$ with $\sigma_0 \in \Sigma_0$, $\lambda_\sigma \in \Lambda$, which induces the equality $\mathbb{A}_\sigma = \mathbb{A}_{\sigma_0} + \lambda_\sigma$ of subsets of $N_{\mathbb{R}}$. This equality, the equality $\mathbb{A}_{\sigma_0} = \bigcap_{i \in I_{\sigma_0}} H_=(m_{\sigma_0,i}, c_{\sigma_0,i})$ from Step III.2.b, and the equality $H_=(m_{\sigma_0,i}, c_{\sigma_0,i}) + \lambda_\sigma = H_=(m_{\sigma_0,i}, c_{\sigma_0,i} + \langle m_{\sigma_0,i}, \lambda_\sigma \rangle)$ for all $i \in J_{\sigma_0}$ show equation (5.11). The equality $\dim_{\mathbb{R}}(\mathbb{A}_\sigma) = \dim_{\mathbb{R}}(\mathbb{A}_{\sigma_0}) = n - |I_{\sigma_0}|$ completes the proof of Step III.2.

In the following two steps we will prove that there exist choices of the convergent sequences (5.9) fulfilling some properties (iii) and (iv), and afterwards we will check in Step III.5 that these properties provide that the polytopal decompositions $\overline{m^{-1}\mathscr{C}_k}$ are $\overline{\Sigma}$-generic for all $m \in \mathbb{Z}\backslash\{0\}$ and $k \in \mathbb{N}$:

Step III.3: For any $\eta_0 \in \mathscr{N}_0$ there exists a choice of the convergent sequence

$$\underbrace{m_{\eta_0,k}}_{\in M_{\mathbb{Q}}} \xrightarrow[k \to +\infty]{k \in \mathbb{N}} m_{\eta_0}$$

fulfilling the following property:

(iii) For any $k \in \mathbb{N}$, for any $\lambda \in \Lambda$, for any $\sigma \in \Sigma$, for any $S \subseteq \mathscr{N}_0$ with $|S| \leq n + 1 - |I_{\sigma_0}|$, for any $\eta_{00} \in S$, and for any function $\zeta \colon S \to \Lambda_0$, the elements

$$(m_i)_{i \in I_{\sigma_0}}, (\Delta m_{\eta_0})_{\eta_0 \in S \backslash \{\eta_{00}\}} \tag{5.12}$$

of $M_{\mathbb{R}}$ are $\mathbb{R}$-linearly independent, where we use for simplicity the notations $m_i := m_{\sigma_0,i}$ and

$$\Delta m_{\eta_0} := m_{\eta_0 + \lambda + \zeta(\eta_0),k} - m_{\eta_{00} + \lambda + \zeta(\eta_{00}),k}. \tag{5.13}$$

Proof of Step III.3: Let $||.||_{M_{\mathbb{R}}}$ be some norm on the n-dimensional $\mathbb{R}$-vector space $M_{\mathbb{R}}$. In order to provide the convergence of the sequence $(m_{\eta_0,k})_{k \in \mathbb{N}}$ (5.9), for any $k \in \mathbb{N}$ and $\eta_0 \in \mathscr{N}_0$ we aim to choose $m_{\eta_0,k} \in M_{\mathbb{Q}}$ such that

the inequality $||m_{\eta_0,k} - m_{\eta_0}||_{M_\mathbb{R}} < \frac{1}{k}$ is true. Let $k \in \mathbb{N}$ and work with this fixed k in the rest of the proof of Step III.3.

Step III.3.a: We may assume $\lambda = 0$, $\lambda_\sigma = 0$ and η_{00} being the minimal element of S with respect to some fixed order on $\mathscr{N}_0$.

Proof of Step III.3.a: $\lambda = 0$: The elements Δm_{η_0} are independent of the choice of $\lambda \in \Lambda$ because the summand $b(.,\lambda)$ in $m_{\eta_0+\lambda,k} = m_{\eta_0,k} + b(.,\lambda)$ (5.8) vanishes in the difference $\Delta m_{\eta_0} = m_{\eta_0+\lambda+\zeta(\eta_0),k} - m_{\eta_{00}+\lambda+\zeta(\eta_{00}),k}$ (5.13).

$\lambda_\sigma = 0$: The translations $\mathbb{A}_\sigma = \mathbb{A}_{\sigma_0} + \lambda_\sigma$ do not affect linear independencies.

η_{00} being the minimal element: Since the elements $\Delta m_{\eta_0} = m_{\eta_0+\lambda+\zeta(\eta_0),k} - m_{\eta_{00}+\lambda+\zeta(\eta_{00}),k}$ (5.13) with $\eta_0 \in S\backslash\{\eta_{00}\}$ span the same $\mathbb{R}$-vector subspace of $M_\mathbb{R}$ for any $\eta_{00} \in S$, it suffices to show the claim for one $\eta_{00} \in S$.

Step III.3.b: Now we will do an induction over the ordered set $\mathscr{N}_0$. The goal in each induction step is to find a suitable definition of $m_{\eta_0,k} \in M_\mathbb{Q}$:

Induction claim: Let $\eta_{\text{now}} \in \mathscr{N}_0$. Then for all $\eta_0 \leq \eta_{\text{now}}$ there exist elements $m_{\eta_0,k} \in M_\mathbb{Q}$ fulfilling $||m_{\eta_0,k} - m_{\eta_0}||_{M_\mathbb{R}} < \frac{1}{k}$ and property (iii) under the additional requirement $S \subseteq \{\eta_0 \in \mathscr{N}_0 \mid \eta_0 \leq \eta_{\text{now}}\}$.

Induction start: Let η_{now} be the minimal element of $\mathscr{N}_0$. We then get either $S = \varnothing$ or $S = \{\eta_{\text{now}}\}$. In both cases there is nothing to prove apart from $||m_{\eta_{\text{now}},k} - m_{\eta_{\text{now}}}||_{M_\mathbb{R}} < \frac{1}{k}$, which can be fulfilled by the density of $\mathbb{Q}$ in $\mathbb{R}$.

Induction step: Let η_{now} be any element of $\mathscr{N}_0$ apart from the minimal one. Assume that the induction claim is true for all $\eta_0 < \eta_{\text{now}}$ in $\mathscr{N}_0$.

Step III.3.b.i: We may assume $\eta_{\text{now}} \in S$ and $\eta_{00} \neq \eta_{\text{now}}$.

Proof of Step III.3.b.i: If $\eta_{\text{now}} \notin S$, the induction claim follows already by the induction hypothesis. If it was $\eta_{00} = \eta_{\text{now}}$, the set $S\backslash\{\eta_{00}\}$ would be empty as η_{00} is the minimal and η_{now} is the maximal element of S. Then there would be nothing to prove.

Step III.3.b.ii: For any choice of σ_0, S and ζ, there exists a proper $\mathbb{R}$-affine subspace $f(\sigma_0, S, \zeta)$ of $M_\mathbb{R}$ such that the vectors $(m_i)_{i \in I_{\sigma_0}}$, $(\Delta m_{\eta_0})_{\eta_0 \in S\backslash\{\eta_{00}\}}$ (5.12) are $\mathbb{R}$-linearly independent if and only if $m_{\eta_{\text{now}},k} \notin f(\sigma_0, S, \zeta)$.

Proof of Step III.3.b.ii: By Step III.3.b.i, we have

$$|I_{\sigma_0}| + |S\backslash\{\eta_{\text{now}}, \eta_{00}\}| = |I_{\sigma_0}| + \underbrace{|S|}_{\leq n+1-|I_{\sigma_0}|} - 2 \leq n-1.$$

Since the induction hypothesis for $S\backslash\{\eta_{\text{now}}\}$ provides the $\mathbb{R}$-linear independence of the at most $n-1$ other vectors $(m_i)_{i\in I_{\sigma_0}}$ and $(\Delta m_{\eta_0})_{\eta_0\in S\backslash\{\eta_{\text{now}},\eta_{00}\}}$ (5.12), Step III.3.b.ii follows by

$$\Delta m_{\eta_{\text{now}}} = \underbrace{m_{\eta_{\text{now}}+\zeta(\eta_{\text{now}}),k}}_{=m_{\eta_{\text{now}},k}+b(.,\zeta(\eta_{\text{now}}))\text{ by (5.8)}} - m_{\eta_{00}+\zeta(\eta_{00}),k}.$$

Step III.3.b.iii: There exists a choice of $m_{\eta_{\text{now}},k} \in M_{\mathbb{Q}}$ fulfilling the inequality $||m_{\eta_{\text{now}},k} - m_{\eta_{\text{now}}}||_{M_{\mathbb{R}}} < \frac{1}{k}$ and property (iii).



Figure 5.3: Sketch of the set P in the case $n = 2$. The set F removes just finitely many lines and points.

Proof of Step III.3.b.iii: Step III.3.b.ii implies that $m_{\eta_{\text{now}},k} \in M_{\mathbb{Q}}$ fulfils property (iii) if and only if $m_{\eta_{\text{now}},k} \notin F := \bigcup_{\sigma_0,S,\zeta} f(\sigma_0, S, \zeta) \subseteq M_{\mathbb{R}}$. Then the set of possible choices $P := \big(B_{1/k}(m_{\eta_{\text{now}}}) \cap M_{\mathbb{Q}}\big)\backslash F$ for $m_{\eta_{\text{now}},k}$ is nonempty because the sets $f(\sigma_0, S, \zeta)$ are proper $\mathbb{R}$-affine subspaces of $M_{\mathbb{R}}$ by Step III.3.b.ii, and there are just finitely many choices for σ_0, S and ζ as the sets Σ_0, $\mathscr{N}_0$ and Λ_0 are all finite. See Figure 5.3 for a visualization. This completes the induction (Step III.3.b) and thus the proof of Step III.3.

Step III.4: After having fixed all choices of $(m_{\eta_0,k})_{k\in\mathbb{N}}$ (5.9), for any $\eta_0 \in \mathscr{N}_0$ there exists a choice of the convergent sequence

$$\underbrace{c_{\eta_0,k}}_{\in\Gamma} \xrightarrow[k\to+\infty]{k\in\mathbb{N}} c_{\eta_0}$$

fulfilling the following property:

(iv) For any $k \in \mathbb{N}$, for any $m \in \mathbb{Z}\backslash\{0\}$, for any $\lambda \in \Lambda$, for any $\sigma \in \Sigma$, for any $S \subseteq \mathscr{N}_0$ with $|S| > n + 1 - |I_{\sigma_0}|$, for any $\eta_{00} \in S$, and for any function $\zeta\colon S \to \Lambda_0$, the inhomogeneous system of linear equations

$$\begin{aligned} \langle m_i, u\rangle &= c_i && \text{for all } i \in I_{\sigma_0} \\ \langle \Delta m_{\eta_0}, u\rangle &= \tfrac{1}{m}\Delta c_{\eta_0} && \text{for all } \eta_0 \in S\backslash\{\eta_{00}\} \end{aligned} \tag{5.14}$$

is not solvable in $N_{\mathbb{R}}$, where we write for simplicity

$$c_i := c_{\sigma_0,i} + \langle m_{\sigma_0,i}, \lambda_\sigma\rangle \tag{5.15}$$

and

$$\Delta c_{\eta_0} := c_{\eta_0+\lambda+\zeta(\eta_0),k} - c_{\eta_{00}+\lambda+\zeta(\eta_{00}),k}. \tag{5.16}$$

Proof of Step III.4: In order to provide the convergence (5.9) we aim to choose $c_{\eta_0,k} \in \Gamma$ such that $|c_{\eta_0,k} - c_{\eta_0}|_{\mathbb{R}} < \frac{1}{k}$ holds for any $k \in \mathbb{N}$ and $\eta_0 \in \mathscr{N}_0$. Let $k \in \mathbb{N}$ and work with this fixed k in the rest of the proof of Step III.4.

Step III.4.a: We may assume η_{00} being the minimal element of S.

Proof of Step III.4.a: When replacing in (iv) the choice $\eta_{00} \in S$ by another choice $\eta'_{00} \in S$, to all linear equations corresponding to some $\eta_0 \in S\backslash\{\eta'_{00}\}$ the linear equation $\Delta m_{\eta_{00}} u = \frac{1}{m}\Delta c_{\eta_{00}}$ will be added, where $\Delta m_{\eta_{00}} \neq 0$ because of (iii) for $S = \{\eta_{00}, \eta'_{00}\}$ and $I_{\sigma_0} = \varnothing$. Since this linear equation is solvable, the replacement does not affect the solvability of the system of linear equations (5.14).

Step III.4.b: Now we will do an induction over the ordered set $\mathscr{N}_0$. The goal in each induction step is to find a suitable definition of $c_{\eta_0,k} \in \Gamma$:

Induction claim: Let $\eta_{\mathrm{now}} \in \mathscr{N}_0$. Then for all $\eta_0 \leq \eta_{\mathrm{now}}$ there exist elements $c_{\eta_0,k} \in \Gamma$ fulfilling the inequality $|c_{\eta_0,k} - c_{\eta_0}|_{\mathbb{R}} < \frac{1}{k}$ and property (iv) under the additional requirement $S \subseteq \{\eta_0 \in \mathscr{N}_0 \mid \eta_0 \leq \eta_{\mathrm{now}}\}$. (Because of the condition $|S| > n + 1 - |I_{\sigma_0}|$, this may become an empty statement for low-order η_{now}, which is by definition always true.)

Induction start: Let η_{now} be the minimal element of $\mathscr{N}_0$. We then get either $S = \varnothing$ or $S = \{\eta_{\mathrm{now}}\}$, but the inequality $|S| > n + 1 - |I_{\sigma_0}|$ demands at least $|S| > 1$. Hence we get an empty statement in property (iv) in both cases, and there is nothing to prove apart from $|c_{\eta_{\mathrm{now}},k} - c_{\eta_{\mathrm{now}}}|_{\mathbb{R}} < \frac{1}{k}$, which can be fulfilled because $\Gamma \subseteq \mathbb{R}$ is dense as it is a $\mathbb{Q}$-vector subspace.

Induction step: Let η_{now} be any element of $\mathscr{N}_0$ apart from the minimal one. Assume that the induction claim is true for all $\eta_0 < \eta_{\mathrm{now}}$.

Step III.4.b.i: We may assume $\eta_{\mathrm{now}} \in S$, $|S| = n+2-|I_{\sigma_0}|$, and $\eta_{00} \neq \eta_{\mathrm{now}}$.

Proof of Step III.4.b.i: $\eta_{\mathrm{now}} \in S$: If $\eta_{\mathrm{now}} \notin S$, the induction claim follows already by the induction hypothesis.

$|S| = n + 2 - |I_{\sigma_0}|$: If we have $|S| > n + 2 - |I_{\sigma_0}|$, then it follows $|S\backslash\{\eta_{\mathrm{now}}\}| > n + 1 - |I_{\sigma_0}|$ and the induction claim follows again by the induction hypothesis for $S\backslash\{\eta_{\mathrm{now}}\}$.

$\eta_{00} \neq \eta_{\mathrm{now}}$: Because of $|S| = n+2-|I_{\sigma_0}| \geq 2$, the minimal element η_{00} and the maximal element η_{now} of the ordered subset $S \subseteq \mathscr{N}_0$ cannot coincide.

Terminology: From now on we fix choices of σ_0, S, ζ. By the word "constants" in the following we mean elements which may depend on the choice of (σ_0, S, ζ) (just finitely many possibilities) but are independent of the choice of $(m, \lambda, \lambda_\sigma) \in \mathbb{Z}\backslash\{0\} \times \Lambda \times \Lambda$ (infinitely many possibilities). We will mark dependencies on some of the latter choices in the following always in brackets.

Step III.4.b.ii: There exist constants $a_1, a_2 \in \mathbb{R}$ and $v_1, v_2 \in M_{\mathbb{R}}$ such that the system of linear equations (5.14) is unsolvable for a choice $(m, \lambda, \lambda_\sigma)$ if and only if $c_{\eta_{\mathrm{now}},k} \neq f(m, \lambda, \lambda_\sigma)$ where

$$f(m, \lambda, \lambda_\sigma) := a_1 + a_2 m + \langle v_1, \lambda \rangle + \langle v_2, m\lambda_\sigma \rangle \in \mathbb{R}.$$

Step III.4.b.ii.α: The system of linear equations (5.14) without the linear equation corresponding to $\eta_{\mathrm{now}} \in S$ has a unique solution $u(m, \lambda, \lambda_\sigma) \in N_{\mathbb{R}}$.

Proof of Step III.4.b.ii.α: By Step III.4.b.i, the system of linear equations (5.14) without the linear equation corresponding to $\eta_{\text{now}} \in S$ has

$$|I_{\sigma_0}| + |S \backslash \{\eta_{\text{now}}\}| - 1 = |I_{\sigma_0}| + \underbrace{|S|}_{=n+2-|I_{\sigma_0}|} - 2 = n$$

linear equations. Since its slopes are $\mathbb{R}$-linearly independent by property (iii), there exists a unique solution of the system of linear equations (5.14), denoted by $u(m, \lambda, \lambda_\sigma) \in N_{\mathbb{R}}$.

Step III.4.b.ii.β: There exist constants $a_3 \in \mathbb{R}$ and $v_3, v_4 \in M_{\mathbb{R}}$, such that the system of linear equations (5.14) is solvable if and only if we have $c_{\eta_{\text{now}},k} = f(m, \lambda, \lambda_\sigma)$ for $f(m, \lambda, \lambda_\sigma) := a_3 + \langle v_3, \lambda \rangle + m\langle v_4, u(m, \lambda, \lambda_\sigma) \rangle \in \mathbb{R}$.

Proof of Step III.4.b.ii.β: By Step III.4.b.ii.α, the system of linear equations (5.14) is solvable if and only if we have

$$\Delta c_{\eta_{\text{now}}} = m\langle \Delta m_{\eta_{\text{now}}}, u(m, \lambda, \lambda_\sigma) \rangle \,. \tag{5.17}$$

By definition (5.8), for any $\eta_0 \in S$ we have the equality $c_{\eta_0+\lambda+\zeta(\eta_0),k} = c_{\eta_0,k} + \langle m_{\eta_0,k}, \lambda + \zeta(\eta_0) \rangle + \frac{1}{2} b(\lambda + \zeta(\eta_0), \lambda + \zeta(\eta_0)) - \frac{1}{2} l(\lambda + \zeta(\eta_0))$. Since the summand $b(\lambda, \lambda)$ vanishes in the difference $\Delta c_{\eta_{\text{now}}} = c_{\eta_{\text{now}}+\lambda+\zeta(\eta_{\text{now}}),k} - c_{\eta_{00}+\lambda+\zeta(\eta_{00}),k}$ (5.16), there exist constants $a_3 \in \mathbb{R}$, $v_3 \in M_{\mathbb{R}}$ such that $\Delta c_{\eta_{\text{now}}} = c_{\eta_{\text{now}},k} - a_3 - \langle v_3, \lambda \rangle$. This, in combination with equation (5.17) for $v_4 := \Delta m_{\eta_{\text{now}}}$, which is independent of the choice of $(m, \lambda, \lambda_\sigma)$, proves Step III.4.b.ii.β.

Step III.4.b.ii.γ: There exist constants $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \hom_{\mathbb{R}-\mathbf{Vec}}(N_{\mathbb{R}}, N_{\mathbb{R}})$ and $v_5, v_6 \in N_{\mathbb{R}}$ such that we get $u(m, \lambda, \lambda_\sigma) = v_5 + \boldsymbol{A}_1 \lambda_\sigma + \frac{1}{m}(v_6 + \boldsymbol{A}_2 \lambda)$.

Proof of Step III.4.b.ii.γ: At first we have to check that there exist constants $a_4, a_5 \in \mathbb{R}$ and $v_7, v_8 \in M_{\mathbb{R}}$, such that the equalities $c_i(\lambda_\sigma) = a_4 + \langle v_7, \lambda_\sigma \rangle$ and $\Delta c_{\eta_0}(\lambda) = a_5 + \langle v_8, \lambda \rangle$ hold true. The first equality follows by the definition $c_i = c_{\sigma_0,i} + \langle m_{\sigma_0,i}, \lambda_\sigma \rangle$ (5.15) and the second by the definition $c_{\eta_0+\lambda,k} = c_{\eta_0,k} + \langle m_{\eta_0}, \lambda \rangle + \frac{1}{2} b(\lambda, \lambda) - \frac{1}{2} l(\lambda)$ (5.8), where it is important that the summand $\frac{1}{2} b(\lambda, \lambda)$ vanishes in the difference $\Delta c_{\eta_0} = c_{\eta_0+\lambda+\zeta(\eta_0),k} - c_{\eta_{00}+\lambda+\zeta(\eta_{00}),k}$ (5.16).
Considering the whole system of linear equations (5.14) without the linear equation corresponding to $\eta_{\text{now}} \in S$, these two equalities yield that there

exist constants $v_9, v_{10} \in \mathbb{R}^n$ and $\boldsymbol{A}_3, \boldsymbol{A}_4 \in \hom_{\mathbb{R}-\mathbf{Vec}}(N_\mathbb{R}, \mathbb{R}^n)$ such that in $\mathbb{R}^n$ we get the equality

$$c(m,\lambda,\lambda_\sigma) := \begin{pmatrix} (c_i)_{i\in I_{\sigma_0}} \\ (\frac{1}{m}\Delta c_{\eta_0})_{\eta_0 \in S\setminus\{\eta_{00},\eta_{\mathrm{now}}\}} \end{pmatrix} = v_9 + \boldsymbol{A}_3\lambda_\sigma + \tfrac{1}{m}(v_{10} + \boldsymbol{A}_4\lambda)\,.$$

Since the vectors m_i, Δm_{η_0} are independent of the choice $(m,\lambda,\lambda_\sigma)$, this behaviour of $c(m,\lambda,\lambda_\sigma)$ transfers to $u(m,\lambda,\lambda_\sigma)$ after multiplying with the inverse $\boldsymbol{M}^{-1} \in \hom_{\mathbb{R}-\mathbf{Vec}}(\mathbb{R}^n, N_\mathbb{R})$ of

$$\boldsymbol{M} := \begin{pmatrix} (m_i)_{i\in I_{\sigma_0}} \\ (\Delta m_{\eta_0})_{\eta_0 \in S\setminus\{\eta_{00},\eta_{\mathrm{now}}\}} \end{pmatrix} \in \hom_{\mathbb{R}-\mathbf{Vec}}(N_\mathbb{R}, \mathbb{R}^n),$$

which exists by property (iii).

Proof of Step III.4.b.ii: Combine Step III.4.b.ii.β and III.4.b.ii.γ.

Step III.4.b.iii: There exists a choice of $c_{\eta_0,k} \in \Gamma$ fulfilling the inequality $|c_{\eta_0,k} - c_{\eta_0}|_\mathbb{R} < \frac{1}{k}$ and property (iv).

Step III.4.b.iii.α: Define $F \subseteq \mathbb{R}$ to be the subset with the property that $c_{\eta_0,k} \in \mathbb{R}$ (later we will need $\in \Gamma$) fulfils property (iv) if and only if it does not lie in the subset $F \subseteq \mathbb{R}$. Then there exists a free $\mathbb{Z}$-submodule W of $\mathbb{R}$ of finite rank containing F and some $\gamma \in \Gamma\setminus\{0\}$.

Proof of Step III.4.b.iii.α: Step III.4.b.ii implies that the subset $F \subseteq \mathbb{R}$ is given by the set of all elements $f(m,\lambda,\lambda_\sigma)$ for all possible choices of (σ_0, S, ζ) and $(m,\lambda,\lambda_\sigma)$. Since $m \in \mathbb{Z}\setminus\{0\} \subseteq \mathbb{Z}$, since Λ is a lattice and hence finitely generated $\mathbb{Z}$-module in $N_\mathbb{R}$, and since there are only finitely many possible choices of (σ_0, S, ζ), Step III.4.b.ii provides the existence of a finitely generated $\mathbb{Z}$-submodule W of $\mathbb{R}$ containing F. By extension, one can assume that W is a free $\mathbb{Z}$-submodule of finite rank and contains some $\gamma \in \Gamma\setminus\{0\}$.

Step III.4.b.iii.β: There exists a $t \in \mathbb{N}\setminus\{0\}$ such that $F_\gamma := F \cap \mathbb{Q}\gamma \subseteq \mathbb{Z}\frac{\gamma}{t}$.

Proof of Step III.4.b.iii.β: Let $t \in \mathbb{N}\setminus\{0\}$ be the greatest common divisor of all coefficients of γ in W for some basis of W, which exists by Step III.4.b.iii.α. Then it follows $F_\gamma \subseteq \mathbb{Z}\frac{\gamma}{t}$. See Figure 5.4 for a visualization.

Step III.4.b.iii.γ: The subset $\Gamma\setminus F \subseteq \mathbb{R}$ is dense.

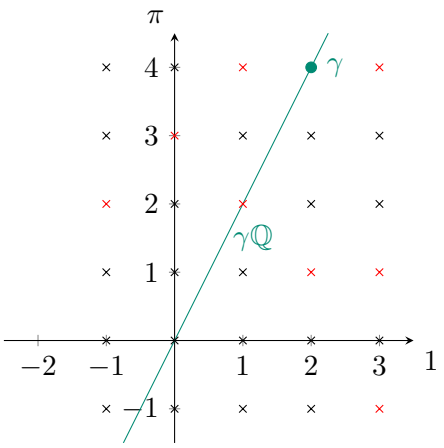


Figure 5.4: Sketch of the construction for $W = \mathbb{Z} + \mathbb{Z}\pi$. The elements of the set F are marked in red. Here we had $t = 2$ because $\gamma = \left(\begin{smallmatrix}2\\4\end{smallmatrix}\right) = 2\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)$.

Proof of Step III.4.b.iii.γ: Since $\gamma \in \Gamma\backslash\{0\}$, it follows $\mathbb{Q}\gamma \subseteq \Gamma$ because the subset $\Gamma \subseteq \mathbb{R}$ is a $\mathbb{Q}$-vector subspace of $\mathbb{R}$. By Step III.4.b.iii.β, we get $\big(\mathbb{Q}\backslash\mathbb{Z}\frac{1}{t}\big)\gamma \subseteq \Gamma\backslash F$, which is a dense subset of $\mathbb{R}$ because of $\gamma \neq 0$.

Proof of Step III.4.b.iii: Since $\Gamma\backslash F \subseteq \mathbb{R}$ is dense by Step III.4.b.iii.γ and because the intersection of an open ball with a dense subset is non-empty, the set of possible choices $P := \big(B_{1/k}(c_{\eta_{\mathrm{now}}}) \cap \Gamma\big)\backslash F = B_{1/k}(c_{\eta_{\mathrm{now}}}) \cap (\Gamma\backslash F)$ for $c_{\eta_{\mathrm{now}},k}$ is non-empty. This completes the induction (Step III.4.b) and hence the proof of Step III.4.

Step III.5: Property (iii) and (iv) imply property (ii).

Proof of Step III.5: In Step II and in Step III.1, we have shown that the convergent sequences $(m_{\eta_0,k})_{k\in\mathbb{N}}$ and $(c_{\eta_0,k})_{k\in\mathbb{N}}$ (5.9) yield a polytopal decomposition $\overline{\mathscr{C}}_k$ of $N_{\mathbb{R}}/\Lambda$ with the additional property: For any $\eta, \check{\eta} \in \mathscr{N}$ the property $\Delta_{\eta,k} \cap \Delta_{\check{\eta},k} \neq \varnothing$ implies $\omega_2(\check{\eta}) - \omega_2(\eta) \in \Lambda_0$ (5.10). Let $k \in \mathbb{N}$, $m \in \mathbb{Z}\backslash\{0\}$, $\sigma \in \Sigma$ and $\Delta \in \mathscr{C}_k$.

Step III.5.a: There exists an element $\lambda \in \Lambda$, a subset $S \subseteq \mathscr{N}_0$, an element $\eta_{00} \in S$, and a function $\zeta\colon S \to \Lambda_0$ such that

$$\mathbb{A}_{m^{-1}\Delta} = \bigcap_{\eta_0 \in S \setminus \{\eta_{00}\}} H_{=}\big(\Delta m_{\eta_0}, \tfrac{1}{m}\Delta c_{\eta_0}\big) \tag{5.18}$$

with $\dim_{\mathbb{R}}(\mathbb{A}_{m^{-1}\Delta}) = n - |S| + 1$.

Proof of Step III.5.a: At first Δ is the intersection of finitely many $\Delta_{\eta_0,k}$, which means that there exists a finite subset $\mathscr{N}' \subseteq \mathscr{N}$ such that we get $\Delta = \bigcap_{\eta' \in \mathscr{N}'} \Delta_{\eta',k}$. Since the relative interior of two different polytopes of $\mathscr{C}_k$ have empty intersection, we may assume the set $\mathscr{N}'$ to fulfil the equality

$$\dim_{\mathbb{R}}(\Delta) = n - |\mathscr{N}'| + 1. \tag{5.19}$$

We denote the image of the finite subset $\mathscr{N}' \subseteq \mathscr{N}$ under the function $\omega\colon \mathscr{N} \to \mathscr{N}_0 \times \Lambda$ (5.5) by Ω. Then this subset $\Omega \subseteq \mathscr{N}_0 \times \Lambda$ is again finite and it is by definition a binary relation. A binary relation is a subset of a product of two sets. Since $\Delta_{\eta',k} \cap \Delta_{\eta'',k} = \varnothing$ for all $\omega_1(\eta') = \omega_1(\eta'')$ with $\eta' \neq \eta''$ in $\mathscr{N}'$, the binary relation Ω is right unique. Choose some $\eta' \in \mathscr{N}'$ and define $\eta_{00} := \omega_1(\eta') \in \mathscr{N}_0$ and $\lambda := \omega_2(\eta') \in \Lambda$. Defining S to be the image of Ω under the first projection ω_1, we have $\Omega \subseteq S \times (\lambda + \Lambda_0)$ and Ω becomes a left total and right unique binary relation, which is by definition a function $\zeta'\colon S \to \lambda + \Lambda_0$. Defining the function $\zeta := \zeta' - \lambda\colon S \to \Lambda_0$, we get the equality

$$\Omega = \{(\eta_0, \lambda + \zeta(\eta_0)) \mid \eta_0 \in S\} \tag{5.20}$$

of subsets of $\mathscr{N}_0 \times \Lambda$.
The $\mathbb{R}$-affine subspace $\mathbb{A}_\Delta$ of $N_{\mathbb{R}}$ is the set of all $u \in N_{\mathbb{R}}$ such that the values of the $\mathbb{R}$-affine functions $A_{\eta'',k}$ coincide at u for all $\eta'' \in \mathscr{N}'$, which means

$$\mathbb{A}_\Delta = \bigcap_{\eta'' \in \mathscr{N}' \setminus \{\eta'\}} \big\{u \in N_{\mathbb{R}} \mid A_{\eta',k}(u) = A_{\eta'',k}(u)\big\}. \tag{5.21}$$

For any $\eta'' \in \mathscr{N}'$ define $\eta_{01} := \omega_1(\eta'') \in \mathscr{N}_0$ and $\lambda_{01} := \omega_2(\eta'') \in \Lambda$. This yields $\lambda + \zeta(\eta_{01}) = \zeta'(\eta_{01}) = \lambda_{01}$, which transforms equation (5.21) into

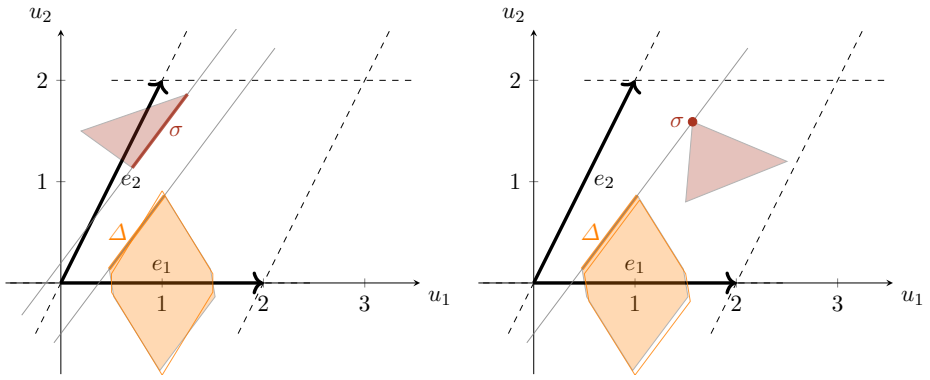
$$\mathbb{A}_\Delta = \bigcap_{\eta_{01} \in S \setminus \{\eta_{00}\}} \overbrace{H_{=}(\Delta m_{\eta_{01}}, \Delta c_{\eta_{01}})}^{= m H_{=}(\Delta m_{\eta_{01}}, \frac{1}{m}\Delta c_{\eta_{01}})}.$$

This formula checks equation (5.18), because $\mathbb{A}_{m^{-1}\Delta} = \frac{1}{m}\mathbb{A}_\Delta$. Finally, the equation $\dim_{\mathbb{R}}(\mathbb{A}_\Delta) = \dim_{\mathbb{R}}(\Delta) = n - |\mathscr{N}'| + 1$ (5.19) in combination with the equality $|\mathscr{N}'| = |\Omega| = |S|$ (5.20) completes the proof of Step III.5.a.

Step III.5.b: We get $D := D(\sigma, m^{-1}\Delta) = n - |I_{\sigma_0}| - |S| + 1$.

Proof of Step III.5.b: By the equation $\dim_{\mathbb{R}}(\mathbb{A}_\sigma) = n - |I_{\sigma_0}|$ (Step III.2), and Step III.5.a, we get the equality $D = \dim_{\mathbb{R}}(\mathbb{A}_\sigma) + \dim_{\mathbb{R}}(\mathbb{A}_{m^{-1}\Delta}) - n = n - |I_{\sigma_0}| - |S| + 1$.

Step III.5.c: For all $m \in \mathbb{N}$ the polytopal decomposition $\overline{m^{-1}\mathscr{C}_k}$ is $\overline{\Sigma}$-generic.



(a) Case $D \geq 0$: Since the affine lines $\mathbb{A}_\sigma$ and $\mathbb{A}_\Delta$ are parallel, we need a small perturbation of the slope of the edge Δ.

(b) Case $D < 0$: Since the vertex $\sigma = \mathbb{A}_\sigma$ of the upper polytope lies at the line $\mathbb{A}_\Delta$, we need a small translation of the lower polytope.

Figure 5.5: Example for $m = 1$, $n = 2$ and $\Lambda = \mathbb{Z}e_1 + \mathbb{Z}e_2$.

Proof of Step III.5.c: Case $D \geq 0$: Since Step III.5.b provides the inequality $|S| \leq n - |I_{\sigma_0}| + 1$, we may apply property (iii). In combination with Step III.5.a, property (iii) shows that the elements $(m_i)_{i\in I_{\sigma_0}}, (\Delta m_{\eta_0})_{\eta_0 \in S\setminus\{\eta_{00}\}}$ (5.12) are $\mathbb{R}$-linearly independent in $M_{\mathbb{R}}$, which implies the equality

$$\dim_{\mathbb{R}}(\mathbb{A}_\sigma \cap \mathbb{A}_{m^{-1}\Delta}) = \dim_{\mathbb{R}}\left(\mathbb{V}_\sigma \cap \mathbb{V}_{m^{-1}\Delta}\right) = n - |I_{\sigma_0}| - |S\setminus\{\eta_{00}\}| = D.$$

See Figure 5.5a for a visualization.

Case $D < 0$: Since Step III.5.b provides the inequality $|S| > n - |I_{\sigma_0}| + 1$, we may apply property (iv). We have to show $\mathbb{A}_\sigma \cap \mathbb{A}_{m^{-1}\Delta} = \varnothing$. Using equation $\mathbb{A}_\sigma = \bigcap_{i \in I_{\sigma_0}} H_=(m_{\sigma_0,i}, c_{\sigma,i})$ (5.11) and Step III.5.a, any element $u \in \mathbb{A}_\sigma \cap \mathbb{A}_{m^{-1}\Delta}$ would be a solution of the system of linear equations (5.14), which is not solvable by property (iv). See Figure 5.5b for a visualization. This completes the proof of Step III.5 and hence of Step III. □

Lemma 5.5.6. *Let Γ be the value group of an algebraically closed non-trivially valued non-archimedean field $(\mathbb{K}, |.|)$, let A be an abelian $\mathbb{K}$-variety, let (L, ρ_L) be an ample rigidified line bundle on A, and let $(z_\lambda \colon N_\mathbb{R} \to \mathbb{R})_{\lambda \in \Lambda}$ be the Λ-cocycle induced by L (see Lemma 5.3.10). Then, in Proposition 5.5.5 there additionally exists a sequence $(\mathscr{L}_k)_{k \in \mathbb{N}}$ of formal $\mathbb{K}^\circ$-models of $L^{\otimes N_k}$ determined on the Mumford model $\mathscr{A}_{\overline{\mathscr{C}_k}}$ such that for all $k \in \mathbb{N}$ the following holds:*

(iii) The metric $||.||_{\mathscr{L}_k}^{\otimes 1/N_k}$ is semi-positive and converges for $k \to +\infty$ uniformly to the toric metric $||.||_\psi$ corresponding to f_ψ (metric-function correspondence, see Proposition 5.4.9).

Proof. The value group $\Gamma \subseteq \mathbb{R}$ is a $\mathbb{Q}$-vector subspace (see Lemma 2.1.6), and the $\mathbb{R}$-quadratic function z is strictly convex because L is ample (see Lemma 5.3.11(iii)). The assumption $b(., \lambda) \in M$ for all $\lambda \in \Lambda$ is given by [BGJK21, Remark 7.1.2] and implies also $b(\lambda, \lambda) \in \Gamma$. Then Lemma 5.3.11(i) implies furthermore $l(\lambda) \in \Gamma$. Note that b is positive definite as L is ample (see Lemma 5.3.11(iii)). Therefore we may apply Proposition 5.5.5:
Since f_k fulfils the cocycle rule (see Proposition 5.5.5), the function $N_k f_k$ fulfils the cocycle rule corresponding to the line bundle $L^{\otimes N_k}$. Moreover, by Proposition 5.5.5(i) the function $N_k f_k$ is also Γ-rational. Now we will do the same construction as described in the last paragraph of the proof of [BGJK21, Proposition 8.3.1]:
By [BGJK21, Remark 8.2.2 and Lemma 8.2.3] there exists a formal $\mathbb{K}^\circ$-model $\mathscr{O}_{\mathscr{E}_{\mathscr{C}_k}}(N_k f_k)$ of the trivial line bundle (structure sheaf) $\mathcal{O}_{E^{\mathrm{an}}}$ of E^{an} such that its special fibre is nef. Since the line bundle L is ample, the line bundle $\mathscr{H}$ is ample (see Lemma 5.3.11(iii)) and hence also its special fibre $\tilde{\mathscr{H}}$ is ample (see Lemma 3.4.26), which implies $\tilde{\mathscr{H}}$ being nef (see Lemma 4.1.9). Consequently, the tensor product of the line bundles $\mathscr{O}_{\mathscr{E}_{\mathscr{C}_k}}(N_k f_k)$

and $\mathscr{H}$ induces a formal $\mathbb{K}^\circ$-model $\mathscr{L}_k$ of $L^{\otimes N_k}$ determined on the Mumford model $\mathscr{A}_{\overline{\mathscr{C}_k}}$ such that its special fibre is nef (see [BGJK21, Remark 8.2.5] for details). According to Definition 4.1.10, this means that the metric $||.||_{\mathscr{L}_k}$ corresponding to the function $N_k f_k$ is semi-positive, and consequently also the metric $||.||_{\mathscr{L}_k}^{\otimes 1/N_k}$ corresponding to the function f_k is semi-positive.
Since the sequence $(f_k)_{k\in\mathbb{N}}$ converges uniformly to f_ψ (see Proposition 5.5.5) and because the metric-function correspondence (see Proposition 5.4.9) is an isomorphism of metric spaces, the sequence $(||.||_{\mathscr{L}_k}^{\otimes 1/N_k})_{k\in\mathbb{N}}$ of metrics converges uniformly to the toric metric $||.||_\psi$ corresponding to f_ψ. □

Remark 5.5.7. Proposition 5.5.5 together with Lemma 5.5.6 is a generalization of [Gub07a, Lemma 8.4]:

- We do not assume that the absolute value $|.|$ on $\mathbb{K}$ is induced by a discrete valuation, which would imply $\Gamma = \mathbb{Q}$.
- We do not assume that the polytopes in $\overline{\Sigma}$ are Γ-rational.
- The polytopal decompositions $\overline{\mathscr{C}_k}$, which we construct, are Γ-rational and not just Γ'-rational for an infinite-dimensional $\mathbb{Q}$-vector subspace $\Gamma' \subseteq \mathbb{R}$ containing Γ.
- According to Theorem 5.4.11, we can approximate any semi-positive toric metric and not just the canonical metric.

Furthermore, Proposition 5.5.5(i) adapted to the situation of Lemma 5.5.6 strengthens the statement of [BGJK21, Proposition 8.2.6] by replacing polyhedral functions with strongly polyhedral functions. Eventually, in Theorem C, we will just apply the case $m = 1$. Originally, the case $m \neq 1$ had been necessary for the proof of [Gub10, Theorem 6.7], whose statement coincides with Corollary D under the additional assumption that the absolute value $|.|$ on $\mathbb{K}$ is induced by a discrete valuation.

6 Strictly poly-stable formal schemes

Let $(\mathbb{K}, |.|)$ be an algebraically closed non-trivially valued non-archimedean field, and Γ be its value group. Gubler [Gub07a; Gub10] figured out that it is quite convenient to work with strictly semi-stable alterations when computing the canonical measure on an abelian $\mathbb{K}$-variety if the absolute value $|.|$ on $\mathbb{K}$ is induced by a discrete valuation. The existence of a strictly semi-stable alteration is guaranteed by de Jong's alteration theorem [Jon96, Theorem 6.5], which is unfortunately just applicable in the case of discrete valuations. Since the main goal of this thesis is to drop this assumption, we have to replace them by strictly poly-stable alterations, whose existence has been proved by Adiprasito, Liu, Pak and Temkin [ALPT19, Theorem 5.2.19]. The goal of this chapter is to introduce all necessary terms and constructions regarding them.

6.1 Strata and poly-simplices

Just in this section let K be a field.

Definition 6.1.1 ([GW10, Definition 6.36 and (6.13.1)])**.** Let X be a reduced scheme of locally finite type over K. Then X is *normal* at a point $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is an integrally closed ring, which means that it is integral and its integral closure in its field of fractions is itself. Moreover, the *normality locus* $\mathrm{Nor}(X)$ is defined to be the subset of X consisting of all points $x \in X$ such that X is normal at x.

Lemma 6.1.2. *Let X be a reduced scheme of locally finite type over K. Then the normality locus $\mathrm{Nor}(X)$ is an open and dense subset of X.*

Proof. Beginning of [Ber99, §2] □

Definition 6.1.3 ([Ber99, §2])**.** Let X be a reduced scheme of locally finite type over K. The definitions $X^{(0)} := X$ and $X^{(j+1)} := X^{(j)} \setminus \mathrm{Nor}(X^{(j)})$ for

all $j \in \mathbb{N}$ yield subsets of X. A *stratum* of X is defined to be an irreducible component of the locally closed subsets $\mathrm{Nor}(X^{(j)})$ of X for some $j \in \mathbb{N}$. The set of all strata of X is denoted by $\mathrm{str}(X)$. We equip it with a partial order $S_1 \leq S_2 :\Leftrightarrow \overline{S}_1 \subseteq \overline{S}_2$.

Definition 6.1.4. A *simplex* (in our sense) is a subset

$$\Delta^{=}(r,a) := \left\{u \in \mathbb{R}^{r+1}_{\geq 0} \mid u_0 + ... + u_r = a\right\}$$

of $\mathbb{R}^{r+1}_{\geq 0}$ for some $r \in \mathbb{N}$ and $a \in \mathbb{R}_{>0}$.

Lemma 6.1.5. *Let $\Delta^{=}(r,a)$ be a simplex. Then the definition*

$$\Delta^{\leq}(r,a) := \left\{u \in \mathbb{R}^{r}_{\geq 0} \mid u_1 + ... + u_r \leq a\right\}$$

and the epimorphism of topological spaces $\kappa\colon \mathbb{R}^{r+1} \twoheadrightarrow \mathbb{R}^r, (u_0, u_1, ..., u_r) \mapsto (u_1, ..., u_r)$ yield the commutative diagram

$$\begin{array}{ccc} \Delta^{=}(r,a) & = & \Delta^{\leq}(r,a) \\ \downarrow & & \downarrow \\ \mathbb{R}^{r+1} & \xrightarrow{\kappa} & \mathbb{R}^{r} \end{array}$$

in the category of topological spaces, which allows us to identify $\Delta^{=}(r,a)$ and $\Delta^{\leq}(r,a)$ and write $\Delta(r,a)$ instead. We may do the analogous for their relative interior and their dimension when considering them as a polytope in the $\mathbb{R}$-vector space $\mathbb{R}^r$ respectively $\mathbb{R}^{r+1}$.

Proof. Take $u_0 := a - u_1 - ... - u_r \geq 0$. □

Definition 6.1.6. A *poly-simplex* (in our sense) is a subset

$$\Delta^{=}(r,a) := \prod_{j=1}^{p} \Delta^{=}(r_j, a_j)$$

of $\mathbb{R}^{r+1}_{\geq 0} := \prod_{j=1}^{p} \mathbb{R}^{r_j+1}_{\geq 0}$, for some $p \in \mathbb{N}$, $r \in \mathbb{N}^p$, $a \in \mathbb{R}^p_{>0}$. Analogously to Lemma 6.1.5, we get a subset $\Delta^{\leq}(r,a)$ of $\mathbb{R}^{r}_{\geq 0} := \prod_{j=1}^{p} \mathbb{R}^{r_j}_{\geq 0}$. Again, we may identify both and denote them by $\Delta(r,a)$.

6.2 Skeletons

The following section is based on [Ber99].

Definition 6.2.1. For any $p \in \mathbb{N}$, $r \in \mathbb{N}^p$, $\pi \in (\mathbb{K}^\circ \backslash \{0\})^p$ we define the $\mathbb{K}^\circ$-algebra $A(r,\pi) := \hat{\otimes}_{j=1}^{p} A(r_j, \pi_j)$ as the completed tensor product of the $\mathbb{K}^\circ$-algebras $A(r_j,\pi_j) := \mathbb{K}^\circ\{T_0, ..., T_{r_j}\}/\langle T_0 \cdots T_{r_j} - \pi_j \rangle$, and its corresponding admissible formal affine $\mathbb{K}^\circ$-scheme $\mathscr{S}(r,\pi) := \mathrm{Spf}(A(r,\pi))$.

Definition 6.2.2. An admissible formal $\mathbb{K}^\circ$-scheme $\mathscr{X}'$ is called *strictly poly-stable* if it has a *strictly poly-stable structure* $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ which is defined to be an atlas $(\mathscr{U}_{i'})_{i' \in I'}$ such that for any $i' \in I'$ there exist $p_{i'} \in \mathbb{N}\backslash\{0\}$, $r_{i'} \in \mathbb{N}^{p_{i'}}$, $\pi_{i'} \in (\mathbb{K}^\circ\backslash\{0\})^{p_{i'}}$, and an étale morphism of admissible formal $\mathbb{K}^\circ$-schemes $\psi_{i'} \colon \mathscr{U}_{i'} \to \mathscr{S}_{i'}$ where we use the notation $\mathscr{S}_{i'} := \mathscr{S}(r_{i'}, \pi_{i'})$. Furthermore, we define the $\mathbb{K}^\circ$-algebra $A_{i'} := A(r_{i'}, \pi_{i'})$, and we will write $||r_{i'}|| := r_{i',1} + ... + r_{i',p_{i'}} \in \mathbb{N}$.

Remark 6.2.3. For an admissible formal $\mathbb{K}^\circ$-scheme, our definition of strictly poly-stable is equal to the definitions of "strongly non-degenerate" and "non-degenerate" strictly poly-stable in the beginning of [Ber04, §4.4] because we require that all entries of the vectors $\pi_{i'}$ are non-zero. Then by [Gub10, Remark 5.19] the results of [Gub10, §5] can be generalized as follows.

Definition 6.2.4 ([Gub10, 5.3])**.** Let $\mathscr{X}'$ be a strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme and $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ be a strictly poly-stable structure on $\mathscr{X}'$. For every $i' \in I'$ the poly-simplex $\Delta_{i'} := \Delta(r_{i'}, -\log|\pi_{i'}|)$ with the notation $-\log|\pi_{i'}| := (-\log|\pi_{i',1}|, ..., -\log|\pi_{i',p_{i'}}|)$ is called *canonical*.

Lemma 6.2.5. *Let $\mathscr{X}'$ be a strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme and $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ be a strictly poly-stable structure on $\mathscr{X}'$. For all $i' \in I'$, we have a monomorphism of topological spaces*

$$\begin{aligned} \xi_{i'} \colon \Delta_{i'}^{=} &\hookrightarrow \mathscr{S}_{i'}^{\mathrm{an}}, \\ u &\mapsto \xi_u \end{aligned}$$

for the definition

$$\begin{aligned} \xi_u \colon A(r_{i'}, \pi_{i'}) \otimes_{\mathbb{K}^\circ} \mathbb{K} &\to \mathbb{R}_{\geq 0}, \\ \sum_{m \in \mathbb{N}_{i'}} a_m T^m &\mapsto \max_{m \in \mathbb{N}_{i'}} |a_m| e^{-m^T u} \end{aligned}$$

where we define the set $\mathbb{N}_{i'} := \prod_{j\in\{1,\dots,p_{i'}\}} \mathbb{N}^{r_{i',j}+1}$*, and use the notation* $m^T u := \sum_{j\in\{1,\dots,p_{i'}\}} m_j^T u_j$ *for* $m = (m_1, \dots, m_{p_{i'}})$*.*

Proof. The condition $u_{j,0} + \dots + u_{j,r_{i',j}} = -\log|\pi_{i',j}|$ corresponds to the equality $\xi_u(T_{j,0}\cdots T_{j,r_{i',j}}) = |\pi_{i',j}|$. □

Definition 6.2.6. Let $\mathscr{X}'$ be a strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme and $(\mathscr{U}_{i'}, \psi_{i'})_{i'\in I'}$ be a strictly poly-stable structure on $\mathscr{X}'$. We define $S(\mathscr{U}_{i'})$ to be the pull-back

$$\begin{array}{ccc} S(\mathscr{U}_{i'}) & \longrightarrow & \Delta_{i'} \\ \big\downarrow & & \big\downarrow{\scriptstyle \xi_{i'}} \\ (\mathscr{U}_{i'})^{\mathrm{an}} & \xrightarrow{\psi_{i'}^{\mathrm{an}}} & \mathscr{S}_{i'}^{\mathrm{an}} \end{array} \tag{6.1}$$

in the category of topological spaces.

Definition 6.2.7. Let $\mathscr{X}'$ be a strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme. We call a strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i'\in I'}$ on $\mathscr{X}'$ *suitable* if it fulfils the following conditions:

(i) For any $i' \in I'$ there exists a stratum $S_{i'}$ of the special fibre $\tilde{\mathscr{X}'}$ of $\mathscr{X}'$ such that for any stratum S of $\tilde{\mathscr{X}'}$, we have $S_{i'} \leq S$ if and only if $\tilde{\mathscr{U}_{i'}} \cap \overline{S} \neq \varnothing$.

(ii) The function $s\colon I' \to \mathrm{str}(\tilde{\mathscr{X}'}), i' \mapsto S_{i'}$ is surjective.

(iii) The morphism of topological spaces $\psi_{i'}^{\mathrm{an}}\colon S(\mathscr{U}_{i'}) \to \Delta_{i'}$ in the pull-back square (6.1) is an isomorphism of topological spaces.

Proposition 6.2.8. *Let* $\mathscr{X}'$ *be a strictly poly-stable admissible formal* $\mathbb{K}^\circ$*-scheme. There exists a suitable strictly poly-stable structure* $(\mathscr{U}_{i'}, \psi_{i'})_{i'\in I'}$ *on* $\mathscr{X}'$*.*

Proof. [Ber99, §5, especially Theorem 5.2, Step 6(b)]; see [Gub10, Proposition 5.2 and 5.3] for the semi-stable case and [Gub10, Proposition A.2] for the poly-stable case □

Lemma 6.2.9. *Let* $\mathscr{X}'$ *be a strictly poly-stable admissible formal* $\mathbb{K}^\circ$*-scheme, and let* $(\mathscr{U}_{i'}, \psi_{i'})_{i'\in I'}$ *and* $(\mathscr{U}_{j'}, \psi_{j'})_{j'\in J'}$ *be two suitable strictly poly-stable structures on* $\mathscr{X}'$*. Let* $S \in \mathrm{str}(\tilde{\mathscr{X}'})$*, and* $i' \in I'$*,* $j' \in J'$ *such that* $s(i') = S = s(j')$*. We then have the following:*

(i) The subsets $S(\mathscr{U}_{i'})$ and $S(\mathscr{U}_{j'})$ of $(\mathscr{X}')^{\mathrm{an}}$ coincide.

(ii) We have $||r_{i'}|| = ||r_{j'}||$ and $|\pi_{i'}| = |\pi_{j'}|$.

(iii) Define $\psi_{i'j'} := \psi_{j'}^{\mathrm{an}} \circ (\psi_{i'}^{\mathrm{an}})^{-1} \colon \Delta_{i'} \xrightarrow{\sim} \Delta_{j'}$ (see Definition 6.2.7(iii)) in the category of topological spaces. Then there exists an automorphism of $\mathbb{Z}$-modules $\psi_{i'j'} \colon \mathbb{Z}^{||r_{i'}||} \xrightarrow{\sim} \mathbb{Z}^{||r_{j'}||}$ such that its $\mathbb{R}$-linear extension $\psi_{i'j'} \colon \mathbb{R}^{||r_{i'}||} \xrightarrow{\sim} \mathbb{R}^{||r_{j'}||}$ coincides with $\psi_{i'j'} \colon \Delta_{i'}^{\leq} \xrightarrow{\sim} \Delta_{j'}^{\leq}$ on $\Delta_{i'}^{\leq}$. In particular we have $|\det(\psi_{i'j'})|_{\mathbb{R}} = 1$.

Proof. Property (i) is given by [Ber99, §5] and the other two properties by [Ber99, Proposition 4.3]. □

Definition 6.2.10. Let $\mathscr{X}'$ be a strictly poly-stable admissible formal $\mathbb{K}^{\circ}$-scheme, and $S \in \mathrm{str}(\tilde{\mathscr{X}'})$. We denote the coinciding subsets of $(\mathscr{X}')^{\mathrm{an}}$ in Lemma 6.2.9(i) by Δ_S. Furthermore, we will denote the equal numbers $||r_{i'}|| = ||r_{j'}||$ in Lemma 6.2.9(ii) by $r_S \in \mathbb{N}$ or equivalently by $\dim_{\mathbb{R}}(\Delta_S)$, because it coincides with $\dim_{\mathbb{R}}(\Delta_{i'}) = \dim_{\mathbb{R}}(\Delta_{j'})$. Finally, we define $\mathrm{relint}(\Delta_S)$ to be the unique subset of Δ_S such that for any suitable strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ on $\mathscr{X}'$, and any $i' \in I'$ with $s(i') = S$, we have $\psi_{i'}^{\mathrm{an}}(\mathrm{relint}(\Delta_S)) = \mathrm{relint}(\Delta_{i'})$ (see Definition 2.4.2).

Definition 6.2.11 ([Ber99, §5])**.** Let $\mathscr{X}'$ be a strictly poly-stable admissible formal $\mathbb{K}^{\circ}$-scheme. We define the *skeleton* of $\mathscr{X}'$ to be the subset

$$S(\mathscr{X}') = \bigcup_{S \in \mathrm{str}(\tilde{\mathscr{X}'})} \Delta_S$$

of $(\mathscr{X}')^{\mathrm{an}}$.

Lemma 6.2.12. *Let $\mathscr{X}'$ be a strictly poly-stable admissible formal $\mathbb{K}^{\circ}$-scheme. The skeleton $S(\mathscr{X}')$ is a closed subset of the topological space $(\mathscr{X}')^{\mathrm{an}}$ and is independent of the choice of a suitable strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ on $\mathscr{X}'$.*

Proof. [Ber99, Theorem 5.2] □

Example 6.2.13. Let $|.|_3$ be the 3-adic absolute value, which means $|3|_3 = \frac{1}{3}$ and $|p|_3 = 1$ for all prime numbers $p \neq 3$, and $\mathbb{C}_3$ the completion of the algebraic closure of the completion $\mathbb{Q}_3$ of the field $\mathbb{Q}$ with respect to $|.|_3$. The admissible formal $\mathbb{C}_3^\circ$-scheme

$$\mathscr{X}' := \mathrm{Spf}(\mathbb{C}_3^\circ\{T_1, ..., T_5\}/\langle T_1T_2 - 9, T_3T_4T_5 - 12\rangle)$$

is strictly poly-stable. In order to see this, let $I' := \{0\}$, $p_0 := 2$, $r_0 := \binom{1}{2}$, and $\pi_0 := \binom{9}{12}$, $\mathscr{U}_0 := \mathscr{X}'$, and $\psi_0 := \mathrm{id}$. This strictly poly-stable structure is not suitable (see Definition 6.2.7) because particularly the function $s\colon I' \to \mathrm{str}(\tilde{\mathscr{X}'})$ could not be surjective: The special fibre $\tilde{\mathscr{X}'}$ has more than one stratum as we will see in the following.

Description of the strata of $\tilde{\mathscr{X}'}$

For simplicity we will make use of the factorization $\mathscr{X}' = \mathscr{X}_1' \times \mathscr{X}_2'$ into the admissible formal $\mathbb{C}_3^\circ$-schemes $\mathscr{X}_1' := \mathrm{Spf}(\mathbb{C}_3^\circ\{T_1, T_2\}/\langle T_1T_2 - 9\rangle)$ and $\mathscr{X}_2' := \mathrm{Spf}(\mathbb{C}_3^\circ\{T_3, T_4, T_5\}/\langle T_3T_4T_5 - 12\rangle)$. The underlying sets of their special fibres $\tilde{\mathscr{X}_1'} = \mathrm{Spec}(\tilde{\mathbb{C}}_3[T_1, T_2]/\langle T_1T_2\rangle)$ respectively $\tilde{\mathscr{X}_2'} = \mathrm{Spec}(\tilde{\mathbb{C}}_3[T_3, T_4, T_5]/\langle T_3T_4T_5\rangle)$ (because of $|9|_3 = \frac{1}{9} < 1$ and $|12|_3 = \frac{1}{3} < 1$) are given by

$$\tilde{\mathscr{X}_1'} = \underbrace{\{(a_1, a_2) \mid a_1a_2 = 0\}}_{\text{closed points}} \cup \underbrace{\{(0, \eta), (\eta, 0)\}}_{\text{generic points}}$$

respectively

$$\tilde{\mathscr{X}_2'} = \underbrace{\{(a_3, a_4, a_5) \mid a_3a_4a_5 = 0\}}_{\text{closed points}} \cup \underbrace{\{(0, \eta, \eta), (\eta, 0, \eta), (\eta, \eta, 0)\}}_{\text{generic points}},$$

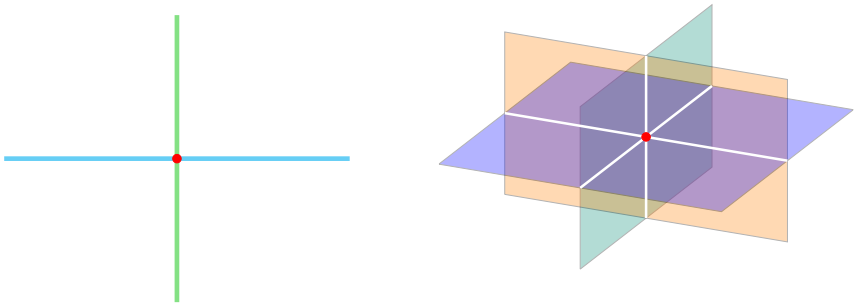
where $a_1, ..., a_5 \in \tilde{\mathbb{C}}_3$ and we use the notations $(a_1, a_2) := \langle T_1 - a_1, T_2 - a_2\rangle$ etc. for the closed points and $(0, \eta) := \langle T_1, 0\rangle$ etc. for the generic points. The local ring of the $\tilde{\mathbb{C}}_3$-variety $\tilde{\mathscr{X}_2'}$ in a closed point (a_3, a_4, a_5) is given by

$$\mathcal{O}_{\tilde{\mathscr{X}_2'}, (a_3, a_4, a_5)} = \begin{cases} \tilde{\mathbb{C}}_3 & \text{if } a_3 = 0, a_4, a_5 \neq 0 \\ (\tilde{\mathbb{C}}_3[T_3, T_4]/\langle T_3T_4\rangle)_{\langle T_3, T_4\rangle} & \text{if } a_3, a_4 = 0, a_5 \neq 0 \\ (\tilde{\mathbb{C}}_3[T_3, T_4, T_5]/\langle T_3T_4T_5\rangle)_{\langle T_3, T_4, T_5\rangle} & \text{if } a_3, a_4, a_5 = 0 \end{cases}$$

and in the generic points it is given by fields, which are in particular integrally closed rings (see Definition 6.1.1). The local ring of $\tilde{\mathscr{X}_2'}$ in the generic

point $(0, \eta, \eta)$, for example, is given by the field $\tilde{\mathbb{C}}_3(T_4, T_5)$. The local rings in the closed points with at least two coordinates equal to zero are not even integral.

The set $\tilde{\mathscr{X}}_2'$ can be decomposed into the subsets $\mathrm{Nor}(\tilde{\mathscr{X}}_2'^{(j)})$ with $j \in \{0, 1, 2\}$, which consists, due to the above computations, of all closed *and* generic points (a_3, a_4, a_5) ($a_j = \eta$ possible) such that exactly $j + 1$ of the elements a_3, a_4, a_5 are equal to zero. The analogous holds for $\tilde{\mathscr{X}}_1'$. The subset $\mathrm{Nor}(\tilde{\mathscr{X}}_2'^{(1)})$, for example, can be further decomposed into three subsets where in each of them exactly the elements a_3, a_4 respectively a_3, a_5 respectively a_4, a_5 are equal to zero (see the three white lines without the red point in Figure 6.1b). These are the irreducible components of $\mathrm{Nor}(\tilde{\mathscr{X}}_2'^{(1)})$, called strata (see Definition 6.1.3). Doing this for all $j \in \{0, 1, 2\}$ and also for $\tilde{\mathscr{X}}_1'$, one gets a decomposition of the special fibres $\tilde{\mathscr{X}}_1'$ and $\tilde{\mathscr{X}}_2'$ into strata, visualized in Figure 6.1.



(a) The three strata $S_{i_1'}$ of $\tilde{\mathscr{X}}_1'$ for $i_1' = \{1, 2\}$, $\{1\}$, $\{2\}$

(b) The seven strata $S_{i_2'}$ of $\tilde{\mathscr{X}}_2'$ for $i_2' = \{3, 4, 5\}$, $\{4, 5\}$, $\{3, 5\}$, $\{4, 5\}$ (white lines), $\{3\}$, $\{4\}$, $\{5\}$

Figure 6.1: Visualizing the $(2^2-1)\cdot(2^3-1) = 21$ strata of $\tilde{\mathscr{X}}'$ would require 5 dimensions but they are all given as products of the depicted strata. Each stratum $S_{(i_1', i_2')}$ of $\tilde{\mathscr{X}}'$ is given by the closed and generic points $(a_1, ..., a_5)$ where exactly the components corresponding to the elements of the subset $i_1' \cup i_2' \subseteq \{1, ..., 5\}$ are zero.

Construction of a suitable strictly poly-stable structure on $\mathscr{X}'$

The next goal is to construct a suitable strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ on $\mathscr{X}'$, which again we will do factor-wise. Define I_1' respectively I_2' to be the set of all non-empty subsets of $\{1,2\}$ respectively $\{3,4,5\}$ equipped with the partial order $\leq$ given by the inclusion $\supseteq$ (inverse direction!). By writing $(i_1', i_2') \leq (i_1'', i_2'')$ if and only if both $i_1' \leq i_1''$ and $i_2' \leq i_2''$ are true, we get a partially ordered set $I' := I_1' \times I_2'$.
Define for any $i_1' \in I_1'$ the ideal $\mathfrak{a}_{i_1'}$ of $\mathbb{C}_3^\circ\{T_1, T_2\}/\langle T_1T_2 - 9\rangle$ which is generated by all T_j with $j \in \{1,2\}\backslash i_1'$. In the case $i_1' = \{1,2\}$, we take $\mathfrak{a}_{i_1'} = \langle 1\rangle = \mathbb{C}_3^\circ\{T_1, T_2\}/\langle T_1T_2 - 9\rangle$. Let $\mathscr{V}(\mathfrak{a}) \subseteq \mathscr{X}_1'$ be the subset of all bounded multiplicative semi-norms $||.||_x$ on $\mathbb{C}_3^\circ\{T_1, T_2\}/\langle T_1T_2 - 9\rangle$ such that $\mathrm{red}(||.||_x) \in \tilde{\mathscr{X}}_1'$ contains $\tilde{\mathfrak{a}} \in \tilde{\mathscr{X}}_1'$ when considering them as subsets (prime ideals) of $\tilde{\mathbb{C}}_3[T_1, T_2]/\langle T_1T_2\rangle$. Since we have the formal topology on $\mathscr{X}_1'$ (see the beginning of Definition 3.4.1), the subset $\mathscr{V}(\mathfrak{a})$ is closed. Define $\mathscr{U}_{i_1'} := \mathscr{X}_1'\backslash\mathscr{V}(\mathfrak{a}_{i_1'})$ as an open subset of $\mathscr{X}_1'$, do the analogous for $\tilde{\mathscr{X}}_2'$, and define the open subsets $\mathscr{U}_{(i_1', i_2')} := \mathscr{U}_{i_1'} \times \mathscr{U}_{i_2'}$ of $\mathscr{X}'$. One can easily check that for any $i' \in I'$ the pair $(\mathscr{U}_{i'}, \mathcal{O}_{\mathscr{X}'}|_{\mathscr{U}_{i'}})$ is isomorphic to the formal affine $\mathbb{C}_3^\circ$-scheme $\mathscr{S}_{i'}$ defined by the parameter $p_{i'} = 2$, $r_{i'} := \binom{|i_1'|-1}{|i_2'|-1}$, and $\pi_{i'} := \binom{9}{12}$, which makes $(\mathscr{U}_{i'})_{i' \in I'}$ being an atlas of $\mathscr{X}'$ (see Definition 3.4.6), and taking $\psi_{i'} := \mathrm{id}$ yields a strictly poly-stable structure on $\mathscr{X}'$.
In the following, we will show that this strictly poly-stable structure is suitable (see Definition 6.2.7) in contrast to the one in the beginning:

(i) For any $i_1' \in I_1'$ we define $S_{i_1'}$ to be the subset of $\tilde{\mathscr{X}}_1'$ consisting of all closed and generic points (a_1, a_2) such that exactly the components belonging to the elements of the subset $i_1' \subseteq \{1,2\}$ are equal to zero. In the case $i_1' = \{2\}$ we have for example $S_{\{2\}} = \{(a_1, 0) \mid a_1 \in \tilde{\mathbb{C}}_3\backslash\{0\} \cup \{\eta\}\}$. After doing the analogous definitions for $\tilde{\mathscr{X}}_2'$, we define $S_{(i_1', i_2')} := S_{i_1'} \times S_{i_2'}$.

It remains to check that for any stratum S of $\tilde{\mathscr{X}}'$ we have $S_{i'} \leq S$ if and only if $\tilde{\mathscr{U}}_{i'} \cap \overline{S} \neq \varnothing$. At first note that the function $s\colon I' \to \mathrm{str}(\tilde{\mathscr{X}}'), i' \mapsto S_{i'}$ is an isomorphism of partially ordered sets by our construction. Therefore we have to check that for any $i_1', i_1'' \in I_1'$ we have $i_1' \leq i_1''$ if and only if $\tilde{\mathscr{U}}_{i_1'} \cap \overline{S}_{i_1''} \neq \varnothing$, and the analogous for $\mathscr{X}_2'$. By the definition of $\mathscr{U}_{i_1'}$, the latter is equivalent to $\overline{S}_{i_1''} \subseteq \mathrm{red}(V(\mathfrak{a}_{i_1'}))$.

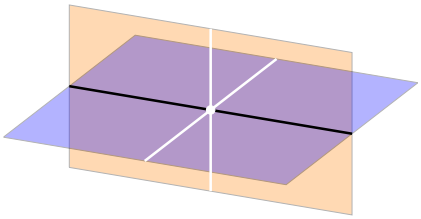


Figure 6.2: $\tilde{\mathscr{U}}_{i'_2}$ for $i'_2 = \{4,5\}$ is the subset of all closed and generic points (a_3, a_4, a_5) with $a_3 \neq 0$, which is the complement of the closure of the stratum $S_{\{3\}}$. The corresponding stratum $S_{\{4,5\}}$ is the depicted black line between the depicted strata $S_{\{4\}}$ and $S_{\{5\}}$ without the middle point $S_{\{3,4,5\}}$. Consequently we have $\tilde{\mathscr{U}}_{\{4,5\}} \cap \overline{S}_{\{4\}} \neq \varnothing$ and $S_{\{4,5\}} \leq S_{\{4\}}$, as well as $\tilde{\mathscr{U}}_{\{4,5\}} \cap \overline{S}_{\{3\}} = \varnothing$ and $S_{\{4,5\}} \not\leq S_{\{3\}}$.

By the above definitions we get that $\mathrm{red}(V(\mathfrak{a}_{i'_1}))$ respectively $\overline{S}_{i''_1}$ is the subset of $\tilde{\mathscr{X}}'_1$ consisting of all closed and generic points (a_1, a_2) such that ~~exactly~~ (because of the closure) the components belonging to the elements of the subset $i'_1 \subseteq \{1,2\}$ respectively $i''_1 \subseteq \{1,2\}$ are equal to zero. By the definition of the partial order on I'_1, the above inclusion is equivalent to $i'_1 \leq i''_1$.

(ii) The function s is here even bijective by (i).

(iii) Since $\psi_{i'} = \mathrm{id}$ is an isomorphism of admissible formal $\mathbb{C}_3^\circ$-schemes, the morphism of topological spaces $\psi_{i'}^{\mathrm{an}} \colon S(\mathscr{U}_{i'}) \to \Delta_{i'}$ in the pullback square (6.1) is also an isomorphism.

Description of the skeleton $S(\mathscr{X}')$

Finally we want to illustrate the canonical poly-simplices $\Delta_{i'}$ (see Definition 6.2.4) and the skeleton $S(\mathscr{X}')$ (see Definition 6.2.11) of $\mathscr{X}'$. The generic fibre of $\mathscr{X}'_1$ is given by the $\mathbb{C}_3$-analytic space $\mathscr{M}(\mathbb{C}_3\{T_1, T_2\}/\langle T_1 T_2 - 9\rangle)$. According to [Ber90, 1.4.4], [Wer15, Satz 3.15] and by using the equality

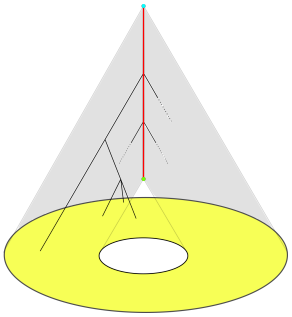
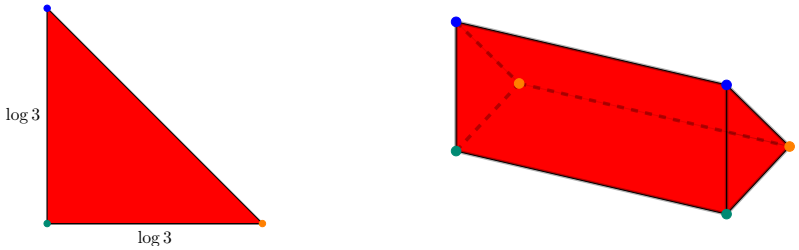


Figure 6.3: The simplices $\Delta_{i'_1}$ yield the skeletons of $\tilde{\mathscr{U}}_{i'_1}$ for $i'_1 = \{1,2\}$, $\{1\}$, $\{2\}$, which form the red subset $\Delta_{\{1,2\}} \subseteq (\mathscr{X}'_1)^{\mathrm{an}}$, where $(\mathscr{X}'_1)^{\mathrm{an}}$ is illustrated here as an upside-down tree. The yellow annulus at the bottom is the set of $x \in \mathbb{C}_3$ with $\frac{1}{9} \leq |x|_3 \leq 1$, which are the points of type (1) of the $\mathbb{C}_3$-analytic space $(\mathscr{X}'_1)^{\mathrm{an}}$, forming the end points of the branches of the tree. (The graphic was created by using the freeware programme *Inkscape*.)

$T_2 = \frac{9}{T_1}$ in the Banach $\mathbb{C}_3$-algebra $\mathbb{C}_3\{T_1, T_2\}/\langle T_1T_2 - 9\rangle$, we get that all points of $(\mathscr{X}'_1)^{\mathrm{an}}$ apart from the points of type (4), which do not affect our visualizations, are given as bounded multiplicative seminorms

$$\begin{aligned} \zeta_{a,r} \colon \mathbb{C}_3\{T_1, T_2\}/\langle T_1T_2 - 9\rangle &\to \mathbb{R}_{\geq 0}, \\ f &\mapsto \sup_{\substack{x \in \mathbb{C}_3 \\ |x-a|_3 \leq r \\ |9/x - 9/a|_3 \leq r}} \left|f(x, \tfrac{9}{x})\right|_3 \end{aligned}$$

for $a \in \mathbb{C}_3$ with $\frac{1}{9} \leq |a|_3 \leq 1$, and $r \in \mathbb{R}$ with $0 \leq |r|_{\mathbb{R}} \leq 1$.
The Berkovich topology (see Definition 3.1.7) on the set $(\mathscr{X}'_1)^{\mathrm{an}}$ can be visualized by a so called Berkovich tree (see Figure 6.3). According to Lemma 6.2.5 it contains the canonical simplex $\Delta^{=}_{\{1,2\}}$ as a subset. The simplex $\Delta^{\leq}_{\{1,2\}} \subseteq \mathbb{R}$ is an interval of length $-\log|9|_3 = 2\log 3$ and its end points are the simplices $\Delta_{\{1\}}$ and $\Delta_{\{2\}}$. The closed subset $\Delta_{\{1,2\}} \subseteq (\mathscr{X}'_1)^{\mathrm{an}}$ is the unique skeleton $S(\mathscr{X}'_1)$ of $\mathscr{X}'_1$ (see Definition 6.2.11). Furthermore, we have visualized the skeletons $S(\mathscr{X}'_2)$ and $S(\mathscr{X}') = S(\mathscr{X}'_1) \times S(\mathscr{X}'_2)$ (see Figure 6.4), where the coloured subsets are the simplices respectively poly-simplices corresponding to the strata in Figure 6.1.



(a) The simplices $\Delta_{i'_2}$ yield the skeletons of $\mathscr{U}_{i'_2}$ for $i'_2 = \{3,4,5\}$, $\{4,5\}$, $\{3,5\}$, $\{4,5\}$ (black lines), $\{3\}$, $\{4\}$, $\{5\}$. The simplex $\Delta^{\leq}_{\{3,4,5\}} \subseteq \mathbb{R}^2$ is visualized following Lemma 6.1.5.

(b) The skeleton of $\mathscr{X}'$ is given as the product of the skeletons of $\mathscr{X}'_1$ and $\mathscr{X}'_2$. It is not a simplex anymore but still a poly-simplex.

Figure 6.4: There is a 1:1-correspondence between strata $S_{i'}$ and the relative interior of the polytopes $\Delta_{i'}$ in $\mathbb{R}^5$ with $i' \in I'$, which is called the stratum-face-correspondence (see [Gub10, Proposition 5.7]). Furthermore we have $\dim(S_{i'}) = 5 - \dim_{\mathbb{R}}(\Delta_{i'})$ (see [Gub10, Corollary 5.9(a)]). The analogous holds for $i'_1 \in I'_1$ respectively $i'_2 \in I'_2$ when replacing 5 by 2 respectively 3.

Definition 6.2.14 ([Ber99, §5])**.** Let $\mathscr{X}'$ be a strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme and $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ be a suitable strictly poly-stable structure on $\mathscr{X}'$. For every $i' \in I'$ let $\mathscr{U}_{i'} = \mathrm{Spf}(B_{i'})$. Then the étale morphism of admissible formal $\mathbb{K}^\circ$-schemes $\psi_{i'}$ corresponds to a morphism of admissible $\mathbb{K}^\circ$-algebras $\psi^{\#}_{i'} \colon A_{i'} \to B_{i'}$. We may define the epimorphism of topological spaces

$$\tau_{i'} \colon (\mathscr{U}_{i'})^{\mathrm{an}} \twoheadrightarrow \Delta^{=}_{i'},$$
$$||.|| \mapsto (-\log ||\psi^{\#}_{i'}(T_{j,k})||)_{j \in \{1,\dots,p_{i'}\}, k \in \{0,\dots,r_{i',j}\}},$$

which we glue to an epimorphism of topological spaces $\tau \colon (\mathscr{X}')^{\mathrm{an}} \twoheadrightarrow S(\mathscr{X}')$ called the *retraction map*.

Definition 6.2.15. A *deformation retraction* of a monomorphism of topological spaces $\iota \colon Y \hookrightarrow X$ is a homotopy $F \colon X \times [0,1] \to X$ between a retraction of ι in the category of topological spaces and the identity morphism id_X in the category of topological spaces. A deformation retraction is

called *strong* if for all $t \in [0,1]$ we have $F(.,t)|_Y = \mathrm{id}_Y$ in the category of sets. A morphism of topological spaces is called *proper* if the preimage of any compact set is compact.

Lemma 6.2.16. *Let $\mathscr{X}'$ be a strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme. The retraction map $\tau\colon (\mathscr{X}')^{\mathrm{an}} \twoheadrightarrow S(\mathscr{X}')$ is well-defined and independent of the choice of a suitable strictly poly-stable structure on $\mathscr{X}'$. Furthermore there exists a proper strong deformation retraction F of the monomorphism of topological spaces $S(\mathscr{X}') \hookrightarrow (\mathscr{X}')^{\mathrm{an}}$ such that $\tau = F(.,0)$.*

Proof. [Ber99, Theorem 5.2] and [Ber04, Theorem 4.4.1] □

Example 6.2.17. In Example 6.2.13 for $\mathscr{X}_1'$, the retraction map sends every point of $(\mathscr{X}_1')^{\mathrm{an}}$ to the unique point of the skeleton $S(\mathscr{X}_1')$ that is reached first when following the branches of the Berkovich tree.

6.3 Subdivisions of the skeleton

In the proof of Theorem C it will be necessary to modify the strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme $\mathscr{X}'$ to $\mathscr{X}''$ by a Γ-rational polytopal subdivision of its skeleton $S(\mathscr{X}')$, which will be explained in the following.

Definition 6.3.1. Let $\mathscr{X}'$ be a strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme. A (Γ-rational) *polytopal subdivision* of the skeleton $S(\mathscr{X}')$ is a set $\mathscr{D}$ consisting of subsets W of the skeleton $S(\mathscr{X}')$ fulfilling the following properties:

(i) For the definition $\mathscr{D}_S := \{W \in \mathscr{D} \mid W \subseteq \Delta_S\}$ of subsets of $\mathscr{D}$, we have the equality of sets $\mathscr{D} = \bigcup_{S \in \mathrm{str}(\tilde{\mathscr{X}'})} \mathscr{D}_S$.

(ii) For any suitable strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ on $\mathscr{X}'$, and any $i' \in I'$, the set $\mathscr{D}_{s(i')}$ is a (Γ-rational) polytopal decomposition of the polytope $\Delta_{i'}^{\leqslant} \subseteq \mathbb{R}^{r_{s(i')}}$ when using the isomorphism of topological spaces $\psi_{i'}^{\mathrm{an}}\colon \Delta_{s(i')} \xrightarrow{\sim} \Delta_{i'}^{\leqslant}$.

A *vertex* of $\mathscr{D}$ is an element $\xi \in S(\mathscr{X}')$ such that for some (hence any) suitable strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ on $\mathscr{X}'$ and some (hence any) $i' \in I'$ the element $\xi \in \Delta_{i'}^{\leqslant}$ is a vertex of the (Γ-rational) polytopal decomposition $\mathscr{D}_{s(i')}$ of the polytopal subset $\Delta_{i'}^{\leqslant} \subseteq \mathbb{R}^{r_{s(i')}}$.

Lemma 6.3.2. *Let $\mathscr{X}'$ be a strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme, and $\mathscr{D}$ be a Γ-rational polytopal subdivision of the skeleton $S(\mathscr{X}')$. There exists a coarsest formal $\mathbb{K}$-analytic variety structure $\mathfrak{X}''$ on $(\mathscr{X}')^{\mathrm{an}}$ which refines $(\mathscr{X}')^{\mathrm{f\text{-}an}}$ in such a way that for any $\Delta \in \mathscr{D}$ the pull-back square*

$$\begin{array}{ccc} \mathfrak{U}''_\Delta & \twoheadrightarrow & \Delta \\ \downarrow & & \downarrow \\ (\mathscr{X}')^{\mathrm{an}} & \xrightarrow{\tau} & S(\mathscr{X}') \end{array}$$

in the category of topological spaces yields a formal open subset of $\mathfrak{X}''$, where τ is the retraction map from Definition 6.2.14.

Proof. See [Gub10, Proposition 5.5] for the semi-stable case, whose proof works analogously for the poly-stable case. □

Example 6.3.3. We continue Example 6.2.13 for $\mathscr{X}'_1$. Let ξ_1 respectively ξ_2 be the elements of the singletons $\Delta_{\{1\}}$ respectively $\Delta_{\{2\}}$, and choose a Γ-rational point $\xi_3 \in \mathrm{relint}(\Delta_{\{1,2\}})$. This induces a Γ-rational polytopal decomposition $\mathscr{D}_1 = \{\{\xi_1\}, \{\xi_2\}, \{\xi_3\}, [\xi_2, \xi_3], [\xi_3, \xi_1]\}$ of the closed interval $\Delta^{\leq}_{\{1,2\}} \subseteq \mathbb{R}$. Then $\mathscr{D}_1$ is a Γ-rational polytopal subdivision of the skeleton $S(\mathscr{X}'_1)$ (see Definition 6.3.1). The points ξ_1, ξ_2, ξ_3 are the vertices of $\mathscr{D}_1$.

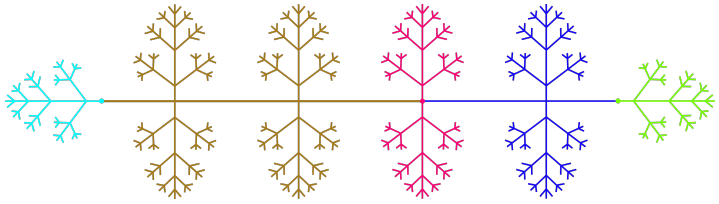


Figure 6.5: The Berkovich tree of $(\mathscr{X}'_1)^{\mathrm{an}}$ (see Figure 6.3) can equivalently be visualized as follows, where the subsets $\mathfrak{U}''_{\{\xi_1\}}$, $\mathfrak{U}''_{\{\xi_2\}}$, $\mathfrak{U}''_{\{\xi_3\}}$, $\mathfrak{U}''_{[\xi_2,\xi_3]}$, $\mathfrak{U}''_{[\xi_3,\xi_1]}$ form the atlas of $\mathfrak{X}''$. Note that the subsets $\mathfrak{U}''_{\{\xi_2\}}$ and $\mathfrak{U}''_{\{\xi_3\}}$ are contained in $\mathfrak{U}''_{[\xi_2,\xi_3]}$, and the subsets $\mathfrak{U}''_{\{\xi_3\}}$ and $\mathfrak{U}''_{\{\xi_1\}}$ are contained in $\mathfrak{U}''_{[\xi_3,\xi_1]}$. (The graphic was created by using the freeware programme *Inkscape*.)

The preimages of the five elements of $\mathscr{D}_1$ with respect to the retraction map τ yield five subsets $\mathfrak{U}''_{\{\xi_1\}}$, $\mathfrak{U}''_{\{\xi_2\}}$, $\mathfrak{U}''_{\{\xi_3\}}$, $\mathfrak{U}''_{[\xi_2,\xi_3]}$, $\mathfrak{U}''_{[\xi_3,\xi_1]}$ of $(\mathscr{X}'_1)^{\mathrm{an}}$. Since the $\mathbb{K}$-analytic space $(\mathscr{X}'_1)^{\mathrm{an}}$ is equal to the formal $\mathbb{K}$-analytic variety $(\mathscr{X}'_1)^{\mathrm{f\text{-}an}}$

as a set, they are also subsets of $(\mathscr{X}_1')^{\text{f-an}}$. By Lemma 6.3.2 they yield formal open subsets of $(\mathscr{X}_1')^{\text{f-an}}$, which form an atlas on $(\mathscr{X}_1')^{\text{f-an}}$ because $\mathscr{D}_1$ fulfils the axioms of a Γ-rational polytopal complex (see Definition 2.4.2). Replacing the atlas of $(\mathscr{X}_1')^{\text{f-an}}$ by this new atlas, we get the described formal $\mathbb{K}$-analytic variety structure $\mathfrak{X}_1''$ on $(\mathscr{X}_1')^{\text{an}}$.

Definition 6.3.4. As by [Gub10, Remark 5.6] the formal $\mathbb{K}$-analytic variety $\mathfrak{X}''$ has reduced special fibre, we may define $\mathscr{X}'' := (\mathfrak{X}'')^{\text{f-sch}}$. Furthermore, we get a unique morphism of admissible formal $\mathbb{K}^\circ$-schemes $\iota' \colon \mathscr{X}'' \to \mathscr{X}'$ extending the identity on the generic fibre.

Proposition 6.3.5. *For an irreducible component Y of $\tilde{\mathscr{X}}''$, let ξ_Y be the unique point (called Shilov point) of $(\mathscr{X}')^{\text{an}}$ with reduction equal to the generic point η_Y of Y. Then $Y \mapsto \xi_Y$ yields a bijective function between the irreducible components of $\tilde{\mathscr{X}}''$ and the vertices of $\mathscr{D}$.*

Proof. [Gub10, Corollary 5.9(g)]. The proof there is based on the so called stratum-face correspondence [Gub10, Proposition 5.7] and was done for the semi-stable case but it works analogously for the poly-stable case when using [Gub10, Proposition A.2(d)]. □

Example 6.3.6. In the situation of Example 6.3.3 the bijective correspondence of Proposition 6.3.5 can be visualized as follows:

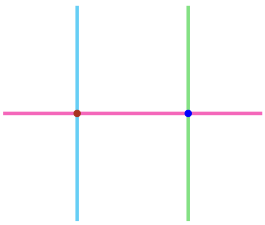


Figure 6.6: The special fibre of $\mathscr{X}_1''$ consists of three irreducible components corresponding to the vertices ξ_1, ξ_2, ξ_3 of $\mathscr{D}_1$.

6.4 Application to a Mumford model

The goal of this section is to find a tropical description of the strictly poly-stable alteration in our situation, which is given as follows: Let A be an abelian $\mathbb{K}$-variety, let $\overline{\mathscr{C}}_{\mathrm{alt}}$ be a Γ-rational polytopal decomposition of $N_{\mathbb{R}}/\Lambda$, let $\mathscr{A}_{\mathrm{alt}}$ be the associated Mumford model of A, let $\mathscr{X}'$ be a strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme, let $\varphi_{\mathrm{alt}}\colon \mathscr{X}' \to \mathscr{A}_{\mathrm{alt}}$ be a morphism of admissible formal $\mathbb{K}^\circ$-schemes, and let $f\colon X' := (\mathscr{X}')^{\mathrm{an}} \to A^{\mathrm{an}}$ be the generic fibre of φ_{alt}. The index "alt" corresponds to the index "0" in [Gub10, 6.2] and has been chosen on the one hand in order to avoid confusion with the object $\mathscr{A}_0$ from Raynaud's uniformization theory and the polytopal decomposition $\overline{\mathscr{C}_k}$ for $k = 0$ (see the proof of Theorem C), and on the other hand to stress the strong link to the later defined strictly poly-stable alteration (see Definition 7.1.4).

Proposition 6.4.1. *There is a unique morphism of topological spaces*

$$\overline{f}_{\mathrm{aff}}\colon S(\mathscr{X}') \to N_{\mathbb{R}}/\Lambda$$

making the diagram in the category of topological spaces

$$\begin{array}{ccc} X' & \xrightarrow{\quad f \quad} & A^{\mathrm{an}} \\ \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle \overline{\mathrm{trop}}} \\ S(\mathscr{X}') & \overset{\exists!\,\overline{f}_{\mathrm{aff}}}{\dashrightarrow} & N_{\mathbb{R}}/\Lambda \end{array}$$

commutative. Furthermore, for any $S \in \mathrm{str}(\tilde{\mathscr{X}}')$ we get the following:

(i) There exists a morphism of topological spaces $f_{S,\mathrm{aff}}\colon \Delta_S \to N_{\mathbb{R}}$ whose composition with the quotient function $\lambda : N_{\mathbb{R}} \twoheadrightarrow N_{\mathbb{R}}/\Lambda$ equals the restriction of $\overline{f}_{\mathrm{aff}}$ to Δ_S.

(ii) For any suitable strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ on $\mathscr{X}'$ and for any $i' \in I'$ with $s(i') = S$, the morphism of topological spaces $f_{i',\mathrm{aff}} := f_{S,\mathrm{aff}} \circ (\psi_{i'}^{\mathrm{an}})^{-1}\colon \Delta_{i'}^{\leq} \to N_{\mathbb{R}}$ (see Definition 6.2.7(iii)) extends uniquely to a Γ-rational $\mathbb{R}$-affine function $f_{i',\mathrm{aff}}\colon \mathbb{R}^{r_S} \to N_{\mathbb{R}}$.

(iii) After the choice of a lift $f_{S,\mathrm{aff}}\colon \Delta_S \to N_{\mathbb{R}}$, there exists a unique element $\Delta_{\mathrm{alt}}^S \in \mathscr{C}_{\mathrm{alt}}$ such that $f_{S,\mathrm{aff}}(\Delta_S) \subseteq \Delta_{\mathrm{alt}}^S$ and $f_{S,\mathrm{aff}}(\mathrm{relint}(\Delta_S)) \subseteq \mathrm{relint}(\Delta_{\mathrm{alt}}^S)$.

Proof. Let $S \in \mathrm{str}(\tilde{\mathscr{X}}')$. Choose an isomorphism of groups $M \xrightarrow{\sim} \mathbb{Z}^n$ during the proof, which induces an isomorphism of $\mathbb{R}$-vector spaces $N_{\mathbb{R}} \xrightarrow{\sim} \mathbb{R}^n$ and leads to $\overline{\mathscr{C}}_{\mathrm{alt}}$ becoming a Γ-rational polytopal decomposition of $\mathbb{R}^n/\Lambda$. This choice will not affect the generality of our proof. See also [Gub10, Proposition 5.11] for the proof in the semi-stable case, which works analogously.

Step 1: Construction of $f_{S,\mathrm{aff}} \colon \Delta_S \to \Delta^S_{\mathrm{alt}}$ and property (iii)

Remember that by Lemma 5.2.1 there exists a suitable choice of the atlas of the admissible formal $\mathbb{K}^\circ$-scheme $\mathscr{B} = (\mathscr{B}, (\mathscr{B}_i = \mathrm{Spf}(A_i))_{i\in I}, \mathcal{O}_{\mathscr{B}})$, which comes from Raynaud's uniformization theory. Furthermore, there exists a suitable strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i'\in I'}$ on $\mathscr{X}'$, an element $i' \in I'$ with $s(i') = S$, and a pair $(i, \Delta^S_{\mathrm{alt}}) \in I \times \mathscr{C}_{\mathrm{alt}}$, such that φ_{alt} is locally given by the lift

$$\begin{array}{ccccc} \mathscr{U}_{i'} & & \xrightarrow{\quad\Phi_{\mathrm{alt}}\quad} & & \mathscr{U}_{i,\Delta^S_{\mathrm{alt}}} \\ \big\downarrow & & & & \big\downarrow \\ \mathscr{X}' & \xrightarrow{\varphi_{\mathrm{alt}}} & \mathscr{A}_{\mathrm{alt}} & \xleftarrow{p_{\mathrm{alt}}} & \mathscr{E}_{\mathrm{alt}} \end{array}$$

in the category of admissible formal $\mathbb{K}^\circ$-schemes. This lift is unique up to the choice of a Λ-translate of Δ^S_{alt}, and the choice $\Delta^S_{\mathrm{alt}} \in \mathscr{C}_{\mathrm{alt}}$ is also unique up to a Λ-translate, which will later provide property (iii). After the choice of a lift Φ_{alt} and defining the function

$$\begin{aligned} \tau^{\mathscr{S}}_{i'} : \mathscr{S}^{\mathrm{an}}_{i'} &\twoheadrightarrow \Delta^{=}_{i'}, \\ ||.|| &\mapsto (-\log ||T_{j,k}||)_{j\in\{1,\dots,p_{i'}\}, k\in\{0,\dots,r_{i',j}\}}, \end{aligned}$$

we get the commutative diagram

$$\begin{array}{ccccc} \mathscr{S}^{\mathrm{an}}_{i'} & \xleftarrow{\psi^{\mathrm{an}}_{i'}} & (\mathscr{U}_{i'})^{\mathrm{an}} & \xrightarrow{\Phi^{\mathrm{an}}_{\mathrm{alt}}} & (\mathscr{U}_{i,\Delta^S_{\mathrm{alt}}})^{\mathrm{an}} \\ & \searrow^{\tau^{\mathscr{S}}_{i'}} & \big\downarrow \tau_{i'} & & \big\downarrow \mathrm{trop}_i \\ & & \Delta_S & \xrightarrow{\exists! f_{S,\mathrm{aff}}} & \Delta^S_{\mathrm{alt}} \end{array} \tag{6.2}$$

in the category of topological spaces (see Definition 6.2.14) without the dashed arrow.

Step 2: Property (ii)

Transferring the upper line of the diagram from the category of admissible formal affine $\mathbb{K}^\circ$-schemes (we have considered their generic fibres) to the dual category of admissible $\mathbb{K}^\circ$-algebras, we get the diagram

$$A_{i'} \xrightarrow{\psi_{i'}^{\#}} B_{i'} \xleftarrow{\Phi_{\mathrm{alt}}^{\#}} \mathbb{K}\langle U_{i,\Delta_{\mathrm{alt}}^S}\rangle.$$

[Gub10, Proposition A.2(e)] is the poly-stable version of [Gub07a, Proposition 2.11]. It proves that for any $l \in \{1,...,n\}$ there exist $a_l \in \mathbb{K}\backslash\{0\}$, $b_l \in (B_{i'})^\times$, and $m_l \in \mathbb{Z}_{i'} := \prod_{j\in\{1,\dots,p_{i'}\}} \mathbb{Z}^{r_{i',j}+1}$ such that $\Phi_{\mathrm{alt}}^{\#}(T_l) = a_l b_l \psi_{i'}^{\#}(T^{m_l})$ in $B_{i'}$. Then the function

$$\begin{aligned} f_{i',\mathrm{aff}} \colon \Delta_{i'}^{=} &\to \Delta_{\mathrm{alt}}^S, \\ u &\mapsto (m_l^T u - \log|a_l|)_{l\in\{1,\dots,n\}} \end{aligned}$$

composed with the isomorphism of topological spaces $\psi_{i'}^{\mathrm{an}} \colon \Delta_S \xrightarrow{\sim} \Delta_{i'}^{=}$ (see Definition 6.2.7(iii)) yields a morphism of topological spaces $f_{S,\mathrm{aff}} \colon \Delta_S \to \Delta_{\mathrm{alt}}^S$ making the diagram (6.2) commutative. It is unique because the morphism $\tau_{i'}$ is epic. The corresponding function $f_{i',\mathrm{aff}} \colon \Delta_{i'}^{\leq} \to \Delta_{\mathrm{alt}}^S$ is Γ-rational $\mathbb{R}$-affine and can thus be extended to a Γ-rational $\mathbb{R}$-affine function $f_{i',\mathrm{aff}} \colon \mathbb{R}^{rs} \to \mathbb{R}^n$.
For arbitrary suitable strictly poly-stable structures $(\mathscr{U}_{i'}, \psi_{i'})_{i'\in I'}$ on $\mathscr{X}'$ and for arbitrary $j' \in I'$ with $s(j') = S$, we define $f_{j',\mathrm{aff}} \colon \Delta_{j'}^{\leq} \to \Delta_{\mathrm{alt}}^S$ as the composition of $f_{i',\mathrm{aff}}$ and $\psi_{j'i'} \colon \Delta_{j'}^{\leq} \xrightarrow{\sim} \Delta_{i'}^{\leq}$ from Lemma 6.2.9(iii).

Step 3: Property (i), uniqueness and existence of $\overline{f}_{\mathrm{aff}}$

Define $\overline{f}_{\mathrm{aff}} \colon S(\mathscr{X}') \to \mathbb{R}^n/\Lambda$ to be the morphism of topological spaces which is given on Δ_S by the composition of the quotient function $\lambda \colon \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n/\Lambda$ and $f_{S,\mathrm{aff}}$. This is well-defined by the construction of the $f_{S,\mathrm{aff}}$ and because a Λ-translation of $\Delta_{\mathrm{alt}}^S \in \mathscr{C}_{\mathrm{alt}}$ and an M-translation of $i \in I$ do not affect $\overline{f}_{\mathrm{aff}}$ as in the codomain, we take the quotient by Λ. By this definition, we get property (i). Together with the surjectivity of the retraction map τ, we get the uniqueness of $\overline{f}_{\mathrm{aff}}$. □

Proposition 6.4.2. *Let $\overline{\mathscr{C}}$ be a Γ-rational polytopal subdivision of $\overline{\mathscr{C}}_{\mathrm{alt}}$ with associated Mumford model $\mathscr{A}$ of A. Then the set*

$$\mathscr{D} := \bigcup_{S \in \mathrm{str}(\tilde{\mathscr{X}}')} \left\{ \Delta_S \cap \overline{f}_{\mathrm{aff}}^{-1}(\overline{\sigma}) \mid \overline{\sigma} \in \overline{\mathscr{C}} \right\}$$

yields a Γ-rational polytopal subdivision of the skeleton $S(\mathscr{X}')$. Let $\mathscr{X}''$ be the admissible formal $\mathbb{K}^\circ$-scheme given by $\mathscr{D}$ (see Lemma 6.3.2). We then get the pull-back square

$$\begin{array}{ccc} \mathscr{X}'' & \xrightarrow{\varphi} & \mathscr{A} \\ \downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota_{\mathrm{alt}}} \\ \mathscr{X}' & \xrightarrow{\varphi_{\mathrm{alt}}} & \mathscr{A}_{\mathrm{alt}} \end{array}$$

in the category of admissible formal $\mathbb{K}^\circ$-schemes.

Proof. See [Gub10, Proposition 5.13] for the semi-stable case, whose proof works analogously for the poly-stable case. □

Lemma 6.4.3. *Let $S \in \mathrm{str}(\tilde{\mathscr{X}}')$. The restricted morphism of $\tilde{\mathbb{K}}$-varieties $\tilde{\varphi}_{\mathrm{alt}} \colon \overline{S} \to \tilde{\mathscr{A}}_{\mathrm{alt}}$ has a lift $\tilde{\Phi}_{\mathrm{alt}} \colon \overline{S} \to \tilde{\mathscr{E}}_{\mathrm{alt}}$ unique up to the M-action on $\tilde{\mathscr{E}}_{\mathrm{alt}}$.*

Proof. [Gub10, Lemma 5.15] □

Definition 6.4.4. Let $S \in \mathrm{str}(\tilde{\mathscr{X}}')$. The poly-simplex Δ_S is called *non-degenerate* with respect to $f \colon X' \to A^{\mathrm{an}}$ if

(i) $\dim_{\mathbb{R}}(f_{S,\mathrm{aff}}(\Delta_S)) = \dim_{\mathbb{R}}(\Delta_S)$ and

(ii) $\dim(\tilde{q}_{\mathrm{alt}} \circ \tilde{\Phi}_{\mathrm{alt}}(S)) = \dim(S)$.

Lemma 6.4.5. *Definition 6.4.4 only depends on the choice of $\mathscr{X}'$ and f, which means that if we have a second Γ-rational polytopal decomposition $\overline{\mathscr{C}}'_{\mathrm{alt}}$ of $N_{\mathbb{R}}/\Lambda$ with associated Mumford model $\mathscr{A}'_{\mathrm{alt}}$ of A, and $\varphi'_{\mathrm{alt}} \colon \mathscr{X}' \to \mathscr{A}'_{\mathrm{alt}}$ is a morphism of admissible formal $\mathbb{K}^\circ$-schemes such that $f \colon X' \to A^{\mathrm{an}}$ is the generic fibre of φ'_{alt}, and $\tilde{\Phi}'_{\mathrm{alt}}$ is a lift of $\tilde{\varphi}'_{\mathrm{alt}}$, then the definition does not change when replacing these objects and morphisms.*

Proof. [Gub10, 6.3] □

Lemma 6.4.6. *Take the assumptions from the beginning of Section 6.4. Furthermore, let L be a line bundle on A, let $\overline{\mathscr{C}}_k$ be a Γ-rational polytopal decomposition of $N_{\mathbb{R}}/\Lambda$ with associated Mumford model $\mathscr{A}_k$ (take the index k just as a notation with no further meaning so far). Let $\overline{\mathscr{C}} := \overline{\mathscr{C}}_k \cap \overline{\mathscr{C}}_{\mathrm{alt}}$ (see Definition 2.4.5) and let $\mathscr{A}$ be the associated Mumford model. Let $\mathscr{D}$ be the Γ-rational polytopal subdivision of the skeleton $S(\mathscr{X}')$ corresponding to the Γ-rational polytopal subdivision $\overline{\mathscr{C}}$ of $\overline{\mathscr{C}}_{\mathrm{alt}}$ (see Proposition 6.4.2). Let $\mathscr{X}''$ be the admissible formal $\mathbb{K}^\circ$-scheme given by $\mathscr{D}$ (see Lemma 6.3.2), and let Y be an irreducible component of $\tilde{\mathscr{X}''}$.*
Choose some $S \in \mathrm{str}(\tilde{\mathscr{X}'})$ such that $\xi_Y \in \mathrm{relint}(\Delta_S)$. Furthermore, define the element $y := f_{S,\mathrm{aff}}(\xi_Y) \in N_{\mathbb{R}}$ (see Proposition 6.4.1). We then get the following consequences:

(i) There exists a unique $\Delta_k \in \mathscr{C}_k$ such that $y \in \mathrm{relint}(\Delta_k)$.

(ii) We have $\{y\} = \Delta_k \cap f_{S,\mathrm{aff}}(\Delta_S)$ as subsets of $N_{\mathbb{R}}$.

(iii) We have $\{\xi_Y\} = (f_{S,\mathrm{aff}})^{-1}(\Delta_k \cap \Delta^S_{\mathrm{alt}})$ as subsets of Δ_S.

(iv) The function $f_{S,\mathrm{aff}} \colon \Delta_S \to N_{\mathbb{R}}$ is injective.

Proof. (i) The disjoint union of the subsets $\mathrm{relint}(\Delta_k) \subseteq N_{\mathbb{R}}$ for $\Delta_k \in \mathscr{C}_k$ is equal to $N_{\mathbb{R}}$.

(ii) We have $\{y\} \subseteq \Delta_k \cap f_{S,\mathrm{aff}}(\Delta_S) \subseteq \Delta_k \cap \Delta^S_{\mathrm{alt}}$, where the first inclusion follows directly by the choice $\xi_Y \in \mathrm{relint}(\Delta_S)$ and (i), and the second inclusion is provided by Proposition 6.4.1(iii). Since the latter is a polytope in $\mathscr{C}_k \cap \mathscr{C}_{\mathrm{alt}}$ and y is vertex in $\mathscr{C}_k \cap \mathscr{C}_{\mathrm{alt}}$ (by Proposition 6.3.5 the element $\xi_Y \in \Delta_S$ is a vertex of $\mathscr{D}$), the above inclusions become equalities.

(iii) Follows by (ii).

(iv) By Proposition 6.4.1(ii), we may equivalently prove for some suitable strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ on $\mathscr{X}'$ and some $i' \in I'$ with $s(i') = S$ that the $\mathbb{R}$-affine function $f_{i',\mathrm{aff}} \colon \mathbb{R}^{rs} \to N_{\mathbb{R}}$ is injective. Assume for a contradiction that the latter is not injective. Then there would exist two different elements $u_1, u_2 \in \mathbb{R}^{rs}$ with the same image.

Let $f^{(l)}_{i',\mathrm{aff}}\colon \mathbb{R}^{r_S} \to N_{\mathbb{R}}$ be the linearization (see Definition 2.4.12) of the $\mathbb{R}$-affine function $f_{i',\mathrm{aff}}$. Then

$$f^{(l)}_{i',\mathrm{aff}}(\alpha(u_1 - u_2)) = 0 \tag{6.3}$$

for all $\alpha \in \mathbb{R}$. If $\Delta_{i'}$ has more than one element, it is possible to choose $\alpha \in \mathbb{R}\backslash\{0\}$ such that $u := \xi_Y + \alpha(u_1 - u_2) \in \Delta_{i'}$. By equation (6.3) we get $f_{i',\mathrm{aff}}(u) = f_{i',\mathrm{aff}}(\xi_Y)$ and hence $u \in f^{-1}_{i',\mathrm{aff}}(\Delta_k \cap \Delta^S_{\mathrm{alt}})$ with $u \neq \xi_Y$, which yields a contradiction to (iii). □

Definition 6.4.7. By Proposition 6.4.2 and because $\overline{\mathscr{C}}$ is a Γ-rational polytopal subdivision of $\overline{\mathscr{C}_k}$, we get the sequence $\mathscr{X}'' \xrightarrow{\varphi} \mathscr{A} \xrightarrow{\iota_k} \mathscr{A}_k$ in the category of admissible formal $\mathbb{K}^\circ$-schemes. We will write

$$\deg_{\mathscr{L}_k}(Y) := \deg_{(\tilde{\iota}_k \circ \tilde{\varphi})^* \tilde{\mathscr{L}}_k}(Y). \tag{6.4}$$

Let $\mathscr{H}$ be a line bundle on $\mathscr{B}$ such that $p^*L^{\mathrm{an}} = q^*\mathscr{H}^{\mathrm{an}}$ as line bundles on E^{an}, which exists by Lemma 5.3.7. After choosing a lift $\tilde{\Phi}_{\mathrm{alt}}\colon \overline{S} \to \tilde{\mathscr{E}}_{\mathrm{alt}}$ of $\tilde{\varphi}_{\mathrm{alt}}\colon \overline{S} \to \tilde{\mathscr{A}}_{\mathrm{alt}}$ and by applying Lemma 5.2.10, we get the sequence $\overline{S} \xrightarrow{\tilde{\Phi}_{\mathrm{alt}}} \tilde{\mathscr{E}}_{\mathrm{alt}} \xrightarrow{\tilde{q}_{\mathrm{alt}}} \tilde{\mathscr{B}}$ in the category of $\tilde{\mathbb{K}}$-varieties. We will write

$$\deg_{\mathscr{H}}(\overline{S}) := \deg_{(\tilde{q}_{\mathrm{alt}} \circ \tilde{\Phi}_{\mathrm{alt}})^* \tilde{\mathscr{H}}}(\overline{S}), \tag{6.5}$$

which is independent of the choice of a rigidification ρ_L on L. Finally we define $d := \dim(X') \in \mathbb{N}$ and $e := d - \dim_{\mathbb{R}}(\Delta_S) \in \mathbb{N}$.

Proposition 6.4.8. *Take the same assumptions as in Lemma 6.4.6 and use the definitions from Definition 6.4.7. Let $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ be a suitable strictly poly-stable structure on $\mathscr{X}'$ and let $i' \in I'$ such that $s(i') = S$. Additionally we take the following assumptions:*

(A) The morphism of good $\mathbb{K}$-analytic spaces $f\colon X' \to A^{\mathrm{an}}$ is proper.

(B) We take the so called transversality assumption

$$\dim_{\mathbb{R}}(\Delta_k) + \dim_{\mathbb{R}}(\Delta_S) = n.$$

(C) L has a formal $\mathbb{K}^\circ$-model $\mathscr{L}_k$ on $\mathscr{A}_k$ such that the function

$$g_{k,i'} := f_{\mathscr{L}_k} \circ f_{i',\mathrm{aff}}\colon \mathbb{R}^{r_S} \to \mathbb{R}$$

is convex in a neighbourhood of ξ_Y.

We then get the equality

$$\deg_{\mathscr{L}_k}(Y) = \frac{d!}{e!} \cdot \deg_{\mathscr{H}}(\overline{S}) \cdot \mathrm{vol}_{r_S}(\nabla g^Y_{k,i'}(\xi_Y)),$$

where vol_{r_S} *denotes the Lebesgue measure on the* $\mathbb{R}$*-vector space* $(\mathbb{R}^{r_S})^* = \mathbb{R}^{r_S}$ *and* $g^Y_{k,i'}$ *is the induced conic convex function in* ξ_Y *by* $g_{k,i'}$*. Furthermore, assumption (C) and this formula do not depend on the choice of the suitable strictly poly-stable structure* $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ *and* $i' \in I'$ *with* $s(i') = S$*.*

Proof. [Gub10, Proposition 5.18] for the semi-stable case, where we have used that the gradient image $\nabla g^Y_{k,i'}(u_0)$ (see Definition 4.3.1) is by definition the same as the dual polytope

$$\{\xi_Y\}^{g^Y_{k,i'}} := \left\{ u \in (\mathbb{R}^{r_S})^* \;\middle|\; \begin{array}{c} \forall(\text{full-dim.}) \text{ polytopes } \Delta \in \mathscr{D}_S \text{ with vertex } \xi_Y, \\ \forall w \in \Delta^{\leq} \colon \langle u, w - \xi_Y \rangle \leq g^Y_{k,i'}(w) - g^Y_{k,i'}(\xi_Y) \end{array} \right\}$$

as a subset of $(\mathbb{R}^{r_S})^*$. By [Gub10, Remark 5.19] it can be generalized to the poly-stable case. By Lemma 6.2.9(iii), another choice of the suitable strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ on $\mathscr{X}'$ and $i' \in I'$ would neither affect the convexity of the function in assumption (C) nor the last factor of the formula. □

7 Main results

The goal of this chapter is to prove the main results Theorem C, Corollary D, and Theorem E, which have already been explained in the introduction.

Let $(\mathbb{K}, |.|)$ be an algebraically closed non-trivially valued non-archimedean field, and Γ be its value group. Let A be an abelian $\mathbb{K}$-variety, L be an ample line bundle on A, let X be an irreducible closed $\mathbb{K}$-subvariety of A, and define $d := \dim(X) \in \mathbb{N}$. Let $\overline{\mathscr{C}}_{\mathrm{alt}}$ be a Γ-rational polytopal decomposition of $N_{\mathbb{R}}/\Lambda$, let $\mathscr{A}_{\mathrm{alt}}$ be the associated Mumford model of A, and denote the closure of X^{an} in $\mathscr{A}_{\mathrm{alt}}$ by $\mathscr{X}_{\mathrm{alt}}$ (see Definition 3.4.20).

7.1 Monge-Ampère measures of semi-positive toric metrics

The goal of this section is to prove a formula for the non-archimedean Monge-Ampère measure of a semi-positive toric metric (Theorem C) and in particular of a canonical metric (Corollary D).

Definition 7.1.1 ([Tem17, 4.1.2], [Jon96, §1])**.** An *alteration* of X is a pair (X', f) consisting of an irreducible smooth $\mathbb{K}$-variety X' and a

- dominant (that means $f(X') \subseteq X$ is dense, see [Har77, Exercise II.3.7])
- proper (that means separated, of finite type, and universally closed, see [Har77, §II.4 on p. 100])
- generically finite (that means the preimage of the generic point η_X of X with respect to f is a finite set, see [Har77, Exercise II.3.7])

morphism of $\mathbb{K}$-varieties $f\colon X' \to X$.

Remark 7.1.2. Since a morphism of separated reduced schemes is epic if and only if it is dominant (in the above sense), the morphism f in Definition 7.1.1 is epic. Furthermore, it is surjective because a proper (hence closed) morphism between irreducible $\mathbb{K}$-varieties is surjective if it is dominant.

Lemma 7.1.3. *There exists an open dense subset $U \subseteq X$ such that the induced surjective epimorphism of $\mathbb{K}$-varieties $f\colon f^{-1}(U) \twoheadrightarrow U$ is finite. Furthermore, the condition generically finite in Definition 7.1.1 is equivalent to the equality* $\dim(X') = \dim(X)$.

Proof. [Har77, Exercise II.3.7], [Jon96, 2.20] □

Definition 7.1.4. In the above setting, a *strictly poly-stable alteration* of $\mathscr{X}_{\mathrm{alt}}$ is a pair $(\mathscr{X}', \varphi_{\mathrm{alt}})$ consisting of a strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme $\mathscr{X}'$ (see Definition 6.2.2 and Remark 6.2.3) and an epimorphism of admissible formal $\mathbb{K}^\circ$-schemes $\varphi_{\mathrm{alt}}\colon \mathscr{X}' \twoheadrightarrow \mathscr{X}_{\mathrm{alt}}$ such that there exists an alteration (X', f) of X with $(\mathscr{X}')^{\mathrm{an}} = (X')^{\mathrm{an}}$ and $\varphi_{\mathrm{alt}}^{\mathrm{an}} = f^{\mathrm{an}}$ in the category of $\mathbb{K}$-analytic spaces.

Proposition 7.1.5. *In the above setting, a strictly poly-stable alteration always exists.*

Proof. Since $\mathbb{K}$ is algebraically closed, this statement follows basically by [ALPT19, Theorem 5.2.19] for the valuation ring $\mathcal{O} = \mathbb{K}^\circ$ but we have to add some remarks: We can translate [ALPT19, Theorem 5.2.19], which is formulated for flat schemes of finite type over $\mathbb{K}^\circ$, to admissible formal $\mathbb{K}^\circ$-schemes by using [Gub07a, 2.7] and [GM19, Lemma 2.4]. Furthermore, in [ALPT19, Theorem 5.2.19] an alteration is defined as in [Tem17, 4.1.2], which is equivalent to our Definition 7.1.1.
Eventually, our definition of a strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme is slightly stronger than in [Ber04, §4.4], where additionally the property "(strongly) non-degenerate" is required (see Definition 6.2.2 and Remark 6.2.3). But the fact that the generic fibre of $\mathscr{X}'$ is always smooth (remark between the statement and the proof of [ALPT19, Theorem 5.2.19]) in combination with [Ber04, Lemma 4.1.3] provides that the strictly poly-stable alteration $(\mathscr{X}', \varphi_{\mathrm{alt}})$ in [ALPT19, Theorem 5.2.19] fulfils our slightly stronger definition. □

Definition 7.1.6. We will denote the composition $(X')^{\mathrm{an}} \xrightarrow{f^{\mathrm{an}}} X^{\mathrm{an}} \hookrightarrow A^{\mathrm{an}}$ of morphisms of $\mathbb{K}$-analytic spaces also by f^{an}, and the composition $\mathscr{X}' \xrightarrow{\varphi_{\mathrm{alt}}} \mathscr{X}_{\mathrm{alt}} \hookrightarrow \mathscr{A}_{\mathrm{alt}}$ of morphisms of admissible formal $\mathbb{K}^\circ$-schemes also by φ_{alt}.

Theorem 7.1.7 (A formula for the non-archimedean Monge-Ampère measure of semi-positive toric metrics)**.** *Let $(\mathscr{X}', \varphi_{\mathrm{alt}})$ be a strictly poly-stable alteration of $\mathscr{X}_{\mathrm{alt}}$, let $||.||_\psi$ be a semi-positive toric metric on L, and take the non-archimedean Monge-Ampère measure*

$$\mu'_{L,\psi} := c_1((f^{\mathrm{an}})^*(L, ||.||_\psi))^{\wedge d}$$

on the measurable space $(X')^{\mathrm{an}}$. We then get the following results:

(i) For the support of the regular Borel measure $\mu'_{L,\psi}$ on $(X')^{\mathrm{an}}$ we have

$$\mathrm{Supp}(\mu'_{L,\psi}) \subseteq \bigcup_{\substack{S \in \mathrm{str}(\tilde{\mathscr{X}}') \\ \Delta_S \text{non-deg.} \\ \text{w.r.t. } f^{\mathrm{an}}}} \Delta_S.$$

(ii) Let Ω be a Borel-measurable subset of $\mathrm{relint}(\Delta_S)$ (see Definition 6.2.10) for some stratum S of $\tilde{\mathscr{X}}'$. We then get the equality

$$\mu'_{L,\psi}(\Omega) = \frac{d!}{(d - \dim_{\mathbb{R}}(\Delta_S))!} \cdot \deg_{\mathscr{H}}(\overline{S}) \cdot \mathrm{MA}(f_\psi \circ f_{S,\mathrm{aff}})(\Omega),$$

where $\deg_{\mathscr{H}}(\overline{S})$ is defined in (6.5), and $\dim_{\mathbb{R}}(\Delta_S)$ in Definition 6.2.10, and f_ψ is given by the metric-function correspondence (see Proposition 5.4.9) for an arbitrary choice of the rigidification on L.

Proof. **Step I:** We may fix the choice of a suitable strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ of $\mathscr{X}'$, of an element $i' \in I'$ with $s(i') = S$, and of the rigidification ρ_L on L.

Firstly, the function f_ψ depends on the choice of a rigidification ρ_L on L. By Lemma 5.4.8, another choice of ρ_L would lead to the addition of a constant in Γ to the function f_ψ, which does not affect the real Monge-Ampère measure.
Secondly, the function $f_{S,\mathrm{aff}}$ for $S \in \mathrm{str}(\tilde{\mathscr{X}}')$ is not defined on $\mathbb{R}^{r_S}$, just on Δ_S. But we need an injection into a finite-dimensional $\mathbb{R}$-vector space in order to apply the real Monge-Ampère measure. For this reason, we have to choose a suitable strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ on $\mathscr{X}'$ (see Definition 6.2.2), which induces for any $i' \in I'$ with $s(i') = S$ an injection $\psi^{\mathrm{an}}_{i'} \colon \Delta_S \hookrightarrow \mathbb{R}^{r_S}$ and a Γ-rational $\mathbb{R}$-affine function $f_{i',\mathrm{aff}} \colon \mathbb{R}^{r_S} \to N_{\mathbb{R}}$ (see

Proposition 6.4.1(ii)). By Lemma 6.2.9(iii), another choice of the suitable strictly poly-stable structure or of $i' \in I'$ with $s(i') = S$ would induce a composition with an automorphism of $\mathbb{R}$-vector spaces $\psi\colon \mathbb{R}^{r_S} \xrightarrow{\sim} \mathbb{R}^{r_S}$ with $|\det(\psi)|_{\mathbb{R}} = 1$. Because of the latter, this does not affect the real Monge-Ampère measure. In the following, we will fix these choices for any $S \in \mathrm{str}(\tilde{\mathscr{X}}')$, which allows us to write $f_{S,\mathrm{aff}}$ instead of $f_{i',\mathrm{aff}}$, and $\Delta_S \subseteq \mathbb{R}^{r_S}$. Note that by Proposition 7.1.5 the existence of a strictly poly-stable alteration $(\mathscr{X}', \varphi_{\mathrm{alt}})$ of $\mathscr{X}_{\mathrm{alt}}$ is guaranteed.

Step II: We may assume f_ψ being strongly polyhedral convex.

Let $f_\psi\colon N_{\mathbb{R}} \to \mathbb{R}$ be the morphism of topological spaces fulfilling the cocycle rule which corresponds by the metric-function correspondence (see Proposition 5.4.9) to the toric metric $||.||_\psi$. By Theorem 5.4.11, the function f_ψ is convex because $||.||_\psi$ is semi-positive and L is ample.
Define $\overline{\Sigma}$ to be the set of simplices $\overline{f_{S,\mathrm{aff}}(\Delta_S)}$ for all $S \in \mathrm{str}(\tilde{\mathscr{X}}')$ together with all their closed faces. Then $\overline{\Sigma}$ is a finite set of polytopes in $N_{\mathbb{R}}/\Lambda$, because the set $\mathrm{str}(\tilde{\mathscr{X}}')$ is finite. After this, Proposition 5.5.5 and Lemma 5.5.6 yield a sequence $(\overline{\mathscr{C}_k})_{k\in\mathbb{N}}$ of $\overline{\Sigma}$-generic Γ-rational polytopal decompositions of $N_{\mathbb{R}}/\Lambda$, a sequence $(f_k)_{k\in\mathbb{N}}$ of strongly $\mathscr{C}_k$-polyhedral convex functions, a sequence $(N_k)_{k\in\mathbb{N}}$ of elements in $\mathbb{N}\backslash\{0\}$, and a sequence $(\mathscr{L}_k)_{k\in\mathbb{N}}$ of formal $\mathbb{K}^\circ$-models of $L^{\otimes N_k}$ determined on the Mumford model $\mathscr{A}_{\overline{\mathscr{C}_k}}$, such that the metrics $||.||_{\mathscr{L}_k}^{\otimes 1/N_k}$ are semi-positive and converge uniformly to $||.||_\psi$ for $k \to +\infty$. Therefore, by the convergence statements Lemma 4.2.7 and Lemma 4.3.6, it suffices to show the theorem for all $||.||_{\mathscr{L}_k}^{\otimes 1/N_k}$, f_k and

$$\mu'_{L,k} := c_1\Big((f^{\mathrm{an}})^*(L, ||.||_{\mathscr{L}_k}^{\otimes 1/N_k})\Big)^{\wedge d}$$

instead of $||.||_\psi$, f_ψ and $\mu'_{L,\psi}$. Due to the multilinearity of the non-archimedean Monge-Ampère measure $\mu'_{L,k}$ (see Lemma 4.2.4), we may assume $N_k = 1$ for all $k \in \mathbb{N}$.

Step III: Subdivision of the polytopal decomposition $\overline{\mathscr{C}_{\mathrm{alt}}}$

We define the Γ-rational polytopal decomposition $\overline{\mathscr{C}} := \overline{\mathscr{C}_{\mathrm{alt}}} \cap \overline{\mathscr{C}_k}$ of $N_{\mathbb{R}}/\Lambda$ (see Definition 2.4.5), and let $\mathscr{A}$ be the corresponding Mumford model of A. Since $\overline{\mathscr{C}}$ is a Γ-rational polytopal subdivision of $\overline{\mathscr{C}_{\mathrm{alt}}}$, we get by Proposition

6.4.2 a Γ-rational polytopal decomposition $\mathscr{D}$ of the skeleton $S(\mathscr{X}')$, a corresponding admissible formal $\mathbb{K}^\circ$-scheme $\mathscr{X}''$, and the pull-back square

$$\begin{array}{ccc} \mathscr{X}'' & \xrightarrow{\varphi} & \mathscr{A} \\ \downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota_{\mathrm{alt}}} \\ \mathscr{X}' & \xrightarrow{\varphi_{\mathrm{alt}}} & \mathscr{A}_{\mathrm{alt}} \end{array}$$

in the category of admissible formal $\mathbb{K}^\circ$-schemes.

Step IV: Reduction of statement (ii)

By the formula for the non-archimedean Monge-Ampère measure in Lemma 4.2.5 and using the notation $\deg_{\mathscr{L}_k}(Y)$ (6.4), we get the equality

$$\mu'_{L,k} = \sum_Y \deg_{\mathscr{L}_k}(Y)\delta_{\xi_Y} \tag{7.1}$$

of measures on $(X')^{\mathrm{an}}$, where Y runs over the irreducible components of $\tilde{\mathscr{X}''}$ and δ_{ξ_Y} is the Dirac measure in the Shilov point $\xi_Y \in (X')^{\mathrm{an}}$ (see Lemma 3.4.13).
Define the morphism of topological spaces $g_{k,S} := f_k \circ f_{S,\mathrm{aff}} \colon \mathbb{R}^{r_S} \to \mathbb{R}$. Since it is the composition of a strongly $\mathscr{C}_k$-polyhedral convex and an $\mathbb{R}$-affine function, it is strongly $\mathscr{C}_k$-polyhedral convex. By the formula (7.1), it remains to check for any irreducible component Y of $\tilde{\mathscr{X}''}$ the equality

$$\deg_{\mathscr{L}_k}(Y) = \frac{d!}{e!} \cdot \deg_{\mathscr{H}}(\overline{S}) \cdot \mathrm{vol}_{r_S}(\nabla g_{k,S}(\xi_Y)) \tag{7.2}$$

for $e := d - \dim_{\mathbb{R}}(\Delta_S)$ in order to prove (ii), where $\nabla g_{k,S}(\xi_Y) = \nabla g^Y_{k,S}(\xi_Y)$, because $g^Y_{k,S}$ is the induced conic convex function in ξ_Y by $g_{k,S}$.

Step V: Proof of equation (7.2) and hence (ii)

Let Y be an irreducible component of $\tilde{\mathscr{X}''}$. We will check the assumptions of Proposition 6.4.8:

(A) The definition of a strictly poly-stable alteration (see Definition 7.1.4 and Definition 7.1.1) provides that f is proper and [Ber90, Proposition 3.4.7] implies that the generic fibre f^{an} of φ_{alt} is proper.

(B) Take the notation from Lemma 6.4.6. Lemma 5.5.4 implies that the $\overline{\Sigma}$-generic polytopal decomposition $\overline{\mathscr{C}_k}$ is also $\overline{\Sigma}$-transversal, which means

$$\dim_{\mathbb{R}}(\Delta_k) + \dim_{\mathbb{R}}(f_{S,\mathrm{aff}}(\Delta_S)) - n = \dim_{\mathbb{R}}(\Delta_k \cap f_{S,\mathrm{aff}}(\Delta_S)) = 0,$$

where the second equality is given by Lemma 6.4.6(ii). By Lemma 6.4.6(iv), the $\mathbb{R}$-affine function $f_{S,\mathrm{aff}}\colon \mathbb{R}^{r_S} \to N_{\mathbb{R}}$ is injective and thus

$$\dim_{\mathbb{R}}(f_{S,\mathrm{aff}}(\Delta_S)) = \dim_{\mathbb{R}}(\Delta_S), \tag{7.3}$$

which transforms the above equation to $\dim_{\mathbb{R}}(\Delta_k) + \dim_{\mathbb{R}}(\Delta_S) = n$.

(C) It has been shown in Step IV that the function $g_{k,S}$ is strongly $\mathscr{C}_k$-polyhedral convex.

Since all the assumptions of Proposition 6.4.8 are fulfilled, equality (7.2) and hence assertion (ii) are proved to be true. Note that the number d is defined differently in Section 6.4 and Section 7.1 but they coincide because of Lemma 7.1.3.

Step VI: Proof of (i)

Equation (7.3) checks condition (i) of Definition 6.4.4 (Δ_S non-degenerate with respect to f^{an}). Since the line bundle $\tilde{\mathscr{H}}$ on $\tilde{\mathscr{B}}$ is ample (see Lemma 5.3.11(iii)), we have $\deg_{\mathscr{H}}(\overline{S}) \neq 0$ if and only if condition (ii) of Definition 6.4.4 is fulfilled (see the end of the proof of [Gub10, Theorem 6.7]). Consequently, equation (7.1), together with Step II, proves assertion (i) of this theorem. □

Definition 7.1.8. For any choice of a rigidification ρ_L on L, the corresponding function (metric-function correspondence, see Proposition 5.4.9) to the canonical metric $||.||_{L,\mathrm{can}}$ is the $\mathbb{R}$-quadratic function $z\colon N_{\mathbb{R}} \to \mathbb{R}$. By Lemma 5.4.8, the canonical function z is independent of the choice of ρ_L.
Let $(\mathscr{X}', \varphi_{\mathrm{alt}})$ be a strictly poly-stable alteration, and let $S \in \mathrm{str}(\tilde{\mathscr{X}}')$. We fix the choice of the suitable strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ of $\mathscr{X}'$ and of an element $i' \in I'$ with $s(i') = S$. The composition $z \circ f_{S,\mathrm{aff}}\colon \mathbb{R}^{r_S} \to \mathbb{R}$ (see Proposition 6.4.1(i)) yields an $\mathbb{R}$-quadratic function, whose associated $\mathbb{R}$-quadratic form is given by $u \mapsto u^T \boldsymbol{Q}_S u$ for some matrix

$\boldsymbol{Q}_S \in \mathbb{R}^{r_S \times r_S} := \mathbb{R}^{r_S} \times \mathbb{R}^{r_S}$. By Lemma 6.2.9(iii) the number $\det(\boldsymbol{Q}_S)$ does not depend on the choice of the strictly poly-stable structure and of $i' \in I'$ with $s(i') = S$.

Corollary 7.1.9 (A formula for the canonical measure)**.** *If in Theorem 7.1.7 the metric is canonical, we get equality in (i) and the formula in (ii) transforms to*

$$\mu'_{L,\mathrm{can}}(\Omega) = \frac{d!}{(d - \dim_{\mathbb{R}}(\Delta_S))!} \cdot \deg_{\mathscr{H}}(\overline{S}) \cdot \det(\boldsymbol{Q}_S) \cdot \mathrm{vol}_{r_S}(\Omega),$$

where Ω is a Borel-measurable subset of $\mathrm{relint}(\Delta_S)$ *for some* $S \in \mathrm{str}(\tilde{\mathscr{X}}')$.

Proof. Lemma 4.3.5 yields $\mathrm{MA}(z \circ f_{S,\mathrm{aff}}) = \det(\boldsymbol{Q}_S) \cdot \mathrm{vol}_{r_S}$. We have $\deg_{\mathscr{H}}(\overline{S}) \neq 0$ respectively $\det(\boldsymbol{Q}_S) \neq 0$ if and only if the canonical poly-simplex Δ_S is non-degenerate with respect to f^{an}, because the line bundle $\tilde{\mathscr{H}}$ is ample respectively the associated $\mathbb{R}$-bilinear form b of z is positive definite (for both see Lemma 5.3.11(iii)). Together with this equivalence, the Lebesgue measure in the new formula provides equality in Theorem 7.1.7(i) in the case of the canonical metric. □

Remark 7.1.10. Corollary 7.1.9 is deduced from [Gub10, Theorem 6.7] by dropping the assumption that the absolute value $|.|$ on $\mathbb{K}$ is induced by a discrete valuation. In the formula of [Gub10, Theorem 6.7], the factor $\det(\boldsymbol{Q}_S)$ is substituted by the fraction $\frac{\mathrm{vol}(\Lambda_S^L)}{\mathrm{vol}(\Lambda_S)}$ if the canonical poly-simplex Δ_S is non-degenerate with respect to f^{an}, where Λ_S^L and Λ_S are some complete lattices in $\mathbb{R}^{r_S}$ defined in [Gub10, 6.6].
But for non-discrete valuations we can not assure that the lattice Λ_S in $\mathbb{R}^{r_S}$ is still complete. For $r_S = 1$, $n = 2$, and $t \in \mathbb{R}$ being a transcendent element over $\mathbb{Q}$, we could have the following situation: If the canonical poly-simplex Δ_S is non-degenerate with respect to f^{an}, the function $f_{S,\mathrm{aff}}$ is injective. Therefore the element $v_1 := f_{S,\mathrm{aff}}^{(l)}(1) \in N$ is different from zero. Let $v_2 \in N_{\mathbb{R}}$ such that v_1, v_2 are $\mathbb{R}$-linearly independent in $N_{\mathbb{R}}$. After defining $e_1 := v_1 + v_2$ and $e_2 := tv_2$ we take $\Lambda := \mathbb{Z}e_1 + \mathbb{Z}e_2$. Finally, we get $\Lambda_S = \{0\}$ because of $\mathbb{R}v_1 \cap \Lambda = \{0\}$. Note that for a concrete counterexample, one would have to contruct an abelian $\mathbb{K}$-variety A and an ample line bundle L inducing all these parameters, which means that in general we do not know whether the lattice Λ_S is complete or not.

7.2 The canonical subset

The referee of the paper [Gub10] suggested that the support of the canonical measure has the structure of a Γ-rational piecewise linear space, which was afterwards proved by Gubler in [Gub10, Theorem 6.12] by using strictly semi-stable alterations. In the meantime, Ducros has introduced so-called c-skeletons [Duc12], which simplify the proof very much. In the following we will introduce the necessary definitions from [Duc12] and then give a new proof of [Gub10, Theorem 6.12] without assuming that the absolute value $|.|$ on $\mathbb{K}$ is induced by a discrete valuation.

Definition 7.2.1 ([Gub10, Remark 6.11])**.** The support of the canonical measure $\mu_{L,\mathrm{can}} := c_1((L, ||.||_{L,\mathrm{can}})|_{X^{\mathrm{an}}})^{\wedge d}$ on X^{an} is called the *canonical subset* of X^{an} and denoted by Σ_X.

Lemma 7.2.2. *The canonical subset Σ_X is compact and independent of the choice of the ample line bundle L on A. Furthermore, for any choice of the strictly poly-stable alteration $(\mathscr{X}', \varphi_{\mathrm{alt}})$ of $\mathscr{X}_{\mathrm{alt}}$ it is equal to the subset*

$$\bigcup_{\substack{S \in \mathrm{str}(\tilde{\mathscr{X}'}) \\ \Delta_S \text{non-deg.} \\ \text{w.r.t. } f^{\mathrm{an}}}} f^{\mathrm{an}}(\Delta_S) \tag{7.4}$$

of X^{an}.

Proof. The canonical subset Σ_X is compact because it is the image of a compact subset (finite union of poly-simplices) of $(X')^{\mathrm{an}}$ with respect to the morphism of topological spaces f^{an}.
By the projection formula in [Gub10, Proposition 3.8(b)] we get the equality

$$(f^{\mathrm{an}})_* c_1((f^{\mathrm{an}})^*(L, ||.||_{L,\mathrm{can}}))^{\wedge d} = c_1((L, ||.||_{L,\mathrm{can}})|_{X^{\mathrm{an}}})^{\wedge d} \tag{7.5}$$

of measures on X^{an}, where the first f^{an} denotes the morphism of $\mathbb{K}$-analytic spaces $f^{\mathrm{an}} \colon (X')^{\mathrm{an}} \to A^{\mathrm{an}}$ and the second one the epimorphism of $\mathbb{K}$-analytic spaces $f^{\mathrm{an}} \colon (X')^{\mathrm{an}} \twoheadrightarrow X^{\mathrm{an}}$ (see Definition 7.1.6). Equation (7.5) in combination with Corollary 7.1.9 proves that Σ_X is equal to the subset (7.4) of X^{an}, and the right-hand side of equation (7.5) shows the independence of the choice of the strictly poly-stable alteration $(\mathscr{X}', \varphi_{\mathrm{alt}})$ of $\mathscr{X}_{\mathrm{alt}}$.
Finally, the equality of Σ_X and (7.4) proves the independence of the choice of the ample line bundle L on A. See also [Gub10, Remark 6.11]. □

Definition 7.2.3 ([Duc12, 0.23])**.** For any $k \in \mathbb{N}$ we define $\mathrm{Aff}_\Gamma(\mathbb{R}^k)$ to be the set of functions $\psi\colon \mathbb{R}^k \to \mathbb{R}$ such that there exists an $N \in \mathbb{N}\backslash\{0\}$ such that $N\psi$ is Γ-rational $\mathbb{R}$-affine (see Definition 2.4.12).
For a Γ-rational polytopal set P in $\mathbb{R}^k$ in the sense of Definition 2.4.2 we denote by $\Lambda_\Gamma(P)$ the set of continuous functions $\zeta\colon P \to \mathbb{R}$ satisfying the following condition: There exists a finite family $(\Delta_i)_{i\in I}$ of Γ-rational polytopes in $\mathbb{R}^k$ in the sense of Definition 2.4.2 such that $\bigcup_{i\in I} \Delta_i = P$, and for every $i \in I$ there exists an element $\psi_i \in \mathrm{Aff}_\Gamma(\mathbb{R}^k)$ such that $\zeta|_{\Delta_i} = \psi_i|_{\Delta_i}$.

Definition 7.2.4 ([Duc12, 0.24.1, 0.24.2])**.** A Γ-*rational polytopal structure* $\Lambda_\Gamma(V)$ on a compact topological space V is a set of continuous functions $V \to \mathbb{R}$ such that there exists a triple (k, P, φ) (called *presentation*) consisting of an element $k \in \mathbb{N}$, a Γ-rational polytopal set $P \subseteq \mathbb{R}^k$ in the sense of Definition 2.4.2, and an isomorphism of topological spaces $\varphi\colon V \xrightarrow{\sim} P$ such that the pull-back of the set $\Lambda_\Gamma(P)$ with respect to φ is equal to $\Lambda_\Gamma(V)$. A morphism of Γ-rational polytopal structures $\varphi\colon \Lambda_\Gamma(W) \to \Lambda_\Gamma(V)$ is a morphism of compact topological spaces $\varphi\colon W \to V$ such that for all elements $\zeta\colon V \to \mathbb{R}$ of the set $\Lambda_\Gamma(V)$ the function $\zeta \circ \varphi\colon W \to \mathbb{R}$ is an element of the set $\Lambda_\Gamma(W)$.

Definition 7.2.5 ([Duc12, 0.25])**.** Let Σ be a locally compact Hausdorff topological space. A Γ-*rational polytopal chart* on Σ is a pair $(V, \Lambda_\Gamma(V))$ consisting of a compact subset V of Σ and a Γ-rational polytopal structure $\Lambda_\Gamma(V)$ on V. Two Γ-rational polytopal charts $(V_1, \Lambda_\Gamma(V_1))$ and $(V_2, \Lambda_\Gamma(V_2))$ on Σ are called *compatible* if $V_1 \cap V_2$ is a Γ-rational polytope of both and the Γ-rational polytopal structures of both induce the same set $\Lambda_\Gamma(V_1 \cap V_2)$. A Γ-*rational polytopal atlas* on Σ is a family $(V_i, \Lambda_\Gamma(V_i))_{i\in I}$ of pairwise compatible Γ-rational polytopal charts on Σ such that the family $(V_i)_{i\in I}$ is a quasi-net on Σ (see Definition 3.3.1). Two Γ-rational polytopal atlases on Σ are called *equivalent* if all Γ-rational polytopal charts on Σ of the first are compatible with the ones of the second.

Definition 7.2.6 ([Duc12, 0.26.1, 0.26.3])**.** A Γ-*rational piecewise linear space* is a locally compact Hausdorff topological space Σ together with an equivalence class of Γ-rational polytopal atlases on Σ (see Definition 7.2.5). A morphism of Γ-rational piecewise linear spaces $\varphi\colon \Sigma' \to \Sigma$ is a morphism of the underlying topological spaces such that there exists a Γ-rational polytopal atlas $(W_j, \Lambda_\Gamma(W_j))_{j\in J}$ on Σ' and a Γ-rational polytopal

atlas $(V_i, \Lambda_\Gamma(V_i))_{i\in I}$ on Σ fulfilling the following condition: For all $i \in I$ there exists a $j \in J$ such that $\varphi(W_j) \subseteq V_i$ and φ induces a morphism of Γ-rational polytopal structures $\varphi\colon \Lambda_\Gamma(W_j) \to \Lambda_\Gamma(V_i)$ (see Definition 7.2.4). The Γ-rational piecewise linear spaces form a category.

Definition 7.2.7 ([Duc12, 4.1.1, 4.2])**.** Let Y be a $\mathbb{K}$-analytic space such that the underlying topological space is Hausdorff and let V be a compact subset of Y. A Γ-rational polytopal structure $\Lambda_\Gamma(V)$ on V (see Definition 7.2.4) is called *analytic* if

(i) there exists a presentation (k, P, φ) (see Definition 7.2.4), a $\mathbb{K}$-analytic domain Y_0 on Y (see Definition 3.3.5), and invertible analytic functions $f_1, ..., f_k$ on Y_0 such that V is a subset of Y_0 and the isomorphism of topological spaces $\varphi\colon V \xrightarrow{\sim} P$ is given by the equality $\varphi(v) = (-\log|f_1(v)|, ..., -\log|f_k(v)|)$ for all $v \in V$,

(ii) and for all $\mathbb{K}$-analytic domains Y' on Y the intersection $Y' \cap V$ is a Γ-rational piecewise linear subspace of V (see Definition 7.2.6), and for all invertible analytic functions f' on Y' the function $-\log|f'|\colon Y' \to \mathbb{R}$ restricted to $Y' \cap V$ induces a morphism of Γ-rational piecewise linear spaces (see Definition 7.2.6).

We call V a Γ-*rational analytic polytope* if

(i) all elements of V are Abhyankar points in Y (see Definition 3.3.18), and

(ii) there exists a Γ-rational analytic polytopal structure $\Lambda_\Gamma(V)$ on V (see Definition 7.2.4).

Definition 7.2.8 ([Duc12, 4.6])**.** Let Y be a $\mathbb{K}$-analytic space such that the underlying topological space is Hausdorff and let Σ be a locally closed subset of Y. A Γ-rational polytopal chart $(V, \Lambda_\Gamma(V))$ on Σ (see Definition 7.2.5) is called *analytic* if the Γ-rational polytopal structure $\Lambda_\Gamma(V)$ is analytic (see Definition 7.2.7). The subset $\Sigma \subseteq Y$ is called a Γ-*rational skeleton* of Y if the set of all Γ-rational analytic polytopal charts $(V, \Lambda_\Gamma(V))$ on Σ with $V \subseteq \Sigma$ being a Γ-rational analytic polytope (see Definition 7.2.7), forms a Γ-rational polytopal atlas on Σ (see Definition 7.2.5).

Remark 7.2.9. Definition 7.2.8 is equivalent to the definition of so-called c-skeletons for $\Gamma = \{1\}$ in [Duc12] because $\mathbb{K}$ is algebraically closed. Note that the object Γ in [Duc12] is not the value group of $(\mathbb{K}, |.|)$. Moreover, we use the additive notation in contrast to the multiplicative notation in [Duc12] by applying the same isomorphism of groups $-\log\colon (\mathbb{R}_{>0}, \cdot) \xrightarrow{\sim} (\mathbb{R}, +)$ which was used in the definition of the value group Γ (see Definition 2.1.5).
The category of Γ-rational piecewise linear spaces introduced in Definition 7.2.6 is equivalent to the category of piecewise c-linear spaces in [Duc12, 0.26]. Both categories are, in turn, equivalent to the category of piecewise R_S-linear spaces in [Ber04, §1.3] for $R := \mathbb{Q} \cap (0, 1]$ and $S := \mathbb{Z}_{\geq 0}$. See also the paragraph before [Gub10, Theorem 6.12] respectively the beginning of [Duc12, 0.23] for these equivalences.

Lemma 7.2.10. *Let Y be a $\mathbb{K}$-analytic space such that the underlying topological space is Hausdorff. Every Γ-rational skeleton Σ of Y yields a Γ-rational piecewise linear space.*

Proof. [Duc12, 4.6] □

Lemma 7.2.11. *Let Y be an integral $\mathbb{K}$-affinoid space. Any finite union of Γ-rational analytic polytopes on Y is a Γ-rational analytic polytope on Y.*

Proof. By induction it suffices to show the case of the union of two Γ-rational analytic polytopes V and W on Y. It suffices to check criterion 2) of [Duc12, Lemme 4.4]:

(i) For all $k \in \mathbb{N}\backslash\{0\}$ and $h_1, ..., h_k \in \mathcal{O}_Y(Y)\backslash\{0\}$ the image of $V \cup W$ with respect to the function

$$\begin{aligned} \mathrm{trop}_h\colon Y &\to \mathbb{R}^k, \\ ||.|| &\mapsto (-\log||h_1||, ..., -\log||h_k||) \end{aligned}$$

is a Γ-rational polytopal set in $\mathbb{R}^k$ in the sense of Definition 2.4.2.

(ii) There exist $k \in \mathbb{N}\backslash\{0\}$ and $h_1, ..., h_k \in \mathcal{O}_Y(Y)\backslash\{0\}$ such that the restriction of the function trop_h to $V \cup W$ is injective.

Condition (i) is clear because the union of two Γ-rational polytopal sets in $\mathbb{R}^k$ is again a Γ-rational polytopal set in $\mathbb{R}^k$. For condition (ii) we may assume that there exist $k_V, k_W \in \mathbb{N}\backslash\{0\}$ and $h_1, ..., h_{k_v+k_W} \in \mathcal{O}_Y(Y)\backslash\{0\}$ such that the restriction of the function trop_h to V respectively W is injective.
Assume for a contradiction that for some $||.||_v \in V$ there exists a $||.||_w \in W$ with $||.||_v \neq ||.||_w$ and $\mathrm{trop}_h(||.||_v) = \mathrm{trop}_h(||.||_w)$. Due to $||.||_v \neq ||.||_w$ there exists a $h_0 \in \mathcal{O}_Y(Y)\backslash\{0\}$ such that $||h_0||_v \neq ||h_0||_w$. Add the element h_0 to the listing $h_1, ..., h_{k_v+k_W} \in \mathcal{O}_Y(Y)\backslash\{0\}$. Then there exists an open neighbourhood U_v of $||.||_v \in V$ in Y such that for any $||.||_u \in U_v$ there does not exist any $||.||_w \in W\backslash\{||.||_u\}$ with $\mathrm{trop}_h(||.||_u) = \mathrm{trop}_h(||.||_w)$. Since the subset $V \subseteq Y$ is compact (see Definition 7.2.7), there exists a finite subset $V_0 \subseteq V$ such that V can be covered by the finite family $(U_v)_{||.||_v \in V_0}$ of open subsets of Y. Hence, after doing the above process (adding h_0 to the listing) for all (finitely many) $||.||_v \in V_0$, we get a contradiction. Together with the property mentioned in the second sentence of the first paragraph, this checks condition (ii). □

Remark 7.2.12. Due to Lemma 7.2.10, Theorem 7.2.13 will be a generalization of [Gub10, Theorem 6.12] by dropping the assumption that the absolute value $|.|$ on $\mathbb{K}$ is induced by a discrete valuation.

Theorem 7.2.13. *The canonical subset Σ_X is a Γ-rational skeleton of X^{an}. Moreover, for any choice of the strictly poly-stable alteration $(\mathscr{X}', \varphi_{\mathrm{alt}})$ of $\mathscr{X}_{\mathrm{alt}}$ and any canonical poly-simplex Δ_S being non-degenerate with respect to f^{an}, the induced morphism of topological spaces $f^{\mathrm{an}}\colon \Delta_S \to f^{\mathrm{an}}(\Delta_S)$ yields an isomorphism of Γ-rational piecewise linear spaces.*

Proof. **Step I:** Every element of the canonical subset Σ_X is an Abhyankar point in X^{an} (see Definition 3.3.18).

Let Δ_S be non-degenerate with respect to f^{an}. By [Duc12, Exemple 4.8], we know that every element of the skeleton $S(\mathscr{X}')$ of $\mathscr{X}'$ is an Abhyankar point. Hence for every $x \in f^{\mathrm{an}}(\Delta_S)$ we get by [Duc18, 1.4.14(4)] the equality

$$d_{\mathbb{K}}(x) = \underbrace{\dim((X')^{\mathrm{an}})}_{=d} - \underbrace{\dim((f^{\mathrm{an}})^{-1}(x))}_{=0}$$

(see Definition 3.3.18), where $\dim((X')^{\mathrm{an}}) = \dim(X')$ is given by [Ber90, Proposition 3.4.3], $\dim(X') = \dim(X) = d$ is given by Lemma 7.1.3, and the last summand is zero because of the following: Let $y \in \Delta_S$. Then there exists an affine subscheme $W = \mathrm{Spec}(\mathscr{A}_W)$ of the $\mathbb{K}$-variety X' for a $\mathbb{K}$-affinoid algebra $\mathscr{A}_W$ such that the image of y under the kernel map $\pi\colon (X')^{\mathrm{an}} \twoheadrightarrow X'$ (see Remark 3.3.15) lies in W. By Lemma 7.1.3, there exists an open dense subset $U \subseteq X$ such that the induced surjective epimorphism of $\mathbb{K}$-varieties $f\colon f^{-1}(U) \twoheadrightarrow U$ is finite. If $\pi(y)$ was in the closed subset $W \backslash f^{-1}(U) \subseteq W$, there would exist an element $g \in \mathscr{A}_W \backslash \{0\}$ such that $g(\pi(y)) = 0$ and hence $g(y) = 0$. Since this would imply $d_{\mathbb{K}}(y) < \dim((X')^{\mathrm{an}})$, it would be a contradiction to y being an Abhyankar point (see above, [Duc12, Exemple 4.8]). Consequently, we get $\pi(y) \in f^{-1}(U)$ and hence $\pi(x) \in U$ for the kernel map $\pi\colon X^{\mathrm{an}} \twoheadrightarrow X$. The finiteness of the epimorphism of $\mathbb{K}$-varieties $f\colon f^{-1}(U) \twoheadrightarrow U$ together with [Ber90, Proposition 3.4.7], which shows that the analytification of this epimorphism is also finite, provides that $(f^{\mathrm{an}})^{-1}(x)$ is a finite subset of $(X')^{\mathrm{an}}$ and hence $\dim((f^{\mathrm{an}})^{-1}(x)) = 0$.

Step II: It suffices to show that for any $i \in I$, $\overline{\Delta} \in \overline{\mathscr{C}_{\mathrm{alt}}}$, for any integral $\mathbb{K}$-affinoid domain $Y \subseteq X^{\mathrm{an}} \cap U_{i,\overline{\Delta}}$, and for any canonical poly-simplex Δ_S being non-degenerate with respect to f^{an} the subset $f^{\mathrm{an}}(\Delta_S) \cap Y \subseteq Y$ is a Γ-rational analytic polytope.

The canonical subset Σ_X is locally closed because it is even compact by Lemma 7.2.2. Since every point $x \in \Sigma_X$ is an Abhyankar point in X^{an} (see Step I), it is also regular and hence there exists an integral $\mathbb{K}$-affinoid domain $Y_x \subseteq X^{\mathrm{an}}$ with $x \in Y_x$. A $\mathbb{K}$-affinoid domain is called integral if the underlying ring of the corresponding $\mathbb{K}$-affinoid algebra (see Lemma 3.2.7 and Lemma 3.2.11) is integral. Note that Y_x is integral because otherwise there would exist $f_1, f_2 \in \mathscr{A}_{Y_x} \backslash \{0\}$ with $f_1 f_2 = 0$, where $\mathscr{A}_{Y_x}$ denotes the corresponding $\mathbb{K}$-affinoid algebra of Y_x. On the other hand the local ring $\mathcal{O}_{X^{\mathrm{an}},x}$ has to be a principal ideal domain because the point x is regular. This would yield a contradiction as every principle ideal domain is an integral ring.

Denote for all $i \in I$, $\overline{\Delta} \in \overline{\mathscr{C}_{\mathrm{alt}}}$ the image of the polytopal domain $U_{i,\Delta}$ (see Definition 5.2.5) with respect to the epimorphism of $\mathbb{K}$-analytic spaces $p\colon E^{\mathrm{an}} \twoheadrightarrow A^{\mathrm{an}}$ for an arbitrary $\Delta \in \mathscr{C}_{\mathrm{alt}}$ with $\lambda(\Delta) = \overline{\Delta}$ (see Definition

2.4.5) by $U_{i,\overline{\Delta}}$. It is independent of the choice of $\Delta \in \mathscr{C}_{\mathrm{alt}}$ and for any choice we get the commutative diagram

$$\begin{array}{ccc} U_{i,\Delta} & = & U_{i,\overline{\Delta}} \\ \downarrow & & \downarrow \\ E^{\mathrm{an}} & \xrightarrow{p} & A^{\mathrm{an}} \end{array}$$

in the category of $\mathbb{K}$-analytic spaces.
Since the intersection of two $\mathbb{K}$-affinoid domains is again a $\mathbb{K}$-affinoid domain (see [Ber90, Corollary 3.1.6] because analytifications of $\mathbb{K}$-varieties are always separated due to [Ber90, Proposition 3.1.5] in combination with [Ber90, Theorem 3.4.8(i)]), we may assume that for any $x \in \Sigma_X$ there exist $i \in I$, $\overline{\Delta} \in \overline{\mathscr{C}_{\mathrm{alt}}}$ such that Y_x is a $\mathbb{K}$-affinoid domain on $X^{\mathrm{an}} \cap U_{i,\overline{\Delta}}$. Consequently, there exists a family of $\mathbb{K}$-analytic domains Y on X^{an} forming an element of $\mathrm{Cov}(\mathbf{Dom}(X^{\mathrm{an}}))_{X^{\mathrm{an}}}$ (see Definition 3.3.5) such that any Y is either an integral $\mathbb{K}$-affinoid domain on $X^{\mathrm{an}} \cap U_{i,\overline{\Delta}}$ for some $i \in I$, $\overline{\Delta} \in \overline{\mathscr{C}_{\mathrm{alt}}}$ or that $\Sigma_X \cap Y$ is empty.
Since the empty subset forms a Γ-rational skeleton by definition, by [Duc12, Proposition 4.9] it suffices to show that for any $i \in I$, $\overline{\Delta} \in \overline{\mathscr{C}_{\mathrm{alt}}}$, for any integral $\mathbb{K}$-affinoid domain $Y \subseteq X^{\mathrm{an}} \cap U_{i,\overline{\Delta}}$ the subset $\Sigma_X \cap Y \subseteq Y$ forms a Γ-rational skeleton of Y. By [Duc12, 4.6.1], we may just show that the subset $\Sigma_X \cap Y \subseteq Y$ forms a Γ-rational analytic polytope, and by Lemma 7.2.11 it suffices to show that for any Δ_S being non-degenerate with respect to f^{an} the subset $f^{\mathrm{an}}(\Delta_S) \cap Y \subseteq Y$ forms a Γ-rational analytic polytope.

Step III: For any $i \in I$, $\overline{\Delta} \in \overline{\mathscr{C}_{\mathrm{alt}}}$, for any integral $\mathbb{K}$-affinoid domain $Y \subseteq X^{\mathrm{an}} \cap U_{i,\overline{\Delta}}$, and for any canonical poly-simplex Δ_S being non-degenerate with respect to f^{an} the subset $f^{\mathrm{an}}(\Delta_S) \cap Y \subseteq Y$ is a Γ-rational analytic polytope.

By Step I and II, it suffices to check criterion 2) of [Duc12, Lemme 4.4]:

(i) For all $k \in \mathbb{N}\backslash\{0\}$ and $h_1, ..., h_k \in \mathcal{O}_{X_G^{\mathrm{an}}}(Y)\backslash\{0\}$, the image of the subset $f^{\mathrm{an}}(\Delta_S) \cap Y$ with respect to the function

$$\begin{aligned} \mathrm{trop}_h \colon Y &\to \mathbb{R}^k, \\ ||.|| &\mapsto (-\log||h_1||, ..., -\log||h_k||) \end{aligned}$$

is a Γ-rational polytopal set in $\mathbb{R}^k$ in the sense of Definition 2.4.2.

(ii) There exist $k \in \mathbb{N}\backslash\{0\}$ and $h_1, ..., h_k \in \mathcal{O}_{X_G^{\mathrm{an}}}(Y)\backslash\{0\}$ such that the restriction of the function trop_h to $f^{\mathrm{an}}(\Delta_S) \cap Y$ is injective.

For all $j \in \{1, ..., k\}$ and $x \in f^{\mathrm{an}}(\Delta_S) \cap Y$ we have $h_j(x) \neq 0$ due to Step I. By the compactness of $f^{\mathrm{an}}(\Delta_S) \cap Y$ in Y (see for the beginning of Step II) there exists a Laurent domain Z (see Definition 3.2.8) in Y such that $f^{\mathrm{an}}(\Delta_S) \cap Y \subseteq Z$ and $h_j \in \mathcal{O}_{X_G^{\mathrm{an}}}(Z)^\times$. As remarked in [Ber93, Definition 1.3.1], the subset $Z' := (f^{\mathrm{an}})^{-1}(Z) \subseteq (X')^{\mathrm{an}}$ forms a $\mathbb{K}$-analytic domain on $(X')^{\mathrm{an}}$ (see Definition 3.3.5) because every $\mathbb{K}$-affinoid domain is a $\mathbb{K}$-analytic domain.
This yields a morphism of rings $(f^{\mathrm{an}})^{\#}\colon \mathcal{O}_{X_G^{\mathrm{an}}}(Z) \to \mathcal{O}_{(X')_G^{\mathrm{an}}}(Z')$, where both times we consider the structure sheaf with respect to the G-site (see Definition 3.3.11). We denote the image of h_j with respect to this morphism by h'_j, which is invertible in $\mathcal{O}_{(X')_G^{\mathrm{an}}}(Z')$ because of $h_j \in \mathcal{O}_{X_G^{\mathrm{an}}}(Z)^\times$ (see Example 3.3.12 for the equality). This allows us to define the function

$$\begin{aligned}\mathrm{trop}_{h'}\colon Z' &\to \mathbb{R}^k,\\ ||.|| &\mapsto (-\log||h'_1||, ..., -\log||h'_k||).\end{aligned}$$

Note that for $y = ||.||_y \in Z'$ we have $||h'_j||_y = |h'_j(y)|_y$ (see Definition 3.1.8). Since, by definition, for all $y \in Z'$ we have $h'_j(y) = h_j(f^{\mathrm{an}}(y))$, we get the equality

$$\mathrm{trop}_h(f^{\mathrm{an}}(\Delta_S) \cap Y) = \mathrm{trop}_{h'}(\Delta_S \cap (f^{\mathrm{an}})^{-1}(Y))$$

of subsets of $\mathbb{R}^k$. We claim that the equality $\Delta_S \cap (f^{\mathrm{an}})^{-1}(Y) = \Delta_S \cap Z'$ of subsets of $(X')^{\mathrm{an}}$ holds true:
"$\subseteq$": Let $y \in \Delta_S \cap (f^{\mathrm{an}})^{-1}(Y)$. Then $f^{\mathrm{an}}(y) \in f^{\mathrm{an}}(\Delta_S) \cap Y \subseteq Z$ and hence $y \in (f^{\mathrm{an}})^{-1}(Z)$.
"$\supseteq$": By the inclusion $Z \subseteq Y$ we have $Z' = (f^{\mathrm{an}})^{-1}(Z) \subseteq (f^{\mathrm{an}})^{-1}(Y)$.
Consequently we get the equality

$$\mathrm{trop}_h(f^{\mathrm{an}}(\Delta_S) \cap Y) = \mathrm{trop}_{h'}(\Delta_S \cap Z') \tag{7.6}$$

of subsets of $\mathbb{R}^k$. By [Duc12, Exemple 4.8] we know that the subset $S(\mathscr{X}') \subseteq (X')^{\mathrm{an}}$ is a Γ-rational skeleton of $(X')^{\mathrm{an}}$. By [Duc12, 4.6.2], the subset $\Delta_S \subseteq (X')^{\mathrm{an}}$ is also a Γ-rational skeleton of $(X')^{\mathrm{an}}$. By [Duc12, Proposition 4.9], the subset $\Delta_S \cap Z' \subseteq Z'$ is a Γ-rational skeleton of Z'. This, in combination with equation (7.6), checks (i).

The monomorphism of $\mathbb{K}$-affinoid spaces $Y \hookrightarrow U_{i,\Delta}$ corresponds by Lemma 3.2.11 to an epimorphism (not necessarily surjective!) of $\mathbb{K}$-affinoid algebras $\mathbb{K}\langle U_{i,\Delta}\rangle \twoheadrightarrow \mathcal{O}_{X_G^{\mathrm{an}}}(Y)$ (see Lemma 5.2.4). Now choose $k := n$ (we may assume $n \neq 0$ because the case of good reduction $n = 0$ is trivial), fix an isomorphism of groups $M \xrightarrow{\sim} \mathbb{Z}^n$, and define h_j to be the image of $1 \otimes T_j \in \mathbb{K}\langle U_{i,\Delta}\rangle^\times$ with respect to this epimorphism, which is then also invertible. Since the function trop_h is equal to the restriction of the function trop_i (see Definition 5.2.2) to $U_{i,\Delta}$, we may conclude the proof by the following:

Proposition 6.4.1 shows that the restriction of $\overline{f}_{\mathrm{aff}}$ to the subset $\Delta_S \subseteq S(\mathscr{X}')$ (see Definition 6.2.10) has a continuous lift $f_{S,\mathrm{aff}} \colon \Delta_S \to \mathbb{R}^n$, which extends uniquely to a Γ-rational $\mathbb{R}$-affine function $f_{i',\mathrm{aff}} \colon \mathbb{R}^{r_S} \to \mathbb{R}^n$ when fixing the choice of a suitable strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ on $\mathscr{X}'$ and of an $i' \in I'$ with $s(i') = S$. This allows us to consider Δ_S as a subset of $\mathbb{R}^{r_S}$ and consequently we may write $f_{S,\mathrm{aff}}$ instead of $f_{i',\mathrm{aff}}$. Proposition 6.4.6(iv) shows that $f_{S,\mathrm{aff}} \colon \Delta_S \to \mathbb{R}^n$ is injective. Furthermore, this lift can be chosen such that the diagram

$$\begin{array}{ccc} \Delta_S & \xrightarrow{f^{\mathrm{an}}} & U_{i,\Delta_{\mathrm{alt}}^S} \\ & {\scriptstyle f_{S,\mathrm{aff}}} \searrow \quad \swarrow {\scriptstyle \mathrm{trop}_i} & \\ & \mathbb{R}^n & \end{array}$$

commutes in the category of topological spaces. Taking just the images of the morphisms instead of the codomains, all morphisms become epic. Since this means that $f_{S,\mathrm{aff}} \colon \Delta_S \hookrightarrow \mathrm{trop}_i(f^{\mathrm{an}}(\Delta_S))$ becomes an isomorphism, the other two epimorphisms also have to be monic. Finally, all three morphisms become isomorphisms in the category of topological spaces:

$$\begin{array}{ccc} \Delta_S & \xrightarrow[\sim]{f^{\mathrm{an}}} & f^{\mathrm{an}}(\Delta_S) \\ & {\scriptstyle f_{S,\mathrm{aff}}}^{\sim} \searrow \quad \swarrow {}^{\sim}{\scriptstyle \mathrm{trop}_i} & \\ & \mathrm{trop}_i(f^{\mathrm{an}}(\Delta_S)) & \end{array}$$

This shows in particular that the function trop_i is injective on $f^{\mathrm{an}}(\Delta_S) \cap Y$, which checks condition (ii). Moreover, since the function $f_{S,\mathrm{aff}} \colon \mathbb{R}^{r_S} \to \mathbb{R}^n$ is Γ-rational $\mathbb{R}$-affine (see Proposition 6.4.1(i)), the latter commutative diagram immediately implies that $f^{\mathrm{an}} \colon \Delta_S \to f^{\mathrm{an}}(\Delta_S)$ yields an isomorphism of Γ-rational piecewise linear spaces. □

Glossary of Notation

We use the following notations:

Notation	Description
$\varnothing$	Empty set
$A \subseteq B$	The set A is a subset of B where equality is allowed
$A \backslash B$	Complement of a set B in a set A
Ring	Commutative ring with 1
$\mathbb{N}$	Natural numbers including 0
$\mathbb{R}_{\geq 0}$	The set $\{r \in \mathbb{R} \mid r \geq 0\}$
$\mathbb{R}_{>0}$	The set $\{r \in \mathbb{R} \mid r > 0\}$
$\lvert . \rvert_{\mathbb{R}}$	The usual archimedean absolute value on $\mathbb{R}$
$(K, \lvert . \rvert)$	Non-trivially valued non-archimedean field (see Section 2.1)
$(\mathbb{K}, \lvert . \rvert)$	Algebraically closed non-trivially valued non-archimedean field (see Section 2.1)
K°, $K^{\circ\circ}$, $\tilde{K}$	See Definition 2.1.4
Γ	Value group of $(\mathbb{K}, \lvert . \rvert)$ (see Definition 2.1.5)
$\mathrm{ob}(\mathcal{C})$	The class of objects in a category $\mathcal{C}$
$\hom_{\mathcal{C}}(A, B)$	The class of morphisms in a category $\mathcal{C}$ between two objects $A, B \in \mathrm{ob}(\mathcal{C})$

$\mathbf{Grp}(\mathcal{C})$	Category of group objects of a category $\mathcal{C}$ with finite products and a terminal object 0 (see Definition 2.2.5)
Monic morphism	Monomorphism
Epic morphism	Epimorphism
M	Free abelian group of rank $n \in \mathbb{N}$ (see Section 2.4)
N, $M_{\mathbb{Q}}$, $M_{\mathbb{R}}$, $N_{\mathbb{Q}}$, $N_{\mathbb{R}}$, $\langle .,. \rangle$	See Definition 2.4.1
$\mathbb{R}-\mathbf{Vec}$	Category of $\mathbb{R}$-vector spaces
v^T	Transpose of a vector $v \in \mathbb{R}^r$ with $r \in \mathbb{N}$
vol_r	Lebesgue measure on the $\mathbb{R}$-vector space $(\mathbb{R}^r)^* = \mathbb{R}^r$ with $r \in \mathbb{N}$
$H_{\square}(m, c)$ for $\square = \geq / > / =$	Closed/open half space respectively hyperplane in $N_{\mathbb{R}}$ (see Definition 2.4.2)
$\mathrm{relint}(\Delta)$	Relative interior of a polyhedron Δ (see Definition 2.4.2)
$\mathbb{A}_{\Delta}$, $\mathbb{V}_{\Delta}$	$\mathbb{R}$-affine subspace and $\mathbb{R}$-vector subspace corresponding to a polyhedron $\Delta \subseteq N_{\mathbb{R}}$ (see Definition 2.4.2)
Λ	Generally: Complete lattice in $N_{\mathbb{R}}$ (see Definition 2.4.4); Beginning from Chapter 5 more specifically: Image of M' under $\mathrm{trop}\colon E^{\mathrm{an}} \twoheadrightarrow N_{\mathbb{R}}$ (see Lemma 5.1.20(ii))
$\mathscr{C}$, $\overline{\mathscr{C}}$	Polytopal complex/subdivision in $N_{\mathbb{R}}$, $N_{\mathbb{R}}/\Lambda$ (see Definition 2.4.2)
$\mathrm{Supp}(\mathscr{C})$, $\mathrm{Supp}(\overline{\mathscr{C}})$	Support of a polytopal complex $\mathscr{C}$, $\overline{\mathscr{C}}$ (see Definition 2.4.2)
$\mathscr{C}_1 \cap \mathscr{C}_2$, $\overline{\mathscr{C}_1} \cap \overline{\mathscr{C}_2}$	Intersection of two polytopal complexes (see Definition 2.4.2)
(Γ-rational) $\mathbb{R}$-affine function	See Definition 2.4.12

$f^{(l)}$	Linearization of an $\mathbb{R}$-affine function f (see Definition 2.4.12)
$\mathscr{N} \subseteq \mathscr{C}$	Subset of n-dimensional polyhedrons (see Definition 2.4.13)
(Strongly) $\mathscr{C}$-polyhedral function	See Definition 2.4.13
d	Distance of uniform convergence of functions (see Definition 2.4.15) or of metrics (see Definition 4.1.3)
(Abelian) K-variety	See Definition 2.5.1 and Definition 2.5.4
(L, ρ)	Rigidified line bundle (see Definition 2.5.7)
$[m]$	See Lemma 2.5.8
Odd/even/ample line bundle	See Definition 2.5.9 and Definition 2.5.12
b, q, l	Associated (symmetric) bilinear form, quadratic form, linear form to a quadratic function (see Definition 2.6.2, Lemma 2.6.3 and Definition 2.6.4)
$\mathscr{M}(R)$	Berkovich spectrum (see Definition 3.1.7)
$\mathscr{H}(x)$, χ_x, $f(x)$	See Definition 3.1.8
π	Kernel map (see Definition 3.1.9 and Remark 3.3.15)
ρ	Spectral radius of a Banach ring (see Definition 3.1.10)
$K-$**affAlg**	Category of K-affinoid algebras, whose objects are mostly denoted by $\mathscr{A}$ (see Definition 3.2.3)
$K-$**affSpc**	Category of K-affinoid spaces, which is dual to $K-$**affAlg** (see Definition 3.2.10 and Lemma 3.2.11)
$X\left(\frac{p^{-1}f}{q^{-1}g}\right)$	Weierstrass/Laurent domain of a K-affinoid space X (see Definition 3.2.8)

$K-\mathbf{An}$	Category of K-analytic spaces $(X, \mathscr{A}, \tau)$ (see Definition 3.3.2 and Remark 3.3.3)
$X(K)$	Set of K-rational points of a K-analytic space X (see Definition 3.3.4)
X, X_G	Berkovich site and G-site of a K-analytic space X (see Definition 3.3.7 and Definition 3.3.7)
$L(x)$	Fibre of a line bundle L in $x \in X$ where X is a K-analytic space (see Definition 3.3.14)
L^{an} on X^{an}	Analytification of a line bundle L on a K-variety X (see Remark 3.3.15)
$\mathfrak{X}$	Formal K-analytic variety (see Definition 3.4.1)
$\mathscr{X}$	Admissible formal K°-scheme (see Definition 3.4.10)
$\mathfrak{X}^{\mathrm{an}}$, $\mathscr{X}^{\mathrm{an}}$	Generic fibre of $\mathfrak{X}$ respectively $\mathscr{X}$ (see Definition 3.4.2 and Definition 3.4.18)
$\tilde{\mathfrak{X}}$, $\tilde{\mathscr{X}}$	Special fibre of $\mathfrak{X}$ respectively $\mathscr{X}$ (see Definition 3.4.2 and Definition 3.4.11)
red	Reduction map (see Definition 3.4.12)
η_Y, ξ_Y	The generic point of an irreducible component Y of the special fibre $\tilde{\mathscr{X}}$ and its corresponding Shilov point (see Lemma 3.4.13)
$\mathfrak{X}^{\text{f-sch}}$, $\mathscr{X}^{\text{f-an}}$	See Definition 3.4.14 and Definition 3.4.16
$f^*\|\|.\|\|$	Pull-back of a metric (see Definition 4.1.1)
$\|\|.\|\|_1 \otimes \|\|.\|\|_2$	Product of two metrics (see Definition 4.1.1)
$(\mathcal{C}(L), d)$	The metric space consisting of the continuous metrics on the line bundle L and the distance of uniform convergence (see Definition 4.1.1 and Definition 4.1.3)
$\|\|.\|\|_{\mathscr{L}}$	The associated formal metric on a line bundle L (see Definition 4.1.5)

$\|\|.\|\|_{L,\mathrm{can}}$	The canonical metric on a rigidified line bundle (L, ρ_L) (see Definition 4.1.16)
$c_1(L_1, \|\|.\|\|_1) \wedge \ldots \wedge c_1(L_d, \|\|.\|\|_d)$	Non-archimedean Monge-Ampère measure (see Definition 4.2.3)
$\mu_{L,\mathrm{can}}$	Canonical measure (see Definition 4.2.8)
$\mathrm{MA}(g)$	Real Monge-Ampère measure associated to a convex function $g\colon \mathbb{R}^r \to \mathbb{R}$ (see Definition 4.3.1)
χ^M, $\mathbb{K}[[M]]$, $\mathbb{K}[M]$, $\mathrm{Supp}(f)$	See Definition 5.1.2
$\mathbb{T}$	$\mathbb{K}$-torus of rank n (see Definition 5.1.3)
trop	Canonical tropicalization map (see Definition 5.1.6 or Lemma 5.1.20)
$\mathbb{T}_0$	Formal torus (see Definition 5.1.9)
$t\colon \mathbb{T}_0^{\mathrm{an}} \hookrightarrow \mathbb{T}^{\mathrm{an}}$	See Lemma 5.1.10
$\mathscr{A}_0$, $\mathscr{B}$, $a\colon \mathscr{A}_0^{\mathrm{an}} \hookrightarrow A^{\mathrm{an}}$, $r\colon \mathbb{T}_0^{\mathrm{an}} \hookrightarrow \mathscr{A}_0^{\mathrm{an}}$, $b\colon \mathscr{A}_0^{\mathrm{an}} \twoheadrightarrow \mathscr{B}^{\mathrm{an}}$	Raynaud's uniformization of an abelian K-variety A (see Proposition 5.1.12)
E, B, $q\colon E \twoheadrightarrow B$, $s\colon \mathbb{T}^{\mathrm{an}} \to A^{\mathrm{an}}$, $p\colon E^{\mathrm{an}} \twoheadrightarrow A^{\mathrm{an}}$	Raynaud's uniformization of an abelian K-variety A (see Proposition 5.1.15)
$s_i\colon B_i \hookrightarrow \mathscr{A}_0$	Local trivializations of $\mathscr{B}$ (see Definition 5.2.2)
$U_{i,\Delta}$, $\mathbb{K}\langle U_{i,\Delta}\rangle$	Polytopal domain associated to (i, Δ), and corresponding strictly $\mathbb{K}$-affinoid algebra (see Lemma 5.2.4 and Definition 5.2.5)
$\mathscr{E}_{\mathscr{C}}$, $\mathscr{A}_{\overline{\mathscr{C}}}$	Mumford model of an abelian K-variety A associated to $\overline{\mathscr{C}}$ (see Definition 5.2.7 and 5.2.9)
$(z_\lambda)_{\lambda\in\Lambda}$	Λ-cocycle (see Definition 5.3.1, Lemma 5.3.10)
z	Canonical function (see Definition 5.3.1(ii), Lemma 5.3.2(i), Lemma 5.3.11, Remark 5.4.10)

H, $\mathscr{H}$	Line bundle on B respectively $\mathscr{B}$ (see Lemma 5.3.7)
α_γ	See Remark 5.3.8
Z_γ	See Lemma 5.3.9
$f_{\mathscr{L}}$	See Proposition 5.4.1
$\|\|.\|\|_\psi$	Toric metric (see Definition 5.4.2)
$\overline{\psi}$	See Lemma 5.4.4
f_ψ	Corresponding function to $\|\|.\|\|_\psi$ (see Definition 5.4.7)
$\overline{\Sigma}$	Finite set of polytopes in $N_\mathbb{R}/\Lambda$ (see Section 5.5)
$D(\Delta_1, \Delta_2)$	Expected dimension of two polyhedrons (see Definition 5.5.1)
$\mathscr{N}_0$, Λ_0, Σ_0	Finite subsets of $\mathscr{N}$, Λ, Σ fulfilling certain properties (see the proof of Proposition 5.5.5)
$\mathrm{Nor}(X)$	Normality locus of a variety X over a field (see Definition 6.1.1)
$\mathrm{str}(X)$	Set of strata of a variety X over a field (see Definition 6.1.3)
$\Delta^=(r,a)$, $\Delta^\leq(r,a)$	(Poly-)simplex as a subset of $\mathbb{R}^{r+1}_{\geq 0}$ respectively $\mathbb{R}^r_{\geq 0}$ (see Definition 6.1.4, Lemma 6.1.5, Definition 6.1.6)
$A(r,\pi)$, $\mathscr{S}(r,\pi)$	$\mathbb{K}^\circ$-algebra and corresponding admissible formal affine $\mathbb{K}^\circ$-scheme (see Definition 6.2.1)
$(\mathscr{U}_{i'}, \psi_{i'})_{i'\in I'}$	(Suitable) strictly poly-stable structure on a strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme $\mathscr{X}'$ (see Definition 6.2.2, Definition 6.2.7)
$p_{i'}$, $r_{i'}$, $\pi_{i'}$, $\psi_{i'}$, $\mathscr{S}_{i'}$, $A_{i'}$, $\Delta_{i'}$, $\xi_{i'}$, $S(\mathscr{U}_{i'})$	Parameter, objects and morphisms for a strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i'\in I'}$ (see Definition 6.2.2, Definition 6.2.4, Lemma 6.2.5, Definition 6.2.6)
$s\colon I' \to \mathrm{str}(\tilde{\mathscr{X}}')$, $i' \mapsto S_{i'}$	Surjective function corresponding to a suitable strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i'\in I'}$

Δ_S, $\dim_{\mathbb{R}}(\Delta_S)$, r_S, $\operatorname{relint}(\Delta_S)$	See Definition 6.2.10
$S(\mathscr{X}')$	Skeleton (see Definition 6.2.11)
$B_{i'}$	$\mathbb{K}^\circ$-algebras corresponding to the $\mathscr{U}_{i'}$ of a strictly poly-stable structure $(\mathscr{U}_{i'}, \psi_{i'})_{i' \in I'}$ (see Definition 6.2.14)
τ	Retraction map (see Definition 6.2.14)
$\mathscr{X}''$, ι'	Admissible formal $\mathbb{K}^\circ$-scheme $\mathscr{X}''$ induced by a Γ-rational polytopal subdivision $\mathscr{D}$ of the skeleton $S(\mathscr{X}')$ together with a unique morphism of admissible formal $\mathbb{K}^\circ$-schemes $\iota' \colon \mathscr{X}'' \to \mathscr{X}'$ extending the identity on the generic fibre (see Definition 6.3.4)
$\overline{\mathscr{C}}_{\mathrm{alt}}$, $\mathscr{A}_{\mathrm{alt}}$	Γ-rational polytopal decomposition of $N_{\mathbb{R}}/\Lambda$, which can be chosen freely, and the associated Mumford model of the given abelian $\mathbb{K}$-variety A (see Section 6.4 and Section 7.1)
$\mathscr{X}_{\mathrm{alt}}$	Closure of X^{an} in $\mathscr{A}_{\mathrm{alt}}$ for a geometrically integral closed $\mathbb{K}$-subvariety X of A (see Section 7.1)
p_{alt}, q_{alt}	Morphisms induced by $\overline{\mathscr{C}}_{\mathrm{alt}}$ (see Lemma 5.2.10)
$(\mathscr{X}', \varphi_{\mathrm{alt}})$	Poly-stable alteration (see Definition 7.1.4). In Section 6.4 $\mathscr{X}'$ respectively φ_{alt} is just a strictly poly-stable admissible formal $\mathbb{K}^\circ$-scheme respectively a morphism of admissible formal $\mathbb{K}^\circ$-schemes $\varphi_{\mathrm{alt}} \colon \mathscr{X}' \to \mathscr{A}_{\mathrm{alt}}$ without further assumptions.
$f^{\mathrm{an}} \colon (X')^{\mathrm{an}} \to A^{\mathrm{an}}$	Generic fibre of $\varphi_{\mathrm{alt}} \colon \mathscr{X}' \to \mathscr{A}_{\mathrm{alt}}$. Note also Definition 7.1.6.
$\overline{f}_{\mathrm{aff}}$, $f_{S,\mathrm{aff}}$, $f_{i',\mathrm{aff}}$, Δ^S_{alt}	See Proposition 6.4.1
$\tilde{\Phi}_{\mathrm{alt}} \colon \overline{S} \to \tilde{\mathscr{E}}_{\mathrm{alt}}$	Lift of $\tilde{\varphi}_{\mathrm{alt}} \colon \overline{S} \to \tilde{\mathscr{A}}_{\mathrm{alt}}$ for $S \in \operatorname{str}(\tilde{\mathscr{X}}')$ (see Lemma 6.4.3)

y	Image of ξ_Y under $f_{S,\mathrm{aff}}$ (see Proposition 6.4.1 and Lemma 6.4.6)
$\deg_{\mathscr{L}_k}(Y)$, $\deg_{\mathscr{H}}(\overline{S})$, d, e	See Definition 6.4.7
Λ_S, Λ_S^L	See Remark 7.1.10 or [Gub10, 6.6]
Σ_X	Canonical subset (see Definition 7.2.1)
$\mathrm{Aff}_\Gamma(\mathbb{R}^k)$, $\Lambda_\Gamma(P)$	See Definition 7.2.3
$(V, \Lambda_\Gamma(V))$	Polytopal chart (see Definition 7.2.5)

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