Universität Regensburg
Master Thesis in Mathematics

Initial Value Spaces in General Relativity

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Introduction

Einstein's general relativity theory has attracted interest from both mathematicians and physicists. Over the years, the problem of understanding the solutions to its underlying field equations has been tackled from different sides: First insight into the nature of its solutions was gained by constructing explicit solutions. Such solutions are known under restrictive assumptions on the matter involved, for example vacuum or a single type of matter as an electromagnetic field, and typically possess a lot of symmetry. In contrast to this, cosmological considerations do not rely on the knowledge of concrete solutions. Instead, the necessary simplification here stems from the fact that virtually all known kinds of matter satisfy certain physically motivated energy conditions. Assuming that all matter does, the matter term in the field equations can be estimated, turning the field equations into an inequality. This then allows, for example, to derive bounds on singularity formation.

An intuitive way of thinking about solutions to the field equations is the following: A time-oriented solution spacetime can be foliated into spacelike hypersurfaces, at least locally around a chosen spacelike hypersurface. The leaves of this foliation can be thought of as representing the state of the universe at a certain point of time. Then the field equations decompose into a system of evolution equations for the dynamical quantities and a set of constraint equations that have to be satisfied by these quantities on every leaf. So, on a chosen leaf called initial hypersurface the solution spacetime determines initial data subject to the constraints. The converse is also true: It can be shown that the evolution equations admit a short-time solution if the initial data comply with the constraints, and that this solution satisfies the constraints on every leaf. Thus, initial data satisfying the constraints determine a solution spacetime of the field equations.

With this in mind, analyzing solutions of the field equations amounts to studying solutions of the constraints. As with the field equations, significant progress has been made in the cases of vacuum or simple matter models, leading, for example, to a complete classification of constant mean curvature solutions to the vacuum constraints by Isenberg [Ise95]. In contrast, the inequality obtained by abstracting from specific matter via the dominant energy condition seems to be scarcely studied. Whereas a solution to this inequality is seen to always exist by simply increasing mean curvature, it is an interesting question whether its space of solutions is contractible. This question is approached in the following by comparing it to the space of positive scalar curvature metrics and invoking recent results by Crowley, Steimle and Schick [CSS18] as well as Botvinnik, Ebert and Randall-Williams [BER14] on the non-contractibility of the latter.

This thesis is organized in the following way: The first chapter provides a rough sketch of the dynamical formulation of general relativity, just enough to establish a connection between the strict dominant energy condition for spacetimes and the corresponding notion for initial values. Then, a comparison map between the space of positive scalar curvature metrics and the space of initial values satisfying the strict dominant energy condition is constructed. This map induces a homomorphism on homotopy groups and it is the aim to show that the homomorphism has non-trivial image.

The second chapter starts with an introduction of $Cl_{n,k}$-linear Fredholm operators along the example of the Cl_n-linear Dirac operator. Afterwards, the family version of the KO-valued index map for these operators is discussed in some detail. The index map is then used to construct the α-invariant. In [CSS18] and [BER14], this particular invariant was shown, under certain assumptions on the manifold, to be non-trivial, so it witnesses that the homotopy groups of the space of positive scalar curvature metrics are non-trivial.

The last chapter mainly deals with the construction of a similar invariant for the space of initial values. It begins with a detailed analysis of the $Cl_{n,1}$-linear hypersurface spinor bundle and its Dirac-Witten operator: It is necessary to observe that these structures fit into the framework of $Cl_{n,1}$-linear Fredholm operators in order to be able to apply the index map. Moreover, a special focus is laid on comparing the hypersurface spinor bundle to the ordinary spinor bundle and the Dirac-Witten operator to the Dirac operator as this will be needed in the proof of the main theorem. The chapter concludes with the definition of the α-invariant for initial values and the proof of the main theorem. This theorem states that the classical α-invariant factors over the new one, via the comparison map from the first chapter. Together with the non-triviality results for the classical α-invariant mentioned above, this implies that, under suitable assumptions on the hypersurface, the space of initial values satisfying the strict dominant energy condition must be homotopically non-trivial.

1. The dominant energy condition for initial values

1.1. Energy conditions in general relativity

According to general relativity theory, the universe can be modeled by a Lorentzian [1] manifold $(N, \overline{g})$ and its large-scale behavior is governed by the Einstein equation [2]

$$\mathrm{ric}^{\overline{g}} - \tfrac{1}{2}\,\mathrm{scal}^{\overline{g}}\,\overline{g} = T,$$

where T is the energy-momentum tensor, a quantity determined by the distribution of matter and fields. For many purposes, it is more appropriate to adopt the following point of view: From a given initial distribution of matter and fields the Einstein equation together with specific matter equations determines the future and the past.

We want to make this more precise. We assume that N is time-oriented and foliated into spacelike hypersurfaces, the foliation being given by $M_t = f^{-1}(t)$ for a function $f\colon N \to \mathbb{R}$ with $\mathrm{grad}(f)$ past-timelike. At least locally, around an initial spacelike hypersurface $M = M_0$, such a function can always be found by patching together the time-variables of Fermi coordinates. Globally, this imposes a condition on N that, for example, rules out closed causal curves.

The Gauß, Codazzi and Mainardi equations compute the curvature of N in terms of quantities of the leaves M_t. For $n = \dim M_t \geq 2$, this leads to a reformulation of the Einstein equation (cf. [BI04]):

Proposition 1.1 (Dynamical formulation of the Einstein equation)**.** *Let e_0 be the future-directed unit normal on M_t, g the induced metric and $\mathrm{I\!I} = K e_0$ its vector-valued second fundamental form. Furthermore, we split up the energy-momentum tensor into components:*

$$\begin{aligned} \textit{energy density} \quad & \rho = T(e_0, e_0), \\ \textit{momentum density} \quad & j = T(e_0, -)_{|TM_t} \quad \textit{and} \\ & \hat{T} = T_{|TM_t \otimes TM_t}. \end{aligned}$$

The Einstein equation is equivalent to the following: The constraint equations

$$\begin{aligned} 2\rho &= \mathrm{scal}^g + (\mathrm{tr}\,K)^2 - \|K\|^2 \\ j &= \mathrm{div}\,K - \mathrm{d}\,\mathrm{tr}\,K \end{aligned}$$

hold on every leaf M_t and the evolution equation

$$\mathcal{L}_{e_0} K = \hat{T} - \tfrac{1}{n-1}\,\mathrm{tr}(T) g - \mathrm{ric}^g + 2K^2 - \mathrm{tr}(K)K + L^{-1}\,\mathrm{Hess}(L).$$

[1] We will use the signature convention $(-, +, \ldots, +)$.

[2] In the literature, there exist various conventions for constants in front of T. As they do not play a role in the mathematical theory, we subsume them under T.

holds at all M_t. Thereby, $L = \sqrt{-\overline{g}(\operatorname{grad} f, \operatorname{grad} f)}$ is the lapse function and $K^2 = K(K^\sharp(-), -)$.

For the vacuum case (i.e. $T = 0$), this leads to the following solution strategy: Given a solution (g_0, K_0) of the vacuum constraints

$$\begin{aligned} 0 &= \operatorname{scal}^{g_0} + (\operatorname{tr} K_0)^2 - \|K_0\|^2 \\ 0 &= \operatorname{div} K_0 - \mathrm{d} \operatorname{tr} K_0 \end{aligned}$$

on the initial hypersurface $M = M_0$, one tries to solve the system of the evolution equations[3]

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} K_t &= -\operatorname{ric}^{g_t} + 2K_t^2 - \operatorname{tr}(K_t) K_t \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\, g_t &= \tfrac{1}{2} K_t \end{aligned} \tag{1}$$

on an open neighborhood N of $M \times \{0\}$ in $M \times \mathbb{R}$. Once one has shown that the solution (g_t, K_t) of (1) to the initial values (g_0, K_0) solves the vacuum constraints for all t, the above Proposition 1.1 shows that $(N, -\mathrm{d}t^2 + g_t)$ solves the vacuum Einstein equation.

With presence of matter, a similar procedure can be thought of. The matter equations will result in additional evolution equations for the fields and there might be further constraints (cf. [Ise95] for constraints in Einstein-Maxwell theory). However, it seems to be impossible to include all kinds of matter, let alone to solve the resulting system. Hence, for cosmological considerations, as the famous singularity theorems of Hawking and Penrose [HE73, Sec. 8.2], a common property of (almost) all matter models is exploited: The energy-momentum tensor satisfies certain energy conditions. An often considered energy condition is the dominant energy condition:

Definition 1.2. The energy-momentum tensor T satisfies the *dominant energy condition* if $T(V, W) \geq 0$ for all future-causal vectors V, W.

We also define two strict versions of the dominant energy condition.

Definition 1.3. The energy-momentum tensor T satisfies the *strict dominant energy condition* if $T(V, W) > 0$ for all future-causal vectors V, W with $\overline{g}(V, W) < 0$. It satisfies the *very strict dominant energy condition* if $T(V, W) > 0$ holds for all future-causal V and W.

Remark 1.4. By the Einstein equation, the energy-momentum tensor is given as a curvature term. So the ((very) strict) dominant energy condition is a curvature condition for $(N, \overline{g})$.

No classical matter model satisfies either of the strict conditions as the special case of vacuum does not. Yet, under the additional condition that the matter density nowhere

[3]Apart from $T = 0$, we fixed the gauge by $L \equiv 1$ resulting in a particularly simple equation.

vanishes, solid matter satisfies the strict dominant energy condition as the following example (adapted from [Mül16]) shows.

Example 1.5. Solid matter is described by a field Φ and its energy-momentum tensor is given by

$$T(V,W) = \partial_V \Phi \partial_W \Phi - \frac{1}{2}(\overline{g}(\operatorname{grad}\Phi, \operatorname{grad}\Phi) + m^2\Phi^2)\overline{g}(V,W)$$

for a constant $m > 0$ (cf. [HE73, (3.6)]). We show that for all $p \in N$, all causal vectors $V, W \in T_pN$ with $\overline{g}(V,W) \leq 0$ and any $X \in T_pN$

$$2\overline{g}(V,X)\overline{g}(W,X) \geq \overline{g}(X,X)\overline{g}(V,W) \tag{2}$$

holds. Then, setting $X = \operatorname{grad}_p \Phi$,

$$T(V,W) \geq -\frac{1}{2}m^2\Phi^2(p)\overline{g}(V,W) > 0$$

if $\Phi(p) \neq 0$ and $\overline{g}(V,W) < 0$.

It suffices to prove (2) for timelike V, then it holds true for all causal vectors by continuity. Moreover, by scaling invariance, we can assume without loss of generality that $\overline{g}(V,V) = -1$. For a vector X, we define its parallel and perpendicular part to be $X^{\|} = -\overline{g}(X,V)V$ and $X^{\perp} = X - X^{\|}$, respectively. Then

$$\begin{aligned} 2\overline{g}(V,X)\overline{g}(W,X) - \overline{g}(X,X)\overline{g}(V,W) &= -2\overline{g}(X,V)^2\overline{g}(V,W) + 2\overline{g}(X,V)\overline{g}(X^{\perp},W) \\ &\quad + \overline{g}(X,V)^2\overline{g}(V,W) - \overline{g}(X,X^{\perp})\overline{g}(V,W) \\ &= -\overline{g}(X,V)^2\overline{g}(V,W) + 2\overline{g}(X,V)\overline{g}(X^{\perp},W^{\perp}) \\ &\quad - \overline{g}(X^{\perp},X^{\perp})\overline{g}(V,W). \end{aligned}$$

As V is timelike, the Lorentzian inner product restricts to a Riemannian one on the orthogonal complement. In particular, we can use the Cauchy-Schwarz inequality along with

$$\begin{aligned} 0 \geq \overline{g}(W,W) &= \overline{g}(W^{\|},W^{\|}) + 2\overline{g}(W^{\|},W^{\perp}) + \overline{g}(W^{\perp},W^{\perp}) \\ &= -\overline{g}(W^{\|},W^{\|}) + 2\overline{g}(W^{\|},W) + \overline{g}(W^{\perp},W^{\perp}) \\ &= \overline{g}(V,W)^2 - 2\overline{g}(V,W)^2 + \overline{g}(W^{\perp},W^{\perp}) \\ &= -\overline{g}(V,W)^2 + \overline{g}(W^{\perp},W^{\perp}) \end{aligned}$$

to get

$$\overline{g}(X^{\perp},W^{\perp})^2 \leq \overline{g}(X^{\perp},X^{\perp})\overline{g}(W^{\perp},W^{\perp}) \leq \overline{g}(X^{\perp},X^{\perp})\overline{g}(V,W)^2.$$

Thus

$$\begin{aligned} 2\overline{g}(V,X)\overline{g}(W,X) - \overline{g}(X,X)\overline{g}(V,W) &\geq -\overline{g}(X,V)^2\overline{g}(V,W) - \overline{g}(X^{\perp},X^{\perp})\overline{g}(V,W) \\ &\quad + 2|\overline{g}(X,V)|\overline{g}(X^{\perp},X^{\perp})^{\frac{1}{2}}\overline{g}(V,W) \\ &= -\left(\overline{g}(X^{\perp},X^{\perp})^{\frac{1}{2}} - |g(X,V)|\right)^2 g(V,W) \geq 0. \end{aligned}$$

The importance of the very strict dominant energy condition lies in the fact that it is an open condition: If it is satisfied in some point $p \in N$, then it is satisfied on a neighborhood of p. First, we examine why this does not hold for either of the other two dominant energy conditions.

Example 1.6. Let (M, g) be a Riemannian manifold of dimension $n > 1$ with constant Ricci curvature $\mathrm{ric} = \lambda g$ for some $\lambda > 0$ that will be determined later. For example, (M, g) could be a rescaled standard sphere. We consider the manifold $N = M \times \mathbb{R}$ with the Lorentzian metric $\overline{g} = -\mathrm{d}t^2 + f(t)^2 g$ with $f(t) = 1 + \frac{1}{2}at^2$ for $a > 0$. By the formulae for the curvature of warped product metrics [ONe83, Cor. 7.43]

$$\mathrm{ric}^{\overline{g}} = -n\frac{f''(t)}{f(t)}\mathrm{d}t^2 + \left(\frac{\lambda}{f(t)^2} - \frac{f''(t)}{f(t)} - (n-1)\frac{f'(t)^2}{f(t)^2}\right) f(t)^2 g.$$

The scalar curvature then computes to

$$\mathrm{scal}^{\overline{g}} = n\frac{\lambda}{f(t)^2} - n(n-1)\frac{f'(t)^2}{f(t)^2}.$$

This implies that the energy-momentum tensor is given by

$$T = \mathrm{ric}^{\overline{g}} - \frac{1}{2}\,\mathrm{scal}^{\overline{g}}\,\overline{g} = -A(t)\mathrm{d}t^2 + B(t)f(t)^2 g$$

for

$$\begin{aligned} A(t) &= n\frac{f''(t)}{f(t)} - \frac{n}{2}\frac{\lambda}{f(t)^2} + \frac{n}{2}(n-1)\frac{f'(t)^2}{f(t)^2} \\ B(t) &= -\frac{f''(t)}{f(t)} + \frac{2-n}{2}\frac{\lambda}{f(t)^2} + \frac{n-2}{2}(n-1)\frac{f'(t)^2}{f(t)^2}. \end{aligned}$$

Setting

$$\lambda = \frac{2(n+1)}{n(n-1)} \qquad \text{and} \qquad a = \frac{2}{n(n-1)},$$

provides $A(0) = -1$ and $B(0) = -1$. Thus, $T = -\overline{g}$ on $M \times \{0\}$, so the (strict) dominant energy condition is satisfied there. As

$$\begin{aligned} f(t)^2(A(t) - B(t)) &= (n+1)f''(t)f(t) - \lambda + (n-1)f'(t)^2 \\ &= (n+1)a\left(1 + \frac{1}{2}at^2\right) - \lambda + (n-1)a^2t^2 \\ &= \frac{3n-1}{2}a^2t^2, \end{aligned}$$

$A(t) - B(t) > 0$ for $t \neq 0$. For $p \in M$ and $t \neq 0$, let $V \in T_{(p,t)}N$ be a future-lightlike vector. Then

$$T(V,V) = -(A(t) - B(t))\mathrm{d}t^2(V,V) + B(t)\overline{g}(V,V) = -(A(t) - B(t))\mathrm{d}t^2(V,V) < 0,$$

but if T satisfied the (strict) dominant energy condition in (p,t), then $T(V,V) \geq 0$ would hold by continuity.

Lemma 1.7. *If the very strict dominant energy condition is satisfied in a point $p \in N$, then it is satisfied on a neighborhood of p.*

Proof. Without loss of generality, we can, by restricting on a small neighborhood of p, assume that TN is trivial. We choose a positive definite scalar product on T_pN and consider the associated compact unit sphere $S \subseteq T_pN$. Then

$$J := \left\{ (V, W) \in S \times S \,\middle|\, V,\, W \text{ are future-causal with respect to } \overline{g}_p \right\} \subseteq S \times S$$

is compact as a closed subset of a compact one and

$$\varepsilon := \min_{(V,W) \in J} T_p(V, W) > 0$$

by the very strict dominant energy condition. Now, we define the compact set

$$C := T_p^{-1}((-\infty, \tfrac{\varepsilon}{2}]) \cap (S \times S).$$

Hence the map

$$\begin{aligned} T_p^*N \otimes T_p^*N &\longrightarrow \mathbb{R} \\ \overline{h} &\longmapsto \min_{V,W \in C} \max\{\overline{h}(V, V),\, \overline{h}(W, W),\, \overline{h}(V, W)\} \end{aligned}$$

is well-defined and continuous. We observe the following: If there is a pair (V, W) of vectors in C which is future-causal with respect to $\overline{h}$, then $\overline{h}$ is mapped to a non-positive number. Conversely, if $\overline{h}$ is in the preimage of $\mathbb{R}_{\leq 0}$, then there is a pair (V, W) of causal vectors in C lying in the same component of the light cone and using the symmetry of C under $(V, W) \mapsto (-V, -W)$, we can assume that they are both future-directed. So a metric $\overline{h}$ maps to a positive number iff all pairs of future-causal vectors are contained in $(S \times S) \setminus C$.

The way ε and C are defined, $\overline{g}_p$ is mapped to a positive number and by continuity the same holds for all metrics in an open neighborhood. Moreover, the energy-momentum tensors T in a sufficiently small neighborhood of T_p satisfy $|T(V, W) - T_p(V, W)| < \frac{\varepsilon}{2}$ for all $(V, W) \in S \times S$. In particular,

$$T(V, W) > 0 \quad \text{for all } (V, W) \in (S \times S) \setminus C.$$

Hence, on the intersection of these neighborhoods, the very strict dominant energy condition holds. $\square$

We now study the following question: Given a spacetime that satisfies the dominant energy condition, what can be said about the initial value pairs (g, K) arising as induced metric and second fundamental form on some spacelike hypersurface?

Definition 1.8. A pair of initial data (g, K) satisfies the *dominant energy condition* if $\rho \geq \|j\|$ holds. Thereby,

$$\begin{aligned} 2\rho &= \operatorname{scal}^{g_t} + (\operatorname{tr} K)^2 - \|K\|^2 \\ j &= \operatorname{div} K - \mathrm{d} \operatorname{tr} K. \end{aligned}$$

It satisfies the *strict dominant energy condition* if this inequality holds strictly.

Lemma 1.9. *If $(N, \overline{g})$ satisfies the strict dominant energy condition, then for any spacelike hypersurface $M \subseteq N$ the induced pair (g, K) satisfies the strict dominant energy condition. The same holds true for the (non-strict) dominant energy condition.*

Proof. Let $p \in M$ be an arbitrary point. First, we study the case $j_p = 0$. As the unit normal e_0 is future-causal, the dominant energy condition implies

$$\rho_p = T_p(e_0, e_0) \geq 0 = \|j_p\|$$

with strict inequality if the strict dominant energy condition is satisfied by (N, g).

So we are left with the case $j_p \neq 0$. Then the vector $V = e_0 - \frac{j_p^\sharp}{\|j_p\|}$ is well-defined. As

$$\begin{aligned} \overline{g}(V, V) &= \overline{g}(e_0, e_0) - \frac{2}{\|j_p\|}\overline{g}(e_0, j_p^\sharp) + \frac{1}{\|j_p\|}\overline{g}(j_p^\sharp, j_p^\sharp) = -1 + 1 = 0 \\ \overline{g}(e_0, V) &= \overline{g}(e_0, e_0) - \frac{1}{\|j_p\|}\overline{g}(e_0, j_p^\sharp) = -1, \end{aligned}$$

V is causal and future-directed. So

$$0 \leq T_p(e_0, V) = T_p(e_0, e_0) - \frac{1}{\|j_p\|}T_p(e_0, j_p^\sharp) = \rho_p - \|j_p\|.$$

Furthermore, this inequality holds strictly if the strict dominant energy condition is satisfied. □

Proposition 1.10. *Let (g, K) be a pair of initial data on M satisfying the strict dominant energy condition. Then there is a spacetime $(N, \overline{g})$ that contains M as spacelike hypersurface, induces (g, K) on M and satisfies the very strict dominant energy condition.*

Proof. We set $\overline{N} = M \times \mathbb{R}$ and identify M with $M \times \{0\}$. In the end, N will be an open neighborhood of M in $\overline{N}$. The strategy is now the following: For a good choice of a symmetric $(0, 2)$-tensor H, we define

$$\begin{aligned} g_t &= g + 2tK + t^2 H \\ \overline{g}_{(p,t)} &= -\mathrm{d}t^2 + g_{t|p}. \end{aligned}$$

Then the induced metric on M is $g_0 = g$ and the induced second fundamental form is $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} g_t = K$ (for the choice of $f(p,t) = t$ as time function), as required. As g is positive definite, $\overline{g}$ defines a Lorentzian metric in a neighborhood of M. The choice of H will be made in such a way that T satisfies the very strict dominant energy condition all over M. Lemma 1.7 then implies that we can find a neighborhood N of M on which the very strict dominant energy condition is satisfied.

It remains to choose H appropriately. First, we show that if we set $\hat{T} = \frac{1}{\rho} j \otimes j$, where ρ and j are determined by (g, K), then

$$T = \rho \mathrm{d}t^2 + j \otimes \mathrm{d}t + \mathrm{d}t \otimes j + \hat{T}$$

satisfies the very strict dominant energy condition on M. Note, that the condition $\rho > \|j\| \geq 0$ ensures that $\hat{T}$ is well-defined and smooth. Let $p \in M$ and $V, W \in T_p\overline{N}$ be future-causal. We write $V = \alpha e_0 + X$, $W = \beta e_0 + Y$ for $X, Y \in T_pM$. As they are future-causal, $\alpha, \beta > 0$ and $\alpha \geq \|X\|, \beta \geq \|Y\|$. Then

$$\begin{aligned}
T(V,W) &= \tfrac{1}{\rho}(\rho\alpha + j(X))(\rho\beta + j(Y)) \\
&= \tfrac{1}{\rho}\Big((\rho - \|j\|)\alpha + \|j\|\alpha + j(X)\Big)\Big((\rho - \|j\|)\beta + \|j\|\beta + j(Y)\Big) \\
&\geq \tfrac{1}{\rho}\Big((\rho - \|j\|)\alpha + \|j\|(\alpha - \|X\|)\Big)\Big((\rho - \|j\|)\beta + \|j\|(\beta - \|Y\|)\Big) > 0.
\end{aligned}$$

We now set

$$H = \hat{T} - \tfrac{1}{n-1}\operatorname{tr}(\hat{T})g + \tfrac{1}{n-1}\rho g - \operatorname{ric}^g + 2K^2 - \operatorname{tr}(K)K$$

with $\hat{T} = \frac{1}{\rho} j \otimes j$ as above. This has the following reason: Let $T_0 = (\operatorname{ric}^{\overline{g}} - \frac{1}{2}\operatorname{scal}^{\overline{g}}\overline{g})_{|M}$. By Proposition 1.1

$$H = \tfrac{1}{2}\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}_{|t=0} g_t = \hat{T}_0 - \tfrac{1}{n-1}\operatorname{tr}(\hat{T}_0)g + \tfrac{1}{n-1}\rho g - \operatorname{ric}^g + 2K^2 - \operatorname{tr}(K)K.$$

Comparing these expressions, we obtain

$$\hat{T} - \tfrac{1}{n-1}\operatorname{tr}(\hat{T})g = \hat{T}_0 - \tfrac{1}{n-1}\operatorname{tr}(\hat{T}_0)g$$

and it follows that $\hat{T}_0 = \hat{T}$. So at M, the energy-momentum tensor is presicely the one we want. $\square$

We have seen that the initial data pairs satisfying the strict dominant energy condition are the ones that give rise to spacetimes with strict dominant energy condition. In the remainder of this work, we want to draw our attention towards the *space* of such initial data pairs. We will study this space by comparing it to the space of metrics of positive scalar curvature. The comparison map will be established in the next section.

1.2. Positive scalar curvature and initial values

In the following, M is a compact smooth manifold of dimension $n \geq 2$. Let $\mathcal{R}(M)$ be the space of smooth metrics endowed with C^∞-topology and $\mathcal{R}^+(M)$ the (possibly empty) subspace of positive scalar curvature metrics. Furthermore, we denote by $\mathcal{I}(M)$ the C^∞-space of pairs (g, K) consisting of a metric g and a symmetric $(0,2)$-tensor K and by $\mathcal{I}^+(M)$ the subspace of those pairs satisfying the strict dominant energy condition defined in the previous section. The aim of this section is to construct a continuous map

$$\Phi \colon \Sigma\,\mathcal{R}^+(M) \longrightarrow \mathcal{I}^+(M)$$

such that $\Phi|_{\mathcal{R}^+(M)\times\{0\}}$ is the inclusion $g \mapsto (g, 0)$.

Lemma 1.11. *For every $C > 0$, the function*

$$\begin{aligned}\tau \colon \mathcal{R}(M) &\longrightarrow \mathbb{R}\\ g &\longmapsto \sqrt{\frac{n}{n-1}\max\{0, \sup_{x\in M} -\mathrm{scal}^g(x)\} + C}\end{aligned}$$

is continuous.

Proof. It suffices to show that the assignment $g \mapsto \sup_{x\in M}\mathrm{scal}^g(x)$ is continuous. This breaks into two pieces: Firstly, the function $C^0(M) \to \mathbb{R}$, $f \mapsto \sup_{x\in M} f(x)$ is (Lipschitz-)continuous, because for all $f, g \in C^0(M)$

$$\begin{aligned}\sup_{x\in M} f(x) - \sup_{x\in M} g(x) &= \sup_{x\in M}\left(f(x) - g(x) + g(x) - \sup_{y\in M} g(y)\right)\\ &\leq \sup_{x\in M}(f(x) - g(x)) \leq \|f - g\|_{C^0}.\end{aligned}$$

and analogously $\sup_{x\in N} g(x) - \sup_{x\in N} f(x) \leq \|f - g\|_{C^0}$. Secondly, the continuity of $\mathcal{R}(M) \to C^0(N)$, $g \mapsto -\mathrm{scal}^g$ follows from the fact that the scalar curvature can be expressed locally as a function of the coefficients of the metric and their first and second derivatives (cf. Theorem A.7). $\square$

Proposition 1.12. *For any $C > 0$ and $I = [-1, 1]$, the following is a well-defined continuous map of pairs*[4]*:*

$$\begin{aligned}\phi \colon (\mathcal{R}(M), \mathcal{R}^+(M)) \times (I, \partial I) &\longrightarrow (\mathcal{I}(M), \mathcal{I}^+(M))\\ (g, t) &\longmapsto \left(g, \frac{\tau(g)}{n} t g\right).\end{aligned}$$

Moreover, its homotopy class $[\phi] \in [(\mathcal{R}(M), \mathcal{R}^+(M)) \times (I, \partial I)\,, (\mathcal{I}(M), \mathcal{I}^+(M))]$ is independent of $C > 0$.

[4]For two pairs (X, A) and (Y, B), we write $(X, A) \times (Y, B) := (X \times Y, X \times B \cup Y \times A)$.

Proof. Continuity directly follows from the lemma above. Moreover, varying the parameter $C > 0$ defines a continuous homotopy between different such maps. Thus, it only remains to prove that $\mathcal{R}(M) \times \partial I \cup \mathcal{R}^+(M) \times I$ is mapped into $\mathcal{I}^+(M)$. To this aim, we first observe that for a pair of the form $(g, \frac{\tau}{n} g)$ with $\tau \in \mathbb{R}$

$$\begin{aligned} 2\rho &= \operatorname{scal} + \frac{n-1}{n}\tau^2 \\ j &= \frac{1-n}{n} \operatorname{grad} \tau = 0 \end{aligned}$$

holds. Hence, such a pair fulfills the strict dominant energy condition if and only if

$$\tau^2 > -\frac{n}{n-1} \operatorname{scal}.$$

But by definition of the function τ, this is the case for $\left(g, \pm\frac{\tau(g)}{n} g\right)$, which shows that $\mathcal{R}(M) \times \partial I$ maps into $\mathcal{I}^+(M)$. Moreover, the condition is automatically satisfied if g has positive scalar curvature, so $\mathcal{R}^+(M) \times I$ is sent to $\mathcal{I}^+(M)$ as well. $\square$

Proposition 1.13. *Let $C > 0$ and $h \in \mathcal{R}(M)$ a Riemannian metric. Then the composition*

$$\Phi \colon \Sigma \mathcal{R}^+(M) \longrightarrow \mathcal{R}(M) \times \partial I \cup \mathcal{R}^+(M) \times I \xrightarrow{\phi} \mathcal{I}^+(M),$$

where the first map is given by

$$[g, t] \longmapsto \begin{cases} ((-2t-1)h + 2(1+t)g, -1) & t \in [-1, -\frac{1}{2}] \\ (g, 2t) & t \in [-\frac{1}{2}, \frac{1}{2}] \\ ((2t-1)h + 2(1-t)g, 1) & t \in [\frac{1}{2}, 1], \end{cases}$$

is a well-defined, continuous map. Its homotopy class is independent of $C > 0$ and $h \in \mathcal{R}(M)$.

Proof. By the previous proposition, we just need to study the first map: Plugging in $t = \pm\frac{1}{2}$, we see that the different definitions agree on the intersections, and for the special values $t = \pm 1$ we observe that the result is independent of g, i.e. the map descends to the suspension. This shows well-definedness. Continuity can now be checked on each domain of definition, where it is obvious. Moreover, this map continuously depends on $h \in \mathcal{R}(M)$, so by connectedness of $\mathcal{R}(M)$, its homotopy class is independent of h. $\square$

Corollary 1.14. *The inclusion $\mathcal{R}^+(M) \to \mathcal{I}^+(M)$, $g \mapsto (g, 0)$ is null-homotopic. In particular, if there exists a metric $g_0 \in \mathcal{R}^+(M)$, the induced map on homotopy groups $\pi_k(\mathcal{R}^+(M), g_0) \to \pi_k(\mathcal{I}^+(M), (g_0, 0))$ is the zero-map for all k.*

Proof. Using the map defined above, we get a factorization of the inclusion map as follows

$$\mathcal{R}^+(M) \hookrightarrow C\,\mathcal{R}^+(M) \hookrightarrow \Sigma\,\mathcal{R}^+(M) \xrightarrow{\Phi} \mathcal{I}^+(M),$$

where the first two maps are the canonical inclusions of a space into the its cone and of the cone into the suspension as upper half. As cones are contractible, the composition is null-homotopic. □

This shows that we cannot find non-trivial elements of homotopy groups in the space initial data with strict dominant energy condition by simply considering the space of positive scalar curvature metrics as subspace. However, the map Φ defined above allows for a better construction: We will show that under certain conditions the composition

$$\pi_k(\mathcal{R}^+(M), g_0) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma\,\mathcal{R}^+(M), [g_0, 0]) \xrightarrow{\Phi_*} \pi_{k+1}(\mathcal{I}^+(M), (g_0, 0))$$

has non-trivial image. First, though, we will take a look at the map that witnesses that there are non-trivial elements in $\pi_k(\mathcal{R}^+(M), g_0)$: the α-invariant.

2. The classical α-invariant

2.1. KO-theory via Fredholm operators

In this section, we introduce certain spaces of Clifford-linear Fredholm operators and relate them to KO-theory. The tools and examples developed here, will be needed later to define the α-invariant. In this presentation, we basically follow Ebert [Ebe13].

Definition 2.1. For $n, k \in \mathbb{Z}_{\geq 0}$, a $\mathbb{Z}/2\mathbb{Z}$*-graded* $Cl_{n,k}$*-Hilbert space* is a separable, real Hilbert space H together with a bounded linear operator $\iota\colon H \to H$ called *grading operator* and a linear map $c\colon \mathbb{R}^{n+k} \to B(H)$ called *Clifford multiplication* satisfying the following properties:

$$\begin{aligned}
\iota^2 &= \mathbb{1} \\
c(v)^2 &= (-\langle v_1, v_1\rangle + \langle v_2, v_2\rangle)\,\mathbb{1} \\
\iota c(v) &= -c(v)\iota \\
\iota^* &= \iota \\
c(v_1 + v_2)^* &= c(-v_1 + v_2)
\end{aligned}$$

for all $v = v_1 + v_2 \in \mathbb{R}^{n+k} = \mathbb{R}^n \oplus \mathbb{R}^k$ with $v_1 \in \mathbb{R}^n \oplus 0$ and $v_2 \in 0 \oplus \mathbb{R}^k$. Thereby, $\langle -, - \rangle$ denotes the standard Euclidean scalar product. $\mathbb{Z}/2\mathbb{Z}$-graded $Cl_{n,k}$-Hilbert spaces form a category, with morphisms $F\colon (H_1, \iota_1, c_1) \to (H_2, \iota_2, c_2)$ being bounded linear maps of the underlying Hilbert spaces that are even (i.e. $\iota_2 F = F\iota_1$) and $Cl_{n,k}$-linear (i.e. $c_2(v)F = Fc_1(v)$ for all $v \in \mathbb{R}^{n,k}$). A finite-dimensional $\mathbb{Z}/2\mathbb{Z}$-graded $Cl_{n,k}$-Hilbert space is called $\mathbb{Z}/2\mathbb{Z}$*-graded* $Cl_{n,k}$*-module.*

For simplicity, we will often write Cl_n instead of $Cl_{n,0}$. As we do not consider the ungraded notions normally, we will drop the term $\mathbb{Z}/2\mathbb{Z}$-graded for convenience.

Example 2.2. The Clifford algebra $Cl_{n,k}$ is a $Cl_{n,k}$-module in the following way:

- The scalar product $\langle -, - \rangle$ on $Cl_{n,k}$ is defined by the requirement that the standard basis $(e_{i_1} \cdots e_{i_l})_{0 \leq l \leq n,\, 1 \leq i_1 < \cdots < i_l \leq n+k}$ is orthonormal.

- The grading operator $\iota = \alpha\colon Cl_{n,k} \to Cl_{n,k}$ is given by the Cliffordization of the map $\mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$, $v \mapsto -v$.

- The Clifford multiplication $c = R\colon \mathbb{R}^{n+k} \to \mathrm{End}(Cl_{n,k})$ is given by multiplication from the right: $v \mapsto (\alpha \mapsto \alpha \cdot v)$.

It is clear that α is an involution, R satisfies the Clifford relations and that they anti-commute. As α is diagonal with respect to the standard basis, we have $\alpha^* = \alpha$. For the last property, we observe that given a standard basis vector $e_I \in Cl_{n,k}$ not containing e_j then

$$R(e_j)e_I = (-1)^l e_J$$

for another standard basis vector e_J and l being the number of swaps necessary to bring e_j to the right position. Then

$$R(e_j)e_J = (-1)^l R(e_j)^2 e_I = -\varepsilon_j(-1)^l e_I$$

where $\varepsilon_j = 1$ for $j \in \{1, \dots n\}$ and $\varepsilon_j = -1$ for $j \in \{n+1, \dots, e_{n+k}\}$. So we see that with the right ordering of basis vectors, $R(e_j)$ is of block diagonal form with blocks

$$\begin{pmatrix} 0 & (-1)^l \\ -\varepsilon_j(-1)^l & 0 \end{pmatrix}.$$

In the case $j \in \{1, \dots, n\}$, this is anti-symmetric, thus $R(e_j)^* = R(-e_j)$, and if $j \in \{n+1, \dots, n+k\}$, then $R(e_j)^* = R(e_j)$.

Note, that the same works with the left multiplication L. In particular,

$$\langle L(e_j)\Phi, \Psi\rangle = -\varepsilon_j \langle \Phi, L(e_j)\Psi\rangle \tag{3}$$

for all $\Phi, \Psi \in Cl_{n,k}$. The use of right multiplication was motivated by the next example.

Example 2.3. Let (M, g) be an n-dimensional Riemannian spin manifold with a spin structure $P_{\mathrm{Spin}(n)}M$. Then the Cl_n*-linear spinor bundle* is the associated bundle

$$\Sigma_{Cl}M = P_{\mathrm{Spin}(n)}M \times_{\mathrm{Spin}(n)} Cl_n,$$

where $\mathrm{Spin}(n)$ acts on Cl_n via Clifford multiplication from the left. As left multiplication by elements of $\mathrm{Spin}(n)$ commutes with both α and R, they give rise to bundle homomorphisms

$$\begin{aligned} \alpha\colon\ \Sigma_{Cl}M &\longrightarrow \Sigma_{Cl}M \\ R\colon\ \underline{\mathbb{R}}^n &\longrightarrow \mathrm{End}(\Sigma_{Cl}M). \end{aligned}$$

Furthermore, (3) implies that $\langle -, -\rangle$ extends to a bundle metric on $\Sigma_{Cl}M$. All these structures along with their relations derived for Cl_n extend to the Hilbert space $H = L^2(M, \Sigma_{Cl}M)$ of L^2-sections, making it a Cl_n-Hilbert space.

The following *Morita equivalences* play an important role in the classification of $Cl_{n,k}$-modules.

Lemma 2.4. *For all $n, k \geq 0$, the categories of*

1. *$Cl_{n,k}$-Hilbert spaces and $Cl_{n+1,k+1}$-Hilbert spaces*

2. *$Cl_{n+4,k}$-Hilbert spaces and $Cl_{n,k+4}$-Hilbert spaces*

are equivalent.

Proof. For the first part, we identify $\mathbb{R}^{n+1+k+1} = \mathbb{R}^{n+1} \oplus \mathbb{R}^{k+1}$ with $\mathbb{R}^{n+k} \oplus \operatorname{span}\{e, \varepsilon\}$, where e is the last basis vector of $\mathbb{R}^{n+1}$ and ε the last basis vector of $\mathbb{R}^{k+1}$. On objects a functor in the one direction is given by mapping a $Cl_{n,k}$-Hilbert space (H, ι, c) to $(H \oplus H, \tilde{\iota}, \tilde{c})$, where

$$\begin{aligned}
\tilde{\iota} &= \begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix} \\
\tilde{c}(v) &= \begin{pmatrix} c(v) & 0 \\ 0 & -c(v) \end{pmatrix} \qquad \text{for all } v \in R^{n+k} \oplus 0 \\
\tilde{c}(e) &= \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \\
\tilde{c}(\varepsilon) &= \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix},
\end{aligned}$$

and on morphisms by mapping F to

$$\tilde{F} = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}.$$

It is easily checked that this defines a $Cl_{n+1,k+1}$-Hilbert space and a morphism of such, respectively.

For the converse direction, we map a $Cl_{n+1,k+1}$-Hilbert space (H, ι, c) to (H_0, ι_0, c_0), whereby $H_0 = \ker(c(\varepsilon)c(e) - \mathbb{1})$ is the 1-eigenspace of the involution $c(\varepsilon)c(e)$ and $\iota_0 \colon H_0 \to H_0$ and $c_0 \colon \mathbb{R}^{n+k} \to B(H_0)$ are appropriate restrictions of ι and c, respectively. Also, a morphism $F \colon H \to H'$ is mapped to its restriction $F_0 \colon H_0 \to H'_0$. All of these restrictions are well-defined, because the operators commute with $c(\varepsilon)c(e)$.

We need to see that these functors are mutually inverse up to natural isomorphism. Starting in the category of $Cl_{n,k}$-Hilbert spaces, we note that

$$\tilde{H}_0 = \ker(\tilde{c}(\varepsilon)\tilde{c}(e) - \mathbb{1}) = \ker \begin{pmatrix} 0 & 0 \\ 0 & -2\,\mathbb{1} \end{pmatrix} = H \oplus 0$$

and $\tilde{\iota}_0 = \iota$ as well as $\tilde{c}_0 = c$ under the identification $H \oplus 0 \cong H$. In the same way, $\tilde{F}_0 = F$ holds.

Now we start with a $Cl_{n+1,k+1}$-Hilbert space (H, ι, c). Since $c(\varepsilon)c(e)$ is a self-adjoint involution, the spectral theorem implies that there is an eigenspace decomposition

$$H = \ker(\tilde{c}(\varepsilon)\tilde{c}(e) - \mathbb{1}) \oplus \ker(\tilde{c}(\varepsilon)\tilde{c}(e) + \mathbb{1}) = H_0 \oplus c(\varepsilon)H_0,$$

where the last equality results from the fact that $c(\varepsilon)$ interchanges the eigenspaces. Under the obvious isomorphism $H_0 \oplus c(\varepsilon)H_0 \cong H_0 \oplus H_0$,

$$\iota = \begin{pmatrix} \iota_{|H_0} & 0 \\ 0 & \iota_{|c(\varepsilon)H_0} \end{pmatrix}$$

gets mapped to

$$\begin{pmatrix} \iota_0 & 0 \\ 0 & c(\varepsilon)\iota c(\varepsilon)_{|H_0} \end{pmatrix} = \begin{pmatrix} \iota_0 & 0 \\ 0 & -\iota_0 \end{pmatrix}$$

and this, along with the same argument for c, implies that $(H, \iota, c) \cong (\widetilde{H_0}, \widetilde{\iota_0}, \widetilde{c_0})$. In the same way a morphism F is identified with $\widetilde{F_0}$.

The second equivalence is easier to describe. Let us regard $\mathbb{R}^{n+k+4}$ as $\mathbb{R}^n \oplus \mathbb{R}^k \oplus \operatorname{span}\{e_1, e_2, e_3, e_4\}$, where $e_1, \dots e_4$ are meant to change their role from being the last four basis vectors of $\mathbb{R}^{n+4}$ to being the last four basis vectors of $\mathbb{R}^{k+4}$ and vice versa. Now, we map a $Cl_{n+4,k}$-Hilbert space (H, ι, c) to $(H, \iota, \tilde{c})$ with $\tilde{c}$ defined by $\tilde{c}_{|\mathbb{R}^{n+k}} = c_{|\mathbb{R}^{n+k}}$ along with $\tilde{c}(e_i) = \eta c(e_i)$ for $\eta = c(e_1) \cdots c(e_4)$. It can be checked that $\eta^2 = \mathbb{1}$ and that $(H, \iota, \tilde{c})$ defines a $Cl_{n,k+4}$-Hilbert space. A morphism is mapped to the morphism defined by the same underlying bounded linear map.

Conversely, for a $Cl_{n,k+4}$-Hilbert space $(H, \iota, \tilde{c})$ we define the corresponding $Cl_{n+4,k}$-Hilbert space (H, ι, c) by $c_{|\mathbb{R}^{n+k}} = \tilde{c}_{|\mathbb{R}^{n+k}}$ and $c(e_i) = \tilde{\eta}\tilde{c}(e_i)$ for $\tilde{\eta} = \tilde{c}(e_1) \cdots \tilde{c}(e_4)$. Again, the image of a morphism is defined by the same underlying bounded linear map. Since in both cases $\eta = \tilde{\eta}$, these functors are seen to be mutually inverse. $\square$

Proposition 2.5. *If $n - k \not\equiv 0 \mod 4$, there is an irreducible $Cl_{n,k}$-module, unique up to isomorphism. If $n - k \equiv 0 \mod 4$, there are two isomorphism classes of irreducible $Cl_{n,k}$-modules, and they are distinguished by whether the volume element $\omega_{n,k} = \iota c(e_1) \dots c(e_{n+k})$ acts as $+1$ or -1.*

Proof. This is the statement of [LM89, Thm I.5.7] along with the discussion of [LM89, Thm I.5.9]. Note that in this reference, $Cl_{n,k}$-modules are ungraded, which results in an index shift: $\mathbb{Z}/2\mathbb{Z}$-graded $Cl_{n,k}$-modules correspond to ungraded $Cl_{n,k+1}$-modules setting $c(e_{n+k+1}) = \iota$. $\square$

Definition 2.6. A $Cl_{n,k}$-Hilbert space is called *ample*, if it contains each irreducible $Cl_{n,k}$-module with infinite multiplicity.

Example 2.7. Let $H = L^2(M, \Sigma_{Cl}M)$ with the Cl_n-Hilbert space structure defined in Example 2.3. H is ample if $n = \dim(M) > 0$: Let $U \subseteq M$ be a non-empty open subset such that there exists a non-vanishing vector field $X \in \mathfrak{X}(U)$ (e.g. U can be a chart neighborhood of a point). We can assume that X is a unit vector field. Let us now consider the inclusion

$$H' = L^2(U, \Sigma_{Cl}M_{|U}) \hookrightarrow H$$

defined by zero continuation. Note that H' is infinite-dimensional and each eigenspace of $\omega_{n,0|H'}$ includes into the corresponding eigenspace of $\omega_{n,0}$. The operator

$$L(X)\colon H' \to H'$$

induced by left Clifford multiplication with X is invertible as $L(X)^2 = -\mathbb{1}$. Left and right Clifford multiplication commute, thus we have

$$L(X)\omega_{n,0} = L(X)\alpha R(e_1)\cdots R(e_n) = -\alpha L(X)R(e_1)\cdots R(e_n) = -\omega_{n,0}L(X).$$

This implies that $L(X)$ maps the 1-eigenspace of $\omega_{n,0|H'}$ to the -1-eigenspace and vice versa. So these eigenspaces both must be infinite-dimensional.

Definition 2.8. Let (H, ι, c) be an ample $Cl_{n,k}$-Hilbert space. Then a *$Cl_{n,k}$-Fredholm operator* is a (bounded) Fredholm operator on H that is self-adjoint, odd with respect to ι, $Cl_{n,k}$-linear and, in the case $n-k \equiv -1 \mod 4$, satisfies the additional condition that $\omega_{n,k}F\iota$ is neither essentially positive nor essentially negative. We denote by $\mathrm{Fred}^{n,k}(H)$ the space of $Cl_{n,k}$-Fredholm operators with operator norm topology. Furthermore, we write $G^{n,k}(H) \subseteq \mathrm{Fred}^{n,k}(H)$ for the subspace of invertible elements.

Remark 2.9. Assume that F is a self-adjoint, odd, $Cl_{n,k}$-linear Fredholm operator on an infinite-dimensional $Cl_{n,k}$-Hilbert space H. If $n-k \equiv 1$ or $2 \mod 4$, then $F \in \mathrm{Fred}^{n,k}(H)$. In the case $n-k \equiv 0 \mod 4$ we have to additionally check that H is ample and in the case $n-k \equiv -1 \mod 4$ the condition concerning the essential spectrum needs to be checked. Now assume that the Clifford action extends, so that H is a $Cl_{n+1,k}$- or a $Cl_{n,k+1}$-Hilbert space. Then the additional generator C of the Clifford action satisfies $\omega_{n,k}C = -C\omega_{n,k}$ for $n-k \equiv 0 \mod 4$, so H is ample as $Cl_{n,k}$-Hilbert space. If, furthermore, F is Clifford-linear with respect to the extended Clifford multiplication, then $\omega_{n,k}F\iota$ anti-commutes with C for $n-k \equiv -1 \mod 4$ and so its spectrum is neither essentially positive nor essentially negative. Thus, in any case, $\mathrm{Fred}^{n+1,k}(H) \subseteq \mathrm{Fred}^{n,k}(H)$ and $\mathrm{Fred}^{n,k+1}(H) \subseteq \mathrm{Fred}^{n,k}(H)$.

Example 2.10. Let $H = L^2(M, \Sigma_{Cl}M)$ be defined as above and assume additionally that M is compact. The Levi-Civita connection on M induces a connection on $P_{\mathrm{Spin}(n)}M$, which defines a connection ∇ on $\Sigma_{Cl}M$. The composition

$$D\colon \Gamma(\Sigma_{Cl}M) \xrightarrow{\nabla} \Gamma(T^*M \otimes \Sigma_{Cl}M) \xrightarrow{\sharp\otimes\mathbb{1}} \Gamma(TM \otimes \Sigma_{Cl}M) \xrightarrow{L} \Gamma(\Sigma_{Cl}M)$$

of connection, musical isomorphism and (left) Clifford multiplication defines the $Cl_{n,k}$-*linear Dirac operator*, which extends to an unbounded operator $D\colon H \to H$. It is immediate that D is odd and Cl_n-linear. Moreover, general results for Dirac operators imply that D is formally self-adjoint and admits a spectral decomposition $D = \sum_{i=0}^{\infty} \lambda_i \pi_{E_i}$ with finite dimensional eigenspaces E_i and discrete eigenvalues λ_i (cf. [Roe99, Thm. 5.27]). Hence the bounded transform

$$F = \frac{D}{\sqrt{1+D^2}} = \sum_{i=0}^{\infty} \frac{\lambda_i}{\sqrt{1+\lambda_i^2}} \pi_{E_i}$$

is a well-defined, self-adjoint Fredholm operator that is odd and Cl_n-linear. In the case $n \not\equiv -1 \mod 4$ this already shows $F \in \mathrm{Fred}^{n,0}(H)$.

For the remaining case, we study the operator $\tilde{D} = \omega_{n,0} D \iota = R(e_1) \cdots R(e_n) D$. As $R(e_1) \cdots R(e_n)$ and D commute, each eigenspace of D can be further decomposed into a 1- and a -1-eigenspace of $R(e_1) \cdots R(e_n)$. So the eigenvalues of the product $\tilde{D}$ accumulate at least at ∞ or $-\infty$. If we can show that both of them are accumulation points, then

$$\frac{\tilde{D}}{\sqrt{1+\tilde{D}^2}} = R(e_1) \cdots R(e_n) \frac{D}{\sqrt{1+D^2}} = \omega_{n,0} F \iota$$

is neither essentially positive nor essentially negative, as needed.

In order to do so, we adopt the argument of [Amm17, Prop 7.21]: We assume that ∞ is the only accumulation point for contradiction, the argumentation for $-\infty$ being analogous. Let λ_0 be the smallest eigenvalue. We cover M by chart neighborhoods $U_1, \dots, U_N$ and choose a partition of unity $\Psi_1^2, \dots, \Psi_j^2$ subordinate to the cover, where $\Psi_i \in C^{\infty}_{\geq 0}(M)$ for all i. Using Gram-Schmidt orthonormalization, there exists an orthonormal frame $e_1^i, \dots, e_n^i$ over U_i for each i. Now, let $\tilde{D}\phi = \lambda\phi$. As $\Psi_i e_j^i$ is a smooth function, $L(\Psi_i e_j^i)\phi$ is in the domain of $\tilde{D}$ and

$$\sum_{i,j} \left(\tilde{D} L(\Psi_i e_j^i)\phi, L(\Psi_i e_j^i)\phi \right) \geq \sum_{i,j} \lambda_0 \left(L(\Psi_i e_j^i)\phi, L(\Psi_i e_j^i)\phi \right) = \sum_{i,j} \Psi_i^2 \lambda_0 (\phi, \phi) = n\lambda_0 \|\phi\|^2.$$

On the other hand,

$$\begin{aligned}
\sum_{i,j} \left(\tilde{D} L(\Psi_i e_j^i)\phi,\, L(\Psi_i e_j^i)\phi \right) &= \sum_{i,j,k} \left(L(e_k^i) R(e_1) \dots R(e_n) \nabla_{e_k^i} L(\Psi_i e_j^i)\phi,\, L(\Psi_i e_j^i)\phi \right) \\
&= \sum_{i,j,k} \left(L(e_k^i) R(e_1) \dots R(e_n) L(\nabla_{e_k^i} \Psi_i e_j^i)\phi,\, L(\Psi_i e_j^i)\phi \right) \\
&\quad + \sum_{i,j,k} \left(L(e_k^i) R(e_1) \dots R(e_n) L(\Psi_i e_j^i) \nabla_{e_k^i}\phi,\, L(\Psi_i e_j^i)\phi \right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i,j,k} \Psi_i \|\nabla_{e_k^i} \Psi_i e_j^i\|_{C^0} \|\phi\|^2 \\
&\quad - \sum_{i,j,k} \Psi_i^2 \left(L(e_j^i) R(e_1) \dots R(e_n) L(e_k^i) \nabla_{e_k^i} \phi,\, L(e_j^i)\phi \right) \\
&\quad + 2 \sum_{i,j=k} \Psi_i^2 \left(L(e_j^i) R(e_1) \dots R(e_n) L(e_k^i) \nabla_{e_k^i} \phi,\, L(e_j^i)\phi \right) \\
&\leq C\|\phi\|^2 - \sum_{i,j} \Psi_i^2 \lambda\,(\phi,\, \phi) + 2\sum_i \Psi_i^2 \lambda\,(\phi,\, \phi) \\
&= (C - (n-2)\lambda)\|\phi\|^2
\end{aligned}
$$

for some C independent of ϕ and λ. Putting those inequalities together,

$$
n\lambda_0 + (n-2)\lambda \leq C
$$

for all eigenvalues λ. As $n \geq 3$ (if $n \equiv -1 \mod 4$) and ∞ is an accumulation point, this is a contradiction. Thus, $F \in \mathrm{Fred}^{n,0}(H)$ for all $n > 0$.

Furthermore, if g is a metric of positive scalar curvature, then the Schrödinger-Lichnerowicz formula (cf. [Roe99, Props. 3.18 and 4.21])

$$
D^2 = \nabla^* \nabla + \frac{\mathrm{scal}}{4}\,\mathbb{1}
$$

implies that $\ker D = 0$ and so $F \in G^{n,0}(H)$.

The following consequence of Kuiper's theorem is proven in [Ebe13]:

Proposition 2.11. *The space $G^{n,k}(H)$ is contractible for all $n, k \geq 0$.*

Proposition 2.12. *The Morita equivalences induce homeomorphisms of pairs*

$$
\begin{aligned}
(\mathrm{Fred}^{n,k}(H), G^{n,k}(H)) &\longrightarrow (\mathrm{Fred}^{n+1,k+1}(H \oplus H), G^{n+1,k+1}(H \oplus H)) \\
F &\longmapsto \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathrm{Fred}^{n+4,k}(H), G^{n+4,k}(H)) &\longrightarrow (\mathrm{Fred}^{n,k+4}(H), G^{n,k+4}(H)) \\
F &\longmapsto F.
\end{aligned}
$$

In particular, there is a homoemorphism

$$
\begin{aligned}
(\mathrm{Fred}^{n,k}(H), G^{n,k}(H)) &\longrightarrow (\mathrm{Fred}^{n+8,k}(H \otimes \mathbb{R}^{16}), G^{n+8,k}(H \otimes \mathbb{R}^{16})) \\
F &\longmapsto F \otimes \mathbb{1}_{\mathbb{R}^{16}}\,.
\end{aligned}
$$

Proof. We use the same notation as in the proof of Lemma 2.4. The first thing to check is that $H \oplus H$ is ample iff H is ample. In the case $n-k \not\equiv 0 \mod 4$, this is just the fact that H is infinite-dimensional iff $H \oplus H$ is. In the other case, we note that $\omega_{n+1,k+1}$ is up to sign given by

$$\begin{pmatrix} \omega_{n,k} & 0 \\ 0 & \omega_{n,k} \end{pmatrix}.$$

Hence, the 1- and -1-eigenspaces of $\omega_{n+1,k+1}$ are both infinite-dimensional iff those of $\omega_{n,k}$ are. It is easily checked that if F is a self-adjoint, odd and $Cl_{n,k}$-linear Fredholm operator, then its image is a self-adjoint, odd and $Cl_{n+1,k+1}$-linear Fredholm operator and if F is invertible, so is its image. This shows the well-definedness of the first map up to the additional condition in case $n-k \equiv -1 \mod 4$. For this, we note that

$$\begin{pmatrix} \omega_{n,k} & 0 \\ 0 & -\omega_{n,k} \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix} = \begin{pmatrix} \omega_{n,k} F \iota & 0 \\ 0 & \omega_{n,k} F \iota \end{pmatrix}$$

is neither essentially positive nor essentially negative iff $\omega_{n,k} F \iota$ satisfies this condition. Clearly the map is continuous and we have to construct a continuous inverse. For this, let

$$F = \begin{pmatrix} F_0 & F_1 \\ F_2 & F_3 \end{pmatrix} \in \mathrm{Fred}^{n+1,k+1}(H \oplus H).$$

As it has to commute with

$$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

we must have $F_0 = F_3$ and $F_1 = F_2 = 0$. So the assignment $F \mapsto F_0$ defines a continuous map of pairs, inverse to the first map.

For the second part, it suffices to note that under the performed change of the Clifford structure, $\tilde{c}(e_i) = \eta c(e_i)$, Clifford-linearity of F is left unchanged and the volume element ω changes at most its sign.

The last part is obtained by applying four times the first map and then once the inverse of the second one. □

We now turn our attention to KO-theory. Classically, KO is defined as follows.

Definition 2.13. Let X be a compact space. The *KO-group of X*, denoted by $KO(X)$, is the Grothendieck group associated to the abelian monoid

$$(\{\text{real vector bundles over } X\}/\text{vector bundle isomorphisms}, \oplus).$$

If $x_0 \in X$ is a base point, then the *reduced KO-group of* (X, x_0) is

$$\widetilde{KO}(X) = \ker(KO(X) \to KO(\{x_0\})),$$

where the map is induced by pullback via $\{x_0\} \hookrightarrow X$. If $Y \subseteq X$ is closed, then the *relative KO-group of the pair* (X, Y) is $KO(X, Y) = \widetilde{KO}(X/Y)$ where X/Y is defined by the pushout

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \{*\} \arrow[d] \\ X \arrow[r] & X/Y \end{tikzcd}$$

and the base point is given by the canonical map $\{*\} \to X/Y$. Moreover, for $n \geq 0$, we define higher KO-groups by

$$\begin{aligned} KO^{-n}(X, Y) &= \widetilde{KO}(\Sigma^n_{red} X/Y) \\ KO^{-n}(X) &= KO^{-n}(X, \varnothing), \end{aligned}$$

where Σ_{red} is the reduced suspension of a pointed space.

There is much that can be said about these groups. For instance, they are homotopy invariant and the tensor product turns $KO^{-*}(\{*\})$ into a graded ring and $KO^{-*}(X, Y)$ into a $KO^{-*}(\{*\})$-module. Hence the following theorem (cf. [LM89, Thms. 9.21 and 9.22]) tells much about the structure of KO-theory.

Theorem 2.14 (Bott periodicity)**.**

$$KO^*(\{*\}) \cong \mathbb{Z}[\eta, y, x]/(2\eta, \eta^3, \eta y, y^2 - 4x)$$

with $\deg(\eta) = -1$, $\deg(y) = -4$, $\deg(x) = -8$. *In particular, for* $n \geq 0$

$$KO^{-n}(\{*\}) \cong \begin{cases} \mathbb{Z} & n \equiv 0, 4 \mod 8 \\ \mathbb{Z}/2\mathbb{Z} & n \equiv 1, 2 \mod 8 \\ 0 & n \equiv 3, 5, 6, 7 \mod 8 \end{cases}.$$

Moreover, multiplication by the generator $x \in KO^{-8}(\{*\})$ induces an isomorphism

$$KO^{-n}(X, Y) \xrightarrow{\cong} KO^{-n-8}(X, Y)$$

for all compact pairs (X, Y).

This allows us to extend the definition of KO to positive degrees in such a way that KO^* is 8-periodic, i.e. $KO^n(X, Y) \cong KO^{n-8k}(X, Y)$ as abelian groups.

Definition 2.15. With abuse of notation, let $KO^*(\{*\})$ be the localization of the version from Definition 2.13 at the generator $x \in KO^{-8}(\{*\})$ and $KO^*(X,Y)$ be the $KO^*(\{*\})$-module obtained by localizing the previously defined version at x.

The connection between KO-groups and the spaces of Fredholm operators is given by the index map.

Theorem 2.16 (Index map)**.** *If H is an ample $Cl_{n,k}$-Hilbert space, then* $\mathrm{Fred}^{n,k}(H)$ *represents KO-theory: For compact relative CW-complexes (X,Y), there is a natural bijection*

$$\mathrm{ind}\colon [(X,Y),(\mathrm{Fred}^{n,k}(H),G^{n,k}(H))] \longrightarrow KO^{k-n}(X,Y)$$

called index map*. In particular, the class of null-homotopic maps is mapped to zero. Moreover,* ind *is invariant under $Cl_{n,k}$-Hilbert space isomorphisms, i.e. if $U\colon H \to H'$ is an isomorphism of $Cl_{n,k}$-Hilbert spaces, then*

$$\begin{tikzcd}[(X,Y),(\mathrm{Fred}^{n,k}(H),G^{n,k}(H))] \arrow[rr, "\cong"] \arrow[dr, "\mathrm{ind}"'] & & [(X,Y),(\mathrm{Fred}^{n,k}(H'),G^{n,k}(H'))] \arrow[dl, "\mathrm{ind}"] \\ & KO^{k-n}(X,Y) & \end{tikzcd}$$

commutes, where the upper map is induced by $\mathrm{Fred}^{n,k}(H) \ni F \mapsto UFU^{-1}$.

The theorem is a consequence of the following two results.

Theorem 2.17. *For compact spaces X, there is a natural bijection*

$$\mathrm{ind}\colon [X,\mathrm{Fred}^{0,0}(H)] \longrightarrow KO(X),$$

which induces, for compact CW-pairs (X,Y), a natural bijection

$$\mathrm{ind}\colon [(X,Y),(\mathrm{Fred}^{0,0}(H),G^{0,0}(H))] \longrightarrow KO^0(X,Y).$$

Moreover, ind *is invariant under Cl_0-Hilbert space isomorphisms.*

Proof. The first part of this theorem can be found in [AS69], with the difference that, instead of $\mathrm{Fred}^{0,0}(H)$, the space of all (bounded) Fredholm operators $F(H_0)$ on a separable, infinite-dimensional real Hilbert space H_0 is considered there. But those two spaces can be easily identified: The ampleness condition for H implies that both eigenspaces of the involution ι are infinite-dimensional. So we can choose an isometric isomorphism $H \cong H_0 \oplus H_0$ for $H_0 = \ker(\iota - \mathbb{1}) \cong \ell^2$ such that ι is given by

$$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

Then F is of the form

$$\begin{pmatrix} F_0 & F_1 \\ F_2 & F_3 \end{pmatrix}.$$

Since F anti-commutes with ι, we must have $F_0 = F_3 = 0$. Moreover, $F^* = F$ implies $F_2 = F_1^*$. So F is of the form

$$\begin{pmatrix} 0 & F_1 \\ F_1^* & 0 \end{pmatrix}$$

with $F_1 \colon H_0 \to H_0$ being a Fredholm operator. Conversely, a Fredholm operator on a separable, infinite-dimensional Hilbert space H_0 defines a $Cl_{0,0}$-Fredholm operator on $H = H_0 \oplus H_0$ by the above formula.

Let us take a short look at how the index map is constructed: Given a map $F_1 \colon X \to F(H_0)$, by compactness of X, a closed subspace $V \subseteq H_0$ of finite codimension can be found such that $V \cap \ker(F_1(x)) = 0$ for all $x \in X$. Then

$$\operatorname{ind}([F_1]) = [\ker F_1 P_V] - [\operatorname{coker} F_1 P_V] \in KO(X)$$

with P_V being the orthogonal projection on V (cf. [Ati67], particularly for well-definedness). Noting that $\operatorname{coker} F_1 P_V \cong \ker(F_1 P_V)^* = \ker P_V F_1^*$, we conclude that in the $\mathbb{Z}/2\mathbb{Z}$-graded picture the index of $F \colon X \to F(H)$ is given by

$$\operatorname{ind}([F]) = [\ker(P_V F P_V)_{|H_0}] - [\ker(P_V F_1 P_V)_{|H_0^\perp}] \in KO(X)$$

where the closed subspace $V \subseteq H_0 \subseteq H$ of finite codimension in H_0 is chosen such that $V \cap \ker(F(x))_{|H_0} = 0$ for all $x \in X$. From this description, we see that the index map is invariant under Cl_0-Hilbert space isomorphisms.

For the second part, we note that for the one-point space $\{x_0\}$, we can take $V = \ker(F_1)^\perp$, so the map is given by

$$\begin{aligned}
[\{x_0\}, F(H_0)] &\to KO(\{x_0\}) \cong \mathbb{Z} \\
[F_1] &\mapsto \operatorname{index} F_1 = \dim\ker F_1 - \dim\operatorname{coker} F_1.
\end{aligned}$$

This implies that it is the path component containing the identity (and all invertible operators) that is mapped to zero. Now, let X be a compact space with non-degenerate base point $x_0 \in X$. We consider

$$\begin{tikzcd}
{[(X, x_0), (F(H_0), \mathbb{1})]} \arrow[r, dashed] \arrow[d, hook] & \widetilde{KO}(X) \arrow[d, hook] \\
{[X, F(H_0)]} \arrow[r, "\cong"] \arrow[d, "res"] & KO(X) \arrow[d, "res"] \\
{[\{x_0\}, F(H_0)]} \arrow[r, "\cong"] & KO(\{x_0\}).
\end{tikzcd}$$

By definition, $\widetilde{KO}(X)$ is the fiber over 0 with respect to the restriction map. Hence, if we can show that on the left hand side $[(X, x_0), (F(H_0), \{\mathbb{1}\})]$ is preimage of the identity component under the restriction map, the dashed map exists and is a bijection.

As $x_0 \in X$ is non-degenerate, the preimage under the restriction map is given by

$$\{[f] \in [X, F(H_0)] \,|\, \exists g \in [f] \colon g(x_0) = \mathbb{1}\} \subseteq [X, F(H_0)].$$

This is almost the set in the left-hand upper corner of the diagram. Indeed, there is a canonical comparison map

$$[(X, x_0), (F(H_0), \mathbb{1})] \longrightarrow \{[f] \in [X, F(H_0)] \,|\, \exists g \in [f] \colon g(x_0) = \mathbb{1}\},$$

which is surjective. For injectivity, we need to show that whenever two pointed maps $f, g \colon (X, x_0) \to (F(H_0), \mathbb{1})$ are homotopic, then they are also pointed homotopic. Let $H \colon X \times [0,1] \to F(H_0)$ be the homotopy. We set $\gamma = H(x_0, -) \colon [0,1] \to F(H_0)$, this is a path in $F(H_0)$ from $\mathbb{1}$ to $\mathbb{1}$. We remember that under the projection $p \colon B(H_0) \to B(H_0)/K(H_0)$ to the Calkin algebra, Fredholm operators are mapped to invertible elements. Moreover, according to the theorem of Bartle and Graves [BG52], p has a continuous (though not linear) section σ. Then $t \mapsto \sigma(p(\gamma(t))^{-1})\gamma(t)$ agrees with the identity up to a compact operator. So

$$\begin{aligned} \widetilde{H} \colon (X, x_0) \times [0,1] &\to (F(H_0), \mathbb{1}), \\ (x,t) &\to \sigma(p(\gamma(t))^{-1}) H(x,t) + \mathbb{1} - \sigma(p(\gamma(t))^{-1})\gamma(t) \end{aligned}$$

is a well-defined pointed homotopy from f to g.

Now, the claim follows from the following line of natural isomorphisms:

$$\begin{aligned} [(X,Y), (\mathrm{Fred}^{0,0}(H), G^{0,0}(H))] &\cong \left[(X,Y), \left(\mathrm{Fred}^{0,0}(H_0 \oplus H_0), \left\{\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\right\}\right)\right] \\ &\cong [(X,Y), (F(H_0), \{\mathbb{1}\})] \\ &\cong [(X/Y, Y/Y), (F(H_0), \{\mathbb{1}\})] \\ &\cong \widetilde{KO}(X/Y) = KO^0(X,Y) \end{aligned}$$

Note that the first isomorphism uses that $G^{0,0}(H)$ is contractible (Proposition 2.11) along with the fact that $Y \hookrightarrow X$ is a cofibration. $\square$

Theorem 2.18 (Bott map)**.** *For compact CW-pairs (X,Y), the map*

$$\begin{aligned} [(X,Y), (\mathrm{Fred}^{n+1,k}(H), G^{n+1,k}(H))] &\longrightarrow [(X,Y) \times (I, \partial I), (\mathrm{Fred}^{n,k}(H), G^{n,k}(H))] \\ [x \mapsto F_x] &\longmapsto [(x,t) \mapsto F_x + tc(e)\iota] \end{aligned}$$

is a natural bijection. Thereby, e is the additional basis vector of $\mathbb{R}^{n+1+k} = \mathbb{R} \oplus \mathbb{R}^{n+k}$ and $I = [-1, 1]$.

Proof. In essence, this is the statement of [AS69, Thm A(k)]. In this source, however, the spaces of Fredholm operators are denoted by $F_*^n(H_0)$ and defined slightly differently: If $n = 0$, $F_*^0(H_0) = F(H_0)$ is the space defined in the previous proof and we have already seen, how this relates to $\mathrm{Fred}^{0,0}(H_0 \oplus H_0)$.

If $n > 0$, for an ample ungraded Cl_{n-1}-Hilbert space H_0, the space $F_*^n(H_0)$ is the space of skew-adjoint, Cl_{n-1}-anti-linear Fredholm operators F_0 on H_0 that satisfy the additional condition that $c(e_1)\cdots c(e_{n-1})F_0$ is neither essentially positive nor essentially negative if $n \equiv -1 \mod 4$. This corresponds to the $\mathrm{Fred}^{n,0}(H)$-spaces by a construction similar to the first Morita equivalence: If $F_0 \in F_*^n(H_0)$, then

$$F = \begin{pmatrix} 0 & F_0 \\ -F_0 & 0 \end{pmatrix} \in \mathrm{Fred}^{n,0}(H_0 \oplus H_0),$$

where the Cl_n-Hilbert space structure on $H_0 \oplus H_0$ is defined by

$$\iota = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix},\ c(e_n) = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix},\ c(e_i) = \begin{pmatrix} c(e_i) & 0 \\ 0 & -c(e_i) \end{pmatrix} \text{ for } i = 1, \dots, n-1.$$

Conversely, an $F \in \mathrm{Fred}^{n,0}(H)$ defines on $H_0 = \ker(\iota c(e_n) - \mathbb{1})$ an operator $F_0 = c(e_n)F_{|H_0} \in F_*^n(H_0)$. These two procedures are mutually inverse.

With these translations at hand, the main theorem of [AS69] together with the contractibility of $G^{n,0}(H)$ implies that

$$\begin{aligned} [X, \mathrm{Fred}^{n+1,0}(H)] &\longrightarrow [X \times (I, \partial I), (\mathrm{Fred}^{n,0}(H), G^{n,0}(H))] \\ [x \mapsto F_x] &\longmapsto \left[(x,t) \mapsto \cos\left(\frac{\pi}{2}t\right) F_x + \sin\left(\frac{\pi}{2}t\right) c(e)\iota\right] \end{aligned} \tag{4}$$

is an isomorphism for all X. This is the map from the claim, as $(x,t) \mapsto F_x + tc(e)\iota$ and $(x,t) \mapsto \cos\left(\frac{\pi}{2}t\right) F_x + \sin\left(\frac{\pi}{2}t\right) c(e)\iota$ are homotopic as maps of pairs via the linear homotopy connecting those. This works as $aF_x + bc(e)\iota$ is invertible if $a \neq 0$ and F_x is invertible or simply $b \neq 0$, since $(aF_x + bc(e)\iota)^2 = a^2F_x^2 + b^2\,\mathbb{1}$.

The occurrence of the term $c(e)\iota$ in (4) is explained as follows: If $n > 1$, then the corresponding term on H_0-level (i.e. in [AS69]) is given by $-c(e)$, which translates into

$$\begin{pmatrix} 0 & -c(e) \\ c(e) & 0 \end{pmatrix} = -\begin{pmatrix} c(e) & 0 \\ 0 & -c(e) \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = -c(e)\iota.$$

Precomposition with the homeomorphism $X \times (I, \partial I) \to X \times (I, \partial I),\ (x,t) \mapsto (x,-t)$ corrects the sign. In the case $n = 1$, the situation is more delicate. On H_0, the term in [AS69] is given by $-\mathbb{1}$, which corresponds to

$$\begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = -\iota$$

by the identification for $n = 0$. However, this construction of the Cl_0-Hilbert space from H_0 yields

$$\left(H, \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\right) = (H, \iota c(e))$$

instead of (H, ι), which would be the result of just forgetting the $c(e)$-action of the Cl_1-Hilbert space (H, ι, c). Hence we apply the isomorphism of ($\mathbb{Z}/2\mathbb{Z}$-graded) Cl_0-Hilbert spaces given by $\frac{1}{\sqrt{2}}(\iota + \iota c(e))$. This transforms the Fredholm operator $-\iota$ into

$$\frac{1}{2}(\iota + \iota c(e))(-\iota)(\iota + \iota c(e)) = -\frac{1}{2}(\iota + \iota c(e) + \iota c(e) - \iota) = c(e)\iota.$$

The generalization to compact CW-pairs (X,Y) works as in the previous theorem. First, we consider the case of a compact pointed space (X, x_0) with non-degenerate base point. Then in the diagram

$$\begin{tikzcd}
{[(X,x_0),(\mathrm{Fred}^{n+1,0}(H),G^{n+1,0}(H))]} \arrow[r, dashed] \arrow[d, hook] & {[(X,x_0)\times(I,\partial I),(\mathrm{Fred}^{n,0}(H),G^{n,0}(H))]} \arrow[d, hook] \\
{[X,\mathrm{Fred}^{n+1,0}(H)]} \arrow[r, "\cong"] \arrow[d, "res"] & {[X\times(I,\partial I),(\mathrm{Fred}^{n,0}(H),G^{n,0}(H))]} \arrow[d, "res"] \\
{[\{x_0\},\mathrm{Fred}^{n+1,0}(H)]} \arrow[r, "\cong"] & {[\{x_0\}\times(I,\partial I),(\mathrm{Fred}^{n,0}(H),G^{n,0}(H))]}
\end{tikzcd}$$

the upper spaces can be identified with fibers of the restriction maps. More precisely, they are seen to be the preimages of the subsets defined by those homotopy classes that contain a representative mapping into $G^{n+1,0}(H)$ or $G^{n,0}(H)$, respectively. As before, this requires to show that unpointed homotopies of maps $(X, x_0) \to (\mathrm{Fred}^{n+1,0}(H), G^{n+1,0}(H))$ can be turned into pointed homotopies. This can be done as in the previous proof, replacing $B(H)$ and $K(H)$ by their subspaces of $\mathbb{Z}/2\mathbb{Z}$-graded Cl_{n+1}-linear operators. By the same procedure, homotopies of maps $(X, x_0) \times (I, \partial I) \to (\mathrm{Fred}^{n,0}(H), G^{n,0}(H))$ relative $X \times \partial I$ give rise to homotopies relative $X \times \partial I \cup \{x_0\} \times I$.

Now, the commutativity of

$$\begin{tikzcd}
{[(X,Y),(\mathrm{Fred}^{n+1,0}(H),G^{n+1,0}(H))]} \arrow[r] \arrow[d, "\cong"] & {[(X,Y)\times(I,\partial I),(\mathrm{Fred}^{n,0}(H),G^{n,0}(H))]} \arrow[d, "\cong"] \\
{[(X/Y,Y/Y),(\mathrm{Fred}^{n+1,0}(H),G^{n+1,0}(H))]} \arrow[r, "\cong"] & {[(X/Y,Y/Y)\times(I,\partial I),(\mathrm{Fred}^{n,0}(H),G^{n,0}(H))]}
\end{tikzcd}$$

shows that the upper map is a bijection. Note that the vertical maps are bijective as $G^{n+1,0}(H)$ is contractible and $Y \subseteq X$ satisfies the homotopy extension property.

The generalization to arbitrary k works by induction, using that the Morita equivalences (Proposition 2.12) induce the commutative diagrams

$$\begin{array}{ccc} [(X,Y), (\mathrm{Fred}^{n+1,k}(H \oplus H), G^{n+1,k}(H \oplus H))] & \longrightarrow & [(X,Y) \times (I, \partial I), (\mathrm{Fred}^{n,k}(H \oplus H), G^{n,k}(H \oplus H))] \\ \downarrow \cong & & \downarrow \cong \\ [(X,Y), (\mathrm{Fred}^{n,k-1}(H), G^{n,k-1}(H))] & \longrightarrow & [(X,Y) \times (I, \partial I), (\mathrm{Fred}^{n-1,k-1}(H), G^{n-1,k-1}(H))] \end{array}$$

and

$$\begin{array}{ccc} [(X,Y), (\mathrm{Fred}^{n+5,k}(H), G^{n+5,k}(H))] & \longrightarrow & [(X,Y) \times (I, \partial I), (\mathrm{Fred}^{n+4,k}(H), G^{n+4,k}(H))] \\ \downarrow \cong & & \downarrow \cong \\ [(X,Y), (\mathrm{Fred}^{n+1,k+4}(H), G^{n+1,k+4}(H))] & \longrightarrow & [(X,Y) \times (I, \partial I), (\mathrm{Fred}^{n,k+4}(H), G^{n,k+4}(H))]. \end{array}$$

$\square$

Proof of Theorem 2.16. The case $(n,k) = (0,0)$ is Theorem 2.17. From this, we can define the map for all $(n,0)$ recursively using the map from Theorem 2.18:

$$\begin{aligned} [(X,Y), (\mathrm{Fred}^{n,0}(H), G^{n,0}(H))] &\xrightarrow{\cong} [(X,Y) \times (I, \partial I), (\mathrm{Fred}^{n-1,0}(H), G^{n-1,0}(H))] \\ &\xrightarrow{\cong} KO^{-(n-1)}((X,Y) \times (I, \partial I)) \\ &= \widetilde{KO}(\Sigma^{n-1}_{red}(X \times I)/(Y \times I \cup X \times \partial I)) \\ &\cong \widetilde{KO}(\Sigma^{n}_{red} X/Y) = KO^{-n}(X,Y) \end{aligned}$$

The first Morita equivalence allows to further generalize the definition to all (n,k) with $0 \leq k \leq n$ inductively:

$$\begin{aligned} [(X,Y), (\mathrm{Fred}^{n+1,k+1}(H), G^{n+1,k+1}(H))] &\xrightarrow{\cong} [(X,Y), (\mathrm{Fred}^{n,k}(H_0), G^{n,k}(H_0))] \\ &\xrightarrow{\cong} KO^{k-n}(X,Y) \end{aligned}$$

Lastly, we extend inductively to all pairs $(n,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ via

$$\begin{aligned} [(X,Y), (\mathrm{Fred}^{n,k}(H), G^{n,k}(H))] &\xrightarrow{\cong} [(X,Y), (\mathrm{Fred}^{n+8,k}(H \otimes \mathbb{R}^{16}), G^{n+8,k}(H \otimes \mathbb{R}^{16}))] \\ &\xrightarrow{\cong} KO^{k-n-8}(X,Y) \cong KO^{k-n}(X,Y), \end{aligned}$$

where the last map is the multiplication with the periodicity element $x^{-1} \in KO^8(\{*\})$.

Note that all the maps involved are natural in (X,Y). The "in particular" part then follows from the fact that for a map being null-homotopic means that it factors up to homotopy over $(\{*\}, *)$. Hence, its class is in the image of the upper left corner in the commutative diagram

$$\begin{array}{ccc} [(\{*\}, *), (\mathrm{Fred}^{n,k}(H), G^{n,k}(H))] & \longrightarrow & KO^{k-n}(\{*\}, *) = 0 \\ \downarrow & & \downarrow \\ [(X, Y), (\mathrm{Fred}^{n,k}(H), G^{n,k}(H))] & \longrightarrow & KO^{k-n}(X, Y). \end{array}$$

Since right hand map is a group homomorphism, its image is zero.

In the case $(n, k) = (0, 0)$, the index map was seen to be invariant under isomorphisms of Cl_0-Hilbert spaces. Now, we observe that the property of being invariant under $Cl_{n,k}$-Hilbert space isomorphisms is preserved in each inductive step – giving invariance for all $(n, k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. □

Remark 2.19. Looking at the proof of Theorem 2.16, we see that

$$\begin{array}{ccc} [(X, Y), (\mathrm{Fred}^{n,k}(H), G^{n,k}(H))] & \longrightarrow & KO^{k-n}(X, Y) \\ \downarrow{\cong} & & \downarrow{\cdot x} \\ [(X, Y), (\mathrm{Fred}^{n+8,k}(H \otimes \mathbb{R}^{16}), G^{n+8,k}(H \otimes \mathbb{R}^{16}))] & \longrightarrow & KO^{k-n-8}(X, Y) \end{array}$$

commutes by definition if $n < k$. The same is true for $k \leq n$ provided that the right generator $x \in KO^{-8}(\{*\})$ is chosen. This follows from the last remark in [AS69]. We will not make use of this fact.

Example 2.20. The index map can be used to define an interesting invariant for a compact Riemannian spin manifold (M, g) of dimension $n > 0$ with chosen spin structure, the so-called *α-index*. As explained above, the bounded transform of the Cl_n-linear Dirac operator

$$F = \frac{D}{\sqrt{1 + D^2}} : H = L^2(M, \Sigma_{Cl} M) \to H$$

defines an element of $\mathrm{Fred}^{n,0}(H)$. So we can define

$$\alpha = \mathrm{ind}(F) \in KO^{-n}(\{*\}).$$

If g has positive scalar curvature, then F is invertible and so $\alpha = 0$. It can be shown that α does only depend on the spin-bordism class of the spin manifold M. In particular, it is independent of the metric chosen. Thus $\alpha \neq 0$ is an obstruction to positive scalar curvature on M.

The α-index allows to detect that $\mathcal{R}^+(M)$ is empty. In the next section, we will define a more refined invariant that is able to detect non-trivial homotopy groups of $\mathcal{R}^+(M)$, provided that this space is non-empty.

2.2. Construction of the α-invariant

Let M be a compactl spin manifold of dimension $n > 0$ that has a positive scalar curvature metric g_0. The α-invariant is the map $\alpha\colon \pi_k(\mathcal{R}^+(M), g_0) \to KO^{-n-k-1}(\{*\})$ that arises in the following way: As $\mathcal{R}(M)$ is contractible, the long exact sequence for homotopy groups implies $\pi_k(\mathcal{R}^+(M), g_0) \cong \pi_{k+1}(\mathcal{R}(M), \mathcal{R}^+(M), g_0)$. For each metric g, the Cl_n-linear Dirac operator D_g defines a Cl_n-linear Fredholm operator

$$F_g = \frac{D_g}{\sqrt{1 + D_g^2}},$$

which is invertible if $g \in \mathcal{R}^+(M)$. The assignment $g \mapsto F_g$ gives rise to a map $(\mathcal{R}(M), \mathcal{R}^+(M)) \to (\mathrm{Fred}^{n,0}, G^{n,0})$, which induces a map to $\pi_{k+1}(\mathrm{Fred}^{n,0}, G^{n,0}, F_{g_0})$. Applying the index map from the last section, we obtain an element in $KO^{-n}(D^{k+1}, S^k) \cong KO^{-n-k-1}(\{*\})$.

In this outline, however, we glossed over the detail that the Cl_n-linear spinor bundles and hence the L^2-spaces, on which the Fredholm operators F_g act, depend on the metric g. These L^2-spaces form a Hilbert bundle over $\mathcal{R}(M)$, which, by Kuiper's theorem, can be trivialized. Such a trivialization allows to define the map $(\mathcal{R}(M), \mathcal{R}^+(M)) \to (\mathrm{Fred}^{n,0}, G^{n,0})$. We will make this more explicit: The Cl_n-linear spinor bundles for different metrics can be identified using the method of generalized cylinders due to Bär, Gauduchon and Moroianu [BGM05]. This gives rise to a specific trivialization of the Hilbert bundle of L^2-spaces.

Let us start with this construction by fixing a topological spin structure on M, i.e. a double covering

$$P_{\widetilde{GL}^+(n)}M \to P_{GL^+(n)}M$$

over the principal bundle of positively oriented frames of TM. This defines, for any $g \in \mathcal{R}(M)$, a spin structure for (M, g) by pullback

$$\begin{array}{ccc} P_{\mathrm{Spin}(n)}(M,g) & \longrightarrow & P_{\widetilde{GL}^+(n)}M \\ \downarrow & & \downarrow \\ P_{\mathrm{SO}(n)}(M,g) & \longrightarrow & P_{GL^+(n)}M, \end{array}$$

where $P_{\mathrm{SO}(n)}(M,g)$ is the principal bundle of positively oriented orthonormal frames with respect to g. Moreover, pulling back over the canonical projection $M \times [0,1] \to M$, we obtain

$$\begin{array}{ccc} P_{\widetilde{GL}^+(n)}M \times [0,1] & \longrightarrow & P_{\widetilde{GL}^+(n)}M \\ \big\downarrow & & \big\downarrow \\ P_{GL^+(n)}M \times [0,1] & \longrightarrow & P_{GL^+(n)}M \\ \big\downarrow & & \big\downarrow \\ M \times [0,1] & \longrightarrow & M. \end{array}$$

This gives rise a topological spin structure $P_{\widetilde{GL}^+(n+1)}M \times [0,1] \to P_{GL^+(n+1)}M \times [0,1]$ on $M \times [0,1]$ by extension along the standard embedding

$$\begin{aligned} GL^+(n) &\longrightarrow GL^+(n+1) \\ A &\longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

and its double covering.

Now, given a metric $g \in \mathcal{R}(M)$, we can define a family of metrics by $g_t = (1-t)g_0 + tg$. Such a family in turn defines the generalized cylinder $(M \times [0,1], g_t + \mathrm{d}t^2)$, t being the variable in $[0,1]$-direction. As above, the topological spin structure induces a spin structure $P_{\mathrm{Spin}(n+1)}(M\times[0,1], g_t+\mathrm{d}t^2) \to P_{\mathrm{SO}(n+1)}(M\times[0,1], g_t+\mathrm{d}t^2)$ on the generalized cylinder. This has the property that for all $t_0 \in [0,1]$ it restricts to the spin structure of (M, g_{t_0}) in the sense that

$$\begin{array}{ccc} P_{\mathrm{Spin}(n)}(M, g_{t_0}) & \longrightarrow & P_{\mathrm{Spin}(n+1)}(M \times [0,1], g_t + \mathrm{d}t^2) \\ \big\downarrow & & \big\downarrow \\ P_{\mathrm{SO}(n)}(M, g_{t_0}) & \longrightarrow & P_{\mathrm{SO}(n+1)}(M \times [0,1], g_t + \mathrm{d}t^2) \end{array}$$

is a pullback, where the lower map is the inclusion $(e_1, \ldots, e_n) \mapsto (e_1, \ldots, e_n, \frac{\partial}{\partial t})$.

The reason, why we do this is that on $P_{\mathrm{Spin}(n+1)}(M \times [0,1], g_t + \mathrm{d}t^2)$ the Levi-Civita connection induces a canonical connection ∇, which provides parallel transports

$$P^{\nabla}_{\gamma_x} \colon P_{\mathrm{Spin}(n+1)}(M \times [0,1], g_t + \mathrm{d}t^2)_{|(x,0)} \longrightarrow P_{\mathrm{Spin}(n+1)}(M \times [0,1], g_t + \mathrm{d}t^2)_{|(x,1)}$$

along the curves $\gamma_x \colon [0,1] \to M \times [0,1],\ t \mapsto (x,t)$ for all $x \in M$. These assemble into an isomorphism of principle bundles

$$P^{\nabla} \colon P_{\mathrm{Spin}(n+1)}(M \times [0,1], g_t + \mathrm{d}t^2)_{|M\times\{0\}} \xrightarrow{\cong} P_{\mathrm{Spin}(n+1)}(M \times [0,1], g_t + \mathrm{d}t^2)_{|M\times\{1\}}.$$

The fact that $\frac{\partial}{\partial t}$ is parallel along the curves γ_x implies that P^{∇} restricts to

$$P^{\nabla} \colon P_{\mathrm{Spin}(n)}(M, g_0) \xrightarrow{\cong} P_{\mathrm{Spin}(n)}(M, g),$$

and this induces an isomorphism on the associated Cl_n-linear spinor bundles

$$\begin{aligned} P^\nabla \colon \Sigma_{Cl}(M, g_0) &\xrightarrow{\cong} \Sigma_{Cl}(M, g). \\ [\tilde{\varepsilon}, \tilde{\phi}] &\longmapsto [P^\nabla \tilde{\varepsilon}, \tilde{\phi}] \end{aligned}$$

Furthermore, it is immediate that P^∇ is a point-wise isometry with respect to the scalar product $\langle -, - \rangle$ defined in Example 2.3.

We want to promote this to a unitary transformation between the associated L^2-spaces. As the L^2-norm also depends on the volume element, we first compare those: There exists a positive function $\beta \in \mathrm{C}^\infty(M)$ such that $\mathrm{dvol}^g = \beta \, \mathrm{dvol}^{g_0}$. Then $\sqrt{\beta} P^\nabla \colon \Sigma_{Cl}(M, g_0) \to \Sigma_{Cl}(M, g)$ induces a unitary transformation

$$\Phi_g \colon H \coloneqq L^2(M, \Sigma_{Cl}(M, g_0)) \xrightarrow{\cong} L^2(M, \Sigma_{Cl}(M, g))$$

as

$$(\Phi_g(\phi), \Phi_g(\psi))_{L^2} = \int_M \langle \sqrt{\beta} P^\nabla(\phi), \sqrt{\beta} P^\nabla(\psi) \rangle \, \mathrm{dvol}^g = \int_M \langle \phi, \psi \rangle \, \mathrm{dvol}^{g_0} = (\phi, \psi)_{L^2}.$$

Moreover, the following compatibilities with the structures from Example 2.3 are immediate:

Lemma 2.21. *The isometric Hilbert space isomorphism Φ_g commutes with the $\mathbb{Z}/2\mathbb{Z}$-grading and the right Clifford multiplication. The left Clifford multiplication satisfies*

$$\Phi_g(X \cdot \phi) = P^\nabla(X) \cdot \Phi_g(\phi),$$

where $P^\nabla(X)$ is the vector field obtained from X by parallel transport along the curves $(\gamma_x)_{x \in M}$.

The following is the main statement of this section:

Theorem 2.22. *The map*

$$\begin{aligned} (\mathcal{R}(M), \mathcal{R}^+(M)) &\longrightarrow (\mathrm{Fred}^{n,0}(H), G^{n,0}(H)) \\ g &\longmapsto \Phi_g^{-1} \circ \frac{D_g}{\sqrt{1 + D_g^2}} \circ \Phi_g \end{aligned}$$

is well-defined and continuous with respect to the C^1-topology on the space of smooth metrics $\mathcal{R}(M)$. In particular, it is continuous if $\mathcal{R}(M)$ carries the C^∞-topology.

Proof. The well-definedness follows from Example 2.10, so we need to check continuity. We split this up into the following steps: Firstly, we reduce to showing continuity in g_0 by estimating the difference of parallel transports. Secondly, we establish continuity of $g \mapsto \Phi_g^{-1} D_g \Phi_g$ in $B(H^1, L^2)$-norm. Lastly, we show that the bounded transform promotes this to continuity in $B(L^2)$-norm.

For the first step, we note that

$$\begin{aligned} \left\| \Phi_g^{-1} \frac{D_g}{\sqrt{1+D_g^2}} \Phi_g - \Phi_h^{-1} \frac{D_h}{\sqrt{1+D_h^2}} \Phi_h \right\| &= \left\| \Phi_h \Phi_g^{-1} \frac{D_g}{\sqrt{1+D_g^2}} \Phi_g \Phi_h^{-1} - \frac{D_h}{\sqrt{1+D_h^2}} \right\| \\ \leq \left\| \Phi_{hg}^{-1} \frac{D_g}{\sqrt{1+D_g^2}} \Phi_{hg} - \frac{D_h}{\sqrt{1+D_h^2}} \right\| &+ \left\| \Phi_{hg}^{-1} \frac{D_g}{\sqrt{1+D_g^2}} \Phi_{hg} - \Phi_h \Phi_g^{-1} \frac{D_g}{\sqrt{1+D_g^2}} \Phi_g \Phi_h^{-1} \right\| \end{aligned}$$

where $\Phi_{hg} \colon L^2(M, \Sigma_{Cl}(M,h)) \xrightarrow{\cong} L^2(M, \Sigma_{Cl}(M,g))$ is defined as Φ_g, but with h instead of g_0. The first term tends to zero for $g \to h$ if continuity holds in the base point of $\mathcal{R}(M)$. This, we will establish in steps two and three. To show that the second term goes to zero, it suffices to show that $\Phi_{hg} - \Phi_g \Phi_h^{-1}$ converges to zero as the bounded operators form a Banach algebra. If $\mathrm{dvol}^g = \beta \, \mathrm{dvol}^{g_0}$ and $\mathrm{dvol}^h = \gamma \, \mathrm{dvol}^{g_0}$, then $\mathrm{dvol}^g = \frac{\beta}{\gamma} \mathrm{dvol}^h$, so we need to provide an appropriate estimate for

$$\| P_{hg}^{\nabla} P_h^{\nabla} - P_g^{\nabla} \|.$$

We do so by adapting the proof of [Wit17, Lem 4.2].

We form[5] $(M \times \triangle, g_{st} + \mathrm{d}s^2 + \mathrm{d}t^2)$, where $\triangle = \{(s,t) \in [0,1]^2 \mid s + t \leq 1\} \subseteq \mathbb{R}^2$ and $g_{st} = (1-s-t)g_0 + sh + tg$. In the same way as we did for $(M \times [0,1], g_t + \mathrm{d}t^2)$, we can define a spin structure on $(M \times \triangle, g_{st} + \mathrm{d}s^2 + \mathrm{d}t^2)$ that restricts to the spin structure of (M, g_{st}) at $M \times \{s\} \times \{t\}$ for all $(s,t) \in \triangle$. Then P_g, P_h and P_{hg} can be obtained by parallel transport along the curves $\gamma_x \colon [0,1] \to M \times \triangle$ defined by $\tau \mapsto (x, 0, \tau)$, $\tau \mapsto (x, \tau, 0)$ and $\tau \mapsto (x, 1-\tau, \tau)$, respectively.

Now let $x \in M$, $\phi \in \Sigma_{Cl}(M, g_0)_{|x}$ and $\psi \in \Sigma_{Cl}(M, g)_{|x}$. We define $\phi(s,0)$ by parallel transport along $\tau \mapsto (x, \tau, 0)$ and $\phi(s,t) \in \Sigma_{Cl}(M, g_{st})_{|x}$ by transporting $\phi(s+t, 0)$ parallelly along $\tau \mapsto (x, s+t-\tau, \tau)$. Note that $\phi(1,0) = P_h^{\nabla} \phi$ and $\phi(0,1) = P_{hg}^{\nabla} P_h^{\nabla} \phi$. Furthermore, let $\psi(0,t) \Sigma_{Cl}(M, g_t)_{|x}$ be defined by parallel transport along $\tau \mapsto (x, 0, 1-\tau)$ and $\psi(s,t) \in \Sigma_{Cl}(M, g_{st})_{|x}$ by transporting $\psi(0, s+t)$ parallelly along $\tau \mapsto (x, \tau, s+t-\tau)$. Then $\psi(0,0) = (P_g^{\nabla})^{-1} \psi$.

[5] $\triangle$ is neither a submanifold nor a submanifold with boundary of $\mathbb{R}^2$, it has corners. But this does not cause problems: The metric can be extended to a small neighborhood of $M \times \triangle$ that can be taken to be a submanifold of $M \times \mathbb{R}^2$. All the operations that we are going to perform can then be defined in terms of this manifold.

Using the parallelism of the scalar product $\langle -, - \rangle$, we can calculate

$$
\begin{aligned}
\left\langle P^{\nabla}_{hg} P^{\nabla}_{h} \phi - P^{\nabla}_{g} \phi, \psi \right\rangle &= \langle \phi(0,1), \psi(0,1) \rangle - \langle \phi(0,0), \psi(0,0) \rangle \\
&= \int_0^1 \tfrac{\mathrm{d}}{\mathrm{d}y} \langle \phi(0,y), \psi(0,y) \rangle \, \mathrm{d}y \\
&= \int_0^1 \left\langle \nabla_{\frac{\partial}{\partial t}} \phi(0,y), \psi(0,y) \right\rangle \mathrm{d}y - \int_0^1 \left\langle \nabla_{\frac{\partial}{\partial s}} \phi(y,0), \psi(y,0) \right\rangle \mathrm{d}y \\
&= \int_0^1 \int_0^1 \tfrac{\mathrm{d}}{\mathrm{d}z} \left\langle \nabla_{(1-z)\frac{\partial}{\partial s} + z\frac{\partial}{\partial t}} \phi((1-z)y, zy), \psi((1-z)y, zy) \right\rangle \mathrm{d}z\mathrm{d}y \\
&= \int_0^1 \int_0^1 \left\langle \nabla_{-y\frac{\partial}{\partial s} + y\frac{\partial}{\partial t}} \nabla_{(1-z)\frac{\partial}{\partial s} + z\frac{\partial}{\partial t}} \phi((1-z)y, zy), \psi((1-z)y, zy) \right\rangle \mathrm{d}z\mathrm{d}y \\
&= \int_0^1 \int_0^1 \left\langle R(-y\tfrac{\partial}{\partial s} + y\tfrac{\partial}{\partial t}, (1-z)\tfrac{\partial}{\partial s} + z\tfrac{\partial}{\partial t}) \phi((1-z)y, zy), \psi((1-z)y, zy) \right\rangle \mathrm{d}z\mathrm{d}y \\
&= \int_0^1 \int_0^1 \left\langle y R(\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial s}) \phi((1-z)y, zy), \psi((1-z)y, zy) \right\rangle \mathrm{d}z\mathrm{d}y.
\end{aligned}
$$

We will show that the curvature operator of the tangent bundle satisfies an estimate of the form

$$
\|R(\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial s})\|_{g_{st}} \leq C \|g - h\|_{C^0}
$$

for all g in a sufficiently small neighborhood of h. Then by the well-known formula for the curvature of the spinor bundle (e.g. [LM89, II.(4.37)]), a similar estimate holds for the spinorial curvature operator. The calculation above then yields the desired estimate

$$
\|P^{\nabla}_{hg} P^{\nabla}_{h} - P^{\nabla}_{g}\| \leq C \|g - h\|_{C^0}.
$$

So let $Y(s,t), Z(s,t) \in \mathfrak{X}(M)$ be vector fields of M smoothly depending on $(s,t) \in \triangle$. As such, they define vector fields on $M \times \triangle$, and the Koszul formula implies

$$
\begin{aligned}
g_{st}(\nabla_{\frac{\partial}{\partial t}} Y, Z) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(g_{st}(Y,Z)) - \frac{1}{2} g_{st}(Y, [\tfrac{\partial}{\partial t}, Z]) - \frac{1}{2} g_{st}(Z, [Y, \tfrac{\partial}{\partial t}]) \\
&= \frac{1}{2}\frac{\partial g_{st}}{\partial t}(Y,Z) + \frac{1}{2} g_{st}(\tfrac{\partial Y}{\partial t}, Z) + \frac{1}{2} g_{st}(Y, \tfrac{\partial Z}{\partial t}) - \frac{1}{2} g_{st}(Y, \tfrac{\partial Z}{\partial t}) + \frac{1}{2} g_{st}(Z, \tfrac{\partial Y}{\partial t}) \\
&= \frac{1}{2}\frac{\partial g_{st}}{\partial t}(Y,Z) + g_{st}(\tfrac{\partial Y}{\partial t}, Z) \qquad (5)
\end{aligned}
$$

and similarly

$$
g_{st}(\nabla_{\frac{\partial}{\partial s}} Y, Z) = \frac{1}{2}\frac{\partial g_{st}}{\partial s}(Y,Z) + g_{st}(\tfrac{\partial Y}{\partial s}, Z). \qquad (6)
$$

If Y, Z are constant in s and t, then this implies

$$\begin{aligned}
g_{st}(\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial s}}Y, Z) &= \frac{\mathrm{d}}{\mathrm{d}t} g_{st}(\nabla_{\frac{\partial}{\partial s}}Y, Z) - \frac{1}{2}\frac{\partial g_{st}}{\partial t}(\nabla_{\frac{\partial}{\partial s}}Y, Z) \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial g_{st}}{\partial s}(Y, Z) - \frac{1}{2}\frac{\partial g_{st}}{\partial t}(\nabla_{\frac{\partial}{\partial s}}Y, Z) \\
&= -\frac{1}{2}(g - g_0)(\nabla_{\frac{\partial}{\partial s}}Y, Z)
\end{aligned}$$

and

$$g_{st}(\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}}Y, Z) = -\frac{1}{2}(h - g_0)(\nabla_{\frac{\partial}{\partial t}}Y, Z).$$

Defining $\tilde{Z}(s,t)$ by $(h - g_0)(-, Z) = g_{st}(-, \tilde{Z}(s,t))$ and $\hat{Z}(s,t)$ by $(h - g)(-, Z) = g_{st}(-, \hat{Z}(s,t))$ and using (5) and (6) again, we get

$$\begin{aligned}
g_{st}(R(\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial s})Y, Z) &= -\frac{1}{2}(g - g_0)(\nabla_{\frac{\partial}{\partial s}}Y, Z) + \frac{1}{2}(h - g_0)(\nabla_{\frac{\partial}{\partial t}}Y, Z) \\
&= \frac{1}{2}g_{st}(\nabla_{\frac{\partial}{\partial s}}Y, \hat{Z}) - \frac{1}{2}g_{st}(\nabla_{\frac{\partial}{\partial s}}Y, \tilde{Z}) + \frac{1}{2}g_{st}(\nabla_{\frac{\partial}{\partial t}}Y, \tilde{Z}) \\
&= \frac{1}{4}\frac{\partial g_{st}}{\partial s}(Y, \hat{Z}) - \frac{1}{4}\frac{\partial g_{st}}{\partial s}(Y, \tilde{Z}) + \frac{1}{4}\frac{\partial g_{st}}{\partial t}(Y, \tilde{Z}) \\
&= \frac{1}{4}(h - g_0)(Y, \hat{Z}) - \frac{1}{4}(h - g)(Y, \tilde{Z}).
\end{aligned}$$

Hence,

$$\begin{aligned}
|g_{st}(R(\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial s})Y, Z)| &\leq \frac{1}{4}\|h - g_0\|\|Y\|\|\hat{Z}\| + \frac{1}{4}\|h - g\|\|Y\|\|\tilde{Z}\| \\
&\leq \frac{1}{4}\|Y\|\left(\|h - g_0\|\|g_{st}^{-1}\|\|h - g\|\|Z\| + \|h - g\|\|g_{st}^{-1}\|\|h - g_0\|\|Z\|\right) \\
&= \frac{1}{2}\|Y\|\|g_{st}^{-1}\|\|h - g_0\|\|Z\|\|h - g\|,
\end{aligned}$$

where all norms are taken with respect to g_0. Now, it only remains to control the norm of the induced metric on co-vectors g_{st}^{-1}. Viewing metrics as maps from vectors to co-vectors and dual metrics as maps from co-vectors to vectors, this amounts to controlling the norm of the inverse of the endomorphism $g_0^{-1}g_{st}$. For $C = \max_{s\in[0,1]}\|g_{s,0}^{-1}\|$ (which only depends on g_0 and h), we consider the neighborhood $U = \left\{g \in \mathcal{R}(M) \,\middle|\, \|h - g\| < \frac{1}{2C}\right\}$ of h. Then for all $g \in U$ and $(s,t) \in \triangle$

$$\|g_{s+t,0}^{-1}(g_{st} - g_{s+t,0})\| \leq \|g_{s+t,0}^{-1}\|\|(tg - th)\| < \frac{1}{2}.$$

The geometric series now shows that $\mathbb{1} + g_{s+t,0}^{-1}(g_{st} - g_{s+t,0}) = g_{s+t,0}^{-1}g_{st}$ is invertible with $\|(g_{s+t,0}^{-1}g_{st})^{-1}\| < 2$. Therefore,

$$\begin{aligned}
\|g_{st}^{-1}\| = \|(g_0^{-1}g_{st})^{-1}\| &= \|(g_0^{-1}g_{s+t,0}g_{s+t,0}^{-1}g_{st})^{-1}\| \\
&= \|(g_{s+t,0}^{-1}g_{st})^{-1}(g_0^{-1}g_{s+t,0})^{-1}\| \leq 2C.
\end{aligned}$$

This completes the first step.

For the second step, we first observe that if $X \in \mathfrak{X}(M)$ and $\psi \in \Gamma(\Sigma_{Cl}(M, g_0))$, then

$$\begin{aligned}\left(\nabla^g_X P^\nabla - P^\nabla \nabla^{g_0}_X\right)\psi &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\left(P^\nabla_{t,1} \nabla^{g_t}_X P^\nabla_{0,t}\psi\right)\mathrm{d}t \\ &= \int_0^1 P^\nabla_{t,1}\, \nabla_{\frac{\partial}{\partial t}} \nabla^{g_t}_X\, P^\nabla_{0,t}\psi\ \mathrm{d}t \\ &= \int_0^1 P^\nabla_{t,1}\, R(\tfrac{\partial}{\partial t}, X)\, P^\nabla_{0,t}\psi\ \mathrm{d}t\end{aligned}$$

where $P^\nabla_{s,t}$ denotes the obvious parallel transport in $M \times [0,1]$ from $M \times \{s\}$ to $M \times \{t\}$. Here, from the first to the second line, we used the definition of the covariant derivative in terms of parallel transport.

Again, we estimate the corresponding curvature term of the tangent bundle as this gives rise to an estimate of the spinorial curvature. For vector fields $X, Y, Z \in \mathfrak{X}(M)$, that are regarded as vector fields on $M \times [0,1]$, constant in t-direction, equation (5) allows to calculate

$$\begin{aligned}g_t(R(\tfrac{\partial}{\partial t}, X)Y, Z) &= g_t(\nabla_{\frac{\partial}{\partial t}} \nabla^{g_t}_X Y, Z) - g_t(\nabla^{g_t}_X \nabla_{\frac{\partial}{\partial t}} Y, Z) \\ &= g_t(\nabla_{\frac{\partial}{\partial t}} \nabla^{g_t}_X Y, Z) + g_t(\nabla_{\frac{\partial}{\partial t}} Y, \nabla^{g_t}_X Z) - \partial_X g_t(\nabla_{\frac{\partial}{\partial t}} Y, Z) \\ &= g_t\left(\frac{\mathrm{d}}{\mathrm{d}t}\nabla^{g_t}_X Y, Z\right) + \frac{1}{2}\frac{\partial g_t}{\partial t}(\nabla^{g_t}_X Y, Z) + \frac{1}{2}\frac{\partial g_t}{\partial t}(Y, \nabla^{g_t}_X Z) - \frac{1}{2}\partial_X \frac{\partial g_t}{\partial t}(Y, Z) \\ &= g_t\left(\lim_{s\to 0}\frac{1}{s}(\nabla^{g_{t+s}}_X - \nabla^{g_t}_X)Y, Z\right) - \frac{1}{2}(g - g_0)\left((\nabla^{g_0}_X - \nabla^{g_t}_X)Y, Z\right) \\ &\quad - \frac{1}{2}(g - g_0)\left(Y, (\nabla^{g_0}_X - \nabla^{g_t}_X)Z\right) - \frac{1}{2}(\nabla^{g_0}_X (g - g_0)).(Y, Z)\end{aligned}$$

As the difference of two covariant derivatives is a tensor, we can calculate its C^0-norm in local coordinates. Thus, from the local formula

$$\Gamma^k_{ij} = \tfrac{1}{2} g^{kl}(\partial_i g_{jl} - \partial_j g_{il} - \partial_l g_{ij})$$

it follows that

$$\|\nabla^{g_t} - \nabla^{g_s}\|_{C^0} \le C\|g_t - g_s\|_{C^1} = C\|g - g_0\|_{C^1}|t - s|$$

for all g in a small neighborhood of g_0. Here, the neighborhood is chosen so small that the geometric series allows us to control $\|g_t^{-1} - g_s^{-1}\|_{C^0}$ by $\|g_t - g_s\|_{C^0}$. Hence, we obtain

$$|g_t(R(\tfrac{\partial}{\partial t}, X)Y, Z)| \le (C\|g_t\|_{C^0} + C\|g - g_0\|_{C^0} + \tfrac{1}{2})\|X\|_{C^0}\|Y\|_{C^0}\|Z\|_{C^0}\|g - g_0\|_{C^1},$$

which implies

$$\|R(\tfrac{\partial}{\partial t}, X)\|_{C^0} \le \tilde{C}\|X\|_{C^0}\|g - g_0\|_{C^1}$$

and

$$\left\| \sqrt{\beta^g} \left(\nabla^g_X P^\nabla - P^\nabla \nabla^{g_0}_X \right) \psi \right\|_{L^2} \leq \tilde{C} \|X\|_{C^0} \|g - g_0\|_{C^1} \|\psi\|_{L^2}$$

for some constant $\tilde{C}$ depending on the choice of neighborhood of g_0.

This serves to show that Φ_g maps H^1 to H^1: As

$$\begin{aligned} \nabla_X \Phi_g \psi &= \partial_X(\sqrt{\beta^g}) P^\nabla \psi + \sqrt{\beta^g} \nabla^g_X P^\nabla \psi \\ &= \tfrac{1}{4} \partial_X \left(\log \left((\beta^g)^2 \right) \right) \Phi_g \psi + \sqrt{\beta^g} \left(\nabla^g_X P^\nabla - P^\nabla \nabla^{g_0}_X \right) \psi + \Phi_g \nabla^{g_0}_X \psi, \end{aligned}$$

we are only left to control $\partial_X(\log{(\beta^g)^2})$. But it follows from the local expression

$$(\beta^g)^2 = \det((g_0^{ik} g_{kj})_{i,j}) = \det(\mathbb{1} + (g_0^{ik}(g_{kj} - g_{0kj}))_{i,j})$$

and the fact that $x \mapsto \frac{1}{x}$ is bounded in a neighborhood of 1 that

$$\left\| \partial_X \log \left((\beta^g)^2 \right) \right\|_{C^0} = \left\| \frac{\partial_X (\beta^g)^2}{(\beta^g)^2} \right\|_{C^0} \leq C \|X\|_{C^0} \|g - g_0\|_{C^1}$$

holds for all g within a certain C^1-neighborhood of g_0.

Now we turn our attention to Dirac operators. We fix an open cover $M = \bigcup_{j \in J} U_j$ and a subordinate partition of unity $(\theta_j)_{j \in J}$ such that for each $j \in J$ there exists a local frame $(e^j_1, \ldots, e^j_n)$, orthonormal with respect to g_0. Then $(P^\nabla e^j_1, \ldots, P^\nabla e^j_n)$ is a local orthonormal frame with respect to g and for any $\psi \in H^1(M, \Sigma_{Cl}(M, g_0))$

$$\begin{aligned} D^g \Phi_g \psi - \Phi_g D^{g_0} \psi &= \sum_{j \in J} \theta_j \left(\sum_i L(P^\nabla e^j_i) \nabla^g_{P^\nabla e^j_i} \Phi_g \psi - \sum_i \Phi_g L(e^j_i) \nabla^{g_0}_{e^j_i} \psi \right) \\ &= \sum_{j \in J} \theta_j \sum_i L(P^\nabla e^j_i) \left(\nabla^g_{P^\nabla e^j_i} \Phi_g \psi - \Phi_g \nabla^{g_0}_{e^j_i} \psi \right) \\ &= \sum_{j \in J} \theta_j \sum_i L(P^\nabla e^j_i) \Big(\tfrac{1}{4} \partial_{P^\nabla e^j_i} (\log{(\beta^g)^2}) \Phi_g \psi \\ &\quad + \sqrt{\beta^g} \left(\nabla^g_{P^\nabla e^j_i} P^\nabla - P^\nabla \nabla^{g_0}_{P^\nabla e^j_i} \right) \psi + \Phi_g \nabla^{g_0}_{P^\nabla e^j_i - e^j_i} \psi \Big). \end{aligned}$$

By the estimates obtained above, it only remains to show

$$\|P^\nabla e^j_i - e^j_i\|_{C^0} \leq \tilde{C} \|g - g_0\|_{C^0}$$

in order to obtain

$$\|\Phi_g^{-1} D^g \Phi_g \psi - D^{g_0} \psi\|_{L^2} = \|D^g \Phi_g \psi - \Phi_g D^{g_0} \psi\|_{L^2} \leq C \|g - g_0\|_{C^1} \|\psi\|_{H^1}.$$

But by setting $Y = P^\nabla e_i^j$ in (5)

$$P^\nabla e_i^j - e_i^j = \int_0^1 \frac{\partial P_t^\nabla e_i^j}{\partial t}\,\mathrm{d}t = -\frac{1}{2}\int_0^1 g_t^{-1}((g-g_0)(e_i^j,-))\,\mathrm{d}t$$

holds, where g_t^{-1} is viewed as mapping 1-forms to vector fields. From this, the desired estimate is easily obtained.

We now turn towards the last step. Let $D = D^{g_0}$, $D' = \Phi_g^{-1} D^g \Phi_g$, $F = f(D)$ and

$$F' = f(D') = \Phi_g^{-1} f(D^g)\Phi_g$$

for a function $f\colon \mathbb{R} \to \mathbb{R}$ with

$$|f(\lambda') - f(\lambda)| \leq \frac{c}{1+|\lambda|}|\lambda' - \lambda| \tag{7}$$

for all $\lambda, \lambda' \in \mathbb{R}$ and a fixed constant $c \in \mathbb{R}$. Of course, we are interested in the case $f(\lambda) = \frac{\lambda}{\sqrt{1+\lambda^2}}$ and we will finish this proof by showing that f is of that kind. The third step now consists of proving that there is a constant C only depending on f such that whenever the Dirac operators satisfy $\|D' - D\|_{B(H^1,L^2)} \leq \varepsilon$, then $\|F' - F\|_{B(L^2)} \leq C\varepsilon$.

Let $(\phi_i)_{i\in\mathbb{N}}$ be an orthonormal Hilbert basis of H consisting of eigenvectors of D with corresponding eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$. Similarly, let $(\psi_i)_{i\in\mathbb{N}}$ be an orthonormal Hilbert basis of eigenvectors of D' corresponding to the eigenvalues $(\lambda_i')_{i\in\mathbb{N}}$. For $\phi, \psi \in H$ with $\|\phi\| = \|\psi\| = 1$ we then have

$$\begin{aligned}
|\langle \psi, (F'-F)\phi\rangle| &= \left|\sum_{i,j} \langle \psi, \psi_j\rangle \langle \phi_i, \phi\rangle (f(\lambda_j') - f(\lambda_i))\langle \psi_j, \phi_i\rangle\right| \\
&\leq \sum_{i,j} \frac{c}{1+|\lambda_i|}|\lambda_j' - \lambda_i|\,|\langle \psi, \psi_j\rangle\langle \phi_i, \phi\rangle\langle \psi_j, \phi_i\rangle| \\
&\leq \sum_{i,j} \frac{c}{1+|\lambda_i|}|\lambda_j' - \lambda_i|\,|\langle \psi, \phi_i\rangle\langle \phi_i, \psi_j\rangle\langle \phi_i, \phi\rangle\langle \psi_j, \phi_i\rangle| \\
&= \sum_{i,j} |\langle \psi, \phi_i\rangle\langle \phi_i, \phi\rangle||\langle \phi_i, \psi_j\rangle| \frac{c}{1+|\lambda_i|}|\langle \psi_j, (D'-D)\phi_i\rangle| \\
&\leq \sum_i |\langle \psi, \phi_i\rangle\langle \phi_i, \phi\rangle| \\
&\qquad \left(\frac{\varepsilon}{2}\sum_j \langle \phi_i, \psi_j\rangle^2 + \frac{1}{2\varepsilon}\sum_j \left(\frac{c}{1+|\lambda_i|}\langle \psi_j, (D'-D)\phi_i\rangle\right)^2\right)
\end{aligned}$$

by Young's enquality.

Now, the desired estimate is obtained as follows:

$$\begin{aligned}
|\langle\psi,(F'-F)\phi\rangle| &\leq \sum_i |\langle\psi,\phi_i\rangle\langle\phi_i,\phi\rangle| \left(\frac{\varepsilon}{2}\|\phi_i\|^2 + \frac{1}{2\varepsilon}\left(\frac{c}{1+|\lambda_i|}\right)^2 \|(D'-D)\phi_i\|^2\right) \\
&\leq \sum_i |\langle\psi,\phi_i\rangle\langle\phi_i,\phi\rangle| \left(\frac{\varepsilon}{2} + \frac{1}{2\varepsilon}\left(\frac{c}{1+|\lambda_i|}\right)^2 \varepsilon^2\|\phi_i\|^2_{H^1}\right) \\
&\leq \sum_i |\langle\psi,\phi_i\rangle\langle\phi_i,\phi\rangle| \left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}c^2C^2\right) \\
&\leq \frac{1+c^2C^2}{2}\varepsilon.
\end{aligned}$$

The second but last inequality thereby used the Gårding inequality:

$$\|\phi_i\|_{H^1} \leq C(\|\phi_i\| + \|D\phi_i\|) = C(1+|\lambda_i|).$$

It now remains to show that $f(\lambda) = \frac{\lambda}{\sqrt{1+\lambda^2}}$ is subject to (7). Since the absolute value of $\frac{\mathrm{d}}{\mathrm{d}\lambda}\sqrt{1+\lambda^2} = \frac{\lambda}{\sqrt{1+\lambda^2}}$ is bounded by 1

$$\left|\sqrt{1+\lambda^2} - \sqrt{1+\lambda'^2}\right| \leq |\lambda'-\lambda|,$$

and thus

$$\begin{aligned}
|\lambda||f(\lambda')-f(\lambda)| &= |\lambda|\left|\frac{\lambda'\sqrt{1+\lambda^2} - \lambda\sqrt{1+\lambda'^2}}{\sqrt{1+\lambda^2}\sqrt{1+\lambda'^2}}\right| \\
&\leq |\lambda||\lambda'|\frac{\left|\sqrt{1+\lambda^2}-\sqrt{1+\lambda'^2}\right|}{\sqrt{1+\lambda^2}\sqrt{1+\lambda'^2}} + |\lambda|\frac{|\lambda'-\lambda|}{\sqrt{1+\lambda^2}} \\
&\leq \left|\sqrt{1+\lambda^2}-\sqrt{1+\lambda'^2}\right| + |\lambda'-\lambda| \\
&\leq 2|\lambda'-\lambda|.
\end{aligned}$$

Boundedness of $\frac{\mathrm{d}}{\mathrm{d}\lambda}f(\lambda) = (1+\lambda^2)^{-\frac{3}{2}}$ by 1 implies

$$|f(\lambda')-f(\lambda)| \leq |\lambda'-\lambda|.$$

Adding up those two inequalties, we obtain the required one:

$$|f(\lambda')-f(\lambda)| \leq \frac{3}{1+|\lambda|}|\lambda'-\lambda|.$$

□

Remark 2.23. The first step of the proof also shows that for $g_0' \in \mathcal{R}(M)$

$$\begin{aligned}
\mathcal{R}(M) &\longrightarrow B(H, L^2(\Sigma_{Cl}(M,g_0'))) \\
g &\mapsto \Phi^{-1}_{g_0'g}\Phi_g
\end{aligned}$$

is continuous, i.e. the Hilbert bundle structure defined on $L^2(M, \Sigma_{Cl}(M, -)) \to \mathcal{R}(M)$ using g_0 is independent of the choice of g_0. Moreover, the continuity of this map ensures that the α-invariant defined next does not depend on g_0.

Definition 2.24. The map from Theorem 2.22 gives rise to the composition

$$\begin{aligned}\alpha\colon \pi_k(\mathcal{R}^+(M), g_0) \cong \pi_{k+1}(\mathcal{R}(M), \mathcal{R}^+(M), g_0) &\to [(D^{k+1}, S^k), (\mathcal{R}(M), \mathcal{R}^+(M))] \\ &\to [(D^{k+1}, S^k), (\mathrm{Fred}^{n,0}, G^{n,0})] \cong KO^{-n-k-1}(\{*\})\end{aligned}$$

called *α-invariant.*

The α-invariant allows to detect non-trivial homotopy groups in the space of metrics of positive scalar curvature. The following two results of this kind were independently obtained by different methods:

Theorem 2.25 (Crowley, Schick, Steimle [CSS18])**.** *Let (M, g_0) be a compact Riemannian spin manifold of positive scalar curvature and $n = \dim(M) \geq 6$. For all $k \geq 0$ with $k + n + 1 \equiv 1, 2 \mod 8$, the α-invariant*

$$\alpha\colon \pi_k(\mathcal{R}^+(M), g_0) \longrightarrow KO^{-n-k-1}(\{*\}) \cong \mathbb{Z}/2\mathbb{Z}$$

is split surjective.

Theorem 2.26 (Botvinnik, Ebert, Randal-Williams [BER14])**.** *Let (M, g_0) be a compact Riemannian spin manifold of positive scalar curvature and $n = \dim(M) \geq 6$. For all $k \geq 0$ with $k + n + 1 \equiv 1, 2 \mod 8$, the α-invariant*

$$\alpha\colon \pi_k(\mathcal{R}^+(M), g_0) \longrightarrow KO^{-n-k-1}(\{*\}) \cong \mathbb{Z}/2\mathbb{Z}$$

is surjective, and for all $k \geq 0$ with $k + n + 1 \equiv 0, 4 \mod 8$, the localized α-invariant

$$\alpha \otimes \mathbb{1}_{\mathbb{Q}}\colon \pi_k(\mathcal{R}^+(M), g_0) \otimes \mathbb{Q} \longrightarrow KO^{-n-k-1}(\{*\}) \otimes \mathbb{Q} \cong \mathbb{Q}$$

is surjective.

We will use these results to construct non-trivial homotopy groups in the space of initial value pairs satisfying the dominant energy condition. The detection of these groups then uses a kind of α-invariant for initial values that will be defined in the next chapter.

3. An α-invariant for initial values

3.1. The $Cl_{n,1}$-linear hypersurface spinor bundle

In this section, we want to study the bundle obtained by restricting the $Cl_{n,1}$-linear spinor bundle of a space- and time-oriented Lorentzian spin manifold $(N, \overline{g})$ to a spacelike hypersurface $M \subseteq N$. Especially, we want to describe it intrinsically, only in terms of quantities of M, the induced metric g and the second fundamental form. This will be use later, when defining the α-invariant for initial values and comparing it to the classical α-invariant.

The first step is to construct compatible spin structures on M and N. Fixing a spin structure on $(N, \overline{g})$, we obtain a spin structure on (M, g) by pulling back the one from N:

$$\begin{array}{ccc} P_{\mathrm{Spin}(n)}(M) & \longrightarrow & P_{\mathrm{Spin}_0(n,1)}(N)_{|M} \\ \downarrow & & \downarrow \\ P_{\mathrm{SO}(n)}(M) & \longrightarrow & P_{\mathrm{SO}_0(n,1)}(N)_{|M}. \end{array} \tag{8}$$

Thereby, the lower map is given by $(e_1, \dots, e_n) \mapsto (e_0, e_1, \dots, e_n)$, where e_0 is the future-pointing unit normal on M. As the right-hand map is a double covering, so is the left-hand one, and it suffices to construct a compatible $\mathrm{Spin}(n)$-action. This, we obtain by pulling back the action maps. More explicitly, there is a commutative diagram

$$\begin{array}{ccc} P_{\mathrm{Spin}(n)}(M) \times \mathrm{Spin}(n) & \longrightarrow & P_{\mathrm{Spin}_0(n,1)}(N)_{|M} \times \mathrm{Spin}_0(n,1) \\ \downarrow & & \downarrow \\ P_{\mathrm{SO}(n)}(M) \times \mathrm{SO}(n) & \longrightarrow & P_{\mathrm{SO}_0(n,1)}(N)_{|M} \times \mathrm{SO}_0(n,1). \end{array} \tag{9}$$

and the desired map is the unique map from its upper-left corner to the upper-left corner of (8) building, together with the other action maps, a commutative cube out of (8) and (9). Note, that this commutative cube shows that $P_{\mathrm{Spin}(n)}(M)$ is not only a $\mathrm{Spin}(n)$-reduction of $P_{\mathrm{SO}(n)}(M)$ but also a reduction of $P_{\mathrm{Spin}_0(n,1)}(N)_{|M}$ with respect to the inclusion $i\colon \mathrm{Spin}(n) \hookrightarrow \mathrm{Spin}_0(n,1)$.

Next, we study associated bundles. The $Cl_{n,1}$-linear spinor bundle

$$\Sigma_{Cl} N = P_{\mathrm{Spin}_0(n,1)}(N) \times_\ell Cl_{n,1}$$

is defined via the representation induced by left multiplication on $Cl_{n,1}$:

$$\ell\colon \mathrm{Spin}_0(n,1) \hookrightarrow Cl_{n,1} \longrightarrow \mathrm{End}(Cl_{n,1}).$$

As noted above, $P_{\mathrm{Spin}(n)}(M) \to P_{\mathrm{Spin}_0(n,1)}(N)_{|M}$ is a $\mathrm{Spin}(n)$-reduction. Hence, from the theory of principal bundles (e.g. [Bau14, Satz 2.18]), it follows that

$$\Sigma_{Cl} N_{|M} = P_{\mathrm{Spin}_0(n,1)}(N)_{|M} \times_\ell Cl_{n,1} \cong P_{\mathrm{Spin}(n)}(M) \times_{\ell i} Cl_{n,1}, \tag{10}$$

so the bundle $\Sigma_{Cl} N_{|M} \to M$ only depends on the Riemannian manifold (M, g) and its chosen spin structure.

Definition 3.1. The bundle $\Sigma_{Cl} N_{|M}$ from above is called *$Cl_{n,1}$-linear hypersurface spinor bundle* and denoted by $\overline{\Sigma}_{Cl} M$.

Remark 3.2. Going a step futher, we can express $\overline{\Sigma}_{Cl} M$ in terms of the Cl_n-linear spinor bundle on M defined by $\Sigma_{Cl} M = P_{\mathrm{Spin}(n)}(M) \times_{\ell'} Cl_n$ (cf. Example 2.3) with ℓ' the left multiplication of $\mathrm{Spin}(n)$ on Cl_n. For this we note that

$$\begin{array}{ccc}
\mathrm{Spin}(n) & \xrightarrow{\qquad\qquad\qquad i \qquad\qquad\qquad} & \mathrm{Spin}_0(n,1) \\
\big\downarrow \ell' & & \big\downarrow \ell \\
\mathrm{End}(Cl_n) & \xrightarrow{-\otimes_{Cl_n} \mathbb{1}_{Cl_{n,1}}} \mathrm{End}(Cl_n \otimes_{Cl_n} Cl_{n,1}) \cong & \mathrm{End}(Cl_{n,1})
\end{array}$$

commutes and so

$$\begin{aligned}
\overline{\Sigma}_{Cl} M &\cong P_{\mathrm{Spin}(n)}(M) \times_{\ell i} Cl_{n,1} \\
&\cong (P_{\mathrm{Spin}(n)}(M) \times_{\ell'} Cl_n) \otimes_{Cl_n} (P_{\mathrm{Spin}(n)}(M) \times_{\mathbb{1}_{Cl_{n,1}}} Cl_{n,1}) \\
&\cong \Sigma_{Cl} M \otimes_{Cl_n} Cl_{n,1},
\end{aligned}$$

because the constant representation $\mathbb{1}_{Cl_{n,1}}$ defines the trivial $Cl_{n,1}$-bundle. The last term is to be understood in the way that Cl_n acts by right multiplication on $\Sigma_{Cl} M$ and by left multiplication on $Cl_{n,1}$.

We need more structure on this bundle to be in the setting of the Fredholm model for KO-theory. The structures we shall define and study in the remainder of this section all arise in a similar way: We first define them on $Cl_{n,1}$ and then show that they are Spin-invariant in the right way so that they generalize to the spinor bundle. They are: The involution

$$\alpha \colon Cl_{n,1} \longrightarrow Cl_{n,1}$$

arising as the Cliffordization of the map $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $v \mapsto -v$, the left and the right Clifford multiplication

$$\begin{aligned}
L \colon \mathbb{R}^{n+1} &\longrightarrow \mathrm{End}(Cl_{n,1}) \\
R \colon \mathbb{R}^{n+1} &\longrightarrow \mathrm{End}(Cl_{n,1})
\end{aligned}$$

as well as the (positive definite) scalar product

$$\langle -, - \rangle \colon Cl_{n,1} \times Cl_{n,1} \longrightarrow \mathbb{R}$$

defined by the requirement that the basis consisting of $e_{i_1} e_{i_2} \cdots e_{i_k}$ for $0 \le k \le n$ and $0 \le i_1 < \cdots < i_k \le n$ is orthonormal.

Lemma 3.3. *The structures satisfy the following:*

1. *Both Clifford multiplications are grading reversing, i.e.*

$$\alpha \circ L(X) = -L(X) \circ \alpha$$
$$\alpha \circ R(X) = -R(X) \circ \alpha$$

   for all $X \in R^{n+1}$.

2. *Left and right Clifford multiplication commute.*

3. *The grading operator is self-adjoint with respect to* $\langle -, - \rangle$.

4. *The adjoints of the Clifford multiplications are given by*

$$\begin{aligned} L(\beta e_0 + X)^* &= L(\beta e_0 - X) \\ R(\beta e_0 + X)^* &= R(\beta e_0 - X) \end{aligned} \tag{11}$$

   for $\beta \in \mathbb{R}$ *and* $X \in \mathrm{span}\{e_1, \ldots, e_n\}$.

Proof. The second part is immediate and all the other parts were covered in Example 2.2. □

From the lemma it is clear that both α and R commute with the left Clifford multiplication by elements in $\mathrm{Spin}_0(n, 1)$. So we get an induced involution

$$\alpha \colon \overline{\Sigma}_{Cl}M \to \overline{\Sigma}_{Cl}M$$

and an induced right Clifford multiplication

$$R \colon \mathbb{R}^{n+1} \to \mathrm{End}(\overline{\Sigma}_{Cl}M).$$

As vectors in $TN_{|M}$ transform via $Y \mapsto \sigma Y \sigma^{-1}$ for $\sigma \in \mathrm{Spin}_0(n, 1)$, L descends to a left Clifford multiplication

$$L \colon TN_{|M} \to \mathrm{End}(\overline{\Sigma}_{Cl}M).$$

For the scalar product, the situation is a bit more subtle. It follows from equation (11) that for $n > 0$ the scalar product is not $\mathrm{Spin}_0(n, 1)$-invariant (e.g. consider $e_0(\cosh(t)e_0 + \sinh(t)e_1) \in \mathrm{Spin}_0(n, 1)$ for $t \neq 0$). However, it is $\mathrm{Spin}(n)$-invariant, and as $P_{\mathrm{Spin}_0(n,1)}(N)_{|M}$ reduces to $P_{\mathrm{Spin}(n)}(M)$ this is sufficient to get a well-defined scalar product

$$\langle -, - \rangle \colon \overline{\Sigma}_{Cl}M \otimes \overline{\Sigma}_{Cl}M \to \mathbb{R}.$$

Remark 3.4. From the viewpoint of semi-Riemannian spin geometry, this scalar product can be understood in the following way. As discussed in [Bau81], in the semi-Riemannian case, the Spin-invariant non-degenerate symmetric bilinear forms on a representation space are no longer positive definite in general. In our case, the bilinear form $\langle e_0 \cdot -, - \rangle$ on $Cl_{n,1}$ is $\mathrm{Spin}_0(n,1)$-invariant, as can be seen with equation (11). So this bilinear form extends to[6]

$$(-,-)\colon \overline{\Sigma}_{Cl}M \otimes \overline{\Sigma}_{Cl}M \to \mathbb{R}.$$

Despite not being positive definite, this has the property that, if T is a timelike vector field, then $(T \cdot -, -)$ is positive definite. In our situation, there is a canonical choice of such a vector field: the future-pointing unit normal e_0. The resulting scalar product $(e_0 \cdot -, -)$ is precisely $\langle -, - \rangle$ constructed above. This is because e_0 defines the reduction to $\mathrm{Spin}(n)$.

The lemma above immediately implies

Lemma 3.5. *The structures satisfy the following:*

1. *Both Clifford multiplications are grading reversing.*
2. *Left and right Clifford multiplication commute.*
3. *The grading operator is self-adjoint with respect to $\langle -, - \rangle$.*
4. *The adjoints of the Clifford multiplications are given by*

$$\begin{aligned} L(\beta e_0 + X)^* &= L(\beta e_0 - X) \\ R(\beta e_0 + X)^* &= R(\beta e_0 - X) \end{aligned}$$

for $\beta \in \mathbb{R}$ and $X \in TM$ or $X \in \mathrm{span}\{e_1, \dots, e_n\}$, respectively.

In particular, α and R define a $Cl_{n,1}$-structure on the Hilbert space $L^2(M, \overline{\Sigma}_{Cl}M)$, where the L^2-scalar product is induced by $\langle -, - \rangle$. We can do even better:

Proposition 3.6. *Setting*

$$\Psi \cdot e_{n+1} \coloneqq e_0 \cdot \alpha(\Psi)$$

for all $\Psi \in \overline{\Sigma}_{Cl}M$, R extends to a $Cl_{n+1,1}$-multiplication

$$\tilde{R}\colon \mathbb{R}^{n+2} \to \mathrm{End}(\overline{\Sigma}_{Cl}M).$$

that commutes with left multiplication by any $X \in TM$. Moreover, $(L^2(M, \overline{\Sigma}_{Cl}M), \alpha, \tilde{R})$ is an ample $Cl_{n+1,1}$-Hilbert space.

[6] It even extends to a bilinear form on $\Sigma_{Cl}N$.

Proof. At first, we have to show that $\tilde{R}(e_{n+1})$ is skew-adjoint, anticommutes with α and $\tilde{R}(e_i)$ for all $i < n+1$ and squares to $-\mathbb{1}$. This is immediately checked:

$$\begin{aligned}
\tilde{R}(e_{n+1})^* &= (L(e_0)\alpha)^* = \alpha^* L(e_0)^* = \alpha L(e_0) = -L(e_0)\alpha = -\tilde{R}(e_{n+1}) \\
\tilde{R}(e_{n+1})\alpha &= L(e_0)\alpha\alpha = -\alpha L(e_0)\alpha = -\alpha\tilde{R}(e_{n+1}) \\
\tilde{R}(e_{n+1})R(e_i) &= L(e_0)\alpha R(e_i) = -R(e_i)L(e_0)\alpha = -R(e_i)\tilde{R}(e_{n+1}) \\
\tilde{R}(e_{n+1})^2 &= L(e_0)\alpha L(e_0)\alpha = -L(e_0)^2\alpha^2 = -\mathbb{1}
\end{aligned}$$

The left multiplication with a vector $X \in TM$ commutes with $\tilde{R}$ as this is true for R and $L(X)L(e_0)\alpha - L(e_0)\alpha L(X) = (L(X)L(e_0) + L(e_0)L(X))\alpha = -2g(X, e_0)\alpha = 0$ because e_0 is a normal vector. Ampleness follows literally as in Example 2.7 replacing $\Sigma_{Cl}M$ by $\overline{\Sigma}_{Cl}M$. □

As a consequence of (10), the $Cl_{n,1}$-linear hypersurface spinor bundle possesses two natural connections: On the one hand, the Levi-Civita connection $(N, \overline{g})$ induces a connection $\overline{\nabla}$ on $P_{\mathrm{Spin}_0(n,1)}N_{|M}$ and $\overline{\Sigma}_{Cl}M$. On the other hand, as bundle associated to $P_{\mathrm{Spin}(n)}M$ the bundle $\overline{\Sigma}_{Cl}M$ carries a connection ∇ induced by the Levi-Civita connection of (M, g). They are related by the Weingarten map (also known as shape operator):

Lemma 3.7. *For all $X \in TM$ and $\psi \in \Gamma(\overline{\Sigma}_{Cl}M)$*

$$\overline{\nabla}_X\psi = \nabla_X\psi - \frac{1}{2}e_0 \cdot W(X) \cdot \psi$$

holds, where $W(X) = \overline{\nabla}_X e_0$ is the Weingarten map[7].

Proof. Let $\tilde{\varepsilon}$ be a local section of $P_{\mathrm{Spin}(n)}M$, and $(e_1, \ldots, e_n)$ its projection to $P_{SO(n)}M$. Writing a spinor locally as $\psi = [\tilde{\varepsilon}, \tilde{\psi}]$ and using the local formula for the spinorial connection, we perform the following local calculation:

$$\begin{aligned}
\overline{\nabla}_X\psi - \nabla_X\psi &= \left[\tilde{\varepsilon}, \partial_X\tilde{\psi} + \frac{1}{2}\sum_{0\le i<j}\varepsilon_i\overline{g}(\overline{\nabla}_X e_i, e_j)e_i \cdot e_j \cdot \tilde{\psi}\right] \\
&\quad - \left[\tilde{\varepsilon}, \partial_X\tilde{\psi} + \frac{1}{2}\sum_{1\le i<j}\overline{g}(\nabla_X e_i, e_j)e_i \cdot e_j \cdot \tilde{\psi}\right] \\
&= \left[\tilde{\varepsilon}, \frac{1}{2}\sum_{0<j}(-1)g(\overline{\nabla}_X e_0, e_j)e_0 \cdot e_j \cdot \tilde{\psi}\right] \\
&= -\frac{1}{2}e_0 \cdot W(X) \cdot \psi.
\end{aligned}$$

□

[7] The sign of W is different than in Riemannian geometry. It is chosen such that for all $X, Y \in TM$, $K(X,Y) = -\overline{g}(\mathrm{II}(X,Y), e_0) = -\overline{g}(\overline{\nabla}_X Y, e_0) = \overline{g}(Y, \overline{\nabla}_X e_0) = g(Y, W(X))$ holds, so $W = K^\sharp$.

The next question is, how these connections interact with the structures defined on $\overline{\Sigma}_{Cl}M$.

Lemma 3.8. *1. The grading operator α is both $\overline{\nabla}$- and ∇-parallel.*

2. *For all $v \in \mathbb{R}^{n+1}$, the right Clifford multiplication $R(v)$ is both $\overline{\nabla}$- and ∇-parallel.*

3. *The left Clifford multiplication $L\colon TN_{|M} \otimes \overline{\Sigma}_{Cl}M \to \overline{\Sigma}_{Cl}M$ is $\overline{\nabla}$-parallel. Both the restricted left Clifford multiplication $L\colon TM \otimes \overline{\Sigma}_{Cl}M \to \overline{\Sigma}_{Cl}M$ and the endomorphism $L(e_0)\colon \overline{\Sigma}_{Cl}M \to \overline{\Sigma}_{Cl}M$ are ∇-parallel.*

4. *The scalar product $\langle -, - \rangle$ is ∇-parallel and satisfies*

$$\partial_X \langle \phi, \psi \rangle = \langle \overline{\nabla}_X \phi, \psi \rangle + \langle \phi, \overline{\nabla}_X \psi \rangle + \langle e_0 \cdot W(X) \cdot \phi, \psi \rangle$$

for all $X \in TM$, $\phi, \psi \in \Gamma(\overline{\Sigma}_{Cl}M)$.

Proof. Let $X \in T_pM$ and $U \subseteq M$ a neighborhood of p such that there is a local section $\tilde{\varepsilon} \in \Gamma(P_{\mathrm{Spin}_0(n,1)}N_{|U})$ with $\overline{\nabla}_X \tilde{\varepsilon} = 0$. For any $\psi \in \Gamma(\overline{\Sigma}_{Cl}M)$, we can write $\psi_{|U} = [\tilde{\varepsilon}, \tilde{\psi}]$ with $\tilde{\psi}\colon U \to Cl_{n,1}$. As α and $R(v)$ for $v \in \mathbb{R}^{n+1}$ are induced by linear maps on $Cl_{n,1}$,

$$\begin{aligned} \overline{\nabla}_X \alpha(\psi) &= [\tilde{\varepsilon}, \partial_X \alpha(\tilde{\psi})] = [\tilde{\varepsilon}, \alpha(\partial_X \tilde{\psi})] = \alpha(\tilde{\nabla}_X \psi) \\ \overline{\nabla}_X R(v)(\psi) &= [\tilde{\varepsilon}, \partial_X(\tilde{\psi} \cdot v)] = [\tilde{\varepsilon}, (\partial_X \tilde{\psi}) \cdot v] = R(v)(\tilde{\nabla}_X \psi) \end{aligned}$$

hold. For $Y \in \Gamma(TN_{|M})$, we can write $Y_{|U} = [\tilde{\varepsilon}, \tilde{Y}]$ with $\tilde{Y}\colon U \to R^{n+1}$, as $TN_{|M}$ is associated to $P_{\mathrm{Spin}_0(n,1)}N_{|M}$. Then

$$\overline{\nabla}_X L(Y)(\psi) = [\tilde{\varepsilon}, \partial_X(\tilde{Y} \cdot \tilde{\psi})] = [\tilde{\varepsilon}, (\partial_X \tilde{Y}) \cdot \tilde{\psi} + \tilde{Y} \cdot \partial_X \tilde{\psi}] = L(\overline{\nabla}_X Y)(\psi) + L(Y)(\overline{\nabla}_X \psi)$$

shows that L is $\overline{\nabla}$-parallel.

Using that $\overline{\Sigma}_{Cl}M$ is associated to $P_{\mathrm{Spin}(n)}M$, a similar reasoning works for ∇ as well. More concretely, we simply have to choose $\tilde{\varepsilon} \in \Gamma(P_{\mathrm{Spin}(n)}M_{|U})$ with $\nabla_X \tilde{\varepsilon} = 0$ and the calculations are literally the same ones. Note, however, that the connection induced on $TN_{|M} = TM \oplus \mathbb{R}e_0$ by ∇ (as bundle associated to $P_{\mathrm{SO}(n)}M$, or $P_{\mathrm{Spin}(n)}M$) is the sum of Levi-Civita connection on TM and the trivial connection[8] on $\mathbb{R}e_0$, which gives the third part.

[8] I.e. trivial with respect to the trivialization defined by e_0. The reason, why e_0 appears here, is that this vector field was used to define the reduction to $P_{\mathrm{SO}(n)}M$.

In the same way, we can prove the ∇-parallelism for $\langle -, - \rangle$: Writing $\phi \in \Gamma(\overline{\Sigma}_{Cl}M)$ as $\phi_{|U} = [\tilde{\varepsilon}, \tilde{\phi}]$, we have

$$\begin{aligned}
\partial_X \langle \phi, \psi \rangle &= \partial_X \langle \tilde{\phi}, \tilde{\psi} \rangle \\
&= \langle \partial_X \tilde{\phi}, \tilde{\psi} \rangle + \langle \tilde{\phi}, \partial_X \tilde{\psi} \rangle \\
&= \langle [\tilde{\varepsilon}, \partial_X \tilde{\phi}], [\tilde{\varepsilon}, \tilde{\psi}] \rangle + \langle [\tilde{\varepsilon}, \tilde{\phi}], [\tilde{\varepsilon}, \partial_X \tilde{\psi}] \rangle \\
&= \langle \nabla_X \phi, \psi \rangle + \langle \phi, \nabla_X \psi \rangle.
\end{aligned}$$

This argument does not translate to the $\overline{\nabla}$-case as the pointwise scalar product $\langle -, - \rangle$ on $Cl_{n,1}$ is not $\mathrm{Spin}_0(n,1)$-invariant in general. Yet, we can use the previous lemma to obtain

$$\begin{aligned}
\partial_X \langle \phi, \psi \rangle &= \langle \nabla_X \phi, \psi \rangle + \langle \phi, \nabla_X \psi \rangle \\
&= \langle \overline{\nabla}_X \phi + \tfrac{1}{2} e_0 \cdot W(X) \cdot \phi, \psi \rangle + \langle \phi, \overline{\nabla}_X \psi + \tfrac{1}{2} e_0 \cdot W(X) \cdot \psi \rangle \\
&= \langle \overline{\nabla}_X \phi, \psi \rangle + \langle \phi, \overline{\nabla}_X \psi \rangle + \tfrac{1}{2} \langle e_0 \cdot W(X) \cdot \phi, \psi \rangle + \tfrac{1}{2} \langle (-W(X)) \cdot e_0 \cdot \phi, \psi \rangle \\
&= \langle \overline{\nabla}_X \phi, \psi \rangle + \langle \phi, \overline{\nabla}_X \psi \rangle + \langle e_0 \cdot W(X) \cdot \phi, \psi \rangle.
\end{aligned}$$

□

In view of the $Cl_{n+1,1}$-structure defined on $\overline{\Sigma}_{Cl}M$, this lemma implies:

Corollary 3.9. $\tilde{R}\colon \overline{\Sigma}_{Cl}M \otimes \mathbb{R}^{n+2} \longrightarrow \overline{\Sigma}_{Cl}M$ *is* ∇*-parallel.*

In the following section we will use the connection $\overline{\nabla}$ to define the Dirac-Witten operator and compare it to the Dirac operator defined in terms of ∇. The Dirac-Witten operator will be used later to construct a kind of α-invariant for initial value pairs, and the comparison results will be a key ingredient in the main theorem, where we relate both kinds of α-invariant.

3.2. The $Cl_{n,1}$-linear Dirac-Witten operator

As before, let M be a spacelike hypersurface of a space- and time-oriented Lorentzian spin manifold $(N, \overline{g})$. The Dirac-Witten operator is a kind of Dirac operator on the hypersurface spinor bundle. In the case of classical spinor bundles, it was first defined by Witten [Wit81] in order to give his spinorial proof of the positive mass theorem and later studied in more detail by Hijazi and Zhang [HZ03]. We are interested in its $Cl_{n,1}$-linear version and compare it to the $Cl_{n,1}$-linear Dirac operator:

Definition 3.10. The composition

$$\overline{D}\colon \Gamma(\overline{\Sigma}_{Cl}M) \xrightarrow{\overline{\nabla}} \Gamma(T^*M \otimes \overline{\Sigma}_{Cl}M) \xrightarrow{\sharp\otimes\mathbb{1}} \Gamma(TM \otimes \overline{\Sigma}_{Cl}M) \xrightarrow{L} \Gamma(\overline{\Sigma}_{Cl}M)$$

defines the *$Cl_{n,1}$-linear Dirac-Witten operator*. The composition (with $\overline{\nabla}$ replaced by ∇)

$$D\colon \Gamma(\overline{\Sigma}_{Cl}M) \xrightarrow{\nabla} \Gamma(T^*M \otimes \overline{\Sigma}_{Cl}M) \xrightarrow{\sharp\otimes\mathbb{1}} \Gamma(TM \otimes \overline{\Sigma}_{Cl}M) \xrightarrow{L} \Gamma(\overline{\Sigma}_{Cl}M)$$

is the *$Cl_{n,1}$-linear Dirac operator*.

The results of Lemmata 3.5 and 3.8 and Corollary 3.9 from before immediately imply the following lemma, which justifies the names of these operators.

Lemma 3.11. *$\overline{D}$ and D are both $Cl_{n,1}$-linear with respect to the right Clifford multiplication R and odd with respect to α. Furthermore, D is $Cl_{n+1,1}$-linear with respect to the extended right Clifford multiplication $\tilde{R}$.*

Lemma 3.12. *$\overline{D} = D - \frac{1}{2}\tau L(e_0)$ holds, where $\tau = \operatorname{tr} W = \operatorname{tr} K$ is the mean curvature of M in N. Both D and $\overline{D}$ are formally self-adjoint.*

Proof. For $\psi \in \Gamma(\overline{\Sigma}_{Cl}M)$ and a local orthonormal frame $e_1, \ldots, e_n$ we perform the following local calculation:

$$\begin{aligned}
\overline{D}\psi - D\psi &= \sum_{i=1}^{n} e_i \cdot (\overline{\nabla}_{e_i} - \nabla_{e_i})\psi \\
&= -\frac{1}{2}\sum_{i=1}^{n} e_i \cdot e_0 \cdot W(e_i) \cdot \psi \\
&= \ \ \frac{1}{2}\sum_{i,j=1}^{n} g(W(e_i), e_j) e_i \cdot e_j \cdot e_0 \cdot \psi \\
&= -\frac{1}{2}\sum_{i=1}^{n} g(W(e_i), e_i) e_0 \cdot \psi \\
&= -\frac{1}{2}\tau e_0 \cdot \psi.
\end{aligned}$$

Here, we used that $g(W(e_i), e_j) = K(e_i, e_j)$ is symmetric in i and j. Being a Dirac operator, D is formally self-adjoint. As $L(e_0)$ is self-adjoint by Lemma 3.5, the same holds true for $\overline{D}$. $\square$

The utility of the Dirac-Witten operator to general relativity results from following observation due to Witten [Wit81]:

Proposition 3.13. *The Dirac-Witten operator satisfies the Schrödinger-Lichnerowicz type formula*

$$\overline{D}^2 = \overline{\nabla}^* \overline{\nabla} + \frac{1}{2}(\rho - L(e_0)L(j^\sharp)),$$

with

$$\begin{aligned} 2\rho &= \operatorname{scal} + \tau - \|K\|^2 \\ j &= -\mathrm{d}\tau + \operatorname{div} K. \end{aligned}$$

Proof. By the previous lemma, we have

$$\begin{aligned} \overline{D}^2 \psi &= \left(D - \frac{1}{2}\tau L(e_0)\right)\left(D - \frac{1}{2}\tau L(e_0)\right)\psi \\ &= D^2\psi - \frac{1}{2}D(\tau e_0 \cdot \psi) - \frac{1}{2}\tau e_0 \cdot D\psi + \frac{1}{4}\tau^2\psi \\ &= D^2\psi + \frac{1}{2}e_0 \cdot \operatorname{grad}\tau \cdot \psi + \frac{1}{4}\tau^2\psi. \end{aligned}$$

The last step of the calculation used that $D(\tau e_0 \cdot \psi) = \operatorname{grad}\tau \cdot e_0 \cdot \psi + \tau D(e_0 \cdot \psi)$ along with the fact that D anti-commutes with $L(e_0) = \tilde{R}(e_{n+1})\alpha$. Applying the Schrödinger-Lichnerowicz formula for D^2, we obtain

$$\overline{D}^2\psi = \nabla^*\nabla\psi + \frac{1}{4}\operatorname{scal}\psi + \frac{1}{2}e_0 \cdot \operatorname{grad}\tau \cdot \psi + \frac{1}{4}\tau^2\psi. \tag{12}$$

Next, we express $\overline{\nabla}^*$ in terms of ∇^*. Calculating point-wise,

$$\langle e_0 \cdot W(-) \cdot \phi, \Psi\rangle_{T^*M\otimes\overline{\Sigma}_{Cl}M} = \sum_{i=1}^n \langle e_0 \cdot W(e_i) \cdot \phi, \Psi(e_i)\rangle = \sum_{i=1}^n \langle \phi, e_0 \cdot W(e_i) \cdot \Psi(e_i)\rangle$$

holds for all $p \in M$, $\phi \in \Sigma_{Cl}N_{|p}$, $\Psi \in T_p^*M \otimes \Sigma_{Cl}N_{|p}$ and an orthonormal basis $e_1, \ldots e_n \in T_pM$. Thus defining

$$\widetilde{W} \colon T^*M \otimes \overline{\Sigma}_{Cl}M \xrightarrow{\sharp\otimes\mathbb{1}} TM \otimes \overline{\Sigma}_{Cl}M) \xrightarrow{W\otimes\mathbb{1}} TM \otimes \overline{\Sigma}_{Cl}M \xrightarrow{L} \overline{\Sigma}_{Cl}M,$$

the adjoint of $\overline{\nabla}$ is given by

$$\overline{\nabla}^* = \left(\nabla - \frac{1}{2}L(e_0)L(W(-))\right)^* = \nabla^* - \frac{1}{2}L(e_0)\tilde{W}.$$

Hence we get

$$\begin{aligned}\overline{\nabla}^*\overline{\nabla}\psi &= \left(\nabla^* - \frac{1}{2}L(e_0)\widetilde{W}\right)\left(\nabla - \frac{1}{2}L(e_0)L(W(-))\right)\psi \\ &= \nabla^*\nabla\psi - \frac{1}{2}\nabla^*(e_0 \cdot W(-) \cdot \psi) - \frac{1}{2}e_0 \cdot \widetilde{W}(\nabla\psi) + \frac{1}{4}e_0 \cdot \widetilde{W}(e_0 \cdot W(-) \cdot \psi).\end{aligned}$$

The last term can be calculated point-wise to be

$$\frac{1}{4}e_0 \cdot \widetilde{W}(e_0 \cdot W(-) \cdot \psi) = \frac{1}{4}\sum_{i=1}^{n} e_0 \cdot W(e_i) \cdot e_0 \cdot W(e_i) \cdot \psi = \frac{1}{4}\sum_{i=1}^{n} \|W(e_i)\|^2\psi = \frac{1}{4}\|K\|^2\psi.$$

The middle two terms can be simplified using a local calculation, $e_1, \dots, e_n$ being a local orthonormal frame:

$$\begin{aligned}&\nabla^*(e_0 \cdot W(-) \cdot \psi) + e_0 \cdot \widetilde{W}(\nabla\psi) \\ &\qquad = -\sum_{i=1}^{n}(\nabla_{e_i}(e_0 \cdot W(-) \cdot \psi))(e_i) + \sum_{i=1}^{n} e_0 \cdot W(e_i) \cdot \nabla_{e_i}\psi \\ &\qquad = -\sum_{i=1}^{n} e_0 \cdot ((\nabla_{e_i}W)(e_i) \cdot \psi + W(e_i) \cdot \nabla_{e_i}\psi - W(e_i) \cdot \nabla_{e_i}\psi) \\ &\qquad = -e_0 \cdot \operatorname{div}(W) \cdot \psi.\end{aligned}$$

So we find

$$\overline{\nabla}^*\overline{\nabla}\psi = \nabla^*\nabla\psi + \frac{1}{2}e_0 \cdot \operatorname{div}(K)^\sharp \cdot \psi + \frac{1}{4}\|K\|^2\psi$$

and inserting this into (12), we obtain

$$\begin{aligned}\overline{D}^2 &= \overline{\nabla}^*\overline{\nabla}\psi - \frac{1}{2}e_0 \cdot \operatorname{div}(K)^\sharp \cdot \psi - \frac{1}{4}\|K\|\psi + \frac{1}{4}\operatorname{scal}\psi + \frac{1}{2}e_0 \cdot \operatorname{grad}\tau \cdot \psi + \frac{1}{4}\tau^2\psi \\ &= \overline{\nabla}^*\overline{\nabla} + \frac{1}{4}(\operatorname{scal} + \tau^2 - \|K\|^2)\psi - \frac{1}{2}e_0 \cdot (\operatorname{div}(K) - \mathrm{d}\tau)^\sharp \cdot \psi\end{aligned}$$

as claimed. □

From now on, we assume that M is compact.

Corollary 3.14. *If the pair (g, K) satisfies the strict dominant energy condition, i.e. if $\rho > \|j\|$, then $\overline{D}$ has empty kernel.*

Proof. For any $\psi \in \Gamma(\overline{\Sigma}_{Cl}M)$ with $\psi \not\equiv 0$

$$\begin{aligned}\|\overline{D}\psi\|_{L^2}^2 = (\psi, \overline{D}\overline{D}\psi) &= \|\overline{\nabla}\psi\|_{L^2}^2 + \frac{1}{2}(\psi, \rho\psi) - \frac{1}{2}(\psi, e_0 \cdot j^\sharp \cdot \psi) \\ &\geq \frac{1}{2}(\psi, \rho\psi) - \frac{1}{2}(\psi, \|j\|\psi) = \frac{1}{2}(\psi, (\rho - \|j\|)\psi) > 0\end{aligned}$$

holds as $|\langle\psi, e_0 \cdot j^\sharp \cdot \psi\rangle| \leq \|j\|\|\psi\|^2$. Here, $\| - \|$ (without subscript L^2) denotes the pointwise norm. □

Proposition 3.15. *$\overline{D}$ and D extend to densely defined operators*

$$D, \overline{D} \colon L^2(M, \overline{\Sigma}_{Cl}M) \supseteq H^1(M, \overline{\Sigma}_{Cl}M) \to L^2(M, \overline{\Sigma}_{Cl}M)$$

admitting a spectral decomposition with discrete spectrum and finite dimensional eigenspaces.

Proof. Recall that a generalized Dirac operator on a vector bundle $\Sigma \to M$ in the sense of Roe [Roe99] is a formally self-adjoint operator $\widetilde{D}$ with

$$\widetilde{D} = \nabla^* \nabla + A$$

for a metric connection ∇ on Σ and some bounded operator $A \colon L^2(M, \Sigma) \to L^2(M, \Sigma)$. $\overline{D}$ is such an operator by (12) and D is by the Schrödiger-Lichnerowicz formula. Now the proposition is just a special case of the corresponding statement for generalized Dirac operators [Roe99, Thm 5.27]. □

Corollary 3.16. *If $n = \dim(M) > 0$ and $\overline{H} := L^2(M, \overline{\Sigma}_{Cl}M)$, then there are well-defined elements*

$$\overline{F} := \frac{\overline{D}}{\sqrt{1 + \overline{D}^2}} \in \mathrm{Fred}^{n,1}(\overline{H})$$

and

$$F := \frac{D}{\sqrt{1 + D^2}} \in \mathrm{Fred}^{n+1,1}(\overline{H}) \subseteq \mathrm{Fred}^{n,1}(\overline{H}).$$

Furthermore, $\overline{F}$ is invertible if (g, K) satisfies the strict dominant energy condition and F is invertible if g has positive scalar curvature.

Proof. H is ample as $Cl_{n+1,1}$-Hilbert space by Proposition 3.6 and any infinite dimensional $Cl_{n+1,1}$-Hilbert space is ample as $Cl_{n,1}$-Hilbert space with the restricted Clifford action. As $\overline{D}$ is odd and $Cl_{n,1}$-linear, so is $\overline{F}$. From the proposition above, we see that $\ker \overline{F} = \ker \overline{D}$ is finite dimensional and that $\overline{F}_{|\ker(\overline{F})^\perp} \colon \ker(\overline{F})^\perp \to \ker(\overline{F})^\perp$ is invertible, so $\operatorname{im} \overline{F} = \ker(\overline{F})^\perp$ is closed and $\operatorname{coker} \overline{F} = \ker \overline{F}$ is finite dimensional. Thus $\overline{F}$ is a Fredholm operator. The additional condition in the case $n - 1 \equiv -1 \mod 4$ follows as in Example 2.10. Note that for the argument to work, $n \geq 3$ is needed, which follows from $n > 0$ and $n - 1 \equiv -1 \mod 4$. Invertibility for (g, K) satisfying the strict dominant energy condition follows from Corollary 3.14 and $\operatorname{coker} \overline{F} = \ker \overline{F}$.

The argumentation for F is completely analogous. Invertibility here uses the classical Schrödinger-Lichnerowicz formula as in Example 2.10. □

If the mean curvature τ is constant, we can relate the spectral decompositions of $\overline{D}$ and D and refine the invertibility result.

Proposition 3.17. *The spectral decomposition of D can be written as*

$$D = \sum_{k=0}^{\infty} \lambda_k \pi_{E_k} + \sum_{k=0}^{\infty} (-\lambda_k) \pi_{\alpha(E_k)}$$

where all $\lambda_k > 0$ are pairwise disjoint and π_{E_k} and $\pi_{\alpha(E_k)}$ are the orthogonal projections on the finite dimensional subspaces E_k and $\alpha(E_k)$, respectively. If the mean curvature τ is constant, then there are decompositions $F_k \oplus \alpha(F_k) = E_k \oplus \alpha(E_k)$ for all $k \geq 0$ and $K \oplus \alpha(K) = \ker D$ such that the spectral decomposition of $\overline{D}$ is given by

$$\overline{D} = \sum_{k=0}^{\infty} \sqrt{\lambda_k^2 + \frac{1}{4}\tau^2}\, \pi_{F_k} + \sum_{k=0}^{\infty} \left(-\sqrt{\lambda_k^2 + \frac{1}{4}\tau^2} \right) \pi_{\alpha(F_k)} + \frac{1}{2}\tau \pi_K - \frac{1}{2}\tau \pi_{\alpha(K)}$$

In particular, $\overline{D}$ is invertible for all constants $\tau \neq 0$.

Proof. As α anti-commutes with D, for any eigenvector ϕ to the eigenvalue λ

$$D\alpha(\phi) = -\alpha(D\phi) = -\alpha(\lambda\phi) = -\lambda\alpha(\phi).$$

So $\alpha(\phi)$ is an eigenvector to the eigenvalue $-\lambda$. This implies that the spectral decomposition can be written in the stated form. With the same argument, we also expect the spectral decomposition of $\overline{D}$ to be of that form.

$\tilde{R}$ anti-commutes with D, so the eigenspaces are invariant under $\tilde{R}(v)$ for all $v \in \mathbb{R}^{n+2}$. In particular,

$$\alpha(E_k) = \tilde{R}(e_{n+1})\alpha(E_k) = L(e_0)(E_k)$$

for all $k \geq 0$. Thus we can identify E_k with $\alpha(E_k)$ via the map $E_k \to \alpha(E_k)$, $\phi \mapsto L(e_0)(\phi)$ and get $E_k \oplus \alpha(E_k) \cong E_k \oplus E_k \cong E_k \otimes \mathbb{R}^2$. Under this identification, by Lemma 3.12, the restriction of the Dirac-Witten operator corresponds to

$$\mathbb{1}_{E_k} \otimes \begin{pmatrix} \lambda_k & -\frac{1}{2}\tau \\ -\frac{1}{2}\tau & -\lambda_k \end{pmatrix}.$$

The characteristic polynomial of the 2×2-matrix is $x^2 - \lambda_k^2 - \frac{1}{4}\tau^2$, so it is diagonalizable with eigenvalues $\pm\sqrt{\lambda_k^2 + \frac{1}{4}\tau^2}$. This gives rise to a diagonalization of $\overline{D}_{|E_k \oplus \alpha E_k}$ with the same eigenvalues, and we call the positive eigenspace F_k.

Now, we turn our attention to $\ker D$. As $L(e_0) = \tilde{R}(e_{n+1})\alpha$ anti-commutes with D, $L(e_0)$ operates on $\ker D$. This operation is self-adjoint and squares to $\mathbb{1}_{\ker D}$, so by the spectral theorem $L(e_0)_{|\ker D}$ is diagonalizable and its eigenvalues must be contained in $\{1, -1\}$. Let K be the -1-eigenspace. Then $\alpha(K)$ is the 1-eigenspace. Due to

$$\overline{D}_{|\ker D} = -\frac{1}{2}\tau L(e_0)_{|\ker D},$$

K and $\alpha(K)$ become the $\frac{1}{2}\tau$- and $-\frac{1}{2}\tau$-eigenspaces of $\overline{D}$, respectively. $\square$

Remark 3.18. That $\overline{D}$ is invertible for constant mean curvature $\tau \neq 0$, can also be seen directly from the fact that D anti-commutes with $L(e_0)$: As $L(e_0)^2 = \mathbb{1}$,

$$\overline{D}^2 = \left(D - \frac{1}{2}\tau L(e_0)\right)^2 = D^2 + \frac{1}{4}\tau^2\,\mathbb{1}$$

and so $\operatorname{coker} \overline{D} = \ker \overline{D} = 0$.

With this knowledge at hand, we can turn towards the definition of the α-invariant for initial values, and prove the comparison result with the classical α-invariant. This will be carried out in the remaining section.

3.3. Comparing the α-invariants

Let M be a compact spin manifold of dimension $n > 0$ and $(g_0, K_0) \in \mathcal{I}^+(M)$ an initial value pair satisfying the strict dominant energy condition. The aim of this section is to define an α-invariant-like map $\overline{\alpha}\colon \pi_k(\mathcal{I}^+(M), (g_0, K_0)) \to KO^{-n-k}(\{*\})$. Then we use the map from Proposition 1.13 to relate this to the classical α-invariant, which then leads to a non-triviality result for $\pi_k(\mathcal{I}^+(M), (g_0, K_0))$.

In analogy to the case of the classical α-invariant, we need to compare the spaces of L^2-sections of the hypersurface spinor bundles for different initial value pairs (g, K). In fact, the $Cl_{n,1}$-linear hypersurface spinor bundle $\overline{\Sigma}_{Cl}(M, g) \cong \Sigma_{Cl}(M, g) \otimes_{Cl_n} Cl_{n,1}$ (cf. Remark 3.2) depends the metric g only, K solely effects its connection $\overline{\nabla}$. So adopting the notation from Section 2.2 there is a bundle map

$$\sqrt{\beta} P^{\nabla} \otimes \mathbb{1}_{Cl_{n,1}} \colon \Sigma_{Cl}(M, g_0) \otimes_{Cl_n} Cl_{n,1} \to \Sigma_{Cl}(M, g) \otimes_{Cl_n} Cl_{n,1},$$

which induces

$$\overline{\Phi}_g \colon \overline{H} := L^2(M, \overline{\Sigma}_{Cl}(M, g_0)) \xrightarrow{\cong} L^2(M, \overline{\Sigma}_{Cl}(M, g)).$$

This allows to produce a continuous map from initial values to the space of Fredholm operators.

Theorem 3.19. *The map*

$$\begin{aligned} (\mathcal{I}(M), \mathcal{I}^+(M)) &\longrightarrow (\mathrm{Fred}^{n,1}(\overline{H}), G^{n,1}(\overline{H})) \\ (g, K) &\longmapsto \overline{\Phi}_g^{-1} \circ \frac{\overline{D}_{(g,K)}}{\sqrt{1 + \overline{D}^2_{(g,K)}}} \circ \overline{\Phi}_g \end{aligned}$$

is well-defined and continuous with respect to the C^1-topology on the space of smooth initial value pairs $\mathcal{I}(M)$. In particular, it is continuous if $\mathcal{I}(M)$ carries the C^∞-topology.

Proof. The well-definedness follows from Corollary 3.16. For the continuity statement we argue as in the proof of Theorem 2.22. The first and third step immediately carry over to the current situation, and the second step provides us with a proof that $\overline{\Phi}_g^{-1} D_g \overline{\Phi}_g \to D_{g_0}$ in $B(H^1, L^2)$-topology if $g \to g_0$ in C^1-topology. But as $L(e_0)$ commutes with P^{∇} and thus with Φ_g, this implies that

$$\begin{aligned} \overline{\Phi}_g^{-1} \overline{D}_{(g,K)} \overline{\Phi}_g = \overline{\Phi}_g^{-1} D_g \overline{\Phi}_g - \frac{1}{2} \operatorname{tr}^g(K) \overline{\Phi}_g^{-1} L(e_0) \overline{\Phi}_g = \overline{\Phi}_g^{-1} D_g \overline{\Phi}_g - \frac{1}{2} \operatorname{tr}^g(K) L(e_0) \\ \longrightarrow D_{g_0} - \frac{1}{2} \operatorname{tr}^{g_0}(K_0) L(e_0) = \overline{D}_{(g_0,K_0)} \end{aligned}$$

in $B(H^1, L^2)$-topology if $(g, K) \longrightarrow (g_0, K_0)$ in C^1-topology. □

Definition 3.20. The *α-invariant for initial values* is defined by the composition

$$\begin{aligned}\overline{\alpha}\colon \pi_k(\mathcal{I}^+(M),(g_0,K_0)) &\cong \pi_{k+1}(\mathcal{I}(M),\mathcal{I}^+(M),(g_0,K_0))\\ &\to [(D^{k+1},S^k),(\mathcal{I}(M),\mathcal{I}^+(M))]\\ &\to [(D^{k+1},S^k),(\mathrm{Fred}^{n,1}(\overline{H}),G^{n,1}(\overline{H}))] \cong KO^{n-k}(\{*\}).\end{aligned}$$

Theorem 3.21 (Main Theorem)**.** *For $g_0 \in \mathcal{R}^+(M)$ and all $k \geq 0$, the diagram*

$$\begin{tikzcd} \pi_k(\mathcal{R}^+(M),g_0) \arrow[r, "\Sigma"] \arrow[dr, "\alpha"'] & \pi_{k+1}(\Sigma\,\mathcal{R}^+(M),[g_0,0]) \arrow[r, "\Phi_*"] & \pi_{k+1}(\mathcal{I}^+(M),(g_0,0)) \arrow[dl, "\overline{\alpha}"] \\ & KO^{n-k-1}(\{*\}) & \end{tikzcd}$$

commutes. Here, Σ is the suspension homomorphism and Φ is the map from Proposition 1.13.

Note that Φ is well-defined since the existence of $g_0 \in \mathcal{R}^+(M)$ implies $n \geq 2$.

Proof. We begin by identifying $\overline{H}$ in terms of $H := L^2(M,\Sigma_{Cl}M)$: By the first Morita equivalence (Lemma 2.4), the $Cl_{n+1,1}$-Hilbert space $\overline{H}$ corresponds to the $Cl_{n,0}$-Hilbert space $\overline{H}_0 = \ker(\tilde{R}(e_0)\tilde{R}(e_{n+1}) - \mathbb{1}) = \ker(R(e_0)L(e_0)\alpha - \mathbb{1})$ with the structure obtained by restriction. $R(e_0)L(e_0)\alpha$ is induced by a map $Cl_{n+1} \to Cl_{n+1}$, which in turn is induced by the endomorphism

$$\begin{aligned}\mathbb{R}^{n+1} &\longrightarrow \mathbb{R}^{n+1}\\ v &\longmapsto -e_0 v e_0\end{aligned}$$

reflecting at the hyperplane orthogonal to the line $\mathbb{R}e_0$. So the 1-eigenspace of the map on Cl_{n+1} is given by $Cl_n \subseteq Cl_{n+1}$ and the -1-eigenspace is $R(e_0)Cl_n \subseteq Cl_{n+1}$, where Cl_n is viewed as the subalgebra generated by $e_1,\ldots,e_n$. This implies that

$$\overline{H}_0 = L^2(M,\ker(R(e_0)L(e_0)\alpha - \mathbb{1})) = L^2(M, P_{\mathrm{Spin}(n)}M \times_\ell Cl_n) = H.$$

So $\overline{H}$ and H correspond to each other under the first Morita equivalence.

Let us now explore the effect of the composition

$$\pi_k(\mathcal{R}^+(M),g_0) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma\,\mathcal{R}^+(M),[g_0,0]) \xrightarrow{\Phi_*} \pi_{k+1}(\mathcal{I}^+(M),(g_0,0)).$$

The claim is that

$$\begin{tikzcd}
\pi_k(\mathcal{R}^+(M), g_0) \arrow[r, "\Sigma"] & \pi_{k+1}(\Sigma\,\mathcal{R}^+(M), [g_0, 0]) \arrow[r, "\Phi_*"] & \pi_{k+1}(\mathcal{I}^+(M), (g_0, 0)) \\
\pi_{k+1}(\mathcal{R}(M), \mathcal{R}^+(M), g_0) \arrow[u, "\cong"] \arrow[rr] \arrow[d] & & \pi_{k+2}(\mathcal{I}(M), \mathcal{I}^+(M), (g_0, 0)) \arrow[u, "\cong"'] \arrow[d] \\
[(D^{k+1}, S^k), (\mathcal{R}(M), \mathcal{R}^+(M))] \arrow[rr] & & [(D^{k+1}, S^k) \times (I, \partial I), (\mathcal{I}(M), \mathcal{I}^+(M))]
\end{tikzcd} \tag{13}$$

commutes, where the middle and the lower map are both induced by

$$\begin{aligned} \phi\colon (\mathcal{R}(M), \mathcal{R}^+(M)) \times (I, \partial I) &\longrightarrow (\mathcal{I}(M), \mathcal{I}^+(M)) \\ (g, t) &\longmapsto \left(g, \frac{\tau(g)}{n} tg\right). \end{aligned}$$

Note that ϕ preserves the base point, if the base point of $(D^{k+1}, S^k) \times (I, \partial I)$ is chosen to be $(*, 0)$ when $*$ is the base point of S^k, so the middle map is well-defined. The lower square obviously commutes. For the upper square, we start with a class $[g] \in \pi_k(\mathcal{R}^+(M), g_0)$. Then the preimage under the boundary isomorphism is represented by

$$\begin{aligned} \tilde{g}\colon (D^{k+1}, S^k, *) &\longrightarrow (\mathcal{R}(M), \mathcal{R}^+(M), g_0) \\ rx &\longmapsto (1-r)g_0 + rg(x) \end{aligned}$$

for $r \in [0, 1]$ and $x \in S^k$. Applying the horizontal map and restricting to the boundary yields the class of

$$\begin{aligned} (\partial(D^{k+1} \times I), (*, 0)) &\longrightarrow (\mathcal{I}^+(M), (g_0, 0)) \\ (x, t) &\longmapsto \left(\tilde{g}(x), -\frac{\tau(\tilde{g}(x))}{n} t\tilde{g}(x)\right). \end{aligned}$$

Using the homeomorphism

$$\begin{aligned} (\Sigma S^k, [*, 0]) &\cong (\partial(D^{k+1} \times I), (*, 0)) \\ [x, t] &\mapsto \begin{cases} (2(1+t)x, -1) & t \in [-1, -\frac{1}{2}] \\ (x, 2t) & t \in [-\frac{1}{2}, \frac{1}{2}] \\ (2(1-t)x, 1) & t \in [\frac{1}{2}, 1], \end{cases} \end{aligned}$$

this precisely gives the formula for $\Phi \circ \Sigma g$ (cf. Proposition 1.13).

The core of the proof is showing that the following diagram commutes:

$$\begin{tikzcd}
{[(D^{k+1},S^k),(\mathcal{R}(M),\mathcal{R}^+(M))]} \arrow[rr] \arrow[d] & & {[(D^{k+1},S^k)\times(I,\partial I),(\mathcal{I}(M),\mathcal{I}^+(M))]} \arrow[d] \\
{[(D^{k+1},S^k),(\mathrm{Fred}^{n,0}(H),G^{n,0}(H))]} \arrow[dr, "\cong"] & & {[(D^{k+1},S^k)\times(I,\partial I),(\mathrm{Fred}^{n,1}(\overline{H}),G^{n,1}(\overline{H}))]} \\
& {[(D^{k+1},S^k),(\mathrm{Fred}^{n+1,1}(\overline{H}),G^{n+1,1}(\overline{H}))].} \arrow[ur, "\cong"] &
\end{tikzcd} \tag{14}$$

The lower maps are the ones from Proposition 2.12 and Theorem 2.18, with $e = -e_{n+1}$.

Before doing so, let us show that

$$\begin{tikzcd}
& & {[(D^{k+1},S^k)\times(I,\partial I),(\mathrm{Fred}^{n,1}(\overline{H}),G^{n,1}(\overline{H}))]} \arrow[dd, "\cong"] \\
& {[(D^{k+1},S^k),(\mathrm{Fred}^{n+1,1}(\overline{H}),G^{n+1,1}(\overline{H}))]} \arrow[ur, "\cong"] \arrow[dr, "\cong"] & \\
{[(D^{k+1},S^k),(\mathrm{Fred}^{n,0}(H),G^{n,0}(H))]} \arrow[ur, "\cong"] \arrow[dr, "\cong"] \arrow[dd, "\cong"'] & & {[(D^{k+1},S^k)\times(I,\partial I),(\mathrm{Fred}^{n,1}(\overline{H}),G^{n,1}(\overline{H}))]} \arrow[dd, "\cong"] \\
& {[(D^{k+1},S^k)\times(I,\partial I),(\mathrm{Fred}^{n-1,0}(H),G^{n-1,0}(H))]} \arrow[ur, "\cong"] \arrow[dr, "\cong"] & \\
KO^{-n}(D^{k+1},S^k) \arrow[rr, "\cong"] \arrow[dr, "\cong"'] & & KO^{-n+1}((D^{k+1},S^k)\times(I,\partial I)) \arrow[dl, "\cong"] \\
& KO^{-n-k-1}(\{*\}) &
\end{tikzcd} \tag{15}$$

commutes, where the maps forming the central diamond are the Bott maps associated to $e = e_n$ along with the maps induced by the Morita equivalences, and the topmost right hand map is induced by a $Cl_{n,1}$-Hilbert space isomorphism to be defined later. Notice that the right hand vertical composition is the index map, which follows from the invariance of the index map under $Cl_{n,1}$-Hilbert space isomorphisms. So stitching the diagrams (13)-(15) together, we obtain the diagram from the claim.

Looking at the proof of Theorem 2.16, we see that the lower half of (15) commutes by definition of the index map. The middle diamond commutes as well, this is obvious from the way its constituting maps are defined. We are left with the upper triangle. Note first that we are dealing with two different $Cl_{n,1}$-Hilbert space structures on $\overline{H}$: Since the map from the center upwards is the Bott map for $e = -e_{n+1}$, the $Cl_{n,1}$-structure is the one obtained by forgetting the $\tilde{R}(e_{n+1})$-action, whereas in the lower Hilbert space, we forget the multiplication by e_n. These are connected by the $Cl_{n,1}$-Hilbert space isomorphism

$$\begin{aligned} U \colon \overline{H} &\longrightarrow \overline{H} \\ \phi &\mapsto \frac{1}{\sqrt{2}}\tilde{R}(e_{n+1})\tilde{R}(e_n + e_{n+1}). \end{aligned}$$

Indeed, $\iota \in B(\overline{H})$ corresponds via U to $\iota = U\iota U^{-1}$, $\tilde{R}(e_i)$ to $\tilde{R}(e_i)$ for $i < n$ and $\tilde{R}(e_n)$ to $\tilde{R}(e_{n+1})$. The right hand map in the triangle is defined to be the map inducd by $\mathrm{Fred}^{n,1}(\overline{H}) \ni F \mapsto UFU^{-1}$. As the analogous map on $\mathrm{Fred}^{n+1,1}(\overline{H})$ is the identity, the diagram relating the Bott maps gets the shape of a triangle rather than a square. Its commutativity follows from

$$\begin{aligned} U\tilde{R}(-e_{n+1})U^{-1} &= \frac{1}{2}\tilde{R}(e_{n+1})\tilde{R}(e_n + e_{n+1})\tilde{R}(-e_{n+1})\tilde{R}(e_n + e_{n+1})\tilde{R}(e_{n+1}) \\ &= \frac{1}{2}(\tilde{R}(e_{n+1}) + \tilde{R}(e_n) + \tilde{R}(e_n) - \tilde{R}(e_{n+1})) = \tilde{R}(e_n). \end{aligned}$$

It only remains prove that (14) commutes. The first two maps of the lower composition map $[g] \in [(D^{k+1}, S^k), (\mathcal{R}(M), \mathcal{R}^+(M))]$ to the class of

$$\begin{aligned} (D^{k+1}, S^k) &\longrightarrow (\mathrm{Fred}^{n+1,1}(\overline{H}), G^{n+1,1}(\overline{H})) \\ x &\longmapsto \Phi^{-1}_{g(x)} \frac{D_{g(x)}}{\sqrt{1 + D^2_{g(x)}}} \Phi_{g(x)}. \end{aligned}$$

This is because it restricts to the correct map on $H = \ker(R(e_0)L(e_0)\alpha - \mathbb{1}) \subseteq \overline{H}$. The remaining map sends it to the class of

$$\begin{aligned} (D^{k+1}, S^k) \times (I, \partial I) &\longrightarrow (\mathrm{Fred}^{n,1}(\overline{H}), G^{n,1}(\overline{H})) \\ (x, t) &\longmapsto \Phi^{-1}_{g(x)} \frac{D_{g(x)}}{\sqrt{1 + D^2_{g(x)}}} \Phi_{g(x)} - t\tilde{R}(e_{n+1})\alpha \\ &= \Phi^{-1}_{g(x)} \left(\frac{D_{g(x)}}{\sqrt{1 + D^2_{g(x)}}} - tL(e_0) \right) \Phi_{g(x)}. \end{aligned}$$

In contrast, the result of the upper composition is represented by

$$\begin{aligned} (D^{k+1}, S^k) \times (I, \partial I) &\longrightarrow (\mathrm{Fred}^{n,1}(\overline{H}), G^{n,1}(\overline{H})) \\ (x, t) &\longmapsto \Phi^{-1}_{g(x)} \frac{\overline{D}_{(g(x), K(x,t))}}{\sqrt{1 + \overline{D}^2_{(g(x), K(x,t))}}} \Phi_{g(x)} \end{aligned}$$

with $K(x, t) = \frac{\tau(g(x))}{n} tg(x)$.

Remembering that $\overline{D}_{(g,K)} = D_g - \frac{1}{2}\tau L(e_0)$, these do not look too much different, and we show that the following is a well-defined homotopy between them:

$$\begin{aligned}(D^{k+1}, S^k) \times (I, \partial I) \times [0,1] &\to (\mathrm{Fred}^{n,1}(\overline{H}), G^{n,1}(\overline{H})) \\ (x,t,s) &\mapsto \Phi_{g(x)}^{-1} \left(a_{(x,t,s)}(D_{g(x)}) D_{g(x)} - b_{(x,t,s)}(D_{g(x)}) t L(e_0) \right) \Phi_{g(x)}\end{aligned}$$

for

$$\begin{aligned}a_{(x,t,s)}(\lambda) &= \frac{s}{\sqrt{1+\lambda^2}} + \frac{1-s}{\sqrt{1+\lambda^2+\frac{1}{4}t^2\tau(g(x))}} \\ b_{(x,t,s)}(\lambda) &= s + \frac{(1-s)\frac{1}{2}\tau(g(x))}{\sqrt{1+\lambda^2+\frac{1}{4}t^2\tau(g(x))}}.\end{aligned}$$

As this operator family is obtained by linearly interpolating between two continuous operator families, it is again continuous. So it remains to see that its target is indeed $(\mathrm{Fred}^{n,1}(\overline{H}), G^{n,1}(\overline{H}))$. It is clear, that all the operators are bounded, self-adjoint, odd and $Cl_{n,1}$-linear. To show that the operator $F_{(x,t,s)}$ associated to (x,t,s) is Fredholm, we use the spectral decomposition of $D_{g(x)}$ from Proposition 3.17: The restriction of $F_{(x,t,s)}$ to $E_k \oplus \alpha(E_k) \cong E_k \otimes \mathbb{R}^2$ is given by

$$\mathbb{1}_{E_k} \otimes \begin{pmatrix} a_{(x,t,s)}(\lambda_k)\lambda_k & -b_{(x,t,s)}(\lambda_k)t \\ -b_{(x,t,s)}(\lambda_k)t & -a_{(x,t,s)}(\lambda_k)\lambda_k \end{pmatrix}.$$

This is diagonalizable with eigenvalues $\pm\sqrt{a_{(x,t,s)}(\lambda_k)^2\lambda_k^2 + b_{(x,t,s)}(\lambda_k)^2t^2}$. Note that due to $\sqrt{a_{(x,t,s)}(\lambda_k)^2\lambda_k^2 + b_{(x,t,s)}(\lambda_k)^2t^2} \geq a_{(x,t,s)}(\lambda_k)|\lambda_k|$ their absolute values, for any $t \in I$ and $s \in [0,1]$, are bounded away from zero by

$$\frac{\lambda_0}{\sqrt{1+\lambda_0^2+\frac{1}{4}\tau(g(x))}} > 0,$$

where $\lambda_0 > 0$ denotes the smallest positive eigenvalue of $D_{g(x)}$. A similar consideration as in Proposition 3.17 shows that $F_{(x,t,s)}$ restricted to $\ker(D_{g(x)})$ is diagonalizable as well, with eigenvalues $\pm b_{(x,t,s)}(0)t$. Putting this together, we find that $F_{(x,t,s)}$ has finite dimensional kernel, co-kernel and closed image (for this, the boundedness away from zero is needed). Furthermore, $F_{(x,t,s)}$ is invertible if $D_{g(x)}$ is invertible or $t > 0$, one of which is the case on $\partial(D^{k+1} \times I)$.

In the case $n-1 \equiv -1 \mod 4$ one more tiny bit of thought is necessary. The space self-adjoint $Cl_{n,1}$-linear Fredholm operators has three components (cf. [AS69]): Those F for which $\omega_{n,1}F\iota$ is essentially positive, those for which it is essentially negative and the rest. As for $s = 0$ (or $s = 1$) all operators $F_{(x,t,s)}$ fall into the last category, the same has to be true for all $s \in [0,1]$ by continuity. $\square$

Together with the non-triviality results for the classical α-invariant from Theorems 2.25 and 2.26, we obtain the following conclusions:

Corollary 3.22. *If M is a closed spin manifold of dimension $n \geq 6$ that carries a metric g_0 of positive scalar curvature, then for all $k \geq 1$ with $k + n \equiv 1, 2 \mod 8$ the α-invariant for initial values $\overline{\alpha}\colon \pi_k(\mathcal{I}^+(M), (g_0, 0)) \to KO^{-n-k}(\{*\}) \cong \mathbb{Z}/2\mathbb{Z}$ is split surjective.*

Corollary 3.23. *If M is a closed spin manifold of dimension $n \geq 6$ that carries a metric g_0 of positive scalar curvature, then for all $k \geq 1$ with $k+n \equiv 1, 2 \mod 8$ the α-invariant for initial values $\overline{\alpha}\colon \pi_k(\mathcal{I}^+(M), (g_0, 0)) \to KO^{-n-k}(\{*\}) \cong \mathbb{Z}/2\mathbb{Z}$ is surjective and for all $k \geq 1$ with $k + n \equiv 0, 4 \mod 8$ the localized α-invariant for initial values $\overline{\alpha} \otimes \mathbb{1}_{\mathbb{Q}}\colon \pi_k(\mathcal{I}^+(M), (g_0, 0)) \otimes \mathbb{Q} \to KO^{n-k}(\{*\}) \otimes \mathbb{Q} \cong \mathbb{Q}$ is surjective.*

In particular, under the assumptions of the corollaries above, $\pi_k(\mathcal{I}^+(M), (g_0, 0)) \neq 0$. Moreover, the main theorem provides an explicit construction of its non-trivial elements provided that in $\pi_{k-1}(\mathcal{R}^+(M), g_0)$ the non-trivial elements detected by the α-invariant are known.

A. On C^k-topologies

In this chapter, we will show that there is a well-defined notion of C^k-topology for sections of a vector bundle E over a compact manifold M. Furthermore, we will prove a criterion that allows us to check the continuity of a map

$$C^k(M,E) \longrightarrow C^l(M,F)$$

by looking at local expressions, where F is another vector bundle over M. We start with two definitions of a C^k-norm.

Definition A.1. Let $(U_i)_{i\in I}$ be an open cover by chart neighborhoods of M such that $E_{|U_i}$ is trivial for all $i \in I$. Let $\phi_i\colon U_i \to W_i \subseteq \mathbb{R}^m$ be corresponding charts and $\Phi_i\colon E_{|U_i} \to U_i \times \mathbb{R}^n$ be chosen trivializations. Furthermore, let $(\psi_i)_{i\in I}$ be a partition of unity subordinate to $(U_i)_{i\in I}$. Then we define the *C^k-norm with respect to ϕ, Φ and ψ* of a section $s \in C^k(M,E)$ by

$$\|s\|_{C^k} \coloneqq \sum_{i\in I} \|(\psi_i s)_{|U_i}\|_{C^k(U_i)} \coloneqq \sum_{i\in I} \| \operatorname{pr}_{\mathbb{R}^n} \circ \Phi_i \circ (\psi_i s) \circ \phi_i^{-1}\|_{C^k(W_i)}$$

where $\| - \|_{C^k(W_i)}$ denotes the usual C^k-norm on functions $\mathbb{R}^m \supseteq W_i \to \mathbb{R}^n$.

Definition A.2. Let g be a Riemannian metric on M, $< -, - >$ be a bundle metric on E and ∇ be a connection on E. Then we define the *C^k-norm with respect to g, $< -, - >$ and ∇* of a section $s \in C^k(M,E)$ by

$$\|s\|_{C^k} \coloneqq \sum_{i=0}^{k} \sup_{p\in M} |\nabla^i s_{|p}|$$

where $| - |$ is the point-wise norm on $T^{0,i}M \otimes E$ induced by g and $< -, - >$.

These definitions depend on a number of choices. Nonetheless, as we will see, they define equivalent norms and hence a unique notion of C^k-topology.

Lemma A.3. *Let $(\psi_i)_{i\in I}$ be as in Definition A.1. Then there is an $\varepsilon > 0$ such that $V_i \coloneqq \{x \in U_i \,|\, \psi_i(x) > \varepsilon\} \subseteq U_i$ define an open cover of M. Furthermore, all but finitely many V_i's are empty.*

Proof. Due to local finiteness of a partition of unity, we can choose for any $x \in M$ an open neighborhood V_x such that $\{i \in I \,|\, V_x \cap \operatorname{supp}\psi_i \neq \varnothing\}$ is finite. Compactness of M allows us to take a finite sub-cover $(V_x)_{x\in J}$ of $(V_x)_{x\in M}$. This shows that $\{i \in I \,|\, \psi_i \not\equiv 0\} = \bigcup_{x\in J} \{i \in I \,|\, V_x \cap \operatorname{supp}\psi_i \neq \varnothing\}$ is finite.

Set $N = \#\{i \in I \mid \psi_i \not\equiv 0\}$ and $\varepsilon = \frac{1}{2N}$. Then, for every $x \in M$, there is an $i \in I$ such that $x \in V_i = \{x \in U_i \mid \psi_i(x) > \varepsilon\}$ as otherwise

$$1 = \sum_{i \in I} \psi_i(x) \leq \sum_{\{i \in I \mid x \in \operatorname{supp} \psi_i\}} \frac{1}{2N} \leq N \frac{1}{2N} = \frac{1}{2}.$$

From the finiteness of $\{i \in I \mid \psi_i \not\equiv 0\}$ it immediately follows that almost all V_i's are empty. $\square$

Lemma A.4. *Suppose we are in the setting of Definition A.1 and choose ε and V_i as in the previous lemma. We equip $\prod_{i \in I} C^k(V_i, E_{|V_i})$ with the product norm (note that all but finitely many factors are zero) of the norms $\| - \|_{C^k(V_i)}$ defined in the same way as the norm $\| - \|_{C^k(U_i)}$. Denote by $\prod'_{i \in I} C^k(V_i, E_{|V_i})$ (notice the prime after the product sign) the subspace defined by those families of sections that coincide on all intersections $V_i \cap V_j$. Then the vector space isomorphism*

$$\begin{aligned} C^k(M, E) &\longrightarrow \prod_{i \in I}{}' C^k(V_i, E_{|V_i}) \\ s &\longmapsto (s_{|V_i})_{i \in I} \end{aligned}$$

is continuous.

Proof. The argument is that, on V_i, ψ_i is bounded away from zero by definition. Hence, $\left\| \frac{1}{\psi_{i|V_i}} \right\|_{C^k(V_i)}$ is bounded, and thus

$$\begin{aligned} \sum_{i \in I} \|s_{|V_i}\|_{C^k(V_i)} &= \sum_{i \in I} \left\| \frac{1}{\psi_{i|V_i}} (\psi_i s)_{|V_i} \right\|_{C^k(V_i)} \leq \sum_{i \in I} \left\| \frac{1}{\psi_{i|V_i}} \right\|_{C^k(V_i)} \|(\psi_i s)_{|V_i}\|_{C^k(V_i)} \\ &\leq C \sum_{i \in I} \|(\psi_i s)_{|V_i}\|_{C^k(V_i)} \leq C \sum_{i \in I} \|(\psi_i s)_{|U_i}\|_{C^k(U_i)} = C\|s\|_{C^k} \end{aligned}$$

for a constant $C > 0$ independent from s. $\square$

Remark A.5. By showing that $C^k(M, E)$ and $\prod'_{i \in I} C^k(V_i, E_{|V_i})$ are Banach spaces, we could conclude that the map above is a homeomorphism. Yet, we argue differently: We will show that different choices of ϕ, Φ and ψ in Definition A.1 lead to equivalent norms. Then we can assume that $\| - \|_{C^k}$ is defined in terms of the restrictions of ϕ_i and Φ_i to V_i and a partition of unity $(\psi_i)_{i \in I}$ subordinate to $(V_i)_{i \in I}$. The estimate

$$\|s\|_{C^k} = \sum_{i \in I} \|(\psi_i s)_{|V_i}\|_{C^k(V_i)} \leq \sum_{i \in I} \|\psi_{i|V_i}\|_{C^k(V_i)} \|s_{|V_i}\|_{C^k(V_i)} \leq C \sum_{i \in I} \|s_{|V_i}\|_{C^k(V_i)}$$

then shows the continuity of the inverse directly.

Theorem A.6. *The norms defined in Definitions A.1 and A.2 are equivalent. In particular, the induced topology, the C^k-*topology*, is independent of the choices made.*

Proof. To begin, let $\phi\colon U \to W$ be a chart, $\Phi\colon E_{|U} \to U \times \mathbb{R}^n$ be a local trivialization, ∇ a connection on E and $s \in \Gamma(U, E_{|U})$ be a local section. We write $\tilde{\Phi} = \mathrm{pr}_{\mathbb{R}^n} \circ \Phi$. Then we prove by induction on$|\alpha| \in \mathbb{N}$ the following formula:

$$\partial^\alpha(\tilde{\Phi} \circ s \circ \phi^{-1}) = \sum_{l \leq |\alpha|} \sum_{i_1, \ldots, i_l \in \{1, \ldots, m\}} C_\alpha^{i_1 \cdots i_l} \cdot \tilde{\Phi} \circ \nabla^l s \left(\frac{\partial}{\partial \phi^{i_1}}, \ldots, \frac{\partial}{\partial \phi^{i_l}} \right) \circ \phi^{-1}), \qquad (16)$$

where $C_{alpha}^{i_1 \cdots i_l} \in \Gamma(W, \mathbb{R}^{n \times n})$ for all $l \leq |\alpha|$ and all $i_1, \ldots, i_l \in \{1, \ldots, m\}$.

The case $k = 0$ is trivial and the case $k = 1$ follows from

$$\tilde{\Phi} \circ \nabla_i s \circ \phi^{-1} = \partial_{x_i}(\tilde{\Phi} \circ s \circ \phi^{-1}) + \Gamma_i \cdot \tilde{\Phi} \circ s \circ \phi^{-1},$$

where $\Gamma_i \in \Gamma(W, \mathbb{R}^{n \times n})$ is a kind of Christoffel symbol for the chosen connection.

Assume now that the formula (16) holds for $k \in \mathbb{N}$ and that $|\alpha| = k + 1$. Let i be the smallest index such that $\alpha_i \neq 0$ and α' be chosen such that $\partial^\alpha = \partial_{x_i} \partial^{\alpha'}$. Then

$$\begin{aligned}
\partial^\alpha(\tilde{\Phi} \circ s \circ \phi^{-1}) &= \sum_{l \leq |\alpha'|} \sum_{i_1, \ldots, i_l \in \{1, \ldots, m\}} \partial_{x_i} \left(C_{\alpha'}^{i_1 \cdots i_l} \cdot \tilde{\Phi} \circ \nabla^l s \left(\frac{\partial}{\partial \phi^{i_1}}, \ldots, \frac{\partial}{\partial \phi^{i_l}} \right) \circ \phi^{-1} \right) \\
&= \sum_{l \leq |\alpha'|} \sum_{i_1, \ldots, i_l} (\partial_{x_i} C_{\alpha'}^{i_1 \cdots i_l}) \cdot \tilde{\Phi} \circ \nabla^l s \left(\frac{\partial}{\partial \phi^{i_1}}, \ldots, \frac{\partial}{\partial \phi^{i_l}} \right) \circ \phi^{-1} \\
&\quad - \sum_{l \leq |\alpha'|} \sum_{i_1, \ldots, i_l} C_{\alpha'}^{i_1 \cdots i_l} \cdot \Gamma_i \cdot \tilde{\Phi} \circ \nabla^l s \left(\frac{\partial}{\partial \phi^{i_1}}, \ldots, \frac{\partial}{\partial \phi^{i_l}} \right) \circ \phi^{-1} \\
&\quad + \sum_{l \leq |\alpha'|} \sum_{i_1, \ldots, i_l} C_{\alpha'}^{i_1 \cdots i_l} \cdot \tilde{\Phi} \circ \nabla^{l+1} s \left(\frac{\partial}{\partial \phi^{i}}, \frac{\partial}{\partial \phi^{i_1}}, \ldots, \frac{\partial}{\partial \phi^{i_l}} \right) \circ \phi^{-1} \\
&\quad + \sum_{l \leq |\alpha'|} \sum_{i_1, \ldots, i_l} \sum_{a=1}^{l} \sum_{j=1}^{m} \Gamma_{i i_a}^j C_{\alpha'}^{i_1 \cdots i_l} \cdot \tilde{\Phi} \circ \nabla^l s \left(\frac{\partial}{\partial \phi^{1}}, \ldots, \frac{\partial}{\partial \phi^{j}}, \ldots, \frac{\partial}{\partial \phi^{i_l}} \right) \circ \phi^{-1},
\end{aligned}$$

where in the last term $\frac{\partial}{\partial \phi^j}$ replaces $\frac{\partial}{\partial \phi^{i_a}}$ and $\Gamma_{i i_a}^j$ is the Christoffel symbol of the Levi-Civita connection. This is seen to be of the desired form.

Having this formula at hand, we can show that the locally defined norm can be estimated by the globally defined one:

$$\begin{aligned}
\sum_{i\in I} \|\psi_i s\|_{C^k(U_i)} &\leq \tilde{C} \sum_{i\in I} \|s\|_{C^k(\operatorname{supp}\psi_i)} = \tilde{C} \sum_{i\in I} \sum_{|\alpha|\leq k} \sup_{x\in\operatorname{supp}\psi_i\circ\phi_i} \|\partial^\alpha \tilde{\Phi}_i \circ s \circ \phi(x)\| \\
&\leq \tilde{C} \sum_{i\in I} \sum_{|\alpha|\leq k} \sum_{l\leq|\alpha|} \sup_{p\in\operatorname{supp}\psi_i} \sum_{i_1,\ldots,i_l} \left\| C_\alpha^{i_1\cdots i_l} \cdot \tilde{\Phi}_i \circ \nabla^l s\left(\frac{\partial}{\partial\phi^{i_1}},\ldots,\frac{\partial}{\partial\phi^{i_l}}\right)_{|p} \right\| \\
&\leq \hat{C} \sum_{i\in I} \sum_{l\leq k} \sup_{p\in\operatorname{supp}\psi_i} \sum_{i_1,\ldots,i_l} \left\| \tilde{\Phi}_i \circ \nabla^l s\left(\frac{\partial}{\partial\phi^{i_1}},\ldots,\frac{\partial}{\partial\phi^{i_l}}\right)_{|p} \right\| \\
&\leq C \sum_{i\in I} \sum_{l\leq k} \sup_{p\in\operatorname{supp}\psi_i} \sqrt{< \nabla^l s_{|p}, \nabla^l s_{|p} >} \\
&\leq C\,\#\{i\in I \mid \psi_i \not\equiv 0\} \sum_{l\leq k} \sup_{p\in M} \sqrt{< \nabla^l s_{|p}, \nabla^l s_{|p} >}.
\end{aligned}$$

Here we used in the second but last inequality that all norms on a finite dimensional space are equivalent and that, furthermore, there is a uniform estimate if the families of norms on $(E_p)_{p\in\operatorname{supp}\psi_i}$ are continuous.

For the opposite direction, we argue analogously. First, by a similar inductive argument as in the beginning, we get for all $i_1,\ldots,i_l \in \{1,\ldots,m\}$ the formula

$$\tilde{\Phi} \circ \nabla^l s\left(\frac{\partial}{\partial\phi^{i_1}},\ldots,\frac{\partial}{\partial\phi^{i_l}}\right) \circ \phi = \sum_{|\alpha|\leq l} C^\alpha_{i_1\cdots i_l} \cdot \partial^\alpha_x(\tilde{\Phi}\circ s\circ\phi^{-1}),$$

where $C^\alpha_{i_1\cdots i_l} \in \Gamma(W, \mathbb{R}^{n\times n})$ for all $|\alpha| \leq l$.

Then, we can establish the estimate

$$\begin{aligned}
\sum_{l\leq k} \sup_{p\in M} \sqrt{< \nabla^l s_{|p}, \nabla^l s_{|p} >} &\leq \sum_{i\in I} \sum_{l\leq k} \sup_{p\in V_i} \sqrt{< \nabla^l s_{|p}, \nabla^l s_{|p} >} \\
&\leq \tilde{C} \sum_{i\in I} \sum_{l\leq k} \sup_{p\in V_i} \sum_{i_1,\ldots,i_l} \left\| \tilde{\Phi}_i \circ \nabla^l s\left(\frac{\partial}{\partial\phi^{i_1}},\ldots,\frac{\partial}{\partial\phi^{i_l}}\right)_{|p} \right\| \\
&\leq \tilde{C} \sum_{i\in I} \sum_{l\leq k} \sum_{i_1,\ldots,i_l} \sum_{|\alpha|\leq l} \sup_{x\in\phi_i^{-1}(V_i)} \left\| C^\alpha_{i_1\cdots i_l} \cdot \partial^\alpha \tilde{\Phi}_i \circ s \circ \phi(x) \right\| \\
&\leq C \sum_{i\in I} \sum_{|\alpha|\leq k} \sup_{x\in\phi_i^{-1}(V_i)} \left\| \partial^\alpha \tilde{\Phi}_i \circ s \circ \phi(x) \right\| = C \sum_{i\in I} \|s_{|V_i}\|_{C^k(V_i)}.
\end{aligned}$$

Here V_i is defined as assumed in Lemma A.4. Using the statement of this lemma, we get the desired result. □

Theorem A.7. *Let F be another vector bundle over M and $D\colon C^k(M,E) \to C^l(M,F)$ a local operator. If there is an atlas of charts $\phi_i\colon U_i \to W_i$ such that $E_{|U_i}$ and $F_{|U_i}$ can be trivialized and the local expressions $D_{|U_i}\colon C^k(U_i, E_{|U_i}) \to C^l(U_i, F_{|U_i})$ are continuous for all $i \in I$, then $D\colon C^k(M,E) \to C^l(M,F)$ is continuous.*

Proof. We define V_i as in Lemma A.4 and look at the decomposition

$$D\colon C^k(M,E) \to {\prod_{i\in I}}' C^k(V_i, E_{|V_i}) \xrightarrow{(D_{|V_i})_{i\in I}} {\prod_{i\in I}}' C^l(V_i, F_{|V_i}) \to C^l(M,F).$$

The first map is continuous by Lemma A.4, the last one by Remark A.5. The middle map is well-defined (i.e. lands in the primed product) by the locality condition. As $\phi_i^{-1}(V_i) = (\psi_i \circ \phi_i)^{-1}((\varepsilon, \infty)) \subseteq W_i$ can be assumed to have a smooth boundary (by slightly varying ε and Sard's theorem), the theory of function spaces on $\mathbb{R}^n$ implies that there exists a continuous extension operator $C^k(V_i, E_{|V_i}) \to C^k(U_i, E_{|U_i})$. The continuity of $D_{|U_i}$ implies the continuity of the middle map, since it is given by

$$D_{|V_i}\colon C^k(V_i, E_{|V_i}) \to C^k(U_i, E_{|U_i}) \xrightarrow{D_{|U_i}} C^l(U_i, F_{|U_i}) \to C^l(V_i, F_{|V_i}).$$

$\square$

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Erklärung

Ich habe die Arbeit selbständig verfasst, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und bisher keiner anderen Prüfungsbehörde vorgelegt. Außerdem bestätige ich hiermit, dass die vorgelegten Druckexemplare und die vorgelegte elektronische Version der Arbeit identisch sind und dass ich von den in §26 Abs. 6 vorgesehenen Rechtsfolgen Kenntnis habe.