Spinors and the Dominant Energy Condition for Initial Data Sets

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Abstract

Initial data sets (g, k) on a manifold M consist of a Riemannian metric g and a symmetric 2-tensor k on M. Their name strives from the fact that they constitute the gravitational initial data for the Cauchy problem of general relativity. The idea is that M should be contained as a spacelike hypersurface within a time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$ and that g is the induced metric and k the induced second fundamental form – the latter being a geometric version of the first derivative of g in the timelike normal direction.

Not all initial data sets can be considered to be physically reasonable. A restriction is given by the dominant energy condition, which implies that $\rho \geq |j|_g$, where the energy and momentum density ρ and j are computed from (g, k) via the so-called constraint equations. We note that in the case $k = 0$ this condition reduces to $\mathrm{scal}^g \geq 0$. Much is known about positive (and non-negative) scalar curvature on compact Riemannian manifolds and many of these results were obtained using Dirac operator methods. It is the idea of this thesis to extend these spinorial techniques to the setting of initial data sets with the aim of understanding the dominant energy condition $\rho \geq |j|_g$ better and, ultimately, being able to say something about general relativity.

The thesis is comprised of three articles. The first one deals with the C^∞-space of initial data sets subject to the strict condition $\rho > |j|_g$. We show that this space often has many non-zero homotopy groups. The non-trivial elements in these groups are obtained via a suspension construction out of certain known non-trivial elements in the homotopy groups of the space of positive scalar curvature metrics. While non-triviality of the latter is detected by Hitchin's index difference, we construct a similar index difference for initial data sets that is able to show that the former are non-zero.

A special case of this argument from the first article shows that if the so-called α-index of the manifold M is non-zero, then initial data sets with $\mathrm{tr}^g(k) > 0$ and the ones with $\mathrm{tr}^g(k) < 0$ belong to different connected components of the space of initial data sets under consideration. This is interesting because, in a certain sense, it rules out Lorentzian manifolds $(\overline{M}, \overline{g})$ with Cauchy hypersurface M satisfying a strict dominant energy condition and having both a big bang and a big crunch singularity. Since a non-zero α-index obstructs positive scalar curvature, one might wonder whether a similar conclusion also holds for other obstructions of positive scalar curvature. In the second article we show that this is indeed the case for Gromov and Lawson's enlargeability obstruction, which is effective for a huge class of examples.

For the third article we switch the perspective and look at a single initial data set. This time also the equality case in the inequality $\rho \geq |j|_g$ is allowed. In fact, we will assume that the manifold M has boundary and impose conditions on the boundary such that we end up in a rigid situation in which the equality has to hold everywhere. One motivation to study this rigid setup is that it is very similar to the settings that need to be ruled out when trying to extend the results from the first article to the non-strict case $\rho \geq |j|_g$.

Another motivation is that it leads to rigidity of the Lorentzian manifold in which the initial data set is contained. This is explained in the introductory chapter, where also the relation of the other results to relativity theory is described.

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Contents

1. Introduction

1.1. Overview of the results

Initial data sets allow to study general relativity using methods of Riemannian geometry. The idea is the following: Given a time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$, an embedded spacelike hypersurface M carries an induced Riemannian metric g and an induced second fundamental form k (taken w. r. t. to the future unit normal e_0 on M). The pair (g, k) is the induced *initial data set* on M. These pairs of a Riemannian metric and a symmetric 2-tensor capture some important physical information of the time-slice M. For instance, there is an *energy density* ρ and a *momentum density* j defined by

$$\begin{aligned}\rho &= \frac{1}{2}(\mathrm{scal}^g + \mathrm{tr}^g(k)^2 - |k|_g^2)\\ j &= \mathrm{div}^g\, k - \mathrm{d}\,\mathrm{tr}^g(k),\end{aligned}$$

respectively.

Apart from *vacuum* initial data sets, where $\rho \equiv 0$ and $j \equiv 0$, especially the more general class of initial data sets subject to the *dominant energy condition* (=DEC) $\rho \geq |j|_g$ is of physical importance. Roughly speaking, this condition follows from the postulate that matter does not propagate faster than light.

It is the general purpose of this thesis to study initial data sets subject to DEC. In order to do so, we first look at the strict DEC $\rho > |j|_g$ and consider the C^∞-space $\mathcal{I}^>(M)$ of all initial data sets satisfying strict DEC on a fixed closed n-manifold M. We wish to show that – similarly to the C^∞-space $\mathcal{R}^>(M)$ of positive scalar curvature (=PSC) metrics on M – this space has a lot of non-trivial homotopy groups. Indeed, there is a connection between PSC and strict DEC stemming from the fact that the initial data set $(g, 0)$ satisfies strict DEC if and only if the metric g has PSC. Analyzing this connection a bit more closely, we find that there is a homotopy equivalence between the suspension of $\mathcal{R}^>(M)$ and strict DEC initial data sets that are *totally umbilical*, i. e. of the form $(g, \tau g)$ for some metric g and $\tau \in \mathbb{R}$, cf. [AG23]. This yields a continuous map $\Phi\colon S\,\mathcal{R}^>(M) \to \mathcal{I}^>(M)$, which is well-defined up to homotopy (cf. Lemma 2.4.1). The following theorem allows to detect many non-trivial homotopy groups of $\mathcal{I}^>(M)$.

Theorem A (Theorem 2.1.2, Item 1)**.** *Let M be a closed spin manifold, $g_0 \in \mathcal{R}^{>}(M)$ and $l \geq 0$. Then there is a commutative diagram*

$$\begin{array}{ccccc} \pi_l(\mathcal{R}^{>}(M), g_0) & \xrightarrow{\text{Susp}} & \pi_{l+1}(S\,\mathcal{R}^{>}(M), [g_0, 0]) & \xrightarrow{\Phi_*} & \pi_{l+1}(\mathcal{I}^{>}(M), (g_0, 0)) \\ & \searrow^{\alpha\text{–diff}} & & \swarrow_{\overline{\alpha}\text{–diff}} & \\ & & \mathrm{KO}^{-n-l-1}(\{*\}). & & \end{array}$$

Here, the first upper map is the usual suspension homomorphism. The homomorphism denoted by α–diff is Hitchin's index difference, cf. Section 2.2.2. The remaining homomorphism $\overline{\alpha}$–diff is constructed in analogy to the index difference α–diff and this construction is one central part of this thesis. That α–diff (for $l > 0$) and $\overline{\alpha}$–diff are homomorphisms follows from an Eckman-Hilton argument as explained in [AG23, Rem. 6.8 (2)]. As indicated above, it is known by the work of various authors that for $n \geq 5$ there are plenty of non-trivial homotopy groups of $\mathcal{R}^{>}(M)$, e. g. cf. [Hit74; HSS14; BER14; CSS18]. In all the cited examples the non-trivial elements are detected by the index difference, i. e. their image under α–diff is non-zero. With this in mind, Theorem A shows that also $\mathcal{I}^{>}(M)$ has many non-trivial homotopy groups.

In a certain way, the statement is still correct when $l = -1$. Then the index difference $\overline{\alpha}$–diff for initial data sets is meant to detect different path-components of $\mathcal{I}^{>}(M)$ and α–diff gets replaced by the α-invariant detecting emptiness of $\mathcal{R}^{>}(M)$. In order to formulate it elegantly, we introduce the notion of *big bang initial data sets*. These are strict DEC initial data sets with $\mathrm{tr}^g(k) > 0$ (cf. Definition 3.1.2). Their name strives from the fact that they provide suitable initial conditions for Hawking's singularity theorem to apply. It turns out that all big bang initial data sets belong to the same path-component of $\mathcal{I}^{>}(M)$, which we call C^+. Dually, all *big crunch initial data sets*, characterized by $\mathrm{tr}^g(k) < 0$, are contained in a single path-component C^- of $\mathcal{I}^{>}(M)$.

Theorem B (Theorem 2.1.2, Item 2)**.** *Let M be a closed spin manifold. Then*

$$\overline{\alpha}\text{–diff}(C^-, C^+) = \alpha(M) \in \mathrm{KO}^{-n}(\{*\}).$$

If big bang and big crunch initial data sets belong to the same path-component of $\mathcal{I}^{>}(M)$, i. e. if $C^+ = C^-$, then index difference $\overline{\alpha}$–diff(C^-, C^+) is zero. Thus Theorem B shows that $\alpha(M) \neq 0$, which is a well-known obstruction to PSC for spin manifolds, also obstructs path-connectedness of $\mathcal{I}^{>}(M)$.

From the viewpoint of relativity theory especially the question whether $C^+ = C^-$ seems to be interesting. By a result from Bernal and Sánchez [BS05], a globally hyperbolic Lorentzian manifold $(\overline{M}, \overline{g})$ admits a foliation $\overline{M} \cong M \times \mathbb{R}$ by spacelike hypersurfaces $M \times \{t\}$, $t \in \mathbb{R}$. If $(\overline{M}, \overline{g})$ satisfies a suitable strict spacetime dominant energy condition, then such a foliation gives rise a path $\mathbb{R} \to \mathcal{I}^>(M)$, $t \mapsto (g_t, k_t)$, where (g_t, k_t) is the induced initial data set on $M \times \{t\}$. Thus $C^+ \neq C^-$ implies that $(\overline{M}, \overline{g})$ cannot have both a big bang and a big crunch singularity.

It is immediate to see that existence of a PSC metric on M implies that $C^+ = C^-$. Theorem B provides a partial converse to this in the sense that if $C^+ = C^-$, then at least a PSC metric on M is not obstructed by the α-index. In a celebrated article Stephan Stolz [Sto92] showed that if M is simply connected and of dimension $n \geq 5$, then being spin with $\alpha(M) \neq 0$ is the only obstruction. Thus in the high-dimensional simply connected case admitting a PSC metric and having $C^+ = C^-$ are equivalent. The same conclusion also holds for orientable 3-manifolds.

Theorem C (Theorem 3.1.3)**.** *Let M be an orientable closed connected 3-manifold. Then the following are equivalent:*

1. *$M \cong M_1 \sharp \cdots \sharp M_k$, where each M_i is either S^3/Γ for a lattice $\Gamma \subseteq \mathrm{SO}(4)$ or $S^2 \times S^1$.*
2. *M admits a metric of positive scalar curvature.*
3. *$C^+ = C^-$, so big bang and big crunch initial data sets belong to the same path-component of $\mathcal{I}^>(M)$.*

In fact, Theorem C follows from the classification of 3-manifolds and a statement about enlargeable manifolds. Enlargeability is a concept invented by Gromov and Lawson [GL80] to show non-existence of PSC metrics on a large class of manifolds. We show that the enlargeability obstruction for PSC implies $C^+ \neq C^-$ in analogy to Theorem B, where the same implication is shown for the α-index obstruction.

Theorem D (Theorem 3.1.5)**.** *If M is an enlargeable spin manifold, then the path-components C^+ and C^- of $\mathcal{I}^>(M)$ are distinct.*

So far, all the results dealt with strict DEC. This fits well with the research on scalar curvature, which focuses very much on PSC. From the point of physics, however, the strict condition seems somewhat artificial, although a cosmological constant or vacuum fluctuations could serve as motivation for studying the strict condition.

In order to extend the results to not necessarily strict DEC, it is important to study the equality case. This was started in joint work with Bernd Ammann [AG23]. There it turned out that for many manifolds M the topological implications for $\mathcal{I}^>(M)$ from Theorems A and B extend to the space of (plain) DEC initial data sets on M. The reason is that this extension only fails if M admits certain special initial data sets and existence of this geometric structure in turn has consequences for the topology of M. The findings from this article are not directly part of this thesis, although they provide an important link to the last main result.

Namely, the equality analysis from [AG23] happens – as everything mentioned so far – in the closed setting. If M is compact with boundary, there are situations where such special initial data sets show up as well. For this, we will consider initial data sets satisfying suitable boundary conditions.

Given an initial data set (g, k) on M and a co-oriented hypersurface $F \subset M$ with unit normal $\tilde{\nu}$, the *future null second fundamental form* (w. r. t. $\tilde{\nu}$) is the symmetric 2-tensor χ^+ on F defined by $\chi^+(X, Y) = g(\nabla_X \tilde{\nu}, Y) + k(X, Y)$ and its trace $\theta^+ = \mathrm{tr}^F(\chi^+)$ is called *future null expansion scalar.* These notions originate from mathematical relativity, where especially *MOTS* – hypersurfaces with $\theta^+ = 0$ – are used to characterize black hole boundaries in initial data sets. They are used in the next theorem, which is similar to a rigidity result by Eichmair, Galloway and Mendes [EGM21] applying in dimensions $3 \leq n \leq 7$.

Theorem E (Theorem 4.1.2)**.** *Let M be a compact connected spin manifold with boundary $\partial M = \partial_+ M \,\dot{\cup}\, \partial_- M$ endowed with an initial data set (g, k). Denote by $\tilde{\nu}$ the unit normal on ∂M that is inward-pointing along $\partial_+ M$ and outward-pointing along $\partial_- M$. Assume that*

- *(g, k) satisfies the dominant energy condition $\rho \geq |j|_g$,*
- *the future null expansion scalar (with respect to $\tilde{\nu}$) satisfies $\theta^+ \leq 0$ on $\partial_+ M$ and $\theta^+ \geq 0$ on $\partial_- M$, and*
- *the $\hat{A}$-genus of $\partial_- M$ is non-zero: $\hat{A}(\partial_- M) \neq 0$.*

Then there is a diffeomorphism $\Phi \colon \partial_- M \times [0, \ell] \to M$ defining a foliation $F_t = \Phi(\partial_- M \times \{t\})$ with $F_0 = \partial_+ M$ and $F_\ell = \partial_- M$. The leaves can be endowed with an induced initial data set, an induced spin structure and a unit normal $\tilde{\nu}$ pointing in the direction of growing t-parameter. The diffeomorphism can be chosen in such a way that the following holds for every leaf F_t:

- *Its future null second fundamental form (with respect to $\tilde{\nu}$) vanishes, $\chi^+ = 0$, in particular it is a MOTS.*
- *It carries a non-trivial parallel spinor, in particular its metric is Ricci-flat.*
- *Its tangent vectors are orthogonal to $j^\sharp$ and $\rho + j(\tilde{\nu}) = 0$, in particular the dominant energy condition holds marginally: $\rho = |j|_g$.*

Looking at totally umbilical initial data sets $(g, \tau g)$ yields an interesting Riemannian corollary. Somewhat surprisingly, its statement seems to be new, although there are very similar theorems by Cecchini and Zeidler [CZ24] as well as Daniel Räde [Räd23].

Theorem F (Corollaries 4.1.1 and 4.1.7)**.** *Let (M, g) be a compact connected Riemannian spin manifold with boundary $\partial M = \partial_+ M \dot\cup \partial_- M$ of dimension n and $\tau \in \{-1, 0, 1\}$. Assume that*

- *the scalar curvature is bounded below by* $\mathrm{scal}^g \geq -n(n-1)\tau^2$,
- *the mean curvature of the boundary with respect to the inward-pointing unit normal is bounded below by $H^g \geq -\tau$ on $\partial_+ M$ and $H^g \geq \tau$ on $\partial_- M$, and*
- *the $\hat{A}$-genus of $\partial_- M$ is non-zero: $\hat{A}(\partial_- M) \neq 0$.*

Then (M, g) is isometric to $(\partial_- M \times [0, \ell], e^{2\tau t}\gamma + \mathrm{d}t^2)$ (with $\partial_+ M$ corresponding to $\partial_- M \times \{0\}$) for a Ricci-flat metric γ on $\partial_- M$ admitting a non-trivial parallel spinor.

Switching back to relativity theory, another interesting consequence of Theorem E is concerned with globally hyperbolic Lorentzian manifolds $(\overline{M}, \overline{g})$ that contain M as spacelike Cauchy hypersurface with induced initial data set (g, k). We suppose that the Lorentzian manifold satisfies the spacetime dominant energy condition, and that the rigidity result is applicable for (M, g, k). Then using Theorem E with Addendum 4.1.3 it follows from Proposition 1.5.3 that $(\overline{M}, \overline{g})$ is uniquely determined up to restriction and isometry (cf. Corollary 1.5.4). The same conclusion also holds in the closed setting when the spacelike Cauchy hypersurface is obtained by gluing the initial data set from Theorem E along the boundary (cf. Corollary 1.5.5). This is surprising insofar that the spacetime dominant energy condition is a "partial differential inequality", more accurately a differential relation (cf. [Gro86, Ch. 1.1.1]) with non-empty interior, for which in general uniqueness cannot be expected.

The outlined main results originate from a series of articles by the author [Glö24b; Glö23a; Glö23b]. They are not only linked by the line of argument depicted above but also through the methods by which they were obtained. The main technical tool we use is the so-called *Dirac-Witten operator*. To define it, let M be a spacelike hypersurface of a time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$ and assume that M is spin. In this case a small open neighborhood of M in $\overline{M}$ is also spin. We may thus form a spinor bundle over such an open neighborhood and then restrict it to M. The resulting bundle does not depend on the choice of open neighborhood and comes by the name *hypersurface spinor bundle* $\overline{\Sigma}M \to M$. Importantly, this bundle carries several structures such as Clifford multiplication, scalar product and connection. All these are naturally induced from the surrounding Lorentzian neighborhood and the orthogonal splitting $T\overline{M}_{|M} = \underline{\mathbb{R}}e_0 \oplus TM$, but as it turns out the hypersurface spinor bundle with these structures may equally be constructed directly from M and the induced initial data set (g,k) on M.

The Dirac-Witten operator $\overline{D}$ is now the composition

$$\Gamma(\overline{\Sigma}M) \xrightarrow{\overline{\nabla}} \Gamma(T^*M \otimes \overline{\Sigma}M) \xrightarrow{\mathbb{1}_{T^*M} \otimes \mathrm{cl}} \Gamma(T^*M \otimes T^*M \otimes \overline{\Sigma}M) \xrightarrow{\mathrm{tr}^g \otimes \mathbb{1}_{\overline{\Sigma}M}} \Gamma(\overline{\Sigma}M)$$

of connection, Clifford multiplication and trace. It is crucial that the connection used here is the one induced from the Levi-Civita connection of $\overline{g}$, while the trace is just formed over directions tangent to M. It was observed by Witten [Wit81] that the so-defined self-adjoint elliptic operator satisfies the following Schrödinger-Lichnerowicz type formula

$$\overline{D}^2 = \overline{\nabla}^*\overline{\nabla} + \frac{1}{2}(\rho - e_0 \cdot j^\sharp \cdot),$$

which ties it to the DEC for initial data sets. Namely, if the strict DEC $\rho > |j|_g$ holds on a closed manifold, then $\overline{D}$ must be invertible. This is the main observation underlying the aforementioned results.

This thesis is structured as follows. The remainder of this introductory chapter is devoted to the two central objects: initial data sets and Dirac-Witten operators. We introduce a variety of associated notions and establish their main properties, laying the common ground for the other chapters. A special emphasis is on the interplay of initial data set perspective and the viewpoint from Lorentzian manifolds. In fact, this is the only part of the thesis where we work with Lorentzian manifolds. On the one hand they motivate many definitions, on the other hand they serve to interpret the main results within relativity theory.

The other chapters each consist of one of the aforementioned articles by the author. Apart from small changes necessary for the coherence of this thesis they were kept close to the referenced published or preprint version (at the expense of certain redundancy). Concretely, the second chapter is basically identical to [Glö24b], which in turn is an outgrowth of the author's master thesis [Glö19]. We use a Clifford-linear version of the hypersurface spinor bundle and its Dirac-Witten operator to construct the index difference $\overline{\alpha}$–diff for initial data sets. Then we compare it to the index difference α–diff for Riemannian metrics first studied by Hitchin, which serves as a blueprint for its construction. Doing so, we obtain Theorems A and B.

The third chapter corresponds to [Glö23a]. It aims at proving Theorem D and its corollary Theorem C. For this, we consider a twisted complex hypersurface spinor bundle. From its Dirac-Witten operator we construct a twisted index difference for initial data sets and prove a corresponding index theorem. Actually, in order to be able to cover the most general notion of enlargeability, we allow for certain non-compact manifolds and consider a pair of twist bundles coinciding outside a compact subset. We then define a relative index difference.

The fourth and last chapter covers [Glö23b]. There, we consider *Dirac-Witten harmonic spinors*, i. e. spinors in the kernel of the (untwisted complex) Dirac-Witten operator. The conclusions of Theorem E are derived from the existence of such spinors on DEC initial data sets with the stated boundary conditions. Their existence follows from an index theorem. As already mentioned, Theorem F is then a rather direct consequence.

1.2. Initial data sets, Lorentzian manifolds and energy conditions

As pointed out, in the remainder of this chapter we introduce initial data sets, Dirac-Witten operators and a variety of connected notions and foundational results. Some of the material presented here is very basic, other pieces are famous theorems of relativity theory and particularly the last section also contains original research. Usually, the deeper theorems presented here are not logically necessary for the main results, but are meant to provide a background for the rest of the thesis. When we study the interaction between initial data sets and the time-oriented Lorentzian manifolds in which they are contained, we will freely use notions of Lorentzian causality theory as they are summarized in [ONe83, Ch. 14]. However notice that in contrast to this reference, when speaking of (Cauchy) hypersurfaces, we shall always mean smooth ones. Throughout, M will be a smooth manifold of dimension $n \geq 2$.

Definition 1.2.1. An *initial data set* is a triple (M, g, k) consisting of a manifold M, a Riemannian metric g on M and a symmetric $(0, 2)$-tensor field k on M. The pair (g, k) is referred to as initial data set on M.

Example 1.2.2. Given a Riemannian manifold (M, g), the pair $(g, 0)$ is an initial data set on M. Initial data sets of this kind (with $k = 0$) are called *time-symmetric.* More generally, $(g, \tau g)$ is an initial data set on M for any $\tau \in \mathbb{R}$. These initial data sets are called *totally umbilical.*

Example 1.2.3. Suppose that M is a spacelike hypersurface of a time-oriented Lorentzian manifold[1] $(\overline{M}, \overline{g})$. In this case the induced metric g on M is Riemannian. The second fundamental form k with respect to the future unit normal e_0 is the symmetric $(0, 2)$-tensor field on M defined by

$$\overline{\nabla}_X Y = \nabla_X Y + k(X, Y) e_0 \tag{1.1}$$

for any $X, Y \in \Gamma(TM)$, where $\overline{\nabla}$ is the Levi-Civita connection of $(\overline{M}, \overline{g})$ and ∇ is the one of (M, g). The pair (g, k) is the initial data set on M *induced* by $(\overline{M}, \overline{g})$.

[1] Throughout this thesis, the overline signifies quantities related to such an ambient Lorentzian manifold. To avoid confusion, we will usually not use it differently, e. g. for topological closures.

Example 1.2.3 shows how we usually think about initial data sets: as embedded into some time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$. Moreover, we usually consider M not only to be some spacelike hypersurface but actually to be a spacelike *Cauchy hypersurface* of $(\overline{M}, \overline{g})$. This means that it is met precisely once by each inextendable timelike curve. Existence of such a hypersurface has strong implications on the causal structure of the time-oriented Lorentzian manifold, namely it has to be *globally hyperbolic* (cf. [ONe83, Cor. 14.39]). Conversely, the following holds:

Theorem 1.2.4 ([BS05])**.** *Suppose a Lorentzian manifold $(\overline{M}, \overline{g})$ is globally hyperbolic. Then there is a* Cauchy temporal function *f on $\overline{M}$, i. e. a smooth function $f\colon \overline{M} \to \mathbb{R}$ such that $T = \mathrm{grad}^{\overline{g}} f$ is past-timelike and each level set of f is a Cauchy hypersurface of $(\overline{M}, \overline{g})$. In particular, taking M to be any Cauchy hypersurface of $\overline{M}$, the map*

$$\Phi\colon \mathbb{R} \times M \to \overline{M}$$

sending a point (t, p) to $\gamma_p(t)$, where γ_p is the flow line of $\frac{-T}{\overline{g}(T,T)}$ through p with $f(\gamma_p(t)) = t$, defines a diffeomorphism and $\Phi^*\overline{g} = -N^2 \mathrm{d}t^2 + g_t$ for a smooth function $N\colon \mathbb{R} \times M \to (0, \infty)$ and a smooth family $(g_t)_{t\in\mathbb{R}}$ of Riemannian metrics on M, which is canonically identified with the leaves $M_t := \{t\} \times M = \Phi^{-1}(f^{-1}(t))$.

Remark 1.2.5. This Cauchy temporal function can also be chosen in such a way that it is *adapted* to a previously chosen Cauchy hypersurface M, meaning that M is one of its level sets. We show this in Appendix A.1.

The reason for us to consider Cauchy hypersurfaces and globally hyperbolic manifolds is that they allow for the presumably best correspondence between properties of the initial data set and the ones of the Lorentzian manifold. For instance, Theorem 1.2.4 shows that their topology (and differential structure) are determined by one another. Another example of such a connection is provided by the singularity theorems that we review in Section 1.4.

On top of that, global hyperbolicity is a natural assumption when considering the Cauchy problem of general relativity. It is in the context of this Cauchy problem where initial data sets have their historic origin and play an important role (cf. [BI04] for an overview). The main observation is that in the situation of Example 1.2.3 certain components of the Einstein curvature are already determined by the induced initial data set on M. This is obtained by a direct calculation using the Gauß-Codazzi equations for the curvature of a hypersurface.

Proposition 1.2.6 (cf. e. g. [BI04, eq. (16-17)])**.** *Suppose that M is a spacelike hypersurface of a time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$ and (g, k) is the induced initial* *data set on M. Then*

$$\begin{aligned}
\mathrm{Ein}^{\overline{g}}(e_0, e_0) &= \frac{1}{2}(\mathrm{scal}^g + \mathrm{tr}^g(k)^2 - |k|_g^2) \\
\mathrm{Ein}^{\overline{g}}(e_0, \text{-})_{|TM} &= \mathrm{div}^g\, k - \mathrm{d}\,\mathrm{tr}^g(k).
\end{aligned}$$

According to the Einstein field equations, the Einstein curvature is (up to a scaling factor) the same as the energy-momentum tensor determined by the matter distribution. It hence has a physical meaning. This motivates the following definition.

Definition 1.2.7. The *energy density* $\rho \in C^\infty(M)$ and the *momentum density* $j \in \Omega^1(M)$ of an initial data set (M, g, k) are defined by

$$\begin{aligned}
\rho &= \frac{1}{2}(\mathrm{scal}^g + \mathrm{tr}^g(k)^2 - |k|_g^2) \\
j &= \mathrm{div}^g\, k - \mathrm{d}\,\mathrm{tr}^g(k).
\end{aligned} \tag{1.2}$$

In the vacuum case, the energy-momentum tensor and hence the Einstein curvature of the Lorentzian manifold $(\overline{M}, \overline{g})$ are zero. This implies that the induced initial data set on any spacelike hypersurface M satisfies the *vacuum constraint equations* $\rho = 0$ and $j = 0$. Conversely, we have the following celebrated result by Yvonne Choquet-Bruhat.

Theorem 1.2.8 (Local solution of the vacuum Cauchy problem, [Fou52, Ch. IV], cf. also [Cho09, Thms. 8.3, 8.4])**.** *Any initial data set (M, g, k) subject to the vacuum constraints sits as a Cauchy hypersurface inside a time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$ satisfying the vacuum Einstein equation* $\mathrm{Ein}^{\overline{g}} = 0$. *This globally hyperbolic Lorentzian manifold is* locally geometrically uniquely *determined by (M, g, k), i. e. for any two such vacuum solutions $(\overline{M}_1, \overline{g}_1)$ and $(\overline{M}_2, \overline{g}_2)$ there are open neighborhoods $U_1 \subseteq \overline{M}_1$ and $U_2 \subseteq \overline{M}_2$ of M and a time-orientation preserving isometry $(U_1, \overline{g}_1) \cong (U_2, \overline{g}_2)$ that restricts to the identity on M.*

Remark 1.2.9. In the non-vacuum case, the Cauchy problem also involves prescribing matter fields (and potentially their first time-derivatives) on M. Again, there are (gravitational) constraint equations relating ρ and j to expressions involving the initial data for matter – and potentially additional non-gravitational constraints. The precise form of all these depends on the type of matter considered; some examples of possible matter models can be found in [HE73, Sec. 3.3]. In specific cases, the Cauchy problem has also been resolved, e. g. for the Einstein-Maxwell [Cho09, Thm. 10.3, Cor. 10.4], the Einstein-Maxwell-Dirac [MN17, Lem. 3.20, 3.21] and the Einstein-Vlasov system [Rin13, Thms. 22.12, 22.14].

In general, however, even spelling out all the constraints and evolution equations for a universe as the one we live in seems to be a hopeless task. At least qualitative statements about the large-scale behavior of solutions to the Einstein equations with matter may be obtained using energy conditions.

Definition 1.2.10. Let $(\overline{M}, \overline{g})$ be a time-oriented Lorentzian manifold. It is said to satisfy

- the *dominant energy condition* (=DEC) if $\mathrm{Ein}^{\overline{g}}(V, W) \geq 0$ for all future-causal vectors $V, W \in T_p\overline{M}$ and all $p \in \overline{M}$,
- the *weak energy condition* if $\mathrm{Ein}^{\overline{g}}(V, V) \geq 0$ for all future-causal vectors $V \in T\overline{M}$,
- the *null energy condition* if $\mathrm{Ein}^{\overline{g}}(V, V) \geq 0$ for all future-lightlike vectors $V \in T\overline{M}$, and
- the *strong energy condition* if $\mathrm{ric}^{\overline{g}}(V, V) \geq 0$ for all future-causal vectors $V \in T\overline{M}$.

The dominant energy condition reflects the physical axiom that matter should not propagate faster than light, as will also become apparent in Proposition 1.5.1. It is thus supposed to be satisfied for any time-oriented Lorentzian manifold subject to the Einstein field equations with physically reasonable matter. It obviously implies the weak energy condition, which in turn implies the null energy condition. On the other hand, there is no a priori reason why the strong energy condition should hold. In fact, despite being satisfied for many matter models it is violated by a massive scalar field. Nonetheless, Hawking and Ellis argue that it makes sense to assume it when considering large-scale effects [HE73, Sec. 4.3, last paragraph]. It also directly implies the null energy condition.

The dominant energy condition can be equivalently stated by requiring that $\mathrm{Ein}^{\overline{g}}(V, \text{-})^\sharp$ is zero or a past-causal vector for all future-causal $V \in T\overline{M}$. In particular, inserting for V the future unit normal e_0 of a spacelike hypersurface and taking into account Proposition 1.2.6 and (1.2), we immediately obtain the following lemma, which motivates the subsequent definition.

Lemma 1.2.11. *Suppose that M is a spacelike hypersurface of a time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$ with induced initial data set (g, k). If $(\overline{M}, \overline{g})$ satisfies the dominant energy condition, then $\rho \geq |j|_g$ holds on M.*

Definition 1.2.12. An initial data set (M, g, k) satisfies the *dominant energy condition* (=DEC) if $\rho \geq |j|_g$. It is said to satisfy the *strict dominant energy condition* if $\rho > |j|_g$.

It is natural to ask for a converse of Lemma 1.2.11, i. e. whether every DEC initial data set (M, g, k) extends to a time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$ subject to DEC. By the solution of the vacuum Cauchy problem (cf. Theorem 1.2.8) such an extension exists in the vacuum case $\rho \equiv 0$, $j \equiv 0$. There is also an extension as desired if the strict DEC $\rho > |j|_g$ is satisfied. We show a slightly stronger version of this statement below, using a variation of the technique originally employed in [Glö19, Prop. 1.10]. In the general case $\rho \geq |j|_g$, however, such an extension does not exist: In [Glö24a] the author constructs DEC initial data sets that do not lie within any time-oriented Lorentzian manifold subject to DEC.

Proposition 1.2.13 (cf. [Glö19, Prop. 1.10])**.** *Let (M, g, k) be an initial data set subject to strict DEC. Then there is a globally hyperbolic Lorentzian manifold $(\overline{M}, \overline{g})$ that contains M as spacelike Cauchy hypersurface with induced initial data set (g, k) such that both the dominant and the strong energy condition are satisfied by $(\overline{M}, \overline{g})$.*

Proof. The idea is to define a suitable symmetric 2-tensor h on M and then to consider

$$\overline{g} = -\mathrm{d}t^2 + g + 2tk + t^2 h$$

on an open neighborhood $\overline{M}$ of $\{0\} \times M$ in $\overline{M}_0 \subseteq \mathbb{R} \times M$, where t denotes the $\mathbb{R}$-coordinate and $\overline{M}_0$ is the open subset on which the symmetric 2-tensor $\overline{g}$ is a Lorentzian metric. Identifying $\{0\} \times M$ with M in the canonical way, the induced initial data set on M in $(\overline{M}_0, \overline{g})$ is precisely (g, k).

The next observation is that in the situation above the map $\Gamma(T^*M \odot T^*M) \to \Gamma(T^*M \odot T^*M)$ sending h to $S := \mathrm{Ein}^{\overline{g}}_{|TM \otimes TM}$ is a bijection. Namely, the formula [BI04, eq. (22)] shows that

$$h = S - \frac{1}{n-1} tr^g(S) g + \frac{1}{n-1} \rho g - \mathrm{ric}^g + 2k^2 - \mathrm{tr}^g(k) k,$$

where ρ is the energy density of (g, k) and $k^2(X, Y) = k(k(X, \text{-})^\sharp, Y)$ for $X, Y \in \Gamma(TM)$. This equation can be solved for S by first solving the traced equation for $\mathrm{tr}^g(S)$ and using the result to get rid of the second S on the right-hand side. Hence we may freely specify S and choose h accordingly.

The choice we make is $S = \frac{1}{\rho} j \otimes j + \alpha g$, where $\alpha \in C^\infty(M)$ with $0 < \alpha < \frac{(\rho - |j|_g)^2}{\rho}$. Note that due to $\rho > |j|_g \geq 0$ such an α exists and the first summand of S is a well-defined and smooth symmetric 2-tensor on M. We claim that for any $p \in M$ and future-causal vectors $V, W \in T_p\overline{M}_0$ the following conditions hold:

$$\mathrm{Ein}^{\overline{g}}(V, W) > 0 \quad \text{and} \quad \mathrm{ric}^{\overline{g}}(V, V) > 0. \tag{1.3}$$

To see this, we write $V = ae_0 + X$ and $W = be_0 + Y$ with $a, b > 0$ and $X, Y \in T_pM$, observe that $|X|_g \leq a$ and $|Y|_g \leq b$. Then we calculate

$$\begin{aligned}
\mathrm{Ein}^{\overline{g}}(V,W) &= ab\rho + aj(Y) + bj(x) + S(X,Y) \\
&= \frac{1}{\rho}(a\rho + j(X))(b\rho + j(Y)) + \alpha g(X,Y) \\
&\geq \frac{1}{\rho}(a\rho - |j|_g|X|_g)(b\rho - |j|_g|Y|_g) - \alpha|X|_g|Y|_g \\
&\geq \frac{ab}{\rho}(\rho - |j|_g)^2 - \alpha ab > 0
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{ric}^{\overline{g}}(V,V) &= \mathrm{Ein}^{\overline{g}}(V,V) - \frac{1}{n-1}\,\mathrm{tr}^g(\mathrm{Ein}^{\overline{g}})g(V,V) \\
&= a^2\rho + 2aj(X) + S(X,X) - \frac{1}{n-1}(-\rho + \mathrm{tr}^g(S))(-a^2 + |X|_g^2) \\
&= \frac{1}{(n-1)\rho}\left((n-1)(a\rho + j(X))^2 - (-\rho^2 + |j|_g^2)(-a^2 + |X|_g^2)\right) \\
&\quad + \frac{\alpha}{n-1}\left((n-1)|X|_g^2 - n(-a^2 + |X|_g^2)\right) \\
&\geq \frac{1}{(n-1)\rho}\left((a\rho - |j|_g|X|_g)^2 - (\rho^2 - |j|_g^2)(a^2 - |X|_g^2)\right) \\
&\quad + \frac{\alpha}{n-1}\left(n|X|_g^2 - a^2 - n(-a^2 + |X|_g^2)\right) \\
&= \frac{1}{(n-1)\rho}\left(-2a\rho|j|_g|X|_g + \rho^2|X|_g^2 + a^2|j|_g^2\right) + \alpha a^2 \\
&= \frac{1}{(n-1)\rho}\left(\rho|X|_g - a|j|_g\right)^2 + \alpha a^2 > 0
\end{aligned}$$

where we used $n - 1 \geq 1$ and $-|X|_g^2 \geq -a^2$.

Now we argue that the conditions (1.3) also hold on an open neighborhood of M. Assume for contradiction that there is a sequence of points $p_i \in \overline{M}_0$ and future-causal vectors $V_i, W_i \in T_{p_i}\overline{M}_0$ with $p_i \longrightarrow p \in M$ for $i \longrightarrow \infty$ such that $\mathrm{Ein}^{\overline{g}}(V_i, W_i) \leq 0$ or $\mathrm{ric}^{\overline{g}}(V_i, V_i) \leq 0$ for all i. Since these inequalities are invariant under rescaling V_i and W_i with a positive factor, we may assume that they are unit vectors with respect to some auxiliary Riemannian metric. Then, after potentially passing to a subsequence, the V_i and W_i converge. The limit vectors $V, W \in T_pM$ will again be future-causal and at least one of the two conditions $\mathrm{Ein}^{\overline{g}}(V,W) \leq 0$ and $\mathrm{ric}^{\overline{g}}(V,V) \leq 0$ holds, yielding the desired contradiction.

We have seen that there is an open neighborhood $\overline{M}_1$ of $\{0\} \times M$ in $\overline{M}_0$ on which the dominant and the strong energy condition are satisfied (even in the strict sense (1.3)).

As a final step, we restrict to the *domain of dependence* of $\{0\} \times M$ in $(\overline{M}_1, \overline{g})$, i. e.

$$D(M) := \left\{ p \in \overline{M}_1 \,\middle|\, \text{every inextendable causal curve in } \overline{M}_1 \text{ through } p \text{ intersects } M \right\}.$$

It follows from [ONe83, Ch. 14, Lem. 42 and 43] that $\overline{M} := D(M)$ is an open neighborhood of $\{0\} \times M$ and globally hyperbolic. □

Remark 1.2.14. If $n = \dim(M) \geq 3$, the calculations in the proof above can be significantly simplified by choosing $\alpha \equiv 0$. In this case the desired strictness in the inequality $\mathrm{ric}^g(V, V) > 0$ can be obtained from $n - 1 > 1$ and does not rely on the second summand of S.

1.3. Spinors and Dirac-Witten operator

In this section we investigate the correspondence between spinors on initial data sets and those on a surrounding Lorentzian manifold. This leads to the notion of a hypersurface spinor bundle that goes back to Hijazi and Zhang [HZ03] and acquired a more precise meaning in the article [AG23] by Bernd Ammann and the author of this thesis. We will see that its structures allow for the definition of a natural first-order differential operator: the Dirac-Witten operator. This operator was first studied by Edward Witten [Wit81], who observed that it is connected to DEC. That connection – through a Schrödinger-Lichnerowicz type formula – will be a central ingredient for all the main results, Theorems A to F.

Before investigating spinor bundles and spin structures, we have a brief look at what might be called hypersurface tangent bundle. Let (M, g, k) be an initial data set. Suppose first that it is the induced initial data set on a spacelike hypersurface of a time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$. The hypersurface tangent bundle is supposed to be (canonically isomorphic to) $T\overline{M}_{|M}$ with its metric $\overline{g}$ and Levi-Civita connection $\overline{\nabla}$. Although there is no obstruction to finding such an $(\overline{M}, \overline{g})$ – to find one, for instance, one can use the ansatz $\overline{g} = -\mathrm{d}t^2 + g + 2tk$ similarly as in the proof of Proposition 1.2.13 – it is a priori not clear that the bundle $T\overline{M}_{|M}$ is independent of this choice.

Definition 1.3.1. The *hypersurface tangent bundle* of an initial data set (M, g, k) is the bundle $\overline{T}M := \underline{\mathbb{R}} \oplus TM$ together with the metric $\overline{g}((a, X), (b, Y)) := -ab + g(X, Y)$ and the connection $\overline{\nabla}_X(b, Y) := (\partial_X b, \nabla_X Y) + (k(X, Y), k(bX, \text{-})^\sharp)$ for $a, b \in C^\infty(M)$ and $X, Y \in \Gamma(TM)$, where ∇ is the Levi-Civita connection of TM.

It is straightforward to see that the map $\overline{T}M \to T\overline{M}_{|M}$, $(a, X) \mapsto ae_0 + X$, where e_0 is the future unit normal as before, defines the requested metric bundle isomorphism.

Its compatibility with the connection follows from (1.1), $\overline{g}(\overline{\nabla}_X e_0, e_0) = \frac{1}{2}\partial_X \overline{g}(e_0, e_0) = 0$ and $\overline{g}(\overline{\nabla}_X e_0, Y) = -\overline{g}(e_0, \overline{\nabla}_X Y) = k(X, Y)$. Note that as a consequence of this isomorphism the connection $\overline{\nabla}$ on $\overline{T}M$ is a metric connection. This can also be verified by direct computation.

In terms of principal bundles, the construction can be understood as follows. For simplicity, we now assume that M is oriented. Then $\overline{M}$ is space- and time-oriented and there is a principal bundle $P_{\mathrm{SO}_0(n,1)}\overline{M}$ of generalized orthonormal frames for the structure group $\mathrm{SO}_0(n, 1)$ – the identity component of $O(n, 1)$. We restrict this bundle to M and observe that the structure group can be reduced to $\mathrm{SO}(n) \subseteq \mathrm{SO}_0(n, 1)$: The $\mathrm{SO}(n)$-principal bundle $P_{\mathrm{SO}(n)}M$ of orthogonal frames of M embeds into $P_{\mathrm{SO}_0(n,1)}\overline{M}_{|M}$ via $(e_1, \ldots, e_n) \mapsto (e_0, e_1, \ldots, e_n)$ and the following diagram involving furthermore the obvious action and projection maps commutes:

$$\begin{array}{ccc} P_{\mathrm{SO}(n)}M \times \mathrm{SO}(n) & \longrightarrow & P_{\mathrm{SO}_0(n,1)}\overline{M}_{|M} \times \mathrm{SO}_0(n,1) \\ \downarrow & & \downarrow \\ P_{\mathrm{SO}(n)}M & \longrightarrow & P_{\mathrm{SO}_0(n,1)}\overline{M}_{|M} \\ & \searrow \quad M. \quad \swarrow & \end{array}$$

The hypersurface tangent bundle now emerges by associating the standard (orthogonal) representation of $\mathrm{SO}_0(n, 1)$ on $\mathbb{R}^{n,1}$. After restricting to $\mathrm{SO}(n)$, this representation decomposes into an orthogonal direct sum of the trivial representation on $\mathbb{R}$ and the standard representation of $\mathrm{SO}(n)$ on $\mathbb{R}^n$, explaining the orthogonal sum in Definition 1.3.1. However, in the general case $k \not\equiv 0$ the Levi-Civita connection on $P_{\mathrm{SO}_0(n,1)}\overline{M}_{|M}$ does not reduce to $\mathrm{SO}(n)$. This results in two connections on the associated bundle: one coming from the Levi-Civita connection of $P_{\mathrm{SO}_0(n,1)}\overline{M}$ and one from the Levi-Civita connection of $P_{\mathrm{SO}(n)}M$. Their difference is the second summand in the definition of $\overline{\nabla}$ in Definition 1.3.1.

The construction of the hypersurface spinor bundle resembles the one of the hypersurface tangent bundle, but here it is more suitable to start from principal bundles right away. We assume that we are in the following situation. Both M and $(\overline{M}, \overline{g})$ are spin and the spin structure on $(\overline{M}, \overline{g})$ restricts to the one on M in the sense that

$$\begin{array}{ccc} P_{\mathrm{Spin}(n)}M & \longrightarrow & P_{\mathrm{Spin}_0(n,1)}\overline{M}_{|M} \\ \downarrow & & \downarrow \\ P_{\mathrm{SO}(n)}M & \longrightarrow & P_{\mathrm{SO}_0(n,1)}\overline{M}_{|M} \end{array}$$

is a pullback of double coverings. Here, the vertical maps are reductions of the structure group to $\mathrm{Spin}(n)$ and $\mathrm{Spin}_0(n, 1)$, respectively; the latter being the identity component of $\mathrm{Spin}(n,1) \subseteq \mathrm{Cl}_{n,1}$. In this case also the following cube involving the action maps commutes:

$$\begin{array}{ccccccc}
P_{\mathrm{Spin}(n)}M \times \mathrm{Spin}(n) & & \longrightarrow & & P_{\mathrm{Spin}_0(n,1)}\overline{M}_{|M} \times \mathrm{Spin}_0(n,1) & & \\
& \searrow & & & \big| & \searrow & \\
\big\downarrow & & P_{\mathrm{Spin}(n)}M & \longrightarrow & \big\downarrow & & P_{\mathrm{Spin}_0(n,1)}\overline{M}_{|M} \\
P_{\mathrm{SO}(n)}M \times \mathrm{SO}(n) & & \big| \longrightarrow & & P_{\mathrm{SO}_0(n,1)}\overline{M}_{|M} \times \mathrm{SO}_0(n,1) & & \big\downarrow \\
& \searrow & \big\downarrow & & & \searrow & \\
& & P_{\mathrm{SO}(n)}M & & \longrightarrow & & P_{\mathrm{SO}_0(n,1)}\overline{M}_{|M}.
\end{array}$$

This is already clear for all sides but the upper one, for which it then follows using the unique lifting property of coverings. We conclude that $P_{\mathrm{Spin}(n)}M \longrightarrow P_{\mathrm{Spin}_0(n,1)}\overline{M}_{|M}$ is a $\mathrm{Spin}(n)$-reduction. Note that if M is spin the described situation can always be achieved by restricting to a small open neighborhood of M in $\overline{M}$.

Let now $\rho\colon \mathrm{Cl}_{n,1} \to \mathrm{End}(W)$ be a representation on a non-zero real or complex vector space W. It restricts to a $\mathrm{Spin}_0(n,1)$-representation and yields a spinor bundle $\Sigma\overline{M}$ on $\overline{M}$ associated to $P_{\mathrm{Spin}_0(n,1)}\overline{M}$. The $\mathrm{Cl}_{n,1}$-action on W gives rise to a Clifford multiplication $T\overline{M} \otimes \Sigma\overline{M} \longrightarrow \Sigma\overline{M}$ and this is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ on $T\overline{M}$ and its induced connection $\overline{\nabla}$ on $\Sigma\overline{M}$. We restrict $\Sigma\overline{M}$ and the described structures to M and observe that they are independent of the Lorentzian manifold $(\overline{M}, \overline{g})$.

Definition 1.3.2. The *hypersurface spinor bundle* $\overline{\Sigma}M$ of (M, g, k) (w. r. t. ρ) is the vector bundle associated to $P_{\mathrm{Spin}(n)}M$ along $\mathrm{Spin}(n) \hookrightarrow \mathrm{Cl}_{n,1} \xrightarrow{\rho} \mathrm{End}(W)$, equipped with

- the Clifford multiplication $\overline{T}M \otimes \overline{\Sigma}M \longrightarrow \overline{\Sigma}M$ induced from the $\mathrm{Cl}_{n,1}$-action on W and

- the connection $\overline{\nabla}$ defined by

$$\overline{\nabla}_X \psi = \nabla_X \psi - \frac{1}{2} e_0 \cdot k(X, \text{-})^\sharp \cdot \psi \tag{1.4}$$

  for $\psi \in \Gamma(\overline{\Sigma}M)$ and $X \in \Gamma(TM)$, where ∇ is the connection associated to the Levi-Civita connection of (M, g) on $P_{\mathrm{Spin}(n)}M$.

Sometimes it is more convenient to use the splitting $\overline{T}M = \underline{R} \oplus TM$ and think of the ($\overline{T}M$-)Clifford multiplication as a Clifford multiplication $TM \otimes \overline{\Sigma}M \longrightarrow \overline{\Sigma}M$ by vectors of TM and an involution $e_0\cdot\colon \overline{\Sigma}M \longrightarrow \overline{\Sigma}M$ anti-commuting with the TM-Clifford multiplication. Both of these operations are parallel with respect to ∇, but in general not $\overline{\nabla}$-parallel. This allows to compute that, on the other hand, the $\overline{T}M$-Clifford multiplication is $\overline{\nabla}$-parallel while in general not being ∇-parallel. The parallelism of

$\overline{T}M \otimes \overline{\Sigma}M \longrightarrow \overline{\Sigma}M$ w. r. t. $\overline{\nabla}$ also follows from the following lemma that motivated the definition above.

Lemma 1.3.3. *The canonical maps $\overline{T}M \longrightarrow T\overline{M}_{|M}$ and $\overline{\Sigma}M \longrightarrow \Sigma\overline{M}_{|M}$ arising from the reductions of the structure group from* $\mathrm{Spin}_0(n,1)$ *to* $\mathrm{Spin}(n)$ *are isomorphisms. These isomorphisms preserve the metric $\overline{g}$ on the hypersurface tangent bundle, the connections $\overline{\nabla}$ on hypersurface tangent and spinor bundle as well as the Clifford multiplication involving both bundles.*

Proof. The claims concerning the hypersurface tangent bundle have already been verified in the discussion following Definition 1.3.1. The isomorphism $\overline{\Sigma}M \cong \Sigma\overline{M}_{|M}$ and the compatibility with the Clifford multiplication follow directly after unraveling the associate bundle constructions involved. It therefore remains to verify (1.4). This is done later in Lemma 2.3.4 for a specific representation ρ, but the proof there does not depend on this choice and literally applies to the more general situation here. □

Definition 1.3.4. The *Dirac-Witten operator* on a hypersurface spinor bundle $\overline{\Sigma}M$ is the composition

$$\overline{D}\colon \Gamma(\overline{\Sigma}M) \xrightarrow{\overline{\nabla}} \Gamma(T^*M \otimes \overline{\Sigma}M) \longrightarrow \Gamma(\overline{\Sigma}M),$$

where the second map is essentially the Clifford multiplication and is obtained from it by precomposing with $T^*M \xrightarrow{\sharp} TM \hookrightarrow \overline{T}M$ in the first tensor factor.

The Dirac-Witten operator is closely related to the usual Dirac operator D on the spinor bundle of M associated to the representation $\mathrm{Cl}_n \hookrightarrow \mathrm{Cl}_{n,1} \xrightarrow{\rho} \mathrm{End}(W)$. In fact, this spinor bundle is just on the nose $\overline{\Sigma}M$ equipped with the TM-Clifford multiplication and the connection ∇ discussed above and hence D is given in terms of the same composition as $\overline{D}$, but with $\overline{\nabla}$ replaced by ∇. A short calculation based on (1.4) (cf. Lemma 2.3.7) leads to the following comparison formula

$$\overline{D}\psi = D\psi - \frac{1}{2}\,\mathrm{tr}^g(k)e_0 \cdot \psi \tag{1.5}$$

for $\psi \in \Gamma(\overline{\Sigma}M)$.

The formula (1.5) allows to transfer many analytical properties from D to $\overline{D}$. One of the most important ones is formal self-adjointness of D. In order to speak of that, we need to introduce a scalar product on $\overline{\Sigma}M$. The starting point is a (positive definite) Euclidean or Hermitian scalar product $\langle\text{-}\,,\text{-}\rangle$ on W that is invariant under multiplication (via ρ) by the standard basis vectors $E_0, E_1, \ldots, E_n$ of $\mathbb{R}^{n,1} \subseteq \mathrm{Cl}_{n,1}$. Such a scalar product can always be obtained from an arbitrary one by an averaging procedure. It is important to note that, in consequence, multiplication by a vector in $\mathbb{R}^n = \operatorname{span}\{E_1, \dots, E_n\}$ is skew-adjoint and $\langle\text{-}, \text{-}\rangle$ is $\mathrm{Spin}(n)$-invariant. Hence there is an induced scalar product, also denoted by $\langle\text{-}, \text{-}\rangle$, on the associated bundle $\overline{\Sigma}M$, with respect to which the TM-Clifford multiplication is skew-adjoint. Moreover, it is ∇-parallel and the involution $e_0\cdot$ is self-adjoint with respect to it. With these compatibilities at hand, the Dirac operator D is formally self-adjoint, cf. e. g. [Roe99, Prop. 3.11], and (1.5) shows that the same is true for the Dirac-Witten operator $\overline{D}$.

The Dirac-Witten operator also satisfies a Schrödinger-Lichnerowicz type formula comparing its square to the connection Laplacian of $\overline{\nabla}$ – the composition of $\overline{\nabla}$ and its formal adjoint. It is a remarkable observation by Edward Witten [Wit81] that the curvature term is made up of the energy and momentum density of (g, k).

Proposition 1.3.5. *Suppose that the hypersurface spinor bundle $\overline{\Sigma}M$ on (M, g, k) is equipped with a scalar product with the properties described above. Then, for all $\psi \in \Gamma(\overline{\Sigma}M)$, the following formula holds:*

$$\overline{D}^2\psi = \overline{\nabla}^*\overline{\nabla}\psi + \frac{1}{2}(\rho - e_0 \cdot j^\sharp\cdot)\psi. \tag{1.6}$$

Proof. We reduce the statement to the well-known Schrödinger-Lichnerwicz formula for the Dirac operator (cf. e. g. [LM89, Thm. II.8.8]):

$$D^2\psi = \nabla^*\nabla\psi + \frac{1}{4}\operatorname{scal}^g \psi \tag{1.7}$$

for $\psi \in \Gamma(\overline{\Sigma}M)$.

In order to do so, we fix $\psi \in \Gamma(\overline{\Sigma}M)$ and a local orthonormal frame $(e_1, \dots, e_n)$ on a neighborhood of a point $p \in M$. Using (1.5), the left hand side computes to

$$\begin{aligned}\overline{D}^2\psi &= D^2\psi - \frac{1}{2}\operatorname{tr}^g(k)e_0 \cdot D\psi - \frac{1}{2}D(\operatorname{tr}^g(k)e_0 \cdot \psi) + \frac{1}{4}\operatorname{tr}^g(k)^2\psi \\ &= D^2\psi + \frac{1}{2}e_0 \cdot \mathrm{d}(\operatorname{tr}^g(k))^\sharp \cdot \psi + \frac{1}{4}\operatorname{tr}^g(k)^2\psi,\end{aligned} \tag{1.8}$$

where we used

$$\begin{aligned}D(\operatorname{tr}^g(k)e_0 \cdot \psi) &= \sum_{i=1}^n e_i \cdot (\partial_{e_i}\operatorname{tr}^g(k))e_0 \cdot \psi + \sum_{i=1}^n e_i \cdot \operatorname{tr}^g(k)e_0 \cdot \nabla_{e_i}\psi \\ &= -\mathrm{d}(\operatorname{tr}^g(k))^\sharp \cdot e_0 \cdot \psi - \operatorname{tr}^g(k)e_0 \cdot D\psi.\end{aligned}$$

For the right hand side, we first recall that the formal adjoint of ∇ is given by $\nabla^*\Psi = -\sum_{i=1}^n \nabla_{e_i}(\Psi(e_i)) + \sum_{i=1}^n \Psi(\nabla_{e_i}e_i)$ for $\Psi \in \Gamma(T^*M \otimes \overline{\Sigma}M)$. Furthermore, taking the formal adjoint of (1.1) yields $\overline{\nabla}^*\Psi = \nabla^*\Psi - \frac{1}{2}\sum_{i=1}^n e_0 \cdot k(e_i, \text{-})^\sharp \cdot \Psi(e_i)$ for all $\Psi \in \Gamma(T^*M \otimes \overline{\Sigma}M)$. Hence,

$$\begin{aligned}\overline{\nabla}^*\overline{\nabla}\psi &= \nabla^*\nabla\psi - \frac{1}{2}\sum_{i=1}^n e_0 \cdot k(e_i, \text{-})^\sharp \cdot \nabla_{e_i}\psi - \frac{1}{2}\nabla^*(e_0 \cdot k(\text{-}\,, \text{-})^\sharp \cdot \psi) + \frac{1}{4}|k|_g^2\psi \\ &= \nabla^*\nabla\psi + \frac{1}{2}\sum_{i=1}^n e_0 \cdot \operatorname{div}^g(k)^\sharp \cdot \psi + \frac{1}{4}|k|_g^2\psi,\end{aligned} \tag{1.9}$$

where $k(\text{-}\,, \text{-})^\sharp$ denotes the endomorphism that corresponds to k via g and we made use of the equations

$$\sum_{i=1}^n e_0 \cdot k(e_i, \text{-})^\sharp \cdot e_0 \cdot k(e_i, \text{-})^\sharp \cdot \psi = -\sum_{i=1}^n e_0 \cdot e_0 \cdot k(e_i, \text{-})^\sharp \cdot k(e_i, \text{-})^\sharp \cdot \psi = \sum_{i,j=1}^n k(e_i, e_j)^2\psi$$

and

$$\begin{aligned}\nabla^*(e_0 \cdot k(\text{-}\,, \text{-})^\sharp \cdot \psi) &= -\sum_{i=1}^n e_0 \cdot \nabla_{e_i}(k(e_i, \text{-})^\sharp) \cdot \psi - \sum_{i=1}^n e_0 \cdot k(e_i, \text{-})^\sharp \cdot \nabla_{e_i}\psi \\ &\quad + \sum_{i=1}^n e_0 \cdot k(\nabla_{e_i}e_i, \text{-})^\sharp \cdot \psi \\ &= -\sum_{i=1}^n e_0 \cdot (\nabla_{e_i}k)(e_i, \text{-})^\sharp \cdot \psi - \sum_{i=1}^n e_0 \cdot k(e_i, \text{-})^\sharp \cdot \nabla_{e_i}\psi.\end{aligned}$$

Keeping in mind the definition of ρ and j in (1.2), the desired result is now obtained by combining (1.7), (1.8) and (1.9). □

Remark 1.3.6. It is interesting to note that, while the Dirac-Witten operator by virtue of (1.5) just depends on g and $\operatorname{tr}^g(k)$, the individual terms on the right hand side of Witten's Schrödinger-Lichnerwicz type formula need more knowledge about k than a mere scalar quantity. In this sense the use of the connection $\overline{\nabla}$ for the connection Laplacian plays a greater role in the appearance of ρ and j than the Dirac-Witten operator.

We have seen that the Clifford multiplication and the connection $\overline{\nabla}$ on $\overline{\Sigma}M$ arise from restricting the respective structures on the spinor bundle of $\overline{M}$. We wish to do something similar for the scalar product. Here, a difficulty arises: the scalar product on W is not $\mathrm{Spin}_0(n,1)$-invariant. What is worse, there does not exist any $\mathrm{Spin}_0(n,1)$-invariant (positive definite) scalar product on W at all, cf. [Bau81, Ch. 1.5, final remark, itm. 2.)]. The solution presented by Helga Baum [Bau81, Ch. 1.5] involves dropping the requirement of positive definiteness. In the notation from above, she defines the indefinite inner product $\langle\!\langle \text{-}\,, \text{-} \rangle\!\rangle \coloneqq \langle \rho(E_0)(\text{-}), \text{-} \rangle$. It has the property that multiplication by any vector in $\mathbb{R}^{n,1}$ is self-adjoint and it is $\mathrm{Spin}_0(n,1)$-invariant. Therefore, there is an induced inner product $\langle\!\langle -, - \rangle\!\rangle$ on $\Sigma\overline{M}$, with respect to which the Clifford multiplication by vectors in $T\overline{M}$ is self-adjoint and which is $\overline{\nabla}$-parallel. Moreover, it has the property that if T is a future-timelike vector field, then $\langle -, - \rangle_T \coloneqq \langle\!\langle T \cdot -, - \rangle\!\rangle$ is a positive definite scalar product. Unlike $\langle\!\langle -, - \rangle\!\rangle$, this scalar product depends on the choice of T and is in general not very well-behaved with respect to Clifford multiplication or connection. On M, however, the future unit normal e_0 provides a canonical choice for T. Unraveling the definitions shows that on $\overline{\Sigma}M \cong \Sigma\overline{M}_{|M}$ the scalar product $\langle -, - \rangle_{e_0}$ coincides with the previously defined scalar product $\langle -, - \rangle$.

The inner product on $\Sigma\overline{M}$ and the Clifford multiplication can be used to define differential forms out of (pairs of) spinors. We will only discuss the special case in which a 1-form (or a vector) is produced out of a single spinor.

Definition 1.3.7. The *Dirac current* of a spinor $\psi \in \Sigma_p\overline{M}$, $p \in \overline{M}$, is the vector $V_\psi \coloneqq -\langle\!\langle - \cdot \psi, \psi \rangle\!\rangle^\sharp \in T_p\overline{M}$.

If W is a complex vector space and $\langle\!\langle -, - \rangle\!\rangle$ a Hermitian inner product, V_ψ a priori lives in the complexification of $T_p\overline{M}$. It is a real vector since $\overline{\langle\!\langle X \cdot \psi, \psi \rangle\!\rangle} = \langle\!\langle \psi, X \cdot \psi \rangle\!\rangle = \langle\!\langle X \cdot \psi, \psi \rangle\!\rangle$ for all $X \in T_p\overline{M}$.

Lemma 1.3.8. *For any future-lightlike vector $T \in T_p\overline{M}$, $p \in \overline{M}$, the Dirac current V_ψ of a spinor $\psi \in \Sigma_p\overline{M}$ satisfies*

$$\langle V_\psi \cdot \psi, V_\psi \cdot \psi \rangle_T = -\overline{g}(V_\psi, V_\psi)\langle \psi, \psi \rangle_T.$$

In particular, V_ψ is either zero or a causal vector. More precisely, we can distinguish the following three cases:

- $V_\psi = 0$ *if and only if* $\psi = 0$,
- V_ψ *is future-lightlike if and only if* $\psi \neq 0$ *and* $V_\psi \cdot \psi = 0$,
- V_ψ *is future-timelike if and only if* $V_\psi \cdot \psi \neq 0$.

Proof. We calculate

$$\begin{aligned}
\langle V_\psi \cdot \psi, V_\psi \cdot \psi \rangle_T &= \langle\!\langle T \cdot V_\psi \cdot \psi, V_\psi \cdot \psi \rangle\!\rangle \\
&= \langle\!\langle V_\psi \cdot T \cdot V_\psi \cdot \psi, \psi \rangle\!\rangle \\
&= -\langle\!\langle T \cdot V_\psi \cdot V_\psi \cdot \psi, \psi \rangle\!\rangle - 2\overline{g}(V_\psi, T)\langle\!\langle V_\psi \cdot \psi, \psi \rangle\!\rangle \\
&= \overline{g}(V_\psi, V_\psi)\langle\!\langle T \cdot \psi, \psi \rangle\!\rangle - 2\langle\!\langle T \cdot \psi, \psi \rangle\!\rangle\overline{g}(V_\psi, V_\psi) \\
&= -\overline{g}(V_\psi, V_\psi)\langle \psi, \psi \rangle_T.
\end{aligned}$$

Since $\langle - , - \rangle_T$ is positive definite, all of the claims immediately follow – except for the statement that V_ψ is a future-directed vector if it is non-zero. But this is a direct consequence from $\overline{g}(T, V_\psi) = -\langle\!\langle T \cdot \psi, \psi \rangle\!\rangle = -\langle \psi, \psi \rangle_T \leq 0$. □

Definition 1.3.9. A non-zero spinor $\psi \in \Sigma\overline{M}$ is called *timelike* or *lightlike* if its Dirac current V_ψ is timelike or lightlike, respectively.

If $\psi \in \Gamma(\Sigma\overline{M})$ is a $\overline{\nabla}$-parallel spinor field, then $V_\psi \in \Gamma(T\overline{M})$ is also $\overline{\nabla}$-parallel since metric, inner product and Clifford multiplication are compatible with the connection. In particular, the "flavor" – being timelike, lightlike or zero – is constant in this case. It is the lightlike case that is of primary interest in the rest of the thesis.

We finally remark that restricting to M yields a Dirac current $V_\psi \in \overline{T}_p M$ for hypersurface spinors $\psi \in \overline{\Sigma}_p M$, $p \in M$, and the statements above directly carry over. With respect to the splitting $\overline{T}M = \underline{R} \oplus TM$, the Dirac current decomposes as $V_\psi = u_\psi e_0 - U_\psi$ with $u_\psi = -\overline{g}(e_0, V_\psi) = \langle\!\langle e_0 \cdot \psi, \psi \rangle\!\rangle = \langle \psi, \psi \rangle$. U_ψ is called *Riemannian Dirac current* (cf. Definition 4.4.2) and is characterized by $g(U_\psi, X) = \langle e_0 \cdot X \cdot \psi, \psi \rangle$ for all $X \in T_p M$.

1.4. Singularity theorems

In the end of Section 1.2, we discussed to what extent DEC initial data sets are the induced initial data sets of "physically reasonable" Lorentzian manifolds – physically reasonable meaning here that they satisfy the dominant (and the strong) energy condition and global hyperbolicity. Given the existence of such a Lorentzian manifold, we may wonder about its properties. Although little can be said in general, it is remarkable that under certain additional assumptions on the initial data, "singularities" have to form.

Theorem 1.4.1 (Hawking's singularity theorem, [ONe83, Ch. 14, Thm. 55A])**.** *Let $(\overline{M}, \overline{g})$ be a globally hyperbolic Lorentzian manifold subject to the strong energy condition. Suppose that $M \subseteq \overline{M}$ is a spacelike Cauchy hypersurface and that the induced initial data set (g, k) on M satisfies $\mathrm{tr}^g(k) \leq -\lambda$ for a constant $\lambda > 0$. Then any future-timelike curve starting at M has proper time at most $\frac{n}{\lambda}$.*

By time-reversal, a similar statement holds under the initial condition $\mathrm{tr}^g(k) \geq \lambda$: In this case it is the proper time of future-timelike curves ending at M that is bounded. Often, this is interpreted as $(\overline{M}, \overline{g})$ having an initial "big bang" singularity – similarly, the situation in Theorem 1.4.1 would be interpreted as "big crunch". This motivates the following definition.

Definition 1.4.2. An initial data set (M, g, k) satisfying strict DEC is called *big bang initial data set* if $\operatorname{tr}^g(k) \geq \lambda$ for some constant $\lambda > 0$. Analogously, it is called *big crunch initial data set* if $\operatorname{tr}^g(k) \leq -\lambda$ for some $\lambda > 0$.

Actually, there is no real need here to restrict to strict DEC initial data sets, except maybe due to considering them as the physically most relevant ones. The condition was included to be compatible with Definition 3.1.2, where the requirement of strict DEC was added to conveniently formulate Lemma 3.2.2. Big bang and big crunch initial data sets play a hidden role in Theorem A, and are of major interest in Theorems B to D since they define the path-components C^+ and C^-.

Example 1.4.3. The past cone $I^-(0) \subseteq \mathbb{R}^{n,1}$ consisting of all the points in the timelike past of the origin in Minkowski spacetime is globally hyperbolic. Any of the hyperboloids $M_t = \{p \in I^-(0) \,|\, \langle p, p\rangle = -t^2\}$ with $t < 0$ is a Cauchy hypersurface of $I^-(0)$. The second fundamental form of M_t is proportional to the induced metric, more precisely $k_t = \frac{1}{t} g_t$. In particular, the hypotheses of Theorem 1.4.1 are satisfied. Note that in this example any future inextendable future-timelike curve starting at M_t has proper time exactly $-t$.

This example shows that in the context of this section the word “singularity” has to be taken with a grain of salt. As $I^-(0)$ is a subset of Minkowski space, the metric gets in no way singular as we approach its boundary. What happens though is that we loose predictability from M_t once we reach $\partial I^-(0) \subseteq \mathbb{R}^{n,1}$: There are past-inextendable future-causal curves to these boundary points that do not cross M_t.

If the Cauchy hypersurface M is compact though (or more generally of finite volume), we can see that Theorem 1.4.1 deals with an overall contracting regime. I learned the core of this argument from Bernd Ammann.

Theorem 1.4.4. *In the setting of Theorem 1.4.1, let $\tau_M \colon J^+(M) \to \mathbb{R}$ be the time separation from the Cauchy hypersurface M. Then, assuming M to be of finite volume,*

$$\limsup_{\varepsilon,\delta\to 0} \frac{1}{\varepsilon+\delta} \operatorname{vol}^{\overline{g}}\left(\tau_M^{-1}\left([t-\varepsilon, t+\delta]\right)\right) \leq \left(1 - \frac{\lambda}{n} t\right)^n \operatorname{vol}^g(M)$$

for any $t > 0$.

Remark 1.4.5. The left hand side of the inequality above is supposed to be an upper bound on the volume of the level set $\tau_M^{-1}(t)$. This is indeed the case when $N \coloneqq \tau_M^{-1}(t)$ is a smooth hypersurface, in which case there is a well-defined notion of its n-dimensional volume. Considering – similarly as in the proof below – the normal exponential map $\Psi\colon \mathbb{R} \times N \supset U \to \overline{M}$, $(t,p) \mapsto \exp_p(te_0)$, we obtain the co-area formula (cf. (1.10))

$$\mathrm{vol}^{\Psi^*\overline{g}}\left(([0,\delta] \times N) \cap U\right) = \int_0^\delta V(s)\,\mathrm{d}s,$$

where $V(s)$ is the volume of the smooth hypersurface $\Psi((\{s\} \times N) \cap U)$ with respect to the induced metric g_s. If U is suitably chosen, then V is continuous on $[0,\delta]$ for sufficiently small $\delta > 0$. Thus the fundamental theorem of calculus implies

$$\mathrm{vol}^{g_0}(N) = V(0) = \lim_{\delta \to 0} \frac{1}{\delta} \mathrm{vol}^{\Psi^*\overline{g}}\left(([0,\delta] \times N) \cap U\right).$$

Now $\Psi(([0,\delta] \times N) \cap U) \subseteq \tau_M^{-1}([t, t+\delta])$ shows that

$$\mathrm{vol}^{g_0}(N) \leq \limsup_{\delta \to 0} \frac{1}{\delta} \mathrm{vol}^{\overline{g}}\left(\tau_M^{-1}\left([t,t+\delta]\right)\right) \leq \limsup_{\varepsilon,\delta \to 0} \frac{1}{\varepsilon + \delta} \mathrm{vol}^{\overline{g}}\left(\tau_M^{-1}\left([t-\varepsilon,t+\delta]\right)\right).$$

Proof. The normal exponential map gives rise to a smooth map $\Phi_0\colon \mathbb{R} \times M \supseteq D_0 \to \overline{M}$, $(t,p) \mapsto \exp_p(te_0)$, where D_0 is the maximal domain of definition. We now restrict this to $D = \left\{(t,p) \in D_0 \cap ([0,\infty) \times M) \,\middle|\, \tau_M(\exp_p(te_0)) = t\right\}$ and obtain $\Phi\colon D \to J^+(M)$. By [ONe83, Ch. 14, Thm. 44], global hyperbolicity of $(\overline{M}, \overline{g})$ implies that for each $q \in I^+(M)$ there is a timelike geodesic from M to q which maximizes proper time, and thus Φ is surjective. Moreover, by maximality, such a geodesic cannot have focal points between S and q and is the unique maximal geodesic from S to any of the intermediate points. Hence, the restriction of Φ to the interior U of D in $[0,\infty) \times M$ is a diffeomorphism.

Let $f\colon M \to (0,\infty]$ be the function assigning to each $p \in M$ the maximal value such that $t \mapsto \exp_p(te_0)$ is a maximal geodesic on $[0, f(p))$. We claim that f is lower semi-continuous. This implies that $\{(t,p) \,|\, p \in M,\, t \in [0, f(p))\}$ is open in $[0,\infty) \times M$ and equal to U. To check lower semi-continuity in $p \in M$, we assume for contradiction that there is a sequence of points $p_i \longrightarrow p$, for $i \longrightarrow \infty$, such that $\lim_{i \longrightarrow \infty} f(p_i) < f(p)$. Set $s := \lim_{i \longrightarrow \infty} f(p_i)$ and $q := \exp_p(se_0)$. Moreover, choose $\hat{s} \in (s, f(p))$ and set $\hat{q} := \exp_p(\hat{s}e_0)$. Since D_0 is open, the normal geodesics starting in a small neighborhood of p in M exist for longer than time $\hat{s}$. We may thus assume without loss of generality that all $q_i := \exp_{p_i}(f(p_i)e_0)$ exist. We may moreover assume that none of the normal geodesics from p_i to q_i contains focal points: Since $\mathrm{d}\Phi_0$ is invertible along the line $[0,\hat{s}] \times \{p\} \subseteq D_0$, the same is true on an open neighborhood of this line. Lastly, we can assume without loss of generality that $q_i \in J^-(\hat{q})$ for all $i \in \mathbb{N}$, since $J^-(\hat{q})$ is a neighborhood of $q = \lim_{i \longrightarrow \infty} q_i$.

Now because $t \mapsto \exp_{p_i}(te_0)$ exists until time $f(p_i)$ and has no focal point on $[0, f(p_i)]$, by definition of f there has to be a second maximal geodesic from M to q_i. Let $p_i' \in M$ be the starting points of such different maximal geodesics. Then note that $J^-(\hat{q}) \cap M$ is compact: Since M is a Cauchy hypersurface, there is a well-defined continuous map assigning to each past-causal vector X in $T_{\hat{q}}\overline{M}$ the unique point where the geodesic starting in $\hat{q}$ in direction X meets M. Since this map is invariant under rescaling X, $J^-(\hat{q}) \cap M$ is the continuous image of a compact space. Since, clearly, $p_i' \in J^-(q_i) \subseteq J^-(\hat{q})$, compactness implies that a subsequence of the p_i''s converges. There are two cases: If the subsequential limit is different from p, then the limiting geodesics give rise to a maximal geodesic from M to q, contradicting maximality till $f(p)$. If, on the other hand, the subsequential limit is p, then Φ_0 would not be injective close to (s,p), contradicting injectivity of $\mathrm{d}\Phi_0$ in this point.

We obtain $D \setminus U \subseteq \{(f(p),p) \,|\, p \in M\}$. Since lower semi-continuous functions are measurable, this is a (Lebesgue) zero set. In consequence,

$$\begin{aligned} \mathrm{vol}^{\overline{g}}\left(\tau_M^{-1}([t-\varepsilon, t+\delta])\right) &= \mathrm{vol}^{\overline{g}}\left(\tau_M^{-1}([t-\varepsilon, t+\delta]) \cap \Phi(U)\right) \\ &= \mathrm{vol}^{\Phi^*\overline{g}}\left(([t-\varepsilon, t+\delta] \times M) \cap U\right). \end{aligned}$$

Now we use that the pull-back metric on U is given by $\Phi^*\overline{g} = -\mathrm{d}t^2 + g_t$, where g_t is a Riemannian metric on $(\{t\} \times M) \cap U$. This can be proved as the Gauß lemma by looking at Jacobi fields along the geodesics $t \mapsto \Phi(t,p)$, cf. the proof of Proposition A.1.1. Setting $M_t := \mathrm{pr}_M((\{t\} \times M) \cap U) \subseteq M$ and

$$V(t) := \mathrm{vol}^{g_t}(\{t\} \times M_t) = \int_{\{t\} \times M_t} \mathrm{dvol}^{g_t}$$

we arrive at

$$\mathrm{vol}^{\Phi^*\overline{g}}\left(([t-\varepsilon, t+\delta] \times M) \cap U\right) = \int_{t-\varepsilon}^{t+\delta} V(s)\,\mathrm{d}s. \tag{1.10}$$

We will show that $V(t) \leq (1 - \frac{\lambda}{n}t)^n V(0)$ for all $t \in \mathbb{R}$. Keeping $V(0) = \mathrm{vol}^g(M)$ in mind, this implies

$$\begin{aligned} \limsup_{\varepsilon,\delta \to 0} \frac{1}{\varepsilon + \delta} \mathrm{vol}^{\overline{g}}\left(\tau_M^{-1}([t-\varepsilon, t+\delta])\right) &= \limsup_{\varepsilon,\delta \to 0} \frac{1}{\varepsilon+\delta} \int_{t-\varepsilon}^{t+\delta} V(s)\,\mathrm{d}s \\ &\leq \limsup_{\varepsilon,\delta \to 0} \frac{1}{\varepsilon+\delta} \int_{t-\varepsilon}^{t+\delta} \left(1 - \frac{\lambda}{n}s\right)^n V(0)\,\mathrm{d}s \\ &= \left(1 - \frac{\lambda}{n}t\right)^n \mathrm{vol}^g(M). \end{aligned}$$

So it remains to show the inequality for V.

In order to do so, we look at the trace of the second fundamental form $k_t = \frac{1}{2}\frac{\partial}{\partial t} g_t$ of the leaves $\{t\} \times M_t$. We have

$$\frac{\partial}{\partial t}\,\mathrm{tr}^{g_t}(k_t) = \mathrm{tr}^{g_t}\left(\frac{\partial}{\partial t} k_t\right) - 2|k_t|^2_{g_t}.$$

The Mainardi equation [BI04, eq. (14)] implies $\operatorname{ric}^{\overline{g}}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -\operatorname{tr}^{g_t}\left(\frac{\partial}{\partial t}k_t\right) + |k_t|^2_{g_t}$, so by the strong energy condition we obtain

$$\frac{\partial}{\partial t}\operatorname{tr}^{g_t}(k_t) \leq -|k_t|^2_{g_t}.$$

Combining this with the Cauchy-Schwarz inequality $\operatorname{tr}^{g_t}(k_t)^2 \leq n|k_t|^2_{g_t}$, we arrive at the differential inequality

$$\frac{\partial}{\partial t}\operatorname{tr}^{g_t}(k_t) \leq -\frac{1}{n}\operatorname{tr}^{g_t}(k_t)^2.$$

Comparing with the ODE-solution $h(t) = (\frac{t}{n} - \frac{1}{\lambda})^{-1}$ to the initial value $h(0) = -\lambda$ shows that

$$\operatorname{tr}^{g_t}(k_t)_{|p} \leq \left(\frac{t}{n} - \frac{1}{\lambda}\right)^{-1}$$

for each $p \in M_t$. Note that the right-hand side tends to $-\infty$ for $t \longrightarrow \frac{n}{\lambda}$, which proves Hawking's singularity theorem. In particular, $M_t = \varnothing$ and $V(t) = 0$ for $t \geq \frac{n}{\lambda}$.

We fix $t > 0$ with $V(t) > 0$ and look at the function $V_t\colon [0,t] \ni s \mapsto \operatorname{vol}^{g_s}(M_t)$, which is well-defined since $M_t \subseteq M_s$ for $s \leq t$. Combining the above inequality with the variation formula yields

$$\frac{\mathrm{d}}{\mathrm{d}s}V_t(s) = \int_{M_t}\operatorname{tr}^{g_s}(k_s)\mathrm{dvol}^{g_s} \leq \left(\frac{s}{n} - \frac{1}{\lambda}\right)^{-1} V_t(s)$$

and thus

$$\frac{\mathrm{d}}{\mathrm{d}s}\log(V_t(s)) \leq n\frac{\mathrm{d}}{\mathrm{d}s}\log\left|\frac{s}{n} - \frac{1}{\lambda}\right|.$$

From this we get the desired inequality

$$V(t) = V_t(t) \leq \left(1 - \frac{\lambda}{n}t\right)^n V_t(0) \leq \left(1 - \frac{\lambda}{n}t\right)^n V(0). \qquad \square$$

Whereas Hawking's singularity theorem deals with the initial and terminal behavior of spacetimes, Penrose's singularity theorem is meant to describe "black holes" – archetypically represented by the Schwarzschild solution and its generalizations. We state a version that discriminates between an inside and an outside region that I learned from [Gal14, Thm. 7.2].

Theorem 1.4.6 (Penrose's singularity theorem, [And+09, Thm. 7.1])**.** *Let $(\overline{M}, \overline{g})$ be a globally hyperbolic Lorentzian manifold subject to the null energy condition and let $M \subseteq \overline{M}$ be a spacelike Cauchy hypersurface with induced initial data set (g, k). Suppose* *there exists a compact hypersurface $F \subseteq M$, which separates M into two pieces – an "inside" M_{in} and an "outside" M_{out} with $M \setminus F = M_{\text{in}} \dot\cup M_{\text{out}}$ – such that the closure of M_{out} is non-compact and $\theta^+ := \operatorname{tr}^F(g(\nabla_{\cdot}\tilde\nu, \cdot) + k) < 0$, where $\tilde\nu$ is the unit-normal of F pointing to the outside. Then at least one of the geodesics starting at F in the (future outgoing) lightlike direction $\ell^+ := e_0 + \tilde\nu$ does not exist for all positive times.*

For a detailed (sketch of) proof, we refer to [Lee19, Thm. 7.29]. Nevertheless, we provide at least the following intuition for the situation described in the theorem. We consider the normal exponential map of F in $\overline{M}$ and restrict it to a map $\Phi\colon M \times \mathbb{R} \supseteq U \to \overline{M}$, $(p, t) \mapsto \exp_p(t\ell^+)$. This describes radiation sent out from F in the (future) outgoing direction. Now we look at the null hypersurface $\overline{F} := \operatorname{im}(\Phi)$ or rather its foliation $(F_t)_{t\in\mathbb{R}}$ with $F_t := \Phi(M \times \{t\})$. Then, by the variation formula, the initial change of volume is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \operatorname{vol}(F_t) = \int_F \operatorname{tr}^F(\overline{\nabla}_{\cdot}\ell^+)\,\mathrm{dvol}^F.$$

The integrand is the quantity appearing in the theorem, since $\operatorname{tr}^F(\overline{\nabla}_{\cdot}\ell^+) = \operatorname{tr}^F(\overline{\nabla}_{\cdot}\tilde\nu) + \operatorname{tr}^F(\overline{\nabla}_{\cdot}e_0) = \operatorname{tr}^F(\nabla_{\cdot}\tilde\nu) + \operatorname{tr}^F(k) = \theta^+$. Due to its influence on the volume growth, it acquires the following name.

Definition 1.4.7. Let (M, g, k) be an initial data set and F a hypersurface in M with unit normal $\tilde\nu$. Then the symmetric 2-tensor on F defined by $\chi^+(X, Y) := g(\nabla_X\nu, Y) + k(X, Y)$ for $X, Y \in TF$ is called *future null second fundamental form* (w. r. t. $\tilde\nu$). The function $\theta^+ := \operatorname{tr}^F(\chi^+)$ is the *future null expansion scalar* (w. r. t. $\tilde\nu$).

Remark 1.4.8. There is also another null second fundamental form[2]: $\chi^-(X, Y) := g(\nabla_X\nu, Y) - k(X, Y)$ for $X, Y \in TF$. Together χ^+ and χ^- determine the second fundamental form of F in $\overline{M}$, namely it is given by $-\frac{1}{2}\chi^- \cdot \ell^+ - \frac{1}{2}\chi^+ \cdot \ell^-$ with $\ell^- = -e_0 + \tilde\nu$.

The initial condition $\theta^+ < 0$ amounts to the volume of the F_t initially decreasing. Moreover, we may define ℓ^+ on all of $\overline{F}$ by setting it to be the velocity vector of the geodesics $\Phi(p, \cdot)$. Then $\theta^+_t = \operatorname{tr}^{F_t}(\overline{\nabla}_{\cdot}\ell^+)$ satisfies a differential equation, known as *Raychaudhuri equation*. Similarly as in the proof of Theorem 1.4.4 the assumed energy condition allows to deduce from this an ordinary differential inequality of the form $\frac{\partial}{\partial t}\theta^+_t(\Phi(p, t)) \leq -\frac{1}{n-1}\theta^+_t(\Phi(p, t))^2$ for all $p \in F$. Thus under the initial condition $\theta^+ < 0$ for each $p \in F$ it monotonously decreases to $-\infty$ within some fixed finite time. This implies that the volume decrease continues. Moreover, the blow-up behavior of θ^+ is an important ingredient for the incompleteness statement of Theorem 1.4.6.

[2]For some authors use the symbols χ^-, $\theta^- = \operatorname{tr}^F(\chi^-)$ and ℓ^- for the quantities with the opposite sign, e. g. [Lee19, Sec. 7.4.2].

Definition 1.4.9. In the situation of Definition 1.4.7 the hypersurface F is called

- *outer trapped* if $\theta^+ < 0$,
- *weakly outer trapped* if $\theta^+ \leq 0$, and
- *marginally outer trapped* or a *MOTS* if $\theta^+ = 0$.

Historically, the term MOTS stands for marginally outer trapped *surface*, which fits in the physical dimension $3+1$, where $\dim(F) = 2$.

Remark 1.4.10. There is a way to understand these conditions in the case where F a hypersurface within a null hypersurface $\overline{F} \subseteq \overline{M}$ as in the discussion above. Suppose that ℓ^+ is a future-lightlike tangent vector field of $\overline{F}$, then

$$\overline{g}(\overline{\nabla}_{\text{-}}\,\ell^+, \text{-})\colon T\overline{F}/\mathbb{R}\ell^+ \otimes T\overline{F}/\mathbb{R}\ell^+ \longrightarrow \mathbb{R}$$

is well-defined, symmetric and called *null second fundamental form* of $\overline{F}$ in $\overline{M}$ w. r. t. ℓ^+. The reason for considering $T\overline{F}$ modulo ℓ^+ is that on the quotient the induced metric is non-degenerate (actually positive definite) and hence we may form the trace of the null second fundamental form, yielding the *null mean curvature.* Both null second fundamental form and null mean curvature depend on the choice of ℓ^+ and, since ℓ^+ is lightlike, there is no natural way of normalization at hand. However, changing ℓ^+ just results in multiplication with a positive function. In particular, the sign of the null mean curvature and thus the trappedness properties are independent of the choice. Note that the choice of a spacelike hypersurface M with $F = M \cap \overline{F}$ allows for the normalization $\overline{g}(\ell^+, e_0) = -1$ and in this case, along F, the null second fundamental form defined here and the null mean curvature essentially coincide with χ^+ and θ^+, respectively.

To a certain extent – under a genericity assumption on the curvature – the arguments leading to Penrose's singularity theorem still work when the assumption of outer trappedness is replaced by weak outer trappedness, cf. [EGP13, Thm. 3.2]. This leads to the notion of *outer trapped region* consisting of all the points of M that lie within some weakly outer trapped hypersurface.

Theorem 1.4.11 ([AM09, Thm. 1.3], [AEM11, Thm. 4.6])**.** *If* $2 \leq \dim(M) \leq 7$*, then the boundary of the outer trapped region is a smooth hypersurface and a MOTS.*

This theorem can already be found in the book of Hawking and Ellis [HE73, Prop. 9.2.9], where they (admittedly) only provide a rough outline as proof. As they discuss, the components of this very MOTS that is the boundary of the outer trapped region deserve the name *apparent horizon*: Under additional assumptions leading to a well-defined notion of *event horizon*, there is one to be found outside of each apparent horizon, which is shown by a variation of the argument of Penrose's singularity theorem. So they give an indication where event horizons lie (in the static case they even agree). Yet – unlike event horizons – apparent horizons are defined purely in terms of the initial data set and hence do not require knowledge of the future in order to be located. We do not plunge into these topics further; instead we just remark that weakly outer trapped and weakly outer untrapped (i. e. $\theta^+ \geq 0$) hypersurfaces, but especially MOTS play an important role in Theorem E.

1.5. Rigidity of initial data sets and uniqueness of DEC Lorentzian manifolds

We have seen that many DEC initial data sets (M, g, k) admit an extension to a time-oriented DEC Lorentzian manifold $(\overline{M}, \overline{g})$ in the sense that M is a spacelike Cauchy hypersurface in $(\overline{M}, \overline{g})$ and (g, k) is the induced initial data set on M. We might wonder about the uniqueness of these extensions. Of course, uniqueness can only hold up to restriction and isometry, but even then asking for it seems to be too much, since the differential relation (in the sense of [Gro86, Ch. 1.1.1]) describing spacetime DEC is the closure of a (non-empty) open differential relation. Yet, there are two situations, where (local geometric) uniqueness holds nonetheless. The first one is the vacuum case and we need the following observation to set forth the argument.

Proposition 1.5.1 (Conservation Theorem, [HE73, Sec. 4.3])**.** *Let $(\overline{M}, \overline{g})$ be globally hyperbolic Lorentzian manifold subject to DEC and assume that the induced initial data set on a spacelike hypersurface M satisfies $\rho \equiv 0$. Then $\mathrm{Ein}^{\overline{g}} = 0$ on the domain of dependence $D(M) \subseteq \overline{M}$. In particular, if M is a spacelike Cauchy hypersurface, then $\mathrm{Ein}^{\overline{g}} = 0$ holds on all of $\overline{M}$.*

Now, suppose that $(\overline{M}, \overline{g})$ satisfies DEC and is globally hyperbolic with spacelike Cauchy hypersurface M. If the induced initial data set on M has $\rho \equiv 0$ (and hence also $j \equiv 0$), then by the above proposition $(\overline{M}, \overline{g})$ is a vacuum solution of the Einstein equations. But those are locally geometrically uniquely determined from the initial data set (cf. Theorem 1.2.8).

The other case, where local geometric uniqueness holds, is when a certain rigidity property for DEC initial data sets holds. The hypotheses are geared to be satisfied the situation of Theorem E. The main point is that these initial data sets carry a future-lightlike $\overline{\nabla}$-parallel (hypersurface) vector field $V \in \Gamma(\overline{T}M)$, in the sense of Definition 1.3.1.

Remark 1.5.2. Using the splitting $\overline{T}M = \underline{\mathbb{R}} \oplus TM$ and writing $V = ue_0 - U$ with $U \in \Gamma(TM)$, we see that being future-lightlike means that U is nowhere vanishing while u is determined by U through $u = |U|_g$. Also, being $\overline{\nabla}$-parallel is equivalent to

$$\nabla_X U = uk(X, \text{-})^\sharp$$
$$\mathrm{d}u(X) = k(U, X)$$

holding for all $X \in TM$. Actually, in the future-lightlike case, the second equation follows from the first one since $\mathrm{d}u(X) = \partial_X \sqrt{g(U,U)} = \frac{1}{u} g(\nabla_X U, U) = k(U, X)$.

Proposition 1.5.3. *Let $(\overline{M}, \overline{g})$ be a time-oriented Lorentzian manifold subject to DEC and assume that the induced initial data set on an achronal spacelike hypersurface M carries a future-lightlike $\overline{\nabla}$-parallel vector field $V = ue_0 - U \in \Gamma(\overline{T}M)$, $U \in \Gamma(TM)$. Suppose that the triple (g, k, V) is* rigid *in the following sense: For every $p \in M$ there is an open neighborhood $W \subsetneq M$ of p such that every DEC initial data set (g', k') coinciding with (g, k) on $M \setminus W$ carries a future-lightlike $\overline{\nabla}$-parallel vector field V' coinciding with V on $M \setminus W$. Then there is an open neighborhood of M in $\overline{M}$ that isometrically embeds into the* Killing development *(cf. [BC96, Sec. II]) $(\mathbb{R} \times M, -U^\flat \otimes \mathrm{d}t - \mathrm{d}t \otimes U^\flat + g)$, mapping M to $\{0\} \times M$ in the canonical way and the future of M into $\mathbb{R}_{>0} \times M$.*

Proof. Let $\overline{\nabla}$ be the Levi-Civita connection of $(\overline{M}, \overline{g})$. The main point of the proof is to construct an extension of V on a neighborhood of M in $\overline{M}$ that is future-lightlike and parallel w. r. t. $\overline{\nabla}$. We then consider the flow of the extension V. Since V lightlike and thus transversal to the spacelike hypersurface M, it defines a local diffeomorphism Φ from an open neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$ to an open neighborhood of M in $\overline{M}$. We assume that Φ is chosen such that its domain is of the form $\bigcup_{p \in M} (I_p \times \{p\})$, where $I_p \subseteq \mathbb{R}$ is an open interval containing 0 for each $p \in M$. Restricting the codomain we may assume surjectivity of Φ, and the achronality condition for M ensures that flow lines of V can cross M at most once (it is important here that the hypersurface has no boundary) so that Φ is injective. So Φ is a diffeomorphism with $\mathrm{d}\Phi(\frac{\partial}{\partial t}) = V$, where t denotes the $\mathbb{R}$-coordinate of $\mathbb{R} \times M$. Along $\{0\} \times M$ the pullback metric is given by

$$\Phi^* \overline{g} = \pi^{TM}(V)^\flat \otimes \mathrm{d}t + \mathrm{d}t \otimes \pi^{TM}(V)^\flat + g,$$

where π^{TM} denotes the orthogonal projection on TM. Since V is parallel and thus in particular a Killing vector field, the formula for $\Phi^* \overline{g}$ holds on the whole domain of Φ.

It remains to construct such a vector field V extending the given vector field on (M, g, k). First of all, we recall that Definition 1.3.1 is made in such a way that indeed every extension V is future-lightlike and $\overline{\nabla}$-parallel at least along M. The idea is now to slightly perturb the hypersurface M in $\overline{M}$ and use the rigidity property to also find a lightlike and $\overline{\nabla}$-parallel vector field V along the perturbed hypersurface. It then requires some technical work to see that these vector fields along the hypersurfaces are just the restrictions of a single vector field V defined on a neighborhood of M in $\overline{M}$, i. e. that different (spacelike) hypersurfaces passing through the same point lead to the same vector in that point.

Consider a point $p \in M$. Let T be a future-timelike vector field defined on an open neighborhood of p in $\overline{M}$. There is an $\varepsilon > 0$ and a small compact neighborhood K of p in M such that the flow of T is defined on $(-\varepsilon, \varepsilon) \times K$. We equip $(-\varepsilon, \varepsilon) \times K$ with the metric obtained by pulling back along the flow. Since K is compact, after possibly making ε smaller, we may assume that there exists some $C > 0$ such that for all $(t, q) \in (-\varepsilon, \varepsilon) \times K$, $\alpha \in \mathbb{R}$ and $X \in T_qM$ with $|\alpha| < C|X|_g$ the vector $\alpha\frac{\partial}{\partial t} + X \in T_{(t,q)}((-\varepsilon, \varepsilon) \times K)$ is spacelike. Consequently, any smooth function $f\colon K \to (\varepsilon, \varepsilon)$ with $|\mathrm{d}f|_g < C$ defines a spacelike hypersurface $\mathrm{Graph}(f) = \{(f(q), q) \,|\, q \in K\}$ of $(-\varepsilon, \varepsilon) \times K$.

We may assume that K was chosen small enough so that it is contained in a neighborhood $W \subsetneq M$ around p with the properties mentioned in the assumption. We now identify $(-\varepsilon, \varepsilon) \times K$ with its diffeomorphic image in $\overline{M}$ and define the vector field V on a neighborhood of p in $(-\varepsilon, \varepsilon) \times K$ as follows. For any compactly supported function $f \in C_c^\infty(K)$ with $|f| < \varepsilon$ and $|\mathrm{d}f|_g < C$ we consider the spacelike hypersurface $\mathrm{Graph}(f)$ extended by $M \setminus K$. This is canonically diffeomorphic to M and the induced initial data set obviously coincides outside of K. The rigidity property allows to extend $V_{|M\setminus K}$ to $\mathrm{Graph}(f)$ such that it is a future-lightlike $\overline{\nabla}$-parallel hypersurface vector field. Since the set of *admissible* functions, i. e. functions $f \in C_c^\infty(K)$ with $|f| < \varepsilon$ and $|\mathrm{d}f|_g < C$, is star-shaped w. r. t. the zero function, this procedure allows to define V on a neighborhood of p – once we have seen that the obtained vector V at a point (t, q) does not depend on the function f with $f(q) = t$ that is used.

To show this independence of f, we observe the following. Since the vector field V along $\mathrm{Graph}(f)$ is $\overline{\nabla}$-parallel, its value at $(f(q), q)$ may be obtained via parallel transport along a curve in $\mathrm{Graph}(f)$ starting outside of $\mathrm{supp}(f)$ and ending at $(q, f(q))$. In particular, the vector $V_{(f(q),q)}$ is the same for two functions f coinciding on a curve from $M \setminus K$ to q. Thus it would suffice to show that for any admissible f_1 and f_2 with $f_1(q) = f_2(q)$ there is an admissible function f that coincides with each of them on such a curve.

Locally around q such a function can be constructed because in the model situation around 0 in $\mathbb{R}^n$ a solution is given by $f_0(x_1, \ldots, x_n) = f_2(x_1, \ldots, x_n) + f_1(x_1, 0, \ldots, 0) - f_2(x_1, 0, \ldots, 0)$. Note that $f_0 = f_1$ on $\mathbb{R} \times \{0\}$ and $f_0 = f_2$ on $\{0\} \times \mathbb{R}^{n-1}$. If the coordinates are chosen orthogonally in q and such that $\frac{\partial}{\partial x_i} \in \ker \mathrm{d}_q f_2$ for all $i = 2, \ldots, n$, then

$$|\mathrm{d}_q f_0|_g \leq |\mathrm{d}_q f_1|_g \leq \max\{|\mathrm{d}_q f_1|_g, |\mathrm{d}_q f_2|_g\}. \tag{1.11}$$

Moreover, there is an estimate $\left|\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} f_0\right| \leq 3\max\{\|f_1\|_{C^2,x}, \|f_2\|_{C^2,x}\}$ for all $j, k = 1, \ldots n$, where the C^2-norm is the one induced by the coordinate system $x = (x_1, \ldots, x_n)$ around q. Let C_0 satisfy $\max_{q' \in K, i=1,2} |\mathrm{d}_{q'} f_i|_g < C_0 < C$. Together with (1.11) it follows that the function f_0 satisfies $|f_0| < \varepsilon$ and $|\mathrm{d}f_0|_g \leq C_0$ in a small open neighborhood U_0 around q, whose size may be determined from $f_1(q)$, C_0, $\max\{|\mathrm{d}_q f_1|_g, |\mathrm{d}_q f_2|_g\}$ and $3\max\{\|f_1\|_{C^2,x}, \|f_2\|_{C^2,x}\}$ alone.

Now, we choose two paths γ_1 and γ_2 in M connecting q with $M\backslash K$. We suppose that they only intersect in q and that in the chosen coordinates around q they lie within $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}^{n-1}$, respectively. Let (χ_0, χ_1, χ_2) be a partition of unity subordinate to the open cover $(U_0, K \setminus \mathrm{im}(\gamma_2), K \setminus \mathrm{im}(\gamma_1))$ of K. By construction, $f := \sum_{i=0}^{2} \chi_i f_i \in C_c^\infty(K)$ satisfies $|f| < \varepsilon$ and coincides with f_i along γ_i for $i = 1, 2$. Noting that $\mathrm{d}\chi_2 = -\mathrm{d}\chi_0 - \mathrm{d}\chi_1$ and $|f_0 - f_2| \leq \|f_1 - f_2\|_\infty$, we can estimate

$$\begin{aligned} |\mathrm{d}f|_g &\leq |\mathrm{d}\chi_0(f_0 - f_2)|_g + |\mathrm{d}\chi_1(f_1 - f_2)|_g + \sum_{i_0}^{2} \chi_i |\mathrm{d}f_i|_g \\ &\leq (|\mathrm{d}\chi_0|_g + |\mathrm{d}\chi_1|_g)\|f_1 - f_2\|_\infty + C_0. \end{aligned}$$

Hence there is some $\delta > 0$ such that if $\|f_1 - f_2\|_\infty < \delta$, then the constructed function f is admissible. So in the case where f_1 and f_2 are close enough, we are done.

In the general case, let $N \in \mathbb{N}$ satisfy $\|f_1 - f_2\|_\infty < \delta N$. Considering the convex combinations $f_{1,j} = \frac{N-j}{N} f_1 + \frac{j}{N} f_2$ and $f_{2,j} = \frac{N-j-1}{N} f_1 + \frac{j+1}{N} f_2$ for $j = 0, \ldots, N-1$ instead of f_1 and f_2, respectively, we run the same construction as above with precisely the same choice of coordinates x around q, neighborhood U_0 and partition of unity $(\chi_i)_{i=0,1,2}$. Notice that the function $f_{0,j}$ constructed in the first step still satisfies $|f_{0,j}| < \varepsilon$ and $|\mathrm{d}f_{0,j}|_g \leq C_0$ on U_0. This is due to the fact that the listed constants determining the size of U_0 play the same role for $f_{0,j}$ as they did for f_0. For instance, (1.11) can be replaced with

$$\begin{aligned} |\mathrm{d}_q f_{0,j}|_g^2 &= \left|\mathrm{d}_q f_{1,j}\left(\frac{\partial}{\partial x_1}\right)\right|^2 + \sum_{i=2}^{n} \left|\mathrm{d}_q f_{2,j}\left(\frac{\partial}{\partial x_i}\right)\right|^2 \\ &\leq \left|\mathrm{d}_q f_{1,j}\left(\frac{\partial}{\partial x_1}\right)\right|^2 + \sum_{i=2}^{n} \left|\mathrm{d}_q f_{1,j}\left(\frac{\partial}{\partial x_i}\right)\right|^2 \\ &= |\mathrm{d}_q f_{1,j}|_g^2 \leq (\max\{|\mathrm{d}_q f_1|_g, |\mathrm{d}_q f_2|_g\})^2, \end{aligned}$$

where we used in the second step $\mathrm{d}_q f_{2,j}\left(\frac{\partial}{\partial x_i}\right) = \frac{N-j-1}{N}\mathrm{d}_q f_1\left(\frac{\partial}{\partial x_i}\right)$ and $\mathrm{d}_q f_{1,j}\left(\frac{\partial}{\partial x_i}\right) = \frac{N-j}{N}\mathrm{d}_q f_1\left(\frac{\partial}{\partial x_i}\right)$ for all $i = 2, \ldots, n$. Now since $\|f_{1,j} - f_{2,j}\|_\infty < \delta$, we obtain an admissible function f_j coinciding with $f_{1,j}$ along γ_1 and with $f_{2,j}$ along γ_2. Thus for all $j = 0, \ldots, N-1$ at $(f_1(q), q)$ the vector field V is the same for $f_{1,j}$ and $f_{2,j} = f_{1,j+1}$. It follows that it is the same for the functions f_1 and f_2 we started with.

Thus the above procedure gives a well-defined future-lightlike vector field V locally around p, which on M coincides with the previously defined vector field. To see that it is $\overline{\nabla}$-parallel, we just observe that in each point it is $\overline{\nabla}$-parallel in the directions tangent to a hypersurface Graph(f) (for an admissible f) passing through that point and that the tangent directions of such hypersurfaces collectively span its whole tangent space in $\overline{M}$. Finally, these local extensions of V glue together to a lightlike and $\overline{\nabla}$-parallel vector field defined on a neighborhood of M in $\overline{M}$ since a $\overline{\nabla}$-parallel extension is necessarily unique. □

Corollary 1.5.4. *Suppose an initial data set (M, g, k) satisfies the assumptions of Theorem E. Then the DEC extension of the initial data set $(g_{|M_0}, k_{|M_0})$ on $M_0 := M \setminus \partial M$ is locally geometrically unique: For any two time-oriented DEC Lorentzian manifolds $(\overline{M}_1, \overline{g}_1)$ and $(\overline{M}_2, \overline{g}_2)$ containing $(M_0, g_{|M_0}, k_{|M_0})$ as achronal spacelike hypersurface there are open neighborhoods $U_1 \subseteq \overline{M}_1$ and $U_2 \subseteq \overline{M}_2$ of M_0 and a time-orientation preserving isometry $(U_1, \overline{g}_1) \cong (U_2, \overline{g}_2)$ fixing M_0.*

Proof. By Theorem E together with Addendum 4.1.3, (M, g, k) carries a lightlike $\overline{\nabla}$-parallel (hypersurface) spinor $\psi \in \Gamma(\overline{\Sigma} M)$. Its Dirac current $V_\psi \in \Gamma(\overline{T} M)$ is then future-lightlike and $\overline{\nabla}$-parallel. We fix a point $q \in \partial_- M$ and consider the normalization $V = \frac{1}{u_\psi(q)} V_\psi$, where $V_\psi = u_\psi e_0 - U_\psi$. As stated in Addendum 4.1.3, we will then have $V = e_0 + \tilde{\nu}$ in q.

We wish to apply Proposition 1.5.3 to $(g_{|M_0}, k_{|M_0}, V_{|M_0})$. In order to do so, we need to check the rigidity property of the triple. Let $p \in M_0$ and choose a neighborhood $W \subseteq M_0$ of p so small that the closure of W in M is disjoint from ∂M and $M \setminus W$ stays connected. Let (g', k') be any DEC initial data set on M_0 coinciding with $(g_{|M_0}, k_{|M_0})$ on $M_0 \setminus W$. By the first condition on W, (g', k') extends uniquely by continuity to a DEC initial data set on M that coincides with (g, k) on $M \setminus W$. Now note that the assumptions of Theorem E are satisfied for this initial data set: It satisfies DEC and the future null expansion scalar θ^+ along the boundary coincides with the one of (g, k) because the initial data sets are the same on ∂M. Since the conditions of Theorem E were assumed to hold for (M, g, k), θ^+ has the required sign and all the needed topological conditions on M are also satisfied. Thus the theorem is applicable and the same construction as above yields a future-lightlike $\overline{\nabla}$-parallel vector field V' on M with $V' = e_0 + \tilde{\nu}$ in q. It should be pointed out that the connection $\overline{\nabla}$ used here corresponds to (g', k'), but on $M \setminus W$ it coincides with the one of (g, k), so there is no ambiguity in using the symbol $\overline{\nabla}$ there. Since $M \setminus W$ is connected and $\overline{\nabla}$-parallel vector fields are uniquely determined along a smooth path by the vector at a single point on the path, V and V' do not only coincide in q but on all of $M \setminus W$. Thus the rigidity property is satisfied.

Now Proposition 1.5.3 yields that both $\overline{M}_1$ and $\overline{M}_2$ contain open neighborhoods of M_0 that isometrically embed into the same Killing development, and the embeddings are the same on M_0. From this, the claim follows. □

Corollary 1.5.5. *Let (g,k) be a DEC initial data set on a closed connected spin manifold M. Assume that there is a* MOTS *F, i. e. a co-oriented hypersurface with unit normal ν such that $\theta^+ := \mathrm{tr}(\nabla\nu) + \mathrm{tr}(k) = 0$, with $\hat{A}(F) \neq 0$. Then the extension of (M,g,k) to a time-oriented DEC Lorentzian manifold is locally geometrically unique.*

Proof. Cutting M along F, we obtain a manifold whose boundary consists of two copies of F – one where ν is inward pointing and another where it points outwards. It comes with an initial data set obtained from (g,k) by cutting and satisfying the assumptions of Theorem E. From Corollary 1.5.4 we thus obtain that the extension of $(M_0, g_{|M_0}, k_{|M_0})$ to a time-oriented DEC Lorentzian manifold is locally geometrically unique for $M_0 := M \backslash F$.

Furthermore, Theorem E implies that the cut initial data set is foliated by MOTS. In particular, there is a MOTS F' in (M,g,k) that is disjoint from F. Running the same argument for F' we find local geometric uniqueness for the extension of the initial data set on $M_0' := M \backslash F'$ to a time-oriented DEC Lorentzian manifold. Together with the first part, we obtain local geometric uniqueness on $M = M_0 \cup M_0'$. This last step uses that the isometry required for local geometric uniqueness is uniquely determined on a connected neighborhood of the fixed spacelike hypersurface. So after potentially shrinking the neighborhoods of M_0 and M_0' the respective isometries obtained from Corollary 1.5.4 glue together. □

Remark 1.5.6. Although the conditions of Corollary 1.5.5 imply that after cutting along F the initial data set carries a future-lightlike $\overline{\nabla}$-parallel vector field, this needs not be the case for the original initial data set (M,g,k): There is no reason to believe that the vector field fulfills the fitting condition necessary for re-gluing. Therefore we cannot apply Proposition 1.5.3 directly. Similarly, (M,g,k) does not need to carry a future-lightlike $\overline{\nabla}$-parallel spinor even though the cut manifold does.

2. On the space of initial data sets satisfying the strict dominant energy condition

The dominant energy condition imposes a restriction on initial data sets found on a spacelike hypersurface of a Lorentzian manifold. In this chapter, we study the space of initial data sets that strictly satisfy this condition. To this aim, we introduce an index difference for initial data sets and compare it to its classical counterpart for Riemannian metrics. Recent non-triviality results for the latter will then imply that this space has non-trivial homotopy groups.

2.1. Introduction

2.1.1. Dominant energy condition for initial data sets

According to general relativity, the universe can be modeled by a time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$ whose large-scale behavior is governed by the Einstein equation

$$T = \mathrm{ric}^{\overline{g}} - \frac{1}{2} \mathrm{scal}^{\overline{g}} \, \overline{g},$$

where T denotes the energy-momentum tensor. This does not only apply to the dynamics, the field equations also constraint the physical quantities experienced on a time-slice. More precisely, suppose that $(\overline{M}, \overline{g})$ contains M as a spacelike hypersurface. On M, the induced Riemannian metric g and the second fundamental form k, defined with respect to the future-pointing unit normal e_0, form a so-called initial data set (g, k). The Gauß-Codazzi equations imply that it is subject to the Einstein constraints (cf. [BI04], Proposition 1.2.6)

$$\begin{aligned} 2\rho &= \mathrm{scal}^g + (\mathrm{tr}\, k)^2 - |k|^2 \\ j &= \mathrm{div}\, k - \mathrm{d}\,\mathrm{tr}\, k, \end{aligned}$$

where energy density $\rho = T(e_0, e_0)$ and momentum density $j = T(e_0, \text{-})_{|TM}$ are components of the energy-momentum tensor.

For physical reasons, the energy-momentum tensor is assumed to always satisfy the dominant energy condition, which implies that $\rho \geq |j|$. We will say that an initial data set (g, k) satisfies the dominant energy condition if $\rho \geq |j|$, when ρ and j are defined by (1.2). This condition plays a vital role in the positive mass theorem [SY81; Wit81] stating that for an asymptotically Euclidean manifold (M, g) with k tending to zero at infinity, the ADM-mass is non-negative if (g, k) satisfies the dominant energy condition.

In this chapter, we consider the case that M is a closed spin manifold of dimension $n \geq 2$. Our aim is to study the space $\mathcal{I}^{>}(M)$ of initial data sets (g, k) on M for which the strict dominant energy condition holds, i. e. $\rho > |j|$ everywhere. This is a subspace of the space $\mathcal{I}(M)$ of all initial data sets on M, with C^∞-topology. The reason for restricting to the strict version of the dominant energy condition is that it nicely connects to positive scalar curvature, which in turn is rather well-studied. In [AG23], Ammann and the author discuss some ideas how to extend the results to the (non-strict) dominant energy condition.

2.1.2. Connection to positive scalar curvature and main result

It is a simple observation that if $k \equiv 0$, then the strict dominant energy condition for (g, k) reduces to the condition that g has positive scalar curvature. However, whereas existence of positive scalar curvature metrics imposes a condition on the manifold, this is not true for the strict dominant energy condition. More precisely, we will see later that taking any metric g, the pair $(g, \frac{1}{n}\tau g)$ satisfies the strict dominant energy condition as long as the absolute value of the constant $\tau \in \mathbb{R}$ is large enough. Moreover, such a τ can be chosen in a way that it continuously depends on the metric g (in C^2-topology). This allows to define a comparison map $\Phi\colon S\mathcal{R}^{>}(M) \simeq \mathcal{R}^{>}(M) \times [-1, 1] \cup \mathcal{R}(M) \times \{-1, 1\} \to \mathcal{I}^{>}(M)$ by $(g, t) \mapsto (g, \frac{1}{n}\tau(g)tg)$, where $\mathcal{R}(M)$ is the C^∞-space of metrics, $\mathcal{R}^{>}(M)$ its subspace of positive scalar curvature metrics and S denotes the suspension.

One of the main approaches to positive scalar curvature is by index theoretic methods. Assume that (M, g) is closed, spin, and of dimension n. Then, there is a spinor bundle $\Sigma_{\mathrm{Cl}} M$ with a right Cl_n-action, called Cl_n-linear spinor bundle of M. Its Dirac operator D commutes with the Cl_n-action and thus gives rise to a Cl_n-Fredholm operator, which has a KO-valued index called α-index $\alpha(M)$. The Schrödinger-Lichnerowicz formula

$$D^2 = \nabla^*\nabla + \frac{1}{4}\,\mathrm{scal}$$

implies that D is invertible if g has positive scalar curvature and so its index vanishes. By homotopy invariance of the index, it is independent of g, and so the α-index provides an obstruction to existence of positive scalar curvature metrics on M if it is non-zero for some spin structure on M.

In the case when there is a positive scalar curvature metric g_0 on M, this invariant can be refined to a secondary invariant known as index difference that allows to detect non-trivial homotopy groups in the space of positive scalar curvature metrics. In order to emphasize that it refines the α-index and to stress its connection with the α-invariant for diffeomorphisms (cf. [CSS18, eq. (2)]), we will call it *α-index difference*, or *α-difference* for short. It is constructed as follows: As before, the Cl_n-linear Dirac operator defines a map assigning to each metric a Cl_n-Fredholm operator, which is invertible if the metric is of positive scalar curvature. Then applying the KO-valued index, we obtain the map

$$\alpha\text{–diff}\colon \pi_l(\mathcal{R}^>(M), g_0) \cong \pi_{l+1}(\mathcal{R}(M), \mathcal{R}^>(M), g_0) \to \mathrm{KO}^{-n-l-1}(\{*\}).$$

A similar invariant exists in the case of initial data sets. For this, the Cl_n-linear spinor bundle has to be replaced by the $\mathrm{Cl}_{n,1}$-linear hypersurface spinor bundle $\overline{\Sigma}_{\mathrm{Cl}}M$. To define it, we embed M as spacelike hypersurface into a time-oriented spin Lorentzian manifold $(\overline{M}, \overline{g})$ such that the pair (g, k) arises as induced metric and second fundamental form. Then $\overline{\Sigma}_{\mathrm{Cl}}M$ is the restriction of the $\mathrm{Cl}_{n,1}$-linear spinor bundle of $\overline{M}$ to M. It turns out that this bundle can be defined intrinsically – without reference to $\overline{M}$ – by $\overline{\Sigma}_{\mathrm{Cl}}M = \Sigma_{\mathrm{Cl}}M \otimes_{\mathrm{Cl}_n} \mathrm{Cl}_{n,1}$, i. e. it is given by two copies of $\Sigma_{\mathrm{Cl}}M$. The role of the Dirac operator is now played by the Dirac-Witten operator $\overline{D}$, which is $\mathrm{Cl}_{n,1}$-linear in our case, and which will be defined in Section 2.3.2 below. There is a Schrödinger-Lichnerowicz type formula for $\overline{D}$

$$\overline{D}^2 = \overline{\nabla}^*\overline{\nabla} + \frac{1}{2}(\rho - e_0 \cdot j^\sharp \cdot),$$

which ensures that $\overline{D}$ is invertible if (g, k) satisfies the strict dominant energy condition. With these changes, the same construction as before yields an index difference for initial data sets

$$\overline{\alpha}\text{–diff}\colon \pi_l(\mathcal{I}^>(M), (g_0, k_0)) \cong \pi_{l+1}(\mathcal{I}(M), \mathcal{I}^>(M), (g_0, k_0)) \to \mathrm{KO}^{-n-l}(\{*\}),$$

where $(g_0, k_0) \in \mathcal{I}^>(M)$. Notice that there is a degree shift in the target compared to α–diff: This results from the $\mathrm{Cl}_{n,1}$-linearity of the Dirac-Witten operator in contrast to the Cl_n-linearity of the Dirac operator.

Notation 2.1.1. To avoid clumsy notation, we often write $\alpha\text{–diff}(g_{-1}, g_1)$ for the α-difference applied to the π_0-class represented by $(S^0, 1) \to (\mathcal{R}^>(M), g_1)$, $t \mapsto g_t$. Likewise, we write $\overline{\alpha}\text{–diff}((g_{-1}, k_{-1}), (g_1, k_1))$ for the $\overline{\alpha}$-difference of the π_0-class defined by $(S^0, 1) \to (\mathcal{I}^>(M), (g_1, k_1))$, $t \mapsto (g_t, k_t)$.

Unlike the situation of the α-difference, where the α-index constitutes an interesting invariant obstructing positive scalar curvature, there is no interesting primary invariant associated with the $\overline{\alpha}$-difference: The index of the Dirac-Witten operator $\overline{D}$ is always zero. This follows for example from the observation that the dominant energy condition is not obstructed, since, as mentioned above, $(g, \frac{1}{n}\tau g) \in \mathcal{I}^{>}(M)$ for $g \in \mathcal{R}(M)$ and suitably large $\tau \in \mathbb{R}$. The $\overline{\alpha}$-difference, however, is an interesting invariant. This is a consequence of the main theorem of this chapter, where we compare it to the α-difference or, in the case of the π_0-part, to the α-index.

Theorem 2.1.2 (Theorems A and B, Theorem 2.4.5). *1. For $g_0 \in \mathcal{R}^{>}(M)$ and all $l \geq 0$, the diagram*

$$\begin{array}{ccccc} \pi_l(\mathcal{R}^{>}(M), g_0) & \xrightarrow{\text{Susp}} & \pi_{l+1}(S\,\mathcal{R}^{>}(M), [g_0, 0]) & \xrightarrow{\Phi_*} & \pi_{l+1}(\mathcal{I}^{>}(M), (g_0, 0)) \\ & \searrow^{\alpha\text{–diff}} & & \swarrow^{\overline{\alpha}\text{–diff}} & \\ & & \mathrm{KO}^{-n-l-1}(\{*\}) & & \end{array}$$

commutes.

2. For $g_0 \in \mathcal{R}(M)$,

$$\overline{\alpha}\text{–diff}\left(\left(g_0, -\frac{1}{n}\tau(g_0)g_0\right), \left(g_0, \frac{1}{n}\tau(g_0)g_0\right)\right) = \alpha(M) \in \mathrm{KO}^{-n}(\{*\}).$$

The idea of the proof is the following: For a pair of the form $(g, \frac{1}{n}\tau(g)tg)$, $t \in \mathbb{R}$, the Cl_{n+1}-linear Dirac-Witten operator is given by $\overline{D} = D \otimes_{\mathrm{Cl}_n} \mathrm{Cl}_{n,1} - \tau(g)tL(e_0)$, where D is the Cl_n-linear Dirac operator from before and $L(e_0)$ is left multiplication with the future-pointing unit normal on M when M is considered as spacelike hypersurface of $\overline{M}$ as above. Now, we observe that the $\mathrm{Cl}_{n,1}$-structure of $\overline{\Sigma}_{\mathrm{Cl}}M$ given by right multiplication can be extended to a $\mathrm{Cl}_{n+1,1}$-structure by setting the right multiplication by the additional basis vector as $\tilde{R}(e_{n+1}) \coloneqq L(e_0)a$, where a is the even-odd grading operator. With this $\mathrm{Cl}_{n+1,1}$-structure, $\overline{\Sigma}_{\mathrm{Cl}}M$ corresponds to $\Sigma_{\mathrm{Cl}}M$ under the Morita equivalence relating Cl_n- and $\mathrm{Cl}_{n+1,1}$-modules. Moreover, under this equivalence $D \otimes_{\mathrm{Cl}_n} \mathrm{Cl}_{n,1}$ is associated to D and, by definition, the index map is invariant under this correspondence. The second summand can be understood as coming from the Bott map, which assigns to a $\mathrm{Cl}_{n+1,1}$-Fredholm operator F the family of $\mathrm{Cl}_{n,1}$-Fredholm operators $[-1, 1] \ni t \mapsto F + t\tilde{R}(e_{n+1})a = F + tL(e_0)$. Again, invariance of the index map under this assignment is a consequence of its definition, but an extra sign has to be taken into account resulting from the fact that in the definition of the index map Morita equivalence and Bott map are applied in the reverse order.

As a consequence of the main theorem, every element in $\pi_l(\mathcal{R}^{>}(M), g_0)$ with non-trivial α-difference gives rise to a non-zero element in $\pi_{l+1}(\mathcal{I}^{>}(M), (g_0, 0))$. Such elements have been constructed for example by Hitchin [Hit74], Hanke, Schick and Steimle [HSS14], Botvinnik, Ebert and Randal-Williams [BER14] as well as Crowley, Schick and Steimle [CSS18] using different techniques. In particular, we obtain the following corollary.

2. *On the space of initial data sets satisfying the strict dominant energy condition*

Corollary 2.1.3. 1. *If M is a closed spin manifold of dimension $n \geq 6$ that admits a metric of positive scalar curvature, then $\mathcal{I}^>(M)$ is not contractible.*

2. *If M is a closed spin manifold of dimension $n \geq 2$ with $\alpha(M) \neq 0$ (in particular, M does not carry a positive scalar curvature metric), then $\mathcal{I}^>(M)$ is not connected.*

The structure of this chapter is as follows. In the first section, we review the KO-valued index map and the construction of the α-difference. Much of this material is owed to Ebert [Ebe17]. The second section is devoted to the construction of the $\overline{\alpha}$-difference. To this end, the $\mathrm{Cl}_{n,1}$-linear hypersurface spinor bundle and its Dirac-Witten operator are introduced. We discuss the $\mathrm{Cl}_{n,1}$-linear version of the Dirac-Witten operator in some detail, as it seems not to have been studied before. In the last section, we construct the comparison map, prove the main theorem and discuss some more of its consequences.

2.2. The classical α-index difference

2.2.1. KO-theory via Fredholm operators

This subsection is devoted to the KO-valued index map, a map that associates to a family of Clifford-linear Fredholm operators an element in KO-theory. In its description, we will stick closely to the framework presented in Ebert [Ebe17] that we briefly recall. All Hilbert spaces are understood as being real and separable. A $\mathrm{Cl}_{n,k}$-*Hilbert space* H is always $\mathbb{Z}/2\mathbb{Z}$-graded. Typically, the $\mathbb{Z}/2\mathbb{Z}$-grading is given in terms of a grading operator $\iota\colon H \to H$, and the Clifford action is determined by a Clifford multiplication $c\colon \mathbb{R}^{n,k} \to \mathrm{End}(H)$, where $\mathbb{R}^{n,k}$ is the pseudo-Euclidean vector space $\mathbb{R}^n \oplus \mathbb{R}^k$ with the standard inner product that is positive definite on the first summand and negative definite on the second one. The convention for the Clifford multiplication is such that $c(v)c(w) + c(w)c(v) = -2\langle v, w\rangle$.

If (H, ι, c) is a $\mathrm{Cl}_{n,k}$-Hilbert space, then c gives rise to a representation $\mathrm{Cl}_{n,k} \to \mathrm{End}(H)$, which can be decomposed into irreducible ones. (H, ι, c) is called *ample*, if it contains each irreducible representation infinitely often. By the structure theory for real Clifford representations, this just means that H is infinite-dimensional if $n - k \not\equiv 0 \mod 4$, and amounts to the condition that both the $+1$- and the -1-eigenspace of the volume element $\omega_{n,k} \coloneqq \iota c(e_1) \cdots c(e_{n+k})$ are infinite-dimensional if $n - k \equiv 0 \mod 4$.

Definition 2.2.1. Let (H, ι, c) be an ample $\mathrm{Cl}_{n,k}$-Hilbert space. Then a $\mathrm{Cl}_{n,k}$-*Fredholm operator* F is a (bounded) Fredholm operator on H that is self-adjoint, odd with respect to ι, $\mathrm{Cl}_{n,k}$-linear and, in the case $n - k \equiv -1 \mod 4$, satisfies the additional condition that $\omega_{n,k}F\iota$ is neither essentially positive nor essentially negative. We denote by $\mathrm{Fred}^{n,k}(H)$ the space of $\mathrm{Cl}_{n,k}$-Fredholm operators with operator norm topology. Furthermore, we write $G^{n,k}(H) \subseteq \mathrm{Fred}^{n,k}(H)$ for the subspace of invertible elements.

Note that we have $\mathrm{Fred}^{n+1,k}(H) \subseteq \mathrm{Fred}^{n,k}(H)$ and $\mathrm{Fred}^{n,k+1}(H) \subseteq \mathrm{Fred}^{n,k}(H)$: In the cases $n-k = 1,2 \mod 4$, this is immediate. If $n-k = 0 \mod 4$, this follows since the additional generator of the extended Clifford action on the $\mathrm{Cl}_{n,k}$-Hilbert space H anti-commutes with $\omega_{n,k}$. Finally, in case $n-k \equiv -1 \mod 4$, we use that for a $\mathrm{Cl}_{n+1,k}$- or $\mathrm{Cl}_{n,k+1}$-linear operator F, the additional generator of the extended Clifford action anti-commutes with $\omega_{n,k}F\iota$.

Remark 2.2.2. As was pointed out by the referee, ampleness of H and the additional condition in the case where $n-k \equiv -1 \mod 4$ are only needed to ensure bijectivity of the index map discussed below and are not necessary for its existence. For instance, the inductive extension of the index map from degree $n-1$ to degree n does not require the left hand vertical map in diagram (2.1) to be an isomorphism. Since in this thesis we will not use that the index map is an isomorphism, all discussions about ampleness and the additional condition are included for the sake of completeness only (and shifted to a large extent to Appendix A.2).

Example 2.2.3. The archetypical example of a $\mathrm{Cl}_{n,0}$-Fredholm operator is (the bounded transform of) the Cl_n-linear Dirac operator on a closed Riemannian spin manifold (M,g) of dimension $n>0$: Let $P_{\mathrm{Spin}(n)}M \to P_{SO(n)}M$ be a spin structure of M. The *Cl_n-linear spinor bundle* is $\Sigma_{\mathrm{Cl}}M := P_{\mathrm{Spin}(n)}M \times_\ell \mathrm{Cl}_n$, where $\ell\colon \mathrm{Spin}(n) \to \mathrm{End}(\mathrm{Cl}_n)$ is given by left multiplication. Its name derives from the fact that right multiplication in Cl_n induces a *right Clifford multiplication* $R\colon \mathbb{R}^n \to \mathrm{End}(\Sigma_{\mathrm{Cl}}M)$, which commutes with the left Clifford multiplication by tangent vectors. Furthermore, it carries a $\mathbb{Z}/2\mathbb{Z}$-grading a induced by $\mathrm{Cl}_n \to \mathrm{Cl}_n$, $\mathbb{R}^n \ni v \mapsto -v$, the *even-odd-grading.* The bundle metric induced by the metric on Cl_n that makes the standard basis $(e_{i_1}\dots e_{i_l})_{0\le l\le n+k, 1\le i_1<\dots<i_l\le n+k}$ orthonormal allows to define an L^2-scalar product and the space of L^2-sections $H := L^2(M,\Sigma_{\mathrm{Cl}}M)$. Both a and R descend to H, turning (H,a,R) into an ample Cl_n-Hilbert space. The *Cl_n-linear Dirac operator* D, i. e. the Dirac operator of $\Sigma_{\mathrm{Cl}}M$ w.r.t. the connection induced by the Levi-Civita connection, can be viewed as unbounded operator on H. By standard results on the analysis of Dirac operators, its bounded transform $F := \frac{D}{\sqrt{1+D^2}}$ is a Fredholm operator on H, and as D is Cl_n-linear (w.r.t. R) and odd (w.r.t. a), so is F. Thus, $F \in \mathrm{Fred}^{n,0}(H)$, whereby the additional condition for $n \equiv -1 \mod 4$ is well-known to be satisfied for Dirac type operators. In order to be self-contained, we recall this in the appendix. It is worth noting that the Schrödinger-Lichnerowicz formula implies that F is invertible, so $F \in G^{n,0}(H)$, if g is a metric of positive scalar curvature.

The following consequence of Kuiper's theorem is proven in [Ebe17]. It is one of the main ingredients for translating the classical results from [AS69] into the present framework.

Proposition 2.2.4. *The space $G^{n,k}(H)$ is contractible for all $n, k \geq 0$.*

Theorem 2.2.5 (Index map)**.** *If H is an ample $\mathrm{Cl}_{n,k}$-Hilbert space, then $\mathrm{Fred}^{n,k}(H)$ represents KO-theory: For compact relative CW-complexes (X, Y), there is a natural (in (X, Y)) bijection*

$$\mathrm{ind}\colon [(X,Y),(\mathrm{Fred}^{n,k}(H), G^{n,k}(H))] \longrightarrow \mathrm{KO}^{k-n}(X,Y)$$

called index map*. Moreover,* ind *is invariant under $\mathrm{Cl}_{n,k}$-Hilbert space isomorphisms, i. e. if $U\colon H \to H'$ is an isomorphism of $\mathrm{Cl}_{n,k}$-Hilbert spaces, then*

$$\begin{array}{ccc} [(X,Y),(\mathrm{Fred}^{n,k}(H), G^{n,k}(H))] & \xrightarrow{\cong} & [(X,Y),(\mathrm{Fred}^{n,k}(H'), G^{n,k}(H'))] \\ & {\scriptstyle \mathrm{ind}} \searrow \quad \swarrow {\scriptstyle \mathrm{ind}} & \\ & \mathrm{KO}^{k-n}(X,Y) & \end{array}$$

commutes, where the upper map is induced by $\mathrm{Fred}^{n,k}(H) \ni F \mapsto UFU^{-1}$.

The index map is constructed inductively, the starting point being the index of a family of $\mathrm{Cl}_{0,0}$-Fredholm operators, i. e. odd Fredholm operators on a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space. Here, the corresponding statement is known as Atiyah-Jänich theorem (cf. [Glö19, Thm. 2.17] for a detailed derivation from the version in [AS69]).

The generalization to arbitrary n (but still with $k = 0$) is provided by the Bott map.

Theorem 2.2.6 (Bott map, [AS69, Thm. A(k)])**.** *For compact CW-pairs (X, Y), the map*

$$\begin{aligned} [(X,Y),(\mathrm{Fred}^{n+1,k}(H), G^{n+1,k}(H))] &\longrightarrow [(X,Y)\times(I,\partial I),(\mathrm{Fred}^{n,k}(H), G^{n,k}(H))] \\ [x \mapsto F_x] &\longmapsto [(x,t)\mapsto F_x + tc(e)\iota] \end{aligned}$$

is a natural bijection.[1] *Here, e is the additional basis vector of $\mathbb{R}^{n+1,k}$ compared to $\mathbb{R}^{n,k}$ and $I = [-1, 1]$.*

As $(X \times I)/(Y \times I \cup X \times \partial I) \cong \Sigma_{red} X/Y$ the right hand isomorphism in the following diagram exists, and the defintion of the index map can be extended inductively by requiring that it commutes:

$$\begin{array}{ccc} [(X,Y),(\mathrm{Fred}^{n,0}(H), G^{n,0}(H))] & \overset{\mathrm{ind}}{\dashrightarrow} & \mathrm{KO}^{-n}(X,Y) \\ \downarrow\cong & & \downarrow\cong \\ [(X,Y)\times(I,\partial I),(\mathrm{Fred}^{n-1,0}(H), G^{n-1,0}(H))] & \overset{\mathrm{ind}}{\longrightarrow} & \mathrm{KO}^{-n+1}(X\times I\, ,\, X\times\partial I\cup Y\times I). \end{array} \tag{2.1}$$

[1]For two pairs (X, A) and (Y, B), we write $(X, A) \times (Y, B) := (X \times Y, X \times B \cup A \times Y)$.

The extension to arbitrary k uses periodicity statements in the theory of $\mathrm{Cl}_{n,k}$-Hilbert spaces known as Morita equivalences. One of them states that the categories of $\mathrm{Cl}_{n,k}$-Hilbert spaces and $\mathrm{Cl}_{n+1,k+1}$-Hilbert spaces are equivalent. Its construction is the following: A $\mathrm{Cl}_{n,k}$-Hilbert space (H,ι,c) defines a $\mathrm{Cl}_{n+1,k+1}$-Hilbert space structure on $H \oplus H$ by

$$\begin{aligned} \tilde{\iota} &= \begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix} \\ \tilde{c}(v) &= \begin{pmatrix} c(v) & 0 \\ 0 & -c(v) \end{pmatrix} \quad \text{for all } v \in R^{n+k} \oplus 0 \\ \tilde{c}(e) &= \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \\ \tilde{c}(\varepsilon) &= \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \end{aligned} \tag{2.2}$$

where we view $\mathbb{R}^{n+1,k+1}$ as $R^{n,k} \oplus \mathbb{R}e \oplus \mathbb{R}\varepsilon$. And a morphism $F\colon H \to H'$ of $\mathrm{Cl}_{n,k}$-Hilbert spaces gives rise to a morphism

$$\tilde{F} = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} : H \oplus H \to H' \oplus H'$$

of the corresponding $\mathrm{Cl}_{n+1,k+1}$-Hilbert spaces. Conversely, for a $\mathrm{Cl}_{n+1,k+1}$-Hilbert space (H,ι,c), the restrictions of the structure maps to $H_0 := \ker(c(\varepsilon)c(e) - \mathbb{1})$ yield a $\mathrm{Cl}_{n,k}$-Hilbert space, and morphisms of $\mathrm{Cl}_{n+1,k+1}$-Hilbert spaces restrict to morphisms of these $\mathrm{Cl}_{n,k}$-Hilbert spaces. These constructions are seen to be mutually inverse up to natural isomorphism.

Another Morita equivalence exists between $\mathrm{Cl}_{n+4,k}$-Hilbert spaces and $\mathrm{Cl}_{n,k+4}$-Hilbert spaces. For this, we regard both $\mathbb{R}^{n+4,k}$ and $\mathbb{R}^{n,k+4}$ as $\mathbb{R}^n \oplus \mathbb{R}^k \oplus \operatorname{span}\{e_1,e_2,e_3,e_4\}$, where $e_1,\dots e_4$ are the last four basis vectors of $\mathbb{R}^{n+4}$ or the last four basis vectors of $\mathbb{R}^{k+4}$, respectively. Given a $\mathrm{Cl}_{n+4,k}$-Hilbert space (H,ι,c), we can define a $\mathrm{Cl}_{n,k+4}$-Hilbert space $(H,\iota,\tilde{c})$ by $\tilde{c}_{|\mathbb{R}^{n,k}} = c_{|\mathbb{R}^{n,k}}$ and $\tilde{c}(e_i) = \eta c(e_i)$ for $\eta = c(e_1)\cdots c(e_4)$. Morphisms are mapped to the morphisms defined by the same underlying bounded linear maps. The inverse procedure is given similarly, by assigning to a $\mathrm{Cl}_{n,k+4}$-Hilbert space $(H,\iota,\tilde{c})$ the $\mathrm{Cl}_{n+4,k}$-Hilbert space (H,ι,c) with $c_{|\mathbb{R}^{n,k}} = \tilde{c}_{|\mathbb{R}^{n,k}}$ and $c(e_i) = \tilde{\eta}\tilde{c}(e_i)$, where $\tilde{\eta} = \tilde{c}(e_1)\cdots\tilde{c}(e_4)$.

These equivalences are accompanied by homeomorphisms between the spaces of Clifford-linear Fredholm operators.

Proposition 2.2.7. *The Morita equivalences discussed above induce homeomorphisms* *of pairs*

$$\begin{aligned}(\mathrm{Fred}^{n,k}(H), G^{n,k}(H)) &\longrightarrow (\mathrm{Fred}^{n+1,k+1}(H \oplus H), G^{n+1,k+1}(H \oplus H))\\ F &\longmapsto \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}(\mathrm{Fred}^{n+4,k}(H), G^{n+4,k}(H)) &\longrightarrow (\mathrm{Fred}^{n,k+4}(H), G^{n,k+4}(H))\\ F &\longmapsto F.\end{aligned}$$

In particular, there is a homeomorphism

$$\begin{aligned}(\mathrm{Fred}^{n,k}(H), G^{n,k}(H)) &\longrightarrow (\mathrm{Fred}^{n+8,k}(H \otimes \mathbb{R}^{16}), G^{n+8,k}(H \otimes \mathbb{R}^{16}))\\ F &\longmapsto F \otimes \mathbb{1}_{\mathbb{R}^{16}}\,.\end{aligned}$$

The index map is then defined inductively for all (n, k) with $0 \leq k \leq n$ by the requirement that

$$\begin{array}{ccc}[(X,Y),(\mathrm{Fred}^{n,k}(H),G^{n,k}(H))] & \xrightarrow{\ \mathrm{ind}\ } & \mathrm{KO}^{k-n}(X,Y)\\ \cong\big\uparrow & & \big\| \\ [(X,Y),(\mathrm{Fred}^{n-1,k-1}(H_0),G^{n-1,k-1}(H_0))] & \xrightarrow{\ \mathrm{ind}\ } & \mathrm{KO}^{k-n}(X,Y)\end{array} \tag{2.3}$$

commutes. Lastly, it is extended to the missing (n, k) with $0 \leq n, k$ by commutativity of

$$\begin{array}{ccc}[(X,Y),(\mathrm{Fred}^{n,k}(H),G^{n,k}(H))] & \xrightarrow{\ \mathrm{ind}\ } & \mathrm{KO}^{k-n}(X,Y)\\ \big\downarrow\cong & & \big\downarrow\cdot x \\ [(X,Y),(\mathrm{Fred}^{n+8,k}(H\otimes\mathbb{R}^{16}),G^{n+8,k}(H\otimes\mathbb{R}^{16}))] & \xrightarrow{\ \mathrm{ind}\ } & \mathrm{KO}^{k-n-8}(X,Y),\end{array} \tag{2.4}$$

where x denotes a generator of $\mathrm{KO}^{-8}(\{*\})$.

Remark 2.2.8. The commutativity of (2.4) does not only hold for $n < k$ (where it is true by definition), but is also true for $k \leq n$ provided that the right generator $x \in \mathrm{KO}^{-8}(\{*\})$ is chosen. This follows from the last remark in [AS69].

Example 2.2.9. In the setting of Example 2.2.3, we can define the α-index of M by $\alpha(M) = \mathrm{ind}(F) \in \mathrm{KO}^{-n}(\{*\})$. This invariant was first defined by Hitchin [Hit74] and is a well-known obstruction to positive scalar curvature: From the continuity of the assignment $g \mapsto F_g$ discussed in the next section, it follows that $\alpha(M)$ is independent of the metric on M (in fact, it is even spin-bordism invariant) and so has to vanish for every spin structure if M carries a positive scalar curvature metric.

2.2.2. Construction of the α-index difference

Let M be a closed spin manifold of dimension $n > 0$ that has a positive scalar curvature metric g_0. The α-index difference, also introduced by Hitchin [Hit74], is a family version of the α-index. More precisely, $\alpha\text{–diff}\colon \pi_l(\mathcal{R}^>(M), g_0) \to \mathrm{KO}^{-n-l-1}(\{*\})$ arises in the following way: As $\mathcal{R}(M)$ is contractible, the long exact sequence for homotopy groups implies $\pi_l(\mathcal{R}^>(M), g_0) \cong \pi_{l+1}(\mathcal{R}(M), \mathcal{R}^>(M), g_0)$. For each metric g, the Cl_n-linear Dirac operator D_g defines a Cl_n-linear Fredholm operator

$$F_g = \frac{D_g}{\sqrt{1 + D_g^2}},$$

which is invertible if $g \in \mathcal{R}^>(M)$. The assignment $g \mapsto F_g$ gives rise to a map $(\mathcal{R}(M), \mathcal{R}^>(M)) \to (\mathrm{Fred}^{n,0}, G^{n,0})$, which induces a map to $\pi_{l+1}(\mathrm{Fred}^{n,0}, G^{n,0}, F_{g_0})$. Applying the index map from the previous subsection, we obtain an element in $\mathrm{KO}^{-n}(D^{l+1}, S^l) \cong \mathrm{KO}^{-n-l-1}(\{*\})$.

In this outline, however, we glossed over the detail that the Cl_n-linear spinor bundles and hence the L^2-spaces, on which the Fredholm operators F_g act, depend on the metric g. These L^2-spaces form a Hilbert bundle over $\mathcal{R}(M)$, which, by Kuiper's theorem, can be trivialized. Such a trivialization allows to define the map $(\mathcal{R}(M), \mathcal{R}^>(M)) \to (\mathrm{Fred}^{n,0}, G^{n,0})$. We will make this more explicit: The Cl_n-linear spinor bundles for different metrics can be identified using the method of generalized cylinders due to Bär, Gauduchon and Moroianu [BGM05]. This gives rise to a specific trivialization of the Hilbert bundle of L^2-spaces.

Let us start with this construction by fixing a topological spin structure on M, i. e. a double covering

$$P_{\widetilde{GL}^+(n)}M \to P_{GL^+(n)}M$$

over the principal bundle of positively oriented frames of TM. This defines, for any $g \in \mathcal{R}(M)$, a spin structure for (M, g) by pullback

$$\begin{array}{ccc} P_{\mathrm{Spin}(n)}(M,g) & \longrightarrow & P_{\widetilde{GL}^+(n)}M \\ \downarrow & & \downarrow \\ P_{\mathrm{SO}(n)}(M,g) & \longrightarrow & P_{GL^+(n)}M, \end{array}$$

where $P_{\mathrm{SO}(n)}(M, g)$ is the principal bundle of positively oriented orthonormal frames with respect to g. Moreover, pulling back over the canonical projection $M \times [0, 1] \to M$, we obtain

$$\begin{array}{ccc} P_{\widetilde{GL}^+(n)}M \times [0,1] & \longrightarrow & P_{\widetilde{GL}^+(n)}M \\ \downarrow & & \downarrow \\ P_{GL^+(n)}M \times [0,1] & \longrightarrow & P_{GL^+(n)}M \\ \downarrow & & \downarrow \\ M \times [0,1] & \longrightarrow & M. \end{array}$$

This gives rise a topological spin structure $P_{\widetilde{GL}^+(n+1)}M \times [0,1] \to P_{GL^+(n+1)}M \times [0,1]$ on $M \times [0,1]$ by extension along the standard embedding

$$GL^+(n) \longrightarrow GL^+(n+1)$$
$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

and its double covering.

Now, given a metric $g \in \mathcal{R}(M)$, we can define a family of metrics by $g_t = (1-t)g_0 + tg$. Such a family in turn defines the generalized cylinder $(M \times [0,1], g_t + \mathrm{d}t^2)$, t being the variable in $[0,1]$-direction. As above, the topological spin structure induces a spin structure $P_{\mathrm{Spin}(n+1)}(M\times[0,1], g_t+\mathrm{d}t^2) \to P_{\mathrm{SO}(n+1)}(M\times[0,1], g_t+\mathrm{d}t^2)$ on the generalized cylinder. This has the property that for all $t_0 \in [0,1]$ it restricts to the spin structure of (M, g_{t_0}) in the sense that

$$\begin{array}{ccc} P_{\mathrm{Spin}(n)}(M, g_{t_0}) & \longrightarrow & P_{\mathrm{Spin}(n+1)}(M \times [0,1], g_t + \mathrm{d}t^2) \\ \downarrow & & \downarrow \\ P_{\mathrm{SO}(n)}(M, g_{t_0}) & \longrightarrow & P_{\mathrm{SO}(n+1)}(M \times [0,1], g_t + \mathrm{d}t^2) \end{array}$$

is a pullback, where the lower map is the inclusion $(e_1, \ldots, e_n) \mapsto (e_1, \ldots, e_n, \frac{\partial}{\partial t})$.

The reason why we do this is that on $P_{\mathrm{Spin}(n+1)}(M \times [0,1], g_t + \mathrm{d}t^2)$ the Levi-Civita connection induces a canonical connection ∇, which provides parallel transports

$$P^{\nabla}_{\gamma_x} \colon P_{\mathrm{Spin}(n+1)}(M \times [0,1], g_t + \mathrm{d}t^2)_{|(x,0)} \longrightarrow P_{\mathrm{Spin}(n+1)}(M \times [0,1], g_t + \mathrm{d}t^2)_{|(x,1)}$$

along the curves $\gamma_x \colon [0,1] \to M \times [0,1],\ t \mapsto (x,t)$ for all $x \in M$. These assemble into an isomorphism of principal bundles

$$P^{\nabla} \colon P_{\mathrm{Spin}(n+1)}(M \times [0,1], g_t + \mathrm{d}t^2)_{|M\times\{0\}} \xrightarrow{\cong} P_{\mathrm{Spin}(n+1)}(M \times [0,1], g_t + \mathrm{d}t^2)_{|M\times\{1\}}.$$

The fact that $\frac{\partial}{\partial t}$ is parallel along the curves γ_x implies that P^{∇} restricts to

$$P^{\nabla} \colon P_{\mathrm{Spin}(n)}(M, g_0) \xrightarrow{\cong} P_{\mathrm{Spin}(n)}(M, g),$$

and this induces an isomorphism on the associated Cl_n-linear spinor bundles

$$\begin{aligned} P^\nabla \colon \Sigma_{\mathrm{Cl}}(M, g_0) &\xrightarrow{\cong} \Sigma_{\mathrm{Cl}}(M, g). \\ [\tilde{\varepsilon}, \tilde{\phi}] &\longmapsto [P^\nabla \tilde{\varepsilon}, \tilde{\phi}] \end{aligned}$$

Furthermore, it is immediate that P^∇ is a point-wise isometry with respect to the standard scalar products $\langle \text{-}\,, \text{-} \rangle$ defined on the Cl_n-linear spinor bundles.

We want to promote this to a unitary transformation between the associated L^2-spaces. As the L^2-norm also depends on the volume element, we first compare those: There exists a positive function $\beta \in C^\infty(M)$ such that $\mathrm{dvol}^g = \beta\,\mathrm{dvol}^{g_0}$. Then $\sqrt{\beta} P^\nabla \colon \Sigma_{\mathrm{Cl}}(M, g_0) \to \Sigma_{\mathrm{Cl}}(M, g)$ induces a unitary transformation

$$\Phi_g \colon H := L^2(M, \Sigma_{\mathrm{Cl}}(M, g_0)) \xrightarrow{\cong} L^2(M, \Sigma_{\mathrm{Cl}}(M, g))$$

as

$$(\Phi_g(\phi), \Phi_g(\psi))_{L^2} = \int_M \langle \sqrt{\beta} P^\nabla(\phi), \sqrt{\beta} P^\nabla(\psi) \rangle \,\mathrm{dvol}^g = \int_M \langle \phi, \psi \rangle \,\mathrm{dvol}^{g_0} = (\phi, \psi)_{L^2}.$$

Moreover, it is clear that Φ_g preserves the $\mathbb{Z}/2\mathbb{Z}$-grading and the right Clifford multiplication. The left Clifford multiplication by a vector field $X \in \Gamma(TM)$ satisfies $\Phi_g(X \cdot \phi) = P^\nabla(X) \cdot \Phi_g(\phi)$ for any $\phi \in H$, where $P^\nabla(X)$ is the vector field on $M = M \times \{1\}$ obtained from X by parallel transport along the curves $(\gamma_x)_{x \in M}$ in the cylinder $(M \times [0,1], g_t + \mathrm{d}t^2)$.

It is not surprising that using this identification of the L^2-spaces (the bounded transforms of) the Dirac operators depend continuously on the metric. For a detailed proof of the following statement see [Glö19, Thm. 2.22].

Theorem 2.2.10. *The map*

$$\begin{aligned} (\mathcal{R}(M), \mathcal{R}^>(M)) &\longrightarrow (\mathrm{Fred}^{n,0}(H), G^{n,0}(H)) \\ g &\longmapsto \Phi_g^{-1} \circ \frac{D_g}{\sqrt{1 + D_g^2}} \circ \Phi_g \end{aligned}$$

is well-defined and continuous with respect to the C^1-topology on the space of smooth metrics $\mathcal{R}(M)$. In particular, it is continuous if $\mathcal{R}(M)$ carries the C^∞-topology.

Definition 2.2.11. The map from Theorem 2.2.10 gives rise to the composition

$$\begin{aligned} \alpha\text{–diff} \colon \pi_l(\mathcal{R}^>(M), g_0) &\cong \pi_{l+1}(\mathcal{R}(M), \mathcal{R}^>(M), g_0) \\ &\to \pi_{l+1}(\mathrm{Fred}^{n,0}(H), G^{n,0}(H), F_{g_0}) \xrightarrow{\mathrm{ind}} \mathrm{KO}^{-n-l-1}(\{*\}) \end{aligned}$$

that we call *α-index difference* or shortly *α-difference.*

The α-difference detects non-trivial homotopy groups in the space of metrics of positive scalar curvature. The following two results of this kind were independently obtained by different methods:

Theorem 2.2.12 (Crowley, Schick, Steimle [CSS18])**.** *Let (M, g_0) be a closed Riemannian spin manifold of positive scalar curvature and $n = \dim(M) \geq 6$. For all $l \geq 0$ with $l + n + 1 \equiv 1, 2 \mod 8$, the α-difference*

$$\alpha\text{–diff}\colon \pi_l(\mathcal{R}^{>}(M), g_0) \longrightarrow \mathrm{KO}^{-n-l-1}(\{*\}) \cong \mathbb{Z}/2\mathbb{Z}$$

is split surjective.

Theorem 2.2.13 (Botvinnik, Ebert, Randal-Williams [BER14])**.** *Let (M, g_0) be a closed Riemannian spin manifold of positive scalar curvature and $n = \dim(M) \geq 6$. For all $l \geq 0$, the α-difference*

$$\alpha\text{–diff}\colon \pi_l(\mathcal{R}^{>}(M), g_0) \longrightarrow \mathrm{KO}^{-n-l-1}(\{*\})$$

is non-trivial whenever the target is non-zero, that is when $l + n + 1 \equiv 0, 1, 2, 4 \mod 8$.

We will use these results to construct non-trivial homotopy groups in the space of initial data sets satisfying the dominant energy condition. The detection of these groups then uses an index difference for initial data sets that will be defined in the next section.

2.3. An index difference for initial data sets

2.3.1. The $\mathrm{Cl}_{n,1}$-linear hypersurface spinor bundle

Throughout this subsection, $(\overline{M}, \overline{g})$ denotes a space- and time-oriented Lorentzian spin manifold. We follow the convention that the metric has signature $(-, +, \ldots, +)$, so that the induced metric g on a spacelike hypersurface $M \subseteq \overline{M}$ is positive definite. The future-pointing unit normal on M will be called e_0. If $\overline{\nabla}$ denotes the Levi-Civita connection of $\overline{g}$ and ∇ the one of g, the second fundamental form with respect to e_0 is the symmetric 2-tensor $k \in \Gamma(T^*M \otimes T^*M)$ defined by $\overline{\nabla}_X Y = k(X, Y)e_0 + \nabla_X Y$ for all $X, Y \in \Gamma(TM)$.

We want to study the bundle obtained by restricting the $\mathrm{Cl}_{n,1}$-linear spinor bundle of $(\overline{M}, \overline{g})$ to the hypersurface $M \subseteq \overline{M}$. Especially, we want to describe it intrinsically, only in terms of the pair (g, k) induced on M. This will be of use later, when defining the $\overline{\alpha}$-difference for initial data sets and comparing it to the α-difference.

The first step is to construct compatible spin structures on M and $\overline{M}$. Fixing a spin structure on $(\overline{M}, \overline{g})$, we obtain a spin structure on (M, g) by pulling back the one from $\overline{M}$:

$$\begin{array}{ccc} P_{\mathrm{Spin}(n)}(M) & \longrightarrow & P_{\mathrm{Spin}_0(n,1)}(\overline{M})_{|M} \\ \downarrow & & \downarrow \\ P_{\mathrm{SO}(n)}(M) & \longrightarrow & P_{\mathrm{SO}_0(n,1)}(\overline{M})_{|M}. \end{array} \tag{2.5}$$

Here, the lower map is given by $(e_1, \dots, e_n) \mapsto (e_0, e_1, \dots, e_n)$, where e_0 is the future-pointing unit normal on M. As the right hand map is a double covering, so is the left hand one, and it suffices to construct a compatible $\mathrm{Spin}(n)$-action. This, we obtain by pulling back the action maps. More explicitly, there is a commutative diagram

$$\begin{array}{ccc} P_{\mathrm{Spin}(n)}(M) \times \mathrm{Spin}(n) & \longrightarrow & P_{\mathrm{Spin}_0(n,1)}(\overline{M})_{|M} \times \mathrm{Spin}_0(n,1) \\ \downarrow & & \downarrow \\ P_{\mathrm{SO}(n)}(M) \times \mathrm{SO}(n) & \longrightarrow & P_{\mathrm{SO}_0(n,1)}(\overline{M})_{|M} \times \mathrm{SO}_0(n,1). \end{array} \tag{2.6}$$

and the desired map is the unique map from its upper-left corner to the upper-left corner of (2.5) building, together with the other action maps, a commutative cube out of (2.5) and (2.6). Note, that this commutative cube shows that $P_{\mathrm{Spin}(n)}(M)$ is not only a $\mathrm{Spin}(n)$-reduction of $P_{\mathrm{SO}(n)}(M)$ but also a reduction of $P_{\mathrm{Spin}_0(n,1)}(\overline{M})_{|M}$ with respect to the inclusion $i\colon \mathrm{Spin}(n) \hookrightarrow \mathrm{Spin}_0(n,1)$.

Next, we study associated bundles. The $\mathrm{Cl}_{n,1}$-*linear spinor bundle*

$$\Sigma_{\mathrm{Cl}}\overline{M} = P_{\mathrm{Spin}_0(n,1)}(\overline{M}) \times_\ell \mathrm{Cl}_{n,1}$$

is defined via the representation induced by left multiplication on $\mathrm{Cl}_{n,1}$:

$$\ell\colon \mathrm{Spin}_0(n,1) \hookrightarrow \mathrm{Cl}_{n,1} \longrightarrow \mathrm{End}(\mathrm{Cl}_{n,1}).$$

As noted above, $P_{\mathrm{Spin}(n)}(M) \to P_{\mathrm{Spin}_0(n,1)}(\overline{M})_{|M}$ is a $\mathrm{Spin}(n)$-reduction. Hence, from the theory of principal bundles (e. g. [Bau14, Satz 2.18]), it follows that

$$\Sigma_{\mathrm{Cl}}\overline{M}_{|M} = P_{\mathrm{Spin}_0(n,1)}(\overline{M})_{|M} \times_\ell \mathrm{Cl}_{n,1} \cong P_{\mathrm{Spin}(n)}(M) \times_{\ell i} \mathrm{Cl}_{n,1}, \tag{2.7}$$

so the bundle $\Sigma_{\mathrm{Cl}}\overline{M}_{|M} \to M$ only depends on the Riemannian manifold (M, g) and its chosen spin structure.

Definition 2.3.1. The bundle $\Sigma_{\mathrm{Cl}}\overline{M}_{|M}$ from above is called $\mathrm{Cl}_{n,1}$-*linear hypersurface spinor bundle* and denoted by $\overline{\Sigma}_{\mathrm{Cl}}M$.

Similarly to the case of the Cl_n-linear spinor bundle, the $\mathrm{Cl}_{n,1}$-linear hypersurface spinor bundle carries a right Clifford multiplication $R\colon \mathbb{R}^{n,1} \to \mathrm{End}(\overline{\Sigma}_{\mathrm{Cl}}M)$ and an even-odd grading $a\colon \overline{\Sigma}_{\mathrm{Cl}}M \to \overline{\Sigma}_{\mathrm{Cl}}M$ as the corresponding notions for $\mathrm{Cl}_{n,1}$ are $\mathrm{Spin}_0(n,1)$-invariant. Despite not being $\mathrm{Spin}_0(n,1)$-invariant, the scalar product $\langle -, - \rangle$ on $\mathrm{Cl}_{n,1}$ for which the basis[2] $(e_{i_1} e_{i_2} \cdots e_{i_k})_{0\leq k\leq n,\, 0\leq i_1<\cdots<i_k\leq n}$ is orthonormal can be extended to $\overline{\Sigma}_{\mathrm{Cl}}M$: Due to (2.7), $\mathrm{Spin}(n)$-invariance of $\langle -, - \rangle$ is sufficient. This scalar product gives rise to a space of L^2-sections $\overline{H} \coloneqq L^2(M, \overline{\Sigma}_{\mathrm{Cl}}M)$, on which R and a define a $\mathrm{Cl}_{n,1}$-Hilbert space structure.

Yet, the trivialization of $T\overline{M}_{|M}$ by e_0 allows us to do better. We immediately obtain the following result:

Proposition 2.3.2. *Setting*

$$\Psi \cdot e_{n+1} \coloneqq e_0 \cdot a(\Psi)$$

for all $\Psi \in \overline{\Sigma}_{\mathrm{Cl}}M$*,* R *extends to a* $\mathrm{Cl}_{n+1,1}$*-multiplication*

$$\tilde{R}\colon \mathbb{R}^{n+1,1} \to \mathrm{End}(\overline{\Sigma}_{\mathrm{Cl}}M).$$

that commutes with left multiplication by any $X \in TM$*. Moreover,* $(\overline{H}, a, \tilde{R})$ *is an ample* $\mathrm{Cl}_{n+1,1}$*-Hilbert space.*

This $\mathrm{Cl}_{n+1,1}$-Hilbert space structure establishes the connection to the space H of L^2-sections of the Cl_n-linear spinor bundle $\Sigma_{\mathrm{Cl}}M$.

Proposition 2.3.3. *The* $\mathrm{Cl}_{n+1,1}$*-Hilbert space* $(\overline{H}, a, \tilde{R})$ *corresponds to the* Cl_n*-Hilbert space* (H, a, R) *under the Morita equivalence described in* (2.2)*.*

Proof. Via this Morita equivalence, the $\mathrm{Cl}_{n+1,1}$-Hilbert space $\overline{H}$ corresponds to the $\mathrm{Cl}_{n,0}$-Hilbert space $\overline{H}_0 = \ker(\tilde{R}(e_0)\tilde{R}(e_{n+1}) - \mathbb{1})$ with the structure obtained by restriction.

Let us look at the endomorphism of $\mathbb{R}^{n,1}$ given by reflection at the hyperplane orthogonal to the line $\mathbb{R}e_0$. Viewing $\mathbb{R}^{n,1}$ as subset of the Clifford algebra $\mathrm{Cl}_{n,1}$, it may be described as

$$\begin{aligned} \mathbb{R}^{n,1} &\longrightarrow \mathbb{R}^{n,1} \\ v &\longmapsto -e_0 v e_0, \end{aligned}$$

since $e_0 e_0 = 1$. This reflection now successively induces an endomorphism: First on the Clifford algebra $\mathrm{Cl}_{n,1}$, then by the associate bundle construction on $\overline{\Sigma}_{\mathrm{Cl}}M$ and finally on

[2] For consistency with Lorentzian geometry, the basis vector of the negative definite part of $\mathbb{R}^{n,1}$ is called e_0 rather than e_{n+1}.

its space of L^2-sections $\overline{H}$. The obtained endomorphism is $\tilde{R}(e_0)\tilde{R}(e_{n+1}) = R(e_0)L(e_0)a$. We are interested in its 1-eigenspace.

On the level of $\mathrm{Cl}_{n,1}$, the 1-eigenspace is given by $\mathrm{Cl}_n \subseteq \mathrm{Cl}_{n,1}$, the subalgebra generated by the fixed vectors $e_1, \ldots, e_n$, whereas the -1-eigenspace is the complement $R(e_0)\,\mathrm{Cl}_n \subseteq \mathrm{Cl}_{n,1}$. This implies that on the level of spinor bundles

$$\overline{\Sigma}_{\mathrm{Cl}} M \supseteq \ker(\tilde{R}(e_0)\tilde{R}(e_{n+1}) - \mathbb{1}) = P_{\mathrm{Spin}(n)} M \times_\ell \mathrm{Cl}_n = \Sigma_{\mathrm{Cl}} M$$

holds. On the level of L^2-sections, we get

$$\overline{H}_0 = L^2(M, \ker(\tilde{R}(e_0)\tilde{R}(e_{n+1}) - \mathbb{1})) = L^2(M, \Sigma_{\mathrm{Cl}} M) = H$$

as required. □

As a consequence of (2.7), the $\mathrm{Cl}_{n,1}$-linear hypersurface spinor bundle possesses two natural connections: On the one hand, the Levi-Civita connection $(\overline{M}, \overline{g})$ induces a connection $\overline{\nabla}$ on $P_{\mathrm{Spin}_0(n,1)} \overline{M}_{|M}$ and $\overline{\Sigma}_{\mathrm{Cl}} M$. On the other hand, as bundle associated to $P_{\mathrm{Spin}(n)} M$, the bundle $\overline{\Sigma}_{\mathrm{Cl}} M$ carries a connection ∇ induced by the Levi-Civita connection of (M, g). They are related by the Weingarten map (also known as shape operator):

Lemma 2.3.4. *For all $X \in TM$ and $\psi \in \Gamma(\overline{\Sigma}_{\mathrm{Cl}} M)$*

$$\overline{\nabla}_X \psi = \nabla_X \psi - \frac{1}{2} e_0 \cdot W(X) \cdot \psi$$

holds, where $W(X) = \overline{\nabla}_X e_0$ is the Weingarten map.

Proof. On the tangent bundle the difference of the connections is given by $\overline{\nabla}_X Y - \nabla_X Y = k(X,Y)e_0$. As $k(X,Y) = -\overline{g}(\overline{\nabla}_X Y - \nabla_X Y, e_0) = -\overline{g}(\overline{\nabla}_X Y, e_0) = \overline{g}(Y, \overline{\nabla}_X e_0) = g(Y, W(X))$ for all $X, Y \in \Gamma(TM)$, the Weingarten map W is the endomorphism associated to the symmetric bilinear form k.

In order to transfer this to the spinor bundle, let $\tilde{\varepsilon}$ be a local section of $P_{\mathrm{Spin}(n)} M$, and $(e_1, \ldots, e_n)$ its projection to $P_{SO(n)} M$. Abusing notation, we denote by $\tilde{\varepsilon}$ also its image in $P_{\mathrm{Spin}_0(n,1)} \overline{M}_{|M}$, projecting to $(e_0, e_1, \ldots, e_n) \in P_{SO_0(n,1)} \overline{M}_{|M}$. As the spinor bundle is associated to these spin principal bundles, we may write a spinor locally as $\psi = [\tilde{\varepsilon}, \tilde{\psi}]$. Using the local formula for the spinorial connection (cf. [BGM05, (2.5)]), we perform the following local calculation:

$$\begin{aligned}\overline{\nabla}_X\psi - \nabla_X\psi &= [\tilde{\varepsilon}, \partial_X\tilde{\psi}] + \frac{1}{2}\sum_{0\le i<j}\varepsilon_i\overline{g}(\overline{\nabla}_X e_i, e_j)e_i\cdot e_j\cdot\psi \\ &\quad - \left([\tilde{\varepsilon}, \partial_X\tilde{\psi}] + \frac{1}{2}\sum_{1\le i<j}\overline{g}(\nabla_X e_i, e_j)e_i\cdot e_j\cdot\psi\right) \\ &= \frac{1}{2}\sum_{0<j}(-1)g(\overline{\nabla}_X e_0, e_j)e_0\cdot e_j\cdot\psi \\ &= -\frac{1}{2}e_0\cdot W(X)\cdot\psi,\end{aligned}$$

where $\varepsilon_i = \overline{g}(e_i, e_i) \in \{\pm 1\}$. □

By the way a, R and $\langle\text{-}\,,\text{-}\rangle$ are defined, it is clear that they are ∇-parallel. The left Clifford multiplication $L\colon T\overline{M}_{|M}\otimes\overline{\Sigma}_{\mathrm{Cl}}M \to \overline{\Sigma}_{\mathrm{Cl}}M$ is ∇-parallel as well, where ∇ is defined on $T\overline{M}_{|M}$ by viewing it as bundle associated to $P_{SO(n)}M$ via the lower map of (2.5). This can be reexpressed by saying that both the restricted left Clifford multiplication $TM\otimes\overline{\Sigma}_{\mathrm{Cl}}M \to \overline{\Sigma}_{\mathrm{Cl}}M$ and the endomorphism $\overline{\Sigma}_{\mathrm{Cl}}M \to \overline{\Sigma}_{\mathrm{Cl}}M$ given by left multiplication with e_0 are ∇-parallel. As a consequence, the extended right Clifford multiplication $\tilde{R}$ is ∇-parallel as well.

With respect to the other connection, the following can be said. a, R and L are $\overline{\nabla}$-parallel. The scalar product $\langle\text{-}\,,\text{-}\rangle$, however, in general is not, as it does not originate from a $\mathrm{Spin}_0(n,1)$-invariant scalar product on $\mathrm{Cl}_{n,1}$. Instead, it satisfies the following formula that follows from ∇-parallelism together with Lemma 2.3.4:

$$\partial_X\langle\phi,\psi\rangle = \langle\overline{\nabla}_X\phi,\psi\rangle + \langle\phi,\overline{\nabla}_X\psi\rangle + \langle e_0\cdot W(X)\cdot\phi,\psi\rangle.$$

2.3.2. $\mathrm{Cl}_{n,1}$-linear Dirac-Witten operator and index difference for initial data sets

As in the previous subsection, let M be a spacelike hypersurface of a space- and time-oriented Lorentzian spin manifold $(\overline{M},\overline{g})$. The Dirac-Witten operator is a kind of Dirac operator on the hypersurface spinor bundle. In the case of classical spinor bundles, it was first defined by Witten [Wit81] in order to give his spinorial proof of the positive mass theorem (cf. [PT82] for a rigorous formulation of the proof) and later studied in more detail by Hijazi and Zhang [HZ03]. We are interested in its $\mathrm{Cl}_{n,1}$-linear version and use it to define a kind of index difference for initial data sets. Furthermore, we compare it to the $\mathrm{Cl}_{n,1}$-linear Dirac operator, which will be of later use.

Definition 2.3.5. The composition

$$\overline{D}\colon \Gamma(\overline{\Sigma}_{\mathrm{Cl}}M) \xrightarrow{\overline{\nabla}} \Gamma(T^*M \otimes \overline{\Sigma}_{\mathrm{Cl}}M) \xrightarrow{\sharp\otimes\mathbb{1}} \Gamma(TM \otimes \overline{\Sigma}_{\mathrm{Cl}}M) \xrightarrow{L} \Gamma(\overline{\Sigma}_{\mathrm{Cl}}M),$$

where L is the left Clifford multiplication, defines the $\mathrm{Cl}_{n,1}$*-linear Dirac-Witten operator*. The composition (with $\overline{\nabla}$ replaced by ∇)

$$D\colon \Gamma(\overline{\Sigma}_{\mathrm{Cl}}M) \xrightarrow{\nabla} \Gamma(T^*M \otimes \overline{\Sigma}_{\mathrm{Cl}}M) \xrightarrow{\sharp\otimes\mathbb{1}} \Gamma(TM \otimes \overline{\Sigma}_{\mathrm{Cl}}M) \xrightarrow{L} \Gamma(\overline{\Sigma}_{\mathrm{Cl}}M)$$

is the $\mathrm{Cl}_{n,1}$*-linear Dirac operator.*

The following lemma justifies the names of these operators. It is a direct consequence of the parallelism discussion at the end of the last subsection.

Lemma 2.3.6. *$\overline{D}$ and D are both $\mathrm{Cl}_{n,1}$-linear with respect to the right Clifford multiplication R and odd with respect to a. Furthermore, D is $\mathrm{Cl}_{n+1,1}$-linear with respect to the extended right Clifford multiplication $\tilde{R}$.*

Lemma 2.3.7. *$\overline{D} = D - \frac{1}{2}\operatorname{tr}(k)L(e_0)$ holds, where $\operatorname{tr} k = \operatorname{tr} W$ is the mean curvature of M in $\overline{M}$. Both D and $\overline{D}$ are formally self-adjoint.*

Proof. For $\psi \in \Gamma(\overline{\Sigma}_{\mathrm{Cl}}M)$ and a local orthonormal frame $e_1, \ldots, e_n$ we perform the following local calculation applying Lemma 2.3.4:

$$\begin{aligned}
\overline{D}\psi - D\psi &= \sum_{i=1}^{n} e_i \cdot (\overline{\nabla}_{e_i} - \nabla_{e_i})\psi \\
&= -\frac{1}{2}\sum_{i=1}^{n} e_i \cdot e_0 \cdot W(e_i) \cdot \psi \\
&= \frac{1}{2}\sum_{i,j=1}^{n} g(W(e_i), e_j) e_i \cdot e_j \cdot e_0 \cdot \psi \\
&= -\frac{1}{2}\sum_{i=1}^{n} g(W(e_i), e_i) e_0 \cdot \psi.
\end{aligned}$$

Here, we used that $g(W(e_i), e_j) = k(e_i, e_j)$ is symmetric in i and j.

The hypersurface spinor bundle $\overline{\Sigma}_{\mathrm{Cl}}M$ together with the connection $\overline{\nabla}$, the (left) Clifford multiplication by TM and scalar product $\langle -, - \rangle$ forms a Clifford bundle, since these structures are compatible as mentioned in the end of the last subsection. Since D is the Dirac operator associated to this Clifford bundle, it is formally self-adjoint (cf. [Roe99, Prop. 3.11]). As left multiplication with e_0 is self-adjoint as well, the same holds true for $\overline{D}$. □

The utility of the Dirac-Witten operator to general relativity results from following observation due to Witten [Wit81, eqs. (24)-(34)]. The proof (cf. also [PT82, Sec. 3]) verbatim applies to the $\mathrm{Cl}_{n,1}$-linear version considered here. We also proved the statement in the introductory chapter, cf. Proposition 1.3.5.

Proposition 2.3.8. *The Dirac-Witten operator satisfies the Schrödinger-Lichnerowicz type formula*

$$\overline{D}^2 = \overline{\nabla}^* \overline{\nabla} + \frac{1}{2}(\rho - e_0 \cdot j^\sharp \cdot),$$

with

$$\begin{aligned} 2\rho &= \mathrm{scal} + \mathrm{tr}(k)^2 - |k|^2 \\ j &= -\mathrm{d}\,\mathrm{tr}(k) + \mathrm{div}\, k. \end{aligned}$$

The Dirac-Witten operator $\overline{D}$ is elliptic, in fact it has the same principal symbol as the Dirac operator D. So it possesses good functional analytic properties, some of which we will state below. From now on, we assume that M is closed.

Corollary 2.3.9. *If the pair (g, k) satisfies the strict dominant energy condition, i. e. if $\rho > |j|$, then $\overline{D}$ has zero kernel.*

Proof. For any smooth section $\psi \in \Gamma(\overline{\Sigma}_{\mathrm{Cl}} M)$ with $\psi \not\equiv 0$

$$\begin{aligned} \|\overline{D}\psi\|^2_{L^2} = (\psi, \overline{D}\overline{D}\psi) &= \|\overline{\nabla}\psi\|^2_{L^2} + \frac{1}{2}(\psi, \rho\psi) - \frac{1}{2}(\psi, e_0 \cdot j^\sharp \cdot \psi) \\ &\geq \frac{1}{2}(\psi, \rho\psi) - \frac{1}{2}(\psi, |j|\psi) = \frac{1}{2}(\psi, (\rho - |j|)\psi) > 0 \end{aligned}$$

holds as $|\langle \psi, e_0 \cdot j^\sharp \cdot \psi \rangle| \leq |j||\psi|^2$. Here (as in the rest of the thesis), $|\text{-}|$ denotes the pointwise norm. The claim follows, since the kernel of the elliptic differential operator $\overline{D}$ consists of smooth sections, see also Proposition 2.3.10 below. □

Proposition 2.3.10. *$\overline{D}$ and D extend to densely defined self-adjoint operators*

$$D, \overline{D} \colon L^2(M, \overline{\Sigma}_{\mathrm{Cl}} M) \supseteq H^1(M, \overline{\Sigma}_{\mathrm{Cl}} M) \to L^2(M, \overline{\Sigma}_{\mathrm{Cl}} M)$$

admitting a spectral decomposition with discrete spectrum and finite dimensional eigenspaces consisting of smooth sections.

Proof. This is true for any formally self-adjoint elliptic differential operator of order one, for example cf. [LM89, Thm. III.5.2 and Thm. III.5.8]. □

Corollary 2.3.11. *If $n = \dim(M) > 0$ and $\overline{H} := L^2(M, \overline{\Sigma}_{\mathrm{Cl}} M)$, then there are well-defined elements*

$$\overline{F} := \frac{\overline{D}}{\sqrt{1+\overline{D}^2}} \in \mathrm{Fred}^{n,1}(\overline{H})$$

and

$$F := \frac{D}{\sqrt{1+D^2}} \in \mathrm{Fred}^{n+1,1}(\overline{H}) \subseteq \mathrm{Fred}^{n,1}(\overline{H}).$$

Furthermore, $\overline{F}$ is invertible if (g,k) satisfies the strict dominant energy condition and F is invertible if g has positive scalar curvature.

Proof. $\overline{H}$ is ample as $\mathrm{Cl}_{n+1,1}$-Hilbert space, so it is ample as $\mathrm{Cl}_{n,1}$-Hilbert space with the restricted Clifford action as well. As $\overline{D}$ is odd and $\mathrm{Cl}_{n,1}$-linear, so is $\overline{F}$. From Proposition 2.3.10 above, we conclude that $\overline{F}$ is a Fredholm operator. The additional condition in the case $n-1 \equiv -1 \mod 4$ is again a consequence of the discussion of the spectral asymptotics in the appendix. Invertibility for (g,k) satisfying the strict dominant energy condition follows from Corollary 2.3.9 and $\operatorname{coker} \overline{F} = \ker \overline{F}$. The argumentation for F is completely analogous. Invertibility here uses the classical Schrödinger-Lichnerowicz formula. □

If the mean curvature $\mathrm{tr}(k)$ is constant, we can relate the spectral decompositions of $\overline{D}$ and D and refine the invertibility result.

Proposition 2.3.12. *The spectral decomposition of D can be written as*

$$D = \sum_{r=0}^{\infty} \lambda_r \pi_{E_r} + \sum_{r=0}^{\infty} (-\lambda_r) \pi_{a(E_r)}$$

where all $\lambda_r > 0$ are pairwise disjoint and π_{E_r} and $\pi_{a(E_r)}$ are the orthogonal projections on the finite dimensional subspaces E_r and $a(E_r)$, respectively. If the mean curvature $\tau := \mathrm{tr}(k)$ is constant, then there are decompositions $F_r \oplus a(F_r) = E_r \oplus a(E_r)$ for all $r \geq 0$ and $K \oplus a(K) = \ker D$ such that the spectral decomposition of $\overline{D}$ is given by

$$\overline{D} = \sum_{r=0}^{\infty} \sqrt{\lambda_r^2 + \frac{1}{4}\tau^2}\, \pi_{F_r} + \sum_{r=0}^{\infty} \left(-\sqrt{\lambda_r^2 + \frac{1}{4}\tau^2} \right) \pi_{a(F_r)} + \frac{1}{2}\tau \pi_K - \frac{1}{2}\tau \pi_{a(K)}$$

In particular, $\overline{D}$ is invertible for all constants $\mathrm{tr}(k) = \tau \neq 0$.

Proof. As a anti-commutes with D, for any eigenvector ϕ to the eigenvalue λ

$$Da(\phi) = -a(D\phi) = -a(\lambda\phi) = -\lambda a(\phi).$$

So $a(\phi)$ is an eigenvector to the eigenvalue $-\lambda$. This implies that the spectral decomposition can be written in the stated form. With the same argument, we observe that the spectral decomposition of $\overline{D}$ to be of that form.

$\tilde{R}$ commutes with D, so the eigenspaces are invariant under $\tilde{R}(v)$ for all $v \in \mathbb{R}^{n+1,1}$. In particular,

$$a(E_r) = \tilde{R}(e_{n+1})a(E_r) = L(e_0)(E_r)$$

for all $r \geq 0$. Thus we can identify E_r with $a(E_r)$ via the map $E_r \to a(E_r),\ \phi \mapsto L(e_0)(\phi)$ and get $E_r \oplus a(E_r) \cong E_r \oplus E_r \cong E_r \otimes \mathbb{R}^2$. Under this identification, by Lemma 2.3.7, the restriction of the Dirac-Witten operator corresponds to

$$\mathbb{1}_{E_r} \otimes \begin{pmatrix} \lambda_r & -\frac{1}{2}\tau \\ -\frac{1}{2}\tau & -\lambda_r \end{pmatrix}.$$

The characteristic polynomial of the 2×2-matrix is $x^2 - \lambda_r^2 - \frac{1}{4}\tau^2$, so it is diagonalizable with eigenvalues $\pm\sqrt{\lambda_r^2 + \frac{1}{4}\tau^2}$. This gives rise to a diagonalization of $\overline{D}_{|E_r \oplus aE_r}$ with the same eigenvalues, and we call the positive eigenspace F_r.

Now, we turn our attention to $\ker D$. As $L(e_0) = \tilde{R}(e_{n+1})a$ anti-commutes with D, $L(e_0)$ operates on $\ker D$. This operation is self-adjoint and squares to $\mathbb{1}_{\ker D}$, so by the spectral theorem $L(e_0)_{|\ker D}$ is diagonalizable and its eigenvalues must be contained in $\{1, -1\}$. Let K be the -1-eigenspace. Then $a(K)$ is the 1-eigenspace. Due to

$$\overline{D}_{|\ker D} = -\frac{1}{2}\tau L(e_0)_{|\ker D},$$

K and $a(K)$ become the $\frac{1}{2}\tau$- and $-\frac{1}{2}\tau$-eigenspaces of $\overline{D}$, respectively. □

Remark 2.3.13. That $\overline{D}$ is invertible for constant mean curvature $\operatorname{tr}(k) \neq 0$, can also be seen directly from the fact that D anti-commutes with $L(e_0)$: As $L(e_0)^2 = \mathbb{1}$,

$$\overline{D}^2 = \left(D - \frac{1}{2}\operatorname{tr}(k)L(e_0)\right)^2 = D^2 + \frac{1}{4}\operatorname{tr}(k)^2\,\mathbb{1}$$

and so $\operatorname{coker}\overline{D} = \ker\overline{D} = 0$.

In the remainder of this section, we want to use the $\mathrm{Cl}_{n,1}$-linear Dirac-Witten operator to define an index difference for initial data sets. For this, let M be closed, spin and of dimension $n > 0$. We need no longer assume that it is embedded into a manifold $\overline{M}$, as we succeeded in expressing all the relevant structures in terms of M and the pair (g, k). In fact, the $\mathrm{Cl}_{n,1}$-linear hypersurface spinor bundle $\overline{\Sigma}_{\mathrm{Cl}}(M, g) \cong \Sigma_{\mathrm{Cl}}(M, g) \otimes_{\mathrm{Cl}_n} \mathrm{Cl}_{n,1}$ depends on the metric g alone, whereas its connection $\overline{\nabla}$ and thus its $\mathrm{Cl}_{n,1}$-linear Dirac-Witten operator $\overline{D}$ is affected by k as well.

In analogy to the case of the α-difference, we need to compare the spaces of L^2-sections of the hypersurface spinor bundles for different initial data sets (g,k). Adopting the notation from Section 2.2.2, there is a bundle map

$$\sqrt{\beta}P^{\nabla}\otimes \mathbb{1}_{\mathrm{Cl}_{n,1}}\colon \Sigma_{\mathrm{Cl}}(M,g_0)\otimes_{\mathrm{Cl}_n}\mathrm{Cl}_{n,1}\to \Sigma_{\mathrm{Cl}}(M,g)\otimes_{\mathrm{Cl}_n}\mathrm{Cl}_{n,1},$$

which induces

$$\overline{\Phi}_g\colon \overline{H}\coloneqq L^2(M,\overline{\Sigma}_{\mathrm{Cl}}(M,g_0))\xrightarrow{\cong} L^2(M,\overline{\Sigma}_{\mathrm{Cl}}(M,g)).$$

This allows to produce a continuous map from initial data sets to the space of Fredholm operators.

Theorem 2.3.14 (cf. [Glö19, Thm. 3.19])**.** *The map*

$$\begin{aligned}(\mathcal{I}(M),\mathcal{I}^{>}(M))&\longrightarrow (\mathrm{Fred}^{n,1}(\overline{H}),G^{n,1}(\overline{H}))\\ (g,k)&\longmapsto \overline{\Phi}_g^{-1}\circ\frac{\overline{D}_{(g,k)}}{\sqrt{1+\overline{D}^2_{(g,k)}}}\circ\overline{\Phi}_g\end{aligned}$$

is well-defined and continuous with respect to the C^1-topology on the space of smooth initial data sets $\mathcal{I}(M)$. In particular, it is continuous if $\mathcal{I}(M)$ carries the C^∞-topology.

Definition 2.3.15. The $\overline{\alpha}$*-difference* is defined by the composition

$$\begin{aligned}\overline{\alpha}\text{–diff}\colon \pi_l(\mathcal{I}^{>}(M),(g_0,k_0))&\cong \pi_{l+1}(\mathcal{I}(M),\mathcal{I}^{>}(M),(g_0,k_0))\\ &\to \pi_{l+1}(\mathrm{Fred}^{n,1}(\overline{H}),G^{n,1}(\overline{H}),\overline{F}_{g_0,k_0})\xrightarrow{\mathrm{ind}}\mathrm{KO}^{n-l}(\{*\}),\end{aligned}$$

where the middle map is the one from Theorem 2.3.14.

In the next section, $\overline{\alpha}$–diff will be compared to the α-difference. The first step will be to establish a comparison map between the space of metrics of positive scalar curvature and the space of initial data sets satisfying the strict dominant energy condition.

2.4. Comparing the index differences

2.4.1. Positive scalar curvature and initial data sets

In the following, M is a closed smooth manifold of dimension $n\geq 2$. The aim of this subsection is to construct a continuous map $\Phi\colon S\mathcal{R}^{>}(M)\longrightarrow \mathcal{I}^{>}(M)$, which will be used later to relate the index differences.

Lemma 2.4.1. *For every $C > 0$, the function*

$$\begin{aligned} \tau\colon \mathcal{R}(M) &\longrightarrow \mathbb{R} \\ g &\longmapsto \sqrt{\frac{n}{n-1}\max\{0, \sup_{x\in M} -\operatorname{scal}^g(x)\} + C} \end{aligned}$$

is continuous.

Proposition 2.4.2. *For any $C > 0$, the following is a well-defined continuous map of pairs:*

$$\begin{aligned} \phi\colon (\mathcal{R}(M), \mathcal{R}^{>}(M)) \times (I, \partial I) &\longrightarrow (\mathcal{I}(M), \mathcal{I}^{>}(M)) \\ (g,t) &\longmapsto \left(g, \frac{\tau(g)}{n} tg\right). \end{aligned}$$

Moreover, its homotopy class $[\phi] \in [(\mathcal{R}(M), \mathcal{R}^{>}(M)) \times (I, \partial I)\,,\, (\mathcal{I}(M), \mathcal{I}^{>}(M))]$ is independent of $C > 0$.

Proof. Continuity directly follows from the lemma above. Moreover, varying the parameter $C > 0$ defines a continuous homotopy between different such maps. Thus, it only remains to prove that $\mathcal{R}(M) \times \partial I \cup \mathcal{R}^{>}(M) \times I$ is mapped into $\mathcal{I}^{>}(M)$. To this aim, we first observe that for a pair of the form $(g, \frac{\tau}{n} g)$ with $\tau \in \mathbb{R}$

$$\begin{aligned} 2\rho &= \operatorname{scal} + \frac{n-1}{n}\tau^2 \\ j &= \frac{1-n}{n}\operatorname{grad}\tau = 0 \end{aligned}$$

holds. Hence, such a pair fulfills the strict dominant energy condition if and only if

$$\tau^2 > -\frac{n}{n-1}\operatorname{scal}.$$

But by definition of the function τ, this is the case for $\left(g, \pm\frac{\tau(g)}{n} g\right)$, which shows that $\mathcal{R}(M) \times \partial I$ maps into $\mathcal{I}^{>}(M)$. Moreover, the condition is automatically satisfied if g has positive scalar curvature, so $\mathcal{R}^{>}(M) \times I$ is sent to $\mathcal{I}^{>}(M)$ as well. ☐

Proposition 2.4.3. *Let $C > 0$ and $h \in \mathcal{R}(M)$ a Riemannian metric. Then the composition*

$$\Phi\colon S\mathcal{R}^{>}(M) \longrightarrow \mathcal{R}(M) \times \partial I \cup \mathcal{R}^{>}(M) \times I \xrightarrow{\phi} \mathcal{I}^{>}(M),$$

where the first map is given by

$$[g,t] \longmapsto \begin{cases} ((-2t-1)h + 2(1+t)g, -1) & t \in [-1, -\frac{1}{2}] \\ (g, 2t) & t \in [-\frac{1}{2}, \frac{1}{2}] \\ ((2t-1)h + 2(1-t)g, 1) & t \in [\frac{1}{2}, 1], \end{cases}$$

is a well-defined, continuous map. Its homotopy class is independent of $C > 0$ and $h \in \mathcal{R}(M)$.

Proof. By the previous proposition, we just need to study the first map: Plugging in $t = \pm\frac{1}{2}$, we see that the different definitions agree on the intersections, and for the special values $t = \pm 1$ we observe that the result is independent of g, i. e. the map descends to the suspension. This shows well-definedness. Continuity can now be checked on each domain of definition, where it is obvious. Moreover, this map continuously depends on $h \in \mathcal{R}(M)$, so by connectedness of $\mathcal{R}(M)$, its homotopy class is independent of h. □

Corollary 2.4.4. *The inclusion $\mathcal{R}^{>}(M) \to \mathcal{I}^{>}(M)$, $g \mapsto (g, 0)$ is null-homotopic. In particular, if there exists a metric $g_0 \in \mathcal{R}^{>}(M)$, the induced map on homotopy groups $\pi_l(\mathcal{R}^{>}(M), g_0) \to \pi_l(\mathcal{I}^{>}(M), (g_0, 0))$ is the zero-map for all l.*

Proof. Using the map defined above, we get a factorization of the inclusion map as follows

$$\mathcal{R}^{>}(M) \hookrightarrow C\,\mathcal{R}^{>}(M) \hookrightarrow S\,\mathcal{R}^{>}(M) \xrightarrow{\Phi} \mathcal{I}^{>}(M),$$

where the first two maps are the canonical inclusions of a space into the its cone and of the cone into the suspension as upper half. As cones are contractible, the composition is null-homotopic. □

This shows that we cannot find non-trivial elements of homotopy groups in the space initial data with strict dominant energy condition by simply considering the space of positive scalar curvature metrics as subspace. However, the map Φ defined above allows for a better construction: In the next subsection, we will show that under certain conditions the composition

$$\pi_l(\mathcal{R}^{>}(M), g_0) \xrightarrow{\text{Susp}} \pi_{l+1}(S\,\mathcal{R}^{>}(M), [g_0, 0]) \xrightarrow{\Phi_*} \pi_{l+1}(\mathcal{I}^{>}(M), (g_0, 0))$$

has non-trivial image.

2.4.2. Main theorem

Let M be a closed spin manifold of dimension $n \geq 2$. The aim of this subsection is to relate the $\overline{\alpha}$-difference for initial data sets $\overline{\alpha}\text{–diff}\colon \pi_l(\mathcal{I}^{>}(M), (g_0, 0)) \to \mathrm{KO}^{-n-l}(\{*\})$, where g_0 is a metric of positive scalar curvature, to the classical α-difference using the map from Proposition 2.4.3. This will lead to a non-triviality result for $\pi_l(\mathcal{I}^{>}(M), (g_0, 0))$. Moreover, the same argument shows that the $\overline{\alpha}$-difference detects that $\mathcal{I}^{>}(M)$ has least two connected components if $\alpha(M) \neq 0$.

Theorem 2.4.5 (Theorems A and B, Theorem 2.1.2)**.** *1. If M carries a metric g_0 of positive scalar curvature, then for all $l \geq 0$, the diagram*

$$\begin{array}{ccccc} \pi_l(\mathcal{R}^>(M), g_0) & \xrightarrow{\mathrm{Susp}} & \pi_{l+1}(S\mathcal{R}^>(M), [g_0, 0]) & \xrightarrow{\Phi_*} & \pi_{l+1}(\mathcal{I}^>(M), (g_0, 0)) \\ & \searrow^{\alpha\text{–diff}} & & \swarrow_{\overline{\alpha}\text{–diff}} & \\ & & \mathrm{KO}^{-n-l-1}(\{*\}) & & \end{array}$$

commutes. Here, Susp *is the suspension homomorphism and Φ is the map from Proposition 2.4.3.*

2. For any metric g_0,

$$\overline{\alpha}\text{–diff}\left(\left(g_0, -\frac{1}{n}\tau(g_0)g_0\right), \left(g_0, \frac{1}{n}\tau(g_0)g_0\right)\right) = \alpha(M) \in \mathrm{KO}^{-n}(\{*\}),$$

where τ is defined as in Lemma 2.4.1.

Proof. For the first part, we start by exploring the effect of the upper composition. The claim is that

$$\begin{array}{ccc} \pi_l(\mathcal{R}^>(M), g_0) \overset{\mathrm{Susp}}{\longrightarrow} \pi_{l+1}(S\mathcal{R}^>(M), [g_0, 0]) & \xrightarrow{\Phi_*} & \pi_{l+1}(\mathcal{I}^>(M), (g_0, 0)) \\ \cong\uparrow & & \cong\uparrow \\ \pi_{l+1}(\mathcal{R}(M), \mathcal{R}^>(M), g_0) & \xrightarrow{\phi_*} & \pi_{l+2}(\mathcal{I}(M), \mathcal{I}^>(M), (g_0, 0)) \\ \downarrow & & \downarrow \\ [(D^{l+1}, S^l), (\mathcal{R}(M), \mathcal{R}^>(M))] & \xrightarrow{\phi_*} & [(D^{l+1}, S^l) \times (I, \partial I), (\mathcal{I}(M), \mathcal{I}^>(M))] \end{array} \tag{2.8}$$

commutes, where the middle and the lower map are both induced by

$$\begin{aligned} \phi\colon (\mathcal{R}(M), \mathcal{R}^>(M)) \times (I, \partial I) &\longrightarrow (\mathcal{I}(M), \mathcal{I}^>(M)) \\ (g, t) &\longmapsto \left(g, \frac{\tau(g)}{n} tg\right). \end{aligned}$$

Note that ϕ preserves the base point, if the base point of $(D^{l+1}, S^l) \times (I, \partial I)$ is chosen to be $(*, 0)$ when $*$ is the base point of S^l, so the middle map is well-defined. The lower square obviously commutes. For the upper square, we start with a class $[g] \in \pi_l(\mathcal{R}^>(M), g_0)$. Then the preimage under the boundary isomorphism is represented by

$$\begin{aligned} \tilde{g}\colon (D^{l+1}, S^l, *) &\longrightarrow (\mathcal{R}(M), \mathcal{R}^>(M), g_0) \\ rx &\longmapsto (1-r)g_0 + rg(x) \end{aligned}$$

for $r \in [0,1]$ and $x \in S^l$. Applying the horizontal map and restricting to the boundary yields the class of

$$\begin{aligned} (\partial(D^{l+1} \times I), (*,0)) &\longrightarrow (\mathcal{I}^{>}(M), (g_0, 0)) \\ (x,t) &\longmapsto \left(\tilde{g}(x), -\frac{\tau(\tilde{g}(x))}{n} t\tilde{g}(x)\right). \end{aligned}$$

Using the homeomorphism

$$\begin{aligned} (S(S^l), [*,0]) &\cong (\partial(D^{l+1} \times I), (*,0)) \\ [x,t] &\mapsto \begin{cases} (2(1+t)x, -1) & t \in [-1, -\frac{1}{2}] \\ (x, 2t) & t \in [-\frac{1}{2}, \frac{1}{2}] \\ (2(1-t)x, 1) & t \in [\frac{1}{2}, 1], \end{cases} \end{aligned}$$

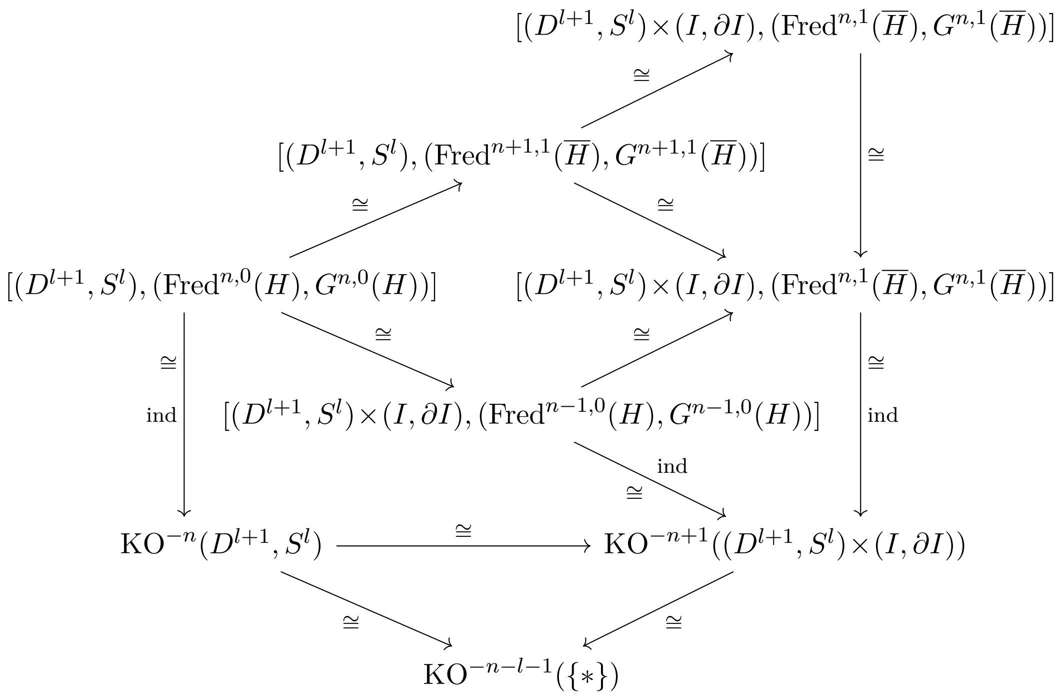
this precisely gives the formula for $\Phi \circ Sg$ (cf. Proposition 2.4.3).

The core of the proof is showing that the following diagram commutes:

$$\begin{array}{ccc} [(D^{l+1}, S^l), (\mathcal{R}(M), \mathcal{R}^{>}(M))] & \xrightarrow{\phi_*} & [(D^{l+1}, S^l) \times (I, \partial I), (\mathcal{I}(M), \mathcal{I}^{>}(M))] \\ \downarrow & & \downarrow \\ [(D^{l+1}, S^l), (\mathrm{Fred}^{n,0}(H), G^{n,0}(H))] & & [(D^{l+1}, S^l) \times (I, \partial I), (\mathrm{Fred}^{n,1}(\overline{H}), G^{n,1}(\overline{H}))] \\ \searrow \cong & & \cong \nearrow \\ & [(D^{l+1}, S^l), (\mathrm{Fred}^{n+1,1}(\overline{H}), G^{n+1,1}(\overline{H}))]. & \end{array} \tag{2.9}$$

Here, the first lower map is associated to the Morita equivalence between $\mathrm{Cl}_{n,0}$- and $\mathrm{Cl}_{n+1,1}$-Hilbert spaces, that is the first map in Proposition 2.2.7. This uses that H and $\overline{H}$ correspond to each other under this Morita equivalence according to Proposition 2.3.3. The second lower map is the Bott map (cf. Theorem 2.2.6), associated to $e = -e_{n+1}$.

Before doing so, let us show that



(2.10)

commutes. Here the central diamond is formed by the Bott maps associated to $e = e_n$ as well as maps induced by Morita equivalences. The topmost right hand map is induced by a $\mathrm{Cl}_{n,1}$-Hilbert space isomorphism to be defined later. Notice that the right hand vertical composition is the index map, which follows from the invariance of the index map under $\mathrm{Cl}_{n,1}$-Hilbert space isomorphisms. So stitching the diagrams (2.8)-(2.10) together, we obtain the diagram from the first claim.

Moreover, setting $l = -1$, the commutative diagram composed of (2.9) and (2.10) implies the second assertion. Then $(D^{l+1}, S^l) = (\{*\}, \varnothing)$ and the upper left corner of the diagram is the one-point set $[\{*\}, \mathcal{R}(M)]$. Now the left hand vertical composition maps this point to the α-index of M, whereas the composition through the upper right corner is seen to map it to the $\overline{\alpha}$-difference of the π_0-class from the claim.

The lower half of (2.10) commutes by the definition of the index map, cf. (2.1) and (2.3). The middle diamond commutes as well, this is obvious from the way its constituting maps are defined. We are left with the upper triangle. Note first that we are dealing with two different $\mathrm{Cl}_{n,1}$-Hilbert space structures on $\overline{H}$: Since the map from the center upwards is the Bott map for $e = -e_{n+1}$, the $\mathrm{Cl}_{n,1}$-structure is the one obtained by forgetting the $\tilde{R}(e_{n+1})$-action, whereas in the lower Hilbert space, we forget the multiplication by e_n.

These are connected by the $\mathrm{Cl}_{n,1}$-Hilbert space isomorphism

$$\begin{aligned} U\colon \overline{H} &\longrightarrow \overline{H} \\ \phi &\mapsto \frac{1}{\sqrt{2}}\tilde{R}(e_{n+1})\tilde{R}(e_n+e_{n+1}). \end{aligned}$$

Indeed, $a \in B(\overline{H})$ corresponds via U to $a = UaU^{-1}$, $\tilde{R}(e_i)$ to $\tilde{R}(e_i)$ for $i < n$ and $\tilde{R}(e_n)$ to $\tilde{R}(e_{n+1})$. The right hand map in the triangle is defined to be the map induced by $\mathrm{Fred}^{n,1}(\overline{H}) \ni F \mapsto UFU^{-1}$. As the analogous map on $\mathrm{Fred}^{n+1,1}(\overline{H})$ is the identity, the diagram relating the Bott maps gets the shape of a triangle rather than a square. Its commutativity follows from

$$\begin{aligned} U\tilde{R}(-e_{n+1})U^{-1} &= \frac{1}{2}\tilde{R}(e_{n+1})\tilde{R}(e_n+e_{n+1})\tilde{R}(-e_{n+1})\tilde{R}(e_n+e_{n+1})\tilde{R}(e_{n+1}) \\ &= \frac{1}{2}(\tilde{R}(e_{n+1})+\tilde{R}(e_n)+\tilde{R}(e_n)-\tilde{R}(e_{n+1})) = \tilde{R}(e_n). \end{aligned}$$

It only remains prove that (2.9) commutes. The first two maps of the lower composition map $[g] \in [(D^{l+1}, S^l), (\mathcal{R}(M), \mathcal{R}^{>}(M))]$ to the class of

$$\begin{aligned} (D^{l+1}, S^l) &\longrightarrow (\mathrm{Fred}^{n+1,1}(\overline{H}), G^{n+1,1}(\overline{H})) \\ x &\longmapsto \Phi_{g(x)}^{-1}\frac{D_{g(x)}}{\sqrt{1+D_{g(x)}^2}}\Phi_{g(x)}. \end{aligned}$$

This is because it restricts to the correct map on $H = \ker(\tilde{R}(e_0)\tilde{R}(e_{n+1}) - \mathbb{1}) \subseteq \overline{H}$, i. e. the Cl_n-Hilbert space associated to $\overline{H}$ via the Morita equivalence (2.2). The remaining map sends it to the class of

$$\begin{aligned} (D^{l+1}, S^l) \times (I, \partial I) &\longrightarrow (\mathrm{Fred}^{n,1}(\overline{H}), G^{n,1}(\overline{H})) \\ (x,t) &\longmapsto \Phi_{g(x)}^{-1}\frac{D_{g(x)}}{\sqrt{1+D_{g(x)}^2}}\Phi_{g(x)} - t\tilde{R}(e_{n+1})a \\ &= \Phi_{g(x)}^{-1}\left(\frac{D_{g(x)}}{\sqrt{1+D_{g(x)}^2}} - tL(e_0)\right)\Phi_{g(x)}. \end{aligned}$$

In contrast, the result of the upper composition is represented by

$$\begin{aligned} (D^{l+1}, S^l) \times (I, \partial I) &\longrightarrow (\mathrm{Fred}^{n,1}(\overline{H}), G^{n,1}(\overline{H})) \\ (x,t) &\longmapsto \Phi_{g(x)}^{-1}\frac{\overline{D}_{(g(x),k(x,t))}}{\sqrt{1+\overline{D}_{(g(x),k(x,t))}^2}}\Phi_{g(x)} \end{aligned}$$

with $k(x,t) = \frac{\tau(g(x))}{n}tg(x)$.

Remembering that $\overline{D}_{(g,k)} = D_g - \frac{1}{2}\tau L(e_0)$, these do not look too much different, and we show that the following is a well-defined homotopy between them:

$$\begin{aligned}(D^{l+1}, S^l) \times (I, \partial I) \times [0,1] &\to (\mathrm{Fred}^{n,1}(\overline{H}), G^{n,1}(\overline{H}))\\ (x,t,s) &\mapsto \Phi_{g(x)}^{-1}\left(a_{(x,t,s)}(D_{g(x)})D_{g(x)} - b_{(x,t,s)}(D_{g(x)})tL(e_0)\right)\Phi_{g(x)}\end{aligned}$$

for

$$\begin{aligned} a_{(x,t,s)}(\lambda) &= \frac{s}{\sqrt{1+\lambda^2}} + \frac{1-s}{\sqrt{1+\lambda^2+\frac{1}{4}t^2\tau(g(x))}}\\ b_{(x,t,s)}(\lambda) &= s + \frac{(1-s)\frac{1}{2}\tau(g(x))}{\sqrt{1+\lambda^2+\frac{1}{4}t^2\tau(g(x))}}. \end{aligned}$$

As this operator family is obtained by linearly interpolating between two continuous operator families, it is again continuous. So it remains to see that its target is indeed $(\mathrm{Fred}^{n,1}(\overline{H}), G^{n,1}(\overline{H}))$. It is clear, that all the operators are bounded, self-adjoint, odd and $\mathrm{Cl}_{n,1}$-linear. To show that the operator $F_{(x,t,s)}$ associated to (x,t,s) is Fredholm, we use the spectral decomposition of $D_{g(x)}$ from Proposition 2.3.12: The restriction of $F_{(x,t,s)}$ to $E_r \oplus a(E_r) \cong E_r \otimes \mathbb{R}^2$ is given by

$$\mathbb{1}_{E_r} \otimes \begin{pmatrix} a_{(x,t,s)}(\lambda_r)\lambda_r & -b_{(x,t,s)}(\lambda_r)t \\ -b_{(x,t,s)}(\lambda_r)t & -a_{(x,t,s)}(\lambda_r)\lambda_r \end{pmatrix}.$$

This is diagonalizable with eigenvalues $\pm\sqrt{a_{(x,t,s)}(\lambda_r)^2\lambda_r^2 + b_{(x,t,s)}(\lambda_r)^2t^2}$. Note that due to $\sqrt{a_{(x,t,s)}(\lambda_r)^2\lambda_r^2 + b_{(x,t,s)}(\lambda_r)^2t^2} \geq a_{(x,t,s)}(\lambda_r)|\lambda_r|$, their absolute values, for any $t \in I$ and $s \in [0,1]$, are bounded away from zero by

$$\frac{\lambda_0}{\sqrt{1+\lambda_0^2+\frac{1}{4}\tau(g(x))}} > 0,$$

where $\lambda_0 > 0$ denotes the smallest positive eigenvalue of $D_{g(x)}$. A similar consideration as in Proposition 2.3.12 shows that $F_{(x,t,s)}$ restricted to $\ker(D_{g(x)})$ is diagonalizable as well, with eigenvalues $\pm b_{(x,t,s)}(0)t$. Putting this together, we find that $F_{(x,t,s)}$ has finite dimensional kernel, co-kernel and closed image (for this, the boundedness away from zero is needed). Furthermore, $F_{(x,t,s)}$ is invertible if $D_{g(x)}$ is invertible or $t > 0$, one of which is the case on $\partial(D^{l+1} \times I)$.

In the case $n - 1 \equiv -1 \mod 4$ one more tiny bit of thought is necessary. The space self-adjoint $\mathrm{Cl}_{n,1}$-linear Fredholm operators has three components (cf. [AS69]): Those F for which $\omega_{n,1}F\iota$ is essentially positive, those for which it is essentially negative and the rest. As for $s = 0$ (or $s = 1$) all operators $F_{(x,t,s)}$ fall into the last category, the same has to be true for all $s \in [0,1]$ by continuity. □

2.4.3. Corollaries and examples

In this final subsection, we explore some of the consequences of Theorem 2.1.2. We start by combining the first part of this main theorem with the non-triviality results for the α-difference from Theorems 2.2.12 and 2.2.13. This gives the following conclusions:

Corollary 2.4.6. *If M is a closed spin manifold of dimension $n \geq 6$ that carries a metric g_0 of positive scalar curvature, then for all $l \geq 1$ with $l + n \equiv 1, 2 \mod 8$ the $\overline{\alpha}$-difference for initial data sets $\overline{\alpha}\text{–diff}\colon \pi_l(\mathcal{I}^>(M), (g_0, 0)) \to \mathrm{KO}^{-n-l}(\{*\}) \cong \mathbb{Z}/2\mathbb{Z}$ is split surjective.*

Corollary 2.4.7. *If M is a closed spin manifold of dimension $n \geq 6$ that carries a metric g_0 of positive scalar curvature, then for all $l \geq 1$ the $\overline{\alpha}$-difference for initial data sets $\overline{\alpha}\text{–diff}\colon \pi_l(\mathcal{I}^>(M), (g_0, 0)) \to \mathrm{KO}^{-n-l}(\{*\})$ is non-trivial whenever the target is non-zero, that is when $l + n \equiv 0, 1, 2, 4 \mod 8$.*

In particular, under the assumptions of the corollaries above, $\pi_l(\mathcal{I}^>(M), (g_0, 0)) \neq 0$, which shows the first part of Corollary 2.1.3. Note that the main theorem Theorem 2.1.2 provides an explicit construction of the non-trivial elements, provided that in $\pi_{l-1}(\mathcal{R}^>(M), g_0)$ the non-trivial elements detected by the α-difference are known.

Particularly much is known about connected components of the space of positive scalar curvature metrics. If there are several components of $\mathcal{R}^>(M)$ that can be distinguished by their α-index difference, the main theorem provides us with non-trivial loops in $\mathcal{I}^>(M)$.

Example 2.4.8. As explained in [LM89, Ex. IV.7.5], there is a sequence of positive scalar curvature metrics $g_k \in \mathcal{R}^>(S^7)$, $k \in \mathbb{Z}$, on the (standard) 7-sphere with the following property: If $V_k \to S^4$ is the real vector bundle with Euler number $\chi = 1$ and Pontrjagin number $p_1 = 4 + 896k$, then, after identifying its sphere bundle $\partial D(V_k)$ with S^7, the metric g_k extends to a positive scalar curvature metric $\hat{g}_k$ on the disk bundle $D(V_k)$ collared along the boundary. All these metrics g_k lie in different path components of $\mathcal{R}^>(S^7)$. More precisely, $\alpha\text{–diff}(g_k, g_l) = l - k$. This can be seen as follows: According to the main result of [Ebe17], $\alpha\text{–diff}(g_k, g_l)$ is equal to the index of the Cl_8-linear Dirac operator on $S^7 \times \mathbb{R}$ equipped with a metric of the form $\hat{h} = h_t + \mathrm{d}t^2$, where $h_t = g_k$ for $t \leq -1$ and $h_t = g_l$ for $t \geq 1$. Under complexification and Bott periodicity $\mathrm{KO}^{-8}(\{*\}) \cong K^{-8}(\{*\}) \cong K^0(\{*\}) \cong \mathbb{Z}$, this corresponds to the index of the classical Dirac operator on $(S^7 \times \mathbb{R}, \hat{h})$. We compute this using the cut-and-paste version version of relative index theorem (cf. [Bun95, Thm. 1.2]). We take the double of $(D(V_k), \hat{g}_k)$ and cut it along the former boundary $\partial D(V_k)$. We also cut $(S^7 \times \mathbb{R}, \hat{h})$ along $S^7 \times \{-1\}$. Then, using the identification $S^7 \cong \partial D(V_k)$, we glue them together in the other way that respects the boundary orientations. For the indices of the associated Dirac operators, we obtain:

$$\begin{aligned}\operatorname{index}(S^7\times\mathbb{R},\hat{h}) = &-\operatorname{index}(D(V_k)\cup(-D(V_k)),\hat{g}_k\cup\hat{g}_k)\\ &+\operatorname{index}((S^7\times(-\infty,-1])\cup(-D(V_k)),\hat{h}\cup\hat{g}_k)\\ &+\operatorname{index}(D(V_k)\cup(S^7\times[-1,\infty)),\hat{g}_k\cup\hat{h}).\end{aligned}$$

Here, the two first indices vanish since the metric has positive scalar curvature. Proceeding similarly at $S^7\times\{1\}$, we get

$$\begin{aligned}\operatorname{index}(S^7\times\mathbb{R},\hat{h}) &= \operatorname{index}(D(V_k)\cup(S^7\times[-1,\infty)),\hat{g}_k\cup\hat{h})\\ &= \operatorname{index}(D(V_k)\cup(S^7\times[-1,1])\cup(-D(V_l)),\hat{g}_k\cup\hat{h}\cup\hat{g}_l).\end{aligned}$$

The latter is of course equal to $\hat{A}(D(V_k)\cup(-D(V_l)))$. Using cut-and-paste once more, the claimed equality with $k-l$ reduces to the statement $\hat{A}(D(V_k)\cup D^8)=k$ from [LM89, Ex. IV.7.5].

Now, the suspension construction from Section 2.4.1 produces an element in $\pi_1(\mathcal{I}^>(S^7))$ out of the π_0-class defined by g_k and g_l. If $k\neq l$, the main theorem shows that its $\overline{\alpha}$-difference is $k-l\neq 0$, hence it is non-trivial. Tracking through the definitions, it is represented by a loop that is concatenated from the following four segments: In the first segment the initial data sets are all of the form $(g,\frac{1}{n}\tau(g)g)$ and the metric g interpolates between g_l and g_k. The second segment is a linear interpolation between $(g_k,\frac{1}{n}\tau(g_k)g_k)$ and $(g_k,-\frac{1}{n}\tau(g_k)g_k)$. In particular, the first component of the initial data set is fixed throughout the second segment. The third piece consists of initial data sets $(g,-\frac{1}{n}\tau(g)g)$, where g runs from g_k to g_l. The final segment is again an interpolation within the second component only, running from $(g_l,-\frac{1}{n}\tau(g_l)g_l)$ to $(g_l,\frac{1}{n}\tau(g_l)g_l)$. We have thus found a rather explicit infinite family of non-trivial elements in $\pi_1(\mathcal{I}^>(S^7))$.

Concerning path components of $\mathcal{I}^>(M)$, we can say the following. It is easy to see that all pairs $(g,\frac{1}{n}\tau(g)g)$, $g\in\mathcal{R}(M)$, lie in the same path component of $\mathcal{I}^>(M)$. The same is true for all pairs of the form $(g,-\frac{1}{n}\tau(g)g)$. If M carries a positive scalar curvature metric, then the components of $(g,\frac{1}{n}\tau(g)g)$ and $(g,-\frac{1}{n}\tau(g)g)$ are actually the same. If on the other hand $\alpha(M)\neq 0$ (and hence M does not admit positive scalar curvature), the second part of the main theorem of this chapter shows that these belong to different path component as their $\overline{\alpha}$-difference is non-zero. This immediately implies the second part of Corollary 2.1.3. It is the purpose of the next chapter (following [Glö23a]) to show that we can still distinguish these two path components if M does not carry a positive scalar curvature metric due to the (also index-theoretic) enlargeability obstruction. In special cases, we may be able to distinguish more components.

Example 2.4.9. Consider the connected sum $M=K3\#K3$, which we decompose into $(K3\setminus D^4)\cup(S^3\times[-L,L])\cup(K3\setminus D^4)$, $L>0$. Choose a metric g on M that is symmetric under the involution σ switching the two $K3$-surfaces and reflecting the $[-L, L]$-component of the connecting neck. We assume moreover that g is the standard product metric on the neck $S^3 \times [-L, L]$. Observe that we can make the neck longer, i. e. L larger, without changing $\tau(g)$. Since $\alpha(M) = 2\alpha(K3) \neq 0$, we already know that $(g, -\frac{1}{n}\tau(g)g)$ and $(g, \frac{1}{n}\tau(g)g)$ lie in different path components.

We now consider the following initial data set (g, k). On the left $K3 \setminus D^4$, it is given by $(g, -\frac{1}{n}\tau(g)g)$. On the right $K3 \setminus D^4$, it is $(g, \frac{1}{n}\tau(g)g)$. Along the neck, we take $(g, \frac{t}{nL}\tau(g)g)$ at $(x, t) \in S^3 \times [-L, L]$. By the definition of τ, the so obtained initial data set satisfies the strict dominant energy condition along the two $K3$-parts. Since the metric on $S^3 \times [-L, L]$ has positive scalar curvature, the estimate

$$\rho - |j| \geq \mathrm{scal}^g - \frac{n-1}{nL}\tau(g)$$

shows that this initial data set also satisfies the strict dominant energy condition in the neck region as long as L is chosen to be large enough. Thus we have constructed an element $(g, k) \in \mathcal{I}^>(M)$ and we claim that it is part of neither of two components mentioned before. Assume that there were a path $t \mapsto (g_t, k_t)$ in $\mathcal{I}^>(M)$ connecting (g, k) to $(g, \frac{1}{n}\tau(g)g)$, say. Then $(\sigma^* g_t, -\sigma^* k_t)$ would be a path in $\mathcal{I}^>(M)$ connecting it also with $(g, -\frac{1}{n}\tau(g)g)$, contradiction.

It might be worth noting that the component of the pair (g, k) constructed above may be detected by the $\overline{\alpha}$-difference. Namely, it is not hard to see that it is additive in the sense

$$\begin{aligned}\overline{\alpha}\text{–diff}\left(\left(g, -\frac{1}{n}\tau(g)g\right), (g,k)\right) &+ \overline{\alpha}\text{–diff}\left((g,k), \left(g, \frac{1}{n}\tau(g)g\right)\right) \\ &= \overline{\alpha}\text{–diff}\left(\left(g, -\frac{1}{n}\tau(g)g\right), \left(g, \frac{1}{n}\tau(g)g\right)\right) = 2\alpha(K3).\end{aligned}$$

Moreover, replacing the endomorphism $L(e_0)$ by $-L(e_0)$ the Dirac-Witten operators defining $\overline{\alpha}$–diff$((g, -\frac{1}{n}\tau(g)g), (g, -k))$ turn on the nose into the Dirac-Witten operators defining $\overline{\alpha}$–diff$((g, \frac{1}{n}\tau(g)g), (g, k))$. Hence,

$$\begin{aligned}\overline{\alpha}\text{–diff}\left((g,k), \left(g, \frac{1}{n}\tau(g)g\right)\right) &= -\,\overline{\alpha}\text{–diff}\left(\left(g, \frac{1}{n}\tau(g)g\right), (g,k)\right) \\ &= \overline{\alpha}\text{–diff}\left(\left(g, -\frac{1}{n}\tau(g)g\right), (g,-k)\right) \\ &= \overline{\alpha}\text{–diff}\left(\left(g, -\frac{1}{n}\tau(g)g\right), (g,k)\right),\end{aligned}$$

where the last step uses the invariance of the $\overline{\alpha}$-difference under the diffeomorphism σ. We obtain

$$\overline{\alpha}\text{–diff}\left((g,k), \left(g, \frac{1}{n}\tau(g)g\right)\right) = \overline{\alpha}\text{–diff}\left(\left(g, -\frac{1}{n}\tau(g)g\right), (g,k)\right) = \alpha(K3) \neq 0.$$

This result can probably also be obtained with the help of a suitable relative index theorem.

3. An enlargeability obstruction for spacetimes with both big bang and big crunch

Given a spacelike hypersurface M of a time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$, the pair (g, k) consisting of the induced Riemannian metric g and the second fundamental form k is known as initial data set. In this chapter, we study the space of all initial data sets (g, k) on a fixed closed n-manifold M that are subject to a strict version of the dominant energy condition. In this space we characterize big bang and big crunch initial data by $\operatorname{tr} k > 0$ and $\operatorname{tr} k < 0$, respectively. It is easy to see that these belong to the same path-component when M admits a positive scalar curvature metric. Conversely, it was observed in the previous chapter (following [Glö24b]) that this is not the case when the existence of a positive scalar curvature metric on M is obstructed in terms of the index in $\mathrm{KO}^{-n}(\{*\})$. In the present chapter we extend this disconnectedness result to Gromov-Lawson's enlargeability obstruction. In particular, for orientable closed 3-manifolds M, we can tell precisely when big bang and big crunch initial data belong to the same path-component. In the context of general relativity theory, this result may be interpreted as excluding the existence of certain globally hyperbolic spacetimes with both a big bang and a big crunch singularity.

3.1. Introduction

One of the guiding questions of mathematical relativity is the one about the ultimate fate of the universe. One answer to this is provided by the celebrated singularity theorem of Hawking (cf. [HE73, Sec. 8.2], Theorem 1.4.1). It states that under suitable initial conditions on a Cauchy hypersurface and assuming the strong energy condition, the spacetime possesses a big crunch singularity. More precisely there is an upper bound for the proper time of every causal curve starting on the Cauchy hypersurface. Via time-reversal the theorem also provides sufficient conditions for a big bang singularity, where there is an upper bound for the age of any observer at the moment specified by the Cauchy hypersurface. A new question arises: Given that a spacetime has a big bang singularity, can it have a big crunch singularity as well?

Before we can give a partial answer to this question, we have to make it mathematically precise. We will address this question in terms of initial data sets. Recall that a if time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$ is *globally hyperbolic*, then by a theorem of Bernal and Sánchez [BS05] (Theorem 1.2.4) there is a diffeomorphism $\Phi\colon M \times \mathbb{R} \xrightarrow{\cong} \overline{M}$ defining a foliation into spacelike Cauchy hypersurfaces. The leaves $M_t = \Phi(M \times \{t\})$ come equipped with a Riemannian metric g_t induced by $\overline{g}$ and a second fundamental form $k_t \in \Gamma(\bigodot^2 T^*M)$ (taken with respect to the future unit normal e_0). The pair (g_t, k_t) is called *initial data set* on M_t. The C^∞-space of all initial data sets, i. e. pairs of metric and symmetric 2-tensor, on a fixed manifold M will be denoted by $\mathcal{I}(M)$. So by pullback we obtain a smooth family $(\Phi^* g_t, \Phi^* k_t)$ of initial data sets on M and in particular a continuous path $\mathbb{R} \to \mathcal{I}(M)$.

Not all spacetimes are physically relevant. Besides causality conditions such as global hyperbolicity, energy conditions are typically assumed. A prominent one – the *dominant energy condition* – informally speaking says that matter cannot move faster than light. From a mathematical point of view this is a curvature condition for $(\overline{M}, \overline{g})$, namely its Einstein curvature should satisfy $\mathrm{Ein}(V, W) \geq 0$ for all future-causal vectors V and W. For the purpose of this chapter its main feature is that it restricts the initial data sets that can be found on spacelike hypersurfaces. One defines the *energy density* $\rho = \mathrm{Ein}(e_0, e_0)$ and the *momentum density* $j = \mathrm{Ein}(e_0, \text{-}) \in \Omega^1(M)$ and observes that the dominant energy condition implies $\rho \geq |j|$. It should be noted that ρ and j can be expressed solely in terms of (g, k) via the *constraint equations* (Proposition 1.2.6)

$$
\begin{aligned}
2\rho &= \mathrm{scal}^g + (\operatorname{tr} k)^2 - |k|^2 \\
j &= \operatorname{div} k - \mathrm{d} \operatorname{tr} k.
\end{aligned}
$$

Thus the following definition (Definition 1.2.12) makes sense.

Definition 3.1.1. An initial data set (g, k) on a manifold M is said to satisfy the *dominant energy condition* (=DEC) if $\rho \geq |j|$ for ρ and j defined by (1.2). It satisfies the *strict dominant energy condition* if $\rho > |j|$.

So if $(\overline{M}, \overline{g})$ satisfies the dominant energy condition, then the pairs (g_t, k_t) constructed above satisfy DEC. If $(\overline{M}, \overline{g})$ satisfies the dominant energy condition in some strict sense, then the path $\mathbb{R} \to \mathcal{I}(M)$ even takes values in the subspace $\mathcal{I}^>(M) \subseteq \mathcal{I}(M)$ of initial data sets satisfying strict DEC.

Before coming to the main result of this chapter, we briefly turn our attention back to Hawking's singularity theorem. As mentioned above it requires a certain initial condition on the Cauchy hypersurface. For a big bang the condition is $\operatorname{tr}(k) \geq \lambda > 0$; in the big crunch case it is $\operatorname{tr}(k) \leq -\lambda < 0$. This leads to the following definition.

Definition 3.1.2. An initial data set (g, k) on a manifold M is called *big bang initial data set* if it satisfies strict DEC and $\operatorname{tr}(k) \geq \lambda$ for some $\lambda > 0$. It is *big crunch initial data set* if it satisfies strict DEC and $\operatorname{tr}(k) \leq -\lambda < 0$ for some $\lambda > 0$.

In Lemma 3.2.2 we will show: When the manifold M is compact, then all big bang initial data belong to the same path-component of $\mathcal{I}^{>}(M)$, which we call C^{+}. Analogously, there is a path-component C^{-} of $\mathcal{I}^{>}(M)$ that contains all big crunch initial data. Now the question becomes whether there is a path in $\mathcal{I}^{>}(M)$ from big bang initial data to big crunch initial data, i. e. whether $C^{+} = C^{-}$.

Theorem 3.1.3 (Theorem C, Theorem 3.5.4)**.** *Let M be an orientable closed connected 3-manifold. Then the following are equivalent:*

1. *$M \cong M_1 \sharp \cdots \sharp M_k$, where each M_i is either S^3/Γ for a lattice $\Gamma \subseteq \mathrm{SO}(4)$ or $S^2 \times S^1$.*
2. *M admits a metric of positive scalar curvature.*
3. *$C^{+} = C^{-}$, so big bang and big crunch initial data sets belong to the same path-component of $\mathcal{I}^{>}(M)$.*

The equivalence of the first two items has been long-since known – up to the eventual solution of the geometrization conjecture (cf. [LM89, Sec. IV.6]). Furthermore, it is easy to see that if g is a positive scalar curvature (=PSC) metric, then $t \mapsto (g, -tg)$ defines a path in $\mathcal{I}^{>}(M)$ from big bang to big crunch initial data. The new contribution thus lies in showing that the vast number of orientable closed connected 3-manifolds that are not in the list of item 1 do not admit such a path.

This provides us with plenty of examples of closed 3-manifolds that do not appear as Cauchy hypersurfaces in globally hyperbolic spacetimes that satisfy the dominant energy condition in a strict sense and have both a big bang and a big crunch singularity. For instance, this applies to quotients of the 3-dimensional torus, which is still considered to be within the range of physical observations of our universe, cf. [GKW22]. A similar conclusion was already known before by work of Gerhardt [Ger83] (cf. also [AG23, Cor. 6.20]). His approach uses the whole spacetime dominant energy condition, which is a stronger assumption than the dominant energy condition for initial data sets. Also, the spinorial and index theoretic methods employed here might be fruitful in the future to say even more about the topology of $\mathcal{I}^{>}(M)$.

The proof of Theorem 3.1.3 relies on the following observation. In the situation of item 2 we do not only know *that* M does not admit a PSC metric, we also know *why* it does not: The reason is that all these manifolds are *enlargeable*. We will show that enlargeability does not only obstruct positive scalar curvature, but also implies $C^{+} \neq C^{-}$.

The concept of enlargeability was first introduced by Gromov and Lawson [GL80]. There are several versions, we will start with the following simple version.

Definition 3.1.4. A smooth map $f\colon (M,g) \to (N,h)$ between Riemannian manifolds is *ε-contracting* for some $\varepsilon > 0$ if $|\mathrm{d}f| \leq \varepsilon$. A closed Riemannian manifold (M,g) of dimension n is called *enlargeable* if for all $\varepsilon > 0$ there exists a Riemannian covering $(M',g') \to (M,g)$ admitting an ε-contracting map $(M',g') \to (S^n, g_{Std})$ that is constant outside a compact set and of non-zero degree. It is called *compactly enlargeable* if the coverings may be chosen to be finite-sheeted.

Despite being geometrically defined, the notion turns out to be homotopy invariant (in particular, independent of g), and thus provides a link between geometry and topology. Moreover, the class of enlargeable manifolds is rich. It contains all closed manifolds that carry a metric of non-positive sectional curvature, especially tori, and further examples may be constructed through products and direct sums. In the aforementioned article [GL80] Gromov and Lawson show that spin manifolds do not carry a PSC metric when they are compactly enlargeable. Later they extended this both to (not necessarily compactly) enlargeable spin manifolds and, in dimensions $n \leq 7$, to compactly enlargeable (not necessarily spin) manifolds (cf. [GL83]). In the present chapter we prove an initial data version of Gromov and Lawson's result:

Theorem 3.1.5 (Theorem D, Theorem 3.5.3)**.** *If M is an enlargeable spin manifold, then $\mathcal{I}^>(M)$ is not path-connected. More precisely, the path-components C^+ and C^- are distinct.*

In fact, we prove this for the more general notion of $\hat{A}$-area-enlargeability, cf. Definition 3.5.1. Similar to what has been stated above, we get a conclusion about the topology of a globally hyperbolic spacetime with both big bang and big crunch singularities and subject to a strict version of the dominant energy condition (also in dimension other than $3+1$): Its Cauchy hypersurface is not an enlargeable spin manifold. For example, it cannot be the quotient of a torus.

The theorem relies on the connection between PSC and strict DEC, which becomes apparent in the time-symmetric case $k \equiv 0$, where $\rho = \frac{1}{2}\,\mathrm{scal}^g$ and $j = 0$. Positive scalar curvature has been a vast field of study over the last decades, see e. g. [Ros07]. Apart from minimal hypersurface techniques, most of the results in this area were obtained by Dirac operator methods. In the previous chapter, we were able to transfer some of these to the dominant energy setting. Namely, it was shown independently in [BER14] and [CSS18] that the C^∞-space $\mathcal{R}^>(M)$ of PSC metrics on a fixed closed spin manifold M of dimension $n \geq 6$ has infinitely many non-trivial homotopy groups if it is non-empty. These non-trivial elements in $\pi_i(\mathcal{R}^>(M))$ give rise to non-trivial elements in

$\pi_{i+1}(\mathcal{I}^{>}(M))$, $i \geq 0$. The proof relied on a suspension construction – associating to a PSC metric a path of initial data sets subject to strict DEC – and the computation of a family index of (Clifford-linear) *Dirac-Witten operators*. These operators are a certain zero-order perturbation of Dirac operators. Their classical version dates back to Witten's spinorial proof of the positive mass theorem [Wit81] (cf. [PT82] for a more detailed account of the proof).

In some way, the comparison arguments between $\pi_i(\mathcal{R}^{>}(M))$ and $\pi_{i+1}(\mathcal{I}^{>}(M))$ still apply in the case $i = -1$. Here, the statement is the following: If a closed spin manifold M has non-zero α-index $\alpha(M) \neq 0$, which is a well-known obstruction for the existence of PSC metrics on M, then $C^+ \neq C^-$, so in particular $\mathcal{I}^{>}(M)$ is not path-connected. However, this result becomes void in the physically relevant spatial dimension $n = 3$, as in this case $\alpha(M)$ is an element of $\mathrm{KO}^{-3}(\{*\}) = 0$. This drawback is overcome in the above theorem, which – as was pointed out – has strong three-dimensional implications.

It would be desirable, especially from the point of view of general relativity, to extend the results from strict DEC to (plain) DEC. This is similar to the passage from positive scalar curvature to non-negative scalar curvature performed by Schick and Wraith [SW21]. For DEC, some ideas in that direction can be found in the recent article [AG23] of Ammann and the author. Further future extensions might include the discussion of non-compact Cauchy hypersurfaces, especially the asymptotically Euclidean setting. All of this is beyond the scope of this thesis.

Finally, let us briefly discuss the idea of the proof in the case where M is even-dimensional and compactly enlargeable. From the (compact) enlargeability, Gromov and Lawson construct a sequence of (finite-sheeted) coverings $M_i \to M$ and complex vector bundles $E_i \to M_i$. They come equipped with a hermitian metric and a metric connection such that the curvatures R^{E_i} tend to 0 for $i \longrightarrow \infty$ and $\hat{A}(M_i, E_i) \neq 0$ for all $i \in \mathbb{N}$. Here, $\hat{A}(M_i, E_i) = \int_{M_i} \hat{A}(TM_i) \wedge \mathrm{ch}(E_i)$ is the E_i-twisted $\hat{A}$-genus, which is equal to the index of the E_i-twisted Dirac operator D^{E_i} by the Atiyah-Singer index theorem. In our initial data setting, we need a suitable analog of D^{E_i}. This is provided by the twisted Dirac-Witten operator $\overline{D}^{E_i} = D^{E_i} - \frac{1}{2}\,\mathrm{tr}(k) e_0 \cdot$ defined on a certain E_i-twisted spinor bundle. This operator is associated to an initial data set (g, k) on M_i (in our case, it is pulled back from M) and satisfies the Schrödinger-Lichnerowicz type formula

$$\left(\overline{D}^{E_i}\right)^2 \psi = \overline{\nabla}^* \overline{\nabla} \psi + \frac{1}{2}(\rho - e_0 \cdot j^\sharp \cdot)\psi + \mathcal{R}^{E_i} \psi.$$

Hence, if the strict dominant energy condition $\rho > |j|$ holds, then $\overline{D}^{E_i}$ is invertible for large enough $i \in \mathbb{N}$.

Choosing a metric g and some sufficiently large $\tau > 0$, we obtain big bang and big crunch initial data sets $(g, \tau g)$ and $(g, -\tau g)$, respectively. We use Dirac-Witten operators to define the *twisted index difference* $\mathrm{ind\text{-}diff}^{E_i}((g, -\tau g), (g, \tau g))$, a spectral-flow-like invariant associated to a path γ of initial data sets but depending only on its end points $(g, -\tau g)$ and $(g, \tau g)$. As a consequence of the Schrödinger-Lichnerowicz type formula, the index difference vanishes for large $i \in \mathbb{N}$ if the path γ may be chosen to lie within $\mathcal{I}^{>}(M)$. On the other hand, an index theorem shows that $\text{ind-diff}^{E_i}((g, -\tau g), (g, \tau g)) = \hat{A}(M_i, E_i) \neq 0$ for all $i \in \mathbb{N}$. Thus it is not possible to connect $(g, -\tau g)$ and $(g, \tau g)$ by a path in $\mathcal{I}^{>}(M)$.

The chapter is structured as follows: In section 2 we start by describing how initial data sets and Dirac-Witten operators arise in a Lorentzian setup. In section 3, we construct the twisted index difference of Dirac-Witten operators using a framework laid out by Ebert [Ebe18]. This is a bit technical, as we also deal with non-compact manifolds so that we can later apply it to infinite covers $M_i \to M$ as well. In fact, we construct a *relative twisted index difference*, relative meaning that it depends on the "difference" of two twist bundles $E_i^{(0)}$ and $E_i^{(1)}$ that coincide outside a compact set. Section 4 is devoted to the proof of the (relative) index theorem $\text{ind-diff}^{E_i}((g, -\tau g), (g, \tau g)) = \hat{A}(M_i, E_i)$. In the last section we put the arguments together to prove the main theorems of this chapter.

3.2. Lorentzian manifolds, initial data sets and Dirac-Witten operators

Within this section, we want to recall several notions from Lorentzian geometry and thereby fix certain notations. We start with a discussion of initial data sets and dominant energy condition, making some of the statements from the introduction more precise. Afterwards we turn our attention to hypersurface spinor bundles and Dirac-Witten operators. Although the rest of the chapter could be understood to a very large extent without knowing how these arise in the Lorentzian setup, they appear more naturally in this context. Thus the Lorentzian setting provides a better understanding.

Let us consider a Lorentzian manifold $(\overline{M}, \overline{g})$ of dimension $n+1$. We use the signature convention such that a generalized orthonormal basis $e_0, e_1, \ldots, e_n$ satisfies $\overline{g}(e_0, e_0) = -1$ and $\overline{g}(e_i, e_i) = 1$ for $1 \leq i \leq n$. In particular, the induced metric g on a spacelike hypersurface $M \subseteq \overline{M}$ will be Riemannian. In this context, we will denote by e_0 a (generalized) unit normal on M. Usually, we will assume that $\overline{M}$ is time-oriented and then we agree that e_0 is future-pointing. Apart from the induced metric, M will carry an induced second fundamental form k with respect to e_0 that we define by

$$\overline{\nabla}_X Y - \nabla_X Y = k(X, Y) e_0$$

for all vectors fields $X, Y \in \Gamma(TM)$. Here, $\overline{\nabla}$ denotes the Levi-Civita connection of $(\overline{M}, \overline{g})$ and ∇ is the one of the hypersurface (M, g). Pairs (g, k) consisting of a Riemannian metric g and a symmetric 2-tensor k will be called *initial data set.* Hence, the above procedure provides an induced initial data set (g, k) on a spacelike hypersurface of a time-oriented Lorentzian manifold.

The *dominant energy condition* is the condition that for all future-causal vectors V, W of $(\overline{M}, \overline{g})$, the Einstein tensor $\mathrm{Ein} = \mathrm{ric} - \frac{1}{2}\,\mathrm{scal}\,\overline{g}$ satisfies $\mathrm{Ein}(V, W) \geq 0$. Equivalently, for every future-causal vector V the metric dual of $\mathrm{Ein}(V, \text{-})$ is required to be past-causal. Applying this to the future-pointing unit normal e_0 of a spacelike hypersurface M, we get $\rho \geq |j|$ for $\rho = \mathrm{Ein}(e_0, e_0)$ and $j = \mathrm{Ein}(e_0, \text{-}) \in \Omega^1(M)$. As explained in the introduction, ρ and j are completely determined by the induced initial data set (g, k) on M via the constraint equations (1.2). This gives rise to the notion of (strict) dominant energy condition for initial data sets (cf. Definition 1.2.12).

Definition 3.2.1. When M is compact, we denote by $\mathcal{I}(M)$ the space of all initial data sets (g, k) on M, equipped with the C^∞-topology of uniform convergence, and by $\mathcal{I}^>(M)$ the subspace of those initial data sets satisfying the strict dominant energy condition.

For the remaining discussion about initial data sets, we will assume that M is compact and of dimension $n \geq 2$. Recall from Definition 3.1.2 that in this case a *big bang initial data set* is some $(g, k) \in \mathcal{I}^>(M)$ with $\mathrm{tr}(k) > 0$. Likewise, a *big crunch initial data set* shows $\mathrm{tr}(k) < 0$.

Lemma 3.2.2. *All big bang initial data sets belong to the same path-component of* $\mathcal{I}^>(M)$ *and this component is non-empty.*

Proof. Given an initial data set (g, k), we decompose $k = \lambda g + A$, where $\lambda \in C^\infty(M)$ and $\mathrm{tr}(A) = 0$. Using $|k|^2 = n\lambda^2 + |A|^2$, the strict dominant energy condition becomes

$$n(n-1)\lambda^2 > -\,\mathrm{scal}^g + |A|^2 + 2|(n-1)\mathrm{d}\lambda - \mathrm{div}\,A|.$$

If (g, k) is big bang initial data, i. e. $\lambda > 0$, adding a positive constant to λ will preserve this inequality. Moreover, increasing it yields a path to an initial data set, where even

$$n(n-1)\lambda^2 > -\,\mathrm{scal}^g + |A|^2 + 2(n-1)|\mathrm{d}\lambda| + 2|\,\mathrm{div}\,A|$$

holds. Now we can deform to an initial data set $(g, \tau g)$ with constant $\tau \in \mathbb{R}$: Take τ to be the maximum of λ and use convex combination. But all initial data sets $(g, \tau g)$ with $\tau \in \mathbb{R}$ lie in the same path-component of $\mathcal{I}^>(M)$ as the space of metrics is contractible and the condition on τ, namely $n(n-1)\tau^2 > -\min \mathrm{scal}^g$, continuously depends on g (even in C^2-topology). Furthermore, if g is some metric, then taking $\tau > 0$ with $n(n-1)\tau^2 > -\min \mathrm{scal}^g$ yields a big bang initial data set $(g, \tau g)$ on M. □

By reversing the sign of k, the analogous statement also holds for big crunch initial data sets.

Definition 3.2.3. The path-component of $\mathcal{I}^{>}(M)$ that contains all big bang initial data sets will be denoted by C^{+}; the one that contains all big crunch initial data will be denoted by C^{-}.

We are interested in the question, whether $C^{+} = C^{-}$. There are two observations. Firstly, when there exists a positive scalar curvature metric g on M, then the path $[-1,1] \to \mathcal{I}^{>}(M)$, $t \mapsto (g, tg)$ shows that $C^{+} = C^{-}$. Secondly, we have a certain stability property:

Lemma 3.2.4. *Let M be a compact manifold such that the path-components C^{+} and C^{-} of $\mathcal{I}^{>}(M)$ coincide. Then for any compact manifold N the path-components C^{+} and C^{-} of $\mathcal{I}^{>}(M \times N)$ agree as well.*

Proof. Let us first consider pairs of product form, i. e. $(g_M + g_N, k_M + k_N) \in \mathcal{I}(M \times N)$ for $(g_M, k_M) \in \mathcal{I}(M)$ and $(g_N, k_N) \in \mathcal{I}(N)$. For these we compute

$$\begin{aligned} 2\rho &= \operatorname{scal}^{g_M} + \operatorname{scal}^{g_N} + (\operatorname{tr}^{g_M}(k_M) + \operatorname{tr}^{g_N}(k_N))^2 - |k_M|^2_{g_M} - |k_N|^2_{g_N} \\ &= 2\rho_M + 2\rho_N + 2\operatorname{tr}^{g_M}(k_M)\operatorname{tr}^{g_N}(k_N) \end{aligned}$$

and

$$\begin{aligned} j &= \operatorname{div}^{g_M}(k_M) + \operatorname{div}^{g_N}(k_N) - d(\operatorname{tr}^{g_M}(k_M)) - d(\operatorname{tr}^{g_N}(k_N)) \\ &= j_M + j_N. \end{aligned}$$

Hence, we obtain

$$|j|^2 = |j_M|^2_{g_M} + |j_N|^2_{g_N} \leq (|j_M|_{g_M} + |j_N|_{g_N})^2$$

and thus

$$\rho - |j| \geq (\rho_M - |j_M|_{g_M}) + (\rho_N - |j_N|_{g_N}) + \operatorname{tr}^{g_M}(k_M)\operatorname{tr}^{g_N}(k_N). \tag{3.1}$$

Now, by assumption, for sufficiently large $\tau > 0$ the pairs $(g_M, -\tau g_M)$ and $(g_M, \tau g_M)$ can be connected by a path $t \mapsto \gamma_M(t) = (g_M(t), k_M(t))$ in $\mathcal{I}^{>}(M)$. By the discussion following Definition 3.2.1, we may assume that τ is a constant. As the interval $[-1,1]$ is compact, $\rho_M - |j_M|_{g_M}$ attains a positive minimum. Replacing γ_M by $\widetilde{\gamma_M} = (C^{-2} g_M, C^{-1} k_M)$ for some suitably chosen $C > 0$, we may assume that this minimum is larger than $-\frac{1}{2}\min_{p\in N} \operatorname{scal}^{g_N}(p)$. This is due to the fact, that in this rescaling $\widetilde{\rho_M} = C^2 \rho_M$ and $\left|\widetilde{j_M}\right|_{\widetilde{g_M}} = C^2 |j_M|_{g_M}$, indicating the rescaled quantities by $\widetilde{\cdot}$.

The required path from $(g_M + g_N, -\tau(g_M + g_N))$ to $(g_M + g_N, \tau(g_M + g_N))$ can be easily pieced together from three segments:

$$\begin{aligned}[-1,1] &\longrightarrow \mathcal{I}^{>}(M \times N) \\ t &\longmapsto \begin{cases} (g_M + g_N, -\tau g_M + (2t+1)\tau g_N) & t \in [-1, -\frac{1}{2}] \\ (g_M(2t) + g_N, k_M(2t)) & t \in [-\frac{1}{2}, \frac{1}{2}] \\ (g_M + g_N, \tau g_M + (2t-1)\tau g_N) & t \in [\frac{1}{2}, 1]. \end{cases}\end{aligned}$$

In the first section, both $\mathrm{tr}^{g_M}(k_M)$ and $\mathrm{tr}^{g_N}(k_N)$ are non-positive, so its product is non-negative. Furthermore, for the pair $(g_N, (2t+1)g_N)$ with $t \in [-1, -\frac{1}{2}]$ we have

$$\rho_N - |j_N|_{g_N} = \frac{1}{2}(\mathrm{scal}^{g_N} + \dim(N)(\dim(N) - 1)(2t+1)^2\tau^2) \geq \frac{1}{2} \min_{p \in N} \mathrm{scal}^{g_N}(p).$$

By choice of the rescaling, we have that $(g_M, -\tau g_M) = \gamma_M(-1)$ satisfies $\rho_M - |j_M|_{g_M} > -\frac{1}{2} \min_{p \in N} \mathrm{scal}^{g_N}(p)$, and so (3.1) shows that the first section lies in $\mathcal{I}^{>}(M \times N)$. The same argument applies for the last section. In the middle section, the last term in (3.1) is zero, $\rho_N - |j_N|_{g_N} \geq \frac{1}{2} \min_{p \in N} \mathrm{scal}^{g_N}(p)$, and so by our choice of rescaling, the pair is in $\mathcal{I}^{>}(M \times N)$ for all $t \in [-\frac{1}{2}, \frac{1}{2}]$. □

We conclude this section by a brief discussion of the Dirac-Witten operator. Assume that $(\overline{M}, \overline{g})$ is a space- and time-oriented Lorentzian *spin* manifold. Let $\Sigma\overline{M} \to \overline{M}$ be the classical spinor bundle of $(\overline{M}, \overline{g})$, i. e. the spinor bundle associated to an irreducible representation of $\mathbb{C}\mathrm{l}_{n,1}$. A short summary of spin geometry in the semi-Riemannian setting can be found in [BGM05]. Restricting this bundle to the spacelike hypersurface M yields the induced *hypersurface spinor bundle*[1] $\Sigma\overline{M}_{|M} \to M$. The Levi-Civita connection of $(\overline{M}, \overline{g})$ induces a connection $\overline{\nabla}$ on $\Sigma\overline{M}_{|M}$, and the associated Dirac type operator $\overline{D}\psi = \sum_{i=1}^{n} e_i \cdot \overline{\nabla}_{e_i}\psi$, where $(e_1, \ldots, e_n)$ is a local orthonormal frame of TM, is known as *Dirac-Witten operator*. It was first observed by Witten [Wit81] that it is linked to the dominant energy condition by a Schrödinger-Lichnerowicz type formula (Proposition 1.3.5):

$$\overline{D}^2\psi = \overline{\nabla}^*\overline{\nabla}\psi + \frac{1}{2}(\rho - e_0 \cdot j^\sharp \cdot)\psi.$$

Here, ψ is any smooth section of $\Sigma\overline{M}$ and the star denotes the formal adjoint. The formula shows that the strict dominant energy condition implies that $\overline{D}$ is positive and hence invertible.

It now turns out, that $\overline{D}$ does not depend on the whole Lorentzian manifold $(\overline{M}, \overline{g})$, but only on the induced initial data set (g, k) on M. In fact, denoting by e_0 the future-pointing unit normal as above, we obtain an induced $\mathrm{Spin}(n)$-principal bundle on M by

[1] Here, we do not use the notation $\overline{\Sigma}M$ for the hypersurface spinor bundle established in the introductory chapter. Instead, we reserve it for the next section, where it essentially denotes the $\mathbb{Z}/2\mathbb{Z}$-graded hypersurface spinor bundle introduced below.

pulling back the $\mathrm{Spin}_0(n,1)$-principal bundle $P_{\mathrm{Spin}_0(n,1)}\overline{M}_{|M}$ along the inclusion of frame bundles

$$\begin{aligned} P_{\mathrm{SO}(n)}M &\longrightarrow P_{\mathrm{SO}_0(n,1)}\overline{M}_{|M} \\ (e_1,\dots,e_n) &\longmapsto (e_0,e_1,\dots,e_n). \end{aligned}$$

In the following, we restrict our attention to the case where $n=2m$ is even. In order to make a statement about $\mathcal{I}^>(M)$ for odd-dimensional manifolds M later, we will invoke Lemma 3.2.4 with $N=S^1$. If n is even, there are two irreducible (ungraded) complex $\mathbb{C}\mathrm{l}_{n,1}$-representations and either restricts to the unique irreducible (ungraded) representation of $\mathbb{C}\mathrm{l}_n$ along the inclusion [2] $\mathbb{C}\mathrm{l}_n \hookrightarrow \mathbb{C}\mathrm{l}_{n,1}$. On associated bundles, this yields an isomorphism $\Sigma\overline{M}_{|M}\cong\Sigma M$, where $\Sigma M\to M$ is the classical spinor bundle on M. The difference between the two representations results in the fact that in one case multiplication by e_0 is given by $\omega=i^m e_1\cdots e_n$, whereas in the other case it is given by $-\omega$.

Apart from $\overline{\nabla}$, there is another canonical connection on $\Sigma\overline{M}_{|M}\cong\Sigma M$: the one induced by the Levi-Civita connection of (M,g), called ∇. Those two connections differ by a term depending on the second fundamental form, namely

$$\overline{\nabla}_X\psi=\nabla_X\psi-\frac{1}{2}e_0\cdot k(X,\text{-})^\sharp\cdot\psi$$

for $\psi\in\Gamma(\Sigma M)$ and $X\in TM$. As a consequence, we obtain

$$\overline{D}\psi=D\psi-\frac{1}{2}\,\mathrm{tr}(k)e_0\cdot\psi$$

for all $\psi\in\Gamma(\Sigma M)$, where D denotes the Dirac operator on ΣM.

Unlike the Dirac operator D, which – for even $n=2m$ – is odd with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading defined by the volume element ω of ΣM, there is no natural grading which is compatible with the Dirac-Witten operator in this case. As for index theory, however, such gradings are very useful, we consider a $\mathbb{Z}/2\mathbb{Z}$-graded version instead. We do so by replacing $\Sigma\overline{M}$ with $\Sigma_{\mathrm{gr}}\overline{M}$, the bundle associated to the unique irreducible $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}\mathrm{l}_{n,1}$-representation[3]. This representation is obtained by summing the two irreducible ungraded $\mathbb{C}\mathrm{l}_{n,1}$-representations and taking an appropriate grading. More precisely, starting with the irreducible representation with $i^m e_0\cdots e_n=\mathbb{1}$, the other one can be obtained by replacing $e_0\cdot$ with $-e_0\cdot$, keeping the multiplication by the other basis vectors the same. We take the direct sum of these two representations and equip it with the grading given by the (eigenspaces of the) grading operator

$$\iota=\begin{pmatrix}0 & \omega\\ \omega & 0\end{pmatrix} \tag{3.2}$$

[2]Some authors identify $\mathbb{C}\mathrm{l}_n$ with the subalgebra $\mathbb{C}\mathrm{l}^0_{n,1}$, the even part of $\mathbb{C}\mathrm{l}_{n,1}$, instead. This is used in [BGM05].

[3]The grading on the Clifford algebra is given by the even-odd-grading. Note that then a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}\mathrm{l}_{n,1}$-representation is nothing else than an ungraded $\mathbb{C}\mathrm{l}_{n,2}$-representation.

for $\omega = i^m e_1 \cdots e_n$ as above. As a consequence, there is an isomorphism $\Sigma_{\mathrm{gr}}\overline{M}_{|M} \cong \Sigma M \oplus \Sigma M$ such that the grading is given by (3.2) and multiplication by e_0 is given by

$$e_0 \cdot \; = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}. \tag{3.3}$$

This isomorphism is to be understood in the way that it respects Clifford multiplication by vectors $X \in TM$, i. e.

$$X \cdot \; = \begin{pmatrix} X\cdot & 0 \\ 0 & X\cdot \end{pmatrix}. \tag{3.4}$$

It should be noted that doubling the spinor bundle creates an additional symmetry:

$$c_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{3.5}$$

defines an odd $\mathbb{C}\mathrm{l}_1$-action on $\Sigma_{\mathrm{gr}}\overline{M}_{|M}$, which has the property that it commutes with the Dirac-Witten operator. For this reason, the Dirac-Witten operator on this bundle is called $\mathbb{C}\mathrm{l}_1$*-linear Dirac-Witten operator*.

3.3. The twisted index difference for Dirac-Witten operators

Let M be a connected spin manifold of even dimension $n = 2m$. We will not assume that M is compact. The necessity for also considering non-compact manifolds – although the main result is only concerned with compact ones – strives from the fact, that we will later be looking at coverings and we do not want to assume them to be finite. The aim of this section is to define a homotopy invariant for a path $\gamma\colon (I, \partial I) \to (\mathcal{I}(M), \mathcal{I}^{>\mathcal{R}^E+c}(M))$, where $I = [-1, 1]$ and $c > 0$. This invariant will have the property of being zero if a representative of its homotopy class takes values only in $\mathcal{I}^{>\mathcal{R}^E+c}(M)$. The way it is constructed is quite similar to the $\overline{\alpha}$-difference (cf. Definition 2.3.15), yet it differs in the way that instead of real Clifford-linear spinors, it uses complex spinors with coefficients in a twist bundle. This is needed to make use of enlargeability.

To be able to define this also when M is non-compact, we need some extra care in the definition of $\mathcal{I}(M)$: It will be the space of pairs (g, k), where g is a *complete* Riemannian metric and $k \in \Gamma(T^*M \otimes T^*M)$ is symmetric. It carries the topology of uniform convergence of all derivatives on compact sets, i. e. the weak C^∞-topology in the sense of Hirsch [Hir76, Ch. 2]. By $\mathcal{I}^{>\mathcal{R}^E+c}(M)$ we denote the subspace of those pairs satisfying the dominant energy condition in the stricter sense $\frac{1}{2}(\rho - |j|) > |\mathcal{R}^E| + c$ (cf. (1.2)), where $\mathcal{R}^E$ is the curvature endomorphism of a twist bundle E.

The twist bundle E arises as a direct sum of two complex vector bundles $E_0, E_1 \to M$ with hermitian metrics and metric connections, such that outside a compactum K

they can be identified by an isometric and connection preserving bundle isomorphism $\Psi\colon E_{0|M\setminus K} \to E_{1|M\setminus K}$. If M is already compact, then, of course, we may take $K = M$ and the compatibility condition becomes void.

The construction of the desired homotopy invariant begins as follows: For a chosen (complete) Riemannian metric g on M, let ΣM be the classical complex spinor bundle associated to the (topological) spin structure on M. We consider the double spinor bundle $\overline{\Sigma}M := \Sigma M \oplus \Sigma M$ with its direct sum hermitian metric. This carries a (self-adjoint) $\mathbb{Z}/2\mathbb{Z}$-grading ι and a (skew-adjoint) odd $\mathbb{C}l_1$-action c_1 given by (3.2) and (3.5), respectively. Moreover, $\overline{\Sigma}M$ admits an operator $e_0\cdot$ defined by (3.3), which is self-adjoint, odd and commutes with the $\mathbb{C}l_1$-action. If D is the Dirac operator of the double spinor bundle, then the $\mathbb{C}l_1$-linear Dirac-Witten operator

$$\overline{D} = D - \frac{1}{2}\operatorname{tr}^g(k)e_0\cdot$$

is formally self-adjoint, odd and commutes with the $\mathbb{C}l_1$-action. We mean to associate a suitable (relative) index to it.

We now bring in the twist bundles. From E_0 and E_1, we form the sum $E = E_0 \oplus E_1$, which we endow with the $\mathbb{Z}/2\mathbb{Z}$-grading

$$\eta = \begin{pmatrix} \mathbb{1}_{E_0} & 0 \\ 0 & -\mathbb{1}_{E_1} \end{pmatrix}.$$

On the twist bundle $\overline{\Sigma}^E M := \overline{\Sigma}M \otimes E$, we have a $\mathbb{Z}/2\mathbb{Z}$-grading $\iota \otimes \eta$ and an odd $\mathbb{C}l_1$-action $c_1 \otimes \eta$. The connections on $\overline{\Sigma}M$, E_0 and E_1 define a connection on $\overline{\Sigma}^E M$, giving rise to a twisted Dirac operator D^E and a twisted $\mathbb{C}l_1$-linear Dirac-Witten operator $\overline{D}^E = D^E - \frac{1}{2}\operatorname{tr}^g(k)e_0 \cdot \otimes \mathbb{1}_E$, which is again formally self-adjoint, odd and commutes with the $\mathbb{C}l_1$-action.

The assumption that E_0 and E_1 agree via Ψ outside a compact set K leads to even more structure on the twisted bundle. Namely, we choose a smooth cut-off function $\theta\colon M \to [0,1]$ with compact support such that $\theta \equiv 1$ on K. Then

$$T = \iota \otimes \begin{pmatrix} 0 & -(1-\theta)i\Psi^{-1} \\ (1-\theta)i\Psi & 0 \end{pmatrix}$$

is a self-adjoint, odd $\mathbb{C}l_1$-linear operator that anti-commutes with $\overline{D}^E$ away from $\operatorname{supp}(\mathrm{d}\theta)$. Hence, for any $\sigma > 0$, the operator $\overline{D}^E + \sigma T$ is again formally self-adjoint, odd and $\mathbb{C}l_1$-linear and satisfies

$$\left(\overline{D}^E + \sigma T\right)^2 = \left(\overline{D}^E\right)^2 + \sigma^2 T^2 \geq \sigma^2 \tag{3.6}$$

outside of $\operatorname{supp}(\theta)$. Note that if M itself is compact, then $\theta \equiv 1$ is a possible choice, in which case $T = 0$.

We now consider a family of such operators associated to a family of pairs (g, k). Here, a technical difficulty arises: The spinor bundle depends on the metric, so the operators act on different bundles. In order to link these bundles we employ a construction similar to the method of generalized cylinders [BGM05] and its further development, the universal spinor bundle [MN17]. We start by recalling from [BGM05] that a topological spin structure is given by a double covering $P_{\widetilde{\mathrm{GL}_+(n)}}M \to P_{\mathrm{GL}_+(n)}M$ of the principal bundle of positively oriented frames. Moreover, the map associating to a given basis the scalar product, for which this basis is orthonormal, gives rise to an $\mathrm{SO}(n)$-principal bundle $P_{\mathrm{GL}_+(n)}M \to \bigodot^2_+ T^*M$, where $\bigodot^2_+ T^*M$ is the bundle of symmetric positive definite bilinear forms. Now, denoting by $g = (g_t)_{t\in I}$ the family of complete Riemannian metrics obtained by looking at the first component of the path $\gamma\colon I = [-1, 1] \to \mathcal{I}(M)$, we form the pullback squares

$$\begin{array}{ccc}
P_{\mathrm{Spin}(n)}(g_\bullet) & \longrightarrow & P_{\widetilde{\mathrm{GL}_+(n)}}M \\
\downarrow & & \downarrow \\
P_{\mathrm{SO}(n)}(g_\bullet) & \longrightarrow & P_{\mathrm{GL}_+(n)}M \\
\downarrow & & \downarrow \\
M \times I & \xrightarrow{(x,t)\mapsto g_t(x)} & \bigodot^2_+ T^*M.
\end{array}$$

Notice that the so-defined principal bundles, the $\mathrm{SO}(n)$-principal bundle $P_{\mathrm{SO}(n)}(g_\bullet) \to M \times I$ and the $\mathrm{Spin}(n)$-principal bundle $P_{\mathrm{Spin}(n)}(g_\bullet) \to M \times I$, are in general just continuous, not smooth, as we assumed the path $\gamma\colon I \to \mathcal{I}(M)$ only to be continuous. However, when we restrict to a certain parameter $t \in I$, these give back the (smooth) principal bundles associated to the metric g_t, i. e. the following is a pullback diagram:

$$\begin{array}{ccc}
P_{\mathrm{Spin}(n)}(M, g_t) & \longrightarrow & P_{\mathrm{Spin}(n)}(g_\bullet) \\
\downarrow & & \downarrow \\
P_{\mathrm{SO}(n)}(M, g_t) & \longrightarrow & P_{\mathrm{SO}(n)}(g_\bullet) \\
\downarrow & & \downarrow \\
M & \xrightarrow{x\mapsto (x,t)} & M \times I.
\end{array}$$

By associating to $P_{\mathrm{Spin}(n)}(g_\bullet)$ the double of the irreducible $\mathbb{C}\mathrm{l}_n$-representation, we obtain a continuous bundle $\overline{\Sigma} g_\bullet \to M \times I$ that restricts for each $t \in I$ to the double spinor bundle $\overline{\Sigma}(M, g_t) \to M$ for the metric g_t. Moreover, the twisted bundle $\overline{\Sigma}^E g_\bullet := \overline{\Sigma} g_\bullet \otimes p^*E$, where $p\colon M \times I \to M$ is the canonical projection, restricts for fixed $t \in I$ to the twisted bundle $\overline{\Sigma}^E(M, g_t)$ considered above.

A single differential operator on a vector bundle is often best considered as an unbounded operator acting on the L^2-space of sections of that bundle. The corresponding notion for families of operators acting on a family of vector bundles is that of a densely defined operator family on a continuous field of Hilbert spaces. This notion was developed by Ebert [Ebe18] building on work by Dixmier and Douady [DD63] and in the following we stick to his notation.

We start by constructing a continuous field of Hilbert spaces with $\mathbb{C}l_1$-structure from the bundle $\overline{\Sigma}^E g_\bullet \to M \times I$. Roughly speaking, this consists of the spaces of L^2-sections of $\overline{\Sigma}^E g_t \to M$, parametrized over $t \in I$, together with the datum of when a family $(u_t)_{t\in I}$ of L^2-sections is continuous. It will be obtained as completion of an appropriate field of pre-Hilbert spaces: For each $t \in I$ let

$$V_t = C_c^\infty\left(M, \overline{\Sigma}^E(M, g_t)\right)$$

be the space of smooth compactly supported sections and denote by

$$\Lambda = \left\{u \in C_c^0\left(M \times I, \overline{\Sigma}^E g_\bullet\right) \,\middle|\, u_{|M\times\{t\}} \in V_t \text{ for all } t \in I\right\} \subseteq \prod_{t\in I} V_t$$

the subset of those families of such sections that assemble into a compactly supported continuous section of the bundle $\overline{\Sigma}^E g_\bullet \to M \times I$.

Lemma 3.3.1. *Together with the L^2-scalar product on V_t, the pair $((V_t)_{t\in I}, \Lambda)$ defines a continuous field of pre-Hilbert spaces.*

Proof. There are two things to show. The first one is that

$$\begin{aligned} I &\longrightarrow \mathbb{R} \\ t &\longmapsto \int_M \langle u(x,t), v(x,t)\rangle \,\mathrm{dvol}^{g_t}(x) \end{aligned}$$

is continuous for $u, v \in \Lambda$. By the definition of Λ, the functions $(x,t) \mapsto |u(x,t)|$ and hence $x \mapsto \max_{t\in I} |u(x,t)|$ are continuous and compactly supported. The same argument applies to v. Thus, using the Cauchy-Schwarz inequality, the theorem of dominated convergence yields the desired continuity.

Secondly, we have to see that the restriction map $\Lambda \to V_t$, $u \mapsto u_{|M\times\{t\}}$ is dense for all $t \in I$. In fact, we will show surjectivity. So let $u_t \in V_t$ be given. This defines a commutative diagram

$$\begin{array}{ccc} & & \overline{\Sigma}^E \\ & \overset{u_t}{\nearrow} & \downarrow \uparrow \tilde{u} \\ M & \xrightarrow{g_t} & \bigodot^2_+ T^*M, \end{array}$$

whereby $\overline{\Sigma}^E$ denotes the twisted double spinor bundle associated to the spin structure $P_{\widetilde{\mathrm{GL}_+(n)}} M \to \bigodot^2_+ T^*M$. The twist is given by the pull back of E along $\bigodot^2_+ T^*M \to M$.

We wish to extend u_t to a smooth compactly supported section $\tilde{u}$, as this gives rise to a section $u \in \Lambda$ restricting to u_t. As u_t is compactly supported, we can turn any smooth extension $\tilde{u}$ into a compactly supported one by multiplying with a suitable cut-off function. We construct $\tilde{u}$ by gluing local pieces using a partition of unity of $\bigodot_+^2 T^*M$: For $x \in M$, let $U \subseteq \bigodot_+^2 T^*M$ be an open neighborhood of $g_t(x)$ with the property that there exists a section ε of $P_{\widetilde{\mathrm{GL}_+(n)}}M_{|U} \to U$. Possibly restricting U, we may assume that $\{g_t(x) \,|\, x \in \pi(U)\} \subseteq U$, where $\pi\colon \bigodot_+^2 T^*M \to M$ is the canonical projection. We now obtain an extension of u_t on U by taking the coefficients with respect to ε in the associated bundle $\overline{\Sigma}^E_{|U} \to U$ to be constant along the fibers of π. □

We will denote by $(L^2(\overline{\Sigma}^E g_\bullet), \overline{\Lambda})$ the continuous field of Hilbert spaces obtained as completion of $((V_t)_{t\in I}, \Lambda)$. It is clear that it carries a $\mathbb{C}\mathrm{l}_1$-structure induced by the $\mathbb{Z}/2\mathbb{Z}$-grading $\iota \otimes \eta$ and the Clifford multiplication $c_1 \otimes \eta$. Next, we want to see how $\overline{D}^E + \sigma T$ defines an unbounded operator family on $(L^2(\overline{\Sigma}^E g_\bullet), \overline{\Lambda})$ and establish its main analytic properties with the goal to associate a suitable index to it.

Lemma 3.3.2. *The operators $\overline{D}^E_t + \sigma_t T\colon V_t \to V_t$ for $t \in I$ assemble to a densely defined operator family on $(L^2(\overline{\Sigma}^E g_\bullet), \overline{\Lambda})$ with initial domain $((V_t)_{t\in I}, \Lambda)$.*

Proof. We have to show that $\overline{D}^E_\bullet + \sigma T$ maps sections $u \in \Lambda$ to sections in $\overline{\Lambda}$. We will show that $(\overline{D}^E_\bullet + \sigma T)u \in \Lambda \subseteq \overline{\Lambda}$. The only thing that is not clear here is that $(\overline{D}^E_\bullet + \sigma T)u$ is a continuous section of $\overline{\Sigma}^E g_\bullet \to M \times I$. This boils down to showing that $D^E_\bullet u$ is a continuous section, where $D^E_\bullet$ is fiberwise the twisted Dirac operator as above. As continuity may be checked locally, we can restrict our attention to an open subset of $M \times I$, where there exists a continuous section ε of $P_{\mathrm{Spin}(n)}(g_\bullet) \to M \times I$. The associated orthonormal frame will be called $(e_1, \ldots, e_n)$. Assuming that u can be written as a tensor product $\Psi \otimes e$ with Ψ a section of $\overline{\Sigma} g_\bullet \to M \times I$ and e a section of $p^*E \to M \times I$ (in general, u will be a sum of such) and expressing $\Psi = [\varepsilon, \psi]$, we have

$$D^E_\bullet u = \sum_{i=1}^n \left[\varepsilon\,,\, e_i \cdot \partial_{e_i}\psi + \sum_{j,k=1}^n \frac{1}{2} g_\bullet(\nabla^{g_\bullet}_{e_i} e_j, e_k)\, e_i \cdot e_j \cdot e_k \cdot \psi\right] \otimes e + \sum_{i=1}^n (e_i \cdot \Psi) \otimes \nabla^E_{e_i} e.$$

This expression is continuous as we assumed that the family $(g_t)_{t\in I}$ and its derivatives to be uniformly continuous on all compact sets, which particularly implies that the Christoffel symbols are uniformly continuous on compact sets. □

The closure of this operator family will be denoted by $\overline{D}^E_\bullet + \sigma T\colon \mathrm{dom}(\overline{D}^E_\bullet + \sigma T) \to (L^2(\overline{\Sigma}^E g_\bullet), \overline{\Lambda})$.

Lemma 3.3.3. *The unbounded operator family* $\overline{D}^E_\bullet + \sigma T$ *is self-adjoint.*

Proof. By definition, this is the case if and only if the operators $\overline{D}^E_t + \sigma T$ are self-adjoint and regular for each $t \in I$. A sufficient criterion for this is the existence of a coercive, i. e. bounded below and proper, smooth function $h_t \colon M \to \mathbb{R}$, such that the commutator $[\overline{D}^E_t + \sigma_t T, h_t]$ is bounded, cf. [Ebe18, Thm. 1.14].

For some fixed $x_0 \in M$, we consider the distance function $d_t(x) = \mathrm{dist}^{g_t}(x_0, x)$. This is bounded below by 0. As g_t is complete, by the theorem of Hopf-Rinow, d_t is proper. Yet d_t is not smooth, only a Lipschitz function, with Lipschitz constant 1. The desired function h_t can now be obtained by suitably smoothing d_t out. For example, using the main theorem of [Aza+07], we obtain the existence of a smooth function h_t with $\sup_{x \in M} |h_t(x) - d_t(x)| \leq \varepsilon$ and Lipschitz constant $1 + \varepsilon$ (for any $\varepsilon > 0$). The first condition ensures that h_t is also bounded below and proper, whereas the second one implies that $[\overline{D}^E_t + \sigma_t T, h_t]$ is bounded by $1 + \varepsilon$, as the principal symbol of $\overline{D}^E_t + \sigma_t T$ is given by Clifford multiplication. □

Proposition 3.3.4. *The self-adjoint unbounded operator family* $\overline{D}^E_\bullet + \sigma T$ *is a Fredholm family.*

Proof. The first thing to note is that $\overline{D}^E_\bullet + \sigma T$ arises as closure of a formally self-adjoint *elliptic* differential operator of order 1 on the bundle $\overline{\Sigma}^E g_\bullet \to M \times I$. In view of [Ebe18, Thm. 2.41], the statement is basically a consequence of the fact that $(\overline{D}^E_\bullet + \sigma T)^2 \geq \sigma^2$ outside the compact set $\mathrm{supp}(\theta) \times I$. However, we are not precisely in the setting of Ebert's article. Namely, the bundle $\overline{\Sigma}^E g_\bullet \to M \times I$ is only continuous and not smooth; but this lower regularity does not affect the proofs. Moreover, we have not yet established the existence of a smooth coercive function $h \colon M \times I \to \mathbb{R}$ such that $[\overline{D}^E_\bullet + \sigma T, h]$ is bounded – and we will not do so.

Instead, we note that h serves only two purposes. Firstly, it (again) shows that the operator family is self-adjoint, as the functions $h(\text{-}, t)$ can play the role of the h_t above. Secondly, it serves as a basis for constructing a compactly supported smooth function $f \colon M \times I \to \mathbb{R}$ such that $(\overline{D}^E_\bullet + \sigma T)^2 + f^2 \geq \sigma^2$ everywhere on $M \times I$ and $\|[\overline{D}^E_\bullet + \sigma T, f]\| \leq \frac{\sigma^2}{2}$, which is needed in the proof of Fredholmness. So we may just construct such a function f directly.

For any $t \in I$, let R_t be chosen such that $\mathrm{supp}(\theta) \subseteq B^{g_t}_{R_t}(x_0)$ and h_t a function as above (for $\varepsilon \leq \frac{1}{3}$). Furthermore, we choose a smooth cut-off function $\Psi \colon \mathbb{R} \to [0, 1]$ with $\Psi(r) = \sigma$ for $r \leq 1$ and $|\Psi'(r)| \leq \frac{\sigma^2}{3}$ for all $r \in \mathbb{R}$. Denote by L a number such that

$\Psi \equiv 0$ on $[L, \infty)$. Now, let $f_t(x) = \Psi(h_t(x) - R_t)$. Note that $f_t \equiv \sigma$ on $\mathrm{supp}(\theta)$, as $h_t(x) \leq d_t(x) + \varepsilon < R_t + 1$ for all $x \in \mathrm{supp}(\theta)$.

Continuity of $(g_s)_{s\in I}$ allows us to choose $\delta_t > 0$ such that $\|g_s^{-1}\|_{g_t} \leq \left(\frac{9}{8}\right)^2$ for all $s \in U_t \coloneqq (t - \delta_t, t + \delta_t)$ on the closed ball[4] $\overline{B^{g_t}_{R_t+L}(x_0)}$. Using $\|\mathrm{d}f_t\|^2_{g_s} \leq \|g_s^{-1}\|_{g_t}\|\mathrm{d}f_t\|^2_{g_t}$, we obtain

$$\begin{aligned} \|[\overline{D}^E_s + \sigma T, f_t]\| &= \|\mathrm{d}f_t\|_{g_s} \\ &\leq \sqrt{\|g_s^{-1}\|_{g_t}}\|\Psi'\|_\infty\|\mathrm{d}h_t\|_{g_t} \\ &\leq \frac{9}{8}\cdot\frac{\sigma^2}{3}\cdot(1+\varepsilon) \leq \frac{\sigma^2}{2} \end{aligned} \tag{3.7}$$

for all $s \in U_t$.

Now, there exists a finite collection $t_1, \ldots, t_n$ such that $U_{t_1}, \ldots, U_{t_n}$ cover I and a smooth partition of unity $\psi_1, \ldots \psi_n$ subordinate to this open cover. We define $f(x,t) = \sum_{i=1}^n \psi_i(t) f_t(x)$. Then $f \equiv \sigma$ on $\mathrm{supp}(\theta)$, which implies $(\overline{D}^E_\bullet + \sigma T)^2 + f^2 \geq \sigma^2$ everywhere on $M \times I$. Moreover, the second property of f immediately follows from (3.7). □

It is important to know when the Fredholm family $\overline{D}^E_\bullet + \sigma T$ is invertible. The next lemma provides a criterion for this.

Lemma 3.3.5. *Let $A \subseteq I$ with*

$$\inf_{t\in A,\, x\in M}\left(\rho_t - |j_t|_{g_t} - 2|\mathcal{R}^E_t|\right) > 0. \tag{3.8}$$

Then there is a $\sigma' > 0$ such that $\overline{D}^E_\bullet + \sigma T$ is invertible over A for all $0 < \sigma < \sigma'$. Here, $\mathcal{R}^E_t$ denotes the curvature endomorphism defined by $\mathcal{R}^E_t(\phi \otimes e) = \sum_{i<j} e_i \cdot e_j \cdot \phi \otimes R^E(e_i, e_j)e$ for $\phi \otimes e \in (\Sigma_p M \oplus \Sigma_p M) \otimes_{\mathbb{C}} E_p$ and an orthonormal basis $(e_1, \ldots, e_n)$ of T_pM, $p \in M$, with respect to g_t.

Proof. We first consider the situation for some fixed metric g. The twisted Dirac-Witten operator associated to g satisfies the following Schrödinger-Lichnerowicz type formula, the proof of which is deferred to Appendix A.3:

$$\left(\overline{D}^E\right)^2\psi = (\overline{\nabla}^E)^*\overline{\nabla}^E\psi + \frac{1}{2}(\rho - e_0 \cdot j^\sharp\cdot)\psi + \mathcal{R}^E\psi.$$

[4] Unlike stated in the very beginning, the overline here does not relate to an ambient Lorentzian manifold.

Here, $\overline{\nabla}^E$ denotes the connection on $\overline{\Sigma}^E M$ induced by $\overline{\nabla}$ and the connection on E, and the star indicates the formal adjoint. Together with

$$\left(\overline{D}^E + \sigma T\right)^2 \psi = \left(\overline{D}^E\right)^2 \psi + \sigma\left(\overline{D}^E T + T\overline{D}^E\right)\psi + \sigma^2(1-\theta)^2\psi$$

this implies

$$\begin{aligned}\left(\left(\overline{D}^E + \sigma T\right)^2 \psi, \psi\right)_{L^2} &\geq \left(\left(\overline{D}^E\right)^2 \psi, \psi\right)_{L^2} - \sigma|\mathrm{d}\theta|_g\|\psi\|^2_{L^2} + \sigma^2(1-\theta)^2\|\psi\|^2_{L^2} \\ &\geq \|\overline{\nabla}^E\psi\|^2_{L^2} + \left(\frac{1}{2}(\rho - |j|_g) - |\mathcal{R}^E|\right)\|\psi\|^2_{L^2} - \sigma|\mathrm{d}\theta|_g\|\psi\|^2_{L^2} \\ &\geq \left(\frac{1}{2}(\rho - |j|_g) - |\mathcal{R}^E|\right)\|\psi\|^2_{L^2} - \sigma|\mathrm{d}\theta|_g\|\psi\|^2_{L^2}.\end{aligned}$$

for any compactly supported smooth section ψ (single norms $|\text{-}|_g$ denote pointwise norms).

As $|d\theta|_{g_t}$ is a continuous compactly supported function on $M \times I$ and (3.8) holds, we may choose $\sigma' > 0$ such that

$$\inf_{t\in A,\, x\in M}\left(\rho_t - |j_t|_{g_t} - 2|\mathcal{R}^E_t|\right) \geq \sigma' \sup_{t\in A,\, x\in M} |d\theta|_{g_t}.$$

Then for all $0 < \sigma < \sigma'$, there exists some constant $c > 0$ with $\left(\overline{D}^E_t + \sigma T\right)^2 \geq c$ for all $t \in A$. From this, the statement follows immediately (cf. [Ebe18, Prop. 1.21, Lem. 2.6]). □

Proposition 3.3.6. *The operator family $\overline{D}^E_\bullet + \sigma T$ is odd with respect to $\iota \otimes \eta$ and $\mathbb{C}\mathrm{l}_1$-linear with respect to $c_1 \otimes \eta$. For suitably small σ, it defines an element*

$$\left[(L^2(\overline{\Sigma}^E g_\bullet), \overline{\Lambda}),\ \iota\otimes\eta,\ -\iota c_1 \otimes \mathbb{1}_E,\ \overline{D}^E_\bullet + \sigma T\right] \in \mathrm{K}^1(I, \partial I),$$

that is independent of the choices of K, θ, Ψ and σ (as long as they fulfill the assumed requirements) and depends only on the relative homotopy class of $\gamma\colon (I, \partial I) \to (\mathcal{I}(M), \mathcal{I}^{>\mathcal{R}^E+c}(M))$. Moreover, this class is zero if γ is homotopic to a path $I \to \mathcal{I}^{>\mathcal{R}^E+c}(M)$.

Note that, as $\mathcal{I}(M)$ is convex, the homotopy class of γ just depends on its endpoints.

Definition 3.3.7. For $(g_{-1}, k_{-1}), (g_1, k_1) \in \mathcal{I}^{>\mathcal{R}^E+c}(M)$, their *E-relative index difference* $\text{ind-diff}^E((g_{-1}, k_{-1}), (g_1, k_1)) \in \mathrm{K}^1(I, \partial I)$ is the class defined in Proposition 3.3.6 using some path $\gamma\colon (I, \partial I) \to (\mathcal{I}(M), \mathcal{I}^{>\mathcal{R}^E+c}(M))$ connecting these two pairs.

If M is compact, then E_0 and E_1 need not be isomorphic anywhere.In particular, we may take $E_0 = \mathbb{C}$ and $E_1 = 0$, which gives the untwisted index difference.

Definition 3.3.8. If M is compact, the *index difference* of (g_{-1}, k_{-1}) and $(g_1, k_1) \in \mathcal{I}^+(M)$ is $\text{ind-diff}((g_{-1}, k_{-1}), (g_1, k_1)) := \text{ind-diff}^E((g_{-1}, k_{-1}), (g_1, k_1)) \in \mathrm{K}^1(I, \partial I)$ for the trivial bundles $E_0 = \underline{\mathbb{C}}$ and $E_1 = \underline{0}$.

Proof. That $\overline{D}^E_\bullet + \sigma T$ is odd and $\mathbb{C}\mathrm{l}_1$-linear follows from the fact that this holds for $\overline{D}^E_t + \sigma T$ for all $t \in I$. As $\overline{D}^E_\bullet + \sigma T$ is, moreover, an unbounded Fredholm family that is by Lemma 3.3.5 invertible over ∂I, we get an element in the K-theory of $(I, \partial I)$. Note that the K-theoretic model described in [Ebe18, Ch. 3], which we are using here, requires a $\mathbb{C}\mathrm{l}_1$-antilinear operator rather than a $\mathbb{C}\mathrm{l}_1$-linear one. But this is no problem as a $\mathbb{C}\mathrm{l}_1$-linear operator is $\mathbb{C}\mathrm{l}_1$-antilinear with respect to the $\mathbb{C}\mathrm{l}_1$-structure defined by $i(\iota \otimes \eta)(c_1 \otimes \eta) = -i\iota c_1 \otimes \mathbb{1}$.

We now show independence of the choices starting with σ. Let $\sigma_0 > 0$ and $\sigma_1 > 0$ be two admissible values, i. e. smaller than σ' from Lemma 3.3.5 applied to $A = \partial I$. We consider the pullback of $\Big((L^2(\overline{\Sigma}^E g_\bullet), \overline{\Lambda}),\ \iota \otimes \eta,\ -i\iota c_1 \otimes \mathbb{1}\Big)$ along the canonical projection $I \times [0, 1] \to I$. This continuous field of $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert spaces with $\mathbb{C}\mathrm{l}_1$-structure carries the odd $\mathbb{C}\mathrm{l}_1$-antilinear Fredholm family $\overline{D}^E_t + ((1 - s)\sigma_0 + s\sigma_1)T$ for $t \in I$ and $s \in [0, 1]$, which is invertible over $\partial I \times [0, 1]$. Thus, we have a concordance between the cycles $\Big((L^2(\overline{\Sigma}^E g_\bullet), \overline{\Lambda}),\ \iota \otimes \eta,\ -i\iota c_1 \otimes \mathbb{1},\ \overline{D}^E_\bullet + \sigma T\Big)$ for $\sigma = \sigma_0$ and $\sigma = \sigma_1$.

Independence of θ, K and Ψ are slightly connected, as we have to have $\theta \equiv 1$ on K and Ψ is defined on the complement of K. Given two such triples (θ_0, K_0, Ψ_0) and (θ_1, K_1, Ψ_1), we first show that we can first replace the θ_0 by some θ with $\theta \equiv 1$ on $K = K_0 \cup K_1$ without changing the K^1-class. Then noting that (θ, K_0, Ψ_0) and $(\theta, K, \Psi_{0|M\setminus K})$ even define the same operator family, it just remains to show that the K^1-class is independent of Ψ for fixed θ and K.

Concering the replacement of θ_0 by θ (similarly for θ_1 by θ), we use the same argumentation as for σ with the difference that this time the operator family is given by $\overline{D}^E_t + \sigma T_s$ with

$$T_s = \iota \otimes \begin{pmatrix} 0 & -(1 - (1 - s)\theta_0 - s\theta)i\Psi_0^{-1} \\ (1 - (1 - s)\theta_0 - s\theta)i\Psi_0 & 0 \end{pmatrix}.$$

For changing $\Psi_{0|M\setminus K}$ to $\Psi_{1|M\setminus K}$, we note that the requirement that these bundle isomorphisms preserve hermitian metric and connection implies that they differ by a single element of $U(k)$ on every connected component of $M \setminus K$, where k is the rank of E_0 (and

E_1). As $U(k)$ is connected, there exists a homotopy $(\Psi_s)_{s\in[0,1]}$ connecting these. Again, the operator family $\overline{D}^E_t + \sigma T_s$ defines a concordance, where this time

$$T_s = \iota \otimes \begin{pmatrix} 0 & -(1-\theta)i\Psi_s^{-1} \\ (1-\theta)i\Psi_s & 0 \end{pmatrix}.$$

Now assume we are given a homotopy $H\colon (I\times[0,1], \partial I \times [0,1]) \to (\mathcal{I}(M), \mathcal{I}^{>\mathcal{R}^E+c}(M))$ between $\gamma_0 = H(\text{-},0)$ and $\gamma_1 = H(\text{-},1)$. In this case we can construct a concordance between the K^1-cycles associated to γ_0 and γ_1 as follows. Let $(g_{t,s})_{t\in I,\, s\in[0,1]}$ be the first components of the pairs $(H(t,s))_{t\in I,\, s\in[0,1]}$. Similarly to before, we may form the pullback

$$\begin{array}{ccc} P_{\mathrm{Spin}(n)}(g_{\bullet,\bullet}) & \longrightarrow & P_{\widetilde{\mathrm{GL}_+}(n)}M \\ \downarrow & & \downarrow \\ P_{\mathrm{SO}(n)}(g_{\bullet,\bullet}) & \longrightarrow & P_{\mathrm{GL}_+(n)}M \\ \downarrow & & \downarrow \\ M \times I \times [0,1] & \xrightarrow{(x,t,s)\mapsto g_{t,s}(x)} & \bigodot^2_+ T^*M \end{array}$$

and obtain a bundle $\overline{\Sigma}^E g_{\bullet,\bullet} \to M \times I \times [0,1]$ and a continuous field of $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert spaces $(L^2(\overline{\Sigma}^E g_{\bullet,\bullet}), \overline{\Lambda})$ with $\mathbb{C}\mathrm{l}_1$-structure. The operators $(\overline{D}^E_{t,s} + \sigma T)_{t\in I,\, s\in[0,1]}$ constitute an unbounded Fredholm family on this continuous field of Hilbert spaces. For suitably small σ, by an analogous statement to Lemma 3.3.5, this is invertible over $\partial I \times [0,1]$. Together, these provide the required concordance.

For the last statement assume that the image of γ is contained in $\mathcal{I}^{>\mathcal{R}^E+c}$. In this case, $\overline{D}^E_\bullet + \sigma T$ is invertible on all of I by Lemma 3.3.5, as long as σ is chosen suitably small. Therefore, the K^1-class under consideration is in the image of the restriction homomorphism $\mathrm{K}^1(I,I) \to \mathrm{K}^1(I,\partial I)$. But as $\mathrm{K}^1(I,I) = 0$, this class has to be zero. ☐

3.4. An index theorem for the twisted index difference

The purpose of this section is to calculate the E-relative index difference for between pairs of the form $(g, \tau g)$ and $(g, -\tau g)$ for some $\tau > 0$. This is done in two steps. The first one is to express the relative index difference as relative index of a suitable Dirac type operator. In the second step, a relative index theorem identifies this index with a twisted version of the $\hat{A}$-genus.

The first thing we realize is the following: If for (g_{-1}, k_{-1}) and (g_1, k_1) there exists some $\sigma' > 0$ such that $\overline{D}^E_\bullet + \sigma T$ is invertible for all $0 < \sigma < \sigma'$, then it makes sense to speak of their E-relative index difference, even if they are not contained in some $\mathcal{I}^{>\mathcal{R}^E+c}(M)$.

Let g be some complete metric on M. Such a metric always exists by a classical result of Greene [Gre78], it is even the case that every conformal class contains such a metric by [MN15]. For $\tau > 0$, we consider the pairs $(g, \tau g)$ and $(g, -\tau g)$. Using

$$\left(\overline{D}^E\right)^2 = \left(D^E \pm \frac{1}{2}\tau e_0 \cdot \otimes \mathbb{1}_E\right)^2 = \left(D^E\right)^2 + \frac{n^2}{4}\tau^2$$

we obtain

$$\begin{aligned}\left(\overline{D}^E + \sigma T\right)^2 &= \left(\overline{D}^E\right)^2 + \sigma\left(\overline{D}^E T + T\overline{D}^E\right) + \sigma^2(1-\theta)^2 \\ &\geq \left(D^E\right)^2 + \frac{n^2}{4}\tau^2 - \sigma\|\mathrm{d}\theta\|_g,\end{aligned}$$

which shows that for these pairs $\overline{D}^E + \sigma T$ is invertible for sufficiently small $\sigma > 0$. Thus it makes sense to speak of $\text{ind-diff}^E((g, -\tau g), (g, \tau g)) \in \mathrm{K}^1(I, \partial I)$.

We denote by D_0^E the Dirac operator on $\Sigma^E M := \Sigma M \otimes E$ for the metric g. With

$$T_0 = \omega \otimes \begin{pmatrix} 0 & -(1-\theta)i\Psi^{-1} \\ (1-\theta)i\Psi & 0 \end{pmatrix},$$

this gives rise to a Fredholm operator $D_0^E + \sigma T_0$, for $\sigma > 0$, which is odd with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading $\omega \otimes \eta$. The proof of Fredholmness uses that g is complete. It is similar to Lemma 3.3.3 and Proposition 3.3.4, but much less delicate.

Proposition 3.4.1. *The isomorphism* $\mathrm{K}^1(I, \partial I) \cong \mathrm{K}^0(\{*\}) \cong \mathbb{Z}$, *induced by the Bott map, maps the class* $\text{ind-diff}^E((g, -\tau g), (g, \tau g))$ *to* $\mathrm{ind}(D_0^E + \sigma T_0)$.

Proof. We start from the class $[L^2(\Sigma^E M),\ \omega \otimes \eta,\ D_0^E + \sigma T_0] \in \mathrm{K}^0(\{*\})$ corresponding to the integer value $\mathrm{ind}(D_0^E + \sigma T_0)$. Using the conventions of Ebert [Ebe18], the Bott map $\mathrm{K}^0(\{*\}) \xrightarrow{\sim} \mathrm{K}^1(\mathbb{R}, \mathbb{R} \setminus \{0\})$ sends this class to

$$\left[\mathbb{C}^2 \otimes p^* L^2(\Sigma^E M), \begin{pmatrix} \omega\otimes\eta & 0 \\ 0 & -\omega\otimes\eta \end{pmatrix}, \begin{pmatrix} 0 & -\omega\otimes\eta \\ \omega\otimes\eta & 0 \end{pmatrix}, \begin{pmatrix} D_0^E + \sigma T_0 & t\omega\otimes\eta \\ t\omega\otimes\eta & D_0^E + \sigma T_0 \end{pmatrix}\right],$$

where $p\colon \mathbb{R} \to \{*\}$ is the projection and t is the $\mathbb{R}$-coordinate. As the inclusion $(I, \partial I) \to (\mathbb{R}, \mathbb{R} \setminus \{0\})$ induces an isomorphism in K-theory, the same formula defines the corresponding element in $\mathrm{K}^1(I, \partial I)$; now assuming $p\colon I \to \{*\}$ and $t \in I$.

Note that we may identify $\left(L^2(\overline{\Sigma}^E g_\bullet), \overline{\Lambda}\right) = \mathbb{C}^2 \otimes p^* L^2(\Sigma^E M)$ for the constant family $g_t = g$. Using the automorphism of $\left(L^2(\overline{\Sigma}^E g_\bullet), \overline{\Lambda}\right)$ given by

$$\begin{pmatrix} \mathbb{1}_{\Sigma M} & 0 \\ 0 & \mathbb{1}_{\Sigma M} \end{pmatrix} \otimes \begin{pmatrix} \mathbb{1}_{E_0} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \mathbb{1}_{\Sigma M} & 0 \\ 0 & -\mathbb{1}_{\Sigma M} \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{E_1} \end{pmatrix},$$

the K^1-class translates into

$$\left[\left(L^2(\overline{\Sigma}^E g_\bullet), \overline{\Lambda}\right), \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \otimes \eta, \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \otimes \mathbb{1}_E, \begin{pmatrix} D_0^E + \sigma T_0 & t\omega \otimes \mathbb{1}_E \\ t\omega \otimes \mathbb{1}_E & D_0^E - \sigma T_0 \end{pmatrix}\right].$$

Applying furthermore the automorphism

$$\frac{1}{\sqrt{2}}\left(\begin{pmatrix} \mathbb{1}_{\Sigma M} & 0 \\ 0 & \mathbb{1}_{\Sigma M} \end{pmatrix} + \begin{pmatrix} 0 & \mathbb{1}_{\Sigma M} \\ -\mathbb{1}_{\Sigma M} & 0 \end{pmatrix}\right) \otimes \mathbb{1}_E,$$

this gets

$$\begin{aligned} &\left[\left(L^2(\overline{\Sigma}^E g_\bullet), \overline{\Lambda}\right), \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix} \otimes \eta, \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \otimes \mathbb{1}_E, \begin{pmatrix} D_0^E - t\omega \otimes \mathbb{1}_E & \sigma T_0 \\ \sigma T_0 & D_0^E + t\omega \otimes \mathbb{1}_E \end{pmatrix}\right] \\ &\quad = \left[\left(L^2(\overline{\Sigma}^E g_\bullet), \overline{\Lambda}\right),\ \iota \otimes \eta,\ -i\iota c_1 \otimes \mathbb{1}_E,\ D^E - te_0 \cdot \otimes \mathbb{1}_E + \sigma T\right]. \end{aligned}$$

Bearing in mind that $\overline{D}^E = D^E - \frac{1}{2}\operatorname{tr}(k)e_0 \cdot \otimes \mathbb{1}_E$, this is the class defined by the Dirac-Witten operators associated to the straight unit speed path from $(g, -2g)$ to $(g, 2g)$. For general $\tau > 0$, the result follows by rescaling, e. g. replacing the inclusion $(I, \partial I) \to (\mathbb{R}, \mathbb{R} \setminus \{0\})$ above with the map $t \mapsto \frac{\tau}{2}t$. $\square$

It now remains to determine $\operatorname{ind}(D_0^E + \sigma T_0)$. This is done by the relative index theorem going back to [GL83].

Theorem 3.4.2 (Relative Index Theorem)**.**

$$\operatorname{ind}(D_0^E + \sigma T_0) = \int_M \hat{A}(TM) \wedge (\operatorname{ch}(E_0) - \operatorname{ch}(E_1)) =: \hat{A}(M, E).$$

Proof. Although probably well-known, there seems not to be an easily citable reference matching the setup here. We therefore provide a proof using cut-and-paste-techniques from [BB11].

First, we note that the $\mathbb{Z}/2\mathbb{Z}$-graded index of $D_0^E + \sigma T_0$ is by definition just the usual Fredholm index of its "positive" part $(D_0^E)^+ + \sigma T_0^+ \colon \Gamma((\overline{\Sigma}^E M)^+) \to \Gamma((\overline{\Sigma}^E M)^-)$ mapping from positive to negative half-spinors. The operator $(D_0^E)^+ + \sigma T_0^+$ is of Dirac type, so we may use the decomposition theorem from [BB11].

In order to do so, let $M = M_1 \cup M_2$ be a decomposition into two smooth manifolds with boundary $\partial M_1 = \partial M_2$ such that M_1 is compact and $\operatorname{supp}\theta \subseteq M_1$. Let

$B \subseteq H^{\frac{1}{2}}(\partial M_1, (\overline{\Sigma}^E M)^+)$ be an elliptic boundary condition and denote by $B^\perp$ its L^2-orthogonal complement. For example, B could be Atiyah-Patodi-Singer boundary conditions. The decomposition theorem states that

$$\operatorname{Ind}\left((D_0^E)^+ + \sigma T_0^+\right) = \operatorname{Ind}\left(((D_0^E)^+ + \sigma T_0^+)_{|M_1,\, B}\right) + \operatorname{Ind}\left(((D_0^E)^+ + \sigma T_0^+)_{|M_2,\, B^\perp}\right). \tag{3.9}$$

The first summand of (3.9) computes to

$$\begin{aligned} \operatorname{Ind}\left(((D_0^E)^+ + \sigma T_0^+)_{|M_1,\, B}\right) &= \operatorname{Ind}\left((D_0^E)^+_{|M_1,\, B}\right) \\ &= \operatorname{Ind}\left((D_0^{E_0})^+_{|M_1,\, B}\right) - \operatorname{Ind}\left((D_0^{E_1})^+_{|M_1,\, B}\right) \\ &= \int_M \hat{A}(TM) \wedge (\operatorname{ch}(E_0) - \operatorname{ch}(E_1)). \end{aligned} \tag{3.10}$$

Here, in the first step, we used the homotopy $[0,1] \ni t \mapsto (D_0^E)^+ + t\sigma T_0^+$. The second step follows from the decomposition

$$(D_0^E)^+ = \begin{pmatrix} (D_0^{E_0})^+ & 0 \\ 0 & (D_0^{E_1})^- \end{pmatrix} = \begin{pmatrix} (D_0^{E_0})^+ & 0 \\ 0 & \left((D_0^{E_1})^+\right)^* \end{pmatrix}.$$

The last step is (contained in the proof of) the relative index theorem [BB11, Thm. 1.21].

It remains to show that the second summand of (3.9) is zero. To see this, we note that there exists a bundle $\widetilde{E}_1 \to M$ admitting a metric and connection preserving isomorphism $\widetilde{\Psi}\colon E_0 \to \widetilde{E}_1$, such that $\widetilde{E}_{1|M_2} = E_{1|M_2}$ and $\widetilde{\Psi}_{|M_2} = \Psi_{|M_2}$. For instance, such a bundle can be obtained by gluing $E_{0|M_1}$ and $E_{1|M_2}$. Similarly as before, we denote by $D_0^{\widetilde{E}}$ the Dirac operator on $\Sigma^{\widetilde{E}} M$ for $\widetilde{E} = E_0 \oplus \widetilde{E}_1$ and define

$$\widetilde{T}_0 = \omega \otimes \begin{pmatrix} 0 & -i\widetilde{\Psi}^{-1} \\ i\widetilde{\Psi} & 0 \end{pmatrix}.$$

Notice, that we are allowed to take $\theta \equiv 0$, as $\widetilde{\Psi}$ is defined on all of M. Again, we have a decomposition

$$\operatorname{Ind}\left((D_0^{\widetilde{E}})^+ + \sigma \widetilde{T}_0^+\right) = \operatorname{Ind}\left(((D_0^{\widetilde{E}})^+ + \sigma \widetilde{T}_0^+)_{|M_1,\, B}\right) + \operatorname{Ind}\left(((D_0^{\widetilde{E}})^+ + \sigma \widetilde{T}_0^+)_{|M_2,\, B^\perp}\right).$$

In this case, the calculation (3.10) shows that the first summand is zero, as $\operatorname{ch}(\widetilde{E}_1) = \operatorname{ch}(E_0)$. The second summand is the second summand from above as the bundles and operators are equal on M_2. Thus, we obtain

$$\begin{aligned} \operatorname{Ind}\left(((D_0^E)^+ + \sigma T_0^+)_{|M_2,\, B^\perp}\right) &= \operatorname{Ind}\left(((D_0^{\widetilde{E}})^+ + \sigma \widetilde{T}_0^+)_{|M_2,\, B^\perp}\right) \\ &= \operatorname{Ind}\left((D_0^{\widetilde{E}})^+ + \sigma \widetilde{T}_0^+\right) = 0, \end{aligned}$$

where we used that $D_0^{\widetilde{E}} + \sigma \widetilde{T}_0$ is invertible as $(D_0^{\widetilde{E}} + \sigma \widetilde{T}_0)^2 \geq \sigma^2 > 0$. □

Corollary 3.4.3 (Relative index theorem for the twisted index difference)**.** *The isomorphism* $\mathrm{K}^1(I, \partial I) \cong \mathrm{K}^0(\{*\}) \cong \mathbb{Z}$ *sends* $\text{ind-diff}^E((g, -\tau g), (g, \tau g))$ *to* $\hat{A}(M, E)$*. If* M *is compact,* $\text{ind-diff}((g, -\tau g), (g, \tau g))$ *is sent to* $\hat{A}(M)$*.*

The relative index theorem allows to obtain an obstruction to the path-connectedness of the space of initial data sets that satisfy the strict dominant energy condition. The following corollary illustrates the general strategy and turns out to be a special case of the enlargeability obstruction Theorem 3.5.3 that we discuss in the remaining section.

Corollary 3.4.4. *Let* M *be a closed manifold,* g *a metric on* M *and* $\tau > 0$ *be chosen such that* $(g, -\tau g), (g, \tau g) \in \mathcal{I}^>(M)$*. If* M *is spin and* $\hat{A}(M) \neq 0$*, then* $(g, -\tau g)$ *and* $(g, \tau g)$ *belong to different path-components of* $\mathcal{I}^>(M)$*.*

Proof. If there were a path $\gamma\colon I \to \mathcal{I}^>(M)$ from $(g, -\tau g)$ to $(g, \tau g)$, then, by Proposition 3.3.6, $\text{ind-diff}((g, -\tau g), (g, \tau g))$ would be zero. But by Corollary 3.4.3 it is mapped to $\hat{A}(M) \neq 0$. □

Remark 3.4.5. The statement of Corollary 3.4.4 also follows from Theorem B. This is due to the fact that, in even dimension n, Hitchin's α-index is mapped to the $\hat{A}$-genus under the complexification map $\mathrm{KO}^{-n}(\{*\}) \to \mathrm{K}^{-n}(\{*\}) \cong \mathbb{Z}$. In fact, also the proof is the same, as complexification turns the α-difference for initial data sets defined in Definition 2.3.15 into the (untwisted) index-difference considered in this chapter, up to Bott periodicity.

3.5. Enlargeability obstruction for initial data sets

Gromov-Lawson's enlargeability obstruction gives a major source of examples of manifolds that do not admit a positive scalar curvature metric. And – as we shall discuss in the end of this section – completely answers the existence question for positive scalar curvature metrics in the case of closed orientable 3-manifolds. Before that, we prove that enlargeability is also an obstruction to path-connectedness of the space of initial data sets satisfying the strict dominant energy condition.

There are many versions of enlargeability. In Definition 3.1.4 in the introduction we introduced the notion of *compact enlargeability.* We will state and prove our main theorem in terms of the more general $\hat{A}$-area-enlargeability:

Definition 3.5.1. A smooth map $f\colon (M, g) \to (N, h)$ between Riemannian manifolds is *ε-area-contracting* for some $\varepsilon > 0$ if the induced map $f_*\colon \Lambda^2 TM \to \Lambda^2 TN$ satisfies

$|f_*| \leq \varepsilon$. A closed Riemannian manifold (M, g) of dimension n is called *area-enlargeable in dimension k* if for all $\varepsilon > 0$ there exists a Riemannian covering $(M', g') \to (M, g)$ admitting an ε-area-contracting map $(M', g') \to (S^k, g_{Std})$ that is constant outside a compact set and of non-zero $\hat{A}$-degree. It is called *$\hat{A}$-area-enlargeable* if it is area-enlargeable in some dimension k.

Recall that the $\hat{A}$-degree of a smooth map $f \colon X \to Y$, where Y is closed and connected and f is constant outside a compact set, may be defined by the requirement that $\int_X \hat{A}(TX) \wedge f^*(\omega) = \hat{\text{A}}\text{-deg}(f) \int_Y \omega$ for all top dimensional forms $\omega \in \Omega^{\dim(Y)}(Y)$. If Y is non-connected, there is one such number for every connected component of Y and the $\hat{A}$-degree is the vector consisting of these. From this definition we see that it can only be non-zero if $\dim(Y) \leq \dim(X)$ and $\dim(Y) \equiv \dim(X) \mod 4$. The $\hat{A}$-degree can be thought of as interpolating between the following two special cases: If $\dim(Y) = \dim(X)$, the $\hat{A}$-degree is just the usual degree. If $\dim(Y) = 0$ and Y is connected, it is the $\hat{A}$-genus $\hat{A}(X)$ of X.

Although the definition of enlargeability uses a Riemannian metric, the property itself is independent of this choice (for closed manifolds). Manifolds that are enlargeable in dimension 0 are precisely the ones having non-zero $\hat{A}$-genus. Another main example is the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$, which is enlargeable in the top dimension n. Furthermore, every closed manifold that admits a metric of non-positive sectional curvature is enlargeable (in the top dimension) by the theorem of Cartan-Hadarmard. If M is enlargeable then the direct sum $M \# N$ with another closed manifold N is again enlargeable. Furthermore, for an enlargeable manifold M the product $M \times S^1$ with a circle is again enlargeable. This, and much more, is discussed in great detail in [LM89, Sec. IV.5].

The proof that enlargeable spin manifolds do not admit PSC metrics [GL83, Thm. 5.21] can be split into two parts. The first part consists of using the enlargeability condition to construct a suitable family of complex vector bundles over coverings of the manifold. The existence of such a sequence of coverings and bundles is also the starting point of Hanke and Schick's proof that enlargeability implies non-triviality of the Rosenberg index $\alpha^{\mathbb{R}}_{\max}(M) \in \mathrm{KO}^{-n}(C^*_{\max,\mathbb{R}}\pi_1(M))$ [HS06; HS07].

Theorem 3.5.2 (Gromov-Lawson)**.** *Let (M, g) be an $\hat{A}$-area-enlargeable manifold of even dimension. Then there exists a sequence of coverings $M_i \to M$ and $\mathbb{Z}/2\mathbb{Z}$-graded hermitian vector bundles $E_i = E_i^{(0)} \oplus E_i^{(1)} \to M_i$ with compatible connection, such that*

- *for all $i \in \mathbb{N}$ the bundles $E_i^{(0)} \to M_i$ and $E_i^{(1)} \to M_i$ are isometrically isomorphic in a connection preserving way outside a compactum K_i,*
- $\hat{A}(M_i, E_i) = \int_{M_i} \hat{A}(TM_i) \wedge (\mathrm{ch}(E_i^{(0)}) - \mathrm{ch}(E_i^{(1)})) \neq 0$ *for all $i \in \mathbb{N}$ and*

- $\|R^{E_i}\|_\infty \longrightarrow 0$ *for* $i \longrightarrow \infty$.

Roughly speaking, the construction is the following. When $\hat{A}(M) \neq 0$, then one can just take the constant sequence consisting of the identity $M \to M$ as covering and the trivial $\mathbb{Z}/2\mathbb{Z}$-graded bundle $\underline{\mathbb{C}} \oplus \underline{0} \to M$ as bundle. Else, if $\hat{A}(M) = 0$, one may take a sequence $M_i \to M$ such that M_i admits a $\frac{1}{i}$-area-contracting map $M_i \to S^{2\ell}$, where $2\ell = k > 0$ is chosen as in the definition of enlargeability. The bundles are obtained by pulling back a bundle $E^{(0)} \oplus E^{(1)} \to S^{2\ell}$, where $E^{(0)} \to S^{2\ell}$ satisfies $c_\ell(E^{(0)}) \neq 0$ and $E^{(1)} \to S^{2\ell}$ is a trivial bundle of the same rank.

The second part consists of calculating the index of the Dirac operator on the twisted spinor bundle $\Sigma M_i \otimes E_i$ in two different ways. On the one hand, by the the relative index theorem, its index is $\hat{A}(M_i, E_i) \neq 0$. On the other hand, assuming that M carries a PSC metric, the twisted Schrödinger-Lichnerowicz formula implies that this Dirac operator is invertible and thus has index zero, for large $i \in \mathbb{N}$. As we are interested in initial data sets, we replace this second step and obtain our main theorem:

Theorem 3.5.3 (Theorem D, Theorem 3.1.5)**.** *Let M be a closed spin manifold that is $\hat{A}$-area-enlargeable. Then the path-components C^- and C^+ of $\mathcal{I}^>(M)$ do not agree, i. e. if g is a metric on M and $\tau > 0$ is chosen so large that $(g, -\tau g), (g, \tau g) \in \mathcal{I}^>(M)$, then $(g, -\tau g)$ and $(g, \tau g)$ belong to different path-components of $\mathcal{I}^>(M)$.*

Proof. We first consider the case where the dimension of M is even. Let g be a metric on M and $\tau > 0$ be large enough that $(g, -\tau g), (g, \tau g) \in \mathcal{I}^>(M)$. We choose a sequence of complex vector bundles $E_i \to M_i$ as in Theorem 3.5.2 and denote by g_i the pull-back metric of g on M_i. From the relative index theorem Corollary 3.4.3, we obtain that for all $i \in \mathbb{N}$ the twisted index difference $\text{ind-diff}^{E_i}((g_i, -\tau g_i), (g_i, \tau g_i))$ corresponds to $\hat{A}(M_i, E_i) \neq 0$ under the isomorphism $\mathrm{K}^1(I, \partial I) \cong \mathbb{Z}$, in particular it is non-zero.

We now assume for contradiction that $(g, -\tau g)$ and $(g, \tau g)$ are connected in $\mathcal{I}^>(M)$ by a path $t \mapsto (g(t), k(t))$. As the interval I is compact, there is a constant $c > 0$, such that $\rho(t) - |j(t)| \geq 4c$ for all $t \in I$. Of course, this holds as well for the pulled-back path in $\mathcal{I}^>(M_i)$, with the same constant. Since for any $\phi \in \overline{\Sigma} M_i$ and $e \in E_i$

$$|\mathcal{R}^{E_i}(\phi \otimes e)| \leq \sum_{j<k} |\phi||R^{E_i}(e_j, e_k)e| \leq \frac{n(n-1)}{2}|R^{E_i}||\phi \otimes e|.$$

and $\|R^{E_i}\|_\infty \longrightarrow 0$ for $i \longrightarrow \infty$, we have $|\mathcal{R}^{E_i}| < c$ as long as $i \in \mathbb{N}$ is large enough. Hence for large $i \in \mathbb{N}$ the pulled back path $t \mapsto (g_i(t), k_i(t))$ lies entirely in $\mathcal{I}^{>\mathcal{R}^{E_i}+c}(M_i)$. Thus by Proposition 3.3.6 $\text{ind-diff}^{E_i}((g_i, -\tau g_i), (g_i, \tau g_i)) = 0$, which is the desired contradiction.

In the odd-dimensional case, we replace M by $M \times S^1$, which will again be $\hat{A}$-area-enlargeable and spin, and is of even dimension. Thus, we conclude that $C^+ \neq C^-$ in $\mathcal{I}^>(M \times S^1)$. By Lemma 3.2.4, the same holds for $\mathcal{I}^>(M)$. □

As a consequence, we obtain the following 3-dimensional result announced in the introduction.

Theorem 3.5.4 (Theorem C, Theorem 3.1.3)**.** *Let M be an orientable closed connected 3-manifold. Then the following are equivalent:*

1. *$M \cong M_1 \sharp \cdots \sharp M_k$, where each M_i is either S^3/Γ for a lattice $\Gamma \subseteq \mathrm{SO}(4)$ or $S^2 \times S^1$.*
2. *M admits a metric of positive scalar curvature.*
3. *$C^+ = C^-$, so big bang and big crunch initial data sets belong to the same path-component of $\mathcal{I}^>(M)$.*
4. *M is not $\hat{A}$-area-enlargeable.*
5. *M is not compactly enlargeable.*

Proof. Clearly, all the summands in item 1 admit a positive scalar curvature metric. As the direct sum operation is defined as a dimension 0 surgery on the disjoint union, M will also carry a positive scalar curvature metric by a theorem independently shown by Gromov-Lawson and Schoen-Yau (cf. [LM89, Prop. IV.4.3]). If g is a positive scalar curvature metric on M, then the path $t \mapsto (g, tg) \in \mathcal{I}^>(M)$ shows that $C^+ = C^-$. The implication from the third to the fourth point is the contrapositive of Theorem 3.5.3. For this, note that orientable 3-manifolds always admit a spin structure. It is trivial that a compactly enlargeable manifold is $\hat{A}$-area-enlargeable.

To show that the last item implies the first, we decompose M into prime factors $M_1, \ldots,$ M_k. As M is not enlargeable, none of the prime factors M_i can be aspherical (cf. [KN13, Thm. 3]). Thus M_i is either diffeomorphic to $S^2 \times S^1$ or it has a finite fundamental group. In the latter case, elliptization provides a spherical metric on M_i, so it is diffeomorphic to a quotient S^3/Γ as claimed. □

4. Initial data rigidity via Dirac-Witten operators

In this chapter we prove an initial data rigidity result à la Eichmair, Galloway and Mendes [EGM21] using Dirac operator techniques. It applies to initial data sets on spin bands that satisfy the dominant energy condition, a boundary condition for the future null expansion scalar and the $\hat{A}$-obstruction for positive scalar curvature on one of the boundary pieces. Interestingly, these bands turn out to carry lightlike imaginary W-Killing spinors, which are connected to Lorentzian special holonomy and moduli spaces of Ricci-flat metrics. We also obtain slight generalizations of known rigidity results on Riemannian bands.

4.1. Introduction

One of the main questions studied for positive scalar curvature is that of existence: Given an n-manifold M, does it carry a metric of positive scalar curvature? This question is answered by either providing a construction or finding a suitable obstruction. Obstructionwise, two main answers have been found: Firstly, Dirac operator techniques. For instance, if M is a closed spin manifold with non-vanishing $\hat{A}$-genus $\hat{A}(M)$, then with respect to any metric it carries a non-trivial Dirac-harmonic spinor and none of those can be of positive scalar curvature. Secondly, minimal hypersurface techniques. If M is a closed oriented manifold of dimension $2 \leq n \leq 7$ and assuming that there are cohomology classes $h_1, \dots, h_{n-2} \in H^1(M, \mathbb{Z})$ such that $[M] \cap (h_1 \cup \dots h_{n-2})$ is not in the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M, \mathbb{Z})$, then M does not carry a positive scalar curvature metric (cf. e. g. [Sch98]). Actually, the upper dimension bound for the minimal hypersurface method can be improved to 10. This is a consequence of a recent result by Chodosh, Mantoulidis and Schulze [CMS23] building on earlier work by Nathan Smale [Sma93] that allowed to raise the bound to 8.

When trying to extend these obstruction results from positive to non-negative scalar curvature, we encounter rigidity phenomena. Most prominently, if a closed manifold M carries a non-negative scalar curvature metric g, but is known not to admit a positive scalar curvature metric, then g must already be Ricci-flat (cf. e. g. [KW75, Lemma 5.2]). Moreover, in the case where M is spin with $\hat{A}(M) \neq 0$, the Riemannian manifold (M, g)

carries a non-trivial parallel spinor. By studying closed Ricci-flat manifolds in more detail, further rigidity may be deduced. For example, if additionally $b_1(M) \geq n$ (or $b_1(M) \geq n-1$ if M is orientable), then by an argument of Bochner (M, g) must even be isometric to a flat torus (cf. e. g. [Pet16, Corollary 9.5.2]).

When dealing with compact manifolds with boundary, even without closely looking at the geometry of Ricci-flat manifolds, there are surprisingly strong rigidity statements for non-negative scalar curvature metrics. To obtain these, it is important to also assume appropriate boundary conditions. Otherwise there are no obstructions to positive scalar curvature by Gromov's h-principle, and hence no rigidity in the sense of this thesis. In [BH23] Bär and Hanke discuss and compare various boundary conditions. A crucial role is played by the condition of mean convexity of the boundary. This means that the mean curvature, defined as $H^g = \frac{1}{n-1} \operatorname{tr}(-\nabla \nu)$ for the inward-pointing unit normal ν, is non-negative. Among other things, they show the following rigidity statement [BH23, Thm. 19]: If M is a compact connected spin manifold with boundary that has stably infinite K-area, then any Riemannian metric g with $\operatorname{scal}^g \geq 0$ and $H^g \geq 0$ is Ricci-flat with $H^g \equiv 0$. In particular, positive scalar curvature metrics with mean convex boundary are obstructed on such manifolds. The theorem remains true for manifolds $M' \times N$, where M' has stably infinite K-area as above and N is a closed spin manifold with non-zero $\hat{A}$-genus; in particular, it applies to $M = [0,1] \times N$.

In this chapter, we will obtain a strengthening of this special case, where we do not a priori assume a cylindrical form, but just suppose that M is what is sometimes called a *band*. This means that the boundary is decomposed into two pieces $\partial M = \partial_+ M \,\dot\cup\, \partial_- M$ as a topological disjoint union, so $\partial_+ M$ and $\partial_- M$ are unions of components of ∂M. Also, we obtain a more explicit description of the metrics in the rigidity case.

Corollary 4.1.1 (Theorem F with $\varepsilon = 0$)**.** *Let (M, g) be a compact connected Riemannian spin manifold with boundary $\partial M = \partial_+ M \,\dot\cup\, \partial_- M$. Assume that*

- *g has non-negative scalar curvature* $\operatorname{scal}^g \geq 0$*,*
- *the boundary is mean convex, i. e. $H^g \geq 0$ with respect to the inward-pointing unit normal on ∂M, and*
- *the $\hat{A}$-genus of $\partial_- M$ is non-zero: $\hat{A}(\partial_- M) \neq 0$.*

Then (M, g) is isometric to $(\partial_- M \times [0, \ell], \gamma + \mathrm{d}t^2)$ for a Ricci-flat metric γ on $\partial_- M$ admitting a non-trivial parallel spinor.

A very similar statement was shown by Räde [Räd23, Thm. 2.14]. The major difference is that he assumes a dimension bound and that any closed embedded hypersurface between $\partial_- M$ and $\partial_+ M$ does not to admit positive scalar curvature so that minimal hypersurface (more precisely: μ-bubble) techniques can be applied. On the other hand he has no need of the spin and the $\hat{A}$-condition on $\partial_- M$ that we use for our spinorial proof.

In our case, Corollary 4.1.1 arises as a byproduct of studying rigidity for initial data sets. By an *initial data set* on a manifold M we understand a pair (g, k) consisting of a Riemannian metric g and a symmetric 2-tensor field k on M. They naturally appear in the following way: If M is a spacelike hypersurface in a time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$, then there is an induced initial data set (g, k) on M, where g is the induced Riemannian metric on M and k is its second fundamental form with respect to the future-pointing unit normal e_0 (cf. (1.1)). They serve as initial data for the Cauchy problem of general relativity, together with initial data for the matter fields under consideration. In particular, the pair (g, k) is all the initial data needed for the Cauchy problem in the vacuum case, where *energy density* $\rho := \frac{1}{2}(\mathrm{scal}^g + \mathrm{tr}^g(k)^2 - |k|_g^2)$ and *momentum density* $j := \mathrm{div}^g(k) - \mathrm{d}\,\mathrm{tr}^g(k)$ vanish identically. More generally, the initial data sets of physical interest are the ones that satisfy the *dominant energy condition* $\rho \geq |j|_g$. Note that if $k \equiv 0$, then (g, k) satisfies the dominant energy condition if and only if $\mathrm{scal}^g \geq 0$.

Let now (g, k) be an initial data set on a manifold M, potentially with boundary ∂M. We consider a co-oriented hypersurface F in M. The co-orientation will be given by a unit normal vector field $\tilde{\nu}$. Following physics literature, we refer to the direction of $\tilde{\nu}$ as *outgoing* and make the following definitions: The *future outgoing null second fundamental form* $\chi^+ \in \Gamma(\bigodot^2 T^*F)$ is defined by $\chi^+ = g(\nabla\tilde{\nu}, -) + k_{|F}$. Its trace is the *future outgoing null expansion scalar* $\theta^+ = \mathrm{tr}^F(\chi^+) = \mathrm{tr}^F(\nabla\tilde{\nu}) + \mathrm{tr}^F(k)$.

Geometrically, its significance is the following. If (g, k) is the induced initial data set on a hypersurface M of a time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$, then the second fundamental form of F in $\overline{M}$ can be expressed in the normal frame of F given by the null vector fields $\tilde{\nu}+e_0$ and $\tilde{\nu}-e_0$. In this case the coefficient in front of $\tilde{\nu}-e_0$ is given by $-\frac{1}{2}\chi^+$. Thus for any compactly supported variation $(F_t)_{t\in(-\varepsilon,\varepsilon)}$ of $F = F_0$ in $\overline{M}$ with variation vector field $f \cdot (\tilde{\nu} + e_0)$, $f \in C_c^\infty(F)$, the first variation of the volume is given in terms of the null expansion scalar, namely it is equal to $\int_F f\, \mathrm{div}^F(\tilde{\nu} + e_0)\mathrm{dvol} = \int_F f\theta^+ \mathrm{dvol}$.

If $\theta^+ \equiv 0$, then F is called a *MOTS* (which stands for *marginally outer trapped surface*). Note that if $k \equiv 0$, then χ^+ and θ^+ reduce to the second fundamental form of F in M and (a multiple of) its mean curvature, respectively. A MOTS is then just a minimal surface.

With these notions at hand, we can formulate the main theorem of this chapter. The role of F will be played at first by the boundary pieces $\partial_+ M$ and $\partial_- M$, later by the leaves F_t of a foliation extending $F_0 = \partial_+ M$ and $F_\ell = \partial_- M$. Notice that, somewhat confusingly, on $\partial_+ M$ the outgoing unit normal $\tilde{\nu}$ is chosen to be the inward-pointing.

Theorem 4.1.2 (Theorem E)**.** *Let M be a compact connected spin manifold with boundary $\partial M = \partial_+ M \,\dot\cup\, \partial_- M$ endowed with an initial data set (g, k). Denote by $\tilde\nu$ the unit normal on ∂M that is inward-pointing along $\partial_+ M$ and outward-pointing along $\partial_- M$. Assume that*

- *(g, k) satisfies the dominant energy condition $\rho \geq |j|_g$,*
- *the future null expansion scalar (with respect to $\tilde\nu$) satisfies $\theta^+ \leq 0$ on $\partial_+ M$ and $\theta^+ \geq 0$ on $\partial_- M$, and*
- *the $\hat{A}$-genus of $\partial_- M$ is non-zero: $\hat{A}(\partial_- M) \neq 0$.*

Then there is a diffeomorphism $\Phi\colon \partial_- M \times [0, \ell] \to M$ defining a foliation $F_t = \Phi(\partial_- M \times \{t\})$ with $F_0 = \partial_+ M$ and $F_\ell = \partial_- M$. The leaves can be endowed with an induced initial data set, an induced spin structure and a unit normal $\tilde\nu$ pointing in the direction of growing t-parameter. The diffeomorphism can be chosen in such a way that the following holds for every leaf F_t:

- *Its future null second fundamental form (with respect to $\tilde\nu$) vanishes, $\chi^+ = 0$, in particular it is a MOTS.*
- *It carries a non-trivial parallel spinor, in particular its metric is Ricci-flat.*
- *Its tangent vectors are orthogonal to $j^\sharp$ and $\rho + j(\tilde\nu) = 0$, in particular the dominant energy condition holds marginally: $\rho = |j|_g$.*

Let us remark that $\hat{A}(\partial_- M) \neq 0$ in particular implies that $\partial_- M$ is non-empty. The same is true for $\partial_+ M$ since $\partial_+ M$ is spin bordant to $\partial_- M$ and thus $|\hat{A}(\partial_+ M)| = |\hat{A}(\partial_- M)|$. We also see that the theorem is symmetric under exchanging $\partial_+ M$ with $\partial_- M$ (i. e. flipping the orientation of $\tilde\nu$) if at the same time k is replaced by $-k$.

Initial data rigidity was first studied by Eichmair, Galloway and Mendes in their recent paper [EGM21]. They did so using minimal hypersurface (or rather: MOTS) techniques, whereas this thesis follows a spinoral approach to the problem. Let us compare Theorem 4.1.2 with their result [EGM21, Thm. 1.2] in little more detail. The main setup is the same: They also consider initial data sets on compact connected manifolds with boundary satisfying the dominant energy condition and the boundary condition for θ^+. Then, there is an assumption that excludes positive scalar curvature on one of the boundary pieces. In our case, this is provided $\hat{A}(\partial_- M) \neq 0$ and the observation that $\partial_- M$ is spin. In their case, it is what they call cohomology condition – existence of classes $h_1, \ldots h_{n-1} \in H^1(\partial_- M, \mathbb{Z})$ with $h_1 \cup \ldots \cup h_{n-1} \neq 0$ – together with the dimension bound $2 \leq n - 1 \leq 6$. As a last assumption some “weak niceness” of the boundary inclusion $\partial_- M \hookrightarrow M$ is needed. We need a spin structure of $\partial_- M$ to extend to M; they require the so-called homotopy condition, i. e. that there is a continuous map $M \to \partial_- M$ so that the composition $\partial_- M \hookrightarrow M \to \partial_- M$ is homotopic to the identity. The conclusions almost coincide: Both theorems show $M \cong \partial_- M \times [0,\ell]$ such that the canonical leaves are Ricci-flat manifolds with $\chi^+ = 0$ and $j^\sharp = -\rho\tilde{\nu}$. In our theorem, we additionally obtain existence of a non-trivial parallel spinor on the leaves – a feature we are going to discuss in more detail below. Since Eichmair, Galloway and Mendes impose strong enough conditions on $\partial_- M$ to make use of the argument by Bochner mentioned above, they are able to further conclude that the leaves are isometric to flat tori.

In a new article [GM24], Galloway and Mendes also discuss initial data rigidity for closed manifolds. An analogue of [GM24, Thm. 4.2] is the following immediate consequence of Theorem 4.1.2: If (g,k) is an initial data set with $\rho \geq |j|_g$ on a closed connected spin manifold M that contains a MOTS F with $\hat{A}(F) \neq 0$, then M is diffeomorphic to a mapping torus $F \times [0,\ell]/(x,\ell) \sim (f(x),0)$, $f \in \mathrm{Diff}(F)$, in such a way that the canonical leaves satisfy all the properties listed in Theorem 4.1.2.

The author's interest into initial data rigidity arose from studying the space of initial data sets subject to the dominant energy condition on a fixed manifold M. In the chapters before (covering the articles [Glö24b] and [Glö23a]), it was shown that for many choices of M this space has non-trivial homotopy groups and, more importantly, different connected components – if the strict version $\rho > |j|_g$ of the dominant energy condition is considered. For statements of increased physical relevance, this condition should be relaxed to the non-strict dominant energy condition $\rho \geq |j|_g$. For instance, in [AG23] Bernd Ammann and the author discuss that for certain manifolds M any spacetime $(\overline{M},\overline{g})$ containing M as Cauchy hypersurface and satisfying the spacetime dominant energy condition cannot have both a big bang and a big crunch singularity. The main step there is the mentioned passage from strict to non-strict inequality. This is done by examining how rigid the equality case is.

More precisely, in the situation of [AG23], a bit more is known about the equality case of interest: There exists a spinor $\phi \not\equiv 0$ that is parallel with respect to the connection $\overline{\nabla}_X\phi = \nabla_X\phi + \frac{1}{2}k(X, \text{-})^\sharp \cdot e_0 \cdot \phi$. Here, ϕ is a section of the *hypersurface spinor bundle* $\overline{\Sigma}M \to M$ (cf. Section 4.2), $e_0\cdot\colon \overline{\Sigma}M \to \overline{\Sigma}M$ is a Clifford-antilinear involution that it comes equipped with and ∇ is induced by the Levi-Civita connection of (M,g). These $\overline{\nabla}$-parallel spinors (which are also known under the name *imaginary W-Killing spinor*) come in two flavors, depending on whether their Lorentzian Dirac current $V_\phi = u_\phi e_0 - U_\phi \in \Gamma(TM \oplus \underline{\mathbb{R}}e_0)$ (cf. Definition 4.4.2) is timelike or lightlike. Especially the lightlike ones have attracted attention, since they play an important role in the study of Lorentzian special holonomy [BLL16].

In the proof of Theorem 4.1.2, one main step will be to show existence of a non-trivial $\overline{\nabla}$-parallel spinor. As it turns out, non-emptiness of the boundary helps since a boundary condition forces the spinor to be lightlike. From there, the other conclusions will be deduced by considering the foliation defined by U_ϕ. Since this intermediate result might be of interest in the future, we formulate it more explicitly.

Addendum 4.1.3. Under the assumptions of Theorem 4.1.2, the initial data set (g,k) on M carries a lightlike $\overline{\nabla}$-parallel spinor ϕ. The foliation $(F_t)_{t\in[0,\ell]}$ in Theorem 4.1.2 may be constructed in such a way that the Riemannian Dirac current U_ϕ of ϕ is orthogonal[1] to the leaves F_t.

One might ask whether even more rigidity can be deduced, similarly as in Corollary 4.1.1 where the metric stays the same on all leaves. There is only little room for this, since Bernd Ammann, Klaus Kröncke and Olaf Müller gave a method for constructing lightlike $\overline{\nabla}$-parallel spinors on cylinders $M\times[0,L]$ in [AKM21] providing many examples of initial data sets as in Theorem 4.1.2. Namely, from a Ricci-flat metric γ_0 on M with a parallel spinor $\phi_0\not\equiv 0$, a smooth curve $([\tilde\gamma_t])_{t\in[0,L]}$ in the moduli space of Ricci-flat metrics on M starting at $[\tilde\gamma_0]=[\gamma_0]$ and a smooth function $f\colon[0,L]\to\mathbb{R}$ they construct an initial data set $(g,k)=(\gamma_t+\mathrm{d}t^2,\frac{1}{2}\frac{\partial}{\partial t}\gamma_t+f(t)\mathrm{d}t^2)$ and a lightlike $\overline{\nabla}$-parallel spinor ϕ on $M\times[0,L]$ such that $[\gamma_t]=[\tilde\gamma_t]$ for all $t\in[0,L]$, $\phi_{|M\times\{0\}}=\phi_0$ and U_ϕ is orthogonal to the canonical leaves. There is one point, where more rigidity could be hidden: All these cylinders feature that $|\phi|^2=|U_\phi|_g$ is constant along the leaves; this norm is given in terms of $|\phi_0|$ and f. To the moment it is not known whether or not this leafwise constancy always holds.

We conclude our discussion with a rather general rigidity statement for Riemannian bands, which essentially follows from Theorem 4.1.2. Though its assumptions might seem rather technical, they nicely fit into the context of warped products.

Example 4.1.4. Consider a warped product $(\tilde M,\tilde g)=(N\times[0,L],w(s)^2\tilde\gamma+\mathrm{d}s^2)$ for a Riemannian manifold $(N,\tilde\gamma)$ and a warping function $w\colon[0,L]\to\mathbb{R}$. Setting $h=\frac{w'}{w}$, its scalar curvature is given by $\mathrm{scal}^{\tilde g}=(w\circ s)^{-2}\,\mathrm{scal}^{\tilde\gamma}-n(n-1)(h\circ s)^2-2(n-1)h'\circ s$, where s denotes the canonical projection on the $[0,L]$-factor. Moreover, the mean curvature of the leaf $N\times\{s\}$ with respect to the unit normal $\frac{\partial}{\partial s}$ is given by $H^{\tilde g}=-h(s)$. Let us now assume that the scalar curvature of $(N,\tilde\gamma)$ is non-negative, the warping function is log-concave, i. e. $\frac{\mathrm{d}^2}{\mathrm{d}s^2}\log(w)=h'\leq 0$, and there exists a 1-Lipschitz map $\tilde\Phi\colon(M,g)\to(\tilde M,\tilde g)$ sending ∂_+M to $N\times\{0\}$ and ∂_-M to $N\times\{L\}$ such that $\mathrm{scal}^g\geq\mathrm{scal}^{\tilde g}\circ\tilde\Phi$ and $H^g\geq H^{\tilde g}\circ\tilde\Phi$ (along ∂M). Then h and $s\circ\tilde\Phi\colon M\to[0,L]$ satisfy the assumptions of Theorem 4.1.5.

With this example in mind, Theorem 4.1.5 may thus be read as a comparison result.

Theorem 4.1.5. *Let (M,g) be a compact connected Riemannian spin manifold of dimension n with boundary $\partial M=\partial_+M\,\dot\cup\,\partial_-M$. Suppose $h\colon[0,L]\to\mathbb{R}$ is a smooth*

[1] Actually, pointing in the direction of $-\tilde\nu$.

function with $h' \leq 0$ *and* $s\colon M \to [0, L]$ *is a smooth map with* $s(\partial_+ M) = \{0\}$ *and* $s(\partial_- M) = \{L\}$ *and such that* $|\mathrm{d}s|_g \leq 1$*. Assume that*

- *the scalar curvature of* g *is bounded below by* $\mathrm{scal}^g \geq -n(n-1)(h \circ s)^2 - 2(n-1)h' \circ s$*,*
- *the mean curvature of the boundary with respect to the inward-pointing unit normal* ν *is bounded below by* $H^g \geq -h(0)$ *on* $\partial_+ M$ *and* $H^g \geq h(L)$ *on* $\partial_- M$*, and*
- *the* $\hat{A}$*-genus of* $\partial_- M$ *is non-zero:* $\hat{A}(\partial_- M) \neq 0$*.*

Then there is an isometry $\Phi\colon (\partial_- M \times [0, \ell], v(t)^2\gamma + \mathrm{d}t^2) \to (M, g)$ *with* $\Phi(\partial_- M \times \{0\}) = \partial_+ M$ *and* $\Phi(\partial_- M \times \{\ell\}) = \partial_- M$*, where* $v\colon [0, \ell] \to \mathbb{R}$ *is a smooth function and* γ *is a Ricci-flat metric on* $\partial_- M$ *admitting a non-trivial parallel spinor. More precisely, the composition* $h \circ s$ *is constant along the leaves of the canonical foliation and – reinterpreting* $h \circ s$ *as function* $[0, \ell] \to \mathbb{R}$ *– the warping function* v *is determined (up to multiplication by a constant) by* $\frac{v'}{v} = h \circ s$*. Moreover,* $\mathrm{d}s = (\Phi^{-1})^*\mathrm{d}t$ *wherever* $h' \circ s \neq 0$ *and the inequalities for* scal^g *and* H^g *are equalities.*

Again, there is a symmetry interchanging $\partial_+ M$ with $\partial_- M$. This involves replacing s by $\sigma \circ s$ and h by $-h \circ \sigma$, where $\sigma\colon [0, L] \to [0, L]$ is the affine linear map switching the boundaries. Furthermore, the assumption on s can be slightly weakened.

Remark 4.1.6. Theorem 4.1.5 still holds true when the condition $|\mathrm{d}s|_g \leq 1$ is only satisfied on the subset of M where $h' \circ s \neq 0$.

Comparable statements were already derived in articles by Cecchini and Zeidler [CZ24] and Räde [Räd23]. The theorem of Cecchini and Zeidler [CZ24, Thm. 8.3 (cf. also Thms. 9.1 and 10.2)] is also derived using spinor techniques. It is more general in the sense that it also allows for non-trivial twist bundles $E \to M$ with the consequence that the index obstruction can be relaxed to $0 \neq \hat{A}(\partial_- M, E) = \int_{\partial_- M} \hat{A}(T\partial_- M) \wedge \mathrm{ch}(E_{|\partial_- M})$. On the other hand, it is more restrictive as it requires the strict inequality $h' < 0$. In this case, the *band width* $\mathrm{dist}^g(\partial_+ M, \partial_- M)$ plays a crucial role as the function s needs to be 1-Lipschitz. Since a priori there does not need to exist a smooth 1-Lipschitz function s realizing the width, meaning $L = \mathrm{dist}^g(\partial_+ M, \partial_- M)$, it is also interesting to allow for non-smooth Lipschitz functions in the case $h' < 0$. Cecchini and Zeidler's article contains some arguments for this.

As already mentioned above, in Räde's work the $\hat{A}$- and the spin condition are replaced by conditions needed for a μ-bubble argument to work. His main theorem captures both the case $h' < 0$ and the case $h' \equiv 0$, but only in the latter case he is able to derive a rigidity statement comparable to Theorem 4.1.5.

Although the general case $h' \leq 0$ seems to be new, the main applications of Theorem 4.1.5 are the ones, where h is such that the lower scalar curvature bound is a constant. Then h is subject to an ordinary differential equation and either $h' < 0$ or $h' \equiv 0$. These functions and the associated corollaries are discussed in [Räd23, Sec. 2.A]. We restrict our attention just to the cases $h \equiv 0$, which yields Corollary 4.1.1, and $h \equiv -1$ yielding Corollary 4.1.7 below, where our theorem supersedes the result of Cecchini and Zeidler. (Setting $h \equiv 1$ yields the statement of Corollary 4.1.7 with interchanged boundary pieces and Theorem F with $\varepsilon = 1$.) Notice that a statement analogous to Corollaries 4.1.1 and 4.1.7 is also contained in the article by Eichmair, Galloway and Mendes [EGM21, Cor. 1.4].

Corollary 4.1.7 (Theorem F with $\varepsilon = -1$)**.** *Let (M, g) be a compact connected Riemannian spin manifold with boundary $\partial M = \partial_+ M \,\dot\cup\, \partial_- M$ of dimension n. Assume that*

- *the scalar curvature is bounded below by $\mathrm{scal}^g \geq -n(n-1)$,*
- *the mean curvature of the boundary with respect to the inward-pointing unit normal is bounded below by $H^g \geq 1$ on $\partial_+ M$ and $H^g \geq -1$ on $\partial_- M$, and*
- *the $\hat{A}$-genus of $\partial_- M$ is non-zero: $\hat{A}(\partial_- M) \neq 0$.*

Then (M, g) is isometric to $(\partial_- M \times [0, \ell], e^{-2t}\gamma + \mathrm{d}t^2)$ (with $\partial_+ M$ corresponding to $\partial_- M \times \{0\}$) for a Ricci-flat metric γ on $\partial_- M$ admitting a non-trivial parallel spinor.

Let us finally discuss the strategy of the proof of Theorem 4.1.2 and the structure of this chapter. The main tool we are going to use is the *Dirac-Witten operator* $\overline{D}$, which lives on the hypersurface spinor bundle $\overline{\Sigma} M \to M$ mentioned above. This bundle is best explained if M is assumed to sit as a spacelike hypersurface in a time-oriented Lorentzian spin manifold $(\overline{M}, \overline{g})$. In this case, $\overline{\Sigma} M \to M$ is just the restriction to M of a spinor bundle on $\overline{M}$. In particular, it carries an involution $e_0 \cdot$ induced by Clifford multiplication with the future unit normal on M and a connection $\overline{\nabla}$ induced by the Levi-Civita connection of $(\overline{M}, \overline{g})$. In Section 4.2, we discuss how to construct this bundle and its additional structures intrinsically, from the initial data set (g, k) and a spin structure on M alone. The Dirac-Witten operator $\overline{D}$ is the Dirac operator of $\overline{\Sigma} M$ with respect to its connection $\overline{\nabla}$. The hypersurface spinor bundle also carries a connection ∇ induced from the Levi-Civita connection of (M, g) and there is a Dirac operator D associated to it. As it turns out, they are related via $\overline{D} = D - \frac{1}{2}\operatorname{tr}^g(k) e_0 \cdot$. This means that $\overline{D}$ is a Callias operator in the sense of Cecchini and Zeidler. Putting *chirality boundary conditions* $\tilde{\nu} \cdot \phi = -e_0 \cdot \phi$ on the sections of $\overline{\Sigma} M \to M$, we are able to invoke their analytical results. This is done in Section 4.3. As a result, we obtain existence of non-trivial Dirac-Witten harmonic spinors ϕ subject to chirality boundary conditions if $\hat{A}(\partial_- M) \neq 0$. These spinors are then further studied in Section 4.4 using an integrated version of the Schrödinger-Lichnerowicz formula $\overline{D}^2 = \overline{\nabla}^*\overline{\nabla} + \frac{1}{2}(\rho - e_0 \cdot j^\sharp \cdot)$. If the dominant energy condition holds and the inequalities for θ^+ are satisfied along ∂M, we are able to conclude that ϕ is a lightlike $\overline{\nabla}$-parallel spinor. We continue by studying the foliation defined by its Riemannian Dirac current U_ϕ. Doing so, we prove the main theorems – up to the observation that this foliation is actually of cylindrical type $\partial_- M \times [0, \ell]$. The remaining piece is provided in Section 4.5, where we look at the flow of $-\frac{U_\phi}{|U_\phi|_g^2}$ in more detail.

4.2. Spinor bundles on hypersurfaces

This section is devoted to the study of spinor bundles on hypersurfaces. This is to be understood in a two-fold manner: First, we are interested in the situation where M is an n-dimensional spacelike hypersurface of a time-oriented Lorentzian manifold $\overline{M}$. Second, we assume that M has boundary ∂M and restrict the spinor bundle further to ∂M.

The first step is known under the name hypersurface spinor bundle, cf. [HZ03; AG23]. The construction is the following: Given a complex, say, representation $\mathbb{C}l_{n,1} \to \mathrm{End}(W)$ and a spin structure $P_{\mathrm{Spin}(n)}M \to M$ of the Riemannian spin n-manifold (M, g), we form the *hypersurface spinor bundle* $\overline{\Sigma}M \to M$ by associating W to $P_{\mathrm{Spin}(n)}M$ via the restricted representation $\mathrm{Spin}(n) \hookrightarrow \mathbb{C}l_n \hookrightarrow \mathbb{C}l_{n,1} \to \mathrm{End}(W)$. To justify the name, we assume that (M, g) is a spacelike hypersurface (with induced metric) of a space- and time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$. We moreover assume that $\overline{M}$ is spin (which can be assured by restricting to a small neighborhood of M) and the spin structure $P_{\mathrm{Spin}_0(n,1)}\overline{M} \to \overline{M}$ restricts to the one of M in the sense that

$$\begin{array}{ccc} P_{\mathrm{Spin}(n)}M & \longrightarrow & P_{\mathrm{Spin}_0(n,1)}\overline{M}_{|M} \\ \downarrow & & \downarrow \\ P_{\mathrm{SO}(n)}M & \longrightarrow & P_{\mathrm{SO}_0(n,1)}\overline{M}_{|M} \\ (e_1, \ldots, e_n) & \longmapsto & (e_0, e_1, \ldots, e_n) \end{array} \tag{4.1}$$

is a pullback, where e_0 denotes the future unit normal of M in $(\overline{M}, \overline{g})$. Then the spinor bundle $\Sigma\overline{M} \to \overline{M}$ associated to the representation $\mathbb{C}l_{n,1} \to \mathrm{End}(W)$ restricts to the hypersurface spinor bundle on M, meaning that the canonical map yields a bundle isomorphism $\overline{\Sigma}M \cong \Sigma\overline{M}_{|M}$.

The hypersurface spinor bundle can be equipped with additional structures. First of all, it comes with a Clifford multiplication $T\overline{M}_{|M} \otimes \overline{\Sigma}M \to \overline{\Sigma}M$. For our purposes, it is more convenient to view it as a Clifford multiplication by vectors of TM and an involution $e_0\cdot\colon \overline{\Sigma}M \to \overline{\Sigma}M$ that anti-commutes with the TM-Clifford multiplication. Secondly, if W admits a $\mathbb{Z}/2\mathbb{Z}$-grading and the representation $\mathbb{C}\mathrm{l}_{n,1} \to \mathrm{End}(W)$ is a graded representation, then $\overline{\Sigma}M$ carries a $\mathbb{Z}/2\mathbb{Z}$-grading with respect to which the Clifford multiplication and the involution $e_0\cdot$ are odd. Thirdly, W can be endowed with a (positive definite) scalar product that is invariant under multiplication by the standard basis vectors $E_0, E_1, \dots, E_n \in \mathbb{R}^{n,1} \subseteq \mathbb{C}\mathrm{l}_{n,1}$. Such a scalar product can always be constructed by an averaging procedure. It can also be made compatible with the $\mathbb{Z}/2\mathbb{Z}$-grading if W carries one. Such a scalar product on W induces a fiberwise scalar product on $\overline{\Sigma}M$ such that the Clifford multiplication by vectors in TM is skew-adjoint, the involution $e_0\cdot$ is self-adjoint and the $\mathbb{Z}/2\mathbb{Z}$-grading is orthogonal.

The last structure we want to consider is the one of a connection. There are two canonical choices. The Levi-Civita connection of (M, g) gives rise to a connection ∇ on $P_{\mathrm{Spin}(n)}M$ and hence on $\overline{\Sigma}M$. On the other hand, the Levi-Civita connection of $(\overline{M}, \overline{g})$ induces a connection $\overline{\nabla}$ on $\Sigma\overline{M}$ and thus also on $\overline{\Sigma}M$. On tangent bundles the Levi-Civita connections of $(\overline{M}, \overline{g})$ and the hypersurface M differ by the second fundamental form k:

$$\overline{\nabla}_X Y = \nabla_X Y + k(X, Y)\, e_0 \tag{4.2}$$

for all $X, Y \in \Gamma(TM)$. It follows that a similar relation also holds for the connections on $\overline{\Sigma}M$:

$$\overline{\nabla}_X \phi = \nabla_X \phi + \frac{1}{2} k(X, \text{-})^\sharp \cdot e_0 \cdot \phi \tag{4.3}$$

for all $X \in \Gamma(TM)$ and $\phi \in \Gamma(\overline{\Sigma}M)$. This formula allows to define the connection $\overline{\nabla}$ even in the case when M is not embedded as a hypersurface. It is only necessary to have a metric g and a symmetric 2-tensor k playing the role of the second fundamental form. Pairs (g, k) of this kind are known as *initial data sets* on M. From the way it is defined, it is clear that grading, Clifford multiplication, the involution $e_0\cdot$ and scalar product are parallel with respect to ∇. Compatibility formulae for $\overline{\nabla}$ may be derived using (4.3).

Setup 4.2.1. Given an initial data set (g, k) on a spin manifold M, we form a hypersurface spinor bundle $\overline{\Sigma}M \to M$ with the structures of a TM-Clifford multiplication, an involution $e_0\cdot$, a (positive definite) scalar product and a connection $\overline{\nabla}$. They satisfy the compatibility conditions described in the previous two paragraphs. When forming a $\mathbb{Z}/2\mathbb{Z}$-graded hypersurface spinor bundle, we also require the grading to be compatible with the other structures in the above-described sense.

For step two, let M furthermore have boundary ∂M. The inward-pointing unit normal along ∂M will be denoted by ν. The hypersurface spinor bundle restricts to $\overline{\Sigma}M_{|\partial M} \to \partial M$, to which we refer as *boundary hypersurface spinor bundle*. It may, similarly as explained above, also be defined on ∂M intrinsically. From that perspective the $TM_{|\partial M}$-Clifford multiplication can be seen as a $T(\partial M)$-Clifford multiplication together with a homomorphism $\nu\cdot$ anti-commuting with this Clifford multiplication and squaring to $-\mathbb{1}$.

The boundary hypersurface spinor bundle carries even more connections of interest. Of course, the connections $\overline{\nabla}$ and ∇ restrict to $\overline{\Sigma}M_{|\partial M} \to \partial M$. Viewing the boundary hypersurface spinor bundle as bundle associated to the induced spin structure on ∂M, we obtain the connection ∇^∂ induced by the Levi-Civita connection of ∂M. We have

$$\nabla_X \phi = \nabla^\partial_X \phi - \frac{1}{2}(\nabla_X \nu) \cdot \nu \cdot \phi \tag{4.4}$$

for all $X \in \Gamma(T(\partial M))$ and $\phi \in \Gamma(\overline{\Sigma}M_{|\partial M})$.

There is another, less obvious choice. For this, we observe that every metric connection on $T\overline{M}_{|\partial M}$ gives rise to a connection on $P_{\mathrm{Spin}_0(n,1)}\overline{M}_{|\partial M}$ and thus on the boundary hypersurface spinor bundle. In this way the Levi-Civita connection of $(\overline{M}, \overline{g})$ induces $\overline{\nabla}$. Equipping $T\overline{M}_{|\partial M} = TM_{|\partial M} \oplus \underline{\mathbb{R}}e_0$ with sum of the Levi-Civita connection of (M, g) and the trivial connection we obtain ∇. The connection ∇^∂ arises when we put on $T\overline{M}_{|\partial M} = T(\partial M) \oplus \underline{\mathbb{R}}\nu \oplus \underline{\mathbb{R}}e_0$ the sum of the Levi-Civita connection of ∂M with the trivial connection on the other summands. Now, instead, let us take on the normal bundle $N(\partial M) = \underline{\mathbb{R}}\nu \oplus \underline{\mathbb{R}}e_0$ the connection induced by $\overline{\nabla}$, i. e. $\pi^{\mathrm{nor}}(\overline{\nabla}_X n)$ with $X \in \Gamma(T(\partial M))$, $n \in \Gamma(N(\partial M))$ and $\pi^{\mathrm{nor}} \colon T\overline{M}_{|\partial M} \to N(\partial M)$ the orthogonal projection. We obtain a connection on the boundary hypersurface spinor bundle, which we denote by $\overline{\nabla}^\partial$.

As before, there is a simple comparison formula with (one of) the other connections. This time, we provide a proof, which should also serve as a blueprint for the other claims made.

Lemma 4.2.2. *The connection $\overline{\nabla}^\partial$ satisfies*

$$\begin{aligned}\overline{\nabla}_X \phi &= \overline{\nabla}^\partial_X \phi + \frac{1}{2}(\overline{\nabla}_X(\nu \cdot e_0 \cdot))\, \nu \cdot e_0 \cdot \phi \\ &= \overline{\nabla}^\partial_X \phi + \frac{1}{2}(\overline{\nabla}_X \nu) \cdot e_0 \cdot \nu \cdot e_0 \cdot \phi + \frac{1}{2}\nu \cdot (\overline{\nabla}_X e_0) \cdot \nu \cdot e_0 \cdot \phi\end{aligned}$$

for all $X \in \Gamma(T(\partial M))$ and $\phi \in \Gamma(\overline{\Sigma}M_{|\partial M})$.

Proof. Abusing notation, we also denote the corresponding connections on $T\overline{M}_{|\partial M}$ by $\overline{\nabla}$ and $\overline{\nabla}^\partial$, respectively. Their difference defines a tensor $A \coloneqq \overline{\nabla} - \overline{\nabla}^\partial \in \Gamma(T^*(\partial M) \otimes \mathfrak{so}(T\overline{M}, \overline{g})_{|\partial M})$. Looking at tangential and normal parts separately, it computes to $A_X(Y) = \pi^{\mathrm{nor}}(\overline{\nabla}_X \pi^{\mathrm{tan}}(Y)) + \pi^{\mathrm{tan}}(\overline{\nabla}_X \pi^{\mathrm{nor}}(Y))$.

Now observe that $T\overline{M}_{|\partial M}$ is associated to $P_{\mathrm{Spin}_0(n,1)}\overline{M}_{|\partial M}$ via the standard representation $\chi \colon \mathrm{Spin}_0(n,1) \longrightarrow \mathrm{SO}_0(n,1) \subseteq \mathrm{End}(\mathbb{R}^{n,1})$. If $\tilde{A} \in \Omega^1(P_{\mathrm{Spin}_0(n,1)}\overline{M}_{|\partial M}, \mathfrak{spin}(n,1))$

denotes the difference of the connection 1-forms of the connections $\overline{\nabla}$ and $\overline{\nabla}^{\partial}$ on the spin principal bundle, then A may be expressed as

$$A_X(Y) = [\varepsilon,\, \mathrm{d}\chi(\tilde{A} \circ \mathrm{d}\varepsilon(X))\, y],$$

where $Y = [\varepsilon, y] \in T_p\overline{M}$ and ε is a local section of $P_{\mathrm{Spin}_0(n,1)}\overline{M}_{|\partial M}$ around p (cf. [Bau14, (3.11)]). Similarly, the difference term that we aim for is given by

$$(\overline{\nabla}_X - \overline{\nabla}^{\partial}_X)\phi = [\varepsilon,\, \mathrm{d}\rho(\tilde{A} \circ \mathrm{d}\varepsilon(X))\, \Phi],$$

where $\phi = [\varepsilon, \Phi]$, ε is as above and $\rho\colon\ \mathbb{C}\mathrm{l}_{n,1} \to \mathrm{End}(W)$ denotes the Clifford multiplication action.

Around some $p \in \partial M$, fix a local orthonormal frame $(e_0, \nu, e_2, \ldots, e_n)$ which admits a lift ε. Denoting the standard basis of $\mathbb{R}^{n,1}$ by $E_0, E_1, \ldots, E_n$, we obtain $\mathrm{d}\chi(\tilde{A}\circ\mathrm{d}\varepsilon(X))(E_i) = \sum_j s_j \overline{g}(e_j, A_X(e_i))E_j$ with $e_1 = \nu$ and $s_j = \overline{g}(e_j, e_j) \in \{\pm 1\}$. Using that the isomorphism $\mathrm{d}\chi$ is given by $E_iE_j \mapsto 2E_j\langle E_i, -\rangle - 2E_i\langle E_j, -\rangle$, we obtain

$$\tilde{A} \circ \mathrm{d}\varepsilon(X) = \frac{1}{2}\sum_{i<j} s_i s_j \overline{g}(e_j, A_X(e_i))E_iE_j.$$

Thus, remains to compute

$$\begin{aligned}
(\overline{\nabla}_X - \overline{\nabla}^{\partial}_X)\phi &= \frac{1}{2}\sum_{i<j} s_i s_j \overline{g}(e_j, A_X(e_i))\, e_i \cdot e_j \cdot \phi \\
&= \frac{1}{2}\sum_{j=2}^{n} -\overline{g}(e_j, \overline{\nabla}_X e_0)\, e_0 \cdot e_j \cdot \phi + \frac{1}{2}\sum_{j=2}^{n} \overline{g}(e_j, \overline{\nabla}_X \nu)\, \nu \cdot e_j \cdot \phi \\
&= -\frac{1}{2} e_0 \cdot (\overline{\nabla}_X e_0) \cdot \phi + \frac{1}{2}\overline{g}(\nu, \overline{\nabla}_X e_0)\, e_0 \cdot \nu \cdot \phi \\
&\quad + \frac{1}{2}\nu \cdot (\overline{\nabla}_X \nu) \cdot \phi + \frac{1}{2}\overline{g}(e_0, \overline{\nabla}_X \nu)\, \nu \cdot e_0 \cdot \phi \\
&= \frac{1}{2}(\overline{\nabla}_X e_0) \cdot e_0 \cdot \phi - \overline{g}(\nu, \overline{\nabla}_X e_0)\, \nu \cdot e_0 \cdot \phi - \frac{1}{2}(\overline{\nabla}_X \nu) \cdot \nu \cdot \phi \\
&= \frac{1}{2}\nu \cdot (\overline{\nabla}_X e_0) \cdot \nu \cdot e_0 \cdot \phi + \frac{1}{2}(\overline{\nabla}_X \nu) \cdot e_0 \cdot \nu \cdot e_0 \cdot \phi. \qquad \square
\end{aligned}$$

Remark 4.2.3. The expression $X \mapsto \overline{\nabla}_X e_0$ defines a section of the endomorphism bundle of TM since $\overline{g}(\overline{\nabla}_X e_0, e_0) = \frac{1}{2}\partial_X \overline{g}(e_0, e_0) = 0$ for any $X \in TM$. Moreover, we have $\overline{g}(\overline{\nabla}_X e_0, Y) = -\overline{g}(e_0, \overline{\nabla}_X Y) = k(X, Y)$ for all $X, Y \in \Gamma(TM)$, so it is the endomorphism associated to k via g. In particular, it only depends on the initial data set (g, k) and the expression also makes sense when the surrounding Lorentzian manifold $(\overline{M}, \overline{g})$ is not at hand.

4.3. Dirac-Witten operators as Callias operators

In this section, we introduce the main player – the Dirac-Witten operator – and study its analytic properties. As ot turns out, the Dirac-Witten operator is a Callias operator, i. e. of the form Dirac operator plus potential. The analytic framework will be borrowed from Cecchini and Zeidler [CZ24], who studied this kind of operators.

The general setup for this section is the following. We consider a compact spin manifold M with potentially empty boundary $\partial M = \partial_+ M \,\dot\cup\, \partial_- M$. We endow M with an initial data set (g,k) and denote by $\overline{\Sigma} M$ a hypersurface spinor bundle on M as in Setup 4.2.1. As explained in the last section, this carries a connection $\overline{\nabla}_X \phi = \nabla_X \phi - \frac{1}{2} e_0 \cdot k(X, \text{-})^\sharp \cdot \phi$ associated to (g,k). Furthermore, let $\tilde{\nu}$ be the unit normal on ∂M that is inward-pointing along $\partial_+ M$ and outward-pointing along $\partial_- M$. The function s will be defined to be $+1$ on $\partial_+ M$ and -1 on $\partial_- M$, so that $\nu \coloneqq s\tilde{\nu}$ is inward-pointing on all of ∂M.

Definition 4.3.1. The *Dirac-Witten operator* $\overline{D}\colon \Gamma(\overline{\Sigma} M) \to \Gamma(\overline{\Sigma} M)$ of a hypersurface spinor bundle $\overline{\Sigma} M \to M$ is defined by the local formula

$$\overline{D} = \sum_{i=1}^{n} e_i \cdot \overline{\nabla}_{e_i}$$

where $e_1, \ldots, e_n$ is a local g-orthonormal frame.

A straightforward calculation shows

$$\overline{D} = D - \frac{1}{2} \operatorname{tr}^g(k)\, e_0 \cdot,$$

where the Dirac operator $D = \sum_{i=1}^n e_i \cdot \nabla_{e_i}$ is defined with respect to the connection ∇. Hence, the Dirac-Witten operator is the sum of a Dirac operator and a potential – a Callias operator.

One of the most important properties of the Dirac-Witten operator is that it satisfies the following Schrödinger-Lichnerowicz type formula (Proposition 1.3.5, cf. [Wit81; PT82]):

$$\overline{D}^2 = \overline{\nabla}^* \overline{\nabla} + \frac{1}{2}(\rho - e_0 \cdot j^\sharp \cdot),$$

where *energy density* ρ and *momentum density* j are defined by

$$\rho = \frac{1}{2}(\operatorname{scal}^g + \operatorname{tr}^g(k)^2 - |k|_g^2)$$
$$j = \operatorname{div}^g(k) - \mathrm{d}\operatorname{tr}^g(k),$$

respectively. We will study an integrated form of this identity. For the boundary terms appearing, we use the following definitions:

Definition 4.3.2. The *boundary chirality operator* $\mathcal{X} \colon \overline{\Sigma}M_{|\partial M} \to \overline{\Sigma}M_{|\partial M}$ of the hypersurface spinor bundle is defined by $\mathcal{X} = \tilde{\nu} \cdot e_0 \cdot = s\nu \cdot e_0 \cdot$. The *boundary Dirac-Witten operator* $\overline{A} \colon \Gamma(\overline{\Sigma}M_{|\partial M}) \to \Gamma(\overline{\Sigma}M_{|\partial M})$ is defined via the local formula

$$\overline{A} = \sum_{i=2}^{n} e_i \cdot \nu \cdot \overline{\nabla}^{\partial}_{e_i},$$

where $\nu, e_2, \ldots, e_n$ is a local g-orthonormal frame.

It is immediate that $\mathcal{X}$ is a self-adjoint involution. We shall need the following properties of $\overline{A}$.

Lemma 4.3.3. *The boundary Dirac-Witten operator anti-commutes with the boundary chirality operator.*

Proof. We first observe that $0 = \overline{\nabla}\, \mathbb{1}_{\overline{\Sigma}M_{|\partial M}} = \overline{\nabla}(\mathcal{X}^2) = \mathcal{X}\overline{\nabla}\mathcal{X} + (\overline{\nabla}\mathcal{X})\mathcal{X}$. For any $\phi \in \Gamma(\overline{\Sigma}M_{|\partial M})$ we hence get

$$\begin{aligned}
\overline{\nabla}^{\partial}(\mathcal{X}\phi) &= \overline{\nabla}(\mathcal{X}\phi) - \frac{1}{2}(\overline{\nabla}\mathcal{X})\mathcal{X}\,\mathcal{X}\phi \\
&= \overline{\nabla}(\mathcal{X})\phi + \mathcal{X}\overline{\nabla}\phi - \frac{1}{2}(\overline{\nabla}\mathcal{X})\mathcal{X}^2\phi \\
&= \mathcal{X}\overline{\nabla}\phi + \frac{1}{2}(\overline{\nabla}\mathcal{X})\mathcal{X}^2\phi \\
&= \mathcal{X}\overline{\nabla}\phi - \frac{1}{2}\mathcal{X}(\overline{\nabla}\mathcal{X})\mathcal{X}\phi \\
&= \mathcal{X}\overline{\nabla}^{\partial}\phi
\end{aligned}$$

using Lemma 4.2.2. Together with $\nu \cdot \mathcal{X} = -\mathcal{X}\nu \cdot$ and $e_i \cdot \mathcal{X} = \mathcal{X} e_i \cdot$ for $e_i \perp \nu$, we obtain $\overline{A}\mathcal{X}\phi = \sum_{i=2}^{n} e_i \cdot \nu \cdot \overline{\nabla}^{\partial}_{e_i}(\mathcal{X}\phi) = \sum_{i=2}^{n} e_i \cdot \nu \cdot \mathcal{X}\overline{\nabla}^{\partial}_{e_i}\phi = -\mathcal{X}\overline{A}\phi$. □

Lemma 4.3.4. *For $\phi \in \Gamma(\overline{\Sigma}M)$, we have*

$$\overline{A}\phi_{|\partial M} = -\nu \cdot (\overline{D}\phi)_{|\partial M} - \overline{\nabla}_{\nu}\phi + \frac{1}{2}s\left(\mathrm{tr}^{\partial M}(-\overline{\nabla}\tilde{\nu}) - \mathrm{tr}^{\partial M}(\overline{\nabla}e_0)\,\mathcal{X}\right)\phi_{|\partial M}.$$

Proof. The necessary calculation is straightforward, using Lemma 4.2.2 when passing from the first to the second line:

$$\begin{aligned}
\overline{A}\phi_{|\partial M} &= \sum_{i=2}^{n} e_i \cdot \nu \cdot \overline{\nabla}^{\partial}_{e_i} \phi_{|\partial M} \\
&= -\nu \cdot \sum_{i=2}^{n} e_i \cdot \overline{\nabla}_{e_i} \phi_{|\partial M} + \frac{1}{2} \nu \cdot \sum_{i=2}^{n} e_i \cdot (\overline{\nabla}_{e_i} \mathcal{X}) \mathcal{X} \phi_{|\partial M} \\
&= -\nu \cdot (\overline{D}\phi)_{|\partial M} + \nu \cdot \nu \cdot \overline{\nabla}_{\nu} \phi \\
&\quad + \frac{1}{2} \sum_{i=2}^{n} s\tilde{\nu} \cdot e_i \cdot (\overline{\nabla}_{e_i} \tilde{\nu}) \cdot e_0 \cdot \mathcal{X} \phi_{|\partial M} + \frac{1}{2} \sum_{i=2}^{n} s\tilde{\nu} \cdot e_i \cdot \tilde{\nu} \cdot (\overline{\nabla}_{e_i} e_0) \cdot \mathcal{X} \phi_{|\partial M} \\
&= -\nu \cdot (\overline{D}\phi)_{|\partial M} - \overline{\nabla}_{\nu} \phi + \frac{1}{2} s \operatorname{tr}^{\partial M}(-\overline{\nabla} \tilde{\nu})\, \tilde{\nu} \cdot e_0 \cdot \mathcal{X} \phi_{|\partial M} - \frac{1}{2} s \operatorname{tr}^{\partial M}(\overline{\nabla} e_0) \mathcal{X} \phi_{|\partial M} \\
&= -\nu \cdot (\overline{D}\phi)_{|\partial M} - \overline{\nabla}_{\nu} \phi + \frac{1}{2} s \left(\operatorname{tr}^{\partial M}(-\overline{\nabla} \tilde{\nu}) - \operatorname{tr}^{\partial M}(\overline{\nabla} e_0)\, \mathcal{X} \right) \phi_{|\partial M}. \qquad \square
\end{aligned}$$

Now, we are ready to state and prove the integrated version of the Schrödinger-Lichnerowicz formula for $\overline{D}$.

Proposition 4.3.5. *For $\phi \in \Gamma(\overline{\Sigma} M)$, the following holds:*

$$\begin{aligned}
\|\overline{D}\phi\|^2_{L^2(M)} &= \|\overline{\nabla}\phi\|^2_{L^2(M)} + \left(\phi, \frac{1}{2}(\rho - e_0 \cdot j^{\sharp} \cdot)\phi \right)_{L^2(M)} \\
&\quad + \left(\phi_{|\partial M}, -\overline{A}\phi_{|\partial M} + \frac{1}{2} s \left(\operatorname{tr}^{\partial M}(-\overline{\nabla}\tilde{\nu}) - \operatorname{tr}^{\partial M}(\overline{\nabla} e_0) \mathcal{X} \right) \phi_{|\partial M} \right)_{L^2(\partial M)}.
\end{aligned}$$

Proof. The formula follows from taking together four formulae. Firstly, there is a partial integration formula for $\overline{D}$. This follows from the well-known one for D, keeping in mind that the difference term $\overline{D} - D$ is a self-adjoint section in $\Gamma(\mathrm{End}(\overline{\Sigma} M))$:

$$\begin{aligned}
(\overline{D}\phi, \psi)_{L^2(M)} - (\phi, \overline{D}\psi)_{L^2(M)} &= (D\phi, \psi)_{L^2(M)} - (\phi, D\psi)_{L^2(M)} \\
&= (\phi_{|\partial M}, \nu \cdot \psi_{|\partial M})_{L^2(\partial M)}
\end{aligned}$$

for all $\phi, \psi \in \Gamma(\overline{\Sigma} M)$. Secondly, the partial integration formula for $\overline{\nabla}$ following from the one for ∇:

$$\begin{aligned}
(\overline{\nabla}\phi, \Psi)_{L^2(M)} - (\phi, \overline{\nabla}^* \Psi)_{L^2(M)} &= (\nabla\phi, \Psi)_{L^2(M)} - (\phi, \nabla^* \Psi)_{L^2(M)} \\
&= -(\phi_{|\partial M}, \Psi(\nu))_{L^2(\partial M)}
\end{aligned}$$

for all $\phi \in \Gamma(\overline{\Sigma} M)$, $\Psi \in \Gamma(T^* M \otimes \overline{\Sigma} M)$. Thirdly, there is the Schrödinger-Lichnerowicz formula (1.6). Together, we get:

$$\begin{aligned}
\|\overline{D}\phi\|^2_{L^2(M)} &= (\phi, \overline{D}^2\phi)_{L^2(M)} + (\phi_{|\partial M}, \nu \cdot (\overline{D}\phi)_{|\partial M})_{L^2(\partial M)} \\
&= (\phi, \overline{\nabla}^*\overline{\nabla}\phi)_{L^2(M)} + \left(\phi, \frac{1}{2}(\rho - e_0 \cdot j^\sharp\cdot)\phi\right)_{L^2(M)} \\
&\quad + (\phi_{|\partial M}, \nu \cdot (\overline{D}\phi)_{|\partial M})_{L^2(\partial M)} \\
&= \|\overline{\nabla}\phi\|^2_{L^2(M)} + \left(\phi, \frac{1}{2}(\rho - e_0 \cdot j^\sharp\cdot)\phi\right)_{L^2(M)} \\
&\quad + \left(\phi_{|\partial M}, \nu \cdot (\overline{D}\phi)_{|\partial M} + \overline{\nabla}_\nu\phi\right)_{L^2(\partial M)}.
\end{aligned}$$

Now the claim follows from the formula of Lemma 4.3.4. □

We now consider the Dirac-Witten operator with chirality boundary conditions, i. e. the Dirac-Witten operator defined on sections $\phi \in \Gamma(\overline{\Sigma}M)$ with $\mathcal{X}\phi_{|\partial M} = \phi_{|\partial M}$. This fits into the framework of chirality boundary conditions discussed in (cf. [BB16, Ex. 4.20]). To see this, we first note that $\overline{A}$ is an adapted boundary operator for $\overline{D}$, i. e. their respective principal symbols $\sigma_{\overline{A}}$ and $\sigma_{\overline{D}}$ satisfy $\sigma_{\overline{A}}(\xi) = \xi^\sharp \cdot \nu\cdot = -\nu \cdot \xi^\sharp\cdot = \sigma_{\overline{D}}(\nu^\flat)^{-1} \circ \sigma_{\overline{D}}(\xi)$ for all $\xi \in T^*\partial M$. Now it just remains to observe that $\mathcal{X}$ is a self-adjoint involution that anti-commutes with $\overline{A}$. In general, chirality boundary conditions are elliptic in the sense of Bär and Ballmann. Moreover, the chirality boundary condition considered here is also self-adjoint. This follows from the fact that $\mathcal{X}$ anti-commutes with $\nu\cdot = \sigma_{\overline{D}}(\nu^\flat)$.

From now on, we assume that the hypersurface spinor bundle $\overline{\Sigma}M \to M$ is $\mathbb{Z}/2\mathbb{Z}$-graded as in Setup 4.2.1. We recall that it carries a scalar product, a metric connection ∇ and a skew-adjoint, ∇-parallel TM-Clifford multiplication such that the $\mathbb{Z}/2\mathbb{Z}$-grading is orthogonal, parallel and the Clifford multiplication is odd – this is what Cecchini and Zeidler call a *$\mathbb{Z}/2\mathbb{Z}$-graded Dirac bundle* over M (cf. [CZ24, Def. 2.1]). Furthermore, they call a $\mathbb{Z}/2\mathbb{Z}$-graded Dirac bundle over M a *relative Dirac bundle* with *support* K if it is endowed with an odd, self-adjoint and parallel involution $\sigma \in \Gamma(M \setminus K, \mathrm{End}(\overline{\Sigma}M))$ that anti-commutes with the TM-Clifford multiplication and admits a smooth extension to a bundle map on an open neighborhood of $\overline{M \setminus K}$ ([CZ24, Def. 2.2]). Hence, $e_0\cdot$ equips $\overline{\Sigma}M$ with the structure of a relative Dirac bundle with empty support. The formula $\overline{D} = D - \frac{1}{2}\operatorname{tr}^g(k)e_0\cdot$ shows that the Dirac-Witten operator is the Callias operator ([CZ24, eq. (3.1)]) of this relative Dirac bundle associated to the potential $-\frac{1}{2}\operatorname{tr}^g(k)$, which has also been observed by Chai and Wan [CW22]. Here, the Dirac-Witten operator will be viewed as bounded operator

$$\overline{D}_{\mathcal{X}} \colon H^1_{\mathcal{X}}(\overline{\Sigma}M) \to L^2(\overline{\Sigma}M),$$

where $H^1_{\mathcal{X}}(\overline{\Sigma}M)$ is the closure of $\{\phi \in \Gamma(\overline{\Sigma}M) \mid \mathcal{X}\phi_{|\partial M} = \phi_{|\partial M}\}$ with respect to the H^1-Sobolev norm. The analytic results from [CZ24, Sec. 3] give the following proposition.

Proposition 4.3.6 ([CZ24, Thm. 3.4])**.** *The Dirac-Witten operator with chirality boundary conditions defines a Fredholm operator* $\overline{D}_{\mathcal{X}}\colon H^1_{\mathcal{X}}(\overline{\Sigma}M) \to L^2(\overline{\Sigma}M)$*, which is self-adjoint when considered as unbounded operator* $L^2(\overline{\Sigma}M) \supseteq H^1_{\mathcal{X}}(\overline{\Sigma}M) \to L^2(\overline{\Sigma}M)$*.*

Since M is compact and by homotopy invariance of the index, we expect its index to be independent of (g,k). In fact, it can be expressed by a topological formula that we discuss in the case of the "classical" hypersurface spinor bundle. If n is even, then $i^{\frac{n+2}{2}}\iota e_0\cdot e_1\cdot\ldots e_n\cdot$, where ι is the $\mathbb{Z}/2\mathbb{Z}$-grading operator, defines an additional symmetry of any $\mathbb{Z}/2\mathbb{Z}$-graded hypersurface spinor bundle $\overline{\Sigma}M$ forcing the Fredholm index to be zero. We can thus restrict our attention to the odd-dimensional case. In this case there is a unique irreducible representation of $\mathbb{C}\mathrm{l}_{n,1}$. It can be endowed with the $\mathbb{Z}/2\mathbb{Z}$-grading induced by the volume form. The *classical hypersurface spinor bundle* is the $\mathbb{Z}/2\mathbb{Z}$-graded hypersurface spinor bundle associated to this representation.

Remark 4.3.7. Let us denote by $\overline{\Sigma}M = \overline{\Sigma}^+M \oplus \overline{\Sigma}^-M$ the decomposition of the classical hypersurface spinor bundle given by the $\mathbb{Z}/2\mathbb{Z}$-grading for n odd. Then we can identify $\overline{\Sigma}^-M \overset{\cong}{\to} \overline{\Sigma}^+M$ via $ie_0\cdot$. The involution $e_0\cdot$ then corresponds to the matrix

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The Clifford multiplication by X gets identified with

$$\begin{pmatrix} 0 & ie_0\cdot X\cdot \\ ie_0\cdot X\cdot & 0 \end{pmatrix}.$$

The operators $ie_0\cdot X\cdot$ define a Clifford multiplication on $\overline{\Sigma}^+M$. In fact, $\overline{\Sigma}^+M \to M$ is associated to a $\mathbb{C}\mathrm{l}_n$-representation obtained by suitably restricting the irreducible $\mathbb{C}\mathrm{l}_{n,1}$-representation. For dimension reasons, this $\mathbb{C}\mathrm{l}_n$-representation is irreducible, so $\overline{\Sigma}^+M$ is associated to one of the two irreducible representations of $\mathbb{C}\mathrm{l}_n$. In the description of the classical hypersurface spinor bundle, we had been vague about what we mean by the volume element, which determines the decomposition $\overline{\Sigma}M = \overline{\Sigma}^+M \oplus \overline{\Sigma}^-M$. We choose it in such a way that $\overline{\Sigma}^+M \to M$ becomes isomorphic to the classical spinor bundle on M considered in [CZ24, Ex. 2.6], which appears to be characterized by $i^{\frac{n+1}{2}}e_1\cdot\ldots e_n\cdot = \mathbb{1}$ for any positively oriented orthonormal frame $e_1,\ldots,e_n$. So the volume element we take is $i^{\frac{n+1}{2}}ie_0\cdot e_1\cdot\ldots ie_0\cdot e_n = i^{\frac{n+1}{2}+1}e_0\cdot e_1\cdot\ldots e_n$. We observe that then $\overline{\Sigma}M \to M$ recovers the (untwisted) relative Dirac bundle of the cited example. Note that while the construction by Cecchini and Zeidler appears rather ad-hoc, the hypersurface spinor bundle has a geometric meaning. In particular, the involution (called σ there) now naturally arises as Clifford multiplication with the unit normal.

Now we are ready to state the index theorem for the Dirac-Witten operator. Note that the formula is specific for the classical hypersurface spinor bundle. As a consequence, we obtain a criterion for the existence of non-trivial elements in the kernel of the Dirac-Witten operator, which also holds independently of the chosen hypersurface spinor bundle (and even when it is not $\mathbb{Z}/2\mathbb{Z}$-graded).

Theorem 4.3.8 (Callias Index Theorem, [CZ24, Cor. 3.10])**.** *The index of the operator* $\overline{D}_{\mathcal{X}} \colon H^1_{\mathcal{X}}(\overline{\Sigma}M) \to L^2(\overline{\Sigma}M)$ *is given by* $\operatorname{ind}(\overline{D}_{\mathcal{X}}) = \hat{A}(\partial_- M)$.

Corollary 4.3.9. *Assume that* $\hat{A}(\partial_- M) \neq 0$*. Then there are non-trivial smooth* Dirac-Witten harmonic spinors *subject to the chirality boundary condition, i. e.* $\phi \in \Gamma(\overline{\Sigma}M)\backslash\{0\}$ *with* $\overline{D}\phi = 0$ *and* $\mathcal{X}\phi_{|\partial M} = \phi_{|\partial M}$.

4.4. The kernel of Dirac-Witten operators

In this section, we investigate some consequences of non-zero kernel of the Dirac-Witten operator on a compact spin manifold with boundary. To a large extent this discussion is similar to the closed case that was treated in [AG23].

The general setup for this section is the following. We consider an initial data set (g, k) on a compact spin manifold M with boundary $\partial M = \partial_+ M \,\dot\cup\, \partial_- M$. We denote by $\tilde{\nu}$ the unit normal on ∂M that is inward-pointing along $\partial_+ M$ and outward-pointing along $\partial_- M$. The function s will be defined to be $+1$ on $\partial_+ M$ and -1 on $\partial_- M$. Furthermore, we consider a hypersurface spinor bundle $\overline{\Sigma}M$ on M with its connection $\overline{\nabla}_X \phi = \nabla_X \phi - \frac{1}{2} e_0 \cdot k(X, \text{-})^\sharp \cdot \phi$ associated to (g, k) (cf. Setup 4.2.1). Its Dirac-Witten operator with respect to chirality boundary conditions $\tilde{\nu} \cdot e_0 \cdot \phi_{|\partial M} = \phi_{|\partial M}$ will be denoted by $\overline{D}_{\mathcal{X}}$.

Proposition 4.4.1. *Let* M *be as above and* (g, k) *an initial data set on* M*. We assume that it is subject to the dominant energy condition* $\rho \geq |j|_g$ *and that the future outgoing null expansion scalar* $\theta^+ = \operatorname{tr}^{\partial M}(\nabla \tilde{\nu}) + \operatorname{tr}^{\partial M}(k)$ *(with respect to* $\tilde{\nu}$*) satisfies* $\theta^+ \leq 0$ *on* $\partial_+ M$ *and* $\theta^+ \geq 0$ *on* $\partial_- M$*. Then any* $\phi \in \ker(\overline{D}_{\mathcal{X}})$ *is* $\overline{\nabla}$*-parallel and satisfies* $(\rho e_0 - j^\sharp) \cdot \phi = 0$*. If, moreover,* M *is connected and* $\phi \not\equiv 0$*, then* ϕ *is nowhere vanishing,* $\rho = |j|_g$ *and* $\theta^+ = 0$*.*

Proof. We consider the integrated Schrödinger-Lichnerowicz type formula from Proposition 4.3.5

$$\begin{aligned} \|\overline{D}\phi\|^2_{L^2(M)} &= \|\overline{\nabla}\phi\|^2_{L^2(M)} + \frac{1}{2}\left(\phi, (\rho - e_0 \cdot j^\sharp \cdot)\phi\right)_{L^2(M)} + \left(\phi_{|\partial M}, -\overline{A}\phi_{|\partial M}\right)_{L^2(\partial M)} \\ &\quad + \frac{1}{2}\left(\phi_{|\partial M}, s\left(\operatorname{tr}^{\partial M}(-\overline{\nabla}\tilde{\nu}) - \operatorname{tr}^{\partial M}(\overline{\nabla} e_0)\mathcal{X}\right)\phi_{|\partial M}\right)_{L^2(\partial M)}. \end{aligned}$$

As ϕ is subject to the boundary condition $\mathcal{X}\phi_{|\partial M} = \phi_{|\partial M}$ and $\overline{A}$ anti-commutes with $\mathcal{X}$, we have

$$\begin{aligned}\left(\phi_{|\partial M}, -\overline{A}\phi_{|\partial M}\right)_{L^2(\partial M)} &= \left(\phi_{|\partial M}, -\overline{A}\mathcal{X}\phi_{|\partial M}\right)_{L^2(\partial M)} = \left(\phi_{|\partial M}, \mathcal{X}\overline{A}\phi_{|\partial M}\right)_{L^2(\partial M)} \\ &= \left(\mathcal{X}\phi_{|\partial M}, \overline{A}\phi_{|\partial M}\right)_{L^2(\partial M)} = \left(\phi_{|\partial M}, \overline{A}\phi_{|\partial M}\right)_{L^2(\partial M)}\end{aligned}$$

and hence the third summand on the right is zero. Recalling in addition that $\theta^+ = \mathrm{tr}^{\partial M}(\nabla\tilde{\nu}) + \mathrm{tr}^{\partial M}(\overline{\nabla}e_0)$ (cf. Remark 4.2.3), the formula simplifies to

$$\|\overline{D}\phi\|^2_{L^2(M)} = \|\overline{\nabla}\phi\|^2_{L^2(M)} + \frac{1}{2}\left(\phi, (\rho - e_0 \cdot j^\sharp \cdot)\phi\right)_{L^2(M)} + \frac{1}{2}\left(\phi_{|\partial M}, -s\theta^+\phi_{|\partial M}\right)_{L^2(\partial M)}.$$

Clearly, the first term is always non-negative, the second one is non-negative if the dominant energy condition holds and the third term is non-negative by our assumptions as well. Hence, all these terms must be zero if $\overline{D}\phi = 0$, in particular $\overline{\nabla}\phi = 0$. It then follows from the Schrödinger-Lichnerowicz formula that $(\rho - e_0 \cdot j^\sharp \cdot)\phi = \overline{D}^2\phi - \overline{\nabla}^*\overline{\nabla}\phi = 0$.

If M is connected and $\phi \not\equiv 0$, then $\overline{\nabla}$-parallelism of ϕ implies that ϕ is nowhere vanishing. Then we get $\rho = |j|_g$ and $\theta^+ = 0$ since the two latter terms in the equation above are zero and $\rho \geq |j|_g$ and $-s\theta^+ \geq 0$, respectively. □

Even more can be deduced by looking at the Dirac current of ϕ. To define it, we use the Lorentzian metric $\overline{g}$ on $TM \oplus \mathbb{R}e_0$ (cf. Definition 1.3.1) defined by $\overline{g}(U + ue_0, U' + u'e_0) = g(U, U') - uu'$ for all $U, U' \in T_pM$, $p \in M$ and $u, u' \in \mathbb{R}$.

Definition 4.4.2. The *(Lorentzian) Dirac current* associated to $\phi \in \overline{\Sigma}_pM$, $p \in M$, is the vector $V_\phi \in T_pM \oplus \mathbb{R}e_0$ uniquely determined by the condition

$$\overline{g}(V_\phi, X) = -\langle e_0 \cdot X \cdot \phi, \phi\rangle$$

for all $X \in T_pM \oplus \mathbb{R}e_0$. Its *Riemannian Dirac current* $U_\phi \in T_pM$ is defined by

$$g(U_\phi, X) = \langle e_0 \cdot X \cdot \phi, \phi\rangle$$

for all $X \in T_pM$.

Since $\overline{g}(V_\phi, X) = \overline{g}(-U_\phi, X)$ for all $X \in TM$ and $\overline{g}(V_\phi, e_0) = -\langle e_0 \cdot e_0 \cdot \phi, \phi\rangle = -|\phi|^2$, the Lorentzian Dirac current splits up as $V_\phi = u_\phi e_0 - U_\phi$ with $u_\phi = |\phi|^2$. Thus V_ϕ is zero if and only if $\phi = 0$. Moreover, a short calculation (cf. Lemma 1.3.8) shows $|V_\phi \cdot \phi|^2 = -\overline{g}(V_\phi, V_\phi)|\phi|^2$. Hence if $\phi \neq 0$, then V_ϕ is either future-timelike or future-lightlike. In the latter case, additionally $V_\phi \cdot \phi = 0$ holds.

If $\phi \in \Gamma(\overline{\Sigma}M)$ is $\overline{\nabla}$-parallel, then $\overline{\nabla}V_\phi = 0$ or, equivalently,

$$\begin{aligned} \nabla_X U_\phi &= u_\phi k(X, \text{-})^\sharp \\ \mathrm{d}u_\phi(X) &= k(U_\phi, X) \end{aligned} \tag{4.5}$$

for all $X \in \Gamma(TM)$. In particular, whether V_ϕ is zero, timelike or lightlike will not change on a connected component of M.

Definition 4.4.3. A $\overline{\nabla}$-parallel spinor $\phi \in \Gamma(\overline{\Sigma}M)$ is called *timelike* or *lightlike* if V_ϕ is timelike or lightlike on all of M, respectively.

Proposition 4.4.4. *Let (g, k) be an initial data set on a compact connected spin manifold M with non-empty boundary $\partial M = \partial_+ M \,\dot\cup\, \partial_- M$. As above, we consider the unit normal $\tilde\nu$ on ∂M that is inward-pointing along $\partial_+ M$ and outward-pointing along $\partial_- M$. Assume that ϕ is a non-zero $\overline{\nabla}$-parallel spinor on M subject to chirality boundary conditions. Then ϕ is a lightlike $\overline{\nabla}$-parallel spinor. Its Riemannian Dirac current U_ϕ satisfies $j^\sharp = \frac{\rho}{u_\phi} U_\phi$ on all of M and $\tilde\nu = -\frac{1}{u_\phi} U_\phi$ on the boundary ∂M. Moreover, U_ϕ is a non-vanishing vector field with $\mathrm{d}U_\phi^\flat = 0$. In particular, $\ker(U_\phi^\flat)$ defines an involutive distribution and hence a foliation of M by Frobenius' theorem. The leaves may be co-oriented by the unit normal $\tilde\nu = -\frac{1}{u_\phi} U_\phi$, and then the future outgoing null second fundamental form $\chi^+ = \nabla \tilde\nu^\flat + k$ satisfies $\chi^+ = 0$, in particular the leaves are MOTS. On the leaves, the restriction of $\frac{\phi}{|\phi|}$ is parallel with respect to the Levi-Civita connection of the induced metric. In particular, the induced metric on every leaf is Ricci-flat.*

We use the following lemma.

Lemma 4.4.5. *Assume $V \in T_pM \oplus \mathbb{R}e_0$ and $\phi \in \overline{\Sigma}_p M \setminus \{0\}$ is some spinor with $V \cdot \phi = 0$. Then V is a scalar multiple of V_ϕ. If additionally $V \neq 0$, then V_ϕ is lightlike.*

Proof. We have $\overline{g}(V_\phi, V_\phi) \leq 0$, $\overline{g}(V, V)\phi = -V \cdot V \cdot \phi = 0$ and $\overline{g}(V_\phi, V) = -\langle e_0 \cdot V \cdot \phi, \phi\rangle = 0$. Hence, $\overline{g}_{|L \times L}$ is negative semi-definite for $L = \mathrm{span}(V, V_\phi) \subseteq T_pM \oplus \mathbb{R}e_0$. Since $\overline{g}$ is a Lorentzian metric, the dimension of L can be at most one, yielding the first part. If now $V \neq 0$, then L is a one-dimensional lightlike subspace and the rest of the claim follows. ☐

Proof of Proposition 4.4.4. On ∂M the boundary condition yields $(e_0 + \tilde\nu) \cdot \phi = 0$. Using the previous lemma and $\partial M \neq \varnothing$ we obtain that the $\overline{\nabla}$-parallel spinor ϕ is lightlike with $V_\phi = u_\phi(e_0 + \tilde\nu)$ along the boundary. A further application of the lemma, this time to $(\rho e_0 - j^\sharp) \cdot \phi = e_0 \cdot (\overline{D}^2 \phi - \overline{\nabla}^* \overline{\nabla} \phi) = 0$, yields $\rho e_0 - j^\sharp = \frac{\rho}{u_\phi} V_\phi$.

As V_ϕ is lightlike, $|U_\phi|_g = u_\phi = |\phi|^2$, so U_ϕ is nowhere vanishing. From $\overline{\nabla}$-parallelism of V_ϕ (cf. (4.5)), we get

$$\begin{aligned} \mathrm{d}U_\phi^\flat(X,Y) &= \partial_X(U_\phi^\flat(Y)) - \partial_Y(U_\phi^\flat(X)) - U_\phi^\flat([X,Y]) \\ &= g(\nabla_X U_\phi, Y) - g(\nabla_Y U_\phi, X) \\ &= u_\phi k(X,Y) - u_\phi k(Y,X) = 0 \end{aligned}$$

for all $X, Y \in \Gamma(TM)$. The expression in the first line of this equation then helps to conclude that $\ker(U_\phi^\flat)$ is an involutive distribution of codimension one.

Let us now calculate χ^+ of the leaves of the associated foliation, where the unit normal on the leaves is given by $\tilde{\nu} := -\frac{1}{u_\phi}U_\phi$. (On the boundary, this coincides with the previously defined $\tilde{\nu}$.) We have

$$\begin{aligned} \nabla_X\tilde{\nu} &= \frac{1}{u_\phi^2}\mathrm{d}u_\phi(X)U_\phi - \frac{1}{u_\phi}\nabla_X U_\phi \\ &= \frac{1}{u_\phi^2}k(U_\phi, X)U_\phi - k(X, \text{-})^\sharp \\ &= -(k(X, \text{-})^\sharp - k(X,\tilde{\nu})\tilde{\nu}) \end{aligned}$$

Thus $\chi^+(X,Y) = g(\nabla_X\tilde{\nu}, Y) + k(X,Y) = 0$ for all X, Y tangential to the leaves of the foliation.

Let now F be a leaf and $X \in TF$. The Levi-Civita connection of F induces on $\overline{\Sigma}M_{|F}$ the connection $\nabla_X^F = \nabla_X + \frac{1}{2}\nabla_X\tilde{\nu}\cdot\tilde{\nu}\cdot$, cf. (4.4). Thus, using $\overline{\nabla}_X\phi = 0$ and $(e_0+\tilde{\nu})\cdot\phi = \frac{1}{u_\phi}V_\phi\cdot\phi = 0$, we obtain

$$\begin{aligned} \nabla_X^F\phi &= \nabla_X\phi + \frac{1}{2}\nabla_X\tilde{\nu}\cdot\tilde{\nu}\cdot\phi \\ &= \overline{\nabla}_X\phi - \frac{1}{2}k(X, \text{-})^\sharp\cdot e_0\cdot\phi - \frac{1}{2}\left(k(X, \text{-})^\sharp - k(X,\tilde{\nu})\tilde{\nu}\right)\cdot\tilde{\nu}\cdot\phi \\ &= -\frac{1}{2}k(X,\tilde{\nu})\phi. \end{aligned}$$

Hence,

$$\begin{aligned} \nabla_X^F\frac{\phi}{|\phi|} &= \nabla_X^F\left(u_\phi^{-\frac{1}{2}}\phi\right) \\ &= -\frac{1}{2}u_\phi^{-\frac{3}{2}}\mathrm{d}u_\phi(X)\phi - \frac{1}{2}u_\phi^{-\frac{1}{2}}k(X,\tilde{\nu})\phi \\ &= 0 \end{aligned}$$

as desired. □

In the next section, we will prove the following general fact about foliations. Although its proof only uses elementary differentialgeometric methods, its setup of foliations on manifolds with boundary is rather special and not covered by standard textbooks.

Theorem 4.4.6. *Let M be a connected manifold with boundary $\partial M = \partial_+ M \,\dot\cup\, \partial_- M$, where $\partial_+ M$ and $\partial_- M$ are unions of components and $\partial_+ M \neq \varnothing$. Let U be a non-vanishing vector field on M that is outward-pointing on $\partial_+ M$ and inward-pointing on $\partial_- M$. We assume that there exists a metric g on M such that U is orthogonal to the boundary and its metric dual satisfies $\mathrm{d}U^\flat = 0$. Then the flow of $X = -\frac{U}{|U|_g^2}$ defines a diffeomorphism*

$$\Phi' \colon \partial_+ M \times [0, \ell] \to M$$

for some $\ell > 0$. Moreover, Φ' maps the leaves of the foliation $(\partial_+ M \times \{t\})_{t \in \mathbb{R}}$ precisely to the leaves of the foliation defined by $U^\flat$.

With this result at hand, we can prove the main theorems of this chapter.

Proof of Theorem 4.1.2 and Addendum 4.1.3. Since $\hat{A}(\partial_- M) \neq 0$, the dimension of $\partial_- M$ is even and the one of M is odd. Let $\overline{\Sigma} M \to M$ be the irreducible hypersurface spinor bundle on M. It follows from the Callias index theorem (cf. Corollary 4.3.9) that there is a spinor $\phi \in \Gamma(\overline{\Sigma} M) \setminus \{0\}$ with $\overline{D}\phi = 0$ and $\mathcal{X}\phi_{|\partial M} = \phi_{|\partial M}$ for the initial data set (g, k). From Proposition 4.4.1, we get that ϕ is a non-zero $\overline{\nabla}$-parallel spinor.

Now note that $\partial_+ M \neq \varnothing$, as otherwise $\partial_- M$ would be spin zero-bordant and $\hat{A}(\partial_- M) = 0$. We apply Proposition 4.4.4 to ϕ and obtain that ϕ is a lightlike $\overline{\nabla}$-parallel spinor and its Riemannian Dirac current U_ϕ is nowhere vanishing, outward-pointing along $\partial_+ M$, inward-pointing along $\partial_- M$ and satisfies $\mathrm{d}U_\phi^\flat = 0$. Now Theorem 4.4.6 provides us with a diffeomorphism $\Phi' \colon \partial_+ M \times [0, \ell] \to M$. Using the identification $\partial_+ M \cong \Phi'(\partial_+ M \times \{\ell\}) = \partial_- M$, $p \mapsto \Phi'(p, \ell)$, we obtain a diffeomorphism $\Phi \colon \partial_- M \times [0, \ell] \to M$ and we claim that this has all the desired properties.

By construction, $(\Phi(\partial_- M \times \{t\}))_{t \in [0, \ell]}$ coincides with the foliation defined by $U_\phi^\flat$. This directly shows the claim of the addendum and allows us to make use of the properties derived in Proposition 4.4.4: Those are that each leaf satisfies $\chi^+ = 0$ with respect to the unit normal $\tilde{\nu} = -\frac{1}{u_\phi} U_\phi$, carries the parallel spinor $\frac{\phi}{|\phi|}$ and is orthogonal to $j^\sharp = -\rho \tilde{\nu}$. This is all that was left to show for the theorem. □

Proof of Theorem 4.1.5. We consider the initial data set $(g, -(h \circ s) \cdot g)$ on M. For this initial data set, we have

$$\begin{aligned} 2\rho &= \mathrm{scal}^g + n(n-1)(h \circ s)^2 \quad \text{and} \\ j &= (n-1)(h' \circ s)\mathrm{d}s. \end{aligned}$$

Since

$$|j|_g = (n-1)|h' \circ s||\mathrm{d}s|_g \leq -(n-1)(h' \circ s),$$

the inequality for scal^g implies the dominant energy condition $\rho \geq |j|_g$. Moreover, keeping in mind that H^g is defined with respect to the inward-pointing unit normal ν, we have

$$\begin{aligned} \theta^+ &= \mathrm{tr}^{\partial_+ M}(\nabla\tilde{\nu} + k) = (n-1)(-H^g - h(0)) \leq 0 \qquad && \text{on } \partial_+ M \text{ and} \\ \theta^+ &= \mathrm{tr}^{\partial_- M}(\nabla\tilde{\nu} + k) = (n-1)\ (H^g - h(L))\ \geq 0 \qquad && \text{on } \partial_- M. \end{aligned}$$

Hence, the assumptions of Theorem 4.1.2 are satisfied for this initial data set.

We get a diffeomorphism $\tilde{\Phi}\colon \partial_- M \times [0, \tilde{\ell}]$ inducing the foliation defined by $U_\phi^\flat$ for a lightlike $\overline{\nabla}$-parallel spinor ϕ. If X is a vector tangential to the leaves, then $\mathrm{d}u_\phi(X) = k(U_\phi, X) = -(h \circ s)g(U_\phi, X) = 0$, so u_ϕ is constant along the leaves. Hence, we can reparameterize the second factor to obtain a diffeomorphism $\Phi\colon \partial_- M \times [0,\ell] \to M$ such that $\Phi_*(\frac{\partial}{\partial t}) = \tilde{\nu}$. Then $\Phi^* g = \gamma_t + \mathrm{d}t^2$ for a family $(\gamma_t)_{t\in[0,\ell]}$ of metrics on $\partial_- M$.

Theorem 4.1.2 provides us with more knowledge about the leaves $F_t = \Phi(\partial_- M \times \{t\})$. First, $j^\sharp = -\rho\tilde{\nu}$ shows that $(n-1)\partial_X(h \circ s) = j(X) = 0$ for all $X \in TF_t$. Thus $h \circ s$ is constant along the leaves and we may define $(h_t)_{t\in[0,\ell]}$ by $\{h_t\} = (h \circ s)(F_t)$ for all $t \in [0,\ell]$. Moreover, since $\rho = |j|_g$ the inequalities used to establish the dominant energy condition must be equalities. This yields the scalar curvature equality as well as $|\mathrm{d}s|_g = 1$ whenever $h' \circ s \neq 0$. Since $-j^\sharp$ and $\tilde{\nu}$ point in the same direction, $\mathrm{d}t = \Phi^*(\mathrm{d}s)$ holds where $h' \circ s \neq 0$.

Second, $0 = \chi^+(X,Y) = \gamma_t(\nabla_X \tilde{\nu}, Y) - h_t \cdot \gamma_t(X,Y)$ for all $X, Y \in TF_t$ implies

$$\frac{\partial}{\partial t}\gamma_t = 2\gamma_t(\nabla\tilde{\nu}, \text{-}) = 2h_t \cdot \gamma_t.$$

The unique solution of this ordinary differential equation starting at $\gamma_0 = \gamma$ is given by $\gamma_t = v(t)^2\gamma$ with $v(t) = \exp(\int_0^t h_\tau \mathrm{d}\tau)$. Moreover, since $\chi^+ = 0$ implies $\theta^+ = 0$, the inequalities for H^g along ∂M turn into equalities.

Third, the metrics on the leaves admit a non-trivial parallel spinor. In particular, this applies to γ, which was left to show. □

4.5. Identifying the product structure

This section is devoted to a proof of Theorem 4.4.6. Throughout, M will be a compact manifold with boundary $\partial M = \partial_+ M \mathbin{\dot\cup} \partial_- M$, where $\partial_+ M$ and $\partial_- M$ are unions of components. Moreover, we will assume that M is connected and that $\partial_+ M \neq \varnothing$. For a manifold with boundary, as usual, notions such as diffeomorphism, foliation etc. should always be understood in the sense that there exists a smooth extension along a small collar neighborhood around the boundary. Notice that with this notion, the leaves of the foliation of M are not necessarily homeomorphic: Even for two diffeomorphic leaves of the collar-extended manifold, their portions lying inside M may be topologically different.

Theorem 4.5.1. *Let M be as above and X a smooth vector field that is transverse to the boundary, inward-pointing on $\partial_+ M$ and outward-pointing on $\partial_- M$. We denote by*

$$\Phi\colon \mathcal{D} \to M,$$

the flow of X, defined on the maximal domain of definition $\mathcal{D} \subseteq M \times \mathbb{R}$. Then Φ restricts to a diffeomorphism

$$\Phi'\colon \mathcal{D}' := (\partial_+ M \times [0,\infty)) \cap \mathcal{D} \to M'.$$

onto an open subset $M' \subseteq M$, and the smooth function $f := \mathrm{pr}_{\mathbb{R}} \circ {\Phi'}^{-1}\colon M' \to \mathbb{R}$ is proper. Moreover, if the foliation $(f^{-1}(t))_{t\in\mathbb{R}}$ of M' extends to a foliation by hypersurfaces of M such that X is transversal to its leaves, then $M' = M$.

Note that the additional condition of X being transversal to some foliation by hypersurfaces forces the vector field X to be nowhere vanishing, which already rules out many pathological examples. Yet X being nowhere vanishing is not sufficient for the full conclusion: Even in nice cases M' might not be all of M as the following example shows.

Example 4.5.2. Consider $M = S^1 \times [-\ell, \ell]$ with $\partial_+ M = S^1 \times \{\ell\}$ and $\partial_- M = S^1 \times \{-\ell\}$ and the vector field $X(s,r) = \frac{\partial}{\partial s} - r^2 \frac{\partial}{\partial r}$, $s \in S^1, r \in [-\ell, \ell]$. In this case $M' = S^1 \times (0, \ell]$ and $f(s,r) = \frac{1}{r} - \frac{1}{\ell}$. Since f is independent on s, the foliation $(f^{-1}(t))_{t\in\mathbb{R}}$ of M' extends to the canonical foliation of M. But note that X is not transversal to the leaf $S^1 \times \{0\}$.

Proof. Let $\hat{M}$ be the manifold (without boundary) that arises by adding collar neighborhoods to M, $\hat{X}$ a smooth extension of X to $\hat{M}$ and $\hat{\Phi}\colon \hat{\mathcal{D}} \to \hat{M}$ be the flow of $\hat{X}$ defined on the maximal domain of definition. Note that Φ is the restriction of $\hat{\Phi}$ to $\hat{\Phi}^{-1}(M) \cap (M \times \mathbb{R})$. This uses that the vector field X is transversal to the boundary, so that flow lines of $\hat{X}$ cannot re-enter M.

We start by showing that Φ', or rather $\hat{\Phi}' := \hat{\Phi}_{|\hat{\mathcal{D}}'}$ for $\hat{\mathcal{D}}' = (\partial_+ M \times \mathbb{R}) \cap \hat{\mathcal{D}}$, is a local diffeomorphism. By definition, for $x \in \partial_+ M$, $Y \in T_x \partial_+ M$ and $a \in \mathbb{R}$ the differential of $\hat{\Phi}'$ is given by $\mathrm{d}_x \hat{\Phi}'(Y + a\frac{\partial}{\partial t}) = Y + aX$. As X is transversal to $\partial_+ M$, this differential is an isomorphism and $\hat{\Phi}'$ is a local diffeomorphism on a neighborhood of $\partial_+ M \times \{0\}$ in $\hat{\mathcal{D}}'$. We now consider arbitrary points $(x,t) \in \hat{\mathcal{D}}'$. Using that, locally around $(x,0)$, $\hat{\Phi}(\,\text{-}\,,t)$ is a diffeomorphism (with inverse $\hat{\Phi}(\,\text{-}\,,-t)$) and the factorization $\hat{\Phi}' = \hat{\Phi}(\,\text{-}\,,t) \circ \hat{\Phi}' \circ ((y,s) \mapsto (y,s-t))$, we conclude that $\hat{\Phi}'$ is also a local diffeomorphism around (x,t).

For the first part of the claim it is now sufficient to see that $\hat{\Phi}'$ is injective. Then $\hat{\Phi}'$ is a diffeomorphism onto its image and Φ' its restriction to $\mathcal{D}'$. Suppose that $\hat{\Phi}'(x,t) = \hat{\Phi}'(y,s)$ and $t \leq s$. Then $x = \hat{\Phi}(\,\text{-}\,,-t) \circ \hat{\Phi}'(x,s) = \hat{\Phi}'(y,s-t)$. As X is inward-pointing at $x \in \partial_+ M$, this implies $s-t=0$ and $x = \hat{\Phi}'(y,0) = y$.

We now show that the subsets $f^{-1}([a,b])$ are compact for any real numbers $a \leq b$. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in $f^{-1}([a,b]) \subseteq M'$. As $\partial_+ M \times [a,b]$ is compact, we may assume without loss of generality that $(y_i,c_i) \coloneqq (\Phi')^{-1}(x_i) \longrightarrow (y,c)$ for some $(y,c) \in \partial_+ M \times [a,b]$. We have to show that $(y,c) \in \mathcal{D}'$. Then $x = \Phi(y,c) \in M'$ is the limit of the sequence $(x_i)_{i\in\mathbb{N}}$.

Suppose for contradiction that $(y,c) \notin \mathcal{D}$. Let d be the maximal value so that $(y,d) \in \mathcal{D}$. Since M is compact, this maximum exists. Furthermore, we may choose an $\varepsilon > 0$ smaller than $c-d$ such that $(y,d+\varepsilon) \in \hat{\mathcal{D}}$ with $\hat{\Phi}(y,d+\varepsilon) \in \hat{M} \setminus M$. Thus $d+\varepsilon \leq c_i$ for almost all $i \in \mathbb{N}$, so that $(y_i,d+\varepsilon) \in \mathcal{D}$ and $\Phi(y_i,d+\varepsilon) \in M$. But since M is compact, thus closed in $\hat{M}$, we deduce from $\hat{\Phi}(y_i,d+\varepsilon) \longrightarrow \hat{\Phi}(y,d+\varepsilon)$ for $i \longrightarrow \infty$ that $\hat{\Phi}(y,d+\varepsilon) \in M$, contradiction.

For the last part, we show that M' is closed in M and invoke that M is connected. So let $x \in \overline{M'}$. We may assume without loss of generality that x is in the interior of M: If $x \in \partial_+ M$, there is nothing to show as $x \in M'$ by definition, and if $x \in \partial_- M$, we may argue with $\Phi(x,-\varepsilon) \in M \setminus \partial M$ for a small $\varepsilon > 0$ instead as this will be in $\overline{M'}$ if x is. By assumption, there is a co-dimension one foliation $\mathcal{F}$ of M that is transversal to X and such that its leaves are level sets of f wherever f is defined. We may choose a chart $\psi\colon U \to (-r,r)^{n-1} \times (-\delta,\delta)$ of M around x so that the leaves of $\mathcal{F}$ correspond to level sets of the last component. After potentially shrinking δ the image of the flow line $\Phi(x,\,\text{-}\,)\colon (a,b) \to U$ through x crosses every level set. As $x \in \overline{M'}$, there is some $y \in M' \cap U$. We consider the level set $(-r,r)^{n-1} \times \{t\}$ containing $\psi(y)$. Within this set, $\psi(f^{-1}(\{f(y)\}) \cap U) = \psi(M' \cap U) \cap ((-r,r)^{n-1} \times \{t\})$ is both open and closed, where the latter follows from properness of f. Thus this whole level set is contained in $\psi(M' \cap U)$. In particular, a point in the flow line of x is contained in M'. But then $x \in M'$ by definition. □

Proof of Theorem 4.4.6. We invoke Theorem 4.5.1 for the vector field $X = -\frac{U}{|U|_g^2}$. In order to make use of its full strength, we show that the foliation defined by the closed 1-form $U^\flat$ extends (or actually coincides with) the foliation $(f^{-1}(t))_{t\in\mathbb{R}}$. Since $U^\flat(X) = 1 \neq 0$, X is transversal to the leaves of this foliation.

So let Y be a vector in $Tf^{-1}(t)$ for some $t \in \mathbb{R}$. We have to show that $g(U,Y) = 0$. We pull Y back to $\mathcal{D}'$ along Φ' and extend this to a vector field on $\mathcal{D}'$ in such a way that it is constant in the $\mathbb{R}$-direction. Then the pushed-forward vector field $\widetilde{Y}$ on M' extends Y, is tangential to the foliation $(f^{-1}(t))_{t\in\mathbb{R}}$ everywhere and satisfies $[\widetilde{Y}, X] = 0$.

Since $g(U, \widetilde{Y}) = 0$ on $\partial_+ M$ and every point in M' can be reached from there via a flow line of X it suffices to show that $\partial_X g(U, \widetilde{Y}) = 0$. Since

$$\begin{aligned}[\widetilde{Y}, X] &= -\left(\partial_{\widetilde{Y}} \frac{1}{|U|_g^2}\right) U - \frac{1}{|U|_g^2}[\widetilde{Y}, U] \\ &= \frac{2g(\nabla_{\widetilde{Y}} U, U)}{|U|_g^4} U - \frac{1}{|U|_g^2}[\widetilde{Y}, U],\end{aligned}$$

the condition $[\widetilde{Y}, X] = 0$ is equivalent to $2g(\nabla_{\widetilde{Y}} U, U)\, U = |U|_g^2\, [\widetilde{Y}, U]$, which implies

$$2g(\nabla_{\widetilde{Y}} U, U) = g([\widetilde{Y}, U], U).$$

Note, moreover, that the condition $\mathrm{d}U^\flat = 0$ is equivalent to

$$g(\nabla_A U, B) = g(\nabla_B U, A)$$

for any vectors A and B. Taking this together, we obtain the desired equation

$$\begin{aligned}-|U|_g^2 \partial_X g(U, \widetilde{Y}) &= \partial_U g(U, \widetilde{Y}) \\ &= g(\nabla_U U, \widetilde{Y}) + g(U, \nabla_U \widetilde{Y}) \\ &= g(\nabla_{\widetilde{Y}} U, U) + g(U, \nabla_{\widetilde{Y}} U) - g(U, [\widetilde{Y}, U]) \\ &= 0.\end{aligned}$$

The previous theorem now establishes that $\Phi' \colon \mathcal{D}' \to M$ is a diffeomorphism and that U is orthogonal to all the level sets of f. Since M and thus $\mathcal{D}'$ is compact, there exists a maximal number $\ell \geq 0$ such that $\partial_+ M \times \{\ell\}$ is contained in $\mathcal{D}'$. Maximality of ℓ implies that there exists some point $\Phi'(x, \ell)$ that lies in $\partial_- M$. In particular, $\ell > 0$. As U is orthogonal to $\partial_- M$, the connected component of $\partial_- M$ that contains $\Phi'(x, \ell)$ is a component of a leaf of the foliation defined by $U^\flat$, i. e. a component of the level set $f^{-1}(\ell)$. But since M and thus $\mathcal{D}' \cong M$ is connected, also $\partial_+ M$ and $f^{-1}(\ell) = \Phi'(\partial_+ M \times \{\ell\})$ are connected. Thus $f^{-1}(\ell)$ is a component of $\partial_- M$. Since flow lines end when they reach $\partial_- M$, this implies that $\mathcal{D}' = \partial_+ M \times [0, \ell]$. □

A. Appendix

A.1. Constructing adapted Cauchy temporal functions

In this appendix we show how to adjust the Cauchy temporal function obtained from Theorem 1.2.4 so that a chosen Cauchy hypersurface is a level set of the new Cauchy temporal function.

Proposition A.1.1. *Let M be a spacelike Cauchy hypersurface of a (globally hyperbolic) Lorentzian manifold $(\overline{M}, \overline{g})$. Then for a any Cauchy temporal function f on $\overline{M}$ and any open neighborhood $U \subseteq \overline{M}$ of M in $\overline{M}$ there is a Cauchy temporal function $\tilde{f}$ with the following properties:*

- *M is a leaf of the foliation defined by $\tilde{f}$, i. e. $M = \tilde{f}^{-1}(t)$ for some $t \in \mathbb{R}$, and*
- *outside of U the involutive distributions corresponding to the foliations by level sets of f and $\tilde{f}$ coincide, i. e. $(\ker \mathrm{d}f)_{|\overline{M}\setminus U} = (\ker \mathrm{d}\tilde{f})_{|\overline{M}\setminus U}$ as subbundles of $T(\overline{M} \setminus U)$.*

In particular, for any finite collection of mutually disjoint spacelike Cauchy hypersurfaces $M_1, \ldots, M_k$ there is a Cauchy temporal function having all of these as level sets.

Proof. We start by defining three temporal functions, i. e. smooth functions with past-timelike gradient. These will be glued together in the second step. Let $\phi\colon \mathbb{R} \to (-1, 1)$ be an orientation-preserving diffeomorphism. Then define on $U_1 = I^+(M)$ – the (timelike) future of M – the function $f_1\colon U_1 \to (1, 3)$, $p \mapsto \phi(f(p)) + 2$ and on the (timelike) past $U_3 = I^-(M)$ of M the function $f_3\colon U_3 \to (-3, -1)$, $p \mapsto \phi(f(p)) - 2$. Clearly, at each point their gradient is a positive multiple of that of f, so they are indeed temporal functions. To define $f_2\colon U_2 \to (-1, 1)$, we consider the function $\mathbb{R} \times M \supseteq V \to \overline{M}$, $(t, p) \mapsto \exp_p(te_0)$ on its maximal domain of definition, which is essentially the normal exponential map of M in $\overline{M}$. Restricting to a suitably chosen small open neighborhood $V_2 \subseteq ((-1, 1) \times M) \cap V$ of $\{0\} \times M$ yields a diffeomorphism $\Phi\colon V_2 \to U_2$ with $U_2 \subseteq U$. Then we set $f_2 := \mathrm{pr}_1 \circ \Phi^{-1}$, where pr_1 is the canonical projection on the first factor. The pullback metric on V_2 is given by $\Phi^*\overline{g} = -\mathrm{d}t^2 + g_t$, where g_t is a Riemannian metric on $V_2 \cap (\{t\} \times M)$. To see this, we first observe that the canonical tangent vector in $\mathbb{R}$-direction at the point (t,p) gets mapped to the tangent vector of the geodesic $\gamma_p\colon s \mapsto \exp_p(se_0)$ at time t. Moreover, the tangent vector at (t,p) that canonically corresponds to $X \in T_pM$ gets mapped to $W(t)$, where W is the Jacobi field along γ_p with $W(0) = X$ and $\frac{\overline{\nabla}}{\mathrm{d}s}W(0) = k(X, \text{-})^\sharp$. The formula for $\Phi^*\overline{g}$ now follows from the facts that W is stays orthogonal to γ_p' for all times and that γ_p has unit speed. It follows that $\mathrm{grad}^{\Phi^*\overline{g}}(\mathrm{pr}_1)$ is past-timelike and via the isometry Φ the same is true for $\mathrm{grad}^{\overline{g}}(f_2)$.

Now we consider the domain of dependence of M inside $(U_2, \overline{g}_{|U_2})$. This is the set

$$D^{U_2}(M) = \{p \in U_2 \,|\, \text{every inextendable causal curve in } U_2 \text{ through } p \text{ intersects } M\}\,.$$

According to [ONe83, Ch. 14, Lem. 42 and 43] $D^{U_2}(M)$ is open and globally hyperbolic. It follows that the same is true for $U^+ := D^{U_2}(M) \cap I^+(M)$ and $U^- := D^{U_2}(M) \cap I^-(M)$. Thus by Theorem 1.2.4 they admit a Cauchy temporal functions f^+ and f^-, respectively. Now let $\chi\colon \mathbb{R} \to [0,1]$ be a smooth monotonous function such that $\mathrm{d}\chi$ is compactly supported, $\chi(t) \longrightarrow 0$ for $t \longrightarrow -\infty$ and $\chi(t) \longrightarrow 1$ for $t \longrightarrow \infty$. We then define (Ψ_1, Ψ_2, Ψ_3) to be the partition of unity subordinate to the open cover (U_1, U_2, U_3) of $\overline{M}$ with $\Psi_1(p) = \chi(f^+(p))$ for $p \in U^+ \subseteq U_2 \cap U_1$ and $\Psi_3(p) = \chi(-f^-(p))$ for $p \in U^- \subseteq U_2 \cap U_3$. Then we set

$$\tilde{f} := \phi^{-1}\left(\frac{1}{3}\sum_{i=1}^{3} f_i \cdot \Psi_i\right).$$

Note that this expression is well-defined since $\mathrm{supp}(\Psi_i) \subseteq U_i$, where f_i is defined, for all $i = 1,2,3$ and the sum gives a number in $(-3,3)$.

We check that $\tilde{f}$ has the desired properties. To see that $\tilde{f}$ is a temporal function, it suffices to prove this for $\sum_{i=1}^{3} f_i \cdot \Psi_i$. We calculate

$$\mathrm{grad}\left(\sum_{i=1}^{3} f_i \cdot \Psi_i\right) = \sum_{i=1}^{3} \mathrm{grad}(f_i) \cdot \Psi_i + (f_1 - f_2) \cdot \mathrm{grad}(\Psi_1) + (f_3 - f_2) \cdot \mathrm{grad}(\Psi_3)$$

and note $\mathrm{grad}(\Psi_1) = (\chi' \circ f^+) \cdot \mathrm{grad}(f^+)$ and $\mathrm{grad}(\Psi_3) = -(\chi' \circ (-f^-)) \cdot \mathrm{grad}(f^-)$. Recalling that $f_1 \geq f_2 \geq f_3$, we find that each summand consists of a non-negative prefactor and the gradient of a temporal function, which makes it at each point either a past-timelike vector or zero. Since (Ψ_1, Ψ_2, Ψ_3) is a partition of unity, at each point at least one of the first three summands is non-zero, thus the sum is indeed a past-timelike vector.

The property that each level set of $\tilde{f}$ is a Cauchy hypersurface amounts to showing that $\mathrm{im}(\tilde{f} \circ \gamma) = \mathbb{R}$ for every inextendable future-causal curve γ. Let γ be such a curve and observe that there is a parameter s_0 for which $\gamma(s_0) \in M$ since M is a Cauchy hypersurface. Then $\gamma(s) \in I^+(M) = U_1$ for all $s > s_0$ and $\gamma(s) \in D^{U_2}(M)$ for all s

in a small neighborhood around s_0. Let $s_+ \in (s_0, \infty]$ be chosen maximally such that $\gamma((s_0, s_+)) \subseteq U^+ = D^{U_2}(M) \cap I^+(M)$. Then $\gamma_{|(s_0,s_+)}$ is an inextendable causal curve in U^+ and thus intersects each level set of f^+. In particular, it reaches a level where $\Psi_1 \equiv 1$. From that point on, $\Psi_1 \circ \gamma$ must stay 1, so that $\tilde{f} \circ \gamma = \phi^{-1}(\frac{1}{3} f_1 \circ \gamma)$ in a neighborhood of ∞. Now, since f was a Cauchy temporal function, we have $f(\gamma(s)) \longrightarrow \infty$ for $s \to \infty$. Hence in the same limit we get $f_1(\gamma(s)) \longrightarrow 3$ and $\tilde{f} \longrightarrow \infty$. Arguing analogously for $s < s_0$, we also obtain $\tilde{f} \longrightarrow -\infty$ for $s \longrightarrow -\infty$.

The other properties are quickly checked: On $I^+(M)$ we have $f_1 > 0$, $f_2 > 0$ whenever $\Psi_2 > 0$, and $\Psi_3 = 0$, so the sum $\sum_{i=1}^{3} f_i \cdot \Psi_i$ is positive there. Similar considerations yield that this sum is negative on $I^-(M)$ and zero on M. Hence $M = \tilde{f}^{-1}(\phi(0))$. Finally, on $U_1 \setminus U_2$ the derivative $\mathrm{d}\tilde{f}$ is pointwise a positive multiple of $\mathrm{d}f_1$, which in turn is pointwise a positive multiple of $\mathrm{d}f$. Together with the same reasoning for $U_3 \setminus U_2$, we see that the condition on the involutive distributions is satisfied on the complement of $U_2 \subseteq U$.

The last part follows by induction over k. For the induction step we simply apply the proposition with $M = M_k$ and $U = \overline{M} \setminus \bigcup_{i=0}^{k-1} M_i$ to the Cauchy temporal function constructed in the previous step of the induction. □

A.2. On the spectral asymptotics

This appendix is devoted to the fact that the spectrum of a formally self-adjoint, first order elliptic differential operator has both infinitely many positive and infinitely many negative eigenvalues. This is used in the text when the operator is $\omega_{n,0} D\iota$ for $n \equiv -1 \mod 4$ or $\omega_{n,1} \overline{D}\iota$ for $n \equiv 0 \mod 4$, where $n > 0$ is as always the dimension of the manifold. Although this statement is probably well-known, it is hard to find a reference in the literature. The following argument was suggested by the anonymous referee of [Glö24b].

Proposition A.2.1. *Let M be a closed Riemannian manifold of dimension $n \geq 1$ and $E \to M$ be a vector bundle with a metric and a metric connection ∇. Assume that $D\colon \Gamma(E) \to \Gamma(E)$ is a formally self-adjoint, first order elliptic differential operator. Then D has infinitely many positive and infinitely many negative eigenvalues.*

Proof. First of all, after potentially passing to the complexification, we may assume that $E \to M$ is a complex vector bundle. Note that the assumptions on D together with the compactness of M guarantee that the spectrum of D is discrete and consists of real eigenvalues with finite multiplicity (cf. [LM89, Thm. III.5.8]). We assume for contradiction that the spectrum is bounded below. Then, replacing D by $D+c$ for some $c \in \mathbb{R}$, we may assume that D is positive.

Now take a covector $\xi \in T_p^*M$ so that $\sigma_D(\xi) \neq 0$, where the principal symbol σ_D of D is defined by $\sigma_D(\mathrm{d}f) = [D, f]$ for any $f \in C^\infty(M)$. In fact, since D is elliptic, any $\xi \neq 0$ will do the job. Since the endomorphism $i\sigma_D(\xi)$ is self-adjoint, we may choose an eigenvector $\Psi_p \in E_p$ of non-zero eigenvalue. Let $f \in C^\infty(M)$ be a function with $\mathrm{d}_p f = \xi$ and $\Psi \in \Gamma(E)$ be a section extending Ψ_p. Since $\langle \Psi_p, i\sigma_D(\xi)\Psi_p \rangle \neq 0$, we will have $(\Psi, i\sigma_D(\mathrm{d}f)\Psi)_{L^2} \neq 0$ – at least after multiplying Ψ with a cut-off function supported near p.

For any $t \in \mathbb{R}$, we have

$$e^{-itf} D(e^{itf}\Psi) = D\Psi + e^{itf}[D, e^{itf}]\Psi = D\Psi + e^{-itf}\sigma_D(\mathrm{d}e^{itf})\Psi = D\Psi + it\sigma_D(\mathrm{d}f)\Psi$$

and thus

$$(e^{itf}\Psi, De^{itf}\Psi)_{L^2} = (\Psi, e^{-itf}De^{itf}\Psi)_{L^2} = (\Psi, D\Psi)_{L^2} + t(\Psi, i\sigma_D(\mathrm{d}f)\Psi)_{L^2}.$$

This yields the desired contradiction since positivity of D implies that the left-hand side $(e^{itf}\Psi, De^{itf}\Psi)_{L^2} \geq 0$ for every $t \in \mathbb{R}$. □

A.3. Schrödinger-Lichnerowicz formula for the twisted Dirac-Witten operator

Theorem A.3.1. *For all* $\psi \in \Gamma((\Sigma M \oplus \Sigma M) \otimes_{\mathbb{C}} E)$

$$\left(\overline{D}^E\right)^2 \psi = \overline{\nabla}^* \overline{\nabla} \psi + \frac{1}{2}(\rho - e_0 \cdot j^\sharp \cdot)\psi + \mathcal{R}^E \psi,$$

where ρ *and* j *are defined as in* (1.2) *in terms of the pair* (g, k) *and* $\mathcal{R}^E(\phi \otimes e) = \sum_{i<j} e_i \cdot e_j \cdot \phi \otimes R^E(e_i, e_j)e$ *for* $\phi \otimes e \in (\Sigma_p M \oplus \Sigma_p M) \otimes_{\mathbb{C}} E_p$ *and an orthonormal basis* $(e_1, \dots, e_n)$ *of* T_pM, $p \in M$.

Proof. We show how to reduce the formula to the Schödinger-Lichnerowicz type formula in the untwisted case (1.6), using a local calculation. For this, let $(e_1, \dots, e_n)$ be a local orthonormal frame. Without loss of generality, we may assume that ψ can be written locally as $\phi \otimes e$ as everything is linear. Then

$$\begin{aligned}
\left(\overline{D}^E\right)^2(\phi \otimes e) &= \sum_{i,j} e_i \cdot \overline{\nabla}_{e_i}(e_j \cdot \overline{\nabla}_{e_j}\phi) \otimes e + \sum_{i,j} e_i \cdot \overline{\nabla}_{e_i}(e_j \cdot \phi) \otimes \nabla^E_{e_j} e \\
&\quad + \sum_{i,j} e_i \cdot e_j \cdot (\overline{\nabla}_{e_j}\phi) \otimes \nabla^E_{e_i} e + \sum_{i,j} e_i \cdot e_j \cdot \phi \otimes \nabla^E_{e_i}\nabla^E_{e_j} e \\
&= (\overline{D}^2\phi) \otimes e + \sum_{i,j} e_i \cdot (\overline{\nabla}_{e_i} e_j) \cdot \phi \otimes \nabla^E_{e_j} e \\
&\quad - 2\sum_i (\overline{\nabla}_{e_i}\phi) \otimes \nabla^E_{e_i} e + \sum_{i,j} e_i \cdot e_j \cdot \phi \otimes \nabla^E_{e_i}\nabla^E_{e_j} e
\end{aligned}$$

and

$$\begin{aligned}
\overline{\nabla}^*\overline{\nabla}(\phi \otimes e) &= \sum_i \overline{\nabla}^*(e_i^* \otimes \overline{\nabla}_{e_i}(\phi \otimes e)) \\
&= -\sum_i \overline{\nabla}_{e_i}\overline{\nabla}_{e_i}(\phi \otimes e) - \sum_i e_0 \cdot k(e_i,-)^\sharp \cdot \overline{\nabla}_{e_i}(\phi \otimes e) + \sum_i \overline{\nabla}_{\nabla_{e_i} e_i}(\phi \otimes e) \\
&= -\sum_i (\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}\phi) \otimes e - 2\sum_i (\overline{\nabla}_{e_i}\phi) \otimes \nabla^E_{e_i} e - \sum_i \phi \otimes \nabla^E_{e_i}\nabla^E_{e_i} e \\
&\quad - \sum_i e_0 \cdot k(e_i,-)^\sharp \cdot (\overline{\nabla}_{e_i}\phi) \otimes e - \sum_i e_0 \cdot k(e_i,-)^\sharp \cdot \phi \otimes \nabla^E_{e_i} e \\
&\quad + \sum_i (\overline{\nabla}_{\nabla_{e_i} e_i}\phi) \otimes e + \sum_i \phi \otimes \nabla^E_{\nabla_{e_i} e_i} e \\
&= (\overline{\nabla}^*\overline{\nabla}\phi) \otimes e - 2\sum_i (\overline{\nabla}_{e_i}\phi) \otimes \nabla^E_{e_i} e - \sum_i \phi \otimes \nabla^E_{e_i}\nabla^E_{e_i} e \\
&\quad - \sum_i e_0 \cdot k(e_i,-)^\sharp \cdot \phi \otimes \nabla^E_{e_i} e + \sum_i \phi \otimes \nabla^E_{\nabla_{e_i} e_i} e
\end{aligned}$$

using $\overline{\nabla}_X\psi = \nabla_X\psi - \frac{1}{2}e_0 \cdot k(X,-)^\sharp \cdot \psi$ and that the formal adjoint of ∇ is given by $\nabla^*\colon \alpha \otimes \psi \mapsto -\sum_j \nabla_{e_j}(\alpha \otimes \psi)(e_j) = -\sum_j \nabla_{e_j}(\alpha(e_j)\psi) + \sum_j \alpha(\nabla_{e_j} e_j)\psi$, $\alpha \in \Omega^1(M)$. Noting that

$$\begin{aligned}
&\sum_{i,j} e_i \cdot (\overline{\nabla}_{e_i} e_j) \cdot \phi \otimes \nabla^E_{e_j} e \\
&\qquad = \sum_{i,j,k} g(\nabla_{e_i} e_j, e_k) e_i \cdot e_k \cdot \phi \otimes \nabla^E_{e_j} e + \sum_{i,j} k(e_i, e_j) e_i \cdot e_0 \cdot \phi \otimes \nabla^E_{e_j} e \\
&\qquad = -\sum_{i,j,k} g(e_j, \nabla_{e_i} e_k) e_i \cdot e_k \cdot \phi \otimes \nabla^E_{e_j} e + \sum_j k(-,e_j)^\sharp \cdot e_0 \cdot \phi \otimes \nabla^E_{e_j} e \\
&\qquad = -\sum_{i,j} e_i \cdot e_j \cdot \phi \otimes \nabla^E_{\nabla_{e_i} e_j} e - \sum_i e_0 \cdot k(e_i,-)^\sharp \cdot \phi \otimes \nabla^E_{e_i} e
\end{aligned}$$

this implies

$$\begin{aligned}
\left(\overline{D}^E\right)^2(\phi \otimes e) - \overline{\nabla}^*\overline{\nabla}(\phi \otimes e) &= (\overline{D}^2\phi) \otimes e - (\overline{\nabla}^*\overline{\nabla}\phi) \otimes e - \sum_{i \neq j} e_i \cdot e_j \cdot \phi \otimes \nabla^E_{\nabla_{e_i} e_j} e \\
&\quad + \sum_{i \neq j} e_i \cdot e_j \cdot \phi \otimes \nabla^E_{e_i}\nabla^E_{e_j} e.
\end{aligned}$$

Using the untwisted Schrödinger-Lichnerowicz type formula (1.6), the first two terms combine to $\frac{1}{2}(\rho - e_0 \cdot j^\sharp \cdot)\phi \otimes e$. Thus it remains to identify the remaining terms with $\mathcal{R}^E(\phi \otimes e)$:

$$\begin{aligned}
\mathcal{R}^E(\phi \otimes e) &= \sum_{i<j} e_i \cdot e_j \cdot \phi \otimes R^E(e_i, e_j)e \\
&= \sum_{i<j} e_i \cdot e_j \cdot \phi \otimes (\nabla^E_{e_i}\nabla^E_{e_j} - \nabla^E_{e_j}\nabla^E_{e_i} - \nabla^E_{\nabla_{e_i}e_j} + \nabla^E_{\nabla_{e_j}e_i})e \\
&= \sum_{i\neq j} e_i \cdot e_j \cdot \phi \otimes (\nabla^E_{e_i}\nabla^E_{e_j} - \nabla^E_{\nabla_{e_i}e_j})e.
\end{aligned}$$

□

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