Mass, Momentum and Energy of Causal Fermion Systems

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für Mama, Papa und Magda

Abstract

Causal fermion systems are a candidate for a unified physical theory, giving relativistic quantum mechanics, general relativity and quantum field theory as limiting cases. They are based on the Dirac equation, a first order differential equation, which describes the fermions, the particles matter consists of. Fundamental for causal fermion systems is the so-called causal action principle. This determines the physically admissible objects like spacetimes defined in the setting of causal fermion systems, similar to the way the Einstein equations determine the relevant Lorentzian manifolds in general relativity. In this thesis the mass and energy of black holes are investigated in the theory of causal fermion systems based on the Euler-Lagrange equations and so-called surface layer integrals. More explicitly, the main goal of this thesis is to introduce the notions "mass" (and to this end "area"), "momentum" and "energy" in the setting of causal fermion systems, where "energy" is given by an energy-momentum four-vector with the energy as first component and momentum in the three spatial directions as the other components. Moreover we will show an analogy to the "Positive Mass Theorem" adapted to the theory of causal fermion systems. Finally these notions are made manifest by calculating the energy vector for a boosted Schwarzschild black hole and we discuss how to generalize these calculations to Lorentzian Manifolds.

Zusammenfassung

Kausale Fermionensysteme sind ein Kandidat für eine vereinheitlichte physikalische Theorie, da sie relativistische Quantenmechanik, allgemeine Relativitätstheorie und Quantenfeldtheorie als Grenzfälle liefert. Sie basieren auf der Dirac-Gleichung, die die Fermionen beschreibt - den Teilchen, aus denen Materie besteht. Grundlegend für kausale Fermionensysteme ist das sogenannte kausale Wirkungsprinzip. Es bestimmt die physikalisch zulässigen Objekte (wie z.B. Raumzeiten) in der Theorie kausaler Fermionensysteme, ähnlich wie die Einsteingleichungen die relevanten Lorentz-Mannigfaltigkeiten in der allgemeinen Relativitätstheorie ermittelt.
Diese Arbeit beschäftigt sich mit der Masse und der Energie Schwarzer Löcher in der Theorie kausaler Fermionensysteme auf der Grundlage der Euler-Lagrange-Gleichungen und sogenannten Oberflächenschichtintegralen. Präziser formuliert ist das Ziel dieser Arbeit, die Konzepte „Masse“ (und dafür „Fläche“), „Impuls“ und „Energie“ (in Form eines Energie-Impuls-Vektors mit der Energie in der ersten Komponente und den Impulsen in die drei räumlichen Richtungen als die anderen drei Komponenten) für kausale Fermionensysteme zu definieren und eine Entsprechung des „Positive-Masse-Theorems“ in der Theorie kausaler Fermionensysteme zu untersuchen. Schließlich werden diese Konzepte greifbarer gemacht, indem explizit der Energie-Vektor für eine geboostete Schwarzschild-Raumzeit bestimmt und die Verallgemeinerung auf Lorentz-Mannigfaltigkeiten diskutiert wird.

Contents

1 Introduction

In general relativity [ADM] Arnowitt, Deser and Misner defined the mass, momentum and energy of a system only depending on the asymptotics of the metric tensor at infinity. Therefore the system had to describe a so-called asymptotically flat spacetime which nearly looks like the Minkowski spacetime in the sense that the deviation of the metric at infinity had to fall off fast enough. They also proved that this energy is non-negative if the local energy density is non-negative. In the static case this energy density is given by the scalar curvature of the metric.
The theory of causal fermion systems describes fundamental physical structures and in a limiting case the Euler-Lagrange equations give the Einstein equations (up to correction terms) making it possible to do general relativity. Therein, spacetime is defined as the support of a measure fulfilling certain constraints instead of a Lorentzian manifold. The ADM mass resp. energy is defined as a surface integral at spatial infinity, measuring the non-linear gravitation, given by this asymptotic deviation of the metric from the flat Minkowski metric. One issue is that for causal fermion systems we do not have the local notion of an induced volume form on the hypersurface, hence we cannot just define the mass as a surface integral. However, in the theory of causal fermion systems there is an equivalent to surface integrals, the so-called surface layer integrals where we integrate over a thickened layer around the hypersurface instead.
For static causal fermion systems in [PMT] a notion of mass is introduced, based on the model of the ADM mass. Not only the surface layer integrals over spheres going to spatial infinity replaced the surface integrals, but lacking the locality of a metric tensor the notion of asymptotical flatness had to be adapted to the causal fermion setting. Then the gravitation as deviation of the metrics at infinity is displaced by comparing the measure for the asymptotically flat spacetime with the measure describing Minkowski spacetime.
However, the notion of mass introduced in [PMT] only works for static causal fermion systems. This thesis wants to generalize this notion to time-dependent causal fermion systems, i.e. systems changing with time like e.g. with moving or rotating black holes. Since for the static causal fermion systems, looking the same at each point in time, there is the possibility of decomposing the measure describing the asymptotically flat spacetime into the change of time and a spatial measure. Anyway, this is not possible for time-dependent spacetimes as there is no such strict separation between space and time, hence the spatial measure for such a decomposition depends on the considered time. This makes it necessary to replace the inner volume constraint in [PMT]. Instead, since the surface area and the volume of balls in Schwarzschild spacetime depend on each other, in this thesis it is replaced by the so-called area constraint.
For causal fermion systems we also prove a statement corresponding to the Positive

Mass Theorem for the ADM mass: Given a certain local condition, i.e. if there is something like non-negative mass density, the spacetime has non-negative mass. Furthermore, in the setting of time-dependent causal fermion systems spacetimes do not only have mass anymore, it is possible to introduce a notion of momentum and energy therein. This momentum is defined by surface layer integrals like for the mass but with contributions on thickened hypersurfaces of a constant spatial coordinate. The energy is defined as a vector consisting of energy and momentum, transforming as a four-vector like the ADM energy and momentum.
These notions are shown to coincide with the corresponding notions from Arnowitt, Deser and Misner (up to a constant) in special cases like Minkowski, Schwarzschild or boosted Schwarzschild spacetimes (i.e. boosting the black hole from Schwarzschild spacetime).
Finally we give a short overview over the content of each section:

- §2: In Section 2 we will recall causal fermion systems and all basic notions concerning causal fermion systems.

- §3: Section 3 will introduce preliminaries for the spacetimes to causal fermion systems as well as the important so-called linearized field equations therein and discuss the freedom in the solutions of these equations.

- §4: Section 4 is the main section of this thesis. Here we will recall the definition of the ADM mass from Arnowitt, Deser and Misner (cf. [ADM]) and the mass definition for static causal fermion systems (cf. [PMT]). (Static causal fermion systems describe time-independent spacetimes.)
  Then, based on the idea of how the ADM mass is defined and based on the model of the mass for static causal fermion systems, the mass for time-dependent spacetimes will be defined.
  Then we will prove a positive mass theorem for this mass definition.

- §5: In Section 5 we will introduce a notion of energy for causal fermion systems. To this end we define a momentum vector of causal fermion systems. Then the mass and the momentum together build the energy in form of an energy-momentum vector.

- §6: Section 6 will make all these definitions more tangible. In this Section we will consider a Schwarzschild black hole moving in a fixed direction with a fixed velocity. We then will calculate all these notions applied to this example.

- §7: In Section 7 we discuss how the calculation of the energy-momentum vector for Lorentzian Manifolds differs from the one for symmetric spacetimes like Schwarzschild spacetime.

- §8: Section 8 concludes with a short summary of the results of this thesis.

2 Preliminaries to Causal Fermion Systems

Causal fermion systems can be used to investigate black holes and their associated basic concepts. We will first recall the crucial notions in the theory of causal fermion systems:

2.1 Causal Fermion Systems

The following basic definitions are taken from [FF]:

Definition 2.1 (Causal Fermion System)**.** *A causal fermion system is a triple* $(\mathcal{H}, \mathcal{F}, \rho)$ *consisting of:*

i) a separable, complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ *of dimension* $f \in \mathbb{N} \cup \{\infty\}$

ii) the set $\mathcal{F}$ *of all linear, self-adjoint operators of finite rank on* $\mathcal{H}$*, which have at most* n *positive and at most* n *negative eigenvalues (counted with multiplicity)*

iii) a positive measure ρ *on a* σ*-algebra of subsets of* $\mathcal{F}$*.*

n *is then called spin dimension and* ρ *is called universal measure.*

Definition 2.2 (Spectral Weight)**.** *For* $x, y \in \mathcal{F}$ *define the spectral weight of the operator products* xy *resp.* $(xy)^2$ *by*

$$|xy| := \sum_{i=1}^{n} |\lambda_i^{xy}| \quad \textit{resp.} \;\; |(xy)^2| := \sum_{i=1}^{n} |\lambda_i^{xy}|^2,$$

where $\lambda_i^{xy} \in \mathbb{C}$ *are the non-trivial eigenvalues of* xy *(counted with multiplicity).*

Definition 2.3 (Lagrangian and Causal Action)**.** *The Lagrangian is defined by*

$$\mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}, \quad \mathcal{L}(x, y) := |(xy)^2| - \frac{1}{2n}|xy|^2.$$

The Causal Action is defined by

$$\mathcal{S} : \mathcal{B} \to \mathbb{R}, \quad \mathcal{S}(\rho) := \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x, y) d\rho(x) d\rho(y),$$

where $\mathcal{B}$ *denotes the space of regular Borel measures on* $\mathcal{F}$*.*

Definition 2.4 (Constraints)**.** *To avoid trivial minimizers of the causal action one considers the following constraints:*
In the case $\dim \mathcal{H} < \infty$ *the constraints are:*

$$\begin{aligned} \textit{Volume Constraint:} \quad & \rho(\mathcal{F}) = \textit{constant} \\ \textit{Trace Constraint:} \quad & \int_{\mathcal{F}} tr(x) d\rho(x) = \textit{constant} \\ \textit{Boundedness Constraint:} \quad & \iint_{\mathcal{F}\times\mathcal{F}} |xy| d\rho(x) d\rho(y) \leq C, \end{aligned}$$

where C *is a given parameter (and* tr *denotes the trace of a linear operator on* $\mathcal{H}$*).*
In the case $\dim \mathcal{H} = \infty$ *one has to replace the volume constraint by*

$$|\rho - \widetilde{\rho}|(\mathcal{F}) < \infty \quad \textit{and} \quad (\rho - \widetilde{\rho})(\mathcal{F}) = 0$$

for all variations $\widetilde{\rho}$ *of* ρ*, where* $|.|$ *denotes the total variation of the measure.*

Definition 2.5 (Causal Action Principle)**.** *The causal action principle means to minimize the causal action by varying the measure* ρ *under the constraints from Definition 2.4, where* ρ *is a regular Borel measure on* $\mathcal{F}$*.*
A measure ρ *is said to be a minimizer of the causal action, if* ρ *fulfills*

$$\mathcal{S}(\widetilde{\rho}) - \mathcal{S}(\rho) \geq 0$$

for all $\widetilde{\rho}$ *satisfying the constraints from Definition 2.4. For a minimizer* ρ *of the causal action the spacetime* M *is defined as its support* $M := supp\ \rho$*.*

Definition 2.6 (Lagrange Multiplier)**.** *In order to take the boundedness constraint into account by positive Lagrange multipliers, one defines the Lagrangian* $\mathcal{L}_\kappa$ *by adding a Lagrange multiplier term,* $\kappa \in \mathbb{R}$*, to the causal Lagrangian,*

$$\mathcal{L}_\kappa : \mathcal{F} \times \mathcal{F} \to \mathbb{R}, \quad \mathcal{L}_\kappa(x, y) := \mathcal{L}(x, y) + \kappa |xy|^2.$$

2.2 Euler-Lagrange Equations

For a minimizer ρ of the causal action one can derive very important equations, the so-called Euler-Lagrange equations.

Definition 2.7 (Functions ℓ and ℓ_κ)**.** *For a reasonable definition of the variation the following assumptions have to be made:*

i) The measure ρ *is locally finite, i.e. every* $x \in \mathcal{F}$ *has an open neighborhood* U *with* $\rho(U) < \infty$*. The topology on* $\mathcal{F}$ *is the one induced by the operator norm*

$$\|x\| := \sup\{\|xu\|_{\mathcal{H}} \textit{ with } \|u\|_{\mathcal{H}} = 1\}$$

(with $\|.\|_{\mathcal{H}}$ *denoting the norm on* $\mathcal{H}$ *induced by the scalar product* $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ *on* $\mathcal{H}$*). We will always consider* $M = supp\ \rho$ *endowed with the induced topology.*

ii) The function $\mathcal{L}_\kappa(x,.)$ is ρ-integrable for all $x \in \mathcal{F}$, giving a bounded continuous function on $\mathcal{F}$.

The bounded and lower semi-continuous functions ℓ and ℓ_κ can then be defined by

$$\ell(x) := \int_{\mathcal{F}} \mathcal{L}(x,y)d\rho(y) - s$$

resp.

$$\ell_\kappa(x) := \int_{\mathcal{F}} \mathcal{L}_\kappa(x,y)d\rho(y) - s$$

(for taking the volume constraint into account) with $s \in \mathbb{R}$ as specified below.

Definition 2.8 (Euler-Lagrange Equations)**.** *For every minimizer ρ of the causal action one has*

$$\ell_\kappa|_{supp\ \rho} \equiv \inf_{\mathcal{F}} \ell_\kappa.$$

Choosing the parameter $s \in \mathbb{R}$ s.t. this infimum is zero one obtains

$$\ell_\kappa|_{supp\ \rho} \equiv \inf_{\mathcal{F}} \ell_\kappa = 0. \tag{1}$$

These are called the Euler-Lagrange equations.

Definition 2.9 (Weak Euler-Lagrange Equations)**.** *Physically restricting the attention to ℓ_κ in a small neighborhood of $M = supp\ \rho$, one has*

$$\ell_\kappa|_M \equiv 0 \quad \text{and hence} \quad D\ell_\kappa|_M \equiv 0, \tag{2}$$

where $D\ell_\kappa(p) : T_p\mathcal{F} \to \mathbb{R}$ is the derivative. To combine these, define:
A jet is a pair $\mathfrak{u} := (a,u) \in \mathfrak{J}^\infty := C^\infty(M,\mathbb{R}) \oplus C^\infty(M,T\mathcal{F})$ of a real-valued function a on M and a vector field u on $T\mathcal{F}$ along M and the related derivative is defined by

$$\nabla_{\mathfrak{u}}\ell_\kappa(x) := a(x)\ell_\kappa(x) + (D_u\ell_\kappa)(x).$$

By equation (2) then $\nabla_{\mathfrak{u}}\ell_\kappa(x)$ vanishes for all $x \in M$ and one obtains the weak Euler-Lagrange equations

$$\nabla_{\mathfrak{u}}\ell_\kappa(x)|_M = 0 \quad \forall \mathfrak{u} \in \mathfrak{J}^{test}, \tag{3}$$

where the subset $\mathfrak{J}^{test} \subset \mathfrak{J}^\infty$ of test jets is defined in detail in [PMT, Section 2.1.1].

2.3 Surface Layer Integrals

In the setting of causal fermion systems the usual integrals over hypersurfaces in spacetime are undefined due to the lack of an induced volume form on the hypersurface. Instead, one considers so-called surface layer integrals, integrals over a thickened surface:

For a given differential operator (...) acting on the Lagrangian $\mathcal{L}_\kappa$, a surface layer integral is a double integral of the form

$$\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\; (...)\mathcal{L}_\kappa(x,y),$$

where $\Omega \subset M$.

Surface layer integrals correspond to the notion of surface integrals for causal fermion systems in the following sense (cf. [NLT, Section 2.3]):
Assume that $\mathcal{L}_\kappa$ is of short range, i.e. it decays on length scale $\delta \in \mathbb{R}$ and vanishes on distances larger than δ: Let d be the metric on M induced by the operator norm on M. Then we assume that

$$\forall x, y \in M \quad d(x,y) > \delta \Rightarrow \mathcal{L}_\kappa(x,y) = 0.$$

This way one only gets a contribution to the double integral if x and y are close together, i.e. close to the boundary $\partial\Omega$, hence the contribution to the integral comes from integrating the a bit expanded surface $\partial\Omega$.

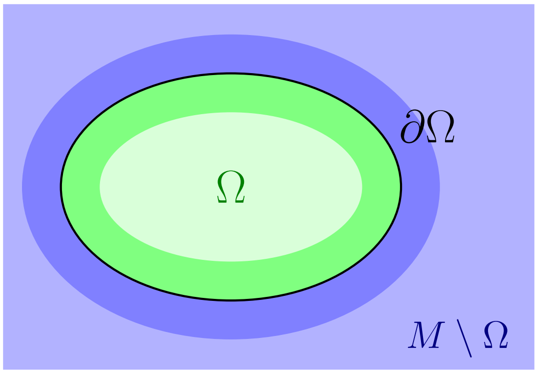

Figure 1: $\mathcal{L}(x,y)$ vanishes, if x or y is not in the strong green resp. blue domain.

Example 2.10. *Four-dimensional Minkowski spacetime can be regarded as a causal fermion system. This is explained in detail in [FF, Section 1.2]. We will always consider Minkowski spacetime with the signature* $(-,+,+,+)$.

3 Preliminaries to the Area, Mass and Energy of Causal Fermion Systems

After having recapped the basic notions for causal fermion systems in Section 2, we now will recall important notions needed for the definition of the mass such as the notion of time and important equations for causal fermion systems.

Situation 3.1. *Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a causal fermion system with spacetime denoted by $M = supp\ \rho$. We will always consider the situation that*

1. *the spacetime M is a four-dimensional smooth submanifold of $\mathcal{F}$.*
2. *the measure ρ is of the form $d\rho = h(x)\, d^4x$ for some $h \in C^\infty(U, \mathbb{R}^+)$ in local coordinates (for local charts U).*

Let $\Omega \subset M$ be compact with smooth boundary $\partial\Omega$. On the boundary define the measure $d\rho(v, x)$ as the contraction of the volume form on M with a vector field v, i.e. in local charts

$$d\rho(v, x) = h\epsilon_{ijkl} v^i dx^j dx^k dx^l,$$

where ϵ_{ijkl} is the totally anti-symmetric Levi-Civita symbol (which is normalized by $\epsilon_{0123} = 1$).

3.1 Past

In this section we will define the notion of time in the setting of causal fermion systems as well as the past of a given time. We will need this e.g. for a representation of the mass using spacetime integrals.

Definition 3.2 (Past)**.** *Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a causal fermion system. Under the smoothness assumption* 3.1*, after choosing a folitation $M = \mathbb{R} \times N$ of the spacetime $M = supp\ \rho$ the measure ρ on M can be decomposed as*

$$d\rho = dt\, d\mu_t \tag{4}$$

on $\mathbb{R} \times N$ (where the subscript t means that the measures μ_t depend on the first component, the time), where μ_t is in local coordinates (t, x^1, x^2, x^3) given by

$$d\mu_t = h(t, x^1, x^2, x^3) dx^1 dx^2 dx^3$$

for h from Situation 3.1. This implies that spacetime points can hence be written as $x = (t_x, \vec{x}) \in \mathbb{R} \times N$, where t_x can be interpreted as the time component. Denoting the projection on the time component by

$$T : M \to \mathbb{R}, \quad (t_x, \vec{x}) \mapsto t_x,$$

the time hypersurface for a fixed time $t_0 \in \mathbb{R}$ *(i.e. a constant-time-hypersurface) can be defined as*

$$N_{t_0} := T^{-1}(t_0)$$

and the past of the fixed time t_0 *by*

$$\Omega_{t_0} := \{x \in M | T(x) \leq t_0\}.$$

We then have the relation $N_{t_0} = \partial\Omega_{t_0}$.

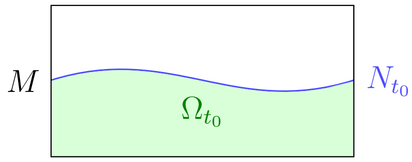

Figure 2: Time hypersurface and past for the time $t_0 \in \mathbb{R}$

We can generalize this to spatial coordinates:

Definition 3.3 (Spatial Hypersurfaces and Halfspaces). *Let* M *be a spacetime from Situation 3.1. Like for the decomposition in Definition 3.2 we can fix another direction* x_i, $i \in \{1, 2, 3\}$ *and we can as well consider a decomposition of the form* $M = \mathbb{R} \times \{x_i = \mathit{constant}\}$ *by constant-*x_i*-hypersurfaces. Analogous to Section 3.2 considering the corresponding projections* $X_i : M \to \mathbb{R}$, $i \in \{1, 2, 3\}$, *define the lower-*x_i*-halfspace* $H_{x_{ir}}$ *of the fixed spatial* x_i*-coordinate* $x_{ir} \in \mathbb{R}$ *by*

$$H_{x_{ir}} := \{x \in M | X_i(x) \leq x_{ir}\}.$$

3.2 Linearized Field Equations

We now give the linearized field equations. However, in the theory of causal fermion systems these are obtained by varying the measure in order to vary the field strengh. For more details on that see [PMT, Section 2.1.2].

Convention 3.4. *A variable provided with a prime always refers to the second component of the Lagrangian* $\mathcal{L}_\kappa$.

Definition 3.5 (Linearized Field Equations). *A jet* $\mathfrak{v} \in \mathfrak{J}^\infty$ *is called a solution of the linearized field equations with inhomogenity* $\mathfrak{w} \in \mathfrak{J}^\infty$ *if*

$$\Delta\mathfrak{v} = \mathfrak{w},$$

i.e. if for all test jets $\mathfrak{u} \in \mathfrak{J}^{test}$ *and for all spacetime points* $x \in M$ *we have with the notation from Definition 3.2*

$$\langle \mathfrak{u}, \Delta\mathfrak{v}\rangle(x) := \nabla_\mathfrak{u} \int_{-\infty}^{\infty} dt' \int_{N_{t'}} d\mu_{t'}(\vec{y}) \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\right) \mathcal{L}_\kappa(x, (t', \vec{y})) - \nabla_\mathfrak{u}\nabla_\mathfrak{v} s = \nabla_\mathfrak{u}\mathfrak{w},$$

where the subscripts 1 *resp.* 2 *refer to the corresponding component of* $\mathcal{L}_\kappa$.

3.3 Inner Solutions

Next we will define a special class of jets which satisfy the linearized field equations. These will allow us to shift solutions of the linearized field equations and obtain solutions with any possible scalar component:

Definition 3.6 (Divergence)**.** *Under the assumptions in Situation 3.1 one can define the divergence of a vector field $v \in C^\infty(M, T\mathcal{F})$ by*

$$div(v) := \frac{1}{h}\partial_i(hv^i)$$

(using the Einstein summation convention) as in common differential geometry. A jet $\mathfrak{v} \in \mathfrak{J}^\infty$ is called inner solution if its scalar component is the divergence of its vector field component, i.e. if it is of the form

$$\mathfrak{v} := (div(v), v).$$

Lemma 3.7. *Every inner solution $\mathfrak{v} = (div(v), v) \in \mathfrak{J}^\infty$ is a solution of the homogeneous linearized field equations, i.e.*

$$\Delta\mathfrak{v} = 0.$$

Proof. We again use the notation from Definition 3.2. We slightly adapt the proof of [PMT, Lemma 2.6]: Integrating by parts in $\vec{y}$ we have for all $x \in M$ that

$$\begin{aligned}
\langle \mathfrak{u}, \Delta\mathfrak{v}\rangle &= \nabla_{\mathfrak{u}}\left(\int_{-\infty}^{\infty} dt' \int_{N_{t'}} d\mu_{t'}(\vec{y})\,(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}})\,\mathcal{L}_\kappa(x,(t',\vec{y})) - \nabla_{\mathfrak{v}} s\right) \\
&= \nabla_{\mathfrak{u}}\left(\int_{-\infty}^{\infty} dt' \int_{N_{t'}} d\mu_{t'}(\vec{y})\nabla_{1,\mathfrak{v}}\mathcal{L}_\kappa(x,(t',\vec{y})) - \nabla_{\mathfrak{v}} s\right) \\
&= \nabla_{\mathfrak{u}}\nabla_{\mathfrak{v}}\ell(x) = \nabla_{\mathfrak{v}}\nabla_{\mathfrak{u}}\ell(x) = 0
\end{aligned}$$

by the weak Euler-Lagrange equations (3). (For further details on integrating by parts in this setting see [BFS, Equation (3.3)].) □

Lemma 3.8. *For a spacetime M as in Situation 3.1 let $I \subset M$ be a relatively compact subset and*

$$\Phi : M \setminus I \to \mathbb{R}^4 \setminus B_R$$

be a diffeomorphism (where $B_R \subset \mathbb{R}^4$ denotes the open ball of radius $R \geq 0$). Then for every $a \in C^\infty(M, \mathbb{R})$ there is a vector field $u \in C^\infty(M, T\mathcal{F})$, s.t. $div(u) = a$.

Proof. We closely follow the proof from [PMT, Lemma 2.7]. Choosing a partition of unity $(\phi_n)_{n\in\mathbb{N}}$ on M with

$$\begin{aligned} &\operatorname{supp}\phi_1 \subset I \cup \Phi^{-1}(B_{R+2}) \\ &\operatorname{supp}\phi_{n+1} \subset \Phi^{-1}(B_{R+n+1} \setminus \overline{B_{R+n-1}}) \quad \text{for } n \geq 2, \end{aligned}$$

by Situation 3.1 there is a volume form $\psi \in \Lambda^k(M)$, s.t.

$$\rho(U) = \int_U \psi \quad \text{for all compact } U \subset M.$$

Also there is a representation of the measure $a\rho$ by a volume form $\omega \in \Lambda^k(M)$, i.e.

$$\int_U a(x)d\rho(x) = \int_U \omega \quad \text{for all compact } U \subset M.$$

Choosing $c_1 \in \mathbb{R}$, s.t.

$$\int_M (\phi_1\omega - c_1\phi_1\psi) = 0,$$

[FDG, Theorem 1.2 in Section XVIII] then yields a compactly supported form $\eta_1 \in \Lambda_0^{k-1}(M)$ with

$$\phi_1\omega - c_1\phi_1\psi = d\eta_1.$$

Inductively by choosing $c_{n+1} \in \mathbb{R}$, s.t.

$$\int_M (\phi_{n+1}\omega + c_n\phi_n\psi - c_{n+1}\phi_{n+1}\psi) = 0,$$

for every $n \in \mathbb{N}$, we obtain forms $\eta_{n+1} \in \Lambda_0^{k-1}(M)$ with

$$\phi_{n+1}\omega + c_n\phi_n\psi - c_{n+1}\phi_{n+1}\psi = d\eta_{n+1}. \tag{5}$$

By theorem [FDG, Theorem 1.2 in Section XVIII] we can demand that the support of this η_{n+1} is always contained in the corresponding connected annulus $\Phi^{-1}(B_{R+n+1} \setminus \overline{B_{R+n-1}})$ (resp. $I \cup \Phi^{-1}(B_{R+2})$ for supp η_1). Adding up the equations of the form (5), we get that

$$\eta := \sum_{n=1}^{\infty} \eta_n \quad \text{satisfies} \quad d\eta = \sum_{n=1}^{\infty} \phi_n\omega = \omega$$

(where the sum converges because of the local finiteness). In order to find a vector field v, s.t. $\operatorname{div}(v)d\rho = d\eta$, choose a Riemannian metric g on M and apply a conformal transformation s.t. the corresponding volume form and ρ coincide. Then the vector field given by

$$v^\alpha = g^{\alpha\beta}(\star\eta)_\beta$$

(with $\star : \Lambda^{k-1}(M) \to \Lambda^1(M)$ the Hodge operator) satisfies the claim. □

Corollary 3.9. *By the Lemmas 3.7 and 3.8 we can restrict all considerations to solutions of the linearized field equations with a certain scalar component later on: Since for every solution $\mathfrak{v} = (a, v) \in \mathfrak{J}^\infty$ of the linearized field equations by Lemma 3.8 and any scalar component $b \in C^\infty(M, \mathbb{R})$ there is a vector field $\widetilde{v}$ together with a corresponding inner solution $\widetilde{\mathfrak{v}} = (b - a, \widetilde{v}) \in \mathfrak{J}^\infty$ and by Lemma 3.7 then $\mathfrak{v} + \widetilde{\mathfrak{v}} = (b, v + \widetilde{v})$ is a solution of the linearized field equations with this given scalar component.*

Convention 3.10. *For inner solutions to a vector field the contraction of a measure with this inner solution always means the contraction with that vector field.*

4 Mass of Causal Fermion Systems

The basic idea for the mass definition for causal fermion systems comes from the so-called *ADM* mass, hence we will shortly recall its setting and definition:

4.1 Motivation: *ADM* Mass and the Mass of Static Causal Fermion Systems

First we will mention the definition of the *ADM* mass in [ADM] giving the idea for the definition of the mass of causal fermion systems and afterwards we will recall the mass definition from [PMT] for causal fermion systems in the static setting:

4.1.1 *ADM* Mass

Arnowitt, Deser and Misner defined in [ADM] mass and energy for spacetimes which in some sense asymptotically look like the Minkowski spacetime. For this they measure the deviation of the geometry at infinity between Minkowski and the other spacetime. We will shortly recall this definition:

Situation 4.1. *Let*

- (N, g) *be a four-dimensional Lorentzian manifold with metric* g *of signature* $(-, +, +, +)$ *and denote the induced Levi-Civita connection by* ∇
- $M \subset N$ *be an oriented spacelike hypersurface with induced metric* g_{ij}
- ν *be a normal vector field on* M *and denote the second fundamental form by* $h_{ij} = (\nabla_i \nu)_j$.

Definition 4.2 (Energy, Momentum and Mass). *i)* (M, g) *is called asymptotically flat if* *a) there is a compact subset $K \subset M$ and for some closed ball in $\mathbb{R}^3$ of radius $r > 0$ a diffeomorphism $\Phi : M \setminus K \to \mathbb{R}^3 \setminus \overline{B_r(0)}$.*

b) under Φ the metric has the form

$$g_{ij}(x) = \delta_{ij} + a_{ij}(x)$$

for $x \in \mathbb{R}^3 \setminus \overline{B_r(0)}$ with the decay properties for the remainder

$$a_{ij} = \mathcal{O}\left(\frac{1}{\|x\|}\right), \quad \partial_k a_{ij} = \mathcal{O}\left(\frac{1}{\|x\|^2}\right) \quad \text{and} \quad \partial_k \partial_l a_{ij} = \mathcal{O}\left(\frac{1}{\|x\|^3}\right)$$

(where $\|.\|$ denotes the Euclidean norm on $\mathbb{R}^3$) as well as for the second fundamental form

$$h_{ij} = \mathcal{O}\left(\frac{1}{\|x\|^2}\right) \quad \text{and} \quad \partial_k h_{ij} = \mathcal{O}\left(\frac{1}{\|x\|^3}\right).$$

ii) For $R > r$ from i) let $S_R \subset \mathbb{R}^3$ denote the sphere with radius R. Then the total energy E of an asymptotically flat manifold (M, g) is defined as

$$E := \frac{1}{16\pi} \lim_{R\to\infty} \sum_{i,j=1}^{3} \int_{S_R} \left(\partial_j g_{ij} - \partial_i g_{jj}\right) \nu^i d\Omega$$

with ν resp. $d\Omega$ the outward normal resp. area form on S_R and the total momentum P is with the components $P_k, k \in \{1,2,3\}$ defined by

$$P_k := \frac{1}{16\pi} \lim_{R\to\infty} \sum_{i,j=1}^{3} \int_{S_R} \left(\partial_j h_{0k} - \partial_0 h_{jk} + \delta_{jk}\partial_0 h_{ii} - \delta_{jk}\partial_i h_{0i}\right) \nu^i d\Omega.$$

For so-called static manifolds, i.e. if the second fundamental form h_{ij} vanishes, so does the momentum P. In this case the energy is then called ADM mass.

With the concept of the *ADM* mass in mind, following [PMT, Section 1] we shortly recall the mass definition for static spacetimes which we will adapt to the time-dependent causal fermion system setting afterwards. Here, static spacetime means a globally hyperbolic spacetime, where a foliation $(N_t)_{t\in\mathbb{R}}$ can be chosen so that the timelike vector field ∂_t is a Killing field which is orthogonal to the hypersurfaces N_t (cf. [PMT, Section 2.3.2]).

4.1.2 Mass of Static Causal Fermion Systems

Definition 4.3 ([PMT], Definition 3.1)**.** *Let $(\mathcal{U}_t)_{t\in\mathbb{R}}$ be a strongly continuous one-parameter group of unitary transformations on the Hilbert space $\mathcal{H}$ (i.e. for all $t, t' \in \mathbb{R}$ we have $\mathcal{U}_t\mathcal{U}_{t'} = \mathcal{U}_{t+t'}$ and $\lim_{t'\to t}\mathcal{U}_{t'} = \mathcal{U}_t$). Then the causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ is called static w.r.t. $(\mathcal{U}_t)_{t\in\mathbb{R}}$ if:*

i) The spacetime $M := \text{supp}\,\rho$ is a topological product $M = \mathbb{R} \times N$, allowing to denote spacetime points $x \in M$ by $x = (t, \vec{x}) \in \mathbb{R} \times N$.

ii) ρ is $(\mathcal{U}_t)_{t\in\mathbb{R}}$-invariant, i.e.

$$\forall t \in \mathbb{R}, \Omega \subset \mathcal{F} \;\rho\text{-measurable} \quad \rho(\mathcal{U}_t\Omega\mathcal{U}_t^{-1}) = \rho(\Omega)$$

and additionally

$$\forall t, t' \in \mathbb{R}, (t, \vec{x}) \in \mathbb{R} \times N \quad \mathcal{U}_{t'}(t, \vec{x})\mathcal{U}_{t'}^{-1} = (t + t', \vec{x}).$$

We will then call the spacetime M static as well. In this case ρ induces a measure μ on N by $\mu(\Omega) := \rho\left([0,1] \times \Omega\right)$ for $\Omega \subset N$ which then fulfills

$$d\rho = dt d\mu. \tag{6}$$

Example 4.4. *The Minkowski spacetime (cf. Example 2.10) is static.*

For the (static) spacetime $\widetilde{M} = \text{supp}\,\widetilde{\rho}$ of a static causal fermion system $(\widetilde{\mathcal{H}}, \widetilde{\mathcal{F}}, \widetilde{\rho})$, where $\widetilde{\rho}$ minimizes the causal action with the same constants s and κ as ρ on M, the mass was as well defined by integrating the deviation of the geometry at infinity of this spacetime $\widetilde{M}$ from the flat spacetime M of the causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ representing Minkowski (cf. 2.10). We exhausted both spacetimes by sets of finite measure giving a definition of the form

$$\begin{aligned}\mathfrak{M}\left(\widetilde{\mu}\right) = \lim_{\Omega\nearrow N}\lim_{\widetilde{\Omega}\nearrow\widetilde{N}} \Big(&-s\left(\widetilde{\mu}(\widetilde{\Omega}) - \mu(\Omega)\right) \\ &+ \int_{\widetilde{\Omega}} d\widetilde{\mu}(x)\int_{N\setminus\Omega} d\mu(y)\mathcal{L}_\kappa(x,y) - \int_{\Omega} d\mu(x)\int_{\widetilde{N}\setminus\widetilde{\Omega}} d\widetilde{\mu}(y)\mathcal{L}_\kappa(x,y)\Big)\end{aligned}$$

(with $\widetilde{M} = \mathbb{R}\times\widetilde{N}$ with $d\widetilde{\rho} = dt d\widetilde{\mu}$ as $\widetilde{M}$ is static) for the static Lagrangian (for more information cf. [PMT, Section 3.2]), where the quantities without tilde always refer to Minkowski spacetime while the ones with tilde refer to the spacetime whose mass is calculated. (In the mass definition for time-dependent causal fermion systems we will later on denote elements of the exhaustion by U as we will denote pasts by Ω.)

Considering only exhaustions $\Omega \nearrow M$ resp. $\widetilde{\Omega} \nearrow \widetilde{M}$ with the same inner volume, i.e. $\mu(\Omega) = \widetilde{\mu}(\widetilde{\Omega})$ for the elements of the exhaustions, then doing the Taylor expansion and taking the difference of the highest order terms the inner volumes drop out and the next order terms give the mass

$$\begin{aligned}\mathfrak{M}(\widetilde{\mu}) = &\lim_{\Omega \nearrow N, \widetilde{\Omega} \nearrow \widetilde{N} \text{ with } \mu(\Omega)=\widetilde{\mu}(\widetilde{\Omega})} \\ &\left(\int_{\widetilde{\Omega}} d\widetilde{\mu}(x) \int_{N \setminus \Omega} d\mu(y) \mathcal{L}_\kappa(x,y) - \int_{\Omega} d\mu(x) \int_{\widetilde{N} \setminus \widetilde{\Omega}} d\widetilde{\mu}(y) \mathcal{L}_\kappa(x,y) \right) \\ = &\lim_{\Omega \nearrow N} \int_{\Omega} d\mu(x) \int_{N \setminus \Omega} d\mu(y) \left(\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}}\right) \mathcal{L}_\kappa(x,y)\end{aligned}$$

for some jet $\mathfrak{v} \in \mathfrak{J}^\infty$ describing the gravitation.

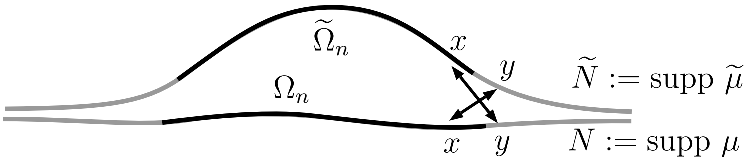

Figure 3: The surface layer integral for the mass in [PMT]

Example 4.5. *By [PMT, Theorem 1.10] for the Schwarzschild spacetime this mass definition coincides with the ADM mass of the Schwarzschild spacetime up to a constant.*

However, in the time-dependent setting there is no longer an obvious notion of inner volume because then the measure μ from a decomposition as in equation (6) depends on the time t (other as in equation (4)), hence the inner volume condition has to be replaced. Since the surface area of balls in Schwarzschild spacetime is related to their mass this suggests to demand that the exhausting subsets have the same surface area instead of the same inner volume.

4.2 Area for Causal Fermion Systems

In this section we will introduce a notion of surface area for causal fermion systems:

Definition 4.6 (Area)**.** *Let $(\mathcal{H}, \mathcal{F}, \rho)$ again be a causal fermion system with four-dimensional critical measure ρ and smooth spacetime $M = supp\ \rho$. Additionally let $\Omega, U \subset M$ be open, s.t. their boundaries $\partial\Omega$ and ∂U intersect transversally, giving a two-dimensional surface $\partial\Omega \cap \partial U$. Then define its area by*

$$A(\Omega, U) = \int_{\Omega \cap U} d\rho(x) \int_{M \setminus (\Omega \cup U)} d\rho(y) \mathcal{L}_\kappa(x,y). \tag{7}$$

Here we integrate in x over the green and in y over the blue domain, where the main contribution comes from integrating over x and y close to the two-dimensional surface $\partial\Omega \cap \partial U$ (marked by the slightly darker regions):

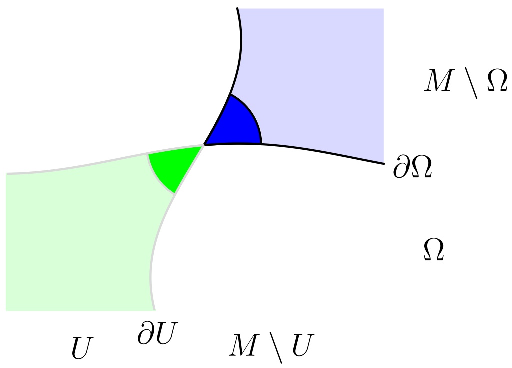

Figure 4: Integration domains for the area in the time-dependent case

Lemma 4.7. *In Definition 4.6 let M be static (cf. Definition 4.3) and $\Omega \subset M$ be the past of a time $t \in \mathbb{R}$ in M. Then the area (7) does not depend on the chosen time t. (Hence we will often drop the chosen time in Minkowski in definitions.)*

Proof. For static spacetimes we can consider the decomposition from Definition 4.3 and shift the time using the one-parameter group $(\mathcal{U}_t)_{t\in\mathbb{R}}$: Consider an arbitrary time $s \in \mathbb{R}$, its past $\Omega_s = (-\infty, s] \times N \subset M$ and let $\Omega_0 = (-\infty, 0] \times N \subset M$ denote the past for time $0 \in \mathbb{R}$. Then we can calculate the area at time s and any $U \subset N$ (as in the static setting, for more details see [PMT]) with boundary ∂U intersecting $N = \partial\Omega$ transversally:

$$\begin{aligned}
A(\Omega_s, U) &= \int_{\Omega_s \cap U} d\rho(x) \int_{M\setminus(\Omega_s\cup U)} d\rho(y) \mathcal{L}_\kappa\left((t,\vec{x}),(t',\vec{y})\right)\\
&= \int_{\Omega_s \cap U} d\rho(x) \int_{M\setminus(\Omega_s\cup U)} d\rho(y) \mathcal{L}_\kappa\left(\mathcal{U}_s(t-s,\vec{x})\mathcal{U}_s^{-1},(t',\vec{y})\right)\\
&= \int_{\Omega_s \cap U} d\rho(x) \int_{M\setminus(\Omega_s\cup U)} d\rho(y) \mathcal{L}_\kappa\left(\mathcal{U}_s^{-1}\mathcal{U}_s(t-s,\vec{x})\mathcal{U}_s^{-1}\mathcal{U}_s,\mathcal{U}_s^{-1}(t',\vec{y})\mathcal{U}_s\right)\\
&= \int_{\Omega_s \cap U} d\rho(x) \int_{M\setminus(\Omega_s\cup U)} d\rho(y) \mathcal{L}_\kappa\left((t-s,\vec{x}),\mathcal{U}_{-s}(t',\vec{y})\mathcal{U}_{-s}^{-1}\right)\\
&= \int_{\Omega_s \cap U} d\rho(x) \int_{M\setminus(\Omega_s\cup U)} d\rho(y) \mathcal{L}_\kappa\left((t-s,\vec{x}),(t'-s,\vec{y})\right)\\
&= \int_{\Omega_0 \cap U} d\rho(x) \int_{M\setminus(\Omega_0\cup U)} d\rho(y) \mathcal{L}_\kappa\left((t,\vec{x}),(t',\vec{y})\right) = A(\Omega_0, U)\,,
\end{aligned}$$

where we used the transformation formula in the penultimate step.
Hence we can conclude that in static spacetimes the area is the same w.r.t. the past of any time $s \in \mathbb{R}$. $\square$

4.3 Mass of Time-Dependent Causal Fermion Systems

We will define the mass of a spacetime also using surface layer integrals measuring the deviation of this spacetime from Minkowski spacetime. The notion of asymptotic flatness will ensure that the geometry of this spacetime approaches Minkowski geometry at infinity and is asymptotically close enough to Minkowski, s.t. these surface layer integrals converge and can be calculated by some linearization.

Convention 4.8. *While defining mass, momentum and energy of time-dependent causal fermion systems we will often refer to spacetimes M and $\widetilde{M}$. This will always mean we consider two causal fermion systems $(\mathcal{H}, \mathcal{F}, \rho)$ resp. $(\widetilde{\mathcal{H}}, \widetilde{\mathcal{F}}, \widetilde{\rho})$ with the spacetimes $M = supp\ \rho$ resp. $\widetilde{M} = supp\ \widetilde{\rho}$ (and decompositions as in Definition 3.2), which are minimizers of the same causal action, i.e. with the same constants s and κ. Moreover, M will from now on always denote the Minkowski spacetime (cf. 2.10) with decomposition $M = \mathbb{R} \times N$, $d\rho = dtd\mu$ from Section 4.3.*

4.3.1 Asymptotic Flatness

First we will express decay properties as in Definition 4.2,i),b) in the theory of causal fermion systems:

Definition 4.9 (Asymptotically Flat)**.** *(This definition follows closely [PMT, Definition 1.5].) In the situation of Convention 4.8 the spacetime $\widetilde{M} = supp\ \widetilde{\rho}$ is called asymptotically flat (we then also say $\widetilde{\rho}$ is asymptotically flat) if*

i) $\rho(M) = \widetilde{\rho}(\widetilde{M}) = \infty$

ii) There is a mapping $F : M \to \widetilde{M}$, s.t.

$$\widetilde{\rho} = F_* \rho$$

(with $F_*\rho$ denoting the pushforward measure of ρ under F), F is a diffeomorphism onto its image outside of a relatively compact open subset $I \subset M$ and the surface layer integral

$$\lim_{U_n \nearrow M, \widetilde{U}_n \nearrow \widetilde{M} \text{ with } A(U_n) = \widetilde{A}\left(\widetilde{\Omega}_{t'}, \widetilde{U}_n\right)} \times$$
$$\left(\int_{N \setminus U_n} d\mu(x) \int_{\widetilde{U}_n} d\widetilde{\rho}(y) - \int_{N \cap U_n} d\mu(x) \int_{\widetilde{M} \setminus \widetilde{U}_n} d\widetilde{\rho}(y) \right) \mathcal{L}_\kappa(x,y)$$

(where $\widetilde{\Omega}_{t'} \subset \widetilde{M}$ is the past for a time $t' \in \mathbb{R}$ for a decomposition as in Definition 3.2) can be linearized, s.t. in the limit we have

$$\begin{aligned}&\left(\int_{N\setminus U_n} d\mu(x) \int_{U_n} d\widetilde{\rho}(y) - \int_{N\cap U_n} d\mu(x) \int_{M\setminus U_n} d\rho(y)\right) \mathcal{L}_\kappa(x, F(y)) \\ &= \left(\int_{N\setminus U_n} d\mu(x) \int_{U_n} d\rho(y) - \int_{N\cap U_n} d\mu(x) \int_{M\setminus U_n} d\rho(y)\right) \nabla_{2,\mathfrak{w}}\mathcal{L}_\kappa(x, y) \qquad (8)\end{aligned}$$

for some jet $\mathfrak{w} \in \mathfrak{J}^\infty$. (This limit will in the next section define the mass of a causal fermion system.)

iii) For the inner solution $\mathfrak{u} \in \mathfrak{J}^\infty$ to the vector field $\frac{\partial}{\partial t}$ (resp. $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ or $\frac{\partial}{\partial z}$) on Minkowski spacetime M there is an inner solution $\widetilde{\mathfrak{u}}$ to the corresponding vector field $\frac{\partial}{\partial t}$ (resp. $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ or $\frac{\partial}{\partial z}$) on the spacetime $\widetilde{M}$, s.t.

$$(\nabla_{1,\mathfrak{u}} + \nabla_{2,\widetilde{\mathfrak{u}}})\,\mathcal{L}(x, y) \in L^1(N \times \widetilde{M})$$

(where N denotes a constant-time-hypersurface or a constant-x_i-hypersurface, cf. Definition 3.2 and Definition 3.3).

4.3.2 Mass Definition for Time-Dependent Causal Fermion Systems

Having a notion of area, we can now define the mass as follows, using the notation from Convention 4.8:

Definition 4.10 (Mass)**.** *In the situation of Convention 4.8 let $\widetilde{\rho}$ be asymptotically flat. Then the mass of $\widetilde{\rho}$ is defined as*

$$\mathfrak{M}(\widetilde{\rho}) := \lim_{U_n \nearrow M, \widetilde{U}_n \nearrow \widetilde{M} \text{ with } A(\Omega, U_n) = \widetilde{A}\left(\widetilde{\Omega}_{t'}, \widetilde{U}_n\right)} O(\widetilde{\rho}, U_n, \widetilde{U}_n) \qquad (9)$$

with

$$O(\widetilde{\rho}, U_n, \widetilde{U}_n) := \left(\int_{N\setminus U_n} d\mu(x) \int_{\widetilde{U}_n} d\widetilde{\rho}(y) - \int_{N\cap U_n} d\mu(x) \int_{\widetilde{M}\setminus \widetilde{U}_n} d\widetilde{\rho}(y)\right) \mathcal{L}_\kappa(x, y),$$

where $U_n \nearrow M$ resp. $\widetilde{U}_n \nearrow \widetilde{M}$ means we exhaust M resp. $\widetilde{M}$ by sets of finite measure and $\Omega \subset M$ resp. $\widetilde{\Omega}_{t'} \subset \widetilde{M}$ are the pasts for times $t \in \mathbb{R}$ resp. $t' \in \mathbb{R}$ for decompositions as in Definition 3.2.

Remark 4.11. *Since $\widetilde{\rho}$ is asymptotically flat, we can linearize the surface layer integrals to obtain in the limit*

$$O(\widetilde{\rho}, U_n, \widetilde{U}_n) = \left(\int_{N\setminus U_n} d\mu(x) \int_{U_n} d\rho(y) - \int_{N\cap U_n} d\mu(x) \int_{M\setminus U_n} d\rho(y)\right) \nabla_{2,\mathfrak{w}}\mathcal{L}_\kappa(x, y).$$

The integration domains for $\nabla_{2,\mathfrak{w}}\mathcal{L}$ are then in Minkowski spacetime and can be pictured as follows:

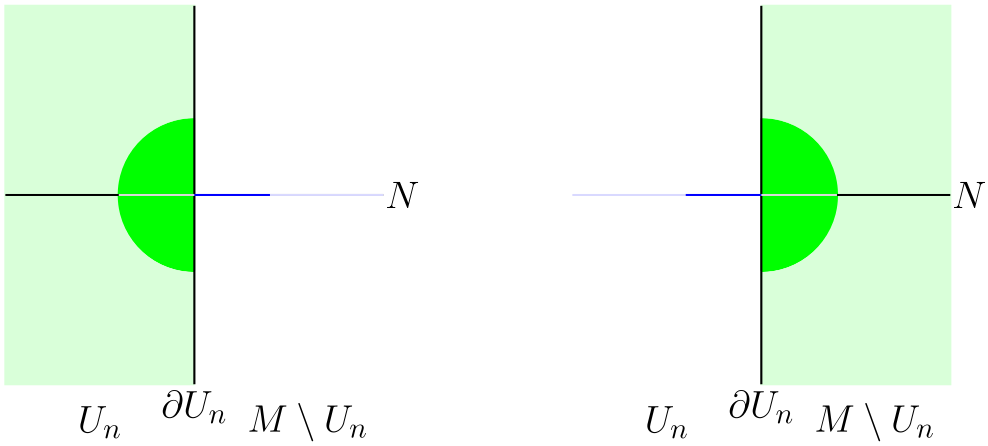

Figure 5: Integration domains for the mass: minuend (left) and subtrahend (right)

Here we again integrate in x over the green and in y over the blue domain, where the main contribution comes from integrating over x and y close to the two-dimensional surface $\partial\Omega \cap \partial U$ marked by the slightly darker regions.

Like the ADM mass this mass definition only depends on the geometry of the spacetime $\widetilde{M}$ at infinity. Moreover we will show that this definition does neither depend on the chosen exhaustions $U \nearrow M$ resp. $\widetilde{U} \nearrow \widetilde{M}$ nor on the choice of the time nor on the chosen pasts Ω_t and $\widetilde{\Omega}_{t'}$, hence we can denote the mass by $\mathfrak{M}(\widetilde{\rho})$.

4.3.3 Rewritten Definition of the Mass

In order to replenish the mass with a notion of momentum to obtain a definition of energy of causal fermion systems we will rewrite the term of the mass. We will also use this to show that the mass from Definition (9) is time-independent.

Lemma 4.12. *In the situation of Definition 4.10 we could also define the mass as follows:*

$$\mathfrak{M}(\widetilde{\rho}) := \lim_{U_n \nearrow M, \widetilde{U}_n \nearrow \widetilde{M} \text{ with } A(U_n) = \widetilde{A}\left(\widetilde{\Omega}_{t'}, \widetilde{U}_n\right)} O_1(\widetilde{\Omega}_{t'}, U_n, \widetilde{U}_n)$$

with

$$O_1(\widetilde{\Omega}_{t'}, U_n, \widetilde{U}_n) :=$$
$$- \left(\int_{\Omega_t \cap U_n} d\rho(x) \int_{\widetilde{M} \setminus (\widetilde{\Omega}_{t'} \cup \widetilde{U}_n)} d\widetilde{\rho}(y) - \int_{\Omega_t \setminus U_n} d\rho(x) \int_{\widetilde{U}_n \setminus \widetilde{\Omega}_{t'}} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x,y)$$
$$+ \left(\int_{U_n \setminus \Omega_t} d\rho(x) \int_{\widetilde{\Omega}_{t'} \setminus \widetilde{U}_n} d\widetilde{\rho}(y) - \int_{M \setminus (\Omega_t \cup U_n)} d\rho(x) \int_{\widetilde{\Omega}_{t'} \cap \widetilde{U}_n} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x,y),$$

where Ω_t is the past of the time hypersurface N from Definition 4.10 w.r.t. a time $t \in \mathbb{R}$ (cf. Definitions 4.3 and 3.2) and $\mathfrak{u}$ is the inner solution corresponding to the time derivative $\frac{\partial}{\partial t}$. ($O_1$ depends on $\widetilde{\rho}$, but we drop $\widetilde{\rho}$ for the shorter notation.) By Lemma 4.7 the mass does not depend on the choice of Ω_t. Moreover, we will show in Proposition 4.14 that this definition is independent of the chosen past $\widetilde{\Omega}_{t'}$, hence we can drop Ω_t and $\widetilde{\Omega}_{t'}$ in the notation of O_1 resp. the mass $\mathfrak{M}$.

Proof. Using $\mathfrak{u} = \left(\operatorname{div}\left(\frac{\partial}{\partial t}\right), \frac{\partial}{\partial t}\right)$ and integrating by parts in the first component yields integrals over the Minkowski time hypersurface $N = N_t = \partial\Omega_t$:

$$- \left(\int_{\Omega_t \cap U_n} d\rho(x) \int_{\widetilde{M} \setminus (\widetilde{\Omega}_{t'} \cup \widetilde{U}_n)} d\widetilde{\rho}(y) - \int_{\Omega_t \setminus U_n} d\rho(x) \int_{\widetilde{U}_n \setminus \widetilde{\Omega}_{t'}} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x,y)$$
$$+ \left(\int_{U_n \setminus \Omega_t} d\rho(x) \int_{\widetilde{\Omega}_{t'} \setminus \widetilde{U}_n} d\widetilde{\rho}(y) - \int_{M \setminus (\Omega_t \cup U_n)} d\rho(x) \int_{\widetilde{\Omega}_{t'} \cap \widetilde{U}_n} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x,y)$$
$$= - \left(\int_{N \cap U_n} d\rho(x) \int_{\widetilde{M} \setminus (\widetilde{\Omega}_{t'} \cup \widetilde{U}_n)} d\widetilde{\rho}(y) - \int_{N \setminus U_n} d\rho(x) \int_{\widetilde{U}_n \setminus \widetilde{\Omega}_{t'}} d\widetilde{\rho}(y) \right) \mathcal{L}_\kappa(x,y)$$
$$+ \left(- \int_{N \cap U_n} d\rho(x) \int_{\widetilde{\Omega}_{t'} \setminus \widetilde{U}_n} d\widetilde{\rho}(y) + \int_{N \setminus U_n} d\rho(x) \int_{\widetilde{\Omega}_{t'} \cap \widetilde{U}_n} d\widetilde{\rho}(y) \right) \mathcal{L}_\kappa(x,y)$$
$$= \left(\int_{N \setminus U_n} d\mu(x) \int_{\widetilde{U}_n} d\widetilde{\rho}(y) - \int_{N \cap U_n} d\mu(x) \int_{\widetilde{M} \setminus \widetilde{U}_n} d\widetilde{\rho}(y) \right) \mathcal{L}_\kappa(x,y),$$

hence $O_1(\widetilde{\Omega}_{t'}, U_n, \widetilde{U}_n)$ coincides with $O(\widetilde{\rho}, U_n, \widetilde{U}_n)$ from Section 4.3 for the past $\widetilde{\Omega}_{t'}$ of any time $t' \in \mathbb{R}$. □

4.3.4 Independence of the Choice of the Exhaustions

In this Section we will prove the following statement:

Proposition 4.13. *The mass from Definition 4.10 is independent of the chosen exhaustions $U \nearrow M$ resp. $\widetilde{U} \nearrow \widetilde{M}$.*

Proof. As in Definition 4.10 we consider exhaustions $U_r \nearrow M$, $\widetilde{U}_r \nearrow \widetilde{M}$ and let $\mathfrak{v}$ denote the inner solution corresponding to the vector field $\frac{\partial}{\partial r}$ on Minkowski spacetime. Since $\widetilde{\rho}$ is asymptotically flat, there is an inner solution $\widetilde{\mathfrak{v}}$ to the vector field $\frac{\partial}{\partial r}$ on the spacetime $\widetilde{M}$, s.t. the integrability condition

$$\left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\widetilde{\mathfrak{v}}}\right) \mathcal{L}(x,y) \in L^1(N \times \widetilde{M}) \tag{10}$$

is fulfilled. An infinitesimal change of the exhausting sets yields

$$\begin{aligned}
&\frac{d}{dr} O(\widetilde{\rho}, U_r, \widetilde{U}_r) \\
&= \frac{d}{dr} \left(\int_{N \setminus U_r} d\mu(x) \int_{\widetilde{U}_r} d\widetilde{\rho}(y) - \int_{N \cap U_r} d\mu(x) \int_{\widetilde{M} \setminus \widetilde{U}_r} d\widetilde{\rho}(y) \right) \mathcal{L}_\kappa(x,y) \\
&= \left(\int_{N \setminus U_r} d\mu(x) \int_{\partial \widetilde{U}_r} d\widetilde{\rho}(\mathfrak{v}, y) + \int_{N \cap U_r} d\mu(x) \int_{\partial \widetilde{U}_r} d\widetilde{\rho}(y) \right. \\
&\quad \left. - \int_{N \cap \partial U_r} d\mu(\mathfrak{v}, x) \int_{\widetilde{U}_r} d\widetilde{\rho}(y) - \int_{N \cap \partial U_r} d\mu(\mathfrak{v}, x) \int_{\widetilde{M} \setminus \widetilde{U}_r} d\widetilde{\rho}(y) \right) \mathcal{L}_\kappa(x,y) \\
&= \left(\int_N d\mu(x) \int_{\partial \widetilde{U}_r} d\widetilde{\rho}(\mathfrak{v}, y) - \int_{N \cap \partial U_r} d\mu(\mathfrak{v}, x) \int_{\widetilde{M}} d\widetilde{\rho}(y) \right) \mathcal{L}_\kappa(x,y)
\end{aligned} \tag{11}$$

(for $d\mu(\mathfrak{v}, x)$ resp. $d\widetilde{\rho}(\mathfrak{v}, y)$ denoting the contraction of μ with $\frac{\partial}{\partial r}$ on Minkowski resp. $\widetilde{\rho}$ with $\frac{\partial}{\partial r}$ on the spacetime $\widetilde{M}$ (for the definition of contraction see Situation 3.1)), where we used integration in parts in the second step.
Next we want to write this with integrand $\left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\widetilde{\mathfrak{v}}}\right) \mathcal{L}(x,y)$ in order to use the integrability condition from equation (10). Therefore we first rewrite

$$\int_{N \cap \partial U_r} d\mu(\mathfrak{v}, x) \int_{\widetilde{M}} d\widetilde{\rho}(y) \mathcal{L}_\kappa(x,y),$$

which will then also cancel out the term $\int_N d\mu(x) \int_{\partial \widetilde{U}_r} d\widetilde{\rho}(\mathfrak{v}, y) \mathcal{L}_\kappa(x,y)$. Using integration by parts (in the second and forth step) and adding a zero (in the third step), we can rewrite this second term as follows:

$$\begin{aligned}
&\int_{N \cap \partial U_r} d\mu(\mathfrak{v}, x) \int_{\widetilde{M}} d\widetilde{\rho}(y) \mathcal{L}_\kappa(x,y) \\
&= \left(\int_{N \cap \partial U_r} d\mu(\mathfrak{v}, x) \int_{\widetilde{M} \setminus \widetilde{U}_r} d\widetilde{\rho}(y) + \int_{N \cap \partial U_r} d\mu(\mathfrak{v}, x) \int_{\widetilde{U}_r} d\widetilde{\rho}(y) \right) \mathcal{L}_\kappa(x,y) \\
&= \left(\int_{N \cap U_r} d\mu(x) \int_{\widetilde{M} \setminus \widetilde{U}_r} d\widetilde{\rho}(y) - \int_{N \setminus U_r} d\mu(x) \int_{\widetilde{U}_r} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{v}} \mathcal{L}_\kappa(x,y)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_{N\cap U_r} d\mu(x) \int_{\widetilde{M}\setminus\widetilde{U}_r} d\widetilde{\rho}(y) - \int_{N\setminus U_r} d\mu(x) \int_{\widetilde{U}_r} d\widetilde{\rho}(y)\right) \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\widetilde{\mathfrak{v}}}\right) \mathcal{L}_\kappa(x,y) \\
&\quad - \left(\int_{N\cap U_r} d\mu(x) \int_{\widetilde{M}\setminus\widetilde{U}_r} d\widetilde{\rho}(y) - \int_{N\setminus U_r} d\mu(x) \int_{\widetilde{U}_r} d\widetilde{\rho}(y)\right) \nabla_{2,\widetilde{\mathfrak{v}}} \mathcal{L}_\kappa(x,y) \\
&= \left(\int_{N\cap U_r} d\mu(x) \int_{\widetilde{M}\setminus\widetilde{U}_r} d\widetilde{\rho}(y) - \int_{N\setminus U_r} d\mu(x) \int_{\widetilde{U}_r} d\widetilde{\rho}(y)\right) \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\widetilde{\mathfrak{v}}}\right) \mathcal{L}_\kappa(x,y) \\
&\quad + \left(\int_{N\cap U_r} d\mu(x) \int_{\partial\widetilde{U}_r} d\widetilde{\rho}(\mathfrak{v},y) + \int_{N\setminus U_r} d\mu(x) \int_{\partial\widetilde{U}_r} d\widetilde{\rho}(\mathfrak{v},y)\right) \mathcal{L}_\kappa(x,y) \\
&= \left(\int_{N\cap U_r} d\mu(x) \int_{\widetilde{M}\setminus\widetilde{U}_r} d\widetilde{\rho}(y) - \int_{N\setminus U_r} d\mu(x) \int_{\widetilde{U}_r} d\widetilde{\rho}(y)\right) \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\widetilde{\mathfrak{v}}}\right) \mathcal{L}_\kappa(x,y) \\
&\quad + \int_N d\mu(x) \int_{\partial\widetilde{U}_r} d\widetilde{\rho}(\mathfrak{v},y) \mathcal{L}_\kappa(x,y).
\end{aligned}
$$

Plugging this into equation (11), we obtain

$$
\begin{aligned}
&\frac{d}{dr} O(\widetilde{\rho}, U_r, \widetilde{U}_r) \\
&= \left(\int_N d\mu(x) \int_{\partial\widetilde{U}_r} d\widetilde{\rho}(\mathfrak{v},y) - \int_N d\mu(x) \int_{\partial\widetilde{U}_r} d\widetilde{\rho}(\mathfrak{v},y)\right) \mathcal{L}_\kappa(x,y) \\
&\quad - \left(\int_{N\cap U_r} d\mu(x) \int_{\widetilde{M}\setminus\widetilde{U}_r} d\widetilde{\rho}(y) - \int_{N\setminus U_r} d\mu(x) \int_{\widetilde{U}_r} d\widetilde{\rho}(y)\right) \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\widetilde{\mathfrak{v}}}\right) \mathcal{L}_\kappa(x,y) \\
&= -\left(\int_{N\cap U_r} d\mu(x) \int_{\widetilde{M}\setminus\widetilde{U}_r} d\widetilde{\rho}(y) - \int_{N\setminus U_r} d\mu(x) \int_{\widetilde{U}_r} d\widetilde{\rho}(y)\right) \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\widetilde{\mathfrak{v}}}\right) \mathcal{L}_\kappa(x,y).
\end{aligned}
$$

Now we can apply the integrability condition (10) in order to show that both these terms vanish in the limit $r \to \infty$. We begin with the first term:

$$
\begin{aligned}
0 &\leq \left| -\int_{N\cap U_r} d\mu(x) \int_{\widetilde{M}\setminus\widetilde{U}_r} d\widetilde{\rho}(y) \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\widetilde{\mathfrak{v}}}\right) \mathcal{L}_\kappa(x,y) \right| \\
&\leq \int_{N\cap U_r} d\mu(x) \int_{\widetilde{M}\setminus\widetilde{U}_r} d\widetilde{\rho}(y) \left| \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\widetilde{\mathfrak{v}}}\right) \mathcal{L}_\kappa(x,y) \right| \\
&\leq \int_N d\mu(x) \int_{\widetilde{M}\setminus\widetilde{U}_r} d\widetilde{\rho}(y) \left| \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\widetilde{\mathfrak{v}}}\right) \mathcal{L}_\kappa(x,y) \right| .
\end{aligned}
$$

As this last line converges to 0 in the limit $r \to \infty$ by the integrability condition (10) this implies that

$$
-\int_{N\cap U_r} d\mu(x) \int_{\widetilde{M}\setminus\widetilde{U}_r} d\widetilde{\rho}(y) \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\widetilde{\mathfrak{v}}}\right) \mathcal{L}_\kappa(x,y) \xrightarrow{r\to\infty} 0
$$

as well. For the second term we can proceed similarly: Since

$$\begin{aligned} 0 &\leq \left| - \int_{N\setminus U_r} d\mu(x) \int_{\widetilde{U}_r} d\widetilde{\rho}(y) \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\widetilde{\mathfrak{v}}}\right) \mathcal{L}_\kappa(x,y) \right| \\ &\leq \int_{N\setminus U_r} d\mu(x) \int_{\widetilde{U}_r} d\widetilde{\rho}(y) \left| \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\widetilde{\mathfrak{v}}}\right) \mathcal{L}_\kappa(x,y) \right| \\ &\leq \int_{N\setminus U_r} d\mu(x) \int_{\widetilde{M}} d\widetilde{\rho}(y) \left| \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\widetilde{\mathfrak{v}}}\right) \mathcal{L}_\kappa(x,y) \right| \end{aligned}$$

and since by condition (10) again the last term converges to 0 in the limit $r \to \infty$ this implies

$$- \int_{N\setminus U_r} d\mu(x) \int_{\widetilde{U}_r} d\widetilde{\rho}(y) \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\widetilde{\mathfrak{v}}}\right) \mathcal{L}_\kappa(x,y) \xrightarrow{r\to\infty} 0.$$

Hence the mass of causal fermion systems does not depend on the chosen exhaustions. □

4.3.5 Independence of the Considered Time

Next we will prove that the mass from Section 4.3 is independent of the chosen time by showing that the time derivative of the mass vanishes in the limit of the exhaustions:

Proposition 4.14. *The mass from the definition in Lemma 4.12 does not depend on the chosen time.*

Proof. With the notation from Lemma 4.12 we consider exhaustions $U \nearrow M$ and $\widetilde{U} \nearrow \widetilde{M}$, pasts $\Omega_t \subset M$, $\widetilde{\Omega}_t \subset \widetilde{M}$ and let $\mathfrak{u}$ denote the inner solution corresponding to the vector field $\frac{\partial}{\partial t}$ on Minkowski spacetime. Since $\widetilde{\rho}$ is asymptotically flat, there is an inner solution $\widetilde{\mathfrak{u}}$ to the vector field $\frac{\partial}{\partial t}$ on the spacetime $\widetilde{M}$, s.t. the integrability condition

$$(\nabla_{1,\mathfrak{u}} + \nabla_{2,\widetilde{\mathfrak{u}}})\, \mathcal{L}(x,y) \in L^1(N \times \widetilde{M}) \tag{12}$$

is fulfilled. To be able to use the integrability condition (12) we first want to rewrite the mass to obtain integrals over the whole of U_n or $M \setminus U_n$ (resp. $\widetilde{U}_n$ or

$\widetilde{M} \setminus \widetilde{U}_n)$, which can be done as follows:

$$\begin{aligned}
&O_1(\Omega_t, \widetilde{\Omega}_t, U_n, \widetilde{U}_n) \\
&= -\left(\int_{\Omega_t \cap U_n} d\rho(x) \int_{\widetilde{M} \setminus (\widetilde{\Omega}_t \cup \widetilde{U}_n)} d\widetilde{\rho}(y) - \int_{\Omega_t \setminus U_n} d\rho(x) \int_{\widetilde{U}_n \setminus \widetilde{\Omega}_t} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x,y) \\
&\quad + \left(\int_{U_n \setminus \Omega_t} d\rho(x) \int_{\widetilde{\Omega}_t \setminus \widetilde{U}_n} d\widetilde{\rho}(y) - \int_{M \setminus (\Omega_t \cup U_n)} d\rho(x) \int_{\widetilde{\Omega}_t \cap \widetilde{U}_n} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x,y) \\
&= -\left(\int_{\Omega_t \cap U_n} d\rho(x) \int_{\widetilde{M} \setminus \widetilde{U}_n} d\widetilde{\rho}(y) - \int_{\Omega_t \setminus U_n} d\rho(x) \int_{\widetilde{U}_n} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x,y) \\
&\quad + \left(\int_{U_n} d\rho(x) \int_{\widetilde{\Omega}_t \setminus \widetilde{U}_n} d\widetilde{\rho}(y) - \int_{M \setminus U_n} d\rho(x) \int_{\widetilde{\Omega}_t \cap \widetilde{U}_n} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x,y) \\
&\quad - \left(-\int_{\Omega_t \cap U_n} d\rho(x) \int_{\widetilde{\Omega}_t \setminus \widetilde{U}_n} d\widetilde{\rho}(y) + \int_{\Omega_t \setminus U_n} d\rho(x) \int_{\widetilde{\Omega}_t \cap \widetilde{U}_n} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x,y) \\
&\quad + \left(-\int_{\Omega_t \cap U_n} d\rho(x) \int_{\widetilde{\Omega}_t \setminus \widetilde{U}_n} d\widetilde{\rho}(y) + \int_{\Omega_t \setminus U_n} d\rho(x) \int_{\widetilde{\Omega}_t \cap \widetilde{U}_n} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x,y) \\
&= -\left(\int_{\Omega_t \cap U_n} d\rho(x) \int_{\widetilde{M} \setminus \widetilde{U}_n} d\widetilde{\rho}(y) - \int_{\Omega_t \setminus U_n} d\rho(x) \int_{\widetilde{U}_n} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x,y) \\
&\quad + \left(\int_{U_n} d\rho(x) \int_{\widetilde{\Omega}_t \setminus \widetilde{U}_n} d\widetilde{\rho}(y) - \int_{M \setminus U_n} d\rho(x) \int_{\widetilde{\Omega}_t \cap \widetilde{U}_n} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x,y) \\
&= -\left(\int_{\Omega_t \cap U_n} d\rho(x) \int_{\widetilde{M} \setminus \widetilde{U}_n} d\widetilde{\rho}(y) - \int_{\Omega_t \setminus U_n} d\rho(x) \int_{\widetilde{U}_n} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x,y),
\end{aligned}$$

where we integrated by parts in the last steps so that the last two integrals dropped out since $\frac{\partial}{\partial t}$ is tangential to ∂U. Deriving by t then yields

$$\begin{aligned}
&\frac{d}{dt} O_1(\Omega_t, \widetilde{\Omega}_t, U_n, \widetilde{U}_n) \\
&= -\left(\int_{N_t \cap U_n} d\mu(x) \int_{\widetilde{M} \setminus \widetilde{U}_n} d\widetilde{\rho}(y) - \int_{N_t \setminus U_n} d\mu(x) \int_{\widetilde{U}_n} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x,y)
\end{aligned}$$

for the boundary $N_t = \partial\Omega_t$. Since the integrals over y do not depend on $\widetilde{\Omega}_t$, we can with partial integration add a zero to obtain

$$\begin{aligned}
&\frac{d}{dt} O_1(\Omega_t, \widetilde{\Omega}_t, U_n, \widetilde{U}_n) \\
&= -\left(\int_{N_t \cap U_n} d\mu(x) \int_{\widetilde{M} \setminus \widetilde{U}_n} d\widetilde{\rho}(y) - \int_{N_t \setminus U_n} d\mu(x) \int_{\widetilde{U}_n} d\widetilde{\rho}(y) \right) \left(\nabla_{1,\mathfrak{u}} + \nabla_{2,\widetilde{\mathfrak{u}}} \right) \mathcal{L}_\kappa(x,y).
\end{aligned}$$

Now the integrability condition (12) yields that both these surface layer integrals vanish in the limit $n \to \infty$: For the first integral we have

$$
\begin{aligned}
0 &\leq \left| - \int_{N_t \cap U_n} d\mu(x) \int_{\widetilde{M} \setminus \widetilde{U}_n} d\widetilde{\rho}(y) \, (\nabla_{1,\mathfrak{u}} + \nabla_{2,\widetilde{\mathfrak{u}}}) \, \mathcal{L}_\kappa(x,y) \right| \\
&\leq \int_{N_t \cap U_n} d\mu(x) \int_{\widetilde{M} \setminus \widetilde{U}_n} d\widetilde{\rho}(y) \, \left| (\nabla_{1,\mathfrak{u}} + \nabla_{2,\widetilde{\mathfrak{u}}}) \, \mathcal{L}_\kappa(x,y) \right| \\
&\leq \int_{N_t} d\mu(x) \int_{\widetilde{M} \setminus \widetilde{U}_n} d\widetilde{\rho}(y) \, \left| (\nabla_{1,\mathfrak{u}} + \nabla_{2,\widetilde{\mathfrak{u}}}) \, \mathcal{L}_\kappa(x,y) \right| .
\end{aligned}
$$

As the last term converges to 0 in the limit $n \to \infty$ by the integrability condition (12) we obtain

$$
- \int_{N_t \cap U_n} d\mu(x) \int_{\widetilde{M} \setminus \widetilde{U}_n} d\widetilde{\rho}(y) \, (\nabla_{1,\mathfrak{u}} + \nabla_{2,\widetilde{\mathfrak{u}}}) \, \mathcal{L}_\kappa(x,y) \xrightarrow{n \to \infty} 0.
$$

Analogously we can consider the second term:

$$
\begin{aligned}
0 &\leq \left| - \int_{N_t \setminus U_n} d\mu(x) \int_{\widetilde{U}_n} d\widetilde{\rho}(y) \, (\nabla_{1,\mathfrak{u}} + \nabla_{2,\widetilde{\mathfrak{u}}}) \, \mathcal{L}_\kappa(x,y) \right| \\
&\leq \int_{N_t \setminus U_n} d\mu(x) \int_{\widetilde{U}_n} d\widetilde{\rho}(y) \, \left| (\nabla_{1,\mathfrak{u}} + \nabla_{2,\widetilde{\mathfrak{u}}}) \, \mathcal{L}_\kappa(x,y) \right| \\
&\leq \int_{N_t \setminus U_n} d\mu(x) \int_{\widetilde{M}} d\widetilde{\rho}(y) \, \left| (\nabla_{1,\mathfrak{u}} + \nabla_{2,\widetilde{\mathfrak{u}}}) \, \mathcal{L}_\kappa(x,y) \right| .
\end{aligned}
$$

Now by the integrability condition (12) the last term converges to 0 in the limit $n \to \infty$ and we have

$$
- \int_{N_t \setminus U_n} d\mu(x) \int_{\widetilde{U}_n} d\widetilde{\rho}(y) \, (\nabla_{1,\mathfrak{u}} + \nabla_{2,\widetilde{\mathfrak{u}}}) \, \mathcal{L}_\kappa(x,y) \xrightarrow{n \to \infty} 0.
$$

Hence the mass of causal fermion systems does not depend on the considered time. □

4.3.6 Independence of the Identification of Hilbert Spaces

As also discussed in [PMT, Section 4.3], for the mass we needed to identify the Hilbert spaces $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ of the causal fermion systems from Convention 4.8 by a unitary transformation $V : \mathcal{H} \to \widetilde{\mathcal{H}}$. Here we have freedom in choosing this transformation, since it is only unique up to transformations of the form $V \to VW$ for some unitary transformation W on $\mathcal{H}$. However, in this section we will show that this freedom does not influence the mass:

Proposition 4.15. *In the situation of Convention 4.8 let W be a unitary transformation on $\mathcal{H}$ and define the measure $W\widetilde{\rho}$ by:*

$$\forall \Omega \subset \mathcal{F} \quad (W\widetilde{\rho})(\Omega) := \widetilde{\rho}(W^{-1}\Omega W). \tag{13}$$

Then if $\widetilde{\rho}$ and $W\widetilde{\rho}$ are asymptotically flat (cf. Definition 4.9), the masses $\mathfrak{M}(\widetilde{\rho})$ and $\mathfrak{M}(W\widetilde{\rho})$ coincide.

Proof. We follow and adapt the proof of [PMT, Theorem 4.7]: If a measure ρ is a minimizer of the causal action, so is $W\rho$ (defined as in equation (13)) due to the invariance of operator eigenvalues under unitary transformations. Now let $(W_s)_{s\in[0,\tau_{\max}]}$ be a smooth and strongly continuous family of unitary transformations and let

$$\mathcal{A} := -i\frac{d}{ds}W_s|_{s=0}$$

be its generator. Then the so-called commutator jet given by

$$\mathfrak{v} := (0, v) \quad \text{with} \quad v(x) := i[\mathcal{A}, x]$$

(with $[.,.]$ denoting the commutator) satisfies after deriving by s due to the unitary invariance of the Lagrangian the equation

$$(D_{1,v} + D_{2,v})\,\mathcal{L}_\kappa(x, y) = 0. \tag{14}$$

In particular, we can follow (using the notation from the decomposition from Definition 3.2)

$$\begin{aligned} &\int_{\mathbb{R}} dt'\,(D_{1,v} + D_{2,v})\,\mathcal{L}_\kappa(x, (t', \vec{y})) = 0 \\ &\qquad \Rightarrow -\int_{\mathbb{R}} dt' D_{2,v}\mathcal{L}_\kappa(x, (t', \vec{y})) = \int_{\mathbb{R}} dt' D_{1,v}\mathcal{L}_\kappa(x, (t', \vec{y})) \end{aligned} \tag{15}$$

as well as

$$\begin{aligned} &\left(\int_{\mathbb{R}} dt + \int_{\mathbb{R}} dt'\right) D_{1,v}\mathcal{L}_\kappa((t, \vec{x}), (t', \vec{y})) \\ &= \int_{\mathbb{R}} dt D_{1,v}\mathcal{L}_\kappa((t, \vec{x}), (t', \vec{y})) + \int_{\mathbb{R}} dt' D_{1,v}\mathcal{L}_\kappa((t, \vec{x}), (t', \vec{y})) \\ &= \int_{\mathbb{R}} dt' D_{2,v}\mathcal{L}_\kappa((t, \vec{x}), (t', \vec{y})) + \int_{\mathbb{R}} dt' D_{1,v}\mathcal{L}_\kappa((t, \vec{x}), (t', \vec{y})) \\ &= \int_{\mathbb{R}} dt'(D_{1,v} + D_{2,v})\mathcal{L}_\kappa((t, \vec{x}), (t', \vec{y})) = 0 \end{aligned} \tag{16}$$

using the symmetry of the Lagrangian in the second step and equation (14) in the last step. This implies that the linearized mass (cf. equation (8)) vanishes, if the jet $\mathfrak{w}$ in equation (8) is such a commutator jet: Let $\mathfrak{v}$ be a commutator jet as above. Then using in the second step the symmetry of the Lagrangian as well as equation (15) and in the last step equation (16) (we abbreviate $(x,y) = ((t,\vec{x}),(t',\vec{y}))$ for shorter notation from the second line) yields

$$
\begin{aligned}
&\left(\int_{N\setminus U_n} d\mu(\vec{x}) \int_{U_n} d\rho(y) - \int_{N\cap U_n} d\mu(\vec{x}) \int_{M\setminus U_n} d\rho(y)\right) D_{2,v}\mathcal{L}_\kappa(x,y) \\
&= \left(\int_{N\setminus U_n} d\mu(\vec{x}) \int_{N\cap U_n} d\mu(\vec{y}) \int_{\mathbb{R}} dt' - \int_{N\cap U_n} d\mu(\vec{x}) \int_{N\setminus U_n} d\mu(\vec{y}) \int_{\mathbb{R}} dt'\right) D_{2,v}\mathcal{L}_\kappa(x,y) \\
&= \int_{N\cap U_n} d\mu(\vec{x}) \int_{N\setminus U_n} d\mu(\vec{y}) \int_{\mathbb{R}} dt D_{1,v}\mathcal{L}_\kappa(x,y) \\
&\quad + \int_{N\cap U_n} d\mu(\vec{x}) \int_{N\setminus U_n} d\mu(\vec{y}) \int_{\mathbb{R}} dt' D_{1,v}\mathcal{L}_\kappa(x,y) \\
&= \int_{N\cap U_n} d\mu(\vec{x}) \int_{N\setminus U_n} d\mu(\vec{y}) \left(\int_{\mathbb{R}} dt + \int_{\mathbb{R}} dt'\right) D_{1,v}\mathcal{L}_\kappa(x,y) \\
&= 0. \qquad (17)
\end{aligned}
$$

This proves the proposition, since: If $\widetilde{\rho}$ and $W\widetilde{\rho}$ are asymptotically flat, by definition the surface layer integrals for the masses can in the limit be linearized w.r.t. some jets $\mathfrak{w}_1$ resp. $\mathfrak{w}_2$ (cf. equation (8)). By the considerations above an infinitesimal unitary transformation corresponds to adding a commutator jet, i.e. $\mathfrak{w}_2 = \mathfrak{w}_1 + \mathfrak{v}$ for a commutator jet $\mathfrak{v}$. However, since the linearized term for the mass vanishes for commutator jets as shown in equation (17), this additional commutator jet does not change the value of the mass. □

4.4 Positive Mass Theorem

Finally we will prove an analogue to the Positive Mass Theorem from [ADM]: If a local condition is fulfilled determining the sign of an expression like a mass density, the mass from Definition 4.10 will be non-negative.
To this end we first will adapt the definitions [PMT, Definition 1.6] as well as [PMT, Definition 1.7] to our setting, allowing to write down the jet $\mathfrak{w}$ from Definition 4.9 explicitly.

Definition 4.16 (κ-extendable)**.** *We again use the notation from Convention 4.8. The measure $\widetilde{\rho}$ is called κ-extendable if*

i) there is a family $(\widetilde{\rho}_\tau)_{\tau\in(-1,1)}$ of measures of the form

$$\widetilde{\rho}_\tau = (\widetilde{F}_\tau)_*\widetilde{\rho}, \qquad (18)$$

(similar to in Definition 4.9) which all satisfy the Euler-Lagrange equations (1) with a parameter $\kappa(\tau)$ *and*

$$\widetilde{F}_0 = id_{\widetilde{M}} \quad \text{as well as} \quad \kappa'(0) \neq 0.$$

ii) For every $x \in \widetilde{M}$*, the curve* $\widetilde{F}_\tau(x)$ *is differentiable at* $\tau = 0$*, giving rise to a vector field*

$$\widetilde{v} := \frac{d}{d\tau}\widetilde{F}_\tau|_{\tau=0} \in \Gamma(\widetilde{M}, T\widetilde{\mathcal{F}}). \tag{19}$$

For convenience choose the parametrization, s.t.

$$\frac{d}{d\tau}\log\kappa(\tau)|_{\tau=0} = -1.$$

Definition 4.17 (κ-scalable)**.** *Let* $\widetilde{\rho}$ *be asymptotically flat. Then* $\widetilde{\rho}$ *is said to be* κ*-scalable if for a suitable choice of the mapping* $\widetilde{F}_\tau$ *in Definition 4.16 the vector field component* w *of* $\mathfrak{w}$ *from Definition 4.9 coincides with the vector field* $\widetilde{v}$ *from equation (19) up to a constant* $g \in \mathbb{R}$*, the so-called gravitational coupling constant, i.e. if we have*

$$w = g\widetilde{v}. \tag{20}$$

For the positive mass theorem we will consider that $\widetilde{\rho}$ is κ-scalable, hence we can always write the jet $\mathfrak{w}$ from equation (8) depending on $\widetilde{v}$ from equation (20). For the proof of the positive mass theorem we will need a jet $\mathfrak{v}$ with this vector field component $\widetilde{v}$ (we then have $\mathfrak{w} = g\mathfrak{v}$ for the gravitational coupling constant g) to fulfill a certain equation, the so-called hypersurface equation. This can be achieved by fixing the last freedom (cf. Corollary 3.9: the jet $\mathfrak{v}$ can be modified by adding inner solutions), demanding that the jet $\mathfrak{v}$ from Definition 4.10 (which is a solution of the linearized field equations by [PMT, Section 2.1.2], since we demanded that the measures of the form (18) satisfy the Euler-Lagrange equations) is shifted by an inner solution (as is described in Corollary 3.9), s.t. it fulfills the so-called hypersurface equation:

Definition 4.18 (Hypersurface Equation)**.** *Let* $N \subset M$ *be a hypersurface. Then we locally choose an embedding* $I \times N \to M$ *for some (time) interval* $I \subset \mathbb{R}$*, denoting points by* $x = (t, \vec{x})$ *and having a measure decomposition* $d\rho = dt d\mu_t$*. We define for all* $\vec{x} \in N$*:*

$$s_N(\vec{x}) := \left(\int_0^\infty \int_{-\infty}^0 + \int_{-\infty}^0 \int_0^\infty\right) dt dt' \int_{N_{t'}} d\mu_{t'}(\vec{y})\, \mathcal{L}_\kappa((t,\vec{x}),(t',\vec{y})). \tag{21}$$

Then a jet $\mathfrak{v} \in \mathfrak{J}^\infty$ *is said to fulfill the hypersurface equation if*

$$\langle \mathfrak{u}, \Delta_N \mathfrak{v}\rangle = 0 \quad \forall \mathfrak{u} \in \mathfrak{J}^{test}, \tag{22}$$

i.e. if for all test jets $\mathfrak{u} \in \mathfrak{J}^{test}$ and for all $\vec{x} \in N$ we have

$$
\begin{aligned}
0 &= \langle \mathfrak{u}, \Delta_N \mathfrak{v} \rangle (\vec{x}) \\
&= \nabla_{\mathfrak{u}} \left(\int_0^\infty \int_{-\infty}^0 + \int_{-\infty}^0 \int_0^\infty \right) dt dt' \int_{N_{t'}} d\mu_{t'}(\vec{y}) \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}} \right) \mathcal{L}_\kappa((t,\vec{x}),(t',\vec{y})) \\
&\quad - \nabla_{\mathfrak{u}} \nabla_{\mathfrak{v}} s_N (\vec{x}) .
\end{aligned}
$$

We also have to implement the area constraint:

Proposition 4.19. *To fulfill the area constraint from equation (9) in Definition 4.10, $\mathfrak{v}$ needs to satisfy the equation*

$$
\int_{\Omega_t \cap U_n} d\rho(x) \int_{M \setminus (\Omega_t \cup U_n)} d\rho(y) \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}} \right) \mathcal{L}_\kappa(x,y) = 0. \tag{23}
$$

Proof. We start from the area constraint

$$
A(\Omega_t, U_n) = \widetilde{A}\left(\widetilde{\Omega}_t, \widetilde{U}_n\right).
$$

Taylor up to first order gives

$$
\begin{aligned}
\widetilde{A}\left(\widetilde{\Omega}_t, \widetilde{U}_n\right) &= \int_{\widetilde{\Omega}_t \cap \widetilde{U}_n} d\widetilde{\rho}(x) \int_{\widetilde{M} \setminus \left(\widetilde{\Omega}_t \cup \widetilde{U}_n\right)} d\widetilde{\rho}(y) \mathcal{L}_\kappa(x,y) \\
&= \int_{\Omega_t \cap U_n} d\rho(x) \int_{M \setminus (\Omega_t \cup U_n)} d\rho(y) \mathcal{L}_\kappa(x,y) \\
&\quad + \int_{\Omega_t \cap U_n} d\rho(x) \int_{M \setminus (\Omega_t \cup U_n)} d\rho(y) \nabla_{1,\mathfrak{v}} \mathcal{L}_\kappa(x,y) \\
&\quad + \int_{\Omega_t \cap U_n} d\rho(x) \int_{M \setminus (\Omega_t \cup U_n)} d\rho(y) \nabla_{2,\mathfrak{v}} \mathcal{L}_\kappa(x,y).
\end{aligned}
$$

Then the area constraint

$$
A(\Omega_t, U_n) = \widetilde{A}\left(\widetilde{\Omega}_t, \widetilde{U}_n\right) \iff \widetilde{A}\left(\widetilde{\Omega}_t, \widetilde{U}_n\right) - A(\Omega_t, U_n) = 0
$$

means that $\mathfrak{v}$ has to satisfy

$$
\int_{\Omega_t \cap U_n} d\rho(x) \int_{M \setminus (\Omega_t \cup U_n)} d\rho(y) \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}} \right) \mathcal{L}_\kappa(x,y) = 0,
$$

i.e. that $\mathfrak{v}$ does not change the surface area. □

Next we introduce the local condition playing the same role as the positive mass density for the positivity of the *ADM* mass:

Definition 4.20 (Local Mass Condition)**.** *In the situation of Definition 4.18 assume that the spatial limit*

$$s_{N,\infty} := \lim_{N \ni \vec{x} \to \infty} s_N(\vec{x})$$

exists. In this case we say a jet $\mathfrak{v} \in \mathfrak{J}^\infty$ fulfills the local energy condition if

$$\nabla_{\mathfrak{v}} \left(s_N\left(\vec{x}\right) - s_{N,\infty} \right) \geq 0. \tag{24}$$

In order to formulate a positive mass theorem we now want to linearize the generalized (i.e. for general measure ρ instead of measure μ) surface layer integrals from the mass definition in κ. For this by adding a zero we will obtain integrals over the whole past (resp. future):

$$\begin{aligned}
&-\frac{d}{d\kappa}\Bigg(\bigg(\int_{U_n\setminus\Omega_t} d\rho(x) \int_{\widetilde{\Omega}_{\widetilde{t}'}\setminus\widetilde{U}_n} d\widetilde{\rho}(y) - \int_{\Omega_t\setminus U_n} d\rho(x) \int_{\widetilde{U}_n\setminus\widetilde{\Omega}_{\widetilde{t}'}} d\widetilde{\rho}(y) \\
&-\int_{\Omega_t\cap U_n} d\rho(x) \int_{\widetilde{M}\setminus\left(\widetilde{\Omega}_{\widetilde{t}'}\cup\widetilde{U}_n\right)} d\widetilde{\rho}(y) - \int_{M\setminus(\Omega_t\cup U_n)} d\rho(x) \int_{\widetilde{\Omega}_{\widetilde{t}'}\cap\widetilde{U}_n} d\widetilde{\rho}(y)\bigg) \,\mathcal{L}_\kappa(x,y)\big|_{\kappa=0} \\
&= -\frac{d}{d\kappa}\bigg(\int_{U_n\setminus\Omega_t} d\rho(x) \int_{\widetilde{\Omega}_{\widetilde{t}'}\setminus\widetilde{U}_n} d\widetilde{\rho}(y) - \int_{\Omega_t\setminus U_n} d\rho(x) \int_{\widetilde{U}_n\setminus\widetilde{\Omega}_{\widetilde{t}'}} d\widetilde{\rho}(y) \\
&\qquad -\int_{\Omega_t\cap U_n} d\rho(x) \int_{\widetilde{M}\setminus\left(\widetilde{\Omega}_{\widetilde{t}'}\cup\widetilde{U}_n\right)} d\widetilde{\rho}(y) - \int_{M\setminus(\Omega_t\cup U_n)} d\rho(x) \int_{\widetilde{\Omega}_{\widetilde{t}'}\cap\widetilde{U}_n} d\widetilde{\rho}(y) \\
&\qquad -\int_{U_n\setminus\Omega_t} d\rho(x) \int_{\widetilde{\Omega}_{\widetilde{t}'}\cap\widetilde{U}_n} d\widetilde{\rho}(y) - \int_{\Omega_t\cap U_n} d\rho(x) \int_{\widetilde{U}_n\setminus\widetilde{\Omega}_{\widetilde{t}'}} d\widetilde{\rho}(y) \\
&\qquad -\int_{\Omega_t\cap U_n} d\rho(x) \int_{\widetilde{U}_n\setminus\widetilde{\Omega}_{\widetilde{t}'}} d\widetilde{\rho}(y) - \int_{U_n\setminus\Omega_t} d\rho(x) \int_{\widetilde{\Omega}_{\widetilde{t}'}\cap\widetilde{U}_n} d\widetilde{\rho}(y)\bigg) \,\mathcal{L}_\kappa(x,y)\big|_{\kappa=0} \\
&= -\frac{d}{d\kappa}\Bigg(\int_{U_n\setminus\Omega_t} d\rho(x) \int_{\widetilde{\Omega}_{\widetilde{t}'}} d\widetilde{\rho}(y) + \int_{\Omega_t\cap U_n} d\rho(x) \int_{\widetilde{M}\setminus\widetilde{\Omega}_{\widetilde{t}'}} d\widetilde{\rho}(y) \\
&\qquad + \bigg(\int_{\Omega_t} d\rho(x) \int_{\widetilde{U}_n\setminus\widetilde{\Omega}_{\widetilde{t}'}} d\widetilde{\rho}(y) + \int_{M\setminus\Omega_t} d\rho(x) \int_{\widetilde{\Omega}_{\widetilde{t}'}\cap\widetilde{U}_n} d\widetilde{\rho}(y)\bigg)\Bigg) \,\mathcal{L}_\kappa(x,y)\big|_{\kappa=0}\,.
\end{aligned}$$

Now we can derive by κ and use the symmetry of $\mathcal{L}_\kappa$ in the second step to obtain:

$$\begin{aligned}
-\int_0^\infty dt \int_{-\infty}^0 dt' \int_{N_t \cap U_n} d\mu_t(\vec{x}) \int_{N_{t'}} d\mu_{t'}(\vec{y}) \, \nabla_{2,\mathfrak{v}} \mathcal{L}_\kappa((t,\vec{x}),(t',\vec{y})) \\
- \int_{-\infty}^0 dt \int_0^\infty dt' \int_{N_t \cap U_n} d\mu_t(\vec{x}) \int_{N_{t'}} d\mu_{t'}(\vec{y}) \, \nabla_{2,\mathfrak{v}} \mathcal{L}_\kappa((t,\vec{x}),(t',\vec{y})) \\
+ \Bigg(\int_{-\infty}^0 dt \int_0^\infty dt' \int_{N_t} d\mu_t(\vec{x}) \int_{N_{t'} \cap U_n} d\mu_{t'}(\vec{y}) \, \nabla_{2,\mathfrak{v}} \mathcal{L}_\kappa((t,\vec{x}),(t',\vec{y})) \\
- \int_0^\infty dt \int_{-\infty}^0 dt' \int_{N_t} d\mu_t(\vec{x}) \int_{N_{t'} \cap U_n} d\mu_{t'}(\vec{y}) \, \nabla_{2,\mathfrak{v}} \mathcal{L}_\kappa((t,\vec{x}),(t',\vec{y})) \Bigg) \\
= \left(\int_0^\infty \int_{-\infty}^0 + \int_{-\infty}^0 \int_0^\infty \right) dt dt' \times \\
\int_{N_t \cap U_n} d\mu_t(\vec{x}) \int_{N_{t'}} d\mu_{t'}(\vec{y}) \left(\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}} \right) \mathcal{L}_\kappa((t,\vec{x}),(t',\vec{y})), \qquad (25)
\end{aligned}$$

where the jet $\mathfrak{v}$ comes from equation (19).
With these preparations we can now formulate our positive mass theorem for the mass of causal fermion systems:

Theorem 4.21 (Positive Mass Theorem)**.** *If $\tilde{\rho}$ is asymptotically flat and κ-scalable, the gravitational coupling constant $g \in \mathbb{R}$ from Definition 4.17 is positive and the shifted $\mathfrak{v}$ to the vector field $\tilde{v}$ from equation (19) fulfills the hypersurface equation (22) and constraint (23) as well as the local mass condition (24), the resulting mass is non-negative.*

Proof. By linearizing in κ we computed the surface layer integrals for the mass up to the (by assumption) positive gravitational coupling constant $g \in \mathbb{R}$ until equation (25). From there we can further calculate for the integrand

$$\begin{aligned}
\left(\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}} \right) \mathcal{L}_\kappa((t,\vec{x}),(t',\vec{y})) = \\
2 \nabla_{1,\mathfrak{v}} \mathcal{L}_\kappa((t,\vec{x}),(t',\vec{y})) - \left(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}} \right) \mathcal{L}_\kappa((t,\vec{x}),(t',\vec{y})).
\end{aligned}$$

Using this and plugging in the hypersurface equation (22) as well as the definition

(21) of s_N into equation (25) yields

$$\left(\int_0^\infty \int_{-\infty}^0 + \int_{-\infty}^0 \int_0^\infty\right) dtdt' \times$$
$$\int_{N_t \cap U_n} d\mu_t\left(\vec{x}\right) \int_{N_{t'}} d\mu_{t'}\left(\vec{y}\right) \left(\nabla_{1,\mathfrak{v}} - \nabla_{2,\mathfrak{v}}\right) \mathcal{L}_\kappa((t,\vec{x}),(t',\vec{y}))$$
$$= 2\nabla_{\mathfrak{v}} \int_{N\cap U_n} d\mu_N\left(\vec{x}\right) s_N\left(\vec{x}\right) - \nabla_{\mathfrak{v}} \int_{N\cap U_n} d\mu_N\left(\vec{x}\right) s_N\left(\vec{x}\right)$$
$$= \nabla_{\mathfrak{v}} \int_{N\cap U_n} d\mu_N\left(\vec{x}\right) s_N\left(\vec{x}\right).$$

Plugging in the local mass condition (24)

$$\nabla_{\mathfrak{v}}\left(s_N\left(\vec{x}\right) - s_{N,\infty}\right) \geq 0$$

proofs the claim. □

5 Energy of Causal Fermion Systems

A time-dependent causal fermion system $(\widetilde{\mathcal{H}}, \widetilde{\mathcal{F}}, \widetilde{\rho})$ (with corresponding non-static spacetime $\widetilde{M} = \text{supp}\, \widetilde{\rho})$ does not only have a mass, but a momentum as well. An example would be boosting the stationary black hole from Schwarzschild spacetime, then moving into a fixed direction with fixed velocity. (This example will be considered in detail in Section 6.) This momentum shall be given by a three-dimensional vector, of which each component describes the momentum along one of the three spatial directions. Together with the (one-dimensional) mass the momentum will build a four-dimensional energy vector to be defined later in this section.

5.1 Momentum of Causal Fermion Systems

The idea of how to define the momentum is - instead of considering the measure μ in Minkowski spacetime as the contraction of ρ with the vector field $\frac{\partial}{\partial t}$ for the time derivative - to contract ρ with the spatial derivatives $\frac{\partial}{\partial x}$ (resp. $\frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ for the corresponding component). (For the definition of contraction see Situation 3.1.) These integrals similar to the one from Section 4.3.3 (but with the corresponding inner solutions for the spatial derivatives) then will not give a contribution from the integration over the constant-$\{t = 0\}$-time-hypersurface N, but now a contribution from the $\{x = 0\}$-domain (resp. $\{y = 0\}, \{z = 0\}$) with a fixed spatial coordinate. For this section we will recall the notation from Section 4:

Convention 5.1. *While defining the mass, momentum and energy of time-dependent causal fermion systems we will often refer to spacetimes M and $\widetilde{M}$. This will always mean we consider two causal fermion systems $(\mathcal{H}, \mathcal{F}, \rho)$ resp. $(\widetilde{\mathcal{H}}, \widetilde{\mathcal{F}}, \widetilde{\rho})$ with the spacetimes $M = supp\ \rho$ resp. $\widetilde{M} = supp\ \widetilde{\rho}$ (and decompositions as in Definition 3.2), which are minimizers of the same causal action, i.e. with the same constants s and κ. Moreover, M will from now on always denote the Minkowski spacetime (cf. 2.10) with decomposition $M = \mathbb{R} \times N$, $d\rho = dtd\mu$ from Section 4.3 and $\widetilde{M}$ will be asymptotically flat. Additionally, $\Omega_t := (-\infty, t] \times N \subset M$ will always denote the past for time $t \in \mathbb{R}$ in Minkowski, whereas $\widetilde{\Omega}_{t'} \subset \widetilde{M}$ will denote the past of time $t' \in \mathbb{R}$ in the spacetime $\widetilde{M}$.*

Now we can define the notion of momentum for causal fermion systems:

Definition 5.2 (Momentum)**.** *In the situation of Convention 5.1 define for every $i \in \{1, 2, 3\}$ the momentum of the causal fermion system w.r.t. the spatial direction x_i (in Minkowski spacetime) by*

$$\mathfrak{P}_i(\widetilde{\rho}) := \lim_{U_n \nearrow M, \widetilde{U}_n \nearrow \widetilde{M} \text{ with } A(U_n) = \widetilde{A}\left(\widetilde{\Omega}_{t'}, \widetilde{U}_n\right)} \mathcal{I}_i(\widetilde{\rho}, \Omega_t, \widetilde{\Omega}_{t'}, U_n, \widetilde{U}_n)$$

with

$$\begin{aligned}
&\mathcal{I}_i(\widetilde{\rho}, \Omega_t, \widetilde{\Omega}_{t'}, U_n, \widetilde{U}_n) := \\
&-\left(\int_{\Omega_t \cap U_n} d\mu(x) \int_{\widetilde{M} \setminus \left(\widetilde{\Omega}_{t'} \cup \widetilde{U}_n\right)} d\widetilde{\rho}(y) - \int_{\Omega_t \setminus U_n} d\mu(x) \int_{\widetilde{U}_n \setminus \widetilde{\Omega}_{t'}} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x, y) \\
&+\left(\int_{U_n \setminus \Omega_t} d\mu(x) \int_{\widetilde{\Omega}_{t'} \setminus \widetilde{U}_n} d\widetilde{\rho}(y) - \int_{M \setminus (\Omega_t \cup U_n)} d\mu(x) \int_{\widetilde{\Omega}_{t'} \cap \widetilde{U}_n} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x, y),
\end{aligned}$$

where $\mathfrak{u}$ denotes the inner solution corresponding to the spatial derivative $\frac{\partial}{\partial x_i}$.

The well-definedness of the momentum can be proven analogously to the well-definedness of the mass, we just need to rotate the setting for the mass (use for example the decomposition as in Definition 3.3 instead of the decomposition from Definition 3.2). This, however, does not only mean we consider for a fixed spatial direction x_i ($i \in \{1, 2, 3\}$) the constant-x_i-hypersurface instead of the constant-time-hypersurface N and an inner solution corresponding to the spatial derivative $\frac{\partial}{\partial x_i}$ instead of the inner solution for the time derivative but also the exhaustions of the spacetimes are considered s.t. this inner solution is tangential.

Lemma 5.3. *The momentum from Definition 5.2 is independent of the chosen exhaustions $U_n \nearrow M$ resp. $\widetilde{U}_n \nearrow \widetilde{M}$.*

Proof. Using the asymptotic flatness of the spacetime $\widetilde{M}$ the limits $U \nearrow M$ resp. $\widetilde{U}_n \nearrow \widetilde{M}$ exist and are independent of the chosen exhaustions analogous to the proof for the mass. ☐

Lemma 5.4. *The momentum from Definition 5.2 is independent of the chosen time $t' \in \mathbb{R}$.*

Proof. The proof is analogous to the proof for the time independence of the mass. However, instead of integrating over time hypersurfaces we integrate over spatial hypersurfaces w.r.t. one spatial component. The integrability over these hypersurfaces is again ensured by the asymptotic flatness. ☐

Lemma 5.5. *The momentum from Definition 5.2 is independent of the identification of Hilbert spaces (cf. Section 4.3.6).*

Proof. The proof is analogous to the proof considering the mass in Section 4.3.6 but we again consider the "rotated setting" with the decomposition as in Definition 3.3 instead of the decomposition from Definition 3.2. ☐

5.2 Energy of Causal Fermion Systems

With the notions of mass and momentum already defined we can now define the energy of causal fermion systems. The energy will be an energy-momentum vector as will be specified in Remark 5.7.

Definition 5.6 (Energy)**.** *We again use the notation from Convention 5.1. Additionally, let $\mathfrak{u}$ be the inner solution corresponding to a vector field in Minkowski space. Then define the energy of $\widetilde{M}$ by*

$$\mathcal{E}(\widetilde{\rho}, \mathfrak{u}) := \lim_{U_n \nearrow M, \widetilde{U}_n \nearrow \widetilde{M} \text{ with } A(U_n) = \widetilde{A}\left(\widetilde{U}_n, \widetilde{\Omega}_{t'}\right)} \mathfrak{E}(\widetilde{\rho}, \widetilde{\Omega}_{t'}, \mathfrak{u}, U_n, \widetilde{U}_n)$$

with

$$\begin{aligned}
&\mathfrak{E}(\widetilde{\rho}, \widetilde{\Omega}_{t'}, \mathfrak{u}, U_n, \widetilde{U}_n) := \\
&\quad - \left(\int_{\Omega_t \cap U_n} d\rho(x) \int_{\widetilde{M} \setminus \left(\widetilde{\Omega}_{t'} \cup \widetilde{U}_n\right)} d\widetilde{\rho}(y) - \int_{\Omega_t \setminus U_n} d\rho(x) \int_{\widetilde{U}_n \setminus \widetilde{\Omega}_{t'}} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x,y) \\
&\quad + \left(\int_{U_n \setminus \Omega_t} d\rho(x) \int_{\widetilde{\Omega}_{t'} \setminus \widetilde{U}_n} d\widetilde{\rho}(y) - \int_{M \setminus (\Omega_t \cup U_n)} d\rho(x) \int_{\widetilde{\Omega}_{t'} \cap \widetilde{U}_n} d\widetilde{\rho}(y) \right) \nabla_{1,\mathfrak{u}} \mathcal{L}_\kappa(x,y),
\end{aligned}$$

giving a four-dimensional energy vector.

Remark 5.7. *By definition the last three components of the energy vector coincide with the definition of the momentum from Definition 5.2. Additionally the first component of the energy coincides with the mass (in the form from the alternative definition in Lemma 4.12).*

Remark 5.8. *By the linearity of the integral and the jet derivatives the energy vector $\mathcal{E}$ from Definition 5.6 is linear in $\mathfrak{u}$. Hence the energy vectors $\mathcal{E}(\mathfrak{u})$ to inner solutions $\mathfrak{u}$ form a four-dimensional vector space over $\mathbb{R}$.*

5.3 Energy-Momentum as a Four-Vector

In this section we will prove the following statement:

Proposition 5.9. *The energy from Definition 5.6 does not depend on the observer, more concretely this energy-momentum vector is a four-vector.*

Proof. Consider a spacetime satisfying Situation 3.1 and a Lorentz boost with fixed velocity $v \in \mathbb{R}$ along a direction x_j for some $j \in \{1, 2, 3\}$. (The case in which the direction is not along an axis then follows from the linearity of the energy in its jet component (cf. Remark 5.8).) Then the boost is of the form

$$\widetilde{t} = \gamma\,(t - vx_j)\,, \quad \text{and} \quad \widetilde{x}_j = \gamma\,(x_j - vt)$$

with

$$\gamma := \frac{1}{\sqrt{1 - v^2}} \tag{26}$$

as well as $\widetilde{x}_i = x_i$ for $i \neq j$. In order to calculate the energy of the boosted spacetime we need the partial derivatives

$$\begin{aligned} \frac{\partial}{\partial \widetilde{t}} &= \gamma \frac{\partial}{\partial t} - v\gamma \frac{\partial}{\partial x_j} \\ \frac{\partial}{\partial \widetilde{x}_j} &= -v\gamma \frac{\partial}{\partial t} + \gamma \frac{\partial}{\partial x_j} \\ \frac{\partial}{\partial \widetilde{x}_i} &= \frac{\partial}{\partial x_i} \quad \text{for } i \neq j. \end{aligned}$$

From now on we will w.l.o.g. consider a boost in x_3-direction (the calculation for the other directions is analogous) and denote the axes by t, x, y and z (for shorter notation and better readability). Since the components of the energy, i.e. the mass $\mathfrak{M}$ and the momentum $\mathfrak{P}_x := \mathfrak{P}_1, \mathfrak{P}_y := \mathfrak{P}_2$ resp. $\mathfrak{P}_z := \mathfrak{P}_3$ in x-, y- resp. z-direction correspond to the partial derivatives $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ resp. $\frac{\partial}{\partial z}$ as vector fields for the inner solution $\mathfrak{u}$ in Definition 5.6, by linearity we have for the boosted spacetime the energy-momentum vector

$$\begin{pmatrix} \widetilde{\mathfrak{M}} \\ \widetilde{\mathfrak{P}}_x \\ \widetilde{\mathfrak{P}}_y \\ \widetilde{\mathfrak{P}}_z \end{pmatrix} = \begin{pmatrix} \gamma\mathfrak{M} - v\gamma\mathfrak{P}_z \\ \mathfrak{P}_x \\ \mathfrak{P}_y \\ -v\gamma\mathfrak{M} + \gamma\mathfrak{P}_z \end{pmatrix}.$$

To prove that the energy-momentum vector from Definition 5.6 is a four-vector, we need to show that this energy-momentum vector of the boosted system has the same "length" w.r.t. the Minkowski norm (with the signature $(-,+,+,+)$) as the vector for the initial system:

$$\begin{aligned} \left\| \left(\widetilde{\mathfrak{M}}, \widetilde{\mathfrak{P}}_x, \widetilde{\mathfrak{P}}_y, \widetilde{\mathfrak{P}}_z \right)^T \right\| &= -\widetilde{\mathfrak{M}}^2 + \widetilde{\mathfrak{P}}_x^2 + \widetilde{\mathfrak{P}}_y^2 + \widetilde{\mathfrak{P}}_z^2 \\ &= -\left(\gamma\mathfrak{M} - v\gamma\mathfrak{P}_z\right)^2 + \mathfrak{P}_x^2 + \mathfrak{P}_y^2 + \left(-v\gamma\mathfrak{M} + \gamma\mathfrak{P}_z\right)^2 \\ &= -\gamma^2\mathfrak{M}^2 + 2v\gamma^2\mathfrak{M}\mathfrak{P}_z - v^2\gamma^2\mathfrak{P}_z^2 + \mathfrak{P}_x^2 + \mathfrak{P}_y^2 \\ &\quad + v^2\gamma^2\mathfrak{M}^2 - 2v\gamma^2\mathfrak{M}\mathfrak{P}_z + \gamma^2\mathfrak{P}_z^2 \\ &= -\left(1 - v^2\right)\gamma^2\mathfrak{M}^2 + \left(1 - v^2\right)\gamma^2\mathfrak{P}_z^2 + \mathfrak{P}_x^2 + \mathfrak{P}_y^2, \end{aligned}$$

where "$.^T$" denotes the transposed vector.
Plugging in γ (cf. equation (26)) we obtain

$$\left\| \left(\widetilde{\mathfrak{M}}, \widetilde{\mathfrak{P}}_x, \widetilde{\mathfrak{P}}_y, \widetilde{\mathfrak{P}}_z \right)^T \right\| = -\mathfrak{M}^2 + \mathfrak{P}_x^2 + \mathfrak{P}_y^2 + \mathfrak{P}_z^2 = \left\| \left(\mathfrak{M}, \mathfrak{P}_x, \mathfrak{P}_y, \mathfrak{P}_z \right)^T \right\|.$$

□

6 Example: Energy of a Boosted Schwarzschild Spacetime

In this section we compute the energy in the example of a boosted Schwarzschild spacetime.

6.1 Coordinates of Boosted Schwarzschild Spacetime

We consider the Schwarzschild spacetime for the causal fermion system with the Dirac sea $\mathcal{H} = \mathcal{H}_-$ (cf. construction in [PMT, Section 6.1]). Taking the Black Hole from Schwarzschild spacetime with mass M_S, but now additionally moving in a fixed direction with a fixed velocity $v \in \mathbb{R}$ (normalized w.r.t. the speed of light $c = 1$), we obtain a (non-rotating, uncharged) boosted Schwarzschild Black Hole. Here we will always consider a Black Hole moving in z-direction (w.r.t. the Cartesian coordinate directions in Minkowski space). In this case, the Cartesian coordinates from Minkowski space can be transformed as follows:

$$\begin{pmatrix} t_{\text{c,new}} \\ x_{\text{c,new}} \\ y_{\text{c,new}} \\ z_{\text{c,new}} \end{pmatrix} = \begin{pmatrix} \gamma\,(t - vx) \\ x \\ y \\ \gamma\,(z - vt) \end{pmatrix}$$

with

$$\gamma = \frac{1}{\sqrt{1 - v^2}}.$$

Hence transforming to spherical coordinates the boost can be written as

$$\Psi : \begin{pmatrix} t \\ r \\ \theta \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} \gamma(t - vr\cos(\theta)) \\ \sqrt{r^2\sin^2(\theta) + \gamma^2(r\cos(\theta) - vt)^2} \\ \arccos\left(\frac{\gamma(r\cos(\theta) - vt)}{\sqrt{r^2\sin^2(\theta) + \gamma^2(r\cos(\theta) - vt)^2}}\right) \\ \phi \end{pmatrix}. \tag{27}$$

(Here θ always denotes the polar angle whereas ϕ denotes the azimuthal angle.) We now can express the Schwarzschild line element (with M_S denoting the mass of the black hole)

$$ds^2_{\text{Schw}} = -\left(1 - \frac{2M_S}{r}\right)dt^2 + \left(1 - \frac{2M_S}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

transformed with these boosted spherical coordinates, giving the line element

$$\begin{aligned} ds^2_{bS} = &-\left(1 - \frac{2M_S}{r}\right)dt^2 - \frac{8M_S v\cos\theta}{r}dtdr + 4M_S v\sin\theta dtd\theta \\ &+ \left(1 - \frac{2M_S}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \end{aligned}$$

for large r for the boosted Schwarzschild metric. In matrix form this can be written as the metric

$$G_{\text{bS}} = \begin{pmatrix} -\left(1 - \frac{2M_S}{r}\right) & -\frac{8M_S v\cos\theta}{r} & 4M_S v\sin\theta & 0 \\ -\frac{8M_S v\cos\theta}{r} & \left(1 - \frac{2M_S}{r}\right)^{-1} & 0 & 0 \\ 4M_S v\sin\theta & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{28}$$

In this section we will always consider $\widetilde{\rho}$ given by $d\widetilde{\rho} = \sqrt{|\det G_{\mathrm{bS}}|}d^4x$. For the calculation of the area (in order to compute the mass thereafter) we will also need the value of the square root of the determinant of this matrix G_{bS}, which is (using Taylor) approximately given by

$$\sqrt{|\det G_{\mathrm{bS}}|} = r^2 \sin\theta. \tag{29}$$

As seen in Section 4, the deviation of the boosted Schwarzschild metric from the Minkowski metric can be taken into account in two possible (equivalent) ways:

- by integrating the second variable of $\mathcal{L}_\kappa$ in the boosted Schwarzschild spacetime
- by linearizing, integrating the second variable in Minkowski spacetime but integrating $\nabla_{2,\mathfrak{w}}\mathcal{L}_\kappa$ for a jet $\mathfrak{w}$, describing the change of the metric. (cf. Definition 4.9)

The latter is what we will do in this thesis as we can then calculate the surface layer integrals. Since we want to consider closed balls (w.r.t. the Minkowski norm) as exhaustion of the Minkowski spacetime (as well as balls for the exhaustion of the boosted Schwarzschild spacetime), for every fixed radius $R \in \mathbb{R}$ we want to therefore choose coordinates in Minkowski space close to the boundary $S_R = \partial B_R$ of B_R, s.t. the metric after the coordinate change coincides with the boosted Schwarzschild metric (given by G_{bS}) near to S_R: (Then the infinitesimal difference between this coordinate system and the usual Minkowski coordinate system will yield the jet $\mathfrak{w}$.)

Lemma 6.1. *Choosing in Minkowski spacetime the coordinates*

$$\begin{pmatrix} \widetilde{t} \\ \widetilde{r} \\ \widetilde{\theta} \\ \widetilde{\phi} \end{pmatrix} = \begin{pmatrix} t + t\frac{M_S}{R} - \frac{8M_S v \cos\theta}{R} r + 4M_S v\cos\theta \\ r - (r-R)\frac{M_S}{R} \\ \theta \\ \phi \end{pmatrix} =: \Phi\left(\begin{pmatrix} t \\ r \\ \theta \\ \phi \end{pmatrix}\right)$$

the metric locally (close to the boundary of the closed ball B_R of radius R) coincides with the boosted Schwarzschild metric G_{bS}.

Proof. Close to the boundary of B_R, i.e. around $r = R$ we can calculate:

$$\begin{pmatrix} t \\ r \\ \theta \\ \phi \end{pmatrix} = \begin{pmatrix} \widetilde{t} - \widetilde{t}\frac{M_S}{R} + \frac{8M_S v\cos\widetilde{\theta}}{R}\widetilde{r} - 4M_S v\cos\widetilde{\theta} \\ \widetilde{r} + (\widetilde{r} - R)\frac{M_S}{R} \\ \widetilde{\theta} \\ \widetilde{\phi} \end{pmatrix}$$

giving the Jacobian

$$J := \frac{\partial(t,r,\theta,\phi)}{\partial(\widetilde{t},\widetilde{r},\widetilde{\theta},\widetilde{\phi})}\bigg|_{r=R} = \begin{pmatrix} 1-\frac{M_S}{R} & \frac{8M_S v\cos\widetilde{\theta}}{R} & -4M_S v\sin\widetilde{\theta} & 0 \\ 0 & 1+\frac{M_S}{R} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and linearly in M_S we have

$$J\cdot G_{\text{Mink}}\cdot J^T = G_{\text{bS}}$$

with the diagonal matrix G_{Mink} describing the Minkowski metric with signature $(-,+,+,+)$ as well as G_{bS} from (28) describing the boosted Schwarzschild metric. (Here " $\cdot$ " resp. "$.^T$" denote the matrix multiplication resp. the transposed matrix.) □

From this we can calculate the jet derivative for the jet $\mathfrak{w}$ describing the deviation of the boosted Schwarzschild metric from Minkowski: An infinitesimal change is given by (the subscript 2 again refers to the second component of the Lagrangian and for shorter notation we use $x_{\text{M}} := (t,r,\theta,\phi)$, $y_{\text{M}} := (t',r',\theta',\phi') \in M$)

$$D_{2,\mathfrak{w}}\mathcal{L}(x_{\text{M}},y_{\text{M}}) = \tag{30}$$

$$\left(\begin{pmatrix} t - t\frac{M_S}{R} + \frac{8M_S v\cos\theta}{R}r - 4M_S v\cos\theta \\ r+(r-R)\frac{M_S}{R} \\ \theta \\ \phi \end{pmatrix} - \begin{pmatrix} t \\ r \\ \theta \\ \phi \end{pmatrix}\right)\circ\begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix}\mathcal{L}_\kappa(x_{\text{M}},y_{\text{M}}) =$$

$$\left(\left(-t'\frac{M_S}{R} + \frac{8M_S v\cos(\theta')}{R}r' - 4M_S v\cos(\theta')\right)\frac{\partial}{\partial t'} + (r'-R)\,\frac{M_S}{R}\frac{\partial}{\partial r'}\right)\mathcal{L}_\kappa(x_{\text{M}},y_{\text{M}}) \tag{31}$$

as in fact coincides with the linear deviation in M_S, i.e.

$$D_{2,\mathfrak{w}}\mathcal{L}(x_{\text{M}},y_{\text{M}}) = M_S\frac{\partial}{\partial M_S}\mathcal{L}(x_{\text{M}},y_{\text{M}})|_{M_S=0}$$

for all $x_{\text{M}}, y_{\text{M}} \in M$.

6.2 Area of Balls in Boosted Schwarzschild Spacetime

Since we will use balls for the exhaustion of the boosted Schwarzschild spacetime when calculating its mass, we need to calculate the area of balls in boosted Schwarzschild spacetime. Then we can compare this result to the area of balls exhausting Minkowski spacetime and adjust the radius of the considered balls s.t. the area constraint is satisfied. However, it turns out that changing the radius of the balls in boosted Schwarzschild spacetime is not necessary:

Lemma 6.2. *The area of a ball with a given radius in the boosted Schwarzschild spacetime from Section 6.1 coincides with the area of a ball with the same radius in Minkowski spacetime.*

Convention 6.3. *Quantities with tilde will always refer to boosted Schwarzschild spacetime.*

Proof. Consider a decomposition of the boosted Schwarzschild spacetime as in Definition 3.2 and let $\widetilde{\Omega}$ be the past of time $0 \in \mathbb{R}$. Additionally, let $\widetilde{U}$ be a closed ball of radius R in the boosted Schwarzschild spacetime with transversally intersecting boundaries $\partial\widetilde{\Omega}$ and $\partial\widetilde{U}$. Then by Definition 4.6 the area can be calculated as follows (to abbreviate the notation, denote in the following $\widetilde{x}_{\mathrm{bS}} = (0, \widetilde{r}, \widetilde{\theta}, \widetilde{\phi})$ resp. $\widetilde{y}_{\mathrm{bS}} = (\widetilde{t}', \widetilde{r}', \widetilde{\theta}', \widetilde{\phi}') \in \widetilde{M}$ and $x_{\mathrm{M}} = (0, r, \theta, \phi)$ resp. $y_{\mathrm{M}} = (t', r', \theta', \phi') \in M$):

$$\begin{aligned}&\int_{\widetilde{\Omega}\cap\widetilde{U}} d\widetilde{\rho}(x) \int_{\widetilde{M}\setminus(\widetilde{\Omega}\cup\widetilde{U})} d\widetilde{\rho}(y)\mathcal{L}_\kappa(\widetilde{x}_{\mathrm{bS}}, \widetilde{y}_{\mathrm{bS}})\\ &= \int_{-\infty}^{0} d\widetilde{t} \int_0^R d\widetilde{r} \int_0^\pi d\widetilde{\theta} \int_0^{2\pi} d\widetilde{\phi} \int_0^\infty d\widetilde{t}' \int_R^\infty d\widetilde{r}' \int_0^\pi d\widetilde{\theta}' \int_0^{2\pi} d\widetilde{\phi}' \\ &\quad \sqrt{|\det G_{\mathrm{bS}}|}\mathcal{L}_\kappa(\widetilde{x}_{\mathrm{bS}}, \widetilde{y}_{\mathrm{bS}}).\end{aligned}$$

Calculating the limits and setting

$$\Psi_1(R) = \sqrt{R^2 \sin^2(\theta) + \gamma^2 \left(R\cos(\theta) - vt\right)^2} =: \widetilde{R} \tag{32}$$

(where Ψ_1 is the second component of Ψ from (27)) as well as using (29) this transforms to

$$\begin{aligned}&\int_{-\infty}^{-\gamma vr\cos(\theta)} dt \int_{\gamma vt}^{\widetilde{R}} dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_{-\gamma vr\cos(\theta)}^{\infty} dt' \int_{\widetilde{R}}^\infty dr' \int_0^\pi d\theta' \int_0^{2\pi} d\phi' \times\\ &r^2 \sin(\theta) r'^2 \sin(\theta')\mathcal{L}_\kappa(x_{\mathrm{M}}, y_{\mathrm{M}}).\end{aligned}$$

This coincides with the area of a ball in Minkowski of radius $\widetilde{R}$ giving the area

$$\widetilde{A}\left(\widetilde{\Omega}, \widetilde{U}_n\right) = 4\pi\widetilde{R}^2 \cdot C$$

with the real constant

$$C := \int_{-\infty}^0 dt \int_0^\infty dt' \int_{-\infty}^0 dx \int_0^\infty dx' \int_{-\infty}^\infty dy' \int_{-\infty}^\infty dz' \mathcal{L}\left((t, x, 0, 0), (t', x', y', z')\right).$$

Plugging in the definition of $\widetilde{R}$ from equation (32) and doing a Taylor expansion in $v = 0$ yields approximately

$$\widetilde{A}\left(\widetilde{\Omega}, \widetilde{U}_n\right) = 4\pi R^2 \cdot C$$

which coincides with the area of a ball of radius R in Minkowski spacetime. □

Hence to take the area condition into account we do not have to change the radius of the balls in the exhaustion of boosted Schwarzschild, i.e. we can now calculate the mass of the boosted Schwarzschild spacetime with the same radius for the balls in the exhaustion of the boosted Schwarzschild spacetime as we consider for the balls in the exhaustion of Minkowski spacetime.

6.3 Mass of Boosted Schwarzschild Spacetime

In this section we will compute the mass for the boosted Schwarzschild spacetime. As one would expect, moving the Schwarzschild black hole in a fixed direction will not influence the mass of this black hole:

Proposition 6.4. *The mass of the boosted Schwarzschild spacetime coincides with the mass of the (unboosted) Schwarzschild spacetime.*

Proof. Since the mass is time-independent, let N be the time-$\{t=0\}$-hypersurface of Minkowski spacetime. Abbreviating the notation using $\widetilde{x}_{\mathrm{bS}} = (0, \widetilde{r}, \widetilde{\theta}, \widetilde{\phi})$ resp. $\widetilde{y}_{\mathrm{bS}} = (\widetilde{t'}, \widetilde{r'}, \widetilde{\theta'}, \widetilde{\phi'}) \in \widetilde{M}$ and $x_{\mathrm{M}} = (0, r, \theta, \phi)$ resp. $y_{\mathrm{M}} = (t', r', \theta', \phi') \in M$ the mass of the boosted Schwarzschild spacetime can be computed from the mass definition in Definition 4.10, taking the exhaustions $(U_n)_{n\in\mathbb{N}} = (B_R)_{R\in\mathbb{N}} \subset M$ resp. $(\widetilde{U}_n)_{n\in\mathbb{N}} = (\widetilde{B}_R)_{R\in\mathbb{N}} \subset \widetilde{M}$ (closed balls with the same radius) of Minkowski resp. boosted Schwarzschild spacetime and the jet $\mathfrak{w}$ from (31) describing the gravitation:

$$
\begin{aligned}
O(\widetilde{\rho}, U_n, \widetilde{U}_n) &= -\left(\int_{N\cap U_n} d\mu(x) \int_{\widetilde{M}\setminus\widetilde{U}_n} d\widetilde{\rho}(y) - \int_{N\setminus U_n} d\mu(x) \int_{\widetilde{U}_n} d\widetilde{\rho}(y)\right) \mathcal{L}_\kappa(x_{\mathrm{M}}, \widetilde{y}_{\mathrm{bS}}) \\
&= -\left(\int_{N\cap U_n} d\mu(x) \int_{M\setminus U_n} d\rho(y) - \int_{N\setminus U_n} d\mu(x) \int_{U_n} d\rho(y)\right) \nabla_{2,\mathfrak{w}} \mathcal{L}_\kappa(x_{\mathrm{M}}, y_{\mathrm{M}}) \\
&= -\Bigg(\int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_{-\infty}^\infty dt' \int_R^\infty dr' \int_0^\pi d\theta' \int_0^{2\pi} d\phi' \\
&\quad - \int_R^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_{-\infty}^\infty dt' \int_0^R dr' \int_0^\pi d\theta' \int_0^{2\pi} d\phi' \Bigg) \times \\
&\quad \left(\left(-t' \frac{M_S}{R} + \frac{8M_S v\cos(\theta')}{R} r' - 4M_S v \cos(\theta')\right) \frac{\partial}{\partial t'} + (r'-R)\, \frac{M_S}{R} \frac{\partial}{\partial r'}\right) \times \\
&\quad r^2 \sin(\theta) r'^2 \sin(\theta') \mathcal{L}_\kappa(x_{\mathrm{M}}, y_{\mathrm{M}})
\end{aligned}
$$

$$
\begin{aligned}
&= -\Bigg(\int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_{-\infty}^\infty dt' \int_R^\infty dr' \int_0^\pi d\theta' \int_0^{2\pi} d\phi' \\
&\quad - \int_R^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_{-\infty}^\infty dt' \int_0^R dr' \int_0^\pi d\theta' \int_0^{2\pi} d\phi' \Bigg) r^2 \sin(\theta) r'^2 \sin(\theta') \times \\
&\quad \left(-t' \frac{M_S}{R} \frac{\partial}{\partial t'} + (r' - R) \frac{M_S}{R} \frac{\partial}{\partial r'}\right) \mathcal{L}_\kappa(x_{\mathrm{M}}, y_{\mathrm{M}}) \\
&\quad - \Bigg(\int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_{-\infty}^\infty dt' \int_R^\infty dr' \int_0^\pi d\theta' \int_0^{2\pi} d\phi' \\
&\quad - \int_R^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_{-\infty}^\infty dt' \int_0^R dr' \int_0^\pi d\theta' \int_0^{2\pi} d\phi' \Bigg) r^2 \sin(\theta) r'^2 \sin(\theta') \times \\
&\quad \cos(\theta') \left(\frac{8 M_S v}{R} r' - 4 M_S v\right) \frac{\partial}{\partial t'} \mathcal{L}_\kappa(x_{\mathrm{M}}, y_{\mathrm{M}}) \\
&\xrightarrow{R \to \infty} \mathfrak{M}_S
\end{aligned}
$$

with $\mathfrak{M}_S$ denoting the mass of Schwarzschild spacetime from [PMT, Equation (6.23)] since the second term vanishes by the symmetry in θ and the first one coincides with the term for the mass in Schwarzschild spacetime calculated in [PMT] to obtain [PMT, Equation (6.23)]. □

6.4 Momentum of Boosted Schwarzschild Spacetime

In order to determine the energy of the boosted Schwarzschild spacetime it remains to compute the momentum in all spatial directions. With the Schwarzschild black hole moving in z-direction it is expected that we have momentum along the z-direction, but the momentum vanishes in the x- resp. y-direction.

6.4.1 Momentum of Boosted Schwarzschild Spacetime w.r.t. the z-direction

We can now calculate the momentum along each direction. (In the computations we will sometimes use symmetry arguments. These can in some cases seem a bit different for surface layer integrals: One has to keep in mind that since $\mathcal{L}(x, y)$ is of short range (cf. Section 2.3), there is contribution only for x and y close together and no contribution (hence no compensating terms) if x and y are far away from each other.)

Proposition 6.5. *The momentum in z-direction of a Schwarzschild black hole from Section 6.1 moving in z-direction is given by*

$$
\mathfrak{P}_3(\tilde{\rho}) = 4\pi M_S v \lim_{R \to \infty} \int_{\mathbb{R}} dt \int_{\mathbb{R}^3} d^3\vec{y}\, R \|\vec{y}\| \left(\frac{2\|\vec{y}\|}{R} - 1\right) \mathcal{L}_\kappa\left(\left(t, (0,1,0)\right), (0, \vec{y})\right),
$$

where $\|.\|$ denotes the Euclidean norm on $\mathbb{R}^3$.

Proof. For the momentum along the z-direction first linearize for the inner solution $\mathfrak{u}$ corresponding to $\frac{\partial}{\partial z}$ the surface layer integral of the form from Definition 5.2 with the notation from Convention 5.1:

$$\begin{aligned}
&\mathcal{I}_3(\tilde{\rho}, \Omega_t, \widetilde{\Omega}_{t'}, U_n, \widetilde{U}_n) = \\
&-\left(\int_{\Omega_t\cap U_n} d\mu(x) \int_{\widetilde{M}\setminus(\widetilde{\Omega}_{t'}\cup\widetilde{U}_n)} d\tilde{\rho}(y) - \int_{\Omega_t\setminus U_n} d\mu(x) \int_{\widetilde{U}_n\setminus\widetilde{\Omega}_{t'}} d\tilde{\rho}(y)\right) \nabla_{1,\mathfrak{u}}\mathcal{L}_\kappa(x_{\mathrm{M}}, \tilde{y}_{\mathrm{bS}}) \\
&+\left(\int_{U_n\setminus\Omega_t} d\mu(x) \int_{\widetilde{\Omega}_{t'}\setminus\widetilde{U}_n} d\tilde{\rho}(y) - \int_{M\setminus(\Omega_t\cup U_n)} d\mu(x) \int_{\widetilde{\Omega}_{t'}\cap\widetilde{U}_n} d\tilde{\rho}(y)\right) \nabla_{1,\mathfrak{u}}\mathcal{L}_\kappa(x_{\mathrm{M}}, \tilde{y}_{\mathrm{bS}}) \\
&= -\Bigg(\int_{\Omega_t\cap U_n} d\mu(x) \int_{M\setminus(\Omega_t\cup U_n)} d\tilde{\rho}(y) - \int_{\Omega_t\setminus U_n} d\mu(x) \int_{U_n\setminus\Omega_t} d\tilde{\rho}(y) \\
&+ \int_{U_n\setminus\Omega_t} d\mu(x) \int_{\Omega_t\setminus U_n} d\rho(y) - \int_{M\setminus(\Omega_t\cup U_n)} d\mu(x) \int_{\Omega_t\cap U_n} d\tilde{\rho}(y)\Bigg) \nabla_{1,\mathfrak{u}}\nabla_{2,\mathfrak{w}}\mathcal{L}_\kappa(x_{\mathrm{M}}, y_{\mathrm{M}})
\end{aligned}$$

with the jet from equation (31) as jet $\mathfrak{w}$. Since we can express the Cartesian derivative in z-direction as spherical derivative via

$$\frac{\partial}{\partial z} = \cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}$$

and the integrals vanish for the part derived by θ as $\frac{\partial}{\partial\theta}$ is tangential to the boundary of U_n (for a proof of this, see [Area, Definition 3.4 and Lemma 3.5]). Plugging in the closed balls B_R of radius R for U_n (resp. $\widetilde{B}_R$ for $\widetilde{U}_n$) as well as the past $\Omega_0 = (-\infty, 0] \times N$ for the time hypersurface N for the static Minkowski spacetime from its decomposition as in Definition 4.3 and carrying out the radial derivative $\frac{\partial}{\partial r}$ yields (we only get boundary terms at $r = R$, since $\mathcal{L}_\kappa$ decays on length scale $\delta \in \mathbb{R}$ (cf. Section 2.3)):

$$
\begin{aligned}
&\mathcal{I}_3(\widetilde{\rho}, \Omega_0, \widetilde{\Omega}_0, B_R, \widetilde{B}_R) \\
&\quad = -\Bigg(\Bigg(\int_{-\infty}^{0} dt \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{\infty} dt' \int_R^{\infty} dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \\
&\qquad - \Bigg(-\int_{-\infty}^{0} dt \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{\infty} dt' \int_0^{R} dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi'\Bigg) \\
&\qquad - \Bigg(\int_0^{\infty} dt \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_{-\infty}^{0} dt' \int_R^{\infty} dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \\
&\qquad - \Bigg(-\int_0^{\infty} dt \int_R^{\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_{-\infty}^{0} dt' \int_0^{R} dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi'\Bigg)\Bigg)\Bigg) \times \\
&\qquad \Bigg(\Bigg(-t'\frac{M_S}{R} + \frac{8M_S v\cos(\theta')}{R} r' - 4M_S v\cos(\theta')\Bigg)\frac{\partial}{\partial t'} + (r'-R)\,\frac{M_S}{R}\frac{\partial}{\partial r'}\Bigg) \times \\
&\qquad \cos(\theta) R^2 \sin(\theta) r'^2 \sin(\theta') \mathcal{L}_\kappa(x_{\mathrm{M}}, y_{\mathrm{M}}). \qquad (33)
\end{aligned}
$$

After multiplying out the brackets in the integrand, the integrals over $-t'\frac{M_S}{R}\frac{\partial}{\partial t'}$ resp. $(r'-R)\,\frac{M_S}{R}\frac{\partial}{\partial r'}$ vanish because of the symmetry of $\cos(\theta)$ with center $\frac{\pi}{2}$, giving

$$
\begin{aligned}
&\mathcal{I}_3(\widetilde{\rho}, \Omega_0, \widetilde{\Omega}_0, B_R, \widetilde{B}_R) \\
&= -\Bigg(\int_{-\infty}^{0} dt \int_0^{\infty} dt' \int_R^{\infty} dr' + \int_{-\infty}^{0} dt \int_0^{\infty} dt' \int_0^{R} dr' - \int_0^{\infty} dt \int_{-\infty}^{0} dt' \int_R^{\infty} dr' \\
&\quad - \int_0^{\infty} dt \int_{-\infty}^{0} dt' \int_0^{R} dr'\Bigg) \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \times \\
&\quad R^2 \sin(\theta) r'^2 \sin(\theta')\cos(\theta)\left(\frac{8M_S v\cos(\theta')}{R} r' - 4M_S v\cos(\theta')\right)\frac{\partial}{\partial t'}\mathcal{L}_\kappa(x_{\mathrm{M}}, y_{\mathrm{M}}).
\end{aligned}
$$

By integrating by parts in t' we obtain (we again only have boundary terms at $t' = 0$ as $\mathcal{L}_\kappa$ decays on length scale $\delta \in \mathbb{R}$ (cf. Section 2.3)):

$$
\begin{aligned}
&\mathcal{I}_3(\widetilde{\rho}, \Omega_0, \widetilde{\Omega}_0, B_R, \widetilde{B}_R) \\
&\quad = -\Bigg(-\int_{-\infty}^{0} dt \int_R^{\infty} dr' - \int_{-\infty}^{0} dt \int_0^{R} dr' \\
&\qquad - \int_0^{\infty} dt \int_R^{\infty} dr' - \int_0^{\infty} dt \int_0^{R} dr'\Bigg) \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \times \\
&\qquad R^2 \sin(\theta) r'^2 \sin(\theta')\cos(\theta)\left(\frac{8M_S v\cos(\theta')}{R} r' - 4M_S v\cos(\theta')\right)\mathcal{L}_\kappa(x_{\mathrm{M}}, y_{\mathrm{M}})
\end{aligned}
$$

$$= \int_{\mathbb{R}} dt \int_0^\infty dr' \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^\pi d\theta' \int_0^{2\pi} d\phi' \times$$
$$M_S v R^2 \sin(\theta) r'^2 \sin(\theta') \cos(\theta) \cos(\theta') \left(\frac{8r'}{R} - 4 \right) \mathcal{L}_\kappa(x_{\mathrm{M}}, y_{\mathrm{M}}).$$

We next want to express the spatial integrals in Cartesian coordinates (x_1, x_2, x_3) resp. (y_1, y_2, y_3) for the first resp. second component. For the transformation we have the factors $R^2 \sin(\theta)$ resp. $r'^2 \sin(\theta')$. The cosines $\cos(\theta)$ resp. $\cos(\theta')$ can in Cartesian coordinates be written as

$$\cos(\theta) = \frac{x_3}{R} \quad \text{resp.} \quad \cos(\theta') = \frac{y_3}{\|\vec{y}\|}$$

(with the Euclidean norm $\|.\|$ on $\mathbb{R}^3$) as we have $\|x\| = R$. Hence transformating to Cartesian coordinates leads to

$$\mathcal{I}_3(\widetilde{\rho}, \Omega_0, \widetilde{\Omega}_0, B_R, \widetilde{B}_R)$$
$$= 2 \int_{\mathbb{R}} dt \int_{-R}^{R} dx_3 \int_{-\sqrt{R^2 - x_3^2}}^{\sqrt{R^2 - x_3^2}} dx_2 \int_{\mathbb{R}^3} d^3\vec{y} \times$$
$$M_S v \frac{x_3 y_3}{R \|\vec{y}\|} \left(\frac{8\|\vec{y}\|}{R} - 4 \right) \mathcal{L}_\kappa \left(\left(t, \sqrt{R^2 - x_2^2 - x_3^2}, x_2, x_3 \right), (0, \vec{y}) \right).$$

(Here the factor 2 comes from the symmetry since we integrate only over the positive-x_1-hemisphere as we only have $x_1 = \sqrt{R^2 - x_2^2 - x_3^2}$ in the argument of the Lagrangian instead of $x_1 \in \{\pm\sqrt{R^2 - x_2^2 - x_3^2}\}$.) Now we have to take into account that the Lagrangian is of short range (cf. Section 2.3). Hence we have

$$|y_3 - x_3| > \delta \Rightarrow \|(y_1, y_2, y_3) - (x_1, x_2, x_3)\| > \delta \Rightarrow \mathcal{L}_\kappa(x, y) = 0.$$

with $|.|$ resp. $\|.\|$ denoting the absolute value on $\mathbb{R}$ resp. Euclidean norm on $\mathbb{R}^3$. Hence we only need to integrate in x_3 over the (one-dimensional) closed ball $\overline{B_\delta(y_3)} = y_3 + \overline{B_\delta(0)} \subset \mathbb{R}$. Since for $\epsilon \in \overline{B_\delta(0)}$ and $x_3 \in [-R, R]$ we have

$$[-R, R] \ni x_3 = y_3 + \epsilon \iff \epsilon \in \overline{B_R(-y_3)},$$

this yields the integral

$$
\begin{aligned}
&\mathcal{I}_3(\widetilde{\rho},\Omega_0,\widetilde{\Omega}_0,B_R,\widetilde{B}_R)\\
&\quad = 2\int_{\mathbb{R}} dt \int_{\mathbb{R}^3} d^3\vec{y} \int_{\overline{B_R(-y_3)}\cap\overline{B_\delta(0)}} d\epsilon \int_{-\sqrt{R^2-(y_3+\epsilon)^2}}^{\sqrt{R^2-(y_3+\epsilon)^2}} dx_2 \times \\
&\qquad M_S v \frac{(y_3+\epsilon)y_3}{R\|\vec{y}\|}\left(\frac{8\|\vec{y}\|}{R}-4\right)\mathcal{L}_\kappa\left(\left(t,\sqrt{R^2-x_2^2-(y_3+\epsilon)^2},x_2,y_3+\epsilon\right),(0,\vec{y})\right)\\
&\quad = 8\int_{\mathbb{R}} dt \int_{\mathbb{R}^3} d^3\vec{y} \int_{\overline{B_R(-y_3)}\cap\overline{B_\delta(0)}} d\epsilon \int_{-\sqrt{R^2-(y_3+\epsilon)^2}}^{\sqrt{R^2-(y_3+\epsilon)^2}} dx_2 \times \\
&\qquad M_S v \frac{y_3^2}{R\|\vec{y}\|}\left(\frac{2\|\vec{y}\|}{R}-1\right)\mathcal{L}_\kappa\left(\left(t,\sqrt{R^2-x_2^2-(y_3+\epsilon)^2},x_2,y_3+\epsilon\right),(0,\vec{y})\right)\\
&\qquad + 8\int_{\mathbb{R}} dt \int_{\mathbb{R}^3} d^3\vec{y} \int_{\overline{B_R(-y_3)}\cap\overline{B_\delta(0)}} d\epsilon \int_{-\sqrt{R^2-(y_3+\epsilon)^2}}^{\sqrt{R^2-(y_3+\epsilon)^2}} dx_2 \times \\
&\qquad M_S v \frac{\epsilon y_3}{R\|\vec{y}\|}\left(\frac{2\|\vec{y}\|}{R}-1\right)\mathcal{L}_\kappa\left(\left(t,\sqrt{R^2-x_2^2-(y_3+\epsilon)^2},x_2,y_3+\epsilon\right),(0,\vec{y})\right).
\end{aligned}
$$

Taking $\vec{x} = (\sqrt{R^2-x_2^2-x_3^2}, x_2, x_3) = (\sqrt{R^2-x_2^2-(y_3+\epsilon)^2}, x_2, y_3+\epsilon) \in S_R^2$ into account (where $S_R^2 \subset \mathbb{R}^3$ denotes the two-dimensional sphere of radius R and because of the positive sign in front of the root we are only considering the points in the upper half of the sphere) rotational symmetry gives for the leading term

$$
\begin{aligned}
&\mathcal{I}_3(\widetilde{\rho},\Omega_0,\widetilde{\Omega}_0,B_R,\widetilde{B}_R)\\
&\quad = 2\pi R^2\cdot 8\int_{\mathbb{R}} dt \int_{\mathbb{R}^3} d^3\vec{y}\, M_S v \frac{y_3^2}{R\|\vec{y}\|}\left(\frac{2\|\vec{y}\|}{R}-1\right)\mathcal{L}_\kappa\left((t,(0,1,0)),(0,\vec{y})\right).
\end{aligned}
$$

Using the symmetry of the integrand about the diagonal $\{y_3 = y_1\}$ and afterwards the symmetry about the diagonal $\{y_3 = y_2\}$ yields

$$
\begin{aligned}
&\mathcal{I}_3(\widetilde{\rho},\Omega_0,\widetilde{\Omega}_0,B_R,\widetilde{B}_R)\\
&\quad = 16\pi M_S v R^2\int_{\mathbb{R}} dt \int_{\mathbb{R}^3} d^3\vec{y}\, \frac{\|\vec{y}\|^2}{2^2 R\|\vec{y}\|}\left(\frac{2\|\vec{y}\|}{R}-1\right)\mathcal{L}_\kappa\left((t,(0,1,0)),(0,\vec{y})\right).
\end{aligned}
$$

Hence the momentum is given by the limit

$$
\mathfrak{P}_3(\widetilde{\rho}) = 4\pi M_S v \lim_{R\to\infty}\int_{\mathbb{R}} dt \int_{\mathbb{R}^3} d^3\vec{y}\, R\|\vec{y}\|\left(\frac{2\|\vec{y}\|}{R}-1\right)\mathcal{L}_\kappa\left((t,(0,1,0)),(0,\vec{y})\right).
$$

□

6.4.2 Momentum of Boosted Schwarzschild Spacetime w.r.t. the x-resp. y-direction

Next we will compute the momentum w.r.t. the x-direction. However, this is as one would expect:

Proposition 6.6. *For a Schwarzschild black hole moving in z-direction the momentum along the x-direction vanishes.*

Proof. The Cartesian derivative by x can be expressed in spherical coordinates as

$$\frac{\partial}{\partial x} = \cos(\phi)\sin(\theta)\frac{\partial}{\partial r} - \frac{\sin(\phi)}{r\sin(\theta)}\frac{\partial}{\partial \phi} + \frac{\cos(\phi)\cos(\theta)}{r}\frac{\partial}{\partial \theta}. \tag{34}$$

As $\frac{\partial}{\partial\phi}$ and $\frac{\partial}{\partial\theta}$ are again tangential to the balls exhausting M, their contributions vanish (just as for the momentum in z-direction).
With the analogous approach as for the z-direction instead of equation (33) taking the radial derivative term from (34) into account we obtain (and can from there calculate):

$$\begin{aligned}
&\mathcal{I}_1(\widetilde{\rho}, \Omega_0, \widetilde{\Omega}_0, B_R, \widetilde{B}_R)\\
&= -\Bigg(\int_{-\infty}^{0} dt \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{\infty} dt' \int_R^{\infty} dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi'\\
&\quad + \int_{-\infty}^{0} dt \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{\infty} dt' \int_0^{R} dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi'\Bigg) \times\\
&\quad \left(\left(-t'\frac{M_S}{R} + \frac{8M_S v\cos(\theta')}{R}r' - 4M_S v\cos(\theta')\right)\frac{\partial}{\partial t'} + (r'-R)\frac{M_S}{R}\frac{\partial}{\partial r'}\right) \times\\
&\quad R^2\sin(\theta)\cos(\phi)\sin(\theta)r'^2\sin(\theta')\mathcal{L}_\kappa(x_{\mathrm{M}}, y_{\mathrm{M}})\\
&= -\Bigg(\int_{-\infty}^{0} dt \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{\infty} dt' \int_R^{\infty} dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi'\\
&\quad + \int_{-\infty}^{0} dt \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{\infty} dt' \int_0^{R} dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi'\Bigg) \times\\
&\quad R^2\sin(\theta)\cos(\phi)\sin(\theta)r'^2\sin(\theta')\left(-t'\frac{M_S}{R}\frac{\partial}{\partial t'} + (r'-R)\frac{M_S}{R}\frac{\partial}{\partial r'}\right)\mathcal{L}_\kappa(x_{\mathrm{M}}, y_{\mathrm{M}})\\
&= -\frac{M}{R}\int_{-\infty}^{0} dt \int_0^{\infty} dt' \int_0^{\infty} dr' \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \times\\
&\quad R^2\sin(\theta)\cos(\phi)\sin(\theta)r'^2\sin(\theta')\left(-t'\frac{M_S}{R}\frac{\partial}{\partial t'} + (r'-R)\frac{M_S}{R}\frac{\partial}{\partial r'}\right)\mathcal{L}_\kappa(x_{\mathrm{M}}, y_{\mathrm{M}}).
\end{aligned}$$

Integration by parts using

$$-\frac{\partial}{\partial t'}\left(r'^2\cdot(-t')\right) = r'^2 \quad \text{and} \quad -\frac{\partial}{\partial r'}\left(r'^2(r'-R)\right) = -\left(3r'^2 - 2r'R\right)$$

yields

$$
\begin{aligned}
&\mathcal{I}_1(\widetilde{\rho}, \Omega_0, \widetilde{\Omega}_0, B_R, \widetilde{B}_R) \\
&= -\frac{M_S}{R} \int_{-\infty}^{0} dt \int_0^{\infty} dt' \int_0^{\infty} dr' \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \times \\
&\qquad R^2 \sin(\theta)\cos(\phi)\sin(\theta)\sin(\theta') \left(r'^2 - 3r'^2 + 2r'R\right) \mathcal{L}_\kappa(x_{\mathrm{M}}, y_{\mathrm{M}}).
\end{aligned}
$$

Since in spatial Cartesian coordinates (x_1, x_2, x_3) (to the spatial spherical coordinates (r, θ, ϕ), where θ denotes the polar angle whereas ϕ denotes the azimuthal angle) we have

$$
\cos(\phi)\sin(\theta) = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \cdot \frac{\sqrt{x_1^2 + x_2^2}}{\sqrt{x_1^2 + x_2^2 + x_3^2}} = \frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} = \frac{x_1}{R}
$$

(with $\sqrt{x_1^2 + x_2^2 + x_3^2} = R$ as we integrate over the sphere of radius R in the first component) as well as the factors $R^2 \sin(\theta)$ resp. $r'^2 \sin(\theta')$, the integral can also be written as

$$
\begin{aligned}
&\mathcal{I}_1(\widetilde{\rho}, \Omega_0, \widetilde{\Omega}_0, B_R, \widetilde{B}_R) \\
&= -2 \cdot \frac{M_S}{R} \int_{-\infty}^{0} dt \int_0^{\infty} dt' \int_{-R}^{R} dx_1 \int_{-\sqrt{R^2 - x_1^2}}^{\sqrt{R^2 - x_1^2}} dx_2 \int_{\mathbb{R}^3} d^3\vec{y} \times \\
&\qquad \frac{x_1}{R} \cdot \frac{-2\|\vec{y}\|^2 + 2R\|\vec{y}\|}{\|\vec{y}\|^2} \mathcal{L}_\kappa\left(\left(t, x_1, x_2, \sqrt{R^2 - x_1^2 - x_2^2}\right), (0, \vec{y})\right),
\end{aligned}
$$

(with $\|.\|$ denoting the Euclidean norm on $\mathbb{R}^3$ and where the factor 2 comes from the symmetry, considering $x_3 = \sqrt{R^2 - x_1^2 - x_2^2}$ instead of $x_3 \in \{\pm\sqrt{R^2 - x_1^2 - x_2^2}\}$) which vanishes because of the symmetry in x_1, therefore the limit $\mathfrak{P}_x := \mathfrak{P}_1(\widetilde{\rho})$ vanishes as well. □

Finally we consider the y-direction:

Proposition 6.7. *For a Schwarzschild black hole moving in z-direction the momentum along the y-direction vanishes.*

Proof. Transforming the Cartesian derivative by y into spherical coordinates gives

$$
\frac{\partial}{\partial y} = \sin(\phi)\sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\phi)}{r\sin(\theta)}\frac{\partial}{\partial \phi} + \frac{\sin(\phi)\cos(\theta)}{r}\frac{\partial}{\partial \theta}. \tag{35}
$$

$\frac{\partial}{\partial \phi}$ and $\frac{\partial}{\partial \theta}$ drop out once again (as for the momentum in z- resp. x-direction) as they are tangential to the boundary of the balls in the exhaustion of Minkowski.

Proceeding analogously to the computation of the momentum along the z- and x-direction because of (the radial derivative in) (35) we obtain instead of equation (33) the term

$$\begin{aligned}
&\mathcal{I}_2(\widetilde{\rho}, \Omega_0, \widetilde{\Omega}_0, B_R, \widetilde{B}_R) \\
&= -\left(\int_{-\infty}^{0} dt \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{\infty} dt' \int_R^{\infty} dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \right. \\
&\quad \left. + \int_{-\infty}^{0} dt \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{\infty} dt' \int_0^{R} dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \right) \times \\
&\quad \left(\left(-t' \frac{M_S}{R} + \frac{8 M_S v \cos(\theta')}{R} r' - 4 M_S v \cos(\theta') \right) \frac{\partial}{\partial t'} + (r' - R) \frac{M_S}{R} \frac{\partial}{\partial r'} \right) \times \\
&\quad R^2 \sin(\theta) \cdot \sin(\phi) \sin(\theta) {r'}^2 \sin(\theta') \mathcal{L}_\kappa(x_{\mathrm{M}}, y_{\mathrm{M}}).
\end{aligned}$$

This vanishes since the integrand is symmetric in ϕ with center π, hence so does the limit $\mathfrak{P}_y := \mathfrak{P}_2(\widetilde{\rho})$. □

6.5 Energy-Momentum Vector of Boosted Schwarzschild Spacetime

Summing up Sections 6.3 and 6.4 the energy of the boosted Schwarzschild spacetime is given by the energy-momentum vector

$$\begin{pmatrix} \widetilde{\mathfrak{M}} \\ \widetilde{\mathfrak{P}}_x \\ \widetilde{\mathfrak{P}}_y \\ \widetilde{\mathfrak{P}}_z \end{pmatrix} = \begin{pmatrix} \mathfrak{M}_S \\ 0 \\ 0 \\ \widetilde{\mathfrak{P}}_z \end{pmatrix}$$

for the mass $\mathfrak{M}_S$ of Schwarzschild spacetime from [PMT, Equation (6.23)] and with $\widetilde{\mathfrak{P}}_z$ given by

$$\widetilde{\mathfrak{P}}_z = \mathfrak{P}_z = 4\pi M_S v \lim_{R\to\infty} \int_{\mathbb{R}} dt \int_{\mathbb{R}^3} d^3\vec{y}\, R \|\vec{y}\| \left(\frac{2\|\vec{y}\|}{R} - 1 \right) \mathcal{L}_\kappa \left((t, (0,1,0)), (0, \vec{y}) \right),$$

which as expected by Section 5.3 equals $\gamma v \mathfrak{M}_S$ (Taylor expanded in $v = 0$). (In [PMT, Equation (6.14)] evaluating the integral $\int_{S^2} d\omega$ gives 4π instead of $\frac{1}{4\pi}$ after [PMT, Lemma 6.1].)

7 Outlook: Energy and Momentum of an Asymptotically Flat Lorentzian Manifold

In this section we discuss the generalization of the example from Section 6 to the setting of asymptotically flat globally hyperbolic Lorentzian Manifolds. For an asymptotically flat Lorentzian manifold $(\mathcal{M}, g)$ the metric g has the form

$$g_{ij}(x) = \delta_{ij} + a_{ij}(x)$$

for $x \in \mathbb{R}^4 \setminus \overline{B_r(0)}$ with the decay properties for the remainder

$$a_{ij} = \mathcal{O}\left(\frac{1}{\|x\|}\right), \quad \partial_k a_{ij} = \mathcal{O}\left(\frac{1}{\|x\|^2}\right) \quad \textit{and} \quad \partial_k \partial_l a_{ij} = \mathcal{O}\left(\frac{1}{\|x\|^3}\right) \tag{36}$$

(where $\|.\|$ denotes the Euclidean norm on $\mathbb{R}^4$). In order to regard this as a spacetime in the theory of causal fermion systems, we need to construct a suitable causal fermion system. In the static case, this was done in [PMT, Section 2.3]: Using the notation from [PMT] for the Hilbert space $\mathcal{H}$ of physical wave functions the minimizing measure was given by $d\rho = (F^\varepsilon)_* d\mu_{\mathcal{M}}$ with $d\mu_{\mathcal{M}} = \sqrt{|det\, g|} d^4x$ the volume measure on $\mathcal{M}$.
However, how to choose $\mathcal{H}$ for the suitable causal fermion system is not obvious for time-dependent causal fermion systems. Missing fundamental symmetries in comparison to static spacetimes, in the time-dependent setting we now have to fix a Hadamard state (for more detail on this topic, see for example [FHP, Section 1]). But it is plausible that choosing the Hadamard state is not of importance for the calculation of mass, momentum and energy, since we only consider the singularity structure of the fermionic projector (for more detail see e.g. [MOP]) for the Lagrangian as is done for the static case in [PMT, Section 6.6] using [LCE, Appendix B]. Summing this up, although it is not known yet if mass, momentum and energy of arbitrary asymptotically flat globally hyperbolic Lorentzian Manifolds can be obtained in the same way as for the boosted Schwarzschild spacetime in Section 6, it is expected to work similarly.

Lacking spherical symmetry, we would expectedly then have to calculate e.g. the mass by averaging over masses calculated for fixed directions. More explicitly:
We denote the matrix (denoting the metric tensor) corresponding to the metric g by G. Consider one direction w in $\widetilde{M}$ with $g(w, w) = 1$ (and stay constant along the other directions). (For example in spherical coordinates one would consider fixed $\theta = \theta_0$ and $\phi = \phi_0$ in order to do this.) This then yields an induced metric on $g_{\mathbb{R}\cdot w}$ on the subspace $\mathbb{R} \cdot w = \{\lambda w | \lambda \in \mathbb{R}\}$. Then choose coordinates $(\widetilde{t}, \widetilde{r}, \widetilde{\theta}, \widetilde{\phi})^T = \Phi((t, r, \theta, \phi)^T)$ in Minkowski space (regarded in spherical coordinates

(t, r, θ, ϕ), with fixed θ, ϕ in spherical coordinates) s.t. the metric in this direction coincides linearly with the metric $g_{\mathbb{R}\cdot w}$ around Rw for radius $R \in \mathbb{R}$ (corresponding to $r = R$ in the boosted Schwarzschild example). This is possible due to the decay properties (36).
For this we can then calculate the derivative describing the gravitation by

$$D_{2,v}\mathcal{L} = \left(\left(\Phi\left((t,r,\theta,\phi)^T\right) - (t,r,\theta,\phi)^T\right) \circ \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \right) \right) \mathcal{L}$$

and can with this derivative (taking the limit $R \to \infty$) calculate the mass $\mathfrak{M}(\widetilde{\rho}, w)$ and momentum $\mathfrak{P}(\widetilde{\rho}, w)$ along this direction.
Finally we have to get rid of the fixed direction by averaging over all directions by

$$\mathfrak{M}(\widetilde{\rho}) = \frac{1}{\widetilde{\rho}(\partial B_1(0))} \int_{w \in B_1(0)} \mathfrak{M}(\widetilde{\rho}, w) d\widetilde{\rho}$$

resp.

$$\mathfrak{P}(\widetilde{\rho}) = \frac{1}{\widetilde{\rho}(\partial B_1(0))} \int_{w \in B_1(0)} \mathfrak{P}(\widetilde{\rho}, w) d\widetilde{\rho}.$$

(This averaging was not necessary for spherical symmetric spacetimes.)

8 Conclusion

Finally we will recap the results proven in this thesis:
We have introduced a notion of mass for causal fermion systems (with the area constraint instead of the inner volume constraint) depending only on the spacetime geometry at infinity, which coincides with the mass for static causal fermion systems for the Minkowski and the Schwarzschild spacetime. Hence it also coincides with the *ADM* mass in the cases of a Minkowski resp. a Schwarzschild spacetime as it does as well for the boosted Schwarzschild spacetime as seen in the previous section.
Furthermore we have defined a notion of momentum for causal fermion systems. This has been used to define the energy as an energy-momentum vector. For this we have shown that the energy vectors build a vector space and moreover that the energy is invariant of the observer.

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