A mathematical model for stress modulated growth and existence theorems in one spatial dimension

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Abstract

We investigate the model of stress modulated growth introduced in [AM02] in one dimension, which describes aspects of a growing tumour. In particular, the interaction of growth - described by an ODE - and the mechanical stress - within non-linear elasticity - is analysed. The key to the model is to understand and handle the "natural configuration" - the tumour grown but not yet elastically deformed. Here, we use it as reference configuration for the elastic problem and afterwards transform all variables to the initial set of the tumour.
In order to state existence and uniqueness by the Picard–Lindelöf theorem, the stress has to depend Lipschitz continuously on the growth. This is discussed in several settings in one dimension for a linear ODE and generalized to arbitrary Lipschitz continuous RHS. Moreover, regularity results are stated.
In addition, the difficulties in higher dimensions are discussed and examples of simple two dimensional problems displayed.

Zusammenfassung

Wir untersuchen das Modell von durch Spannungen beeinflusstes Wachstums, welches in [AM02] eingeführt wurde. Es beschreibt Aspecte von wachsenden Tumoren. Genauer wird das Zusammenspiel von Wachstum - beschrieben von einer gewöhnlichen DGL - und den mechanischen Spannungen - im Rahmen nichtlinearer Elastizität - analysiert. Der Kernpunkt ist, die "natürliche Konfiguration" - den gewachsenen aber noch nicht elastisch deformierten Tumor - zu verstehen und damit umzugehen. Wir benutzen diese Konfiguration als die Referenzkonfiguration des elastischen Problems und transformieren alle Variablen auf die initiale Menge des Tumors.
Um Existenz und Eindeutigkeit mit dem Picard–Lindelöf Theorem zu beweisen, müssen die Spannungen lipschitzstetig von dem Wachstum abhängen. Dies wir in mehreren Situationen in einer Dimension für eine lineare, gewöhnliche DGL diskutiert und auf beliebige lipschitzstetige rechte Seiten verallgemeinert. Außerdem werden Regularitätsresultate diskutiert.
Zusätzlich werden die Schwierigkeiten des Modells in höheren Dimensionen diskutiert und Beispiele für einfache, zweidimensionale Situationen aufgeführt.

Contents

1 Introduction

Recently, the importance of tumour growth research has increased. Moreover, the interest in investigating growth processes in general grew in the last century. It ranges from minerals densifying in bones, over tumour growth, to the development of embryos. Many studies and some numerical simulations have been done over that time period, however mathematical analysis of complex structures is rather rare to find in literature. The basis of this thesis is the "stress modulated growth" model, developed since 1930s e.g. by Huxley [Hux31], Hsu [Hsu68], Skalak [Ska81], Rodriguez et al [RHM94] and Ambrosi and Mollica [AM02]. Only for very specific settings, analytical tools as well-posedness, existence and uniqueness can be established. For an overview over growth in general, see for example[JC12], [Gor17] and [Tab20], where a model on a two-dimensional artery wall is illustrated in [Tab20].

Why investigating tumour growth better?

Almost everybody has had contact with cancer, in one way or another, since it is a very common disease in our time. The WHO notes that approximately half of people who have cancer die because of it, see [WHO22]. Therefore, we want to understand tumour growth as well as possible with the final goal to be able to heal cancer. The more that is known about the growth of tumours, the better it can be diagnosed, predicted or even healed.

When did research for growth models start?

In the second half of the 20th century, there was a great increase in interest in growth processes. However, the first modern work [Tho17] on describing growth and what influences it was written by Thompson in 1917. This well-known book in the biology of growth shows the interest in all kinds of growth, which have been or are still under investigation today.
So far, many influences of growth have been investigated: Some chemicals, known as morphogens, can have a strong influence on the growth, see [Nel09]; by lack of nutrients, the centre of a tumour becomes necrotic and the total growth - of at least in vitro tumours - is bounded, see [RCM07]; another broad field is the study or embryonic development, which is not further discussed here. These studies have not only the goal to find how growth is stimulated, but they also are dedicated to the following research question:

Why does something grow?

The main influences of growth are genetic factors, nutrients, chemical factors, and mechanics, which act on different scales in the body. For instance, a certain level of nutrients is needed in order to grow while too low concentration of nutrients inhibits growth. In addition, diseases or injuries can support growth. There are many papers from biologists about growth, seen in nature: In [Law92], it is observed that a starfish regrows its arm after it was lost. If a lung is removed from a rabbit, the remaining lung grows bigger to compensate for the missing volume, because more mechanical requirements are asked of it, see [CC75]. In a study, it was found that tennis players have different properties in the bones of their playing arms than their non-playing arms, see [Haa+00]. Similar effects are observed by [VM15] in a study on training hands for martial arts. The training increases the bone mineral density of the trained hands. Other studies vary from investigating the effects of weightlessness due to space flights, see [Goo+98], over the problem of restricted movements in plaster casts, see [JLKU80], or simple physical exercise influencing the muscle, see [JZ00] [NPL98], as almost everybody has experienced already. Moreover, a broad field in the research of growth is the remodelling of bones, see [GC01], [Sto+06], and below. No matter which object is under observation, on the macroscopic scale, mechanical effects play into the process of growth.

Why are mechanical influences important?

Grossman [Gro80] found in 1980 that a vascular hypertrophy - thickening of the cardiac muscle - is at least in some part caused by higher pressure of the blood against the wall and the wall's internal stress as reaction to the increased requirements. In 1983, Fung et al. [Fun90] found that without residual stress arteries have a non-constant stress in the wall with high stress at the inner surface. Fung did not approve of that and found it to be wrong by proving the existence of residual stress, [CF83] and [LF88]. To do so, he cut into an artery perpendicularly to its length and observed that it opened up over time, because the residual stress was released along the cut line.

While Grassman, Fung et al. investigated soft tissue, another important series of experiments was done by Cowin et al., investigating the influence of stress on the growth of bones. In [CH77], [Cow83], [Cow86] and [GC92], stress in the limb influencing the growth of the long bone is observed. Among other things, they found how torsion influences the surface growth of a bone. A first prediction of remodelling of the diaphyseal - the shaft of a long bone - was made in [CF81] by describing it as a function of the strain. Also, [Pau80] investigates an equilibrium state between stress and growth.

Both fields have the question in common: How do stress and growth influence each other? This is still an interesting question and the answer highly depends on the material under observation. It is certain that stress plays an important role when it comes to growth processes. It influences the growth to become non-uniform, it can accelerate growth, or prohibit it, it ensures the continuity of the grown material.

What grows?

As they are mentioned above, mostly soft tissue and bones have been investigated so far. The influence of stress is bigger for soft material than for bones. That motivates us to continue to investigate soft tissue. Later on it will be assumed to consist of an elastic material, setting the model into the terms of continuum mechanics. For the comparable small growth of bones, the model would lead to linear elasticity theory, whereas in this thesis, we would like to investigate non-linear theory. In the setting of continuum mechanics follows the question:

How to add mass?

In continuum mechanics a body is assumed to consist of particles. Hence, a naive idea is to simply add particles over time. This has the downside that the motion for these added particles cannot be defined for all times, i.e. there is no possibility to track them. One way to circumvent this is to work in the Euler frame, but that raises problems with the boundary conditions. Another idea is to assume growth to be the exchange of mass. In particular, there exists a function describing the material, e.g. if the function is equal to 1, there is tumour tissue, while for the value -1 there is healthy tissue. To describe growth in such settings, the theory of phase fields is applied. However, we assume the growth as "increase of mass of already existing particles" as written in [AM02]. This idea was first introduced by Hsu in [Hsu68] in case of linear elasticity. More precisely, he approached the question: If the form of the body after growth is known, how do mechanics affect the growth of the body? Skalak [Ska81] introduced the idea of discontinuities after growth, which are evened out by the elastic deformation. This relates to the idea of a growth gradient introduced by Huxley [Hux31] in 1931. The term "growth gradient" is used to describe the growth in each direction and not to consider only isotropic - equal in each direction - growth. In [Ska+82], Skalak et al. pointed out the importance of residual stress and concentrated on tracking particles during growth and elastic deformation, namely if neighbouring particles before the process are neighbours afterwards again. These works led to the formulation as multiplicative decomposition introduced in [RHM94], which we will investigate in this thesis with the additional modelling done in [AM02]. The decomposition goes along with "natural configuration" - what Skalak described by discontinuities after the pure growth. More generally, the natural configuration is the body after free growth without taking stresses into account. This is the basic idea of the "stress modulated growth" model, and a very short answer to the question:

How to model growth and stress simultaneously?

Consider a tumour described by a domain evolving over time. This process F is decomposed into growth G and elastic deformation F_e, in a formula $F = F_eG$. The mass increases during the growth process, which is assumed to be unrestricted by mechanics. Therefore, the natural configuration is not necessarily physically admissible, e.g. the body is not necessarily continuous or it might be self-penetrating. The following elastic deformation ensures that the body is admissible and obeys the possible external loads again. Here, the mass is assumed to be preserved. Thus, the decomposition is strictly divided into growth and elastic deformation, which allows us to find equations, describing each process individually. Growth is assumed to be given by an ODE, while nonlinear elasticity theory is used to describe the elastic deformation. The key in these equations is that the stress is included in the ODE for growth, and the growth needs to be known for the reference configuration of elastic deformation. Moreover, the influence of nutrients shall not be neglected. Those obey to a reaction-diffusion equation and are of course also included into the ODE for the growth. This model, including constitutive equations for the elastic deformation, are in [AM02], but the explicit formulas for the ODE and the elastic strain energy density are highly dependent on the particular situation. For example, [Amb+17] argues to consider the tumour as a porous material with elastic cellular component and states corresponding equations.

What is the state of the art?

Next to the rather biological works mentioned above, growth is also investigated from a mathematical point of view:
Numerics: As stated before, there exist some first attempts on specific situations to understand the relation between growth and stress. One of the most investigated objects is the wall of an artery. In [TE96], the growth in angular and radial direction are simulated under the assumption of the growth rate to rely linearly on the local stress. Also for the artery wall, [Liu+14] concentrate on the deformation and the induced stress using a non-linear finite element model. In [Yan+15] and [Yan+16], the concept of stress modulated growth is used to model and treat plaques in blood vessels numerically. Furthermore, the surface growth can be taken into account, see [KM18], where the difference between isotropic and anisotropic surface growth is considered in combination with stress modulated growth. Numerical simulations show the expected different behaviours.
Buckling in cylindrical setting: A first model of a growing artery wall can be found in [RHM94] and will be recalled and broadened in Examples 2.4.2 and 2.4.3 to more layers. In [Li+11] and [MG11], cylindrical objects are under investigation, too. The first paper models and simulates numerically the wrinkling of the mucosa surface motivated from what is found in studies. The second paper considers single and bi-layered tubes in two dimensions and their buckling. In [Mac+12] a Blatz-Ko type material is used to see buckling effects in two dimensions and deformation of a tube by numerical simulations. This is expanded to non-linear elasticity in [Mac+13] using boundary layer analysis.
Buckling of a shell: In [AG05], the behaviour of a shell in finite elasticity is modelled and analysed. The difference of fibre growth and area growth is highlighted and numerically investigated resulting in the competing of geometry and mechanics for buckling effects. Finally the critical

strains are achieved for the different modes and the buckling displayed. On the other hand, in [AM04] an avascular tumour - a tumour with necrosis - is modelled and by prohibiting buckling effects, the experimental data are matched.
Stability: An important question is the stability of the deformation. In [AG05], it is shown that residual stress is important for the stability, because a shell can become unstable if residual stress is neglected. This example is in linear elasticity without external forces and considers anisotropic growth.
For specific settings, analysis and numerical results can be found throughout the literature. For instance, in the recent paper [DNS21] an existence result for morphoelastic tumours via the Rothe method is given. However, analysis of a general setting is still missing, as are existence theorems. There are many difficulties mentioned already, as buckling and not connected or even overlapping natural configurations. For a more precise overview over open problems as well as missing results in computation we refer to [Amb+11]. It also displays the missing understanding of biology due to the lack of experiments.

What are the goals of this thesis?

We restrict ourselves to one dimension in order to bypass several difficulties of the stress modulated growth model. Nonetheless, many questions arise: What is the mathematical setting? What is a solution? Is the model well-posed? Does a solution exist? Is the solution unique? How do we define a solution? This leads to different settings in which the problem has a (unique) solution. What can be stated about the solution? What regularity can be achieved? Does it behave as physically expected? What physical behaviour can be deduced from the model? Is the solution always physically admissible? What do we have to assume to ensure it? Are the deformations good enough to solve equations on the resulting domains? We will give an answer to all these questions.
Concerning the last question, Stefanelli et al. deal with a similar problem: In plasticity theory via multiplicative decomposition, the intermediate configuration has to be good enough to treat an elastic problem on it. To obtain this, they resort to (ε, δ)-domain, see [KMS20] and [Ste19]. The plastic deformation in terms of a Sobolev function is good enough to work with, which is not necessarily the case for the "stress modulated growth" model. Here, we would like to avoid that problem, in order to understand the other challenges of the model. This is why we work in one dimension only.

How will we proceed?

At first, we recall the model of "stress modulated growth" including a discussion on where to solve which equation and discuss the deformation of an artery wall with one and two layers, see Chapter 2. Unlike many of the mentioned papers, we consider the material to obey non-linear elasticity theory, because we discuss an example of linear elasticity, see Chapter 3.1.1, which contradicts the physical restrictions after finite time. By including only the stress, but not yet the nutrients, we prove existence to several settings:

(i) a body consisting of two different materials, see Chapters 3.2.1 and 4.2,

(ii) a body consisting of m different materials, see Chapter 3.2.2,

(iii) continuous material properties, leading to continuous solutions, see Chapter 4.3,

(iv) more general, yielding an L^∞-solution, see Chapter 4.4.

The Settings (i) and (ii) are considered differently in order to provide simpler notation. For example, in Setting (i), we obtain a scalar equation while in Setting (ii) we obtain a $(m-1)$ equations. Therefore, the splitting of those two cases separates the idea from the technicalities. In order to prove the concept, we apply the Picard–Lindelöf theorem in Setting (i) first. Settings (iii) and (iv) include more technicalities, but also give better results. To be precise, in Setting (iii) we need the material properties to be continuous in the material, but also obtain better regularity, namely continuity of the solution in space, see Theorem 4.3.9. On the other

hand, Setting (iv) does not demand the continuity in space and nor yields continuity in space, see Theorem 4.4.7.
Furthermore, physical behaviour as (un)bounded stress after (un)bounded growth and compression for growing or stretching for shrinking material are proved. In Chapter 5, the regularity of the solution in time and space is discussed. In Chapter 6, the idea of the artery wall geometry and the results in one dimension are used to obtain an existence and uniqueness result of a geometrically simple two dimensional setting. Here, restrictions are applied in order to avoid buckling effects and the difficulties, discussed throughout the one dimensional analysis, as the growth tensor to be a gradient.

2 About ”Stress modulated Growth”

This chapter is dedicated to a general introduction to the model under investigation in this thesis. We want to investigate tumour growth taking mechanics into account. In particular, we discuss the mathematical description of a growing tumour with mechanical properties which splits the process into growth (or resorption) and mechanical response. In the mathematical framework, the initial configuration $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, of the body at time $t = 0$ is called reference configuration. Suppose that at time $t > 0$ the body has undergone growth and elastic deformation due to application of external loads or internal compatibility conditions. We denote the map describing this process by $X(t, \cdot)$ with gradient $F(t, \cdot)$ and the image by $\Omega_t \subset \mathbb{R}^n$. Furthermore, we introduce a third configuration, the natural configuration: Take an infinitesimal part of Ω_t and release its stresses while its mass stays constant. It reaches its natural state which is by construction stress-free and describes the body after unconstrained growth. Uniting natural states of all parts gives the natural configuration Ω_{nat}. The deformation of the natural configuration to the final configuration Ω_t is described by the local deformation tensor F_e and the deformation from the reference configuration Ω to the natural configuration Ω_{nat} by the local deformation tensor G. Hence, the decomposition

$$F = F_e G$$

holds.

In our model we assume that mass is preserved going from Ω_{nat} to Ω_t, hence, F_e is not related to growth but to the mechanical stress, while the tensor G is directly related to growth, and thus, named growth tensor. To summarize, we split the process into the separated contributions of pure growth and pure elastic deformation.
Moreover, we assume the tumour to be hyperelastic, i.e. the mechanical response is hyperelastic going from Ω_{nat} to Ω_t. The elastic strain energy density is called W and we introduce the Cauchy stress tensor T and the second Piola–Kirchhoff tensor S of the total deformation

$$T = F_e(D_F W(F_e))^T \text{ and } S = \det(F)F^{-1}TF^{-T},$$

see Section 2.1 below for a brief introduction. In addition, we assume the growth tensor to obey to the ODE

$$\dot{G} = \mathcal{G}(G, t, n, S) \text{ on } \Omega,$$

where n are the nutrients and $\mathcal{G}$ a RHS describing the growth. Taking the Piola–Kirchhoff tensor is the natural choice, because the growth tensor is invariant under change of frame, see Remark 2.3.1 below, and the second Piola–Kirchhoff tensor does not change upon frame

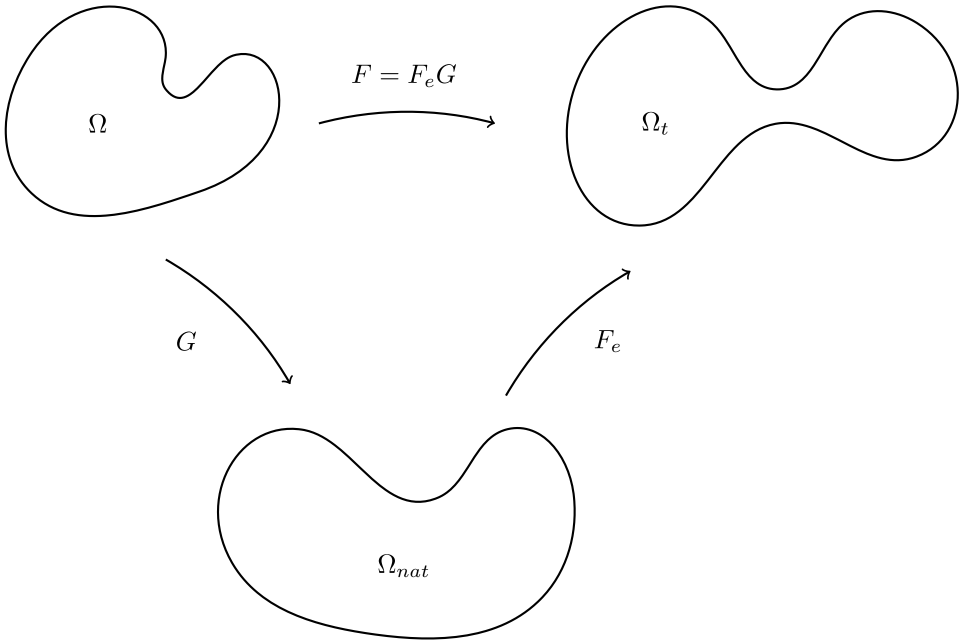

Figure 2.1.: Schematic picture of the decomposition of stress modulated growth into growth and elastic deformation.

changes as well. We need this property of G to differentiate with respect to time.
For the elastic deformation, we already introduced the elastic strain energy density W. Therefore, we are able to prescribe the elastic deformation to minimize the elastic energy by solving the corresponding Euler–Lagrange equation

$$\text{div } D_F W(F_e) = 0 \text{ on } \Omega_{nat}.$$

Since growth is strongly depending on the availability of nutrients and influences of chemical signals, these two factors are already coupled to the growth tensor in the ODE. The nutrients diffuse into the tumour. Therefore, they are supposed to fulfil the reaction diffusion equation

$$\text{div } (nv) - \text{div } (D(n)\nabla n) = -\gamma n \text{ on } \Omega_t,$$

where $v = \partial_t X$. Observe that, on the one hand, the ODE for the growth tensor involves the stress induced by the elastic deformation and the nutrients. On the other hand, the mechanical response needs to know the grown tissue, namely the natural configuration, and the equation of nutrients is defined on the elastically deformed configuration. Hence, we can not study one of the three parts separately but have to solve them simultaneously.
The next subsections provide a basic vocabulary in continuum mechanics, including the Cauchy and the Piola–Kirchhoff stress tensors and hyperelasticity, which are used throughout the whole thesis. Afterwards, we display the modelling of the stress modulated growth problem, following [RHM94] and [AM02], setting it into our language and giving more detailed calculations.

2.1. Basics of continuum mechanics

General setting and definitions

A general setting is considered to introduce vocabulary of continuum mechanics, see [Gur81] for more information. Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be the domain the body covers at time $t = 0$. It is called the reference configuration and its points $p \in \Omega$ are called material points. A motion of the body Ω is a map $X\colon [0,T]\times\Omega\to\mathbb{R}^d$ such that $X(t,.)$ is one-to-one for all $t\in[0,T]$, $\nabla X(t,p)\in Lin^+ := \{M\in\mathbb{R}^{d\times d} | \det(M)>0\}$ for all $p\in\Omega$ and all $t\in[0,T]$, and $x:=X(t,p)$ is the position of the point $p\in\Omega$ at time t. At time $t\in[0,T]$ the body covers the domain

$$\Omega_t := \{X(t,p) |\; p\in\Omega\}\subset\mathbb{R}^d.$$

It is called its material configuration or final configuration. By

$$\mathcal{T} := \{(t,x) |\; t\in[0,T],\; x\in\Omega_t\},$$

the trajectory is defined.
For each $t\in[0,T]$, $X(t,\cdot)$ is assumed to be one-to-one. That is why it has an inverse $P(t,\cdot)\colon\Omega_t\to\Omega$ such that $X(t,P(t,x))=x$ and $P(t,X(t,p))=p$ for all $x\in\Omega_t$ and $p\in\Omega$. The point $P(t,x)$ is the material point that is at x at time t. The map $P\colon\mathcal{T}\to\Omega$ is called the reference map.
In addition, we define the spatial and material description. For a set M, the spatial description $\varphi_s\colon\mathcal{T}\to M$ of a function $\varphi\colon[0,T]\times\Omega\to M$ is defined by

$$\varphi_s(t,x) := \varphi(t,P(t,x))$$

and the material description $\phi_m\colon[0,T]\times\Omega\to M$ of a function $\phi\colon[0,T]\times\Omega_t\to M$ is defined by

$$\phi_m(t,p) := \phi(t,X(t,p)).$$

Furthermore, we define the physical quantities velocity $\dot{X}\colon[0,T]\times\Omega\to\mathbb{R}^d,\; \dot{X}(t,p)=\partial_t X(t,p)$ and acceleration $\ddot{X}\colon[0,T]\times\Omega\to\mathbb{R}^d,\; \ddot{X}(t,p)=\partial_t^2 X(t,p)$. Especially, we define the spatial description of the velocity $v\colon\mathcal{T}\to\mathbb{R}^d$ by

$$v(t,x) := \dot{X}_s(t,x) = \dot{X}(t,P(t,x)).$$

A special type of deformation is the isochoric motion, which preserves volume, see [Gur81].

Definition 2.1.1 (Isochoric motion)**.** *A motion $X\colon[0,T]\times\Omega\to\mathbb{R}^d$ is isochoric (volume preserving) if*

$$\mathcal{L}^d(X(\mathcal{P})) = \mathcal{L}^d(\mathcal{P}) \quad \forall\mathcal{P}\subset\Omega,$$

where $\mathcal{L}^d$ denotes the d-dimensional Lebesgue-measure.

This definition is equivalent to the determinant of the deformation gradient being 1, see [Gur81]. That is the condition, which is applied later on.

Lemma 2.1.2. *A motion X is isochoric if and only if* $\det F = 1$ *with* $F=\nabla X$.

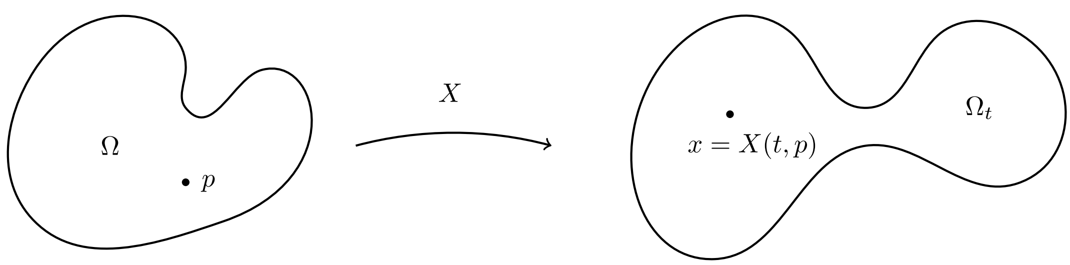

Figure 2.2.: Motion of the domain Ω und the map X.

Forces and momentum of the body are important properties to describe its motion. The external forces are the surface force $s\colon \partial\Omega_t \to \mathbb{R}^d$ and the body force $b\colon \mathcal{T} \to \mathbb{R}^d$. For a part $\Omega' \subset \Omega$, the force f and momentum m on $\Omega'_t := X(t,\Omega')$ at a time t is given by

$$f(t,\Omega') = \int\limits_{\partial\Omega'_t} s(n)\ dA + \int\limits_{\Omega'_t} b\ dV,$$
$$m(t,\Omega') = \int\limits_{\partial\Omega'_t} x \times s(n)\ dA + \int\limits_{\Omega'_t} x \times b\ dV.$$

Furthermore, with the velocity $v = \dot{x}$, the linear ℓ and angular momentum a are defined by

$$\ell(\Omega',t) := \int\limits_{\Omega'_t} v\rho\ dV,$$
$$a(\Omega',t) := \int\limits_{\Omega'_t} x \times v\rho\ dV$$

with time derivates

$$\dot{\ell}(\Omega',t) := \int\limits_{\Omega'_t} \dot{v}\rho\ dV,$$
$$\dot{a}(\Omega',t) := \int\limits_{\Omega'_t} x \times \dot{v}\rho\ dV$$

Then, the balance of linear momentum

$$f(\Omega',t) = \dot{l}(\Omega',t)$$

and the balance of angular momentum

$$m(\Omega',t) = \dot{a}(\Omega',t)$$

hold.

Cauchy stress tensor

In the following, we discuss the elastic deformation of a body and refer to [Gur81] for more details. While a body is deformed, e.g. due to external forces, parts can move against each other, which induces stress. For this stress, an energy exists, for which we need a description of how the energy depends on the deformation in order to handle it analytically. The stress is describe by a stress tensor. Its existence is stated in the next theorem, which also states that only the external force in normal direction has impact on the elastic deformation, see [Gur81] for a proof.

Theorem 2.1.3 (Existence of the Cauchy-stress tensor)**.** *Let $\Omega \subset \mathbb{R}^d$ be a body with s the surface force and b the body force. Then, the balance of momentum holds if and only if there exists a tensor $T\colon \mathcal{T} \to \mathbb{R}^{d\times d}$, called Cauchy stress tensor, such that*

(i) for each $\nu \in \partial\Omega_t$

$$s(\nu) = T\nu,$$

(ii) T is symmetric,

$$T = T^T, \tag{2.1}$$

(iii) T satisfies the equation of motion

$$div\ T + b = \rho \dot{v}. \tag{2.2}$$

As mentioned before, the stress inside the body depends on the displacement of the particles. Therefore, the response functions are introduced expressing the Cauchy stress tensor as a function of the gradient $F = \nabla X : \Omega \to Lin^+$.

Definition 2.1.4 (Response function)**.** *A body Ω is elastic, if there exists a function $\hat{T} : Lin^+ \times \Omega_t \to Sym(d)$ with*

$$T(t, x) = \hat{T}(x, F(t, x)).$$

Then, $\hat{T}$ is named the response function.

Another important property is independence of observer, which states that the stress inside a body is independent of where the deformation takes place but instead depends of the displacement.

Definition 2.1.5. *A elastic body is independent of observer if its response function $\hat{T}$ satisfies*

$$Q\hat{T}(F)Q^T = \hat{T}(QF) \quad \forall F \in Lin^+, Q \in SO(d).$$

Remark 2.1.6. *In one dimension, $SO(1) = 1$. Therefore, the condition of independence of observer is redundant.*

Piola–Kirchhoff stress tensor and hyperelasticity

Above, the equation of motion was introduced to hold in the elastically deformed configuration Ω_t. This might be a problem, since Ω_t is a priori not known. A solution to handle this is to transform the formula to the reference configuration, namely to hold on Ω. Applying this change of variables causes the stress tensor to coincide with the Piola–Kirchhoff stress tensor, see [Gur81]. To do so, we need the change of variables formula, see [Gur81]:

Lemma 2.1.7 (Change of variables)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set and $f : \Omega \to f(\Omega) \subset \mathbb{R}^d$ an integrable diffeomorphism. Then for any continuous function $\varphi : f(\Omega) \to \mathbb{R}$,*

$$\int\limits_{f(\Omega)} \varphi(x)\ dx = \int\limits_{\Omega} \varphi(f(y))|\det Df(y)|\ dy \tag{2.3}$$

holds. Further, with the outward unit normal fields ν_1 on $\partial\Omega$ and ν_2 on $\partial\varphi(\Omega)$ it holds

$$\int\limits_{f(\Omega)} \varphi(x)\nu_1(x)\ dx = \int\limits_{\Omega} \varphi(f(y))|\det Df(y)| Df^{-T}(y)\nu_2(y)\ dy. \tag{2.4}$$

Using change of variables 2.1.7, we write the surface force of a part $\Omega' \subset \Omega$ as

$$\int\limits_{\partial\Omega'_t} T\mu\ dA = \int\limits_{\partial\Omega'} \det F\ T_m F^{-T} \nu\ dA$$

with μ, ν the outward normal of Ω_t' and Ω' respectively. This motivates to define the first Piola–Kirchhoff stress tensor

$$S\colon [0,T] \times \Omega \to Lin,\ S(t,p) := \det F(t,p)\ T_m(t,p) F^{-T}(t,p)$$

which is the stress tensor depending on the reference configuration. The equation of motion hence, transforms (2.2) to

$$\operatorname{div} S + b_0 = \rho_0 \ddot{x}$$

with $b_0 = b_m \det F$ the body force measured in the reference configuration and $\rho_0 = \rho_m \det F$ the density in the reference configuration. Instead of symmetry (2.1), the Piola–Kirchhoff stress tensor fulfils

$$SF^T = FS^T.$$

Similarly, for an elastic material, there exists the response function corresponding to S, more specifically, there exists a function

$$\hat{S}\colon Lin^+ \times \Omega \to Lin,\ \hat{S} = \det F\ \hat{T}(F) F^{-T}$$

satisfying $\hat{S}(QF) = Q\hat{S}(F)$ for all $F \in Lin^+$ and all $Q \in SO(n)$ (independence of observer) such that

$$S(t,p) = \hat{S}(p, F(t,p)).$$

The Piola–Kirchhoff stress tensor has its advantages: if the elastically deformed configuration is not known, but the reference configuration is, the equation of motion is well-defined. Another advantage is that if the material is hyperelastic, see the next definition, there exists a formula for the energy the elastic deformation consumes.

Definition 2.1.8 (Hyperelastic material)**.** *A material is called hyperelastic, if there exists a map $W\colon \Omega \times Lin^+ \to \mathbb{R}$ called the elastic strain energy density such that*

$$W(p, QF) = W(p, F) \quad \forall p \in \Omega, F \in Lin^+, Q \in SO(d)$$

and

$$\hat{S}(p, F) = D_F W(p, F)$$

with $\hat{S}$ the response function corresponding to the material's Piola–Kirchhoff stress tensor S. The total strain-energy of the deformation of a hyperelastic body is

$$E\colon \mathrm{C}^1(\Omega; \mathbb{R}^d) \to \mathbb{R},\ E(X) = \int_\Omega W(p, \nabla X(p))\ dV.$$

A further special kind of material is the homogeneous one, which means that the elastic strain energy density W is independent of the particle.

Definition 2.1.9 (Homogeneous material)**.** *A material is homogeneous, if the density ρ_0 and the elastic strain energy density W are independent of $p \in \Omega$. Then, the response functions are independent of $p \in \Omega$ as well.*

If the process is time-independent, the theory is called elastostatics. The goal is to find a deformation $X\colon [0,T] \times \Omega \to \mathbb{R}^d$ with $F = \nabla X$ such that for a given strain-energy density $W\colon \Omega \times Lin^+ \to \mathbb{R}$ holds

$$S(t,p) = \hat{S}(p, F(t,p)) = D_F W(p, F(t,p)),$$

$$\text{div } S + b_0 = 0 \text{ in } \Omega$$

and boundary conditions hold. These boundary conditions can e.g. be $Sn = 0$ on $\partial\Omega$ or $X = 0$ on $\partial\Omega$.
Using change of variables, the problem of elastostatics in Cauchy theory is transformed to: For a given strain-energy density $W\colon \Omega \times Lin^+ \to \mathbb{R}$ find a map $X\colon [0,T] \times \Omega \to \mathbb{R}^d$ with $F = \nabla X$ such that

$$T(t,x) = \hat{T}(x, F_m(t,x)) = (\det F_m(t,x))^{-1} D_F W(P(t,x), F_m(t,x)) F_m(t,x)^T,$$
$$\text{div } T + b = 0 \text{ in } \Omega_t$$

and boundary condition, e.g. $Tn = 0$ on $\partial\Omega_t$ or $X = 0$ on $\partial\Omega$.
In the following, we denote the response functions without hat, because we always use the response functions.

2.2. Useful theorems in continuum mechanics

We will list some useful theorems, which will be used within this thesis, see [Gur81].

Lemma 2.2.1 (Divergence theorem/Gauß/partial integration)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. For continuous differentiable functions $\varphi\colon \Omega \to \mathbb{R}$, $v\colon \Omega \to \mathbb{R}^d$ and $S\colon \Omega \to \mathbb{R}^{d\times d}$ holds*

$$\int_\Omega \nabla\varphi(y)\ dy = \int_{\partial\Omega} \varphi(x)\nu(x)\ dx, \tag{2.5}$$

$$\int_\Omega div\ v(y)\ dy = \int_{\partial\Omega} v(x){\cdot}\nu(x)\ dx, \tag{2.6}$$

$$\int_\Omega div\ S(y)\ dy = \int_{\partial\Omega} S(x)\nu(x)\ dx, \tag{2.7}$$

with ν the outward unit normal field on $\partial\Omega$.

Lemma 2.2.2 (Reynold's transport theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set, $T > 0$ and X a smooth enough motion of Ω. Further, let $f\colon \Omega\times[0,T] \to \mathbb{R}$ and $\varphi\colon \Omega\times[0,T] \to \mathbb{R}^d$ be continuous differentiable. Then, for every subset $\Omega' \subset \Omega$ with Lipschitz boundary and each time $t \in [0,T]$, it holds*

$$\frac{d}{dt}\int_{\Omega'_t} f(x)\ dx = \int_{\Omega'_t} \dot{f}(y) + f(y) div\ v(y)\ dy, \tag{2.8}$$

$$\frac{d}{dt}\int_{\Omega'_t} \varphi(x)\ dx = \int_{P_t} \varphi'(y)\ dy + \int_{\Omega'_t} \varphi(y)v(y)\cdot\nu(y)\ dy, \tag{2.9}$$

where $\Omega'_t \coloneqq X(t,\Omega')$ denotes the set Ω' after the motion at time t and ν denotes the outward unit normal field on $\partial\Omega'_t$.

2.3. Modelling stress modulated growth

After the introduction on the basic vocabulary, we introduce the model of stress modulated growth. It developed over many years, e.g. in [Hsu68], [Ska81] and [RHM94], and later on, a system of equations is introduced in [AM02].
It is motivated for example by the following: Consider a growing tumour surrounded by tissue right under the skin. When the tumour grows, it pushes the tissue and the skin outwards, while the tissue and the skin provides resistance, and therefore, pushes the tumour inside. Another example is a tumour growing next to a bone which restricts the growth. We want to generalize this and derive a mathematical model.
Let the tumour at time 0 cover the domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$. The problem, we want to study, considers the growth of the tumour, the stresses arising due to the growth and the nutrients diffusing through the tumour motivating growth. Hence, the goal is to find a growth map, an elastic deformation, a nutrients map and consequently the domain $\Omega_t \subset \mathbb{R}^d$, which is the domain covered by the tumour at time t. Since the grown tumour is elastically deformed, it is stressed. The idea of the model is to separate the growth from the elastic deformation. To do so, a new configuration is introduced: The natural configuration. Take an infinitesimal part form Ω_t and release its stress. Take another infinitesimal part and also release its stress. Continue for every part of Ω_t. All unstressed parts together we call natural configuration $\Omega_{nat} \subset \mathbb{R}^d$. Due to the construction, it is locally stress free and describes the body after unconstrained growth. A problem that arises is that the natural configuration must not be physically admissible, e.g. in the natural state the body can penetrate itself or be ripped apart and contain separated parts, see picture 2.3.
The map, which maps the natural configuration Ω_{nat} into the elastically deformed configuration Ω_t, is an elastic deformation we call $\phi\colon \Omega_{nat} \to \Omega_t$. In contrast to the natural configuration, the elastically deformed configuration is physically admissible again. Because we constructed the natural configuration locally, we have a local deformation gradient of the elastic deformation denoted by F_e. What is left is the growth map $g\colon \Omega \to \Omega_{nat}$ with local deformation gradient G. By construction,

$$X = \phi \circ g,$$

and therefore, $F = (F_e)_m G$. Hence, we split the transformation of Ω into first a growth, where all change in mass going from Ω to Ω_{nat} takes place, and then, all elastic deformation going form Ω_{nat} to Ω_t.

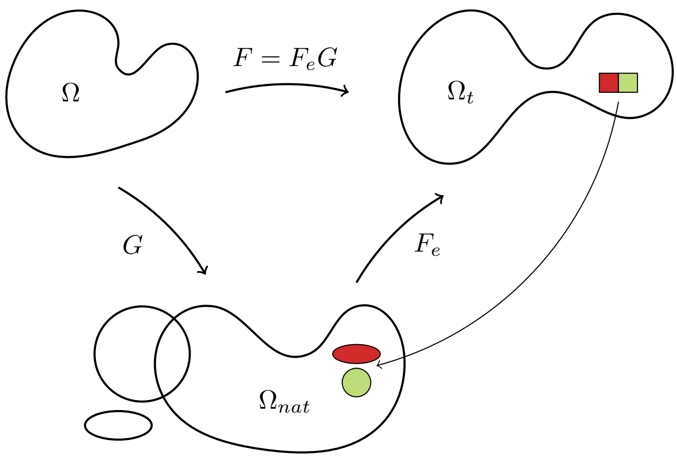

Figure 2.3.: Schematic picture of stress relief: Two small pieces (red and green squares) are cut out of Ω_t and their stresses are relieved, which yields their natural configuration (red and green ellipses).

For the constructed conjunction of growth and elastic deformation, we will derive equations describing the growth, the elastic deformation, the nutrients and their interaction. For the rest of this subsection, we follow the paper [AM02].
Let $\rho_0(p)$ denote the density in $p \in \Omega$ and $\rho(x,t)$ the density in $x \in \Omega_t$. Take an infinitesimal part of Ω with volume dV_0 and mass $dM_0 = \rho_0 dV_0$. To this part corresponds the mass in the natural configuration dM_{nat} and dM_t in Ω_t. Since the growth is unconstrained, the density is conserved, hence, the mass of the grown part is

$$dM_{nat} = \rho_0 dV_n$$

with dV_{nat} the volume of the part. With the assumption of conservation of mass between Ω_{nat} and Ω_t $dM_{nat} = dM_t$ we get

$$\frac{dM_t}{dM_0} = \frac{dM_{nat}}{dM_0} = \frac{\rho_0 dV_{nat}}{\rho_0 dV_0} = \frac{dV_{nat}}{dV_0} = \frac{\det G dV_0}{dV_0} = \det G.$$

For $\det G > 1$ we say growth takes place, while for $\det G < 1$ resorption takes place.

Remark 2.3.1 (Uniqueness in the decomposition of F)**.** *See [RHM94]. Let $G \in \mathbb{R}^{d\times d}$ be the growth tensor. By the polar decomposition for matrices there exist unique $R_G \in SO(d)$ and $U_G \in Sym^+$ with $G = R_G U_G$. If we rotate the natural state Ω_{nat} in a suitable way, the rotation R_G is absorbed into the elastic deformation as the elastic setting is independent of observer, i.e. $G = U_G$, which guarantees the uniqueness of the decomposition of F. Possible shifts of the natural configuration do not change G, since G is the local gradient of the growth map.*

Mass balance

Classic mass balance reads as

$$\partial_t \rho + \text{div }(\rho v) = 0 \text{ in } \Omega_t.$$

We do not consider mass balance, but additional mass to appear due to the growth. This increase in mass is described by the growth rate Γ. Hence, in our model mass balance reads as

$$\partial_t \rho + \text{div }(\rho v) = \dot{\rho} + \rho \text{div } v = \Gamma \rho \text{ in } \Omega_t$$

with the material time derivative $\dot{\rho} = \rho' + v \cdot \nabla \rho$. By change of variables and with $J = \det F$ we get

$$\dot{\rho}_m J + \rho_m (\text{div } v)_m J = \Gamma \rho_m J \text{ in } \Omega.$$

Because the material time derivative of a function on Ω is the time derivative, with the time derivative of the determinant $(\det F)' = \det F \text{tr }(F' F^{-1})$ and chain rule holds

$$\begin{aligned}
\dot{(\rho_m J)} = (\rho_m J)' &= \rho_m' J + \rho_m J' = \rho_m' J + \rho_m J \text{tr }(F' F^{-1}) \\
&= \rho_m' J + \rho_m J \text{tr }(\partial_t \nabla X (\nabla X)^{-1}) \\
&= \rho_m' J + \rho_m J \text{tr }(\nabla (v_m)(\nabla X)^{-1}) \\
&= \rho_m' J + \rho_m J \text{tr }((\nabla v)_m) \\
&= \rho_m' J + \rho_m J (\text{div } v)_m \\
&= \rho_m \dot{J} + \rho_m J (\text{div })_m = \Gamma \rho_m J,
\end{aligned}$$

and thus,

$$\dot{(\rho_m J)} = \Gamma \rho_m J \text{ in } \Omega. \tag{2.10}$$

As mentioned above, the mass is conserved between Ω_{nat} and Ω_t, i.e.

$$dM_{nat} = \rho_m dV_0 \det F = \rho_0 dV_0 \det G = dM_t,$$

and therefore,

$$\rho_0 = \rho_m \det F_e \text{ in } \Omega. \tag{2.11}$$

Balance of linear momentum

The equation of balance of linear momentum reads as follows

$$\partial_t(\rho v) + \text{div } (\rho v \otimes v) - \text{div } T = \rho b + \Gamma \rho v \text{ in } \Omega_t$$

where $\partial_t(\rho v) + \text{div } (\rho v \otimes v)$ is the basic momentum of a moving body, $-\text{div } T$ is the momentum due to the elastic deformation, ρb due to the body forces and $\Gamma \rho v$ the momentum of the new mass. We calculate by inserting the balance of mass

$$\begin{aligned}\partial_t(\rho v) + \text{div } (\rho v \otimes v) &= \rho' v + \rho v' + \rho v \nabla v + v \text{div } (\rho v) \\ &= \Gamma \rho v + \rho \dot{v}\end{aligned}$$

which yields

$$-\text{div } T + \rho \dot{v} = \rho b.$$

Since the velocity of growth and, hence, the change in the elastic deformation is very small compared to the stress, we assume $\dot{v} = 0$ which leads to quasi static elasticity theory. This leaves the equation of motion for the elastic deformation in elastostatics as introduced above, i.e.

$$-\text{div } T = b \text{ in } \Omega_t \tag{2.12}$$

to solve. As last step, we write the equation in the reference configuration like the equation for the growth which can only be in Ω. Therefore, we use the first Piola–Kirchhoff stress tensor P in Ω and get

$$-\text{div } P = b_m \text{ in } [0, T] \times \Omega$$

and for hyperelastic material with elastic energy density W

$$-\text{div } D_F W(\nabla X) = b_m \text{ in } [0, T] \times \Omega.$$

If the natural configuration Ω_{nat} and the map ϕ are nice enough, the equation (2.12) can be written in the term of the Piola–Kirchhoff stress tensor P_{nat} of the deformation ϕ, namely

$$-\text{div } P_{nat} = -\text{div } D_F W_{nat}(\nabla X G^{-1}) = b_{nat},$$

where $b_{nat} = b_m g^{-1}$ is the body force and $W_{nat}(y, F) = W(g^{-1}(y), F)$ the elastic strain energy density in the natural configuration. With the decomposition $X = \phi \circ g$ follows

$$-\text{div } D_F W_{nat}(\nabla \phi) = b_{nat} \text{ in } \Omega_{nat}, \tag{2.13}$$

which is the equation, we shall solve in Chapters 4 and 5.

Concerning the growth rate Γ

Above we stated the equation (2.11). By applying the material time derivative we obtain

$$0 = \dot{\rho}_m \det F_e + \rho_m \dot{\overline{\det F_e}}.$$

Inserting this into (2.10) gives

$$\begin{aligned}\Gamma\rho_m \det F = \dot{\overline{(\rho_m \det F)}} &= \dot{\rho}_m \det F + \rho_m \dot{\overline{\det F}}\\ &= -\rho_m \dot{\overline{\det F_e}} \det G + \rho_m \dot{\overline{\det F_e}} \det G + \rho_m \det F_e \dot{\overline{\det G}}\\ &= \rho_m \det F_e \dot{\overline{\det G}}.\end{aligned}$$

With the time derivative of the determinant, this gives

$$\Gamma = \dot{\overline{\det G}} \det G^{-1} = tr(\dot{G}G^{-1}),$$

since ρ_m and $\det F_e$ are strictly positive.
For a simple example, we consider isotropic material, i.e. $G = gI$. In this case, the equation simplifies to the ODE

$$\Gamma = \text{tr } D_G = \text{tr } (\dot{g}Ig^{-1}I) = \frac{3\dot{g}}{g}.$$

Nutrients

We assume nutrients $n\colon \Omega_t \to \mathbb{R}$ to diffuse into the tumour, i.e. to obey to the reaction diffusion equation

$$\partial_t n + \text{div } (nv) - \text{div } (D(n)\nabla n) = -\beta n\rho \text{ in } \Omega_t \tag{2.14}$$

where $D(n)$ is the diffusion coefficient and the RHS describes the nutrients the tumour absorbs in order to grow.
To be consistent with the pulled-back of the equation for the elastic deformation to Ω, we transform the equation for the nutrients to Ω. Let $\Omega' \subset \Omega$ be a measurable subset with Lipschitz boundary and we denote by Ω'_t the set that is covered by all particles from Ω' at time t. Then, the diffusion equation for the part Ω'_t is

$$\int\limits_{\Omega'_t} \partial_t n + \text{div } (nv)\ dV_t - \int\limits_{\Omega'_t} \text{div } (D(n)\nabla n)\ dV_t = -\int\limits_{\Omega'_t} \beta n\rho\ dV_t. \tag{2.15}$$

For the RHS change of variables and (2.10) yield

$$-\int\limits_{\Omega'_t} \gamma n\rho\ dV_t = -\int\limits_{\Omega'} \gamma_m n_m \rho_m J\ dV_0 = -\int\limits_{\Omega'} \beta n_m \rho_0 \det G\ dV_0.$$

Further, the material derivative of n is $\dot{n} = \partial_t n + v \cdot \nabla n = \partial_t n + \text{div } (nv) - n\text{div } v$, see [Gur81]. With this the first integral on the LHS is

$$\begin{aligned}\int\limits_{\Omega'_t} \partial_t n + \text{div } (nv)\ dV_t &= \int\limits_{\Omega'_t} \dot{n} + n\text{div } v\ dV_t\\ &= \frac{d}{dt}\int\limits_{\Omega'_t} n\ dV_t\end{aligned}$$

$$= \frac{d}{dt} \int_{\Omega'} n_m J \; dV_0.$$

For the second equation we used Reynold's transport Theorem 2.2.2 and for the last equation change of variables 2.1.7.
For the second term of (2.15) on the LHS we use the divergence Theorem 2.2.1 with normal ν_t on Ω'_t and change of variables with outward normal ν_0 of Ω' and get

$$\begin{aligned} \int_{\Omega'_t} \text{div } (D(n)\nabla n) \; dV_t &= \int_{\partial P_t} D(n)\nabla n \cdot \nu_t \; dV_t \\ &= \int_{\partial\Omega'_t} D_m(n_m)(\nabla n)_m \cdot JF^{-T}\nu_0 \; dV_0 \\ &= \int_{\partial\Omega'_t} (\nabla n)_m \cdot D_m(n_m)^T JF^{-T}\nu_0 \; dV_0 \\ &= \int_{\partial\Omega'_t} \nabla(n_m) \cdot F^{-1} D_m(n_m)^T JF^{-T}\nu_0 \; dV_0 \\ &= \int_{\partial\Omega'} JF^{-1} D_m(n_m) F^{-T} \nabla n_m \cdot \nu_0 \; dV_0 \\ &= \int_{\Omega'} \text{div } (F^{-1} D_m(n_m) \text{Div } (JF^{-T} n_m)) \; dV_0, \end{aligned} \tag{2.16}$$

where we used $(\nabla n)_m = F^{-T}\nabla(n_m)$, which can be calculated with the definition, see [Gur81]. Finally, with the localisation theorem, the equation

$$\partial_t(n_m J) - \text{div } (F^{-1} D_m(n_m) \text{Div } (JF^{-T} n_m)) = -\beta_m n_m \rho_0 \det G \text{ in } [0,T] \times \Omega \tag{2.17}$$

is obtained.
Here, we can neglect the time derivative, too, since the diffusion is much quicker than the growth, i.e. we assume the nutrients to diffuse immediately compared to the slow growth. Hence,

$$-\text{div } (F^{-1} D_m(n_m) \text{Div } (JF^{-T} n_m)) = -\beta_m n_m \rho_0 \det G \text{ in } [0,T] \times \Omega. \tag{2.18}$$

Lastly, by the calculation (2.16) this is equivalent to

$$-\text{div } (D(n)\nabla n) = -\beta n \rho \text{ in } \Omega_t. \tag{2.19}$$

This is the equation, we consider in Section 4.4.

Constitutive Equations

Motivated by biology, we assume the growth to obey to an equation which may depend on many variables such as the nutrients n, the stress in the body in terms of the second Piola–Kirchhoff stress tensor $S_2 = JF^{-1}TF^{-T}$, the time t and the material point p. Mathematically speaking, the growth tensor $G\colon [0,T] \times \Omega \to \mathbb{R}^{d\times d}$ obeys to the ODE

$$\dot{G}(t,p) = \mathcal{G}(t, p, S_2(t,p), n_m(t,p), G(t,p)) \text{ in } \Omega \times [0,T] \tag{2.20}$$

with a suitable function $\mathcal{G}$. Here, we consider the second Piola–Kirchhoff stress tensor S_2, because G as a deformation tensor is independent of a change of frame, see Remark 2.3.1. Therefore, the terms on the RHS also have to be independent under change of frame, which the second Piola–Kirchhoff stress tensor fulfils.

Remark 2.3.2. *(i) In Chapter 4, we do not need to use the second Piola–Kirchhoff stress tensor, because the independence of observer is automatically fulfilled, see Remark 2.1.6.*

(ii) In the ODE (2.20), the functions of the nutrients n_m and the stress tensor S_2 are considered as functions on the reference configuration Ω. This is necessary, since the growth tensor G is a function on the reference configuration Ω. But as seen in the modelling above, the equation for the nutrients (2.14) is a equation on the deformed configuration Ω_t. Therefore, and due to the fact that the growth map g and elastic deformation ϕ will be good enough, we can pull back the equation for the nutrients or, after solving the equation on Ω_t, the nutrients. Because the equation is better to solve on Ω_t, we will do that in the Section 4.2 and 4.4. Similarly, we solve the elastic problem between the natural configuration Ω_{nat} and deformed configuration Ω_t, and afterwards, we pull it back to the reference configuration Ω to insert it into the ODE.

A more explicit $\mathcal{G}$ depends on the material under consideration and is not discussed here, since it is motivated by biology. Later on, we simplify the ODE to a linear ODE for a proof using time discretization, see Theorem 3.2.5, and by the Picard–Lindelöf theorem, see 4.4.7. Concerning the stress tensor S, we assume the material to be hyperelastic, see Definition 2.1.8. Hence, there exists a strain-energy density function W depending, due to independence of observer, only on $C_e = F_e^T F_e$, in one dimension on $F_e \in (0, \infty)$. If the material is isotropic, W does only depend on the main invariants I_e, II_e, III_e of C_e, i.e.

$$W = W(I_e, II_e, III_e)$$

with

$$\begin{aligned} I_e &= \text{tr } C_e = \text{tr } F_e^T F_e = \|F_e\|^2, \\ II_e &= \frac{1}{2}(\text{tr } (C_e^2) - (\text{tr } C_e)^2), \\ III_e &= \det C_e. \end{aligned}$$

For example an often used strain energy density is the one for the general Blatz-Ko material, which is hyperelastic and compressible. It is given by

$$W = \frac{\nu f}{2}\left((I_e - 3) - \frac{2}{q}(III_e^{q/2} - 1)\right) + \frac{\nu(1-f)}{2}\left(\left(\frac{II_2}{III_e} - 3\right) - \frac{2}{q}(III_e^{-q/2} - 1)\right)$$

with $\nu > 0$, $0 < f \leq 1$ and $q < 0$.

2.4. Some simple examples

In the modelling papers [RHM94] and [AM02], some simple, time-independent examples were introduced, which are discussed here in more detail. First, the situation of a ductal carcinoma is geometrically simplified and the elastic deformation, depending on the growth factor, calculated. Then, the growth of blood vessel is analysed assuming a simplified geometry. Furthermore, the blood vein example is considered a little bit more complicated by considering the vessel wall to consist of two layers.

Example 2.4.1 (Ductal carcinoma in breast duct)**.** *In this first example, we consider a ductal carcinoma in a breast duct, see [AM02]. The breast duct is assumed to be a cylinder with immovable walls. Furthermore, let the tumour have cylindrical shape, too, see the picture.*

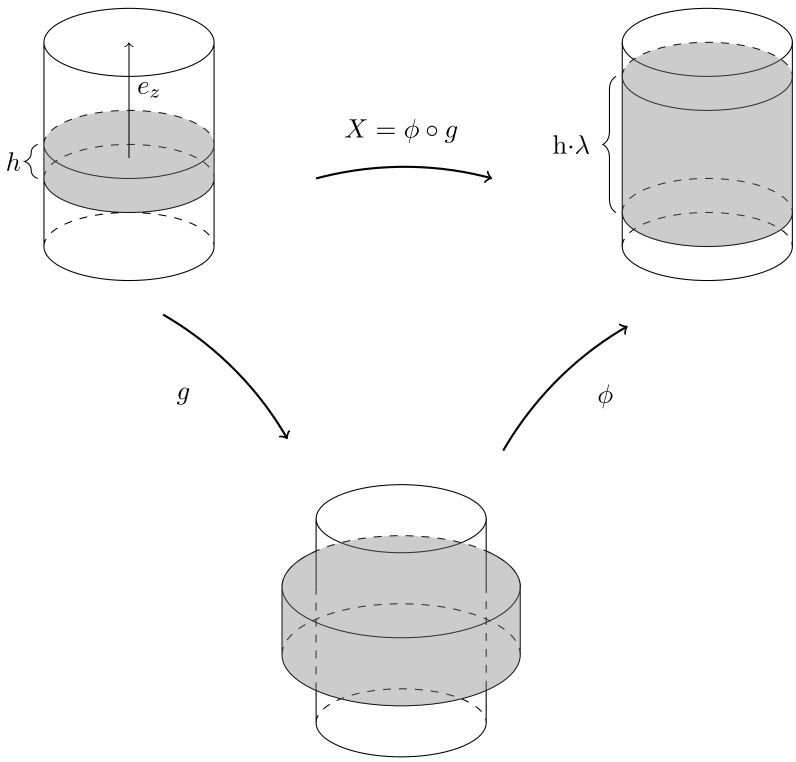

Figure 2.4.: Stress modulated growth of a breast conduct. In the natural configuration the material is more expanded than there is space in the breast duct. Therefore, this configuration is not physically admissible.

For simplicity the material is isotropic, i.e. $G = \tilde{g}I$ *for a constant* $\tilde{g} > 1$ *for a fixed time step, which sets the discussion into the frame of elastostatics. When the tumour grows, it expands in all directions in the same way. Thus, the natural state is the cylinder multiplied by* $\tilde{g}$. *Hence, some parts are outside the breast duct like in the picture (remember that the natural configuration has not to be physically admissible). Since this is physically impossible, the tumour has to be pressed into the cylinder again by the elastic deformation. Due to the incompressibility of the material, the tumour evades in* e_z *direction. Therefore, there exist an* $\lambda \in (0, \infty)$ *with*

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

The goal is to find λ. *Since* G *and* F *are given, it follows from* $F = F_e G$ *that*

$$F_e = \begin{pmatrix} \tilde{g}^{-1} & 0 & 0 \\ 0 & \tilde{g}^{-1} & 0 \\ 0 & 0 & \lambda\tilde{g}^{-1} \end{pmatrix}.$$

We use the assumption that the tumour is a Blatz-Ko material. Choosing $f = 1$ *simplifies* W *to*

$$\begin{aligned} W &= \frac{\nu}{2}\left((I_e - 3) - \frac{2}{q}(III_e^{q/2} - 1)\right) \\ &= \frac{\nu}{2}\left((\|F_e\|^2 - 3) - \frac{2}{q}(\det F_e^q - 1)\right). \end{aligned}$$

Hence,

$$\begin{aligned} T &= \det F_e^{-1} D_{F_e} W(F_e) F_e^T \\ &= \nu \det F_e^{-1} \left(F_e F_e^T - \det F_e^q I \right) \\ &= \nu \frac{g^3}{\lambda} \left(\begin{pmatrix} \tilde{g}^{-2} & 0 & 0 \\ 0 & \tilde{g}^{-2} & 0 \\ 0 & 0 & \lambda^2 \tilde{g}^{-2} \end{pmatrix} - \begin{pmatrix} \lambda^q \tilde{g}^{-3q} & 0 & 0 \\ 0 & \lambda^q \tilde{g}^{-3q} & 0 \\ 0 & 0 & \lambda^q \tilde{g}^{-3q} \end{pmatrix} \right). \end{aligned}$$

At last, the body is stress free in direction of e_z*, otherwise it would deform further in this direction. Due to this,*

$$0 = T_{zz} = \nu \frac{\tilde{g}^3}{\lambda} \left(\frac{1}{\tilde{g}^2} - \frac{\lambda^q}{\tilde{g}^{3q}} \right)$$

which implies

$$\lambda = \tilde{g}^{\frac{2-3q}{2-q}}.$$

Therefore, the elastic deformation can be computed, if the growth is known.

To consider a more complex example, the growth rate is considered to be different in each direction of space other than isochoric growth. To simplify, the growth rate in one direction is set equal to 1, but the following example shows that this simple setting requires more effort.

Example 2.4.2 (Blood vein with constant growth rate)**.** *This example is introduced in [RHM94]. Consider a blood vessel as a ring domain in* $\mathbb{R}^2$*. The goal is to model the elastic deformation after a given growth. We assume the material to be volume preserving* ($\det \nabla\phi = 1$, *see Lemma 2.1.2) and to obey the Guccione model (see below).*

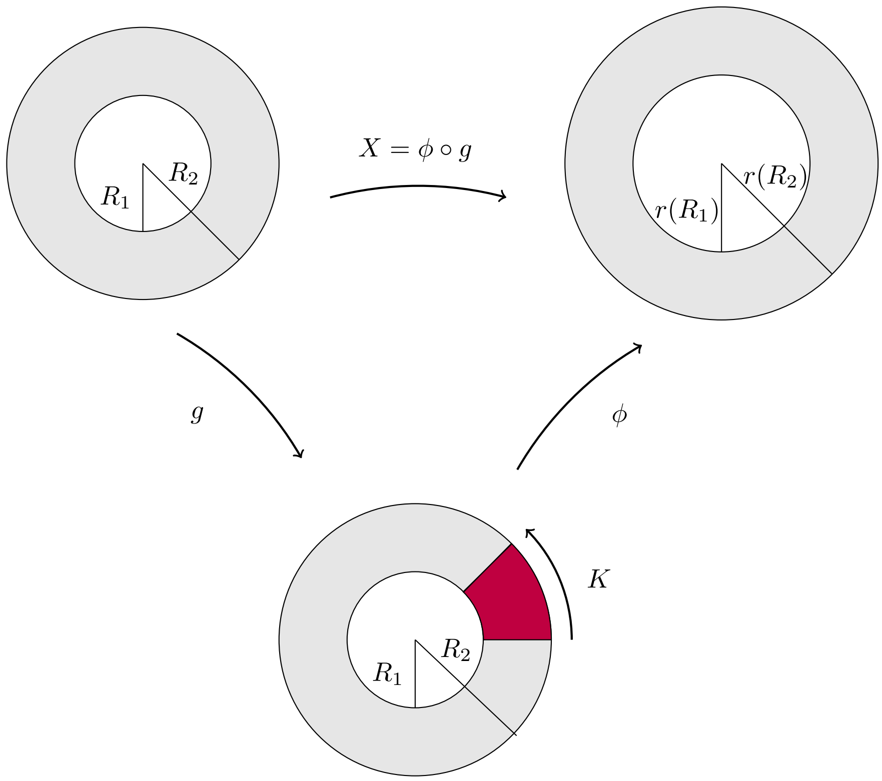

Figure 2.5.: Stress modulated growth in a blood vein with constant growth rate $K > 1$. The part marked red is where the material overlaps itself, what is not physically admissible.

For $0 < R_1 < R_2 < \infty$ define the ring $\Omega := [R_1, R_2] \times [0, 2\pi)$ in polar coordinates. We consider growth in angular direction since growth in radial direction induces no problems, hence, no elastic deformation and no stresses. Consider the growth to be given, namely, for $K > 0$ the growth in angular direction by the factor K is given by the map

$$g \colon \Omega \to \mathbb{R}^2, \ (R, \varphi) \mapsto (R, K\varphi)$$

and name the natural state $\Omega_{nat} := g(\Omega)$. This state is physically not possible, because the material penetrates itself. Thus, in the elastically deformed configuration of the body, the angle has to range from 0 to 2π, again, which is achieved by varying the radius. Here, we assume the material to stay in a perfect ring shape, because otherwise buckling effects can appear for shrinking material. Using the new radius $r \colon [R_1, R_2] \mapsto \mathbb{R}_{>0}$ we model the deformation by $\phi \colon \Omega_{nat} \mapsto \mathbb{R}_{>0} \times [0, 2\pi), (R, \varphi) \mapsto (r(R), \frac{1}{K}\varphi)$, with the elastically deformed configuration $\Omega_t := \phi(g(\Omega)) = [r(R_1), r(R_2)] \times [0, 2\pi)$. The aim is to determine the map r. Then, the total deformation $X \colon \Omega \to \Omega_t$ is determined. To do so, we have to solve the elasticity problem to find ϕ as the solution of

$$\begin{aligned} &\textit{equation of motion:} \quad \operatorname{div} T = 0 \ \textit{in} \ \Omega_t, &\text{(2.21)}\\ &\textit{boundary condition:} \quad Tn = 0 \ \textit{on} \ \partial\Omega_t, &\text{(2.22)}\\ &\textit{material property:} \quad \det \nabla\phi = 1 \ \textit{in} \ \Omega_t, \end{aligned}$$

where $T \colon \Omega_t \to \mathbb{R}^{2\times 2}$ denotes the Cauchy stress tensor and the material is isochoric, see Lemma 2.1.2. The first equation is the equation of motion, the second is the Neumann boundary condition, which states that the blood vessel can move without restrictions from the surrounding tissue, but is stress-free on the boundary, and the last equation is that the material is volume preserving, see lemma 2.1.2. The gradient of the deformation ϕ in polar coordinates is

$$F_e := \nabla u = \begin{pmatrix} \partial_R r(R) & 0 \\ 0 & \frac{r(R)}{RK} \end{pmatrix}$$

and its determinant is $\det \nabla\phi = \partial_R r(R) \frac{r(R)}{RK} = 1$. This is equivalent to $\partial_R r(R) = \frac{RK}{r(R)}$ and the solution of this ODE is

$$r(R) = \sqrt{KR^2 + \alpha} \tag{2.23}$$

for a constant $\alpha \in \mathbb{R}$, which depends on the boundary conditions. Hence, the remaining task is to determine α. Then, the deformation $\phi \circ g$ of Ω under the given growth g is found.
We will deduce α from the equation of motion and boundary conditions. First, we discuss the relation between the Cauchy stress tensor T and the deformation ϕ. Due to the assumption on the material to obey the constitutive equation of the Guccione model ($W(F_e) = \frac{1}{2}(e^Q - 1)$ with $Q = tr(\frac{1}{2}(F_e^T F_e - I))$),the Cauchy stress tensor is

$$\begin{aligned} T =& \det \nabla\phi^{-1} D_F W(\nabla\phi) \nabla\phi^T - pI \\ =& e^Q \nabla\phi\nabla\phi^T - pI = e^Q \begin{pmatrix} \frac{R^2K^2}{r^2(R)} - p(r(R)) & 0 \\ 0 & \frac{r^2(R)}{R^2K^2} - p(r(R)) \end{pmatrix} \end{aligned}$$

where $p \colon \Omega_t \to \mathbb{R}$ is the hydrostatic pressure playing in as Lagrange multiplier. Thus, the equation of motion (2.21) reads in polar coordinates as

$$\begin{aligned} 0 =& (\textit{div } T)_R = \partial_r T_{RR} + \partial_\varphi T_{\varphi R} + \frac{1}{r}(T_{RR} - T_{\varphi\varphi}) = \partial_r T_{RR} + \frac{1}{r}(T_{RR} - T_{\varphi\varphi}) \\ 0 =& (\textit{div } T)_\varphi = \partial_r T_{R\varphi} + \partial_\varphi T_{\varphi\varphi} + \frac{1}{r}(T_{R\varphi} - T_{\varphi R}) = 0. \end{aligned}$$

By integration of the first equation and naming $r(R_1) =: r_1$ we get

$$T_{RR}(r) = \int_{r_1}^{r} \frac{T_{RR}(\tilde{r}) - T_{\varphi\varphi}(\tilde{r})}{\tilde{r}} \, d\tilde{r} + T_{RR}(r_1).$$

Considering the boundary condition (2.22) on the inner boundary $\{r_1\} \times [0, 2\pi)$, it follows that

$$0 = T\nu|_{r=r_1} = -T(r_1, \varphi)e_R = T_{RR}(r_1),$$

which implies the formula

$$T_{RR}(r) = \int_{r_1}^{r} \frac{T_{RR}(\tilde{r}) - T_{\varphi\varphi}(\tilde{r})}{\tilde{r}} \, d\tilde{r}.$$

By substitution it follows

$$\begin{aligned} \tilde{T}_{RR}(R) = T_{RR}(r(R)) &= \int_{r(R_1)}^{r(R)} \frac{T_{RR}(\tilde{r}) - T_{\varphi\varphi}(\tilde{r})}{\tilde{r}} \, d\tilde{r} \\ &= \int_{R_1}^{R} \frac{T_{RR}(r(\tilde{R})) - T_{\varphi\varphi}(r(\tilde{R}))}{r(\tilde{R})} \frac{\tilde{R}K}{r(\tilde{R})} \, d\tilde{R}. \end{aligned}$$

Using the formula for T we conclude

$$\begin{aligned} \tilde{T}_{RR}(R) &= \int_{R_1}^{R} \frac{T_{RR}(r(\tilde{R})) - T_{\varphi\varphi}(r(\tilde{R}))}{r(\tilde{R})} \frac{\tilde{R}K}{r(\tilde{R})} \, d\tilde{R} \\ &= \int_{R_1}^{R} e^{Q} \left(\frac{\tilde{R}^2 K^2}{r^2(\tilde{R})} - \frac{r^2(\tilde{R})}{\tilde{R}^2 K^2} \right) \frac{\tilde{R}K}{r^2(\tilde{R})} \, d\tilde{R}. \end{aligned}$$

By naming $r_2 = r(R_2)$, taking the boundary condition (2.22) on the outer boundary of the ring

$$0 = T\nu|_{r=r_2} = T(r_2, \varphi)\mathbf{e_R} = T_{RR}(r_2) = \tilde{T}_{RR}(R_2),$$

and the formula for $\tilde{T}_{RR}(R)$ the search for α has turned into the search for the zero α of the equation $\tilde{T}_{RR}(R_2) = 0$. In an explicit example this can be solved numerically. With α calculated, T_{RR} and $r(R)$ are determined and by the constitutive equation the hydrostatic pressure p is calculable. Therefore, $T_{\varphi\varphi}$ is determined and the total deformation X of Ω to Ω_t is found.

The previous example considers a homogeneous material, what is not consistent with reality. Hence, the next example gives a first idea of a non-homogeneous material. In our next example, the blood vein consists of two different layers. In a more profound analysis, the goal is to consider more general materials.

Example 2.4.3 (Blood vein with different growth rates)**.** *In the last example the growth had a gradient in the classical sense, and thus, the deformation ϕ as well. Now, we want to study an example without classical gradients. In the following example we consider the growth g as a growth in angular direction, but with two different rates on different domains. That is why g does not have a gradient in the classical sense. In addition, we demand that the two parts stick together after the total process again. Like in the previous example, we do not allow bucking to take place by prescribing the material to stay in a perfect ring shape.*

In more detail, consider for $0 < R_1 < R_2 < R_3 < \infty$ *the domains*

$$\Omega^1 := [R_1, R_2] \times [0, 2\pi),$$
$$\Omega^2 := (R_2, R_3] \times [0, 2\pi),$$
$$\Omega := \Omega^1 \cup \Omega^2 = [R_1, R_3] \times [0, 2\pi).$$

For $K_1, K_2 \in \mathbb{R}_{>0}$ *we consider the growth rate* $K\colon [R_1, R_3] \to \mathbb{R}_{>0}$,

$$K(R) := \begin{cases} K_1, & R \leq R_2, \\ K_2, & R > R_2, \end{cases}$$

and the corresponding growth described by the function

$$g\colon \Omega \to \mathbb{R}^2,\ (R, \varphi) \mapsto (R, K(R)\varphi).$$

This growth function does not have a gradient in the classical sense. Thus, we have to circumvent this.

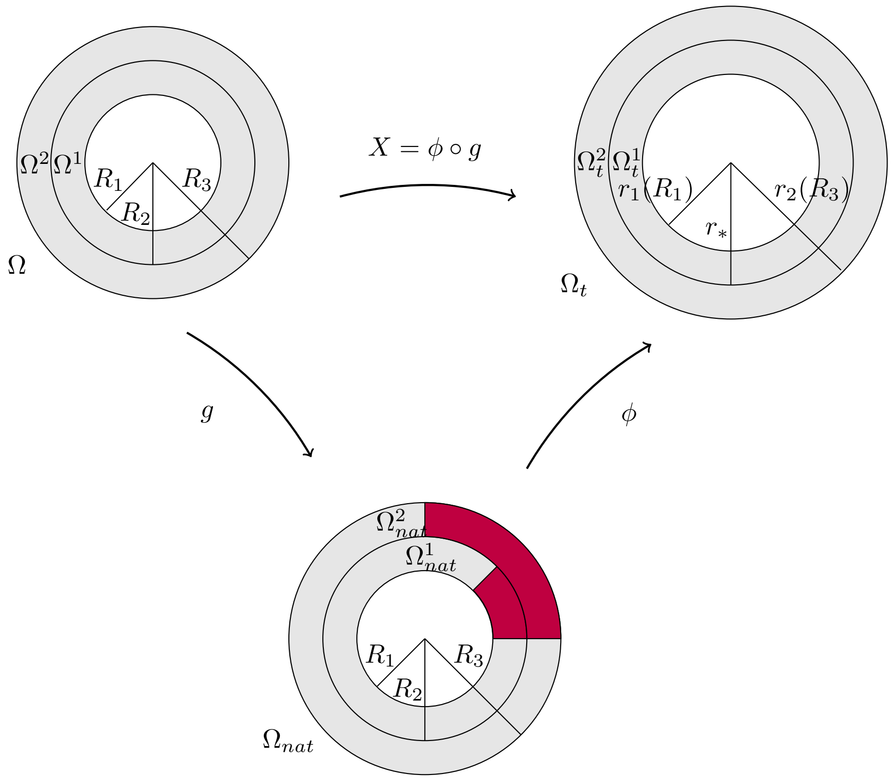

Figure 2.6.: Stress modulated growth in a blood vein with two different growth rates $K_2 > K_1 > 1$. Marked red is where the material penetrates itself, which is why this state is not physically admissible.

First, we introduce some notations: $\Omega^1_{nat} := g(\Omega^1)$, $\Omega^2_{nat} := g(\Omega^2)$ *and* $\Omega_{nat} := \Omega^1_{nat} \cup \Omega^2_{nat}$. *Again, for the elastic deformation it must hold that the angle in the elastically deformed state has to range from 0 to* 2π, *but the radius may vary. In addition, the two domains shall stick together after the whole process where they were together before. Therefore, we need to find a continuous map* $r\colon [R_1, R_3] \to \mathbb{R}_{>0}$ *such that the elastic deformation*

$$\phi\colon \Omega_{nat} \to \mathbb{R}^2,\ (R, \varphi) \mapsto \left(r(R), \frac{1}{K(R)}\varphi \right)$$

obeys to the equation of motion with Neumann boundary conditions for an incompressible material.
We define $\Omega_t^1 := u(\Omega_{nat}^1)$, $\Omega_t^2 := u(\Omega_{nat}^2)$ *and* $\Omega_t := \Omega_t^1 \cup \Omega_t^2$.

Since ϕ *does not have a gradient in the classical sense on* Ω_{nat}, *but on* Ω_{nat}^1 *and* Ω_{nat}^2 *respectively, we can calculate the gradient on* Ω_{nat}^1 *and* Ω_{nat}^2 *separately. With the notation* $r_1\colon [R_1, R_2] \to \mathbb{R}_{>0}$, $r_1(R) = r(R)$ *and* $r_2\colon (R_2, R_3] \to \mathbb{R}_{>0}$, $r_2(R) = r(R)$ *we find*

$$F_e^1(R,\varphi) := \nabla\phi|_{[R_1,R_2]}(R,\varphi) = \begin{pmatrix} \partial_R r_1(R) & 0 \\ 0 & \frac{r_1(R)}{RK} \end{pmatrix} = \partial_R r_1(R) e_R \otimes e_R + \frac{r_1(R)}{RK} e_\phi \otimes e_\phi,$$
$$F_e^2(R,\varphi) := \nabla\phi|_{(R_2,R_3]}(R,\varphi) = \begin{pmatrix} \partial_R r_2(R) & 0 \\ 0 & \frac{r_2(R)}{RK} \end{pmatrix} = \partial_R r_2(R) e_R \otimes e_R + \frac{r_2(R)}{RK} e_\phi \otimes e_\phi.$$

Like in the previous example the material is considered to be isochoric, see Lemma 2.1.2. That means $\det F_e^1 = \det F_e^2 = 1$, *which implies for some* $\alpha_1, \alpha_2 \in \mathbb{R}$

$$r_1(R) = \sqrt{K_1 R^2 + \alpha_1};$$
$$r_2(R) = \sqrt{K_2 R^2 + \alpha_2}.$$

Assuming the two rings to stick together like they did before the growth takes place, for a suitable radius $r_* \in \mathbb{R}$ *the condition*

$$r_1(R_2) = r_2(R_2) = r_*$$

must hold. Hence, α_1 *and* α_2 *depend on* r_* *and the new radii are*

$$r_1(R) = \sqrt{K_1(R^2 - R_2^2) + r_*^2},$$
$$r_2(R) = \sqrt{K_1(R^2 - R_2^2) + r_*^2}.$$

Since r_* *is unknown and the elastic deformation is determined if* r_* *is known, the aim is to find* r_* *such that for the Cauchy stress tensor* $T^i\colon \Omega_t \to \mathbb{R}^{2\times 2}$, $i = 1, 2$, *the equations*

$$div\ T^i = 0 \ \ in\ \Omega_t^i,$$
$$T\nu = 0 \ \ on\ \partial\Omega_t$$

hold. Furthermore, we consider the material to obey to the Guccione model ($W = \frac{1}{2}(e^Q - 1)$ *with* $Q = tr\ (\frac{1}{2}(F^T F - I))$*) and get the Cauchy strain tensor*

$$T^1 = (\det F_e^1)^{-1} D_F W(\nabla\phi)\nabla\phi^T - pI$$
$$= e^Q \nabla\phi\nabla\phi^T - pI = e^Q \begin{pmatrix} \frac{R^2 K_1^2}{r_1^2(R)} - p & 0 \\ 0 & \frac{r_1^2(R)}{R^2 K_1^2} - p \end{pmatrix},$$
$$T^2 = (\det F_e^1)^{-1} D_F W(\nabla\phi)\nabla\phi^T - pI =$$
$$e^Q \nabla\phi\nabla\phi^T - pI = e^Q \begin{pmatrix} \frac{R^2 K_1^2}{r_1^2(R)} - p & 0 \\ 0 & \frac{r_1^2(R)}{R^2 K_1^2} - p \end{pmatrix}.$$

The equations of motion are

$$div\ T^1 = 0 \ \ in\ \Omega_t^1,$$
$$div\ T^2 = 0 \ \ in\ \Omega_t^2.$$

In respect of cylindrical coordinates they transform to

$$0 =(div\ T^1)_r = \partial_r T^1_{rr} + \partial_\varphi T^1_{\varphi r} + \frac{1}{r}(T^1_{rr} - T^1_{\varphi\varphi}) = \partial_r T^1_{rr} + \frac{1}{r}(T^1_{rr} - T^1_{\varphi\varphi})$$
$$0 =(div\ T^1)_\varphi = \partial_r T^1_{r\varphi} + \partial_\varphi T^1_{\varphi\varphi} + \frac{1}{r}(T^1_{r\varphi} - T^1_{\varphi r}) = 0,$$

and by integration like above for $r \in [r_1(R_1), r_*]$

$$T^1_{RR}(r) = \int\limits_{r_1(R_1)}^{r} \frac{T^1_{RR}(\tilde{r}) - T^1_{\varphi\varphi}(\tilde{r})}{\tilde{r}}\ d\tilde{r} + T^1_{RR}(r_1(R_1)).$$

With the boundary condition at the inner boundary we get

$$0 = T^1\nu|_{r=r_1(R_1)} = -T^1(r_1(R_1), \varphi)\mathbf{e_R} = T^1_{RR}(r_1(R_1)),$$

and thus, the formula simplifies to

$$T^1_{RR}(r) = \int\limits_{r_1}^{r} \frac{T^1_{RR}(\tilde{r}) - T^1_{\varphi\varphi}(\tilde{r})}{\tilde{r}}\ d\tilde{r}.$$

To determine a formula for $T^2(r)$ *with* $r \in (r_*, r_2(R_3)]$ *we look at the divergence of* T^2

$$0 =(div\ T^2)_r = \partial_r T^2_{RR} + \partial_\varphi T^2_{\varphi R} + \frac{1}{r}(T^2_{RR} - T^2_{\varphi\varphi}) = \partial_r T^2_{RR} + \frac{1}{r}(T^2_{RR} - T^2_{\varphi\varphi})$$
$$0 =(div\ T^2)_\varphi = \partial_r T^2_{R\varphi} + \partial_\varphi T^2_{\varphi\varphi} + \frac{1}{r}(T^2_{R\varphi} - T^2_{\varphi R}) = 0$$

and integrate the first equation from r *to* $r_2(R_3)$ *with respect to* $\tilde{r}$ *and get*

$$T^2_{RR}(r) = -\int\limits_{r}^{r_2(R_3)} \frac{T^2_{RR}(\tilde{r}) - T^2_{\varphi\varphi}(\tilde{r})}{\tilde{r}}\ d\tilde{r} + T^2_{RR}(r_2(R_3)).$$

Again, with the boundary condition on the outer boundary

$$0 = T^2\nu|_{r=r_2(R_3)} = -T^2(r_2(R_3), \varphi)\mathbf{e_R} = T^2_{RR}(r_2(R_3))$$

the formula for T^2 *simplifies to*

$$T^2_{RR}(r) = -\int\limits_{r}^{r_2(R_3)} \frac{T^2_{RR}(\tilde{r}) - T^2_{\varphi\varphi}(\tilde{r})}{\tilde{r}}\ d\tilde{r}.$$

In addition to the boundary condition on the outer boundary, the forces on the boundary between the two parts of Ω_t *have to be in balance, i.e.*

$$T^1 e_R|_{r=r1(R_2)} = T^2 e_R|_{r=r_2(R_2)},$$

which is equivalent to

$$T^1_{RR}(r_*) = T^2_{RR}(r_*).$$

Due to the formulas for T^1 *and* T^2 *from above, the constitutive equations and by substitution*

($\tilde{r} = r(\tilde{R})$ with $r_* = r_1(R_2) = r_2(R_2)$), we obtain

$$\begin{aligned} 0 &= \int\limits_{r_1(R_1)}^{r_*} \frac{T^1_{RR}(\tilde{r}) - T^1_{\varphi\varphi}(\tilde{r})}{\tilde{r}}\, d\tilde{r} + \int\limits_{r_*}^{r_2(R_3)} \frac{T^2_{RR}(\tilde{r}) - T^2_{\varphi\varphi}(\tilde{r})}{\tilde{r}}\, d\tilde{r} \\ &= \int\limits_{R_1}^{R_2} \frac{T^1_{RR}(r(\tilde{R})) - T^1_{\varphi\varphi}(r(\tilde{R}))}{r(\tilde{R})} \frac{\tilde{R}K}{r(\tilde{R})}\, d\tilde{R} + \int\limits_{R_2}^{R^3} \frac{T^2_{RR}(r(\tilde{R})) - T^2_{\varphi\varphi}(r(\tilde{R}))}{r(\tilde{R})} \frac{\tilde{R}K}{r(\tilde{R})}\, d\tilde{R} \\ &= \int\limits_{R_1}^{R_2} e^Q \left(\frac{\tilde{R}^2 K_1^2}{r^2(\tilde{R})} - \frac{r^2(\tilde{R})}{\tilde{R}^2 K_1^2} \right) \frac{\tilde{R}K_1}{r^2(\tilde{R})}\, d\tilde{R} + \int\limits_{R_2}^{R_3} e^Q \left(\frac{\tilde{R}^2 K^2}{r^2(\tilde{R})} - \frac{r^2(\tilde{R})}{\tilde{R}^2 K^2} \right) \frac{\tilde{R}K}{r^2(\tilde{R})}\, d\tilde{R}. \end{aligned}$$

This equation can be solved numerically for r_*. Then, T^1_{RR}, T^2_{RR}, $r_1(R)$ and $r_2(R)$ are calculable and with them the hydrostatic pressure p, and therefore, $T^1_{\varphi\varphi}$ as well as $T^2_{\varphi\varphi}$ are determined. As a result, the total deformation X of Ω to Ω_t is found.

3 Discussion of Elasticity and Rothe method

In the previous Examples 2.4.1-2.4.3, the growth is considered to be given. Hence, the elastic deformation is time independent and no ODE is solved. This chapter is devoted to the inclusion of time, and thus, to solve the ODE for the growth. To do so, we restrict ourselves to the problem in one dimension. After a discussion whether to consider linear or non-linear elasticity, a first existence result for a material consisting of two bodies with different properties is obtained by the Rothe method. Throughout the chapter, we remark what the challenges are to bring this setting into two dimensions.

3.1. One-dimensional setting and discussion of elasticity

In general, assume a one-dimensional material covering $(0,1) \subset \mathbb{R}$ subdivided into two bodies $(0,\ell)$ and $(\ell,1)$, $\ell \in (0,1)$ with different properties such as growth behaviours and elastic energy densities. Each part is assumed to be homogeneous, see Definition 2.1.9.
As proof of concept, we consider the ODE for the growth tensor to be

$$\begin{aligned} \dot{G}(t,x) &= \gamma(x)G(t,x), \\ G(0,x) &= G_0(x) := 1, \end{aligned}$$

with

$$\gamma(x) = \begin{cases} \gamma_1, & x \in (0,\ell), \\ \gamma_2, & x \in (\ell,1). \end{cases}$$

This leads to the exponential law

$$G(t,x) = \exp(\gamma(x)t), \tag{3.1}$$

which is the gradient of the deformation

$$g(t,x) = \int_0^x G(t,\tilde{x})\, d\tilde{x}. \tag{3.2}$$

The situation after growth is given by the two natural states $(0, \lambda_1)$ and $(\lambda_1, \lambda_1 + \lambda_2)$ with

$$\begin{aligned} \lambda_1 &:= g(t, \ell) - g(t, 0) = \int_0^\ell G(t, \tilde{x})\, d\tilde{x} = \ell \exp(\gamma_0 t), \\ \lambda_2 &:= g(t, 1) - g(t, \ell) = \int_\ell^1 G(t, \tilde{x})\, d\tilde{x} = (1 - \ell) \exp(\gamma_1 t). \end{aligned} \tag{3.3}$$

Strictly speaking, here a decomposition into two parts does not seem to be necessary since growth is not confined by any restrictions. Hence, the decomposition is only introduced as a proof of concept at this point and because it is needed later on.

Remark 3.1.1 (Decomposition of growing materials)**.** *In two or three dimensions, we have to decompose the material, because self-penetration may not be avoidable, e.g. two squares that we need to match together after growth which are incompatible along the common interface. In this case, there is no global gradient.*
Another advantage of our one-dimensional case is that g is injective by construction which must not be true for more dimensions, but it may be enforced by applying a rigid motion, i.e. a shift of parts of the natural configuration (recall that rigid motions have no impact on the elastic energy).

3.1.1. Quadratic elastic potential

We restrict the material to its initial length. Hence, after the material has grown freely, the elastic deformation has to push it back into $(0, 1)$. In the following, the material is considered to be hyperelastic according to Definition 2.1.8 and the notation S is used instead of $\hat{S}$. We start by considering quadratic potentials, which penalize the deviation from the uniform stress free state $\varphi(x) = x$. Namely, for the elastic moduli $\kappa_1, \kappa_2 > 0$ consider the elastic strain energy density $W : (0, 1) \times \mathbb{R} \to \mathbb{R}$ be given in the reference configuration by

$$W(x, F) = \begin{cases} \frac{\kappa_1}{2}(F - 1)^2, & x \in (0, \ell), \\ \frac{\kappa_2}{2}(F - 1)^2, & x \in (\ell, 1). \end{cases}$$

Since the growth map g is known, $W_{nat} \colon (0, \lambda_1 + \lambda_2) \times (0, \infty) \to \mathbb{R}$ is the elastic strain energy density for the grown material and is defined as the elastic strain energy density of the particle before growth by

$$W_{nat}(y, F) := W(g^{-1}(y), F) = \begin{cases} \frac{\kappa_1}{2}(F - 1)^2, & y \in (0, \lambda_1), \\ \frac{\kappa_2}{2}(F - 1)^2, & y \in (\lambda_1, \lambda_1 + \lambda_2). \end{cases} \tag{3.4}$$

Remark 3.1.2. *(i) Here, the elastic strain energy density is considered on the reference configuration. To determine the elastic strain energy density in the natural configuration, take the property $W(g^{-1}(y), \cdot)$ of the particle before the growth g and neglect the change of volume. This is motivated from the assumption that there is more material, but with the same properties. This is consistent with the mathematical modelling. The same holds for the nutrients, see below. Only in the elastic deformation the properties change due to change of volume. To take into account the change of volume for the growth process in this step results in a different model, which we will not consider here.*
Notation: For a function f with domain Ω the subset "nat" denotes the function forwarded to the natural configuration Ω_{nat}, i.e. f_{nat} has the domain Ω_{nat} by the definition $f_{nat}(y) := f(g^{-1}(y))$ for $y \in \Omega_{nat}$.

(ii) Another question is whether the nutrient density n influences the mechanical properties. It does indeed have a huge impact on the growth and, thus, are in included in the ODE in

the modelling, see (2.20). As the nutrients account only for a small part of the material, the difference the nutrients make in the elastic strain energy density W can be neglected compared to the change of material. Hence, W is assumed to be independent of n.

(iii) Note that in one dimension the independence of observer, see Definition 2.1.5, is automatically fulfilled.

With the elastic strain energy density given, the elastic deformation can be determined. To do so, find $\xi \in (0,1)$ and $\phi\colon [0, \lambda_1 + \lambda_2] \to [0,1]$ (notation: $\phi_1 := \phi|_{[0,\lambda_1]}$ and $\phi_2 := \phi|_{[\lambda_1,\lambda_1+\lambda_2]}$) minimizer of

$$\begin{aligned} E_{\lambda_1,\lambda_2}(\phi) &= \int_0^{\lambda_1} W(g^{-1}(y), \partial_y \phi_1(y))\, dy + \int_{\lambda_1}^{\lambda_1+\lambda_2} W(g^{-1}(y), \partial_y \phi_2(y))\, dy \\ &= \int_0^{\lambda_1} \frac{\kappa_1}{2}(\partial_y \phi_1(y) - 1)^2\, dy + \int_{\lambda_1}^{\lambda_1+\lambda_2} \frac{\kappa_2}{2}(\partial_y \phi_2(y) - 1)^2\, dy \end{aligned}$$

subject to the boundary conditions

$$\phi_1(0) = 0,\ \phi_1(\lambda_1) = \phi_2(\lambda_1) = \xi,\ \phi_2(\lambda_1 + \lambda_2) = 1.$$

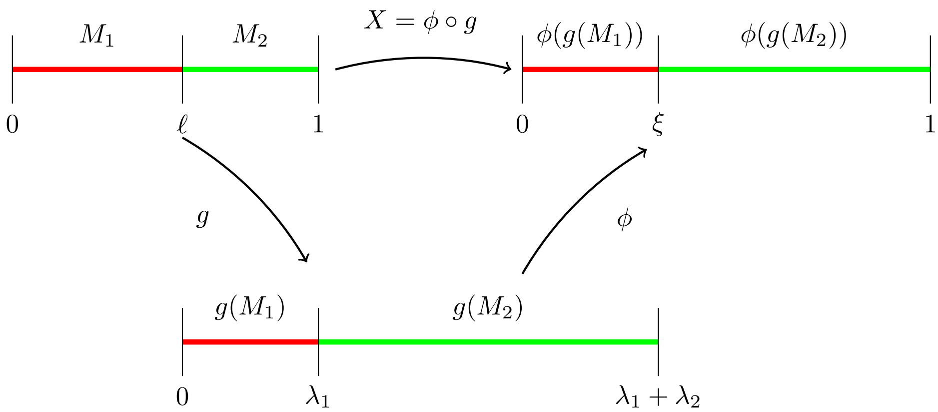

Figure 3.1.: Schematic picture of the growth g and elastic deformation ϕ of a one-dimensional material consisting of two components, marked in red and green respectively.

To continue the discussion, the Euler–Lagrange equation of this energy is needed. In contrast to the quadratic potential, the term of admissible elastic strain energy density is introduced in Definition 3.1.3 below. For those the Theorem A.1.1 states the existence of the Euler–Lagrange equation. Even though the quadratic potential does not fulfil the following definition, the Euler–Lagrange equations are derived similarly to the proof of A.1.1. That is the reason for the introduction of the non-linear elastic strain energy density at this point. For more detail on that topic refer to [Bal81].

Definition 3.1.3 (Admissible elastic strain energy density)**.** *We call $W\colon [0,L] \times (0,\infty) \to \mathbb{R}_{\geq 0}$ an admissible elastic strain energy density, if the following holds:*

(EL1) W is Carathéodory, i.e.

$$\begin{aligned} &W(\cdot, F) \text{ is measurable for all } F \in (0,\infty), \\ &W(x, \cdot) \text{ is continous for all } x \in [0,L]. \end{aligned}$$

Further, let $W(x,\cdot)\colon (0,\infty)\to\mathbb{R}$ be C^2 and strictly convex for all $x\in[0,L]$ and the derivative $D_F W\colon [0,L]\times(0,\infty)\to\mathbb{R}$ be continuous.

(EL2) *Assume that $W(x,1)=0$ and $W(x,F)\to\infty$ for $F\to 0$ and every $x\in[0,L]$. (One defines $W(x,F)=\infty$ for $F\leq 0$.)*

(EL3) *There exist functions $\theta\colon(0,\infty)\to(0,\infty)$ and $a\colon[0,L]\to\mathbb{R}$ with θ convex, $\lim_{F\searrow 0}\theta(F)=\infty$, $a\in L^1(0,L)$ and*

$$\frac{\theta(F)}{F}\to\infty \text{ for } F\to\infty,$$

such that

$$W(x,F)\geq\theta(F)+a(x) \text{ for almost all } x\in[0,L] \text{ and all } F\in(0,\infty).$$

Remark 3.1.4. *(i) The elastic strain energy density W_{nat} defined in (3.4) fulfils (EL1)-(EL3), except for the condition*

$$W_{nat}(y,F)\to\infty \text{ for } F\to 0 \text{ for all } x\in(0,\lambda_1+\lambda_2).$$

The other properties are fulfilled, since W_{nat} is a product of a function in y with two values and a convex C^2-function in F. Further, $W_{nat}(y,1)=0$ holds and

$$W_{nat}(y,F)\geq c\tilde{\theta}(F)=:\theta(F),$$

where $c=\frac{1}{2}\min\{\kappa_1,\kappa_2\}$ and $\tilde{\theta}(F)=(F-1)^2$ with

$$\frac{\theta(F)}{F}=\frac{c(F-1)^2}{F}=c\left(1-\frac{1}{F}\right)\cdot(F-1)\to\infty \text{ for } F\to\infty.$$

(ii) A typical non-linear elastic strain energy density is $\tilde{W}(F)=\frac{1}{2}(F-\frac{1}{F})^2$. It is $C^1(0,\infty)$ as composition of C^2-functions. Furthermore, it holds $\tilde{W}(1)=\frac{1}{2}(1-\frac{1}{1})^2=0$ and

$$\tilde{W}(F)\to\infty \text{ for } F\to 0.$$

To check (EL3), define

$$\theta(F)=\begin{cases}0 & \text{for } F\in(0,1),\\ \tilde{W}(F) & \text{for } F\geq 1\end{cases}$$

and observe that

$$\frac{\tilde{W}(F)}{F}=\frac{(F-\frac{1}{F})^2}{2F}=\left(1-\frac{1}{F^2}\right)\cdot\left(F-\frac{1}{F}\right)\to\infty \text{ for } F\to\infty.$$

(iii) For each W fulfilling the Definition 3.1.3 the following statements hold for each $x\in(0,1)$:

(a) $D_F W(x,\cdot)$ is strictly increasing and, hence, invertible,

(b) $D_F W(x,1)=0$,

(c) $D_F W(x,F)\to-\infty$ for $F\to 0$ and $D_F W(x,F)\to\infty$ for $F\to\infty$.

(iv) The Definition 3.1.3 is for arbitrary length $L>0$, whereas the definitions in the following sections refer to this definition in the case of $L=1$, as we consider the material to cover $(0,1)$ in the beginning of the process and at the end. Nonetheless, the natural configuration $g(t,(0,1))$ is an interval $(0,L(t))$ for a time-dependent $L(t)>0$ for all $t\in[0,T]$ and the elastic problem is solved for each time $t\in[0,T]$ on $(0,L(t))$.

(v) *The definition is no contradiction to the growth condition (G), see Definition 3.1.11 below. See the Example 3.1.12 (i) for W being an admissible elastic strain energy density as well as fulfilling the growth condition (G).*

(vi) *Due to (EL1), $D_F W(x,\cdot)$ exists and is strictly monotonously increasing and due to (EL3) it is bijective from $(0,\infty)$ to $\mathbb{R}$.*

In this general setting as well as in the quadratic elastic energy setting, the minimizer φ of the elastic energy exists, see [Bal81] Theorem 1. Furthermore, it is unique due to strict convexity and fulfils the Euler–Lagrange equation, see the Theorem A.1.1. This theorem cites [Bal81] Theorem 2, proves the statement in more detail and extends it by the case of Neumann boundary conditions.

Remark 3.1.5. *Since the elastic strain energy density W_{nat}, defined in (3.4), does not fulfil all conditions for the Theorem A.1.1, note that the sets Ω_n are only constructed and discussed to interchange the integral and derivative in calculating the Euler–Lagrange equation. But when calculating the Euler–Lagrange equation for W_{nat} evaluating in 0 is not a problem, since $W_{nat}(y,0) = \frac{\kappa(y)}{2}$. Therefore, the Euler–Lagrange equations in Theorem A.1.1(ii) hold for W_{nat}. Furthermore, it does not hold that the derivative of the minimizer is bounded away from 0, see Lemma 3.1.7.*

Applying Theorem A.1.1 to the Dirichlet setting, for a constant $C_D \in \mathbb{R}$ the Euler–Lagrange equation reads as

$$C_D = D_F W_{nat}(y, \partial\phi(y)) = \kappa_{nat}(y) D_F \tilde{W}(\partial\phi(y)) = \kappa(g^{-1}(y)) D_F \tilde{W}(\partial\phi(y)).$$

For ϕ_i, we get

$$\begin{aligned} \frac{C_D}{\kappa_1} &= D_F \tilde{W}(\partial_y \phi_1(y)) \text{ for } y \in (0,\lambda_1), \\ \frac{C_D}{\kappa_2} &= D_F \tilde{W}(\partial_y \phi_2(y)) \text{ for } y \in (\lambda_1, \lambda_1+\lambda_2). \end{aligned} \tag{3.5}$$

Since $\tilde{W}$ is strictly convex, $D_F\tilde{W}$ is strictly monotonously increasing, and therefore, invertible, see Remark 3.1.4(iv). This yields

$$\begin{aligned} \partial_y \phi_1(y) &= (D_F\tilde{W})^{-1}\left(\frac{C_D}{\kappa_1}\right) \in \mathbb{R} \text{ for } y \in (0,\lambda_1), \\ \partial_y \phi_2(y) &= (D_F\tilde{W})^{-1}\left(\frac{C_D}{\kappa_2}\right) \in \mathbb{R} \text{ for } y \in (\lambda_1,\lambda_1+\lambda_2). \end{aligned}$$

Hence, we get the affine linear deformations $\phi_1(y) = \frac{\xi}{\lambda_1} y$ and $\phi_2(y) = \frac{1-\xi}{\lambda_2}(y-\lambda_1)+\xi$. The energy is the quadratic function of ξ:

$$E_{\lambda_1,\lambda_2}(\xi) = \frac{\kappa_1}{2}\lambda_1\left(\frac{\xi}{\lambda_1}-1\right)^2 + \frac{\kappa_2}{2}\lambda_2\left(\frac{1-\xi}{\lambda_2}-1\right)^2$$

having a unique minimizer $\xi \in (0,1)$, because: Differentiating w.r.t. ξ we get

$$0 = \partial_\xi E_{\lambda_1,\lambda_2}(\xi) = \kappa_1\left(\frac{\xi}{\lambda_1}-1\right) - \kappa_2\left(\frac{1-\xi}{\lambda_2}-1\right)$$

and solving for ξ yields

$$\xi = \frac{\kappa_1 - \kappa_2 + \frac{\kappa_2}{\lambda_2}}{\frac{\kappa_1}{\lambda_1} + \frac{\kappa_2}{\lambda_2}}.$$

(In the symmetric case $\kappa_1 = \kappa_2$ and $\lambda_1 = \lambda_2$ is $\xi = 1/2$ as expected.)

Remark 3.1.6. *In higher dimensions a problem concerning injectivity occurs: In elasticity theory the injectivity of the elastic deformation is difficult to ensure. Additionally, we might not solve the problem as easily as in our one-dimensional setting. There are several issues: Self-penetration, complex compatibility conditions inside the material, the natural configuration not being good enough to solve the elasticity problem by standard argumentation or the elastically deformed configuration not being good enough to solve the diffusion problem.*

The question is whether $\xi \in (0,1)$. Otherwise the elastically deformed configuration is not physically admissible, because the interface between the two domains is either a boundary point which corresponds to infinite compression or outside the interval which means interpenetration of matter. Hence, we need to ensure $\xi \in (0,1)$. For $\xi > 0$ we need

$$0 < \kappa_1 - \kappa_2 + \frac{\kappa_2}{\lambda_2} = \kappa_1 + \kappa_2 \left(\frac{1}{\lambda_2} - 1 \right).$$

This is always the case if λ_2, the deformed length of the second part, is less than one, i.e. less than the full interval. Even if the material was almost rigid, a compression of the left material is possible. If $\lambda_2 > 1$, then, the ratio of spring constants κ_1 and κ_2 matters. If $\kappa_2 >> \kappa_1$, then the left material may be pushed out of the domain and the deformation of the right material has a negative derivative.
Depending on the growth constant γ_2 and spring constants κ_1 and κ_2, we can give the explicit time when ξ becomes 0.

Lemma 3.1.7. *In linear elasticity with $\kappa_2 > \kappa_1$ and $\gamma_2 > \gamma_1$ there exists a time $t^* > 0$ such that the elastically deformed configuration is not physically admissible.*

Proof. As discussed before, for $\xi = 0$ the elastically deformed configuration is not physically admissible. Hence, we calculate

$$\begin{aligned} 0 = \xi &= \frac{\kappa_1 - \kappa_2 + \frac{\kappa_2}{\lambda_2}}{\frac{\kappa_1}{\lambda_1} + \frac{\kappa_2}{\lambda_2}} \\ \Leftrightarrow 0 &= \kappa_1 - \kappa_2 + \frac{\kappa_2}{\lambda_2} \\ \Leftrightarrow \lambda_2 &= \frac{\kappa_2}{\kappa_2 - \kappa_1}. \end{aligned}$$

With the definition of λ_2 we continue

$$\begin{aligned} \lambda_2 = (1-\ell)\exp(\gamma_2 t^*) &= \frac{\kappa_2}{\kappa_2 - \kappa_1} \\ \Leftrightarrow t^* &= \frac{1}{\gamma_2} \ln \left(\frac{\kappa_2}{(1-\ell)(\kappa_2 - \kappa_1)} \right) \end{aligned}$$

and get $t^* > 0$, because $\frac{\kappa_2}{(1-l)(\kappa_2-\kappa_1)} > 1$ by the assumptions on κ_1, κ_2 and ℓ. ♣

We get the same result for interchanging the roles of the two materials due to symmetry. This is always the case. Thus, from now on we will consider the situation $0 < \gamma_1 < \gamma_2$.
This example shows the necessity to use an elastic energy which ensures that the interface ξ is at a physically admissible position, i.e. in $(0,1)$. That is why linear elasticity is not sufficient and we will take into account non-linear elastic energy for further considerations.
Finally, we see that the forces at the interface point are in balance. More precisely, for each $t \in [0,T]$ the pressure inside the material is constant: Let $y \in (0, \lambda_1]$, then

$$S_{nat}(t,y) = D_F W(g^{-1}(y), \partial_y \phi(t,y)) = \kappa_1 \left(\frac{\xi}{\lambda_1} - 1 \right)$$

and for $\tilde{y} \in (\lambda_1, \lambda_1 + \lambda_2)$ use the formula for ξ to obtain

$$S_{nat}(t, \tilde{y}) = D_F W(g^{-1}(\tilde{y}), \partial_y \phi(t, \tilde{y})) = \kappa_2 \left(\frac{1-\xi}{\lambda_2} - 1 \right) = \kappa_1 \left(\frac{\xi}{\lambda_1} - 1 \right).$$

Remark 3.1.8. *Here we used that the Piola–Kirchhoff stress tensor S on the reference configuration is equal to the Piola–Kirchhoff stress tensor S_{nat} on the natural configuration due to change of variables. Let $y = g(x)$. Then, it holds with change of variables A.4.2 and the fact that the stress is constant in the material that*

$$\begin{aligned} S_{nat} &= D_F W_{nat}(y, \partial_y \phi(y)) = D_F W(g^{-1}(y), \partial_y \phi(y)) \\ &= \det G D_F W(x, \partial_y \phi(g(x))) G^{-T} = D_F W(x, \partial_y \phi(g(x))) = S, \end{aligned}$$

where

$$\det G = G = G^T$$

holds. In higher dimensions the terms in G do neither commutate nor cancel out. Hence, we have to distinguish between S and S_{nat}, see Chapter 6.

3.1.2. Physical elastic potentials

It was shown in the previous section that interpenetration of matter may occur in a model with quadratic potentials. Therefore, we need to consider elastic energies, which rule out this non-physical behaviour of the material.
In the following we assume that

$$\begin{aligned} &\tilde{W} \colon (0, \infty) \to [0, \infty), \ \tilde{W}(1) = 0, \\ &\tilde{W} \text{ is strictly convex,} \\ &\tilde{W}(F) \to \infty \text{ for } F \to 0 \text{ and } F \to \infty. \end{aligned}$$

Let $\kappa_1, \kappa_2 > 0$ be fixed. Define $W \colon (0,1) \times (0, \infty) \to (0, \infty)$ by

$$W(x, F) = \begin{cases} \kappa_1 \tilde{W}(F), & \text{if } y \in (0, \ell], \\ \kappa_2 \tilde{W}(F), & \text{if } y \in (\ell, 1). \end{cases} \tag{3.6}$$

A typical example is $\tilde{W}(F) = \frac{1}{2}(F - \frac{1}{F})^2$ or $\tilde{W}(F) = (\ln |F|)^2$.
In the framework of the previous section, but with non-linear elastic strain energy density, the elastic energy is now given by

$$E_{\lambda_1, \lambda_2}(\phi) = \int_0^{\lambda_1} \kappa_1 \tilde{W}(\partial_y \phi_1(y)) \, dy + \int_{\lambda_1}^{\lambda_1 + \lambda_2} \kappa_2 \tilde{W}(\partial_y \phi_2(y)) \, dy$$

subject to the boundary conditions

$$\phi_1(0) = 0, \quad \phi_1(\lambda_1) = \phi_2(\lambda_1) = \xi, \quad \phi_2(\lambda_1 + \lambda_2) = 1.$$

The Euler–Lagrange equations are as above, see derivation of (3.5), and hence, we find the same affine linear elastic deformations

$$\begin{aligned} \phi_1(y) &= \frac{\xi}{\lambda_1} y, \\ \phi_2(y) &= \frac{1-\xi}{\lambda_2}(y - \lambda_1) + \xi \end{aligned}$$

and the energy depending on ξ is

$$E_{\lambda_1,\lambda_2}(\xi) = \lambda_1\kappa_1\tilde{W}\left(\frac{\xi}{\lambda_1}\right) + \lambda_2\kappa_2\tilde{W}\left(\frac{1-\xi}{\lambda_2}\right). \tag{3.7}$$

Due to the strict convexity of $\tilde{W}$ and $E_{\lambda_1,\lambda_2}(s) \to \infty$ for $s \to 0$ and $s \to 1$, there exists a unique minimizer $\xi \in (0,1)$. The necessary condition of a minimizer is

$$0 = \partial_\xi E_{\lambda_1,\lambda_2}(\xi) = \kappa_1 D_F\tilde{W}\left(\frac{\xi}{\lambda_1}\right) - \kappa_2 D_F\tilde{W}\left(\frac{1-\xi}{\lambda_2}\right). \tag{3.8}$$

Similar to the case of linear elasticity at the interface point holds balance of forces (see Remark 3.1.8). In fact, we calculate for $y \in (0,\lambda_1]$

$$S(t,y) = \kappa_1 D_F\tilde{W}(\partial_y\phi_1(t,y)) = \kappa_1 D_F\tilde{W}\left(\frac{\xi}{\lambda_1}\right)$$

and for $\tilde{y} \in (\lambda_1, \lambda_1+\lambda_2)$ using (3.8)

$$S(t,\tilde{y}) = \kappa_2 D_F\tilde{W}(\partial_y\phi(t,y)) = \kappa_2 D_F\tilde{W}\left(\frac{1-\xi}{\lambda_2}\right) = \kappa_1 D_F\tilde{W}\left(\frac{\xi}{\lambda_1}\right). \tag{3.9}$$

As we already discussed the existence of ξ, we are also interested in the dependence of ξ on λ_0 and λ_1.

Lemma 3.1.9 (Dependence of the interface of the length)**.** *Suppose the Setting 3.2.1 to hold. Then, there exists a* C^1*-map* $\xi\colon (0,\infty)^2 \to (0,1)$, $(\lambda_1,\lambda_2) \mapsto \xi(\lambda_1,\lambda_2)$ *such that for all* $\lambda_1,\lambda_2 \in (0,\infty)^2$ $\xi(\lambda_1,\lambda_2)$ *minimizes the elastic energy* E_{λ_1,λ_2} *from (3.7).*

Proof. To get the desired result we use the implicit function theorem, see Theorem A.3.5,. We use the Theorem A.3.5 for $n=2, m=1$, $U=[0,\infty)^2$, $V=\mathbb{R}$ and $f\colon [0,\infty)^2 \times \mathbb{R}, (\lambda_1,\lambda_2,\xi) \mapsto f(\lambda_1,\lambda_2,\xi) = \kappa_1 D_F\tilde{W}(\frac{\xi}{\lambda_1}) - \kappa_2 D_F\tilde{W}(\frac{1-\xi}{\lambda_2})$. Let $(\lambda_1^*,\lambda_2^*) \in (0,\infty)^2$ be arbitrary. Due to the uniqueness discussed above, there exists a unique $\xi^* \in (0,1)$ with $F(\lambda_1^*,\lambda_2^*,\xi^*) = 0$. Furthermore,

$$\partial_\xi f(\lambda_1^*,\lambda_2^*,\xi^*) = \kappa_1 D_F^2\tilde{W}\left(\frac{\xi^*}{\lambda_1^*}\right)\frac{1}{\lambda_1^*} + \kappa_2 D_F^2\tilde{W}\left(\frac{1-\xi^*}{\lambda_2^*}\right)\frac{1}{\lambda_2^*} > 0$$

due to strict convexity. Hence, $\partial_\xi f(\lambda_1^*,\lambda_2^*,\xi^*)$ is invertible. The implicit function theorem gives the existence of neighbourhoods $U_0 \subset [0,\infty)^2$ of $(\lambda_1^*,\lambda_2^*)$, $V_0 \subset \mathbb{R}$ of ξ^* and a continuously differentiable map $\xi\colon U_0 \to V_0, (\lambda_1,\lambda_2) \mapsto \xi(\lambda_1,\lambda_2)$ with $\xi(\lambda_1^*,\lambda_2^*) = \xi^*$ and $f(\lambda_1,\lambda_2,\tilde{\xi}) = 0$ if and only if $\xi(\lambda_1,\lambda_2) = \tilde{\xi}$.
Since $(\lambda_1^*,\lambda_2^*)$ was arbitrary in $(0,\infty)^2$, we get a map $\xi\colon (0,\infty)^2 \to \mathbb{R}$ with the same properties. Due to the uniqueness of the minimizer of the energy and its property to be in $(0,1)$, the image of ξ is contained in $(0,1)$. ♣

In Chapter 4, estimates on the interface point are needed. We can prove what is physically expected: The interface point, as we have shown, depends on the grown lengths in the following way: If λ_2 becomes bigger, that is, the right material is bigger, the interface point moves to the left. On the other hand, if λ_2 is smaller, the interface point moves to the right. For λ_1, it holds the other way around: If λ_1 is smaller, the interface point moves to the left and for bigger λ_2 to the right.

Lemma 3.1.10 (Monotonicity of the interface point)**.** *Assume Setting 3.2.1 to hold. Let* $0 < \lambda_1^- < \lambda_1^+ < \infty$ *and* $0 < \lambda_2^- < \lambda_2^+ < \infty$ *be given. Then, it holds for the map* ξ *from Lemma 3.1.9 for* $\lambda_1 \in (\lambda_1^-,\lambda_1^+)$ *and* $\lambda_2 \in (\lambda_2^-,\lambda_2^+)$

$$0 < \xi(\lambda_1^-,\lambda_2^+) \le \xi(\lambda_1,\lambda_2) \le \xi(\lambda_1^+,\lambda_2^-) < 1.$$

Proof. Consider the following two natural states:

(i) The two parts have grown to the new stress-free lengths λ_1 and λ_2 respectively and $\xi := \xi(\lambda_1, \lambda_2)$ is the equilibrium of the interface point.

(ii) The two parts have grown to the new stress-free length λ_1 and $\lambda_2 + \Delta\lambda$ respectively with $\Delta\lambda$ small.

We investigate whether ξ is still an optimal interface in (ii) or whether ξ is not optimal as soon as $\Delta\lambda \neq 0$. Furthermore, we investigate the behaviour of the elastic energy for a slightly perturbed interface point to obtain the statement.
Since ξ is the optimal interface point, it holds

$$0 = \partial_\xi E_{\lambda_1,\lambda_2}(\xi) = \frac{\kappa_1}{2} D_F \tilde{W}\left(\frac{\xi}{\lambda_1}\right) - \frac{\kappa_2}{2} D_F \tilde{W}\left(\frac{1-\xi}{\lambda_2}\right). \tag{3.10}$$

Let $\delta > 0$ be a small perturbation of the interface point. We need to compare the energies

$$E_{\lambda_1,\lambda_2}(\xi) = \frac{\lambda_1\kappa_1}{2}\tilde{W}\left(\frac{\xi}{\lambda_1}\right) + \frac{\lambda_2\kappa_2}{2}\tilde{W}\left(\frac{1-\xi}{\lambda_2}\right)$$

and

$$E_{\lambda_1,\lambda_2+\Delta\lambda}(\xi - \delta) = \frac{\lambda_1\kappa_1}{2}\tilde{W}\left(\frac{\xi-\delta}{\lambda_1}\right) + \frac{(\lambda_2+\Delta\lambda)\kappa_2}{2}\tilde{W}\left(\frac{1-\xi+\delta}{\lambda_2+\Delta\lambda}\right).$$

If ξ is the equilibrium interface point in situation (i), then, the variation of the energy $E_{\lambda_1,\lambda_2+\Delta\lambda}$ in ξ would vanish for $\Delta\lambda = 0$. This derivative is given by

$$\begin{aligned}\partial_\delta E_{\lambda_1,\lambda_2+\Delta\lambda}(\xi-\delta)|_{\delta=0} &= -\frac{\kappa_1}{2} D_F \tilde{W}\left(\frac{\xi}{\lambda_1}\right) + \frac{\kappa_2}{2} D_F \tilde{W}\left(\frac{1-\xi}{\lambda_2+\Delta\lambda}\right) \\ &= -\partial_\xi E_{\lambda_1,\lambda_2+\Delta\lambda}(\xi).\end{aligned} \tag{3.11}$$

It can not vanish for $\Delta\lambda \neq 0$ due to the strict convexity of W: We use (3.11) and add the optimality condition (3.10) of ξ in situation (i) and get

$$\begin{aligned}&-\partial_\xi E_{\lambda_1,\lambda_2+\Delta\lambda}(\xi) = 0 + \partial_\delta E_{\lambda_1,\lambda_2+\Delta\lambda}(\xi-\delta)|_{\delta=0} \\ &= \partial_\xi E_{\lambda_1,\lambda_2}(\xi) + \partial_\delta E_{\lambda_1,\lambda_2+\Delta\lambda}(\xi-\delta)|_{\delta=0} \\ &= \frac{\kappa_1}{2} D_F \tilde{W}\left(\frac{\xi}{\lambda_1}\right) - \frac{\kappa_2}{2} D_F \tilde{W}\left(\frac{1-\xi}{\lambda_2}\right) - \frac{\kappa_1}{2} D_F \tilde{W}\left(\frac{\xi}{\lambda_1}\right) + \frac{\kappa_2}{2} D_F \tilde{W}\left(\frac{1-\xi}{\lambda_2+\Delta\lambda}\right) \\ &= -\frac{\kappa_2}{2} D_F \tilde{W}\left(\frac{1-\xi}{\lambda_2}\right) + \frac{\kappa_2}{2} D_F \tilde{W}\left(\frac{1-\xi}{\lambda_2+\Delta\lambda}\right) \\ &= \frac{\kappa_2}{2} D_F^2 \tilde{W}(\eta)\left(\frac{1-\xi}{\lambda_2+\Delta\lambda} - \frac{1-\xi}{\lambda_2}\right) \\ &= -\frac{\kappa_2}{2} D_F^2 \tilde{W}(\eta)\frac{(1-\xi)\Delta\lambda}{(\lambda_2+\Delta\lambda)\lambda_2},\end{aligned} \tag{3.12}$$

where we used the mean value theorem to find a suitable $\eta \in [\frac{1-\xi}{\lambda_2+\Delta\lambda}, \frac{1-\xi}{\lambda_2}]$ for $\Delta\lambda > 0$ and $\eta \in [\frac{1-\xi}{\lambda_2}, \frac{1-\xi}{\lambda_2+\Delta\lambda}]$ for $\Delta\lambda < 0$ respectively. Since $\tilde{W}$ is strictly convex, $D_F^2\tilde{W}(\eta) > 0$ holds. In addition, the fraction only becomes 0 if $\Delta\lambda = 0$, because $\xi \in (0,1)$. Hence, ξ is the optimal interface point if and only if $\Delta\lambda = 0$.
The question is how the energy behaves in δ. Consider Taylor's expansion

$$E_{\lambda_1,\lambda_2+\Delta\lambda}(\xi-\delta) = E_{\lambda_1,\lambda_2+\Delta\lambda}(\xi) - \partial_\xi E_{\lambda_1,\lambda_2+\Delta\lambda_2}(\xi)\delta + O(\delta^2).$$

The negative of the first derivative on the left hand side is equal to the last term in (3.12). Hence, we have the following statements: If $\Delta\lambda > 0$ the sign is negative, that is, the function

$E_{\lambda_1,\lambda_2+\Delta\lambda}(\xi-\delta)$ is decreasing in δ. That means the energy is smaller if δ is positive, and hence, the interface is shifted to the left. If $\Delta\lambda < 0$, then, the sign is positive, that is, the function $E_{\lambda_1,\lambda_2+\Delta\lambda}(\xi-\delta)$ is increasing in δ. That means the energy is smaller if δ is negative, and therefore, the interface is shifted to the right.
A similar behaviour can be shown by comparing (i) and the situation

(iii) The two parts have grown to the new stress-free length $\lambda_1 + \Delta\lambda$ and λ_2 with $\Delta\lambda$ small.

The argumentation from above implies

$$\xi(\lambda_1, \lambda_2^+) \leq \xi(\lambda_1, \lambda_2) \leq \xi(\lambda_1, \lambda_2^-)$$

and comparison with (iii) implies the desired result

$$\xi(\lambda_1^-, \lambda_2^+) \leq \xi(\lambda_1, \lambda_2^+) \leq \xi(\lambda_1, \lambda_2) \leq \xi(\lambda_1, \lambda_2^-) \leq \xi(\lambda_1^+, \lambda_2^-).$$

♣

Again the question arises, if ξ might become 0. To give an answer we first discuss a growth condition on the elastic strain energy density.

Definition 3.1.11 (Growth condition (G) for the elastic strain energy density)**.** *We say that a $\tilde{W} \in C^2((0,\infty))$ satisfies the growth condition (G), if the following holds: There exists an $F_0 \in (0,1)$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $a, b \in (0, F_0)$ with $\frac{a}{b} < \delta$ holds $|\frac{D_F\tilde{W}(b)}{D_F\tilde{W}(a)}| < \varepsilon$.*

Example 3.1.12. *(a) Standard case: The standard case is a multiple of $\tilde{W}(F) = (F - \frac{1}{F})^2$ with derivative*

$$D_F\tilde{W}(F) = 2\left(F - \frac{1}{F}\right)\left(1 + \frac{1}{F^2}\right).$$

For $\varepsilon > 0$ and $F_0 \in (0,1)$ define $\delta := \sqrt[3]{\frac{\varepsilon(1-F_0^2)}{2}}$ and let $0 \leq a, b < F_0$ be arbitrary with $\frac{a}{b} < \delta$. Then, we calculate

$$\frac{D_F\tilde{W}(b)}{D_F\tilde{W}(a)} = \frac{\left(b - \frac{1}{b}\right)\left(1 + \frac{1}{b^2}\right)}{\left(a - \frac{1}{a}\right)\left(1 + \frac{1}{a^2}\right)} = \frac{a^3}{b^3} \cdot \frac{\left(1 - b^2\right)\left(b^2 + 1\right)}{\left(1 - a^2\right)\left(a^2 + 1\right)} \leq \frac{a^3}{b^3}\frac{2}{1 - F_0^2} < \delta^3 \frac{2}{1 - F_0^2} = \varepsilon.$$

(b) Logarithmic potential: Consider $\tilde{W}(F) = (\ln|F|)^2$ with derivative

$$D_F\tilde{W}(F) = 2\frac{\ln|F|}{F}.$$

For $\varepsilon > 0$ and $F_0 \in (0,1)$ define $\delta := \min\{1, \varepsilon\}$. Further, let $0 < a, b < F_0$ be arbitrary with $\frac{a}{b} < \delta$. Then, we obtain

$$\frac{D_F\tilde{W}(b)}{D_F\tilde{W}(a)} = \frac{a}{b} \cdot \frac{\ln(b)}{\ln(a)} < \delta \cdot \frac{\ln(b)}{\ln(\frac{a}{b}) + \ln(b)} < \delta \cdot \frac{\ln(b)}{\ln(b)} < \delta \leq \varepsilon.$$

(c) A counterexample for the growth condition (G) is $W(F) = \frac{1}{2}(F-1)^2$. Set $\varepsilon = \frac{1}{2}$. Assume there exist $F_0 > 0$ and $\delta > 0$ such that for all $a, b \in (0, F_0)$ with $\frac{a}{b}$, it holds $|\frac{D_F W(b)}{D_F W(a)}| < \varepsilon$. WloG, $\delta < 1$. Choose $a = \min\{\frac{\delta}{8}, \frac{\delta F_0}{4}\}$, $b = \min\{\frac{1}{4}, \frac{F_0}{2}\}$. Then $a, b \in (0, F_0)$ and

$$\frac{a}{b} = \frac{\min\{\frac{\delta}{8}, \frac{\delta F_0}{4}\}}{\min\{\frac{1}{4}, \frac{F_0}{2}\}} = \frac{\delta}{2} < \delta.$$

In addition,

$$\varepsilon > |\frac{D_F W(b)}{D_F W(a)}| = |\frac{b-1}{a-1}| = \frac{1-b}{1-a} > 1-b = 1-\min\{\frac{1}{4}, \frac{F_0}{2}\} = \max\{\frac{3}{4}, 1-\frac{F_0}{2}\} \geq \frac{3}{4}.$$

This shows that the growth condition (G) does not allow the case of linear elasticity.

Under the growth condition (G), the following physically motivated statement holds: In the setting $\gamma_2 > \gamma_1$, the right interval grows faster than the left one and it is expected that the right interval will grow and deform at the expense of the left interval, i.e. $\xi(t) \to 0$ for $t \to \infty$.

Proposition 3.1.13 (Long time behaviour of the interface point)**.** *Assume the Setting 3.2.1 holds and $\tilde{W}$ fulfils the growth condition (G). Then, it holds for the grown lengths λ_1, λ_2 from (3.3) and the map ξ from Lemma 3.1.9 that*

$$\xi(t) = \xi(\lambda_1(t), \lambda_2(t)) \to 0 \text{ for } t \to \infty.$$

Proof. We prove the statement by contradiction. Assume there exists a $\xi_0 > 0$ such that for all $t \in [0, \infty)$ it holds $\xi_0 \leq \xi(t)$. It is equivalent to $1 - \xi(t) \leq 1 - \xi_0$ for all $t \in [0, \infty)$. We recall that

$$0 = \kappa_1 D_F \tilde{W}\left(\frac{\xi}{\lambda_1}\right) - \kappa_2 D_F \tilde{W}\left(\frac{1-\xi}{\lambda_2}\right)$$

holds, which is equivalent to

$$D_F \tilde{W}\left(\frac{\xi}{\lambda_1}\right) = \frac{\kappa_1}{\kappa_2} D_F \tilde{W}\left(\frac{1-\xi}{\lambda_2}\right). \tag{3.13}$$

Furthermore, the growth condition on $\tilde{W}$ states that for all $\varepsilon > 0$ a suitable $F_0 \in (0,1)$ and $\delta > 0$ exist such that for $a, b \in (0, F_0)$ with $\frac{a}{b} < \delta$ holds

$$D_F \tilde{W}(b) \leq D_F \tilde{W}(a).$$

Hence, define $a := \frac{1-\xi}{\lambda_2}$ and $b := \frac{\xi}{\lambda_1}$. Due to $\lambda_1 = \ell \exp(\gamma_1 t)$, $\lambda_2 = (1-\ell)\exp(\gamma_2 t)$ and $\xi \in (0,1)$, we get

$$\begin{aligned} b &= \frac{\xi}{\lambda_1} = \ell \exp(-\gamma_1 t) \to 0 \\ a &= \frac{1-\xi}{\lambda_2} \leq (1-\ell)\exp(-\gamma_2 t) \to 0, \\ \frac{a}{b} &= \frac{\ell}{1-\ell} \exp((\gamma_1 - \gamma_2)t) \to 0 \end{aligned}$$

for $t \to \infty$. Choose $\varepsilon < \frac{\kappa_1}{\kappa_2}$ and δ and F_0 according to the growth condition (G) for W. Then, there exists a $t_0 > 0$ such that $a, b < F_0$ and $\frac{a}{b} < \delta$ for all $t > t_0$. Thus, (3.13) is a contradiction to the growth condition (G). ♣

3.2. Rothe method for material consisting of ...

Next, we include stresses into our model, since large compressive stresses inhibit growth, while large tensile strains favour growth in many soft tissues. Thus, we seek a function $\mu\colon \mathbb{R} \to \mathbb{R}, s \mapsto \mu(s)$ that is bounded. In addition, we assume $\mu(0) = 1$, e.g.

$$\mu(s) = 1 + \arctan(s).$$

The function μ plays the role of an amplification factor and the growth rate is given by the product $\gamma(x)\mu(S(t))$ in the material point x at time t, where S is the Piola–Kirchhoff stress tensor. Recall that the second Piola-Kirchhoff stress tensor is not needed, see Remark 2.3.2. The ODE for the growth is then given as

$$\dot{G}(t,x) = \mathcal{G}(t,x,G(t,x),S(t)) = \gamma(x)\mu(S(t))G(t,x)$$

for $t \in [0,T]$ and $x \in (0,1)$.
Note that this ODE is a special ODE, since it is a linear ODE including the stress tensor in the term $\mu(S(t))$. Furthermore, it does not include the nutrients. Later on, we address a general ODE including the stress, see Section 4.5, and the nutrients, see Sections 4.2 and 4.4.

3.2.1. ... two parts

In this thesis, there are two fundamentally different proofs for the existence of the model of stress modulated growth stated. One is by a time discretization and one by using the Picard–Lindelöf theorem. In this section the first mentioned is presented.
Before discussing existence, the setting and the equations have to be specified. Therefore, the following two definitions clarify the setting and the expression to be a solution of the AMP for two materials.

Definition 3.2.1 (Setting with two materials)**.** *Let $\ell \in (0,1)$ be fixed. Define the function spaces*

$$\begin{aligned} S^0 :=& \{f\colon (0,1) \to \mathbb{R} |\ f \textit{ is constant on } (0,\ell] \textit{ and } (\ell,1)\}, \\ S^1 :=& \{f\colon (0,1) \to \mathbb{R} |\ f \textit{ affine linear on } (0,l] \textit{ and } [l,1),\ f \textit{ continuous}\}. \end{aligned}$$

Let $\gamma_1, \gamma_2 > 0$ be the growth rates and $\kappa_1, \kappa_2 > 0$ the stress moduli. Define

$$\gamma\colon [0,1] \to \mathbb{R},\ \gamma(x) := \begin{cases} \gamma_1, & \textit{if } x \le \ell, \\ \gamma_2, & \textit{if } x > \ell, \end{cases}$$

and

$$\kappa\colon [0,1] \to \mathbb{R},\ \kappa(x) := \begin{cases} \kappa_1, & \textit{if } x \le \ell, \\ \kappa_2, & \textit{if } x > \ell. \end{cases}$$

Then, $\gamma, \kappa \in S^0$. Let $\mu \in \mathrm{C}^0(\mathbb{R})$ be Lipschitz continuous. Choose $-\infty < (\gamma\mu)_{min} < 0 < (\gamma\mu)_{max} < \infty$ with

$$\begin{aligned} (\gamma\mu)_{max} &\ge \operatorname*{ess\,sup}_{x\in(0,1),y\in\mathbb{R}} \{\gamma(x)\mu(y)\} \\ (\gamma\mu)_{min} &\le \operatorname*{ess\,inf}_{x\in(0,1),y\in\mathbb{R}} \{\gamma(x)\mu(y)\}. \end{aligned}$$

Let $\tilde{W} \in \mathrm{C}^2(\mathbb{R}_{>0})$ be strictly convex with $\tilde{W}(1) = 0$, $W(F) \to \infty$ for $F \to 0$ and

$$\frac{\tilde{W}(F)}{F} \to \infty \textit{ for } F \to \infty.$$

The elastic strain energy density W is then defined by

$$W : (0,1) \times \mathbb{R} \to \mathbb{R},\ W(x,F) = \kappa(x)\tilde{W}(F).$$

Further, let $p \in (1,\infty)$ and for a bijective growth map $g\colon [0,1] \to \mathbb{R}$, define the admissible set of elastic deformation $\mathcal{A}_{two}$ by

$$\mathcal{A}_{two} := \{v \in W^{1,p}(g((0,1)))|\ v(g(0)) = 0,\ v(g(1)) = 1\}$$

and the elastic energy E_{two} *of a deformation* $\phi \in \mathcal{A}_{two}$ *by* $E_{two}\colon \mathcal{A}_{two} \to \mathbb{R}$,

$$E_{two}(\phi) \coloneqq \int_0^{\lambda_1} \kappa_1 \tilde{W}(\partial_y \phi(y))\ dy + \int_{\lambda_1}^{\lambda_1+\lambda_2} \kappa_2 \tilde{W}(\partial_y \phi(y))\ dy.$$

Finally, let $T > 0$ *be the time horizon and* $G_0 = 1 \in S^0$ *the initial datum for the ODE for the growth tensor.*

Remark 3.2.2. *(i) In the following, the initial datum is assumed to be 1, but the arguments also hold for initial data* $G_0 \in S^0$ *(or the according spaces in the later sections), if there exists a constant* $c > 0$ *such that*

$$G_0(x) \geq c \text{ for all } x \in (0,1).$$

Similarly, this can be assumed for all $x \in [0,1]$ *in the* C^0*-setting 4.3.1, for almost all* $x \in (0,1)$ *in the generalized elastic energy density setting 4.4.1 and in the higher regularity setting 5.1.1. This condition is crucial for the growth map to be physically admissible.*

(ii) We assume $(\gamma\mu)_{min} < 0$*, because we do not need to distinguish between* $(\gamma\mu)_{min}$ *being positive or negative in calculations. This also applies for* $(\gamma\mu)_{max} > 0$ *and for the other settings, see Settings 3.2.8, 4.3.1, 4.4.1, 5.1.1.*

Definition 3.2.3 (Definition of AMP with two materials)**.** *Assume Setting 3.2.1 to hold. A couple* $(G,S) \in \mathrm{C}^1([0,T];S^0) \times \mathrm{C}^0([0,T])$ *is called a solution of the AMP with two materials if the following is fulfilled:*

(i) The growth tenor G *fulfils the ODE*

$$\begin{aligned} \dot{G}(t,x) &= \gamma(x)\mu(S(t))G(t,x), \\ G(0,x) &= 1, \end{aligned}$$

for all $t \in [0,T]$ *and almost all* $x \in (0,1)$.

(ii) For

$$g(t,x) \coloneqq \int_0^x G(t,\tilde{x})\ d\tilde{x}$$

and for given $t \in [0,T]$ *let* $\phi(t,\cdot) : g((0,1)) \to \mathbb{R}$ *be the unique minimizer of* E_{two} *in* $\mathcal{A}_{two}$*. The Piola–Kirchhoff stress tensor to the elastic deformation* $\phi(t,\cdot)$ *is* S *by*

$$S(t) = D_F W(g^{-1}(y), \partial_y \phi(t,y)).$$

Remark 3.2.4. *(i) Here,* $\phi(t,\cdot)$ *is unique due to the strict convexity of* W *in* F *and injective due to Theorem A.1.1, see [Bal81].*

(ii) Note that Dirichlet boundary conditions are considered in this setting. In the Neumann boundary condition, the elastic deformation is the identity, since it fulfils the boundary conditions and with gradient 1, it has energy 0 which is the absolute minimum. Furthermore, the stress is constant in time and the ODE simplifies to a standard ODE. Hence, the more interesting case is the setting with Dirichlet boundary conditions.

Now, the main theorem can be proved. Note that there is a further condition, namely the growth condition (G). This is needed to use the lemmas stating properties of the interface point ξ, see Lemmas 3.1.9 and 3.1.10.
Note that the following result does not state uniqueness, because the Arzelà–Ascoli theorem is

applied and states the existence of a converging subsequence, but not uniqueness for the limit. Therefore, no uniqueness is obtained at this point. For uniqueness statements see Chapter 4.

Theorem 3.2.5 (Existence for AMP with two materials)**.** *Consider the setting for two materials 3.2.1 and assume that $\tilde{W}$ fulfils the growth condition (G). Then, there exists a solution of the AMP with two materials according to definition 3.2.3.*

Proof. The concept of the proof is a time discretization. We start by the construction of the time step solutions of the growth tensor and its convergence in a suitable space using Arzelà–Ascoli theorem. Using estimates on the growth tensor we show convergence of the grown length which leads to the convergence of the interface points. To complete the proof we define the elastic deformation and prove the compatibility with the convergence of the stress tensor which implies that the equation for the growth holds.
Step 0: Time discretization. For $N \in \mathbb{N}$ and $n \in \{1, \dots, N\}$ define the equidistant times

$$t_0^N := 0,$$
$$t_n^N := \frac{n}{N} T$$

with distance $\Delta t = \frac{T}{N}$. Further for each time discrete problem the initial values are

$$S_0^N := S_0 = 0,$$
$$G_0^N := G_0 = 1.$$

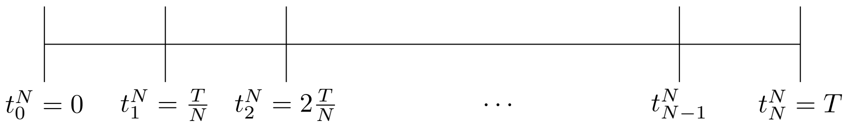

Figure 3.2.: Scheme of time discretization.

Step 1: Construction of time-step growth tensors and growth maps. Let $N \in \mathbb{N}$ and $n \in \{0, \dots, N-1\}$ and $G_n^N(t_n^N)\colon (0,1) \to \mathbb{R}$ and $S_n^N(t_n^N)$ be given. We want to find $G_{n+1}^N \colon [t_n^N, t_{n+1}^N] \times [0,1] \to \mathbb{R}$ as solution of the ODE

$$\begin{aligned} \dot{G}_{n+1}^N(t,x) = \gamma(x)\mu(S_n^N(t_n^N))G_{n+1}^N(t,x) \text{ for all } t \in [t_n^N, t_{n+1}^N], \\ G_{n+1}^N(t_n^N, x) = G_n^N(t_n^N, x). \end{aligned} \tag{3.14}$$

Here, $S_n^N(t_n^N)$ is the Piola–Kichhoff stress tensor to the elastic deformation belonging to the stress tensor $G_n^N(t,\cdot)$. Due to the time constant growth factor the solution is given by the exponential law

$$G_{n+1}^N(t,x) = G_n^N(t_n^N, x)\exp(\gamma(x)\mu(S_n^N(t_n^N))(t - t_n^N)).$$

In the following we notate by $G^N \colon [0,T] \times [0,1] \to \mathbb{R}$ the growth tensor defined by $G^N(t) := G_n^N(t,x)$ for $t \in [t_{n-1}^N, t_n^N)$.
Step 2: Estimates and convergence of growth tensor. To get a strong convergence of the time step solutions we will use the Arzelà–Ascoli theorem A.3.1. Let $x \in (0,1)$ be arbitrary. In our case is $X = [0,T]$, $Y = \mathbb{R}$ and $F := \{G^N(\cdot,x) | N \in \mathbb{N}\} \subset C([0,T])$.
Concerning (i) in A.3.1: We get the statement if we have a uniform estimate on the derivatives of G^N with respect to time. We start with uniform estimates on G^N. Then, we calculate

$$\begin{aligned} &\|G^N(\cdot,x)\|_{C([0,T])} \\ &= \sup_{t\in[0,T]} |G^N(t,x)| \end{aligned}$$

$$
\begin{aligned}
&= \sup_{n\in\{1,\dots,N\}} \sup_{t\in[t_n^N,t_{n+1}^N]} |G_n^N(t,x)| \\
&= \sup_{n\in\{1,\dots,N\}} \sup_{t\in[t_n^N,t_{n+1}^N]} |G_{n-1}^N(t_{n-1}^N,x)| \exp(\gamma(x)\mu(S_n^N(t_n^N))(t-t_n^N)) \\
&\le \sup_{n\in\{1,\dots,N\}} \sup_{t\in[t_n^N,t_{n+1}^N]} |G_0| \exp((\gamma\mu)_{max}\Delta t)^n \\
&\le \exp((\gamma\mu)_{max}T).
\end{aligned}
$$

This and the ODE (3.14) imply the following estimate on the time derivative of $G^N(\cdot,x)$

$$
\begin{aligned}
\sup_{t\in[0,T]} |\dot{G}^N(t,x)| &= \sup_{n\in\{1,\dots,N\}} \sup_{t\in[t_n^N,t_{n+1}^N]} |\dot{G}_n^N(t,x)| \\
&\le \sup_{n\in\{1,\dots,N\}} \sup_{t\in[t_n^N,t_{n+1}^N]} (\gamma\mu)_{max}|G_n^N(t,x)| \\
&\le (\gamma\mu)_{max} \exp((\gamma\mu)_{max}T).
\end{aligned}
$$

This is the uniform estimate on the derivative which implies that (i) holds.
Concerning (ii)in A.3.1: The set $\{G^N(t,x)|N\in\mathbb{N}\}\subset\mathbb{R}$ is relatively compact in $\mathbb{R}$ for each $t\in[0,T]$, since we found a uniform in $N\in\mathbb{N}$ estimate on $G^N(x,t)$.
For $x\in(0,\ell)$ the Arzelà–Ascoli theorem states the existence of a subsequence $(N_M)_{M\in\mathbb{N}}$ and a limit $G(\cdot,x)\in C([0,T])$ with $G^{N_M}(\cdot,x)\to G(\cdot,x)$ in $C([0,t])$, $M\to\infty$. Also, for $x\in(\ell,1)$ the theorem applied on the subset $\{G^{N_M}(\cdot,x)|M\in\mathbb{N}\}$ gives a subsequence $(N_{M_L})_{L\in\mathbb{N}}$ and a limit $G(\cdot,x)\in C([0,T])$ such that $G^{N_{M_l}}(\cdot,x)\to G(\cdot,x)$ in $C([0,T])$, $L\to\infty$. In the following we consider the subsequence $(N_{M_L})_{L\in\mathbb{N}}$ and denote it by $N\in\mathbb{N}$.
For a $t\in[0,T]$ and $N\in\mathbb{N}$ we define the map

$$\tau^N\colon t\mapsto\tau^N(t) := \max\{t_n^N|\ t_n^N\le t,\ n\in\{0,\dots,N\}\}$$

such that $\tau^N(t)\le t$ and $\tau^N(t)\to t$ for $N\to\infty$.
Taking a look at the integral equation of the ODE

$$G^N(t,x) = 1 + \int_0^t \gamma(x)\mu(S^N(\tau^N(s)))G^N(s,x)\ ds,$$

we see that we already can pass to the limit on the left hand side as well as for the term $G^N(s,x)$ on the right hand side. Hence, we need to find a limit of $\mu(S^N(\tau^N(s)))$.
Step 3: Convergence of the grown lengths. We get the growth map by integration of the growth tensors G^N over x: $g^N\colon[0,T]\times[0,1]\to\mathbb{R}$,

$$g^N(t,x) := \int_0^x G^N(t,x)\ dx \tag{3.15}$$

and the lengths of the grown material, which are now time dependent, $\lambda_1^N,\lambda_2^N\colon[0,T]\to\mathbb{R}$,

$$
\begin{aligned}
\lambda_1^N(t) &:= g^N(t,\ell)-g^N(t,0) = \int_0^\ell G^N(t,x)\ dx, \\
\lambda_2^N(t) &:= g^N(t,1)-g^N(t,\ell) = \int_\ell^1 G^N(t,x)\ dx.
\end{aligned}
$$

As above, $G^N(x,t)\ge\exp((\gamma\mu)_{min}T)$ holds for each $N\in\mathbb{N}$, $t\in[0,T]$ and $x\in(0,1)$, hence, $\lambda_i^N(t)\ge\ell_i\exp((\gamma\mu)_{min}T)$ for all $t\in[0,T]$, $i\in\{1,2\}$ and all $N\in\mathbb{N}$ with $\ell_1=\ell$ and $\ell_2=(1-\ell)$.

Further the lengths are estimated from above by

$$\lambda_1^N(t) = \int_0^\ell G^N(t,x)\ dx \le \ell \exp((\gamma\mu)_{max}T),$$
$$\lambda_2^N(t) = \int_\ell^1 G^N(t,x)\ dx \le (1-\ell)\exp((\gamma\mu)_{max}T).$$

Since for each $N \in \mathbb{N}$ the growth tensor G^N is continuous in t and uniformly estimated, dominated convergence implies the continuity of λ_i^N, $i \in \{1,2\}$. Let $t_k \to t$ for $k \to \infty$. Then

$$\begin{aligned}\lim_{k\to\infty} \lambda_1^N(t_k) &= \lim_{k\to\infty} \int_0^\ell G^N(t_k,x)\ dx \\ &= \int_0^\ell \lim_{k\to\infty} G^N(t_k,x)\ dx = \int_0^\ell G^N(t,x)\ dx = \lambda_1^N(t).\end{aligned}$$

The same calculation holds for λ_2^N. We define

$$\lambda_1^*(t) \coloneqq g(t,\ell) - g(t,0) = \int_0^\ell G(t,x)\ dx,$$
$$\lambda_2^*(t) \coloneqq g(t,1) - g(t,\ell) = \int_\ell^1 G(t,x)\ dx,$$

which are the limits of the sequences $(\lambda_1^N)_{N\in\mathbb{N}}$ and $(\lambda_2^N)_{N\in\mathbb{N}}$ respectively, because

$$\begin{aligned}\|\lambda_1^N - \lambda_1^*\|_{C([0,T])} &= \sup_{t\in[0,T]} |\lambda_1^N(t) - \lambda_1^*(t)| \\ &= \sup_{t\in[0,T]} \left| \int_0^\ell G^N(t,x)\ dx - \int_0^\ell G(t,x)\ dx \right| \\ &= \sup_{t\in[0,T]} \int_0^\ell |G^N(t,x) - G(t,x)|\ dx\end{aligned}$$

which converges to 0 for $N \to \infty$ due to the convergence of $G^N(\cdot,x)$ to $G(\cdot,x)$ in $C([0,T])$ for all $x \in (0,1)$ and dominated convergence. The same holds for $(\lambda_2^N)_{N\in\mathbb{N}}$, and therefore, the convergences

$$\lambda_i^N \to \lambda_i^* \text{ in } C([0,T]), N \to \infty, \tag{3.16}$$

hold for $i \in \{1,2\}$.

Step 4: Convergence of interface points. For $t \in [0,T]$ define $\xi^N(t) \coloneqq \xi(\lambda_1^N(t), \lambda_2^N(t))$ and $\xi^*(t) \coloneqq \xi(\lambda_1^*(t), \lambda_2^*(t))$. For the grown length we showed uniform estimates, i.e. for all $t \in [0,T]$, $N \in \mathbb{N}$ and $i \in \{1,2\}$ holds $\lambda_i^N(t) \in [\ell_i \exp((\gamma\mu)_{min}T), \ell_i \exp((\gamma\mu)_{max}T)]$ and due to the uniform convergence (3.16) follows

$$\lambda_i^*(t) \in [\ell_i \exp((\gamma\mu)_{min}T), \ell_i \exp((\gamma\mu)_{max}T)]. \tag{3.17}$$

In Lemma 3.1.9 the dependence of the interface point of the grown length was determined to be C^1. As a C^1-function on a compact set the map ξ is Lipschitz continuous with Lipschitz constant $L_\xi > 0$. We prove the convergence $\xi^N \to \xi^*$ in $C([0,T])$ for $N \to \infty$ directly. Let $N \in \mathbb{N}$ be arbitrary. Then

$$\begin{aligned}\|\xi^N - \xi^*\|_{C([0,T])} &= \sup_{t\in[0,T]} |\xi^N(t) - \xi^*(t)| \\ &= \sup_{t\in[0,T]} |\xi(\lambda_1^N(t), \lambda_2^N(t)) - \xi(\lambda_1^*(t), \lambda_2^*(t))| \\ &\le L_\xi \sup_{t\in[0,T]} \left(|\lambda_1^N(t) - \lambda_1^*(t)| + |\lambda_2^N(t) - \lambda_2^*(t)|\right) \\ &\le L_\xi \left(\|\lambda_1^N - \lambda_1^*\|_{C([0,T])} + \|\lambda_2^N - \lambda_2^*\|_{C([0,T])}\right).\end{aligned}$$

The right hand side converges to 0 for $N \to \infty$, which shows the desired convergence.
Step 5: Elastic deformation. We define the elastic deformation for each $t \in [0,T]$ by $\phi(t,\cdot)\colon (0, \lambda_1^*(t) + \lambda_2^*(t)) \to \mathbb{R}$,

$$\phi(t,y) = \begin{cases} \frac{\xi(t)}{\lambda_1(t)} y, & \text{if } y \le \lambda_1(t), \\ \frac{1-\xi(t)}{\lambda_2(t)}(y - \lambda_1(t)) + \xi(t), & \text{if } y > \lambda_1(t). \end{cases} \tag{3.18}$$

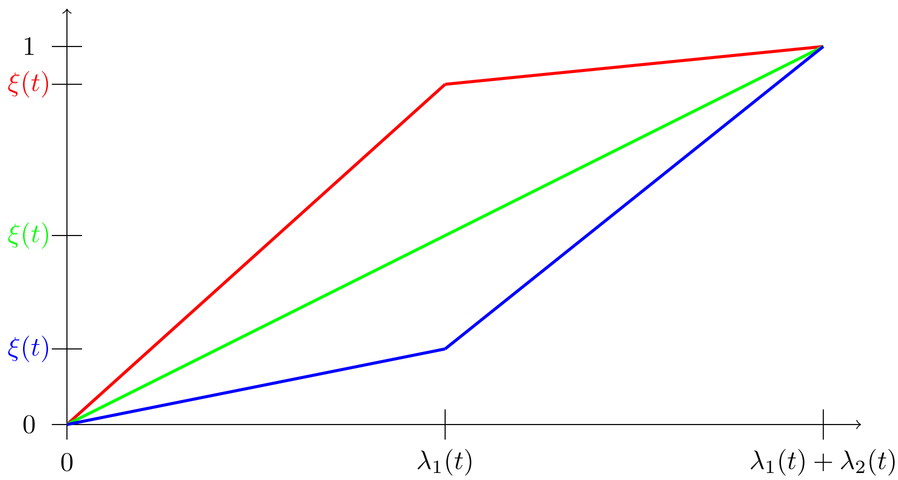

Figure 3.3.: Schematic picture of $\phi(t,y)$ for three different $\xi(t)$.

Step 6: Convergence of stress tensors. We want to prove the convergence of the Piola–Kirchhoff stress tensors in $C([0,T])$. Recall that $S^N = \kappa_1 D_F \tilde{W}(\frac{\xi^N}{\lambda_1^N})$. For $t \in [0,T]$ and $y \in [0, \lambda_1(t)]$ define

$$S^*(t) := D_F W(y, \partial_y \phi(t,y)) = \kappa_1 D_F \tilde{W}(\partial_y \phi(t,y)) = \kappa_1 D_F \tilde{W}\left(\frac{\xi(t)}{\lambda_1(t)}\right).$$

Due to the estimates on λ_i^N and λ_i^* and the argumentation of ξ being a continuous function on a compact set and, hence, having a minimum and a maximum, there exist $0 < \xi_{min} < \xi_{max} < \infty$ such that for each $N \in \mathbb{N}$ and all $t \in [0,T]$ it holds

$$\frac{\xi^N(t)}{\lambda_i^N(t)}, \frac{\xi^*(t)}{\lambda_i^*(t)} \in \left[\frac{\xi_{min}}{\lambda_{max}}, \frac{\xi_{max}}{\lambda_{min}}\right]. \tag{3.19}$$

Then

$$\begin{aligned}\|S^N - S^*\|_{C([0,T])} &= \sup_{t\in[0,T]} \left|S^N(t) - S^*(t)\right| \\ &= \sup_{t\in[0,T]} \left|\kappa_1 D_F\tilde{W}\left(\frac{\xi^N(t)}{\lambda_1^N(t)}\right) - \kappa_1 D_F\tilde{W}\left(\frac{\xi^*(t)}{\lambda_1^*(t)}\right)\right| \\ &\leq \sup_{t\in[0,T]} \kappa_1 L_{D_F\tilde{W}} \left|\frac{\xi^N(t)}{\lambda_1^N(t)} - \frac{\xi^*(t)}{\lambda_1^*(t)}\right|\end{aligned}$$

where we used that the values of the argument of $D_F\tilde{W}$ are in the compact set $\left[\frac{\xi_{min}}{\lambda_{max}}, \frac{\xi_{max}}{\lambda_{min}}\right]^2$ for all $t \in [0,T]$ and $D_F\tilde{W}$ is C^1, hence, Lipschitz continuous on this compact set with Lipschitz constant $L_{D_F\tilde{W}} > 0$. The convergences of ξ^N to ξ^* and λ_1^N to λ_1^* in $C([0,T])$ for $N \to \infty$ now imply the desired convergence.

Step 7: Solution of the ODE. We know that each time step the solution solves an ODE, i.e. in integral version

$$G^N(t,x) = 1 + \int_0^t \gamma(x)\mu(S^N(\tau^N(s)))G^N(s,x)\; ds$$

for each $t \in [0,T]$ and $x \in [0,1]$. The left hand side converges to $G(t,x)$ for $N \to \infty$. Also,

$$\begin{aligned}&|\mu(S^N(\tau^N(t))) - \mu(S(t))| \\ &\leq |\mu(S^N(\tau^N(t))) - \mu(S(\tau^N(t)))| + |\mu(S(\tau^N)) - \mu(S(t))| \to 0,\end{aligned}$$

for $N \to \infty$, because of the uniform convergence $S^N \to S$, $N \to \infty$ and the continuity of S as uniform limit of continuous functions and due to the continuity of μ. With dominated convergence, since μ and G^N are uniformly estimated in $N \in \mathbb{N}$ and the convergence of $G^N(s,x)$ to $G(s,x)$ for $N \to \infty$, we get that the right hand side converges for $N \to \infty$ and that the formula

$$G(t,x) = 1 + \int_0^t \gamma(x)\mu(S(s))G(s,x)\; ds$$

holds. ♣

Remark 3.2.6. *Note that for this proof the growth condition (G) is needed since the existence and properties of ξ require it. Later on, we prove the existence theorem with the Picard–Lindelöf theorem, which does not require the growth condition (G).*

After the existence of a solution of the AMP with two materials is stated, the regularity shall be discussed. For the following other settings, refer to the Chapter 5 for regularity results.

Proposition 3.2.7 (Lipschitz regularity of solution of the AMP with two materials)**.** *Consider the Setting 3.2.1. Let (G,S) be the solution of the AMP with two materials with growth map g and elastic deformation $\phi\colon \mathcal{T}(g) \to (0,1)$ from Definition 3.18. Moreover, suppose $\lambda_1(G), \lambda_2(G)\colon [0,T] \to \mathbb{R}$ be the grown length and $\xi(G)\colon [0,T] \to (0,1)$ the interface point. Then, $\lambda_1(G), \lambda_2(G), \xi(G)$ and ϕ are Lipschitz continuous.*

Proof. By definition

$$\lambda_1(G)(t) = \int_0^\ell G(t,x)\; dx$$

and, hence, with the ODE for G

$$\partial_t \lambda_1(G)(t) = \int_0^\ell \dot{G}(t,x)\ dx = \int_0^\ell \gamma(x)\mu(S(t))G(t,x)\ dx$$

and

$$|\partial_t \lambda_1(G)(t)| = |\int_0^\ell \gamma(x)\mu(S(t))G(t,x)\ dx| \leq (\gamma\mu)_{max} G_{max}.$$

Due to the formula $d_t\lambda_1$ is continuous in t and with the uniform estimate, the Lipschitz continuity follows. A similar argumentation holds for λ_2.
Since λ_1, λ_2 are Lipschitz continuous on the compact set $[0,T]$, there exists $0 < \lambda_{min} < \lambda_{max} < \infty$ such that $\lambda_1(t), \lambda_2(t) \in [\lambda_{min}, \lambda_{\max}]$ for all $t \in [0,T]$. Due to the Lemma 3.1.9 $\xi\colon [\lambda_{min}, \lambda_{max}]^2 \to (0,1)$ is Lipschitz continuous as a C^1-map on a compact set. It follows that the composition

$$\xi \circ (\lambda_1, \lambda_2)\colon [0,T] \to (0,1) \text{ is Lipschitz continous.}$$

The map

$$\tilde{\phi}(x, \lambda_1, \lambda_2, \xi) := \begin{cases} \frac{\xi}{\lambda_1} x, & x \leq \lambda_1, \\ \frac{1-\xi}{\lambda_2}(x - \lambda_1) + \xi, & x > \lambda_1 \end{cases}$$

is Lipschitz continuous on compacts set of $\lambda_1, \lambda_2, \xi$ and x arbitrary, since it is a composition of on compacts sets Lipschitz continuous functions. Further the composition

$$\phi = \tilde{\phi} \circ (id_x, \lambda_1(G), \lambda_2(G), \xi(G))\colon \mathcal{T} \to \mathbb{R}$$

is Lipschitz continuous since the single maps are Lipschitz continuous and map into a compact set when needed in the argumentation. ♣

3.2.2. ... finitely many parts

The before stated existence result is not resticted to material consisting of two parts. It holds also for a material consisting of m parts, what we will prove in this section. The steps of the proofs are the same, but with more parts, one has to take into consideration that the parts influence each other and, hence, there is not one equation to solve but a system of equations and not one interface point to find, but $m-1$. Therefore, in this section, we point out the differences to the setting above. Moreover, we considered the setting of two materials, to focus on the basic idea without concerning about notation requiring indices and deal with one instead of $m-1$ equations.
The situation, we consider now, displays as follows: the material consists of m parts and is homogeneous on each part, see the following schematic picture and for details see Definition 2.1.9. To be precise on the setting:

Definition 3.2.8 (Setting with m materials)**.** *Let $m \in \mathbb{N}$ be the number of parts, the material consists of, and let $0 = \ell_0 < \ell_1 < \ldots < \ell_m = 1$ be such that the material i occupies the domain $(\ell_{i-1}, \ell_i]$. Define the function spaces*

$$\begin{aligned} S_m^0 :=&\{f\colon (0,1) \to \mathbb{R} |\ f \text{ is constant on } (\ell_{i-1}, \ell_i] \text{ for all } i \in \{1, \ldots, m\}\}, \\ S_m^1 :=&\{f\colon (0,1) \to \mathbb{R} |\ f \text{ affine linear on } (\ell_{i-1}, \ell_i] \text{ for all } i \in \{1, \ldots, m\}, \\ & f \text{ continuous}\}. \end{aligned}$$

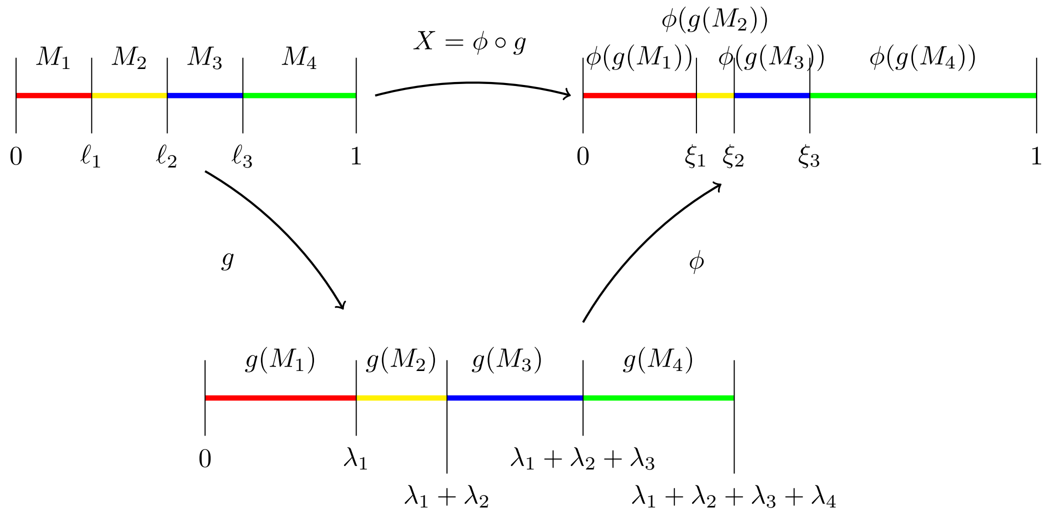

Figure 3.4.: Schematic picture of the growth g and elastic deformation ϕ of an one-dimensional material consisting of four components, marked red, yellow, blue and green respectively.

Let $\gamma_1, \dots, \gamma_m > 0$ be the growth rates and $\kappa_1, \dots, \kappa_m > 0$ the elastic moduli. Define

$$\gamma\colon [0,1] \to \mathbb{R},\ \gamma(x) \coloneqq \begin{cases} \gamma_1, & \text{if } x \in (0, \ell_1], \\ \vdots & \\ \gamma_m, & \text{if } x \in (\ell_{m-1}, 1], \end{cases}$$

and

$$\kappa\colon [0,1] \to \mathbb{R},\ \kappa(x) \coloneqq \begin{cases} \kappa_1, & \text{if } x \in (0, \ell_1], \\ \vdots & \\ \kappa_m, & \text{if } x \in (\ell_{m-1}, 1]. \end{cases}$$

Then, $\gamma, \kappa \in S_m^0$. Let $\mu \in \mathrm{C}^0(\mathbb{R})$ be Lipschitz continuous and bounded. Define $-\infty < (\gamma\mu)_{min} < 0 < (\gamma\mu)_{max} < \infty$ with

$$(\gamma\mu)_{max} \geq \operatorname*{ess\,sup}_{x\in(0,1), y\in\mathbb{R}} \{\gamma(x)\mu(y)\},$$
$$(\gamma\mu)_{min} \leq \operatorname*{ess\,inf}_{x\in(0,1), y\in\mathbb{R}} \{\gamma(x)\mu(y)\}.$$

Let $\tilde{W} \in \mathrm{C}^2(\mathbb{R}_{>0})$ be strictly convex with $\tilde{W}(1) = 0$, $W(F) \to \infty$ for $F \to 0$ and

$$\frac{\tilde{W}(F)}{F} \to \infty \text{ for } F \to \infty.$$

Define the elastic strain energy density by

$$W\colon (0,1) \times (0,\infty) \to (0,\infty),\ W(x,F) = \kappa(x)\tilde{W}(F).$$

Further, let $p \in (1,\infty)$ and for a given growth map g define the admissible set for the elastic deformation

$$\mathcal{A}_m \coloneqq \{v \in W^{1,p}(g((0,1)))|\ v(g(0)) = 0,\ v(g(1)) = 1\}$$

and the elastic energy for a deformation $\phi \in \mathcal{A}_m$ *by* $E_m\colon \mathcal{A}_m \to \mathbb{R}$,

$$E_m(\phi) := \sum_{i=1}^{m} \int_{g(\ell_{i-1})}^{g(\ell_i)} \kappa_i \tilde{W}(\partial_y \phi(y))\ dy.$$

Finally, let $T > 0$ *be the time horizon and* $G_0 = 1 \in S_m^0$ *be the initial value for the ODE for the growth tensor.*

Goal is to find a growth map $g\colon [0,T] \times (0,1) \to \mathbb{R}$ with gradient $G\colon [0,T] \times (0,1) \to \mathbb{R}$, $G(t,\cdot) \in S_m^0$ and for each $t \in [0,T]$ the elastic deformation $\phi(t,\cdot)\colon g(t,(0,1)) \to (0,1)$ as solution of the stress modulated growth problem, namely to prove the existence Theorem 3.2.11. A solution of the AMP is defined as follows:

Definition 3.2.9 (AMP with m materials)**.** *Suppose that the Setting 3.2.8 holds. We call a tuple* $(G,S) \in \mathrm{C}^1([0,T]; S_m^0) \times \mathrm{C}^0([0,T])$ *a solution of the AMP with m materials if the following is fulfilled:*

(i) The growth tenor G fulfils the ODE

$$\begin{aligned} \dot{G}(t,x) &= \gamma(x)\mu(S(t))G(t,x), \\ G(0,x) &= 1, \end{aligned}$$

for all $t \in [0,T]$ *and almost all* $x \in (0,1)$.

(ii) For fixed $t \in [0,T]$ *and*

$$g(t,x) := \int_0^x G(t,\tilde{x})\ d\tilde{x}$$

let $\phi(t,\cdot) : g(t,(0,1)) \to \mathbb{R}$ *be a minimizer of* E_m *in* $\mathcal{A}_m$. *The Piola–Kirchhoff stress tensor to the elastic deformation* ϕ *is S by*

$$S(t) = D_F W(g^{-1}(y), \partial_y \phi(t,y)).$$

Remark 3.2.10. *The minimizer* $\phi(t,\cdot)$ *of* E_m *in* $\mathcal{A}_m$ *is unique due to the strict convexity of* $\tilde{W}$.

Then, the existence and uniqueness result for the AMP with m materials reads as:

Theorem 3.2.11 (Existence and uniqueness of AMP with m materials)**.** *Consider the Setting 3.2.8 and assume that* $\tilde{W}$ *fulfils the growth condition (G). Then for the initial values* $G_0(\cdot) = 1 \in S_m^0$, *there exists a solution* (g,ϕ) *of the AMP with m materials as defined in Definition 3.2.9.*

Proof. The proof is analogous to the proof of the existence Theorem 3.2.5 in the case $m = 2$, hence, some steps are processed quickly.
We start by a time discretization. Let $N \in \mathbb{N}$ and define $\Delta t := T/N$ and $t_n^N := n\Delta t = nT/N$, $n \in \{0,\dots,N\}$. Further define $G_0^N := G_0$ and $S_0^N := S_0$.
Recall that the solution of the ODE

$$\begin{aligned} \dot{G}(t,x) &= a(x)G(t,x), \\ G(0,x) &= 1 \end{aligned}$$

for each $x \in (0,1)$, where $G_0 = 1$ is the initial value and $a\colon (0,1) \to \mathbb{R}$ is piecewise constant, is given by

$$G(t,x) = \exp(a(x)t).$$

For $n \in \{1, \dots, N\}$ let G^N_{n-1} and S^N_{n-1} be given. Then, define $G^N_n : [t^N_{n-1}, t^N_n] \to \mathbb{R}$ as the solution of

$$\begin{aligned} \dot{G}^N_n(t,x) &= \gamma(x)\mu(S^N_{n-1}(t^N_{n-1}))G^N_n(t,x), \\ G^N_n(t_{n-1}) &= G^N_{n-1}(t_{n-1}), \end{aligned}$$

namely one has

$$G^N_n(t,x) = \exp(\gamma(x)\mu(S^N_{n-1}(t^N_{n-1}))t)$$

and the map $G^N : [0,T] \to \mathbb{R}$ with $G^N(t,x) = G^N_n(t,x)$ for $t \in [t^N_{n-1}, t^N_n]$. Let $S^N_n(t^N_n)$ denote the Piola–Kichhoff stress tensor to the elastic deformation belonging to the stress tensor $G^N_n(t,\cdot)$. Further define $g^N : [0,T] \times (0,1) \to \mathbb{R}$,

$$g^N(t,x) := \int_0^x G^N(t,\tilde{x})\, d\tilde{x}$$

and for $i \in \{1, \dots, m\}$ the grown lengths $\lambda_i : [0,T] \to \mathbb{R}$ by

$$\lambda^N_i(t) := g(t,\ell_i) - g(t,\ell_{i-1}) = \int_{\ell_{i-1}}^{\ell_i} G^N(t,\tilde{x})\, d\tilde{x} = (\ell_i - \ell_{i-1})\exp(\gamma_i\mu(S^N_i(\tau^N(t)))t).$$

(Remember $\tau^N(t) = \max\{t^N_n \mid n \in \{0, \dots, N\}, t^N_n \leq t\}$.) Hence, the uniform estimate

$$\lambda^N_i(t) \in [\Delta\ell_{min}\exp((\gamma\mu)_{min}T), \Delta\ell_{max}\exp((\gamma\mu)_{max}T)] \tag{3.20}$$

holds for each $i \in \{1, \dots, m\}$ and each $N \in \mathbb{N}$. Here is $\Delta\ell_{min} := \min_{i\in\{1,\dots,m\}}(\ell_i - \ell_{i-1})$ and $\Delta\ell_{max} := \max_{i\in\{1,\dots,m\}}(\ell_i - \ell_{i-1})$.
As in Step 2 of the existence result for $m = 2$ we can choose a subsequence and find a limit $G(\cdot,x) \in \mathrm{C}^0([0,T])$ for each $x \in (0,1)$ such that $G^N(\cdot,x) \to G(\cdot,x)$ uniformly. This is possible since we have finitely many parts and, hence, apply the theorem finitely many times. To do so, we apply the Arzelà–Ascoli theorem m times. With this we prove the uniform convergence of the grown lengths to

$$\lambda^*_i(t) := g(t,\ell_i) - g(t,\ell_{i-1}) = \int_{\ell_i}^{\ell_{i-1}} G(t,\tilde{x})\, d\tilde{x},$$

where $i \in \{1, \dots, m\}$, and the continuity of them analogously.
The next step is to find interface points and discuss their properties and convergence. Define $\lambda^N(t) = \sum_{i=1}^m \lambda^N_i(t)$ and the grown interface points $Y^N_i(t) := g^N(t,\ell_i)$. The elastic energy of a deformation ϕ^N (notation: $\phi^N_i : (Y^N_{i-1}, Y^N_i] \to \mathbb{R}$) is given by

$$E_m(\phi^N) := \int_0^{\lambda^N} W_{nat}(y, \partial_y\phi^N(y))\, dy = \sum_{i=1}^m \int_{Y^N_{i-1}}^{Y^N_i} \kappa_i\tilde{W}(\partial_y\phi^N_i(y))\, dy.$$

The boundary conditions on the deformations φ^N_i are

$$\phi^N_1(0) = 0,\ \phi^N_i(Y^N_i) = \phi^N_{i+1}(Y^N_i) = \xi^N_{i+1},\ \phi^N_m(\lambda^N) = 1,$$

where $\xi^N_i \in (0,1)$ for all $i \in \{1, \dots, m-1\}$ and $\xi^n_m = 1$. This energy shall be minimized. By Theorem A.1.1, there exists a minimizer ϕ^N and a $C_D \in \mathbb{R}$ such that the Euler–Lagrange equation

$$\kappa_i D_F \tilde{W}(\partial_y \phi_i^N(y)) = C_D, \ i \in \{1, \dots, m\},$$

is fulfilled. Due to the strict convexity of $\tilde{W}$, $D_F\tilde{W}$ is invertible and it follows

$$\partial_y \phi_i^N(y) = (D_F\tilde{W})^{-1}\left(\frac{C_D}{\kappa_i}\right) \in \mathbb{R}, \ i \in \{1, \dots, m\}.$$

This states that the ϕ_i^N are affine linear functions and from the boundary conditions, we conclude

$$\begin{aligned} \phi_1^N(y) &= \frac{\xi_1^N}{\lambda_0^N} y, \\ \phi_i^N(y) &= \frac{\xi_i^N - \xi_{i-1}^N}{\lambda_i^N}(y - \lambda_i^N) + \xi_{i-1}^N, \ i = 2, \dots, m. \end{aligned} \tag{3.21}$$

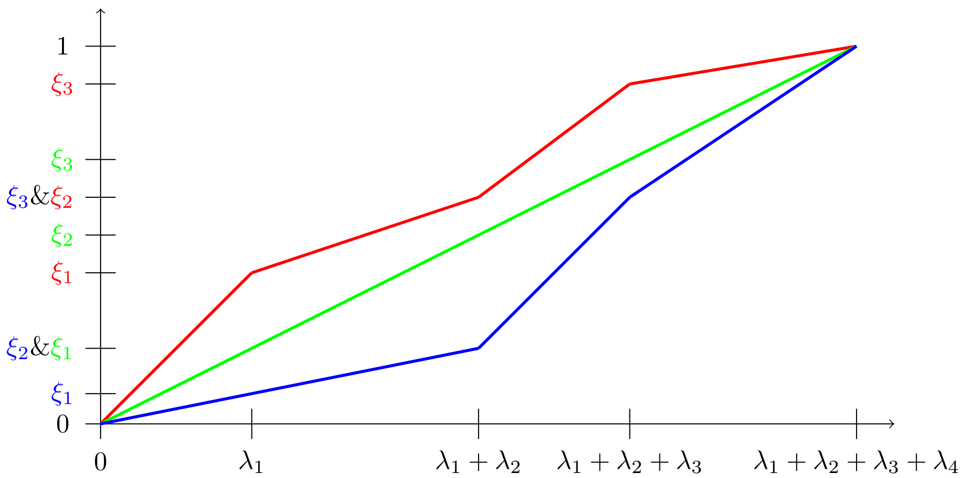

Figure 3.5.: Schematic picture of $\phi(t,\cdot)$ for three different ξ in a material consisting of four parts.

Inserting this into the energy yields

$$\begin{aligned} E_{\lambda_1^N,\dots,\lambda_m^N}(\xi_1^N, \dots, \xi_{m-1}^N) &= \sum_{i=1}^{m} \int_{Y_{i-1}^N}^{Y_i^N} \kappa_i \tilde{W}\left(\frac{\xi_i^N - \xi_{i-1}^N}{\lambda_i^N}\right) \, dy \\ &= \sum_{i=1}^{m} \lambda_i^N \kappa_i \tilde{W}\left(\frac{\xi_i^N - \xi_{i-1}^N}{\lambda_i^N}\right), \end{aligned} \tag{3.22}$$

with $\xi_0^N = 0$. Let $j \in \{1, \dots, m-1\}$ be fixed. Further fix $\xi_1^N, \dots, \xi_{j-1}^N, \xi_{j+1}^N, \dots, \xi_{m-1}^N$ Then, the necessary condition of ξ_j^N to minimize $E_{\lambda_1^N,\dots,\lambda_m^N}(\xi_1^N, \dots, \xi_{j-1}^N, \cdot, \xi_{j+1}^N, \dots, \xi_{m-1}^N)$ is

$$0 = \partial_{\xi_j^N} E^N(\xi_1^N, \dots, \xi_{m-1}^N) = \kappa_{j+1} D_F\tilde{W}\left(\frac{\xi_{i+1}^N - \xi_j^N}{\lambda_{j+1}^N}\right) - \kappa_j D_F\tilde{W}\left(\frac{\xi_j^N - \xi_{j-1}^N}{\lambda_j^N}\right). \tag{3.23}$$

Due to the conditions on $\tilde{W}$ from the Setting 3.2.8 it holds $\xi_j^N \in (\xi_{j-1}^N, \xi_{j+1}^N)$ for all $j \in \{1, \dots, m-1\}$.

Lemma 3.2.12. *In the Setting 3.2.8 there exists a* C^1*-mapping* $\xi\colon (0,\infty)^m \to (0,1)^{m-1}$, $(\lambda_1,\dots,\lambda_m) \mapsto \xi(\lambda_1,\dots,\lambda_m)$ *such that for all* $\lambda_1^N,\dots,\lambda_m^N$ $\xi(\lambda_1^N,\dots,\lambda_m^N)$ *minimizes the energy* $E_{\lambda_1^N,\dots,\lambda_m^N}$ *from (3.22), i.e.* ϕ^N *constructed by (3.21) and using* $\xi^N = \xi(\lambda_1^N,\dots,\lambda_m^N)$ *minimizes* E_m*. Moreover, the* ξ*'s do not cross, i.e.* $\xi_i \in (\xi_{i-1},\xi_{i+1})$*,* $i = 1,\dots,m$*.*

Proof. We obtain the result by applying the implicit function theorem for $f\colon (0,\infty)^m \times \mathbb{R}^{m-1} \to \mathbb{R}^{m-1}$

$$f(\lambda_1,\dots,\lambda_m,\xi_1,\dots,\xi_{m-1}) = \begin{pmatrix} \kappa_2 D_F\tilde{W}\left(\frac{\xi_2-\xi_1}{\lambda_2}\right) - \kappa_1 D_F\tilde{W}\left(\frac{\xi_1}{\lambda_1}\right) \\ \vdots \\ \kappa_m D_F\tilde{W}\left(\frac{1-\xi_{m-1}}{\lambda_m}\right) - \kappa_{m-1} D_F\tilde{W}\left(\frac{\xi_{m-1}-\xi_{m-2}}{\lambda_{m-1}}\right) \end{pmatrix}$$

For each $\lambda = (\lambda_1,\dots,\lambda_m) \in (0,\infty)^m$ there exists a solution $\xi = (\xi_1,\dots,\xi_{m-1}) \in (0,1)^{m-1}$ of $f(\lambda,\xi) = 0$, because to λ there exists a minimizer ϕ_λ of E_m, see Theorem A.1.1, of the form (3.21). Those ξ's solve $f(\lambda,\xi) = 0$. The derivative of f with respect to $\xi = (\xi_1,\dots,\xi_{m-1})$ is of the form

$$\begin{pmatrix} -a_2 - a_1 & 0 & 0 & \dots & 0 \\ a_2 & -a_3 - a_2 & 0 & & \\ 0 & a_3 & -a_4 - a_3 & 0 & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & & a_{m-1} & -a_m - a_{m-1} \end{pmatrix}$$

where each $a_i = \frac{\kappa_i}{\lambda_i} D_F^2\tilde{W}\left(\frac{\xi_i-\xi_{i-1}}{\lambda_i}\right) \in \mathbb{R}_{>0}$ with $\xi_0 = 0$ and $\xi_m = 1$. Since $a_i > 0$ due to the strict convexity of $\tilde{W}$, this matrix is invertible and the implicit function theorem, see Theorem A.3.5, yields that the solution map $\xi\colon (0,\infty)^m \to (0,1)^{m-1}$ is continuous. More precisely, it holds $\xi_i \in (\xi_{i-1},\xi_{i+1})$, $i = 1,\dots,m$: The energy $E_{\lambda_1,\dots,\lambda_m}$ is minimized for $\xi = \xi(\lambda_1,\dots,\lambda_m)$ and it holds $E_{\lambda_1,\dots,\lambda_m}(\xi(\lambda_1,\dots,\lambda_m)) < \infty$. Therefore, (3.22) and the properties of $\tilde{W}$ imply $\frac{\xi_i-\xi_{i-1}}{\lambda_i} \in (0,\infty)$, $i = 1,\dots,m$, and hence $\xi_{i-1} < \xi_i$ holds, $i = 1,\dots,m$. ♣

From the definition of the Piola–Kirchhoff stress tensor and the relation (3.23), we conclude that the stresses are constant in the whole material.

Lemma 3.2.13 (Constant stress in m parts)**.** *In the setting above for each time step solution holds for each* $y \in (Y_{i-1}^N, Y_i^N)$*,* $\tilde{y} \in (Y_{j-1}^N, Y_j^N)$ *that*

$$S^N(t,y) = \kappa_i D_F\tilde{W}\left(\frac{\xi_i^N(t) - \xi_{i-1}^N(t)}{\lambda_i^N(t)}\right) = \kappa_j D_F\tilde{W}\left(\frac{\xi_j^N - \xi_{j-1}^N}{\lambda_j^N}\right) = S^N(t,\tilde{y}).$$

In addition, we can prove the monotonicity of ξ, namely:

Lemma 3.2.14 (Monotonicity of interface points for m parts)**.** *Let* $W \in \mathrm{C}^2(\mathbb{R})$ *fulfil the growth condition (G). Let* $0 < \lambda_1^-,\dots,\lambda_m^-,\lambda_1^+,\dots,\lambda_m^+ < 1$ *be given with* $\lambda_i^- < \lambda_i^+$ *and let* $\lambda_i \in (\lambda_i^-,\lambda_i^+)$ *for all* $i \in \{1,\dots,m\}$*. Then for all* $i \in \{1,\dots,m\}$*, it holds*

$$\begin{aligned} 0 &< \xi_i(\lambda_1^-,\dots,\lambda_i^-,\lambda_{i+1}^+,\dots,\lambda_m^+) \\ &\le \xi_i(\lambda_1,\dots,\lambda_m) \\ &\le \xi_i(\lambda_1^+,\dots,\lambda_i^+,\lambda_{i+1}^-,\dots,\lambda_m^-) < 1. \end{aligned}$$

Proof. The idea is like above to successively replace λ's. For all λ's but one fixed, one can prove an equation like in Lemma 3.1.10, namely a statement in one entry of ξ. This illustrates the following: For all but one λ fixed, shortening the not fixed λ lowers the interface points. The same holds for increasing. Appling this statement in each entry of ξ gives the statement.

More precisely, fix $j \in \{1, \dots, m-1\}$ and fix $\lambda_2, \dots, \lambda_m$. With the same argumentation as in the proof of Lemma 3.1.10 we can prove

$$\begin{aligned}&\xi_j(\lambda_1^-, \lambda_2, \dots, \lambda_m)\\ &\leq \xi_j(\lambda_1, \dots, \lambda_m)\\ &\leq \xi_j(\lambda_1^+, \lambda_2 \dots, \lambda_m),\end{aligned}$$

since the equations only contain the neighbouring λ's. Using this inequality we obtain

$$\begin{aligned}&\xi_j(\lambda_1^-, \dots, \lambda_j^-, \lambda_{j+1}, \dots, \lambda_m)\\ &\leq \xi_j(\lambda_1, \dots, \lambda_m)\\ &\leq \xi_j(\lambda_1^+, \dots, \lambda_j^+, \lambda_{j+1}, \dots, \lambda_m).\end{aligned}$$

Same argumentation holds to obtain

$$\begin{aligned}&\xi_j(\lambda_1^-, \dots, \lambda_j^-, \lambda_{j+1}, \dots, \lambda_{m-1}, \lambda_m^+)\\ &\leq \xi_j(\lambda_1, \dots, \lambda_m)\\ &\leq \xi_j(\lambda_1^+, \dots, \lambda_j^+, \lambda_{j+1}, \dots, \lambda_{m-1}, \lambda_m^-),\end{aligned}$$

and hence,

$$\begin{aligned}&\xi_j(\lambda_1^-, \dots, \lambda_j^-, \lambda_{j+1}^+, \dots, \lambda_m^+)\\ &\leq \xi_j(\lambda_1, \dots, \lambda_m)\\ &\leq \xi_j(\lambda_1^+, \dots, \lambda_j^+, \lambda_{j+1}^-, \dots, \lambda_m^-).\end{aligned}$$

♣

We can continue with the proof of the existence theorem now. So far, we know the grown length $\lambda_i^N(t)$ which converge uniformly to λ_i^*. Due to the uniform estimates on λ_i^N holds

$$\lambda_i^* \in [\Delta\ell_{min} \exp((\gamma\mu)_{min} T), \Delta\ell_{max} \exp((\gamma\mu)_{max} T)].$$

The C^1-regularity of ξ implies its Lipschitz continuity on the compact set $[\Delta\ell_{min} \exp((\gamma\mu)_{min} T), \Delta\ell_{max} \exp((\gamma\mu)_{max} T)]^m$. Hence, we get the uniform convergence of $\xi_i^N = \xi_i(\lambda_1^N, \dots, \lambda_m^N)$ to $\xi_i^*(\lambda_1, \dots, \lambda_m) =: \xi_i^*$. For the convergence of the stresses holds the same argumentation as above and the equation for G is derived by applying the limit $N \to \infty$ to the integral formulation of the ODE of G^N with dominated convergence, hence, the desired solution is found. ♣

Remark 3.2.15. *Due to the lemmas 3.2.13-3.2.14, we were able to generalise the existence Theorem 3.2.5 to the case of m parts. We can consider Neumann boundary conditions.*
Also, uniqueness can be proved analogous to the proof of uniqueness for the case of two materials by the Picard–Lindelöf theorem, see the following chapter. The a priori estimates on the growth tensor goes analogously as in Lemma 4.2.4 and the Lipschitz continuity of S is shown analogously to the proof of Theorem 4.2.1.

4 Existence and Uniqueness

In the Subsection 3.2 we discussed the existence of a solution of the AMP for material consisting of two or $m \in \mathbb{N}$ parts. In this chapter, we discuss uniqueness of such a solution using the uniqueness of the Picard–Lindelöf theorem. It also gives existence. Here, the ODE is interpreted in a Banach space which induces some challenges as pointwise evaluation afterwards. Furthermore, the Piola–Kirchhoff stress tensor needs to be Lipschitz continuous. The proofs in the previous chapter, however, use the time discretization method which is more direct and needs less theory.
First question to answer here is: In which Banach space X to interpret the ODE? From modelling, we want to solve the ODE for the growth tensor pointwise in $x \in (0,1)$, where there is the problem that the solution G needs to be known for all x to calculate the stress S. To circumvent this, we search for a formulation in a Banach space, namely in $L^\infty(0,1)$.
So far, we want to solve the ODE

$$\dot{G}(t,x) = \mathcal{G}(t,x,S(t),G(t,x)) \text{ for all } t \in [0,T],\ x \in (0,1).$$

Define $H(t) \coloneqq G(t,\cdot)$. What is the ODE in $L^\infty(0,1)$ for H? Use the pointwise ODE to reformulate as follows:

$$\dot{H}(t) = \dot{G}(t,\cdot) = \mathcal{G}(t,\cdot,S(t),G(t,\cdot)) = \mathcal{G}(t,\cdot,S(t),H(t)) =: \mathcal{H}(t,S(t),H(t)),$$

where this equation holds pointwise for almost all $x \in (0,1)$ and

$$\mathcal{H}\colon [0,T] \times \mathbb{R} \times L^\infty(0,1) \to L^\infty(0,1),\ \mathcal{H}(t,S,G)(x) \coloneqq \mathcal{G}(t,x,S,G(x)). \tag{4.1}$$

Then, $\mathcal{H}$ inherits the continuity in (t,S,G) and the Lipschitz continuity in S and G from $\mathcal{G}$, see Lemma A.2.2 (iv). Using the ODE for H, we can prove existence and uniqueness for the AMP using the Picard–Lindelöf theorem. In the following, the RHS is denoted by $\mathcal{G}$ and refers to $\mathcal{H}$ as well as to $\mathcal{G}$ via (4.1) depending on the arguments.

4.1. Pointwise ODE vs. ODE in a Banach Space

By interpreting the ODE in a Banach Space, it is also required to interpret the solution of the ODE in the Banach space pointwise again. Especially, the path over time t of a particle $x \in \Omega$ has to be evaluable e.g. to prove a priori estimates via Gronwall lemma. Hence, the goal of this section is to prove existence of a solution for every $x \in \Omega$ which has continuous paths over time and that these pointwise solutions for all $x \in \Omega$ together are the same as the solution in $L^\infty(0,1)$ (in the L^∞ sense).
We first start off with two lemmas, used for the proof of the main Lemma 4.1.4 of this section: The composition of a continuous and a measurable function is measurable and a continuous map to $L^\infty(0,1)$ is measurable on the crossproduct of the domains, which is called strong measurable.

Lemma 4.1.1 (Composition of continuous and measurable is measurable)**.** *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ with the following Carathéodory conditions:*

(i) for all $x \in \Omega$

$$y \mapsto f(x,y) \text{ is continuous,}$$

(ii) for all $y \in \mathbb{R}^n$

$$x \mapsto f(x,y) \text{ is measurable.}$$

Then, for each measurable $u : \Omega \to \mathbb{R}^n$ the map

$$f(\cdot, u(\cdot))\colon \Omega \to \mathbb{R},\ x \mapsto f(x, u(x))$$

is measurable.

Proof. We follow the proof of measurability in [Růž04], p. 69, Lemma 1.20.
Since u is measurable there exists a sequence of simple functions $(u_k)_{k\in\mathbb{N}}$ such that

$$u_k \to u \text{ almost everywhere in } \Omega,\ k \to \infty.$$

It follows with the continuity of f in the second argument (i) that for almost all $x \in \Omega$

$$f(x, u(x)) = \lim_{k\to\infty} f(x, u_k(x)) \tag{4.2}$$

holds. Since for each $k \in \mathbb{N}$ the function u_k is simple, there exist a finite number $M(k) \in \mathbb{N}$ of values $c_k^n \in \mathbb{R}$, $n = 1, \dots, M(k)$, and measurable sets $\Omega_k^n \subset \Omega$, $n = 1, \dots, M(k)$, with $\Omega_k^i \cap \Omega_k^j = \emptyset$ for all $i, j = 1, \dots, M(k)$, $i \neq j$, such that

$$u_k(x) = \sum_{n=1}^{M(k)} c_k^n \chi_{\Omega_k^n}(x).$$

It follows

$$f(x, u_k(x)) = f(x, \sum_{n=1}^{M(k)} c_k^n \chi_{\Omega_k^n}(x)) = \sum_{n=1}^{M(k)} f(x, c_k^n) \chi_{\Omega_k^n}(x),$$

where $f(x, c_k^n)$ is measurable due to the assumption (ii). This proves the measurability of $f(\cdot, u_k(\cdot))$ for all $k \in \mathbb{N}$, since the characteristic function of a measurable set is measurable and sums and products of measurable functions are measurable. To conclude, use that the pointwise limit of measurable functions is measurable and the pointwise convergence (4.2). ♣

Lemma 4.1.2. *Let $f\colon [0,T] \to L^\infty(0,1)$ be continuous. Then, f is $\mathcal{L}^2$ measurable on $[0,T] \times (0,1)$.*

Proof. Since $f\colon [0,T] \to L^\infty(0,1)$ is continuous, there exists a sequence of simple function $(f_k)_{k\in\mathbb{N}}$, $f_k \coloneqq [0,T] \to L^\infty(0,1)$, $k \in \mathbb{N}$, with representation

$$f_k(t) = \sum_{n=1}^{N(k)} f_k^n \chi_{\Omega_k^n}(t),$$

where $f_k^n \in L^\infty(0,1)$ and $\Omega_k^n \subset [0,T]$ measurable, $\Omega_k^n \cap \Omega_k^m = \emptyset$ for all $k \in \mathbb{N},\ n,m = 1,\dots,N(k)$ with $n \neq m$, such that

$$\sup_{t\in[0,T]} \|f_k(t) - f(t)\|_{L^\infty(0,1)} \to 0,\ k \to \infty. \tag{4.3}$$

Furthermore, since f_k^n is measurable, there exists a sequence $((f_k^n)_l)_{l\in\mathbb{N}}$ of simple functions $(f_k^n)_l \coloneqq (0,1) \to \mathbb{R}$ such that for almost all $x \in (0,1)$

$$(f_k^n)_l(x) \to f_k^n(x),\ l \to \infty. \tag{4.4}$$

Moreover, for all $k \in \mathbb{N},\ n = 1,\dots,N(k),\ l \in \mathbb{N}, m = 1,\dots,M(k,n,l)$ there exist constants $(c_k^n)_l^m \in \mathbb{R}$ and measurable sets $(U_k^n)_l^m \subset (0,1)$, such that

$$(f_k^n)_l(x) = \sum_{m=1}^{M(k,n,l)} (c_k^n)_l^m \chi_{(U_k^n)_l^m}(x).$$

Let $(t,x) \in [0,T] \times (0,1)$ be fixed and $\varepsilon > 0$ arbitrary. In view of the convergence (4.4), for all $k \in \mathbb{N}$ and $n = 1,\dots,N(k)$ there exists an $L(k,n) \in \mathbb{N}$ such that for all $l \geq L(k,n)$

$$\left| \sum_{m=1}^{M(k,n,l)} (c_k^n)_l^m \chi_{(U_k^n)_l^m}(x) - f_k^n(x) \right| \leq \frac{\varepsilon}{2}$$

holds. Define

$$\tilde{f}_k(t,x) \coloneqq \sum_{n=1}^{N(k)} \sum_{m=1}^{M(k,n,L(k,n))} (c_k^n)_{L(k,n)}^m \chi_{(U_k^n)_{L(k,n)}^m}(x) \chi_{\Omega_k^n}(t), \tag{4.5}$$

which is for all $k \in \mathbb{N}$ a simple function on $[0,T]\times(0,1)$, since $\Omega_k^n \times (U_k^n)_{L(k,n)}^m$ is measurable for all $k \in \mathbb{N},\ n = 1,\dots,N(k),\ m = 1,\dots,M(k,n,L(k,n))$. It is left to prove that $\tilde{f}_k(t,x) \to f(t,x)$ for $k \to \infty$: With triangle inequality and inserting a zero,

$$\begin{aligned}
&|\tilde{f}_k(t,x) - f(t,x)| \\
&= \left| \sum_{n=1}^{N(k)} \sum_{m=1}^{M(k,n,L(k,n))} (c_k^n)_{L(k,n)}^m \chi_{(U_k^n)_{L(k,n)}^m}(x) \chi_{\Omega_k^n}(t) - f(t,x) \right| \\
&\leq \left| \sum_{n=1}^{N(k)} \sum_{m=1}^{M(k,n,L(k,n))} (c_k^n)_{L(k,n)}^m \chi_{(U_k^n)_{L(k,n)}^m}(x) \chi_{\Omega_k^n}(t) - \sum_{n=1}^{N(k)} f_k^n(x) \chi_{\Omega_k^n}(t) \right| \\
&\quad + \left| \sum_{n=1}^{N(k)} f_k^n(x) \chi_{\Omega_k^n}(t) - f(t,x) \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left| \sum_{n=1}^{N(k)} \left(\sum_{m=1}^{M(k,n,L(k,n))} (c_k^n)^m_{L(k,n)} \chi_{(U_k^n)^m_{L(k,n)}}(x) - f_k^n(x) \right) \chi_{\Omega_k^n}(t) \right| \\
&\quad + \sup_{t\in[0,T]} \|f_k(t) - f(t)\|_{L^\infty(0,1)} \\
&\leq \sum_{n=1}^{N(k)} \left| \sum_{m=1}^{M(k,n,L(k,n))} (c_k^n)^m_{L(k,n)} \chi_{(U_k^n)^m_{L(k,n)}}(x) - f_k^n(x) \right| \cdot |\chi_{\Omega_k^n}(t)| \\
&\quad + \|f_k - f\|_{\mathrm{C}^0([0,T];L^\infty(0,1))} \\
&\leq \left| \sum_{m=1}^{M(k,n,L(k,n))} (c_k^n)^m_{L(k,n)} \chi_{(U_k^n)^m_{L(k,n)}}(x) - f_k^n(x) \right| \cdot \sum_{n=1}^{N(k)} \chi_{\Omega_k^n}(t) \\
&\quad + \sup_{t\in[0,T]} \|f_k(t) - f(t)\|_{L^\infty(0,1)}.
\end{aligned}
$$

Now, choosing $k \in \mathbb{N}$ sufficiently large that $\sup_{t\in[0,T]} \|f_k(t) - f(t)\|_{L^\infty(0,1)} \leq \varepsilon/2$ and the fact that $\sum_{n=1}^{N(k)} \chi_{\Omega_k^n}(t) = 1$ yields that

$$
|\tilde{f}_k(t,x) - f(t,x)| \leq \frac{\varepsilon}{2} \cdot \sum_{n=1}^{N(k)} \chi_{\Omega_k^n}(t) + \frac{\varepsilon}{2} = \varepsilon.
$$

This proves the convergence of a sequence of simple function on $[0,T] \times (0,1)$, which gives the desired measurability. ♣

To prove uniqueness of a solution of an ODE, we need the following Gronwall lemma.

Lemma 4.1.3 (Gronwall lemma in integral form)**.** *Let $I := [a,b]$ be an interval, $u \in \mathrm{C}^0(I)$ and $\alpha, \beta : I \to \mathbb{R}$ be continuous functions with $\beta(t) \geq 0$. Further assume that for all $t \in I$ the inequality*

$$
u(t) \leq \alpha(t) + \int_a^t \beta(s) u(s)\ ds \tag{4.6}
$$

holds. Then, it follows

$$
u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(\tau)\ d\tau \right)\ ds \text{ for all } t \in I.
$$

Proof. The idea of the proof is to test the given inequality with a suitble term and conclude the desired estimate.
Define

$$
v\colon [a,b] \to \mathbb{R},\ v(t) = \exp\left(-\int_a^t \beta(s)\ ds \right) \int_a^t \beta(s) u(s)\ ds.
$$

By chain and product rule and by applying the inequality (4.6), it follows

$$
v'(t) = -\beta(t) \exp\left(-\int_a^t \beta(s)\ ds \right) \int_a^t \beta(s)u(s)\ ds + \exp\left(-\int_a^t \beta(s)\ ds \right) \beta(t) u(t)
$$

$$= \beta(t) \exp\left(-\int_a^t \beta(s)\ ds\right)\left(-\int_a^t \beta(s)u(s)\ ds + u(t)\right)$$
$$\leq \beta(t) \exp\left(-\int_a^t \beta(s)\ ds\right)\alpha(t).$$

By definition, $v(a) = 0$. This together with the fundamental theorem of calculus and the previous calculation yields

$$v(t) = v(t) - v(a) = \int_a^t v'(s)\ ds \leq \int_a^t \alpha(s)\beta(s) \exp\left(-\int_a^s \beta(\tau)\ d\tau\right)\ ds.$$

By reformulating the definition of v and the last calculation, we obtain

$$\int_a^t \beta(s)u(s)\ ds = \exp\left(\int_a^t \beta(s)\ ds\right)v(t)$$
$$\leq \exp\left(\int_a^t \beta(s)\ ds\right)\int_a^t \alpha(s)\beta(s) \exp\left(-\int_a^s \beta(\tau)\ d\tau\right)\ ds$$
$$= \int_a^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(\tau)\ d\tau\right)\ ds.$$

Finally, we insert the lastly calculated inequality into Inequality (4.6) and obtain the statement. ♣

After proving these lemmas, the main statement of this section can be proved: The path of any $x \in (0,1)$ of a pointwise solution is continuous in time.

Lemma 4.1.4 (Pointwise evaluation of a standard ODE)**.** *Let $T, R, L > 0$ be fixed. Let $I = [0,T) \subset \mathbb{R}$ be an interval and $G \subset \mathbb{R}^n$ an open set. Suppose $y_0 : [0,L] \to \mathbb{R}$ is measurable and bounded with $B_R(y_0(x)) \subset G$ for all $x \in [0,L]$. Furthermore, assume that $f : I \times [0,L] \times G \to \mathbb{R}^n$, $f = f(t,x,y)$ fulfils the following:*

(i) For each $x \in [0,L]$: The map $f(\cdot,x,\cdot) : I \times G \to \mathbb{R}^n$ is continuous,

(ii) For all $t \in I$ and all $x \in [0,L]$: The map $f(t,x,\cdot) : G \to \mathbb{R}^n$ is Lipschitz continuous with Lipschitz constant $L_f > 0$,

(iii) For all $y \in G$ the map $f(\cdot,\cdot,y) : I \times [0,L] \to \mathbb{R}^n$ is measurable,

(iv) There exists a constant $M_f > 0$ such that for all $t \in I, x \in [0,L], y \in G$ holds: $|f(t,x,y)| \leq M_f$.

Define $t_0 \coloneqq \min\{T, R/M_f\}$. Then, for all $x \in [0,L]$ the parameter dependent ODE

$$\dot{y}(t,x) = f(t,x,y(t,x))$$

has a unique solution on $[0,t_0]$ and the function $y : [0,t_0] \times [0,L] \to \mathbb{R}^n$ is $\mathcal{L}^2$-measurable and for all $t \in [0,t_0]$ holds that the function

$$x \mapsto y_0(x) + \int_0^t f(\tau,x,y(\tau,x))\ d\tau$$

with domain $[0,L]$ is $\mathcal{L}^1$-measurable.

Proof. The idea is to approximate the solution of the ODE as in the Euler method by iteration, see (4.7) below, show convergence of the approximation and obtain the measurability from Fubini's theorem and the fact that all functions in the iteration process are suitably measurable.
Step 1: Approximation of solution by iteration. In the following, we define an iteration to obtain approximation functions y_k of the actual solution of the ODE. As starting point of the iteration, identify y_0 as a function in $\mathrm{C}^1(0,T;L^\infty(0,1))$ by constant extension in time, namely, for $t\in[0,T]$ and $x\in(0,1)$ holds

$$y_0(t,x)=y_0(x).$$

For given, $\mathcal{L}^2$-measurable on $[0,T]\times[0,L]$ function y_k, $k\in\mathbb{N}$, with $|y_k(t,x)-y_0(t,x)|\le R$, define the next step y_{k+1} by

$$y_{k+1}(t,x)=y_0(x)+\int_0^t f(\tau,x,y_k(\tau,x))\ d\tau. \tag{4.7}$$

Step 2: Iteration is well-defined and approximating functions are $\mathcal{L}^2$-measurable. Next, we prove the iteration is well-defined, i.e. that y_{k+1} meets the requirements again and the integral in (4.7) is well-defined. Concerning (4.7) to be well-defined, from the assumption f to be measurable in (t,x) and continuous in y and the condition that y_k is $\mathcal{L}^2$-measurable, it follows that the map

$$(t,x)\mapsto f(t,x,y_k(t,x))$$

is again $\mathcal{L}^2$-measurable on $[0,t_0]\times[0,L]$, see Lemma 4.1.1. With the boundedness of f, see condition (iv), the integrability follows and the formula for y_{k+1} is well-defined. To prove that y_{k+1} meets the initial conditions again, note that for each $t\in[0,t_0]$ the characteristic function $\chi_{[0,t]\times\mathbb{R}}$ is measurable as well as the product of measurable functions is measurable again. Hence, $\chi_{[0,t]\times\mathbb{R}}f(\cdot,\cdot,y_k(\cdot,\cdot))$ is $\mathcal{L}^2$-measurable on $[0,t_0]\times[0,L]$ and due to the bound on f also $\mathcal{L}^2(t,x)$-integrable. For fixed $t\in[0,t_0]$ by Fubini's theorem, the map

$$x\mapsto\int_0^{t_0}\chi_{[0,t]\times\mathbb{R}}(\tau,x)f(\tau,x,y_k(\tau,x))\ d\tau=\int_0^t f(\tau,x,y_k(\tau,x))\ d\tau \tag{4.8}$$

is $\mathcal{L}^1(x)$-integrable on $[0,L]$, since

$$\int_0^L\int_0^{t_0}\chi_{[0,t]\times\mathbb{R}}(\tau,x)f(\tau,x,y_k(\tau,x))\ d\tau\ dx=\int_{[0,t]\times[0,L]} f(\tau,x,y_k(\tau,x))\ d(\tau,x),$$

and the integrability of the integrand, as stated before. Thus, the map in (4.8) is $\mathcal{L}^1$-measurable on $[0,L]$ and lies in $L^\infty(0,L)$. It follows that for each $t\in[0,t_0]$

$$y_{k+1}(t,\cdot)\colon[0,L]\to\mathbb{R},\ x\mapsto y_0(x)+\int_0^t f(\tau,x,y_k(\tau,x))\ d\tau$$

is $\mathcal{L}^1(x)$-measurable. Furthermore, the map

$$y_{k+1}\colon[0,t_0]\to L^\infty(0,L),\ t\mapsto y_{k+1}(t,\cdot)$$

is continuous: Let $t\in[0,t_0]$ be arbitrary and $(t_n)_{n\in\mathbb{N}}\subset[0,t_0]$ an arbitrary sequence with $t_n\to t$ for $n\to\infty$. W.l.o.g., assume $t_n\le t$ for all $n\in\mathbb{N}$. From the $\mathcal{L}^2$-integrability of the

$f(\cdot,\cdot,y_k(\cdot,\cdot))$, it follows that

$$\begin{aligned}
&\|y_{k+1}(t_n,\cdot) - y_{k+1}(t,\cdot)\|_{L^\infty(0,L)} \\
&= \operatorname*{ess\,sup}_{x\in[0,L]} |y_0(x) + \int_0^{t_n} f(\tau,x,y_k(\tau,x))\ d\tau - y_0(x) - \int_0^{t} f(\tau,x,y_k(\tau,x))\ d\tau| \\
&= \operatorname*{ess\,sup}_{x\in[0,L]} |\int_{t_n}^{t} f(\tau,x,y_k(\tau,x))\ d\tau| \\
&\leq \operatorname*{ess\,sup}_{x\in[0,L]} \int_{t_n}^{t} |f(\tau,x,y_k(\tau,x))|\ d\tau \\
&\leq |t_n - t| \cdot M_f \to 0,
\end{aligned}$$

for $n \to \infty$. This is the continuity of $y_{k+1}\colon [0,t_0] \to L^\infty(0,L)$ and from Lemma 4.1.2 the $\mathcal{L}^2$-measurability of y_{k+1} on $[0,t_0]\times(0,L)$ follows. Finally, we prove the condition that the distance to y_0 is estimated by R. For all $t\in[0,t_0]$ and all $x\in[0,L]$, the bound on f yields

$$\begin{aligned}
|y_{k+1}(t,x) - y_0(x)| &= |y_0(x) + \int_0^{t} f(\tau,x,y_k(\tau,x))\ d\tau - y_0(x)| \\
&= |\int_0^{t} f(\tau,x,y_k(\tau,x))\ d\tau| \\
&\leq \int_0^{T} |f(\tau,x,y_k(\tau,x))|\ d\tau \\
&\leq T\cdot M_f \leq \frac{R}{M_f}\cdot M_f = R.
\end{aligned}$$

This concludes that the iteration is well-defined.
Step 3: Approximation fulfils a "Cauchy property". For fixed $x\in[0,L]$, we prove by mathematical induction that the sequence $(y_k(\cdot,x))_{k\in\mathbb{N}}$ satisfies the "Cauchy property" that for all $t\in[0,t_0]$

$$|y_{k+1}(t,x) - y_k(t,x)| \leq M_f L_f^k \frac{t^{k+1}}{(k+1)!} \tag{4.9}$$

holds. For the base case $k=0$, it is for $t\in[0,t_0]$

$$|y_1(t,x) - y_0(t,x)| = |\int_0^{t} f(\tau,x,y_0(\tau,x))\ d\tau| \leq M_f\cdot t = M_f L_f^0 \frac{t^{0+1}}{(0+1)!}.$$

To show the inductive step, consider the assertion has been established for $k\geq 0$. Then, for all $t\in[0,t_0]$,

$$\begin{aligned}
&|y_{k+2}(t,x) - y_{k+1}(t,x)| \\
&= |y_0(x) + \int_0^{t} f(\tau,x,y_{k+1}(\tau,x))\ d\tau - y_0(x) - \int_0^{t} f(\tau,x,y_k(\tau,x))\ d\tau|
\end{aligned}$$

$$
\begin{aligned}
&= \left|\int_0^t f(\tau,x,y_{k+1}(\tau,x)) - f(\tau,x,y_k(\tau,x))\ d\tau\right| \\
&\leq \int_0^t |f(\tau,x,y_{k+1}(\tau,x)) - f(\tau,x,y_k(\tau,x))|\ d\tau \\
&\leq \int_0^t L_f |y_{k+1}(\tau,x) - y_k(\tau,x)|\ d\tau \\
&\leq \int_0^t L_f \cdot M_f L_f^k \frac{\tau^k}{(k+1)!}\ d\tau \\
&= M_f L_f^{k+1} \frac{t^{k+2}}{(k+2)!},
\end{aligned}
$$

which states the claim.
Step 4: Finding a limit. Define the sequence

$$
(g_k)_{k\in\mathbb{N}},\ g_k \coloneqq \sum_{n=0}^{k} (y_{n+1}(\cdot,x) - y_n(\cdot,x)) + y_0(x) = y_{k+1}(\cdot,x).
$$

These functions are continuous, since for any $k \in \mathbb{N}$ and any $t \in [0,t_0]$ and any sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to t$ for $n \to \infty$ and $t_n \leq t$ for all $n \in \mathbb{N}$ holds with formula (4.7) and the bound on f (iv)

$$
\begin{aligned}
|g_k(t_n) - g_k(t)| &= |y_{k+1}(t_n,x) - y_{k+1}(t,x) + y_0(x) - y_0(x)| \\
&= \left|\int_0^{t_n} f(\tau,x,y_k(\tau,x))\ d\tau - \int_0^t f(\tau,x,y_k(\tau,x))\ d\tau\right| \\
&= \left|\int_{t_n}^t f(\tau,x,y_k(\tau,x))\ d\tau\right| \\
&\leq \int_{t_n}^t |f(\tau,x,y_k(\tau,x))|\ d\tau \\
&\leq |t_n - t| \cdot M_f,
\end{aligned}
$$

which converges to 0 for $n \to \infty$. Moreover, it follows from (4.9) that $(g_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{C}^0([0,t_0])$ and therefore, there exists a continuous limit function

$$
y(\cdot,x)\colon [0,t_0] \to \mathbb{R}^n
$$

and the uniform convergence for a subsequence, again denoted by g_k,

$$
g_k \to y(\cdot,x),\ k \to \infty.
$$

Moreover,

$$
y(\cdot,x) = \lim_{k\to\infty}\left(y_0(x) + \sum_{n=0}^{k}(y_{n+1}(\cdot,x) - y_n(\cdot,x))\right) = \lim_{k\to\infty} y_{k+1}(\cdot,x).
$$

By Lipschitz continuity, for each $t \in [0, t_0]$ holds

$$|f(t, x, y(t, x)) - f(t, x, y_k(t, x))| \leq L_f |y(t, x) - y_k(t, x)|$$

and consequently the sequence $(f(\cdot, x, y_k(\cdot, x)))_{k\in\mathbb{N}}$ converges uniformly to $f(\cdot, x, y(\cdot, x))$ for $k \to \infty$. With this, one can pass to the limit $k \to \infty$ in the definition of y_{k+1} and obtains that

$$\begin{aligned} y(t, x) &= \lim_{k\to\infty} y_{k+1}(t, x) \\ &= \lim_{k\to\infty} \left(y_0(x) + \int_0^t f(\tau, x, y_k(\tau, x))\, d\tau \right) \\ &= y_0(x) + \int_0^t f(\tau, x, y(\tau, x))\, d\tau. \end{aligned}$$

Step 5: Limit is measurable. Finally, we prove the stated measurability. Above we stated that the functions $y_k := [0, t_0] \times [0, L] \to \mathbb{R}^n$ are $\mathcal{L}^2$-measurable for all $k \in \mathbb{N}$. Furthermore, the last calculations shows the pointwise almost everywhere convergence $y_k \to y$. That is why y is $\mathcal{L}^2$-measurable on $[0, t_0] \times [0, L]$. Since y is $\mathcal{L}^2$-measurable and the properties of f, with the argumentation from above follows that for each $t \in [0, t_0]$ the map

$$x \mapsto y_0(x) + \int_0^t f(\tau, x, y(\tau, x))\, d\tau$$

is $\mathcal{L}^1(x)$-integrable and $\mathcal{L}^1$-measurable on $[0, L]$. Finally, $y \in \mathrm{C}^0([0, T]; L^\infty(0, L))$ follows.
Step 6: Uniqueness of the solution. Assume y^1, y^2 to be solutions of the ODEs

$$\begin{aligned} &\dot{y}^1(t, x) = f(t, x, y^1(t, x)),\ t \in [0, t_0], x \in [0, L], \\ &\dot{y}^2(t, x) = f(t, x, y^2(t, x)),\ t \in [0, t_0], x \in [0, L], \\ &y^1(0, x) = y^2(0, x) = y_0(x), x \in [0, L]. \end{aligned}$$

Then, the integral equations

$$\begin{aligned} y^1(t, x) &= y_0^1(x) + \int_0^t f(s, x, y^1(s, x))\, ds, \\ y^2(t, x) &= y_0^2(x) + \int_0^t f(s, x, y^2(s, x))\, ds \end{aligned}$$

hold for all $t \in [0, t_0]$ and all $x \in (0, 1)$. From the Lipschitz continuity of f, it follows

$$\begin{aligned} |y^1(t, x) - y^2(t, x)| &= |\int_0^t f(s, x, y^1(s, x)) - f(s, x, y^2(s, x))\, ds| \\ &\leq \int_0^t |f(s, x, y^1(s, x)) - f(s, x, y^2(s, x))|\, ds \\ &\leq \int_0^t L_f |y^1(s, x) - y^2(s, x)|\, ds. \end{aligned}$$

We apply the Gronwall Lemma 4.1.3 for $u(t) = |y^1(t,x) - y^2(t,x)|$, $\alpha(t) = 0$ and $\beta(t) = L_f > 0$. It yields

$$\begin{aligned} |y^1(t,x) - y^2(t,x)| &\leq 0 + \int\limits_0^t 0 \cdot L_f \exp\left(\int\limits_s^t L_f \, d\tau\right) \, ds \\ &= 0. \end{aligned}$$

Therefore, $y^1(t,x) = y^2(t,x)$ for all $t \in [0,t_0]$ and all $x \in (0,1)$ follows and the uniqueness is shown. ♣

4.2. ... for material consisting of two parts

The goal of this section is to prove existence and uniqueness of the AMP with two materials by the Picard–Lindelöf theorem A.2.6. To do so, the ODE is interpreted in the Banach space $L^\infty(0,1)$. To be precise, the following states the main theorem.

Theorem 4.2.1 (Existence and uniqueness of AMP with two materials)**.** *Consider the conditions from Definition 3.2.1 to hold. Then, there exists a unique solution (G,S) of the AMP with two materials.*

Proof. **Idea:** We want to use the Picard–Lindelöf theorem A.2.6 to obtain short time existence and uniqueness, namely we want to find a fixed $t_0 > 0$ such that, for given bounded initial datum at time t, we get existence and uniqueness for the interval $[t, t+t_0]$. Because the solution is bounded (see below) we can take the solution evaluated in $t + t_0$ as new initial datum for the next time interval $[t+t_0, t+2t_0]$. Starting with $t = 0$ and repeating this procedure T/t_0-times, we get existence and uniqueness on $[0,T]$. To do so, it is important, that t_0 does not depend on the initial data, radius chosen in Theorem A.2.6 or the time.
Step 1: A priori estimates on the solution. Assume (G,S) to be a solution of the AMP with two materials defined in Definition 3.2.3. Then, Lemma 4.2.4 below states that this solution is bounded, namely

$$G(t,x) \in [G_{min}, G_{max}] \quad \forall t \in [0,T], x \in [0,1], \tag{4.10}$$

where $0 < G_{min} < G_{max} < \infty$ from (4.16) and (4.17). In the following, we do not consider the interval $[G_{min}, G_{max}] \subset \mathbb{R}$ but the ball $\overline{B_{R^*}(G^*)} \subset L^\infty(0,1)$, where

$$\begin{aligned} R^* &:= \frac{G_{max} - G_{min}}{2} \in \mathbb{R}, \\ G^* &:= \frac{G_{max} + G_{min}}{2} \in L^\infty(0,1), \end{aligned} \tag{4.11}$$

because we solve the ODE in the Banach space $L^\infty(0,1)$ as discussed in the beginning of this section. With that definition, it holds

$$\begin{aligned} G^* - R^* &= \frac{G_{max} + G_{min}}{2} - \frac{G_{max} - G_{min}}{2} = G_{min}, \\ G^* + R^* &= \frac{G_{max} + G_{min}}{2} + \frac{G_{max} - G_{min}}{2} = G_{max}, \end{aligned}$$

and (4.10) implies

$$G(t,\cdot) \in \overline{B_{R^*}(G^*)} \subset L^\infty(0,1)$$

for all $t \in [0,T]$.

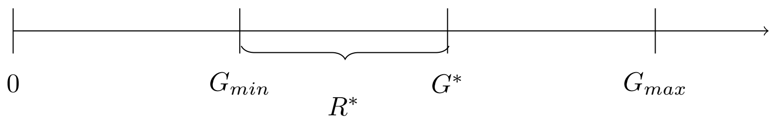

Figure 4.1.: Schematic picture of G^* and R^*.

Step 2: RHS is Lipschitz continuous. First, we prove that $\mathcal{G}\colon [0,T] \times \mathbb{R} \times \overline{B_{R^*}(G^*)} \to \overline{B_{R^*}(G^*)}$, $(t,S,G) \mapsto \mathcal{G}(t,S,G)$ is Lipschitz continuous in S and G. Let $t \in [0,T]$ let $L_\mu > 0$ denote the Lipschitz constant of μ. Then, for $S_1, S_2 \in \mathbb{R}$ and $G_1, G_2 \in \overline{B_{R^*}(G^*)}$ it holds

$$\begin{aligned}
&\|\mathcal{G}(t,\cdot,S_1,G_1(\cdot)) - \mathcal{G}(t,\cdot,S_2,G_2(\cdot))\|_{L^\infty(0,1)}\\
=&\|\gamma\mu(S_1)G_1 - \gamma\mu(S_2)G_2\|_{L^\infty(0,1)}\\
\le&\|\gamma\mu(S_1)G_1 - \gamma\mu(S_1)G_2\|_{L^\infty(0,1)} + \|\gamma\mu(S_1)G_2 - \gamma\mu(S_2)G_2\|_{L^\infty(0,1)}\\
\le&|\gamma\mu(S_1)|\|G_1 - G_2\|_{L^\infty(0,1)} + |\gamma\mu(S_1) - \gamma\mu(S_2)|\|G_2\|_{L^\infty(0,1)}\\
\le&(\gamma\mu)_{max}\|G_1 - G_2\|_{L^\infty(0,1)} + \gamma_{max}L_\mu\|S_1 - S_2\|_{L^\infty(0,1)}G_{max}.
\end{aligned}$$

Moreover, $\mathcal{H}$ inherits the Lipschitz continuity in S and G from $\mathcal{G}$. Let $S_1, S_2 \in \mathbb{R}$ and $G_1, G_2 \in \overline{B_{R^*}(G^*)}$ be arbitrary. Then, it follows with the Lipschitz continuity of $\mathcal{G}$ and the definition of $\mathcal{H}$ in (4.1)

$$\begin{aligned}
&\|\mathcal{H}(t,S_1,G_1) - \mathcal{H}(t,S_2,G_2)\|_{L^\infty(0,1)}\\
&= \operatorname*{ess\,sup}_{x\in(0,1)} |\mathcal{G}(t,x,S_1,G_1(x)) - \mathcal{G}(t,x,S_2,G_2(x))|\\
&\le \operatorname*{ess\,sup}_{x\in(0,1)} L_{\mathcal{G}}\left(|S_1 - S_2| + |G_1(x) - G_2(x)|\right)\\
&= L_{\mathcal{G}}\left(|S_1 - S_2| + \|G_1 - G_2\|_{L^\infty(0,1)}\right),
\end{aligned}$$

which is the Lipschitz-continuity for $\mathcal{H}$ in S and G. Further, Proposition 4.2.5 below states the Lipschitz continuity of

$$S : \overline{B_{\tilde{R}}(G^*)} \subset L^\infty(0,1) \to \mathbb{R},\ G \mapsto S(G),$$

where $\tilde{R}$ is defined in (4.12). From Lemma A.2.2 (iii), the composition of S and $\mathcal{H}$ is again Lipschitz continuous, namely

$$\begin{aligned}
\mathcal{F} &\coloneqq \mathcal{H} \circ S\colon [0,T] \times \overline{B_{\tilde{R}}(G^*)} \to L^\infty(0,1),\\
&\mathcal{F}(t,G)(x) \coloneqq \mathcal{H}(t,S(G),G)(x) = \mathcal{G}(t,x,S(G),G(x))
\end{aligned}$$

is Lipschitz continuous.

Step 3: Suitable ball. When we apply the Picard–Lindelöf theorem to $\mathcal{F}$, the existence time t_0 depends on the ball, we apply it to. This ball has to be centred around the initial datum, see the conditions of A.2.6, what seems to contradict t_0 to be independent of the solution. To overcome this, we apply the Picard–Lindelöf theorem to balls with different centres but the same radius, namely $G_{min}/2$, and find a bigger bound M on the RHS $\mathcal{F}$, but which is independent of the centre. We will see that then the existence time t_0 becomes smaller but independent of the initial data.

Define the radius

$$\tilde{R} \coloneqq \frac{G_{min}}{2} + R^* = \frac{G_{max}}{2}. \tag{4.12}$$

Then, for all $\tilde{G}_0 \in \overline{B_{R^*}(G^*)}$, we can apply the Picard–Lindelöf theorem A.2.6 with the ball

$$B_{\frac{G_{min}}{2}}(\tilde{G}_0) \subset B_{\tilde{R}}(G^*),$$

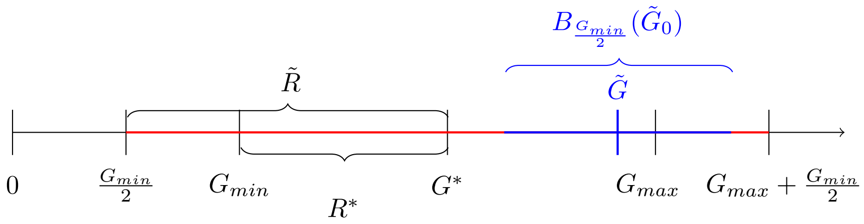

Figure 4.2.: Schematic picture of the balls. Red marks the ball $\overline{B_{\tilde{R}}(G^*)}$, blue gives an example of the ball $B_{\frac{G_{min}}{2}}(\tilde{G}_0)$.

because we know that the initial data, evaluated solution, lies in that ball as seen in Step 1. Furthermore, it holds

$$\begin{aligned} M\left(\tilde{G}_0, \frac{G_{min}}{2}\right) &= \max\left\{\|\mathcal{F}(t,G)\| \mid t \in [0,T], G \in \overline{B_{\frac{G_{min}}{2}}(\tilde{G}_0)}\right\} \\ &\leq \max\left\{\|\mathcal{F}(t,G)\| \mid t \in [0,T], G \in \overline{B_{\tilde{R}}(G^*)}\right\} \\ &= M(G^*, \tilde{R}), \end{aligned}$$

where we used Theorem A.2.6 for the definition of M. Using this, we obtain

$$\begin{aligned} t_0\left(\tilde{G}_0, \frac{G_{min}}{2}\right) &= \min\left\{T, \frac{\frac{G_{min}}{2}}{M(\tilde{G}_0, \frac{G_{min}}{2})}\right\} \\ &\geq \min\left\{T, \frac{G_{min}}{2M(G^*, \tilde{R})}\right\} \\ &=: t_0(G_{min}, G_{max}), \end{aligned} \tag{4.13}$$

where we used that, by definitions, $\tilde{R}$ and G^* depend only on G_{min} and G_{max}.
Step 4: Short time solution. We proved in Steps 1 to 3 that all conditions for the Picard–Lindelöf theorem A.2.6 are fulfilled. Hence, for any initial datum $\tilde{G}_0 \in \overline{B_{R^*}(G^*)}$ at time $\tilde{t} \in [0,T]$, applying it to the RHS $\mathcal{F}$ and the ball $B_{G_{min}/2}(\tilde{G}_0)$ yields the existence time $t_0 = t_0(G_{min}, G_{max})$ and a solution $G \in \mathrm{C}^1([t, t+t_0]; B_{G_{min}/2}(\tilde{G}_0))$ of the ODE

$$\begin{aligned} \dot{G}(t) &= \mathcal{F}(t, G(t)), \\ G(\tilde{t}) &= \tilde{G}_0. \end{aligned}$$

To obtain the bounds on the solution as done in Step 1, take the stress tensor $S(t)$ corresponding to the solution G and apply Lemma 4.1.4 to the RHS

$$f(t,x,G) := \mathcal{G}(t,x,S(t),G)$$

to obtain that the paths are continuous in time in order to apply the Gronwall Lemma 4.2.2 and obtain the estimates on the solution. That way, the argumentation explained in the beginning of the proof, can be applied.
As proved in Step 3, we can shorten the existence time to $t_0(G_{min}, G_{max})$ which is independent of the initial datum. To obtain a long time solution, we start with the given initial data G_0 and get the existence and uniqueness of the solution of

$$\begin{aligned} \dot{G}(t) &= \mathcal{F}(t, G(t)), \\ G(0) &= 1 \end{aligned}$$

on the time interval $[0, t_0(G_{min}, G_{max})]$. Next, apply the Picard–Lindelöf theorem to the ODE with initial datum given by the last value of the former solution, namely $\tilde{G}_0 := G(t_0)$, which is in $\overline{B_{R^*}(G^*)}$ due to Step 1, and hence, an admissible initial value. We get existence and uniqueness for the next t_0 time. Together we found the solution on $[0, 2t_0]$. Applying the theorem $\lceil T/t_0 \rceil$ times yields the solution on $[0,T]$. The C^1-property in the points kt_0 for $k = 1, \ldots, \lfloor T \rfloor$ is obtained since the solutions of the ODE is C^1-regular especially in the start and end points. ♣

The above proof uses lemmas which are left to prove. To start off with the estimates on the growth tensor, the following Gronwall lemma 4.2.2 will be applied. To meet the condition of continuous functions, the paths for each particle $x \in (0,1)$ has to be continuous. This is discussed in the previous section, see lemma 4.1.4.

Lemma 4.2.2 (Gronwall lemma)**.** *Let $I \coloneqq [a,b]$ be an interval, $u \in \mathrm{C}^1(I)$ and $\alpha, \beta\colon I \to \mathbb{R}$ be continuous functions. Further, assume for all $t \in I$ the inequality*

$$\dot{u}(t) \leq \alpha(t) + \beta(t)u(t) \tag{4.14}$$

to hold. Then, it follows

$$u(t) \leq \exp\left(\int_a^t \beta(\tau)\ d\tau\right) u(a) + \int_a^t \exp\left(\int_s^t \beta(\tau)\ d\tau\right) \alpha(s)\ ds$$

for all $t \in I$.

Proof. The idea is to test the given equation with a suitable term to derive an equation from which we can conclude the desired statement.
The term to test with is defined as

$$v\colon [a,b] \to \mathbb{R},\ v(t) \coloneqq \exp\left(-\int_a^t \beta(\tau)\ d\tau\right).$$

We now take the derivative of vu by product rule and use (4.14) to get

$$(vu)' = v'u + vu' = -\beta vu + vu' \leq -\beta vu + v\alpha + v\beta u = v\alpha. \tag{4.15}$$

Integrating from a to $t \in [a,b]$ yields

$$\begin{aligned} v(t)u(t) &\leq v(a)u(a) + \int_a^t v(s)\alpha(s)\ ds \\ &= u(a) + \int_a^t v(s)\alpha(s)\ ds \end{aligned}$$

and last, by dividing by $v(t)$ yields the desired result, namely

$$\begin{aligned} u(t) &\leq v(t)^{-1}u(a) + v(t)^{-1}\int_a^t v(s)\alpha(s)\ ds \\ &= \exp\left(\int_a^t \beta(\tau)\ d\tau\right) u(t) + \exp\left(\int_a^t \beta(\tau)\ d\tau\right)\int_a^t \exp\left(-\int_a^s \beta(\tau)\ d\tau\right)\alpha(s)\ ds \\ &= \exp\left(\int_a^t \beta(\tau)\ d\tau\right) u(t) + \int_a^t \exp\left(\int_s^t \beta(\tau)\ d\tau\right)\alpha(s)\ ds. \end{aligned}$$

♣

Remark 4.2.3. *If (4.14) holds with "$\geq$" instead, it follows*

$$u(t) \geq \exp\left(\int_a^t \beta(\tau)\ d\tau\right) u(a) + \int\limits_a^t \exp\left(\int_s^t \beta(\tau)\ d\tau\right) \alpha(s)\ ds.$$

This one can see in the calculation in (4.15), where "$\geq$" would appear instead. And the rest of the proof goes analogously.

Lemma 4.2.4 (A priori estimate on growth tensor with two materials)**.** *Assume the Setting 3.2.1 to hold. Further suppose that (G, S) is a solution to the AMP with two materials. Then, there exist $0 < G_{min} < G_{max} < \infty$ such that for all $t \in [0, T]$ and almost all $x \in (0, 1)$*

$$G(t, x) \in [G_{min}, G_{max}]$$

holds.

Proof. Assume there exists a solution of the AMP with two materials. Lemma 4.1.4 yields the existence of a pointwise solution G of the ODE, i.e. for all $x \in (0, 1)$ the solution $G(\cdot, x)\colon [0, T] \to \mathbb{R}$ fulfils the ODE

$$\dot{G}(t, x) = \gamma(x)\mu(S(t))G(t, x)$$

for all $t \in [0, T]$, where stress tensor S is a part of the solution (G, S). Furthermore, the pointwise solutions has continuous paths.
Case 1: $G(t, x) \geq 0$ for all $x \in (0, 1)$ and all $t \in [0, T]$. Using the upper estimate on γ and μ, it follows

$$\dot{G}(t, x) \leq (\gamma\mu)_{max} G(t, x).$$

Applying Gronwall lemma 4.2.2 with $\alpha = 0$ leads to

$$\begin{aligned} G(t, x) &\leq G(0, x) \exp\left(\int_0^t (\gamma\mu)_{max}\ d\tau\right) + \int\limits_0^t \alpha(\tau)(\gamma\mu)_{max} \exp\left(\int_\tau^t (\gamma\mu)_{max}\ ds\right)\ d\tau \\ &= \exp((\gamma\mu)_{max} t) \\ &\leq \exp\left((\gamma\mu)_{max} T\right) =: G_{max}. \end{aligned} \tag{4.16}$$

To obtain a lower estimate, define $H(t, x) := -G(t, x)$. It fulfils the ODE

$$\dot{H}(t, x) = -\dot{G}(t, x) = \gamma(x)\mu(S(t))(-G(t, x)) = \gamma(x)\mu(S(t))H(t, x).$$

With the bounds on γ and μ, we obtain

$$\dot{H}(t, x) \leq (\gamma\mu)_{min} H(t, x).$$

Gronwall lemma 4.2.2 with $\alpha = 0$ implies

$$\begin{aligned} H(t, x) &\leq H(0, x) \exp\left(\int_0^t (\gamma\mu)_{min}\ d\tau\right) + \int\limits_0^t \alpha(\tau)(\gamma\mu)_{min} \exp\left(\int_\tau^t (\gamma\mu)_{min}\ ds\right)\ d\tau \\ &= -G(0, x) \exp\left((\gamma\mu)_{min} t\right) \\ &\leq -\exp\left((\gamma\mu)_{min} T\right) =: -G_{min}. \end{aligned} \tag{4.17}$$

Finally, we derive

$$G(t,x) = -H(t,x) \geq G_{min} > 0.$$

Case 2: $G(t,x) < 0$ for one $x \in (0,1)$ and one $t \in [0,T]$. Assume, there exists an $x \in (0,1)$ such that for a $t^- \in (0,T]$ $G(t^-,x) \leq 0$, then, there exists a time interval I such that for all $t^* \in I$ holds

$$G(t^*,x) \in [G_{min}/2, G_{min}). \tag{4.18}$$

But since $G_{min}/2 > 0$, we can define $H(t,x) := -G(t,x)$ for all $t \in I$ and obtain $H(t,x) \leq -G_m in$ on I. This is a contradiction to (4.18). ♣

Furthermore, the proof of Theorem 4.2.1 uses that the stress tensor S depends Lipschitz continuously on the growth tensor $G \in \overline{B_{\tilde{R}}(G^*)}$. This is stated in the following lemma by using the implicit function theorem, see Theorem A.3.5.

Proposition 4.2.5 (Lipschitz continuity of the stress tensor)**.** *Consider the conditions 3.2.1 to hold. Let $\tilde{R}$ be given by (4.12) and G^* by (4.11). Then, the map*

$$S\colon \overline{B_{\tilde{R}}(G^*)} \subset L^\infty(0,1) \to \mathbb{R},\ G \mapsto S(G),$$

where $(G, S(G))$ solves the AMP with two materials, is Lipschitz continuous.

Proof. **Step 1: Concerning the growth map g.** Let $G \in \overline{B_{\tilde{R}}(G^*)}$ be fixed. Using (3.15), we define the growth map g by

$$g\colon \overline{B_{\tilde{R}}(G^*)} \to W^{1,\infty}(0,1),\ G \mapsto g(G) \text{ with } g(G)(x) := \int_0^x G(\tilde{x})\ d\tilde{x}.$$

To show Lipschitz continuity, we check for $G_1, G_2 \in \overline{B_{\tilde{R}}(G^*)}$ that

$$\begin{aligned}
&\|g(G_1) - g(G_2)\|_{W^{1,\infty}((0,1))} \\
&\leq \sup_{x\in(0,1)} \left| \int_0^x G_1(\tilde{x})\ d\tilde{x} - \int_0^x G_1(\tilde{x})\ d\tilde{x} \right| + \|\partial_x g(G_1) - \partial_x g(G_2)\|_\infty \\
&\leq \sup_{x\in(0,1)} \int_0^x |G_1(\tilde{x}) - G_1(\tilde{x})|\ d\tilde{x} + \|G_1 - G_2\|_\infty \\
&\leq \int_0^1 \sup_{x\in(0,1)} |G_1(\tilde{x}) - G_1(\tilde{x})|\ d\tilde{x} + \|G_1 - G_2\|_\infty \\
&= 2\|G_1 - G_2\|_\infty
\end{aligned}$$

holds.

Step 2: Concerning grown interface point Y. The grown interface point depends on G, namely

$$Y\colon \overline{B_{\tilde{R}}(G^*)} \to \mathbb{R},\ Y(G) := g(G)(\ell).$$

Hence, the Lipschitz continuity is obtained by the following calculation for $G_1, G_2 \in \overline{B_{\tilde{R}}(G^*)}$ using the Lipschitz continuity of g

$$\begin{aligned}
|Y(G_1) - Y(G_2)| &= |g(G_1)(\ell) - g(G_2)(\ell)| \\
&\leq \|g(G_1) - g(G_2)\|_\infty
\end{aligned}$$

$$\leq \|G_1 - G_2\|_\infty.$$

Step 3: Concerning grown lengths. The grown lengths are given by the distance between two positions of the interfaces after stress-free growth, in dependence of G it is

$$\lambda\colon \overline{B_{\tilde{R}}(G^*)} \to \mathbb{R}^2_{>0},\ \lambda_1(G) = Y(G),\ \lambda_2(G) = g(G)(1) - Y(G).$$

Again with supremums norm on $\mathbb{R}^2$ we check the Lipschitz continuity with $G_1, G_2 \in \overline{B_{\tilde{R}}(G^*)}$:

$$\begin{aligned}|\lambda_2(G_1) - \lambda_2(G_2)| &= |g(G_1)(1) - Y(G_1) - g(G_2)(1) + Y(G_2)| \\ &\leq \|g(G_1) - g(G_2)\|_\infty + |Y(G_1) - Y(G_2)| \\ &\leq 2\|G_1 - G_2\|_\infty.\end{aligned}$$

A similar calculation holds for λ_2.

Step 4: Concerning the elastic deformation. Using the growth map $g = g(G)$ the variational problem to minimize for the elastic deformation ϕ is

$$E_G(\partial_y\phi) = \int_0^{g(G)(1)} W(g(G)^{-1}(y), \partial_y\phi(y))\, dy = \int_0^{g(G)(1)} \kappa(g(G)^{-1}(y))\tilde{W}(\partial_y\phi(y))\, dy$$

with boundary conditions as above. It has a unique solution, see Remark 3.2.4. The Euler–Lagrange equation $\kappa(g(G)^{-1}(y))D_F W(\partial_y\phi(y)) = C_D$ yields that $\partial_y\phi$ is constant on $(0, Y(G))$ and on $(Y(G), g(G)(1))$ respectively. It is determined by the boundary conditions and continuity as

$$\partial_y\phi(y) = \begin{cases} \frac{\xi(G)}{\lambda_1(G)}, & y \in (0, \lambda_1(G)), \\ \frac{1-\xi(G)}{\lambda_2(G)}, & y \in (\lambda_1(G), g(G)(1)), \end{cases}$$

where the interface point ξ is characterized from the equilibrium condition

$$\kappa_1 D_F\tilde{W}\left(\frac{\xi_1(G)}{\lambda_1(G)}\right) = \kappa_2 D_F\tilde{W}\left(\frac{1-\xi_1(G)}{\lambda_2(G)}\right).$$

The map

$$\xi\colon \mathbb{R}^2_{>0} \to (0,1),\ (\lambda_1, \lambda_2) \mapsto \xi(\lambda_1, \lambda_2)$$

is C^1, see Lemma 3.1.9. Hence, it is Lipschitz continuous on any compact set. Furthermore, we have uniform bounds on $\lambda(G)$: Let $G \in \overline{B_{\tilde{R}}(G^*)})$. Then,

$$\begin{aligned}\lambda_1(G) &= g(G)(\ell) - g(G)(0) \\ &= \int_0^\ell G(x)\, dx \\ &\geq \ell G_{min} =: \lambda_{min}(\ell),\end{aligned}$$

and

$$\begin{aligned}\lambda_1(G) &= \int_0^\ell G(x)\, dx \\ &\leq \ell G_{max} =: \lambda_{max}(\ell).\end{aligned}$$

Similar calculations hold for λ_2. Hence, $\lambda\colon \overline{B_{\tilde{R}}(G^*)} \to [\lambda_{min}, \lambda_{max}]^2$ and

$$\xi \circ \lambda\colon \overline{B_{\tilde{R}}(G^*)} \to (0,1) \text{ is Lipschitz continuous.}$$

Further, $\xi\colon [\lambda_{min}, \lambda_{max}] \to (0,1)$ is a continuous mapping of a compact set, see Lemma 3.1.9. Consequently, there exists $\xi_{min}, \xi_{max} \in (0,1)$ such that

$$0 < \xi_{min} \leq \xi \leq \xi_{max} < 1.$$

Step 5: Concerning the derivatives of the elastic deformation. Stated above, $\partial_y \phi_i := \partial_y \phi|_{(Y_i, Y_{i+1})} = \frac{\xi_{i+1} - \xi_i}{\lambda_i}$ holds, where we again use $\xi_0 = 0$ and $\xi_2 = 1$. Hence, the maps $\partial_y \phi_i\colon \overline{B_{\tilde{R}}(G^*)} \to \mathbb{R}$, $i = 1, 2$ are Lipschitz continuous: Consider $G_1, G_2 \in \overline{B_R(G^*)}$. Then, using the estimate on ξ and λ and their Lipschitz continuity with constants L_ξ and L_λ, holds

$$\begin{aligned}
|\partial_y \phi_1(G_1) - \partial_y \phi_1(G_2)| &= \left| \frac{\xi_1(G_1) - \xi_0(G_1)}{\lambda_1(G_1)} - \frac{\xi_1(G_2) - \xi_0(G_2)}{\lambda_1(G_2)} \right| \\
&= \left| \frac{\xi_1(G_1)\lambda_1(G_2) - \xi_1(G_2)\lambda_1(G_1)}{\lambda_1(G_1)\lambda_1(G_2)} \right| \\
&= \left| \frac{(\xi_1(G_1) - \xi_1(G_2))\lambda_1(G_2) + \xi_1(G_2)(\lambda_1(G_2) - \lambda_1(G_2))}{\lambda_1(G_1)\lambda_1(G_2)} \right| \\
&\leq \frac{L_\xi \|G_1 - G_2\| \lambda_1(G_2)}{\lambda_1(G_1)\lambda_1(G_2)} + \frac{\xi_1(G_2) L_\lambda \|G_1 - G_2\|}{\lambda_1(G_1)\lambda_1(G_2)} \\
&\leq \left(\frac{L_\xi}{\lambda_{min}} + \frac{\xi_{max} L_\lambda}{\lambda_{min}^2} \right) \|G_1 - G_2\|.
\end{aligned}$$

An analogue calculation holds for $\partial_y \phi_2$.
Step 6: Lipschitz continuity of S. For $G \in \overline{B_{\tilde{R}}(G^*)}$ holds

$$\partial_y \phi_0(G) = \frac{\xi_1(G) - \xi_0(G)}{\lambda_1(G)} \leq \frac{\xi_{max}}{\lambda_{min}}$$

and

$$\partial_y \phi_0(G) = \frac{\xi_1(G) - \xi_0(G)}{\lambda_2(G)} \geq \frac{\xi_{min}}{\lambda_{max}} > 0.$$

This together with $\tilde{W} \in C^2((0,\infty))$ yields that $D_F \tilde{W}\colon [\frac{\xi_{min}}{\lambda_{max}}, \frac{\xi_{max}}{\lambda_{min}}] \to \mathbb{R}$ is Lipschitz continuous and since the composition of Lipschitz continuous functions is Lipschitz continuous, see Lemma A.2.2, we obtain that

$$S\colon \overline{B_{\tilde{R}}(G^*)} \to \mathbb{R},\ G \mapsto S(G) = \kappa_1 D_F \tilde{W}(\partial_y \phi_1(G))$$

is Lipschitz continuous. ♣

Concerning nutrients

After stating the existence and uniqueness 3.2.5 of the simplified model of the AMP, the nutrients shall be included again. The idea is to use the same method to include nutrients as on the inclusion of the stress tensor: Prove Lipschitz dependence on the growth tensor in a suitable space, include a term in the ODE including the nutrients Lipschitz continuously and prove existence with the Picard–Lindelöf theorem. This needs a setting including the nutrients into the ODE as well as an equation for the nutrients.
In Setting 3.2.1 with only two different materials, it is consistent to assume the material to have the same diffusion coefficient everywhere on each part respectively, i.e. for $D_1, D_2 > 0$ the

diffusion coefficient D defined by

$$D\colon (0,1) \to \mathbb{R},\ D(x) := \begin{cases} D_1, & \text{if } x < \ell, \\ D_2, & \text{if } x > \ell. \end{cases} \tag{4.19}$$

Similarly, define for $\beta_1, \beta_2 > 0$ the absorption rate β by

$$\beta\colon (0,1) \to \mathbb{R},\ \beta(x) := \begin{cases} \beta_1, & \text{if } x < \ell, \\ \beta_2, & \text{if } x > \ell. \end{cases} \tag{4.20}$$

Remark 4.2.6 (Where to solve the equation for the nutrients?)**.** *In one dimension, the equation for the elastic problem is solved between the sets $\Omega_{nat} = g([0,1])$ and $\Omega_t = [0,1]$, and therefore, the equation 2.13 is solved. By using the argument of local Lipschitz continuity, the growth map g and elastic deformation ϕ are often discussed, where the gradient of the total map ∇X is not used, yet. This is the reason to consider the equation (2.19) for the nutrients in the deformed configuration $\Omega_t = \phi(g(0,1))$, which is a standard elliptic equation, and not on the reference configuration with many terms of gradients, which do not simplify in one dimension, see Equation (2.18). Also, see Remark 2.3.2(ii).*

When we want to solve the equation for the nutrients in the final configuration for a fixed time $t \in [0,T]$, the coefficients have to be transformed to it. Let $g(t,\cdot)$ denote a growth map and $\phi(t,\cdot)$ an elastic deformation and λ_i denote the grown length and ξ the according interface point. We denote D_t and β_t here by $D_{\xi(t)}$ and $\beta_{\xi(t)}$ as they are constant on $(0,\xi(t))$ and $(\xi,1)$, respectively. From the assumption that the growth does not change the properties of the materials we have

$$D_{nat}(y) = D(g^{-1}(y)) \text{ and } \beta_{nat}(y) = \beta(g^{-1}(y)). \tag{4.21}$$

Moreover, we obtain the coefficients in the deformed configuration by testing the equation for the nutrients with a test function $\psi \in H_0^1(0,1)$. Then, the LHS is

$$\begin{aligned} LHS &= \int_{\Omega_t} D_{\xi(t)}(z)\partial_z n(z)\partial_z\psi(z)\ dz \\ &= \int_{\phi(\Omega_{nat})} D_{\xi(t)}(z)\partial_z n(z)\partial_z\psi(z)\ dz \\ &= \int_{\Omega_{nat}} D_{\xi(t)}(\phi(y))(\partial_z n)(\phi(y))(\partial_z\psi)(\phi(y)) \det|\partial_y\phi(y)|dy \\ &= \int_{\Omega_{nat}} D_{\xi(t)}(\phi(y))\partial_y(n\circ\phi)(y)\partial_y(\psi\circ\phi)(y)(\partial_y\phi(y))^{-1}dy \\ &= \int_{\Omega_{nat}} D_{nat}(y)\partial_y(n\circ\phi)(y)\partial_y(\psi\circ\phi)(y)\ dy. \end{aligned}$$

Thus by the fundamental lemma of calculus,

$$\begin{aligned} D_{\xi(t)}(z) &:= D_{nat}(\phi^{-1}(t,z))\partial_y\phi(t,\phi^{-1}(t,z)) \\ &= D(g^{-1}(t,\phi^{-1}(t,z)))\partial_y\phi(t,\phi^{-1}(t,z)) \\ &= \begin{cases} D_1\frac{\xi(t)}{\lambda_1(t)}, & \text{for } z < \xi(t), \\ D_2\frac{1-\xi(t)}{1-\lambda_2(t)}, & \text{for } z > \xi(t), \end{cases} \end{aligned} \tag{4.22}$$

Analogously, the RHS tested with $\psi \in H^1(0,1)$ and after change of variables with $g(t,\cdot)$, see

Lemma A.4.2, is

$$
\begin{aligned}
RHS &= \int\limits_{\Omega_t} \beta_{\xi(t)}(z)n(z)\psi(z)\ dz \\
&= \int\limits_{\phi(\Omega_{nat})} \beta_{\xi(t)}(z)n(z)\psi(z)\ dz \\
&= \int\limits_{\Omega_{nat}} \beta_{\xi(t)}(\phi(y))n(\phi(y))\psi(\phi(y))\partial_y\phi(y)\ dy.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\beta_t(z) &= \beta_{nat}(\phi^{-1}(t,z))(\partial_y\phi)^{-1}(t,\phi^{-1}(t,z)) \\
&= \beta(g^{-1}(t,\phi^{-1}(t,z)))(\partial_y\phi)^{-1}(t,\phi^{-1}(t,z)) \\
&= \begin{cases} \beta_1\frac{\lambda_1(t)}{\xi(t)}, & \text{for } z < \xi(t), \\ \beta_2\frac{1-\lambda_2(t)}{1-\xi(t)}, & \text{for } z > \xi(t). \end{cases}
\end{aligned} \tag{4.23}
$$

Furthermore, assume the tumour to be in an infinite nutrients bath, i.e. there exists n_0, n_1 as Dirichlet boundary conditions. Then, we seek for a solution $n(t,\cdot) \in H^1(0,1)$ such that the equation

$$
\begin{aligned}
-\partial_z(D_{\xi(t)}(z)\partial_z n(t,z)) = -\beta_{\xi(t)}n(t,z) \text{ on } (0,1), \\
n(t,0) = n_0,\ n(t,1) = n_1
\end{aligned} \tag{4.24}
$$

holds for each $t \in [0,T]$ on the deformed configuration $\Omega_t = g(t,\phi(t,(0,1))) = (0,1)$. Moreover, the nutrients have to be included in the mathematical framework of Setting 3.2.1.

Definition 4.2.7 (Setting with nutrients I)**.** *Suppose Setting 3.2.1 holds. Let* $\eta\colon \mathrm{C}^0([0,1]) \to S^0$ *be Lipschitz continuous with Lipschitz constant* $L_\eta > 0$ *and bounded, i.e. there exists* $0 < \eta_{max} < \infty$ *such that*

$$
\|\eta(f)\|_{L^\infty(0,1)} \le \eta_{max} \tag{4.25}
$$

for all $f \in \mathrm{C}^0([0,1])$*. Further, let* $D_1, D_2 \in (0,\infty)$ *define the diffusion coefficient* $D\colon (0,1) \to \mathbb{R}$ *by (4.19) and denote* $D_{min} \coloneqq \min\{D_1, D_2\} > 0$ *and* $D_{max} \coloneqq \max\{D_1, D_2\}$*. Let* $\beta_1, \beta_2 \in (0,\infty)$ *be the absorption rates and define* $\beta\colon (0,1) \to (0,\infty)$ *as in (4.20) with* $\beta_{min} \coloneqq \min\{\beta_1,\beta_2\} > 0$ *and* $\beta_{max} \coloneqq \max\{\beta_1,\beta_2\}$*.*
Moreover, let $n_0, n_1 \in \mathbb{R}$ *be the Dirichlet boundary conditions for the nutrients.*

Example 4.2.8. *Let* $\tilde{\eta}\colon \mathrm{C}^0([0,1]) \to S^0$ *be defined by*

$$
\tilde{\eta}(n)(x) \coloneqq \begin{cases} \frac{1}{\ell}\int\limits_0^\ell n(z)\ dz, & \text{for } x \in (0,\ell), \\ \frac{1}{1-\ell}\int\limits_\ell^1 n(z)\ dz, & \text{for } x \in (\ell,1). \end{cases}
$$

Then, define $\eta\colon \mathrm{C}^0([0,1]) \to S^0$ *by*

$$
\eta(n)(x) \coloneqq 1 - \exp(-\tilde{\eta}(n)(x)^2).
$$

Here, $1-\exp(-\cdot^2)\colon S^0 \to S^0$ *is defined for* $\tilde{\eta} = \tilde{\eta}^1\mathbb{1}_{(0,\ell]} + \tilde{\eta}^2\mathbb{1}_{(\ell,1]}$*,* $\tilde{\eta}^1, \tilde{\eta}^2 \in \mathbb{R}$*, by*

$$
\tilde{\eta} \mapsto \begin{cases} 1-\exp(-(\tilde{\eta}^1)^2) & \text{on } (0,\ell], \\ 1-\exp(-(\tilde{\eta}^2)^2) & \text{on } (\ell,1), \end{cases}
$$

and it is Lipschitz continuous. Therefore, η is Lipschitz continuous as a function composed of Lipschitz continuous functions. Furthermore, holds (4.25), since $1 - \exp(-x^2) \in [0, 1)$ for all $x \in \mathbb{R}$. In addition, this function η states that for low mean values of nutrients the growth is slow, and when there are many nutrients, the growth is faster.

In the new situation, a solution of the AMP is the following:

Definition 4.2.9 (Solution of the AMP with nutrients with two materials)**.** *Assume the Setting 4.2.7 to hold. Then, we say (G, S, n) is a solution of the AMP with general elastic strain energy density and nutrients if the following conditions hold:*

(i) The growth tensor $G \in \mathrm{C}^1([0, T]; S^0)$ fulfils the ODE

$$\begin{aligned} \dot{G}(t, x) = \gamma(x)\mu(S(t))\eta(n_m(t, \cdot))(x)G(t, x), \\ G(0, x) = 1, \end{aligned} \tag{4.26}$$

for all $t \in [0, T]$ and almost all $x \in [0, 1]$.

(ii) For the growth map

$$g(t, x) := \int_0^x G(t, \tilde{x})\ d\tilde{x}$$

and $t \in [0, T]$, let $\phi(t, \cdot) \colon g(t, [0, 1]) \to \mathbb{R}$ be the unique minimizer of E_{two} in $\mathcal{A}_{two}$. The Piola–Kirchhoff stress tensor $S(t) \in \mathrm{C}^0([0, T])$ to the elastic deformation $\phi(t, \cdot)$ is given by

$$S(t) = D_F W(g^{-1}(t, y), \partial_y \phi(t, y)).$$

(iii) For all $t \in [0, T]$, it is $n(t, \cdot) \in H^1(0, 1)$ and $n(t, \cdot)$ fulfils the equation

$$\begin{aligned} -\partial_z(D_{\xi(t)}(z)\partial_z n(z)) = -\beta_{\xi(t)}(z)n(z) \ \text{on}\ (0, 1), \\ n(0) = n_0, \\ n(1) = n_1, \end{aligned} \tag{4.27}$$

where $D_{\xi(t)}$ and $\beta_{\xi(t)}$ are defined by (4.22) and (4.23).

In general, the idea of the proof of Theorem 4.2.1 for the setting without nutrients can be applied, but in addition, Lipschitz continuity of n is required. We can follow the same steps and obtain λ_i, ξ_i, $i = 1, 2$, as before and without influence from the nutrients, therefore we will refer to the formulas in the proof in the following.

Proposition 4.2.10 (Nutrients in two materials are Lipschitz continuous)**.** *Let the conditions from 4.2.7 hold. Further, let (4.19) define the diffusion coefficient D for $D_1, D_2 > 0$ and (4.20) the absorption rate β for $\beta_1, \beta_2 > 0$. Let n_0, $n_1 \in \mathbb{R}$ be the Dirichlet boundary conditions for the nutrients. Further, let ξ, ξ_{min}, ξ_{max} be as in (3.19) in the proof of existence and uniqueness Theorem 3.2.5. Then for each $\xi \in [\xi_{min}, \xi_{max}]$, there exists a unique solution $n_\xi \in H^1(0, 1)$ of*

$$\begin{aligned} &- \partial_z(D_\xi \partial_z n_\xi(z)) = -\beta_\xi n_\xi(z) \ \text{in}\ H^1(0, 1), \\ &n(0) = n_0,\ n(1) = n_1, \end{aligned} \tag{4.28}$$

where D_ξ and β_ξ are defined in (4.22) and (4.23) respectively, and the map

$$n_m \colon [\xi_{min}, \xi_{max}] \to \mathrm{C}^0([0, 1]),\ \xi \mapsto (n_\xi)_m$$

is Lipschitz continuous.

Proof. **Step 1: Finding the solution.** Let $D \in \mathbb{R}$ and $\alpha \in \mathbb{R}_{>0}$. Consider the equation

$$-\partial_z(D\partial_z n(z)) = -\beta n(z) \tag{4.29}$$

on an arbitrary interval $I \subset \mathbb{R}$. It has the solution n given by

$$n(z) = A\exp(\alpha z) + B\exp(-\alpha z), \tag{4.30}$$

where $A, B \in \mathbb{R}$ and $\alpha = \sqrt{\beta/D}$.
We consider the equation (4.28) on the set $(0,\xi)$ and $(\xi,1)$ separately, knowing the solutions n_1, n_2 on each set is of the form of (4.30), namely there exists $A_1, B_1, A_2, B_2 \in \mathbb{R}$ such that

$$n_i(z) = A_i\exp(\alpha_i z) + B_i\exp(-\alpha_i z),\ i = 1,2, \tag{4.31}$$

with $\alpha_i = \alpha_i(\xi) = \sqrt{\beta_i(\xi)/D_i(\xi)}$. The coefficients A_1, B_1, A_2, B_2 are determined by the boundary conditions in (4.28), the continuity condition

$$n_1(\xi) = n_2(\xi)$$

and the condition that the derivatives are the same,

$$\tilde{D}_1\partial_z n_i(\xi) = \tilde{D}_2\partial_z n_2(\xi),$$

where $\tilde{D}_i = D_i\partial_y\phi(t,\phi^{-1}(t,\cdot))$, see (4.22).The last condition is obtained by integrating the equation and continuity of the integral.
Step 2: Lipschitz continuity of n in ξ. These conditions read as

$$\begin{aligned}
&A_1 + B_1 = n_0,\\
&A_2\exp(\alpha_2(\xi)) + B_2\exp(-\alpha_2(\xi)) = n_1\\
&A_1\exp(\alpha_1(\xi)\xi) + B_1\exp(-\alpha_1(\xi)\xi) - A_2\exp(\alpha_2(\xi)\xi) - B_2\exp(-\alpha_2(\xi)\xi) = 0\\
&\tilde{D}_1\alpha_1(\xi)A_1\exp(\alpha_1(\xi)\xi) - \tilde{D}_1\alpha_1(\xi)B_1\exp(-\alpha_1(\xi)\xi)\\
&-\tilde{D}_2\alpha_2(\xi)A_2\exp(\alpha_2(\xi)\xi) + \tilde{D}_2\alpha_2(\xi)B_2\exp(-\alpha_2(\xi)\xi) = 0.
\end{aligned}$$

For every $\xi \in (0,1)$ there exists an explicit solution $(A_1, B_1, A_2, B_2)^T \in \mathbb{R}^4$. The goal is to show that this solution $(A_1, B_1, A_2, B_2)^T$ depends Lipschitz continuously on ξ, by applying the Implicit Function theorem, see Theorem A.3.5. Define $f\colon [0,1]\times\mathbb{R}^4 \to \mathbb{R}^4$ by (we use the notation $\alpha_i = \alpha_i(\xi)$, $i = 1,2$)

$$\begin{aligned}
&f\left(\xi, (A_1, B_1, A_2, B_2)^T\right)\\
&:= \begin{pmatrix} A_1 + B_1 - n_0 \\ A_2\exp(\alpha_2) + B_2\exp(-\alpha_2) - n_1 \\ A_1\exp(\alpha_1\xi) + B_1\exp(-\alpha_1\xi) - A_2\exp(\alpha_2\xi) - B_2\exp(-\alpha_2\xi) \\ \tilde{D}_1\alpha_1\left(A_1\exp(\alpha_1\xi) - B_1\exp(-\alpha_1\xi)\right) - \tilde{D}_2\alpha_2\left(A_2\exp(\alpha_2\xi) + B_2\exp(-\alpha_2\xi)\right) \end{pmatrix}.
\end{aligned}$$

From this definition and the formulas (4.23) and (4.22), we see that f is C^1 and we need to check if the derivative with respect to $(A_1, B_1, A_2, B_2)^T$ is invertible. The derivative of f with respect to $(A_1, B_1, A_2, B_2)^T$ is (we use the notation $\alpha_i = \alpha_i(\xi)$, $i = 1,2$)

$$\begin{aligned}
&D_{(A_1,B_1,A_2,B_2)^T} f(\xi, (A_1, B_1, A_2, B_2)^T)\\
&= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & \exp(\alpha_2) & \exp(-\alpha_2) \\ \exp(\alpha_1\xi) & \exp(-\alpha_1\xi) & -\exp(\alpha_2\xi) & -\exp(-\alpha_2\xi) \\ \tilde{D}_1\alpha_1\exp(\alpha_1\xi) & -\tilde{D}_1\alpha_1\exp(-\alpha_1\xi) & -\tilde{D}_2\alpha_2\exp(\alpha_2\xi) & \tilde{D}_2\alpha_2\exp(-\alpha_2\xi) \end{pmatrix},
\end{aligned}$$

We have to prove that this matrix is invertible. To do so, we show that the determinant is not zero. First, we multiply the matrix by diag $(1,1,1,(\tilde{D}_1\alpha_1)^{-1})$ and name $D = \frac{\tilde{D}_2\alpha_2}{\tilde{D}_1\alpha_1}$. We obtain

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & \exp(\alpha_2) & \exp(-\alpha_2) \\ \exp(\alpha_1\xi) & \exp(-\alpha_1\xi) & -\exp(\alpha_2\xi) & -\exp(-\alpha_2\xi) \\ \exp(\alpha_1\xi) & -\exp(-\alpha_1\xi) & -D\exp(\alpha_2\xi) & D\exp(-\alpha_2\xi) \end{pmatrix}.$$

Subtracting the third row from the forth yields

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & \exp(\alpha_2) & \exp(-\alpha_2) \\ \exp(\alpha_1\xi) & \exp(-\alpha_1\xi) & -\exp(\alpha_2\xi) & -\exp(-\alpha_2\xi) \\ 0 & -2\exp(-\alpha_1\xi) & (1-D)\exp(\alpha_2\xi) & (D+1)\exp(-\alpha_2\xi) \end{pmatrix}.$$

Expanding the determinant along the first column, gives

$$\begin{aligned}
&\det D_{(A_1,B_1,A_2,B_2)^T} f \\
&= -(\tilde{D}_1\alpha_1)^{-1} \left(1 \cdot \det \begin{pmatrix} 0 & \exp(\alpha_2) & \exp(-\alpha_2) \\ \exp(-\alpha_1\xi) & -\exp(\alpha_2\xi) & -\exp(-\alpha_2\xi) \\ -2\exp(-\alpha_1\xi) & (1-D)\exp(\alpha_2\xi) & (D+1)\exp(-\alpha_2\xi) \end{pmatrix} \right. \\
&\qquad \left. + \exp(\alpha_1\xi) \cdot \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(\alpha_2) & \exp(-\alpha_2) \\ -2\exp(-\alpha_1\xi) & (1-D)\exp(\alpha_2\xi) & (D+1)\exp(-\alpha_2\xi) \end{pmatrix} \right) \\
&= -(\tilde{D}_1\alpha_1)^{-1} \left(1 \cdot 0 \cdot (-\exp(\alpha_2\xi)) \cdot (D+1)\exp(-\alpha_2\xi) \right. \\
&\qquad + 1 \cdot \exp(\alpha_2) \cdot (-\exp(-\alpha_2\xi)) \cdot (-2\exp(-\alpha_1\xi)) \\
&\qquad + 1 \cdot \exp(-\alpha_2) \cdot \exp(-\alpha_1\xi) \cdot (1-D)\exp(\alpha_2\xi) \\
&\qquad - 1 \cdot (-2\exp(-\alpha_1\xi)) \cdot (-\exp(\alpha_2\xi)) \cdot \exp(-\alpha_2) \\
&\qquad - 1 \cdot (1-D)\exp(\alpha_2\xi) \cdot (-\exp(-\alpha_2\xi)) \cdot 0 \\
&\qquad - 1 \cdot (D+1)\exp(-\alpha_2\xi) \cdot \exp(-\alpha_1\xi) \cdot \exp(\alpha_2) \\
&\qquad + \exp(\alpha_1\xi) \cdot 1 \cdot \exp(\alpha_2) \cdot (D+1)\exp(-\alpha_2\xi) \\
&\qquad + \exp(\alpha_1\xi) \cdot 0 \cdot \exp(-\alpha_2) \cdot (-2\exp(-\alpha_1\xi)) \\
&\qquad + \exp(\alpha_1\xi) \cdot 0 \cdot 0 \cdot (1-D)\exp(\alpha_2\xi) \\
&\qquad - \exp(\alpha_1\xi) \cdot (-2\exp(-\alpha_1\xi)) \cdot \exp(\alpha_2) \cdot 0 \\
&\qquad - \exp(\alpha_1\xi) \cdot (1-D)\exp(\alpha_2\xi) \cdot \exp(-\alpha_2) \cdot 1 \\
&\qquad \left. - \exp(\alpha_1\xi) \cdot (D+1)\exp(-\alpha_2\xi) \cdot 0 \cdot 0 \right) \\
&= -(\tilde{D}_1\alpha_1)^{-1} \left(1 \cdot \exp(\alpha_2) \cdot (-\exp(-\alpha_2\xi)) \cdot (-2\exp(-\alpha_1\xi)) \right. \\
&\qquad + 1 \cdot \exp(-\alpha_2) \cdot \exp(-\alpha_1\xi) \cdot (1-D)\exp(\alpha_2\xi) \\
&\qquad - 1 \cdot (-2\exp(-\alpha_1\xi)) \cdot (-\exp(\alpha_2\xi)) \cdot \exp(-\alpha_2) \\
&\qquad - 1 \cdot (D+1)\exp(-\alpha_2\xi) \cdot \exp(-\alpha_1\xi) \cdot \exp(\alpha_2) \\
&\qquad + \exp(\alpha_1\xi) \cdot 1 \cdot \exp(\alpha_1) \cdot (D+1)\exp(-\alpha_2\xi) \\
&\qquad \left. - \exp(\alpha_1\xi) \cdot (1-D)\exp(\alpha_2\xi) \cdot \exp(-\alpha_2) \cdot 1 \right) \\
&= -(\tilde{D}_1\alpha_1)^{-1} \left(\exp(\alpha_2) \cdot \exp(-\alpha_2\xi) \cdot \exp(-\alpha_1\xi) \right. \\
&\qquad - \exp(-\alpha_2) \cdot \exp(-\alpha_1\xi) \cdot D\exp(\alpha_2\xi) \\
&\qquad - \exp(-\alpha_1\xi) \cdot \exp(\alpha_2\xi) \cdot \exp(-\alpha_2) \\
&\qquad - D\exp(-\alpha_2\xi) \cdot \exp(-\alpha_1\xi) \cdot \exp(\alpha_2) \\
&\qquad + \exp(\alpha_1\xi) \cdot \exp(\alpha_1) \cdot (D+1)\exp(-\alpha_2\xi) \\
&\qquad \left. - \exp(\alpha_1\xi) \cdot (1-D)\exp(\alpha_2\xi) \cdot \exp(-\alpha_2) \right) \\
&= -(\tilde{D}_1\alpha_1)^{-1} \exp(-\alpha_2)\exp(-\alpha_1\xi)\exp(-\alpha_2\xi) \left(\exp(2\alpha_2) \right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad - D\exp(2\alpha_2\xi)\\
&\qquad - \exp(2\alpha_2\xi)\\
&\qquad - D\exp(2\alpha_2)\\
&\qquad + \exp(2\alpha_1\xi)\cdot\exp(2\alpha_2)\cdot(D+1)\\
&\qquad - \exp(2\alpha_1\xi)\cdot(1-D)\exp(2\alpha_2\xi))\\
&= -(\tilde{D}_1\alpha_1)^{-1}\exp(-\alpha_2(1+\xi)-\alpha_1\xi)\cdot\\
&\qquad ((-1+\exp(2\alpha_1\xi))\,((D-1)\exp(2\alpha_2\xi)+(D+1)\exp(2\alpha_2))\\
&\qquad +2(\exp(2\alpha_2)-\exp(2\alpha_2\xi)))
\end{aligned}
$$

Now, we see that $\tilde{e} \coloneqq \exp(2\alpha_2) - \exp(2\alpha_2\xi)$ is strictly positive. Further,

$$(D-1)\exp(2\alpha_2\xi) + (D+1)\exp(2\alpha_2) = D(\exp(2\alpha_2\xi) + \exp(2\alpha_2)) + \tilde{e} > 0$$

and as $2\alpha_1\xi > 0$, also $(-1 + \exp(2\alpha_1\xi)) > 0$. Therefore, $\det D_{(A_1,B_1,A_2,B_2)^T} f < 0$ and $D_{(A_1,B_1,A_2,B_2)^T} f$ is invertible.
Let $\tilde{\xi} \in (0,1)$ be fixed. Then, the implicit function theorem, see Theorem A.3.5, yields that there exists an open neighbourhood $V(\tilde{\xi})$ of $\tilde{\xi}$ such that the map

$$(A_1, B_1, A_2, B_2)^T\colon V(\tilde{\xi}) \to \mathbb{R}^4,\ \xi \mapsto (A_1(\xi), B_1(\xi), A_2(\xi), B_2(\xi))^T,$$

with $(A_1(\xi), B_1(\xi), A_2(\xi), B_2(\xi))^T$ the solution of

$$f(\xi, (A_1(\xi), B_1(\xi), A_2(\xi), B_2(\xi))^T) = 0, \tag{4.32}$$

is $C^1(V(\tilde{\xi}))$. Moreover, the solution of Equation (4.32) exists and is unique for all $\xi \in (0,1)$, because the equation can be written as (we use the notation $\alpha_i = \alpha_i(\xi)$, $i = 1,2$)

$$\begin{pmatrix} 1 & 1 & 0 & 0\\ 0 & 0 & \exp(\alpha_2) & \exp(-\alpha_2)\\ \exp(\alpha_1\xi) & \exp(-\alpha_1\xi) & -\exp(\alpha_2\xi) & -\exp(-\alpha_2\xi)\\ \tilde{D}_1\alpha_1\exp(\alpha_1\xi) & -\tilde{D}_1\alpha_1\exp(-\alpha_1\xi) & -\tilde{D}_2\alpha_2\exp(\alpha_2\xi) & \tilde{D}_2\alpha_2\exp(-\alpha_2\xi) \end{pmatrix} \begin{pmatrix} A_1(\xi)\\ B_1(\xi)\\ A_2(\xi)\\ B_2(\xi) \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0 \end{pmatrix},$$

where the matrix has full rank for each $\xi \in (0,1)$ as the determinant is strictly negative, see above. Especially, $(A_1(\cdot), B_1(\cdot), A_2(\cdot), B_2(\cdot))^T \in C^1([\xi_{min}, \xi_{max}])$ and the Lipschitz continuity follows.
Step 3: Uniform Lipschitz continuity of n in space. Let $z_1, z_2 \in (0,\xi)$. Then by formula (4.31)

$$
\begin{aligned}
&|n_1(\xi)(z_1) - n_1(\xi)(z_2)|\\
&\quad = |A_1(\xi)\exp(\alpha_1(\xi)z_1) + B_1(\xi)\exp(-\alpha_1(\xi)z_1) - A_1(\xi)\exp(\alpha_1(\xi)z_2) - B_1(\xi)\exp(-\alpha_1(\xi)z_2)|\\
&\quad \le |A_1(\xi)|\cdot|\exp(\alpha_1(\xi)z_1) - \exp(\alpha_1(\xi)z_2)| + |B_1(\xi)|\cdot|\exp(-\alpha_1(\xi)z_1) - \exp(-\alpha_1(\xi)z_2)|
\end{aligned}
$$

Since $\exp(\alpha_1(\xi)\cdot)$ and $\exp(-\alpha_1(\xi)\cdot)$ are Lipschitz continuous to the constants $\alpha_1(\xi)\exp(\lambda_1\xi) > 0$ and $\alpha_1(\xi)$ on $(0,l)$, we obtain

$$|n_1(\xi)(z_1) - n_1(\xi)(z_2)| \le (|A_1(\xi)|\alpha_1(\xi)\exp(\alpha_1(\xi)\xi) + |B_1(\xi)|\alpha_1(\xi))|z_1 - z_2|.$$

The bounds on A_i and B_i imply that $n(\xi)$ is Lipschitz continuous in z on $(0,\xi)$ for all $\xi \in (0,1)$ to the constant

$$L_{n_1;x} = |A_{max}|\alpha_1(\xi)\exp(\lambda_1\ell) + |B_{max}|\alpha_1(\xi).$$

A similar argument shows the Lipschitz continuity of $n(\xi)$ in z on $[\xi,1)$ to the Lipschitz constant

$L_{n_2;x}$. Let $z_1 \in (0,\xi)$ and $z_2 \in [\xi, 1)$. Then

$$\begin{aligned}
&|n(\xi)(z_2) - n(\xi)(z_2)| \\
&= |n_2(\xi)(z_2) - n_1(\xi)(z_1)| \\
&\leq |n_2(\xi)(z_2) - n_2(\xi)(\xi)| + |n_2(\xi)(\xi) - n_1(\xi)(\xi)| + |n_1(\xi)(\xi) - n_1(\xi)(z_1)| \\
&= |n_2(\xi)(z_2) - n_2(\xi)(\xi)| + |n_1(\xi)(\xi) - n_1(\xi)(z_1)| \\
&\leq L_{n_2;x}|z_2 - \xi| + L_{n_1;x}|\xi - z_1| \\
&\leq L_{n;x}|z_2 - z_1|,
\end{aligned}$$

where $L_{n;x} = \max\{L_{n_1;x}, L_{n_2;x}\}$. This proves the statement.

Step 4: Lipschitz continuity of $(n_\xi)_m$. Let $G_1, G_2 \in \overline{B_{\tilde{R}}(G^*)}$ be arbitrary and $x \in [0,1]$ be fixed. Then, from the Lipschitz continuity of n_ξ in ξ, see Step 2, and in z, see Step 3, it follows

$$\begin{aligned}
&|(n_{\xi(G_1)})_m(x) - (n_{\xi(G_2)})_m(x)| \\
&= |n_{\xi(G_1)} \circ X(G_1)(x) - n_{\xi(G_2)} \circ X(G_2)(x)| \\
&\leq |n_{\xi(G_1)} \circ X(G_1)(x) - n_{\xi(G_1)} \circ X(G_2)(x)| + |n_{\xi(G_1)} \circ X(G_2)(x) - n_{\xi(G_2)} \circ X(G_2)(x)| \\
&\leq L_{n;x}|X(G_1)(x) - X(G_2)(x)| + L_{n;\xi}\|G_1 - G_2\|_{L^\infty(0,1)}.
\end{aligned}$$

The map $X \colon S^0 \to \mathrm{C}^0([0,1])$ is Lipschitz continuous in G with constant $L_{X;G}$: By the Formula (3.21), the Lipschitz continuity of ϕ in G with Lipschitz constant $L_{\phi;G}$ follows. Moreover, we proved that g is Lipschitz continuous in G with constant $L_{g;G}$, see proof of Proposition 4.2.5. Thus, X is Lipschitz continuous, because

$$\begin{aligned}
&|X(G_1)(x) - X(G_2)(x)| \\
&= |\phi(G_1) \circ g(G_1)(x) - \phi(G_2) \circ g(G_2)(x)| \\
&\leq |\phi(G_1) \circ g(G_1)(x) - \phi(G_1) \circ g(G_2)(x)| + |\phi(G_1) \circ g(G_2)(x) - \phi(G_2) \circ g(G_2)(x)| \\
&\leq \max\{\frac{\xi_{max}}{\lambda_{min}}; \frac{1-\xi_{min}}{1-\lambda_{max}}\}|g(G_1)(x) - g(G_2)(x)| + \|\phi(G_1) - \phi(G_2)\|_\infty \\
&\leq \max\{\frac{\xi_{max}}{\lambda_{min}}; \frac{1-\xi_{min}}{1-\lambda_{max}}\}L_{g;G}\|G_1 - G_2\| + L_{\phi;G}\|G_1 - G_2\|_\infty.
\end{aligned}$$

Using this, the Lipschitz continuity of ξ in G, see proof of Proposition 4.2.5, we obtain

$$\|(n_{\xi(G_1)})_m - (n_{\xi(G_2)})_m\|_{\mathrm{C}^0([0,1])} \leq (L_{n;x}L_{X;G} + L_{n;\xi}L_\xi)\|G_1 - G_2\|_{L^\infty(0,1)}.$$

♣

With this theorem, a similar argumentation as in the proof of the existence theorem of the AMP 3.2.5 yields the existence of the AMP with the ODE (4.26) and n as solution of (4.24).

Theorem 4.2.11 (Existence and uniqueness of the AMP with nutrients with two materials)**.** *Assume Setting 4.2.7 holds. Then, there exists a unique solution (G, S, n) of the AMP with two materials with nutrients, defined in Definition 4.2.9.*

Remark 4.2.12. *(i) Above the assumption of an infinite nutrient bath was made. This must not be the case, e.g. one could consider the Dirichlet boundary conditions to be time dependent given functions n_0, $n_1 \colon [0,T] \to \mathbb{R}$.*

(ii) In Setting 4.2.7 the assumption is made that η takes only two values depending on the material. This is needed to use Picard–Lindelöf theorem to obtain existence and uniqueness, but it is not biological reasonable in view of necrosis, i.e. that parts of the tumour die due to lack of nutrients, e.g. in a spherical setting. In [MM78], this process is mathematically modelled. There, it is assumed, in contrast to our setting, that cells die if the nutrient concentration falls below a critical point, only living cells consume nutrients and geometric assumptions.

(iii) The proof of Theorem 4.2.11 is analogous to the proof of Theorem 3.2.5. Note that the time step t_0 in (4.13) depends on G_{min} and G_{max}, which now also depend on η. Therefore, the time step is different for both proofs.

4.3. ... for material with C^0-stress modulus κ

In the previous sections, a body consisting of finitely many parts is considered, which is expanded by the case of material, which has properties being continuous in space. It is that the coefficients γ and κ are assumed to be continuously differentiable. This yields also higher regularity of the growth tensor G. Again for an existence result Picard–Lindelöf theorem A.2.6 will be used, which will also give uniqueness. Hence, the critical step is the Lipschitz continuity of the mapping $G \mapsto S$.
To be precise, we consider the following setting, where (4.34) is the important assumption.

Definition 4.3.1 (Setting with C^0-coefficients)**.** *Let $\gamma \in C^0([0,1])$ and $\kappa \in C^0([0,1];\mathbb{R})$ be the growth multiplier and the elastic stress modulus with $\kappa_{min} := \inf_{x\in[0,1]} \kappa(x) > 0$ and $\kappa_{max} := \sup_{x\in[0,1]} \kappa(x) < \infty$. Let $\mu \in C^0(\mathbb{R})$ be Lipschitz continuous. Define $-\infty < (\gamma\mu)_{min} < 0 < (\gamma\mu)_{max} < \infty$ with*

$$(\gamma\mu)_{max} \geq \sup_{x\in[0,1],y\in\mathbb{R}} \{\gamma(x)\mu(y)\},$$
$$(\gamma\mu)_{min} \leq \inf_{x\in[0,1],y\in\mathbb{R}} \{\gamma(x)\mu(y)\}.$$

Let $\tilde{W} \in C^2(\mathbb{R}_{>0})$ be an elastic strain energy density that fulfils the conditions (EL1)-(EL3) from Definition 3.1.3. Further, the uniform convexity, namely there exists a $c_c \in \mathbb{R}$ such that for all $F \in (0,\infty)$ it holds that

$$D_F^2 \tilde{W}(F) > c_c > 0. \tag{4.33}$$

Then, define the elastic energy density by

$$W\colon [0,1] \times (0,\infty) \to [0,\infty),\ W(x,F) := \kappa(x)\tilde{W}(F) \tag{4.34}$$

and for a given growth map g the admissible set for elastic deformations

$$\mathcal{A}_{C^0} := \{v \in W^{1,\infty}(g((0,1)))|\ v(g(0)) = 0,\ v(g(1)) = 1\}$$

and for an elastic deformation $\phi \in \mathcal{A}_{C^0}$ the elastic energy by $E_{C^0}\colon \mathcal{A}_{C^0} \to \mathbb{R}$,

$$E_{C^0}(\phi) := \int_{g((0,1))} \kappa(g^{-1}(y))\tilde{W}(\partial_y \phi(y))\ dy.$$

Finally, let $T > 0$ be the time horizon and $G_0 = 1 \in C^0([0,1])$ the initial datum for the ODE for the growth tensor.

In order of a changed setting, the solution has to be defined again. Due to the $C^0([0,1])$-regularity of the coefficients γ and μ, the ODE is supposed to hold in $C^0(0,1)$ instead of $L^\infty(0,1)$, i.e. the solution $G(t,\cdot)$ is of $C^0(0,1)$-regularity for each $t \in [0,T]$. The definition of a solution of the AMP with C^0-coefficients reads as:

Definition 4.3.2 (Definition of the AMP with C^0-coefficients)**.** *Consider the Setting 4.3.1. Then, we call a couple $(G,S) \in C^1([0,T];C^0([0,1])) \times C^0([0,T])$ a solution of the AMP with C^0-coefficients if the following is fulfilled:*

(i) The growth tensor G fulfils the ODE

$$\dot{G}(t,x) = \gamma(x)\mu(S(t))G(t,x),$$

$$G(0,x) = 1,$$

for all $t \in [0,T]$ and all $x \in [0,1]$.

(ii) For $t \in [0,T]$ and

$$g(t,x) := \int_0^x G(t,\tilde{x})\ d\tilde{x},$$

let $\phi(t,\cdot) : g(t,[0,1]) \to \mathbb{R}$ be a minimizer of E_{C^0} in $\mathcal{A}_{C^0}$. The Piola–Kirchhoff stress tensor to the elastic deformation ϕ is S by

$$S(t) = \kappa(g^{-1}(t,y))D_F\tilde{W}(\partial_y\phi(t,y)).$$

Remark 4.3.3. *The minimizer $\phi(t,\cdot)$ is unique due to the convexity of $\tilde{W}$.*

We adjust the ball $\overline{B_{\tilde{R}}(G^*)} \subset L^\infty(0,1)$ with $\tilde{R}$ defined in (4.12) and G^* in (4.11) to $\overline{B_{\tilde{R}}(G^*)} \subset C^0([0,1])$ and start with the definition of the growth map as before

$$g\colon \overline{B_{\tilde{R}}(G^*)} \to C^1([0,1]),\ G \mapsto g(G)(x) = \int_0^x G(\tilde{x})\ d\tilde{x}, \tag{4.35}$$

which is Lipschitz continuous, see proof of 4.2.5, step 1.
Since G is not piecewise constant, there are no grown length to calculate. But the elastic deformation is for a given growth map $g = g(G) \in C^1([0,1])$ with $g(0) = 0$ still the minimizer of the functional

$$\begin{aligned} E(\phi) &= \int_0^{g(1)} W(g^{-1}(y), \partial_y\phi(y))\ dy \\ &= \int_0^{g(1)} \kappa(g^{-1}(y))\tilde{W}(\partial_y\phi(y))\ dy \end{aligned} \tag{4.36}$$

subject to the boundary conditions $\phi(0) = 0$ and $\phi(g(1)) = 1$. With Theorem A.1.1(i) and in the Setting 4.3.1, the minimizer ϕ is in $C^1(0,g(1))$ and $\min_{y\in[0,g(1)]}\partial_y\phi(y) > 0$. The Theorem A.1.1 is applicable, since the g^{-1} exists and is invertible. Therefore, the function $W(g^{-1}(y),F)$ fulfils the conditions (EL1)-(EL3) from Definition 3.1.3.
To get a uniqueness and existence result, we proceed as before: Showing a priori estimates on the growth tensor G and proving that S depends Lipschitz continuously on G. For the a priori estimates on G we use as before the Gronwall Lemma 4.2.2, namely by an analogous proof as above, it holds

Lemma 4.3.4 (A priori estimates on growth tensor with C^0-coefficients)**.** *Assume the Setting 4.3.1 to hold and assume a solution (G,S) of the AMP 4.3.2 exists. Then, for*

$$\begin{aligned} G_{min} &:= \exp\left((\gamma\mu)_{min}T\right), \\ G_{max} &:= \exp\left((\gamma\mu)_{max}T\right), \end{aligned}$$

holds for each $t \in [0,T]$ and every $x \in [0,1]$

$$G(t,x) \in [G_{min}, G_{max}],$$

which is equivalent to

$$G(t,\cdot) \in \overline{B_{\tilde{R}}(G^*)} \subset C^0([0,1])$$

for all $t \in [0, T]$ *by*

$$R^* := \frac{G_{max} - G_{min}}{2}, \qquad G^* := \frac{G_{max} + G_{min}}{2} \tag{4.37}$$

In order to prove the Lipschitz dependence of S on G, we need bounds on S. In addition, the following lemma is what we expect from the mechanics: If the growth is bounded, the stress in the material is bounded.

Lemma 4.3.5 (Stress is uniformly estimated with C^0-coefficients)**.** *Consider the Setting 4.3.1 with* (G, S) *the solution from AMP with* C^0*-coefficients with elastic deformation* ϕ*. Then, there exist constants* $-\infty < S_{min} < S_{max} < \infty$ *such that*

$$S_{min} \le S(t) \le S_{max} \text{ for all } t \in [0, T]. \tag{4.38}$$

Proof. Recall the equation (4.40)

$$1 = \int_0^1 (D_F \tilde{W})^{-1} \left(\frac{S}{\kappa(x)} \right) G(x)\, dx.$$

Since $D_F \tilde{W}$ is strictly increasing, $(D_F \tilde{W})^{-1}$ is strictly monotonous increasing and we get

$$\int_0^1 (D_F \tilde{W})^{-1} \left(\frac{S}{\kappa_{max}} \right) G_{min}\, dx \le 1 \le \int_0^1 (D_F \tilde{W})^{-1} \left(\frac{S}{\kappa_{min}} \right) G_{max}\, dx$$

which is equivalent to

$$S_{max} := D_F \tilde{W} \left(\frac{1}{G_{min}} \right) \kappa_{max} \ge S \ge D_F \tilde{W} \left(\frac{1}{G_{max}} \right) \kappa_{min} =: S_{min},$$

where G_{min} and G_{max} are given in Lemma 4.3.4. ♣

For proving the Lipschitz continuity of the stress tensor S, we do not know the explicit dependence of S on G. Therefore, use the implicit function theorem, see Theorem A.3.5, to obtain C^1-regularity in a neighbourhood for each $G \in \overline{B_{\tilde{R}}(G^*)}$, where $\tilde{R} = R^* + \frac{G_{min}}{2}$, and hence, the local Lipschitz continuity.

Proposition 4.3.6 (Lipschitz continuity of the stress tensor with C^0-coefficients)**.** *In the Setting 4.3.1 with the growth map* g *from (4.35) and the elastic deformation* ϕ *the minimizer the elastic Problem (4.36) with Dirichlet boundary condition the map*

$$S \colon \overline{B_{\tilde{R}}(G^*)} \to \mathbb{R}, \; G \mapsto S(G)$$

is Lipschitz continuous, where $\overline{B_{\tilde{R}}(G^*)} \subset L^\infty(0,1)$ *is defined in (4.37).*

Remark 4.3.7. *Due to the conditions on* W*, the minimizer of the elastic Problem (4.36) is unique, see Theorem A.1.1.*

Proof. The idea is to use the Euler–Lagrange equation of the elastic problem and the implicit function theorem.
From Theorem A.1.1 we know the Euler–Lagrange equation

$$S_g = D_F W(g^{-1}(y), \partial_y \phi(y)) = \kappa(g^{-1}(y)) D_F \tilde{W}(\partial_y \phi(y)) = C_g \text{ for a.e. } y \in [0, g(1)]. \tag{4.39}$$

Since $\tilde{W}$ is strictly convex, $D_F\tilde{W}$ is strictly increasing and, hence, invertible. Solving (4.39) for ϕ, we obtain

$$\partial_y\phi(y) = (D_F\tilde{W})^{-1}\left(\frac{S_g}{\kappa(g^{-1}(y))}\right).$$

Using the fundamental theorem of calculus and applying the boundary conditions yields

$$\begin{aligned} 1 = \phi(g(1)) - \phi(g(0)) &= \int_{g(0)}^{g(1)} \partial_y\phi(\tilde{y})\ d\tilde{y} \\ &= \int_0^{g(1)} (D_F\tilde{W})^{-1}\left(\frac{S_g}{\kappa(g^{-1}(\tilde{y}))}\right)\ d\tilde{y}. \end{aligned}$$

Change of variables, see Proposition A.4.2, gives

$$\begin{aligned} 1 &= \int_0^1 (D_F\tilde{W})^{-1}\left(\frac{S_g}{\kappa(x)}\right)\partial_x g(x)\ dx \\ &= \int_0^1 (D_F\tilde{W})^{-1}\left(\frac{S_g}{\kappa(x)}\right)G(x)\ dx, \end{aligned} \tag{4.40}$$

which is an implicit equation for S_g. Define the function

$$f\colon \overline{B_{\tilde{R}}(G^*)}\times\mathbb{R}\to\mathbb{R},\ (G,S)\mapsto \int_0^1 (D_F\tilde{W})^{-1}\left(\frac{S}{\kappa(x)}\right)G(x)\ dx - 1.$$

For $G(x) = 1$, it is $\phi(y) = y$, and thus, $D_F\tilde{W}(\partial_y\phi(y)) = 0$ for all $y \in g([0,1])$. Consequently, $S = 0$ and $f(1,0) = 0$. To apply the implicit function theorem, we check the conditions, i.e. $\partial_S f(1,0)$ is a bounded and linear functional and has a bounded inverse. We calculate for $\tilde{S}\in\mathbb{R}$

$$\langle\partial_S f(G,S),\tilde{S}\rangle = \frac{d}{ds}f(G,S+s\tilde{S})|_{s=0} = \int_0^1 ((D_F\tilde{W})^{-1})'\left(\frac{S}{\kappa(x)}\right)\frac{G(x)\tilde{S}}{\kappa(x)}\ dx.$$

Since $\tilde{W}\in \mathrm{C}^2(0,\infty)$ and $D_F\tilde{W}$ is strictly increasing, the inverse $(D_F\tilde{W})^{-1}\colon\mathbb{R}\to(0,\infty)$ is differentiable with derivative

$$((D_F\tilde{W})^{-1})'(z) = \frac{1}{D_F^2\tilde{W}(D_F\tilde{W}^{-1}(z))}.$$

It follows

$$\langle\partial_S f(G,S),\tilde{S}\rangle = \int_0^1 \left(D_F^2\tilde{W}\left(D_F\tilde{W}^{-1}\left(\frac{S}{\kappa(x)}\right)\right)\right)^{-1}\frac{G(x)\tilde{S}}{\kappa(x)}\ dx$$

and

$$\langle\partial_S f(1,0),\tilde{S}\rangle = \tilde{S}\int_0^1 \left(D_F^2\tilde{W}\left(D_F\tilde{W}^{-1}\left(\frac{0}{\kappa(x)}\right)\right)\right)^{-1}\frac{1}{\kappa(x)}\ dx$$

$$= \frac{\tilde{S}}{D_F^2\tilde{W}(1)} \int_0^1 \frac{1}{\kappa(x)}\ dx,$$

where $D_F^2\tilde{W}(1) > c_c > 0$ due to strict convexity (4.33). Since $0 < \int_0^1 \frac{1}{\kappa(x)}\ dx < \infty$, $\partial_S f(1,0)$ is a bounded, linear functional and has a bounded inverse. Applying implicit function theorem, see Theorem A.3.5, yields that $G \mapsto S(G)$ is C^1 on a neighbourhood of $G_0(\cdot) = 1$. We can apply the argumentation to any $G' \in \overline{B_{\tilde{R}}(G^*)}$ (since the elasticity problem can be solved uniquely for each given $G' > 0$ almost everywhere) and since the uniqueness of the elastic deformation ϕ implies the uniqueness of S, we conclude that

$$S\colon \overline{B_{\tilde{R}}(G^*)} \to \mathbb{R} \text{ is } C^1.$$

Let $G \in \overline{B_{\tilde{R}}(G^*)}$ be fixed. Then, for all $\tilde{G} \in L^\infty(0,1)$ and small enough $s \in \mathbb{R}$, it holds

$$0 = f(G + s\tilde{G}, S(G + s\tilde{G})).$$

and due to the regularity of f,

$$\begin{aligned}
0 &= \frac{d}{ds} f(G + s\tilde{G}, S(G + s\tilde{G}))|_{s=0} \\
&= \langle \partial_G f(G, S(G)), \tilde{G}\rangle + \langle \partial_S f(G, S(G)) D_G S(G), \tilde{G}\rangle \\
&= \int_0^1 (D_F\tilde{W})^{-1}\left(\frac{S(G)}{\kappa(x)}\right)\tilde{G}(x)\ dx + \int_0^1 \left(D_F^2\tilde{W}\left(D_F\tilde{W}^{-1}\left(\frac{S(G)}{\kappa(x)}\right)\right)\right)^{-1} \frac{1}{\kappa(x)}\ d\tilde{x} \langle D_G S(G), \tilde{G}\rangle.
\end{aligned}$$

By Lemma 4.3.5, there exist $S_{min}, S_{max} \in \mathbb{R}$ such that $S \in [S_{min}, S_{max}]$ for all $G \in \overline{B_{\tilde{R}}(G^*)}$. Furthermore, by assumption $\kappa_{inf} \le \kappa(x) \le \kappa_{max}$ for all $x \in [0,1]$. Therefore,

$$\frac{S(G)}{\kappa(x)} \in \left[\frac{S_{min}}{\kappa_{max}}, \frac{S_{max}}{\kappa_{min}}\right] \text{ for all } G \in \overline{B_{\tilde{R}}(G^*)} \text{ and all } x \in [0,1].$$

Since $D_F\tilde{W}^{-1}$ is a continuous function on the compact set $[\frac{S_{min}}{\kappa_{max}}, \frac{S_{max}}{\kappa_{min}}]$, there exist constants $D_FW_{min}, D_FW_{max} \in \mathbb{R}$ with $D_FW_{max} > 0$ such that

$$D_F\tilde{W}\left(\frac{S(G)}{\kappa(x)}\right) \in [D_FW_{min}, D_FW_{max}] \text{ for all } G \in \overline{B_{\tilde{R}}(G^*)} \text{ and all } x \in [0,1].$$

In addition, $D_F^2\tilde{W}$ is continuous on the compact set $[D_FW_{min}, D_FW_{max}]$. Hence, there exist constants $W_{min}, W_{max} > 0$ such that

$$\left(D_F^2\tilde{W}\left(D_F\tilde{W}^{-1}\left(\frac{S(G)}{\kappa(x)}\right)\right)\right) \in [W_{min}, W_{max}] \text{ for all } G \in \overline{B_{\tilde{R}}(G^*)} \text{ and all } x \in [0,1].$$

Therefore, we obtain

$$\langle D_G S(G), \tilde{G}\rangle = \frac{-\int_0^1 (D_F\tilde{W})^{-1}\left(\frac{S(G)}{\kappa(x)}\right)\tilde{G}(x)\ dx}{\int_0^1 \left(D_F^2\tilde{W}\left(D_F\tilde{W}^{-1}\left(\frac{S(G)}{\kappa(x)}\right)\right)\right)^{-1}\frac{1}{\kappa(x)}\ d\tilde{x}}$$

It follows

$$\begin{aligned}
|\langle D_G S(G), \tilde{G}\rangle| &= \left| \frac{-\int_0^1 (D_F\tilde{W})^{-1}\left(\frac{S(G)}{\kappa(x)}\right)\tilde{G}(x)\ dx}{\int_0^1 \left(D_F^2\tilde{W}\left(D_F\tilde{W}^{-1}\left(\frac{S(G)}{\kappa(x)}\right)\right)\right)^{-1}\frac{1}{\kappa(x)}\ d\tilde{x}} \right| \\
&\leq \left| \frac{\int_0^1 (D_F\tilde{W})^{-1}\left(\frac{S(G)}{\kappa(x)}\right)\tilde{G}(x)\ dx}{\int_0^1 \frac{1}{W_{max}}\frac{1}{\kappa_{max}}\ d\tilde{x}} \right| \\
&\leq W_{max}\kappa_{max}\int_0^1 \left|(D_F\tilde{W})^{-1}\left(\frac{S(G)}{\kappa(x)}\right)\tilde{G}(x)\right|\ dx \\
&\leq W_{max}\kappa_{max}D_F W_{max}\|\tilde{G}(x)\|_{L^\infty(0,1)}.
\end{aligned}$$

This proves a uniform estimate on $D_G S(G)$ on $\overline{B_{\tilde{R}}(G^*)}$ and the Lipschitz continuity follows. ♣

By this proof, a formula for the derivative of S in $G = 1$ can be obtained easily.

Remark 4.3.8. *The derivative of S in $G = 1$ with respect to G can be determined by calculating the derivative implicitly:*

$$\begin{aligned}
0 &= \frac{d}{ds} f(1 + s\tilde{G}, S(1 + s\tilde{G}))|_{s=0} \\
&= \langle \partial_G f(1, S(1)), \tilde{G}\rangle + \langle \partial_S f(1, S(1)) D_G S(1), \tilde{G}\rangle \\
&= \int_0^1 (D_F\tilde{W})^{-1}\left(\frac{0}{\kappa(x)}\right)\tilde{G}(x)\ dx + \int_0^1 \frac{1}{D_F^2\tilde{W}(1)}\frac{1}{\kappa(x)}\ d\tilde{x}\langle D_G S(1), \tilde{G}\rangle \\
&= \int_0^1 \tilde{G}(x)\ dx + \int_0^1 \frac{1}{D_F^2\tilde{W}(1)}\frac{1}{\kappa(x)}\ d\tilde{x}\langle D_G S(1), \tilde{G}\rangle,
\end{aligned}$$

which yields

$$\langle D_G S(1), \tilde{G}\rangle = -\int_0^1 \tilde{G}(x)\ dx\ D_F^2\tilde{W}(1)\left(\int_0^1 \frac{1}{\kappa(\tilde{x})}\ d\tilde{x}\right)^{-1}.$$

We state the following existence and uniqueness theorem for the AMP with C^0-coefficients. The proof is analogous to the proof of the existence and uniqueness Theorem 4.2.1 of the AMP with two materials. The Banach space, the Picard–Lindelöf theorem is applied to, is now $C^0([0,1])$. The interpretation of the ODE in that Banach space is analogous to the discussion in the beginning of Chapter 4. Furthermore, an a priori estimate on the growth tensor is needed and stated in Lemma 4.3.4. The local Lipschitz continuity of the stress tensor S is proved in Theorem 4.3.6 for the setting with C^0-coefficients. Bounds on the Lipschitz constant are not needed, since the Picard–Lindelöf theorem A.2.6 requires the RHS to be locally Lipschitz continuously dependent on $G \in \overline{B_{\tilde{R}}(G^*)}$ and to be bounded. The latter is the case, because the stress S appears only as $\mu(S)$ and μ is assumed to be bounded.

Theorem 4.3.9 (Existence and uniqueness of AMP with C^0-coefficients)**.** *Consider the Setting 4.3.1 to hold. Then, there exists a unique solution (G, S) of the AMP with C^0-coefficients.*

4.4. ... for material with general energy density

In the last sections we considered material with energy density which is a multiplication of stress modulus depending on space with a general potential, i.e. $W(x,F) = \kappa(x)\tilde{W}(F)$. In this chapter we investigate the AMP with an energy density potential which may vary for each particle, i.e. $W = W(x,F)$. To be precise, we consider the following setting:

Definition 4.4.1 (Setting with general elastic strain energy density)**.** *Let $\gamma \in L^\infty(0,1)$ be the growth multiplier and $\mu \in C^0(\mathbb{R})$ be Lipschitz continuous with $\mu_{min} \coloneqq \inf_{x\in\mathbb{R}} \mu(x)$ and $\mu_{max} \coloneqq \sup_{x\in\mathbb{R}} \mu(x)$. Define $-\infty < (\gamma\mu)_{min} < 0 < (\gamma\mu)_{max} < \infty$ with*

$$(\gamma\mu)_{max} \geq \operatorname*{ess\,sup}_{x\in(0,1),y\in\mathbb{R}} \{\gamma(x)\mu(y)\},$$
$$(\gamma\mu)_{min} \leq \operatorname*{ess\,inf}_{x\in(0,1),y\in\mathbb{R}} \{\gamma(x)\mu(y)\}.$$

Let $W\colon (0,1)\times(0,\infty) \to \mathbb{R}$ be an elastic strain energy density which fulfils the conditions (EL1)-(EL3) in Definition 3.1.3 and there exists a $c_c > 0$ and a continuous function $C_W\colon (0,\infty) \to (0,\infty)$ such that

$$D_F^2 W(x,F) \geq c_c \text{ for all } F \in (0,\infty) \text{ and almost all } x \in (0,1), \tag{4.41}$$
$$D_F^2 W(x,F) \leq C_W(F) \text{ for all } F \in (0,\infty) \text{ and almost all } x \in (0,1). \tag{4.42}$$

Let the following hold: For each $F \in (0,\infty)$, it is

$$\inf_{x\in(0,1)} D_F W(x,F) > -\infty,$$
$$\sup_{x\in(0,1)} D_F W(x,F) < +\infty.$$

For a given growth map g, the admissible set for elastic deformations is given by

$$\mathcal{A}_{gen} \coloneqq \{v \in W^{1,\infty}(g((0,1)))|\ v(g(0)) = 0,\ v(g(1)) = 1\},$$

and for an elastic deformation $\phi \in \mathcal{A}_{gen}$, the elastic energy is defined by $E_{gen}\colon \mathcal{A}_{gen} \to \mathbb{R}$,

$$E_{gen}(\phi) \coloneqq \int_{g((0,1))} W(g^{-1}(y), \partial_y\phi(y))\ dy. \tag{4.43}$$

Finally, let $T > 0$ be given and $G_0 = 1 \in L^\infty(0,1)$ the initial value for the ODE for the growth tensor.

Example 4.4.2. *We assumed that there exists a constant $c_c > 0$ such that for all $x \in (0,1)$ and all $F \in (0,\infty)$, it hold that*

$$D_F^2 W(x,F) > c_c.$$

Define

$$W\colon (0,1)\times(0,\infty) \to [0,\infty),\ W(x,F) = \frac{\kappa(x)}{2}(F + \frac{1}{F})^2 - \kappa(x)$$

with $\kappa \in L^\infty(0,1)$ with $\kappa_{inf} = \operatorname{ess\,inf}_{x\in(0,1)}\kappa(x) > 0$ Then,

$$D_F W(x,F) = \kappa(x)(F - \frac{1}{F^3}),$$
$$D_F^2 W(x,F) = \kappa(x)(1 + \frac{3}{F^4}) > \kappa_{inf}$$

for all $F \in (0,\infty)$ and almost all $x \in (0,1)$. Therefore, W fulfils the conditions with $c_c = \kappa_{inf}$.

With the setting specified, we define the solution of the AMP with general elastic strain energy density as follows.

Definition 4.4.3 (Solution of the AMP with general elastic strain energy density)**.** *Assume the Setting 4.4.1 to hold. Then, we call a couple $(G,S) \in \mathrm{C}^1([0,T];L^\infty(0,1)) \times \mathrm{C}^0([0,T])$ a solution of the AMP with general elastic strain energy density if the following is fulfilled:*

(i) The growth tenor G fulfils the ODE

$$\begin{aligned}\dot{G}(t,x) &= \gamma(x)\mu(S(t))G(t,x),\\ G(0,x) &= 1,\end{aligned} \tag{4.44}$$

for all $t \in [0,T]$ and almost all $x \in (0,1)$.

(ii) For

$$g(t,x) := \int_0^x G(t,\tilde{x})\; d\tilde{x}$$

and $t \in [0,T]$ let $\phi(t,\cdot) : g(t,(0,1)) \to \mathbb{R}$ be a minimizer of E_{gen} in $\mathcal{A}_{gen}$. The Piola–Kirchhoff stress tensor $S(t)$ to the elastic deformation $\phi(t,\cdot)$ is given by

$$S(t) = D_F W(g^{-1}(t,y), \partial_y \phi(t,y)) \text{ for all } y \in [g(t,(0,1))].$$

Remark 4.4.4. *Note that the minimizer $\phi(t,\cdot)$ of E_{gen} exists, refer to [Bal81] Theorem 1, and is unique due to the strict convexity of W in F.*

As before, with the Gronwall Lemma 4.2.2, we can prove a priori estimates on a solution G of the AMP, which yields that

$$G(t,x) \in [G_{min}, G_{max}]$$

holds for all $t \in [0,T]$ and almost all $x \in (0,1)$, and hence,

$$G(t,\cdot) \in \overline{B_{\tilde{R}}(G^*)} \subset L^\infty(0,1),$$

where $G_{min}, G_{max}, \tilde{R}$ and G^* are analogous to the ones in Lemma 4.3.4.
For the Lipschitz continuity, we again use the implicit function theorem. Let $\tilde{R} > 0$ be defined as in (4.12). We start with the definition of the growth map

$$g \colon \overline{B_{\tilde{R}}(G^*)} \to W^{1,\infty}(0,1),\; G \mapsto g(G) \text{ with } g(G)(x) = \int_0^x G(\tilde{x})\; d\tilde{x}, \tag{4.45}$$

which is Lipschitz continuous, see proof of Proposition 4.2.5, Step 1. Given a growth map g, the elastic energy of a deformation ϕ of the natural configuration $(g(0), g(1))$ is given as

$$E(\phi) = \int_{g(0)}^{g(1)} W(g^{-1}(y), \partial_y \phi(y))\; dy. \tag{4.46}$$

Further, the Dirichlet boundary conditions are $\phi(g(0)) = 0$ and $\phi(g(1)) = 1$. Due to Theorem A.1.1(i) the unique minimizer ϕ of this problem in the Setting 4.4.1 is $\mathrm{C}^1(g(0), g(1))$ and $\min_{y\in[g(0),g(1)]} \phi(y) > 0$.

Lemma 4.4.5 (Uniform estimates on deformation gradient). *Assume Setting 4.4.1 holds. Let $G \in \overline{B_{\tilde{R}}(G^*)}$ with growth map $g(G)$ defined by (4.45) and elastic deformation $\phi(G)$ as the unique minimizer of the elastic energy E_{gen}, see Remark 4.4.4. Then, there exist two constants $0 < \phi_{min} < \phi_{max}$ such that*

$$\partial_y \phi(t,y) \in [\phi_{min}, \phi_{max}] \tag{4.47}$$

for all $t \in [0,T]$ and all $y \in g(t,(0,1))$.

Proof. **Step 1: $|g(t,(0,1))| = 1$.** In this case, the elastic deformation is $\phi(t,y) = y$, since $W_{nat}(y, \partial_y \phi(t,y)) = W_{nat}(y,1) = 0$ for all $y \in g(t,(0,1))$ and nothing is to prove.
Step 2: $|g(t,(0,1))| > 1$.
Step 2a): Upper bound. By the mean value theorem, there exists a $\xi \in g(t,(0,1))$ with

$$\partial_y \phi(t,\xi) = \frac{\phi(t,g(t,1)) - \phi(t,g(t,0))}{g(t,1) - g(t,0)} = \frac{1}{|g(t,(0,1))|} \in (0,1).$$

By the strict monotonicity of $D_F W_{nat}(y,\cdot)$ and due to $D_F W_{nat}(y,1) = 0$, it holds that

$$D_F W_{nat}(\xi, \partial_y \phi(t,\xi)) = S(t) < 0.$$

Hence, $D_F W_{nat}(y, \partial_y \phi(t,y)) = S(t) < 0$. Since $D_F W_{nat}(y,\cdot)$ is strictly monotonously increasing and $D_F W_{nat}(y,(0,1)) = (-\infty, 0)$, it follows $\partial_y \phi(t,y) \in (0,1)$ for all $y \in g(t,(0,1))$. Hence, the upper bound 1 is proven.
Step 2b): Lower bound. Due to the assumption on W, $D_F W(x,\cdot)$ is negative and strictly increasing on $[\partial_y \phi(t,x), 1)$. With this, (EL2), $D_F W(x,1) = 0$ and the fundamental theorem of calculus, it follows

$$\begin{aligned}
\theta(\partial_y \phi(t,x)) &\le W(x, \partial_y \phi(t,x)) = W(x, \partial_y \phi(t,x)) - W(x,1) \\
&\le \int_1^{\partial_y \phi(t,x)} D_F W(x,F)\, dF = -\int_{\partial_y \phi(t,x)}^{1} D_F W(x,F)\, dF \\
&\le -\int_{\partial_y \phi(t,x)}^{1} S(t)\, dF = -(1 - \partial_y \phi(t,x)) S(t) \le -S(t) = |S(t)|.
\end{aligned} \tag{4.48}$$

Due to the conditions on θ, define $\theta_{min} \coloneqq \min_{F \in (0,\infty)} \theta(F)$ and the set $M \coloneqq \{F \in (0,\infty) | \, \theta(F) = \theta_{min}\}$. The set M is bounded from below, because $\theta(F) \to \infty$ for $F \searrow 0$. Therefore, define $F_{min} \coloneqq \min M > 0$. Furthermore, the convexity of θ implies that θ is strictly decreasing on $(0, F_{min}]$: Let $F_0, F_1 \in (0, F_{min})$ be with $F_0 < F_1$ and $\theta(F_0) = \theta(F_1) > \theta(F_{min})$. Then there exists a $\lambda \in (0,1)$ such that $F_1 = \lambda F_0 + (1-\lambda) F_{min}$ and from the convexity of θ it follows

$$\begin{aligned}
\theta(F_1) &= \theta(\lambda F_0 + (1-\lambda) F_{min}) \\
&\le \lambda \theta(F_0) + (1-\lambda)\theta(F_{min}) \\
&= \lambda \theta(F_1) + (1-\lambda)\theta(F_{min}) < \theta(F_1),
\end{aligned}$$

which is a contradiction. Therefore, θ is strictly decreasing on $(0, F_{min}]$. Thus, θ is invertible on $(0, F_{min}]$ and we define $F^*(S(t)) \coloneqq (\theta|_{(0,F_{min}]})^{-1}(S(t))$. Since θ is strictly decreasing on $(0, F_{min}]$ and $\theta(F^*(S(t))) = S(t)$, it holds $\theta(F) > |S(t)|$ for all $F \in (0, F^*(S(t)))$. From (4.48), it follows that $\partial_y \phi(t,x) > F^*(S(t))$. By Proposition 4.4.8, there exist $0 < S_{min} < S_{max} < \infty$ such that $S(t) \in [S_{min}, S_{max}]$ for all $t \in [0,T]$. It follows from the definition of F^*, that F^* is continuous in S and therefore it has a minimum on the compact interval $[S_{min}, S_{max}]$ that we denote byϕ_{min}.
Step 3: $|g(t,(0,1))| < 1$.
Step 3a): Lower bound. The argumentation as in Step 2a) also holds for this case and leads

to $S(t) > 0$ and $\partial_y\phi(t,y) \geq 1$.
Step 3b): Upper bound. By the conditions on W, $D_F W(x,\cdot)$ is positive and strictly increasing on $(1, \partial_y\phi(t,x)]$. Together with $W(x,1) = 0$ and the assumptions on θ, we obtain

$$\begin{aligned}\theta(\partial_y\phi(t,y)) &\leq W(x,\partial_y\phi(t,y)) = W(x,\partial_y\phi(t,y)) - W(x,1)\\ &= \int_1^{\partial_y\phi(t,x)} D_F W(x,F)\;dF\\ &\leq (\partial_y\phi(t,y)-1) D_F W(x,\partial_y\phi(t,y)) \leq \partial_y\phi(t,y) S(t)\end{aligned}$$

Define the set $N \coloneqq \{F \in (0,\infty) | \; \frac{\theta(F)}{F} = S(t)\}$. Since $\frac{\theta(F)}{F} \to \infty$ for $F \to \infty$, the set N is bounded and we define $F_{max}(S(t)) \coloneqq \sup N < \infty$. From the assumptions on W, $\theta(\partial_y\phi(t,y))/\partial_y\phi(t,y) \leq S(t)$ and therefore $\partial_y\phi(t,y) \in N$ and thus $\partial_y\phi(t,y) \leq F_{max}$. Moreover, θ is continuous in F and therefore is F_{max} continuous in S and takes the maximum ϕ_{max} on the compact interval $[S_{min}, S_{max}]$. ♣

With the implicit function theorem, see Theorem A.3.5, we prove that the Piola–Kirchhoff stress tensor depends Lipschitz continuously on the growth tensor, which will later prove existence and uniqueness of a solution of the AMP. In this setting, W is not a product, but nonetheless is invertible due to the assumptions (EL1)-(EL3) in 3.6, see Remark 3.1.4(iii)(a).

Proposition 4.4.6 (Lipschitz continuity of the stress tensor with general elastic strain energy density)**.** *Consider the Setting 4.4.1. Let g be the growth map from (4.45) and E elastic energy from (4.46). Then, the map*

$$S\colon \overline{B_{\tilde{R}}(G^*)} \to \mathbb{R},\; G \mapsto S(G),$$

where $\tilde{R}$ and G^* are defined by (4.11), is Lipschitz continuous.

Proof. Theorem A.1.1 states that for the elastic map g given there exists a unique minimizer ϕ of E w.r.t. the Dirichlet boundary conditions and that the Euler–Lagrange equation is

$$D_F W(g^{-1}(y), \partial_y\phi(y)) = S_g \text{ for a.e. } y \in [g(0), g(1)] \tag{4.49}$$

for some $S_g \in \mathbb{R}$. Since $W(g^{-1}(y),\cdot)$ is strictly convex, $D_F W(g^{-1}(y),\cdot)$ is invertible from $(0,\infty)$ to $\mathbb{R}$, see Remark **??**(vi), such that

$$\partial_y\phi(y) = (D_F W(g^{-1}(y),\cdot))^{-1}(S_g) \tag{4.50}$$

holds. The fundamental theorem of calculus and the Dirichlet boundary condition imply that

$$\begin{aligned}1 = \phi(g(1)) - \phi(g(0)) &= \int_{g(0)}^{g(1)} \partial_y\phi(y)\;dy\\ &= \int_{g(0)}^{g(1)} (D_F W(g^{-1}(y),\cdot))^{-1}(S_g)\;dy\\ &= \int_0^1 (D_F W(x,\cdot))^{-1}(S_g)\partial_x g(x) dx\\ &= \int_0^1 (D_F W(x,\cdot))^{-1}(S_g) G(x) dx\end{aligned}$$

holds. This is an implicit formula for S_g depending on G. Hence, we use the implicit function theorem, see Theorem A.3.5, for the function

$$f\colon \overline{B_{\tilde{R}}(G^*)} \times \mathbb{R} \to \mathbb{R},\ (G,S) \mapsto 1 - \int_0^1 (D_F W(x,\cdot))^{-1}(S) G(x)\, dx. \tag{4.51}$$

For $G = 1$ and $S = 0$, it holds $0 = S = D_F W(x,1)$ and, hence, $f(1,0) = 0$. Further, we have to check that the Fréchet derivative $\partial_S f(1,0)$ exists and is a bounded, linear functional with bounded inverse. For the existence we use Lemma A.3.4. We start by calculating the Gâteaux derivative and will prove then that it is continuous in $S = 0$. For $\tilde{S} \in \mathbb{R}_{>0}$ we calculate

$$\langle D_S f(G,S), \tilde{S}\rangle = \frac{d}{ds} f(G, S + s\tilde{S})|_{s=0} = \int_0^1 ((D_F W(x,\cdot))^{-1})'(S) G(x)\tilde{S}\, dx.$$

Here, the derivative and integral can be interchanged by dominated convergence, because the difference quotient converges to the derivative which is bounded, as can be seen in the following: Since $W(x,\cdot) \in \mathrm{C}^2(0,\infty)$ for each $x \in [0,1]$, $D_F W(x,\cdot) \in \mathrm{C}^1(0,\infty)$ and $D_F W(x,\cdot)$ is strictly increasing, due to the strict convexity of $W(x,\cdot)$, and, hence, an isomorphism. Due to Lemma A.3.6 the inverse $(D_F W(x,\cdot))^{-1}\colon \mathbb{R} \to (0,\infty)$ is differentiable with derivative

$$((D_F W(x,\cdot))^{-1})'(\hat{S}) = \frac{1}{D_F^2 W(x, W(x,\cdot)^{-1}(\hat{S}))},\ \ \hat{S} \in \mathbb{R}.$$

It follows

$$\langle D_S f(G,S), \tilde{S}\rangle = \int_0^1 \left(D_F^2 W(x, W(x,\cdot)^{-1}(S))\right)^{-1} G(x)\tilde{S}\, dx,$$

which is continuous in S. Moreover, $(D_F^2 W(x, W(x,\cdot)^{-1}(S)))^{-1} G(x)\tilde{S}$ is a bounded $L^1(0,1)$-function due to the assumptions on W and the conditions for dominated convergence are fulfilled.

Due to Lemma A.3.4 the Fréchet derivative $\partial_S f(G,S)$ exists. Further,

$$\begin{aligned}\langle \partial_S f(1,0), \tilde{S}\rangle &= \tilde{S}\int_0^1 \left(D_F^2 W(x, W(x,\cdot)^{-1}(0))\right)^{-1}\, dx\\ &= \tilde{S}\int_0^1 \frac{1}{D_F^2 W(x,1)}\, dx,\end{aligned}$$

where $D_F^2 W(x,1) > c_c > 0$ due to the uniform strict convexity of W, see (4.41). Hence, the conditions of the implicit functions theorem A.3.5 are fulfilled and the theorem yields the existence of an open neighbourhood $V \subset L^\infty(0,1)$ of $G_0(\cdot) = 1$ such the map

$$S\colon V \to \mathbb{R},\ G \mapsto S(G)$$

is C^1. We can apply the argumentation from above to all $G' \in \overline{B_{\tilde{R}}(G^*)}$, because for each such G' the Theorem A.1.1(i) states that for the minimizer ϕ of the elastic energy E_{gen} exists an $C_{G'}(t) > 0$ such that $\partial_y \phi(t,y) > C_{G'}(t)$ for all $y \in [0, g(t,1)]$. Hence, the derivative of f in (G', S'), where $S' = S(G')$, is

$$\begin{aligned}\langle \partial_S f(G',S'), \tilde{S}\rangle &= \tilde{S}\int_0^1 \left(D_F^2 W(x, W(x,\cdot)^{-1}(S'))\right)^{-1} G'(x)\, dx\\ &= \tilde{S}\int_0^1 \frac{1}{D_F^2 W(x, W(x,\cdot)^{-1}(S'))} G'(x)\, dx.\end{aligned}$$

This term is strictly positive, because $D_F^2 W(x,F) \geq c_c$ for all $F \in (0,\infty)$ and $W^{-1}(x,\cdot)(S') > 0$ for all $x \in (0,1)$. Thus, the implicit function theorem is applicable and it gives that

$$S\colon \overline{B_{\tilde{R}}(G^*)} \to \mathbb{R},\ G \mapsto S(G)$$

exists and is C^1.
In order to prove Lipschitz continuity, a bound on $D_G S$ is needed. Let $G \in \overline{B_{\tilde{R}}(G^*)}$ be fixed. Then, it holds for each $\tilde{G} \in L^\infty(0,1)$ and small enough $s \in \mathbb{R}$ that

$$0 = f(G + s\tilde{G}, S(G + s\tilde{G})).$$

Differentiating in s yields

$$\begin{aligned}
0 =& \frac{d}{ds} f(G + s\tilde{G}, S(G + s\tilde{G}))|_{s=0} \\
=& \langle \partial_G f(G, S(G)), \tilde{G}\rangle + \langle \partial_S f(G, S(G)) D_G S(G), \tilde{G}\rangle \\
=& \int_0^1 (D_F W(x,\cdot))^{-1}(S(G))\tilde{G}(x)\ dx \\
&+ \int_0^1 (D_F^2 W(x, (D_F W(x,\cdot))^{-1}(S(G))))^{-1}\ d\tilde{x} \langle D_G S(G), \tilde{G}\rangle \\
=& \int_0^1 \partial_y \phi(g(x))\tilde{G}(x)\ dx \\
&+ \int_0^1 (D_F^2 W(x, \partial\phi(g(x))))^{-1}\ d\tilde{x} \langle D_G S(G), \tilde{G}\rangle,
\end{aligned}$$

where we used (4.50) in the last step. Lemma 4.4.5 states that there exist $\phi_{min}, \phi_{max} > 0$ such that

$$\partial_y \phi(y) \in [\phi_{min}, \phi_{max}] \text{ for all } y \in g((0,1)).$$

Due to the condition (4.42) on $D_F^2 W$, there exist $D_F^2 W_{min}, D_F^2 W_{max} > 0$ such that

$$D_F^2 W(x, \partial\phi(g(x))) \in [D_F^2 W_{min}, D_F^2 W_{max}] \text{ for all } x \in (0,1).$$

Therefore, we conclude

$$\langle D_G S(G), \tilde{G}\rangle = \frac{-\int_0^1 \partial_y \phi(g(x))\tilde{G}(x)\ dx}{\int_0^1 (D_F^2 W(x, \partial\phi(g(x))))^{-1}\ d\tilde{x}}$$

It follows

$$|\langle D_G S(G), \tilde{G}\rangle| = \left| \frac{-\int_0^1 \partial_y \phi(g(x))\tilde{G}(x)\ dx}{\int_0^1 (D_F^2 W(x, \partial\phi(g(x))))^{-1}\ d\tilde{x}} \right|$$

$$\leq \frac{\int_0^1 \left|\partial_y \phi(g(x))\tilde{G}(x)\right| \; dx}{\int_0^1 \frac{1}{D_F^2 W_{max}} \; d\tilde{x}}$$
$$\leq D_F^2 W_{max} \phi_{max} \|\tilde{G}(x)\|_{L^\infty(0,1)}.$$

This proves a uniform estimate on $D_G S(G)$ on $\overline{B_{\tilde{R}}(G^*)}$ and the Lipschitz continuity follows. ♣

The above theorem together with the a priori estimates on the growth tensor and an analogous proof as of the existence and uniqueness theorem 4.2.1 yields the following existence and uniqueness theorem for the AMP with general elastic strain energy density:

Theorem 4.4.7 (Existence and uniqueness of the AMP with general elastic strain energy density)**.** *Consider the Setting 4.4.1 to hold. Then, there exists a unique solution* (G, S) *of the AMP with general elastic strain energy density.*

Physical statements

After stating the existence and uniqueness of the solution of the AMP with general elastic strain energy, we show statements, which are expected in a consistent model. We prove that the material's stress is finite for finite growth and, vice versa, that the stress goes to infinity if the material is growing infinitely big. Moreover, we discuss the sign of the stress in correspondence with the growth and prove that the elastic deformation has finite derivative for all times.

Proposition 4.4.8 (Bounded stress with general elastic strain energy density)**.** *In the setting of 4.4.1, let* (G, S) *be a solution of the AMP with general elastic strain energy density. Further, suppose that* G *satisfies on* $[0, T] \times (0, 1)$ *the uniform estimates*

$$0 < G_{min} \leq G(t, x) \leq G_{max} < \infty.$$

Then, the stress is uniformly bounded on $[0, T]$*, namely there exist* $S_{min}, S_{max} \in \mathbb{R}$ *such that for all* $t \in [0, T]$

$$S_{min} \leq S(t) \leq S_{max}.$$

Proof. In the proof of the Lipschitz continuity of S we derived for S the implicit formula

$$1 = \int_0^1 (D_F W(x, \cdot))^{-1}(S) G(x) \; dx,$$

which we will use as basic idea to get estimates on S. We will use the intermediate value theorem to obtain a similar equation which we compare with the help of the above equation.
Step 1: Define function for intermediate value problem. For $G \in (0, \infty)$, define

$$f_G \colon \mathbb{R} \to \mathbb{R}, \; f_G(S) := \int_0^1 (D_F W(x, \cdot))^{-1}(S) G \; dx.$$

Since $W(x, \cdot)$ is strictly convex and $C^2(0, \infty)$ and due to the growth conditions, see Definition 3.1.3, $D_F W(x, \cdot) \colon (0, \infty) \to \mathbb{R}$ is invertible with $(D_F W(x, \cdot))^{-1} \colon \mathbb{R} \to (0, \infty)$ strictly increasing. Hence, it holds

$$f_G(S) = \int_0^1 (D_F W(x, \cdot))^{-1}(S) G \; dx$$

$$< \int_0^1 (D_F W(x,\cdot))^{-1}(S')G \; dx$$
$$= f_G(S') \text{ for all } S < S'.$$

Step 2: Limit of f_G for $S \to -\infty$. For $S = 0$ holds

$$f_G(S) = f_G(0) = \int_0^1 (D_F W(x,\cdot))^{-1}(0)G \; dx = \int_0^1 1 \cdot G \; dx = G.$$

Consequently, $(D_F W(x,\cdot))^{-1}(0)G$ dominates $(D_F W(x,\cdot))^{-1}(S)G$ in $L^1(0,1)$ for all $S < 0$. Further $(D_F W(x,\cdot))^{-1}(S) \to 0$ for $S \to -\infty$. Therefore, the dominated convergence theorem yields

$$\lim_{S \to -\infty} f_G(S) = \lim_{S \to -\infty} \int_0^1 (D_F W(x,\cdot))^{-1}(S)G \; dx$$
$$= \int_0^1 \lim_{S \to -\infty} (D_F W(x,\cdot))^{-1}(S)G \; dx = 0.$$

Step 3: Limit of f_G for $S \to \infty$. On the other hand, $(D_F W(x,\cdot))^{-1}$ converges pointwise to ∞ for $S \to \infty$. Further, it is a non-negative function for each $S \in \mathbb{R}$ and strictly increasing in S. Hence, the monotone convergence theorem yields

$$\lim_{S \to \infty} f_G(S) = \lim_{S \to \infty} \int_0^1 (D_F W(x,\cdot))^{-1}(S)G \; dx$$
$$= \int_0^1 \lim_{S \to \infty} (D_F W(x,\cdot))^{-1}(S)G \; dx = \infty.$$

Step 4: Applying intermediate value theorem. In addition, the continuity of $(D_F W(x,\cdot))^{-1}$ implies that F_G is continuous in S. Hence, the application of the intermediate value theorem for $G = G_{min}$ and $G = G_{max}$ yields the existence of $S_{max}, S_{min} \in \mathbb{R}$ respectively such that

$$1 = f_{G_{min}}(S_{max}) = \int_0^1 (D_F W(x,\cdot))^{-1}(S_{max})G_{min} \; dx$$

and

$$1 = f_{G_{max}}(S_{min}) = \int_0^1 (D_F W(x,\cdot))^{-1}(S_{min})G_{max} \; dx$$

Step 5: Conclusion. From this and the initial equation we conclude

$$\begin{aligned}\int_0^1 (D_F W(x,\cdot))^{-1}(S_{min})G_{max} \; dx = 1 &= \int_0^1 (D_F W(x,\cdot))^{-1}(S(t))G(t,x) \; dx \\ &\leq \int_0^1 (D_F W(x,\cdot))^{-1}(S(t))G_{max} \; dx\end{aligned} \tag{4.52}$$

and

$$\int_0^1 (D_F W(x,\cdot))^{-1}(S_{max}) G_{min}\, dx = 1 = \int_0^1 (D_F W(x,\cdot))^{-1}(S(t)) G(t,x)\, dx$$
$$\geq \int_0^1 (D_F W(x,\cdot))^{-1}(S(t)) G_{min}\, dx.$$

Now assume that $S_{min} > S(t)$ for a $t \in [0,T]$. Then with the fact that $(D_F W(x,\cdot))^{-1}$ is strictly increasing, we get $(D_F W(x,\cdot))^{-1}(S_{min}) > (D_F W(x,\cdot))^{-1}(S(t))$ and after integration

$$\int_0^1 (D_F W(x,\cdot))^{-1}(S_{min}) G_{max}\, dx > \int_0^1 (D_F W(x,\cdot))^{-1}(S(t)) G_{max}\, dx,$$

which is a contradiction to (4.52). For S_{max} and $S(t)$ one can argue analogously, hence, it must hold

$$S_{max} \geq S(t) \geq S_{min}.$$

♣

On the other hand, if the material grows infinite and with an additional assumption, we can prove that the stress to push it back to $(0,1)$ goes to infinity, too.

Proposition 4.4.9 (Infinite growth generates infinite stress)**.** *Assume the setting of 4.4.1 to hold except the Lipschitz condition of $\mathcal{G}$ and let (G,S) be the solution of the AMP with Dirichlet boundary conditions on $[0,T)$ with $G(t,x) > 0$ for all $t \in [0,T]$ and $x \in (0,1)$. Further, assume one of the following assumptions to hold:*

(i) for all $S \in \mathbb{R}$ there exists a constant $C(S) > 0$ such that

$$\forall x \in (0,1) : (D_F W(x,\cdot))^{-1}(S) \geq C(S).$$

(ii) the growth is homogenous, i.e. that for all $t \in [0,T)$ and all $x \in (0,1)$ holds

$$G(t,x) = G(t).$$

(iii) the elastic deformation is homogenous, i.e. for all $x \in (0,1)$ and $F \in (0,\infty)$ holds

$$W(x,F) = W(F).$$

Suppose that the material grown infinitely big, i.e.

$$|g(t,(0,1))| \to \infty \text{ for } t \to T,$$

where $|\cdot|$ denotes the $\mathcal{L}^1$-measure. Then,

$$S(t) \to -\infty \text{ for } t \to T.$$

Proof. **Concerning (i):** Assume the additional condition (i) to hold. For a contradiction, assume that $S \nrightarrow -\infty$ for $t \to T$. Due to the continuity of S, there exists an $S^* \in \mathbb{R}$ such that $S(t) \geq S^*$ for all $t \in [0,T)$. Then, due to monotonicity it follows

$$(D_F W(x,\cdot))^{-1}(S(t)) \geq (D_F W(x,\cdot))^{-1}(S^*) \text{ for all } t \in [0,T).$$

From the equation (4.40) from the proof of the Lipschitz continuity of S of G and the a priori estimates on $G(t,\cdot)$ to be almost every positive we get

$$\begin{aligned}
1 &= \int_0^1 (D_F W(x,\cdot))^{-1}(S(t))G(t,x)\ dx \\
&\geq \int_0^1 (D_F W(x,\cdot))^{-1}(S^*)G(t,x)\ dx \\
&\geq C(S^*)\int_0^1 G(t,x)\ dx \\
&= |g(t,(0,1))| \to \infty, t \to T,
\end{aligned}$$

which is a contradiction.
Concerning (ii): Assume the growth tensor to be independent of x and that $S(t)$ is bounded from below by $S^* \in \mathbb{R}$ for all $t \in [0,T)$. Then from (4.40), it follows

$$\begin{aligned}
1 &= \int_0^1 (D_F W(x,\cdot))^{-1}(S(t))G(t,x)\ dx \\
&= \int_0^1 (D_F W(x,\cdot))^{-1}(S(t))G(t)\ dx \\
&= G(t)\int_0^1 (D_F W(x,\cdot))^{-1}(S(t))\ dx \\
&\geq G(t)\int_0^1 (D_F W(x,\cdot))^{-1}(S^*)\ dx.
\end{aligned}$$

Since $\int_0^1 (D_F W(x,\cdot))^{-1}(S^*)\ dx$ is constant and $G(t) \to \infty$ for $t \to T$, we get a contradiction to the estimates on $S(t)$.
Concerning (iii): Assume the elastic strain energy density to be independent of x. Then due to (4.40), it holds

$$\begin{aligned}
1 &= \int_0^1 (D_F W(x,\cdot))^{-1}(S(t))G(t,x)\ dx \\
&= \int_0^1 (D_F W)^{-1}(S(t))G(t,x)\ dx \\
&= (D_F W)^{-1}(S(t))\int_0^1 G(t,x)\ dx \\
&= (D_F W)^{-1}(S(t))|g(t,(0,1))|.
\end{aligned}$$

It follows that $(D_F W)^{-1}(S(t)) \to 0$ for $t \to T$, because $|g(t,(0,1))| \to \infty$ for $t \to T$, and due to the conditions on W, $S(t) \to -\infty$ for $t \to T$. ♣

Remark 4.4.10. *(i) The analogous result holds for $|g(t,(0,1))| \to 0$ for $t \to T$: $S(t) \to +\infty$ for $t \to T$. Here, one of conditions (i) (with " $\leq$ " instead of " $\geq$ "), (ii) or (iii) must hold.*

(ii) *We have to assume $G(t,x) > 0$ for all $t \in [0,T]$ and $x \in (0,1)$, because we got this condition from the Lipschitz continuity of $\mathcal{G}$, but now do not assume $\mathcal{G}$ to be Lipschitz continuous.*

(iii) *The statement finds application for the case of the ODE $\dot{G} = G^2$ with $G_0 = 1$ and $T = 1$. The solution $G(t) = \frac{1}{1-t}$ has a blow up for $t \to T$. The blow up is caused by the not Lipschitz continuous RHS. When we prove the existence of a solution for the AMP, the Lipschitz continuity assures the boundedness of G, which is not assured in the Proposition 4.4.9.*

As last here mentioned physically expected property, we prove that the stress is negative if the material is grown bigger than the interval $(0,1)$ and positive if the material shrinks.

Lemma 4.4.11. *Let (G,S) be the solution of the AMP in the Setting 4.4.1 and let g denote the growth map corresponding to G by (4.45). Then, the following are equivalent:*

(i) $S(t) \leq 0$,

(ii) $|g(t,(0,1))| \geq 1$.

Especially, $S(t) = 0$ iff $|g(t,(0,1))| = 1$.

Proof. **Step 1: (i)⇒ (ii).** From the relation (4.51) and the strict monotonicity of $(D_F W(x,\cdot))^{-1}$ follows

$$\begin{aligned} 1 &= \int_0^1 (D_F W(x,\cdot))^{-1}(S(t))G(t,x)\ dx \\ &\leq \int_0^1 (D_F W(x,\cdot))^{-1}(0)G(t,x)\ dx \\ &= \int_0^1 1 \cdot G(t,x)\ dx = \int_{g(t,(0,1))} 1\ dx = |g(t,(0,1))|. \end{aligned}$$

Step 2: (ii)⇒(i). Let $\phi(t,\cdot)$ be the elastic deformation corresponding to the solution (G,S). By the mean value theorem, there exists an $\eta \in g(t,(0,1))$ such that

$$\partial_y \phi(t,\eta) = \frac{\phi(t,g(t,1)) - \phi(t,g(t,0))}{g(t,1) - g(t,0)} = \frac{1}{|g(t,(0,1))|} \in (0,1].$$

Since the stress $S(t)$ is constant in space, it follows

$$S(t) = D_F W_{nat}(\eta, \partial_y \phi(t,\eta)) \leq 0.$$

♣

The next result is not only useful later on, see Theorem 5.4.1 and Proposition A.4.9, but it states also that the material is never infinitely compressed or stretched.

Concerning nutrients

As in the setting with two components, we want to investigate how to include nutrients in the system. In general, the idea is the same as before, namely including one more term into the ODE like in the ODE (4.26) above. Again, we show that the nutrients are locally Lipschitz continuously depending on the growth tensor G and the Picard–Lindelöf Theorem A.2.6 yields existence and uniqueness. What is changed for arbitrary composed material is the diffusion coefficient and the absorption rate.

More precisely, consider $\eta\colon H^1(0,1) \to L^\infty(0,1)$ to be bounded and Lipschitz continuous and consider for $t \in [0,T]$ and $x \in (0,1)$ the ODE

$$\dot{G}(t,x) = \gamma(x)\mu(S(t))\eta(n(t,\cdot))(x)G(t,x).$$

Further, let $D\colon [0,1] \to \mathbb{R}$ be the diffusion coefficient in the reference configuration and $\beta\colon [0,1] \to \mathbb{R}$ be the absorption rate in the reference configuration. Again, the equation of the nutrients shall be solved in the deformed configuration, see Remark 2.3.2(ii) and Remark 4.2.6. After each given growth g, the diffusion coefficient $D_{nat}\colon \Omega_{nat} \to \mathbb{R}$ and the absorption rate $\beta_{nat}\colon \Omega_{nat} \to \mathbb{R}$ are defined by

$$D_{nat}(y) = D(g^{-1}(y)) \text{ and } \beta_{nat}(y) = \beta(g^{-1}(y)). \tag{4.53}$$

This definition resembles the definition of W_{nat}, motivated by the idea of the model: The volume increase due to growth does not affect the properties of the material. Furthermore, the diffusion coefficient D_t is obtained by the following calculation: The equation in the deformed configuration for the nutrients is given in the modelling, see 2.19, where we neglect the density. For a test function $\psi \in H^1(0,1)$, the LHS is

$$\begin{aligned} LHS &= \int_{\Omega_t} D_t(z)\partial_z n(z)\partial_z\psi(z)\ dz \\ &= \int_{\phi(\Omega_{nat})} D_t(z)\partial_z n(z)\partial_z\psi(z)\ dz \\ &= \int_{\Omega_{nat}} D_t(\phi(y))(\partial_z n)(\phi(y))(\partial_z\psi)(\phi(y)) \det|\partial_y\phi(y)|dy \\ &= \int_{\Omega_{nat}} D_t(\phi(y))\partial_y(n\circ\phi)(y)\partial_y(\psi\circ\phi)(y)(\partial_y\phi(y))^{-1}dy \\ &= \int_{\Omega_{nat}} D_{nat}(y)\partial_y(n\circ\phi)(y)\partial_y(\psi\circ\phi)(y)\ dy. \end{aligned}$$

This motivates the definition

$$\begin{aligned} D_t(z) &:= D_{nat}(\phi^{-1}(t,z))\partial_y\phi(t,\phi^{-1}(t,z)) \\ &= D(g^{-1}(t,\phi^{-1}(t,z)))\partial_y\phi(t,\phi^{-1}(t,z)) \end{aligned} \tag{4.54}$$

for $z \in (0,1)$. Similarly, the RHS tested with ψ and after change of variables with $g(t,\cdot)$, see Lemma A.4.2, is

$$\begin{aligned} RHS &= \int_{\Omega_t} \beta_t(z)n(z)\psi(z)\ dz \\ &= \int_{\phi(\Omega_{nat})} \beta_t(z)n(z)\psi(z)\ dz \\ &= \int_{\Omega_{nat}} \beta_t(\phi(y))n(\phi(y))\psi(\phi(y))\partial_y\phi(y)\ dy, \end{aligned}$$

which motivates the definition,

$$\begin{aligned} \beta_t(z) &= \beta_{nat}(\phi^{-1}(t,z))(\partial_y\phi)^{-1}(t,\phi^{-1}(t,z)) \\ &= \beta(g^{-1}(t,\phi^{-1}(t,z)))(\partial_y\phi)^{-1}(t,\phi^{-1}(t,z)) \end{aligned} \tag{4.55}$$

for $z \in (0,1)$, where now the change in volume affects the diffusion.

Definition 4.4.12 (Setting with nutrients II)**.** *Assume Setting 4.4.1. Further, let* $\eta\colon \mathbb{R} \to \mathbb{R}$ *be Lipschitz continuous with Lipschitz constant* $L_\eta > 0$ *and bounded, i.e. there exist* $-\infty < \eta_{min} < 0 < \eta_{max} < \infty$ *such that*

$$\eta_{min} \leq \eta(x) \leq \eta_{max} \text{ for all } x \in \mathbb{R}.$$

Let the diffusion coefficient $D \in W^{1,\infty}(0,1)$ *be with Lipschitz constant* $L_D > 0$ *and bounded such that there exist* $0 < D_{min} < D_{max} < \infty$ *such that for all* $x \in (0,1)$ *holds that*

$$D_{min} \leq D(x) \leq D_{max}.$$

Moreover, assume the absorption rate $\beta \in W^{1,\infty}(0,1)$ *to be with Lipschitz constant* $L_\beta > 0$ *and bounded by* $0 < \beta_{min} < \beta_{max} < \infty$*, namely, for all* $x \in (0,1)$ *holds that*

$$\beta_{min} \leq \beta(x) \leq \beta_{max}.$$

Moreover, let $D_F W \in \mathrm{C}^1([0,1] \times (0,\infty))$*. Let* $n_0, n_1 \in \mathbb{R}$ *be the Dirichlet boundary conditions for the nutrients.*

Then, a solution of the AMP has to include the nutrients as well.

Definition 4.4.13 (Solution of the AMP with nutrients)**.** *Assume the Setting 4.4.12 to hold. Then, we say* (G, S, n) *is a solution of the AMP with general elastic strain energy density and nutrients if the following conditions are fulfilled:*

(i) The growth tensor $G \in \mathrm{C}^1([0,T]; L^\infty(0,1))$ *fulfils the ODE*

$$\begin{aligned} \dot{G}(t,x) = \gamma(x)\mu(S(t))\eta(n(t,x))G(t,x), \\ G(0,x) = 1, \end{aligned}$$

for all $t \in [0,T]$ *and almost all* $x \in (0,1)$*.*

(ii) For the growth map

$$g(t,x) := \int_0^x G(t,\tilde{x})\, d\tilde{x}$$

and $t \in [0,T]$*, let* $\phi(t,\cdot)\colon g(t,(0,1)) \to \mathbb{R}$ *be the unique minimizer of* E_{gen} *in* $\mathcal{A}_{gen}$*. The Piola–Kirchhoff stress tensor* $S(t) \in \mathrm{C}^0([0,T])$ *to the elastic deformation* $\phi(t,\cdot)$ *is given by*

$$S(t) = D_F W(g^{-1}(t,y), \partial_y \phi(t,y)).$$

(iii) For all $t \in [0,T]$*, it holds* $n(t,\cdot) \in H^1(0,1)$ *and the equation*

$$\begin{aligned} \partial_z(D_t(z)\partial_z n(z)) &= \beta_t(z)n(z) \text{ on } (0,1), \\ n(0) &= n_0, \\ n(1) &= n_1, \end{aligned} \tag{4.56}$$

where D_t *and* β_t *are defined by (4.53), (A.5) and (A.6), is fulfilled.*

In order to obtain existence and uniqueness of the AMP in the sense of Definition 4.4.13, the nutrients n have to depend Lipschitz continuously on the growth tensor G, such that we can apply the Picard–Lindelöf existence result A.2.6 as before. To prove the needed Lipschitz continuity, we first prove several Lemmas.

The proof of the existence of a unique solution of the AMP with nutrients uses the same steps as in the case for the AMP with generalized elastic strain energy density. The a priori bounds G_{min}, G_{max} on G include the bounds on η, but are still strictly positive. Therefore the definition of the ball $\overline{B_{\tilde{R}}(G^*)}$ as well as the proofs of the bounds on ϕ and S, see Lemma 4.4.5 and Lemma 4.4.6, work the same. Therefore, we reference them in the following.

Lemma 4.4.14. *Consider the Setting 4.4.12 to hold. Then $(x, S) \mapsto (D_F W(x, \cdot))^{-1}(S)$ is separately Lipschitz continuous, i.e. there exist two constants $L_{D_F W^{-1};x}, L_{D_F W^{-1};S} > 0$ such that for all $x, x_1, x_2 \in [0, 1]$ and all $S, S_1, S_2 \in [S_{min}, S_{max}]$, it holds that*

$$|(D_F W(x_1, \cdot))^{-1}(S) - (D_F W(x_2, \cdot))^{-1}(S)| \leq L_{D_F W^{-1};x}|x_1 - x_2|,$$
$$|(D_F W(x, \cdot))^{-1}(S_1) - (D_F W(x, \cdot))^{-1}(S_2)| \leq L_{D_F W^{-1};S}|S_1 - S_2|,$$

where S_{min}, S_{max} are from (4.4.6).

Proof. For shorter notation, define

$$F\colon [0, 1] \times [S_{min}, S_{max}] \to \mathbb{R},\ (x, S) \mapsto F(x, S) = (D_F W(x, \cdot))^{-1}(S). \tag{4.57}$$

Step 1: Lipschitz continuity in x.
Step 1a): Inverse function. Let $S \in [S_{min}, S_{max}]$ be fixed. We prove that $F(\cdot, S)$ is differentiable on $(0, 1)$. Let $(\tilde{x}, \tilde{F}) \in (0, 1) \times (0, \infty)$ be a solution of $D_F W(\tilde{x}, \tilde{F}) = S$. Since $D_F W \in \mathrm{C}^1([0, 1] \times (0, \infty))$ with $D_F^2 W > 0$, we can apply the implicit function theorem to obtain an open neighbourhood $U(\tilde{x}) \times V(\tilde{F}) \subset (0, 1) \times (0, \infty)$ and a differential function $\hat{F}(\cdot, S)\colon U(\tilde{x}) \to V(\tilde{F})$ such that $D_F W(x, \hat{F}(x, S)) = S$ for all $x \in U(\tilde{x})$. Since for fixed S the equation $D_F W(x, F) = S$ has a unique solution F for each $x \in U(\tilde{x})$, it follows $F(\cdot, S) = \hat{F}(\cdot, S)$. Because $\tilde{x} \in (0, 1)$ is arbitrary, we conclude $F(\cdot, S) \in \mathrm{C}^1((0, 1))$.
Step 1b): Regularity of F_0. By implicit differentiation of $D_F W(x, F(x, S)) = S$, we obtain

$$0 = d_x S = d_x D_F W(x, F(x, S))) = \partial_x D_F W(x, F(x, S)) + D_F^2 W(x, F(x, S))\partial_x F(x, S).$$

By rearranging and using the fact that $D_F^2 W \geq c_c$, we get

$$\partial_x F(x, S) = \frac{-\partial_x D_F W(x, F(x, S))}{D_F^2 W(x, F(x, S))}.$$

Furthermore, $D_F W \in \mathrm{C}^1([0, 1] \times (0, \infty))$ and by Lemma 4.4.5 $F(x, S) \in [\phi_{min}, \phi_{max}]$ for all $x \in (0, 1)$ and all $S \in [S_{min}, S_{max}]$, thus there exists a maximum $W_{max} \in \mathbb{R}$ such that $|\partial_x D_F W(x, F(x, S))| \leq W_{max}$. Hence,

$$|\partial_x F(x, S)| = \left|\frac{-\partial_x D_F W(x, F(x, S))}{D_F^2 W(x, F(x, S))}\right| \leq \frac{W_{max}}{c_c}. \tag{4.58}$$

From this bound, it follows that the limits $\lim_{x \searrow 0} F(x, S)$ and $\lim_{x \nearrow 1} F(x, S)$ fulfil a Cauchy property and therefore the limits $\lim_{x \searrow 0} F(x, S)$ and $\lim_{x \nearrow 1} F(x, S)$ exist. This yields $F(\cdot) \in \mathrm{C}^0([0, 1]) \cap \mathrm{C}^1(0, 1)$.
Step 1c): Lipschitz continuity in x. Let $x_1, x_2 \in [0, 1]$ be arbitrary. Then, by the mean value theorem, there exists a $\eta \in (0, 1)$ such that

$$\begin{aligned} |(D_F W(x_1, \cdot))^{-1}(S) - (D_F W(x_2, \cdot))^{-1}(S)| &= |F(x_1, S) - F(x_2, S)| \\ &\leq \partial_x F(\eta, S)|x_1 - x_2| \\ &\leq L_{D_F W^{-1};x}|x_1 - x_2|, \end{aligned}$$

where $L_{D_F W^{-1};x} := \frac{W_{max}}{c_c}$, the uniform bound from (4.58).
Step 2: Lipschitz continuity in S.
Step 2a): $(D_F W(x, \cdot))^{-1}$ is continuous in S. Let $x \in [0, 1]$ be fixed and let $(S_k)_{k \in \mathbb{N}}$ be a sequence in $[S_{min}, S_{max}]$ with $S_k \to S$ for $k \to \infty$ but $F(x, S_k)$ does not converge to

$F(x,S)$. Since $(S_k)_{k\in\mathbb{N}}$ is bounded, $(F(x,S_k))_{k\in\mathbb{N}}$ is uniformly bounded, see Lemma 4.4.5. Therefore, there exist a subsequence, denoted by $(S_k)_{k\in\mathbb{N}}$ again, and a limit $F^* \in \mathbb{R}$ such that $F(x,S_k) \to F^* \neq F(x,S)$. However, the continuity of $D_F W(x,\cdot)$ implies that

$$D_F W(x,F^*) = \lim_{k\to\infty} D_F W(x,F(x,S_k)) = \lim_{k\to\infty} S_k = S.$$

From the uniqueness of the solutions of $D_F W(x,F) = S$, it follows $F^* = F(x,S)$, which is a contradiction. Furthermore, this argument holds for each subsequence and we therefore conclude the continuity of $(D_F W(x,\cdot))^{-1}$ in S.

Step 2b): $(D_F W(x,\cdot))^{-1}$ is differentiable in S. Let $S,S' \in \mathbb{R}$, $S' \neq 0$ with $D_F W(x,F(x,S)) = S$ and $D_F W(x,F(x,S+S')) = S+S'$. Then, $D_F W(x,\cdot) \in \mathrm{C}^1(0,\infty)$ yields

$$\begin{aligned} S' &= S+S'-S = D_F W(x,F(x,S+S')) - D_F W(x,F(x,S)) \\ &= \int_0^1 D_F^2 W(x,tF(x,S+S')+(1-t)F(x,S))(F(x,S+S')-F(x,S))\,dt. \end{aligned}$$

Rearranging and $D_F^2 W > c_c$ imply

$$\frac{F(x,S+S')-F(x,S)}{S'} = \left(\int_0^1 D_F^2 W(x,tF(x,S+S')+(1-t)F(x,S))\,dt\right)^{-1}.$$

For a compact interval $I_S \subset \mathbb{R}$ and $S,S' \in I_S$, $tF(x,S+S')+(1-t)F(x,S)$ is uniformly bounded for all $t\in[0,1]$ and due to the continuity of $D_F^2(x,\cdot)$ the integrand is uniformly bounded. Thus, dominated convergence allows us to pass to the limit $S' \searrow 0$ in the integral, such that we obtain

$$\begin{aligned} \partial_S F(x,S) &= \left(\int_0^1 \lim_{S'\searrow 0} D_F^2 W(x,tF(x,S+S')+(1-t)F(x,S))\,dt\right)^{-1} \\ &= \left(\int_0^1 D_F^2 W(x,tF(x,S)+(1-t)F(x,S))\,dt\right)^{-1} \\ &= \left(\int_0^1 D_F^2 W(x,F(x,S))\,dt\right)^{-1} \\ &= \left(D_F^2 W(x,F(x,S))\right)^{-1}. \end{aligned}$$

This expression is uniformly bounded on I_S, especially on $[S_{min},S_{max}]$ and the desired Lipschitz continuity follows. ♣

Lemma 4.4.15. *Assume Setting 4.4.12 to hold. Then there exists constants $L_{X;G}, L_{\partial_y\phi\circ g;G} > 0$ such that the following hold:*

(i) The map $X\colon L^\infty(0,1) \to \mathrm{C}^0([0,1])$, $G \mapsto X(G) = \phi(G)\circ g(G)$ is Lipschitz continuous, i.e. for all $G_1,G_2 \in \overline{B_{\tilde{R}}(G^*)}$ it holds

$$\begin{aligned} \|X(G_1)-X(G_2)\|_{C^0([0,1])} &= \|\phi(G_1)\circ g(G_1) - \phi(G_2)\circ g(G_2)\|_{C^0([0,1])} \\ &\le L_{X;G}\|G_1-G_2\|_{L^\infty(0,1)}. \end{aligned}$$

(ii) The map $\phi'\circ g\colon \overline{B_{\tilde{R}}(G^*)} \to \mathrm{C}^0([0,1])$ is Lipschitz continuous, i.e. for all $G_1,G_2 \in \overline{B_{\tilde{R}}(G^*)}$

it holds

$$\|\partial_y\phi(G_1)\circ g(G_1) - \partial_y\phi(G_2)\circ g(G_2)\|_{C^0([0,1])} \leq L_{\partial_y\phi\circ g;G}\|G_1 - G_2\|_{L^\infty(0,1)}.$$

Proof. **Concerning (ii).** Let $G \in \overline{B_{\tilde{R}}(G^*)}$ be given. Then the Euler–Lagrange equation reads as

$$D_F W(g^{-1}(G)(y), \partial_y\phi(G)(y)) = S(G) \text{ for all } y \in \Omega_{nat}.$$

Furthermore, $g(G)\colon [0,1] \to [g(G)(0), g(G)(1)]$ is bijective and from the uniqueness of the solution of $D_F W(x,F) = S$ we conclude

$$(\partial_y\phi(G)\circ g(G)) = F(\cdot, S(G)) \text{ on } [0,1],$$

where F is the map defined in (4.57). From Lemma 4.4.14, we know that F is Lipschitz continuous in S and from Lemma 4.4.6 $S(G)$ is Lipschitz continuous in G with Lipschitz constant L_S. Therefore, for $G_1, G_2 \in \overline{B_{\tilde{R}}(G^*)}$, it holds

$$\begin{aligned}
\|\phi(G_1)\circ g(G_1) - \phi(G_2)\circ g(G_2)\|_{C^0([0,1])} &= \sup_{x\in[0,1]} |\phi(G_1)\circ g(G_1)(x) - \phi(G_2)\circ g(G_2)(x)| \\
&= \sup_{x\in[0,1]} |F(x, S(G_1)) - F(x, S(G_2))| \\
&\leq L_{D_F W^{-1};S}|S(G_1) - S(G_2)| \\
&\leq L_{D_F W^{-1};S} L_S \|G_1 - G_2\|_{L^\infty(0,1)}.
\end{aligned}$$

Concerning (i). In view of the boundary conditions, $\phi(G_i)\circ g(G_i)(0) = 0$, $i = 1,2$, and that $\phi(G_i)\circ g(G_i) \in W^{1,\infty}(0,1)$, $i = 1,2$, due to the C^1-regularity of $\phi(G_i)$ on $[g(G_i)(0), g(G_i)(1)]$ and $g(G_i) \in W^{1,\infty}(0,1)$. Hence, the fundamental theorem of calculus and the chain rule yield for $x \in [0,1]$

$$\begin{aligned}
|X(G_1)(x) - X(G_2)(x)| =& |\phi(G_1)\circ g(G_1)(x) - \phi(G_2)\circ g(G_2)(x)| \\
=& \left| \int_0^x \partial_x(\phi(G_1)\circ g(G_1))(\tilde{x}) - \partial_x(\phi(G_2)\circ g(G_2))(\tilde{x})\, d\tilde{x} \right| \\
\leq& \int_0^1 |\partial_y\phi(G_1)\circ g(G_1)(\tilde{x})G_1(\tilde{x}) - \partial_y\phi(G_2)\circ g(G_2)(\tilde{x})G_2(\tilde{x})|\, d\tilde{x} \\
\leq& \int_0^1 |\partial_y\phi(G_1)\circ g(G_1)(\tilde{x})G_1(\tilde{x}) - \partial_y\phi(G_1)\circ g(G_1)(\tilde{x})G_2(\tilde{x})|\, d\tilde{x} \\
&+ \int_0^1 |\partial_y\phi(G_1)\circ g(G_1)(\tilde{x})G_2(\tilde{x}) - \partial_y\phi(G_2)\circ g(G_2)(\tilde{x})G_2(\tilde{x})|\, d\tilde{x} \\
\leq& \|\partial_y\phi(G_1)\circ g(G_1)\|_{C^0([0,1])}\|G_1 - G_2\|_{L^\infty(0,1)} \\
&+ \|\partial_y\phi(G_1)\circ g(G_1) - \partial_y\phi(G_2)\circ g(G_2)\|_{C^0([0,1])}\|G_2\|_{L^\infty(0,1)}.
\end{aligned}$$

With $G_2 \leq G_{max} + \frac{G_{min}}{2}$, $\partial_y\phi(G_1)\circ g(G_1)(x) \leq \phi_{max}$ for all $x \in [0,1]$, see Lemma 4.4.5, the Lipschitz continuity of $\partial_y\phi\circ g$ in G, see Step 1, and taking the supremum over $x \in [0,1]$, we obtain the desired Lipschitz continuity. ♣

Lemma 4.4.16. *Suppose that Setting 4.4.12 holds. Then there exist constants $L_{X;x}, L_{\partial_y\phi\circ g;x}$, $L_{\partial_y\phi\circ\phi^{-1};z} > 0$ such that the following hold:*

(i) For all $G \in \overline{B_{\tilde{R}}(G^*)}$ and all $x_1, x_2 \in (0,1)$ it holds

$$|X(G)(x_1) - X(G)(x_2)| \leq L_{;x}|x_1 - x_2|.$$

(ii) For all $G \in \overline{B_{\tilde{R}}(G^*)}$ and all $x_1, x_2 \in (0,1)$ it holds

$$|(\partial_y \phi(G) \circ g(G))(x_1) - (\partial_y \phi(G) \circ g(G))(x_2)| \leq L_{\partial_y \phi \circ g;x}|x_1 - x_2|.$$

(iii) For all $G \in \overline{B_{\tilde{R}}(G^*)}$ and all $z_1, z_2 \in (0,1)$ it holds

$$|(\partial_y \phi(G) \circ \phi^{-1}(G))(z_1) - (\partial_y \phi(G) \circ \phi^{-1}(G))(z_2)| \leq L_{\partial_y \phi \circ \phi^{-1};z}|z_1 - z_2|.$$

Proof. Let $G \in \overline{B_{\tilde{R}}(G^*)}$.
Concerning (i). By Lemma 4.4.5, $\partial_y \phi(G)(y) \in [\phi_{min}, \phi_{max}]$ for all $y \in [g(G)(0), g(G)(1)]$ and by definition $\partial_x g(G)(x) = G(x) \in [\frac{G_{min}}{2}, G_{max} + \frac{G_{min}}{2}]$ for almost all $x \in (0,1)$. Therefore, $\phi(G)$ and $g(G)$ are Lipschitz continuous and the Lipschitz continuity of $X(G) = \phi(G) \circ g(G)$ follows.
Concerning (ii). As in the proof of Lemma 4.4.15, $\partial_y \phi(G) \circ g(G)(x) = F(x, S(G))$ and due to the Lipschitz continuity of F in x, see Lemma 4.4.14, the Lipschitz continuity of $\partial_y \phi(G) \circ g(G)$ in x follows.
Concerning (iii). It holds

$$\partial_y \phi(G) \circ \phi^{-1}(G) = \partial_y \phi(G) \circ g(G) \circ g^{-1}(G) \circ \phi^{-1}(G) = \partial_y \phi(G) \circ g(G) \circ X^{-1}(G). \tag{4.59}$$

Since $X(G)(0) = 0$, with the fundamental theorem of calculus and chain-rule, it follows

$$X(G)(x) = X(G)(x) - X(G)(0) = \int_0^x \partial_x X(G)(\tilde{x})\, d\tilde{x} = \int_0^x \partial_y \phi(G) \circ g(G)(\tilde{x}) G(\tilde{x})\, d\tilde{x}.$$

Because $\partial_y \phi(G)$ is uniformly bounded for all $G \in \overline{B_{\tilde{R}}(G^*)}$ and $G \in \overline{B_{\tilde{R}}(G^*)}$ is bounded and by Lemma A.2.3, $X^{-1}(G)$ is Lipschitz continuous in x. Hence, (4.59) implies that the map $z \mapsto \partial_y \phi(G) \circ \phi^{-1}(G)(z)$ is Lipschitz continuous on $[0,1]$. ♣

Lemma 4.4.17. *Suppose that the Setting 4.4.12 holds. Then,*

$$\begin{aligned} &D\colon \overline{B_{\tilde{R}}(G^*)} \to \mathrm{C}^0([0,1]),\ G \mapsto D(G) = (\partial_y \phi(G) \circ \phi^{-1}(G))(D \circ X^{-1}(G)), \\ &\beta\colon \overline{B_{\tilde{R}}(G^*)} \to L^\infty(0,1),\ G \mapsto \beta(G) = (\partial_y \phi(G) \circ \phi^{-1}(G))(\beta \circ X^{-1}(G)) \end{aligned}$$

are Lipschitz continuous with Lipschitz constants L_D and L_β respectively.

Proof. Let $G_1, G_2 \in \overline{B_{\tilde{R}}(G^*)}$ and $z \in [0,1]$ be fixed. Then

$$\begin{aligned} &|\partial_y \phi(G_1) \circ \phi^{-1}(G_1)(z) - \partial_y \phi(G_2) \circ \phi^{-1}(G_1)(z)| \\ &\quad = |\partial_y \phi(G_1) \circ g(G_1) \circ X^{-1}(G_1)(z) - \partial_y \phi(G_2) \circ g(G_2) \circ X^{-1}(G_2)(z)| \\ &\quad = |\partial_y \phi(G_1) \circ g(G_1) \circ X^{-1}(G_1)(z) - \partial_y \phi(G_1) \circ g(G_1) \circ X^{-1}(G_2)(z)| \\ &\qquad + |\partial_y \phi(G_1) \circ g(G_1) \circ X^{-1}(G_2)(z) - \partial_y \phi(G_2) \circ g(G_2) \circ X^{-1}(G_2)(z)|. \end{aligned} \tag{4.60}$$

For the first term, we use Lemma A.4.5 and obtain

$$\begin{aligned} &|\partial_y \phi(G_1) \circ g(G_1) \circ X^{-1}(G_1)(z) - \partial_y \phi(G_1) \circ g(G_1) \circ X^{-1}(G_2)(z)| \\ &\quad \leq L_{\partial_y \phi \circ g;x}|X^{-1}(G_1)(z) - X^{-1}(G_2)(z)| \\ &\quad \leq L_{\partial_y \phi \circ g;x}\|X^{-1}(G_1) - X^{-1}(G_2)\|_{\mathrm{C}^0([0,1])}. \end{aligned}$$

From Lemma 4.4.15 (i), we know $\|X(G_1) - X(G_2)\|_{C^0([0,1])} \leq L_{X;G}\|G_1 - G_2\|_{L^\infty(0,1)}$ and therefore, with Lemma A.2.3,it follows that

$$L_{\partial_y\phi\circ g;x}\|X^{-1}(G_1) - X^{-1}(G_2)\|_{C^0([0,1])} \leq L_{\partial_y\phi\circ g;x}\frac{L_{X;G}\|G_1 - G_2\|_{L^\infty(0,1)}}{G_{min}}.$$

The second term in (4.59) is estimated by Lemma 4.4.15 (ii) as follows

$$\begin{aligned}
&|\partial_y\phi(G_1)\circ g(G_1)\circ X^{-1}(G_2)(z) - \partial_y\phi(G_2)\circ g(G_2)\circ X^{-1}(G_2)(z)| \\
&\quad \leq \|\partial_y\phi(G_1)\circ g(G_1)\circ X^{-1}(G_2) - \partial_y\phi(G_2)\circ g(G_2)\circ X^{-1}(G_2)\|_{C^0([0,1])} \\
&\quad = \|\partial_y\phi(G_1)\circ g(G_1) - \partial_y\phi(G_2)\circ g(G_2)\|_{C^0([0,1])} \\
&\quad \leq L_{\partial_y\phi\circ g;G}\|G_1 - G_2\|_{L^\infty(0,1)}.
\end{aligned}$$

Thus, $G \mapsto \partial_y\phi(G)\circ\phi^{-1}(G)$ is Lipschitz continuous. As before $G \mapsto X^{-1}(G)$ is Lipschitz continuous and hence $G \mapsto D\circ X^{-1}(G)$. The product of two bounded Lipschitz continuous function is Lipschitz continuous, which shows the statement. The argumentation for β is similar: We have that $G \mapsto \partial_y\phi(G)\circ\phi^{-1}(G)$ is Lipschitz continuous and from Lemma 4.4.5, we know that $\partial_y\phi(G) \geq \phi_{min}$ for all $G \in \overline{B_{\tilde{R}}(G^*)}$. Hence we can apply Lemma 4.4.15 to obtain that $G \mapsto \big(\partial_y\phi(G)\circ\phi^{-1}(G)\big)$ is Lipschitz continuous. Again, Lemma A.2.3 finishes the proof. ♣

Lemma 4.4.18. *Assume the Setting 4.4.12 to hold. Then, for each $G \in \overline{B_{\tilde{R}}(G^*)}$, there exists a unique solution $n(G) \in H^1(0,1)$ of (4.56) where we use the notation $D_t = D(G)$ and $\beta_t = \beta(G)$. Furthermore, there exists a constant $M_n > 0$ such that*

$$\|n(G)\|_{H^2(0,1)} \leq M_n \text{ for all } G \in \overline{B_{\tilde{R}}(G^*)}.$$

Proof. **Step 1: Existence and uniqueness.** To define an alternative problem, assume n to be a solution of (4.56) and define for $z \in [0,1]$

$$\begin{aligned}
\hat{n}(z) &:= (n_1 - n_0)z + n_0, \\
\tilde{n}(z) &:= n(z) - \hat{n}(z).
\end{aligned}$$

Then, $\tilde{n} \in H^1_0(0,1)$ and it fulfils the equation

$$\begin{aligned}
&-\partial_z(D_t(z)\partial_z\tilde{n}(z)) + \beta_t(z)\tilde{n}(z) \\
&\quad = -\partial_z(D_t(z)\partial_z n(x)) + \beta_t(z)n(z) + \partial_z(D_t(z)\partial_z\hat{n}(z)) - \beta_t(z)\hat{n}(z) \\
&\quad = \partial_z(D_t(z)\partial_z\hat{n}(z)) - \beta_t(z)\hat{n}(z) \\
&\quad = \partial_z f_t(z) + h_t(z),
\end{aligned} \tag{4.61}$$

where $f_t(z) := D_t(z)\partial_z\hat{n}(z)$ and $h_t(z) := -\beta_t(z)\hat{n}(z)$. We want to solve this equation with the Lax–Milgram theorem. The RHS fulfils the conditions, since $f_t, h_t \in L^2(0,1)$ due to the assumption on D, the uniform bound on $\partial_y\phi$, see Lemma 4.4.5, and the definition of $\hat{n}$. Furthermore,

$$D_t(z) = \partial_y\phi(t,\phi^{-1}(t,z))D(g^{-1}(t,z)) \in [\phi_{min}D_{min}, \phi_{max}D_{max}] \text{ for all } z \in [0,1],$$

and D_t is elliptic. In addition,

$$\beta_t(z) = \big(\partial_y\phi(t,\phi^{-1}(t,z))\big)^{-1}\beta(g^{-1}(t,z)) \in \left[\frac{\beta_{min}}{\phi_{max}}, \frac{\beta_{max}}{\phi_{min}}\right] \text{ for all } z \in [0,1].$$

As a consequence, we can apply the Lax–Milgram theorem to obtain existence and uniqueness of a solution $\tilde{n} \in H^1_0(0,1)$ and with $n = \tilde{n} + \hat{n}$ the statement follows.
Step 2: Uniform estimate in $H^1(0,1)$. In order to obtain a uniform estimate in $H^1(0,1)$,

test the equation (4.60) with the solution $\tilde{n} \in H^1(0,1)$ and obtain

$$\begin{aligned}
&\phi_{min} D_{min} \|\partial_z \tilde{n}(z)\|^2_{L^2(0,1)} + \frac{\beta_{min}}{\phi_{max}} \|\tilde{n}(z)\|^2_{L^2(0,1)} \\
&\leq \int_0^1 D_t(z)(\partial_z \tilde{n}(z))^2 \; dz + \int_0^1 \beta_t(z)\tilde{n}(z)^2 \; dz \\
&= \int_0^1 f_t(z)\partial_z \tilde{n} + h_t(z)\tilde{n}(z) \; dz \\
&= \int_0^1 D_t(z)\partial_z \hat{n}(z)\partial_z \tilde{n} - \beta_t(z)\hat{n}(z)\tilde{n}(z) \; dz \\
&\leq \int_0^1 \phi_{max} D_{max} |n_1 - n_0||\partial_z \tilde{n}| + \frac{\beta_{max}}{\phi_{min}} |\hat{n}(z)||\tilde{n}(z)| \; dz \\
&\leq \phi_{max} D_{max} \|n_1 - n_0\| \|\partial_z \tilde{n}\|_{L^2(0,1)} + \frac{\beta_{max}}{\phi_{min}} \|\hat{n}(z)\|_{L^2(0,1)} \|\tilde{n}(z)\|_{L^2(0,1)} \\
&\leq C(\phi_{min}, \phi_{max}, D_{max}, \beta_{max}, n_0, n_1) \|\tilde{n}\|_{H^1(0,1)}.
\end{aligned}$$

By rearranging the terms, we obtain the uniform estimate

$$\|\tilde{n}\|_{H^1(0,1)} \leq C'(\phi_{min}, \phi_{max}, D_{min}, D_{max}, \beta_{min}, \beta_{max}, n_0, n_1)$$

and by $n = \tilde{n} + \hat{n}$, the uniform estimate follows for n.
Step 3: Uniform estimate in $H^2(0,1)$. By elliptic regularity theory, it follows $n \in H^2(0,1)$. We are left to prove the uniform bound in $H^2(0,1)$.
Let $h \in \mathbb{R}$ with $|h| > 0$ small enough. Define the difference quotient D_h for a function $u \colon [0,1] \to \mathbb{R}$ via

$$D_h u(z) \coloneqq \frac{u(z+h) - u(z)}{h}.$$

It follows that $D_h \partial_z u = \partial_z D_h u(z)$ for sufficiently smooth functions u.
Let $\zeta \in C_c^\infty(0,1)$ and define the test function $\psi \coloneqq D_{-h}(-\zeta^2 D_h \tilde{n}) \in H_0^1(0,1)$. Testing the equation (4.60) with ψ yields on the LHS

$$\begin{aligned}
&\int_0^1 D_t(z)\partial_z \tilde{n}(z)\partial_z \psi(z) + \beta_t(z)\tilde{n}(z)\phi(z) \; dz \\
&= \int_0^1 D_t(z)\partial_z \tilde{n}(z)\partial_z D_{-h}(-\zeta^2(z)D_h \tilde{n}(z)) + \beta_t(z)\tilde{n}(z)D_{-h}(-\zeta^2(z)D_h \tilde{n}(z)) \; dz \\
&= \int_0^1 D_h(D_t(z)\partial_z \tilde{n}(z))\partial_z(\zeta^2(z)D_h \tilde{n}(z)) + D_h(\beta_t(z)\tilde{n}(z))\zeta^2(z)D_h \tilde{n}(z) \; dz \\
&= \int_0^1 (D_h D_t(z+h)\partial_z \tilde{n}(z) + D_t(z)D_h \partial_z \tilde{n}(z)) \big(\partial_z(\zeta^2)(z)D_h \tilde{n}(z) + \zeta^2(z)\partial_z D_h \tilde{n}(z)\big) \; dz \\
&\quad + \int_0^1 D_h \beta_t(z)\tilde{n}(z)\zeta^2(z)D_h \tilde{n}(z) + \beta_t(z)D_h \tilde{n}(z)\zeta^2(z)D_h \tilde{n}(z) \; dz
\end{aligned}$$

and on the RHS

$$\begin{aligned}
&\int_0^1 f_t(z)\partial_z\psi(z) + h_t(z)\psi(z)\ dz \\
&\quad= \int_0^1 f_t(z)\partial_z D_{-h}(-\zeta^2(z)D_h\tilde{n}(z)) + h_t(z)D_{-h}(-\zeta^2(z)D_h\tilde{n}(z))\ dz \\
&\quad= \int_0^1 D_hf_t(z)\partial_z(\zeta^2(z)D_h\tilde{n}(z)) + D_hh_t(z)\zeta^2(z)D_h\tilde{n}(z)\ dz \\
&\quad= \int_0^1 D_hf_t(z)\partial_z(\zeta^2)(z)D_h\tilde{n}(z) + D_hf_t(z)\zeta^2(z)\partial_z D_h\tilde{n}(z) + D_hh_t(z)\zeta^2(z)D_h\tilde{n}(z)\ dz.
\end{aligned}$$

Rearranging these terms and using Hölders inequality yield

$$\begin{aligned}
&\phi_{min}D_{min}\|\zeta D_h\partial_z\tilde{n}(z)\|^2_{L^2(0,1)} \\
&\quad\leq \int_0^1 D_t(z)|\zeta D_h\partial_z\tilde{n}(z)|^2\ dz \\
&\quad= \left|\int_0^1 (D_hD_t(z+h)\partial_z\tilde{n}(z)\partial_z(\zeta^2)(z)D_h\tilde{n}(z) + D_hD_t(z)\partial_z\tilde{n}(z)\zeta^2(z)\partial_z D_h\tilde{n}(z)\ dz\right| \\
&\qquad+ \left|\int_0^1 D_t(z)D_h\partial_z\tilde{n}(z)\partial_z(\zeta^2)(z)D_h\tilde{n}(z)\ dz\right| \\
&\qquad+ \left|\int_0^1 D_h\beta_t(z)\tilde{n}(z)\zeta^2(z)D_h\tilde{n}(z) + \beta_t(z)D_h\tilde{n}(z)\zeta^2(z)D_h\tilde{n}(z)\ dz\right| \\
&\qquad+ \left|\int_0^1 D_hf_t(z)\partial_z(\zeta^2)(z)D_h\tilde{n}(z) + D_hf_t(z)\zeta^2(z)\partial_z D_h\tilde{n}(z) + D_hh_t(z)\zeta^2(z)D_h\tilde{n}(z)\ dz\right| \\
&\quad\leq \|D_hD_t\|_{L^\infty(0,1)}\|\partial_z\tilde{n}\|_{L^2(0,1)}\|\partial_z(\zeta^2)\|_\infty\|D_h\tilde{n}\|_{L^2(0,1)} \\
&\qquad+ \|D_hD_t\|_{L^\infty(0,1)}\|\partial_z\tilde{n}\|_{L^2(0,1)}\|\zeta^2\|_\infty\|D_h\partial_z\tilde{n}\|_{L^2(0,1)} \\
&\qquad+ \|D_t\|_{L^\infty(0,1)}\|D_h\partial_z\tilde{n}\|_{L^2(0,1)}\|\partial_z(\zeta^2)\|_\infty\|D_h\tilde{n}\|_{L^2(0,1)} \\
&\qquad+ \|D_h\beta_t\|_{L^\infty(0,1)}\|\tilde{n}\|_{L^2(0,1)}\|\zeta^2\|_\infty\|D_h\tilde{n}\|_{L^2(0,1)} \\
&\qquad+ \|\beta_t\|_{L^\infty(0,1)}\|D_h\tilde{n}\|_{L^\infty(0,1)}\|\zeta^2\|_\infty\|D_h\tilde{n}\|_{L^2(0,1)} \\
&\qquad+ \|D_hf_t\|_{L^2(0,1)}\|\partial_z(\zeta^2)\|_\infty\|D_h\tilde{n}\|_{L^2(0,1)} \\
&\qquad+ \|D_hf_t\|_{L^2(0,1)}\|\zeta^2\|_\infty\|D_h\partial_z\tilde{n}\|_{L^2(0,1)} \\
&\qquad+ \|D_hh_t\|_{L^2(0,1)}\|\zeta^2\|_\infty\|D_h\tilde{n}\|_{L^2(0,1)}.
\end{aligned}$$

Next, we use that for a function $u \in W^{1,2}(0,1)$ and all $V \subset\subset (0,1)$ it holds

$$\|D_hu\|_{L^2(V)} \leq \|\partial_z u\|_{L^2(0,1)},$$

see [Eva10] Section 5.8.2 Theorem 3 (i). With this, the uniform bound on $\tilde{n}$ in $H^1(0,1)$ by C and $C_\zeta := \max\{\|\zeta^2\|_\infty, \|\partial_z(\zeta)^2\|_\infty\}$, we obtain

$$\phi_{min}D_{min}\|\zeta D_h\partial_z\tilde{n}(z)\|^2_{L^2(0,1)}$$

$$
\begin{aligned}
\le & C_\zeta\big(C^2\|D_hD_t\|_{L^\infty(0,1)} + C\|D_hD_t\|_{L^\infty(0,1)}\|D_h\partial_z\tilde n\|_{L^2(0,1)} \\
& + C\|D_t\|_{L^\infty(0,1)}\|D_h\partial_z\tilde n\|_{L^2(0,1)} + C^2\|D_h\beta_t\|_{L^\infty(0,1)} + C^2\|\beta_t\|_{L^\infty(0,1)} \\
& + C\|D_hf_t\|_{L^2(0,1)} + \|D_hf_t\|_{L^2(0,1)}\|D_h\partial_z\tilde n\|_{L^2(0,1)} + C\|D_hh_t\|_{L^2(0,1)}\big) \\
= & C_\zeta\|D_h\partial_z\tilde n\|_{L^2(0,1)}\left(C\|D_hD_t\|_{L^\infty(0,1)} + C\|D_t\|_{L^\infty(0,1)} + \|D_hf_t\|_{L^2(0,1)}\right) \\
& + C_\zeta C\left(C\|D_hD_t\|_{L^\infty(0,1)} + C\|D_h\beta_t\|_{L^\infty(0,1)} + C\|\beta_t\|_{L^\infty(0,1)} + \|D_hf_t\|_{L^2(0,1)} + \|D_hh_t\|_{L^2(0,1)}\right).
\end{aligned}
$$

By Lemma A.1.1, $\phi(G) \in C^1([g(G)(0), g(G)(1)])$ and therefore is $D_t, \beta_t \in W^{1,\infty}(0,1)$. By definition of f_t, h_t, $\|D_hf_t\|_{L^2(0,1)}, \|D_hh_t\|_{L^2(0,1)}$ are bounded by a constant C'' depending only on n_0, n_1, ϕ_{min}, ϕ_{max},D_{min},D_{max}, β_{min}, β_{max}. With a suitable constants $\overline{C}, \tilde C > 0$, depending only on n_0, n_1, $\phi_{min}, \phi_{max}, D_{min}, D_{max}, \beta_{min}, \beta_{max}$, we obtain

$$
\|\zeta D_h\partial_z\tilde n(z)\|^2_{L^2(0,1)} \le \overline{C}\|\zeta D_h\partial_z\tilde n(z)\|_{L^2(0,1)} + \tilde C
$$

and therefore the uniform bound on $D_h\partial_z n$ in $L^2(0,1)$. With [Eva10] Section 5.8.2 Theorem 3 (ii), it follows that $\tilde n \in H^2_{loc}(0,1)$.
To prove the global estimate, rearrange

$$
\partial_z f_t(z) + h_t(z) = -\partial_z(D_t(z)\partial_z\tilde n(z)) + \beta_t(z)\tilde n(z) = -\partial_z D_t(z)\partial_z\tilde n(z) - D_t\partial_z^2\tilde n(z) + \beta_t(z)\tilde n(z)
$$

to

$$
D_t\partial_z^2 n(z) = -\partial_z D_t(z)\partial_z\tilde n + \beta_t(z)\tilde n(z) - \partial_z f_t(z) - h_t(z).
$$

Taking the $L^2(0,1)$-norm, pulling out the minimum of D_t, with the triangle inequality and suitable estimates on D_t, β_t, f_t, h_t, it follows

$$
D_{min}\|\partial_z^2\tilde n\|_{L^2(0,1)} \le \hat C(n_0, n_1, \phi_{min}, \phi_{max}, D_{min}, D_{max}, \beta_{min}, \beta_{max})
$$

and the global estimate is shown. ♣

Proposition 4.4.19. *Suppose the Setting 4.4.12 holds. Then, there exists a constant* $L_n > 0, M_{n,0} > 0$ *such that the map*

$$
n_0\colon \overline{B_{\tilde R}(G^*)} \to \mathrm{C}^0([0,1]),\ G \mapsto n_0(G) = n(G)\circ X(G)
$$

is Lipschitz continuous with Lipschitz constant L_n *and bounded by* $M_{n,0}$*, i.e. for all* $G, G_1, G_2 \in \overline{B_{\tilde R}(G^*)}$ *it holds*

$$
\begin{aligned}
&\|n_0(G)\|_{C^0([0,1])} \le M_{n,0}, \\
&\|n_0(G_1) - n_0(G_2)\|_{C^0([0,1])} \le L_n\|G_1 - G_2\|_{L^\infty(0,1)}.
\end{aligned}
$$

Proof. Let $G, G_1, G_2 \in \overline{B_{\tilde R}(G^*)}$ be fixed. Then there exist $n(G), n(G_1), n(G_2) \in H^1(0,1)$ the weak solutions of the reaction-diffusion equation (4.56) to the diffusion $D(G), D(G_1), D(G_2)$ and absorption rates $\beta(G), \beta(G_1), \beta(G_2)$, respectively, and

$$
\|n(G')\|_{H^2(0,1)} \le M_n, G' = G, G_1, G_2,
$$

according to Lemma A.4.7.
Step 1: C^0-bound. By the Sobolev embedding $H^2(0,1) \hookrightarrow \mathrm{C}^0([0,1])$, it follows

$$
\|n_0(G)\|_{C^0([0,1])} = \|n(G)\circ X(G)\|_{C^0([0,1])} = \|n(G)\|_{C^0([0,1])} \le \tilde C\|n(G)\|_{H^2(0,1)} \le \tilde C M_n.
$$

Since G is arbitrary in $\overline{B_{\tilde R}(G^*)}$ the stated bound is proven.
Step 2: $G \mapsto n(G)$ is Lipschitz continuous. Since $n(G_1), n(G_2) \in H^1(0,1)$ we can use

$n(G_1) - n(G_2)$ as a test function in the weak formulation and obtain

$$0 = \int_0^1 D(G_i)(z)\partial_z n(G_i)(z)(\partial_z n(G_1)(z) - \partial_z n(G_2)(z))\ dz + \int_0^1 \beta(G_i)(z)n(G_i)(n(G_1)(z) - n(G_2)(z))\ dz, i = 1, 2.$$

Summing up these two equations, we get

$$0 = \int_0^1 \left(D(G_1)(z)\partial_z n(G_1)(z) - D(G_2)(z)\partial_z n(G_2)(z)\right)\left(\partial_z n(G_1)(z) - \partial_z n(G_2)(z)\right)\ dz + \int_0^1 \left(\beta(G_1)(z)n(G_1) - \beta(G_2)(z)n(G_2)\right)\left(n(G_1)(z) - n(G_2)(z)\right)\ dz.$$

By adding 0's on both sides, we can rewrite it as

$$\begin{aligned} 0 = &\int_0^1 D(G_1)(z)\left(\partial_z n(G_1)(z) - \partial_z n(G_2)(z)\right)^2\ dz \\ &+ \int_0^1 \left(D(G_1)(z) - D(G_2(z))\right)\partial_z n(G_2)(z)\left(\partial_z n(G_1)(z) - \partial_z n(G_2)(z)\right)\ dz \\ &+ \int_0^1 \beta(G_1)(z)\left(n(G_1)(z) - n(G_2)(z)\right)^2\ dz \\ &+ \int_0^1 \left(\beta(G_1)(z) - \beta(G_2)(z)\right) n(G_2)(z)\left(n(G_1)(z) - n(G_2)(z)\right)\ dz. \end{aligned}$$

Since $D(x) > 0$ and $\beta(x) >$ for all $x \in (0, 1)$, and therefore $D(G_i)(z)$, $\beta(G_i)(z) > 0$, $i = 1, 2$, for all $z \in (0, 1)$. It follows with the Hölder inequality

$$\begin{aligned} D_{min}\|\partial_z(n(G_1) - n(G_2))\|^2_{L^2(0,1)} + \beta_{min}\|n(G_1) - n(G_2)\|^2_{L^2(0,1)} \\ \leq \|D(G_1) - D(G_2)\|_{L^\infty(0,1)}\|\partial_z n(G_2)\|_{L^2(0,1)}\|\partial_z(n(G_1) - n(G_2))\|_{L^2(0,1)} \\ + \|\beta(G_1) - \beta(G_2)\|_{L^\infty(0,1)}\|n(G_2)\|_{L^2(0,1)}\|n(G_1) - n(G_2)\|_{L^2(0,1)} \end{aligned}$$

With $C := \min\{D_{min}, \beta_{min}\}$, we get

$$\begin{aligned} C\|n(G_1) - n(G_2)\|^2_{H^1(0,1)} \leq &\|D(G_1) - D(G_2)\|_{L^\infty(0,1)}\|\partial_z n(G_2)\|_{L^2(0,1)}\|n(G_1) - n(G_2)\|_{H^1(0,1)} \\ &+ \|\beta(G_1) - \beta(G_2)\|_{L^\infty(0,1)}\|n(G_2)\|_{L^2(0,1)}\|n(G_1) - n(G_2)\|_{H^1(0,1)}, \end{aligned}$$

and therefore with the bound M_n on $n(G_2)$ in $H^2(0, 1)$ it follows

$$\|n(G_1) - n(G_2)\|_{H^1(0,1)} \leq \frac{M_n}{C}\left(\|D(G_1) - D(G_2)\|_{L^\infty(0,1)} + \|\beta(G_1) - \beta(G_2)\|_{L^\infty(0,1)}\right).$$

The Lipschitz continuity of D and β, see Lemma A.4.6, yields

$$\|n(G_1) - n(G_2)\|_{H^1(0,1)} \leq \frac{M_n}{C}(L_D + L_\beta)\|G_1 - G_2\|_{L^\infty(0,1)},$$

and from the Sobolev embedding $H^1(0,1) \hookrightarrow \mathrm{C}^0([0,1])$,

$$\|n(G_1) - n(G_2)\|_{\mathrm{C}^0([0,1])} \leq C' \|n(G_1) - n(G_2)\|_{H^1(0,1)} \leq \frac{C' M_n}{C}(L_D + L_\beta)\|G_1 - G_2\|_{L^\infty(0,1)},$$

the desired Lipschitz continuity of n with Lipschitz constant $L_{n;G} := \frac{C'M_n}{C}(L_D + L_\beta)$.
Step 3: Lipschitz continuity of n_0. Let $x \in [0,1]$. Then with the fundamental theorem of calculus, the uniform H^2 bound on n and the Lipschitz continuity of n and X in G, see Step 2 and Lemma 4.4.15 (i), it follows

$$\begin{aligned}
&|n_0(G_1)(x) - n_0(G_2)(x)| \\
&\quad = |n(G_1) \circ X(G_1)(x) - n(G_2) \circ X(G_2)(x)| \\
&\quad = |n(G_1) \circ X(G_1)(x) - n(G_1) \circ X(G_2)(x)| + |n(G_1) \circ X(G_2)(x) - n(G_2) \circ X(G_2)(x)| \\
&\quad \leq \left| \int_{X(G_1)(x)}^{X(G_2)(x)} \partial_z n(G_1)(z)\, dz \right| + \|n(G_1) \circ X(G_2) - n(G_2) \circ X(G_2)\|_{C^0([0,1])} \\
&\quad \leq \|\partial_z n(G_1)\|_{\mathrm{C}^0([0,1])} |X(G_1)(x) - X(G_2)(x)| + \|n(G_1) - n(G_2)\|_{\mathrm{C}^0([0,1])} \\
&\quad \leq M_n \|X(G_1) - X(G_2)\|_{C^0([0,1])} + L_{n;G}\|G_1 - G_2\|_{L^\infty(0,1)} \\
&\quad \leq (M_n L_{X;G} + L_{n;G})\|G_1 - G_2\|_{L^\infty(0,1)}.
\end{aligned}$$

By taking the supremum over $x \in [0,1]$ the stated Lipschitz continuity with Lipschitz constant $L_n := M_n L_{X;G} + L_{n;G}$ follows. ♣

Theorem 4.4.20 (Existence of AMP with nutrients with general elastic strain energy density)**.** *Assume Setting 4.4.12 holds. Then, there exists a unique solution of the AMP with nutrients with general elastic strain energy density in the sense of Definition 4.4.13.*

4.5. Generalisations on the growth equation

After a discussion on different settings for the elastic problem, in this section, we discuss the generalisations of the right hand side $\mathcal{G}$ of the ODE, justifying the assumption of a bounded μ or as seen later global Lipschitz continuity. We investigate the combination of μ and powers of G in a homogeneous setting and give some simple other settings for $\mathcal{G}$ and end with a theorem, stating the needed estimates for a generalised RHS $\mathcal{G}$ to obtain existence and uniqueness of the AMP with the corresponding generalised ODE.
Firstly, a very simple counterexample states that the Lipschitz continuity in an arbitrary ODE is needed to avoid blow ups of the solution in finite time.

Example 4.5.1 (Counterexample: Locally Lipschitz and unbounded is not enough for long time existence)**.** *Consider $\mathcal{G}(t, x, S, G) := G^2$. Then, the solution of the ODE can be determined without considering the elastic deformation first, because it does not depend on S and there exists a time $T_0 > 0$ such that*

$$G(t) \to \infty \text{ for } t \nearrow T_0.$$

Proof. Using separation of variables the ODE transforms to

$$\frac{1}{G^2} dG = dt$$

and after integration we have

$$-\frac{1}{G} = C + t.$$

Consequently, the solution is

$$G(t) = \frac{-1}{C+t},$$

where the constant $C = \frac{-1}{G_0}$ is given from the initial value G_0. This function has a blow up for $t \nearrow \frac{1}{G_0} =: T_0$. ♣

This example proves that we do not find a global solution for a superlinear ODE with bounded μ. In the next example we want to discuss unbounded, but increasing μ combined with the quadratic terms of G in the ODE, i.e. can the μ weight out the superlinearity in the ODE?

Example 4.5.2 (Combining μ and powers of G)**.** *We consider the special case of a homogeneous material with $\mathcal{G}(t,x,S,G) = \mu(S)G^2$. As seen above, if $\mu(S(G))G$ has superlinear growth as a function of G, then, the ODE is expected to have a finite time of existence while for sublinear or linear growth the ODE has a long time solution.*
Consider an arbitrary, homogeneous elastic energy density $W\colon (0,\infty) \to [0,\infty)$ fulfilling

$$\begin{aligned} W(F) \to \infty \text{ for } F \to 0 \text{ or } F \to \infty, \\ W \in \mathrm{C}^2, \text{ strictly convex.} \end{aligned}$$

Since we consider homogeneous material the growth tensor is independent of x, hence, the growth map is

$$g(t,x) = \int_0^x G(t)\; d\tilde{x} = G(t)x.$$

Since the elastic energy density is independent of x for homogeneous material, the elastic deformation as minimizer of the elastic energy is given as

$$\phi(t,y) = \frac{1}{G(t)} y$$

with Piola–Kirchhoff stress tensor

$$S(t) = D_F W(\partial_y \phi(t,y)) = D_F W\left(\frac{1}{G(t)}\right).$$

Now, we want to find two functions μ_1, $\mu_2\colon \mathbb{R} \to [0,\infty)$ such that $\mu_1(S(G))G^2 = G^{1/2}$ and $\mu_2(S(G))G^2 = G^{3/2}$, namely long-time existence of a solution and blow up after finite time. We know that W is C^2 and strictly increasing, hence, $D_F W$ is C^1 and invertible. Define

$$\mu_i(S) := ((D_F W)^{-1}(S))^{\beta_i}$$

with $\beta_1 = \frac{3}{2}$ and $\beta_2 := \frac{1}{2}$. Then, we calculate

$$\begin{aligned} \dot{G}(t) &= \mu_1(S(G))G^2 \\ &= ((D_F W)^{-1}(S(G)))^{1/2} G^2 \\ &= \left((D_F W)^{-1}\left(D_F W\left(\frac{1}{G}\right)\right)\right)^{1/2} G^2 \\ &= G^{-1/2}G^2 = G^{3/2} \end{aligned}$$

and

$$\dot{G}(t) = \mu_2(S(G))G^2$$

$$= ((D_F W)^{-1}(S(G)))^{3/2} G^2$$
$$= \left((D_F W)^{-1} \left(D_F W \left(\frac{1}{G} \right) \right) \right)^{3/2} G^2$$
$$= G^{-3/2} G^2 = G^{1/2}.$$

Hence, we found examples for unbounded but still increasing (from the strict convexity of W) μ leading to different existence times.

The two previous example state that for superlinear RHS or for unbounded μ and linear ODE no solution can be obtained for arbitrary times $T > 0$. This implies that a generalised RHS still has to be Lipschitz continuous in order to avoid blow ups in finite times. We want to generalise the RHS $\mathcal{G}$ from only the linear functions, we considered so far, to arbitrary but Lipschitz continuous functions. But an arbitrary RHS does not yield a positive lower bound on the solution. Therefore, additional conditions are needed, which are fulfilled in the linear setting.
For a function $f\colon [0,T] \times (0,1) \times \mathbb{R} \to \mathbb{R}$, which is Lipschitz continuous in the third argument, Lemma A.2.2 (i) yields the existence of two functions $C(t,x)$, $\tilde{C}(t,x) > 0$ such that the formula

$$|f(t,x,\underline{G})| \leq \tilde{C}(t,x)|\underline{G}| + C(t,x)$$

holds. Motivated by this, the following theorem implies that the existence and uniqueness of the AMP can be generalised to more general RHS.

Proposition 4.5.3 (Bounded, arbitrary $\mathcal{G}$)**.** *Let $T > 0$ and $G_0 = 1 \in L^\infty(0,1)$ be given. Let $\mathcal{G}\colon [0,T] \times (0,1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\mathcal{G} = \mathcal{G}(t,x,S,G)$ be continuous and the right hand side of the growth ODE. Assume $\mathcal{G}$ to be Lipschitz continuous in S and G with Lipschitz constant $L_{\mathcal{G}} > 0$. Further, let*

$$\underline{\mathcal{G}}, \overline{\mathcal{G}}\colon [0,T] \times (0,1) \times \mathbb{R} \to \mathbb{R}$$

be continuous functions and Lipschitz continuous in the third argument with Lipschitz constant $L_{\underline{\mathcal{G}}}$ and $L_{\overline{\mathcal{G}}}$ respectively and assume there exist functions $\overline{C}$, $\overline{\tilde{C}}\colon [0,T] \to \mathbb{R}_{\geq 0}$ such that

$$|\overline{\mathcal{G}}(t,x,\overline{G})| \leq \overline{C}(t)|\overline{G}| + \overline{\tilde{C}}(t). \tag{4.62}$$

Assume for all $t \in [0,T], x \in (0,1), S \in \mathbb{R}, G \in \mathbb{R}$ to hold

$$\underline{\mathcal{G}}(t,x,G) < \mathcal{G}(t,x,S(G),G) < \overline{\mathcal{G}}(t,x,G). \tag{4.63}$$

Furthermore, assume one of the following conditions to hold:

(i) There exists functions $\underline{C}$, $\underline{\tilde{C}}\colon [0,T] \to \mathbb{R}_{\geq 0}$ such that

$$|\underline{\mathcal{G}}(t,x,\underline{G})| \leq \underline{C}(t)|\underline{G}| + \underline{\tilde{C}}(t) \tag{4.64}$$

for all $t \in [0,T]$, $x \in (0,1)$ and $G \in \mathbb{R}$. Furthermore, assume there exists an $G^*_{min} > 0$ such that

$$\exp\left(-\int_0^t \underline{C}(s)\ ds \right) - \int_0^t \underline{\tilde{C}}(s)\ ds =: G^*_{min}(t) > G^*_{min} \tag{4.65}$$

holds for $t \in [0,T]$, where $\underline{C}$ and $\underline{\tilde{C}}$ are as in (4.64).

(ii) For all $t \in [0,T]$ and $x \in (0,1)$ holds

$$\underline{\mathcal{G}}(t,x,0) = 0.$$

Let $t_0 > 0$ be the existence time, such that there exists a $G \in \mathrm{C}([0,t_0]; L^\infty(0,1))$ as the unique solution of the ODE in $L^\infty(0,1)$ induced by $\mathcal{G}$ on $[0,t_0]$. Then, $G(t,\cdot)$ is bounded in $L^\infty(0,1)$ for all $t \in [0,t_0]$, i.e. there exist $G_{min}, G_{max} \in \mathbb{R}_{>0}$ such that

$$G(t,x) \in [G_{min}, G_{max}]$$

for all $t \in [0,t_0]$ and almost all $x \in (0,1)$.

Proof. The argument of this proof is to use super- and subsolutions of ODEs, namely the solution G of the ODE with RHS $\mathcal{G}$ is between the solutions of the ODEs with RHS $\underline{\mathcal{G}}$ and $\overline{\mathcal{G}}$. Those are bounded and, hence, G is bounded.

Step 1: Solutions of ODEs. Since $\overline{\mathcal{G}}$ and $\underline{\mathcal{G}}$ are global Lipschitz continuous in G, the solutions of the according ODEs to the initial datum $G_0 = 1$ exist for all times. More precisely, there exist solutions

$$\begin{aligned} \underline{G}&\colon [0,T] \to L^\infty(0,1), \\ \overline{G}&\colon [0,T] \to L^\infty(0,1), \\ G&\colon [0,t_0] \to L^\infty(0,1) \end{aligned}$$

of the ODEs

$$\begin{aligned} \dot{\underline{G}}(t) &= \underline{\mathcal{G}}(t,\cdot,\underline{G}(t)), \\ \dot{\overline{G}}(t) &= \overline{\mathcal{G}}(t,\cdot,\overline{G}(t)), \\ \dot{G}(t) &= \mathcal{G}(t,\cdot,S(G(t)),G(t)) \end{aligned}$$

to the initial condition $G_0 = 1$ in $L^\infty(0,1)$, where $S(G)$ is the Piola–Kirchhoff stress tensor induced by G in the AMP, see Proposition 4.4.6. The solution for the ODE with RHS $\mathcal{G}$ exist for a short time due to the Picard–Lindelöf Theorem A.2.6. Only long time existence is at question in the setting of this proposition. It is achieved by the same argumentation as in the proof of Theorem 4.2.1.

Step 2: Comparison of the solution. The idea is that if G and $\overline{G}$ meet, due to the bigger (or equal) increase of $\overline{G}$, a moment later $\overline{G}$ has to be bigger (or equal) than G. Let $x \in (0,1)$ be fixed. By Lemma 4.1.4, the paths of the functions $G(\cdot,x)$, $\underline{G}(\cdot,x)$, $\overline{G}(\cdot,x)$ are continuously differentiable. Name $t \in [0,t_0]$ the time at which $G(\cdot,x)$ and $\overline{G}(\cdot,x)$ meet, namely $G(t,x) = \overline{G}(t,x)$.

To prove the statement, consider $\varepsilon > 0$ small. Use Taylor expansion theorem in time for the C^1-functions $G(\cdot,x)$ and $\overline{G}(\cdot,x)$ with h and $\overline{h}$ as Peano form of remainder respectively and the notation $\mathcal{G}_{dist} \coloneqq \overline{\mathcal{G}}(t,x,\overline{G}(t,x)) - \mathcal{G}(t,x,S(G(t)),G(t,x)) \in \mathbb{R}$ to calculate

$$\begin{aligned} \overline{G}(t+\varepsilon,x) &= \overline{G}(t,x) + \varepsilon\dot{\overline{G}}(t,x) + \overline{h}(\varepsilon)\varepsilon \\ &= G(t,x) + \varepsilon\overline{\mathcal{G}}(t,x,\overline{G}(t,x)) + \overline{h}(\varepsilon)\varepsilon \\ &= G(t,x) + \varepsilon\mathcal{G}(t,x,S(G(t)),G(t,x)) + \varepsilon\mathcal{G}_{dist} + \overline{h}(\varepsilon)\varepsilon \\ &= G(t,x) + \varepsilon\dot{G}(t,x) + \varepsilon(\mathcal{G}_{dist} + \overline{h}(\varepsilon)) \\ &= G(t+\varepsilon,x) + \varepsilon(\mathcal{G}_{dist} + \overline{h}(\varepsilon) - h(\varepsilon)). \end{aligned}$$

Due to the assumption on $\mathcal{G}$ and $\overline{\mathcal{G}}$, $\mathcal{G}_{dist} > 0$ holds and there exists an $\varepsilon^* > 0$ such that $\mathcal{G}_{dist} + \overline{h}(\varepsilon) - h(\varepsilon) > 0$ for all $\varepsilon < \varepsilon^*$. Hence, it follows

$$G(t+\varepsilon,x) \le \overline{G}(t+\varepsilon,x) \text{ for all } 0 < \varepsilon < \varepsilon^*.$$

For the lower bound, we need it to be strictly positive, because otherwise the injectivity of the growth map is lost. Therefore, we cannot apply the same proof for the lower bound.

Step 3a): Lower estimate under condition (i). First, $\underline{\mathcal{G}}$ has at most linear growth by assumption (4.64). This implies that

$$\underline{\mathcal{G}}(t,x,\underline{G}) \geq -\underline{C}(t)|\underline{G}| - \tilde{C}(t).$$

Next, use the substitution $H \coloneqq -\underline{G}$ to obtain from the ODE and the inequality that

$$\dot{H} = -\dot{\underline{G}} = -\underline{\mathcal{G}}(t,x,\underline{G}) \leq \underline{C}(t)|\underline{G}| + \tilde{\underline{C}}(t) = \underline{C}(t)|H| + \tilde{\underline{C}}(t) \tag{4.66}$$

holds.
By investigating whether the function H is strictly negative for all $t \in [0,T]$, we get a lower, time and space independent, estimate for $G(t,x)$. Let $t' \in [0,t_0]$ be arbitrary. Due to continuity of H in t, two cases may occur: $H(t',x) < -G^*_{min}(t')$ and there exists an $\varepsilon > 0$ such that $H(t,x) < 0$ for all $t \in [t', t'+\varepsilon]$ or not.
Case 1: $H(t',x) \leq -G^*_{min}(t')$ and $H(\cdot,x) < 0$ on $[t', t'+\varepsilon]$. In this case the inequality (4.66) becomes

$$\dot{H} \leq -\underline{C}(t)H + \tilde{\underline{C}}(t) \text{ for all } t \in [t', t'+\varepsilon].$$

From the Gronwall lemma, we conclude

$$H(t,x) \leq H(t',x)\exp\left(\int_{t'}^{t} -\underline{C}(s)\ ds\right) + \int_{t'}^{t} \exp\left(\int_{s}^{t} -\underline{C}(\tau)\ d\tau\right)\tilde{\underline{C}}(s)\ ds.$$

Resubstitution yields

$$\begin{aligned}
G(t,x) &= -H(t,x)\\
&\geq -H(t',x)\exp\left(\int_{t'}^{t} -\underline{C}(s)\ ds\right) - \int_{t'}^{t} \exp\left(\int_{s}^{t} -\underline{C}(\tau)\ d\tau\right)\tilde{\underline{C}}(s)\ ds\\
&= G(t',x)\exp\left(\int_{t'}^{t} -\underline{C}(\tau)\ d\tau\right) - \int_{t'}^{t} \exp\left(\int_{s}^{t} -\underline{C}(\tau)\ d\tau\right)\tilde{\underline{C}}(s)\ ds\\
&\geq G^*_{min}(t')\exp\left(\int_{t'}^{t} -\underline{C}(\tau)\ d\tau\right) - \int_{t'}^{t} \tilde{\underline{C}}(s)\ ds\\
&= \left(\exp\left(\int_{0}^{t'} -\underline{C}(\tau)\ d\tau\right) - \int_{0}^{t'} \tilde{\underline{C}}(s)\ ds\right)\exp\left(\int_{t'}^{t} -\underline{C}(\tau)\ d\tau\right) - \int_{t'}^{t} \tilde{\underline{C}}(s)\ ds\\
&\geq \exp\left(\int_{0}^{t} -\underline{C}(\tau)\ d\tau\right) - \int_{0}^{t} \tilde{\underline{C}}(s)\ ds = G^*_{min}(t) \geq G^*_{min},
\end{aligned}$$

which states the desired strictly positive lower bound.
Case 2: $H(t',x) > -G^*_{min}(t')$. This case can not occur. Since $H(\cdot,x)$ is continuous and for $t = 0$, $H(0,x) < 0$, there exists a time $t^* \in [0,t']$ such that $0 > H(t^*,x)) > -G^*_{min}(t')$ and $H(\cdot,x)$ negative on $[0,t^*]$. For this interval the above calculation states that $H(t^*,x) \leq -G^*_{min}(t)$. A contradiction.
Step 3b): Lower estimate under condition (ii). Note that $G(t,x) = 0$ is the unique solution of

$$\begin{aligned}
\dot{\underline{G}} &= \underline{\mathcal{G}}(t,x,\underline{G}(t,x)),\\
\underline{G}(0) &= 1.
\end{aligned}$$

Due to uniqueness, solutions can not intersect, see Lemma 4.5.5, and $\underline{G}(t,x) > 0$ for all $t \in [0,T]$. Due to the assumption (4.63),

$$\begin{aligned}\dot{\underline{G}}(0,x) &= \underline{\mathcal{G}}(0,x,\underline{G}(0,x)) \\ &= \underline{\mathcal{G}}(0,x,1) \\ &< \mathcal{G}(0,x,S(1),\underline{G}(0,x)) \\ &= \mathcal{G}(0,x,S(1),G(0,x)) = \dot{G}(0,x).\end{aligned}$$

As in step 2, there exists an $\varepsilon > 0$ such that $\underline{G}(t,x) < G(t,x)$ for all $t \in (0,\varepsilon)$. Prove this to hold on $(0,T]$ by contradiction. Assume otherwise, namely there exists an $t^* \in (\varepsilon, T]$, such that $\underline{G}(t^*,x) = G(t^*,x)$ and $\underline{G}(t,x) < G(t,x)$ for all $t \in [0,t^*)$. Then

$$\begin{aligned}\dot{\underline{G}}(t^*,x) &= \lim_{s\nearrow t^*} \frac{\underline{G}(t^*,x) - \underline{G}(s,x)}{t^* - s} \\ &= \lim_{s\nearrow t^*} \frac{G(t^*,x) - \underline{G}(s,x)}{t^* - s} \\ &\geq \lim_{s\nearrow t^*} \frac{G(t^*,x) - G(s,x)}{t^* - s} \\ &= \dot{G}(t^*,x).\end{aligned}$$

On the other hand,

$$\begin{aligned}\dot{\underline{G}}(t^*,x) &= \underline{\mathcal{G}}(t^*,x,\underline{G}(t^*,x)) \\ &< \mathcal{G}(t^*,x,\underline{G}(t^*,x)) \\ &= \mathcal{G}(t^*,x,G(t^*,x)) = \dot{G}(t^*,x),\end{aligned}$$

which yields a contradiction.

Step 4: Estimate on supersolution. It is left to prove an estimate on the solution $\overline{G}$. From the ODE, the assumption (4.62) and the fact that $0 \leq \underline{G} \leq G \leq \overline{G}$ it follows that

$$\begin{aligned}\dot{\overline{G}}(t,x) &= \overline{\mathcal{G}}(t,x,\overline{G}(t,x)) \\ &\leq \overline{C}(t)|\overline{G}(t,x)| + \overline{\tilde{C}}(t) \\ &= \overline{C}(t)\overline{G}(t,x) + \overline{\tilde{C}}(t).\end{aligned}$$

With Gronwall Lemma 4.2.2 we get

$$\begin{aligned}\overline{G}(t,x) &= \exp\left(\int_0^t \overline{C}(\tau)\ d\tau\right) + \int_0^t \exp\left(\int_s^t \overline{C}(\tau)\ d\tau\right) \overline{\tilde{C}}(s)\ ds \\ &\leq \exp\left(\int_0^T \overline{C}(\tau)\ d\tau\right) + \int_0^T \exp\left(\int_s^T \overline{C}(\tau)\ d\tau\right) \overline{\tilde{C}}(s)\ ds < \infty.\end{aligned}$$

Consequently, for finite time T, $\overline{G}(t,x)$ is bounded for each $x \in (0,1)$ and together with the first step, the claim is derived. ♣

Remark 4.5.4. *For the linear ODE (4.44) considered in the Setting 4.4.1, recall that the RHS is given as $\mathcal{G}(t,x,S,G) = \gamma(x)\mu(S)G$, the conditions of Proposition 4.5.3 are fulfilled by the functions*

$$\begin{aligned}\underline{\mathcal{G}}(t,x,G) &:= (\gamma\mu)_{min} G, \\ \overline{\mathcal{G}}(t,x,G) &:= (\gamma\mu)_{max} G.\end{aligned}$$

Obviously, $\underline{\mathcal{G}}(t,x,G) \leq \mathcal{G}(t,x,S(G),G) \leq \overline{\mathcal{G}}(t,x,G)$ *is fulfilled by all* $G > 0$*. By definition of* $\overline{\mathcal{G}}$*, the functions* $\overline{C}(t) = (\gamma\mu)_{max}$ *and* $\overline{\tilde{C}}(t) = 0$ *fulfil the condition (4.62).*

(i) By definition of $\underline{\mathcal{G}}$ *the condition (4.64) is fulfilled by* $\underline{C}(t) = |(\gamma\mu)_{min}|$ *and* $\underline{\tilde{C}}(t) = 0$*. In addition, for each* $x \in (0,1)$ *and* $t \in [0,T]$ *holds*

$$\exp\left(-\int_0^t \underline{C}(s)\ ds\right) - \int_0^t \underline{\tilde{C}}(s)\ ds$$
$$= \exp\left(-t|(\gamma\mu)_{min}|\right) - 0 \geq \exp(T(\gamma\mu)_{min}) =: G^*_{min},$$

which states (4.65).

(ii) For all $t \in [0,T]$ *and all* $x \in (0,1)$ *holds*

$$\underline{\mathcal{G}}(t,x,0) = (\gamma\mu)_{min} \cdot 0 = 0,$$

consequently, (ii) is fulfilled.

The following lemma is used in the proof of Proposition 4.5.3.

Lemma 4.5.5. *Let* $f\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function and Lipschitz continuous in the second argument with constant* $L > 0$*. Further, let* $G_{1,0} < G_{2,0}$ *be initial values and* $G_1,\ G_2\colon [0,T] \to \mathbb{R}$ *be the solutions of the ODEs*

$$\begin{aligned} \dot{G}_1(t) &= f(t, G_1(t)), & \dot{G}_2(t) &= f(t, G_2(t)), \\ G_1(0) &= G_{1,0}, & G_2(0) &= G_{2,0}. \end{aligned} \tag{4.67}$$

Then, it follows

$$G_1(t) < G_2(t) \text{ for all } t \in [0,T].$$

Proof. Assume there exists a $t^* \in (0,T]$ and a $G^* \in \mathbb{R}$ such that $G_1(t^*) = G_2(t^*) = G^*$. Then G_1 and G_2 are two solutions to the same ODE to the initial value G^*. Uniqueness contradicts the original initial condition.
More precisely, we define the functions

$$\tilde{G}_1, \tilde{G}_2\colon [0,t^*] \to \mathbb{R}, \tilde{G}_i(t) = G_i(t^* - t),\ i = 1,2.$$

Then, the ODEs imply

$$\begin{aligned} &\dot{\tilde{G}}_i(t) = -\dot{G}_i(t^* - t) = -f(t^* - t, G_i(t^* - t)) = -f(t^* - t, \tilde{G}_i(t)),\ i = 1,2, \\ &\tilde{G}_i(0) = G^*. \end{aligned}$$

Since $\tilde{G}_1, \tilde{G}_2$ are solutions to the same ODE with the same initial value G^*, the uniqueness yields that $\tilde{G}_1(t) = \tilde{G}_2(t)$ for all $t \in [0,t^*]$. Especially, it follows

$$G_{1,0} = G_1(t^* - t^*) = \tilde{G}_1(t^*) = \tilde{G}_2(t^*) = G_2(t^* - t^*) = G_{2,0}$$

which is a contradiction to $G_{1,0} < G_{2,0}$. Therefore, the solutions G_1, G_2 can not cross and from the initial values follows $G_1(t) < G_2(t)$ for all $t \in [0,T]$. ♣

5 Regularity for stress modulated growth, no nutrients

After proving existence and uniqueness of the AMP in various settings, the regularity shall be discussed. By construction of the AMP, the evolution of a point $x \in [0,1]$ under the stress modulated growth over time is described by the total mapping $X\colon [0,T] \times [0,1] \to [0,1]$, given by

$$X(t,x) = \phi(t, g(t,x)). \tag{5.1}$$

The regularity of this map is induced by the regularities of the growth map g, for regularity in space see Section 5.1 and for regularity in time Section 5.3, and of the elastic deformation ϕ, for regularity in space see Section 5.2 and for regularity in time see Section 5.4.
The results are stated below with the use of the results from the following sections. For the total map X, the following regularity is obtained in space:

Proposition 5.0.1 (Regularity in space of the total mapping)**.** *Let $k \in \mathbb{N},\ k > 2$, be fixed. In the Setting 5.1.1 assume one of the following two conditions:*

(i) W has the structure $W(x,F) = \kappa(x)\tilde{W}(F)$ with $\kappa \in \mathrm{C}^{k-1}([0,1])$ and $\tilde{W} \in \mathrm{C}^k((0,\infty))$ and $\kappa_{min} := \inf_{x\in[0,1]} \kappa(x) > 0$,

(ii) $W \in \mathrm{C}^k([0,1] \times (0,\infty))$.

Then, for each $t \in [0,T]$ it holds that

$$X(t,\cdot) \in \mathrm{C}^k([0,1]).$$

Proof. By chain rule,

$$\partial_x X(t,x) = \partial_y \phi(t, g(t,x))\partial_x g(t,x) = \partial_y \phi(t, g(t,x))G(t,x),$$

where $G(t,\cdot) \in \mathrm{C}^{k-1}([0,1])$, see Theorem 5.1.5 below. See Remark 5.2.5 for the needed regularity of W_{nat}. Thus, $\partial_y \phi(t,\cdot)$ is $\mathrm{C}^{k-1}(g(t,[0,1]))$ for $t \in [0,T]$ due to the assumptions, see Proposition 5.2.3 in case (i) or Proposition 5.2.4 in case (ii). ♣

Moreover, we can prove the following result on the regularity in time for the total mapping X:

Proposition 5.0.2 (Regularity in time of the total mapping)**.** *Let $k \in \mathbb{N}$ be fixed and let the conditions in Setting 5.1.1 be fulfilled. Furthermore, assume one of the following:*

(i) $W(x,F) = \kappa(x)\tilde{W}(F)$ *for all* $x \in [0,1]$ *and* $F \in (0,\infty)$*, where* $\kappa \in \mathrm{C}^{k-1}([0,1])$*,* $\kappa_{min} \coloneqq \inf_{x\in[0,1]} \kappa(x) > 0$ *and* $\tilde{W} \in \mathrm{C}^{k+1}(0,\infty)$*,*

(ii) $W \in \mathrm{C}^k([0,1]\times(0,\infty))$ *and* $W(x,\cdot) \in \mathrm{C}^{k+1}$ *for all* $x \in [0,1]$.

Let $\mu \in \mathrm{C}^k(\mathbb{R})$*. Assume further that for all* $F \in (0,\infty)$ *the condition*

$$\inf_{x\in[0,1]} D_F W(x,F) > -\infty$$

holds. Then, it holds for each $x \in [0,1]$ *that*

$$X(\cdot,x) \in \mathrm{C}^k(0,T).$$

Proof. Let $x \in [0,1]$ be fixed. Further let $t \in (0,T)$ be arbitrary and define $y \coloneqq g(t,x)$. By Proposition 5.4.1, it holds

$$\phi(\cdot,y) \in \mathrm{C}^k(U),$$

where $U \subset (0,T)$ is defined in (5.12). Furthermore, Proposition 5.2.3 states

$$\phi(t,\cdot) \in \mathrm{C}^k(g(t,[0,1]))$$

for each $t \in [0,T]$. Since the partial derivatives in space and time are continuous, we conclude

$$\phi \in \mathrm{C}^k(\mathring{\mathcal{T}}_g).$$

Moreover, Proposition 5.3.2 leads to $G(\cdot,x) \in \mathrm{C}^k([0,T])$ for all $x \in [0,1]$ and by dominated convergence for any $n \in \{1,\dots,k\}$ we compute

$$\begin{aligned}\partial_t^n g(t,x) &= \partial_t^n \int_0^x G(t,\tilde{x})\ d\tilde{x}\\ &= \int_0^x \partial_t^n G(t,\tilde{x})\ d\tilde{x}\end{aligned}$$

and $\partial_t^n g(\cdot,x)$ is continuous. Hence, $g(\cdot,x) \in C^k([0,T])$. Finally, in view of (5.1), the composition X of the $\mathrm{C}^k(\mathring{\mathcal{T}}_g)$-function ϕ and the $\mathrm{C}^k([0,T])$-function $g(\cdot,x)$ is $C^k(0,T)$. ♣

5.1. Regularity in space of the growth tensor

In view of the existence result Theorem 4.4.7, the regularity of the growth tensor in space is given by the Banach space, in which we solve the ODE. In the case of the Theorem 4.4.7 this is $L^\infty(0,1)$. Here, we consider a space with more regularity, namely $\mathrm{C}^k([0,1])$. This motivates the definition of the following setting.

Definition 5.1.1 (Setting for C^k-regularity of growth tensor in space)**.** *Let* $k \in \mathbb{N}$ *be given. Let* $\gamma \in \mathrm{C}^k([0,1])$ *be the growth multiplier and* $\mu \in \mathrm{C}(\mathbb{R})$*, such that*

$$\begin{aligned}(\gamma\mu)^l_{max} &\coloneqq \sup_{x\in[0,1],y\in\mathbb{R}} \{\gamma^{(l)}(x)\mu(y)\} < \infty,\\ (\gamma\mu)^l_{min} &\coloneqq \inf_{x\in[0,1],y\in\mathbb{R}} \{\gamma^{(l)}(x)\mu(y)\} > -\infty\end{aligned}$$

for $0 \le l \le k$*, where the exponent* (l) *denotes the* l*-th derivative. Furthermore, assume* $\mu_{min} \coloneqq \inf_{x\in\mathbb{R}} \mu(x) > -\infty$ *and* $\mu_{max} \coloneqq \sup_{x\in\mathbb{R}} \mu(x) < \infty$.

Let $W\colon [0,1]\times(0,\infty)\to\mathbb{R}$ be the energy density which fulfils the conditions (EL1)-(EL3) from Definition 3.1.3 and suppose there exists a $c_c>0$ such that

$$D_F^2W(x,F)>c_c \tag{5.2}$$

for all $x\in[0,1]$ and all $F\in(0,\infty)$. Furthermore, assume that there exists a continuous function $C_W\colon(0,\infty)\to(0,\infty)$ such that for all $x\in[0,1]$ and all $F\in(0,\infty)$

$$D_F^2W(x,F)\le C_W(F) \text{ for all } F\in(0,\infty) \text{ and almost all } x\in(0,1).$$

Let the following hold: For each $F\in(0,\infty)$, it is

$$\inf_{x\in(0,1)} D_FW(x,F)>-\infty,$$
$$\sup_{x\in(0,1)} D_FW(x,F)<+\infty.$$

For a given growth map g define the admissible set for elastic deformations

$$\mathcal{A}_{\mathrm{C}^k}\coloneqq\{v\in W^{1,\infty}(g((0,1)))|\ v(g(0))=0,\ v(g(1))=1\}$$

and for an elastic deformation $\phi\in\mathcal{A}_{gen}$ the elastic energy $E_{\mathrm{C}^k}\colon\mathcal{A}_{\mathrm{C}^k}\to\mathbb{R}$ is given by

$$E_{\mathrm{C}^k}(\phi)\coloneqq\int\limits_{g((0,1))} W(g^{-1}(y),\partial_y\phi(y))\ dy.$$

Finally, let $T>0$ be the time horizon and let $G_0=1\in\mathrm{C}^k([0,1])$ be the initial datum for the ODE for the growth tensor.

Suitable to this setting, we adjust the definition of a solution of the AMP in the following way.

Definition 5.1.2 (Definition of AMP in C^k)**.** *Consider the Setting 5.1.1. Then, we call a couple $(G,S)\in\mathrm{C}^1([0,T];\mathrm{C}^k([0,1]))\times\mathrm{C}^0([0,T])$ a solution of the AMP with general elastic energy density if the following is fulfilled:*

(i) The growth tenor G fulfils the ODE

$$\begin{aligned}\dot G(t,x)&=\gamma(x)\mu(S(t))G(t,x),\\ G(0,x)&=1,\end{aligned} \tag{5.3}$$

for all $t\in[0,T]$ and all $x\in[0,1]$.

(ii) For

$$g(t,x)\coloneqq\int\limits_0^x G(s,\tilde x)\ d\tilde x$$

and $t\in[0,T]$ let $\phi(t,\cdot):g(t,[0,1])\to\mathbb{R}$ be the unique minimizer of E_{C^k} in $\mathcal{A}_{\mathrm{C}^k}$ (for existence and uniqueness see Remark 3.2.4. The Piola–Kirchhoff stress tensor S to the elastic deformation ϕ is given by

$$S(t)=D_FW(g^{-1}(y),\partial_y\phi(t,y)).$$

With similar arguments as before, here applied to the space of $\mathrm{C}^k([0,1])$, the solution of the AMP exists with the desired regularity in space. In order to prove that statement, again a priori estimates on the growth tensor are needed. It follows from the fact, that the conditions of Lemma 4.2.4 are fulfilled by the Setting 5.1.1.

Lemma 5.1.3 (A priori estimates for the values of the growth tensor)**.** *Let (G, S) be a solution of the AMP in the sense of Definition 5.1.2. Then, for all $t \in [0,T]$ and $x \in [0,1]$*

$$G(t,x) \in [G_{min}, G_{max}]$$

holds, where

$$\begin{aligned} G_{min} &:= \exp\left(\min_{t\in[0,T]}\{(\gamma\mu)_{min}t\}\right), \\ G_{max} &:= \exp\left(\max_{t\in[0,T]}\{(\gamma\mu)_{max}t\}\right). \end{aligned} \tag{5.4}$$

To continue, we show auxiliary results needed for the proof of the existence and uniqueness, see Theorem 5.1.5 below, the Lipschitz continuity of the stress tensor S in dependence of G is shown. The proof of it will be quite similar to that of Theorem 4.4.6. The only difference is that the Fréchet derivative is now considered within a different space, but yields the analogous result.

Proposition 5.1.4 (The stress tensor is Lipschitz continuous)**.** *In the Setting 5.1.1, the map*

$$S\colon V \to \mathbb{R},\ G \mapsto S(G) \text{ is Lipschitz continuous,}$$

where $S(G)$ is the Piola–Kirchhoff stress tensor to the minimizer of the elastic energy and V the subset of $\mathrm{C}^k([0,1])$ defined by

$$V := \{G \in \mathrm{C}^k([0,1]) |\ G(x) \in [G_{min}, G_{max}] \text{ for all } x \in [0,1]\}. \tag{5.5}$$

Proof. Similar as in the proof of Theorem 4.4.6, we get the formula

$$1 = \int_0^1 (D_F W(x,\cdot))^{-1}(S) G(x)\ dx$$

and we apply the implicit function theorem A.3.5 for the function

$$f\colon \mathring{V} \times \mathbb{R} \to \mathbb{R},\ (G,S) \mapsto 1 - \int_0^1 (D_F W(x,\cdot))^{-1}(S) G(x)\ dx.$$

In view of the intermediate value theorem, for each $G \in \mathring{V}$ there exists a unique $S(G) \in \mathbb{R}$ such that $f(G, S(G)) = 0$. The $S(G)$ is unique due to the strict monotonicity of $(D_F W(x,\cdot))^{-1}$. The argumentation to obtain existence was done in the proof of Theorem 4.2.1. Therefore, we are left to prove that the derivative $D_S f$ is invertible, which indeed can be proven as in the proof of Proposition 4.4.6. Therefore, the implicit function theorem yields the C^1-regularity of S on a neighbourhood of G.
Let us now consider $G \in \partial V$. By Lemma 5.1.3, there exists the radius $r = G_{min}/2$ such S is well-defined on $B_r(G)$, since the values of all $\tilde{G} \in B_r(G)$ are uniformly bounded away from 0 by $G_{min}/2$, and the argumentation for a G is the same as in the interior of V.
The bound on $D_G S$ is established with the same arguments as in Proposition 4.4.6. ♣

To prove existence and uniqueness of a solution in the sense of definition 5.1.2, we follow the same strategy as for the $L^\infty(0,1)$-regularity for G, involving the Picard–Lindelöf theorem A.2.6. The difficulty is that the stress tensor $S(t)$ is defined only if almost all values of $G(t,\cdot)$ are strictly positive. Hence, we need to restrict the admissible set of the ODE. To do so, the set V is introduced, making sure that the values of any G are bounded from below away from zero. With this preparation and by following the same ideas as before, the Lipschitz continuity of S and a priori estimates on G, the existence follows.

Theorem 5.1.5 (Existence and uniqueness in C^k)**.** *In the Setting 5.1.1, there exists a unique solution* (G, S) *of the AMP in* C^k *according to Definition 5.1.2.*

Proof. **Step 1: The set V.** Refer to Lemma 5.1.3 for an a priori estimate on the growth tensor which solves the ODE (5.3), obtaining the lower and upper estimates G_{min} and G_{max}, see (5.4). For the definition of the RHS, we define the set $V \subset \mathrm{C}^k([0,1])$ by

$$V \coloneqq \{G \in \mathrm{C}^k([0,1]) |\ G(x) \in [G_{min}, G_{max}] \text{ for all } x \in [0,1]\}, \tag{5.6}$$

which is convex and closed in $\mathrm{C}^k([0,1])$.
Convex: For $\lambda \in (0,1)$ and $G_1, G_2 \in V$ holds

$$(\lambda G_1 + (1-\lambda)G_2)(x) \leq \lambda G_{max} + (1-\lambda)G_{max} = G_{max},$$

and

$$(\lambda G_1 + (1-\lambda)G_2)(x) \geq \lambda G_{min} + (1-\lambda)G_{min} = G_{min}.$$

In consequence, V is convex.
Closed: Let $(G_k) \subset V$ be a sequence with $G_k \to G,\ k \to \infty$ in $\mathrm{C}^k([0,1])$. Then, $G \in \mathrm{C}^k([0,1])$. We have to prove that $G \in V$. From the convergence in $\mathrm{C}^k([0,1])$ follows convergence in $\mathrm{C}^0([0,1])$, and thus, the pointwise convergence. Hence, for any $x \in [0,1]$, it follows that

$$G_{max} \geq \lim_{k\to\infty} G_k(x) = G(x),$$
$$G_{min} \leq \lim_{k\to\infty} G_k(x) = G(x),$$

hence, V is closed in $\mathrm{C}^k([0,1])$.
Step 2: Conclusion. In Proposition 5.1.4, the local Lipschitz continuity of $S : V \to \mathbb{R}$ is shown. This together with the a priori estimates in Lemma 5.1.3 gives all necessary tools to treat the ODE as in the proofs above, which yields a unique solution $G \in \mathrm{C}^1([0,T]; \mathrm{C}^k([0,1]))$. ♣

Remark 5.1.6. *Theorem 5.1.5 is consistent with the theory of parameter dependent ODEs, namely* C^k *dependence of the parameter if the RHS is* C^k*, see e.g. [Tes12], Theorem 2.11. However, we do not have the* C^k *regularity in time, yet, in order to apply the mentioned theorem.*

Furthermore, an a priori estimate on the growth tensor in $\mathrm{C}^k([0,1])$ can be established. It is not necessary for the proof of existence, but listed here for completeness of results.

Lemma 5.1.7 (A priori estimates for the growth tensor in C^k)**.** *Consider the Setting 5.1.1 and let* (G, S) *be a solution of the AMP in* C^k*. Then, there exists a constant* $C > 0$ *such that*

$$\|G(t,.)\|_{\mathrm{C}^k([0,1])} \leq C$$

for all $t \in [0,T]$.

Proof. We prove the statement by induction, since, in order to find estimates for the l-th derivative, we need the $(l-1)$-th derivative to be bounded.
Step 1: Estimate in C^0. We already proved that the values of G can be uniformly estimated. Hence, for $C \coloneqq G_{max} = G_{max}(\gamma, \mu, T, G_0)$, it holds for all $t \in [0,T]$ that

$$\|G(t,\cdot)\|_{\mathrm{C}^0([0,1])} \leq C.$$

Step 2: First derivative. Since G solves the ODE in 5.1.2(i), it also holds that G takes the integral form

$$G(t,x) = 1 + \int_0^t \gamma(x)\mu(S(\tau))G(\tau,x)\ d\tau.$$

Since $G(t,\cdot) \in \mathrm{C}^k([0,1])$, we differentiate with respect to x, use that the paths $G(\cdot,x)$ are continuous in time, see Lemma 4.1.4, to use dominated convergence to interchange derivative and integration and obtain

$$\begin{aligned}\partial_x G(t,x) &= \partial_x 1 + \partial_x \int_0^t \gamma(x)\mu(S(\tau))G(\tau,x)\ d\tau\\ &= \int_0^t \partial_x(\gamma(x)\mu(S(\tau))G(\tau,x))\ d\tau\\ &= \int_0^t \partial_x\gamma(x)\mu(S(\tau))G(\tau,x)\ d\tau + \int_0^t \gamma(x)\mu(S(\tau))\partial_x G(\tau,x)\ d\tau.\end{aligned}$$

Here, we used the fact that all functions are C^1 on a compact set, and hence, the derivative with respect to x of the product takes an absolute maximum. Moreover, the term on the RHS is differentiable in t. Therefore, differentiating with respect to t yields

$$\begin{aligned}\partial_t\partial_x G(t,x) &= \partial_x\gamma(x)\mu(S(t))G(t,x) + \gamma(x)\mu(S(t))\partial_x G(t,x)\\ &= \alpha(t,x) + \beta(t,x)\partial_x G(t,x)\end{aligned}$$

with α and β chosen suitably. Since $G \in \mathrm{C}^1([0,T];\mathrm{C}^k([0,1]))$, $\mu \in \mathrm{C}^0(\mathbb{R})$ and $S \in C^0(\mathbb{R})$, see Proposition 5.1.4, the functions α and β are continuous in their first argument and applying the Gronwall Lemma 4.2.2 yields

$$\begin{aligned}\partial_x G(t,x) \leq& \partial_x G(0,x)\exp\left(\int_0^t \beta(\tau,x)\ d\tau\right) + \int_0^t \exp\left(\int_\tau^t \beta(s,x)\ ds\right)\alpha(\tau,x)\ d\tau\\ =&\partial_x 1\exp\left(\int_0^t \gamma(x)\mu(S(\tau))\ d\tau\right)\\ &+\int_0^t \exp\left(\int_\tau^t \gamma(x)\mu(S(\tau))\ ds\right)\partial_x\gamma(x)\mu(S(\tau))G(\tau,x)\ d\tau\\ \leq& \int_0^t \exp\left((t-\tau)(\gamma\mu)_{max}\right)\|\gamma\|_{\mathrm{C}^1([0,1])}\mu_{max}G_{max}\ d\tau\\ =&\left(\exp\left(T(\gamma\mu)_{max}\right)-1\right)\frac{\|\gamma\|_{\mathrm{C}^1([0,1])}\mu_{max}G_{max}}{(\gamma\mu)_{max}}\\ =:& C_1.\end{aligned}$$

For an estimate from below, note that the Gronwall lemma for $-\partial_x G(t,x)$ yields a lower bound with $-\alpha$ instead of α, see Remark 4.2.3.

Step 3: l-th derivative. Let $l \in \{2,\dots,k\}$. Assume that we already found a bound $C_{l-1} > 0$ such that

$$\|G(t,\cdot)\|_{\mathrm{C}^{l-1}([0,1])} \leq C_{l-1}.$$

Since $G \in \mathrm{C}^1([0,T];C^k([0,1]))$ as part of the solution of the AMP, the following derivative exists and with the ODE and the product rule, we obtain

$$\partial_t \partial_x^l G(t,x) = \sum_{k=0}^{l} \binom{l}{k} \mu(S(t)) \partial_x^{l-k} \gamma(x) \partial_x^k G(t,x) \leq \alpha_l(t,x) + \beta_l(t,x) \partial_x^l G(t,x)$$

with suitable α_l and β_l. Thus, by similar arguments as in step 2, the fact that α_l contains up to the $(l-1)$-st derivative of G, and the estimate on G in $\mathrm{C}^{l-1}([0,1])$ yields that there exists a constant $C_l > 0$ such that

$$\|G(t,\cdot)\|_{\mathrm{C}^l([0,1])} \leq C_l,$$

which proves the claim. ♣

5.2. Regularity in space of the elastic deformation

In this subsection, the regularity in space of the elastic deformation is investigated throughout the different settings. In the case of the material consisting of two or finitely many parts, the best regularity obtained is naturally given as $W^{1,\infty}$, see Remark 5.2.1. For the other settings, additional assumptions on the properties describing functions lead to higher regularity of the solution of the AMP.

Remark 5.2.1. *In the Setting 3.2.1 with two materials, the elastic deformation is affine linear on both grown parts, see (3.18), and continuous in the interface point due to the conditions of the admissible set, see Theorem A.1.1. Hence, $\phi(t,\cdot) \in W^{1,\infty}(g(t,(0,1))$ for all $t \in [0,T]$.*
The derivative of $X = \phi \circ g$, omitting the time t in the notation, is given by

$$\partial_x X(x) = \begin{cases} \frac{\xi}{\lambda_1} G_1 & = \frac{\xi}{\ell}, \quad \text{for } x \in (0,\ell], \\ \frac{1-\xi}{\lambda_2} G_2 & = \frac{1-\xi}{1-\ell}, \text{ for } x \in (\ell,1). \end{cases}$$

Then, $X(t,\cdot) \in \mathrm{C}^1([0,1])$ only if

$$\frac{\xi}{\ell} = \frac{1-\xi}{1-\ell},$$

This is equivalent to $\xi = \ell$, which again is equivalent to $X = id$. Physically interpreted, the process $X(t,\cdot)$ is $\mathrm{C}^1([0,1])$ for each t if and only if the material looks (without considering the stress) the same.
When is $\xi = \ell$? From the equation of balance of forces 3.8, we get by inserting $\xi = \ell$ and $\lambda_i = \ell_i G_i$, $i = 1,2$, that

$$\kappa_1 D_F \tilde{W}(\frac{1}{G_1}) = \kappa_2 D_F \tilde{W}(\frac{1}{G_2}).$$

If $\tilde{W}(F) = \frac{1}{2}(\frac{1}{F} + F)^2 - 4$, this yields

$$\kappa_1(\frac{1}{G_1} - G_1^3) = \kappa_2(\frac{1}{G_2} - G_2^3). \tag{5.7}$$

Furthermore, assume $\mu(S) = 1$ for all $S \in \mathbb{R}$. Then, the growth tensors are given by

$$G_i(t,x) = \exp(\gamma_i t),$$

see Equation (3.1). With this formula, we see that the equation 5.7 is not fulfilled for every $t \in [0,T]$ except for the case $\gamma_1 = \gamma_2$ and $\kappa_1 = \kappa_2$. This would describe a body consisting of one material only. A similar statement holds for the Setting 3.2.8 with m parts.

Assume (G, S) is a solution of the AMP in the Setting 4.3.1 with C^0-coefficient and let g denote the growth map with gradient G and ϕ the elastic deformation corresponding to the stress S. Fix a time $t \in [0, T]$. By Theorem A.1.1, $\phi(t, \cdot) \in C^1(\Omega_{nat})$ and the Euler–Lagrange equation is given by

$$\kappa_{nat}(y) D_F \tilde{W}(\partial_y \phi(t, y)) = S(t) \in \mathbb{R}.$$

It is equal to

$$\partial_y \phi(t, y) = (D_F \tilde{W})^{-1} \left(\frac{S(t)}{\kappa_{nat}(y)} \right), \tag{5.8}$$

since $\kappa(y) > 0$ for all $y \in g(t, [0, 1])$ and $D_F \tilde{W}$ is strictly increasing in the second argument due to the strict convexity of $\tilde{W}$.
To obtain $\phi(t, \cdot) \in C^k(\Omega_{nat})$, the formula (5.8) states that the RHS has to be of $C^k(\Omega_{nat})$-regularity.

Remark 5.2.2. *In the following the y-derivative of κ_{nat} is calculated. Note that by definition*

$$\kappa_{nat}(y) = \kappa(g^{-1}(t, y)),$$

the y-derivative is

$$\partial_y \kappa_{nat}(y) = \partial_y(\kappa(g^{-1}(t, y))) = \partial_x \kappa(g^{-1}(t, y)) \partial_y g^{-1}(t, y).$$

On the RHS, $\partial_y g^{-1}(t, y)$ exists and is continuous if $g(t, \cdot)$ is continuously differentiable and invertible by the Lemma A.3.6. Thus, it holds $\kappa_{nat} \in C^k(g(t, [0, 1]))$, if $\kappa \in C^k([0, 1])$ and $g(t, \cdot) \in C^k([0, 1])$, which is the case in Setting 5.1.1 according to Theorem 5.1.5. Therefore, the assumption on the regularity of the growth map are considered.

We obtain $\phi(t, \cdot) \in C^2(g([0, 1]))$, if the RHS of (5.8) is continuously differentiable. Calculating the derivative of the RHS with chain rule and the formula for the derivative of the inverse A.3.6, it yields that

$$\begin{aligned} \partial_y \left((D_F \tilde{W})^{-1} \left(\frac{S}{\kappa_{nat}(y)} \right) \right) &= D_F (D_F \tilde{W})^{-1} \left(\frac{S}{\kappa_{nat}(y)} \right) \partial_y \frac{S}{\kappa_{nat}(y)} \\ &= \frac{1}{D_F^2 \tilde{W} \left((D_F W)^{-1} \left(\frac{S}{\kappa_{nat}(y)} \right) \right)} \cdot \frac{S \partial_y \kappa_{nat}(y)}{\kappa_{nat}^2(y)} \end{aligned} \tag{5.9}$$

holds. The term $D_F^2 W$ can be estimated well, see conditions of Proposition 5.2.3. Here, $\tilde{W} \in C^2(0, \infty)$ as well as $\kappa_{nat} \in C^1(g([0, 1]))$ is used.
It holds $\phi \in C^3(g([0, 1]))$, if the RHS of (5.8) is C^2. By differentiating (5.9),

$$\begin{aligned} &\partial_y^2 \left((D_F \tilde{W})^{-1} \left(\frac{S}{\kappa_{nat}(y)} \right) \right) \\ =&\partial_y \left(\frac{1}{D_F^2 \tilde{W} \left((D_F \tilde{W})^{-1} \left(\frac{S}{\kappa_{nat}(y)} \right) \right)} \frac{S \partial_y \kappa_{nat}(y)}{\kappa_{nat}^2(y)} \right) \\ =&\frac{\partial_y \left(D_F^2 \tilde{W} \left((D_F \tilde{W})^{-1} \left(\frac{S}{\kappa_{nat}(y)} \right) \right) \right)}{D_F^2 \tilde{W}^2 \left((D_F \tilde{W})^{-1} \left(\frac{S}{\kappa_{nat}(y)} \right) \right)} \frac{S \partial_y \kappa_{nat}(y)}{\kappa_{nat}^2(y)} \\ &+ \frac{1}{D_F^2 \tilde{W} \left((D_F \tilde{W})^{-1} \left(\frac{S}{\kappa_{nat}(y)} \right) \right)} \partial_y \frac{S \partial_y \kappa_{nat}(y)}{\kappa_{nat}^2(y)} \end{aligned}$$

$$
\begin{aligned}
=&\frac{D_F^3\tilde{W}\left((D_F\tilde{W})^{-1}\left(\frac{S}{\kappa_{nat}(y)}\right)\right)}{D_F^2\tilde{W}^2\left((D_F\tilde{W})^{-1}\left(\frac{S}{\kappa_{nat}(y)}\right)\right)}\cdot\frac{S\partial_y\kappa_{nat}(y)}{\kappa_{nat}^2(y)}\\
&+\frac{1}{D_F^2\tilde{W}\left((D_F\tilde{W})^{-1}\left(\frac{S}{\kappa_{nat}(y)}\right)\right)}\cdot\frac{S\partial_y^2\kappa_{nat}(y)\kappa_{nat}(y)+S(\partial_y\kappa_{nat})^2}{\kappa_{nat}^3(y)}.
\end{aligned}
$$

Hence, assuming that $\tilde{W}\in \mathrm{C}^3(0,\infty)$ and $\kappa_{nat}\in \mathrm{C}^2(g([0,1]))$ yields that the RHS is continuous. The derivative of $(D_F\tilde{W})^{-1}$ appears again, which is continuous under the stricter conditions on $\tilde{W}$ and κ_{nat}. Motivated by these observations, we conclude the following proposition on the regularity in space.

Proposition 5.2.3 (Regularity in space of elastic deformation)**.** *Additional to the Setting 5.1.1 let $k\in N_{>0}$, $W(x,F)=\kappa(x)\tilde{W}(F)$ with $\tilde{W}\in \mathrm{C}^k(0,\infty)$ and $\kappa\in \mathrm{C}^{k-1}([0,1])$ with $\kappa_{min}:=\inf_{x\in[0,1]}\kappa(x)>0$. Then for the elastic deformation ϕ corresponding to the solution (G,S) of the AMP in C^k, it holds for each $t\in[0,T]$ that*

$$\phi(t,\cdot)\in \mathrm{C}^k(g(t,[0,1])),$$

where g denotes the growth map corresponding to the solution (G,S).

The regularity of the elastic deformation in space shall also be discussed in the Setting 4.4.1 with generalized elastic strain energy density. From Theorem A.1.2, from the book [BGH99], states the regularity: To apply the theorem for the setting of the AMP, fix a time $t\in[0,T]$ and define $\bar{W}(x,z,F)=W_{nat}(y,F)$, where $x=g^{-1}(t,y)$, which is independent of z. Condition (i) from the theorem is not fulfilled in our setting, but the condition is only used to prove the existence of a minimizer. We get the existence by Theorem A.1.1 for $m=1$. Note that if the derivative of the elastic deformation is bounded, the condition (ii) from the Theorem A.1.2 can be fulfilled on the interval of the estimates on $\partial_y\phi(t,\cdot)$. Furthermore, condition (iii) is fulfilled due to the strict convexity assumed for $\tilde{W}$. Theorem A.1.1 also provides a uniform estimate on the derivative of the minimizer, i.e.

$$\min_{y\in g(t,[0,1])}\partial_y\phi(t,y)\geq c(t)>0.$$

This implies the following result:

Proposition 5.2.4 (Regularity in space of elastic deformation)**.** *Assume in the Setting 5.1.1 the elastic strain energy density $W\in \mathrm{C}^k([0,1]\times(0,\infty))$ for $k\in\mathbb{N}_{>1}$. Then for each time $t\in[0,T]$, it holds for the elastic deformation*

$$\phi(t,\cdot)\in \mathrm{C}^k(g(t,[0,1])).$$

Remark 5.2.5. *Note that the growth map g of class C^k implies $W_{nat}\in \mathrm{C}^k(\Omega_{nat}\times(0,\infty))$, since by chain rule*

$$\partial_yW_{nat}(y,F)=\partial_yW(g^{-1}(y),F)=\partial_xW(g^{-1}(y),F)\partial_yg^{-1}(y)$$

and g^{-1} is as regular as g.

5.3. Regularity in time of the growth tensor

Since the ODE depends in two terms on t on the RHS, the regularity of the growth tensor G in time depends on their regularity. In the proof of the Lipschitz continuity of the stress map S, Proposition 4.2.5, we get $S\in \mathrm{C}^1$ locally. From the following calculation we infer local C^2-regularity in time: Let G be a solution of the AMP (in any setting). Hence, it fulfils the

ODE

$$\dot{G}(t,\cdot) = \gamma\mu(S(G(t,\cdot)))G(t,\cdot)$$

for all $t \in [0,T]$ in a suitable Banach space X. Since the RHS is C^1 in time, differentiating in time yields

$$\begin{aligned} d_t(\gamma\mu(S(G(t,\cdot)))G(t,\cdot)) =& \gamma\mu'(S(G(t,\cdot)))S'(G(t,\cdot))\dot{G}(t,\cdot)G(t,\cdot) + \gamma\mu(S(G(t,\cdot)))\dot{G}(t,\cdot) \\ =& \gamma\mu'(S(G(t,\cdot)))S'(G(t,\cdot))\gamma\mu(S(G(t,\cdot)))G(t,\cdot)G(t,\cdot) \\ &+ \gamma\mu(S(G(t,\cdot)))\gamma\mu(S(G(t,\cdot)))G(t,\cdot) \\ =& \gamma^2\mu'(S(G(t,\cdot)))S'(G(t,\cdot))\mu(S(G(t,\cdot)))G^2(t,\cdot) \\ &+ \gamma^2\mu^2(S(G(t,\cdot)))G(t,\cdot), \end{aligned}$$

where the last term is in $\mathrm{C}^0([0,T];X)$. Hence, we obtain

$$G \in \mathrm{C}^2([0,T];X).$$

We do not get better regularity in time without further assumptions, by the following consideration. Let $G \in \mathrm{C}^3([0,T],X)$, we apply the time derivative to the LHS and by chain rule formally obtain

$$\begin{aligned} &d_t(\gamma^2\mu'(S(G(t,\cdot)))S'(G(t,\cdot))\mu(S(G(t,\cdot)))G^2(t,\cdot) + \gamma^2\mu^2(S(G(t,\cdot)))G(t,\cdot)) \\ =& \gamma^2\mu''(S(G(t,\cdot)))(S'(G(t,\cdot)))^2\dot{G}(t,\cdot)\mu(S(G(t,\cdot)))G^2(t,\cdot) \\ &+ \gamma^2\mu'(S(G(t,\cdot)))S''(G(t,\cdot))\dot{G}(t,\cdot)\mu(S(G(t,\cdot)))G^2(t,\cdot) \\ &+ \gamma^2(\mu'(S(G(t,\cdot))))^2(S'(G(t,\cdot)))\dot{G}(t,\cdot)G^2(t,\cdot) \\ &+ 2\gamma^2\mu'(S(G(t,\cdot)))S'(G(t,\cdot))\mu(S(G(t,\cdot)))G(t,\cdot)\dot{G}(t,\cdot) \\ &+ 2\gamma^2\mu(S(G(t,\cdot)))\mu'(S(G(t,\cdot)))S'(G(t,\cdot))\dot{G}(t,\cdot)G(t,\cdot) \\ =& \gamma^3\mu''(S(G(t,\cdot)))(S'(G(t,\cdot)))^2\mu^2(S(G(t,\cdot)))G^3(t,\cdot) \\ &+ \gamma^3\mu'(S(G(t,\cdot)))S''(G(t,\cdot))\mu^2(S(G(t,\cdot)))G^3(t,\cdot) \\ &+ \gamma^3(\mu'(S(G(t,\cdot))))^2(S'(G(t,\cdot)))\mu(S(G(t,\cdot)))G^3(t,\cdot) \\ &+ 2\gamma^3\mu'(S(G(t,\cdot)))S'(G(t,\cdot))\mu^2(S(G(t,\cdot)))G^2(t,\cdot) \\ &+ 2\gamma^3\mu^2(S(G(t,\cdot)))\mu'(S(G(t,\cdot)))S'(G(t,\cdot))G^2(t,\cdot). \end{aligned} \tag{5.10}$$

This involves the second derivative of the stress tensor S, which corresponds to the third derivative of the elastic strain energy density W. Whereas we already consider the strong assumption of C^2-regularity, but for higher regularity of G, higher regularity of W is assumed.

Proposition 5.3.1 (Stress is locally C^k)**.** *Let $k \in \mathbb{N}_{>1}$ be an integer. In the Setting 4.4.1, assume in addition $W(x,\cdot)$ to be a $\mathrm{C}^{k+1}(0,\infty)$-function for each $x \in [0,1]$. Then,*

$$S\colon \overline{B_{\tilde{R}}(G^*)} \to \mathbb{R} \text{ is } \mathrm{C}^k_{loc}.$$

Here, $\overline{B_{R^*}(G^*)} \subset X$.

Proof. Define

$$f\colon \overline{B_{\tilde{R}}(G^*)} \times \mathbb{R} \to \mathbb{R},\ f(G,S) = 1 - \int_0^1 (D_F W(x,\cdot))^{-1}(S)G(x)\ dx.$$

Then, the implicit formula (4.51) for the stress is described by $f(G,S) = 0$ for $S \in \mathbb{R}$ and $G \in L^\infty(0,1)$. This will be used to prove the local regularity of S in dependence of G by calculating the Gâteaux derivative and using Lemma A.3.4 to obtain the Fréchet derivatives and with Theorem A.3.5 the desired regularity follows.

Step 1: The k-th Gâteaux derivative of f. By mathematical induction, for $k \in \mathbb{N}_{>0}$ and directions $G_i \in \overline{B_{\tilde{R}}(G^*)},\ S_i \in \mathbb{R},\ i = 1,\dots,k$, holds

$$\begin{aligned} D^k f(G,S)(G_1,S_1)\dots(G_k,S_k) = &\int_0^1 ((D_F W(x,\cdot))^{-1})^{(k)}(S) \prod_{i=1}^k S_i G(x)\ dx \\ &+ \sum_{m=1}^k \int_0^1 ((D_F W(x,\cdot))^{-1})^{(k-1)}(S) \prod_{\substack{i=1\\ i\neq m}}^k S_i G_m(x)\ dx \end{aligned} \tag{5.11}$$

For the initial case $k = 1$, let $S_1 \in \mathbb{R}$ and $G_1 \in \overline{B_{\tilde{R}}(G^*)}$ be fixed directions and $s \in [-1,1]$. Then, the first Gâteaux derivative in direction (G_1,S_1) is given through

$$\begin{aligned} Df(G,S)(G_1,S_1) =& \frac{d}{ds} f(G+sG_1, S+sS_1)|_{s=0} \\ =& \frac{d}{ds}\int_0^1 (D_F W(x,\cdot))^{-1}(S+sS_1)(G+sG_1)(x)\ dx|_{t=0} \\ =& \int_0^1 \frac{d}{ds}(D_F W(x,\cdot))^{-1}(S+sS_1)(G+sG_1)(x)\ dx|_{s=0} \\ =& \int_0^1 ((D_F W(x,\cdot))^{-1})'(S+sS_1)S_1(G+sG_1)(x)\ dx|_{s=0} \\ &+ \int_0^1 (D_F W(x,\cdot))^{-1}(S+sS_1)G_1(x)\ dx|_{s=0} \\ =& \int_0^1 ((D_F W(x,\cdot))^{-1})'(S)S_1 G(x)\ dx + \int_0^1 (D_F W(x,\cdot))^{-1}(S)G_1(x)\ dx, \end{aligned}$$

where the integration and differentiation can be interchanged since $\{S + sS_1|\ s \in [-1,1]\}$ is a compact set, $(D_F W(x,.))^{-1}$ a continuous function, and $G + sG_1 \in L^\infty(0,1)$. Then, the formula (5.11) is proven.

Assume (5.11) holds for $k \in \mathbb{N}_{>0}$ and the directions $G_1,\dots,G_k \in \overline{B_{\tilde{R}}(G^*)},\ S_1,\dots,S_k \in \mathbb{R}$. Further, let $G_{k+1} \in \overline{B_{\tilde{R}}(G^*)}$ and $S_{k+1} \in \mathbb{R}$ be directions for the $(k+1)$-th Gâteaux derivative. Then, it holds

$$\begin{aligned} D^{k+1} f(G,S) =& D(D^k f(G,S)(G_1,S_1)\dots(G_k,S_k))(G_{k+1},S_{k+1}) \\ =& \frac{d}{dt} D^k f(G+tG_{k+1}, S+tS_{k+1})(G_1,S_1)\dots(G_k,S_k)|_{t=0} \end{aligned}$$

$$
\begin{aligned}
=&\frac{d}{dt}\int_0^1 ((D_F W(x,\cdot))^{-1})^{(k)}(S+tS_{k+1})\prod_{i=1}^{k} S_i(G+tG_{k+1})(x)\ dx|_{t=0}\\
&+\frac{d}{dt}\sum_{m=1}^{k}\int_0^1 ((D_F W(x,\cdot))^{-1})^{(k-1)}(S+tS_{k+1})\prod_{\substack{i=1\\ i\neq m}}^{k} S_i G_m(x)\ dx|_{t=0}\\
=&\int_0^1 \frac{d}{dt}((D_F W(x,\cdot))^{-1})^{(k)}(S+tS_{k+1})\prod_{i=1}^{k} S_i(G+tG_{k+1})(x)\ dx|_{t=0}\\
&+\sum_{m=1}^{k}\int_0^1 \frac{d}{dt}((D_F W(x,\cdot))^{-1})^{(k-1)}(S+tS_{k+1})\prod_{\substack{i=1\\ i\neq m}}^{k} S_i G_m(x)\ dx|_{t=0}\\
=&\int_0^1 ((D_F W(x,\cdot))^{-1})^{(k+1)}(S+tS_{k+1})S_{k+1}\prod_{i=1}^{k} S_i(G+tG_{k+1})(x)\ dx|_{t=0}\\
&+\int_0^1 ((D_F W(x,\cdot))^{-1})^{(k)}(S+tS_{k+1})\prod_{i=1}^{k} S_i G_{k+1}(x)\ dx|_{t=0}\\
&+\sum_{m=1}^{k}\int_0^1 ((D_F W(x,\cdot))^{-1})^{(k-1+1)}(S+tS_{k+1})S_{k+1}\prod_{\substack{i=1\\ i\neq m}}^{k} S_i G_m(x)\ dx|_{t=0}\\
=&\int_0^1 ((D_F W(x,\cdot))^{-1})^{(k+1)}(S)S_{k+1}\prod_{i=1}^{k} S_i G(x)\ dx\\
&+\int_0^1 ((D_F W(x,\cdot))^{-1})^{(k)}(S)\prod_{i=1}^{k} S_i G_{k+1}(x)\ dx\\
&+\sum_{m=1}^{k}\int_0^1 ((D_F W(x,\cdot))^{-1})^{(k)}(S)\prod_{\substack{i=1\\ i\neq m}}^{k+1} S_i G_m(x)\ dx\\
=&\int_0^1 ((D_F W(x,\cdot))^{-1})^{(k+1)}(S)\prod_{i=1}^{k+1} S_i G(x)\ dx\\
&+\sum_{m=1}^{k+1}\int_0^1 ((D_F W(x,\cdot))^{-1})^{(k)}(S)\prod_{\substack{i=1\\ i\neq m}}^{k+1} S_i G_m(x)\ dx,
\end{aligned}
$$

where again differentiation and integration can be interchanged since W is C^{k+1} in the second argument and, hence, its inverse, too. Therefore, dominated convergence justifies the interchanging of differentiation and integration. Hence, formula (5.11) is demonstrated.
Step 2: Regularity of S. To obtain that the Gâteaux derivative is the Fréchet derivative, observe that the Gâteaux derivative in (5.11) is continuous in G and S due to $W(x,\cdot) \in \mathrm{C}^{k+1}$. Then, Lemma A.3.4 states the equality of Fréchet and Gâteaux derivative. As a last step, applying the implicit function theorem, see Theorem A.3.5, yields the local C^k-regularity of S. Note that all conditions for the theorem except for the differentiability are already proven in Proposition 4.4.6. ♣

With this stated, the following regularity result for G in time can be obtained.

Proposition 5.3.2 (Regularity in time of growth tensor)**.** *Let $k \in \mathbb{N}$, $k \geq 3$. Assume the Setting 4.4.1 to hold. Further, let $\mu \in \mathrm{C}^{k-1}(\mathbb{R})$ and $W(x,\cdot) \in \mathrm{C}^k((0,\infty))$ for each $x \in [0,1]$. Then,*

$$G \in \mathrm{C}^k([0,T];X).$$

Proof. In view of (5.10), it is to prove that $\dddot{G} \in \mathrm{C}^{k-3}([0,T],X)$, which is the case for $S \in \mathrm{C}^{k-1}$ and $\mu \in \mathrm{C}^{k-1}(\mathbb{R})$. Proposition 5.3.1 yields the needed regularity of S. ♣

Remark 5.3.3. *Proposition 5.3.2 can also be shown for the Setting 5.1.1, which yields regularity for mixed derivatives. In case the of $G \in \mathrm{C}^k([0,T];\mathrm{C}^k([0,1]))$, the statement of Theorem 2.11 in [Tes12], a regularity statement for parameter dependent ODEs, is met.*

5.4. Regularity in time of the elastic deformation

Next, we investigate the regularity of the elastic deformation in time. Since the ϕ is defined on the natural configuration $g(t,[0,1])$, $t \in [0,T]$, the domain of ϕ is depending on time. Therefore, only local regularity on the trajectory of g can be stated.
Let (t,y) be an interior point of the trajectory $\mathcal{T}_g$ of g, i.e. there exists an $x \in (0,1)$ such that $(t,y) = (t, g(t,x))$. Further, define the open neighbourhood

$$U := \{t \in (0,T) | (t,y) \in \mathring{\mathcal{T}}_g\} \subset (0,T) \tag{5.12}$$

of t. Note that $U \neq \emptyset$ due to the continuity of g. Recalling the formula (4.51), we find

$$D_F W_{nat}(y, \partial_y \phi(t,y)) = S(t),$$

which, due to the strict convexity of W, is equivalent to

$$\partial_y \phi(t,y) = (D_F W_{nat}(y,\cdot))^{-1}(S(t)).$$

Integrating from 0 to t and the boundary condition $\phi(t,0) = 0$ yield

$$\phi(t,y) = \phi(t,y) - \phi(t,0) = \int_0^y (D_F W_{nat}(\tilde{y},\cdot))^{-1}(S(t))\, d\tilde{y}. \tag{5.13}$$

For $t \in U$, this is differentiable in time, since U ensures that for small enough pertubations $\varepsilon > 0$ also $(t-\varepsilon, t+\varepsilon) \subset U$. Differentiating the RHS of (5.13) with respect to time yields

$$
\begin{aligned}
\partial_t \int_0^y (D_F W_{nat}(\tilde{y},\cdot))^{-1}(S(t))\, d\tilde{y} &= \int_0^y \partial_t (D_F W_{nat}(\tilde{y},\cdot))^{-1}(S(t))\, d\tilde{y} \\
&= \int_0^y ((D_F W_{nat}(\tilde{y},\cdot))^{-1})'(S(t))\partial_t S(t)\, d\tilde{y} \\
&\quad + \int_0^y (D_F W_{nat}(\tilde{y},\cdot))^{-1}(S(t))\partial_t g^{-1}(t,\tilde{y})\, d\tilde{y} \\
&= \partial_t S(t) \int_0^y \frac{1}{D_F^2 W_{nat}(\tilde{y}, (D_F W_{nat}(\tilde{y},\cdot))^{-1}(S(t)))}\, d\tilde{y} \\
&\quad + \int_0^y (D_F W_{nat}(\tilde{y},\cdot))^{-1}(S(t))\partial_t g^{-1}(t,\tilde{y})\, d\tilde{y} \\
&=: T_1 + T_2.
\end{aligned}
\tag{5.14}
$$

The term T_1 is continuous in t, since S is C^1-depending on G, see Proposition 5.3.1, $G \in \mathrm{C}^1([0,T], L^\infty(0,1))$ and $W_{nat}(\tilde{y},\cdot) = W(g^{-1}(t,\tilde{y}),\cdot) \in \mathrm{C}^2(0,\infty)$. For T_2 note that in the Setting 5.1.1 for $k=1$ the time derivative of g^{-1} is continuous. For higher derivatives we can prove the following.

Proposition 5.4.1 (Regularity in time of elastic deformation)**.** *Assume the Setting 5.1.1 and for each $F \in (0,\infty)$ that*

$$\inf_{x\in[0,1]} D_F W(x,F) > -\infty$$

holds. Further, let $W(x,\cdot) \in \mathrm{C}^{k+1}(0,\infty)$ for all $x \in [0,1]$, $\mu \in \mathrm{C}^k(\mathbb{R})$ and $(t,y) \in \mathring{\mathcal{T}}_g$. Then,

$$\phi(\cdot,y) \in \mathrm{C}^k(U),$$

where U is defined in (5.12).

Proof. In view of formula (5.14), S, G and the integrand have to be suitable regular. By Proposition 5.3.1, $W \in \mathrm{C}^{k+1}(0,\infty)$ implies $S \in \mathrm{C}^k$ and, by Proposition 5.3.2, $G \in \mathrm{C}^{k+1}([0,T], L^\infty(0,1))$. Thus, $S \in \mathrm{C}^k([0,T])$.
Note that

$$((D_F W_{nat}(y,\cdot))^{-1})' = \frac{1}{D_F^2 W_{nat}(y, (D_F W_{nat}(y,\cdot))^{-1}(z)} \tag{5.15}$$

and $D_F^2 W_{nat} > c_c > 0$ for all arguments due to the assumption (5.2) hold. Then by chain rule, it holds for the integral term that

$$
\begin{aligned}
&\partial_t \int_0^y \frac{1}{D_F^2 W(\tilde{y}, (D_F W(\tilde{y}, \cdot))^{-1}(S(t)))}\, d\tilde{y} \\
&\quad = \int_0^y \partial_t \frac{1}{D_F^2 W(\tilde{y}, (D_F W(\tilde{y}, \cdot))^{-1}(S(t)))}\, d\tilde{y} \\
&\quad = \int_0^y \frac{D_F^3 W(\tilde{y}, (D_F W(\tilde{y}, \cdot))^{-1})(S(t))(D_F W(\tilde{y}, \cdot)^{-1}(S(t)))'}{D_F^2 W^2(\tilde{y}, (D_F W(\tilde{y}, \cdot))^{-1}(S(t)))}\, d\tilde{y} + C(t,y) \\
&\quad = \int_0^y \frac{D_F^3 W(\tilde{y}, (D_F W(\tilde{y}, \cdot))^{-1})(S(t))}{D_F^2 W^3(\tilde{y}, (D_F W(\tilde{y}, \cdot))^{-1}(S(t)))} \partial_t S(t)\, d\tilde{y} + C(t,y).
\end{aligned}
\tag{5.16}
$$

Here, $C(t, y)$ denotes the term

$$
\int_0^y \frac{1}{D_F^2 W(\tilde{y}, (D_F W(\tilde{y}, \cdot))^{-1}(S(t)))} \partial_t g^{-1}(t, \tilde{y})\, d\tilde{y}
$$

which is continuous by similar arguments as before. To establish continuity of the expression (5.16) use dominated convergence. For that, use the estimate in Lemma 4.4.5 below and the fact that $D_F^3 W$ is continuous and, hence, has a maximum on the compact interval $[\phi_{min}, \phi_{max}]$ of the estimates on $\partial_y \phi$, see Lemma 4.4.5 below. For the term $C(t, x)$ note that the time derivative of g^{-1}, in the Setting we assume 5.1.1, is continuously differentiable.
For higher derivatives, note that the fraction is a composition of differential functions, recall (5.15), and obtain the differentiability of the fraction. ♣

6 Concerning higher dimensions

After a detailed discussion on the analysis of the stress modulated growth model in one dimension, a first attempt to set the analysis into two dimensions is made here as well as a discussion on the challenges to be faced. Since the structure of the model provides some problems, as seen in the Examples 2.4.2 and 2.4.3, we restrict ourselves to a geometrically simple setting, namely we consider the tumour to consist of a circle and a ring around it and moreover to always stay in this geometry, and derive an existence theorem.

6.1. Discussion on challenges in higher dimensions

To discuss the challenges of the analysis of the stress modulated growth model, the setting has to be stated first.

Definition 6.1.1 (Setting in d dimensions)**.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, an open, bounded Lipschitz domain be the given reference configuration at time $t = 0$. Let $\Gamma := \partial\Omega$ be divided into the Dirichlet and the Neumann boundary part Γ_D and Γ_N respectively, namely $\overline{\Gamma_D \dot{\cup} \Gamma_N} = \Gamma$. Let $1 < p < \infty$.*
We assume the stress S and the nutrients n to influence the growth by the ODE, namely let the RHS of the ODE be given by

$$\mathcal{G} : \mathbb{R} \times \Omega \times \mathbb{R}^{d\times d} \times \mathbb{R} \times \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d},\ (t, x, S, n, G) \mapsto \mathcal{G}(t, x, S, n, G).$$

Let $W : \Omega \times Lin^+ \to [0, \infty)$ be the elastic energy density with

$$\begin{aligned} &\forall x \in \Omega : W(x, \cdot) \text{ is continuous, independent of the observer and } W(x, \mathbb{1}_d) = 0,\\ &\forall x \in \Omega : W(x, F) \to \infty \text{ for } \det F \to 0 \text{ and for } \det F \to \infty, \\ &\forall F \in Lin^+ : W(\cdot, F) \text{ is measurable on } \Omega. \end{aligned} \tag{6.1}$$

For $t \in [0, T]$ and a given injective growth map $g(t, \cdot)$ with $\Omega_{nat} := g(t, \Omega)$ an open, bounded Lipschitz domain and given Dirichlet boundary condition $\phi_D \in L^p(\Gamma_D)$ define the admissible set for the elastic deformation by

$$\mathcal{A}^d := \{\phi \in W^{1,p}(\Omega_{nat}; \mathbb{R}^d) |\ \phi = \phi_D \circ g^{-1}(t, \cdot) \text{ on } g(t, \Gamma_D)\}$$

and the elastic energy is given by

$$E^d(\phi) := \int_{\Omega_{nat}} W(g^{-1}(y), \nabla\phi(y))\ dy.$$

Let $D : \Omega \times \mathbb{R} \to \mathbb{R}$ be the diffusion coefficient for the nutrients, $\gamma > 0$ the absorption term and $n_D \in L^2(\Gamma)$ the prescribed nutrients on the boundary.

Let $T > 0$ be the time horizon and $G_0 = \mathbb{1}_d$ the initial datum for the growth ODE.

Remark 6.1.2. *Note that by the conditions on W (6.1) the existence and uniqueness of a minimizer is not ensured, but the non-linearity. See [Bal76] for discussion on convexity and existence for the non-linear elastic problem.*

This is the setting for the general higher dimensional AMP. To be able to write down the equation, we also assume to have compatibility for the growth, i.e. the growth tensor is a gradient. Otherwise, one has to circumvent the notation of a growth map and identify the particles in the possible overlaps in the natural configuration. We define a solution of the AMP in the higher dimensional setting as follows:

Definition 6.1.3 (Solution of the AMP in d dimensions)**.** *In the setting of (6.1.1), we call a couple $(G, S, n_0) \in \mathrm{C}^1([0,T]; L^\infty(\Omega; \mathbb{R}^d)) \times \mathrm{C}^0([0,T]; L^\infty(\Omega; \mathbb{R}^d)) \times \mathrm{C}^0([0,T]; H^1(\Omega; \mathbb{R}^d))$ a solution of the AMP in d dimensions, if the following is fulfilled:*

(i) The growth tensor G is a gradient and the growth map $g : [0,T] \to W^{1,\infty}(\Omega; \mathbb{R}^d)$ corresponding to G by

$$\nabla g(t,x) = G(t,x) \text{ for all } t \in [0,T] \text{ and almost all } x \in \Omega$$

is injective.

(ii) The growth tensor G fulfils the ODE

$$\begin{aligned} \dot{G}(t,x) &= \mathcal{G}(t,x,n_0(t,x),S(t,x),G(t,x)), \\ G(0,x) &= \mathbb{1}_d, \end{aligned}$$

for all $t \in [0,T]$, almost all $x \in \Omega$ and S and n given by (iii) and (iv).

(iii) For each $t \in [0,T]$, the map $\phi(t,\cdot) \in \mathcal{A}^d$ is a minimizer of E^d in $\mathcal{A}^d$. Then, it holds that $S_{nat}(t,\cdot) \in L^\infty(\Omega_{nat})$ is the Piola–Kirchhoff stress tensor corresponding to ϕ, namely

$$S_{nat}(t,x) = D_F W(x, \nabla\phi(t,x)).$$

Then, define the pulled-back stress $S(t,\cdot) \in L^\infty(\Omega)$ via the transformation

$$S(t,x) \coloneqq \det G(t,x) S_{nat}(t, g(t,x)) G^{-T}(t,x).$$

Further, $\phi(t,\cdot)$ is injective and $\Omega_t \coloneqq \phi(t, \Omega_{nat})$ is an open, bounded Lipschitz domain for all $t \in [0,T]$.

(iv) The nutrients $n_0(t,\cdot) \in H^1(\Omega)$ fulfil the equation

$$\begin{aligned} -\mathit{div}\,(F^{-1}(t,\cdot)D(n_0(t,\cdot))\mathit{Div}\,(J(t,\cdot)F^{-T}(t,\cdot)n_0(t,\cdot))) = -\gamma n_0(t,\cdot)\det G(t,\cdot) \text{ in } H^1(\Omega), \\ n_0(t,\cdot) = n_D \text{ on } \Gamma, \end{aligned}$$

where $F(t,x) = \partial_x(\phi(t,g(t,x))) = \partial_y\phi(t,g(t,x))G(t,x)$ and $J(t,x) = \det F(t,x)$, for all $t \in [0,T]$.

Remark 6.1.4. *The definition of the pulled-back stress is motivated from the transformations between stress tensors, see Section 2.1.*

As highlighted throughout the previous chapters, dealing with the AMP in more than one dimension yields many problems, which are summarized here:

(i) **Ensured gradient structure?** As mentioned before, we need the growth map g to be injective, but moreover, we need it to be well-defined, namely that G is a gradient such that we can deduce the growth map. This must not be the case and highly depends on the RHS of the ODE $\mathcal{G}$, see discussion below.

(ii) **Dependence of the stress of the growth tensor:** First of all, we need to discuss that Remark 3.1.8 states that the Piola–Kirchhoff stress tensor in the reference configuration is equal to the one in the elastically deformed configuration. In a higher dimensional setting, $\det G$ and G^{-T} do not cancel out, hence, we have to be careful in the ODE which stress tensor to consider. The modelling paper [AM02] does not give a hint on that, but since the growth is defined on the reference configuration, the stress tensor in the reference configuration should be considered.

(iii) **Natural and elastically deformed configurations "nice"?** The core of the AMP is the decomposition into growth and elastic deformation, which gives two problems, which are solvable if the solution of the other is known. But it has the downside that we have to think about the well-posedness of both problems. The biggest issue here is that the domains have to be "nice", which is not ensured by just solving the equations or by constructions (as in the one dimensional setup). Hence, we have to demand it from the solution.
First, the question of openness of the domains Ω_{nat} and when considering nutrients also of Ω_t has to be discussed, e.g. if the material in two dimensions is folded by the growth, the natural configuration Ω_{nat} is not open.
Further problems which can occur are not connected configurations Ω_{nat} and Ω_t or that the boundaries may be not nice enough to solve the problems. I.e. the maps g and ϕ are not injective. We have to demand it explicitly for the solution. For the elastic problem, this can be circumvented by considering higher derivatives in the elastic energy, gaining more information about the solution, see [HK09], which of course relies on the paper [CN87] by Ciarlet and Nečas. Here, it is not clear if the additional assumption are still physically justifiable. For the growth, injectivity in general can not be obtained by the equation as can be see in the two dimensional Example 2.4.2, where already a very simple setting is considered.
Similar problems appear in plasticity theory with multiplicative decomposition, see for example [KMS20], where the (ε, δ)-domains, introduced by Jones [JC12], are obtained as intermediate configuration by the regularity of the plastic deformation. Another example of dealing with the difficulties of the intermediate configuration is [Gra+20], where the the solution includes that the intermediate configuration is a Lipschitz domain.
In Remark 3.1.6, we already discussed the advantages of the one dimensional setting.

(iv) **Buckling effect:** In the very simple example of a growing ring in Example 2.4.2, bucking effects were explicitly excluded by demanding the shape after growth as well as after elastic deformation to be a perfect ring. Without this assumption, for shrinking rings, buckling might appear due to bifurcation theory, see [Li+11],[Mac+12],[MG11]. This is important to consider when applying the theory to higher dimensional settings.

Is G a gradient?

The idea to answer this question is to use the Helmholtz decomposition. It implies: If G is curl-free, there exists a potential g with $G = \nabla g$. In order to show that G is curl -free, prove that curl G fulfils an ODE with unique solution 0.
Consider the G to be a curl-free solution of the pointwise ODE

$$\dot{G}(t, x) = \mathcal{G}(t, x, G(t, x)),$$

which does not include the stress tensor S, to describe the growth. To find out what $\mathcal{G}$ has to fulfil in order to G being curl-free, apply curl to the ODE. Then, it holds for curl G

$$0 = \partial_t(\text{curl } G(t, x))_{ij} = (\text{curl } \mathcal{G}(t, x, G(t, x)))_{ij}. \tag{6.2}$$

The curl of a matrix is the row wise curl, i.e.

$$\text{curl } G = \begin{pmatrix} \partial_2 G_{13} - \partial_3 G_{12} & \partial_3 G_{11} - \partial_1 G_{13} & \partial_1 G_{12} - \partial_2 G_{11} \\ \partial_2 G_{23} - \partial_3 G_{22} & \partial_3 G_{21} - \partial_1 G_{23} & \partial_1 G_{22} - \partial_2 G_{21} \\ \partial_2 G_{33} - \partial_3 G_{32} & \partial_3 G_{31} - \partial_1 G_{33} & \partial_1 G_{32} - \partial_2 G_{31} \end{pmatrix}.$$

Then, the entries of the first row of the curl-matrix of $\mathcal{G}$ are calculated as follows:

$$\begin{aligned}
(\text{curl } \mathcal{G})_{11} =& \partial_2 \mathcal{G}_{13} - \partial_3 \mathcal{G}_{12} \\
=& \frac{\partial \mathcal{G}_{13}}{\partial G_{kl}} \frac{\partial G_{kl}}{\partial x_2} + \frac{\partial \mathcal{G}_{13}}{\partial x_2} - \frac{\partial \mathcal{G}_{12}}{\partial G_{mn}} \frac{\partial G_{mn}}{\partial x_3} - \frac{\partial \mathcal{G}_{12}}{\partial x_3} \\
=& \frac{\partial \mathcal{G}_{13}}{\partial G_{11}} \frac{\partial G_{11}}{\partial x_2} + \frac{\partial \mathcal{G}_{13}}{\partial G_{12}} \frac{\partial G_{12}}{\partial x_2} + \frac{\partial \mathcal{G}_{13}}{\partial G_{13}} \frac{\partial G_{13}}{\partial x_2} \\
&+ \frac{\partial \mathcal{G}_{13}}{\partial G_{21}} \frac{\partial G_{21}}{\partial x_2} + \frac{\partial \mathcal{G}_{13}}{\partial G_{22}} \frac{\partial G_{22}}{\partial x_2} + \frac{\partial \mathcal{G}_{13}}{\partial G_{23}} \frac{\partial G_{23}}{\partial x_2} \\
&+ \frac{\partial \mathcal{G}_{13}}{\partial G_{31}} \frac{\partial G_{31}}{\partial x_2} + \frac{\partial \mathcal{G}_{13}}{\partial G_{32}} \frac{\partial G_{32}}{\partial x_2} + \frac{\partial \mathcal{G}_{13}}{\partial G_{33}} \frac{\partial G_{33}}{\partial x_2} \\
&- \frac{\partial \mathcal{G}_{12}}{\partial G_{11}} \frac{\partial G_{11}}{\partial x_3} - \frac{\partial \mathcal{G}_{12}}{\partial G_{12}} \frac{\partial G_{12}}{\partial x_3} - \frac{\partial \mathcal{G}_{12}}{\partial G_{13}} \frac{\partial G_{13}}{\partial x_3} \\
&- \frac{\partial \mathcal{G}_{12}}{\partial G_{21}} \frac{\partial G_{21}}{\partial x_3} - \frac{\partial \mathcal{G}_{12}}{\partial G_{22}} \frac{\partial G_{22}}{\partial x_3} - \frac{\partial \mathcal{G}_{12}}{\partial G_{23}} \frac{\partial G_{23}}{\partial x_3} \\
&- \frac{\partial \mathcal{G}_{12}}{\partial G_{31}} \frac{\partial G_{31}}{\partial x_3} - \frac{\partial \mathcal{G}_{12}}{\partial G_{32}} \frac{\partial G_{32}}{\partial x_3} - \frac{\partial \mathcal{G}_{12}}{\partial G_{33}} \frac{\partial G_{33}}{\partial x_3}
\end{aligned}$$

and

$$\begin{aligned}
(\text{curl } \mathcal{G})_{12} =& \partial_3 \mathcal{G}_{11} - \partial_1 \mathcal{G}_{13} \\
=& \frac{\partial \mathcal{G}_{11}}{\partial G_{kl}} \frac{\partial G_{kl}}{\partial x_3} - \frac{\partial \mathcal{G}_{13}}{\partial G_{mn}} \frac{\partial G_{mn}}{\partial x_1} \\
=& \frac{\partial \mathcal{G}_{11}}{\partial G_{11}} \frac{\partial G_{11}}{\partial x_3} + \frac{\partial \mathcal{G}_{11}}{\partial G_{12}} \frac{\partial G_{12}}{\partial x_3} + \frac{\partial \mathcal{G}_{11}}{\partial G_{13}} \frac{\partial G_{13}}{\partial x_3} \\
&+ \frac{\partial \mathcal{G}_{11}}{\partial G_{21}} \frac{\partial G_{21}}{\partial x_3} + \frac{\partial \mathcal{G}_{11}}{\partial G_{22}} \frac{\partial G_{22}}{\partial x_3} + \frac{\partial \mathcal{G}_{11}}{\partial G_{23}} \frac{\partial G_{23}}{\partial x_3} \\
&+ \frac{\partial \mathcal{G}_{11}}{\partial G_{31}} \frac{\partial G_{31}}{\partial x_3} + \frac{\partial \mathcal{G}_{11}}{\partial G_{32}} \frac{\partial G_{32}}{\partial x_3} + \frac{\partial \mathcal{G}_{11}}{\partial G_{33}} \frac{\partial G_{33}}{\partial x_3} \\
&- \frac{\partial \mathcal{G}_{13}}{\partial G_{11}} \frac{\partial G_{11}}{\partial x_1} - \frac{\partial \mathcal{G}_{13}}{\partial G_{12}} \frac{\partial G_{12}}{\partial x_1} - \frac{\partial \mathcal{G}_{13}}{\partial G_{13}} \frac{\partial G_{13}}{\partial x_1} \\
&- \frac{\partial \mathcal{G}_{13}}{\partial G_{21}} \frac{\partial G_{21}}{\partial x_1} - \frac{\partial \mathcal{G}_{13}}{\partial G_{22}} \frac{\partial G_{22}}{\partial x_1} - \frac{\partial \mathcal{G}_{13}}{\partial G_{23}} \frac{\partial G_{23}}{\partial x_1} \\
&- \frac{\partial \mathcal{G}_{13}}{\partial G_{31}} \frac{\partial G_{31}}{\partial x_1} - \frac{\partial \mathcal{G}_{13}}{\partial G_{32}} \frac{\partial G_{32}}{\partial x_1} - \frac{\partial \mathcal{G}_{13}}{\partial G_{33}} \frac{\partial G_{33}}{\partial x_1}
\end{aligned}$$

and

$$\begin{aligned}
(\text{curl } \mathcal{G})_{13} =& \partial_1 \mathcal{G}_{12} - \partial_2 \mathcal{G}_{11} \\
=& \frac{\partial \mathcal{G}_{12}}{\partial G_{kl}} \frac{\partial G_{kl}}{\partial x_1} - \frac{\partial \mathcal{G}_{11}}{\partial G_{mn}} \frac{\partial G_{mn}}{\partial x_2} \\
=& \frac{\partial \mathcal{G}_{12}}{\partial G_{11}} \frac{\partial G_{11}}{\partial x_3} + \frac{\partial \mathcal{G}_{12}}{\partial G_{12}} \frac{\partial G_{12}}{\partial x_3} + \frac{\partial \mathcal{G}_{12}}{\partial G_{13}} \frac{\partial G_{13}}{\partial x_3} \\
&+ \frac{\partial \mathcal{G}_{12}}{\partial G_{21}} \frac{\partial G_{21}}{\partial x_3} + \frac{\partial \mathcal{G}_{12}}{\partial G_{22}} \frac{\partial G_{22}}{\partial x_3} + \frac{\partial \mathcal{G}_{12}}{\partial G_{23}} \frac{\partial G_{23}}{\partial x_3} \\
&+ \frac{\partial \mathcal{G}_{12}}{\partial G_{31}} \frac{\partial G_{31}}{\partial x_3} + \frac{\partial \mathcal{G}_{12}}{\partial G_{32}} \frac{\partial G_{32}}{\partial x_3} + \frac{\partial \mathcal{G}_{12}}{\partial G_{33}} \frac{\partial G_{33}}{\partial x_3}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\partial \mathcal{G}_{11}}{\partial G_{11}}\frac{\partial G_{11}}{\partial x_1}-\frac{\partial \mathcal{G}_{11}}{\partial G_{12}}\frac{\partial G_{12}}{\partial x_1}-\frac{\partial \mathcal{G}_{11}}{\partial G_{13}}\frac{\partial G_{13}}{\partial x_1}\\
&-\frac{\partial \mathcal{G}_{11}}{\partial G_{21}}\frac{\partial G_{21}}{\partial x_1}-\frac{\partial \mathcal{G}_{11}}{\partial G_{22}}\frac{\partial G_{22}}{\partial x_1}-\frac{\partial \mathcal{G}_{11}}{\partial G_{23}}\frac{\partial G_{23}}{\partial x_1}\\
&-\frac{\partial \mathcal{G}_{11}}{\partial G_{31}}\frac{\partial G_{31}}{\partial x_1}-\frac{\partial \mathcal{G}_{11}}{\partial G_{32}}\frac{\partial G_{32}}{\partial x_1}-\frac{\partial \mathcal{G}_{11}}{\partial G_{33}}\frac{\partial G_{33}}{\partial x_1}.
\end{aligned}
$$

Since the goal is to ensure that the solution G is curl-free, for each curl-free test function G^* the entries of curl $\mathcal{G}$ have to be 0. By using the formula for the curl of G, equation (6.2) and test functions of the type

$$
G^*\colon \mathbb{R}^3 \to \mathbb{R}^{3\times 3},\ G^*(x_1,x_2,x_3) = \begin{pmatrix} -x_2 & x_1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}
$$

we get conditions on for $\mathcal{G}$. More precisely,

$$
\begin{aligned}
\text{curl } G^* &= \begin{pmatrix} \partial_2 G^*_{13} - \partial_3 G^*_{12} & \partial_3 G^*_{11} - \partial_1 G^*_{13} & \partial_1 G^*_{12} - \partial_2 G^*_{11}\\ \partial_2 G^*_{23} - \partial_3 G^*_{22} & \partial_3 G^*_{21} - \partial_1 G^*_{23} & \partial_1 G^*_{22} - \partial_2 G^*_{21}\\ \partial_2 G^*_{33} - \partial_3 G^*_{32} & \partial_3 G^*_{31} - \partial_1 G^*_{33} & \partial_1 G^*_{32} - \partial_2 G^*_{31} \end{pmatrix}\\
&= \begin{pmatrix} 0 & 0 & 1-1\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Plugging G^* into the formula for $(\text{curl } \mathcal{G})_{11}$ and using equation (6.2), we obtain

$$
0 = (\text{curl } \mathcal{G})_{11} = \frac{\partial \mathcal{G}_{11}}{\partial x_2}\cdot(-1) = -\frac{\partial \mathcal{G}_{11}}{\partial x_2}.
$$

For the test function

$$
G'\colon \mathbb{R}^3 \to \mathbb{R}^{3\times 3},\ G'(x_1,x_2,x_2) = \begin{pmatrix} x_1 & x_3 & x_2\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}
$$

with

$$
\begin{aligned}
\text{curl } G' &= \begin{pmatrix} \partial_2 G'_{13} - \partial_3 G'_{12} & \partial_3 G'_{11} - \partial_1 G'_{13} & \partial_1 G'_{12} - \partial_2 G'_{11}\\ \partial_2 G'_{23} - \partial_3 G'_{22} & \partial_3 G'_{21} - \partial_1 G'_{23} & \partial_1 G'_{22} - \partial_2 G'_{21}\\ \partial_2 G'_{33} - \partial_3 G'_{32} & \partial_3 G'_{31} - \partial_1 G'_{33} & \partial_1 G'_{32} - \partial_2 G'_{31} \end{pmatrix}\\
&= \begin{pmatrix} 1-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix},
\end{aligned}
$$

it follows from the equation (6.2) that

$$
0 = (\text{curl } \mathcal{G})_{13} = \frac{\partial \mathcal{G}_{12}}{\partial G_{12}} - \frac{\partial \mathcal{G}_{11}}{\partial G_{11}}.
$$

Using test functions of the types of G^* and G', we conclude the conditions

$$
\begin{aligned}
\frac{\partial \mathcal{G}_{11}}{\partial G_{11}} &= \frac{\partial \mathcal{G}_{12}}{\partial G_{12}} = \frac{\partial \mathcal{G}_{13}}{\partial G_{13}} = \alpha_{11},\\
\frac{\partial \mathcal{G}_{11}}{\partial G_{21}} &= \frac{\partial \mathcal{G}_{12}}{\partial G_{22}} = \frac{\partial \mathcal{G}_{13}}{\partial G_{23}} = \alpha_{12},
\end{aligned}
$$

$$\frac{\partial \mathcal{G}_{11}}{\partial G_{31}} = \frac{\partial \mathcal{G}_{12}}{\partial G_{32}} = \frac{\partial \mathcal{G}_{13}}{\partial G_{33}} = \alpha_{13},$$

where α_{11}, α_{12} and α_{13} may depend on t, and

$$\begin{aligned}
\frac{\partial \mathcal{G}_{11}}{\partial G_{12}} = \frac{\partial \mathcal{G}_{11}}{\partial G_{22}} = \frac{\partial \mathcal{G}_{11}}{\partial G_{32}} = \frac{\partial \mathcal{G}_{11}}{\partial G_{13}} = \frac{\partial \mathcal{G}_{11}}{\partial G_{23}} = \frac{\partial \mathcal{G}_{11}}{\partial G_{33}} = 0,\\
\frac{\partial \mathcal{G}_{12}}{\partial G_{12}} = \frac{\partial \mathcal{G}_{12}}{\partial G_{22}} = \frac{\partial \mathcal{G}_{12}}{\partial G_{32}} = \frac{\partial \mathcal{G}_{12}}{\partial G_{13}} = \frac{\partial \mathcal{G}_{12}}{\partial G_{23}} = \frac{\partial \mathcal{G}_{12}}{\partial G_{33}} = 0,\\
\frac{\partial \mathcal{G}_{13}}{\partial G_{12}} = \frac{\partial \mathcal{G}_{13}}{\partial G_{22}} = \frac{\partial \mathcal{G}_{13}}{\partial G_{32}} = \frac{\partial \mathcal{G}_{13}}{\partial G_{13}} = \frac{\partial \mathcal{G}_{13}}{\partial G_{23}} = \frac{\partial \mathcal{G}_{13}}{\partial G_{33}} = 0
\end{aligned}$$

follow. Therefore and because the curl $\nabla f = 0$ for every smooth enough function f, the RHS $\mathcal{G}$ displays as

$$\begin{aligned}
\mathcal{G}_{11}(G) &= \mathcal{G}_{11}(G_{11}, G_{21}, G_{31}) = \alpha_{11}G_{11} + \alpha_{12}G_{21} + \alpha_{13}G_{31} + c_{11} + \nabla(f_1)_1\\
\mathcal{G}_{12}(G) &= \mathcal{G}_{12}(G_{12}, G_{22}, G_{32}) = \alpha_{11}G_{12} + \alpha_{12}G_{22} + \alpha_{13}G_{32} + c_{12} + \nabla(f_1)_2\\
\mathcal{G}_{13}(G) &= \mathcal{G}_{13}(G_{13}, G_{23}, G_{33}) = \alpha_{11}G_{13} + \alpha_{12}G_{23} + \alpha_{13}G_{33} + c_{13} + \nabla(f_1)_3
\end{aligned} \tag{6.3}$$

where c_{11}, c_{12} and c_{13} depend on t and $f_1 : [0,T] \times \Omega \to \mathbb{R}^3$ smooth enough. This holds for every row, because curl is the row wise curl. Consequently, for $i,j \in \{1,2,3\}$, it is

$$\begin{aligned}
\mathcal{G}_{ij} = \mathcal{G}_{ij}(G_{1j}, G_{2j}, G_{3j}) &= \alpha_{i1}G_{1j} + \alpha_{i2}G_{2j} + \alpha_{i3}G_{3j} + c_{ij} + \nabla f_{ij}\\
&= \sum_{k=1}^{3} \alpha_{ik}G_{kj} + c_{ij} + \nabla f_{ij},
\end{aligned} \tag{6.4}$$

where $c_{ij} : [0,T] \to \mathbb{R}$ and $f_i : [0,T] \times \Omega \to \mathbb{R}^3$ smooth enough.
As a conclusion, if $\mathcal{G}$ has the form like in (6.4) the ODE (6.2) becomes

$$\begin{aligned}
(\text{curl } G)^{\cdot}_{ij} &= (\text{curl } \mathcal{G}(G,t,p))_{ij}\\
&= \alpha_{i1}(\text{curl } G)_{1j} + \alpha_{i2}(\text{curl } G)_{2j} + \alpha_{i3}(\text{curl } G)_{3j} =: \mathcal{F}_{ij}(\text{curl } G, t),
\end{aligned}$$

where $\mathcal{F}$ is Lipschitz continuous in curl G. Thus, the existence theorem of Picard–Lindelöf A.2.6 states that there exists a unique solution for the initial datum curl $G(0) = 0$. Since 0 is a solution, it is the only one and curl $G(t,x) = 0$ for all $t \in [0,T]$ and all $x \in \Omega$.
In the beginning of this section, we searched for $\mathcal{G}$ such that the solution $G(t,\cdot)$ is a gradient, i.e. curl-free. The only $\mathcal{G}$ fulfilling this have the form of (6.4). Consequently, the growth equation become linear in order to also fulfil the condition to be locally Lipschitz continuous in G.

6.2. A circular setting in two dimensions

Even though the previous discussion points out troubles in higher dimensions, we investigate a two dimensional situation with strong assumptions on the geometry to achieve a compatible growth map. The downside of those assumptions are that the physicality is lost.
We consider a ring in two dimensions, namely let $0 < R_1 < R_2 < \infty$ be given radii describing the ring

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 |\ R_1 < \sqrt{x_1^2 + x_2^2} < R_2\} \subset \mathbb{R}^2.$$

In the following, we assume the ring to always stay a ring centred around 0. This prohibits buckling effects and simplifies the geometry. Furthermore, we suppose the growth to only change the distance of the particle to 0, but not to change its angle. This means that the growth map has the form

$$g(t,\cdot)\colon \Omega \to \mathbb{R}^2,\ g(t,x) = \rho(t,|x|)\frac{x}{|x|} = \rho(t,R)e_R \tag{6.5}$$

with $R = |x|$, e_R the radial unit vector in polar coordinates and $\rho\colon [0,T]\times[R_1,R_2] \to \mathbb{R}_{>0}$ a strictly increasing, continuous differentiable function. In consistency with the one dimensional case, the initial growth map is the identity, namely

$$g(0,x) = x\rho(0,R)e_R = x = Re_R,$$

which gives the initial condition

$$\rho(0,R) = R. \tag{6.6}$$

The gradient of $g(t,\cdot)$ calculates as

$$\nabla g(t,x) = G(t,x) = \partial_R\rho(t,R)e_R \otimes e_R + \frac{\rho(t,R)}{R}e_\phi \otimes e_\phi. \tag{6.7}$$

A first approach is to assume an ODE similar to the one we assumed in the one dimensional case. Later on, we will see that we do not obtain the dependence of the Piola–Kirchhoff stress tensor S of the growth gradient as in the one dimensional setting, see Remark 6.2.1. Therefore and for simplicity, we assume the ODE $\dot{G} = \gamma G$ to be independent of S. Here, $\gamma = \gamma(R)$ is a strictly increasing and continuously differentiable function. Then, the ODE writes as

$$\begin{aligned}\dot{G}(t,x) &= \partial_t\partial_R\rho(t,R)e_R\otimes e_R + \frac{\partial_t\rho(t,R)}{R}e_\phi\otimes e_\phi\\ &= \gamma(R)\left(\partial_R\rho(t,R)e_R\otimes e_R + \frac{\rho(t,R)}{R}e_\phi\otimes e_\phi\right)\\ &= \gamma(R)G(t,R).\end{aligned}$$

This matrix identity is equivalent to the two scalar equations

$$\begin{aligned}&\partial_t\partial_R\rho(t,R) = \gamma(R)\partial_R\rho(t,R),\\ &\partial_t\rho(t,R) = \gamma(R)\rho(t,R).\end{aligned}$$

To simplify the calculations, we suppose that we may use separation of variables for ρ. Let $\alpha := [0,T] \to \mathbb{R}$ and $\beta\colon [R_1,R_2] \to \mathbb{R}$ be continuously differentiable functions such that $\rho(t,R) = \alpha(t)\beta(R)$. Then, it follows

$$\begin{aligned}&\alpha'(t)\beta'(R) = \gamma(R)\alpha(t)\beta'(R),\\ &\alpha'(t)\beta(R) = \gamma(R)\alpha(t)\beta(R).\end{aligned}$$

In view of the initial condition (6.6), we obtain

$$\rho(t,R) = R\exp(\gamma(R)t)$$

as one solution, and therefore, the growth map g is given as

$$g(t,x) = R\exp(\gamma(R)t)e_R. \tag{6.8}$$

Note that $g(t,\cdot)$ is an injective and continuously differentiable function with gradient given by (6.7).

For the elastic deformation $\phi(t,\cdot)$, we assume the same restriction as for the growth map, namely to only depend on and change the radius and not to depend on the angle nor change it. Define the natural configuration

$$\Omega_{nat} = g(t,\Omega) = \{y = (y_1, y_2) \in \mathbb{R}^2 |\ R_1 \exp(\gamma(R_1)t) < |y| < R_2 \exp(\gamma(R_2)t)\}$$

and suppose the elastic deformation $\phi(t,\cdot)$ to have the structure

$$\phi(t,\cdot)\colon \Omega_{nat} \to \mathbb{R}^2,\ \phi(t,y) = r(t,|y|)\frac{y}{|y|} = r(t,\rho)e_R,$$

where $\rho = |y|$ and $r(t,\cdot) \coloneqq [R_1 \exp(\gamma(R_1)t), R_2 \exp(\gamma(R_2)t)] \to \mathbb{R}_{>0}$ strictly increasing and continuously differentiable. Again, the gradient is calculated as

$$\nabla\phi(t,y) = \partial_\rho r(t,\rho)e_R \otimes e_R + \frac{r(t,\rho)}{\rho} e_\phi \otimes e_\phi.$$

In analogy to the one dimensional case, we assume the tumour to cover Ω. Consequently, the boundary conditions are Dirichlet boundary conditions and they are given by

$$\phi(t, g(t,R_1)) = r(t,\rho(t,R_1))e_R = R_1 e_R,$$
$$\phi(t, g(t,R_2)) = r(t,\rho(t,R_2))e_R = R_2 e_R.$$

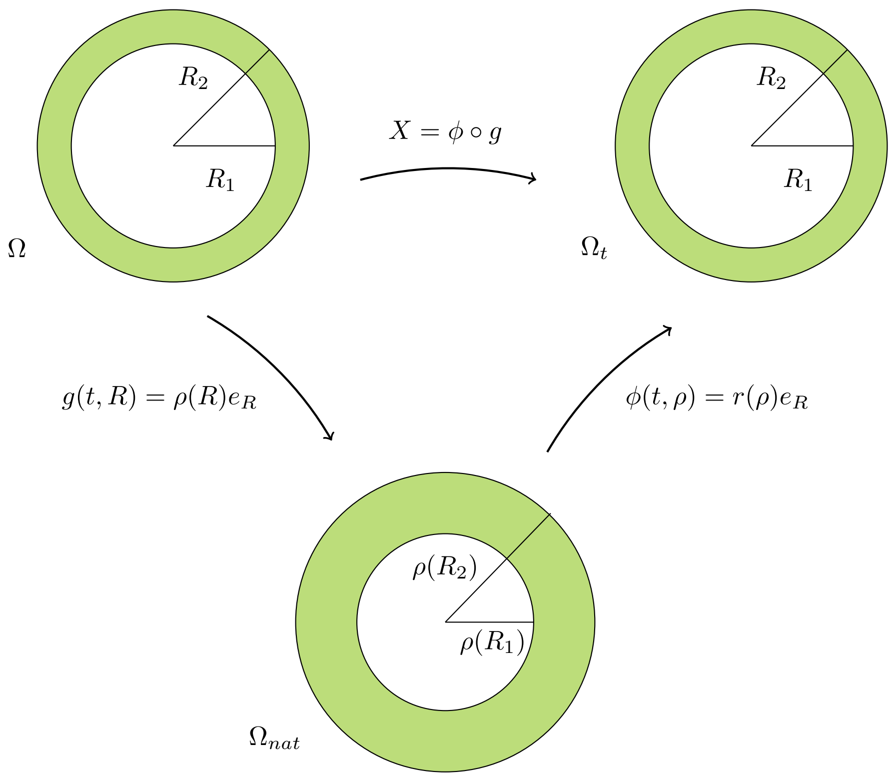

Figure 6.1.: Strongly restricted two dimensional setting.

To determine $\phi(t,\cdot)$ the elastic problem has to be stated and solved. As for the one dimensional setting, we consider hyperelastic, homogeneous material within non-linear elasticity. As a reference for well-posedness and existence refer for example to [Bal76].

Remark 6.2.1. *Assume there exists a real minimizer $\phi(t,\cdot)$ of the elastic problem fulfilling the conditions of Theorem 7.13 in [Bal76]. Then this theorem states that $\phi(t,\cdot)$ fulfils the equation*

$$\int_{\Omega_{nat}} D_{F_{ij}} W(\nabla\phi(t,y)) D^j v^i(y)\; dy = 0$$

for all $v \in C^\infty(\Omega_{nat};\mathbb{R}^2)$. Here, W denotes the elastic strain energy density. This equation cannot be used, in the same way as in the one dimensional setting, to obtain an implicit formula for the Piola-Kirchhoff stress tensor S in dependence of the growth tensor G. Therefore, we assumed the ODE independent of S. To solve the AMP for an on S depending RHS of the ODE, this problem has to be overcome.

Remark 6.2.2. *In the above very strong assumptions on the elastic deformations were made. By changing the inner boundary condition to Neumann boundary condition buckling effects can appear, see [Li+11] and [MG11].*

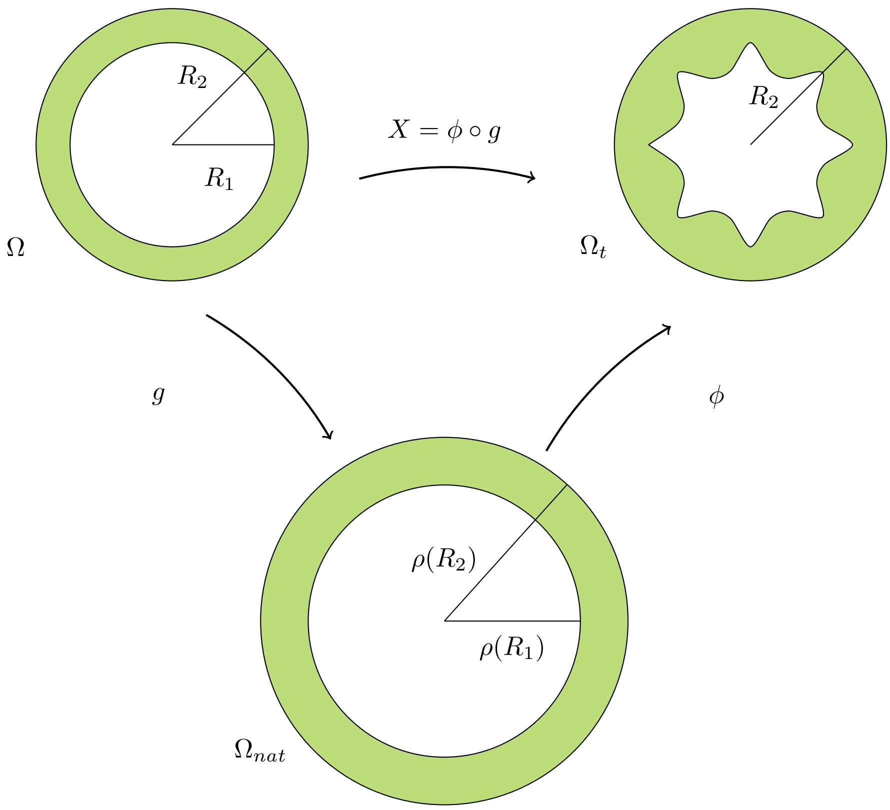

Figure 6.2.: Schematic picture of a possible situation with buckling.

Remark 6.2.3. *In the above considerations, we assumed γ to be continuously differentiable and strictly increasing. This is compared to $\gamma \in L^\infty(0,1)$, the assumption in one dimension, a strong assumption. To start with, we look at the situation with γ taking two values, namely there exist an $R_* \in (R_1,R_2)$ and $\gamma_1,\gamma_2 \in \mathbb{R},\ \gamma_1 \neq \gamma_2$, such that*

$$\gamma(R) = \begin{cases} \gamma_1, & R \leq R^*, \\ \gamma_2, & R > R^*. \end{cases}$$

With a similar procedure as above, we get in analogy to (6.8) the growth map

$$g(t,R) = R\exp(\gamma(R)t) = \begin{cases} R\exp(\gamma_1 t), & R \le R^*, \\ R\exp(\gamma_2 t), & R > R^*. \end{cases}$$

For different relations between γ_1 *and* γ_2 *there are different situations.*
If $\gamma_1 < \gamma_2$*, the ring grows into two parts, which the elastic deformation has to glue together again.*
For $\gamma_1 > \gamma_2$ *and short times, the ring grows into two rings which are overlapping (displayed in brown in the Picture 6.3). This is not consistent with the Definition 6.1.3 of the solution of the AMP in higher dimensions. Therefore, the definition of solution has to be adjusted and the overlap has to be handled. For example, the reference configuration can be described by two sets such that the growth map on each is injective. In such a simple setting, this seems to be easy, but with more complicated setting, e.g. for a continuous but not strictly increasing* γ *splitting up of the reference configuration and, hence, the identification of points* $y \in \Omega_{nat}$ *becomes more complicated. Furthermore, the elastic deformation has to put the grown parts into a physically admissible set again.*
For $\gamma_1 > \gamma_2$ *and large enough times, the ring grew into two rings with changed order. As for the first case* $\gamma_1 < \gamma_2$*, the rings have to be reordered and glued together by the elastic deformation. In all three cases, there is a topological change or overlap of the natural configuration, which the elastic deformation has to even out. In one dimension, however, we were able to shift the natural configuration to obtain a connected, non-overlapping natural configuration, which is not possible in the setting of a ring growing with different rates.*

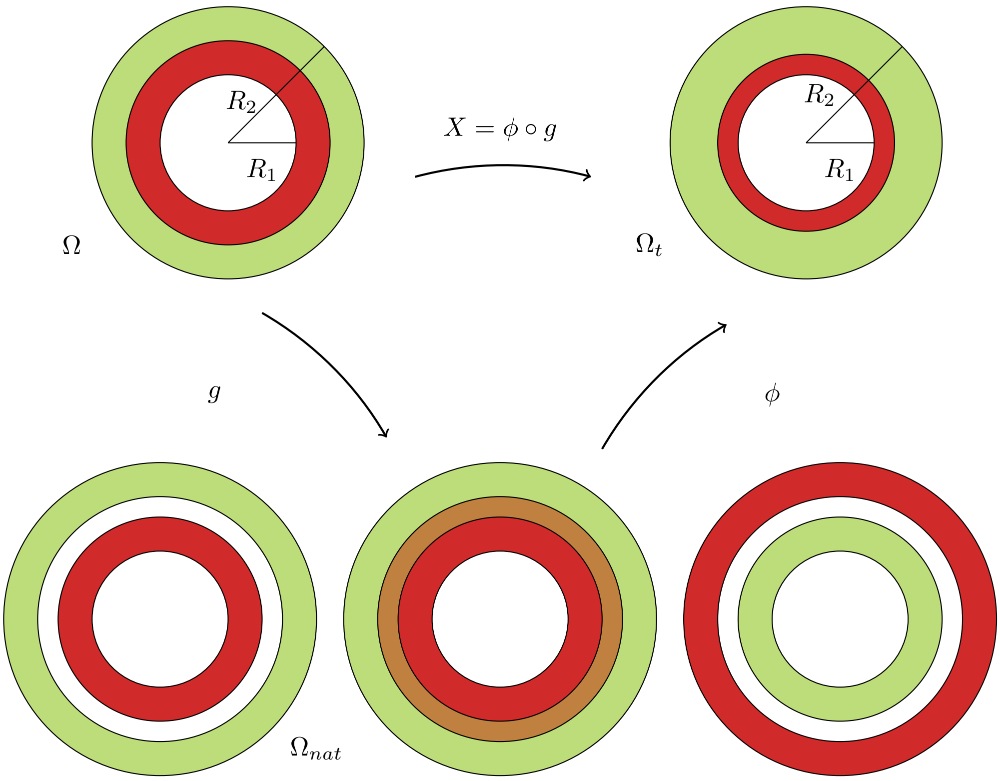

Figure 6.3.: Schematic picture of the different natural configurations. Left: $\gamma_1 < \gamma_2$. Middle: $\gamma_1 > \gamma_2$ and short time; brown displays the overlap. Right: $\gamma_1 > \gamma_2$ and large time.

Remark 6.2.4. *Before, only growth in radial direction was considered. Another thing to consider is growth in angular direction. For positive growth, the ring describes more than 2π angle, namely it overlaps. In the Picture 6.4, this is marked in dark green colour. If, however, the growth is a resorption, the ring opens up. In this situation, the description of the growth map causes problems as complicates the further calculations in the elastic problem. Furthermore, if the ring overlaps, the natural configuration is not consistent with the Definition 6.1.3, which has to be overcome.*

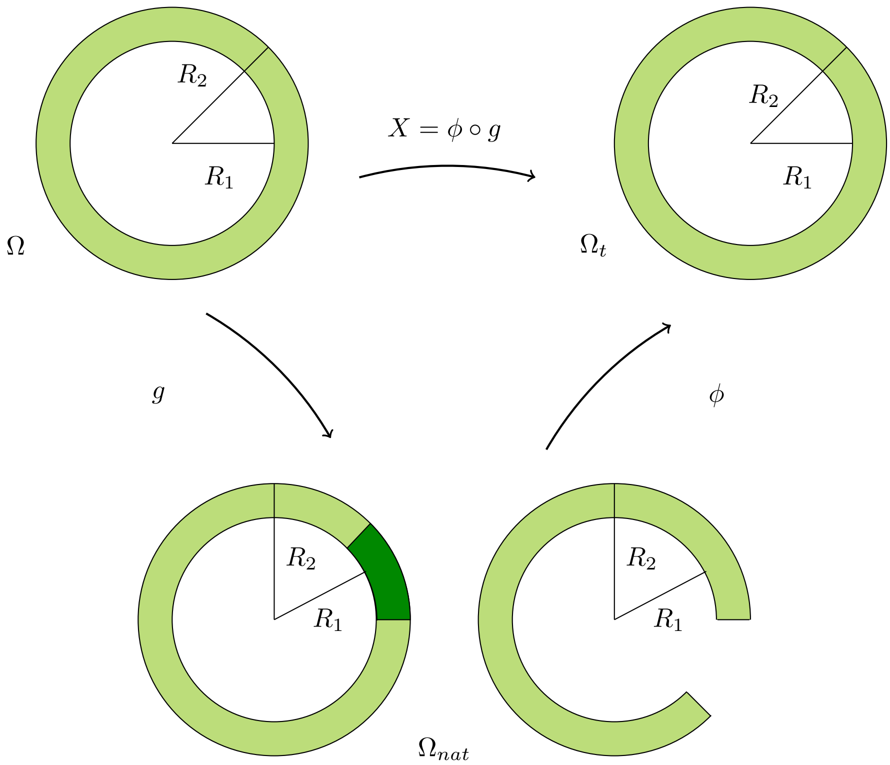

Figure 6.4.: Schematic picture with angular growth. Right: In the natural configuration, the purple area displays the overlap. Left: The ring opens up, if the angular growth is negative.

The Remarks 6.2.2-6.2.4 give simple examples of growth in a circular setting, a setting with strong assumptions on the geometry. However, one does not want to prescribe those restrictions on the geometry, but solve for a general setting. This yields many, in this thesis displayed, difficulties and is yet to be done.

A Appendix

A.1. On one-dimensional elasticity

The following theorem can be found in [Bal81]. Here it is extended by the case of Neumann boundary conditions and proved in more detail to also justify Remark 3.1.4.

Theorem A.1.1 (Euler–Lagrange equation in one dimension)**.** *Let $L > 0$ be fixed and W an elastic strain energy density as in Definition 3.1.3. Further, let $p > 1$ and*

$$\phi_D \in \mathcal{A}_D \coloneqq \{v \in W^{1,p}(0,L)|\ v(0) = 0, v(L) = 1\},$$
$$\phi_N \in \mathcal{A}_N \coloneqq \{v \in W^{1,p}(0,L)|\ v(0) = 0\}$$

be minimizers of the elastic energy $I : W^{1,p}(0,L) \to \mathbb{R} \cup \{\infty\}$,

$$I(v) = \int_0^L W(y, \partial_y v(y))\ dy.$$

over $\mathcal{A}_D$ and $\mathcal{A}_N$ respectively. Then, it holds:

(i) $\phi_D, \phi_N \in \mathrm{C}^1([0,L])$ and it holds

$$\min_{x\in[0,L]} \partial_x \phi_D(x) > 0,$$
$$\min_{x\in[0,L]} \partial_x \phi_N(x) > 0.$$

(ii) The Euler–Lagrange equations are

$$D_F W(y, \partial_y \phi_D(y)) = C_D,$$
$$D_F W(y, \partial_y \phi_N(y)) = C_N$$

for constants $C_D, C_N \in \mathbb{R}$ respectively.

Proof. Concerning (i): See [Bal81].
Concerning (ii): For the minimizer ϕ_D of I in $\mathcal{A_D}$ define for $n \in \mathbb{N}$

$$\Omega_n \coloneqq \{y \in [0,L]| \sup_{F\in B_{\frac{1}{n}}(\partial_y \phi_D(y))} |D_F W(y,F)| \leq n\}$$

and χ_n the characteristic function on Ω_n. It holds that $\Omega_n \subset \Omega_{n+1}$, since for each $y \in \Omega_n$ holds

$$\sup_{F\in B_{\frac{1}{n}}(\partial_y \phi_D(y))} |D_F W(y,F)| \leq n \leq n+1$$

and $B_{\frac{1}{n}}(\partial_y\phi_D(y)) \supset B_{\frac{1}{n+1}}(\partial_y\phi_D(y))$.
Next we prove that $\partial_y\phi_D(y) = 0$ if and only if $y \in [0, L] \setminus \bigcup_{n\in\mathbb{N}} \Omega_n$. Assume that $\partial_y\phi_D(y) = 0$ for an $y \in \bigcup_{n\in\mathbb{N}} \Omega_n$. Then, there exists an $n \in \mathbb{N}$ such that $y \in \Omega_n$. Let $(a_k)_{k\in\mathbb{N}}$ be a sequence with $a_k \to 0 = \partial_y\phi_D(y)$ for $k \to \infty$ and $a_k \in B_{\frac{1}{n}}(\partial_y\phi_D(y))$ for all $k \in \mathbb{N}$. Then

$$D_F W(y, \partial_y\phi_D(y)) \leftarrow D_F W(y, a_k) \leq n \text{ for } k \to \infty.$$

But as $k \to \infty$ also $D_F W(y, a_k) \to \infty$ due to the assumptions on W. A contradiction to $y \in \Omega_n$. Further, the set $[0, L] \setminus \bigcup_{n\in\mathbb{N}} \Omega_n$ is a zero set, because $I(\phi) < \infty$ and the property $W(F) \to \infty$ for $F \to 0$.
To prove (ii) assume the following variation of ϕ: Fix $n \in \mathbb{N}$. Let $v \in L^\infty(0, L)$ be fixed with $\int_{\Omega_n} v(y)\, dy = 0$. Define $w \in W^{1,p}(0, L)$ by $w(0) = 0$ and

$$\partial_y w(x) = \partial_y\phi_D(y) + t\chi_n(y)v(y)$$

for sufficiently small $t \in \mathbb{R}$. Then, the Euler–Lagrange equation is

$$\begin{aligned}
0 &= d_t I(y)|_{t=0} \\
&= d_t \int_0^L W(y, \partial_y\phi_D(y) + t\chi_n(y)v(y))\, dy|_{t=0} \\
&= \int_0^L D_F W(y, \partial_y\phi_D(y))\chi_n(y)v(y)\, dy \\
&= \int_{\Omega_n} D_F W(y, \partial_y\phi_D(y))v(y)\, dy.
\end{aligned}$$

Here we can exchange the derivative and integral, since for small enough t the argument of W is bounded on a bounded interval and, hence, is the function bounded. Further the limit is bounded due to the construction of Ω_n. Hence, we can apply dominated convergence.
Since $D_F W$ is bounded on the set Ω_n and v is arbitrary in $L^\infty(0, 1)$, there exists a constant C such that

$$D_F W(x, \partial_y\phi_D(y)) = C \text{ for almost every } y \in \Omega_n. \tag{A.1}$$

Since $\Omega_n \subset \Omega_{n+1}$, the constant C is independent of n and the equation (A.1) holds for a.e. $y \in \bigcup_{n\in\mathbb{N}} \Omega_n$. Because $[0, L] \setminus \bigcup_{n\in\mathbb{N}} \Omega_n$ is a zero set, (A.1) holds for a.e. $y \in [0, L]$.
A similar argumentation applies to the case with Neumann boundary condition on the left end of the interval. For $v \in L^\infty(0, 1)$ define the variation of w by

$$\partial w(y) = \partial_y\phi_N(y) + t\chi_n(y)v(y).$$

Then, $w \in \mathcal{A}_N$ since the Neumann boundary condition is natural. With the same argumentation as for the Dirichlet setting, we get a constant $C \in \mathbb{R}$ such that

$$D_F W(y, \partial_y\phi_N(y)) = C \text{ almost everywhere in } [0, L].$$

♣

To establish regularity in space the following theorem is cited from [BGH99].

Theorem A.1.2 (Regularity of variational minimizer)**.** *Let $I = (a, b)$ be an open, bounded interval in $\mathbb{R}$ and let $\bar{W}\colon \bar{I} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}_{\geq 1}$, be a C^2-function satisfying the following conditions* *(i) There exist constants $c_0, c_1 > 0$ and $p \in (1, \infty)$ such that*

$$c_0|F|^p \leq \bar{W}(y, z, F) \leq c_1(1 + |F|^p)$$

for all $(y, z, F) \in \overline{I} \times \mathbb{R}^n \times \mathbb{R}^n$,

(ii) There exist function $M\colon (0, \infty) \to (0, \infty)$ such that for all $R > 0$ holds that

$$|D_z\bar{W}(y, z, F)| + |D_F\bar{W}(y, z, F)| \leq M(R)(1 + |F|^2)$$

for all $(y, z, F) \in \overline{I} \times \mathbb{R}^n \times \mathbb{R}^n$ with $y^2 + |z|^2 \leq R^2$,

(iii) for all $(y, z, F) \in \overline{I} \times \mathbb{R}^n \times \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n \setminus \{0\}$ holds

$$D_F^2\bar{W}(y, z, F)\xi \cdot \xi > 0.$$

Further, suppose there exists a minimizer ϕ of

$$\mathcal{E}(\phi) \coloneqq \int_a^b \bar{W}(y, \phi(y), \partial_y\phi(y))\, dy$$

in the class

$$\mathcal{C} \coloneqq \{v \in H^{1,m}(I, \mathbb{R}^n)|\ v(a) = \alpha, v(b) = \beta\},$$

where $\alpha < \beta$ are given boundary conditions. Then, $\phi \in \mathrm{C}^2(\overline{I}; \mathbb{R}^n)$. Moreover, if $\bar{W}$ is of class C^k, $2 \leq k \leq \infty$, then, $\phi \in \mathrm{C}^k(\overline{I}; \mathbb{R}^n)$.

A.2. Lipschitz continuity and Picard–Lindelöf theorem

This section is dedicated to the discussion of Lipschitz continuity and the Picard–Lindelöf existence theorem. The later is of essence for the existence of a solution of the AMP in any setting and it requires Lipschitz continuity. That is the reason, why Lipschitz continuity is often discussed in this thesis and many basic properties are used. Those shall be collected here. First, start with the definition of Lipschitz continuity and its variations.

Definition A.2.1 (Lipschitz-continuity)**.** *Let X, Y be Banach spaces. A function $f\colon X \to Y$ is called Lipschitz continuous if there exists a constant $L > 0$ such that*

$$\|f(x) - f(x')\|_Y \leq L\|x - x'\|_X$$

for all $x, x' \in X$.

Moreover, properties of Lipschitz continuous functions are used, as the composition and product of Lipschitz continuous functions and that a Lipschitz continuous function growth at most linear.

Lemma A.2.2. *Let X, Y, Z be Banach spaces.*

(i) Let $f\colon X \to Y$ be Lipschitz continuous, then, f has at most linear growth.

(ii) Let $f\colon Y \to Z$ and $g\colon X \to Y$ be Lipschitz continuous with constants L_f and L_g respectively. Then, is $f \circ g\colon X \to Z$ Lipschitz continuous with constant $L_{f\circ g} = L_f L_g$.

(iii) Let $g\colon \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Then, $f\colon L^\infty(0, 1) \to L^\infty(0, 1)$, $f(u)(x) \coloneqq g(u(x))$ is Lipschitz continuous.

Proof. Concerning (i): Let $x \in \mathbb{R}$ be arbitrary. Then, it holds

$$\begin{aligned}\|f(x)\|_Y &= \|f(x) - f(0) + f(0)\|_Y \\ &\leq \|f(x) - f(0)\|_Y + \|f(0)\|_Y \\ &\leq L_f\|x - 0\|_X + \|f(0)\|_Y = L_f\|x\|_X + \|f(0)\|_Y.\end{aligned}$$

Concerning (ii): Let $x', x^* \in \mathbb{R}$ be arbitrary. Then, it holds with the Lipschitz continuity of f and g that

$$\begin{aligned}\|f(g(x')) - f(g(x^*))\|_Z &\leq L_f\|g(x') - g(x^*)\|_Y \\ &\leq L_f L_g\|x' - x^*\|_X.\end{aligned}$$

Concerning (iii): Let $u', u^* \in L^\infty(0,1)$ be arbitrary. Then, it holds with the Lipschitz continuity of g with constant L_g follows

$$\begin{aligned}\|f(u') - f(u^*)\|_{L^\infty(0,1)} &= \operatorname*{ess\,sup}_{x\in(0,1)}|f(u')(x) - f(u^*)(x)| \\ &= \operatorname*{ess\,sup}_{x\in(0,1)}|g(u'(x)) - g(u^*(x))| \\ &\leq \operatorname*{ess\,sup}_{x\in(0,1)} L_g|u'(x) - u^*(x)| \\ &= L_g\|u' - u^*\|_{L^\infty(0,1)}.\end{aligned}$$

♣

Lemma A.2.3. *Let $a, b \in (0,\infty)$ and let $f_1, f_2 \in W^{1,\infty}([0,a];[0,b])$ be strictly increasing, bijective, bi-Lipschitz continuous. Further assume there exist $\delta, \varepsilon > 0$ such that $\|f_1 - f_2\|_{C^0([0,a])} \leq \varepsilon$ and $\partial_x f_i > \delta$, $i = 1,2$, almost everywhere on $[0,a]$. Then*

$$\|f_2^{-1} - f_2^{-1}\|_{C^0([0,1])} \leq \frac{\varepsilon}{\delta}.$$

Proof. Let $z \in [0,b]$ and define $x_i = f_i^{-1}(z)$, $i = 1,2$. Then, $f_1(x_1) = f_2(x_2)$ and

$$|f_1(x_1) - f_2(x_1)| = |f_2(x_2) - f_2(x_1)| = |\int_{x_1}^{x_2} \partial_x f_2(\tilde{x})\, d\tilde{x} \geq \delta|x_1 - x_2|.$$

Therefore, it follows

$$|f_1^{-1}(z) - f_2^{-1}(z)| = |x_1 - x_2| \leq \frac{1}{\delta}|f_1(x_1) - f_2(x_1)| \leq \frac{\varepsilon}{\delta}$$

and the statement follows by taking the supremum over $z \in [0,b]$. ♣

Lemma A.2.4. *Let X, Y be Banach spaces. Let $f\colon U \subset X \to Y$ and $f'\colon U \to Y'$ be Lipschitz continuous and bounded by f_{max} and f'_{max}, respectively. Then,*

$$g\colon U \to \mathbb{R},\ x \mapsto \langle f(x), f'(x)\rangle_{Y,Y'}$$

is Lipschitz continuous with Lipschitz constant $L_g \leq f'_{max}L_f + f_{max}L_{f'}$.
In addition, if Y is a Banach algebra, $f, g\colon U \to Y$ and $h(x) := f(x) \cdot g(x)$, then h is Lipschitz continuous.

Proof. Let $x_1, x_2 \in U$ be fixed. Then

$$|\langle f(x_1), f'(x_1)\rangle_{Y,Y'} - \langle f(x_2), f'(x_2)\rangle_{Y,Y'}|$$

$$\leq |\langle f(x_1), f'(x_1)\rangle_{Y,Y'} - \langle f(x_1), f'(x_2)\rangle_{Y,Y'}| + |\langle f(x_1), f'(x_2)\rangle_{Y,Y'} - \langle f(x_2), f'(x_2)\rangle_{Y,Y'}|$$
$$= |\langle f(x_1), f'(x_1) - f'(x_2)\rangle_{Y,Y'}| + |\langle f(x_1) - f(x_2), f'(x_2)\rangle_{Y,Y'}|$$
$$\leq f_{max} L_{f'} \|x_1 - x_2\|_X + f'_{max} L_f \|x_1 - x_2\|_X .$$

Analogously, the statement for the case of Y a Banach algebra is shown. ♣

Lemma A.2.5. *Let $I_1 = [a_1, b_1] \subset \mathbb{R}$, $I_2 = [a_2, b_2] \subset \mathbb{R}_{>0}$, closed intervals and $A_0 \in \mathbb{R}$. Further let $a \in L^\infty(I_1; I_2)$. Then,*

$$A\colon I_1 \to \mathbb{R},\ A(x) = \int_{a_1}^{x} a(\tilde{x})\ d\tilde{x} + A_0$$

is bi-Lipschitz continuous, i.e. A^{-1} exists and A are Lipschitz continuous with Lipschitz constants $L_A \leq b_2$ and $L_{A^{-1}} \leq \frac{1}{a_2}$.

Proof. As $a > 0$ almost everywhere, A is strictly monotonically increasing and $A^{-1}\colon A(I_1) \to I_1$ exists. Let $x_1, x_2 \in I_1$. Then

$$|A(x_1) - A(x_2)| = |\int_{x_2}^{x_1} a(\tilde{x})\ d\tilde{x}| \leq b_2 |x_1 - x_2|.$$

For $y_1, y_2 \in [A(a_1), A(b_2)]$ with $y_i = A(x_i)$, $i = 1, 2$, it follows from $a(x) \geq a_2$ for almost all $x \in I_1$

$$|A^{-1}(y_1) - A^{-1}(y_2)| = |x_1 - x_2| \leq |\int_{x_2}^{x_1} \frac{a(\tilde{x})}{a_2}\ d\tilde{x}| = \frac{1}{a_2}|A(x_1) - A(x_2)| = \frac{1}{a_2}|y_1 - y_2|.$$

♣

Of key importance is the existence theorem from Picard and Lindelöf, which states the existence of a solution of an ODE with a Lipschitz continuous RHS. In more detail:

Theorem A.2.6 (Local Picard–Lindelöf existence theorem)**.** *Let X be a Banach space, $A \subset \mathbb{R} \times X$, $y_0 \in X$ and $R > 0$ such $[a, b] \times B_R(y_0) \subset A$ for $a, b \in \mathbb{R}$. Further let $f\colon A \to X$ be continuous and Lipschitz continuous in the second argument. Define*

$$M \coloneqq \max\left\{ \|f(t, y)\|_X \mid (t, y) \in [a, b] \times \overline{B_R(y_0)} \right\},$$
$$t_0 \coloneqq \min\left\{ b - a, \frac{R}{M} \right\}.$$

Then, there exists a solution $y \in \mathrm{C}^1([a, a + t_0]; X)$ of

$$y'(t) = f(t, y(t)) \qquad \forall t \in [a, a + t_0],$$
$$y(a) = y_0$$

such that $(t, y(t)) \in A$.

For example, this theorem can be found in [Emm04].

A.3. Implicit function theorem

In this chapter, the definitions of Fréchet and Gâteaux derivatives are repeated as well as the connection, see [Růž04]. Those are used to prove the Lipschitz continuity of the stress tensor

S in dependence of the growth tensor G by using the implicit function theorem, see [Zei88].

Theorem A.3.1 (Arzelà–Ascoli). *Let X be a compact Hausdorff space and Y a metric space. Then, $M \subset C(X;Y)$ is relatively compact if and only if the following holds:*

(i) Equicontinuity: For every $\varepsilon > 0$ exists an $\delta > 0$ such that $d(f(x_1), f(x_2)) < \varepsilon$ for all $f \in M$ and all $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$.

(ii) pointwise relatively compact: For every $x \in X$, the set $\{f(x)|f \in M\}$ is relatively compact in Y.

Definition A.3.2 (Fréchet differentiable). *Let X, Y be Banach spaces, $x_0 \in X$ and $U \subset X$ a neighbourhood of x_0. Further, let $f\colon U \to Y$ be a map. We say f is Fréchet differentiable in x_0, if and only if there exists a continuous, linear map $A\colon X \to Y$ with*

$$f(x_0 + h) - f(x_0) = Ah + o(\|h\|_X), h \to 0.$$

If the map A exists, we call it Fréchet derivative of f in x_0 and use the notation $f'(x_0) := A$.

Definition A.3.3 (Gâteaux differentiable). *Let X, Y be Banach spaces, $x_0 \in X$ and $U \subset X$ a neighbourhood of x_0. Further, let $f\colon U \to Y$ be a map and let $h \in X$ be given. Define $\varphi\colon (-\delta, \delta) \to Y$ for $\delta > 0$ small enough by*

$$\phi(t) := f(x_0 + th).$$

If φ is differentiable in $t = 0$, e.g.

$$\partial_y \phi(0) = \frac{d}{dt} f(x_0 + th)|_{t=0} = \lim_{t \to 0} \frac{f(x_0 - th) - f(x_0)}{t} \in Y,$$

we say f has a derivate in direction h and define $\delta f(x_0, h) := \varphi'(0)$. If $\delta f(x_0, h)$ exists for all $h \in X$ and the map

$$Df(x_0)\colon X \to Y, h \mapsto \delta f(x_0, h)$$

is continuous and linear, we say that f is Gâteaux differentiable in x_0 and $Df(x_0)$ is the Gâteaux derivative.

Lemma A.3.4. *Let X, Y be Banach spaces and let $f\colon X \to Y$ be Gâteaux differentiable in a neighbourhood $U \subset X$ of $x_0 \in X$. In addition, let $Df(x_0)$ be continuous in x_0. Then, f is Fréchet differentiable in x_0 with*

$$Df(x_0)h = f'(x_0)h.$$

Theorem A.3.5 (Implicit function theorem). *Suppose that X, Y and Z are Banach spaces and $U(x_0, y_0) \subset X \times Y$ is an open neighbourhood of $x_0 \in X$ and $y_0 \in Y$. Further, let $f : U(x_0, y_0) \to Z$ a function with $f(x_0, y_0) = 0$, such that the partial Fréchet derivative F_y exists on $U(x_0, y_0)$, $f_y(x_0, y_0) : Y \to Z$ is bijective and f and f_y are continuous in (x_0, y_0). Then, there exists a neighbourhood $V \subset X$ of x_0 and a map $y : X \to Y$ such that*

$$f(x, y(x)) = 0.$$

Furthermore, if $f \in \mathrm{C}^m$ on a neighbourhood of (x_0, y_0) for $m \in \mathbb{N}$, then, $y(\cdot)$ is also C^m on a neighbourhood of x_0.

Lemma A.3.6 (Derivative of inverse). *Let $x_0 \in \mathbb{R}$ and $U \subset X$ be an open neighbourhood of x_0. Further, let $f\colon U \to \mathbb{R}$ be of class C^1 with $f'(x_0)\colon \mathbb{R} \to \mathbb{R}$ an isomorphism. Then, there exists a neighbourhood $V \subset U$ of x_0 and a neighbourhood $W \subset \mathbb{R}$ of $f(x_0)$ such that $f^{-1} \in \mathrm{C}^1(W'; \mathbb{R})$ for an open set $W' \subset W$ and for each $x \in f^{-1}(W')$ holds*

$$(Df^{-1})(f(x)) = Df(x)^{-1}.$$

A.4. Change of Variables for Sobolev functions

In the chapter 4, an implicit formula for the stress S in dependence of G is obtained. This uses a change of variables formula for an injective $W^{1,\infty}$-function. Furthermore, the calculation takes place in one dimension. In the following the used formula is derived.
The idea is to use the following theorem from [MM73] for a change of variables formula of $W^{1,p}$-functions for higher space dimensions.

Proposition A.4.1 (Change of variables for $W^{1,p}$ functions)**.** *Assume $d, d' \in \mathbb{N}_{\geq 2}$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Further, consider $\varphi \in W^{1,p}(\Omega; \mathbb{R}^{d'})$ with $p > d$ and $f : \varphi(\Omega) \to \mathbb{R}$ be $\mathcal{H}^d$-measurable in $\mathbb{R}^{d'}$. Then, for each $\mathcal{L}^d$-measurable subset $A \subset \Omega$ holds*

$$\int_{\varphi(A)} f(y) N(\varphi, A, y)\ d\mathcal{H}^d(y) = \int_A f(\varphi(x)) |\det \nabla\varphi(x)|\ d\mathcal{L}^d(x)$$

whenever one of the two sides is well-defined.

By applying the above theorem to a suitable function, the desired change of variables formula is obtained.

Proposition A.4.2 (Change of variables for $W^{1,p}$-functions in one dimension)**.** *Let $I.J \subset \mathbb{R}$ be intervals. Further let $\varphi : I \to J$ be bijective almost everywhere and $\varphi \in W^{1,p}(I)$ with $p > 1$. Then, for each measurable $f : J \to \mathbb{R}$ and each interval $I' \subset I$ holds*

$$\int_{\varphi(I')} f(y)\ dy = \int_{I'} f(\varphi(x)) |\partial_x \varphi(x)|\ dx$$

Proof. To prove this theorem, we use the change of variables A.4.1 for $d = d' = 2$ and define the functions clever.
Step 1: Definition of function to $d = d' = 2$ setting. Let $I' \subset I, \tilde{I} \subset [0, 1]$ be arbitrary intervals. Define $A := I' \times \tilde{I}$. Further, define

$$\tilde{\varphi} \colon I \times [0, 1] \to J \times [0, 1],\ \tilde{\varphi}(x, y) := (\varphi(x), y)$$

which is an almost everywhere bijective function and, hence, it holds

$$N(\tilde{\varphi}, A, y) = 1 \text{ for almost every } y \in \tilde{\varphi}(A). \tag{A.2}$$

Define

$$\tilde{f} \colon J \times [0, 1] \to \mathbb{R}^2,\ \tilde{f}(x, y) := (f(x), y).$$

Then, $\tilde{\phi}, \tilde{f}$ and A fulfil the conditions of the change of variables A.4.1. Hence, it holds

$$\int_{\tilde{\varphi}(A)} \tilde{f}(y) N(\tilde{\varphi}, A, y)\ d\mathcal{H}^2(y) = \int_A \tilde{f}(\tilde{\varphi}(x)) |\det \nabla\tilde{\varphi}(x)|\ d\mathcal{L}^2(x) \tag{A.3}$$

The LHS of (A.3) writes as

$$\begin{aligned} \int_{\tilde{\varphi}(A)} \tilde{f}(y) N(\tilde{\varphi}, A, y)\ d\mathcal{H}^2(y) &= \int_{\tilde{\varphi}(A)} \tilde{f}(y)\ d(x, y) \\ &= \int_{\varphi(I') \times \tilde{I}} \tilde{(}f(x), y)\ d(x, y) \end{aligned}$$

$$= \int_{\varphi(I')} \int_{\tilde{I}} (f(x), y) \; dy \; dx.$$

For the RHS of (A.3), we first calculate the gradient of $\tilde{\varphi}$

$$\nabla\tilde{\varphi}(x,y) = \begin{pmatrix} \partial_x \varphi(x) & \partial_y \varphi(x) \\ \partial_x y \varphi & \partial_y y \end{pmatrix} = \begin{pmatrix} \partial_x \varphi(x) & 0 \\ 0 & 1 \end{pmatrix}$$

and with that its determinant

$$\det \nabla\tilde{\varphi}(x,y) = \det \begin{pmatrix} \partial_x \varphi(x) & 0 \\ 0 & 1 \end{pmatrix} = \partial_x \varphi(x).$$

Then, the RHS of (A.3) calculates as

$$\begin{aligned} \int_A \tilde{f}(\tilde{\varphi}(x)) |\det \nabla\tilde{\varphi}(x)| \; d\mathcal{L}^2(x) &= \int_A \tilde{f}(\tilde{\varphi}(x)) |\partial_x \varphi(x)| \; d(x,y) \\ &= \int_{I' \times \tilde{I}} (f(\varphi(x)), y) |\partial_x \varphi(x)| \; d(x,y). \end{aligned}$$

The first argument of the simplified integrals yield

$$\int_{\varphi(I')} \int_{\tilde{I}} f(x) \; dy \; dx = \int_{I'} \int_{\tilde{I}} f(\varphi(x)) |\partial_x \varphi(x)| \; d(x,y).$$

Dividing by the measure of $\tilde{I}$ provides the desired statement. ♣

Lemma A.4.3. *Let $(a,b) \subset \mathbb{R}$ be an interval and $f \in W^{1,\infty}(a,b)$ and for almost all $x, y \in (a,b)$ with $x>y$ holds*

$$f(x) > f(y).$$

Then, then inverse f^{-1} exists and has a weak derivative $(f^{-1})'$ with

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \quad \text{for almost all } y \in (f(a), f(b)). \tag{A.4}$$

Proof. Let $\psi \in C_C^\infty([a,b])$ be arbitrary. Then, by Proposition A.4.2 and due to the monotonicity of f,

$$\begin{aligned} \int_{f(a)}^{f(b)} f^{-1}(y) \psi'(y) \; dy &= \int_a^b f^{-1}(f(x)) \psi'(f(x)) f'(x) \; dx \\ &= \int_a^b x (\psi \circ f)'(x) \; dx \\ &= -\int_a^b 1 \psi \circ f(x) \; dx \\ &= -\int_a^b \psi \circ f(x) \frac{f'(x)}{f'(x)} \; dx \end{aligned}$$

$$= -\int_{f(a)}^{f(b)} \psi(y) \frac{1}{f'(f^{-1}(y))}\, dy,$$

which proves the existence of a weak derivative of f^{-1} and the formula (A.4). ♣

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