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ORIGINAL PAPER

The spectrum of simplicial volume with fixed fundamental group

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Abstract
We study the spectrum of simplicial volume for closed manifolds with fixed fundamental group and relate the gap problem to rationality questions in bounded (co)homology. In particular, we show that in many cases this spectrum has a gap at zero. For such groups, this leads to corresponding gap results for the minimal volume entropy semi-norm and for the minimal volume entropy in dimension 4.

Keywords Simplicial volume · Bounded cohomology · Rationality in (co)homology

Mathematics Subject Classification 57N65 · 53C23 · 20F65

1 Introduction

The simplicial volume of an oriented closed connected manifold is the ℓ^1-semi-norm of its $\mathbb{R}$-fundamental class [9] (Sect. 2.1). The simplicial volume is connected to amenability, negative curvature, and Riemannian volume estimates [9].

Definition 1.1 (*Spectrum of simplicial volume*) Let $d \in \mathbb{N}$ and let Mfd_d denote the class of all oriented closed connected d-manifolds. The *spectrum of simplicial volume in dimension d* is the set

$$\mathrm{SV}(d) := \bigl\{ \|M\| \bigm| M \in \mathrm{Mfd}_d \bigr\} \subset \mathbb{R}_{\geq 0}.$$

Given a group Γ, we write

$$\mathrm{SV}_\Gamma(d) := \bigl\{ \|M\| \bigm| M \in \mathrm{Mfd}_d,\ \pi_1(M) \cong \Gamma \bigr\} \subset \mathbb{R}_{\geq 0}.$$

A subset $V \subset \mathbb{R}_{\geq 0}$ *has a gap at* 0 if there exists a $c \in \mathbb{R}_{>0}$ with $V \cap (0, c) = \emptyset$. The sets $\mathrm{SV}(d)$ are known not to have a gap at zero whenever $d \geq 4$ (Sect. 1.1). However, the problem is open for the spectrum with fixed fundamental group:

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Question 1.2 (*Gap problem with fixed fundamental group*) Let $d \in \mathbb{N}$ and let Γ be a finitely presented group with $\dim_{\mathbb{R}} H_d(\Gamma; \mathbb{R}) < \infty$. Does the set $\mathrm{SV}_\Gamma(d)$ have a gap at zero?

Fundamental groups of closed manifolds are finitely presented. In the context of homological properties of groups, it is reasonable to further restrict the class of groups: We say that a group Γ has *type* FH_d if it is finitely presented and satisfies $\dim_{\mathbb{R}} H_d(\Gamma; \mathbb{R}) < \infty$.

In the present article, we give partial positive answers to Question 1.2 and put this problem into a geometric context.

1.1 The spectrum of simplicial volume

We first recall known results on the spectrum of simplicial volume. On the one hand, we have generic structural results:

Theorem 1.3 (General structure [11, Remark 2.3] [13, Theorem B/E]) *Let* $d \in \mathbb{N}$.

1. *The set* $\mathrm{SV}(d)$ *is countable and closed under addition.*
2. *The set* $\mathrm{SV}(d)$ *is contained in the set of right-computable real numbers; in particular, if* $A \subset \mathbb{N}$ *is a subset that is recursively enumerable but not recursive, then* $\sum_{n \in A} 2^{-n}$ *is* not *in* $\mathrm{SV}(d)$.

On the other hand, classification results in low dimensions and stable commutator length, respectively, can be used to exhibit concrete real numbers as simplicial volumes:

Theorem 1.4 ((no) gap [11, Example 2.4/2.5, Theorem A])

1. *The sets* $\mathrm{SV}(0), \dots, \mathrm{SV}(3)$ *have a gap at zero.*
2. *If* $d \in \mathbb{N}_{\geq 4}$, *then* $\mathrm{SV}(d)$ *is dense in* $\mathbb{R}_{\geq 0}$.

The most specific information is available in dimension 4:

Theorem 1.5 (Dimension 4) *The set* $\mathrm{SV}(4)$ *contains*

- *all non-negative rationals [11, Theorem B];*
- *a dense set of transcendental numbers that is linearly independent over the field of algebraic numbers [13, Theorem A, Theorem C];*
- *certain irrational algebraic numbers [6, Theorem 1.10].*

The constructions from Theorem 1.5 can be performed with fundamental groups with a bounded number of generators and relations [11, Section 8.4], but it is not clear from the constructions whether it is possible to fix the group.

In contrast to the closed case, the spectrum of the (locally finite) simplicial volume of oriented connected not necessarily compact manifolds without boundary in dimensions ≥ 4 coincides with $\mathbb{R}_{\geq 0} \cup \{\infty\}$ [12].

1.2 Gaps and rationality

We show that the gap behaviour of a given fundamental group is driven by the rationality properties of the zero-norm subspace of singular homology.

Definition 1.6 Let $d \in \mathbb{N}$ and let X be a topological space or a group.

- Then we write

$$N_d(X;\mathbb{R}) := \{\alpha \in H_d(X;\mathbb{R}) \mid \|\alpha\|_1 = 0\} \subset H_d(X;\mathbb{R}),$$
$$B^d(X;\mathbb{R}) := \{\varphi \in H^d(X;\mathbb{R}) \mid \varphi \text{ is bounded}\} \subset H^d(X;\mathbb{R}).$$

- A subspace $V \subset H_d(X;\mathbb{R})$ is *rational* if $V \cap H_d(X;\mathbb{Q})$ generates V over $\mathbb{R}$. A subspace $V \subset H^d(X;\mathbb{R})$ is *rational* if $V \cap H^d(X;\mathbb{Q})$ generates V over $\mathbb{R}$.

Theorem 1.7 (Section 3) *Let $d \in \mathbb{N}_{\geq 4}$ and let Γ be a group of type* FH_d. *Then the following are equivalent:*

1. *The set* $\mathrm{SV}_\Gamma(d)$ *has a gap at zero.*
2. *The set* $\{\|\alpha\|_1 \mid \alpha \in H_d(\Gamma;\mathbb{R}) \text{ is integral}\}$ *has a gap at zero.*
3. *The subspace $N_d(\Gamma;\mathbb{R})$ is rational in $H_d(\Gamma;\mathbb{R})$.*
4. *The subspace $B^d(\Gamma;\mathbb{R})$ is rational in $H^d(\Gamma;\mathbb{R})$.*

The proof in Sect. 3 shows that the implication $2 \Longrightarrow 1$ as well as the equivalence of the properties 2, 3, and 4 also hold for $d \in \{0, 1, 2, 3\}$.

The rationality property 4 is related to a problem of Frigerio and Sisto in the context of quasi-isometrically trivial extensions [8, Question 16].

1.3 Examples

The characterisation in Theorem 1.7 allows us to establish that many groups admit a positive answer to Question 1.2. Let $d \in \mathbb{N}$. We write $\mathrm{Gap}(d)$ for the class of all groups Γ of type FH_d such that $\mathrm{SV}_\Gamma(d)$ has a gap at zero.

If $N_d(\,\cdot\,;\mathbb{R})$ is trivial or the full homology, then $N_d(\,\cdot\,;\mathbb{R})$ is rational in $H_d(\,\cdot\,;\mathbb{R})$ (and similarly for cohomology). Therefore, we obtain:

Example 1.8 (*Base cases*) Let $d \in \mathbb{N}_{\geq 4}$. The class $\mathrm{Gap}(d)$ contains the following groups:

- all amenable groups of type FH_d because they have trivial bounded cohomology [9, 14] (and thus $\mathrm{SV}_\Gamma(d) = \{0\}$ [9]);
- more generally, all boundedly acyclic groups of type FH_d; this includes the Thompson group F [22];
- all hyperbolic groups because they are of finite type and the ℓ^1-semi-norm is a norm by the duality principle and Mineyev's results [19];
- all finitely presented groups with $\dim_\mathbb{R} H_d(\Gamma;\mathbb{R}) \leq 1$;
- all groups Γ of type FH_d whose comparison map $H_b^d(\Gamma;\mathbb{R}) \longrightarrow H^d(\Gamma;\mathbb{R})$ is trivial; this includes all groups of type FH_d whose classifying space admits an amenable open cover of multiplicity at most d [9, 14, 18]. Good bounds for such amenable multiplicities are, e.g., known for right-angled Artin groups [16]. More generally, one can also consider multiplicities of (uniformly) boundedly acyclic open covers [15, 17].

Example 1.9 (*Thompson group T*) The Thompson group T lies in $\mathrm{Gap}(d)$ for all $d \in \mathbb{N}_{\geq 4}$: It is well-known that T is finitely presented and has finite-dimensional cohomology in every degree [10]. Moreover, $B^*(T;\mathbb{R})$ is generated by the cup-powers of the Euler class [7, 20]. Because the Euler class is rational, we see that $B^*(T;\mathbb{R})$ is rational. We can thus apply Theorem 1.7 to conclude.

We have the following inheritance properties (proofs are given in Sect. 4):

Example 1.10 (*Inheritance properties*) For $d \in \mathbb{N}_{\geq 4}$, we have:

- The class $\mathrm{Gap}(d)$ is closed under taking (finite) free products.
  More generally, there is an inheritance principle for graphs of groups with amenable edge groups and vertex groups in $\mathrm{Gap}(d)$ (Lemma 4.2).
- Let $\Gamma \in \bigcap_{k \in \{2,\dots,d\}} \mathrm{Gap}(k)$ and $\Lambda \in \bigcap_{k \in \{2,\dots,d\}} \mathrm{Gap}(k)$. Then
$$\Gamma \times \Lambda \in \mathrm{Gap}(d).$$
- If Γ is a group that contains a finite index subgroup in $\mathrm{Gap}(d)$, then also $\Gamma \in \mathrm{Gap}(d)$.
- Let $1 \longrightarrow A \longrightarrow \Gamma \longrightarrow \Lambda \longrightarrow 1$ be an extension of groups with boundedly acyclic (e.g., amenable) kernel A. If $\Lambda \in \mathrm{Gap}(d)$ and Γ is of type FH_d, then $\Gamma \in \mathrm{Gap}(d)$.
- More generally: Let $f \colon \Gamma \longrightarrow \Lambda$ be a group homomorphism that induces a surjection $H_b^d(f;\mathbb{R}) \colon H_b^d(\Lambda;\mathbb{R}) \longrightarrow H_b^d(\Gamma;\mathbb{R})$. If $\Lambda \in \mathrm{Gap}(d)$ and Γ is of type FH_d, then also $\Gamma \in \mathrm{Gap}(d)$.

However, it remains an open problem whether for all groups Γ of type FH_d the space $N_d(\Gamma;\mathbb{R})$ is rational or not.

If we drop the finiteness conditions, then, in general, we cannot expect a gap on integral classes:

Example 1.11 There exists a countable group Γ such that $\{\|\alpha\|_1 \mid \alpha \in H_2(\Gamma;\mathbb{R}) \text{ is integral}\}$ has *no* gap at zero: For each $n \in \mathbb{N}_{>0}$, there exists a finitely presented group Γ_n with an integral class $\alpha_n \in H_2(\Gamma_n;\mathbb{R})$ satisfying
$$0 < \|\alpha_n\|_1 < \frac{1}{n};$$
for example, such groups and elements can be constructed via stable commutator length [11, Theorem C]. Then the infinite free product Γ of the $(\Gamma_n)_{n\in\mathbb{N}}$ has the claimed property. Clearly, this example Γ is not finitely generated and $\dim_{\mathbb{R}} H_2(\Gamma;\mathbb{R}) = \infty$.

Taking products with fundamental groups of oriented closed connected hyperbolic manifolds and the standard cross-product estimates for $\|\cdot\|_1$ [11, Proposition 2.9] show that such examples also exist in all degrees ≥ 4.

1.4 Gap phenomena for geometric volumes

In dimensions $d \geq 4$, it does not seem to be known whether the set of minimal volumes of all oriented closed connected smooth d-manifolds has a gap at 0 or not. For a smooth manifold M, the *minimal volume* is defined by
$$\operatorname{minvol}(M) := \inf\bigl\{\operatorname{vol}(M,g) \bigm| g \in \operatorname{Riem}_1(M)\bigr\},$$
where $\operatorname{Riem}_1(M)$ denotes the set of all complete Riemannian metrics on M whose sectional curvature lies everywhere in $[-1,1]$. The following connections with the simplicial volume are classical [9, Section 0.5]:

- *Main inequality* For all oriented closed connected smooth d-manifolds M, we have
$$\|M\| \leq (d-1)^d \cdot d! \cdot \operatorname{minvol}(M).$$
- *Isolation theorem* For each $d \in \mathbb{N}$, there exists a constant $\varepsilon_d \in \mathbb{R}_{>0}$ with the following isolation property: If M is an oriented closed connected smooth d-manifold with $\operatorname{minvol}(M) < \varepsilon_d$, then $\|M\| = 0$.

It is not known whether the vanishing of simplicial volume implies the vanishing of the minimal volume. Therefore, the gap results from Sect. 1.3 do not directly give gap results for the minimal volume with fixed fundamental group.

Similarly, the corresponding gap problem for the minimal volume entropy is open. For $d \in \mathbb{N}$, we write $\mathrm{Gap}_{\mathrm{minvolent}}(d)$ for the class of all groups Γ of type FH_d such that the set of minimal volume entropies $\mathrm{minvolent}(M)$ of oriented closed connected smooth d-manifolds M with fundamental group isomorphic to Γ has a gap at 0. In dimension 4, gaps for simplicial volume lead to gaps for minimal volume entropy:

Corollary 1.12 (Minimal volume entropy gaps in dimension 4)

1. *We have* $\mathrm{Gap}(4) \subset \mathrm{Gap}_{\mathrm{minvolent}}(4)$.
2. *In particular, all the examples of groups in* $\mathrm{Gap}(4)$ *listed in Sect.* 1.3 *lie in* $\mathrm{Gap}_{\mathrm{minvolent}}(4)$.

Proof The second part is clear. For the first part, on the one hand, we use that the minimal volume entropy is a linear upper bound for the simplicial volume [4]; on the other hand, in dimension 4, the vanishing of simplicial volume implies the vanishing of the minimal entropy [25, Theorem A] and whence of the minimal volume entropy [4]. □

The volume entropy semi-norm $\|\cdot\|_E$ is equivalent to the ℓ^1-semi-norm on singular homology [5]. Let $\mathrm{Gap}_{\mathrm{E}}(d)$ be the class of all groups Γ of type FH_d such that the set of volume entropy semi-norms $\|[M]_{\mathbb{R}}\|_E$ of oriented closed connected smooth d-manifolds M with fundamental group isomorphic to Γ has a gap at 0.

Corollary 1.13 (Volume entropy semi-norm gaps) *Let* $d \in \mathbb{N}$.

1. *We have* $\mathrm{Gap}(d) \subset \mathrm{Gap}_{\mathrm{E}}(d)$.
2. *In particular, all the examples of groups in* $\mathrm{Gap}(d)$ *listed in Sect.* 1.3 *lie in* $\mathrm{Gap}_{\mathrm{E}}(d)$.

Proof The first part follows from the fact that $\|\cdot\|_E$ and $\|\cdot\|_1$ are equivalent on singular homology [5, Theorem 1.3], whence on fundamental classes of smooth manifolds. The second part is clear. □

The smooth Yamabe invariant can be viewed as a curvature integral sibling of the minimal volume, defined in terms of scalar curvature instead of sectional/Riemannian curvature. If $d \in \mathbb{N}_{\geq 5}$ and Γ is of type FH_d, then it is known that the truncated smooth Yamabe invariant on oriented closed connected smooth spin d-manifolds with fundamental group isomorphic to Γ has a gap at 0; this is implicitly contained in the surgery inheritance results for this version of the Yamabe invariant [1, Section 1.4].

Organisation of this article

Basic notions are recalled in Sect. 2. In Sect. 3, we prove Theorem 1.7. Finally, Sect. 4 treats the inheritance properties listed in Sect. 1.3.

2 Preliminaries

We collect basic terminology and properties on simplicial volume/bounded cohomology [9].

2.1 The ℓ^1-semi-norm and simplicial volume

Definition 2.1 (ℓ^1*-semi-norm*) Let X be a space or a group and let $d \in \mathbb{N}$. For $\alpha \in H_d(X;\mathbb{R})$, we set

$$\|\alpha\|_1 := \inf\bigl\{|c|_1 \bigm| c \in C_d(X;\mathbb{R}),\ \partial c = 0,\ [c] = \alpha\bigr\} \in \mathbb{R}_{\geq 0}.$$

Here, $C_*(X;\mathbb{R})$ denotes the singular chain complex if X is a space; if X is a group, $C_*(X;\mathbb{R})$ can be taken to be the chain complex of the simplicial resolution or the singular chain complex of a classifying space $B\Gamma$ (these chain complexes are boundedly chain homotopy equivalent with respect to $|\cdot|_1$). Moreover, $|\cdot|_1$ denotes the ℓ^1-norm on $C_*(X;\mathbb{R})$ with respect to the basis given by all singular simplices (or all simplicial tuples, respectively).

The ℓ^1-semi-norm on $H_*(\,\cdot\,;\mathbb{R})$ is functorial in the following sense: If $f\colon X \longrightarrow Y$ is a continuous map (or group homomorphism, respectively) and $\alpha \in H_d(X;\mathbb{R})$, then

$$\bigl\|H_d(f;\mathbb{R})(\alpha)\bigr\|_1 \leq \|\alpha\|_1.$$

Definition 2.2 (*Simplicial volume* [9, 23]) The *simplicial volume* of an oriented closed connected d-manifold M is defined as

$$\|M\| := \bigl\|[M]_{\mathbb{R}}\bigr\|_1,$$

where $[M]_{\mathbb{R}} \in H_d(M;\mathbb{R})$ denotes the $\mathbb{R}$-fundamental class of M.

2.2 Bounded cohomology and duality

The bounded cohomology of groups or spaces is $H_b^*(\,\cdot\,;\mathbb{R}) := H^*\bigl(C_*(\,\cdot\,;\mathbb{R})^\#\bigr)$, where $C_*(\,\cdot\,;\mathbb{R})^\#$ denotes the topological dual with respect to $|\cdot|_1$ (the latter is introduced in Definition 2.1). Forgetting boundedness induces a natural transformation $\operatorname{comp}^*\colon H_b^*(\,\cdot\,;\mathbb{R}) \Longrightarrow H^*(\,\cdot\,;\mathbb{R})$, the *comparison map*. Classes in the image of the comparison map are called *bounded*. Evaluating cocycles on cycles induces a Kronecker product $\langle\,\cdot\,,\,\cdot\,\rangle$, which is compatible with the comparison map.

Proposition 2.3 (Duality principle [9, p. 16]) *Let $d \in \mathbb{N}$, let X be a space/group, and let $\alpha \in H_d(X;\mathbb{R})$. Then*

$$\|\alpha\|_1 = \sup\Bigl\{\frac{1}{\|\varphi\|_\infty} \Bigm| \varphi \in H_b^d(X;\mathbb{R}),\ \langle\varphi,\alpha\rangle = 1\Bigr\}.$$

We will also use the following version of the duality principle:

Corollary 2.4 *Let $d \in \mathbb{N}$, let X be a space/group with $\dim_{\mathbb{R}} H_d(X;\mathbb{R}) < \infty$. Then*

$$B^d(X;\mathbb{R}) = \bigl\{\varphi \in H^d(X;\mathbb{R}) \bigm| \forall_{\alpha \in N_d(X;\mathbb{R})}\quad \langle\varphi,\alpha\rangle = 0\bigr\}.$$

Proof By the duality principle (Proposition 2.3), we have

$$\begin{aligned} N_d(X;\mathbb{R}) &= \bigl\{\alpha \in H_d(X;\mathbb{R}) \bigm| \forall_{\varphi \in H_b^d(X;\mathbb{R})}\quad \langle\varphi,\alpha\rangle = 0\bigr\} \\ &= \bigl\{\alpha \in H_d(X;\mathbb{R}) \bigm| \forall_{\varphi \in B^d(X;\mathbb{R})}\quad \langle\varphi,\alpha\rangle = 0\bigr\}. \end{aligned}$$

Because $H_d(X;\mathbb{R})$ is finite-dimensional and $H^d(X;\mathbb{R}) \cong_{\mathbb{R}} \operatorname{Hom}_{\mathbb{R}}(H_d(X;\mathbb{R}),\mathbb{R})$ via the evaluation map, the annihilator

$$\bigl\{\varphi \in H^d(X;\mathbb{R}) \bigm| \forall_{\alpha \in N_d(X;\mathbb{R})}\quad \langle\varphi,\alpha\rangle = 0\bigr\}$$

of this null space coincides with $B^d(X;\mathbb{R})$. $\square$

2.3 Normed Thom realisation

Classical Thom realisation and surgery allow us to construct manifolds from group homology classes with controlled simplicial volume:

Theorem 2.5 ([11, (proof of) Theorem 8.1]) *Let $d \in \mathbb{N}_{\geq 4}$. Then, there exists a constant $K_d \in \mathbb{N}_{>0}$ with the following property: If Γ is a finitely presented group and $\alpha \in H_d(\Gamma; \mathbb{R})$ is an integral class, then there exists an oriented closed connected d-manifold M with $\pi_1(M) \cong \Gamma$ and a $K \in \{1, \dots, K_d\}$ such that*

$$\|M\| = K \cdot \|\alpha\|_1.$$

3 Gaps via rationality

In this section, we prove Theorem 1.7. More precisely, we show:

- the equivalence 1 $\Longleftrightarrow$ 2 in Sect. 3.1 via the mapping theorem and normed Thom realisation;
- the equivalence 2 $\Longleftrightarrow$ 3 in Sect. 3.2 through basic properties of integer lattices in vector spaces;
- the equivalence 3 $\Longleftrightarrow$ 4 in Sect. 3.3 by the duality principle;

3.1 The integral lattice

Let X be a space or a group. A class in $H_d(X; \mathbb{R})$ is called *integral* if it is in the image of the change of coefficients map $H_d(X; \mathbb{Z}) \longrightarrow H_d(X; \mathbb{R})$. We write

$$Z_d(X) := \left\{\alpha \in H_d(X; \mathbb{R}) \;\middle|\; \alpha \text{ is integral}\right\}$$

for the $\mathbb{Z}$-submodule of $H_d(X; \mathbb{R})$ of integral classes. Normed Thom realisation shows that $\mathrm{SV}_\Gamma(d)$ is roughly the same as $\{\|\alpha\|_1 \mid \alpha \in Z_d(\Gamma)\}$:

Proof of Theorem 1.7, 2 $\Longrightarrow$ 1 Let $M \in \mathrm{Mfd}_d$ satisfying $\pi_1(M) \cong \Gamma$ and let $f : M \longrightarrow B\Gamma$ be the classifying map. As f induces an isomorphism on the level of fundamental groups, we obtain from the mapping theorem [9, Section 3.1] and the duality principle (Proposition 2.3) that

$$\|M\| = \big\|[M]_\mathbb{R}\big\|_1 = \big\|H_d(f; \mathbb{R})([M]_\mathbb{R})\big\|_1.$$

Moreover, $[M]_\mathbb{R} \in H_d(M; \mathbb{R})$ is an integral class and so $H_d(f; \mathbb{R})([M]_\mathbb{R}) \in Z_d(\Gamma)$.

Hence, if $\|\cdot\|_1$ has a gap at zero on $Z_d(\Gamma)$, then also $\mathrm{SV}_\Gamma(d)$ has a gap at zero. □

Proof of Theorem 1.7, 1 $\Longrightarrow$ 2 Let $\mathrm{SV}_\Gamma(d)$ have a gap c at zero and let $K_d \in \mathbb{N}_{>0}$ be a constant for normed Thom realisation in dimension d (Theorem 2.5). Then c/K_d is a gap for $\|\cdot\|_1$ on $Z_d(\Gamma)$:

Let $\alpha \in Z_d(\Gamma)$ with $\|\alpha\|_1 \neq 0$. Normed Thom realisation shows that there exists an $M \in \mathrm{Mfd}_d$ with $\pi_1(M) \cong \Gamma$ and $\|M\| = K \cdot \|\alpha\|_1$ with $K \in \{1, \dots, K_d\}$. In particular, we obtain $\|\alpha\|_1 \geq \|M\|/K \geq c/K_d$, as claimed. □

Remark 3.1 (*Lattices*) Let V be a finite-dimensional $\mathbb{R}$-vector space. Then V carries a canonical topology (induced by any Euclidean inner product on V). A *lattice* in V is a $\mathbb{Z}$-submodule that is discrete with respect to the canonical topology. We recall two basic facts on lattices:

- If $\|\cdot\|$ is a norm on V and $L \subset V$ is a lattice, then $\{\|x\| \mid x \in L \setminus \{0\}\}$ has a gap at zero. (The corresponding statement for *semi*-norms is false, in general: The semi-norm $x \mapsto |x_1 - \sqrt{2} \cdot x_2|$ on $\mathbb{R}^2$ does not have a gap on the standard lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. Even worse, this semi-norm is non-degenerate on $\mathbb{Z}^2$.)
- If $L \subset V$ is a cocompact lattice, then V has an $\mathbb{R}$-basis consisting of elements of L.

Our main example is: Let $d \in \mathbb{N}$ and let X be a space/group satisfying $\dim_{\mathbb{R}} H_d(X;\mathbb{R}) < \infty$. Then, by the universal coefficient theorem, $Z_d(X)$ is a cocompact lattice in $H_d(X;\mathbb{R})$.

3.2 Rationality of the zero-norm subspace

In the following, we consider the quotient space $Q_d(\Gamma;\mathbb{R}) := H_d(\Gamma;\mathbb{R})/N_d(\Gamma;\mathbb{R})$. By construction, the quotient semi-norm of $\|\cdot\|_1$ on $Q_d(\Gamma;\mathbb{R})$ is a norm and the canonical projection $\pi\colon H_d(\Gamma;\mathbb{R}) \longrightarrow Q_d(\Gamma;\mathbb{R})$ is isometric. We denote the quotient norm also by $\|\cdot\|_1$.

*Proof of Theorem*1.7, $3 \Longrightarrow 2$ Let $N_d(\Gamma;\mathbb{R})$ be rational in $H_d(\Gamma;\mathbb{R})$. Because $N_d(\Gamma;\mathbb{R})$ is rational and $Z_d(\Gamma)$ is a lattice in $H_d(\Gamma;\mathbb{R})$ (Remark 3.1), the image $\pi(Z_d(\Gamma))$ is a lattice in the finite-dimensional $\mathbb{R}$-vector space $Q_d(\Gamma;\mathbb{R})$ [2, Corollary 10.3]. In particular, the norm $\|\cdot\|_1$ has a gap at 0 on $\pi(Z_d(\Gamma))$ (Remark 3.1). Therefore, also

$$\bigl\{\|\alpha\|_1 \bigm| \alpha \in Z_d(\Gamma)\bigr\} = \bigl\{\|\pi(\alpha)\|_1 \bigm| \alpha \in Z_d(\Gamma)\bigr\} = \bigl\{\|\beta\|_1 \bigm| \beta \in \pi(Z_d(\Gamma)\bigr\}$$

has a gap at zero. □

*Proof of Theorem*1.7, $2 \Longrightarrow 3$ Let $\|\cdot\|_1$ have a gap c at zero on $Z_d(\Gamma)$. We show that $N_d(\Gamma;\mathbb{R})$ is rational in $H_d(\Gamma;\mathbb{R})$:

Because $Z_d(\Gamma)$ is a lattice in $H_d(\Gamma;\mathbb{R})$ (Remark 3.1), there exists a tuple $(v_1,\dots,v_n)$ of elements of $Z_d(\Gamma)$ that is an $\mathbb{R}$-basis for $H_d(\Gamma;\mathbb{R})$ (Remark 3.1). Let $\alpha \in N_d(\Gamma;\mathbb{R})$. We write

$$\alpha = \sum_{j=1}^{n} \lambda_j \cdot v_j$$

with $\lambda_1,\dots,\lambda_n \in \mathbb{R}$. Given $N \in \mathbb{N}_{>0}$, simultaneous Dirichlet approximation [24, Theorem II.1.A] shows that there exist $p_{N,1},\dots,p_{N,n} \in \mathbb{Z}$ and $q_N \in \{1,\dots,N\}$ with

$$\forall_{j\in\{1,\dots,n\}} \quad \left|\lambda_j - \frac{p_{N,j}}{q_N}\right| < \frac{1}{q_N \cdot N^{1/n}}.$$

Then the class $\alpha_N := \sum_{j=1}^{n} p_{N,j} \cdot v_j$ lies in $Z_d(\Gamma)$ and

$$\|q_N \cdot \alpha - \alpha_N\|_1 \le \sum_{j=1}^{n} |q_N \cdot \lambda_j - p_{N,j}| \cdot \|v_j\|_1 \le \sum_{j=1}^{n} \frac{1}{N^{1/n}} \cdot \|v_j\|_1.$$

Because $q_N \cdot \alpha \in N_d(\Gamma;\mathbb{R})$, we obtain $\|\alpha_N\|_1 = \|q_N \cdot \alpha - \alpha_N\|_1$ and so the previous estimate and the gap c show that $\|\alpha_N\|_1 = 0$ for all large enough N. Hence, $\alpha_N \in N_d(\Gamma;\mathbb{R}) \cap Z_d(\Gamma)$ and $1/q_N \cdot \alpha_N \in N_d(\Gamma;\mathbb{R}) \cap H_d(\Gamma;\mathbb{Q})$.

We now consider the standard topology on the finite-dimensional $\mathbb{R}$-vector space $H_d(\Gamma;\mathbb{R})$. Then the choice of the approximating coefficients shows that $\lim_{N\to\infty} 1/q_N \cdot \alpha_N = \alpha$.

In conclusion, α lies in the closure of $N_d(\Gamma;\mathbb{R}) \cap H_d(\Gamma;\mathbb{Q})$ with respect to the standard topology. As $\mathbb{R}$-subspaces of finite-dimensional $\mathbb{R}$-vector spaces are closed in the standard topology, α lies in the $\mathbb{R}$-subspace generated by $N_d(\Gamma;\mathbb{R}) \cap H_d(\Gamma;\mathbb{Q})$. This shows that $N_d(\Gamma;\mathbb{R})$ indeed is rational. □

3.3 Rationality of the bounded subspace

Proof of Theorem 1.7, 3 $\iff$ 4 This is a consequence of Corollary 2.4: By linear algebra over $\mathbb{Q}$, an $\mathbb{R}$-subspace of $H_d(\Gamma;\mathbb{R})$ is rational if and only if its annihliator is rational in the dual $\mathbb{R}$-vector space. Thus, $N_d(\Gamma;\mathbb{R})$ is rational in $H_d(\Gamma;\mathbb{R})$ if and only if $B^d(\Gamma;\mathbb{R})$ is rational in the dual $H^d(\Gamma;\mathbb{R})$ of $H_d(\Gamma;\mathbb{R})$. □

4 Inheritance properties

We prove the inheritance properties from Sect. 1.3.

Lemma 4.1 (Free products) *Let* $d \in \mathbb{N}_{\geq 4}$. *Then* $\operatorname{Gap}(d)$ *is closed under taking (finite) free products.*

Proof Let $\Gamma, \Lambda \in \operatorname{Gap}(d)$. We show that $\Gamma * \Lambda \in \operatorname{Gap}(d)$:

With Γ and Λ also $\Gamma * \Lambda$ is of type FH_d (finitely presented groups are closed under free products and the homology is finite-dimensional by the Mayer–Vietoris sequence). By Theorem 1.7, we know that $N_d(\Gamma;\mathbb{R})$ and $N_d(\Lambda;\mathbb{R})$ are rational and it suffices to show that $N_d(\Gamma * \Lambda;\mathbb{R})$ is rational:

The inclusions/projections i, j and p, q, respectively, of the summands of the free product $\Gamma * \Lambda$ induce the Mayer–Vietoris $\mathbb{R}$-isomorphism $\varphi\colon H_d(\Gamma * \Lambda;\mathbb{R}) \longrightarrow H_d(\Gamma;\mathbb{R}) \oplus H_d(\Lambda;\mathbb{R})$. Under this isomorphism, $N_d(\Gamma * \Lambda;\mathbb{R})$ corresponds to $N_d(\Gamma;\mathbb{R}) \oplus N_d(\Lambda;\mathbb{R})$: If $\alpha \in N_d(\Gamma * \Lambda;\mathbb{R})$, then

$$\left\| H_d(p;\mathbb{R})(\alpha) \right\|_1 \leq \|\alpha\|_1 = 0 \quad \text{and} \quad \left\| H_d(q;\mathbb{R})(\alpha) \right\|_1 \leq \|\alpha\|_1 = 0$$

and so $\varphi(\alpha) \in N_d(\Gamma;\mathbb{R}) \oplus N_d(\Lambda;\mathbb{R})$. Conversely, if $(\alpha, \beta) \in N_d(\Gamma;\mathbb{R}) \oplus N_d(\Lambda;\mathbb{R})$, then

$$\left\| \varphi^{-1}(\alpha, \beta) \right\|_1 = \left\| H_d(i;\mathbb{R})(\alpha) + H_d(j;\mathbb{R})(\beta) \right\|_1 \leq \|\alpha\|_1 + \|\beta\|_1 \leq 0$$

and thus $\varphi^{-1}(\alpha, \beta) \in N_d(\Gamma * \Lambda;\mathbb{R})$.

Because φ maps rational subspaces to rational subspaces, also $N_d(\Gamma * \Lambda;\mathbb{R})$ is rational. □

Lemma 4.2 (Graphs of groups) *Let* $d \in \mathbb{N}_{\geq 4}$, *let* G *be a graph of groups on a finite graph* (V, E), *whose vertex groups* $(G_v)_{v \in V}$ *lie in* $\operatorname{Gap}(d)$ *and whose edge groups* $(G_e)_{e \in E}$ *are amenable. Let* Γ *be the fundamental group of* G. *If* Γ *is of type* FH_d, *then also* $\Gamma \in \operatorname{Gap}(d)$.

Proof By Theorem 1.7, $N_d(G_v;\mathbb{R})$ is rational for all $v \in V$ and it suffices to show that $N_d(\Gamma;\mathbb{R})$ is rational.

We consider the following commutative diagram:

$$\begin{array}{ccc} H_b^d(\Gamma;\mathbb{R}) & \xrightarrow{F_b} & \bigoplus_{v \in V} H_b^d(G_v;\mathbb{R}) \\ \Big\downarrow \mathrm{comp}_\Gamma^d & & \Big\downarrow \bigoplus_{v \in V} \mathrm{comp}_{G_v}^d \\ H^d(\Gamma;\mathbb{R}) & \xrightarrow[F]{} & \bigoplus_{v \in V} H^d(G_v;\mathbb{R}) \end{array}$$

Here, F_b and F denote the maps induced by the inclusions of the vertex groups on bounded cohomology and cohomology, respectively. The upper horizontal arrow F_b is surjective [3]. Hence, the diagram implies that

$$F\big(B^d(\Gamma;\mathbb{R})\big) = \bigoplus_{v \in V} B^d(G_v;\mathbb{R}).$$

The hypothesis that $G_v \in \mathrm{Gap}(d)$ for all $v \in V$ shows that the right-hand side is rational. Moreover, the map F is rational because it is induced by group homomorphisms; in particular, the kernel of F is rational. Therefore, also $B^d(\Gamma; \mathbb{R})$ is rational. □

The statement of Lemma 4.2 can be generalised to uniformly boundedly acyclic edge groups by using the corresponding result on bounded cohomology of such graphs of groups [17, Theorem 8.11].

In the situation of Lemma 4.2, we have the following sufficient condition for the group Γ to be of type FH_d: By hypothesis, all vertex groups are of type FH_d. If all edge groups are of type FH_{d+1}, then the Mayer–Vietoris sequence for graphs of groups shows that Γ is of type FH_d.

Lemma 4.3 *(products) Let* $d \in \mathbb{N}_{\geq 4}$ *and let* $\Gamma \in \bigcap_{k\in\{2,\dots,d\}} \mathrm{Gap}(k)$ *and* $\Lambda \in \bigcap_{k\in\{2,\dots,d\}} \mathrm{Gap}(k)$. *Then* $\Gamma \times \Lambda \in \mathrm{Gap}(d)$.

Proof As Γ and Λ are of type FH_d, also $\Gamma \times \Lambda$ is of type FH_d (finitely presented groups are closed under finite products; and the cohomological Künneth theorem).

By Theorem 1.7, we know that $N_k(\Gamma; \mathbb{R})$ and $N_k(\Lambda; \mathbb{R})$ are rational for all $k \in \{2, \dots, d\}$ and it suffices to show that $N_d(\Gamma \times \Lambda; \mathbb{R})$ is rational:

More precisely, we show that, under the Künneth isomorphism, $N_d(\Gamma \times \Lambda; \mathbb{R})$ corresponds to

$$N := \sum_{j=0}^{d} \bigl(N_j(\Gamma; \mathbb{R}) \otimes_{\mathbb{R}} H_{d-j}(\Lambda; \mathbb{R}) + H_j(\Gamma; \mathbb{R}) \otimes_{\mathbb{R}} N_{d-j}(\Lambda; \mathbb{R})\bigr).$$

Because the Künneth isomorphism preserves rational subspaces and because $N_0(\,\cdot\,; \mathbb{R}) = 0$ and $N_1(\,\cdot\,; \mathbb{R}) = H_1(\,\cdot\,; \mathbb{R})$ are always rational, this would finish the proof.

The standard estimate for the homological cross-product (via the shuffle description of the Eilenberg–Zilber map) shows that $N \subset N_d(\Gamma \times \Lambda; \mathbb{R})$. In order to prove the converse inclusion $N_d(\Gamma \times \Lambda; \mathbb{R}) \subset N$, we proceed as follows:

We consider the bilinear form

$$\langle\,\cdot\,,\,\cdot\,\rangle \colon B^{\leq d}(\Gamma; \mathbb{R}) \times H_{\leq d}(\Gamma; \mathbb{R}) \longrightarrow \mathbb{R}.$$

The description of the bounded part from Corollary 2.4 and elementary finite-dimensional linear algebra show that there exist families $(\varphi_i)_{i\in I_1}$ in $B^{\leq d}(\Gamma; \mathbb{R})$ and $(\alpha_i)_{i\in I_1\sqcup I_0}$ in $H_{\leq d}(\Gamma; \mathbb{R})$ with the following properties:

- The family $(\alpha_i)_{i\in I_0}$ is an $\mathbb{R}$-basis of $N_{\leq d}(\Gamma; \mathbb{R})$.
- The family $(\alpha_i)_{i\in I_0\sqcup I_1}$ is an $\mathbb{R}$-basis of $H_{\leq d}(\Gamma; \mathbb{R})$.
- The family $(\varphi_i)_{i\in I_1}$ is an $\mathbb{R}$-basis of $B^{\leq d}(\Gamma; \mathbb{R})$.
- For all $i, j \in I_1$, we have

$$\langle \varphi_i, \alpha_j \rangle = \delta_{ij}.$$

Similarly, we obtain such families $(\psi_j)_{j\in J_1}$ and $(\beta_j)_{j\in J_1\sqcup J_0}$ for Λ.

Let $\alpha \in N_d(\Gamma \times \Lambda; \mathbb{R})$. Using the Künneth isomorphism, we write (where $I := I_1 \sqcup I_0$ and $J := J_1 \sqcup J_0$)

$$\alpha = \sum_{(i,j)\in I\times J} \lambda_{ij} \cdot \alpha_i \times \beta_j$$

for suitable real coefficients λ_{ij}. Let $(i_1, j_1) \in I_1 \times J_1$. Then $\lambda_{i_1,j_1} = 0$ as the following computation shows:

$$\begin{aligned}|\lambda_{i_1,j_1}| &= \Bigl|\Bigl\langle \varphi_{i_1} \times \psi_{j_1}, \sum_{(i,j)\in I\times J} \lambda_{ij} \cdot \alpha_i \times \beta_j \Bigr\rangle\Bigr| \\ &= \bigl|\langle \varphi_{i_1} \times \psi_{j_1}, \alpha\rangle\bigr| \\ &\le \|\varphi_{i_1}\|_\infty \cdot \|\psi_{j_1}\|_\infty \cdot \|\alpha\|_1 \\ &= 0\end{aligned}$$

Therefore, $\alpha \in N$. □

Lemma 4.4 (Finite index supergroups) *Let $d \in \mathbb{N}_{\ge 4}$ and let Γ be a group that contains a finite index subgroup Λ with $\Lambda \in \operatorname{Gap}(d)$. Then $\Gamma \in \operatorname{Gap}(d)$.*

Proof By Theorem 1.7, $N_d(\Lambda; \mathbb{R})$ is rational and it suffices to show that $N_d(\Gamma; \mathbb{R})$ is rational and that Γ has type FH_d:

Let $i: \Lambda \longrightarrow \Gamma$ denote the inclusion. Because $[\Gamma : \Lambda] < \infty$ and $[\Gamma : \Lambda]$ is a unit in $\mathbb{R}$, there is a homological transfer map $t_d: H_d(\Gamma; \mathbb{R}) \longrightarrow H_d(\Lambda; \mathbb{R})$, which satisfies

$$H_d(i; \mathbb{R}) \circ t_d = [\Gamma : \Lambda] \cdot \mathrm{id}_{H_d(\Gamma;\mathbb{R})} .$$

In particular, $\dim_{\mathbb{R}} H_d(\Gamma; \mathbb{R}) \le \dim_{\mathbb{R}} H_d(\Lambda; \mathbb{R}) < \infty$. Moreover, because Γ contains a finitely presented subgroup of finite index (namely Λ), also Γ is finitely presented.

We now show that $N_d(\Gamma; \mathbb{R}) = H_d(i; \mathbb{R})\bigl(N_d(\Lambda; \mathbb{R})\bigr)$: Clearly, the right-hand side is contained in $N_d(\Gamma; \mathbb{R})$. Conversely, let $\alpha \in N_d(\Gamma; \mathbb{R})$. We consider $\widetilde{\alpha} := 1/[\Gamma : \Lambda] \cdot t_d(\alpha) \in H_d(\Lambda; \mathbb{R})$. The explicit construction of the transfer t_d through lifts of singular simplices shows that

$$\|\widetilde{\alpha}\|_1 \le \frac{1}{[\Gamma : \Lambda]} \cdot [\Gamma : \Lambda] \cdot \|\alpha\|_1 = 0.$$

Hence, $\widetilde{\alpha} \in N_d(\Lambda; \mathbb{R})$. By construction,

$$\alpha = \frac{1}{[\Gamma : \Lambda]} \cdot H_d(i; \mathbb{R})\bigl(t_d(\alpha)\bigr) = H_d(i; \mathbb{R})(\widetilde{\alpha}).$$

This proves the claimed description of $N_d(\Gamma; \mathbb{R})$.

Finally, because $H_d(i; \mathbb{R})$ preserves rational subspaces, the rationality of the subspace $N_d(\Lambda; \mathbb{R})$ implies the rationality of $N_d(\Gamma; \mathbb{R})$.

Alternatively, one could also use the cohomological transfer in (bounded) cohomology. □

Lemma 4.5 (Epis on bounded cohomology) *Let $d \in \mathbb{N}_{\ge 4}$, let $f: \Gamma \longrightarrow \Lambda$ be a group homomorphism that induces a surjection $H_b^d(f; \mathbb{R}): H_b^d(\Lambda; \mathbb{R}) \longrightarrow H_b^d(\Gamma; \mathbb{R})$, let $\Lambda \in \operatorname{Gap}(d)$, and let Γ be of type FH_d. Then $\Gamma \in \operatorname{Gap}(d)$.*

Proof By Theorem 1.7, $B^d(\Lambda; \mathbb{R})$ is rational and it suffices to show that $B^d(\Gamma; \mathbb{R})$ is rational. The commutative diagram

$$\begin{array}{ccc} H_b^d(\Lambda; \mathbb{R}) & \xrightarrow{H_b^d(f;\mathbb{R})} & H_b^d(\Gamma; \mathbb{R}) \\ {\scriptstyle \mathrm{comp}_\Lambda^d} \downarrow & & \downarrow {\scriptstyle \mathrm{comp}_\Gamma^d} \\ H^d(\Lambda; \mathbb{R}) & \xrightarrow[H^d(f;\mathbb{R})]{} & H^d(\Gamma; \mathbb{R}) \end{array}$$

and the surjectivity of the upper arrow $H_b^d(f; \mathbb{R})$ imply that

$$B^d(\Gamma; \mathbb{R}) = H^d(f; \mathbb{R})\big(B^d(\Lambda; \mathbb{R})\big).$$

As $B^d(\Lambda; \mathbb{R})$ is rational in $H^d(\Lambda; \mathbb{R})$ and as the induced homomorphism $H^d(f; \mathbb{R})$ preserves rationality, we obtain that also $B^d(\Gamma; \mathbb{R})$ is rational. □

Lemma 4.6 (Boundedly acyclic extensions) *Let* $d \in \mathbb{N}_{\geq 4}$*, let* $1 \longrightarrow A \longrightarrow \Gamma \longrightarrow \Lambda \longrightarrow 1$ *be an extension of groups with boundedly acyclic kernel* A*, let* $\Lambda \in \mathrm{Gap}(d)$*, and let* Γ *be of type* FH_d*. Then* $\Gamma \in \mathrm{Gap}(d)$.

Proof Let $\pi\colon \Gamma \longrightarrow \Lambda$ be the epimorphism of the given short exact sequence. Because $\ker \pi \cong A$ is boundedly acyclic, the map $H_b^d(\pi; \mathbb{R})\colon H_b^d(\Lambda; \mathbb{R}) \longrightarrow H_b^d(\Gamma; \mathbb{R})$ is an isomorphism; this can be seen from the Hochschild–Serre spectral sequence in bounded cohomology [21, Chapter 12]. Therefore, Lemma 4.5 applies. □

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