Verdier duality for general coefficient systems

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Local to global phenomena are omnipresent in mathematics, and since the appearance of the work of Grothendieck and his school it has been settled that the best way to treat such problems formally is via the theory of sheaves. It has been noticed already many years ago that sheaves are natural coefficients for cohomology theories defined on geometric objects of any kind, which means that they show their full power when set within a homotopical context. Therefore, one is somehow forced to move to the world of higher categories to work efficiently on the subject. This thesis essentially revolves around the theory of sheaves with values in ∞-categories, with a particular attention to its manifestations in topology and differential geometry.

The initial sparkle that paved the way for the making of the present work was the intuition that Lurie's version of *Verdier duality*, expressed as an equivalence between sheaves and cosheaves, would have to be the starting point in the setup of the whole theory. The term Verdier duality is also often used in more classical literature to refer to an equivalence between some derived categories of constructible sheaves and their opposite: we will conclude this thesis by providing a generalization of this result. For these two reasons, we felt obliged to pay tribute to Jean-Louis Verdier in the choice of the title of this work.

Verdier duality will make its initial appearence in the first chapter, which is devoted to the construction of the *six functor formalism* for sheaves on locally compact Hausdorff spaces. We will use it to transport the pushforward and pullback of cosheaves to sheaves, thus providing an abstract description of the well-known *pushforward with proper support* and *exceptional pullback*. Alongside with this interaction between sheaves and cosheaves, we will consider Lurie's *tensor product of cocomplete ∞-categories* as a second fundamental tool to build our six operations. This will allow us to work with a surprisingly vast class of stable coefficients, and to formulate Künneth and projection formulas in a both unusual and convenient way.

In the second chapter, Verdier duality will be used to produce an actual duality on certain ∞-categories of *constructible sheaves*. More precisely, we will consider stratified spaces equipped with *conically smooth atlases*, an extension of smooth atlases to the setting of singular spaces introduced by Ayala, Francis and Tanaka. The theory of conically smooth structures provides a procedure that allows to functorially resolve singularities to smooth manifolds with corners. We will make use of this procedure to perform some computations of *exit path ∞-categories* needed to obtain the aforementioned duality theorem. The reader who is not familiar with conically smooth spaces might wonder how our results interplay with more classical approaches to smoothly stratified spaces. To resolve this reasonable doubt, we also provide a proof of a conjecture of Ayala, Francis and Tanaka written in collaboration with Guglielmo Nocera, which asserts that any *Whitney stratified space* admits a conically smooth structure.

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Chapter 1

Six functor formalism

1.1 Introduction

Tell all the truth but tell it slant –
Success in Circuit lies
Too bright for our infirm Delight
The Truth's superb surprise
As Lightning to the Children eased
With explanation kind
The Truth must dazzle gradually
Or every man be blind –

Emily Dickinson

One of the most complete and general reference dealing with the *six functor formalism* for sheaves on topological spaces is Masaki Kashiwara and Pierre Schapira's seminal book *Sheaves on Manifolds* [KS90]. However, for technical reasons related to the construction of derived functors, the authors there restrict themselves to bounded derived categories of sheaves of R-modules, where R is assumed to have finite global dimension. From a modern perspective, considering ∞-categorical enhancements of derived categories, this can be regarded as the full subcategory of hypercomplete sheaves with values in $D(R)$ spanned by bounded objects.

For many applications though, one would like to be able to consider *non-hypercomplete* sheaves with values in *unbounded* derived categories of *any* ring. On the other hand, more recent papers such as [RS18], [JT17], [Jin19] and [Jin20], justify the need of even further generalizations to sheaves of modules over *ring spectra*, in order to apply the power of six functors to generalized cohomology theories. The work of Voevodsky on *stable motivic homotopy theory* has provided an analog of such constructions in the world of algebraic geometry (see [CD19] for a textbook source on the subject), whereas topologists have succeded only partially in this direction by introducing *parametrized spectra* (see [MS06]), which correspond to *locally constant* sheaves of spectra. In this thesis we exploit the power of the now established theory of ∞-categories (as developed for example in [Lur09] or [Cis19]) to extend the six functor formalism on locally compact Hausdorff spaces to a much broader setting.

Let $f : X \to Y$ be a continuous map between locally compact Hausdorff topological spaces and $\mathcal{C}$ is any stable bicomplete (i.e. complete and cocomplete) ∞-category. For any such map, we construct adjunctions

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}(Y;\mathcal{C}) \arrow[r, bend left, "f^*_{\mathcal{C}}"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \arrow[l, bend left, "f_*^{\mathcal{C}}"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \arrow[r, bend left, "f_!^{\mathcal{C}}"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathrm{hv}(Y;\mathcal{C}). \arrow[l, bend left, "f^!_{\mathcal{C}}"] \end{tikzcd}$$

To do this, we make use of Lurie's Verdier duality equivalence ([Lur17, Theorem 5.5.5.1])

$$\mathbb{D}_{\mathcal{C}} : \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \xrightarrow{\simeq} \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C})$$

where the target is the ∞-category of $\mathcal{C}$-valued *cosheaves* on X, i.e. $\mathcal{S}\mathrm{hv}(X;\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. The adjunctions above are then defined to satisfy natural equivalences

$$\mathbb{D}_{\mathcal{C}} f_!^{\mathcal{C}} \simeq (f_*^{\mathcal{C}^{\mathrm{op}}})^{\mathrm{op}} \mathbb{D}_{\mathcal{C}} \quad \mathbb{D}_{\mathcal{C}} f^!_{\mathcal{C}} \simeq (f^*_{\mathcal{C}^{\mathrm{op}}})^{\mathrm{op}} \mathbb{D}_{\mathcal{C}}.$$

Notice that, since we do not require $\mathcal{C}$ to be presentable, the existence of $f^*_{\mathcal{C}}$ is not at all obvious (see the discussion in Remark 1.2.34), and so we will have to work a bit harder than one might expect. Nevertheless, even though the presentability assumption will be enough for applications, we would like to point out that our efforts to make the results in this chapter as general as possible are not vain, and actually lead to many advantages. First of all, with our definition of $f_!^{\mathcal{C}}$, it is basically immediate to verify that the lower shrieks are functorial with respect to compositions of continuous maps (see Lemma 1.6.2). Moreover, by working with a class of coefficients closed under the operation of passing to the opposite category, we do not break the symmetry which comes from Verdier duality, meaning that whenever we prove some result involving the functors $f_*^{\mathcal{C}}$ and $f^*_{\mathcal{C}}$ that is true for all $\mathcal{C}$ stable and bicomplete, we immediately obtain a dual theorem involving the functors $f_!^{\mathcal{C}}$ and $f^!_{\mathcal{C}}$, and viceversa. Another way to put it is that our formalism applies with no distinctions to sheaves or cosheaves: this will be used in the second chapter in which we will prove a duality theorem for constructible sheaves on a conically smooth stratified space, as we will need to extend some results about constructible sheaves, such as homotopy invariance (see [Hai20]) or the exodromy equivalence (see [PT22]), to constructible cosheaves. It is also worth noticing that, even if one would restrict to presentable coeffients, it would nevertheless be desirable to have formulas such as $f_!^{\mathcal{S}\mathrm{p}} \otimes \mathcal{C} \simeq f_!^{\mathcal{C}}$, and to get this one would still need to verify everything we prove in Section 5.

We try to outline the key ingredients that allow us to work with non-presentable coefficients. As we explained above, the main difficulty lies in showing the existence of the pullback functor $f^*_{\mathcal{C}}$. The main tool we employ to carry out this purpose is Lurie's tensor product of cocomplete ∞-categories as defined in [Lur17, 4.8.1]: in particular, a property of this tensor product that we will use over and over is that it preserves adjunctions between cocontinuous functors (see Remark 1.2.2). We show in Lemma 1.2.12 that it restricts to a monoidal structure on $\mathcal{C}\mathrm{ocont}^{st}_{\infty}$ (i.e the ∞-category of stable cocomplete ∞-categories with cocontinuous funtors between them) and use it to formulate the following theorem.

Theorem 1.1.1 (Corollary 1.5.16)**.** Let $\mathcal{C}$ be a stable bicomplete ∞-category. Then there is an equivalence

$$\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \simeq \mathcal{S}\mathrm{hv}(X;\mathcal{C})$$

where $\otimes$ on the left-hand side denotes the tensor product of stable cocomplete ∞-categories.

Theorem 1.1.1 will play a crucial role in what follows, because it will allow us to reduce a lot of arguments involving cocontinous functors to the case of sheaves of spectra. To prove

Theorem 1.1.1, we start by observing in Theorem 1.5.15 that the model of $\mathcal{K}$-sheaves (see [Lur09, Theorem 7.3.4.9]) implies that $\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})$ is a strongly dualizable object in $\mathcal{C}\mathrm{ocont}_\infty^{st}$, where $\mathcal{S}\mathrm{p}$ denotes the ∞-category of spectra (see also [Lur16, Proposition 21.1.7.1]). Hence, for $\mathcal{C}$ any stable and cocomplete ∞-category, we get an equivalence

$$\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})\otimes\mathcal{C}\simeq\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C}).$$

Combined with Verdier duality gives Theorem 1.1.1.

Having Theorem 1.1.1 at hand, the question of constructing $f^*_{\mathcal{C}}$ can be reduced to the case of sheaves of spectra, where we can directly use the existence of sheafification by the presentability of $\mathcal{S}\mathrm{p}$. More precicely, we first observe that any map f can be factored as the composition of a closed immersion, an open immersion and a proper map (see factorization (1.5.1)), and then prove the existence of a left adjoint to $f^{\mathcal{C}}_*$ in these separate cases. When f is an open immersion this is done easily in Lemma 1.5.19, and the only non-trivial part constists in the following theorem.

Theorem 1.1.2 (Lemma 1.5.14, Proposition 1.5.18)**.** Let $f : X \to Y$ be a proper map between locally compact Hausdorff topological spaces. Then the pushforward

$$f^{\mathcal{C}}_* : \mathcal{S}\mathrm{hv}(X;\mathcal{C})\to\mathcal{S}\mathrm{hv}(Y;\mathcal{C})$$

preserves colimits. Furthermore, there is a commutative square

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})\otimes\mathcal{C} \arrow[r, "\simeq"] \arrow[d, "f^{\mathcal{S}\mathrm{p}}_*\otimes\mathcal{C}"] & \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \arrow[d, "f^{\mathcal{C}}_*"] \\
\mathcal{S}\mathrm{hv}(Y;\mathcal{S}\mathrm{p})\otimes\mathcal{C} \arrow[r, "\simeq"] & \mathcal{S}\mathrm{hv}(Y;\mathcal{C}).
\end{tikzcd}$$

The proof of Theorem 1.1.2 essentially consists of providing a convenient description of $f^{\mathcal{C}}_*$ through the model of $\mathcal{K}$-sheaves, which is easily seen to preserve colimits and to be compatible with Verdier duality. We then achieve our final goal observing that, since $f^{\mathcal{S}\mathrm{p}}_*\otimes\mathcal{C}$ admits a left adjoint of the form $f^*_{\mathcal{S}\mathrm{p}}\otimes\mathcal{C}$, the same is true for $f^{\mathcal{C}}_*$. In particular by taking f to be the projection $X\to *$, we see that the global section functor

$$\mathcal{S}\mathrm{hv}(X;\mathcal{C})\to\mathcal{C}$$

admits a left adjoint. As a consequence, using the results in [Cis19, 6.7], we show in Theorem 1.5.21 that the inclusion of $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ in $\mathcal{C}$-valued presheaves on X admits a left adjoint.

The discussion above involves only the four functors $f^{\mathcal{C}}_*$, $f^*_{\mathcal{C}}$, $f^{\mathcal{C}}_!$ and $f^!_{\mathcal{C}}$, but what about the other two? Our first observation is that, a priori, there is no need to require our category of coefficients to have a monoidal structure to make sense of things like projection formulas or Künneth formulas. To be more precise, one can show that there is a functor

$$\begin{aligned}
\mathcal{S}\mathrm{hv}(X;\mathcal{C})\times\mathcal{S}\mathrm{hv}(Y;\mathcal{D}) &\longrightarrow \mathcal{S}\mathrm{hv}(X\times Y;\mathcal{S}\mathrm{p})\otimes(\mathcal{C}\otimes\mathcal{D}) \\
(F,G) &\longmapsto F\boxtimes G
\end{aligned}$$

which preserves colimits in both variables and induces an equivalence

$$\mathcal{S}\mathrm{hv}(X;\mathcal{C})\otimes\mathcal{S}\mathrm{hv}(Y;\mathcal{D})\simeq\mathcal{S}\mathrm{hv}(X\times Y;\mathcal{S}\mathrm{p})\otimes(\mathcal{C}\otimes\mathcal{D}).$$

Taking $X = Y$ and composing with $\Delta^*_{\mathcal{S}\mathrm{p}}\otimes(\mathcal{C}\otimes\mathcal{D})$, where $\Delta : X\hookrightarrow X\times X$ is the diagonal embedding, we get a variablewise colimit preserving functor denoted as

$$\begin{aligned}
\mathcal{S}\mathrm{hv}(X;\mathcal{C})\times\mathcal{S}\mathrm{hv}(Y;\mathcal{D}) &\longrightarrow \mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})\otimes(\mathcal{C}\otimes\mathcal{D}) \\
(F,G) &\longmapsto F\otimes G
\end{aligned}$$

(see Construction 1.2.25 and Remark 1.5.22 for more details). For this kind of tensor product of sheaves, we prove the following formulas.

Theorem 1.1.3 (Corollary 1.2.31, Proposition 1.6.12, Proposition 1.6.11)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be stable and bicomplete ∞-categories. Then we have the following functorial identifications

$$f^*_{\mathcal{C}\otimes\mathcal{D}}(F \otimes G) \simeq f^*_{\mathcal{C}}F \otimes f^*_{\mathcal{D}}G$$
$$f_!^{\mathcal{C}\otimes\mathcal{D}}(F \otimes f^*_{\mathcal{D}}G) \simeq f_!^{\mathcal{C}}F \otimes G$$
$$(f \times g)_!^{\mathcal{C}\otimes\mathcal{D}}(F \boxtimes G) \simeq f_!^{\mathcal{C}}F \boxtimes g_!^{\mathcal{D}}G.$$

In particular, when $\mathcal{C}$ admits a monoidal structure whose tensor preserves colimits in both variables, one obtains a cocontinuous functor

$$\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes (\mathcal{C} \otimes \mathcal{C}) \to \mathcal{S}\mathrm{hv}(X;\mathcal{C})$$

whose composition with (1.2.27) induces a monoidal structure on $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$: this way we can deduce all the analogous formulas in the monoidal setting. If $\mathcal{C}$ is also closed, we deduce their dual versions involving the internal homomorphism functor (see Remark 1.2.29).

We describe one last advantage of our general rendition of the six functor formalism. In Definition 1.3.12 we define *locally contractible geometric morphisms* (see also [AC21, Definition 3.2.1]). Later, in Definition 1.3.21, we specify a vast class of continuous maps between topological spaces called *shape submersions* which induce a locally contractible geometric morphism (see Corollary 1.3.26). Topological submersions are examples of such morphisms, but our definition is much more general in the sense that it does not force the fibers to be topological manifolds. Another illustrating example to keep in mind is that of the unique map $X \to *$, when X is any CW-complex. An easy implementation of our machinery generalizes [KS90, Proposition 3.3.2] and [Ver65, Section 5] beyond the case of submersive maps.

Theorem 1.1.4 (Proposition 1.6.16)**.** Let $f : X \to Y$ be a map which induces a locally contractible geometric morphism, and let $\mathcal{C}$ be a stable and bicomplete ∞-category. Then $f^!_{\mathcal{C}}$ admits a right adjoint and we have a formula

$$f^!_{\mathcal{C}}F \otimes f^*_{\mathcal{D}}G \simeq f^!_{\mathcal{C}\otimes\mathcal{D}}(F \otimes G).$$

Then, inspired by parallel results in motivic homotopy theory, we conclude the chapter by formulating and proving a relative version of Atiyah duality.

Theorem 1.1.5 (Corollary 1.7.14)**.** Let $f : X \to Y$ be a proper submersion between smooth manifolds. Denote by $\mathbb{S}_X \in \mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})$ the constant sheaf at the sphere spectrum, and by $\mathrm{Th}(-Tf)$ the sheaf the *Thom spectrum* of the virtual vector bundle $-Tf$. Then $f_\sharp(\mathbb{S}_X)$ is strongly dualizable with dual $\mathrm{Th}(-Tf)$.

1.1.1 Linear overview

We now give a linear overview of the contents of the chapter.

In section 2 we recall the definition of Lurie's tensor product of cocomplete ∞-categories and prove some of its basic properties. In particular, we will interpret the results in [Cis19, 6.7] in terms of this tensor product in Theorem 1.2.10, show that it preserves the property of being stable in Lemma 1.2.12, and show that compactly generated stable ∞-category is a strongly dualizable object in the symmetric monoidal ∞-category $\mathcal{C}\mathrm{ocont}^{st}_\infty$ in Proposition 1.2.14. Afterwards we recall the definition of sheaves and cosheaves with values in a general ∞-category and explain how Lurie's tensor product can be used to conveniently describe $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ at least when $\mathcal{C}$ is presentable. Most of the results in this section are not original, but we still felt the necessity to spend some time writing them up to make our discussion as self contained and reader friendly as possible.

In section 3 we define for any geometric morphism $\mathcal{Y} \to \mathcal{X}$ the *relative shape* $\Pi_\infty^{\mathcal{X}}(\mathcal{Y})$ as a pro-object of $\mathcal{X}$ and describe explicitly in Proposition 1.3.3 how this construction can be enhanced to a functor

$$\Pi_\infty^{\mathcal{X}} : \mathcal{T}\mathrm{op}_{/\mathcal{X}} \to \mathrm{Pro}(\mathcal{X})$$

and, even more, to a lax natural transformation between functors $\mathcal{T}\mathrm{op}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ (see Remark 1.3.7): these coherent structures with which we equip the shape will be used to prove easily that the shape is homotopy invariant in Corollary 1.3.4 and later in Proposition 1.7.7 to show that the Thom spectrum gives a natural transformation of sheaves of commutative monoids in spaces. Later we define locally contractible geometric morphisms and give a characterization in Proposition 1.3.11 which mimics the one in [Joh02, C3.3] for the locally connected case. We also show that, when f is a geometric morphism induced by a continuous map of topological spaces, the property of being locally contractible is checked more easily. Then we define shape submersions and prove in Lemma 1.3.25 a base change formula which will imply that they induce locally contractible geometric morphisms (see Corollary 1.3.26).

In section 4 we follow the approach of [Kha19] to obtain the localization sequences associated to a decomposition of a topological space into an open subset and its closed complement. Also the results here are not so new but, after section 5, they will imply that there is a recollement of $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ associated to any open-closed decomposition of X whenever $\mathcal{C}$ is stable and bicomplete, while this was previously known only for $\mathcal{C}$ presentable.

Section 5 is devoted to Verdier duality, and how it can be used to show that the pushforward $f_*^{\mathcal{C}}$ admits a left adjoint for any $\mathcal{C}$ stable and bicomplete in the way we have sketched at the beginning of the introduction.

In section 6 we develop the six functor formalism: as usual, we prove base change (Proposition 1.6.9), projection (Proposition 1.6.12) and Künneth (Proposition 1.6.11) formulas for $f_!^{\mathcal{C}}$, and discuss the properties of $f^!_{\mathcal{C}}$ when f is a shape submersion in Proposition 1.6.16.

At last, in section 7 we show how the six functor formalism can be used to express a relative version of Atiyah duality for any proper submersion between smooth manifolds.

1.2 Sheaves and tensor products

The goal of this section will be twofold: first we are going to introduce Lurie's tensor product of cocomplete ∞-categories as defined in [Lur17], and secondly we will recall the definition of sheaves and cosheaves with values in general ∞-categories. The reason why we want to spend some time discussing this matter, aside from it being interesting on its own, is that in the following sections this tensor product will prove to be an extremely convenient tool to describe some categories of sheaves and functors between them: through it we will be able to produce a vast class of *essential geometric morphisms*, we will extend easily some results regarding sheaves of spaces to sheaves with values in any presentable ∞-category, and later prove the existence of a sheafification functor when the ∞-category of coefficients is stable and bicomplete with no presentability assumption, and construct the full six-functor formalism in this setting. Most of the results in this section are not at all original and can be found for example in [Cis19], in [Lur17], or are already well known.

1.2.1 Tensor product of cocomplete ∞-categories

For the whole section we will fix two universes $\mathbf{V}$, $\mathbf{U}$ such that $\mathbf{V}$ is $\mathbf{U}$-small, and denote by $\mathcal{C}\mathrm{ocont}_\infty$ the ∞-category of $\mathbf{U}$-small ∞-categories admitting $\mathbf{V}$-small colimits, with $\mathbf{V}$-cocontinuous functors between them. For short, we will call an object of $\mathcal{C}\mathrm{ocont}_\infty$ a cocomplete ∞-category and, for any two cocomplete ∞-categories $\mathcal{C}$ and $\mathcal{D}$ we will denote by $\mathrm{Fun}_!(\mathcal{C},\mathcal{D})$ the ∞-category of functors preserving $\mathbf{V}$-small colimits.

Let $\mathcal{C}$ and $\mathcal{D}$ be cocomplete. Recall that, by [Lur09, 5.3.6], there exists a cocomplete ∞-category, denoted by $\mathcal{C}\otimes\mathcal{D}$, and a functor

$$\boxtimes : \mathcal{C}\times\mathcal{D}\to\mathcal{C}\otimes\mathcal{D},$$

which preserves colimits in both variables and such that precomposing with $\boxtimes$ gives an equivalence

$$\mathrm{Fun}_{!\times!}(\mathcal{C}\times\mathcal{D},\mathcal{E})\simeq\mathrm{Fun}_!(\mathcal{C}\otimes\mathcal{D},\mathcal{E}) \tag{1.2.1}$$

functorial on $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ cocomplete, where $\mathrm{Fun}_{!\times!}$ indicates the ∞-category of bifunctors preserving $\mathbf{V}$-small colimits in each variable. More precicely, [Lur17, Corollary 4.8.1.4] shows that this operation provides $\mathcal{C}\mathrm{ocont}_\infty$ with the structure of a symmetric monoidal ∞-category, and the inclusion of $\mathcal{C}\mathrm{ocont}_\infty$ in $\mathcal{C}\mathrm{at}_\infty$ is lax monoidal, where the latter if equipped with the cartesian monoidal structure. Since we obviously have a functorial equivalence

$$\mathrm{Fun}_{!\times!}(\mathcal{C}\times\mathcal{D},\mathcal{E})\simeq\mathrm{Fun}_!(\mathcal{C},\mathrm{Fun}_!(\mathcal{D},\mathcal{E})),$$

this monoidal structure is closed.

Remark 1.2.2. As usual, one may regard $\mathcal{C}\mathrm{ocont}_\infty$ as a $(\infty,2)$-category. It follows by (1.2.1) that, for any cocomplete ∞-category $\mathcal{C}$, tensoring with $\mathcal{C}$ actually gives rise to a 2-functor. An important consequence of this observation is that tensoring with $\mathcal{C}$ preserves adjunctions of cocontinuous functors, since any adjunction is characterized by the classical triangular identities (see for example [Cis19, Theorem 6.1.23, (v)]).

We will now present a list of results about the tensor product of cocomplete ∞-categories that will turn out to be very useful later.

Let A be a small ∞-category, $\mathcal{C}$ cocomplete. For any two objects $a\in A$ and $c\in\mathcal{C}$, denote by $a\boxtimes c=a_!c$ the left Kan extension of c along a (here we are considering a and c as functors $A^{\mathrm{op}}\leftarrow\Delta^0\to\mathcal{C}$). Thus we get a functor

$$\begin{aligned} A\times\mathcal{C} &\xrightarrow{y_{A/\mathcal{C}}} \mathrm{Fun}(A^{\mathrm{op}},\mathcal{C}) \\ (a,c) &\longmapsto a\boxtimes c \end{aligned}$$

which preserves colimits on the $\mathcal{C}$ variable that we will call the *relative Yoneda embedding.* By definition, we have a functorial equivalence

$$\mathrm{Hom}(a\boxtimes c,F)\simeq\mathrm{Hom}(c,F(a))$$

for any $F\in\mathrm{Fun}(A^{\mathrm{op}},\mathcal{C})$.

Remark 1.2.3. Recall that in a closed symmetric monoidal ∞-category, an object x is *strongly dualizable* if and only if the canonical map

$$y\otimes\underline{\mathrm{Hom}}(x,1)\to\underline{\mathrm{Hom}}(x,y) \tag{1.2.4}$$

obtained as adjoint to

$$y \otimes \underline{\mathrm{Hom}}(x, 1) \otimes x \longrightarrow y \otimes 1 \xrightarrow{\simeq} y$$

is an equivalence. In the case of $\mathcal{C}\mathrm{ocont}_\infty$, one sees easily that the map (1.2.4) can be described as induced by

$$\mathcal{D} \times \mathrm{Fun}_!(\mathcal{C}, \mathcal{S}) \simeq \mathrm{Fun}_!(\mathcal{S}, \mathcal{D}) \times \mathrm{Fun}_!(\mathcal{C}, \mathcal{S}) \to \mathrm{Fun}_!(\mathcal{C}, \mathcal{D})$$

where the last functor is given by composition.

Remark 1.2.5. By the naturality on x of the map (1.2.4), we see that the full subcategory spanned by strongly dualizable objects is closed under retracts.

Remark 1.2.6. Consider the variablewise cocontinuous functor

$$\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}) \times \mathcal{C} \to \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{C}) \tag{1.2.7}$$

obtained as the extension by colimits of the relative Yoneda embedding, and denote by $F \boxtimes c$ the image of a pair $(F, c) \in \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}) \times \mathcal{C}$. This induces a cocontinous functor

$$\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}) \otimes \mathcal{C} \to \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{C}). \tag{1.2.8}$$

By definition one has identifications

$$\mathrm{Hom}(F \boxtimes c, G) \simeq \mathrm{Hom}(F, \mathrm{Hom}_{\mathcal{C}}(c, G(-)))$$

functorially on $F \in \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S})$, $G \in \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{C})$ and $c \in \mathcal{C}$, where the hom-space on the right-hand side is taken on the ∞-category of presheaves of $\mathbf{U}$-small spaces.

A very convenient way to model the functor (1.2.7) is as follows. Let $y : \Delta^0 \hookrightarrow \mathcal{S}$ be the Yoneda embedding. Copmbining the fact that y is fully faithful and [Cis19, Proposition 6.4.12], we have

$$y_! c \circ a_! y \simeq a_!(y_! c \circ y) \simeq a_! c$$

and hence we get a commutative triangle

$$\begin{tikzcd}
\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}) \times \mathcal{C} \arrow[r, "\boxtimes"] \arrow[d, "\mathrm{id} \times y_!"'] & \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{C}) \\
\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}) \times \mathrm{Fun}_!(\mathcal{S}, \mathcal{C}) \arrow[ur, "\circ"'] &
\end{tikzcd}$$

where the vertical arrow is an equivalence and the diagonal one is given by composition. In particular, one deduces that the functor (1.2.8) can be seen as an instance of (1.2.4).

For any cocomplete ∞-category $\mathcal{D}$, precomposition with $y_{A/\mathcal{C}}$ induces a functor

$$\mathrm{Fun}(\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(A \times \mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(A, \mathcal{D}))$$

and since colimits are computed pointwise in functor categories, it restricts to

$$\mathrm{Fun}_!(\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{C}), \mathcal{D}) \to \mathrm{Fun}_!(\mathcal{C}, \mathrm{Fun}(A, \mathcal{D})). \tag{1.2.9}$$

Theorem 1.2.10. The functor (1.2.9) is an equivalence. In particular, $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S})$ is strongly dualizable in the monoidal ∞-category $\mathcal{C}\mathrm{ocont}_\infty$ with dual $\mathrm{Fun}(A, \mathcal{S})$, and thus the functor (1.2.8) is an equivalence.

Proof. A complete proof of the first statement can be found in [Cis19, 6.7]. The main ingredient of the proof is that, by [Cis19, Lemma 6.7.7], any $F \in \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{C})$ can be written canonically as

$$F = \varinjlim_{c \to F(a)} a \boxtimes c$$

where the colimit is indexed by the Grothendieck construction of the functor $(a, c) \mapsto \mathrm{Hom}_{\mathcal{C}}(c, F(a))$. Furthermore, even though this indexing category is not small a priori, [Cis19, Lemma 6.7.5] proves that it is finally small. From this one may deduce easily the theorem, in a similar spirit to how one proves that $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S})$ is the free cocompletion under small colimits of A.

To prove the last statement, we just observe that we have canonical equivalences

$$\begin{aligned}\mathrm{Fun}_!(\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{C}), \mathcal{D}) &\simeq \mathrm{Fun}_!(\mathcal{C}, \mathrm{Fun}(A, \mathcal{D}))\\ &\simeq \mathrm{Fun}_!(\mathcal{C}, \mathrm{Fun}_!(\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}), \mathcal{D}))\\ &\simeq \mathrm{Fun}_!(\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}) \otimes \mathcal{C}, \mathcal{D})\end{aligned}$$

whose composition is given by precomposing with (1.2.8), and so we may conclude by Remark 1.2.6. □

Corollary 1.2.11. Let $u : A \to B$ be a functor between small ∞-categories, and let $\mathcal{C}$ be any cocomplete ∞-category. Then we have equivalences $u_! \otimes \mathcal{C} \simeq u_!$ and $u^* \otimes \mathcal{C} \simeq u^*$. Here by an abuse of notation we write $u_!$ (u^*) to indicate both left Kan extension (restriction) along u for functors with values in $\mathcal{S}$ and in $\mathcal{C}$.

Proof. By Remark 1.2.2 and Theorem 1.2.10, we have an adjunction $u_! \otimes \mathcal{C} \dashv u^* \otimes \mathcal{C}$ of cocontinuous functors between $\mathcal{C}$-valued presheaves. By uniqueness of adjoints, it suffices to show that $u^* \otimes \mathcal{C} \simeq u^*$. But this is clear because by Remark 1.2.6 we have a commutative square

$$\begin{array}{ccc}\mathrm{Fun}(B^{\mathrm{op}}, \mathcal{S}) \times \mathcal{C} & \xrightarrow{u^* \times \mathrm{id}} & \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}) \times \mathcal{C}\\ \Big\downarrow{\scriptstyle\boxtimes} & & \Big\downarrow{\scriptstyle\boxtimes}\\ \mathrm{Fun}(B^{\mathrm{op}}, \mathcal{C}) & \xrightarrow{\quad u^* \quad} & \mathrm{Fun}(B^{\mathrm{op}}, \mathcal{C}).\end{array}$$

□

Denote by $\mathcal{C}\mathrm{ont}_\infty^{pt}$ $(\mathcal{C}\mathrm{ont}_\infty^{st})$ the full subcategory of $\mathcal{C}\mathrm{ocont}_\infty$ spanned by pointed (stable) cocomplete ∞-categories.

Lemma 1.2.12. Let $\mathcal{C}$ be any pointed (respectively stable) cocomplete ∞-category, and let $\mathcal{D}$ be cocomplete. Then $\mathcal{C} \otimes \mathcal{D}$ is pointed (respectively stable). In particular, $\mathcal{C}\mathrm{ont}_\infty^{pt}$ (respectively $\mathcal{C}\mathrm{ont}_\infty^{st}$) inherits an obvious monoidal structure from $\mathcal{C}\mathrm{ocont}_\infty$ and the inclusion in $\mathcal{C}\mathrm{ocont}_\infty$ admits a left adjoint given by tensoring with $\mathcal{S}_*$ (respectively $\mathcal{S}\mathrm{p}$).

Proof. First of all, notice that $\Delta^0 \otimes \mathcal{D} \simeq \Delta^0$. Since $\mathcal{C}$ is pointed, the zero object $\Delta^0 \to \mathcal{C}$ is simultaneously a right and a left adjoint of the unique functor $\mathcal{C} \to \Delta^0$, and thus one may tensor these two adjunctions with $\mathcal{D}$ and obtain by Corollary 1.2.11 that $\mathcal{C} \otimes \mathcal{D}$ is pointed.

Assume now that $\mathcal{C}$ is stable. Since $\mathcal{C} \otimes \mathcal{D}$ is pointed, one sees easily that the suspension functor for $\mathcal{C} \otimes \mathcal{D}$ is obtained by applying $- \otimes \mathcal{D}$ to the suspension of $\mathcal{C}$, and so it is in particular an equivalence.

To prove that last part of the statement, it suffices to show that, for any pointed (stable) and cocomplete ∞-category $\mathcal{C}$, the evaluation at S^0 (respectively $\mathbb{S}$) induces an equivalence $\mathrm{Fun}_!(\mathcal{S}_*, \mathcal{C}) \simeq \mathcal{C}$ (respectively $\mathrm{Fun}_!(\mathcal{S}\mathrm{p}, \mathcal{C}) \simeq \mathcal{C}$), but this follows easily by noticing that $\mathcal{S}_* \simeq$

$\mathrm{Ind}(\mathcal{S}_*^{fin})$ (respectively $\mathcal{S}\mathrm{p} \simeq \mathrm{Ind}(\mathcal{S}\mathrm{p}^{fin})$) and that evaluation at S^0 (respectively $\mathbb{S}$) induces an equivalence between finitely cocontinuous functors from $\mathcal{S}_*$ (respectively $\mathcal{S}\mathrm{p}^{fin}$) to $\mathcal{C}$ and $\mathcal{C}$. □

Remark 1.2.13. Let A be any small category and $\mathcal{C}$ any object of $\mathcal{C}\mathrm{ocont}_\infty^{st}$. By the previous lemma, the functor (1.2.7) factors as

$$\begin{tikzcd} \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}) \times \mathcal{C} \arrow[r, "\boxtimes"] \arrow[d, "\Sigma^\infty_+ \times \mathcal{C}"'] & \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{C}) \\ \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \times \mathcal{C} \arrow[ur, dotted, "\boxtimes^{st}"'] & \end{tikzcd}$$

inducing an equivalence

$$\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \otimes \mathcal{C}.$$

Moreover, Remark 1.2.6 implies that one has identifications

$$\mathrm{Hom}(F \boxtimes^{st} c, G) \simeq \mathrm{Hom}(F, \underline{\mathrm{Hom}}_{\mathcal{C}}(c, G(-)))$$

functorially on $F \in \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}\mathrm{p})$, $G \in \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{C})$ and $c \in \mathcal{C}$, where $\underline{\mathrm{Hom}}_{\mathcal{C}}(c, -)$ denotes the canonical enrichment of $\mathcal{C}$ in $\mathbf{U}$-small spectra, and we have a commutative triangle

$$\begin{tikzcd} \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \times \mathcal{C} \arrow[r, "\boxtimes^{st}"] \arrow[d, "\mathrm{id} \times y_!"'] & \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{C}) \\ \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \times \mathrm{Fun}_!(\mathcal{S}\mathrm{p}, \mathcal{C}) \arrow[ur, "\circ"'] & \end{tikzcd}$$

Proposition 1.2.14. Let $\mathcal{C}$ be a compactly generated stable ∞-category. Then $\mathcal{C}$ is a strongly dualizable object of $\mathcal{C}\mathrm{ocont}_\infty^{st}$.

Proof. Since $\mathcal{C}$ is stable and compactly generated, it follows that there exists a small stable ∞-category A with finite colimits such that $\mathcal{C} \simeq \mathrm{Fun}_{ex}(A^{\mathrm{op}}, \mathcal{S}\mathrm{p})$. Thus, since $\mathrm{Fun}_!(\mathcal{C}, \mathcal{S}\mathrm{p}) \simeq \mathrm{Fun}_{ex}(A, \mathcal{S}\mathrm{p})$, to prove the proposition we have to show that, for any $\mathcal{D}$ stable and cocomplete, the canonical functor

$$\mathrm{Fun}_{ex}(A^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \otimes \mathcal{D} \to \mathrm{Fun}_{ex}(A^{\mathrm{op}}, \mathcal{D})$$

is an equivalence. We first prove that the inclusion $i : \mathrm{Fun}_{ex}(A^{\mathrm{op}}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{D})$ admits a left adjoint L.

For any $a \in A$, denote by $y^{st}(a)$ the spectrally enriched representable functor associated to a, obtained as usual through the equivalence

$$\mathrm{Fun}_{ex}(A^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \simeq \mathrm{Fun}_{lex}(A^{\mathrm{op}}, \mathcal{S}). \tag{1.2.15}$$

We define L as the unique (up to a contractible space of choices) cocontinuous functor extending

$$\begin{aligned} A \times \mathcal{D} &\longrightarrow \mathrm{Fun}_{ex}(A^{\mathrm{op}}, \mathcal{D}) \\ (a, x) &\longmapsto y^{st}(a) \boxtimes^{st} x. \end{aligned}$$

Indeed, $y^{st}(a) \boxtimes^{st} x$ is exact as it can be modelled by the composition of two finite colimit preserving functors. When $\mathcal{D} = \mathcal{S}\mathrm{p}$, by (1.2.15) and the Yoneda lemma, one sees that L is left adjoint to i.

Let $\mathcal{D}$ be any stable cocomplete ∞-category. To see that L is the desired left adjoint, we observe that for any $F \in \mathrm{Fun}_{ex}(A^{\mathrm{op}}, \mathcal{D})$, $x \in \mathcal{D}$, we have functorial identifications

$$\begin{aligned}\mathrm{Hom}(y^{st}(a) \boxtimes^{st} x, F) &\simeq \mathrm{Hom}(y^{st}(a), \underline{\mathrm{Hom}}_{\mathcal{D}}(x, F(-))) \\ &\simeq \mathrm{Hom}(y(a), \mathrm{Hom}_{\mathcal{D}}(x, F(-))) \\ &\simeq \mathrm{Hom}(a \boxtimes x, F)\end{aligned}$$

where the hom-space on the right-hand side is taken on the ∞-category of presheaves of $\mathbf{U}$-small spectra on A, and the second equivalence follows by the fact that F, and hence $\underline{\mathrm{Hom}}_{\mathcal{D}}(x, F(-))$, is exact.

Now notice that $i : \mathrm{Fun}_{ex}(A^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \hookrightarrow \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}\mathrm{p})$ preserves colimits, and so, by tensoring with $\mathcal{D}$, one obtains an adjunction between cocontinuous functors

$$\begin{tikzcd} \mathrm{Fun}(A^{\mathrm{op}}, \mathcal{D}) \arrow[r, bend left, "L\otimes\mathcal{D}"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}_{ex}(A^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \otimes \mathcal{D} \arrow[l, bend left, "i\otimes\mathcal{D}"] \end{tikzcd}$$

where $i \otimes \mathcal{D}$ is fully faithful. Since $\mathrm{Fun}_{ex}(A^{\mathrm{op}}, \mathcal{S}\mathrm{p}) \otimes \mathcal{D}$ and $\mathrm{Fun}_{ex}(A^{\mathrm{op}}, \mathcal{D})$ can be respectively identified with the essential images of $(Li) \otimes \mathcal{D}$ and Li, to conclude the proof it suffices to show that the two functors are naturally equivalent, but this is true because they coincide on objects of the type $a \boxtimes x$. $\square$

Recall that an ∞-category $\mathcal{C}$ is called ***V**-presentable* (for short, when there is no possibility of confusion we will only write presentable) if there exists a $\mathbf{V}$-small ∞-category A such that $\mathcal{C}$ is a left Bousfield localization of $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S})$ by a $\mathbf{V}$-small set of morphism in $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S})$. If we furthermore assume that the localization functor $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}) \to \mathcal{C}$ is left exact, we will say that $\mathcal{C}$ is an *∞-topos*. It follows easily by this definition that any presentable ∞-category is complete and cocomplete. Presentable categories are equivalently defined as follows. Recall that, for $\mathcal{C}$ any ∞-category and S a class of morphisms in $\mathcal{C}$, we define an object $X \in \mathcal{C}$ to be *S-local* if, for every morphim $f : A \to B$ in S, the induced morphism

$$\mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$$

is invertible. Then we say that an ∞-category $\mathcal{C}$ is ***V**-presentable* is there exists a $\mathbf{V}$-small class S of morphisms in $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S})$ such that $\mathcal{C}$ is equivalent to the full subcategory of $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S})$ spanned by S-local objects.

We denote by Pr_L the full subcategory of $\mathcal{C}\mathrm{ocont}_\infty$ spanned by presentable ∞-categories and $\mathrm{Pr}_R = \mathrm{Pr}_L^{\mathrm{op}}$. Notice that, by the adjoint functor theorem (see for example [Cis19, Proposition 7.11.8]), the morphisms in Pr_L are functors which admit a right adjoint and, consequently, morphisms in Pr_R are functors which admit a left adjoint. We also denote by $\mathcal{T}\mathrm{op}$ the non full subcategory of Pr_R whose objects are ∞-topoi and morphisms are functors which admit a left exact left adjoint (such functors are called *geometric morphisms*).

Proposition 1.2.16. Let $\mathcal{C}$ and $\mathcal{D}$ be two presentable ∞-categories. Then $\mathcal{C} \otimes \mathcal{D}$ is presentable and there a canonical equivalence $\mathcal{C} \otimes \mathcal{D} \simeq \mathrm{RFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$. In particular, Pr_L inherits a symmetric monoidal structure.

Proof. Let A and B be two small ∞-categories, S and S' two small sets of morphisms of $\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S})$ and $\mathrm{Fun}(B^{\mathrm{op}}, \mathcal{S})$ respectively such that $\mathcal{C}$ and $\mathcal{D}$ are equivalent the full subcategories of S and S'-local objects. By Theorem 1.2.10 we have

$$\begin{aligned}\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}) \otimes \mathrm{Fun}(B^{\mathrm{op}}, \mathcal{S}) &\simeq \mathrm{Fun}(A^{\mathrm{op}}, \mathrm{Fun}(B^{\mathrm{op}}, \mathcal{S})) \\ &\simeq \mathrm{Fun}((A \times B)^{\mathrm{op}}, \mathcal{S}).\end{aligned}$$

It then follows from the proof of [Lur17, Proposition 4.8.1.15] that $\mathcal{C} \otimes \mathcal{D}$ can be identified with the full subcategory of $\mathrm{Fun}((A \times B)^{\mathrm{op}}, \mathcal{S})$ spanned by $S \otimes S'$-local objects, where $S \otimes S'$ is the image of $S \times S'$ through the canonical functor

$$\mathrm{Fun}(A^{\mathrm{op}}, \mathcal{S}) \times \mathrm{Fun}(B^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}((A \times B)^{\mathrm{op}}, \mathcal{S}).$$

The proof of the last assertion follows by [Lur17, Lemma 4.8.1.16] and [Lur17, Proposition 4.8.1.17]. □

Remark 1.2.17. Let $\mathcal{C}$ be a presentable ∞-category. One can deduce easily for Proposition 1.2.16 identifications $\mathcal{C} \otimes \mathcal{S}_* \simeq \mathcal{C}_*$ and $\mathcal{C} \otimes \mathcal{S}\mathrm{p} \simeq \mathrm{Sp}(\mathcal{C})$, where $\mathcal{C}_*$ denotes the ∞-category of *pointed objects* of $\mathcal{C}$, and $\mathrm{Sp}(\mathcal{C})$ denotes the ∞-category of *spectrum objects* of $\mathcal{C}$, i.e. the limit of the tower

$$\dots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*$$

where Ω is the usual loop functor. Both these constructions come with canonical functors $\mathcal{C}_* \to \mathcal{C}$ and $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$, and since $\mathcal{C}$ is presentable one can show that these admit left adjoints $(-)_+ : \mathcal{C} \to \mathcal{C}_*$ and $\Sigma^\infty_+ : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$. By construction, we have a factorization

$$\mathcal{C} \xrightarrow{(-)_+} \mathcal{C}_* \xrightarrow{\Sigma^\infty} \mathrm{Sp}(\mathcal{C}).$$

In particular we see that, if $\mathcal{C}$ is presentable and pointed (stable), by tensoring $(-)_+ : \mathcal{S} \to \mathcal{S}_*$ ($\Sigma^\infty_+ : \mathcal{S} \to \mathcal{S}\mathrm{p}$) with $\mathcal{C}$ we obtain an equivalence $\mathcal{C} \otimes \mathcal{S}_* \simeq \mathcal{C}$ ($\mathcal{C} \otimes \mathcal{S}\mathrm{p} \simeq \mathrm{Sp}(\mathcal{C})$). Thus, if $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are presentable ∞-categories where $\mathcal{D}$ is pointed and $\mathcal{E}$ is stable, we get functorial identifications

$$\mathrm{Fun}_!(\mathcal{C} \otimes \mathcal{S}_*, \mathcal{D}) \simeq \mathrm{Fun}_!(\mathcal{C}, \mathrm{Fun}_!(\mathcal{S}_*, \mathcal{D})) \simeq \mathrm{Fun}_!(\mathcal{C}, \mathcal{D})$$

$$\mathrm{Fun}_!(\mathcal{C} \otimes \mathcal{S}\mathrm{p}, \mathcal{E}) \simeq \mathrm{Fun}_!(\mathcal{C}, \mathrm{Fun}_!(\mathcal{S}\mathrm{p}, \mathcal{E})) \simeq \mathrm{Fun}_!(\mathcal{C}, \mathcal{E})$$

induced respectively by precomposing with $(-)_+$ and Σ^∞_+. Furthermore, we see that $(-)_+ : \mathcal{S} \to \mathcal{S}_*$ and $\Sigma^\infty_+ : \mathcal{S} \to \mathcal{S}\mathrm{p}$ make $\mathcal{S}_*$ and $\mathcal{S}\mathrm{p}$ into idempotent cocomplete ∞-categories with respect to Lurie's tensor product: by [Lur17, Proposition 4.8.2.9] this implies that there are canonical variablewise cocontinuous symmetric monoidal structures on $\mathcal{S}_*$ and $\mathcal{S}\mathrm{p}$ with unit objects given by $S^0 := (*)_+$ and $\mathbb{S} := \Sigma^\infty_+(*)$, and one can show that these coincide with the usual smash products of pointed spaces and spectra. In particular, we see that the functors

$$\mathcal{S} \xrightarrow{(-)_+} \mathcal{S}_* \xrightarrow{\Sigma^\infty} \mathcal{S}\mathrm{p}$$

are all monoidal, where $\mathcal{S}$ is equipped with the cartesian monoidal structure.

1.2.2 Sheaves and cosheaves

We now pass to recalling the definition of sheaves with values in an ∞-category. Let X be a small ∞-category equipped with a Grothendieck topology. Recall that there is a small ∞-category $\mathrm{Cov}(X)$, as defined in [Lur09, Notation 6.2.2.8], which can be described informally as having for objects pairs (x, R), where $x \in X$ and $R \hookrightarrow y(x)$ is a sieve covering x, and morphisms between (x, R) and (y, R') are just maps $f : x \to y$ in X such that the restriction of $y(f)$ to R factors through R'. There is an obvious projection $\rho : \mathrm{Cov}(X) \to X$ which has a section $s : X \to \mathrm{Cov}(X)$ defined on objects by sending x to $(x, y(x))$.

Definition 1.2.18. Let $\mathcal{C}$ be a complete ∞-category. With the same notations as above, we say that a functor $F \in \mathrm{Fun}(X^{\mathrm{op}}, \mathcal{C})$ is a *sheaf* if the unit morphism

$$\rho^* F \to s_* s^* \rho^* F \simeq s_* F$$

is an equivalence. Dually, for a cocomplete ∞-category $\mathcal{C}$, we say that a functor $F \in \mathrm{Fun}(X, \mathcal{C})$ is a *cosheaf* if the counit morphism

$$s_! F \simeq s_! s^* \rho^* F \to \rho^* F$$

is an equivalence. We denote by $\mathcal{S}\mathrm{hv}(X; \mathcal{C})$ ($\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X; \mathcal{C})$) the full subcategory of $\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{C})$ ($\mathrm{Fun}(X, \mathcal{C})$) spanned by (co)sheaves. When $\mathcal{C}$ is the ∞-category of spaces $\mathcal{S}$, we will simply write $\mathcal{S}\mathrm{hv}(X)$.

Remark 1.2.19. More concretely, one can describe a sheaf as a functor F such that for any covering sieve $R \hookrightarrow y(x)$ the canonical morphism

$$F(x) \to \varprojlim_{y(x') \to R} F(x')$$

is an equivalence. Notice also that we clearly have an equivalence $\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X; \mathcal{C}) \simeq \mathcal{S}\mathrm{hv}(X; \mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$.

Remark 1.2.20. It is well known that, for any ∞-site X, the category $\mathcal{S}\mathrm{hv}(X)$ is an ∞-topos. Unlike the case of 1-topoi, it's still unclear whether any ∞-topos is equivalent to $\mathcal{S}\mathrm{hv}(X)$ for some ∞-site X (see [Rez19]).

We now give another description of categories of sheaves and cosheaves.

Lemma 1.2.21. Let X be an ∞-site, $\mathcal{C}$ be any cocomplete ∞-category. Then the restrictiong along the functor $X \xrightarrow{y} \mathrm{Fun}(X^{\mathrm{op}}, \mathcal{S}) \xrightarrow{L} \mathcal{S}\mathrm{hv}(X)$ defines an equivalence

$$\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X; \mathcal{C}) \simeq \mathrm{Fun}_!(\mathcal{S}\mathrm{hv}(X), \mathcal{C}),$$

where y is the Yoneda embedding and L is the sheafification functor. Equivalently, a functor $X \to \mathcal{C}$ is a cosheaf if and only if its extension by colimits $\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{S}) \to \mathcal{C}$ factors through L. Dually, for any complete ∞-category $\mathcal{C}$, we have an equivalence

$$\mathcal{S}\mathrm{hv}(X; \mathcal{C}) \simeq \mathrm{Fun}_*(\mathcal{S}\mathrm{hv}(X)^{\mathrm{op}}, \mathcal{C}).$$

Proof. Since L commutes with colimits, by the universal property of localizations composition with L embeds $\mathrm{Fun}_!(\mathcal{S}\mathrm{hv}(X), \mathcal{C})$ in $\mathrm{Fun}_!(\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{S}), \mathcal{C})$ as the full subcategory of functors sending covering sieves $R \hookrightarrow y(x)$ to equivalences in $\mathcal{C}$. On the other hand, a functor $F : X \to \mathcal{C}$ is a cosheaf precisely if there is an equivalence

$$\varinjlim_{y(x') \to R} F(x') \simeq F(x)$$

for any sieve R on $x \in X$, thus precisely if its extension by colimits $\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{S}) \to \mathcal{C}$ lies in $\mathrm{Fun}_!(\mathcal{S}\mathrm{hv}(X), \mathcal{C})$. □

We provide a couple of examples of cosheaves.

Example 1.2.22. (i) Let $f : X \to Y$ be a continuous map between topological spaces. Recall that this induces a geometric morphism $f : \mathcal{S}\mathrm{hv}(X) \to \mathcal{S}\mathrm{hv}(Y)$, which amounts to an adjunction $f^* \dashv f_*$, where $f_* : \mathcal{S}\mathrm{hv}(X) \to \mathcal{S}\mathrm{hv}(Y)$ is defined by $\Gamma(U; f_* F) = \Gamma(f^{-1}(U); F)$ for any $U \subseteq Y$. By Lemma 1.2.21, one may characterize $f^* : \mathcal{S}\mathrm{hv}(Y) \to \mathcal{S}\mathrm{hv}(X)$ as the essentially unique $\mathcal{S}\mathrm{hv}(X)$-valued cosheaf on Y with the property that $f^*(y(U)) = y(f^{-1}(U))$.

(ii) Let Top be the 1-category of topological spaces, and $\mathrm{Kan} \hookrightarrow \mathrm{sSet}$ be the full subcategory of all simplicial sets consisting of Kan complexes. Recall that there is a functor $\mathrm{Top} \to \mathrm{Kan}$ defined by assigning to each topological space X its *singular complex*, i.e. the simplicial set defined by $n \mapsto \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X)$, where Δ^n is the standard n-simplex. Recall also that, by [Cis19, Theorem 7.8.9], there is a functor $\mathrm{sSet} \to \mathcal{S}$ which identifies $\mathcal{S}$ as a localization of sSet at the class of weak homotopy equivalences. We define $\mathrm{Sing} : \mathrm{Top} \to \mathcal{S}$ as the composition of the two functors defined above. It is proven in [Lur17, A.3] that, for any topological space X, the restriction of Sing to $\mathcal{U}(X)$ is indeed a cosheaf: this may be regarded as a non-truncated version of the classical Seifert-Van Kampen theorem. Furthermore, one can also show that Sing is a *hypercomplete cosheaf* (see [Lur17, Lemma A.3.10]): this means that, as cocontinuous functor $\mathcal{S}\mathrm{hv}(X) \to \mathcal{S}$, Sing factors through the *hypercompletion* of $\mathcal{S}\mathrm{hv}(X)$.

Corollary 1.2.23. Let X be an ∞-site, $\mathcal{C}$ be any presentable ∞-category. Then the inclusion $\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \hookrightarrow \mathrm{Fun}(X,\mathcal{C})$ admits a right adjoint.

Proof. By [Lur09, Proposition 5.5.3.8] and the previous lemma, the ∞-category $\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ is presentable. Thus, since $\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \hookrightarrow \mathrm{Fun}(X,\mathcal{C})$ obviously preserves colimits, we may conclude by the adjoint functor theorem. ☐

Corollary 1.2.24. Let X be an ∞-site, $\mathcal{C}$ be any presentable ∞-category. Then we have an equivalence $\mathcal{S}\mathrm{hv}(X) \otimes \mathcal{C} \simeq \mathcal{S}\mathrm{hv}(X;\mathcal{C})$.

Proof. It follows by the adjoint functor theorem that $\mathrm{Fun}_*(\mathcal{S}\mathrm{hv}(X)^{\mathrm{op}},\mathcal{C}) \simeq \mathrm{RFun}(\mathcal{S}\mathrm{hv}(X)^{\mathrm{op}},\mathcal{C})$. Thus, by the previous lemma and by Proposition 1.2.16, we get the conclusion. ☐

Construction 1.2.25. Let X and Y be two topological spaces. The functor

$$\begin{aligned} \mathcal{U}(X) \times \mathcal{U}(Y) &\longrightarrow \mathcal{S}\mathrm{hv}(X \times Y) \\ (U,V) &\longmapsto y(U \times V) \end{aligned}$$

extends by colimits to a functor

$$\mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}},\mathcal{S}) \times \mathrm{Fun}(\mathcal{U}(Y)^{\mathrm{op}},\mathcal{S}) \to \mathcal{S}\mathrm{hv}(X \times Y).$$

Since it clearly sends covering sieves to equivalences in both variables, we obtain a functor

$$\begin{aligned} \mathcal{S}\mathrm{hv}(X) \times \mathcal{S}\mathrm{hv}(Y) &\longrightarrow \mathcal{S}\mathrm{hv}(X \times Y) \\ (F,G) &\longmapsto F \boxtimes G. \end{aligned}$$

More generally, by Corollary 1.2.24, tensoring with two presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$ gives

$$\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \times \mathcal{S}\mathrm{hv}(Y;\mathcal{D}) \to \mathcal{S}\mathrm{hv}(X \times Y;\mathcal{C} \otimes \mathcal{D}) \tag{1.2.26}$$

for which the image of a pair (F,G) in the domain will still be denoted as $F \boxtimes G$. Let $\Delta : X \to X \times X$ be the diagonal. By post composing with Δ^* we get a functor denoted by

$$\begin{aligned} \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \times \mathcal{S}\mathrm{hv}(X;\mathcal{D}) &\longrightarrow \mathcal{S}\mathrm{hv}(X;\mathcal{C} \otimes \mathcal{D}) \\ (F,G) &\longmapsto F \otimes G := \Delta^*(F \boxtimes G). \end{aligned} \tag{1.2.27}$$

Suppose now that $\mathcal{C}$ is equipped with a monoidal structure $\otimes_{\mathcal{C}}$ such that the functor $\otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves colimits in each variable. Then we have a cocontinuous functor $\mathcal{S}\mathrm{hv}(X; \mathcal{C} \otimes \mathcal{C}) \to \mathcal{S}\mathrm{hv}(X; \mathcal{C})$ and by composing with (1.2.27) we obtain an induced monoidal structure on $\mathcal{S}\mathrm{hv}(X; \mathcal{C})$, which will still be denoted by $\otimes_{\mathcal{C}}$. It is straightforward to check that the functor

$$\otimes_{\mathcal{C}} : \mathcal{S}\mathrm{hv}(X; \mathcal{C}) \times \mathcal{S}\mathrm{hv}(X; \mathcal{C}) \to \mathcal{S}\mathrm{hv}(X; \mathcal{C}) \tag{1.2.28}$$

can be described as

$$(F, G) \mapsto L^{\mathcal{C}}(U \mapsto \Gamma(U; F) \otimes_{\mathcal{C}} \Gamma(U; G))$$

where $L^{\mathcal{C}}$ is the sheafification for $\mathcal{C}$-valued presheaves. In particular, we have that any $F \in \mathcal{S}\mathrm{hv}(X; \mathcal{C})$ induces a colimit preserving functor

$$- \otimes_{\mathcal{C}} F : \mathcal{S}\mathrm{hv}(X; \mathcal{C}) \to \mathcal{S}\mathrm{hv}(X; \mathcal{C}).$$

Since $\mathcal{S}\mathrm{hv}(X; \mathcal{C})$ is presentable, this has a right adjoint denoted by

$$\underline{\mathrm{Hom}}_X(F, -) : \mathcal{S}\mathrm{hv}(X; \mathcal{C}) \to \mathcal{S}\mathrm{hv}(X; \mathcal{C}).$$

This functor supplies $\mathcal{S}\mathrm{hv}(X; \mathcal{C})$ with a self-enrichment and for this reason will be called *internal Hom sheaf* functor.

Remark 1.2.29. Let X and Y be two topological spaces, $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be presentable ∞-categories, and let $\alpha : \mathcal{S}\mathrm{hv}(X) \to \mathcal{S}\mathrm{hv}(Y)$ and $\Phi : \mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ be cocontinuous functors. Notice that the functoriality in each variable of the tensor product of cocomplete ∞-categories gives a commutative diagram

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}(X \times X; \mathcal{C} \otimes \mathcal{D}) \arrow[r, "\Delta^*"] \arrow[d, "\Delta^*"] & \mathcal{S}\mathrm{hv}(X; \mathcal{C} \otimes \mathcal{D}) \arrow[r, "\alpha \otimes (\mathcal{C} \otimes \mathcal{D})"] & \mathcal{S}\mathrm{hv}(X; \mathcal{C} \otimes \mathcal{D}) \arrow[d, "\mathcal{S}\mathrm{hv}(Y) \otimes \Phi"] \\
\mathcal{S}\mathrm{hv}(X; \mathcal{C} \otimes \mathcal{D}) \arrow[r, "\mathcal{S}\mathrm{hv}(X) \otimes \Phi"] & \mathcal{S}\mathrm{hv}(X; \mathcal{E}) \arrow[r, "\alpha \otimes \mathcal{E}"] & \mathcal{S}\mathrm{hv}(Y; \mathcal{E}).
\end{tikzcd}$$

In particular, the diagram above shows that whenever we prove a formula involving the functor (1.2.27) and operations on sheaves coming from some continuous map, then we may deduce immediately a corresponding formula for the functor (1.2.28).

We may now formulate the following proposition, which could be interpreted as a sort of Künneth formula (we will actually see later in Proposition 1.6.11 how one can deduce the Künneth formula from this).

Proposition 1.2.30. Let X and Y be topological spaces, $\mathcal{C}$ and $\mathcal{D}$ two presentable ∞-categories, and assume that one of the two is locally compact. Then the functor (1.2.26) induces an equivalence

$$\mathcal{S}\mathrm{hv}(X; \mathcal{C}) \otimes \mathcal{S}\mathrm{hv}(Y; \mathcal{D}) \simeq \mathcal{S}\mathrm{hv}(X \times Y; \mathcal{C} \otimes \mathcal{D}).$$

Moreover, let $f : X \to X'$ and $g : Y \to Y'$ be two continuous maps and assume that at least one among X' and Y' is locally compact. Then we have an equivalence

$$(f \times g)^*_{\mathcal{C} \otimes \mathcal{D}}(F \boxtimes G) \simeq f^*_{\mathcal{C}} F \boxtimes g^*_{\mathcal{D}} G$$

which is functorial on $F \in \mathcal{S}\mathrm{hv}(X'; \mathcal{C})$ and $G \in \mathcal{S}\mathrm{hv}(Y'; \mathcal{D})$.

Proof. By [Lur09, Proposition 7.3.3.9], for any topological space X and any ∞-topos $\mathcal{Y}$, $\mathcal{S}\mathrm{hv}(X;\mathcal{Y})$ is a product of $\mathcal{S}\mathrm{hv}(X)$ and $\mathcal{Y}$ in the ∞-category $\mathcal{T}\mathrm{op}$. Thus, by the previous corollary combined with [Lur09, Proposition 7.3.1.11], if Y is a locally compact topological space, we have an equivalence

$$\mathcal{S}\mathrm{hv}(X) \otimes \mathcal{S}\mathrm{hv}(Y) \simeq \mathcal{S}\mathrm{hv}(X \times Y).$$

For the second part of the statement, we first observe that by Corollary 1.2.24 and Proposition 1.2.16 it suffices to prove the case when $\mathcal{C} = \mathcal{D} = \mathcal{S}$, which amounts to providing a commutative square

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}(X') \times \mathcal{S}\mathrm{hv}(Y') \arrow[r] \arrow[d, "f^* \times g^*"] & \mathcal{S}\mathrm{hv}(X' \times Y') \arrow[d, "(f\times g)^*"] \\
\mathcal{S}\mathrm{hv}(X) \times \mathcal{S}\mathrm{hv}(Y) \arrow[r] & \mathcal{S}\mathrm{hv}(X \times Y).
\end{tikzcd}$$

Since both the top right and the down left composition commute with colimits in both variables, one then gets this by Lemma 1.2.21, Example 1.2.22 (i) and by observing that

$$(f \times g)^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V)$$

for any U and V open subsets of X' and Y' respectively. □

Corollary 1.2.31 (Monoidality)**.** Let $f : X \to Y$ be a morphism of locally compact topological spaces, $\mathcal{C}$ and $\mathcal{D}$ two presentable ∞-categories. Then we have a canonical identification

$$f^*(F \otimes G) \simeq f^*F \otimes f^*G$$

and in particular, when $\mathcal{C} = \mathcal{D}$ is monoidal, by transposition

$$f_*\underline{\mathrm{Hom}}_X(f^*H, K) \simeq \underline{\mathrm{Hom}}_Y(H, f_*K).$$

Proof. The commutativity of the diagram

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}(Y;\mathcal{C}) \times \mathcal{S}\mathrm{hv}(Y;\mathcal{D}) \arrow[r, "(f^*,f^*)"] \arrow[d, "\otimes"'] & \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \times \mathcal{S}\mathrm{hv}(X;\mathcal{D}) \arrow[d, "\otimes"] \\
\mathcal{S}\mathrm{hv}(Y;\mathcal{C}\otimes\mathcal{D}) \arrow[r, "f^*"] & \mathcal{S}\mathrm{hv}(X;\mathcal{C}\otimes\mathcal{D})
\end{tikzcd}$$

follows from the commutativity of

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Delta"'] & Y \arrow[d, "\Delta"] \\
X \times X \arrow[r, "(f,f)"] & Y \times Y
\end{tikzcd}$$

that is trivially verified. The last part follows directly from the previous lemma. □

Consider now the ∞-category $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$, where X is any ∞-site and $\mathcal{C}$ is complete and cocomplete. It is natural to ask oneselves whether at this level of generality one is still able to obtain a result like Corollary 1.2.24, at least when the inclusion $\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \hookrightarrow \mathrm{Fun}(X^{\mathrm{op}},\mathcal{C})$ admits a left adjoint. In the rest of the section we will briefly outline the reason why the answer to this question doesn't seem to be affirmative. We start with a general proposition concerning left Bousfield localizations and categories of local objects.

Proposition 1.2.32. Let $\mathcal{C}$ be an ∞-category and S a class of morphism in $\mathcal{C}$. Denote by $\mathcal{C}_S$ the full subcategory of $\mathcal{C}$ spanned by S-local objects, and assume that the inclusion $i : \mathcal{C}_S \hookrightarrow \mathcal{C}$ admits a left adjoint L. Thus, composition with L gives a fully faithful functor

$$\mathrm{LFun}(\mathcal{C}_S, \mathcal{D}) \hookrightarrow \mathrm{LFun}(\mathcal{C}, \mathcal{D}) \tag{1.2.33}$$

whose essential image is given by left adjoints $\mathcal{C} \to \mathcal{D}$ sending all morphisms in S to equivalences.

Proof. Let W be the class of morphisms in $\mathcal{C}$ which are sent by L to equivalences and denote by $\mathcal{A}$ and $\mathcal{A}'$ the full subcategories of $\mathrm{LFun}(\mathcal{C}, \mathcal{D})$ spanned respectively by left adjoints $\mathcal{C} \to \mathcal{D}$ sending all morphisms in W to equivalences and left adjoints $\mathcal{C} \to \mathcal{D}$ sending all morphisms in S to equivalences. By [Cis19, Proposition 7.1.18], we already know that (1.2.33) is fully faithful and that its essential image is given by $\mathcal{A}$. It follows immediately by the definition of a local object that L sends all morphisms in S to equivalences, thus we just need to show that $\mathcal{A}'$ is contained in $\mathcal{A}$.

Consider a functor $F : \mathcal{C} \to \mathcal{D}$ in $\mathcal{A}'$ with right adjoint $G : \mathcal{D} \to \mathcal{C}$. By definition of $\mathcal{A}'$, we have that for every morphisms in $f \in S$ and every $d \in \mathcal{D}$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(c, G(d)) \arrow[r, "f"] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathcal{C}}(c', G(d)) \arrow[d, "\simeq"] \\
\mathrm{Hom}_{\mathcal{D}}(F(c), d) \arrow[r, "F(f)"] & \mathrm{Hom}_{\mathcal{D}}(F(c'), d).
\end{tikzcd}$$

Thus $G(d)$ is S-local, and hence there exists a functor $G' : \mathcal{D} \to \mathcal{C}_S$ suche that $G = iG'$. Let now f be a morphism in W. By definition of W we have, functorially on $d \in \mathcal{D}$,

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(c), d) \arrow[r, "F(f)"] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathcal{D}}(F(c'), d) \arrow[d, "\simeq"] \\
\mathrm{Hom}_{\mathcal{C}}(c, iG'(d)) \arrow[r, "f"] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathcal{C}}(c', iG'(d)) \arrow[d, "\simeq"] \\
\mathrm{Hom}_{\mathcal{C}_S}(L(c), G'(d)) \arrow[r, "L(f)", "\simeq"'] & \mathrm{Hom}_{\mathcal{C}_S}(L(c'), G'(d)),
\end{tikzcd}$$

and hence $F(f)$ is invertible, and so we may conclude. $\square$

We now claim that there exists a class of morphisms S of $\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{C})$ such that $\mathcal{S}\mathrm{hv}(X; \mathcal{C})$ can be identified with the full subcategory of S-local objects of $\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{C})$. We define S as the class of morphisms

$$S = \{R \boxtimes M \to x \boxtimes M \mid R \hookrightarrow y(x) \,\text{is a sieve}, M \in \mathcal{C}\}.$$

For any sieve $R \hookrightarrow y(x)$, $M \in \mathcal{C}$ and $F \in \mathrm{Fun}(X^{\mathrm{op}}, \mathcal{C})$, since $R \simeq \varinjlim_{y(x') \to R} y(x')$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(-, F(x)) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathcal{C}}(-, \varprojlim_{y(x') \to R} F(x')) \arrow[d, "\simeq"] \\
\mathrm{Hom}_{\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{C})}(x \boxtimes -, F) \arrow[r, "i"] & \mathrm{Hom}_{\mathrm{Fun}(X^{\mathrm{op}}, \mathcal{C})}(R \boxtimes -, F)
\end{tikzcd}$$

where the upper horizontal arrow is induced by the canonical map $F(x) \to \varprojlim_{y(x')\to R} F(x')$. Thus, we see that the upper horizontal arrow is invertible if and only if the lower horizontal one is, and so F is a sheaf if and only if it is S-local. In paricular, by the previous proposition, whenever the inclusion $\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \hookrightarrow \mathrm{Fun}(X^{\mathrm{op}},\mathcal{C})$ admits a left adjoint, the ∞-category $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ is characterized by the universal property

$$\mathrm{LFun}(\mathcal{S}\mathrm{hv}(X;\mathcal{C}),\mathcal{D}) \hookrightarrow \mathrm{LFun}_S(\mathrm{Fun}(X^{\mathrm{op}},\mathcal{C}),\mathcal{D})$$

where the right-hand side denotes the ∞-category of left adjoint functors sending all morphisms in S to equivalence. On the other hand, tensoring the usual sheafification $\mathrm{Fun}(X^{\mathrm{op}},\mathcal{S}) \to \mathcal{S}\mathrm{hv}(X)$ with $\mathcal{C}$ gives a colimit preserving functor

$$L' : \mathrm{Fun}(X^{\mathrm{op}},\mathcal{C}) \simeq \mathrm{Fun}(X^{\mathrm{op}},\mathcal{S}) \otimes \mathcal{C} \to \mathcal{S}\mathrm{hv}(X) \otimes \mathcal{C}.$$

Combining the universal property of the tensor product of cocomplete categories and Theorem 1.2.10, we see that, for any cocomplete ∞-category $\mathcal{D}$, precomposition with L' may be factored as

$$\begin{aligned}\mathrm{Fun}_!(\mathcal{S}\mathrm{hv}(X)\otimes\mathcal{C},\mathcal{D}) &\simeq \mathrm{Fun}_!(\mathcal{C},\mathrm{Fun}_!(\mathcal{S}\mathrm{hv}(X),\mathcal{D})) \\ &\hookrightarrow \mathrm{Fun}_!(\mathcal{C},\mathrm{Fun}_!(\mathrm{Fun}(X^{\mathrm{op}},\mathcal{S}),\mathcal{D})) \\ &\simeq \mathrm{Fun}_!(\mathrm{Fun}(X^{\mathrm{op}},\mathcal{C}),\mathcal{D})\end{aligned}$$

and hence indentifies $\mathrm{Fun}_!(\mathcal{S}\mathrm{hv}(X)\otimes\mathcal{C},\mathcal{D})$ with the full subcategory of $\mathrm{Fun}_!(\mathrm{Fun}(X^{\mathrm{op}},\mathcal{C}),\mathcal{D})$ spanned by those functors sending maps in S to equivalences. Hence we obtain a comparison functor

$$\mathcal{S}\mathrm{hv}(X)\otimes\mathcal{C} \to \mathcal{S}\mathrm{hv}(X;\mathcal{C})$$

but unless $\mathcal{C}$ is presentable, there is no evident reason why one should expect this to be an equivalence.

Remark 1.2.34. A close inspection of the proof of [Lur09, Proposition 6.2.2.7] shows that the usual formula for sheafifcation provides the desired left adjoint whenever $\mathcal{C}$ is bicomplete and, for every $x \in X$ and every sieve $R \hookrightarrow y(x)$, the functor

$$\begin{aligned}\mathrm{Fun}(X^{\mathrm{op}},\mathcal{C}) &\longrightarrow \mathcal{C} \\ F &\longmapsto (s_*F)(x,R) \simeq \varprojlim_{y(x')\to R} F(x')\end{aligned}$$

is accessible: this will be true automatically for example when $\mathcal{C}$ is presentable, since any functor between presentable ∞-categories which is a right adjoint is automatically accessible. A similar observation in the case of sheaves with values in ordinary 1-categories can be found in [KS06, 17.4]. However, if we drop the presentability assumption for $\mathcal{C}$, it is not clear a priori why the inclusion $\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \hookrightarrow \mathrm{Fun}(X^{\mathrm{op}},\mathcal{C})$ should admit a left adjoint.

Remark 1.2.35. Suppose that $\mathcal{C}$ is such that $\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \hookrightarrow \mathrm{Fun}(X^{\mathrm{op}},\mathcal{C})$ admits a left adjoint L. Hence, for any $x \in X$ an $M \in \mathcal{C}$, by applying L to $x \boxtimes M$ gives an object denoted by M_x with the property that, for any other sheaf F, we have a functorial identification

$$\mathrm{Hom}(M_x,F) \simeq \mathrm{Hom}(M,F(x)).$$

It follows by [Cis19, Proposition 7.1.18] and Theorem 1.2.10 that, for any cocomplete ∞-category $\mathcal{D}$, we have a fully faithful functor

$$\mathrm{Fun}_!(\mathcal{S}\mathrm{hv}(X;\mathcal{C}),\mathcal{D}) \hookrightarrow \mathrm{Fun}_!(\mathcal{C},\mathrm{Fun}(A,\mathcal{D}))$$

and thus any cocontinuous functor with domain $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ is uniquely determined by its values on objects of the type M_x.

1.3 Shape theory and shape submersions

In this section we will deal with questions related to shape theory from the perspective of higher topos theory: we recommend [Lur17, Appendix A] and [Hoy18] for some good introductory accounts to this subject. We will start by defining a version of shape which is relative to a geometric morphism, and give a detailed description of its functoriality as well as a proof of its homotopy invariance. After that we will define essential and locally contractible geometric mophisms: the first notion refers to morphisms $f : \mathcal{X} \to \mathcal{Y}$ whose relative shape is constant (as a pro-object on $\mathcal{Y}$) locally on $\mathcal{X}$, while the second to essential geometric morphisms satisfying an additional push-pull formula. After that we will define shape submersions, i.e. continuous maps which are locally given by projections $X \times Y \to Y$, where X is such that the unique geometric morphism $\mathcal{S}\mathrm{hv}(X) \to \mathcal{S}$ is essential. These are proven to satisfy a base change formula, which will imply that they induce locally contractible geometric morphisms.

1.3.1 Relative shape

For any ∞-category $\mathcal{C}$, denote by $\mathrm{Pro}(\mathcal{C})$ the ∞-category of pro-objects in $\mathcal{C}$, i.e. the free completion of $\mathcal{C}$ under cofiltered limits. When $\mathcal{C}$ is accessible and admits finite limits, one shows ([Lur09, Proposition 3.1.6]) that $\mathrm{Pro}(\mathcal{C})$ is in fact equivalent to the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$ spanned by the left exact functors.

Let $F : \mathcal{C} \to \mathcal{D}$ be an accessible functor between presentable ∞-categories, and let $G : \mathcal{D} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$ be the composition of the Yoneda embedding $\mathcal{D} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$ with $F^* : \mathrm{Fun}(\mathcal{D}, \mathcal{S})^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$. By the adjoint functor theorem, G factors through $\mathcal{C}$ if and only if F commutes with limits, and when this condition is verified G is a left adjoint to F. If F is only left exact, by the characterization stated above G factors through $\mathrm{Pro}(\mathcal{C})$: in this case, we say that G is the *left pro-adjoint* of F. Notice that, in this situation, G is a genuine left adjoint of the functor $\mathrm{Pro}(F) : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{D})$.

Specializing to the case of a geometric morphism between ∞-topoi $f : \mathcal{X} \to \mathcal{Y}$, we see that the pullback f^* admits a pro-left adjoint, that we will denote by $f_\sharp : \mathcal{X} \to \mathrm{Pro}(\mathcal{Y})$. More explicitly, for every object $U \in \mathcal{X}$, $f_\sharp(U)$ is the pro-object on $\mathcal{Y}$ defined by the assignment

$$V \mapsto \mathrm{Hom}_{\mathcal{X}}(U, f^*(V)).$$

Definition 1.3.1. Let $\mathcal{X}$ be an ∞-topos, $f : \mathcal{Y} \to \mathcal{X}$ a geometric morphism. We define the *shape of* $\mathcal{Y}$ *relative to* $\mathcal{X}$ as

$$\Pi_\infty^{\mathcal{X}}(\mathcal{Y}) := f_\sharp 1_{\mathcal{Y}},$$

where $1_{\mathcal{Y}}$ is a terminal object of $\mathcal{Y}$. We will say that f is *constant shape* if $\Pi_\infty^{\mathcal{X}}(\mathcal{Y})$ belongs to $\mathcal{Y}$. In the case where f is the unique geometric morphism $a : \mathcal{Y} \to \mathcal{S}$, $a_\sharp 1_{\mathcal{Y}}$ will be denoted just by $\Pi_\infty(\mathcal{Y})$ and will be called the *shape* or *fundamental pro-∞-groupoid* of $\mathcal{Y}$.

Remark 1.3.2. Notice that, as a left exact functor $\mathcal{X} \to \mathcal{S}$, $f_\sharp 1_{\mathcal{Y}}$ can be identified with the functor $a_* f_* f^*$, where $a : \mathcal{X} \to \mathcal{S}$ is the unique geometric morphism.

Proposition 1.3.3. There exists a functor

$$\Pi_\infty^{\mathcal{X}} : \mathcal{T}\mathrm{op}_{/\mathcal{X}} \to \mathrm{Pro}(\mathcal{X})$$

whose values on objects coincides with the shape relative to $\mathcal{X}$ and whose values on morphisms
$$\begin{tikzcd} \mathcal{Y} \arrow[rr, "g"] \arrow[dr, "f"'] & & \mathcal{Y}' \arrow[dl, "f'"] \\ & \mathcal{X} & \end{tikzcd}$$
is given by the transformation

$$a_* f'_* f'^* \to a_* f'_* g_* g^* f'^* \simeq a_* f_* f^*$$

induced by the unit of the adjunction $g^* \dashv g_*$.

Proof. Since we clearly have a functor

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{X}, \mathcal{X}) \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{X}, \mathcal{S}) \simeq \mathrm{Pro}(\mathcal{X})^{\mathrm{op}}$$

given by post composition with a_*, it suffices to prove that there is a functor

$$T : \Pi_\infty^{\mathcal{X}} : (\mathcal{T}\mathrm{op}_{/\mathcal{X}})^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{X}, \mathcal{X})$$

that assigns $f_* f^*$ to any $f : \mathcal{Y} \to \mathcal{X}$ and at the level of morphisms $\begin{tikzcd}[column sep=small] \mathcal{Y} \arrow[rr, "g"] \arrow[dr, "f"'] & & \mathcal{Y}' \arrow[dl, "f'"] \\ & \mathcal{X} & \end{tikzcd}$ is given by the transformation

$$f'_* f'^* \to f'_* g_* g^* f'^* \simeq f_* f^*$$

induced by the unit of the adjunction $g^* \dashv g_*$. We will proceed through some reduction steps.

First of all, the Yoneda embedding induces a fully faithful functor

$$\mathrm{Fun}(\mathcal{X}, \mathcal{X}) \hookrightarrow \mathrm{Fun}(\mathcal{X}, \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathcal{S})) \simeq \mathrm{Fun}(\mathcal{X}^{\mathrm{op}} \times \mathcal{X}, \mathcal{S})$$

and thus it suffices to construct a functor

$$(\mathcal{T}\mathrm{op}_{/\mathcal{X}})^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{X}^{\mathrm{op}} \times \mathcal{X}, \mathcal{S})$$

whose image lies in $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{X}, \mathcal{X})$. By [Cis19, Remark 6.1.5], standard computations with adjunctions of 1-categories show that we also have, functorially on $F, G \in \mathcal{X}$, a commutative square

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{Y}}(f^*F, f^*G) \arrow[r, "\simeq"] \arrow[d, "g^*"'] & \mathrm{Hom}_{\mathcal{X}}(F, f_*f^*G) \arrow[d, "\mathrm{unit}"] \\ \mathrm{Hom}_{\mathcal{Y}'}(g^*f^*F, g^*f^*G) \arrow[r, "\simeq"] & \mathrm{Hom}_{\mathcal{X}}(F, f_*g_*g^*f^*G), \end{tikzcd}$$

so it suffices to show that the transformation on the left hand side can be enanched to a functor $(\mathcal{T}\mathrm{op}_{/\mathcal{X}})^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{X} \times \mathcal{X}^{\mathrm{op}}, \mathcal{S})$. Recall also that we have a forgetful functor

$$\begin{array}{ccc} (\mathcal{T}\mathrm{op}_{/\mathcal{X}})^{\mathrm{op}} \simeq (\mathcal{T}\mathrm{op}^{\mathrm{op}})_{\mathcal{X}/} & \longrightarrow & \mathcal{C}\mathrm{at}_{\infty\mathcal{X}/} \\ (f : \mathcal{Y} \to \mathcal{X}) & \longmapsto & (f^* : \mathcal{X} \to \mathcal{Y}). \end{array}$$

Hence, we will construct a functor

$$\mathcal{C}\mathrm{at}_{\infty\mathcal{X}/} \to \mathrm{Fun}(\mathcal{X}^{\mathrm{op}} \times \mathcal{X}, \mathcal{S}).$$

Let Cat_∞ be the full subcategory of sSet spanned by ∞-categories, and let $\mathrm{LFib}(\mathcal{X}^{\mathrm{op}} \times \mathcal{X})$ be the full subcategory of $\mathrm{s}\mathcal{S}\mathrm{et}_{/\mathcal{X}^{\mathrm{op}} \times \mathcal{X}}$ spanned by the left fibrations. Since Cat_∞ is the category of fibrant objects in the Joyal model structure on sSet, it follows by [Cis19, Theorem 7.5.18], [Cis19, Example 7.10.14] and [Cis19, Theorem 3.9.7] that one may regard $\mathcal{C}\mathrm{at}_\infty$ as a localization of Cat_∞ by the class W of fully faithful and essentially surjective functors. Thus, by [Cis19, Corollary 7.6.13] and [Cis19, Proposition 7.1.7], we get an equivalence $\mathcal{C}\mathrm{at}_{\infty\mathcal{X}/} \simeq \mathrm{Cat}_{\infty\mathcal{X}/}[W^{-1}]$. On the other hand, $\mathrm{LFib}(\mathcal{X}^{\mathrm{op}} \times \mathcal{X})$ is the category of fibrant objects in the covariant model structure on $\mathrm{s}\mathcal{S}\mathrm{et}_{/\mathcal{X}^{\mathrm{op}} \times \mathcal{X}}$, and so by [Cis19, Theorem 7.5.18], [Cis19, Theorem 7.8.9] and [Cis19, Theorem 4.4.14] we may regard $\mathrm{Fun}(\mathcal{X}^{\mathrm{op}} \times \mathcal{X}, \mathcal{S})$ as the localization of

$\mathrm{LFib}(\mathcal{X}^{\mathrm{op}} \times \mathcal{X})$ by the class W' of fibrewise equivalences. Thus, to produce T it will suffice to provide a functor $\mathrm{Cat}_{\infty\mathcal{X}/} \to \mathrm{LFib}(\mathcal{X}^{\mathrm{op}} \times \mathcal{X})$ of 1-categories which maps W into W'.

Recall that, for any ∞-category $\mathcal{C}$, we have a left fibration $\mathrm{S}(\mathcal{C}) \xrightarrow{(s,t)} \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, called the *twisted diagonal*, classifying the hom-bifunctor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ (as it is defined in [Cis19, 5.6.1]). For any functor $f : \mathcal{X} \to \mathcal{Y}$, consider the left fibration defined by the pullback

$$\begin{tikzcd} T(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{S}(\mathcal{Y}) \arrow[d, "(s,t)"] \\ \mathcal{X}^{\mathrm{op}} \times \mathcal{X} \arrow[r, "f^{\mathrm{op}} \times f"] & \mathcal{Y}^{\mathrm{op}} \times \mathcal{Y} \end{tikzcd}$$

which classifies the functor

$$\mathrm{Hom}_{\mathcal{X}}(f(-), f(-)) : \mathcal{X}^{\mathrm{op}} \times \mathcal{X} \to \mathcal{S}.$$

The functoriality on $\mathcal{Y}$ of the twisted diagonal and the universal property of pullbacks imply that T defines a functor $\mathrm{Cat}_{\infty\mathcal{X}/} \to \mathrm{LFib}(\mathcal{X}^{\mathrm{op}} \times \mathcal{X})$, as illustrated by the diagram corresponding to a morphism $\begin{tikzcd}[column sep=small] & \mathcal{X} \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ \mathcal{Y} \arrow[rr, "g"'] & & \mathcal{Y}' \end{tikzcd}$

$$\begin{tikzcd} T(f) \arrow[r] \arrow[d, dotted] \arrow[dd, bend right] & \mathrm{S}(\mathcal{Y}) \arrow[d, dotted] \arrow[dd, bend right, "(s,t)"'] \arrow[dr, "\mathrm{S}(g)"] & \\ T(f') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T(g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{S}(\mathcal{Y}') \arrow[d, "(s,t)"] \\ \mathcal{X}^{\mathrm{op}} \times \mathcal{X} \arrow[r, "f^{\mathrm{op}} \times f"] & \mathcal{Y}^{\mathrm{op}} \times \mathcal{Y} \arrow[r, "g^{\mathrm{op}} \times g"] & (\mathcal{Y}')^{\mathrm{op}} \times \mathcal{Y}'. \end{tikzcd}$$

Moreover, [Cis19, Corollary 5.6.6] implies that T sends any fully faithful functor to a fibrewise equivalence, and hence we can conclude. $\square$

Recall that, for two maps $Y \overset{f_0}{\underset{f_1}{\rightrightarrows}} Y'$ over a topological space X, we say that f_0 *is homotopic to* f_1 *over* X if there exists a map $h : Y \times I \to Y'$ over X such that $f_t = hi_t$ $t = 0, 1$, where $i_t : Y \hookrightarrow Y \times I$ is the inclusion corresponding to $t \in I$.

Corollary 1.3.4 (Homotopy invariance)**.** Let Y, Y' be two topological spaces over X, and let $Y \overset{f_0}{\underset{f_1}{\rightrightarrows}} Y'$ be two homotopic maps over X. Then the functor T induces an equivalence $T(f_0) \simeq T(f_1)$. In particular, T sends homotopy equivalences over X to invertible morphisms in $\mathrm{Fun}(\mathcal{X}, \mathcal{X})^{\mathrm{op}}$.

Proof. Let $p : Y \times I \to Y$ be the canonical projection. By [Lur17, Lemma A.2.9], we know that p^* is fully faithful, and hence $T(p)$ is invertible. Since $pi_0 = pi_1 = \mathrm{id}_Y$ and T is functorial, we get an equivalence $T(i_0) \simeq T(i_1)$. Thus, since there exists a homotopy h over X such that $f_t = hi_t$ $t = 0, 1$, the functoriality of T gives the desired $T(f_0) \simeq T(f_1)$. $\square$

Remark 1.3.5. Recall that, for any ∞-topos $\mathcal{X}$, there is a fully faithful functor

$$\begin{aligned} \mathcal{X} &\hookrightarrow \mathcal{T}\mathrm{op}_{/\mathcal{X}} \\ x &\longmapsto \mathcal{X}_{/x}. \end{aligned} \tag{1.3.6}$$

Since $\mathcal{T}\mathrm{op}_{/\mathcal{X}}$ has small cofiltered limits, the latter can be extended to a functor

$$\beta : \mathrm{Pro}(\mathcal{X}) \to \mathcal{T}\mathrm{op}_{/\mathcal{X}}.$$

It is possible to construct the functor

$$\Pi_\infty^{\mathcal{X}} : \mathcal{T}\mathrm{op}_{/\mathcal{X}} \to \mathrm{Pro}(\mathcal{X})$$

directly by showing that there is an equivalence

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{X})}(\Pi_\infty^{\mathcal{X}}(\mathcal{Y}), Z) \simeq \mathrm{Hom}_{\mathcal{T}\mathrm{op}_{/\mathcal{X}}}(\mathcal{Y}, \beta(Z))$$

which is functorial on $Z \in \mathrm{Pro}(\mathcal{X})$, as it is done in [Lur16, Proposition E.2.2.1]. However, we preferred to prove directly the functoriality of the relative shape, namely because the approach mentioned above leaves unclear how the functor $\Pi_\infty^{\mathcal{X}}$ would behave at the level of morphisms, which is needed to have a proof of homotopy invariance as clean and immediate as the one above.

Remark 1.3.7. As usual, since $\mathcal{T}\mathrm{op}$ has pullbacks, the slice $\mathcal{T}\mathrm{op}_{/\mathcal{X}}$ can be equipped with a contravariantly functorial structure

$$\mathcal{T}\mathrm{op}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$$

that can be described for any geometric morphism $g : \mathcal{X} \to \mathcal{Y}$ by sending an object $(f : \mathcal{Y}' \to \mathcal{Y}) \in \mathcal{T}\mathrm{op}_{/\mathcal{Y}}$ to the resulting arrow over $\mathcal{X}$ obtained by performing the pullback of f along g. Thus, since by [Lur09, Remark 6.3.5.8] we have for any $y \in \mathcal{Y}$ a canonical pullback square

$$\begin{tikzcd}
\mathcal{X}_{/f^*y} \arrow[r] \arrow[d] & \mathcal{Y}_{/y} \arrow[d] \\
\mathcal{X} \arrow[r, "f"] & \mathcal{Y}
\end{tikzcd}$$

in $\mathcal{T}\mathrm{op}$, we see that the functor (1.3.6) is actually natural in $\mathcal{X}$, where the left hand side is functorial by the usual forgetful $\mathcal{T}\mathrm{op}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ sending a geometric morphism f to f^*. In particular we obtain that

$$\beta : \mathrm{Pro}(\mathcal{X}) \to \mathcal{T}\mathrm{op}_{/\mathcal{X}}$$

is actually natural in $\mathcal{X}$. Notice that, if one regards $\mathcal{C}\mathrm{at}_\infty$ as an $(\infty, 2)$-category, the universal property of $\mathrm{Pro}(\mathcal{X})$ and the definition of the slice imply that β can be seen as a natural transformation between 2-functors. Hence, by [Hau20, Theorem 3.22], by adjunction we may regard the relative shape as a lax natural transformation, where the 2-cells involved may be described as follows: any geometric morphism induces an adjunction

$$\begin{tikzcd}
\mathrm{Pro}(\mathcal{X}) \arrow[r, bend left, "g_\sharp"] \arrow[r, phantom, "\perp"] & \mathrm{Pro}(\mathcal{Y}) \arrow[l, bend left, "g^*"]
\end{tikzcd}$$

and for any commutative square of topoi

$$\begin{tikzcd}
\mathcal{X}' \arrow[r, "g'"] \arrow[d, "f'"] & \mathcal{Y}' \arrow[d, "f"] \\
\mathcal{X} \arrow[r, "g"] & \mathcal{Y}
\end{tikzcd}$$

applying g'^* to the unit of the adjunction $f_\sharp \dashv f^*$ induces a natural transformation

$$g'^* \to g'^* f^* f_\sharp \simeq f'^* g^* f_\sharp$$

and hence by transposition

$$f'_\sharp g'^* \to g^* f_\sharp$$

called the *base change transformation*, which when evaluated at $1_{\mathcal{Y}'}$ gives the desired

$$f'_\sharp g'^* 1_{\mathcal{Y}'} \simeq f'_\sharp 1_{\mathcal{X}'} \to g^* f_\sharp 1_{\mathcal{Y}'}.$$

1.3.2 Locally contractible geometric morphisms

We start by recalling the definition of a locally cartesian closed ∞-category.

Definition 1.3.8. An ∞-category $\mathcal{C}$ is *cartesian closed* if it admits finite products and, for any object $c \in \mathcal{C}$, the functor $- \times c : \mathcal{C} \to \mathcal{C}$ admits a right adjoint.

An ∞-category $\mathcal{C}$ is *locally cartesian closed* if it has pullbacks and, for any object $c \in \mathcal{C}$, the slice $\mathcal{C}_{/c}$ is cartesian closed, or equivalently, if for any arrow $f : c \to d$ in $\mathcal{C}$, the functor $\mathcal{C}_{/d} \to \mathcal{C}_{/c}$ given by pulling back along f admits a right adjoint called the *dependent product along* f and denoted by $\prod_f : \mathcal{C}_{/c} \to \mathcal{C}_{/d}$. A functor $F : \mathcal{C} \to \mathcal{D}$ between locally cartesian ∞-categories if locally cartesian closed if F commutes with pullbacks and dependent products, i.e. for any arrow $f : c \to d$ in $\mathcal{C}$, we have a commutative square

$$\begin{tikzcd}
\mathcal{C}_{/c} \arrow[r, "\prod_f"] \arrow[d, "F"'] & \mathcal{C}_{/d} \arrow[d, "F"] \\
\mathcal{D}_{/Fc} \arrow[r, "\prod_{Ff}"'] & \mathcal{D}_{/Fd}.
\end{tikzcd}$$

We will denote $\mathcal{C}\mathrm{at}_\infty^{lcc}$ the subcategory of $\mathcal{C}\mathrm{at}_\infty$ whose objects are locally cartesian closed ∞-categories with locally cartesian closed functors between them.

Example 1.3.9. By universality of colimits and adjoint functor theorem, any ∞-topos is locally cartesian closed.

Let $F \to G$ be a morphism in $\mathrm{Pro}(\mathcal{Y})$ and let $H \to f^*G$ be a morphism in $\mathrm{Pro}(\mathcal{X})$. Then we have a canonical commutative square

$$\begin{tikzcd}
f_\sharp(f^*F \times_{f^*G} H) \arrow[r] \arrow[d] & f_\sharp H \arrow[d] \\
f_\sharp f^* F \arrow[r] \arrow[d] & f_\sharp f^* G \arrow[d] \\
F \arrow[r] & G
\end{tikzcd}$$

which determines a unique morphism

$$f_\sharp(f^*F \times_{f^*G} H) \to F \times_G f_\sharp H. \tag{1.3.10}$$

Proposition 1.3.11. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism between ∞-topoi. Consider the conditions

(i) f^* admits a left adjoint and, for every $F \to G$ in $\mathcal{Y}$ and $H \to f^*G$ in $\mathcal{X}$, the associated projection morphism is invertible;

(ii) f^* is locally cartesian closed;

(iii) f^* admits a left adjoint and, for every F in $\mathcal{Y}$ and H in $\mathcal{X}$, the associated projection morphism

$$f_\sharp(f^*F \times H) \to F \times f_\sharp H$$

is invertible.

Then (i) and (ii) are equivalent. Moreover, if f is induced by a continuous map between topological spaces, then these are also equivalent to (iii).

Proof. We first show that (i) implies (ii). Since f^* commutes with finite limits, it suffices to show that it commutes with dependent products. But, for any $\alpha : F \to G$ in $\mathcal{Y}$, the square

$$\begin{tikzcd}
\mathcal{Y}_{/F} \arrow[r, "\prod_\alpha"] \arrow[d, "f^*"'] & \mathcal{Y}_{/G} \arrow[d, "f^*"] \\
\mathcal{X}_{/f^*F} \arrow[r, "\prod_{f^*\alpha}"'] & \mathcal{X}_{/f^*G}
\end{tikzcd}$$

commutes if and only if the square given by the corresponding left adjoints commutes. This last assertion is equivalent to requiring the projection morphims to be invertible, and so we are done.

We now show that (ii) implies (i). By the same argument as above, it suffices to prove that f^* admits a let adjoint. Since f^* is cocontinuous and preserves finite limits, we are only left to prove that f^* commutes with infinite products: we will do this by exhibiting products (more generally, limits indexed by small ∞-groupoids) in any ∞-topos as a special case of dependent products, so that the result will follow by assumption (ii). Let $\mathcal{X}$ be an ∞-topos, $\pi : \mathcal{X} \to \mathcal{S}$ the unique geometric morphism. First of all, we observe that the cocontinuous functors

$$\begin{aligned} \mathcal{S} &\longrightarrow \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \\ A &\longmapsto \mathrm{Fun}(A, \mathcal{X}) \end{aligned} \qquad\qquad \begin{aligned} \mathcal{S} &\longrightarrow \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \\ A &\longmapsto \mathcal{X}_{/\pi^*A} \end{aligned}$$

are naturally equivalent, since have equivalences

$$\mathrm{Fun}(\Delta^0, \mathcal{X}) \simeq \mathcal{X} \simeq \mathcal{X}_{/\pi^*\Delta^0}.$$

In particular, if $\alpha : A \to \Delta^0$ is the unique map, we have a corresponding commutative square

$$\begin{tikzcd}
\mathcal{X}_{/\pi^*\Delta^0} \arrow[r, "\pi^*A\times -"] \arrow[d, "\simeq"] & \mathcal{X}_{/\pi^*A} \arrow[d, "\simeq"] \\
\mathrm{Fun}(\Delta^0, \mathcal{X}) \arrow[r, "const"] & \mathrm{Fun}(A, \mathcal{X})
\end{tikzcd}$$

where the lower horizontal arrow assigns to an object F of $\mathcal{X}$ the constant functor at F. Thus we obtain an identification of the respective right adjoints, i.e. a commutative square

$$\begin{tikzcd}
\mathcal{X}_{/\pi^*A} \arrow[r, "\prod_{\pi^*\alpha}"] \arrow[d, "\simeq"] & \mathcal{X}_{/\pi^*\Delta^0} \arrow[d, "\simeq"] \\
\mathrm{Fun}(A, \mathcal{X}) \arrow[r, "\varprojlim"] & \mathrm{Fun}(\Delta^0, \mathcal{X})
\end{tikzcd}$$

which is what we wanted.

Assume now that we have an essential geometric morphism $f : \mathcal{X} = \mathcal{S}\mathrm{hv}(X) \to \mathcal{Y} = \mathcal{S}\mathrm{hv}(Y)$ induced by a continuous map $f : X \to Y$. Clearly (iii) is a special case of (i). Assume then that f satisyes the hypothesis (iii). Let $\alpha : F \to G$ be a morphism in $\mathcal{Y}$ and let $H \to f^*G$ be a morphism in $\mathcal{X}$. Since the projection morphism is a natural transformation between colimit preserving functors and since for any $H \in \mathcal{X}_{/f^*G}$ we have an equivalence $H \simeq \varinjlim_{y(U)\to G} (H \times_{f^*G} y(f^{-1}U))$, we may assume that $H \to f^*G$ factors as $H \to y(f^{-1}U) \to f^*G$ for some open $U \in \mathcal{U}(Y)$, and hence by the pasting properties of pullbacks we may also assume that $G = y(U)$. Notice that, for any (-1)-truncated object V in a topos $\mathcal{Z}$ and for any other two objects $A, B \in \mathcal{Z}_{/V}$, we have an identification $A \times_V B \simeq A \times B$: this follows because for any other object C mapping both to A and B, we have that $\mathrm{Hom}_{\mathcal{Z}}(C, V)$ is contractible, and thus $\mathrm{Hom}_{\mathcal{Z}}(C, A \times_V B) \simeq \mathrm{Hom}_{\mathcal{Z}}(C, A \times B)$. Thus, since both $y(U)$ and $y(f^{-1}U)$ are (-1)-truncated, we are only left to prove that

$$f_\sharp(f^*F \times H) \to F \times f_\sharp H$$

is invertible, which is true by assumption. $\square$

Definition 1.3.12. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. We say that f is *essential* if $f_\sharp$ factors through $\mathcal{Y}$, or equivalently if f^* admits a left adjoint. Furthermore, we say that an essential geometric morphism is *locally contractible* if it satisfies the equivalent conditions (i) and (ii) in Proposition 1.3.11. We say that a geometric morphism is of *trivial shape* if f^* is fully faithful, or equivalently if the unit transformation $\mathrm{id}_{\mathcal{Y}} \to f_*f^*$ is an equivalence. When f is the unique geometric morphism $\mathcal{X} \to \mathcal{S}$, we will say that $\mathcal{X}$ is locally contractible.

Remark 1.3.13. For a continuous map $f : X \to Y$ be a continuous map inducing a, essential geometric motphism, one may interpret the condition of being locally contractible geometric morphism as the requirement of a base change for $f_\sharp$ along open immersions. More precicely, let U be an open subset of Y, and consider the pullback square

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f'"] \arrow[d, hook, "j'"] & U \arrow[d, hook, "j"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

For any $F \in \mathcal{S}\mathrm{hv}(X)$, we have natural equivalences

$$j_\sharp j^* f_\sharp F \xrightarrow{\simeq} f_\sharp F \times y(U) \xrightarrow{\simeq} f_\sharp(F \times y(f^{-1}(U))) \xrightarrow{\simeq} f_\sharp j'_\sharp (j')^* F \xrightarrow{\simeq} j_\sharp f'_\sharp (j')^* F$$

where the second morphism is (1.3.10). Therefore, since $j_\sharp$ is fully faithful, we obtain a natural equivalence

$$j^* f_\sharp \simeq f'_\sharp (j')^*,$$

and, by transposition, an equivalence

$$f^* j_* \simeq j'_* (f')^*.$$

Example 1.3.14. (i) Recall that any object $U \in \mathcal{X}$ of an ∞-topos determines a geometric morphism $j : \mathcal{X}_{/U} \to \mathcal{X}$. By [Lur09, Proposition 6.3.5.1] j is locally contractible, and $j_\sharp : \mathcal{X}_{/U} \to \mathcal{X}$ can be described as the usual forgetful functor.

(ii) By [Lur17, Proposition A.1.9], an ∞-topos is locally contractible if and only if the unique geometric morphism $\mathcal{X} \to \mathcal{S}$ is essential, since in this case the projection morphism is automatically invertible.

(iii) Let $f : \mathcal{X} \to \mathcal{Y}$ be an essential geometric morphism. For any the ∞-topos $\mathcal{Z}$, by Remark 1.2.2 we obtain a geometric morphism $f \otimes \mathcal{Z} : \mathcal{X} \otimes \mathcal{Z} \to \mathcal{Y} \otimes \mathcal{Z}$ by applying to f the functor $- \otimes \mathcal{Z}$. Since both f^* and $f_\sharp$ commute with colimits, the adjunction $f_\sharp \dashv f^*$ is preserved by $- \otimes \mathcal{Z}$, and so $f \otimes \mathrm{id}_{\mathcal{Z}}$ is an essential geometric morphism.

Remark 1.3.15. Let $f : X \to Y$ be a continuous map. Notice that, since the functor

$$f^{-1} : \mathcal{U}(Y) \to \mathcal{U}(X)$$

preserves open coverings, for any complete ∞-category $\mathcal{C}$ we still have a well defined push-forward $f_* : \mathcal{S}\mathrm{hv}(X; \mathcal{C}) \to \mathcal{S}\mathrm{hv}(Y; \mathcal{C})$ given as usual by $\Gamma(U; f_*F) = \Gamma(f^{-1}(U); F)$ for all $U \in \mathcal{U}(Y)$. Although at this level of generality there is no reason to expect f_* to have a left adjoint, if f induces an essential geometric morphism at the level of sheaves of spaces, then it actually does. Indeed, recall that there is an equivalence

$$\mathcal{S}\mathrm{hv}(X; \mathcal{C}) \simeq \mathrm{Fun}_*(\mathcal{S}\mathrm{hv}(X)^{\mathrm{op}}, \mathcal{C}).$$

Through this equivalence and Example 1.2.22 (i), we can identify $f_* : \mathrm{Fun}_*(\mathcal{S}\mathrm{hv}(X)^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}_*(\mathcal{S}\mathrm{hv}(Y)^{\mathrm{op}}, \mathcal{C})$ with precomposition with the opposite of the pullback $f^* : \mathcal{S}\mathrm{hv}(Y) \to \mathcal{S}\mathrm{hv}(X)$. Thus, similarly to Remark 1.2.2, by applying the 2-functor $\mathrm{Fun}_*((-)^{\mathrm{op}}, \mathcal{C})$ to the adjunction between cocontinuous functors $f_\sharp \dashv f^*$, we obtain the desired left adjoint.

It is straightforward to check that the composition of two locally contractible geometric morphism is again locally contractible (see [AC21, Corollary 3.2.5]). We observe that the properties of being essential or locally contractible can be checked locally on the source.

Lemma 1.3.16. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism, and let $\mathcal{B} \subseteq \mathcal{X}$ which generates $\mathcal{X}$ under colimits. For any object $U \in \mathcal{B}$, consider the composite geometric morphism

$$\mathcal{X}_{/U} \longrightarrow \mathcal{X} \xrightarrow{f} \mathcal{Y}. \tag{1.3.17}$$

We have the following

(i) f is essential if and only if (1.3.17) is of constant shape for any $U \in \mathcal{B}$;

(ii) f is locally contractible if and only if (1.3.17) is locally contractible for any $U \in \mathcal{B}$.

Proof. A proof can be found in [AC21, Proposition 3.1.5] and [AC21, Proposition 3.2.6]. □

Remark 1.3.18. The content of part (i) in Lemma 1.3.16 suggests that a valid alternative way to call a geometric morphism whose pullback has a left adjoint could have been *locally of constant shape*. This is actually the approach taken by Lurie in [Lur17, Appendix A]; however, we have decided to stick with the more concise nomenclature which appears also in [Joh02] and [AC21].

Corollary 1.3.19. Let X be a locally contractible topological space, $a : X \to *$ the unique map, and assume that $\mathcal{X} = \mathcal{S}\mathrm{hv}(X)$ is hypercomplete. Then $\mathcal{S}\mathrm{hv}(X)$ is locally contractible and $a_\sharp$ is equivalent to the extension by colimits of the cosheaf Sing. Consequently, for sheaves of spectra, the functor $a_\sharp : \mathcal{S}\mathrm{hv}(X; \mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{p}$ obtained by applying $- \otimes \mathcal{S}\mathrm{p}$ is uniquely determined by the formula $a_\sharp(\mathbb{S}_U) = \Sigma^\infty_+ U$ for any $U \in \mathcal{U}(X)$.

Proof. Let B be the poset of all contractible open subsets of X. Notice that, even though B in general is not a sieve, by the hypercompleteness assumption of $\mathcal{S}\mathrm{hv}(X)$ we get an equivalence

$$\varinjlim_{U \in B_{/V}} y(U) \simeq y(V)$$

for any $V \in \mathcal{U}(X)$ that can be easily checked on stalks. In particular, we see that the full subcategory $\mathcal{S}\mathrm{hv}(B) \subseteq \mathcal{S}\mathrm{hv}(X)$ generates $\mathcal{S}\mathrm{hv}(X)$ under colimits. Thus by Lemma 1.3.16 it suffices to check that $\mathcal{S}\mathrm{hv}(X)_{/U} \simeq \mathcal{S}\mathrm{hv}(U)$ is of constant shape for any $U \in B$, but this is true by homotopy invariance of the shape. The last assertion follows immediately by noticing that through the equivalence $\mathcal{S} \otimes \mathcal{S}\mathrm{p} \simeq \mathcal{S}\mathrm{p}$, an object $A \otimes \mathbb{S}$ corresponds (functorially on A) to $\Sigma^\infty_+ A$. ☐

Remark 1.3.20. Beware that the viceversa of Corollary 1.3.19 is not true. For this reason, to avoid confusion, from now on we will say that a topological space X is essential if $\mathcal{S}\mathrm{hv}(X)$ is (or equivalently, if $\mathcal{S}\mathrm{hv}(X)$ is locally contractible by Example 1.3.14 part (ii)).

1.3.3 Shape submersions

Definition 1.3.21. A continuous map $f : X \to Y$ between topological spaces is a *shape submersion* if for every point $x \in X$ there exist an open neighbourhood U of x and a space X' which is essential, such that $f(U)$ is open in Y, U is homeomorphic to $f(U) \times X'$ and the diagram

$$\begin{tikzcd} f(U) \times X' \arrow[r, "\cong"] \arrow[dr, "p"'] & U \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \\ & f(U) \arrow[r, hook] & Y \end{tikzcd}$$

commutes, where p is the obvious projection.

Example 1.3.22. (i) By Corollary 1.3.19, if X is locally contractible and hypercomplete, then $X \to *$ is a shape submersion.

(ii) Any topological submersion of fiber dimension n is a shape submersion.

Remark 1.3.23. (i) It follows easily from the definition that shape submersions are stable under pullbacks of topological spaces.

(ii) If $f : X \to Y$ is a shape submersion, then the set of open subsets of X of the type $X' \times V$, where V is open in Y and X' is essential, forms a basis for the topology of X. Although this basis is not closed under finite intersections, the set of representable sheaves corresponding to open subsets homeomorphic to the product of an open in Y and an essential space generates $\mathcal{S}\mathrm{hv}(X)$ under colimits. To see this, consider

$$\mathcal{B} = \{U \in \mathcal{U}(X) \mid U \cong W, \text{with}\, W \in \mathcal{U}(V \times S)\, \text{for some}\, V \in \mathcal{U}(Y), S\, \text{essential}\}.$$

The set $\mathcal{B}$ clearly forms a basis closed under finite intersections, and then we have $\mathcal{S}\mathrm{hv}(X) \simeq \mathcal{S}\mathrm{hv}(\mathcal{B})$ by [Aok20, Appendix A]. Moreover, since open immersions induce essential geometric morphisms, and since $\mathcal{U}(V) \times \mathcal{U}(S)$ forms a basis of $V \times S$ which is closed under finite intersections, we get our claim.

In the particular case of a topological submersion of fiber dimension n, since $\mathbb{R}^n$ is hypercomplete, we have an equivalence $\mathcal{S}\mathrm{hv}(\mathbb{R}^n) \simeq \mathcal{S}\mathrm{hv}(\mathcal{W})$ where $\mathcal{W} \subseteq \mathcal{U}(\mathbb{R}^n)$ is the poset of open balls inside $\mathbb{R}^n$, and thus, since $\mathbb{R}^n$ is locally compact, we have

$\mathcal{S}\mathrm{hv}(V \times \mathbb{R}^n) \simeq \mathcal{S}\mathrm{hv}(V) \otimes \mathcal{S}\mathrm{hv}(\mathcal{W})$. In particular, the set of representable sheaves corresponding to open subsets homeomorphic to the product of an open in Y and an open ball in $\mathbb{R}^n$ generates $\mathcal{S}\mathrm{hv}(X)$ under colimits.

For technical reasons that will be justified in a moment, from now on whenever we have a shape submersion $f : X \to Y$ we will assume that either Y is locally compact or all spaces of locally constant shape appearing in the basis of X are locally compact.

Lemma 1.3.24. Any shape submersion $f : X \to Y$ induces an essential geometric morphism. Thus, for any presentable ∞-category $\mathcal{C}$, we obtain an adjunction $f_\sharp \dashv f^*$ for $\mathcal{C}$-valued sheaves.

Proof. Since f^* is a left adjoint, it is in particular accessible. Hence, by adjoint functor theorem, it suffices to prove that f^* commutes with limits. For any functor $I \to \mathcal{S}\mathrm{hv}(Y)$ we have a canonical map

$$f^*(\varprojlim_{i\in I} F_i) \to \varprojlim_{i\in I} f^* F_i$$

and it suffices to check that this is an equivalence after restricting to any open subset in the basis of X associated to f. Hence, since the operation of restricting a sheaf to an open subset commutes with limits, we can assume that f is a projection $f : X \times Y \to Y$ where either X or Y is locally compact and X is essential. Thus, by point (iii) in Example 1.3.14 and Proposition 1.2.30, we get that f is essential. The last assertion follows immediately by Corollary 1.2.24. $\square$

Lemma 1.3.25 (Smooth base change)**.** Let $\mathcal{C}$ be a presentable ∞-category. For every given pullback square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\ Y' \arrow[r, "f"] & Y \end{tikzcd}$$

of topological spaces where g and g' are shape submersions, there is a natural equivalence

$$f^* g_\sharp \simeq g'_\sharp f'^*$$

and, by transposition, also

$$g^* f_* \simeq f'_* g'^*$$

for $\mathcal{C}$-valued sheaves.

Proof. First of all, the base change transformation as defined in Remark 1.3.7 defnes a comparison natural transformation. Since all functors appearing are colimit preserving, it suffices to check that the morphism is an equivalence only on a family of objects generating $\mathcal{S}\mathrm{hv}(X;)$ by colimits. Hence, applying Remark 1.3.23 (ii), we see that we can assume that the pullback square is of the type

$$\begin{tikzcd} X \times Y' \arrow[r, "\mathrm{id}_X \times f"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \times Y \arrow[d, "g"] \\ Y' \arrow[r, "f"] & Y \end{tikzcd}$$

where X is essential, g and g' are the canonical projections. By Example 1.3.14 point (iii) and Proposition 1.2.30, we have $(\mathrm{id}_X \times f)^* \simeq (\mathrm{id}_X)^* \otimes f^*$, $g_\sharp \simeq a_\sharp \otimes (\mathrm{id}_Y)_\sharp$ and $g'_\sharp \simeq a_\sharp \otimes (\mathrm{id}_{Y'})_\sharp$, where $a : X \to *$, and so, since $\otimes$ is a bifunctor, we have

$$\begin{aligned} f^*g_\sharp &\simeq f^*(a_\sharp \otimes (\mathrm{id}_Y)_\sharp) \\ &\simeq a_\sharp \otimes f^* \\ &\simeq (a_\sharp \otimes (\mathrm{id}_{Y'})_\sharp)(\mathrm{id}_X^* \otimes f^*) \\ &\simeq g'_\sharp(\mathrm{id}_X \times f)^*. \end{aligned}$$

$\square$

Corollary 1.3.26 (Smooth projection formula)**.** Let $f : X \to Y$ be a shape submersion, $\mathcal{C}$ and $\mathcal{D}$ two presentable ∞-categories. Then for any $F \in \mathcal{S}\mathrm{hv}(X; \mathcal{C})$ and $G, H \in \mathcal{S}\mathrm{hv}(Y; \mathcal{D})$, we have a canonical equivalence

$$f_\sharp(F \otimes f^*G) \simeq f_\sharp F \otimes G$$

and hence, by transposition, when $\mathcal{C} = \mathcal{D}$ is monoidal, equivalences

$$f_*\underline{\mathrm{Hom}}_X(F, f^*G) \simeq \underline{\mathrm{Hom}}_Y(f_\sharp F, G)$$

and

$$f^*\underline{\mathrm{Hom}}_Y(G, H) \simeq \underline{\mathrm{Hom}}_X(f^*G, f^*H).$$

In particular, any shape submersion induces a locally contractible geometric morphism.

Proof. Let $\Gamma_f : X \to Y$ be the graph of f. We have

$$\begin{aligned} f_\sharp(F \otimes f^*G) &\simeq f_\sharp\Gamma_f^*(F \boxtimes G) \\ &\simeq \Delta^*(f \times \mathrm{id}_Y)_\sharp(F \boxtimes G) \\ &\simeq f_\sharp F \otimes G \end{aligned}$$

where the second equivalence follows by applying Lemma 1.3.25 to the pullback square

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"] & X \times Y \arrow[d, "f \times \mathrm{id}_Y"] \\ Y \arrow[r, "\Delta"] & Y \times Y. \end{tikzcd}$$

The last assertion follows by specializing (1.2.28) to the case when $\mathcal{C}$ is $\mathcal{S}$ equipped with the cartesian monoidal structure and by the commutativity of the diagram

$$\begin{tikzcd} F \times f^*G \arrow[r] \arrow[d, "\simeq"] & f^*(f_\sharp F \times G) \arrow[d, "\simeq"] \\ \Gamma_f^*(F \boxtimes G) \arrow[r] & \Gamma_f^*(f \times \mathrm{id}_Y)^*(f \times \mathrm{id}_Y)_\sharp(F \boxtimes G) \end{tikzcd}$$

where the upper horizontal arrow is the one which transposes to the projection morphism and the lower horizontal one transposes to the smooth base change transformation. $\square$

1.4 Localization sequences

We will now prove a version of the localization theorem for sheaves of spaces (and of spectra) on a topological space X: this essentially states that, for any closed immersion $i : Z \to X$ with open complement $j : U \to X$, the inclusions

$$\mathcal{S}\mathrm{hv}(Z) \xhookrightarrow{i_*} \mathcal{S}\mathrm{hv}(X) \xhookleftarrow[j_*]{} \mathcal{S}\mathrm{hv}(U)$$

form a *recollement* (in the sense of [Lur17, Definition A.8.1]). Achieving this goal in our context will be slightly more complicated than in the case of [KS90, Proposition 2.3.6], namely because we don't want to assume all our spaces to be hypercomplete. We will follow instead the strategy outlined in [Kha19]: the main ingredient will be to show that the pushforward $i_* : \mathcal{S}\mathrm{hv}(Z) \to \mathcal{S}\mathrm{hv}(X)$ commutes with *contractible* colimits, i.e. colimits indexed by contractible simplicial sets. From this we will be able to reduce to checking the theorem in the case of representable sheaves, which is almost straightforward.

We start by reporting [Kha19, Definition 3.1.5] and [Kha19, Lemma 3.1.6].

Definition 1.4.1. Let X and Y be essentially small ∞-sites, and assume that Y admits an initial object $\emptyset_Y$. A functor $u : X \to Y$ is *topologically quasi-cocontinuous* if for every covering sieve $R' \hookrightarrow y(u(x))$ in Y, the sieve $R \hookrightarrow y(x)$, generated by morphisms $x' \to x$ such that either $u(x')$ is initial or $y(u(x')) \to y(u(x))$ factors through $R' \hookrightarrow y(u(x))$, is a covering in X.

Lemma 1.4.2. With notation as in the previous definition, let $u : X \to Y$ be a topologically quasicocontinuous functor. Assume that the initial object $\emptyset_Y$ is strict in the sense that for any object $y \in Y$, any morphism $d \to \emptyset_Y$ is invertible. Assume also that, for any object $y \in Y$, the sieve $\emptyset_{\mathrm{Fun}(Y,\mathcal{S})} \hookrightarrow y(d)$ is a covering in Y if and only if y is initial (where $\emptyset_{\mathrm{Fun}(Y,\mathcal{S})}$ denotes the initial object of $\mathrm{Fun}(Y, \mathcal{S})$). Then the functor $\mathcal{S}\mathrm{hv}(Y) \to \mathcal{S}\mathrm{hv}(X)$, given by the assignment $F \mapsto L_X(u^*(F))$, where $L_X : \mathrm{Fun}(X, \mathcal{S}) \to \mathcal{S}\mathrm{hv}(X)$ denotes the sheafification functor, commutes with contractible colimits.

Lemma 1.4.3. Let $i : Z \hookrightarrow X$ be a closed immersion. Then i_* commutes with contractible colimits.

Proof. By the lemma above and by unraveling the definition of topologically quasi-cocontinuous functor, this amounts to check that, for any $V \in \mathcal{U}(X)$ and any open covering $\{W_i\}_{i \in I} \subseteq \mathcal{U}(Z)$ of $V \cap Z$, the family

$$T = \{U \subseteq V \mid U \cap Z \ = \emptyset \text{ or } U \cap Z \subseteq W_i \,\text{for some}\, i \in I\} \subseteq \mathcal{U}(X)$$

covers V. But this is clear, because $V \setminus Z \in T$ and any W_i can be written as $W_i' \cap Z$ for some $W_i' \in \mathcal{U}(V)$. ☐

Corollary 1.4.4. Let $\mathcal{C}$ be any pointed presentable ∞-category. Then the pushforward $i_*^{\mathcal{C}} : \mathcal{S}\mathrm{hv}(Z; \mathcal{C}) \to \mathcal{S}\mathrm{hv}(X; \mathcal{C})$ commutes with all colimits, and thus admits a right adjoint $i^!_{\mathcal{C}} : \mathcal{S}\mathrm{hv}(X; \mathcal{C}) \to \mathcal{S}\mathrm{hv}(Z; \mathcal{C})$.

Proof. It suffices to prove the corollary for $\mathcal{C} = \mathcal{S}_*$. Note that it suffices to check that i_* preserves the initial object and commutes with contractible colimits: any $F : I \to \mathcal{D}$ from a simplicial set I to an ∞-category $\mathcal{D}$ with an initial object $\emptyset_{\mathcal{D}}$ may be seen as $I \to \mathcal{D}_{\emptyset_{\mathcal{D}}/}$ and thus corresponds to a functor $\Delta^0 \star I \to \mathcal{D}$ with the same colimit as F but indexed by weakly contractible simplicial set. But i_* preserves the initial object because $\mathcal{S}\mathrm{hv}(Z; \mathcal{S}_*)$ is pointed, and thus we may conclude by the previous lemma. ☐

Let $i : Z \hookrightarrow X$ be a closed immersion with open complement $j : U \hookrightarrow X$. For any $F \in \mathcal{S}\mathrm{hv}(X)$, consider the functorial commutative square

$$\begin{tikzcd} j_\sharp j^*(F) \arrow[r] \arrow[d] & F \arrow[d] \\ j_\sharp j^* i_* i^*(F) \arrow[r] & i_* i^*(F), \end{tikzcd}$$

where all the morphisms are given by the obvious units and counits. Notice that for any $G \in \mathcal{S}\mathrm{hv}(X)$ and $V \in \mathcal{U}(U)$, we have

$$\Gamma(V; i_* G) = \Gamma(U \cap Z; G) \simeq *,$$

and so we can identify $j^* i_* : \mathcal{S}\mathrm{hv}(Z) \to \mathcal{S}\mathrm{hv}(U)$ with a constant functor with value the terminal object $y(U) \in \mathcal{S}\mathrm{hv}(U)$. Hence the previous square may be written as

$$\begin{tikzcd} j_\sharp j^*(F) \arrow[r] \arrow[d] & F \arrow[d] \\ j_\sharp(y(U)) \arrow[r] & i_* i^*(F). \end{tikzcd} \tag{1.4.5}$$

Theorem 1.4.6. The canonical square (1.4.5) is a pushout.

Proof. Since all functors appearing in (1.4.5) commute with contractible colimits and any sheaf on X is canonically written as colimit indexed by the contractible category $\mathcal{U}(X)_{/F} = \mathcal{S}\mathrm{hv}(X)_{/F} \times_{\mathcal{S}\mathrm{hv}(X)} \mathcal{U}(X)$ (it has an initial object), it suffices to prove the theorem when $F = y(V)$ for some $V \in \mathcal{U}(X)$, and hence we just need to show that $i_* i^*(y(V)) \simeq y(U \cup V)$. For any $W \in \mathcal{U}(X)$, we have

$$\begin{aligned} \Gamma(W; i_* i^*(y(V))) &\simeq \Gamma(W; i_*(y(V \cap Z))) \\ &= \Gamma(W \cap Z; y(V \cap Z)) \\ &= \mathrm{Hom}_{\mathcal{U}(Z)}(W \cap Z, V \cap Z) \\ &= \mathrm{Hom}_{\mathcal{U}(X)}(W, V \cup U) \\ &= \Gamma(W; y(V \cup U)), \end{aligned}$$

where the second to last identification follows by the usual exponential adjunction in the boolean algebra of all subsets of X. $\square$

Corollary 1.4.7. Let $i : Z \hookrightarrow X$ be a closed immersion with open complement $j : U \hookrightarrow X$, and let $i_*^{\mathcal{C}}$, $i_{\mathcal{C}}^*$, $i_{\mathcal{C}}^!$, $j_\sharp^{\mathcal{C}}$ and $j_{\mathcal{C}}^*$ be the induced pushforward and pullback functors at the level of $\mathcal{C}$-valued sheaves, where $\mathcal{C}$ is any pointed presentable ∞-category. Then we get a canonical cofiber sequence

$$j_\sharp^{\mathcal{C}} j_{\mathcal{C}}^* F \to F \to i_*^{\mathcal{C}} i_{\mathcal{C}}^* F \tag{1.4.8}$$

and dually a fiber sequence

$$i_*^{\mathcal{C}} i_{\mathcal{C}}^! F \to F \to j_*^{\mathcal{C}} j_{\mathcal{C}}^* F. \tag{1.4.9}$$

denote

Proof. It suffices to treat only the case of the sequence (1.4.8). Furthermore, we only need to prove the case of sheaves of pointed spaces, since all functors appearing are colimit preserving and we have a canonical equivalence $\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \simeq \mathcal{S}\mathrm{hv}(X;\mathcal{S}_*) \otimes \mathcal{C}$, where $\mathcal{C}$ is any presentable pointed ∞-category and X is any topological space. We define the canonical morphisms in (1.4.8) through counit and unit of the appropriate adjunctions, and we see immediately that the composition of those two morphisms is null-homotopic because $i^*_{\mathcal{S}_*} j^{\mathcal{S}_*}_\sharp$ is equivalent to a constant functor with value the zero object in $\mathcal{S}\mathrm{hv}(Z;\mathcal{S}_*)$.

Let $\alpha : \mathcal{S}\mathrm{hv}(X;\mathcal{S}_*) \to \mathcal{S}\mathrm{hv}(X)$ be the forgetful functor. A close inspection of the appropriate universal properties shows that, for any $F \in \mathcal{S}\mathrm{hv}(X;\mathcal{S}_*)$, there is a canonical pushout square

$$\begin{tikzcd}
j_\sharp(y(U)) \arrow[r] \arrow[d] & y(X) \arrow[d] \\
j_\sharp j^*\alpha(F) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \alpha(j^{\mathcal{S}_*}_\sharp j^*_{\mathcal{S}_*} F),
\end{tikzcd}$$

where the left vertical map is induced by the point of F. Thus, since α reflects pushouts and we have an equivalence $\alpha i^{\mathcal{S}_*}_* i^*_{\mathcal{S}_*} \simeq i_* i^* \alpha$, it suffices to prove that the canonical square

$$\begin{tikzcd}
\alpha(j^{\mathcal{S}_*}_\sharp j^*_{\mathcal{S}_*} F) \arrow[r] \arrow[d] & \alpha(F) \arrow[d] \\
y(X) \arrow[r] & i_* i^* \alpha(F).
\end{tikzcd}$$

induced by applying α to the sequence (1.4.8) is a pushout. For this purpose, consider the commutative diagram

$$\begin{tikzcd}
j_\sharp(y(U)) \arrow[r] \arrow[d] & y(X) \arrow[d] & \\
j_\sharp j^*\alpha(F) \arrow[r] \arrow[d] & \alpha(j^{\mathcal{S}_*}_\sharp j^*_{\mathcal{S}_*} F) \arrow[r] \arrow[d] & \alpha(F) \arrow[d] \\
j_\sharp j^*(y(X)) \arrow[r] & y(X) \arrow[r] & i_* i^* \alpha(F).
\end{tikzcd}$$

denote The upper left square and the left vertical rectangle are both pushouts, and so also the lower left square is a pushout. But the lower horizontal rectangle is a pushout, and so we can conclude. ☐

Corollary 1.4.10. Consider a pullback square

$$\begin{tikzcd}
Z' \arrow[r, "f'"] \arrow[d, hook, "s'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, hook, "s"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

where f and f' are shape submersions and s (and consequently s') is a closed immersion. Then, for sheaves with values in a pointed presentable ∞-category, we have a canonical equivalence

$$s_! f'_\sharp \simeq f_\sharp s'_!$$

or equivalently

$$f'^* s^! \simeq s'^! f^*.$$

Proof. Let j and j' be the complement open immersions associated respectively to s and s'. By the localization sequences and by smooth base change, we have a commutative diagram where all the rows are cofiber sequences and all the vertical arrows are invertible

$$\begin{tikzcd} j_\sharp j^* f_\sharp \arrow[r] \arrow[d, "\simeq"] & f_\sharp \arrow[r] \arrow[d, "\simeq"] & s_! s^* f_\sharp \arrow[d, "\simeq"] \\ j_\sharp f'_\sharp j'^* \arrow[r] \arrow[d, "\simeq"] & f_\sharp \arrow[r] \arrow[d, "\simeq"] & s_! f'_\sharp s'^* \arrow[d, dashed, "\simeq"] \\ f_\sharp j'_\sharp j'^* \arrow[r] & f_\sharp \arrow[r] & f_\sharp s'_! s'^*. \end{tikzcd}$$

Hence we may conclude by precomposing the dotted equivalendenotece with $s'_!$, since $s'_!$ is fully faithful. ☐

Remark 1.4.11. It is not hard to see that one may deduce from Theorem 1.4.6 that i_* is fully faithful (this was already proven in [Lur09, Corollary 7.3.2.10]). From this follows immediately that, in the case of sheaves of pointed spaces or of spectra, one has the identification

$$i^! \simeq \mathrm{fib}\ (i^* \xrightarrow{i^*(unit)} i^* j_* j^*).$$

Remark 1.4.12. From Theorem 1.4.6 one deduces immediately that, at least when $\mathcal{C}$ is presentable stable, the functors i^* and j^* are *jointly conservative*, i.e. a morphism $\alpha : F \to G$ in $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ is invertible if and only if both $i^*(\alpha)$ and $j^*(\alpha)$ are invertible. This implies in particular that the fully faithful functors i_* and j_* make $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ a *recollement* of $\mathcal{S}\mathrm{hv}(Z;\mathcal{C})$ and $\mathcal{S}\mathrm{hv}(U;\mathcal{C})$, in the sense of [Lur17, Definition A.8.1]. However, this is true in a much greater generality: see [Hai21] for a proof in the cases when $\mathcal{C}$ is an ∞-topos or compactly generated. After Corollary 1.5.16, for stable coefficients, we will also be able to relax the presentability assumption to the more general requirement for $\mathcal{C}$ to admit both limits and colimits.

1.5 Pullbacks with stable bicomplete coefficients

From now on, unless otherwise specified, all the topological spaces we will deal with will be assumed to be *locally compact and Hausdorff.* This implies that the following are equivalent

1. $f : X \to Y$ is proper (i.e. the preimage of any compact subset of Y is compact);
2. f is closed with compact fibers;
3. f is universally closed.

Another important consequence of the previous assumption is that any map $X \xrightarrow{f} Y$ can be factored as a composition of a closed immersion (which is in particular proper by the characterization above), an open immersion and a proper map as follows

(1.5.1)
$$\begin{tikzcd} X \times Y \arrow[r, "j \times id_Y"] & \overline{X} \times Y \arrow[d, "p"] \\ X \arrow[u, "\Gamma_f"] \arrow[r, "f"] & Y \end{tikzcd}$$

where Γ_f is the graph of f, $X \xrightarrow{j} \overline{X}$ is the inclusion of X into its one point compactification and p is the projection to the second coordinate: this factorization will be used very often later. Notice that one may also use the Stone-Cech compactification β to produce a functorial functorization

(1.5.2)
$$\begin{tikzcd}
X \arrow[drr, bend left, "f"] \arrow[dr, hook, dotted, "j"] \arrow[ddr, hook, bend right] & & \\
& \overline{X} \arrow[r, "p"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, hook] \\
& \beta X \arrow[r] & \beta Y
\end{tikzcd}$$

where j is an open immersion and p is proper. However, in many cases it will turn out to be more convenient to have the proper map in the factorization to be a product projection.

This section contains the central technical ingredients of this chapter. We will explain what are the two fundamental facts that allow us to construct pullbacks with non-presentable coefficients: in brief, these these are *covariant Verdier duality* (see [Lur17, Theorem 5.5.5.1]) and the dualizability of $\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})$ in $\mathcal{C}\mathrm{ocont}_\infty^{st}$.

We will start by giving an exposition of covariant Verdier Duality. Following [Lur17, Theorem 5.5.5.1], this is an equivalence between the categories of sheaves and cosheaves on a locally compact Hausdorff space. We will essentially review the proof of [Lur17, Theorem 5.5.5.1], and try to clarify a bit the last step. The reader who is aware of this result may safely skip the first part of this section. Later, we will prove that $\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})$ is strongly dualizable in the symmetric monoidal ∞-category $\mathcal{C}\mathrm{ocont}_\infty^{st}$, by explicitely exhibiting it as a retract in $\mathcal{C}\mathrm{ocont}_\infty^{st}$ of a compactly generated ∞-category. This result is not new, as it could be deduced from [Lur16, Proposition 21.1.7.1] (see also [Hoy]): we would like to thank Peter Haine for pointing this out. We then use the above retraction to provide a convenient description of pushforwards along proper maps, that is compatible with tensor products in $\mathcal{C}\mathrm{ocont}_\infty^{st}$. We conlude the section by showing that $f^*_{\mathcal{S}\mathrm{p}} \otimes \mathcal{C}$ is a left adjoint of $f^{\mathcal{C}}_*$, which is not at all immediate. It will require a use of the factorizations mentioned in the beginning, and a careful combination of all the previously mentioned results.

1.5.1 Recollections on covariant Verdier duality

We start by recalling the definition of $\mathcal{K}$-sheaves.

Definition 1.5.3. Let $\mathcal{K}(X)$ be the poset of compact subsets of a topological space X. Consider the following conditions:

(i) $F(\emptyset)$ is a terminal object,

(ii) For every $K, K' \in \mathcal{K}(X)$ the square

$$\begin{tikzcd}
\Gamma(K \cup K'; F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma(K; F) \arrow[d] \\
\Gamma(K'; F) \arrow[r] & \Gamma(K \cap K'; F)
\end{tikzcd}$$

is pullback,

(iii) For every $K \in \mathcal{K}(X)$, the canonical map

$$\varinjlim_{K \Subset K'} \Gamma(K'; F) \to \Gamma(K; F)$$

is invertible, where $K \Subset K'$ means that K' contains an open neighbourhood of K.

Notice that (i) and (ii) together are equivalent to the sheaf condition for the Grothendieck topology on $\mathcal{K}(X)$ given by finite covering. Hence, we will denote by $\mathcal{S}\mathrm{hv}(\mathcal{K}(X);\mathcal{C})$ the full subcategory spanned by presheaves satifying (i) and (ii). Moreover, we will say that a functor $F : \mathcal{K}(X) \to \mathcal{C}$ is a $\mathcal{K}$-*sheaf* if it satisfies (i), (ii) and (iii). We will denote by $\mathcal{S}\mathrm{hv}_{\mathcal{K}}(X;\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}},\mathcal{C})$ the full subcategory spanned by $\mathcal{K}$-sheaves.

It is possible to relate $\mathcal{K}$-sheaves with usual sheaves. Let M be the union $\mathcal{U}(X) \cup \mathcal{K}(X)$ considered as a poset contained in the power set of X, and let $i : \mathcal{U}(X) \hookrightarrow M$ and $j : \mathcal{K}(X) \hookrightarrow M$ be the corresponding inclusion. We thus get two adjunctions

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}},\mathcal{C}) \arrow[r, bend left, "i_!"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(M^{\mathrm{op}},\mathcal{C}) \arrow[l, bend left, "i^*"] \arrow[r, bend left, "j^*"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}},\mathcal{C}). \arrow[l, bend left, "j_*"] \end{tikzcd}$$

More explicitly, at the level of objects the functors are given by the formulas

$$\begin{aligned} \mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}},\mathcal{C}) &\xrightarrow{\quad\theta\quad} \mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}},\mathcal{C}) \\ F &\longmapsto (K \mapsto \varinjlim_{K\subseteq U} \Gamma(U;F)) \end{aligned}$$

$$\begin{aligned} \mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}},\mathcal{C}) &\xrightarrow{\quad\psi\quad} \mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}},\mathcal{C}) \\ G &\longmapsto (U \mapsto \varprojlim_{K\subseteq U} \Gamma(K;G)). \end{aligned}$$

These two functors actually restrict to an equivalence, assuming $\mathcal{C}$ has limits and colimits, and filtered colimits are exact.

Theorem 1.5.4. Let $\mathcal{C}$ be a bicomplete ∞-category where filtered colimits are exact. Then the functors θ and ψ defined above restrict to an equivalence

$$\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \simeq \mathcal{S}\mathrm{hv}_{\mathcal{K}}(X;\mathcal{C}).$$

Proof. A proof can be found in [Lur09, Theorem 7.3.4.9]. □

Remark 1.5.5. Since in any stable ∞-category $\mathcal{C}$ filtered colimits are exact, and since the opposite of any stable ∞-category is again stable, by Theorem 1.5.4 we get equivalences

$$\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \simeq \mathcal{S}\mathrm{hv}_{\mathcal{K}}(X;\mathcal{C})$$

and

$$\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \simeq \mathcal{S}\mathrm{hv}(X;\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}_{\mathcal{K}}(X;\mathcal{C})$$

where we define $\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}_{\mathcal{K}}(X;\mathcal{C})$ to be $\mathcal{S}\mathrm{hv}_{\mathcal{K}}(X;\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$.

Definition 1.5.6. Let $F \in \mathcal{S}\mathrm{hv}(X;\mathcal{C})$, $U \in \mathcal{U}(X)$, and K any closed subset of X. We define the *sections of* F *supported at* K and *compactly supported sections of* F *over* U respectively as

$$\Gamma_K(X;F) \coloneqq \mathrm{fib}(\Gamma(X;F) \to \Gamma(X \setminus K;F))$$
$$\Gamma_c(U;F) \coloneqq \varinjlim_{K\subseteq U} \Gamma_K(X;F),$$

where the colimit ranges over all compact subsets of U.

Remark 1.5.7. Notice that, if $K \subseteq U$ for some open U, we get a pullback square

$$\begin{tikzcd} \Gamma(X;F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma(X \setminus K;F) \arrow[d] \\ \Gamma(U;F) \arrow[r] & \Gamma(U \setminus K;F) \end{tikzcd}$$

and hence the fibers of the two horizontal maps coincide: for this reason, we will often also $\Gamma_K(X;F)$ by $\Gamma_K(U;F)$. Furthermore, if $S \subseteq X$ is locally closed, we define

$$\Gamma_S(X;F) \coloneqq \Gamma_Z(U;j^*F)$$

where $S = U \cap Z$, with U open, Z closed and $j : U \hookrightarrow X$ the inclusion, but we will also use the notation $\Gamma_S(U;F)$.

Remark 1.5.8. The definition of the sections of a sheaf F on a compact K is functorial both in F and in K: since we have an obvious functor

$$\begin{aligned} \mathcal{K}(X) &\longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{U}(X)^{\mathrm{op}}) \\ K &\longmapsto (X \to X \setminus K) \end{aligned}$$

we get

$$\begin{tikzcd} \mathcal{K}(X) \times \mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}}, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[r, "\mathrm{fib}"] & \mathcal{C} \\ \mathrm{Fun}(\Delta^1, \mathcal{U}(X)^{\mathrm{op}}) \times \mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}}, \mathcal{C}) \arrow[ur, "\circ"] & & \end{tikzcd}$$

where the diagonal arrow is the composition of functors and the right horizontal arrow is given by taking the fiber of an arrow in $\mathcal{C}$, and so by adjunction we get the desired

$$\begin{aligned} \mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}}, \mathcal{C}) &\longrightarrow \mathrm{Fun}(\mathcal{K}(X), \mathcal{C}) \\ F &\longmapsto (K \mapsto \Gamma_K(X;F)). \end{aligned}$$

Finally, by further composing with the functor ψ defined in Theorem 1.5.4, we get

$$\begin{aligned} \mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}}, \mathcal{C}) &\xrightarrow{\mathbb{D}_{\mathcal{C}}} \mathrm{Fun}(\mathcal{U}(X), \mathcal{C}) \\ F &\longmapsto (U \mapsto \Gamma_c(U;F)). \end{aligned} \tag{1.5.9}$$

Theorem 1.5.10. The functor (1.5.9) restricts to an equivalence

$$\mathbb{D}_{\mathcal{C}} : \ \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \xrightarrow{\simeq} \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C}).$$

Proof. We first prove that, if F is a sheaf, then $\mathbb{D}_{\mathcal{C}}(F)$ is a cosheaf. By virtue of Theorem 1.5.4, it suffices to prove that the functor

$$K \mapsto \Gamma_K(X;F)$$

is a $\mathcal{K}$-cosheaf.

- $\Gamma_\emptyset(X;F) \simeq 0$ since $F(X) \to F(X \setminus \emptyset)$ is an equivalence.

- Let $K, K' \in \mathcal{K}(X)$. The square

$$\begin{tikzcd} \Gamma_{K\cap K'}(X;F) \arrow[r] \arrow[d] & \Gamma_K(X;F) \arrow[d] \\ \Gamma_{K'}(X;F) \arrow[r] & \Gamma_{K\cup K'}(X;F) \end{tikzcd}$$

is the fiber of the obvious map between the pullback squares

$$\begin{tikzcd} \Gamma(X;F) \arrow[r] \arrow[d] & \Gamma(X;F) \arrow[d] \\ \Gamma(X;F) \arrow[r] & \Gamma(X;F) \end{tikzcd} \qquad \begin{tikzcd} \Gamma(X\setminus(K\cap K');F) \arrow[r] \arrow[d] & \Gamma(X\setminus K;F) \arrow[d] \\ \Gamma(X\setminus K';F) \arrow[r] & \Gamma(X\setminus(K\cup K');F), \end{tikzcd}$$

and so it is a pullback. Thus, since $\mathcal{C}$ is stable, it's also a pushout.

- For any $K \in \mathcal{K}(X)$, we have a map of fiber sequences

$$\begin{tikzcd} \Gamma_K(X;F) \arrow[r, "a"] \arrow[d] & \varprojlim_{K\Subset K'} \Gamma_{K'}(X;F) \arrow[d] \\ \Gamma(X;F) \arrow[r, "b"] \arrow[d] & \varprojlim_{K\Subset K'} \Gamma(X;F) \arrow[d] \\ \Gamma(X\setminus K;F) \arrow[r, "c"] & \varprojlim_{K\Subset K'} \Gamma(X\setminus K';F). \end{tikzcd}$$

To prove that a is an equivalence, it suffices to prove that b and c are. But b is an equivalence because the poset $\{K' \in \mathcal{K}(X) \mid K \Subset K'\}$ has a contractible nerve (since it is filtered) and c is an equivalence because $\{U \in \mathcal{U}(X) \mid U = X \setminus K' \text{ for some } K \Subset K'\}$ gives an open covering of $X \setminus K$.

We will now prove that $\mathbb{D}^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}}$ is an inverse of $\mathbb{D}_{\mathcal{C}}$. By symmetry, it suffices to show that it is a left inverse. Unraveling the definitions and using the equivalence of Theorem 1.5.4, this amounts to check that we have a cofiber sequence

$$\Gamma_c(X\setminus K;F) \longrightarrow \Gamma_c(X;F) \longrightarrow \Gamma(K;F) \tag{1.5.11}$$

natural in K and F.

First of all, we show that it suffices to prove that, for any fixed $K \in \mathcal{K}(X)$, $U \in \mathcal{U}(X)$ containing K and with compact closure, and $K' \in \mathcal{K}(X)$ containing U, the sequence

$$\Gamma_{K'\setminus U}(X;F) \longrightarrow \Gamma_{K'}(X;F) \longrightarrow \Gamma(U;F), \tag{1.5.12}$$

where the first morphism is given by the functoriality of sections supported on a compact and the second one is given by Remark 1.5.7, is a cofiber sequence. To see this, we start by noticing that the sequence is natural in K' and F, since both morphisms are canonically induced by the restrictions of F. Thus we can pass to the colimit ranging over all compacts $K' \supseteq U$ and get a fiber sequence

$$\varinjlim_{K'\supseteq U} \Gamma_{K'\setminus U}(X;F) \longrightarrow \Gamma_c(X;F) \longrightarrow \Gamma(U;F),$$

since the poset $\{K' \in \mathcal{K}(X) \mid K' \supseteq U\}$ is filtered (it is non-empty because it contains the closure of U) and the inclusion $\{K' \in \mathcal{K}(X) \mid K' \supseteq U\} \subseteq \mathcal{K}(X)$ is cofinal. Since any $K' \supseteq U$ is contained in $(K' \cup \overline{U}) \setminus U$, we get an equivalence

$$\varinjlim_{K' \supseteq U} \Gamma_{K' \setminus U}(X; F) \simeq \varinjlim_{\{K' \mid K' \cap U = \emptyset\}} \Gamma_{K'}(X; F),$$

and hence, adding everything up, we obtain a fiber sequence

$$\varinjlim_{\{K' \mid K' \cap U = \emptyset\}} \Gamma_{K'}(X; F) \longrightarrow \Gamma_c(X; F) \longrightarrow \Gamma(U; F),$$

which is natural in U, since the morphism $\Gamma_c(X; F) \to \Gamma(U; F)$ clearly is. Hence we can get the desired sequence (1.5.11) by passing to the colimit ranging over $P = \{U \in \mathcal{U}(X) \mid \overline{U} \in \mathcal{K}(X) \text{ and } U \supseteq K\}$ because $\Gamma(K; F) = \varinjlim_{U \in P} \Gamma(U; F)$ (since open subsets with compact closure form a basis of X), and because we have equivalences

$$\begin{aligned} \varinjlim_{U \in P} \varinjlim_{\{K' \mid K' \cap U = \emptyset\}} \Gamma_{K'}(X; F) &\simeq \varinjlim_{\bigcup_{U \in P} \{K' \mid K' \cap U = \emptyset\}} \Gamma_{K'}(X; F) \\ &\simeq \varinjlim_{K' \subseteq X \setminus K} \Gamma_{K'}(X; F) \\ &\simeq \Gamma_c(X \setminus K; F) \end{aligned}$$

where the first one follows by [Lur09, Remark 4.2.3.9] and [Lur09, Corollary 4.2.3.10].

We are now left to show that (1.5.12) is a cofiber sequence. Consider the commutative diagram

$$\begin{tikzcd}
\Gamma_{K' \setminus U}(X; F) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
\Gamma_{K'}(X; F) \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Gamma(X; F) \arrow[r] & \Gamma(X \setminus (K' \setminus U); F) \arrow[r] \arrow[d] & \Gamma(X \setminus K'; F) \arrow[d] \\
& \Gamma(U; F) \arrow[r] & 0
\end{tikzcd}$$

where $Z :- \mathrm{fib}(\Gamma(X \setminus (K' \setminus U); F) \to \Gamma(X \setminus K'; F))$. Since the middle big horizontal rectangle is a pullback, it follows that also the left middle square is. But since the left vertical rectangle is pullback, then also the upper left square is. Then it suffices to prove that the composition

$$Z \to \Gamma(X \setminus (K' \setminus U); F) \to \Gamma(U; F)$$

is an equivalence, but this is clear since the lower right square is pullback because F is a sheaf. $\square$

1.5.2 Dualizability of spectral sheaves

Lemma 1.5.13. Let $\mathcal{C}$ be a bicomplete ∞-category where filtered colimits are exact. Then there exists a functor $\varphi : \mathcal{S}\mathrm{hv}(\mathcal{K}(X);\mathcal{C}) \to \mathcal{S}\mathrm{hv}_{\mathcal{K}}(X;\mathcal{C})$ satisfying the following properties

1. for any $F \in \mathcal{S}\mathrm{hv}(\mathcal{K}(X);\mathcal{C})$ and $K \in \mathcal{K}(X)$, we have

$$\Gamma(K;\varphi F) \simeq \varinjlim_{K \Subset K'} \Gamma(K';F),$$

2. φ preserves filtered colimits,

3. it is right inverse to the inclusion $\mathcal{S}\mathrm{hv}_{\mathcal{K}}(X;\mathcal{C}) \hookrightarrow \mathcal{S}\mathrm{hv}(\mathcal{K}(X);\mathcal{C})$.

Proof. Let M be the set whose elements are pairs (K,i) with $K \in \mathcal{K}(X)$ and $i = 0,1$, where we define $(K,i) \leq (K',j)$ if $K' \subseteq K$ and $i = j$ or $K' \Subset K$ and $i < j$. It is easy to see that $\leq$ actually defines a partial order on M. We have two functors

$$\begin{array}{ccc} \mathcal{K}(X)^{\mathrm{op}} \xrightarrow{\ i_0\ } M & \qquad & \mathcal{K}(X)^{\mathrm{op}} \xrightarrow{\ i_1\ } M \\ K \longmapsto (K,0) & & K \longmapsto (K,1) \end{array}$$

and consider the cocontinuous functor

$$\varphi := (i_1)^*(i_0)_! : \mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}},\mathcal{C}) \to \mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}},\mathcal{C}).$$

Unravelling the definition, we see that

$$\Gamma(K;\varphi F) \simeq \varinjlim_{(K',0)\to(K,1)} \Gamma(K';F) \simeq \varinjlim_{K \Subset K'} \Gamma(K';F),$$

and so $\varphi F \simeq F$ whenever F is a $\mathcal{K}$-sheaf. Hence, to conclude the proof, it suffices to show that the essential image of $\varphi_{|_{\mathcal{S}\mathrm{hv}(\mathcal{K}(X);\mathcal{C})}}$ is contained in $\mathcal{S}\mathrm{hv}_{\mathcal{K}}(X;\mathcal{C})$.

Indeed, for any $F \in \mathcal{S}\mathrm{hv}(\mathcal{K}(X);\mathcal{C})$ we have

$$\begin{aligned} \varinjlim_{K \Subset K''} \Gamma(K'';\varphi F) &\simeq \varinjlim_{K \Subset K''} \varinjlim_{K'' \Subset K'} \Gamma(K';F) \\ &\simeq \varinjlim_{K \Subset K'} \Gamma(K';F) \\ &\simeq \Gamma(K;\varphi F), \end{aligned}$$

where the second equivalence holds by [Lur09, Remark 4.2.3.9] and [Lur09, Corollary 4.2.3.10] since for any K compact, the full subposet of $\mathcal{K}(X)^{\mathrm{op}}$ spanned by those K' such that $K \Subset K'$ is filtered, and we have

$$\bigcup_{\{K''|K \Subset K''\}} \{K'|K'' \Subset K'\} = \{K'|K \Subset K'\}.$$

Moreover, since filtered colimits are exact in $\mathcal{C}$, φF belongs to $\mathcal{S}\mathrm{hv}(\mathcal{K}(X);\mathcal{C})$, and so it is a $\mathcal{K}$-sheaf. ☐

Lemma 1.5.13 is useful to describe conveniently pushforwards of sheaves along proper maps. Let $f : X \to Y$ be a proper continuous map between topological spaces, and let again

$\mathcal{C}$ be a bicomplete ∞-category where filtered colimits are exact. Since for any $K \in \mathcal{K}(Y)$, the preimage $f^{-1}(K)$ is compact, we obtain a functor

$$\begin{aligned}\mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}}, \mathcal{C}) &\xrightarrow{f_+} \mathrm{Fun}(\mathcal{K}(Y)^{\mathrm{op}}, \mathcal{C}) \\ F &\longmapsto (K \mapsto \Gamma(f^{-1}(K); F)).\end{aligned}$$

Notice that the restriction of f_+ to $\mathcal{S}\mathrm{hv}_{\mathcal{K}}(X; \mathcal{C})$ lands in $\mathcal{S}\mathrm{hv}(\mathcal{K}(Y); \mathcal{C})$, but a priori not in $\mathcal{S}\mathrm{hv}_{\mathcal{K}}(Y; \mathcal{C})$, therefore we define

$$f_*^{\mathcal{K}} : \mathcal{S}\mathrm{hv}_{\mathcal{K}}(X; \mathcal{C}) \to \mathcal{S}\mathrm{hv}_{\mathcal{K}}(Y; \mathcal{C})$$

as the composition of f_+ restricted to $\mathcal{S}\mathrm{hv}_{\mathcal{K}}(X; \mathcal{C})$ and $\varphi : \mathcal{S}\mathrm{hv}(\mathcal{K}(Y); \mathcal{C}) \to \mathcal{S}\mathrm{hv}_{\mathcal{K}}(Y; \mathcal{C})$.

Lemma 1.5.14. Let $f : X \to Y$ be a proper continuous map between topological spaces, and let $\mathcal{C}$ be bicomplete ∞-category where filtered colimits are exact. Then there is a commutative diagram

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}(X; \mathcal{C}) \arrow[r, "\theta"] \arrow[d, "f_*"] & \mathcal{S}\mathrm{hv}_{\mathcal{K}}(X; \mathcal{C}) \arrow[d, "f_*^{\mathcal{K}}"] \\ \mathcal{S}\mathrm{hv}(Y; \mathcal{C}) \arrow[r, "\theta"] & \mathcal{S}\mathrm{hv}_{\mathcal{K}}(Y; \mathcal{C}), \end{tikzcd}$$

where θ is as in Theorem 1.5.4. In particular, f_* preserves filtered colimits, and when $\mathcal{C}$ is stable it preserves all colimits.

Proof. For any $K \in \mathcal{K}(Y)$, we define

$$T = \{U \in \mathcal{U}(X) \mid \exists K' \Supset K \text{ with } f^{-1}(K') \subseteq U\}.$$

Notice that if $V \in \mathcal{U}(Y)$ contains K, then there exists an open neighbourhood W of K with compact closure, and thus, since f is proper, $f^{-1}(V) \in T$. In particular we obtain a functor

$$\alpha : \{V \in \mathcal{U}(Y) \mid K \subseteq V\} \to T$$

which is obviously final. We have

$$\begin{aligned}\Gamma(K; f_*^{\mathcal{K}}\theta F) &\simeq \varinjlim_{K \Subset K'} \varinjlim_{f^{-1}(K') \subseteq U} \Gamma(U; F) \\ &\simeq \varinjlim_{U \in T} \Gamma(U; F) \\ &\simeq \varinjlim_{K \subseteq V} \Gamma(f^{-1}(V); F) \\ &\simeq \Gamma(K; \theta f_* F)\end{aligned}$$

where the second equivalence follows by [Lur09, Remark 4.2.3.9] and [Lur09, Corollary 4.2.3.10], and the third one since α is final. $\square$

Another straightforward application of Lemma 1.5.13 is the following theorem, that will be of crucial importance for what follows.

Theorem 1.5.15. The ∞-category $\mathcal{S}\mathrm{hv}(X; \mathcal{S}\mathrm{p})$ is a strongly dualizable object in $\mathcal{C}\mathrm{ocont}_\infty^{st}$. In particular, for any $\mathcal{C} \in \mathcal{C}\mathrm{ocont}_\infty^{st}$, the canonical functor

$$\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X; \mathcal{S}\mathrm{p}) \otimes \mathcal{C} \to \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X; \mathcal{C})$$

is an equivalence.

Proof. By Proposition 1.2.14, Theorem 1.5.4 and Remark 1.2.5, it suffices to show that $\mathcal{S}\mathrm{hv}_{\mathcal{K}}(X;\mathcal{S}\mathrm{p})$ is a retract in $\mathcal{C}\mathrm{ocont}^{st}_{\infty}$ of a compactly generated ∞-category. We see that $\mathcal{S}\mathrm{hv}(\mathcal{K}(X);\mathcal{S}\mathrm{p})$ is clearly compactly generated, as it is equivalent to $\mathrm{Fun}_{lex}(\mathcal{K}(X)^{\mathrm{op}},\mathcal{S}\mathrm{p})$. The proof is then concluded by noticing that the inclusion $\mathcal{S}\mathrm{hv}_{\mathcal{K}}(X;\mathcal{S}\mathrm{p}) \subseteq \mathcal{S}\mathrm{hv}(\mathcal{K}(X);\mathcal{S}\mathrm{p})$ and $\varphi : \mathcal{S}\mathrm{hv}(\mathcal{K}(X);\mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{hv}_{\mathcal{K}}(X;\mathcal{S}\mathrm{p})$ are exact and preserve filtered colimits since filtered colimits in $\mathcal{S}\mathrm{p}$ are exact, and thus preserves all colimits since $\mathcal{S}\mathrm{p}$ is stable. One then concludes the proof by identifying $\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})$ with the dual of $\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})$, via Lemma 1.2.21. ☐

Corollary 1.5.16. There is a unique equivalence

$$\eta : \mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \to \mathcal{S}\mathrm{hv}(X;\mathcal{C}). \tag{1.5.17}$$

making the diagram

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \arrow[r, "\eta"] \arrow[d, "\mathbb{D}_{\mathcal{S}\mathrm{p}}\otimes\mathcal{C}"] & \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \arrow[d, "\mathbb{D}_{\mathcal{C}}"] \\
\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \arrow[r] & \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C})
\end{tikzcd}$$

Proof. The functor η is obtained by composing

$$\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \xrightarrow{\mathbb{D}_{\mathcal{S}\mathrm{p}}\otimes\mathcal{C}} \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \xrightarrow{\simeq} \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \xrightarrow{\mathbb{D}_{\mathcal{C}}^{-1}} \mathcal{S}\mathrm{hv}(X;\mathcal{C}).$$

where the middle map is the one in Theorem 1.5.15. More concretely, for any $F \in \mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})$ and $M \in \mathcal{C}$, we have $\eta(F \boxtimes^{st} M) \simeq \mathbb{D}_{\mathcal{C}}^{-1}(M \circ \mathbb{D}_{\mathcal{S}\mathrm{p}}F)$, where M on the right-hand side denotes the essentially unique colimit preserving functor $\mathcal{S}\mathrm{p} \to \mathcal{C}$ corresponding to M. ☐

1.5.3 The pullback $f^*_{\mathcal{C}}$

Proposition 1.5.18. Let $f : X \to Y$ be a proper map, and denote by $f^{\mathcal{C}}_* : \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \to \mathcal{S}\mathrm{hv}(Y;\mathcal{C})$ the pushforward. Then we have a commutative square

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \arrow[r, "\eta"] \arrow[d, "f^{\mathcal{S}\mathrm{p}}_*\otimes\mathcal{C}"] & \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \arrow[d, "f^{\mathcal{C}}_*"] \\
\mathcal{S}\mathrm{hv}(Y;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \arrow[r, "\eta"] & \mathcal{S}\mathrm{hv}(Y;\mathcal{C}).
\end{tikzcd}$$

In particular, $f^{\mathcal{C}}_*$ admits a left adjoint which is identified through η with $f^*_{\mathcal{S}\mathrm{p}} \otimes \mathcal{C}$.

Proof. By a slight abuse of notation, denote as $f^{\mathcal{C}}_* : \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \to \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ the pushforward for cosheaves (i.e. $(f^{\mathcal{C}^{\mathrm{op}}}_*)^{\mathrm{op}}$). We have that the square

$$\begin{tikzcd}
\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \arrow[r, "\simeq"] \arrow[d, "f^{\mathcal{S}\mathrm{p}}_*\otimes\mathcal{C}"] & \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \arrow[d, "f^{\mathcal{C}}_*"] \\
\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(Y;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \arrow[r, "\simeq"] & \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(Y;\mathcal{C})
\end{tikzcd}$$

commutes since the horizontal arrows can be modelled by a composition of functors, and thus we are only left to show that the square

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \arrow[r, "\mathbb{D}"] \arrow[d, "f_*"] & \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \arrow[d, "f_*"] \\
\mathcal{S}\mathrm{hv}(Y;\mathcal{C}) \arrow[r, "\mathbb{D}"] & \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(Y;\mathcal{C})
\end{tikzcd}$$

commutes.

First of all, we show that there exists a natural transformation $f_*\mathbb{D} \to \mathbb{D}f_*$ even when f is not proper. Fix $V \in \mathcal{U}(Y)$ and a compact $K \subseteq f^{-1}(V)$, so that $f(K)$ is a compact subset of V. For any $F \in \mathcal{S}\mathrm{hv}(X;\mathcal{C})$, the commutative triangle

$$\begin{tikzcd} \Gamma(X \setminus K; F) \arrow[rr] & & \Gamma(X \setminus f^{-1}(f(K)); F) \\ & \Gamma(X;F) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

provides a morphism

$$\Gamma_K(X;F) \to \Gamma_{f(K)}(Y; f_*F) \to \Gamma_c(V; f_*F).$$

Since all morphisms are induced by the restrictions of F, the resulting map is natural in K and V, and hence gives rise to the desired transformation as K varies. Furthermore, when f is proper, each compact $K \subseteq X$ is contained in the compact $f^{-1}f(K)$, so by cofinality we obtain an equivalence

$$\varinjlim_{K \subseteq f^{-1}(V)} \Gamma_K(X;F) \simeq \varinjlim_{C \subseteq V} \Gamma_C(Y; f_*F)$$

where C varies over the compact subsets of V, and thus we may conclude. $\square$

Lemma 1.5.19. Let $j : U \hookrightarrow X$ be an open immersion, and denote by $j^*_{\mathcal{C}} : \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \to \mathcal{S}\mathrm{hv}(U;\mathcal{C})$ the restriction. Then we have a commutative square

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \arrow[r, "\eta"] \arrow[d, "j^*_{\mathcal{S}\mathrm{p}} \otimes \mathcal{C}"] & \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \arrow[d, "j^*_{\mathcal{C}}"] \\ \mathcal{S}\mathrm{hv}(U;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \arrow[r, "\eta"] & \mathcal{S}\mathrm{hv}(U;\mathcal{C}). \end{tikzcd}$$

In particular, $j^*_{\mathcal{C}}$ admits a left adjoint which is identified through η with $j^{\mathcal{S}\mathrm{p}}_\sharp \otimes \mathcal{C}$.

Proof. By an abuse of notation, denote by $j^*_{\mathcal{C}} : \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \to \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(U;\mathcal{C})$ the restriction for cosheaves. Again we see that the square

$$\begin{tikzcd} \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \arrow[r, "\simeq"] \arrow[d, "j^*_{\mathcal{S}\mathrm{p}} \otimes \mathcal{C}"] & \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \arrow[d, "j^*_{\mathcal{C}}"] \\ \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(U;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \arrow[r, "\simeq"] & \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(U;\mathcal{C}) \end{tikzcd}$$

is obviously commutative, and thus we only have to show that the square

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \arrow[r, "\mathbb{D}"] \arrow[d, "j^*"] & \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \arrow[d, "j^*"] \\ \mathcal{S}\mathrm{hv}(U;\mathcal{C}) \arrow[r, "\mathbb{D}"] & \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(U;\mathcal{C}) \end{tikzcd}$$

commutes. But this follows immediately because by Remark 1.5.7, and by observing that $\Gamma(V; j^*F) \simeq \Gamma(V;F)$ for any $V \subseteq U$, we have that

$$\Gamma_c(V; j^*F) \simeq \Gamma_c(V;F)$$

functorially on F and V. $\square$

Corollary 1.5.20. Let $f : X \to Y$ be any continuous map. Then the pushforward $f_*^{\mathcal{C}} : \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \to \mathcal{S}\mathrm{hv}(Y;\mathcal{C})$ admits a left adjoint $f^*_{\mathcal{C}}$ such that there exists a commutative square

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}(Y;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \arrow[r, "\eta"] \arrow[d, "f^*_{\mathcal{S}\mathrm{p}}\otimes\mathcal{C}"] & \mathcal{S}\mathrm{hv}(Y;\mathcal{C}) \arrow[d, "f^*_{\mathcal{C}}"] \\
\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \arrow[r, "\eta"] & \mathcal{S}\mathrm{hv}(X;\mathcal{C}).
\end{tikzcd}$$

Proof. Using the factorization 1.5.1, this follows immediately by Proposition 1.5.18 and Lemma 1.5.19. ☐

Theorem 1.5.21. Let $i_{\mathcal{C}} : \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}},\mathcal{C})$ be the inclusion functor, and let $L_{\mathcal{S}\mathrm{p}} : \mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}},\mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})$ be the left adjoint of $i_{\mathcal{S}\mathrm{p}}$. Denote by $L_{\mathcal{C}}$ the composition

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}},\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \arrow[r, "L_{\mathcal{S}\mathrm{p}}\otimes\mathcal{C}"] & \mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \arrow[d, "\eta"] \\
\mathrm{Fun})(\mathcal{U}(X)^{\mathrm{op}},\mathcal{C}) \arrow[u, "\simeq"] \arrow[r, dotted, "L_{\mathcal{C}}"] & \mathcal{S}\mathrm{hv}(X;\mathcal{C}).
\end{tikzcd}$$

Then $L_{\mathcal{C}}$ is left adjoint to $i_{\mathcal{C}}$.

Proof. From the proof of Theorem 1.2.10 and Remark 1.2.13, we see that we may write any $F \in \mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}},\mathcal{C})$ as a colimit

$$F \simeq \varinjlim_{M \to \Gamma(U;F)} U \boxtimes^{st} M$$

where the indexing category is Grothendieck construction of the functor

$$(U, M) \mapsto \mathrm{Hom}_{\mathcal{C}}(M, \Gamma(U;F)).$$

By definition, the presheaf of spectra $U \boxtimes^{st} M$ represents the functor taking sections at U, and thus we get an equivalence

$$\mathbb{S}_U = L_{\mathcal{S}\mathrm{p}}(U \boxtimes \mathbb{S}) \simeq j_\sharp^{\mathcal{S}\mathrm{p}} a^*_{\mathcal{S}\mathrm{p}} \mathbb{S}$$

which is natural in U, where $j : U \hookrightarrow X$ denotes an open inclusion, $a : U \to *$ the unique map. We have equivalences

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{S}\mathrm{hv}(X;\mathcal{C})}(L_{\mathcal{C}}F, G) &\simeq \varprojlim_{M \to \Gamma(U;F)} \mathrm{Hom}_{\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})\otimes\mathcal{C}}(j_\sharp^{\mathcal{S}\mathrm{p}} a^*_{\mathcal{S}\mathrm{p}} \mathbb{S} \boxtimes^{st} M, \eta^{-1}G) \\
&\simeq \varprojlim_{M \to \Gamma(U;F)} \mathrm{Hom}_{\mathcal{S}\mathrm{hv}(X;\mathcal{C})}(j_\sharp^{\mathcal{C}} a^*_{\mathcal{C}} M, G) \\
&\simeq \varprojlim_{M \to \Gamma(U;F)} \mathrm{Hom}_{\mathcal{C}}(M, \Gamma(U;G)) \\
&\simeq \varprojlim_{M \to \Gamma(U;F)} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}},\mathcal{C})}(U \boxtimes M, i^{\mathcal{C}} G) \\
&\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}},\mathcal{C})}(F, i^{\mathcal{C}} G)
\end{aligned}$$

where the first equivalence follows by the observations above, and the second is a consequence of Lemma 1.5.19 and Corollary 1.5.20. Since all identifications are functorial on F and G, we obtain the thesis. ☐

Remark 1.5.22. After the results in this section, we are now able to extend everything we have proven so far for sheaves with presentable coefficients to sheaves with values in a stable bicomplete ∞-category. The only detail we have to handle with more care is the functor (1.2.27): since the tensor product of two stable bicomplete ∞-categories is not again complete in general, (1.2.27) will now take values in $\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes (\mathcal{C} \otimes \mathcal{D})$. Nevertheless, when $\mathcal{C}$ has a monoidal structure such that its tensor $\otimes_{\mathcal{C}}$ preserves colimits in both variables, the composition of (1.2.27) with the obvious functor

$$\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes (\mathcal{C} \otimes \mathcal{C}) \to \mathcal{S}\mathrm{hv}(X;\mathcal{C})$$

still gives the usual monoidal structure on $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$. As a consequence, we see that the equivalences in Corollary 1.2.31 and Corollary 1.3.26 still hold in $\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes (\mathcal{C} \otimes \mathcal{D})$ and in $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ when (1.2.27) is exchanged with the tensor product in $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ described above.

1.6 Six functor formalism

In this section, we will define the operations $f_!$ and $f^!$, and prove all the usual formulas that one expects for these functors. A first attempt towards these results for sheaves of spectra can be found in the paper [BL96], even though it almost totally lacks proofs. A proof of the proper base change theorem with unstable coefficients was provided in [Lur09, Corollary 7.3.1.18], and later extended to spectral coefficients in [Hai21]: our only contribute to this theorem is to explain how to extend it to general stable bicomplete coefficients. A novelty of the approach presented in this section is our expression of the formulas involving tensor products, such as projection or Künneth formula. Here we do not a priori require the coefficients of our sheaves to be equipped with a monoidal structure, but rely instead on the tensor product of stable cocomplete ∞-categories. The advantage of this perspective is that, using the observations in Remark 1.5.22, it clarifies how to obtain all these formulas for a general stable bicomplete ∞-category equipped with a closed symmetric monoidal structure. At the end of the section, building up on the our discussion of Section 3 related to shape theory, we will explain how to prove the formula

$$f^!(\mathbb{1}) \otimes f^* \simeq f^!$$

for any map f which induce locally contractible geometric morphisms.

1.6.1 The formulas for $f_!^{\mathcal{C}}$

Throughout this section, $\mathcal{C}$ is going to be any stable and bicomplete ∞-category. For any continuous map $f : X \to Y$, consider the functor

$$(f_*^{\mathcal{C}^{op}})^{\mathrm{op}} : \mathcal{S}\mathrm{hv}(X;\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \to \mathcal{S}\mathrm{hv}(Y;\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(Y;\mathcal{C}).$$

Since taking opposite categories switches left with right adjoints, the functor

$$(f^*_{\mathcal{C}^{\mathrm{op}}})^{\mathrm{op}} : \mathcal{S}\mathrm{hv}(Y;\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(Y;\mathcal{C}) \to \mathcal{S}\mathrm{hv}(X;\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C})$$

is right adjoint to $(f_*^{\mathcal{C}^{\mathrm{op}}})^{\mathrm{op}}$. Hence, by Theorem 1.5.10, we get a corresponding adjunction at the level of sheaves

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \arrow[r, bend left, "f_!^{\mathcal{C}}"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathrm{hv}(Y;\mathcal{C}). \arrow[l, bend left, "f^!_{\mathcal{C}}"] \end{tikzcd}$$

where $f_!^{\mathcal{C}}$ and $f^!_{\mathcal{C}}$ are defined to be the unique functors fitting in commutative squares

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}(Y;\mathcal{C}) \arrow[r, "f_!^{\mathcal{C}}"] \arrow[d, "\mathbb{D}"'] & \mathcal{S}\mathrm{hv}(Y;\mathcal{C}) \arrow[d, "\mathbb{D}"] \\ \mathcal{C}o\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \arrow[r, "(f_*^{\mathcal{C}^{op}})^{\mathrm{op}}"] & \mathcal{C}o\mathcal{S}\mathrm{hv}(Y;\mathcal{C}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}\mathrm{hv}(Y;\mathcal{C}) \arrow[r, "f^!_{\mathcal{C}}"] \arrow[d, "\mathbb{D}"'] & \mathcal{S}\mathrm{hv}(Y;\mathcal{C}) \arrow[d, "\mathbb{D}"] \\ \mathcal{C}o\mathcal{S}\mathrm{hv}(Y;\mathcal{C}) \arrow[r, "(f^*_{\mathcal{C}^{\mathrm{op}}})^{\mathrm{op}}"] & \mathcal{C}o\mathcal{S}\mathrm{hv}(X;\mathcal{C}). \end{tikzcd}$$

Definition 1.6.1. The functors $f_!^{\mathcal{C}}$ and $f^!_{\mathcal{C}}$ constructed as above are called respectively *push-forward with proper support* and *exceptional pullback*. Unless it is required from the context, we will often omit to include $\mathcal{C}$ in subscripts or superscripts in our notation.

More concretely, $f_!$ is the functor uniquely determined by the formula

$$\Gamma_c(U; f_!F) = \Gamma_c(f^{-1}(U); F)$$

for all $U \in \mathcal{U}(Y)$. In particular, when $a : X \to *$ is the unique map, we get

$$a_!F \simeq \Gamma_c(X; F).$$

In different geometric contexts, when one deals with six functor formalisms, it is common to define the shriek operations making use of appropriate compactifications of maps analogous to (1.5.1). This approach, however, makes it a bit tricky to verify that $f_!$ behaves well under compositions. An advantage of our definition of $f_!$ is that its functoriality is more or less immediate, as illustrated by the following lemma.

Lemma 1.6.2. Let LCH be the category of locally compact Hausdorff spaces. Then, for any $\mathcal{C}$ stable and bicomplete, there is a functor

$$\mathcal{S}\mathrm{hv}_!(-;\mathcal{C}) : \mathrm{LCH} \to \mathcal{C}\mathrm{ocont}^{st}_{\infty}$$

whose values on an object X is given by $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$, and on a morphism f is given by $f_!^{\mathcal{C}}$.

Proof. We first observe that there is a functor

$$\mathcal{S}\mathrm{hv}_*(-;\mathcal{C}) : \mathrm{LCH} \to \mathcal{C}\mathrm{ocont}^{st}_{\infty}$$

whose value on a morphism $f : X \to Y$ is given by the pushforward $f_*^{\mathcal{C}}$. By definition, $f_*^{\mathcal{C}}$ is given by precomposing with the functor $f^{-1} : \mathcal{U}(Y) \to \mathcal{U}(X)$. Thus, the functoriality of $\mathcal{S}\mathrm{hv}_*(-;\mathcal{C})$ descends from the functoriality of internal-homs in $\mathcal{C}\mathrm{at}_\infty$, which is straightforward in the model of quasi-categories. By passing to opposite categories, we obtain a similar functor $\mathcal{C}o\mathcal{S}\mathrm{hv}_*(-;\mathcal{C})$.

Let J be the interval object for the Joyal model structure (see [Cis19, Definition 3.3.3]), and consider the monomorphism of simplicial sets $Ob(\mathrm{LCH}) \hookrightarrow \mathrm{LCH}$. By Theorem 1.5.10, we have a functor

$$Ob(\mathrm{LCH}) \times J \cup \mathrm{LCH} \times \{1\} \xrightarrow{\mathbb{D}_{\mathcal{C}} \cup \mathcal{C}o\mathcal{S}\mathrm{hv}_*(-;\mathcal{C})} \mathcal{C}\mathrm{at}_\infty.$$

which admits a lifting

$$\begin{tikzcd} Ob(\mathrm{LCH}) \times J \cup \mathrm{LCH} \times \{1\} \arrow[r] \arrow[d, hook] & \mathcal{C}\mathrm{at}_\infty \\ \mathrm{LCH} \times J \arrow[ur, dotted] & \end{tikzcd}$$

since the vertical arrow is a categorical anodyne extension. Thus we get the desired functor by restricting the dotted arrow to $\mathrm{LCH} \times \{0\}$. ☐

Lemma 1.6.3. There exists a natural transformation $f_! \to f_*$ which is an equivalence when f is proper.

Proof. By Verdier duality, it suffices to construct a natural tranformation between $\mathbb{D}f_! \to \mathbb{D}f_*$ and show it is an equivalence when f is proper, which is the content of the second part of the proof of Proposition 1.5.18. □

Corollary 1.6.4. Let $i : Z \hookrightarrow X$ be a closed immersion. Then the functor $i^!_{\mathcal{S}\mathrm{p}} : \mathcal{S}\mathrm{hv}(X; \mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{hv}(Z; \mathcal{S}\mathrm{p})$ coincides with the one defined in Corollary 1.4.4.

Proof. This follows immediately by the previous lemma, since any closed immersion is proper. □

Lemma 1.6.5. Let $j : U \hookrightarrow X$ be an open immersion. Then we have $j_! \dashv j^*$ or equivalently $j^* \simeq j^!$.

Proof. By the proof of Lemma 1.5.19, we have a natural equivalence $\mathbb{D}j^* \simeq \mathbb{D}j^!$, and thus we may conclude. □

Remark 1.6.6. A useful consequence of Lemma 1.6.2, Lemma 1.6.5, Lemma 1.6.3 and (1.5.1) is that the functor $f_!^{\mathcal{C}}$ is uniquely determined by the fact that it is right adjoint to $f^*_{\mathcal{C}}$ when f is proper and left adjoint to $f^*_{\mathcal{C}}$ when f is an open immersion.

From Remark 1.6.6 we also see that $f_!$ behaves well with respect to tensor products in $\mathcal{C}\mathrm{ocont}^{st}_{\infty}$.

Corollary 1.6.7. Let $f : X \to Y$ be any continuous map. Then there is a commutative square

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}(X; \mathcal{S}\mathrm{p}) \otimes \mathcal{C} \arrow[r, "\eta"] \arrow[d, "f_!^{\mathcal{S}\mathrm{p}} \otimes \mathcal{C}"] & \mathcal{S}\mathrm{hv}(X; \mathcal{C}) \arrow[d, "f_!^{\mathcal{C}}"] \\
\mathcal{S}\mathrm{hv}(Y; \mathcal{S}\mathrm{p}) \otimes \mathcal{C} \arrow[r, "\eta"] & \mathcal{S}\mathrm{hv}(Y; \mathcal{C}).
\end{tikzcd}$$

Proof. This follows immediately by Remark 1.6.6, Proposition 1.5.18 and Lemma 1.5.19. □

Remark 1.6.8. Let $j : U \hookrightarrow X$ be the inclusion of any open subset with compact closure. Then a simple computation involving Lemma 1.6.3 and the closure of U shows that, for any sheaf F on X, one has

$$\Gamma(U; F) \simeq \Gamma_c(X; j_* j^* F).$$

Since U has compact closure, any closed subset of U can be written as the intersection of U with some compact subset of X, and thus we obtain

$$\begin{aligned}
\Gamma_c(X; j_* j^* F) &\simeq \varinjlim_{K \subseteq X} \Gamma_{U \cap K}(U; F) \\
&\simeq \varinjlim_{S \subseteq U} \Gamma_S(U; F),
\end{aligned}$$

where the last colimit ranges over all closed subsets of U.

Proposition 1.6.9 (Base change)**.** For every given pullback square

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "g"] \\
Y' \arrow[r, "f"] & Y
\end{tikzcd}$$

of topological spaces, there is a natural equivalence

$$f^* g_! \simeq g'_! f'^*$$

and, by transposition, also

$$g^! f_* \simeq f'_* g'^!.$$

Proof. First of all we observe that, since all functors appearing are colimit preserving, by Corollary 1.6.7 it suffices to prove the proposition in the case of sheaves of spectra.

By Remark 1.6.6, it suffices to prove the statement in the separate cases when g is an open immersion or a proper map. The open immersion case follows immediately by Lemma 1.6.5 and Lemma 1.3.25, while the proper case follows by [Hai21, Corollary 3.2]. For the reader's convenience, let us briefly summarize the strategy of [Hai21].

By [Lur09, Corollary 7.3.1.18] we know that the statement of the theorem is true for sheaves of spaces. Since $\mathcal{S}\mathrm{p}$ is compactly generated, by Proposition 1.2.16 we

$$\begin{aligned}\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) &\simeq \mathrm{Fun}_*(\mathcal{S}\mathrm{p}^{\mathrm{op}}, \mathcal{S}\mathrm{hv}(X)) \\ &\simeq \mathrm{Fun}_{lex}((\mathcal{S}\mathrm{p}^{\omega})^{\mathrm{op}}, \mathcal{S}\mathrm{hv}(X))\end{aligned}$$

where $\mathcal{S}\mathrm{p}^{\omega}$ denotes the full subcatgory of $\mathcal{S}\mathrm{p}$ spanned by all compact objects. One checks easily that, for any continuous map $h : W \to T$ of topological spaces, there is a commutative square

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}(W;\mathcal{S}\mathrm{p}) \arrow[r, "\simeq"] \arrow[d, "h_*^{\mathcal{S}\mathrm{p}}"] & \mathrm{Fun}_{lex}((\mathcal{S}\mathrm{p}^{\omega})^{\mathrm{op}}, \mathcal{S}\mathrm{hv}(W)) \arrow[d, "h_*^{\mathcal{S}}\circ(-)"] \\ \mathcal{S}\mathrm{hv}(T;\mathcal{S}\mathrm{p}) \arrow[r, "\simeq"] & \mathrm{Fun}_{lex}((\mathcal{S}\mathrm{p}^{\omega})^{\mathrm{op}}, \mathcal{S}\mathrm{hv}(T)), \end{tikzcd}$$

where the right hand vertical square denotes a post-composition with the pushforward $h_*^{\mathcal{S}}$ of sheaves of spaces. Reasoning analogously to Remark 1.2.2, we see that the functor

$$\mathrm{Fun}_{lex}((\mathcal{S}\mathrm{p}^{\omega})^{\mathrm{op}}, -)$$

preserves adjunctions between left exact functors, and so we get a similar commutative square involving the pullbacks $h^*_{\mathcal{S}\mathrm{p}}$ and $h^*_{\mathcal{S}}$. Hence, we obtain base change for spectral sheaves by applying $\mathrm{Fun}_{lex}((\mathcal{S}\mathrm{p}^{\omega})^{\mathrm{op}}, -)$ to Lurie's nonabelian proper base change in [Lur09, Corollary 7.3.1.18]. ☐

Remark 1.6.10. It follows from Remark 1.2.17 and the definition of the tensor (1.2.27) that for any topological space X and any bicomplete stable ∞-category $\mathcal{C}$, $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ is tensored over $\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})$. When there is no possibility of confusion we will denote by $F \otimes G$ the image through the canonical variablewise colimit preserving functor

$$\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \times \mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{hv}(X;\mathcal{C})$$

of a pair (F, G), and, when $G \in \mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})$ by $\underline{\mathrm{Hom}}_X(G, F)$ the image of any $F \in \mathcal{S}\mathrm{hv}(X;\mathcal{C})$ through the right adjoint of $- \otimes G$.

Let $f : X \to X'$ and $g : Y \to Y'$ be morphisms of topological spaces, and let $f \times g : X \times Y \to X' \times Y'$ be the induced map on the products. For any two stable bicomplete ∞-categories $\mathcal{C}$ and $\mathcal{D}$, the variable-wise colimit preserving functor

$$\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \times \mathcal{S}\mathrm{hv}(Y;\mathcal{D}) \xrightarrow{f_! \times g_!} \mathcal{S}\mathrm{hv}(X';\mathcal{C}) \times \mathcal{S}\mathrm{hv}(Y';\mathcal{D}) \xrightarrow{\boxtimes} \mathcal{S}\mathrm{hv}(X' \times Y';\mathcal{S}\mathrm{p}) \otimes (\mathcal{C} \otimes \mathcal{D})$$

induces a functor

$$f_! \boxtimes g_! : \mathcal{S}\mathrm{hv}(X \times Y;\mathcal{S}\mathrm{p}) \otimes (\mathcal{C} \otimes \mathcal{D}) \to \mathcal{S}\mathrm{hv}(X' \times Y';\mathcal{S}\mathrm{p}) \otimes (\mathcal{C} \otimes \mathcal{D}).$$

Proposition 1.6.11 (Künneth formula)**.** We have a natural equivalence

$$f_! \boxtimes g_! \simeq (f \times g)_!.$$

Notice that, since X and Y are locally compact, $f_! \boxtimes g_!$ is the image of the pair $(f_!, g_!)$ through the bifunctor given by the tensor product of cocomplete ∞-categories.

Proof. By the factorization (1.5.1), it will suffice to prove the statement when both f, g and $f \times g$ are either open immersions or proper maps. By uniqueness of adjoints and Remark 1.2.2, both cases will then follow by Proposition 1.2.30. More precisely, in the open immersions case we use that $f_! \boxtimes g_!$ is left adjoint to $f^* \boxtimes g^*$, while in the proper case we use that $f_! \boxtimes g_!$ is right adjoint to $f^* \boxtimes g^*$, and by Proposition 1.2.30 we always have an equivalence $f^* \boxtimes g^* \simeq (f \times g)^*$. ☐

Proposition 1.6.12 (Projection formula)**.** Let $f : X \to Y$ be a morphism of topological spaces, and let $\mathcal{C}$ and $\mathcal{D}$ be two stable and bicomplete ∞-categories. Then, for any $F \in \mathcal{S}\mathrm{hv}(X; \mathcal{C})$ and $G, H \in \mathcal{S}\mathrm{hv}(Y; \mathcal{D})$, we have a canonical equivalence

$$f_! F \otimes G \simeq f_!(F \otimes f^* G)$$

or, when $\mathcal{C} = \mathcal{D}$ has a closed symmetric monoidal structure, by transposition

$$f_* \underline{\mathrm{Hom}}_X(F, f^! G) \simeq \underline{\mathrm{Hom}}_Y(f_! F, G)$$

and

$$f^! \underline{\mathrm{Hom}}_Y(G, H) \simeq \underline{\mathrm{Hom}}_X(f^* G, f^! H).$$

Proof. Exactly as for the Corollary 1.3.26, one may deduce this result from Proposition 1.6.9 and Proposition 1.6.11 applied to $(f \times \mathrm{id}_Y)_!$. ☐

Corollary 1.6.13. Let $k : Z \to X$ be the inclusion of a locally closed subset of X, $F \in \mathcal{S}\mathrm{hv}(X; \mathcal{C})$ with $\mathcal{C}$ stable and bicomplete. Let $\mathcal{D}$ be another stable bicomplete ∞-category, $M \in \mathcal{D}$ any object. Then we have a canonical equivalence

$$k_! k^*(F \otimes M) \simeq F \otimes M_Z.$$

Moreover, when $\mathcal{C}$ has a closed symmetric monoidal structure, we have

$$k_! k^* F \simeq F \otimes_{\mathcal{C}} \mathbb{1}_Z$$

or equivalently

$$k_* k^! F \simeq \underline{\mathrm{Hom}}_X(\mathbb{1}_Z, F),$$

where $\mathbb{1}$ and is the monoidal unit of $\mathcal{C}$.

Proof. This is an immediate consequence of Proposition (1.6.12) and Corollary (1.2.31). ☐

Corollary 1.6.14. For every given pullback square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\ Y' \arrow[r, "f"] & Y \end{tikzcd}$$

of topological spaces where f and f' are shape submersions, there is a natural equivalence

$$g_! f'_\sharp \simeq f_\sharp g'_!$$

or equivalently

$$f'^* g^! \simeq g'^! f^*.$$

Proof. First of all we construct a natural transformation. Applying $g'_!$ to the unit of the adjunction $f'_\sharp \dashv f'^*$ and using base change we obtain

$$g'_! \to g_! f'^* f_\sharp \simeq f^* g_! f'_\sharp$$

and hence by transposition the desired transformation. By factorization (1.5.1) and Corollary 1.4.10, we are only left to prove the case of a pullback square

$$\begin{tikzcd} X \times Y' \arrow[r, "\mathrm{id}_X \times f"] \arrow[d, "a \times \mathrm{id}_{Y'}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \times Y \arrow[d, "a \times \mathrm{id}_Y"] \\ Y' \arrow[r, "f"] & Y \end{tikzcd}$$

where X is compact and $a : X \to *$ is the unique map, but this follows by Proposition 1.6.11 and by the functoriality of the tensor product of cocomplete ∞-categories as follows

$$\begin{aligned}(a \times \mathrm{id}_Y)_!(\mathrm{id}_X \times f)_\sharp &\simeq (a_! \otimes (\mathrm{id}_Y)_!)((\mathrm{id}_X)_\sharp \otimes f_\sharp) \\ &\simeq a_! f_\sharp \\ &\simeq ((\mathrm{id}_*)_\sharp a_!) \otimes (f_\sharp (\mathrm{id}_{Y'})_!) \\ &\simeq ((\mathrm{id}_*)_\sharp \otimes f_\sharp)(a_! \otimes (\mathrm{id}_{Y'})_!).\end{aligned}$$

□

Remark 1.6.15. Using (1.5.2), Lemma 1.3.25, Corollary 1.3.26, Corollary 1.5.20, Proposition 1.6.9, Proposition 1.6.12, Corollary 1.6.14, and [Man22, Proposition A.5.10], one sees that the functor

$$\begin{aligned} \mathrm{LCH}^{\mathrm{op}} &\longrightarrow \mathcal{C}\mathrm{at}_\infty \\ X &\longmapsto \mathcal{S}\mathrm{hv}(X; \mathcal{C}) \\ (f : X \to Y) &\longmapsto f^*_{\mathcal{C}} \end{aligned}$$

defines a six functor formalism in the sense of [Man22, Definition A.5.9].

1.6.2 $f^!_{\mathcal{C}}$ when f is a locally contractible geometric morphism

Let $f : X \to Y$ be a continuous map inducing an essential geometric morphism (see Definition 1.3.12). In particular, we have an adjunction $f_\sharp \dashv f^*$ for $\mathcal{C}^{\mathrm{op}}$-valued sheaves, and thus, after passing to opposite categories and applying Theorem 1.5.10, we get an adjunction

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}(Y; \mathcal{C}) \arrow[r, bend left, "f^!"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathrm{hv}(X; \mathcal{C}). \arrow[l, bend left, "f_\circ"] \end{tikzcd}$$

We now want to show that, in the special case when f induces a locally contractible geometric morphism (see Definition 1.3.12), one the exceptional pullaback coincides with the usual pullback up to a twist. In this way, we will vastly generalize the classical formula relating the $f^!$ and f^* when f is a topological submersion.

Proposition 1.6.16. Let $f : X \to Y$ be a continuous map inducing a locally contractible geometric morphism, and let $\mathcal{C}$ and $\mathcal{D}$ be two stable and bicomplete ∞-categories. Then, for any $F \in \mathcal{S}\mathrm{hv}(Y; \mathcal{C})$ and $G \in \mathcal{S}\mathrm{hv}(Y; \mathcal{D})$, we have a natural equivalence

$$f^!_{\mathcal{C}} F \otimes f^*_{\mathcal{D}} G \simeq f^!_{\mathcal{C} \otimes \mathcal{D}}(F \otimes G)$$

of functors $\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \times \mathcal{S}\mathrm{hv}(Y;\mathcal{D}) \to \mathcal{S}\mathrm{hv}(X \times Y;\mathcal{C}\otimes\mathcal{D})$. Equivalently, when $\mathcal{C} = \mathcal{D}$ has a closed symmetric monoidal structure, a canonical equivalence for any $K \in \mathcal{S}\mathrm{hv}(X;\mathcal{C})$

$$\underline{\mathrm{Hom}}_Y(F, f_\circ K) \to f_*\underline{\mathrm{Hom}}_X(f^! F, K).$$

Proof. For any $F \in \mathcal{S}\mathrm{hv}(Y;\mathcal{C})$, $G \in \mathcal{S}\mathrm{hv}(Y;\mathcal{D})$ and $H \in \mathcal{S}\mathrm{hv}(Y;\mathcal{C}\otimes\mathcal{D})$, we have the map

$$\mathrm{counit} \otimes G : f_! f^! F \otimes G \to F \otimes G$$

which by adjunction and Proposition 1.6.12 gives the desired natural transformation

$$f^!(-) \otimes f^*(-) \to f^!(- \otimes -).$$

Since all functors appearing are cocontinuous, by Remark 1.6.10 it suffices to prove the invertibility of the map when $\mathcal{C} = \mathcal{D} = \mathcal{S}\mathrm{p}$, and after evaluation on pairs of the type $(F, \mathbb{S}_U)$, where $F \in \mathcal{S}\mathrm{hv}(Y;\mathcal{S}\mathrm{p})$, $\mathbb{S}$ denotes the sphere spectrum, and $j : U \hookrightarrow Y$ is an open subset. Keeping the same notations ad in Remark 1.3.13, we see that by Theorem 1.5.10 we have an equivalence

$$f^!_{\mathcal{C}} j^{\mathcal{C}}_! \simeq (j')^{\mathcal{C}}_! (f')^!_{\mathcal{C}}.$$

Thus, we get

$$\begin{aligned} f^!(F) \otimes f^*(\mathbb{S}_U) &\simeq f^!(\mathbb{S}_Y) \otimes \mathbb{S}_{f^{-1}(U)} \\ &\simeq j'_!(f')^! j^*(F) \\ &\simeq f^! j_! j^*(F) \\ &\simeq f^!(F \otimes \mathbb{S}_U), \end{aligned}$$

where the second equivalence follows by Corollary 1.6.13 and Lemma 1.6.5, the third by Remark 1.3.13 and the fourth again by Corollary 1.6.13. ☐

Remark 1.6.17. We thank Marc Hoyois for pointing out that a result similar to Proposition 1.6.16 can be found in [Ver65, Section 5]. By adapting our proof of Corollary 1.3.26, one can actually deduce the theorem in [Ver65] from Proposition 1.6.16.

Proposition 1.6.18. Let $f : X \to Y$ be a topological submersion of fiber dimension n, $\mathcal{C}$ be any stable bicomplete ∞-category. Then:

(i) if f is a trivial submersion, then there is a canonical equivalence

$$f^! \simeq \Sigma^n f^*;$$

(ii) $f^!$ preseves locally constant sheaves;

(iii) for all $F \in \mathcal{S}\mathrm{hv}(Y;\mathcal{C})$ there is a canonical equivalence

$$f^!\mathbb{S}_Y \otimes f^* F \simeq f^! F$$

or equivalently by adjunction, for every $G \in \mathcal{S}\mathrm{hv}(X;\mathcal{C})$

$$f_\sharp G \simeq f_!(G \otimes f^!\mathbb{S}_Y).$$

Proof. Since f^* preserves locally constant sheaves, we see that (ii) follows from (i). Hence, we now prove (i). By assumption, f is a projection $p : X \times \mathbb{R}^n \to X$. By Proposition 1.6.16, it suffices to show that $p^!\mathbb{S}_X \simeq \Sigma^n\mathbb{S}_{X\times\mathbb{R}^n}$. Since p induces a locally contractible geometric morphism, we have that $p^!$ preserves colimits. Hence, by Proposition 1.6.11 and the uniqueness of adjoints, we may assume that p is the unique map $a : \mathbb{R}^n \to *$.

We first show that $p^!M$ is locally constant for each $M \in \mathcal{S}\mathrm{p}$. By a standard argument (see [Hai20, Proposition 3.1] or Proposition 2.5.13) it suffices to show that for any $U \subseteq \mathbb{R}^n$ euclidean chart, the restriction

$$\Gamma(\mathbb{R}^n; p^!M) \to \Gamma(U; p^!M)$$

is an equivalence. By adjunction, we know that for any open V there is an equivalence

$$\Gamma(V; p^!M) \simeq \underline{\mathrm{Hom}}_{\mathcal{S}\mathrm{p}}(\Gamma_c(V; \mathbb{S}_{\mathbb{R}^n}), M).$$

Thus, it will suffice to prove that the canonical map

$$\Gamma_c(U; \mathbb{S}_{\mathbb{R}^n}) \to \Gamma_c(\mathbb{R}^n; \mathbb{S}_{\mathbb{R}^n})$$

is invertible. Since any compact subset of a vector space is contained in some compact closed ball, it suffices to show that, for any K compact closed ball in U, the map

$$\Gamma_K(U; \mathbb{S}_{\mathbb{R}^n}) \to \Gamma_K(\mathbb{R}^n; \mathbb{S}_{\mathbb{R}^n})$$

is invertible. But this follows by homotopy invariance of the shape, since the inclusion $U \setminus K \hookrightarrow \mathbb{R}^n \setminus K$ is a homotopy equivalence. In particular, we also see that the canonical map

$$\Gamma_{\{0\}}(\mathbb{R}; \mathbb{S}_{\mathbb{R}}) \to \Gamma_c(\mathbb{R}; \mathbb{S}_{\mathbb{R}})$$

is an equivalence.

To conclude the proof of (ii) it suffices to check that the global sections of $p^!M$ are equivalent to Σ^nM. We start with the case $n = 1$. Arguing as above, this amounts to proving that we have an equivalence

$$\Gamma_{\{0\}}(\mathbb{R}; \mathbb{S}_{\mathbb{R}}) \simeq \Omega\mathbb{S}.$$

By Corollary 1.3.4, we may identify $\Gamma_{\{0\}}(\mathbb{R}; \mathbb{S}_{\mathbb{R}})$ with the fiber of the diagonal map

$$\Gamma(\mathbb{R}; \mathbb{S}_{\mathbb{R}}) \to \Gamma(\mathbb{R}; \mathbb{S}_{\mathbb{R}}) \oplus \Gamma(\mathbb{R}; \mathbb{S}_{\mathbb{R}}).$$

It is then a straightforward exercise in pasting pushouts to verify that, for any object A of a stable ∞-category, the fiber of the diagonal map is equivalent to ΩA. For $n > 1$, we see that by Proposition 1.6.11 we have

$$\Gamma_c(\mathbb{R}^n; \mathbb{S}_{\mathbb{R}^n}) \simeq \Gamma_c(\mathbb{R}; \mathbb{S}_{\mathbb{R}}) \otimes \cdots \otimes \Gamma_c(\mathbb{R}; \mathbb{S}_{\mathbb{R}}),$$

and thus what we wanted.

To conclude, we observe that (iii) follows by Proposition 1.6.16 and the fact that any locally constant sheaf whose stalks are invertible spectra is an invertible object with respect to the smash product of sheaves of spectra. □

1.7 Relative Atiyah Duality

Recall that, if X is a compact smooth manifold, then the *Atiyah duality* states that $\Sigma^\infty_+ X$ is strongly dual to the *Thom spectrum* associated to the virtual vector bundle given by the inverse of the tangent bundle of X, which is denoted by $\mathrm{Th}(-TX)$. In this setion we will revisit Atiyah duality using the six functor formalism, in the spirit of motivic homotopy theory.

By what we have achieved up to now, we can see very easily that whenever $f : X \to Y$ is a proper map inducing a locally contractible geometric morphism, $f_\sharp(\mathbb{S}_X) \in \mathcal{S}\mathrm{hv}(Y; \mathcal{S}\mathrm{p})$ is strongly dualizable with dual $f_!(\mathbb{S}_X)$. The question is then about identifying $f_!(\mathbb{S}_X)$ with some sheaf theoretic construction reminiscent of Thom spectra, at least in more geometric situations (e.g. when X and Y are manifolds). We will provide a natural trasformation

$$\mathrm{Th} : \mathrm{Vect}(X) \to \mathrm{Pic}(\mathcal{S}\mathrm{hv}(X; \mathcal{S}\mathrm{p}))$$

where the left-hand side denotes the ∞-groupoid of real vector bundles over X, the right-hand side is the ∞-groupoid of invertible sheaves of spectra, and X ranges through all paracompact Hausdorff spaces. By observing that the left-hand side is a constant sheaf on the site of all paracompact Hausdorff spaces, the natural transformation above will automatically be induced by a functor

$$\mathrm{Vect}(*) \to \mathrm{Pic}(\mathcal{S}\mathrm{hv}(X; \mathcal{S}\mathrm{p}))$$

given by one-point compactification. After that, we will show that for any vector bundle E, $\mathrm{Th}(E)$ can be described through six operations analogously to the definition in motivic homotopy theory. The advantage of our perspective on the definition of $\mathrm{Th}(E)$ is that it makes the verification of all its expected properties, such as compatibility with pulling back vector bundles or short exact sequences, essentially trivial. We will then conclude the section by showing that for a submersion f between smooth manifolds, there is an equivalence

$$f^!(\mathbb{S}_Y) \simeq \mathrm{Th}(-T_f)$$

and hence obtaining a generalization of Atiyah duality.

1.7.1 Thom spaces and the J-homomorphism

Let Vect be the ∞-groupoid obtained by taking the coherent nerve of the topological groupoid whose objects are finite dimensional real vector spaces, and morphisms are spaces of linear isomorphisms between. Here we consider $\mathrm{GL}_n(\mathbb{R})$ equipped with the usual topology of a manifold (or equivalently the compact-open topology). As usual, one may equip Vect with a symmetric monoidal structure given by sum, and hence we may regard it as an object of $\mathrm{CMon}(\mathcal{S})$. Notice that we have an homotopy equivalence $\mathrm{Vect}_{\mathbb{R}} \simeq \coprod_{n \in \mathbb{N}} \mathrm{BGL}_n(\mathbb{R})$.

Let PH be the category of paracompact Hausdorff spaces with continuous maps between them. Throughout this section, we will assume all topological spaces appearing to be in PH. Recall also that PH can be equipped with a Grothendieck topology with the usual open coverings. Let $\mathbf{U}$ be a universe such that PH is $\mathbf{U}$-small, and denote by $\mathcal{S}'$ the ∞-category of $\mathbf{U}$-small spaces. Hence, the inclusion of $\mathcal{S}\mathrm{hv}(\mathrm{PH}; \mathcal{S}') \hookrightarrow \mathrm{Fun}(\mathrm{PH}^{\mathrm{op}}, \mathcal{S}')$ admits a left adjoint.

Definition 1.7.1. We denote by

$$\mathrm{Vect}_{\mathrm{PH}} : \mathrm{PH}^{\mathrm{op}} \to \mathrm{CMon}(\mathcal{S}')$$

the sheafification of the constant presheaf with value Vect.

The next lemma will justify our choice of notation.

Lemma 1.7.2. There have an equivalence

$$\mathrm{Vect}_{\mathrm{PH}}(X) \simeq \mathrm{Sing}(\mathrm{Map}(X, |\mathrm{Vect}|)),$$

which is natural on $X \in \mathrm{PH}$, and Map denotes the mapping space equipped with the compact-open topology. In particular, $\pi_0(\mathrm{Vect}_{\mathrm{PH}}(X))$ is in bijection with the set of equivalence classes of real vector bundles over X.

Proof. This is a consequence of [Lur09, Theorem 7.1.0.1]. The last part of the statement is standard: see for example [Hat17, Theorem 1.16]. □

Consider the map

$$\mathrm{GL}_n(\mathbb{R}) \to \mathrm{Hom}_{\mathcal{S}_*}(\mathrm{S}^n, \mathrm{S}^n) \simeq \Omega^n \mathrm{S}^n$$

given by the functoriality of the one-point compactification. The restriction of the map above to O_n is what's known as the *J-homomorphism*. We thus obtain a functor

$$\mathrm{Vect} \to \mathcal{S}_*^{\simeq}$$

that at the level of objects sends a finite dimensional real vector space V to its one-point compactification $\overline{V}$. Moreover, this is easily seen to be symmetric monoidal, where the right hand side is equipped with the smash product. By post composing with Σ^∞, we get

$$\mathrm{Vect} \to \mathcal{S}\mathrm{p}^{\simeq}$$

and since $\overline{V}$ is homoemorphic to a sphere, this factors as

$$\mathrm{Vect} \to \mathrm{Pic}(\mathcal{S}\mathrm{p}),$$

where $\mathrm{Pic}(\mathcal{S}\mathrm{p})$ is the *Picard ∞-groupoid* of $\mathcal{S}\mathrm{p}$, i.e. the full subcategory of $\mathcal{S}\mathrm{p}^{\simeq}$ spanned by the objects which are invertible with respect to the smash product of spectra.

Let $f : X \to Y$ be any map of topological spaces. Since the pullback functor $f^* : \mathcal{S}\mathrm{hv}(Y; \mathcal{S}\mathrm{p}) \to \mathcal{S}\mathrm{hv}(X; \mathcal{S}\mathrm{p})$ is monoidal, we obtain consequently a functor

$$\begin{aligned} \mathrm{PH}^{\mathrm{op}} &\longrightarrow \mathrm{CMon}(\mathcal{S}) \\ X &\longmapsto \mathrm{Pic}(\mathcal{S}\mathrm{hv}(X; \mathcal{S}\mathrm{p})) \end{aligned}$$

whose global sections are $\mathrm{Pic}(\mathcal{S}\mathrm{hv}(*; \mathcal{S}\mathrm{p})) \simeq \mathrm{Pic}(\mathcal{S}\mathrm{p})$. Since taking spectrum objects and Picard ∞-groupoids both commute with limits, such functor is a sheaf. Thus, Lemma 1.7.2 yields a map

$$\mathrm{Th} : \mathrm{Vect}_{\mathrm{PH}} \to \mathrm{Pic}(\mathcal{S}\mathrm{hv}(-; \mathcal{S}\mathrm{p})). \tag{1.7.3}$$

in $\mathcal{S}\mathrm{hv}(\mathrm{PH}; \mathrm{CMon}(\mathcal{S}))$.

Definition 1.7.4. For each vector bundle $E \to X$, we define $\mathrm{Th}(E)$ to be the image of E through the morphism (1.7.3).

Remark 1.7.5. Denote by $\mathrm{CMon}^{gp}(\mathcal{S})$ the full subcategory of $\mathrm{CMon}(\mathcal{S})$ spanned by those commutative monoids M such that $\pi_0(M)$ is a group. The inclusion $\mathrm{CMon}^{gp}(\mathcal{S}) \hookrightarrow \mathrm{CMon}(\mathcal{S})$ admits a left adjoint, denoted by $(-)^{gp}$, which is called the *group completion.* Since

$$\pi_0(\mathrm{Pic}(\mathcal{S}\mathrm{hv}(X; \mathcal{S}\mathrm{p})))$$

is a group, for any $X \in \mathrm{PH}$ we obtain a morphism

$$\mathrm{Vect}_{\mathrm{PH}}(X)^{gp} \to \mathrm{Pic}(\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})).$$

In particular we see that, for any vector bundle $E \to X$ it makes sense to define $\mathrm{Th}(-E)$ as the tensor inverse of $\mathrm{Th}(E)$, where $-E$ denotes the inverse of the class of E in $\pi_0(\mathrm{Vect}_{\mathrm{PH}}(X))^{gp}$. Moreover, for any split exact sequence

$$0 \to E \to V \to E' \to 0$$

we get an equivalence

$$\mathrm{Th}(V) \simeq \mathrm{Th}(E) \otimes \mathrm{Th}(E').$$

Out next goal is to describe $\mathrm{Th}(E)$ in terms of the six operations. Let $p : E \to X$ be a real vector bundle over X, and denote by $s : X \hookrightarrow E$ its zero section and by $j : E^\times \hookrightarrow E$ its open complement. Consider the sheaf

$$p_\sharp \mathrm{cofib}(j_\sharp j^* \mathbb{S}_E \to \mathbb{S}_E) \in \mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})$$

where the morphism $j_\sharp j^* \mathbb{S}_E \to \mathbb{S}_E$ is the counit. Notice that $p_\sharp$ exists since any vector bundle is obviously a shape submersion, so indeed the definition above makes sense. Notice also that, using Theorem 1.4.6, one has $p_\sharp \mathrm{cofib}(j_\sharp j^* \mathbb{S}_E \to \mathbb{S}_E) \simeq p_\sharp s_! \mathbb{S}_X$.

We will need the following lemma.

Lemma 1.7.6. Let

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

be any pullback square of locally compact Hausdorff spaces. Then the corresponding diagram

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}(X') \arrow[r] \arrow[d] & \mathcal{S}\mathrm{hv}(X) \arrow[d] \\ \mathcal{S}\mathrm{hv}(Y') \arrow[r] & \mathcal{S}\mathrm{hv}(Y) \end{tikzcd}$$

is a pullback of ∞-topoi.

Proof. Using the factorization (1.5.1), we just need to prove seperately the case of an open immersion and a proper map. The first is treated in [Lur09, Remark 6.3.5.8], while the second follows by the proof of [Lur09, Corollary 7.3.1.18]. □

Proposition 1.7.7. The assignment $E \mapsto p_\sharp \mathrm{cofib}(j_\sharp j^* \mathbb{S}_E \to \mathbb{S}_E) \in \mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p})$ defines a natural transformation

$$\mathrm{Vect}_{\mathrm{PH}} \to \mathrm{Pic}(\mathcal{S}\mathrm{hv}(-;\mathcal{S}\mathrm{p}))$$

which is naturally equivalent to (1.7.3).

Proof. First of all, we show that the Thom spectrum induces a natural transformation

$$\mathrm{Vect}_{\mathrm{PH}} \to \mathcal{S}\mathrm{hv}(-;\mathcal{S}\mathrm{p}) \tag{1.7.8}$$

of presheaves of ∞-categories. Let $p : E \to X$ be a vector bundle, $p^\times : E^\times \to X$ be the induced map on the complement of the zero section. Since one can write $\mathrm{Th}(E)$ as

$\mathrm{cofib}(p_\sharp^\times \mathbb{S}_{E^\times} \to p_\sharp \mathbb{S}_E)$, it will suffice to show that the associations $E \mapsto p_\sharp \mathbb{S}_E$ and $E \mapsto p_\sharp^\times \mathbb{S}_{E^\times}$ induce natural transformations. Consider a pullback square in Top

$$\begin{tikzcd} f^*E \arrow[r, "f'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

where p is a vector bundle. By Lemma 1.7.6

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}(f^*E) \arrow[r, "f'"] \arrow[d, "p'"'] & \mathcal{S}\mathrm{hv}(E) \arrow[d, "p"] \\ \mathcal{S}\mathrm{hv}(X') \arrow[r, "f"] & \mathcal{S}\mathrm{hv}(X) \end{tikzcd}$$

is a pullback square in $\mathcal{T}\mathrm{op}$. Similarly one has that the square

$$\begin{tikzcd} \mathcal{S}\mathrm{hv}(f^*E^\times) \arrow[r, "f'"] \arrow[d, "p'^\times"'] & \mathcal{S}\mathrm{hv}(E^\times) \arrow[d, "p^\times"] \\ \mathcal{S}\mathrm{hv}(X') \arrow[r, "f"] & \mathcal{S}\mathrm{hv}(X) \end{tikzcd}$$

is a pullback. Hence we have two natural transformations

$$\mathrm{Vect}_{\mathrm{PH}} \to \mathcal{T}\mathrm{op}_{/\mathcal{S}\mathrm{hv}(-)}$$

given respectively by sending a vector bundle $E \to X$ to $\mathcal{S}\mathrm{hv}(E) \to \mathcal{S}\mathrm{hv}(X)$ and to $\mathcal{S}\mathrm{hv}(E^\times) \to \mathcal{S}\mathrm{hv}(X)$, and thus, by further composing with the relative shape, by Remark 1.3.7 we obtain lax natural transformations

$$\mathrm{Vect}_{\mathrm{PH}} \to \mathrm{Pro}(\mathcal{S}\mathrm{hv}(-))$$

which factors as

$$\mathrm{Vect}_{\mathrm{PH}} \to \mathcal{S}\mathrm{hv}(-)$$

since any shape submersion induces a locally contractible geometric morphism by Corollary 1.3.26. Furthermore, by Lemma 1.3.25 and [Hau20, Theorem 3.22], we see that these are actually natural transformations, and thus, composing with

$$\mathcal{S}\mathrm{hv}(-) \to \mathcal{S}\mathrm{hv}(-; \mathcal{S}\mathrm{p})$$

we get the natural transformation (1.7.8).

We now prove that (1.7.8) is symmetric monoidal and that it factors through $\mathrm{Pic}(\mathcal{S}\mathrm{hv}(-;\mathcal{S}\mathrm{p}))$. Since $\mathrm{Vect}_{\mathrm{PH}}$ is the constant sheaf associated to Vect, for any sheaf $F \in \mathcal{S}\mathrm{hv}(\mathrm{PH}; \mathrm{CMon}(\mathcal{S}))$, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(\mathrm{Vect}_{\mathrm{PH}}, \mathrm{Pic}(F)) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Hom}_{\mathrm{CMon}}(\mathrm{Vect}, \mathrm{Pic}(F(*))) \arrow[d] \\ \mathrm{Hom}(\mathrm{Vect}_{\mathrm{PH}}, F) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Hom}_{\mathrm{CMon}}(\mathrm{Vect}, F(*)) \arrow[d] \\ \mathrm{Hom}(\mathrm{Vect}_{\mathrm{PH}}, F) \arrow[r, "\simeq"] & \mathrm{Hom}_{\mathcal{S}}(\mathrm{Vect}, F(*)) \end{tikzcd}$$

where the horizontal arrows are induced by taking global sections, the upper vertical arrows by the natural transformation $\mathrm{Pic}(F) \to F$ and the lower vertical arrows by the forgetful functor $\mathrm{CMon} \to \mathcal{S}$. By Corollary 1.7.9 we know that, after taking global sections, (1.7.8) factors through $\mathrm{Pic}(\mathcal{S}\mathrm{p})$, and thus we may conclude by a simple diagram chase. $\square$

Corollary 1.7.9. Assume that X is essential. Then, for any vector bundle E over X, $a_\sharp \mathrm{Th}(E)$ is equivalent to the Thom spectrum of E as classically defined.

Proof. Let $b : E \to *$ and $c : E^\times \to *$ be the unique maps. By Corollary 1.3.19, we have equivalences

$$\begin{aligned} a_\sharp \mathrm{Th}(E) &\simeq b_\sharp \mathrm{cofib}(j_\sharp j^* \mathbb{S}_E \to \mathbb{S}_E) \\ &\simeq \mathrm{cofib}(c_\sharp \mathbb{S}_{E^\times} \to b_\sharp \mathbb{S}_E) \\ &\simeq \mathrm{cofib}(\Sigma^\infty_+ E^\times \to \Sigma^\infty_+ E) \end{aligned}$$

and the spectrum on the last line coincides with the usual Thom spectum of E. □

Corollary 1.7.10. Let $p : E \to X$ be a real vector bundle over X, and denote $s : X \hookrightarrow E$ its zero section. Then $\mathrm{Th}(E)$ is invertible with inverse given by $s^! \mathbb{S}_E$.

Proof. By definition, we already know that $\mathrm{Th}(E)$ is invertible, and thus to compute its inverse we just need to look at its dual $\underline{\mathrm{Hom}}_X(\mathrm{Th}(E), \mathbb{S}_X)$. Then we may conclude by Corollary 1.3.26 and Proposition 1.6.12 since

$$\begin{aligned} \underline{\mathrm{Hom}}_X(p_\sharp s_! \mathbb{S}_X, \mathbb{S}_X) &\simeq p_* \underline{\mathrm{Hom}}_E(s_! \mathbb{S}_X, \mathbb{S}_E) \\ &\simeq p_* s_* s^! \mathbb{S}_E \\ &\simeq s^! \mathbb{S}_E. \end{aligned}$$

□

1.7.2 Relative Atiyah duality

Theorem 1.7.11. Let $f : X \to Y$ be a submersion between smooth manifolds. Then we have an equivalence $f^! \mathbb{S}_Y \simeq \mathrm{Th}(T_f)$, where T_f is the relative tangent bundle of f, defined by the short exact sequence of vector bundles

$$0 \to T_f \to TX \to f^* TY \to 0. \tag{1.7.12}$$

Proof. First of all we prove the case when f is the unique map $a : X \to *$ and X is a smooth manifold. Choose a closed embedding $i : X \hookrightarrow \mathbb{R}^n$, and let $a' : \mathbb{R}^n \to *$ be the unique map. By Proposition 1.6.18, we have $a'^! \mathbb{S} \simeq \Sigma^n \mathbb{S}_{\mathbb{R}^n}$ and thus, since $T\mathbb{R}^n$ is a trivial vector bundle of fiber dimension n, we have $\mathrm{Th}(T\mathbb{R}^n) \simeq \Sigma^n \mathbb{S}_{\mathbb{R}^n} \simeq a'^! \mathbb{S}$. Let $p : N_i \to X$ be the conormal bundle of the embedding i, defined by the short exact sequence

$$0 \to TX \to i^* T\mathbb{R}^n \to N_i \to 0, \tag{1.7.13}$$

$s : X \hookrightarrow N_i$ its zero section. Let $k : U \hookrightarrow N_i$ be a tubular neighbourhood of X in $\mathbb{R}^n$. Thus, we get a commutative triangle

$$\begin{tikzcd} & X \arrow[dl, hook, "\tilde{s}"'] \arrow[dr, hook, "i"] & \\ U \arrow[rr, hook, "g"] & & \mathbb{R}^n \end{tikzcd}$$

where g is an open immersion and $\tilde{s}$ is a closed immersion. Hence we get an equivalence

$$i^! \mathbb{S}_{\mathbb{R}^n} \simeq \tilde{s}^! \mathbb{S}_U.$$

Then, by Proposition 1.6.18 and (1.7.13) we have

$$
\begin{aligned}
a^!\mathbb{S} &\simeq a'^! i^! \mathbb{S} \\
&\simeq a'^! \mathbb{S} \otimes i^* \mathrm{Th}(T\mathbb{R}^n) \\
&\simeq \mathrm{Th}(N_i)^{-1} \otimes \mathrm{Th}(i^* T\mathbb{R}^n) \\
&\simeq \mathrm{Th}(TX).
\end{aligned}
$$

Suppose now that $f : X \to Y$ is any submersion between smooth manifolds, $a : X \to *$ and $b : Y \to *$ be the unique maps. Then, by Proposition 1.6.18 and Remark 1.7.5, we have

$$
\begin{aligned}
\mathrm{Th}(T_f) &\simeq \mathrm{Th}(TX) \otimes \mathrm{Th}(f^* TY)^{-1} \\
&\simeq a^!\mathbb{S} \otimes (f^* b^! \mathbb{S})^{-1} \\
&\simeq a^!\mathbb{S} \otimes (f^! b^! \mathbb{S})^{-1} \otimes f^! \mathbb{S}_Y \\
&\simeq f^! \mathbb{S}_Y
\end{aligned}
$$

and thus we can conclude. □

Corollary 1.7.14 (Relative Atiyah Duality)**.** Let $f : X \to Y$ be a proper map inducing a locally contractible geometric morphism. Then $f_\sharp \mathbb{S}_X \in \mathcal{S}\mathrm{hv}(Y; \mathcal{S}\mathrm{p})$ is strongly dualizable with dual $f_! \mathbb{S}_X$. Moreover, if X and Y are smooth manifolds and f is a proper submersion, then $f_\sharp \mathbb{S}_X$ is strongly dualizable with dual $f_\sharp \mathrm{Th}(-T_f)$.

Proof. Since f is proper, by Corollary 1.3.26 and Proposition 1.6.12, we have, functorially on $F \in \mathcal{S}\mathrm{hv}(Y; \mathcal{S}\mathrm{p})$

$$
\begin{aligned}
\underline{\mathrm{Hom}}_Y(f_\sharp \mathbb{S}_X, F) &\simeq f_* \underline{\mathrm{Hom}}_X(\mathbb{S}_X, f^* F) \\
&\simeq f_* f^* F \\
&\simeq f_! f^* F \\
&\simeq f_! \mathbb{S}_X \otimes F.
\end{aligned}
$$

In particular, when f is a submersion of smooth manifolds, by Proposition 1.6.18 and the previous theorem, we have $f_! \mathbb{S}_X \simeq f_\sharp \mathrm{Th}(-T_f)$. □

Remark 1.7.15. Let X be a smooth manifold, $a : X \to *$ the unique map. By specializing the previous corollary to a and Corollary 1.7.9, we see that we recover the classical Atiyah duality.

Chapter 2

Verdier duality on conically smooth stratified spaces

2.1 Introduction

Constructible sheaves are of great interest both in algebraic and differential geometry, as they provide tools to study invariants for singular spaces (such as *intersection cohomology*, following the approach of [BBDG18]) and have surprising and beautiful relations with D-modules (see [Kas84]). A fundamental feature of constructible sheaves is that, assuming a certain kind of finiteness properties on the stalks and considering nicely behaved stratifications, they carry a duality (sometimes referred to as *Verdier duality*) which enables one, through abstract trace methods, to associate to any such sheaf a class in Borel–Moore homology. One of the interests of these classes is that they can be related to Euler characteristics via routine computations with the six functor formalism (see [KS90, Chapter 9] for a discussion on classical index formulas and their microlocal enhancements).

As far the author knows, it was an idea of MacPherson that the duality should actually be thought of as a combination of two different equivalences of categories. The former, given by the usual construction of sections with compact support, was expected to identify constructible sheaves with constructible *cosheaves* (without constructibility assumptions, this was proven by Lurie in [Lur17, Theorem 5.5.5.1], and there named after Verdier); the latter maps back contravariantly construtible cosheaves to sheaves, and obtained using a foreseen combinatorial description of constructible (co)sheaves, similar in spirit to the monodromy for local systems, which is nowadays referred to as *exodromy* (see [Lur17, Theorem A.9.3], [BGH18]). In this chapter we make use of the language of ∞-categories to realize the vision of MacPherson and prove the expected duality result in a very general setting.

Let us spend a few words to specify more precisely the framework in which we are working. Relying on the theory developed in the first chapter, we will be able to deal with sheaves valued in any stable bicomplete ∞-category $\mathcal{C}$, equipped with a closed symmetric monoidal structure. The machinery of six functors developed in the first chapter supplies us with a *dualizing sheaf* $\omega_X^{\mathcal{C}}$ for any $\mathcal{C}$ as above and X locally compact Hausdorff stratified space. Our duality functor will thus be defined as usual by taking an internal-hom into $\omega_X^{\mathcal{C}}$, and denoted by $D_X^{\mathcal{C}}$.

Following the nomenclature of [BGH18], we will define a sheaf with values in $\mathcal{C}$ to be *formally constructible*[1] if its restriction to each stratum is locally constant, and *constructible* if furthermore all its stalks are dualizable. Similar definitions can be given for $\mathcal{C}$-valued

[1]Notice that we will only deal with sheaves which are constructible with respect to a fixed stratification, as opposed to [KS90, Chapter 8], for example.

cosheaves by observing that, up to passing to an opposite category, these are just $\mathcal{C}^{op}$-valued sheaves. The requirement of dualizability for stalks is really unavoidable mainly because, when X has a unique point, we see that $\omega_X^{\mathcal{C}}$ is the monoidal unit of $\mathcal{C}$, and thus the duality functor coincides with the one coming from the monoidal structure on $\mathcal{C}$. Moreover it is also extremely reasonable, since if $\mathcal{C} = D(R)$ where R is a ring (or more generally, modules over any E_∞-ring spectrum), it is well-known that a complex is dualizable if and only if it is perfect, and hence our assumptions allow us to recover the classical setting as a special case.

For the geometric side of the story, we will consider *conically smooth stratified spaces*. These were introduced by Ayala, Francis and Tanaka in [AFT17], and provide a natural extension of C^∞-structures in the stratified/singular setting. The definition of conically smooth atlases is rather involved, as it relies on a elaborate inductive construction based on the *depth*[2] of a stratification. We suggest the reader to have a look at the introduction of [AFT17] or alternatively, to have a look at the short outilne of the main steps to build up the construction that we provide in section 2. To convince the reader of the soundness of this definition, we will also provide a vast class of examples of conically smooth spaces via the following theorem, that was proven in collaboration with Guglielmo Nocera.

Theorem 2.1.1. Any Whitney stratified space admits a conically smooth structure.

We are now ready to give a precise statement of our duality theorem.

Theorem 2.1.2. Let X be a conically smooth stratified space, and let $\mathcal{S}\mathrm{hv}^c(X;\mathcal{C})$ be the full subcategory of $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ spanned by constructible sheaves. Then the restriction to $\mathcal{S}\mathrm{hv}^c(X;\mathcal{C})^{op}$ of the functor $D_X^{\mathcal{C}}$ factors through an equivalence

$$D_X^{\mathcal{C}} : \mathcal{S}\mathrm{hv}^c(X;\mathcal{C})^{op} \xrightarrow{\simeq} \mathcal{S}\mathrm{hv}^c(X;\mathcal{C}).$$

To conclude this introduction, let us make a short comment on how our proof strategy goes. Our first observation, as anticipated above, is that the functor $D_X^{\mathcal{C}}$ factors through Lurie's equivalence

$$\mathbb{D}_{\mathcal{C}} : \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \to \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C}),$$

and most of the work then lies in proving that the restriction of $\mathbb{D}_{\mathcal{C}}$ to constructible sheaves factors through constructible cosheaves. We first show in Proposition 2.6.1 that $\omega_X^{\mathcal{C}}$ is constructible when $\mathcal{C} = \mathcal{S}\mathrm{p}$ (the ∞-category of spectra), and from the techniques developed in the first chapter we deduce immediately that

$$a_{\mathcal{C}}^! : \mathcal{C} \to \mathcal{S}\mathrm{hv}(X;\mathcal{C})$$

factors through formally constructible sheaves. As a consequence of this and some easy properties of constructible sheaves that follow from *homotopy invariance* (see Theorem 2.5.3), one gets that $\mathbb{D}_{\mathcal{C}}$ maps formally constructible sheaves into formally constructible cosheaves. We want to stress that the possibility of working with such a general class of coefficients, which is closed under passing to opposite categories, makes this step extremely formal.

The missing piece then consists in showing that $\mathbb{D}_{\mathcal{C}}$ preserves the property of having dualizable stalks, and this is the point where we will actually have to employ the geometry of conically smooth structures. More specifically, we will use the unzip constuction to prove in

[2] Recall that, for a stratified space $s : X \to P$, the depth is defined as

$$\mathrm{depth}(X) = \sup_{x \in X} \dim_x(X) - \dim_x(X_{s(x)}),$$

where dim denotes the covering dimension and $X_{s(x)}$ is the stratum of X corresponding to $s(x) \in P$.

Proposition 2.4.15, through an inductive argument on the depth, that any compact stratified space equipped with a conically smooth structure has a finite exit path ∞-category. The key observation then is that, if $X = C(Z)$ with Z compact, $x \in X$ is the cone point and F is any constructible sheaf on X, then there is a fiber sequence

$$\Gamma_x(X;F) \to F_x \to \Gamma(Z;F), \tag{2.1.3}$$

where $\Gamma_x(X;F)$ denotes the sections of F supported at x (i.e. the stalk of the associated cosheaf of compactly supported sections of F) and F_x is the stalk of F at x. By Proposition 2.4.15 and the exodromy equivalence (which will be show to hold also for our general class of coefficients in Theorem 2.5.16), one gets that $\Gamma(Z;F)$ must be dualizable, and thus F_x is dualizable if and only if $\Gamma_x(X;F)$ is, which proves our claim.

2.1.1 Linear overview

We now give a linear overview of the results in this chapter.

In section 2 we recall the definitions of stratified spaces, Whitney conditions and conically smooth structures. The reader who knows already about all this may directly skip the section.

In Section 3 we prove that any Whitney stratified space admits a conically smooth structure. We will first recall how to get conical charts using compatible systems of tubular neighbourhood following [Mat70] and [Mat73], and then check that these charts satisy the compatibility conditions required for conically smooth atlases in Theorem 2.3.7.

Section 4 is mainly devoted to prove Proposition 2.4.15. In the first part we will recall the definition of a finite ∞-category, and show how these can be described in the model of quasi-categories. None of these results or definitions are new, but we decided to include a few words on the subject since we could not find any reference dealing with it in our preferred fashion. In the second part we recall Lurie's definition of the simplicial set of exit-paths of a stratified topological space, and show in Lemma 2.4.10 how one can conveniently compute the exit-paths of the cone of a proper stratified fiber bundle $L \to X$ in terms of L and X. The main idea for proving Proposition 2.4.15 is then to cover a conically smooth stratified space X into the open subset U given by the locus of points of depth zero and a tubular neighbourhood of the complement of U. By induction and Lemma 2.4.10 one is then left to show that $\mathrm{Exit}(U)$ is finite, but this follows by observing that, through the unzip construction, U can be identified with the interior of compact manifold with corners.

In Section 5 we deal with extending the results of [HPT20] to sheaves values in stable bicomplete ∞-categories. This is very simple and formal, after one has the six operations. As a consequence, we show in Corollary 2.5.6 that the stalk at a point x of a constructible sheaf is the same as sections at any conical chart around x, and compute in Corollary 2.5.9 the restriction of a construtible sheaf along a stratum in a convenient way. These two results are essential and are used very often in what follows (for example, the first implies immediately the existence of (2.1.3)). In the second part we will then characterize constructible sheaves by the property of being homotopy invariant (see Proposition 2.5.13), and use this to deduce exodromy for conically smooth spaces with general stable bicomplete coefficients.

Section 6 deals with proving our main result. We start by showing, through an inductive argument on the depth, that $\omega_X^{\mathcal{C}}$ is constructible whenever X is C^0-stratified. We first reduce to proving the statement in the case $\mathcal{C} = \mathcal{S}\mathrm{p}$ by employing the techniques developed in the first chapter, and then the only non-trivial part consists in showing that, when $X = C(Z)$, the stalk of the dualizing sheaf at the cone-point is a finite spectrum. We then conclude by proving Theorem 2.6.3. As explained at the beginning of the introduction, our argument starts by observing that the duality functor factors through Lurie's Verdier duality, and the hard part then consists in showing that the latter restricts to an equivalence between constructible (co)sheaves, for which we have to implement all the results obtained previously.

2.2 Background on stratified spaces

In the present section, we recall notations and results from the theory of Whitney stratifications and of conically smooth stratifications.

2.2.1 Whitney stratifications (Thom, Mather)

For our review of the theory of Whitney stratifications, we follow [Mat70], with minimal changes made in order to connect the classical terminology to the one used in [AFT17].

Definition 2.2.1. Let M be a smooth manifold, and $Z \subset M$ a subset. A **smooth stratification** of Z is a partition of Z into subsets $\{Z_\alpha\}_{\alpha \in A}$, such that each Z_α is a smooth submanifold of M. More generally, if M is a C^μ-manifold, then a C^μ stratification of a subset Z of M is a partition of Z into C^μ-submanifolds of M.

Remark 2.2.2. In particular, all strata of a smoothly stratified space $Z \subset M$ are locally closed subspaces of Z.

Definition 2.2.3 (Whitney's Condition B in $\mathbb{R}^n$)**.** Let X, Y be smooth submanifolds of $\mathbb{R}^n$, and let $y \in Y$ be a point. The pair (X, Y) is said to satisfy **Whitney's Condition B** at y if the following holds. Let $(x_i) \subset X$ be a sequence converging to y, and $(y_i) \subset Y$ be another sequence converging to y. Suppose that $T_{x_i}X$ converges to some vector space τ in the r-Grassmannian of $\mathbb{R}^n$ and that the lines $x_i y_i$ converge to some line l in the 1-Grassmannian (projective space) of $\mathbb{R}^n$. Then $l \subset \tau$.

Definition 2.2.4 (Whitney's condition B)**.** Let X, Y be smooth submanifolds of a smooth n-dimensional manifold M, and $y \in Y$. The pair (X, Y) is said to satisfy Whitney's Condition B at y if there exist a chart of M $\phi : U \to \mathbb{R}^n$ around y such that $(\phi(U \cap X), \phi(U \cap Y))$ satisfies Whitney's Condition B at $\phi(y)$.

Recall (e.g. from [Pfl01, Lemma 1.4.4]) that Whitney's Condition B is invariant under change of charts, and therefore Definition 2.2.4 is well-posed.

Definition 2.2.5 (Whitney stratification)**.** Let M be a smooth manifold of dimension n. A smooth stratification $(Z, \mathcal{S})$ on a subset Z of M is said to satisfy the Whitney conditions if

- (local finiteness) each point has a neighbourhood intersecting only a finite number of strata;
- (condition of the frontier) if Y is a stratum of $\mathcal{S}$, consider its closure $\bar{Y}$ in M. Then we require that $(\bar{Y} \setminus Y) \cap Z$ is a union of strata, or equivalently that $S \in \mathcal{S}, S \cap \bar{Y} \neq \varnothing \Rightarrow S \subset \bar{Y}$;
- (Whitney's condition B) Any pair of strata of $\mathcal{S}$ satisfies Whitney's condition B when seen as smooth submanifolds of M.

Case (1). We define the ne Given two strata of a Whitney stratification X and Y, we say that $X < Y$ if $X \subset \bar{Y}$. This is a partial order on $\mathcal{S}$.

A feature of Whitney stratified spaces, perhaps not very evident from the definition, is the existence of compatible tubular neighbourhoods around strata, in a sense that we will now recall. We refer to [Mat70, Section 6] for more details.

Definition 2.2.6. Let M be a manifold and $X \subset M$ be a submanifold. A **tubular neighborhood** T of X in M is a triple (E, ε, ϕ), where $\pi : E \to X$ is a vector bundle with an inner product $\langle , \rangle$, ε is a positive smooth function on X, and ϕ is a diffeomorphism of $B_\varepsilon = \{e \in E \mid \langle e, e\rangle < \varepsilon(\pi(e))\}$ onto an open subset of M, which commutes with the zero section ζ of E:

$$\begin{tikzcd} B_\varepsilon \arrow[dr, "\phi"] & \\ X \arrow[u, "\zeta"] \arrow[r, hook] & M. \end{tikzcd}$$

From [Mat73, Corollary 6.4] we obtain that any stratum W of a Whitney stratified space $(M, \mathcal{S})$ has a tubular neighbourhood, which we denote by (T_W, ε_W); the relationship with the previous notation is the following: T_W is $\phi(B_\varepsilon) \cap M$ (recall that a priori $\phi(B_\varepsilon) \subset M$, the ambient manifold). We also denote by ρ the tubular (or distance) function

$$T_W \to \mathbb{R}_{\geq 0}$$

$$v \mapsto \langle v, v\rangle$$

with the notation as in Definition 2.2.6. Note that $\rho(v) < \varepsilon(\pi(v))$.

A final important feature of the tubular neighbourhoods of strata constructed in Mather's proof is that they satisfy the so-called "control conditions" or "commutation relations". Namely, consider two strata $X < Y$ of a Whitney stratified space M. Then, if T_X and T_Y are the tubular neighbourhoods relative to X and Y as constructed by Mather, one has that

$$\pi_Y \pi_X = \pi_Y$$

$$\rho_X \pi_Y = \rho_X.$$

We explain the situation with an example.

Example 2.2.7. Let M be the real plane $\mathbb{R}^2$ and $\mathcal{S}$ the stratification given by

$$X = \{(0, 0)\}$$

$$Y = \{x = 0\} \setminus \{(0, 0)\}$$

$$Z = M \setminus \{x = 0\}.$$

We take $\mathbb{R}^2$ itself as the ambient manifold. Then Mather's construction of the tubular neighbourhoods associated to the strata gives a result like in Figure 2.1. Here the circle is T_X, and the circular segment is a portion of T_Y around a point of Y. We can see here that T_Y is not a "rectangle" around the vertical line, as one could imagine at first thought, because the control conditions impose that the distance of a point in T_W from the origin of the plane is the same as the distance of its "projection" to Y from the origin.

Keeping this example in mind (together with its upper-dimensional variants) for the rest of the treatment may be a great help for the visualization of the arguments used in our proofs.

The notion of having a compatible system of tubular neighbourhoods around strata is axiomatized in the definition of *abstract stratified set* ([Mat70, Definition 8.1]. Roughly, this consists of a triple $(V, \mathcal{S}, \mathcal{I})$ where V is a locally compact Hausdorff topological space, $\mathcal{S}$ is a partition of V into locally closed subsets (called strata) each equipped "abstractly" with a smooth structure, and $\mathcal{I}$ is a collection of neighbourhoods around each stratum with some structure abstractly axiomatizing the notion of "tubular" neighbourhood.

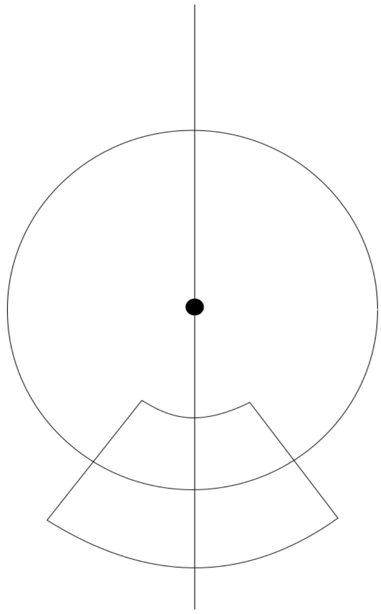

Figure 2.1: Tubular neighbourhoods in $(\mathbb{R}^2, (X, Y, Z))$.

Definition 2.2.8. Let $(V, \mathcal{S}, \mathcal{I})$ be an abstract stratified set. If P is a smooth manifold and $f : V \to P$ is a continuous mapping, we will say that f is a *controlled submersion* if the following conditions are satisfied.

- $f|_X : X \to P$ is a smooth submersion for each stratum X of V.
- For any stratum X, there is a neighborhood $T_{X'}$ of X in T_X such that $f(v) = f\pi_X(v)$ for all $v \in T_{X'}$.

The following result follows from [Mat70, Proposition 7.1].

Proposition 2.2.9. Let $(M, \mathcal{S})$ be a Whitney stratified space. Then it admits a canonical structure of abstract stratified set (up to equivalence of abstract stratified sets, see again [Mat70, Definition 8.1]).

Conversely, we have:

Theorem 2.2.10 ([Nat80, Theorem on page 3])**.** Every paracompact abstract stratified set $(V, \mathcal{S}, \mathcal{I})$ with $\dim V = n$ can be topologically embedded in $\mathbb{R}^{2n+1}$ such that the image of the embedding is a stratified space satisfying the Whitney condition, and the smooth structures on strata coincide with the ones inherited from $\mathbb{R}^{2n+1}$. All such "realizations" as Whitney stratified spaces in $\mathbb{R}^N$ are isotopic if $N > 2n + 2$.

2.2.2 Conical and conically smooth stratifications (Lurie, Ayala-Francis-Tanaka)

We now turn to the more recent side of the story, namely the theory of conically smooth stratified spaces. In order to do that, we briefly recall the treatment of stratified sets given by Jacob Lurie in [Lur17], which is the base of the formalism used in [AFT17].

Definition 2.2.11. Let P be a partially ordered set. The Alexandrov topology on P is defined as follows. A subset $U \subset P$ is open if it is closed upwards: if $p \leq q$ and $p \in U$ then $q \in U$.

With this definition, closed subsets are downward closed subsets and locally closed subsets are "convex" subsets: $p \leq r \leq q, p, q \in U \Rightarrow r \in U$.

Definition 2.2.12 ([Lur17, Definition A.5.1])**.** A stratification on a topological space X is a continuous map $s : X \to P$ where P is a poset endowed with the Alexandrov topology. The fibers of the points $p \in P$ are subspaces of X and are called the strata. We denote the fiber at p by X_p and by $\mathcal{S}$ the collection of these strata.

Remark 2.2.13. Note that, by continuity of s, the strata are locally closed subsets of X. However, in this definition we do not assume any smooth structure, neither on the ambient space nor on the strata. Furthermore, the condition of the frontier in Definition 2.2.5 need not hold for stratifications in the sense of Definition 2.2.12: for example, the map $\mathbb{R} \to \{0, 1\}$ where $0 < 1$ and the map is given by mapping the interval $(0, 1)$ to 1 and the rest to 0 is a stratification in the sense of Definition 2.2.12, but the two strata do not satisfy the condition of the frontier.

Note however that the condition of the frontier implies that any Whitney stratified space is stratified in the sense of Definition 2.2.12: indeed, one obtains a map towards the poset $\mathcal{S}$ defined by $S < T \iff S \subset \overline{T}$, which is easily seen to be continuous by the condition of the frontier.

Definition 2.2.14. A stratified map between stratified spaces (X, P, s) and (Y, Q, t) is the datum of a continuous map $f : X \to Y$ and an order-preserving map $\phi : P \to Q$ making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"] & Y \arrow[d, "t"] \\ P \arrow[r, "\phi"] & Q \end{tikzcd}$$

commute.

Definition 2.2.15. Let (Z, P, s) be a stratified topological space. We define $C(Z)$ (as a set) as

$$\frac{Z \times [0, 1)}{\{(z, 0) \sim (z', 0)\}}.$$

Its topology and stratified structure are defined in [Mat73, Definition A.5.3]. When Z is compact, then the topology is the quotient topology. Note that the stratification of $C(Z)$ is over $P^{\triangleleft}$, the poset obtained by adding a new initial element to P: the stratum over this new point is the vertex of the cone, and the other strata are of the form $X \times (0, 1)$, where X is a stratum of Z.

Definition 2.2.16 ([Lur17, Definition A.5.5])**.** Let (X, P, s) be a stratified space, $p \in P$, and $x \in X_p$. Let $P_{>p} = \{q \in P \mid q > p\}$. A **conical chart** at x is the datum of a stratified space $(Z, P_{>p}, t)$, an unstratified space Y, and a P-stratified open embedding

$$\begin{tikzcd} Y \times C(Z) \arrow[rr, hook] \arrow[dr] & & X \arrow[dl] \\ & P & \end{tikzcd}$$

whose image hits x. Here the stratification of $Y \times C(Z)$ is induced by the stratification of $C(Z)$, namely by the maps $Y \times C(Z) \to C(Z) \to P_{\geq p} \to P$ (see Definition 2.2.15).

A stratified space is **conically stratified** if it admits a covering by conical charts.

More precisely, the conically stratified spaces we are interested in are the so-called **C^0-stratified spaces** defined in [AFT17, Definition 2.1.15]. Here we recall the two important properties of a C^0-stratified space $(X, s : X \to P)$:

- every stratum X_p is a *topological manifold*;
- there is a *basis* of the topology of X formed by conical charts

$$\mathbb{R}^i \times C(Z) \to X$$

  where Z is a *compact* C^0-stratified space over the relevant $P_{>p}$. Note that Z will have depth strictly less than X; this observation will be useful in order to make many inductive arguments work.

Hence the definition of [AFT17] may be interpreted as a possible analogue of the notion of topological manifold in the stratified setting: charts are continuous maps which establish a stratified homeomorphism between a small open set of the stratified space and some "basic" stratified set.

Following this point of view, one may raise the question of finding an analogue of "smooth manifold" (or, more precisely, "smoothly differentiable structure") in the stratified setting. We refer to [AFT17, Definition 3.2.21] for the definition of a **conically smooth structure** (and to the whole Section 3 there for a complete understanding of the notion), which is a very satisfying answer to this question. The definition is rather involved, as it relies essentially on a technical inductive construction based on the *depth* of a stratification. We sketch the main steps of the definition below.

Definition 2.2.17 ([AFT17, Definition 2.4.4])**.** Let $s : X \to P$ be a stratified topological space. We define

$$\operatorname{depth} X = \sup_{x \in X} (\dim_x(X) - \dim_x(X_{s(x)})),$$

where dim denotes the covering dimension and $X_{s(x)}$ is the stratum of X corresponding to $s(x) \in P$.

Remark 2.2.18. Let Z be an unstratified space of Lebesgue covering dimension n. Then the depth of the cone $C(Z)$ at the cone point is $n + 1$.

Remark 2.2.19. Note that there is an alternative natural definition of depth at a point x: namely, the maximal k such that there exists a chain of strata of the form $X_0 < \cdots < X_k$ such that $x \in X_0$. The depth of a stratification at a point in the sense of Definition 2.2.17 is always greater or equal than the latter, and they coincide if and only if there is a chain of maximal length $X_0 < \cdots < X_k$ such that $x \in X_1$ and $\dim X_{i+1} = \dim X_i + 1$ for $i = 0, \ldots, k - 1$.

The following observation will be useful for the proof of our main result.

Remark 2.2.20. Let (X, P, s) be a C^0 stratified space whose strata are all finite dimensional, and suppose that $\operatorname{depth}(X, P, s) = 0$. Then P is a discrete poset, i.e. for any two $p, q \in P$ we have that $p \leq q \iff p = q$. Indeed, by assumption, we know that for any $x \in X$ the following formula holds:

$$\dim_x X_p = \dim_x X$$

where X_p is the stratum containing x. Up to taking a conical chart centered at x, this translates into

$$\dim_x(\mathbb{R}^n) = \dim_{(x,*)}(\mathbb{R}^n \times C(Z))$$

i.e.

$$n = n + \dim C(Z)$$

where $*$ is the cone point of $C(Z)$. This implies that Z is empty, hence the conclusion.

In particular, if $X \to P$ is a C^0 stratified space such that $\operatorname{depth} X = 0$, then a conically smooth atlas on X is just the usual notion of a smooth atlas which defines C^∞-manifolds. If $\operatorname{depth} X > 0$, roughly a conically smooth atlas is a collection of stratified open subsets $\{U_i\}_{i \in I}$ of X satisfying the following conditions

(i) for each U_i there exists a stratified open embedding $\varphi_i : \mathbb{R}^{n_i} \times C(Z_i) \hookrightarrow X$ whose image is U_i, where Z_i is a compact stratified space equipped with a conically smooth atlas (notice that $\operatorname{depth} Z_i < \operatorname{depth} X$, so that by induction the notion of an atlas on Z_i is well defined): an object of the type $\mathbb{R}^{n_i} \times C(Z_i)$ where Z_i is as above will be called a basic conically smooth stratified space;

(ii) for any i, j such that $U_i \cap U_j \neq \emptyset$, there exists some $k \in I$ and *conically smooth open embeddings* $\mathbb{R}^{n_k} \times C(Z_k) \hookrightarrow \mathbb{R}^{n_i} \times C(Z_i)$ and $\mathbb{R}^{n_k} \times C(Z_k) \hookrightarrow \mathbb{R}^{n_j} \times C(Z_j)$ such that the square

$$\begin{tikzcd} \mathbb{R}^{n_k} \times C(Z_k) \arrow[r, hook] \arrow[d, hook] & \mathbb{R}^{n_i} \times C(Z_i) \arrow[d, hook] \\ \mathbb{R}^{n_j} \times C(Z_j) \arrow[r, hook] & X \end{tikzcd}$$

commutes.

For the above definition to make sense, one has to explain what a conically smooth map between basics is. By playing again with inductive arguments on the depth of the target, the main new conceptual input one has to give to formulate precisely this definition is the notion of *differentiability along a cone locus* (see [AFT17, Definition 3.1.4]): for a map $f : \mathbb{R}^n \times C(Z) \to \mathbb{R}^m \times C(S)$ between basics, this amounts to requiring the existence of a continuous extension

$$\begin{tikzcd} \mathbb{R}_{\geq 0} \times \mathrm{T}\mathbb{R}^n \times C(Z) \arrow[r, dashed] & \mathbb{R}_{\geq 0} \times \mathrm{T}\mathbb{R}^m \times C(S) \\ \mathbb{R}_{>0} \times \mathrm{T}\mathbb{R}^n \times C(Z) \arrow[u] \arrow[r, "\gamma^{-1} \circ f_\Delta \circ \gamma"] & \mathbb{R}_{>0} \times \mathrm{T}\mathbb{R}^m \times C(S) \arrow[u] \end{tikzcd}$$

where the lower horizontal map is built out of f and an appropriate use of the action of scaling and translating on the conical charts such that, in case in which $Z = S = \emptyset$, this recovers the usual notion of differentiability.

One main feature of conically smooth structures is the *unzip* construction, that allows one to functorially resolve any conically smooth stratified space into a manifold with corners. For example, if $X_k \hookrightarrow X$ is the inclusion of a stratum of maximal depth, there is a square

$$\begin{tikzcd} \mathrm{Link}_k(X) \arrow[r, hook] \arrow[d, "\pi_X"] & \mathrm{Unzip}_k(X) \arrow[d] \\ X_k \arrow[r, hook] & X \end{tikzcd} \tag{2.2.21}$$

which is both pushout and pullback, and $\mathrm{Unzip}_k(X)$ is a conically smooth manifold *boundary* given by $\mathrm{Link}_k(X)$ such that both its interior and $\mathrm{Link}_k(X)$ have depth strictly smaller than the one of X. An interesting consequence of the existence of pushout/pullback square (Equation (2.2.21)) is that the notion of conically smooth map is completely determined by the one of smooth maps between manifolds with corners.

Definition 2.2.22. A C^0-stratified space together with a conically smooth structure is called a **conically smooth stratified space**.

Remark 2.2.23. Let us list some useful properties of conically smooth structures:

- any conically smooth stratified space is a C^0-stratified space;
- all strata have an induced structure of *smooth* manifold, like in the case of Whitney stratifications;
- the definition of conically smooth space is intrinsic, in the sense that it does not depend on a given embedding of the topological space into some smooth manifold, in contrast to the case of Whitney stratifications (see Definition 2.2.1 and Definition 2.2.5);
- the notion of conically smooth map (which is a map inducing conically smooth maps between basics in charts) differs substantially from the "naive" requirement of being stratified and smooth along each stratum that one has in the case of Whitney stratifications. The introduction of this notion defines a category $\mathcal{S}$trat of conically smooth stratified spaces. In this setting, [AFT17] are able to build up a very elegant theory and prove many desirable results such as a functorial resolution of singularities to smooth manifolds with corners and the existence of tubular neighbourhoods of conically smooth submanifolds. These results allow to equip $\mathcal{S}$trat with a Kan-enrichment (and hence, a structure of ∞-category); also, the hom-Kan complex of conically smooth maps between two conically smooth spaces has the "correct" homotopy type (we refer to the introduction to [AFT17] for a more detailed and precise discussion on this topic), allowing to define a notion of tangential structure naturally extending the one of a smooth manifold and to give a very simple description of the exit-path ∞-category of a conically smooth stratified space.

Up to now, the theory of conically smooth spaces has perhaps been in need of a good quantity of explicit examples, especially of topological nature. The following result (conjectured in [AFT17, Conjecture 1.5.3]) goes in the direction of providing a very broad class of examples coming from differential geometry and topology.

Theorem 2.2.24. Let $(M, \mathcal{S})$ be a Whitney stratified space. Then it admits a conically smooth structure in the sense of [AFT17].

The rest of the section is devoted to the proof of this theorem (Theorem 2.3.7).

2.3 Whitney stratifications are conically smooth

2.3.1 Whitney stratifications are conical

Lemma 2.3.1. Let $(M, \mathcal{S}, \mathcal{I})$ be an abstract stratified set, T a smooth unstratified manifold, and let $f : M \to T$ be a controlled submersion. Then for every $p \in T$ the fiber of f at p has a natural structure of abstract stratified set inherited from M.

Proof. We may assume that for any stratum, up to shrinking tubular neighbourhoods (in such a way to obtain an equivalent abss),

$$f(v) = f\pi_X(v) \; \forall v \in T_X.$$

We only need to prove that for any other stratum Y

$$(T_X \setminus X) \cap Y \to (X \cap f^{-1}(x)) \times \mathbb{R}_{\geq 0}$$

is a submersion.

Therefore, it suffices to show that the square

$$\begin{array}{ccc} T_X \cap f^{-1}(x) & \longrightarrow & X \cap f^{-1}(x) \\ \downarrow & & \downarrow \\ T_X & \longrightarrow & X \end{array}$$

is a pullback. But the pullback is, by definition,

$$\{v \in T_X \mid f\pi(v) = x\}$$

which is equal to

$$\{v \in T_X \mid f(v) = x\} = T_X \cap f^{-1}(x).$$

□

Lemma 2.3.2. Any open subset U of a space M endowed with a structure of abstract $(M, \mathcal{S}, \mathcal{I})$ stratified set inherits a natural structure of abstract stratified set obtained by intersecting the elements of $\mathcal{I}$ with U.

Proof. This follows from the immediate observation that an open embedding of smooth manifolds is a submersion, applied to the open embeddings $Y \cap U \subset Y$ for every stratum Y. □

Now we closely review the proof of [Mat73, Theorem 8.3], but adapted to the context of abstract stratified sets (whereas the original proof is given for Whitney stratified spaces). This review is also useful to fix some notations.

Lemma 2.3.3 (Thom's first isotopy lemma, [Mat70, Corollary 10.2])**.** Let $(V, \mathcal{S}, \mathcal{I})$ be an abstract stratified set, P be a manifold, and $f : V \to P$ be a proper, controlled submersion. Then f is a locally trivial fibration.

Theorem 2.3.4 ([Mat73, Theorem 8.3])**.** Let $(V, \mathcal{S}, \mathcal{I})$ be an abstract stratified set. Then M admits a covering by conical charts of the type $\mathbb{R}^i \times C(Z)$, where Z is a compact topological space endowed with a natural structure of abstract stratified set.

Proof. Let W be a stratum of V, and let x be a point of W. Denote by $\pi_W : T_W \to W$ the projection, $\rho_W : T_W \to \mathbb{R}_{\geq 0}$ the "distance" function. Let us choose a "closed tubular subneighboorhood" of T_W, i.e. a subset N of T_W which is of the form

$$\{x \in T_W \mid \rho_W(x) \leq \varepsilon(\pi_W(x))\}$$

for some smooth function $\varepsilon : W \to \mathbb{R}_{>0}$ such that for every $x \in W$ we have $\varepsilon(x) \in \rho_W(\pi_W^{-1}(x))$. Let also

$$A = \{x \in T_W \mid \rho_W(x) = \varepsilon(\pi_W(x))\}$$

and $f = \pi_W|_A : A \to W$. Note that f is a proper controlled submersion, since π_W is a proper controlled submersion and for any stratum S of M the differential of $\pi_W|_S$ vanishes on the normal to $A \cap S$. Hence by Lemma 2.3.1 the restriction of the stratification of M to any fiber of f has a natural structure of abstract stratified set. Consider the mapping

$$g : N \setminus W \to W \times (0, 1]$$

defined by

$$g(x) = \left(\pi_W(x), \frac{\rho_W(x)}{\varepsilon(\pi_W(x))}\right).$$

The space $N \setminus W$ inherits from M a structure of abstract stratified set (see Lemma 2.3.2) and, by [Mat70, Lemma 7.3 and above], the map g is a proper controlled submersion. Thus, since $A = g^{-1}(W \times \{1\})$, by Lemma 2.3.3 one gets a stratified homeomorphism h (with respect to the stratification induced on A, see Lemma 2.3.1) fitting in the commuting triangle

$$\begin{tikzcd} N \setminus W \arrow[rr, "h"] \arrow[dr, "g"'] & & A \times (0,1] \arrow[dl, "f \times \mathrm{id}"] \\ & W \times (0,1] & \end{tikzcd}.$$

Furthermore, since $W = \rho^{-1}(0) \subseteq N$, h extends to a homeomorphism of pairs

$$(N, W) \xrightarrow{(h,\mathrm{id})} (\bar{C}(f), W),$$

where $\bar{C}(f)$ is the closed mapping cone of f (we recall that $f : A \to W$ is the projection $(\pi_W|_A)$).

Now for any euclidean chart $j : \mathbb{R}^i \hookrightarrow W$ around x, the pullback of f along j becomes a projection $\mathbb{R}^i \times Z \to \mathbb{R}^i$. Note that Z is compact by properness of f, and has an induced structure of abstract stratified set being a fiber of f, as we have noticed above. Finally,

$$\bar{C}(f) \simeq \bar{C}(\mathbb{R}^i \times Z \xrightarrow{\mathrm{pr}_1} \mathbb{R}^i) \simeq \bar{C}(Z) \times \mathbb{R}^i.$$

The open cone $C(f)$ is thus of the form $C(Z) \times \mathbb{R}^i$, and this provides the sought conical chart around x. $\square$

As a consequence, by [AFT17, Definition 2.1.15, Axiom (5)], every Whitney stratified space is C^0-stratified.

2.3.2 Whitney stratifications are conically smooth

Construction 2.3.5. Let $(M, \mathcal{S}, \mathcal{I})$ be an abstract stratified set. The procedure explained in the proof of Theorem 2.3.4 provides a family

$$\mathcal{T} = \{\phi : \mathbb{R}^i \times C(Z) \hookrightarrow M\}$$

indexed by the choice of

- $W \in \mathcal{S}$
- $\psi : \mathbb{R}^i \hookrightarrow W$ smooth chart in W
- $\varepsilon : W \to \mathbb{R}_{>0}$.

We will say that a chart is "centered at the stratum W" and "induced by the smooth chart ψ and the function ε". The image of such a chart is a tubular neighbourhood which we denote by $N(W, \psi, \varepsilon)$. The inverse of the chart is (up to composing with the map induced by ψ^{-1}) exactly the map h appearing in Theorem 2.3.4.

We already know that the family $\mathcal{T}$ is a covering of M, and it will be referred to as the family of **Thom-Mather charts** associated to the abstract stratified set $(V, \mathcal{S}, \mathcal{I})$.

Remark 2.3.6. If two charts are centered at the same stratum W, up to changing the ψ's and the ε's one can assume that their sources are the same basic $\mathbb{R}^n \times C(Z)$ (for the same n and Z).

In particular, if $(M, \mathcal{S})$ is a Whitney stratified space, by Proposition 2.2.9 we can endow it with the structure of an abstract stratified set, and thus obtain a family of Thom-Mather charts. The rest of this section will be devoted to prove that these charts form a conically smooth atlas in the sense of [AFT17, Definition 3.2.10] for M, as conjectured in [AFT17, Conjecture 1.5.3 (3)]. We will then prove (Remark 2.3.9) that equivalent structures of abstract stratified set on the same Whitney stratified space induce equivalent conically smooth atlases, again in the sense of [AFT17, Definition 3.2.10].

Theorem 2.3.7 (Main Theorem)**.** Let $(M, \mathcal{S}, \mathfrak{I})$ be an abstract stratified set, and assume that M has finite dimension. Then the Thom-Mather charts form a conically smooth atlas on $(M, \mathcal{S})$.

Proof. Since M has finite dimension, the depth of $(M, \mathcal{S})$ must be finite. The proof will therefore proceed by induction on the depth of $(M, \mathcal{S})$ (see Definition 2.2.17).

In the case of depth 0, by Remark 2.2.20 we know that M is just a disconnected union of strata which are smooth manifolds. Therefore, the claim follows easily, since the family of Thom-Mather charts reduces to a collection of smooth atlases for each stratum.

Thus, we may assume by induction that for any abstract stratified set $(M', \mathcal{S}', \mathcal{I}')$ with

$$\operatorname{depth}(M', \mathcal{S}') < \operatorname{depth}(M, \mathcal{S})$$

the Thom-Mather charts form a conically smooth atlas on $(M', \mathcal{S}')$.

Fix a Thom-Mather chart $\phi : \mathbb{R}^i \times C(Z) \hookrightarrow M$: then by the proof of Theorem 2.3.4 we know that $\operatorname{depth}(Z) < \operatorname{depth}(M, \mathcal{S})$ when considering Z with its induced structure of abstract stratified set; thus, by the inductive hypothesis, the Thom-Mather charts on Z form a conically smooth atlas, and this implies that the $\mathbb{R}^i \times C(Z)$ is a basic in the sense of [AFT17, Definition 3.2.4].

Hence it remains to prove that the "atlas" axiom is satisfied: that is, if $m \in M$ is a point, $u : \mathbb{R}^i \times C(Z) \to M$ and $v : \mathbb{R}^j \times C(W) \to M$ are Thom-Mather charts with images U and V, such that $m \in U \cap V$, then there exist a basic $\mathbb{R}^k \times C(T)$ and a commuting diagram

(2.3.8)
$$\begin{tikzcd}
\mathbb{R}^{n_3} \times C(Z_3) \arrow[r, "f_1"] \arrow[d, "f_2"] & \mathbb{R}^{n_1} \times C(Z_1) \arrow[d, "\phi_1"] \\
\mathbb{R}^{n_2} \times C(Z_2) \arrow[r, "\phi_2"] & M
\end{tikzcd}$$

such that $x \in \operatorname{Im}(\phi_1 f_1) = \operatorname{Im}(\phi_2 f_2)$ and that f_1 and f_2 are maps of basics in the sense of [AFT17, Definition 3.2.4].

Let $d = \operatorname{depth} M$, and let M_d be the union of all strata X of depth exactly d (i.e. such that $\sup_{x \in X} \operatorname{depth}_x M = d$). Since $M \setminus M_d$ has depth strictly less than d and it is open in M, by the inductive hypothesis we know that the induced structure of abstract stratified set on $M \setminus M_d$ satisfies the claim (i.e. the Thom-Mather charts form a conically smooth atlas). Therefore, it will suffice to examine the following two cases:

(1) the two charts are centered at the same stratum W of depth d (and m may or may not be contained in W or belong to M_d)

(2) one chart is centered at a stratum Y of depth $< d$ and the other is centered at a stratum X of depth d and lying in the closure of Y. (One may use Example 2.2.7 as a guiding example, with m a point on $\{x = 0\} \setminus \{(0, 0)\}$.)

Indeed, in the remaining case when the two charts are both centered at strata of depth less than d, then m must lie in $M \setminus M_d$, and therefore, up to choosing smaller ε's, we can assume that both charts lie in $M \setminus M_d$ and apply the inductive hypothesis.

Case (1). We define the new chart as follows. Suppose that the chart $\phi_1 : \mathbb{R}^n \times C(Z) \to M$ is centered at W and induced by the smooth chart $\psi_1 : \mathbb{R}^n \to W$ and the function $\varepsilon_1 : W \to \mathbb{R}_{>0}$. Analogously, suppose that $\phi_2 : \mathbb{R}^n \times C(Z) \to M$ is centered at W and induced by the smooth chart $\psi_2 : \mathbb{R}^n \to W$ and the function $\varepsilon_1 : W \to \mathbb{R}_{>0}$. Note that we can suppose that the basic has the same form in both cases by Remark 2.3.6. Choose a smooth chart ψ_3 for X and a smooth function $\varepsilon_3 : W \to \mathbb{R}_{>0}$ such that

- $\mathrm{Im}(\psi_3) \subset \mathrm{Im}(\psi_1) \cap \mathrm{Im}(\psi_2) \subset W$
- $\varepsilon_3(w) \leq \min(\varepsilon_1(w), \varepsilon_2(w))$ for any $w \in W$
- the image of the Thom-Mather chart ϕ_3 associated to $(W, \psi_3, \varepsilon_3)$ contains m.

We call i_1, i_2 the transition maps (coming from the smooth structure of X) fitting into the diagram

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, hook, "i_1"] \arrow[d, hook, "i_2"'] \arrow[dr, "\psi_3"] & \mathbb{R}^n \arrow[d, hook, "\psi_1"] \\
\mathbb{R}^n \arrow[r, hook, "\psi_2"'] & W.
\end{tikzcd}$$

Let us define the following two maps:

$$f_1 : \mathbb{R}^n \times C(Z) \to \mathbb{R}^n \times C(Z)$$

$$(v, t, z) \mapsto (i_1(v), \frac{\varepsilon_3(\psi_1 i_1(v))}{\varepsilon_1(\psi_1 i_1(v))} t, z)$$

$$f_2 : \mathbb{R}^n \times C(Z) \to \mathbb{R}^n \times C(Z)$$

$$(v, t, z) \mapsto (i_2(v), \frac{\varepsilon_3(\psi_2 i_2(v))}{\varepsilon_2(\psi_2 i_2(v))} t, z).$$

These are maps of basics. Indeed, let us call, for $j = 1, 2$, $\eta_j = \frac{\varepsilon_3 \psi_j i_j}{\varepsilon_j \psi_j i_j} : \mathbb{R}^n \to \mathbb{R}_{>0}$. This is a smooth function. For $j = 1, 2$, the verification of the condition that f_j is a map of basics amounts to check that:

- f_j is conically smooth along $\mathbb{R}^n$. That is, that the map
$$\mathbb{R}_{>0} \times T\mathbb{R}^n \times C(Z) \to \mathbb{R}_{>0} \times T\mathbb{R}^n \times C(Z)$$
$$(t, v, x, [s, z]) \mapsto (t, \frac{i_j(tv + x) - i_j(x)}{t}, i_j(x), [\eta_j(x)s, z])$$
extends to $t = 0$. This follows from smoothness of i_j.
- The differential
$$T_x\mathbb{R}^n \times C(Z) \to T_{i_j(x)}\mathbb{R}^n \times C(Z)$$
$$(v, [s, z]) \mapsto (D_v i_j(x), [\eta_j(x)s, z])$$
is injective. This follows from the fact that i_j is a smooth open embedding.

- the pullback of the atlas $\mathcal{A}$ that we are considering on $C_1 = \mathbb{R}^n \times \mathbb{R}_{>0} \times Z$ to $C_2 = \mathbb{R}^n \times \mathbb{R}_{>0} \times Z$ along $f_j|_{\mathbb{R}^n \times \mathbb{R}_{>0} \times Z} = i_j \times \eta \cdot \mathrm{id} \times \mathrm{id}$ coincides with $\mathcal{A}$. Indeed, for any chart $c_1 = j_1 \times l_1 \times w_1 : \mathbb{R}^n \times \mathbb{R}_{>0} \times W_1 \to C_1$ there exists a chart $c_2 = i_j \times \eta \cdot \mathrm{id} \times w : \mathbb{R}^n \times \mathbb{R}_{>0} \times W_2 \to C_2$ and a commutative diagram

$$\begin{tikzcd}
\mathbb{R}^n \times \mathbb{R}_{>0} \times W_1 \arrow[r, dotted, "j_1 \times l_1 \times \mathrm{id}"] \arrow[d, "c_1"'] & \mathbb{R}^n \times \mathbb{R}_{>0} \times W_2 \arrow[d, dotted, "c_2"] \\
\mathbb{R}^n \times \mathbb{R}_{>0} \times Z \arrow[r, "f|_{\mathbb{R}^n \times \mathbb{R}_{>0} \times Z}"'] & \mathbb{R}^n \times \mathbb{R}_{>0} \times Z.
\end{tikzcd}$$

The last thing to do is to check that, with the notations of the proof of Theorem 2.3.4 and of Construction 2.3.5, the front square in the diagram

$$\begin{tikzcd}
N(W, \psi_3, \varepsilon_3) \arrow[r, hook] \arrow[d, hook] \arrow[dr, "h_3", "\sim"'] \arrow[drr, dashed, hook, bend right, "\sim"'] & N(W, \psi_1, \varepsilon_1) \arrow[dr, "h_1", "\sim"'] & \\
N(W, \psi_2, \varepsilon_2) \arrow[dr, "h_2", "\sim"'] & \mathbb{R}^n \times C(Z) \arrow[r, hook, "f_1"] \arrow[d, hook, "f_2"'] & \mathbb{R}^n \times C(Z) \arrow[d, hook, "\phi_1"] \\
& \mathbb{R}^n \times C(Z) \arrow[r, hook, "\phi_2"'] & M.
\end{tikzcd}$$

commutes. This follows from the fact that, by an easy computations, for $j = 1, 2$ the diagram

$$\begin{tikzcd}
N(W, \psi_3, \varepsilon_3) \arrow[r, "h_3", "\sim"'] \arrow[d, hook] & \mathbb{R}^n \times C(Z) \arrow[d, hook, "f_j"'] \\
N(W, \psi_j, \varepsilon_j) \arrow[r, "h_j", "\sim"'] & \mathbb{R}^n \times C(Z)
\end{tikzcd}$$

commutes.

Case (2). We can assume that the point m lies outside M_d, and that the image of ϕ_2 is contained in $M \setminus M_d$. Therefore we have the following diagram:

$$\begin{tikzcd}
& \mathbb{R}^n \times \mathbb{R}_{>0} \times Z_1 \arrow[r, hook] \arrow[d, hook] & \mathbb{R}^n \times C(Z_1) \arrow[d, hook] \\
\mathbb{R}^m \times C(Z_2) \arrow[r, hook] & M \setminus M_d \arrow[r, hook] & M
\end{tikzcd}$$

By [AFT17, Lemma 3.2.9] ("basics form a basis for basics") there exists a map of basiscs

$$i : \mathbb{R}^{n'} \times C(Z_1') \hookrightarrow \mathbb{R}^n \times C(Z_1)$$

whose image is contained in $\mathbb{R}^n \times \mathbb{R}_{>0} \times Z_1$ and contains m. This induces a commutative diagram

$$\begin{tikzcd}
\mathbb{R}^{n'} \times C(Z_1') \arrow[d, hook] \arrow[dr, hook, "i"] & \\
\mathbb{R}^n \times \mathbb{R}_{>0} \times Z \arrow[r, hook] \arrow[d, hook] & \mathbb{R}^n \times C(Z) \arrow[d, hook, "\phi_1"] \\
M \setminus M_d \arrow[r, hook] & M.
\end{tikzcd}$$

and therefore a span

$$\begin{tikzcd} & \mathbb{R}^{n_1'} \times C(Z_1') \arrow[d] \\ \mathbb{R}^{n_2} \times C(Z_2) \arrow[r] & M \setminus M_d. \end{tikzcd}$$

To this span we can apply the inductive hypothesis, since depth $M \setminus M_d < d$. This yields maps of basics $f_1' : \mathbb{R}^{n_3} \times C(Z_3) \to \mathbb{R}^{n_1'} \times C(Z_1')$ and $f_2 : \mathbb{R}^{n_3} \times C(Z_3) \to \mathbb{R}^{n_2} \times C(Z_2)$ fitting into the diagram

$$\begin{tikzcd} \mathbb{R}^{n_3} \times C(Z_3) \arrow[r, dotted, "f_1'"] \arrow[d, dotted, "f_2"] & \mathbb{R}^{n_1'} \times C(Z_1') \arrow[r, "i"] \arrow[d] & \mathbb{R}^{n_1} \times C(Z_1) \arrow[d] \\ \mathbb{R}^{n_2} \times C(Z_2) \arrow[r] & M \setminus M_d \arrow[r] & M \end{tikzcd}$$

Finally, we define $f_1 = i \circ f_1'$, which is a map of basics because both i and f_1' are. □

Remark 2.3.9. Let $(M, \mathcal{S})$ be a Whitney stratified space. By [Mat70, Proposition 6.1], different choices of structures of abstract stratified sets over $(M, \mathcal{S})$ (and hence different families of Thom-Mather charts) induce equivalent conically smooth atlases in the sense of [AFT17, Definition 3.2.10]. Indeed, the construction of a Thom-Mather atlas $\mathcal{A}$ depends on the choice of a tubular neighbourhood for each stratum X, along with its distance and projection functions ρ_X, π_X. Thus, let $\mathcal{A}, \mathcal{A}'$ be two conically smooth atlases induced by different choices of a system of tubular neighbourhoods as above. We want to prove that $\mathcal{A} \cup \mathcal{A}'$ is again an atlas. The nontrivial part of the verification is the following. Let us fix two strata X, Y, and a point $y \in Y$; take ϕ_X a Thom-Mather chart associated to the $\mathcal{A}$-tubular neighbourhood T_X of X, and ψ_Y' a Thom-Mather associated to the $\mathcal{A}'$-tubular neighbourhood T_Y' of Y. We want to verify the "atlas condition" (2.3.8); let T_Y be the $\mathcal{A}$-tubular neighbourhood of Y. Now by [Mat70, Proposition 6.1] there is an isotopy between T_Y' and T_Y fixing Y. By pulling back ψ_Y' to T_Y along this isotopy, we obtain an $\mathcal{A}$-Thom Mather chart ψ_Y around y; we are now left with two $\mathcal{A}$-charts ϕ_X and ψ_Y and we finally can apply the fact that $\mathcal{A}$ is an atlas.

2.4 Finite exit paths

This first section contains the main geometric input needed to achieve our goal. Namely, we show the exit-paths ∞-category of a compact conically smooth stratified space is finite (Proposition 2.4.15).

2.4.1 Finite ∞-categories

This short section is devoted to recalling the definition of a finite ∞-category. Before going into that, we say a few words about what an ∞-category is for us. We work in the model of quasicategories. Following [Cis19, Example 7.10.14], we define $\mathcal{C}\mathrm{at}_\infty$ as the localization of the 1-category of simplicial sets at the class of Joyal equivalences. The class of Joyal equivalences and fibrations equip sSet with the structure of a category with weak equivalences and fibrations in the sense of [Cis19, Definition 7.4.12], and so by [Cis19, Theorem 7.5.18] it follows that any object in $\mathcal{C}\mathrm{at}_\infty$ is equivalent to the image through the localization functor $\mathrm{sSet} \to \mathcal{C}\mathrm{at}_\infty$ of a fibrant object in the Joyal model structure. For this reason, objects of $\mathcal{C}\mathrm{at}_\infty$ will be called ∞-categories.

The first definition we propose is expressed internally to the ∞-category $\mathcal{C}\mathrm{at}_\infty$ in terms of pushouts, and so in a kind of model-independent fashion. Later we prove that this is actually equivalent to a notion of finiteness that one might expect in the simplicial model. All the results appearing here are not at all original, but we still felt the need to write this section as, in the process of completing this chapter, we could not locate a reference dealing with the subject.

Definition 2.4.1. An ∞-category is said to be *finite* if it belongs to the smallest full subcategory of $\mathcal{C}\mathrm{at}_\infty$ which contains $\emptyset$, Δ^0, and Δ^1 and is closed under pushouts. An ∞-groupoid is said to be finite if it is so as an ∞-category. We will denote by $\mathcal{C}\mathrm{at}_\infty^f$ and $\mathcal{S}^f$ respectively the full subcategories of $\mathcal{C}\mathrm{at}_\infty$ and $\mathcal{S}$ spanned by finite objects.

Remark 2.4.2. Recall that the inculsion $\mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ admits both a left and a right adjoint, and one may describe the left adjoint on objects by sending and ∞-category $\mathcal{C}$ to the localization $\mathcal{C}[\mathcal{C}^{-1}]$. Thus, since $\mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ preserves colimits and $\Delta^1[(\Delta^1)^{-1}] \simeq \Delta^0$, one may identify the class of finite ∞-groupoids with the objects of the smallest full subcategory of $\mathcal{S}$ which contains $\emptyset$ and Δ^0 and it's closed under pushouts. This implies in particular that, for any finite ∞-category $\mathcal{C}$, the localization $\mathcal{C}[\mathcal{C}^{-1}]$ is again finite.

Lemma 2.4.3. Let $\mathcal{C}$ be a finite ∞-category, and let W be a finite subcategory of $\mathcal{C}$. Then the localization $\mathcal{C}[W^{-1}]$ is again finite.

Proof. We have a pushout square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ W[W^{-1}] \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \mathcal{C}[W^{-1}], \end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$, thus it suffices to show that $W[W^{-1}]$ is finite. This follows immediately by Remark 2.4.2. ☐

Recall that a simplicial set is said to be *finite* if it has a finite number of non-degenerate simplices. In the next proposition we reconcile this notion of finiteness with the one in Definition 2.4.1. We will need the following lemma, whose proof was explained to us by Sebastian Wolf.

Lemma 2.4.4. Let $\mathcal{C}$ be an ∞-category, and let $f : K \to \mathcal{C}$ be a map of simplicial set, where K is finite. Moreover, suppose that there exists a finite simplicial set K' and a Joyal equivalence $g : K' \to \mathcal{C}$. Then there exists a finite simplicial set L and a commutative diagram of simplicial sets

$$\begin{tikzcd} & \mathcal{C} \\ K \arrow[ur, "f"] \arrow[r, hook, "k"'] & L \arrow[u, "j"'] \end{tikzcd}$$

where k is a monomorphism.

Proof. We define inductively a sequence of finite simplicial sets $\{K'_n\}_{n\in\mathbb{N}}$. We set $K'_0 = K'$, and we define K'_n via the pushout

$$\begin{tikzcd} \coprod\limits_{\Lambda^n_j \to K'_{n-1}} \Lambda^n_j \arrow[r] \arrow[d, hook] & K'_{n-1} \arrow[d, hook] \\ \coprod\limits_{\Delta^n \to K'_{n-1}} \Delta^n \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & K'_n. \end{tikzcd}$$

Now set

$$K'_\infty \coloneqq \mathrm{colim}(K'_0 \hookrightarrow K'_1 \hookrightarrow \cdots \hookrightarrow K'_n \hookrightarrow \dots).$$

Furthermore, since all horns are finite simplicial sets, any map $\Lambda^n_j \to K'_\infty$ factors through some K'_m, and by construction of the sequence we get a commutative diagram

$$\begin{tikzcd}
\Lambda^n_j \arrow[r] \arrow[d] & K'_m \arrow[r, hook] \arrow[d, hook] & K'_\infty \\
\Delta^n \arrow[r, dotted] & K'_{m+1} \arrow[ur, hook] &
\end{tikzcd}$$

which implies that K'_∞ is an ∞-category. Since the class of categorical anodyne extensions is saturated (see [Cis19, Definition 3.3.3]), we see that $K' \hookrightarrow K'_\infty$ is a categorical anodyne extension, and in particular a Joyal equivalence. Hence, by the assumption that $\mathcal{C}$ is an ∞-category, we get a commutative triangle

$$\begin{tikzcd}
K' \arrow[r, "g"] \arrow[d, hook] & \mathcal{C} \\
K'_\infty \arrow[ur, dotted, "\phi"'] &
\end{tikzcd}$$

where ϕ is a Joyal equivalence by the 2-out-of-3. Since K'_∞ is also an ∞-category, ϕ admits a quasi-inverse

$$\psi : \mathcal{C} \to K'_\infty.$$

By the finiteness of K, we get that the composition

$$\psi f : K \to \mathcal{C} \to K'_\infty$$

factors through some $\delta : K \to K'_n$. Thus, we get a triangle

$$\begin{tikzcd}
& \mathcal{C} & \\
K \arrow[ur, "f"] \arrow[r, "\delta"] & K'_n \arrow[r, hook] & K'_\infty \arrow[ul, "\phi"']
\end{tikzcd}$$

which commutes up to J-homotopy, where J is the interval object for the Joyal model structure as defined in [Cis19, Definition 3.3.3].

Let now L be the mapping cylinder of δ. Since J is a finite simplicial set, L must be finite as well. By the usual factorizations obtained via mapping cylinders, we get a triangle

$$\begin{tikzcd}
& \mathcal{C} \\
K \arrow[ur, "f"] \arrow[r, hook, "i"] & L \arrow[u, "p"']
\end{tikzcd}$$

commuting up to J-homotopy, where i is a monomorphism and p is a Joyal equivalence. If H is a J-homotopy between f and pi, we may find a map $\widetilde{H}$ fitting in the diagram

$$\begin{tikzcd}
K \times J \cup L \times \{1\} \arrow[r, "H \cup p"] \arrow[d, hook] & \mathcal{C} \\
L \times J \arrow[ur, dotted, "\widetilde{H}"'] &
\end{tikzcd}$$

since $K \times J \cup L \times \{1\} \to L \times J$ is categorical anodyne extension. Hence, by restricting $\widetilde{H}$ to $L \times \{0\}$, we get the desired commutative triangle, and the proof is then concluded. $\square$

Proposition 2.4.5. Let $\gamma : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{C}\mathrm{at}_\infty$ be the localization functor. Then an ∞-category $\mathcal{C}$ is finite if and only if there exists a finite simplicial set K and an equivalence $\mathcal{C} \simeq \gamma(K)$.

Proof. Let $\mathrm{s}\mathcal{S}\mathrm{et}^f$ be the full subcategory of $\mathrm{s}\mathcal{S}\mathrm{et}$ spanned by the finite simplicial sets, and denote by $\mathcal{F}$ the essential image of the restriction of γ to $\mathrm{s}\mathcal{S}\mathrm{et}^f$. We need to show that $\mathcal{C}\mathrm{at}_\infty^f$ coincides with $\mathcal{F}$.

Let K be any finite simplicial set, so that in particular there exists some finite n such that $K = \mathrm{sk}_n(K)$. By induction on n and using the cellular decomposition

$$\begin{tikzcd}
\coprod\limits_{\partial\Delta^n \to K} \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathrm{sk}_{n-1}(K) \arrow[d] \\
\coprod\limits_{\Delta^n \to K} \Delta^n \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{sk}_n(K)
\end{tikzcd}$$

we see that to prove that $L(K)$ belongs to $\mathcal{C}\mathrm{at}_\infty^f$, it suffices to show that each Δ^n does. But this is clear because n-simplex is Joyal equivalent to the n-spine. Thus we have $\mathcal{F} \subseteq \mathcal{C}\mathrm{at}_\infty^f$.

Since $\mathcal{F}$ contains $\emptyset$, Δ^0, and Δ^1, we are now only left to show that $\mathcal{F}$ is closed under pushouts. Let

$$\mathcal{D} \leftarrow \mathcal{C} \to \mathcal{E}$$

be any cospan of ∞-categories in $\mathcal{F}$, and let $K \to \mathcal{C}$ be any Joyal equivalence, where K is a finite simplicial set. Thus, by applying Lemma 2.4.4 twice, we get a map of cospans

$$\begin{tikzcd}
L \arrow[d] & K \arrow[l, hook'] \arrow[r, hook] \arrow[d] & M \arrow[d] \\
\mathcal{D} & \mathcal{C} \arrow[l] \arrow[r] & \mathcal{E}
\end{tikzcd}$$

where the vertical arrows are Joyal equivalences and the upper horizontal arrows are monomorphisms. We then get a Joyal equivalence between the respective homotopy pushouts, and thus the desired conclusion. $\square$

2.4.2 Finiteness properties of compact conically smooth spaces

The main goal of this section is to show that the exit path ∞-category of a compact conically smooth stratified space is finite (Proposition 2.4.15). For this purpose, we make use of Lurie's model of the exit path ∞-category, of which we now recall the definition for the reader's convenience. By a slight abuse of notation, for a poset P we still denote by P the topological space obtained by equipping the poset with the Alexandroff topology. If $X \to P$ is a stratified topological space, then we define $\mathrm{Exit}_P(X)$ by forming the pullback

$$\begin{tikzcd}
\mathrm{Exit}_P(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sing}(X) \arrow[d] \\
N(P) \arrow[r] & \mathrm{Sing}(P)
\end{tikzcd}$$

in the category of simplicial sets. Lurie showed that if the stratification $X \to P$ is conical, then $\mathrm{Exit}_P(X)$ is an ∞-category ([Lur17, Theorem A.6.4]). The reason why we prefer to focus on this perspective is that, as opposed to the model given in [AFT17], it has an evident much richer functoriality: by the functoriality of Sing, Exit is functorial with respect to general stratified maps. On the other hand the one in [AFT17] is only functorial with respect to conically smooth open embeddings. We also see immediately that, if we stratify P over itself through the identity, then $\mathrm{Exit}_P(P) = N(P)$. We will also need the following easy lemma.

Definition 2.4.6. Let $f : (X \to P) \to (Y \to Q)$ be a map of stratified spaces. We say that f is a *full inclusion of strata* if the underlying map of posets $P \to Q$ is injective and full, and the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ P \arrow[r] & Q \end{tikzcd}$$

is a pullback of topological spaces.

Lemma 2.4.7. Let $X \to P$ and $Y \to Q$ be stratified spaces, and assume that the stratification on X is conical. Assume that we have a stratified embedding $Y \hookrightarrow X$ which is a full inclusion of strata. Then $\mathrm{Exit}_Q(Y)$ is an ∞-category and the induced functor $\mathrm{Exit}_Q(Y) \to \mathrm{Exit}_P(X)$ is fully faithful.

Proof. Since the functor Sing from topological spaces to simplicial sets preserves limits and since $Y \hookrightarrow X$ is inclusion of strata, we get a pullback square

$$\begin{tikzcd} \mathrm{Exit}_Q(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Exit}_P(X) \arrow[d] \\ N(Q) \arrow[r] & N(P) \end{tikzcd} \tag{2.4.8}$$

of simplicial sets. By [Lur17, Theorem A.6.4, (1)] the functor $\mathrm{Exit}_P(X) \to N(P)$ is an inner fibration, and so by the pullback square (2.4.8) also $\mathrm{Exit}_Q(Y) \to N(Q)$ is an inner fibration, which implies that $\mathrm{Exit}_Y(Y))$ is an ∞-category. Moreover, since the inclusion is full, we know that the functor $N(Q) \to N(P)$ is fully faithful, and thus we may conclude again by (2.4.8). □

Recall that for a proper conically smooth fiber bundle $\pi : L \to X$, we define the *fiberwise cone* of π as the pushout

$$\begin{tikzcd} L \arrow[r] \arrow[d] & L \times \mathbb{R}_{\geq 0} \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C(\pi) \end{tikzcd} \tag{2.4.9}$$

taken in the category of conically smooth stratified spaces. By definition, we get a new fiber bundle $C(\pi) \to X$ whose fibers are isomorphic to basics. We now show how to compute the exit paths of $C(\pi)$ in terms of L and X.

Lemma 2.4.10. Let $\pi : L \to X$ be a proper conically smooth fiber bundle. Then the commutative square

$$\begin{tikzcd} \mathrm{Exit}(L) \arrow[r] \arrow[d] & \mathrm{Exit}(L \times \mathbb{R}_{\geq 0}) \arrow[d] \\ \mathrm{Exit}(X) \arrow[r] & \mathrm{Exit}(C(\pi)) \end{tikzcd}$$

in $\mathcal{C}\mathrm{at}_\infty$ induced by (2.4.9) is a pushout.

Proof. By the Van Kampen theorem for exit paths [Lur17, Theorem A.7.1], we may assume that X is a basic. Thus, by [AFT17, Corollary 7.1.4], we may also assume that π is a trivial bundle. Since Exit commutes with finite products, we may assume that $X = *$, and hence we are only left to prove that for any compact conically smooth space L the square

$$\begin{tikzcd} \mathrm{Exit}(L) \arrow[r] \arrow[d] & \mathrm{Exit}(L \times \mathbb{R}_{\geq 0}) \arrow[d] \\ \Delta^0 \arrow[r] & \mathrm{Exit}(C(L)) \end{tikzcd}$$

is a pushout in $\mathcal{C}\mathrm{at}_\infty$. This follows, for example, by [AFT17, Lemma 6.1.4], or can be easily checked by hand even without any conically smooth assumption on L. $\square$

We will also need to use the *unzip* and *link* construction, as defined in [AFT17, Definition 7.3.11]. For any proper constructible embedding $X \hookrightarrow Y$ we have a pullback square

$$\begin{tikzcd} \mathrm{Link}_X(Y) \arrow[r, hook] \arrow[d, "\pi_X"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Unzip}_X(Y) \arrow[d] \\ X \arrow[r, hook] & Y. \end{tikzcd} \tag{2.4.11}$$

Here $\mathrm{Unzip}_X(Y)$ is a conically smooth manifold with corners whose interior is identified with $Y \setminus X$, and $\mathrm{Unzip}_X(Y) \to Y$ and $\pi_X : \mathrm{Link}_X(Y) \to X$ are proper constructible bundles. To get a feeling of how $\mathrm{Unzip}_X(Y)$ works, one may think of it as a generalization of the spherical blow-up (see [AK10]): more precisely, when Y is a smooth manifold stratified with a closed submanifold X and its open complement, then the unzip of $X \hookrightarrow Y$ coincides with the spherical blow-up of X in Y, and the link is diffeomorphic to the boundary of any normalized tubular neighbourhood of X in Y.

The link of a proper constructible embedding is used to provide tubular neighbourhoods in the stratified setting. In [AFT17, Proposition 8.2.5], the authors show that there is a conically smooth embedding

$$C(\pi_X) \hookrightarrow Y \tag{2.4.12}$$

under X which is a refinement onto its image and whose image is open in Y (here we are using the same notations are in (2.4.11)). Hence, if we denote by $\widetilde{\mathrm{Link}_X(Y)} \times \mathbb{R}_{>0}$ and $\widetilde{C(\pi_X)}$ the respective refinements of $\mathrm{Link}_X(Y) \times \mathbb{R}_{>0}$ and $C(\pi_X)$ through the embedding (2.4.12), a straightforward application of Van Kampen theorem gives the following.

Corollary 2.4.13. Let $X \hookrightarrow Y$ be a proper conically smooth constructible embedding. Then the square

$$\begin{tikzcd} \mathrm{Exit}(\widetilde{\mathrm{Link}_X(Y)} \times \mathbb{R}_{>0}) \arrow[r] \arrow[d] & \mathrm{Exit}(Y \setminus X) \arrow[d] \\ \mathrm{Exit}(\widetilde{C(\pi_X)}) \arrow[r] & \mathrm{Exit}(Y) \end{tikzcd}$$

is a pushout in $\mathcal{C}\mathrm{at}_\infty$.

Remark 2.4.14. Notice that, for the existence of tubular neighbourhoods, one may relax the assumption of properness for a constructible embedding $i : X \hookrightarrow Y$ to just requiring that there is a factorization

$$\begin{tikzcd} & Y' \arrow[dr, hook, "j"] & \\ X \arrow[ur, hook, "i'"] \arrow[rr, "i"] & & Y \end{tikzcd}$$

where i' is a proper constructible embedding and j is a conically smooth open embedding. For example, if P is the stratifying poset of Y and $X = Y_\alpha$ for some $\alpha \in P$, one may pick $Y' = Y_{\geq\alpha}$ and thus get a tubular neighbourhood of Y_p.

We are now ready to prove the main result of the section.

Proposition 2.4.15. Let X be any compact conically smooth stratified space. Then $\mathrm{Exit}(X)$ is a finite ∞-category.

Proof. Since X is compact, X is finite dimensional, and hence also has finite depth. We then argue by induction on $\mathrm{depth}(X) = k$.

If $k = 0$, it is well known that that $\mathrm{Exit}(X) = \mathrm{Sing}(X)$ is a finite ∞-groupoid. For example, this follows by Van Kampen theorem [Lur17, Theorem A.3.1] and the existence of finite good covers for X.

Assume now that k is positive. Denote by X_0 the union of strata of minimal depth, and by $X_{>0}$ its complement in X. Clearly $X_{>0} \hookrightarrow X$ is a proper constructible embedding, and hence by Corollary 2.4.13 we get a pushout

$$\begin{tikzcd}
\mathrm{Exit}(\widetilde{\mathrm{Link}_{>0}(X)} \times \mathbb{R}_{>0}) \arrow[r] \arrow[d] & \mathrm{Exit}(X_0) \arrow[d] \\
\mathrm{Exit}(\widetilde{C(\pi_{>0})}) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Exit}(X).
\end{tikzcd}$$

By Lemma 2.4.10, to conclude the proof it suffices to show that $\mathrm{Exit}(\widetilde{\mathrm{Link}_{>0}(X)})$, $\mathrm{Exit}(\widetilde{C(\pi_{>0})})$, and $\mathrm{Exit}(X_0)$ are finite.

Being a closed subset of X, the space $X_{>0}$ is compact. By the pullback square (2.4.11) $\mathrm{Link}_{>0}(X)$ is compact too. Since the depths of both are stricly less than $\mathrm{depth}(X)$, by the inductive hypothesis we get that $X_{>0}$ and $\mathrm{Link}_{>0}(X)$ both have finite exit path ∞-categories. Notice that we have a stratified embedding

$$\widetilde{\mathrm{Link}_{>0}(X)} \times \mathbb{R}_{>0} \hookrightarrow X_0$$

and X_0 is a smooth manifold. In particular, we see that the stratification on $\widetilde{\mathrm{Link}_{>0}(X)}$ is trivial. Thus $\mathrm{Exit}(\widetilde{\mathrm{Link}_{>0}(X)})$ is a localization of $\mathrm{Exit}(\mathrm{Link}_{>0}(X))$ at all maps, and by Lemma 2.4.3 we see that $\mathrm{Exit}(\widetilde{\mathrm{Link}_{>0}(X)})$ is finite.

By Lemma 2.4.10 and the inductive hypothesis, we also know that $\mathrm{Exit}(C(\pi_{>0}))$ is finite. By [AFT17, Proposition 1.2.13] we know that the canonical functor

$$\phi : \mathrm{Exit}(C(\pi_{>0})) \to \mathrm{Exit}(\widetilde{C(\pi_{>0})})$$

is a localization at the class of exit paths that are inverted by ϕ. Since the inclusion $C(\pi_{>0}) \hookrightarrow X$ lies under $X_{>0}$, the same argument as above shows that an non-invertible exit path is inverted ϕ is and only if it lies inside $\mathrm{Link}_{>0}(X) \times \mathbb{R}_{>0} \hookrightarrow C(\pi_{>0})$. Furthermore, since $\mathrm{Link}_{>0}(X) \times \mathbb{R}_{>0} \hookrightarrow C(\pi_{>0})$ is a full inclusion of strata, by Lemma 2.4.7 the induced functor on exit paths is the inclusion of a full subcategory. This implies that one can identify $\mathrm{Exit}(\widetilde{C(\pi_{>0})})$ with the localization of $\mathrm{Exit}(C(\pi_{>0}))$ at $\mathrm{Exit}(\mathrm{Link}_{>0}(X) \times \mathbb{R}_{>0})$. By Lemma 2.4.3, $\mathrm{Exit}(\widetilde{C(\pi_{>0})})$ is finite.

We know that X_0 is the interior of the compact manifold with corners $\mathrm{Unzip}_{>0}(X)$. By an argument completely analogous to [AMGR19, Lemma 2.1.3], one can show that the existence of collaring for corners ([AFT17, Lemma 8.2.1]) implies that the inclusion $X_0 \hookrightarrow$

$\mathrm{Unzip}_{>0}(X)$ is an homotopy equivalence. Hence to conclude the proof it suffices to show that $\mathrm{Sing}(\mathrm{Unzip}_{>0}(X))$ is finite. This follows by the existence of good covers for manifolds with corners. □

Corollary 2.4.16. Let X be a finitary conically smooth stratified space (see [AFT17, Definition 8.3.6]). Then $\mathrm{Exit}(X)$ is a finite ∞-category. In particular, if X is the interior of a compact conically smooth manifold with corners, then $\mathrm{Exit}(X)$ is a finite ∞-category.

Proof. By Proposition 2.4.15 and Lemma 2.4.10, the class of conically smooth spaces with finite exit path ∞-category contains all basics. Thus it suffices to show that it is closed under taking collar glueings. But this is clear, since for any collar glueing $f\colon Y \to [-1,1]$ we get an open covering of Y given by $f^{-1}([-1,1))$ $f^{-1}((-1,1])$ and $f^{-1}(0)$ are finitary.

The last part of the statement follows by [AFT17, Theorem 8.3.10 (1)]. □

2.5 Homotopy invariance and exodromy with general coefficients

In this section we explain how to use our development of six operations to extend *homotopy invariance* and the *exodromy equivalence* to (formally) contructible sheaves (see Definition 2.5.2) valued in stable and bicomplete ∞-categories (Theorem 2.5.16). With these at hand, we show that global sections of constructible sheaves on compact conically smooth stratified spaces are dualizable (Corollary 2.5.18).

2.5.1 Homotopy invariance of constructible sheaves

From now on, all ∞-categories appearing as coefficients for sheaves will be assumed to be stable and bicomplete, all topological spaces locally compact Hausdorff and all posets Noetherian.

Remark 2.5.1. Recall that, by [AFT17, Lemma 2.2.2], any C^0-stratified space admits a basis given by its open subsets homeomorphic to $\mathbb{R}^n \times C(Z)$, where Z is a compact C^0-stratified space. A simple inductive argument shows that the stratifying poset of Z, and hence of $\mathbb{R}^n \times C(Z)$, is finite. Therefore the stratifying poset of any C^0-stratified space is locally finite, and therefore Noetherian.

Definition 2.5.2. Let $X \to P$ be a stratified space. We say that a sheaf $F \in \mathcal{S}\mathrm{hv}(X;\mathcal{C})$ is *formally constructible* if for any $\alpha \in P$ the restriction of F to the stratum X_α is locally constant.

Assume now that $\mathcal{C}$ admits a closed symmetric monoidal structure, and denote by $\mathcal{C}^{\mathrm{dual}}$ the full subcategory of $\mathcal{C}$ spanned by dualizable objects. We say that F is *constructible* if F is formally constructible and each stalk of F belongs to $\mathcal{C}^{\mathrm{dual}}$.

We denote by $\mathcal{S}\mathrm{hv}^{fc}(X;C)$ and $\mathcal{S}\mathrm{hv}^{c}(X;\mathcal{C})$ the full subcategories of $\mathcal{S}\mathrm{hv}(X;\mathcal{C})$ spanned respectively by formally constructible and constructible sheaves. Dually, we define formally constructible and constructible cosheaves on X just as $\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}^{fc}(X;\mathcal{C}) := \mathcal{S}\mathrm{hv}^{fc}(X;\mathcal{C}^{op})^{op}$ and $\mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}^{c}(X;\mathcal{C}) := \mathcal{S}\mathrm{hv}^{c}(X;\mathcal{C}^{op})^{op}$.

In this chapter we only deal with constructible sheaves with respect to a specified stratification. Therefore, we will take the liberty of omitting the stratifying poset from our notation for constructible sheaves.

We first recall the proof of the homotopy invariance of constructible sheaves. Our argument follows precisely the one in [HPT20]. Nevertheless we will try to quickly outline the main steps just to convince the reader that all the results in [HPT20], after what we achieved in the first chapter, generalize to the setting of stable bicomplete coefficients, at least if we restrict ourselves to locally compact Hausdorff spaces.

Theorem 2.5.3 (Homotopy invariance)**.** Let $X \to P$ be a stratified space. Let $p : X \times [0,1] \to X$ be the canonical projection. Then $p^* : \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \to \mathcal{S}\mathrm{hv}(X \times [0,1];\mathcal{C})$ restricts to an equivalence

$$\mathcal{S}\mathrm{hv}^{fc}(X;\mathcal{C}) \simeq \mathcal{S}\mathrm{hv}^{fc}(X \times [0,1];\mathcal{C}).$$

As a consequence, if $Y \to P$ is another stratified space and $f : X \to Y$ is a stratified homotopy equivalence, then the functor

$$f^* : \mathcal{S}\mathrm{hv}^{fc}(Y;\mathcal{C}) \to \mathcal{S}\mathrm{hv}^{fc}(X;\mathcal{C}).$$

is an equivalence.

Proof. We first treat the case of locally constant sheaves, i.e. when $P = *$. By [Lur17, Lemma A.2.9] we know that p^* is fully faithful. We first show that f_* preserves constant sheaves: if $a : X \to *$ and $b : X \times [0,1] \to *$ are the unique maps then, for any object $M \in \mathcal{C}$, the fully faithfulness of f^* implies

$$\begin{aligned} p_* b^* M &\simeq p_* p^* a^* M \\ &\simeq a^* M. \end{aligned}$$

Moreover, since I is compact, Proposition 1.6.9 provides a base change formula for p_*. Using basechange and [HPT20, Lemma 4.9], we then see that p_* preserves locally constant sheaves. Hence to conclude we only need to show that, for any locally constant sheaf F on $X \times I$, the counit map $p^*p_*F \to F$ is an equivalence. Again by basechange and [HPT20, Lemma 4.9], we may reduce to $F \simeq b^*M$ constant. In this case we have a commutative diagram

$$\begin{tikzcd}
p^*p_*b^*M \arrow[r, "\mathrm{counit}_{b^*M}"] \arrow[d, "\simeq"] & b^*M \arrow[d, "\simeq"] \\
p^*p_*p^*a^*M \arrow[r, "\mathrm{counit}_{p^*a^*M}"] & p^*a^*M \\
p^*a^*M \arrow[u, "p^*(\mathrm{unit}_{a^*M})", "\simeq"'] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

which implies the desired result.

Now assume that P is any Noetherian poset. Using basechange, one sees that p^* preserves constructible sheaves, and thus we are left to show that for any F constructible $p^*p_*F \to F$ is an equivalence. By Corollary 1.4.7 any stable and bicomplete ∞-category respects glueing in the sense of [HPT20, Definition 5.17]. Hence [HPT20, Lemma 5.19] implies that the functor given by restricting to the strata of X are jointly conservative. By base change we may thus assume that F is locally constant, in which case the counit is known to be an equivalence by the previous step.

The last part of the statement then follows by a standard argument analogous to the proof of Theorem 2.5.3. $\square$

We now present a couple of useful corollaries of homotopy invariance.

Corollary 2.5.4. Let $f : X \to Y$ be a stratified homotopy equivalence, $F \in \mathcal{S}\mathrm{hv}^{fc}(Y;\mathcal{C})$. The the natural map

$$\Gamma(Y;F) \to \Gamma(X;f^*F)$$

is an equivalence.

Proof. The commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "a"'] & & Y \arrow[dl, "b"] \\ & * & \end{tikzcd}$$

induces an invertible natural tranformation $f^*b^* \simeq a^*$. Since both a^* and b^* factor through formally constructible sheaves, we get $b^* \simeq \eta a^*$, where η is any adjoint inverse of the restriction of f^* to $\mathcal{S}\mathrm{hv}^{fc}(Y;\mathcal{C})$. Thus, by passing to right adjoints, we get the desired equivalence. □

We will need the following easy lemma.

Lemma 2.5.5. Let Z be a compact topological space, and let $C(Z)$ be the cone of Z. Then the family of open subsets

$$\{C_\varepsilon(Z) \mid \varepsilon \in \mathbb{R}_{>0}\}$$

forms a basis at the cone point.

Proof. Let $\mathbb{R}_{\geq 0} \times Z \to C(Z)$ be the quotient map, and for each $\varepsilon > 0$ denote by $C_\varepsilon(Z)$ the image of the open subset $[0,\varepsilon) \times Z$. We will prove that for every open subset W of $\mathbb{R}_{\geq 0} \times Z$ containing $\{0\} \times Z$ there exists some $\varepsilon > 0$ such that $[0,\varepsilon) \times Z \subseteq W$. Since Z is compact, one can obtain a finite covering of $\{0\} \times Z$ with opens of the type $V_i \times [0,\varepsilon_i) \subseteq W$, and thus by taking ε to be the minimum of the ε_i we get the claim. □

Corollary 2.5.6. Let X be a C^0-stratified space and $F \in \mathcal{S}\mathrm{hv}^{fc}(X;\mathcal{C})$. For any point $x \in X$ and any conical chart $\mathbb{R}^n \times C(Z)$ centered in x, the natural map

$$\Gamma(\mathbb{R}^n \times C(Z); F) \to F_x$$

is an equivalence.

Proof. By Lemma 2.5.5 and a standard cofinality argument, we see that

$$\varinjlim_{\varepsilon>0} \Gamma(\mathbb{R}^n \times C_\varepsilon(Z); F) \simeq F_x$$

and by Corollary 2.5.4 we have

$$\Gamma(\mathbb{R}^n \times C(Z); F) \simeq \varinjlim_{\varepsilon>0} \Gamma(\mathbb{R}^n \times C_\varepsilon(Z); F)$$

which concludes our proof. □

Now ler $X \to P$ be a conically smooth stratified space, and let $\alpha \in P$. By Remark 2.4.14, we get commutative triangle

$$\begin{tikzcd} & X_\alpha \arrow[dl, hook, "i"'] \arrow[dr, hook, "i_\alpha"] & \\ C(\pi_\alpha) \arrow[rr, hook, "j"'] & & X \end{tikzcd}$$

where π_α is the fiber bundle $\mathrm{Link}_{X_\alpha}(X_{\geq\alpha}) \to X_\alpha$, i is the cone-point section of the fiber bundle $p : C(\pi_\alpha) \to X_\alpha$ and j is a conically smooth open immersion. For any $F \in \mathcal{S}\mathrm{hv}(X;\mathcal{C})$, the unit of the adjunction $i^* \dashv i_*$ gives a natural map

(2.5.7) $$p_*j^*F \to p_*i_*i^*j^*F \simeq i_\alpha^*F.$$

Furthermore the map (2.5.7) can be obtained by applying the sheafification functor to

$$p_* j^* F \to p_* i_* (i^*)^{pre} j^* F \simeq (i_\alpha^*)^{pre} F \tag{2.5.8}$$

where $(i^*)^{pre}$ and $(i_\alpha^*)^{pre}$ denote the corresponding presheaf pullback functors.

Corollary 2.5.9. Let $X_\alpha \hookrightarrow X$ be the inclusion of a stratum in a conically smooth stratified space, and let F be any formally constructible sheaf on X. Then the map (2.5.7) is an equivalence.

Proof. Since F is a sheaf, it suffices to show that (2.5.8) is an equivalence. As usual, it suffices to prove that it is an equivalence after taking sections on any euclidean chart U of X_α. For any such U, by [AFT17, Corollary 7.1.4] we have

$$\Gamma(U; p_* j^* F) = \Gamma(U \times C(Z); F)$$

for some compact conically smooth stratified space Z. Thus we are left to show that the natural map

$$\Gamma(U \times C(Z); F) \to \varinjlim_{U \subseteq V} \Gamma(V; F) \tag{2.5.10}$$

is an equivalence.

By a cofinality argument, the map (2.5.10) factors through an equivalence

$$\varinjlim_{\varepsilon > 0} \Gamma(C_\varepsilon(Z); F) \simeq \varinjlim_{U \subseteq V} \Gamma(V; F),$$

and thus we are left to show that

$$\Gamma(U \times C(Z); F) \to \varinjlim_{\varepsilon > 0} \Gamma(C_\varepsilon(Z); F)$$

is an equivalence. This last assertion then follows by Corollary 2.5.4. □

2.5.2 Exodromy

This subsection is devoted to extending the proof of the exodromy equivalence on conically smooth stratified spaces to constructible sheaves valued in stable and bicomplete ∞-categories. To do this we will use the model of the exit path ∞-category of a conically smooth stratified space given in [AFT17, Definition 1.1.3], so we briefly recall how this is defined.

Let Strat be the 1-category whose objects are conically smooth stratified spaces and morphisms are conically smooth maps, Snglr the subcategory of Strat with the same objects and with morphisms given by open immersions, and Bsc the full subcategory of Snglr spanned by *basic* conically smooth stratified spaces, i.e. those which are isomorphic in Strat to one of the type $\mathbb{R}^n \times C(Z)$, where Z is compact. Thus, one has functors

$$\mathrm{Bsc} \to \mathrm{Snglr} \to \mathrm{Strat}.$$

In [AFT17, Lemma 4.1.4], the authors show that all the categories appearing in (2.5.2) admit enrichments in Kan complexes. By passing to homotopy coherent nerves, one gets ∞-categories that we denote by

$$\mathcal{B}\mathrm{sc} \to \mathcal{S}\mathrm{nglr} \to \mathcal{S}\mathrm{trat}. \tag{2.5.11}$$

For any conically smooth stratified space, the authors of [AFT17] then define $\mathrm{Exit}(X)$ as the opposite of the slice $\mathcal{B}\mathrm{sc}_{/X}$. Their proof of the exodromy equivalence for constructible sheaves of spaces, combined with the one in [Lur17], implies that Lurie's exit paths ∞-category has to be equivalent to $\mathcal{B}\mathrm{sc}_{/X}$ (see [AFT17, Corollary 1.2.10]).

One has obvious functors

$$\begin{tikzcd} \mathrm{Bsc}_{/X} \arrow[r, "\gamma"] \arrow[d, "\mathrm{im}"] & \mathcal{B}\mathrm{sc}_{/X} \\ \mathcal{U}(X) & \end{tikzcd}$$

where im sends an open immersion to its image in X. In [AFT17, Lemma 4.5.1], the authors show that γ is a localization at the class W of open immersions of basics $U \hookrightarrow V$ such that U and V are abstractly isomorphic in Strat. That is, precomposing with γ gives an equivalence

$$\gamma^* : \mathrm{Fun}(\mathrm{Exit}(X), \mathcal{C}) \to \mathrm{Fun}_W(\mathrm{Bsc}^{op}_{/X}, \mathcal{C}) \tag{2.5.12}$$

where the right-hand side denotes the full subcategory of $\mathrm{Fun}(\mathrm{Bsc}^{op}_{/X}, \mathcal{C})$ spanned by functors which send all morphisms in W to equivalences. In the next proposition we show that W coincides with the class of open immersions which are stratified homotopy equivalences, and then characterize the property of being formally constructible through these maps.

Proposition 2.5.13. Let $X \to P$ be a conically smooth stratified space, and let $F \in \mathcal{S}\mathrm{hv}(X; \mathcal{C})$. Then the following assertions are equivalent:

(i) F is formally constructible;

(ii) for any inclusion $V \hookrightarrow U$ of basic open subsets of X which is a stratified homotopy equivalence, then
$$\Gamma(U; F) \to \Gamma(V; F)$$
is an equivalence;

(iii) for any inclusion $V \hookrightarrow U$ of basic open subsets of X which are abstractly isomorphic, then
$$\Gamma(U; F) \to \Gamma(V; F)$$
is an equivalence.

Proof. We first prove that (ii) is equivalent to (iii) by showing that an open immersion $j : V \hookrightarrow U$ of basic open subsets of X is a stratified homotopy equivalence if and only if U and V are abstractly isomorphic.

If j is a stratified homotopy equivalence, it follows that U and V are stratified over the same subposet of P, and so by [AFT17, Lemma 4.3.7] U and V are isomorphic. Viceversa, if $j : \mathbb{R}^n \times C(Z) \hookrightarrow \mathbb{R}^n \times C(Z)$ is a conically smooth open embedding, then by [AFT17, Lemma 4.3.6] j is homotopy equivalent to $D_0 j$, where $D_0 j$ denotes the differential of j at the point $(0, \text{cone pt})$ (see [AFT17, Definition 3.1.4]). Since $D_0 j$ is a stratified homotopy equivalence, then the same is true for j.

By Theorem 2.5.3 we have that (i) implies (ii), so we are left to show that (iii) implies (i). Let $i : Y \to X$ be the inclusion of a stratum, and let $V \hookrightarrow U$ be an inclusion of euclidean charts of U. By Corollary 2.5.9, the horizontal arrows in the commutative square

$$\begin{tikzcd} \Gamma(U \times C(Z); F) \arrow[r] \arrow[d] & \Gamma(U; i^* F) \arrow[d] \\ \Gamma(V \times C(Z); F) \arrow[r] & \Gamma(V; i^* F) \end{tikzcd}$$

are invertible, and thus $\Gamma(U; i^*F) \to \Gamma(V; i^*F)$ is invertible too. Hence, we may assume that X is a smooth manifold. The result is now a very special case of [HPT20, Proposition 3.1]. For the reader's convenience, we review and adapt the proof of [HPT20, Proposition 3.1] to our setting in the following proposition. □

Proposition 2.5.14. Let X be a smooth manifold, and let $F \in \mathcal{S}\mathrm{hv}(X; \mathcal{C})$. Then the following assertions are equivalent:

(i) F is locally constant;

(ii) for any inclusion $V \hookrightarrow U$ of euclidean charts of X, the restriction

$$\Gamma(U; F) \to \Gamma(V; F)$$

is an equivalence.

Proof. Since the question is local, we may assume that $X = \mathbb{R}^n$, in which case we will prove that condition (ii) implies that F is constant. More precisely, we will show that, if $a : \mathbb{R}^n \to *$ is the unique map, the counit morphism

$$a^* a_* F \to F \tag{2.5.15}$$

is an equivalence. Since $\mathbb{R}^n$ is hypercomplete and admits a basis given by those open subsets diffeomorphic to itself, it then suffices to check that for any such open $j : U \hookrightarrow \mathbb{R}^n$, the map $a_* j_* j^* a^* a_* F \to a_* j_* j^* F$ obtained by applying to (2.5.15) the functor of sections at U is invertible. Notice that we have a commutative triangle

$$\begin{tikzcd} a_* j_* j^* a^* a_* F \arrow[r, "a_* j_* j^* \mathrm{counit}_F"] \arrow[d, "\mathrm{counit}_{a_* F}"'] & a_* j_* j^* F \\ a_* F \arrow[ur, "\simeq", "a_*(\mathrm{unit}_F)"'] & \end{tikzcd}$$

where the diagonal map is invertible by the assumption in (iii), and thus to conclude the proof it suffices to show that a^* is fully faithful, which follows by the homotopy invariance of the shape (see Corollary 1.3.4). □

Theorem 2.5.16 (Exodromy)**.** The composition

$$\mathrm{Fun}(\mathrm{Exit}(X), \mathcal{C}) \xrightarrow{\gamma^*} \mathrm{Fun}_W(\mathrm{Bsc}_{/X}, \mathcal{C}) \xrightarrow{\mathrm{im}_*} \mathrm{Fun}(\mathcal{U}(X)^{op}, \mathcal{C})$$

is fully faithful with essential image $\mathcal{S}\mathrm{hv}^{fc}(X; \mathcal{C})$. Moreover, if we assume that $\mathcal{C}$ has a closed symmetric monoidal structure, the statement remains true if we replace $\mathcal{C}$ by $\mathcal{C}^{\mathrm{dual}}$ and $\mathcal{S}\mathrm{hv}^{fc}(X; \mathcal{C})$ by $\mathcal{S}\mathrm{hv}^{c}(X; \mathcal{C})$.

Proof. By Proposition 2.5.13, it suffices to show that the restriction of im_* to $\mathrm{Fun}_W(\mathrm{Bsc}_{/X}, \mathcal{C})$ factors through $\mathcal{S}\mathrm{hv}^{fc}(X; \mathcal{C})$.

Let U be basic open subset of X and let $\kappa : T \hookrightarrow \mathrm{Bsc}_{/U}$ be a covering sieve. Clearly there is at least one $V \in T$ whose image in U intersects the locus of maximal depth. By [AFT17, Lemma 4.3.7], U and V are abstractly isomorphic. Then for any $F \in \mathrm{Fun}_W(\mathrm{Bsc}_{/X}, \mathcal{C})$ we have a commutative triangle

$$\begin{tikzcd} \Gamma(U; F) \arrow[r] \arrow[dr, "\simeq"'] & \varprojlim_{O \in T} \Gamma(O; F) \arrow[d] \\ & \Gamma(V; F) \end{tikzcd}$$

where the diagonal map is invertible by assumption. By [AFT17, Proposition 3.2.23] and [Aok20, Theorem A.6], to conclude the proof it suffices to show that the vertical map is invertible. Let $\delta : T \to \mathcal{T}$ be the localization of T at W. Since δ is final and κ^* sends maps in X to equivalences, the result then follows by observing that V is a terminal object in $\mathcal{T}$.

The second part of the statement is obvious after Corollary 2.5.6. $\square$

Remark 2.5.17. Notice that even though we assumed from the beginning that the coefficients are stable and bicomplete, all the arguments we have discussed work whenever $\mathcal{S}\mathrm{hv}(X;\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{U}(X)^{op}, \mathcal{C})$ admits a left adjoint and $\mathcal{C}$ respects glueings in the sense of [HPT20, Definition 5.17]. In particular, our proof recovers also the case $\mathcal{C} = \mathcal{S}$. A proof of the exodromy equivalence with presentable coefficients but on a much bigger class of stratified spaces will appear soon in [PT22].

Corollary 2.5.18. Let Z be any compact conically smooth stratified space, and let $F \in \mathcal{S}\mathrm{hv}^c(X;\mathcal{C})$. Then $\Gamma(Z;F)$ is dualizable.

Proof. Since γ is final, by Theorem 2.5.16 we know that there exists a functor $G : \mathrm{Exit}(Z) \to \mathcal{C}^{\mathrm{dual}}$ such that global sections of F are equivalent to the limit of G. Therefore the proof is concluded by applying Proposition 2.4.15 and observing that, since $\mathcal{C}$ is stable and its monoidal structure is closed, $\mathcal{C}^{\mathrm{dual}}$ is is itself stable. $\square$

Remark 2.5.19. Let Z be a stratified space such that $\mathrm{Exit}(Z)$ is a retract in $\mathcal{C}\mathrm{at}_\infty$ of a finite ∞-category, and assume that the exodromy equivalence holds for constructible sheaves on Z. Since $\mathcal{C}^{\mathrm{dual}}$ is idempotent complete, the same argument of Corollary 2.5.18 shows that for any $F \in \mathcal{S}\mathrm{hv}^c(X;\mathcal{C})$, the object $\Gamma(Z;F)$ is dualizable.

2.6 Verdier duality

This final section is devoted to proving Verdier duality for conically smooth spaces (Theorem 2.6.3). For this reason, from now on our ∞-categories of coefficients are assumed to be equipped with a closed symmetric monoidal structure.

We first introduce the usual duality functor. For any stratified space X, we will denote by $\omega_X^{\mathcal{C}}$ the sheaf of $a^!(\mathbb{1}_{\mathcal{C}})$, where $a : X \to *$ is the unique map and $\mathbb{1}_{\mathcal{C}}$ is the monoidal unit in $\mathcal{C}$. The sheaf $\omega_X^{\mathcal{C}}$ will be called the $\mathcal{C}$-valued *dualizing sheaf* of X. We denote the functor

$$\underline{\mathrm{Hom}}_X(-, \omega_X^{\mathcal{C}}) : \mathcal{S}\mathrm{hv}(X;\mathcal{C})^{op} \to \mathcal{S}\mathrm{hv}(X;\mathcal{C})$$

simply by $D_X^{\mathcal{C}}$ and, when $X = *$, we will only write $D^{\mathcal{C}} : \mathcal{C}^{op} \to \mathcal{C}$.

Proposition 2.6.1. For any C^0-stratified space X, the dualizing sheaf $\omega_X^{\mathcal{C}}$ is constructible.

Proof. First of all, notice that, by Proposition 1.6.16, we have an equivalence

$$\omega_X^{\mathcal{C}} \simeq \omega_X^{\mathcal{S}\mathrm{p}} \otimes \mathbb{1}_{\mathcal{C}},$$

where the right-hand side denotes the image of the pair $(\omega_X^{\mathcal{S}\mathrm{p}}, \mathbb{1}_{\mathcal{C}})$ under the variablewise cocontinous functor

$$\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \times \mathcal{C} \to \mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \otimes \mathcal{C} \simeq \mathcal{S}\mathrm{hv}(X;\mathcal{C}). \tag{2.6.2}$$

When $X = *$, (2.6.2) corresponds with the usual tensoring over spectra given by cocompleteness and stability of $\mathcal{C}$. Moreover, by Proposition 1.5.18, for each point $x : * \to X$ we have commutative square

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}(X;\mathcal{S}\mathrm{p}) \times \mathcal{C} \arrow[r] \arrow[d, "x^*_{\mathcal{S}\mathrm{p}} \times \mathcal{C}"] & \mathcal{S}\mathrm{hv}(X;\mathcal{C}) \arrow[d, "x^*_{\mathcal{C}}"] \\
\mathcal{S}\mathrm{p} \times \mathcal{C} \arrow[r] & \mathcal{C}.
\end{tikzcd}$$

Since a spectrum is dualizable if and only if it is finite, and since dualizable objects in $\mathcal{C}$ are closed under finite colimits, we see that it suffices to prove the proposition in the case in which $\mathcal{C} = \mathcal{S}\mathrm{p}$.

We will proceed by induction on the depth of X. If X has depth 0, then X is a topological manifold, and hence ω_X is locally constant of rank the dimension of X. Now assume that X has finite non-zero depth. Since the question is local on X, by [AFT17, Lemma 2.2.2] we may assume that $X = \mathbb{R}^n \times C(Z)$, where Z is a compact C^0-stratified space with $\mathrm{depth}(Z) < \mathrm{depth}(X)$.

Let $p : \mathbb{R}^n \times C(Z) \to C(Z)$ be the projection, $b : C(Z) \to *$ the unique map. By Proposition 1.6.18, for any sheaf F on $C(Z)$ we have a functorial equivalence $p^! F \simeq \Sigma^n p^* F$, so it suffices to show that $b^! \mathbb{S} = \omega_{C(Z)}$ is constructible. Hence we may assume that $X = C(Z)$.

Let $i : \{x\} \hookrightarrow X$ be the inclusion of the cone point and $j : U \hookrightarrow X$ its open complement, which is evidently homeomorphic to $\mathbb{R} \times Z$. We have a localization sequence

$$j_\sharp j^* \omega_X \to \omega_X \to i_* i^* \omega_X.$$

By the inductive hypothesis $j^*\omega_X$ is constructible, and thus for every stratum $T \subseteq X$ which does not contain the cone point, the restriction of ω_X is locally constant. Hence it remains to prove that the stalk of ω_X at the cone point is a finite spectrum. But by a routine cofinality argument we get

$$\begin{aligned}
(\omega_X)_x &= \varinjlim_{x \in U} \Gamma(U; \omega_X) \\
&\simeq \varinjlim_{\varepsilon > 0} \Gamma(C_\varepsilon(Z); \omega_X) \\
&\simeq \varinjlim_{\varepsilon > 0} \underline{\mathrm{Hom}}(\Gamma_c(C_\varepsilon(Z); \mathbb{S}_X), \mathbb{S}).
\end{aligned}$$

To conclude the proof, we will now show that the canonical extension map $\Gamma_c(C_\varepsilon(Z); \mathbb{S}_X) \to \Gamma_c(X; \mathbb{S}_X)$ is an equivalence for each ε and that $\Gamma_c(X; \mathbb{S}_X)$ is a finite spectrum.

First of all, notice that for any $K \subseteq C_\varepsilon(Z)$ compact containing the cone point, there exists a $T \geq 0$ such that $K \subseteq \overline{C_T(Z)}$ (namely, take T to be the maximum in the image of K through the projection $C_\varepsilon(Z) \to \mathbb{R}_{\geq 0}$). Hence we have a commutative square

$$\begin{tikzcd}
\Gamma_c(C_\varepsilon(Z); \mathbb{S}_X) \arrow[r, "\simeq"] \arrow[d] & \varinjlim_{0 \leq T < \varepsilon} \Gamma_{\overline{C_T(Z)}}(X; \mathbb{S}_X) \arrow[d] \\
\Gamma_c(X; \mathbb{S}_X) \arrow[r, "\simeq"] & \varinjlim_{T \geq 0} \Gamma_{\overline{C_T(Z)}}(X; \mathbb{S}_X).
\end{tikzcd}$$

Since both colimits on the right-hand side of the square are indexed by filtered posets, to prove that the horizontal maps are invertible it will suffice to show that the functors of which we compute both colimits are constant. By definition, there is a fiber sequence

$$\Gamma_{\overline{C_T(Z)}}(X; \mathbb{S}_X) \to \Gamma(X; \mathbb{S}_X) \to \Gamma(X \setminus \overline{C_T(Z)}; \mathbb{S}_X).$$

By homotopy invariance of the shape, since X is contractible and $X \setminus \overline{C_T(Z)} \simeq Z$ there is a commutative diagram

$$\begin{tikzcd}
\Gamma(X;\mathbb{S}_X) \arrow[r] \arrow[d, "\simeq"] & \Gamma(X \setminus \overline{C_T(Z)};\mathbb{S}_X) \arrow[d, "\simeq"] \\
\underline{\mathrm{Hom}}_{\mathcal{S}\mathrm{p}}(\Sigma^\infty_+ X, \mathbb{S}) \arrow[r] \arrow[d, "\simeq"] & \underline{\mathrm{Hom}}_{\mathcal{S}\mathrm{p}}(\Sigma^\infty_+(X \setminus \overline{C_T(Z)}), \mathbb{S}) \arrow[d, "\simeq"] \\
\mathbb{S} \arrow[r] & \underline{\mathrm{Hom}}_{\mathcal{S}\mathrm{p}}(\Sigma^\infty_+ Z, \mathbb{S})
\end{tikzcd}$$

where the lower horizontal arrow is induced by applying the functor $\underline{\mathrm{Hom}}_{\mathcal{S}\mathrm{p}}(\Sigma^\infty_+(-), \mathbb{S})$ to the unique arrow $Z \to *$. Thus we get

$$\Gamma_{\overline{C_T(Z)}}(X;\mathbb{S}_X) \simeq \mathrm{fib}(\mathbb{S} \to \underline{\mathrm{Hom}}_{\mathcal{S}\mathrm{p}}(\Sigma^\infty_+ Z, \mathbb{S}))$$

for any $T \geq 0$. Hence to conclude the proof it suffices to show that $\Sigma^\infty_+ Z$ is a finite spectrum. This is clear since, for any spectrum M, we have a functorial equivalence

$$\underline{\mathrm{Hom}}(\Sigma^\infty_+ Z, M) \simeq a_! a^* M$$

and $a_! a^*$ preserves (filtered) colimits: since the class of compact objects in $\mathcal{S}\mathrm{p}$ coincides with the one of finite spectra, we may conclude. Otherwise, one may see this as a consequence of Lemma 2.4.3, Proposition 2.4.15 and the fact that Σ^∞_+ preserves finite objects. $\square$

Theorem 2.6.3. Let X be a conically smooth stratified space. Then the restriction to $\mathcal{S}\mathrm{hv}^c(X;\mathcal{C})^{op}$ of the functor $D_X^{\mathcal{C}}$ factors through an equivalence

$$D_X^{\mathcal{C}} : \mathcal{S}\mathrm{hv}^c(X;\mathcal{C})^{op} \xrightarrow{\simeq} \mathcal{S}\mathrm{hv}^c(X;\mathcal{C}).$$

Proof. Let $j : U \hookrightarrow X$ be any open subset of X. Then, for any $F \in \mathcal{S}\mathrm{hv}(X;\mathcal{C})$, by applying Corollary 1.3.26, Lemma 1.6.5 and Proposition 1.6.12 we get functorial equivalences

$$\begin{aligned}
\Gamma(U;\underline{\mathrm{Hom}}_X(F,\omega_X)) &\simeq \Gamma(U;\underline{\mathrm{Hom}}_U(j^*F, j^*\omega_X^{\mathcal{C}})) \\
&\simeq \Gamma(U;\underline{\mathrm{Hom}}_U(j^*F, \omega_U^{\mathcal{C}})) \\
&\simeq \underline{\mathrm{Hom}}_{\mathcal{C}}(\Gamma_c(U;F), \mathbb{1}_{\mathcal{C}})
\end{aligned}$$

and thus there is a factorization

$$\begin{tikzcd}
\mathcal{S}\mathrm{hv}(X;\mathcal{C})^{op} \arrow[rr, "\underline{\mathrm{Hom}}_X(-{,}\omega_X)"] \arrow[dr, "\mathbb{D}"', "\simeq"] & & \mathcal{S}\mathrm{hv}(X;\mathcal{C}). \\
& \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}(X;\mathcal{C})^{op} \arrow[ur, "D_\bullet^{\mathcal{C}}"'] &
\end{tikzcd}$$

where $D_\bullet^{\mathcal{C}}$ denotes the functor obtained by postcomposing with $D^{\mathcal{C}} : \mathcal{C}^{op} \to \mathcal{C}$.

We start by showing that $\mathbb{D}$ restricts to an equivalence

$$\mathbb{D} : \mathcal{S}\mathrm{hv}^{fc}(X;\mathcal{C}) \simeq \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}^{fc}(X;\mathcal{C}). \tag{2.6.4}$$

First of all, we prove that if $F \in \mathcal{S}\mathrm{hv}(X;\mathcal{C})$ is locally constant, then $\mathbb{D}F$ is a formally constructible cosheaf. Since both properties are local, it suffices to show that $\mathbb{D}$ sends constant sheaves to formally constructible sheaves, so assume $F \simeq a^*M$ where $a : X \to *$ is the unique map. In this case we have that $\mathbb{D}F \simeq a^!_{\mathcal{C}^{op}}M$, and so by Proposition 2.6.1 we get what we want.

Assume now that F is any formally constructible sheaf, and let $i : X_\alpha \hookrightarrow X$ by the inclusion of a stratum of X, with complement $j : U \hookrightarrow X$. Thus, we need to show that $i^*_{\mathcal{C}^{op}}\mathbb{D}F \simeq \mathbb{D}i^!_{\mathcal{C}}F$ is locally constant. Since the question is local, by restring to $X_{\geq\alpha}$ one may assume that X_α is a closed stratum, so that we have a fiber sequence

$$i^!_{\mathcal{C}}F \to i^*_{\mathcal{C}}F \to i^*_{\mathcal{C}}j^{\mathcal{C}}_*j^*_{\mathcal{C}}F. \tag{2.6.5}$$

Hence to conclude, since we have already shown that $\mathbb{D}$ preserves locally constant sheaves on smooth manifolds, it suffices to show that $j^{\mathcal{C}}_*j^*_{\mathcal{C}}F$ is formally constructible. For any inclusion $V \hookrightarrow W$ of basic open subsets of X, we have a commutative square

$$\begin{tikzcd}
\Gamma(V; j^{\mathcal{C}}_*j^*_{\mathcal{C}}F) \arrow[r, "\simeq"] \arrow[d] & \Gamma(V \cap U; F) \arrow[d] \\
\Gamma(W; j^{\mathcal{C}}_*j^*_{\mathcal{C}}F) \arrow[r, "\simeq"] & \Gamma(W \cap U; F),
\end{tikzcd}$$

where the right vertical map is an equivalence by Corollary 2.5.4, since $V \cap U \hookrightarrow W \cap U$ is still a stratified homotopy equivalence. Hence, by Proposition 2.5.13 we conclude that $j^{\mathcal{C}}_*j^*_{\mathcal{C}}F$ is formally constructible.

We now prove that $\mathbb{D}$ restricts to an equivalence

$$\mathbb{D} : \mathcal{S}\mathrm{hv}^c(X; \mathcal{C}) \simeq \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}^c(X; \mathcal{C}). \tag{2.6.6}$$

For this it suffices to show that, for any $x \in X$ and $F \in \mathcal{S}\mathrm{hv}^{fc}(X; \mathcal{C})$, $F_x \in \mathcal{C}$ is dualizable if and only if $(\mathbb{D}F)_x$ is dualizable, where the latter denotes the costalk of $\mathbb{D}F$ at x. Let $x : * \hookrightarrow X$ be the inclusion of a point $x \in X$. By definition, there are equivalences $(\mathbb{D}F)_x \simeq x^*_{\mathcal{C}^{op}}(\mathbb{D}F) \simeq x^!_{\mathcal{C}}F$. Thus, by applying global sections to the localization sequence associated to i and j, we obtain a fiber sequence

$$x^!F \simeq \Gamma(X; i_*i^!F) \to \Gamma(X; F) \to \Gamma(U; F)$$

and hence an equivalence

$$\Gamma_{\{x\}}(X; F) \simeq i^!F.$$

Thus, by choosing a conical chart $\mathbb{R}^n \times C(Z)$ around x and by applying Corollary 2.5.6, we get a fiber sequence

$$(\mathbb{D}F)_x \to F_x \to \Gamma((\mathbb{R}^n \times C(Z)) \setminus (0, *); F)$$

where $* \in C(Z)$ denotes the cone point. Therefore by Corollary 2.5.18, to conclude it will suffice to show that $\mathrm{Exit}((\mathbb{R}^n \times C(Z)) \setminus (0, *))$ is finite. But by Van Kampen for exit paths, one has a pushout

$$\begin{tikzcd}
\mathrm{Exit}(\mathbb{R}^n \setminus \{0\} \times \mathbb{R}_{>0} \times Z) \arrow[r] \arrow[d] & \mathrm{Exit}(\mathbb{R}^n \times \mathbb{R}_{>0} \times Z) \arrow[d] \\
\mathrm{Exit}(\mathbb{R}^n \setminus \{0\} \times C(Z)) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Exit}(\mathbb{R}^n \times C(Z)) \setminus (0, *))
\end{tikzcd}$$

and so the result follows from Proposition 2.4.15 and Lemma 2.4.10.

We finally show that

$$D^{\mathcal{C}}_\bullet : \mathcal{C}\mathrm{o}\mathcal{S}\mathrm{hv}^c(X; \mathcal{C})^{op} \to \mathcal{S}\mathrm{hv}^c(X; \mathcal{C})$$

is an equivalence. The diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathrm{Exit}(X), (\mathcal{C}^{\mathrm{dual}})^{op}) \arrow[r, "D^{\mathcal{C}}_{\bullet}"] \arrow[d, "\simeq"', "\gamma^*"] & \mathrm{Fun}(\mathrm{Exit}(X), \mathcal{C}^{\mathrm{dual}}) \arrow[d, "\simeq"', "\gamma^*"] \\
\mathrm{Fun}_W(\mathrm{Bsc}_{/X}, (\mathcal{C}^{\mathrm{dual}})^{op}) \arrow[r, "D^{\mathcal{C}}_{\bullet}"] & \mathrm{Fun}_W(\mathrm{Bsc}_{/X}, \mathcal{C}^{\mathrm{dual}}) \\
\mathcal{C}o\mathcal{S}\mathrm{hv}^c(X;\mathcal{C}) \arrow[r, "D^{\mathcal{C}}_{\bullet}"] \arrow[u, "\simeq", "\mathrm{im}^*"'] & \mathcal{S}\mathrm{hv}^c(X;\mathcal{C}) \arrow[u, "\simeq", "\mathrm{im}^*"']
\end{tikzcd}$$

commutes since the horizontal arrows are given by postcompositions and the vertical arrows by precompositions. Moreover, since the restriction of $D^{\mathcal{C}}$ induces a duality on $\mathcal{C}^{\mathrm{dual}}$, the upper horizontal arrows are equivalences, and thus we get the desired conclusion. □

Remark 2.6.7. Since the functor (2.6.4) can be promoted to a natural transformation of sheaves of ∞-categories on X, to prove that it is actually an equivalence one could have assumed from the beginning that X is a basic. In this case the stratifying poset of X is finite, and hence one could have deduced that (2.6.4) is an equivalence by [AMGR19, Corollary 7.4.25] (see [AMGR19, Example 1.10.8]). However, we preferred to give a more concrete and independent proof to make our discussion as self contained as possible.

Remark 2.6.8. Notice that the equivalences (2.6.4) and (2.6.6) are already interesting on their own, because they imply that for any stratified map $f : X \to Y$, $f_*^{\mathcal{C}}$ or $f^*_{\mathcal{C}}$ preserves (formal) constructibility if and only if $f_!^{\mathcal{C}}$ or $f^!_{\mathcal{C}}$ does. In particular, we see that $f^!_{\mathcal{C}}$ always preserves (formally) constructible sheaves.

Remark 2.6.9. Any μ-stratification of an analytic manifold in the sense of [KS90] satisfies the Whitney conditions, and hence by Theorem 2.3.7 defines a conically smooth structure. Thus, Theorem 2.6.3 recovers and generalizes the duality on constructible sheaves on analytic manifolds as defined in [KS90] (i.e. sheaves which are constructible in our sense with respect to *some* μ-stratification).

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