Nonlinearity

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To cite this article: Helmut Abels and Yadong Liu 2023 *Nonlinearity* **36** 537

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IOP Publishing | London Mathematical Society Nonlinearity
Nonlinearity **36** (2023) 537–583 https://doi.org/10.1088/1361-6544/aca5e1

On a fluid–structure interaction problem for plaque growth

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Received 11 November 2021; revised 1 November 2022
Accepted for publication 24 November 2022
Published 9 December 2022

Recommended by Dr Nader Masmoudi

Abstract
We study a free-boundary fluid–structure interaction problem with growth, which arises from the plaque formation in blood vessels. The fluid is described by the incompressible Navier–Stokes equations, while the structure is considered as a viscoelastic incompressible neo-Hookean material. Moreover, the growth due to the biochemical process is taken into account. Applying the maximal regularity theory to a linearization of the equations, along with a deformation mapping, we prove the well-posedness of the full nonlinear problem via the contraction mapping principle.

Keywords: fluid–structure interaction, two-phase flow, growth, free boundary value problem, maximal regularity
Mathematics Subject Classification numbers:
Primary: 35R35; Secondary: 35Q30, 74F10, 74L15, 76T99

1. Introduction

In this paper, we consider a free-boundary fluid–structure interaction problem with growth, which is used to describe the plaque formation in a human artery. The motion of the blood is assumed to be represented by the incompressible Navier–Stokes equations and the artery is modeled by an elastic equation with viscosity. Based on [46], where the model was proposed and simulated in a cylindrical domain, we analyse such problem in a bounded domain $\Omega^t \subset \mathbb{R}^n$, $n \geqslant 2$. See figure 1. Here, $\Omega^t = \Omega_f^t \cup \Omega_s^t \cup \Gamma^t$, where Ω^t is divided by the interface Γ^t into two

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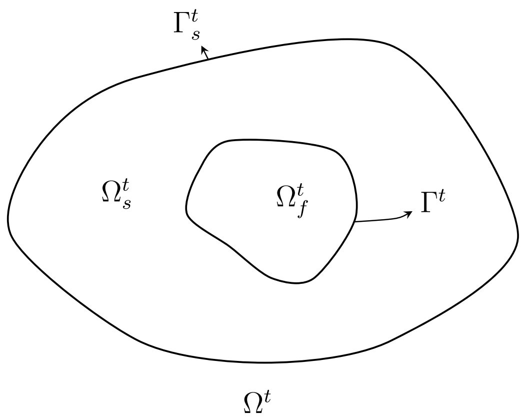

Figure 1. Domain of the problem.

disjoint parts, fluid domain Ω_{f}^t and solid domain Ω_{s}^t. Γ_{s}^t denotes the outer boundary of Ω^t, which is also a free boundary.

For $t \in (0,T)$, $T > 0$, the PDE system describing the problem reads as

$$\rho_{\mathrm{f}}\left(\partial_t + \mathbf{v}_{\mathrm{f}}\cdot\nabla\right)\mathbf{v}_{\mathrm{f}} = \mathrm{div}\boldsymbol{\sigma}_{\mathrm{f}}, \quad \text{in } \Omega_{\mathrm{f}}^t \tag{1.1a}$$

$$\mathrm{div}\mathbf{v}_{\mathrm{f}} = 0, \quad \text{in } \Omega_{\mathrm{f}}^t, \tag{1.1b}$$

$$\rho_{\mathrm{s}}\left(\partial_t + \mathbf{v}_{\mathrm{s}}\cdot\nabla\right)\mathbf{v}_{\mathrm{s}} = \mathrm{div}\boldsymbol{\sigma}_{s}, \quad \text{in } \Omega_{\mathrm{s}}^t, \tag{1.1c}$$

$$\left(\partial_t + \mathbf{v}_{\mathrm{s}}\cdot\nabla\right)\rho_s + \rho_s\mathrm{div}\mathbf{v}_{\mathrm{s}} = f_s^g, \quad \text{in } \Omega_{\mathrm{s}}^t, \tag{1.1d}$$

$$\partial_t c_{\mathrm{f}} + \mathrm{div}\left(c_{\mathrm{f}}\mathbf{v}_{\mathrm{f}}\right) - D_{\mathrm{f}}\Delta c_{\mathrm{f}} = 0, \quad \text{in } \Omega_{\mathrm{f}}^t, \tag{1.1e}$$

$$\partial_t c_{\mathrm{s}} + \mathrm{div}\left(c_s\mathbf{v}_{\mathrm{s}}\right) - D_s\Delta c_s = -f_{\mathrm{s}}^r, \quad \text{in } \Omega_{\mathrm{s}}^t, \tag{1.1f}$$

$$\partial_t c_{\mathrm{s}}^* + \mathrm{div}(c_{\mathrm{s}}^*\mathbf{v}_{\mathrm{s}}) = f_{\mathrm{s}}^r, \quad \text{in } \Omega_{\mathrm{s}}^t, \tag{1.1g}$$

where $\rho_{f/s}$ are the densities and $\mathbf{v}_{\mathrm{f/s}}$ are the velocities of the fluid and the solid respectively, $\boldsymbol{\sigma}_{\mathrm{f}}(\mathbf{v}_{\mathrm{f}},\pi_{\mathrm{f}}) = -\pi_{\mathrm{f}}\mathbb{I} + \nu_{\mathrm{f}}(\nabla\mathbf{v}_{\mathrm{f}} + \nabla^\top\mathbf{v}_{\mathrm{f}})$ denotes the Cauchy stress tensor of the fluid, π_{f} is the unknown fluid pressure and ν_{f} represents the fluid viscosity, while $\boldsymbol{\sigma}_{\mathrm{s}}$ is the Cauchy stress tensor of the solid that includes viscoelastic effects and will be discussed with more details in section 1.1. f_{s}^g is called the growth function, representing the rate of mass growth per unit volume due to the formation of plaque (see e.g. [7, 27, 46]), which will be specified together with the growth. In addition, c_{f}, c_{s}, c_{s}^* denote the concentrations of the monocytes, the macrophages and the foam cells, respectively. The constant $D_{\mathrm{f}/s} > 0$ are the diffusion coefficients in the blood and vessel, which are assumed to be constants. f_{s}^r is the reaction functions, modeling the rate of conversion from macrophages c_{s} into foam cells c_{s}^*.

Moreover, the system (1.1) is subjected to the boundary and initial conditions

$$[\mathbf{v}] = 0, \quad [\boldsymbol{\sigma}]\,\mathbf{n}_{\Gamma^t} = 0, \quad \text{on } \Gamma^t, \tag{1.1h}$$

$$[D\nabla c]\cdot\mathbf{n}_{\Gamma^t} = 0, \quad \zeta\,[c] - D_{\mathrm{s}}\nabla c_{\mathrm{s}}\cdot\mathbf{n}_{\Gamma^t} = 0, \quad \text{on } \Gamma^t, \tag{1.1i}$$

$$\boldsymbol{\sigma}_{\mathrm{s}}\mathbf{n}_{\Gamma_{\mathrm{s}}^t} = 0, \quad D_{\mathrm{s}}\nabla c_{\mathrm{s}}\cdot\mathbf{n}_{\Gamma_{\mathrm{s}}^t} = 0, \quad \text{on } \Gamma_{\mathrm{s}}^t, \tag{1.1j}$$

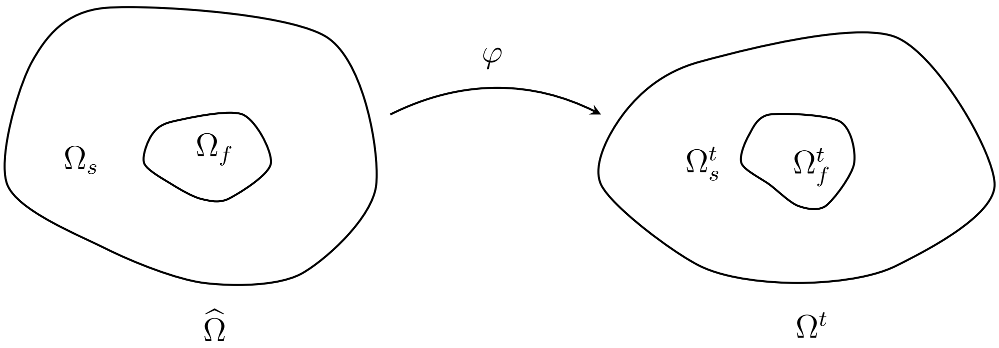

Figure 2. Deformation φ mapping from $\widehat{\Omega}$ into Ω^t.

$$\mathbf{v}_\mathrm{f}|_{t=0} = \mathbf{v}_\mathrm{f}^0, \quad \mathbf{v}_\mathrm{s}|_{t=0} = \mathbf{v}_\mathrm{s}^0, \quad c|_{t=0} = c^0, \quad c_\mathrm{s}^*|_{t=0} = 0, \tag{1.1k}$$

where $\mathbf{n}_{\Gamma^t}$ stands for the outer unit normal vector on Γ^t pointing from Ω_f^t to Ω_s^t and $\mathbf{n}_{\Gamma_\mathrm{s}^t}$ is the unit outer normal vector on $\Gamma_\mathrm{s}^t = \partial\Omega^t$. The constant ζ denotes the permeability of the interface Γ^t between blood and vessel regarding the cells. For a quantity f, $[\![f]\!]$ denotes the jump defined on Ω_f^t and Ω_s^t across Γ^t, namely,

$$[\![f]\!](x) := \lim_{\theta\to 0} f(x + \theta\mathbf{n}_{\Gamma^t}(x)) - f(x - \theta\mathbf{n}_{\Gamma^t}(x)), \quad \forall x \in \Gamma^t.$$

Before giving a precise explanation of the model, we introduce the setting of Lagrangian coordinates. For convenience, we define the moving domain at initial time $t = 0$ as $\widehat{\Omega} = \Omega_\mathrm{f} \cup \Omega_\mathrm{s} \cup \Gamma$, where $\Omega_\mathrm{f} = \Omega_\mathrm{f}^0$, $\Omega_\mathrm{s} = \Omega_\mathrm{s}^0$ and $\Gamma = \Gamma^0$. From the viewpoint of material deformation (see e.g. [14, 23]), we set the so-called reference configuration at $t = 0$ and the deformed configuration at time t. Moreover, we denote the spatial variable at $t = 0$ by the Lagrangian variable X, and by the Eulerian variable x the spatial variable at t. The velocities of displacements are $\hat{\mathbf{v}}(X,t)$ and $\mathbf{v}(x,t)$ respectively. In the sequel, without a special statement, the quantities or operators with a hat '$\hat{\cdot}$' will indicate those in Lagrangian coordinates. To formulate the model, we define the deformation as (see figure 2)

$$\varphi : \widehat{\Omega} \to \Omega^t,$$

with

$$x = \varphi(X,t) = X + \int_0^t \hat{\mathbf{v}}(X,\tau)\mathrm{d}\tau, \quad \forall X \in \widehat{\Omega},$$

and $x|_{t=0} = \varphi(X,0) = X$.

Subsequently, we denote by $\hat{\mathbf{F}}$ the deformation gradient

$$\hat{\mathbf{F}} = \frac{\partial}{\partial X}\varphi(X,t) = \hat{\nabla}\varphi(X,t) = \mathbb{I} + \int_0^t \hat{\nabla}\hat{\mathbf{v}}(X,\tau)\mathrm{d}\tau, \quad \forall X \in \widehat{\Omega}, \tag{1.2}$$

with initial deformation $\hat{\mathbf{F}}|_{t=0} = \mathbb{I}$ and by $\hat{J} = \det\hat{\mathbf{F}}$ its determinant. Conversely, the inverse deformation gradient is defined by $\mathbf{F} = \hat{\mathbf{F}}^{-1}$. In the following, quantities in fluid and structure domain will be distinguished by subscript 'f' and 's' respectively, quantities without subscript are defined both in fluid and structure domain.

1.1. Fluid–solid interaction with a free interface

The motion of the blood is modeled by the classical incompressible Navier–Stokes equations (1.1*a*) and (1.1*b*), while the solid part is described by an incompressible solid equation (1.1*c*) and (1.1*d*). The crucial difference comes from the stress tensor $\boldsymbol{\sigma}_{\mathrm{s}}$, which is defined as $\boldsymbol{\sigma}_{\mathrm{s}} := \boldsymbol{\sigma}^e_{\mathrm{s}} + \boldsymbol{\sigma}^v_{\mathrm{s}}$ with

$$\boldsymbol{\sigma}^e_{\mathrm{s}} = -\pi_{\mathrm{s}}\mathbb{I} + \mu_{\mathrm{s}}\left((\mathbf{F}^e_{\mathrm{s}})^{-1}(\mathbf{F}^e_{\mathrm{s}})^{-\top} - \mathbb{I}\right),$$
$$\boldsymbol{\sigma}^v_{\mathrm{s}} = \nu_{\mathrm{s}}\left(\partial_t \mathbf{F}_{\mathrm{s}}{}^{-1} + \partial_t \mathbf{F}_{\mathrm{s}}{}^{-\top}\right)\mathbf{F}_{\mathrm{s}}{}^{-\top}.$$

Here π_{s} is the unknown solid pressure, μ_{s} denotes the Lamé coefficient and ν_{s} represents the solid viscosity, which are all positive constants. (1.1*c*) is the balance equation of the momentum, wherein $\boldsymbol{\sigma}^e_{\mathrm{s}}$ is given by the constitutive relation of an incompressible neo-Hookean material as above, which is hyperelastic, isotropic and incompressible. This relationship has been widely used to describe blood vessels by many investigators, see e.g. [44, 46]. The tensor $\mathbf{F}^e_{\mathrm{s}}$ is the inverse elastic deformation gradient under the assumption of growth and will be assigned later in section 1.3. We consider not only the elastic stress tensor $\boldsymbol{\sigma}^e_{\mathrm{s}}$, but also the viscoelastic stress tensor $\boldsymbol{\sigma}^v_{\mathrm{s}}$, which could be deduced by linearizing the Kelvin–Voigt stress tensor, see Mielke and Roubíček [35].

Remark 1.1. For short time existence, the Kelvin–Voigt viscous stress tensor $\boldsymbol{\sigma}^v_{\mathrm{s}}$ we introduced brings the parabolicity to the system for the solid, which dominates the regularity of solutions. Moreover, after linearization one obtains a two-phase Stokes type problem, which allows us to get the solvabilities and regularities of fluid and solid velocities by maximal regularity theory. In a recent work [9], a similar stress tensor of the solid part was also considered to investigate weak solutions of the interaction between an incompressible fluid and an incompressible immersed viscous-hyperelastic solid structure.

Remark 1.2. In [44, 46], some numerical simulations are carried out by considering that μ_{s} depends on the concentration of some chemical species, and hence varies from healthy vessel to plaque area. In the case of viscoelasticity, ν_{s} may also vary over the solid domain. However, to simplify the model for the analysis, we assume that these coefficients are constant over the solid domain.

The interaction between the fluid and solid is modeled by transmission conditions (1.1*h*) on the interface Γ^t, which consists of the continuity of velocity and the balance of normal stresses. Moreover, to ensure the compatibility between growth and incompressibility, the boundary condition on Γ^t_{s} is assumed to be the so-called 'stress-free' boundary condition (1.1*j*).

Remark 1.3. We choose the 'stress-free' boundary condition for the velocity in (1.1*j*) to obtain physical compatibility. Since we consider the growth of the solid part and both the fluid and solid are incompressible, we can not impose some types of boundary conditions. For example, the no-slip condition $\mathbf{v}_{\mathrm{s}} = 0$ on Γ^t_{s} (correspondingly, $\mathbf{v}_{\mathrm{s}} = \partial_t \mathbf{u}_{\mathrm{s}} = 0$ on Γ^t_{s} with $\mathbf{u}_{\mathrm{s}}$ being the solid displacement) is incompatible with the incompressible growth assumption (see later in section 1.3). Namely,

$$0 = \underbrace{\int_{\partial\Omega^t} \mathbf{v}_{\mathrm{s}} \cdot \mathbf{n}\, \mathrm{d}\sigma = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^t} \mathrm{d}x}_{\text{by the Reynolds transport theorem}} = \frac{\mathrm{d}}{\mathrm{d}t}|\Omega^t| = \int_{\Omega^t_{\mathrm{s}}} \mathrm{div}\mathbf{v}_{\mathrm{s}}\, \mathrm{d}x = \int_{\Omega^t_{\mathrm{s}}} f^g_{\mathrm{s}}/\rho_{\mathrm{s}}\, \mathrm{d}x \neq 0,$$

due to growth.

Remark 1.4. In this work, the fluid part is supposed to be surrounded by the solid part. In fact, if the solid is immersed in the fluid domain, there will be no essential difference in our framework of analysis. Specifically, the outer boundary will still be a Neumann-type boundary, which is a 'do-nothing' outer boundary condition for fluid.

1.2. Biochemical processes

The formation of plaque is usually caused by the accumulation of foam cells resulting from biochemical processes in the blood flow and vessel wall. To describe the phenomenon properly, we follow the assumptions and modeling in [46, 47]. More precisely, (1.1*e*) and (1.1*f*) imply the dynamics of monocytes c_f in the blood and of macrophages c_s in the vessel respectively, which are both advection–reaction–diffusion equations. This is because the transport and diffusion of cells are happening along with the motion of the blood and vessel, while the reaction is caused by the conversion from macrophages c_s into foam cells c_s^* indicated by the function $-f_\mathrm{s}^r$. Note that vessels could be inhomogeneous, which reveals different diffusion rates in healthy and diseased vessels, see e.g. [46]. To simplify the model, we assume that the solid is a homogeneous material, and thus D_s is a constant. Furthermore, the foam cells are considered to be transported by $\mathbf{v}_\mathrm{s}$ inside the solid material leading to the equation (1.1*g*). Here again, we have the reaction term f_s^r meaning the gain of mass from macrophages, which is supposed to depend on the concentration of macrophages c_s linearly,

$$f_\mathrm{s}^r = \beta c_\mathrm{s}, \quad \text{in } \Omega_\mathrm{s}^t,$$

where $\beta > 0$ is assumed to be a constant. In reality, it is way more complicated and may depend on the concentration of other chemical species. We just assume a linear relation for the sake of analysis. Then, we give another linear dependence of f_s^g, which is

$$f_\mathrm{s}^g = \gamma f_\mathrm{s}^r = \gamma\beta c_\mathrm{s}, \quad \text{in } \Omega_\mathrm{s}^t, \tag{1.3}$$

with a positive constant γ. (1.3) describes the growth of the elastic solid wall as mentioned in (1.1*d*), resulting from the accumulation of foam cells.

In addition to the process in the fluid or solid domain, one needs to specify the interfacial laws for the cell interactions in (1.1*i*). The first one denotes the balance of the normal concentration flux at the interface, while due to the flux, cells move across the interface (penetration), which is the second equation in (1.1*i*). Here the permeability ζ of the interface Γ^t in general should depend on the hemodynamical stress $\boldsymbol{\sigma}_\mathrm{f} \cdot \mathbf{n}_{\Gamma^t}$, which, however, is supposed to be a constant for simplicity. The outer concentration flux is assumed to vanish on Γ_s^t as in (1.1*j*).

1.3. Description of growth

Normally, prescribing the rate of growth function f_s^g is not enough to capture the full effect of the tissue growth. Specifically, the real deformation and corresponding deformation gradient $\hat{\mathbf{F}}_\mathrm{s}$ are influenced by both growth and mechanics. Namely, the deformation gradient $\hat{\mathbf{F}}_\mathrm{s}$ alone does not determine the stress tensor of the solid $\boldsymbol{\sigma}_\mathrm{s}^e$.

As in [46], Yang *et al* took the idea of a deformation gradient decomposition based on the theory of multiple natural configurations. In this formulation, one needs a new configuration, which is usually called *natural configuration*, so that one can decompose the whole process into a pure growth and a pure elastic one. For more details, one is referred to [7, 27, 39, 46, 47]. Therefore, we assume the decomposition of the deformation gradient $\hat{\mathbf{F}}_\mathrm{s}$ as

$$\hat{\mathbf{F}}_\mathrm{s} = \hat{\mathbf{F}}_\mathrm{s}^e \hat{\mathbf{F}}_\mathrm{s}^g, \quad \text{in } \Omega_\mathrm{s},$$

where $\hat{\mathbf{F}}_s^g$ is the so-called growth tensor and $\hat{\mathbf{F}}_s^e$ represents the elastic tensor. The associated determinants are

$$\hat{J}_s^g = \det \hat{\mathbf{F}}_s^g, \quad \hat{J}_s^e = \det \hat{\mathbf{F}}_s^e, \quad \text{in } \Omega_s,$$

respectively. Then we have

$$\hat{J}_s = \hat{J}_s^g \hat{J}_s^e.$$

Growth may happen in different ways. In applications, two assumptions were most commonly employed: *constant-density growth*, which stands for adding new material with the same density; *constant-volume growth*, by which the total mass is added and density varies. Since constant-density growth is usually coupled with the assumption of an incompressible tissue, see e.g. [27, 39], we take this kind of growth into consideration in this work. Then the equation (1.1*d*) reduces to

$$\rho_s \mathrm{div} \mathbf{v}_s = f_s^g \quad \text{in } \Omega_s^t.$$

Moreover, we assume that plaques grows *isotropically*:

$$\hat{\mathbf{F}}_s^g = \hat{g}\mathbb{I}, \quad \text{in } \Omega_s,$$

where $\hat{g} = \hat{g}(X,t)$ is the metric of growth, a scalar function depending on the concentration of macrophages. Hence,

$$\hat{\mathbf{F}}_s^e = \frac{1}{\hat{g}}\hat{\mathbf{F}}_s, \quad \hat{J}_s^g = \hat{g}^n,$$

where n is the dimension of space. As mentioned in [7], $\hat{g}$ describes the deformation state of the material, either growing if $\hat{g} > 1$ or resorbing if $0 < \hat{g} < 1$. Consequently, under the assumption of constant-density growth, one deduces the equation for growth in Lagrangian coordinates

$$\partial_t \hat{g} = \frac{\gamma\beta}{n\hat{\rho}_s}\hat{c}_s \hat{g}, \quad \text{in } \Omega_s. \tag{1.4}$$

This shows the specific dependence of $\hat{g}$ on $\hat{c}_s$. At initial state, the growth tensor $\hat{\mathbf{F}}_s^g$ is supposed to be the identity, i.e.

$$\hat{g}(X,0) = 1, \quad \text{in } \Omega_s,$$

without growth or resorption of the material.

1.4. Literature

During the last decades, fluid–structure interaction problems attracted much attention from mathematicians due to their strong applications in various areas, e.g. biomechanics, hemodynamics, aeroelasticity and hydroelasticity. Studies can be divided into two types depending on the dimensions of the fluid and the solid. They are for example 3d–3d coupled and 3d–2d coupled systems, where the solid is contained in the fluid and one part of the fluid's boundary, respectively.

In the case of a 3d–3d model, which is exactly our consideration, let us recall some existence results of strong solutions. Well-posedness of such model was first established by Coutand and Shkoller [15], where they investigated the interaction problem between the Navier–Stokes equation and a linear Kirchhoff elastic material. The results were extended to the quasilinear elastodynamics case by them, where they regularised the hyperbolic elastic equation by a particular artificial viscosity to obtain a parabolic system. Then they proved the existence of strong solutions together with the *a priori* estimates in [16]. Thereafter, Ignatova *et al* [24, 25] investigated the coupled system of the incompressible Navier–Stokes equations and a wave equation from different aspects. More specifically, in [24], static damping and velocity internal damping were added in the wave equation and boundary friction was considered, by which exponential decay was obtained. Later, the boundary friction was removed in [25] by introducing the tangential and time-tangential energy estimates. The coupling of the Navier–Stokes equations and the Lamé system was analysed by Kukavica and Tuffaha [28] with initial regularity $(v_0, \xi_1) \in H^3(\Omega_f) \times H^2(\Omega_s)$, while Raymond and Vanninathan [38] further proved the existence and uniqueness of local strong solutions with a weaker initial regularity $(v_0, \xi_1) \in H^{3/2+\varepsilon}(\Omega_f) \times H^{1+\varepsilon}(\Omega_s)$, for $\varepsilon > 0$ arbitrarily small, with periodic boundary conditions. Later, Boulakia *et al* [11] showed a similar result for the Navier–Stokes–Lamé system in a smooth domain with reduced demand on the initial regularity.

There are also other variants of free-boundary fluid–structure interaction models. For a compressible fluid coupled with elastic bodies, we refer to [10], where Boulakia and Guerrero addressed the local in time existence and the uniqueness of regular solutions with the initial data $(\rho_0, u_0, w_0, w_1) \in H^3(\Omega_f) \times H^4(\Omega_f) \times H^3(\Omega_s) \times H^2(\Omega_s)$. This results was later improved by Kukavica and Tuffaha [29] with a weaker initial regularity $(\rho_0, u_0, w_1) \in H^3(\Omega_f) \times H^{3/2+r}(\Omega_f) \times H^{3/2+r}(\Omega_s)$, $r > 0$. More recently, Shen *et al* [40] considered the magnetohydrodynamics (MHD)-structure interaction system, where the fluid is described by the incompressible viscous non-resistive MHD equation and the structure is modeled by the wave equation with a superconductor material. They showed the existence of local strong solutions with penalization and regularization techniques.

For 3d–2d/2d–1d systems where the structure is seen as one part of the fluid's boundary, we just mention several works on the existence and uniqueness of strong solutions to be concise. The mostly investigated case is the fluid-beam/plate systems where the beam/plate equation was imposed with different mechanical mechanisms (rigidity, stretching, friction, rotation, etc). Readers are refer to [8, 17, 21, 22, 30, 31, 34, 36] and references therein. Moreover, the fluid-structure interaction problems with nonlinear shells were studied in [12, 13, 33]. It has to be mentioned that in the recent works [17, 34], a maximal regularity framework, which requires lower initial regularity and fewer compatibility conditions compared to the energy method, was employed.

1.5. Mathematical strategy and features

The new difficulties arise from the plaque formation in the blood vessels, along with the interaction between the fluid and the solid separated by a free interface, the reaction and the diffusion of different cells and the growth of the vessel wall. Numerical computations were carried out in recent years [20, 46, 47] to simulate the plaque formation and test the effects of different parameters. To our best knowledge, this is the first work concerning the existence of strong solutions to the fluid–structure interaction problems with growth. Unlike most of the literature above, where L^2-Sobolev spaces and energy methods are used, we establish our local strong solutions in the framework of maximal L^q-regularity for any space dimension ($n \geqslant 2$). The method is based on the Banach fixed-point theorem, for which we rewrite the free boundary problem established in Eulerian coordinates in Lagrangian coordinates, linearise the system at the initial configuration, construct a contraction mapping in a suitable ball and show the local existence and uniqueness of strong solutions. Throughout the proof, we point out the following features:

(a) We adapt the maximal L^q-regularity theory for the Stokes system to solve our problem. Hence, there will be no 'regularity loss' from the data to the solution spaces and only a few compatibility conditions are needed.
(b) The growth is considered to be of constant-density type. Then under the assumption of isotropy, the growth will be described by the metric function $\hat{g}$. An ordinary differential equation for $\hat{g}$ provides the regularity of $\hat{g}$ needed for the solid velocity and the concentration of macrophages.
(c) The Kelvin–Voigt viscous stress tensor $\boldsymbol{\sigma}_s^v$, we introduced, brings parabolicity to the solid equation. For the linearization, we can use a two-phase Stokes type problem for the fluid-structure interaction problem. This ensures that we can get the solvabilities and regularities of fluid and solid velocities by maximal L^q-regularity theory.
(d) The transformed two-phase Stokes problem is endowed with a stress-free (Neumann-type) outer boundary condition, cf remark 1.3. One of our aims is to obtain the solvability of such system. To this end, reduction and truncation arguments are applied. More specifically, we first reduce the inhomogeneous linear system to a semi-homogeneous problem (with inhomogeneous the boundary terms), in order to obtain the pressure regularities. Then by choosing a cutoff function (see (3.13)) which is supported in a subset $U \subseteq \widehat{\Omega}$ and imposing an artificial vanishing Dirichlet boundary on $\Gamma_s = \partial\widehat{\Omega}$, one obtains the solvability of the linear system since the two-phase Stokes problem with Dirichlet boundary is solved in appendix A.1.

1.6. Outline of the paper

In section 2 we briefly introduce some notations and function spaces along with several preliminary results. The transformation from the deformed configuration to the reference one is shown in the last subsection, as well as the main theorem for the transformed system. Section 3 is devoted to the analysis of the underlying linear problems, where three separate parts of the analysis are treated. The main results of this section are the maximal L^q-regularities for these linear problems. The first one is the two-phase Stokes problems with Neumann boundary conditions, to which reduction and truncation (localization) arguments are applied. The second problem consists of two reaction–diffusion systems with Neumann boundary conditions due to the decoupling of the transmission problem, while the last one is an ordinary differential equation for the growth of the foam cells. In section 4, we first give some estimates related to the deformation gradient, which are of much importance when proving that the constructed nonlinear terms are well-defined and Lipschitz continuous. Then the full nonlinear system is shown to be well-posed locally in time via the Banach fixed-point theorem. Moreover, the cell concentrations are proved to be always nonnegative, provided that the initial data is nonnegative. Additionally, we introduce some maximal L^q-regularity results of several linear systems in appendix A and establish uniform extension operators of the Sobolev–Slobodeckij spaces in appendix B.

2. General settings and main results

2.1. Mathematical notations

For matrices $A, B \in \mathbb{R}^{n\times n}$, let $A : B = \mathrm{tr}(B^\top A)$. The corresponding induced modulus of A is denoted by $|A| = \sqrt{A : A}$. The set of invertible matrices in $\mathbb{R}^{n\times n}$ is $GL(n,\mathbb{R})$. For a differentiable $A : \mathbb{R}_+ \to GL(n,\mathbb{R})$, we have two useful formulas

$$\frac{\mathrm{d}}{\mathrm{d}t}\det A = \operatorname{tr}\left(A^{-1}\frac{\mathrm{d}}{\mathrm{d}t}A\right)\det A \tag{2.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}A^{-1} = -A^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}A\right)A^{-1}, \tag{2.2}$$

which can be found in [19, 23]. Furthermore, for a vector function $\mathbf{u}$ and a tensor matrix $\mathbf{T}$, we give an identity, which will be used later (see e.g. [23, (3.20)]):

$$\operatorname{div}\left(\mathbf{T}^\top\mathbf{u}\right) = \mathbf{T} : \nabla\mathbf{u} + \mathbf{u}\cdot\operatorname{div}\mathbf{T}. \tag{2.3}$$

For metric spaces X, $B_X(x,r)$ represents the open ball with radius $r>0$ around $x\in X$. For normed spaces X, Y over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, the set of bounded, linear operators $T : X\to Y$ is denoted by $\mathcal{L}(X,Y)$ and in particular, $\mathcal{L}(X)=\mathcal{L}(X,X)$.

Throughout the paper, unless we give a special declaration, the letter C will denote a generic positive constant that may change its value from line to line, or even in the same line.

2.2. *Function spaces*

If $M\subseteq\mathbb{R}^d$, $d\in\mathbb{N}_+$ is measurable, $L^q(M)$, $1\leqslant q\leqslant\infty$ denotes the usual Lebesgue space and $\|\cdot\|_{L^q(M)}$ its norm, as well as the mean value zero Lebesgue space

$$L^q_{(0)}(M) := \left\{ f\in L^q(M) : \int_M f\mathrm{d}\mu = 0\right\},$$

with $|M|<\infty$. Moreover, $L^q(M;X)$ denotes its vector-valued variant of strongly measurable q-integrable functions/essentially bounded functions, where X is a Banach space. If $M=(a,b)$, we write for simplicity $L^q(a,b)$ and $L^q(a,b;X)$. By simple computation, we have

$$\|f\|_{L^q(a,b)} \leqslant |a-b|^{\frac{1}{q}}\,\|f\|_{L^\infty(a,b)}\,. \tag{2.4}$$

Let $\Omega\subseteq\mathbb{R}^n$ be a open and nonempty domain, $W^m_q(\Omega)$ denotes the usual Sobolev space with $m\in\mathbb{N}$ and $1\leqslant q\leqslant\infty$, and $L^q(\Omega)=W^0_q(\Omega)$. Moreover, we set

$$W^m_{q,0}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^m_q(\Omega)}, \quad W^{-m}_q(\Omega) := [W^m_{q',0}(\Omega)]',$$
$$W^m_{q,(0)}(\Omega) = W^m_q(\Omega)\cap L^q_{(0)}(\Omega), \quad W^{-m}_{q,(0)}(\Omega) := [W^m_{q',(0)}(\Omega)]',$$

where q' is the conjugate exponent to q satisfying $\frac{1}{q}+\frac{1}{q'}=1$.

For $k,k'\in\mathbb{N}$ with $k<k'$, we consider the standard definition of the Besov spaces by real interpolation of Sobolev spaces (see Lunardi [32])

$$B^s_{q,p}(\Omega) = \left(W^k_q(\Omega), W^{k'}_q(\Omega)\right)_{\theta,p},$$

where $s=(1-\theta)k+\theta k'$, $\theta\in(0,1)$. In the special case $q=p$, we also have Sobolev–Slobodeckij spaces

$$W^s_q(\Omega) = B^s_{q,q}(\Omega) = \left(W^k_q(\Omega), W^{k'}_q(\Omega)\right)_{\theta,q},$$

which is endowed with norm $\|\cdot\|_{W^s_q(\Omega)} = \|\cdot\|_{L^q(\Omega)} + [\cdot]_{W^s_q(\Omega)}$, where

$$[f]^q_{W^s_q(\Omega)} = \int_\Omega\int_\Omega \left(\frac{|f(x)-f(y)|}{|x-y|^s}\right)^q \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n}.$$

The multiplication property of such space is given in the next lemma.

Lemma 2.1 (Multiplication). *Let Ω be a bounded Lipschitz domain. For $f,g\in W_q^s(\Omega)$ and $sq>n$ with $s>0$. We have the multiplication property for all $f,g\in W_q^s(\Omega)$*

$$\|fg\|_{W_q^s(\Omega)}\leqslant M_q\,\|f\|_{W_q^s(\Omega)}\,\|g\|_{W_q^s(\Omega)}\,,$$

where M_q is a constant depending on q, s and Ω.

Proof. For the case $s\in\mathbb{N}_+$, we refer to [45, theorem 1]. For the other cases, since $W_q^s=B_{q,q}^s$ for every $s\in\mathbb{R}_+\backslash\mathbb{N}$, [26, theorem 6.6] implies the statement. □

Next, for an interval $I\subset\mathbb{R}$ and a Banach space X, we recall the definition of vector-valued Sobolev–Slobodeckij space as

$$W_q^s(I;X):=\left\{f\in L^q(I;X):\|f\|_{W_q^s(I;X)}<\infty\right\},$$

whose norm is $\|\cdot\|_{W_q^s(I;X)}=\|\cdot\|_{L^q(I;X)}+[\cdot]_{W_q^s(I;X)}$ with

$$[f]_{W_q^s(I;X)}^q=\int_I\int_I\left(\frac{\|f(t)-f(\tau)\|_X}{|t-\tau|^s}\right)^q\frac{\mathrm{d}t\mathrm{d}\tau}{|t-\tau|}.$$

Then we define ${}_0W_q^s(0,T;X)$ with $0<T\leqslant\infty$ as the linear subspace with a vanishing trace at $t=0$, i.e.

$${}_0W_q^s(0,T;X):=\left\{u\in W_q^s(0,T;X):u|_{t=0}=0\right\}.$$

In addition, we introduce one embedding result from Simon [43, corollary 17].

Lemma 2.2. *Suppose $0<r\leqslant s<1$ and $1\leqslant q\leqslant\infty$. Then*

$$W_q^s(I;X)\hookrightarrow W_q^r(I;X)$$

and, for all $f\in W_q^s(I;X)$,

$$[f]_{W_q^r(I;X)}\leqslant\begin{cases}|I|^{s-r}[f]_{W_q^s(I;X)} & \text{for bounded } I,\\ [f]_{W_q^s(I;X)}+\dfrac{4}{r}\|f\|_{L^q(I;X)} & \text{for all } I.\end{cases}$$

For $r,s\geqslant0$, the anisotropic Sobolev–Slobodeckij spaces $W_q^{r,s}$ is defined as

$$W_q^{r,s}(\Omega\times I):=L^q\left(I;W_q^r(\Omega)\right)\cap W_q^s\left(I;L^q(\Omega)\right).\qquad(2.5)$$

Based on the trace interpolation method [37, section 3.4.6] and [6, chapter III, theorem 4.10.2], we give some useful embeddings, which will be employed later.

Lemma 2.3. *Let X_1,X_0 be two Banach spaces and $X_1\hookrightarrow X_0$. Define $X_T=L^q(0,T;X_1)\cap W_q^1(0,T;X_0)$ for all $1<q<\infty$ and $0<T<\infty$. Then*

$$X_T\hookrightarrow C([0,T];X_\gamma),$$

where

$$X_\gamma=(X_0,X_1)_{1-\frac{1}{q},q}=\left\{u|_{t=0}:u\in X_T\right\}$$

is the trace space. Moreover, if X_T is endowed with the norm

$$\|u\|_{X_T}:=\|u\|_{L^q(0,T;X_1)}+\|u\|_{W_q^1([0,T];X_0)}+\left\|u|_{t=0}\right\|_{X_\gamma},$$

then there is some $C>0$ independent of T such that for $T\in[0,\infty)$ and $u\in X_T$,

$$\|u\|_{C(0,T;X_\gamma)}\leqslant C\|u\|_{X_T}.$$

In particular, if $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, *is a bounded domain,* $n < q < \infty$, *and if* $X_1 = W_q^2(\Omega)$, $X_0 = L^q(\Omega)$, *then* $X_\gamma = W_q^{2-\frac{2}{q}}(\Omega)$ *and*

$$W_q^{2,1}(\Omega \times (0,T)) \hookrightarrow C\left([0,T]; W_q^{2-\frac{2}{q}}(\Omega)\right) \hookrightarrow C\left([0,T]; W_q^1(\Omega)\right), \tag{2.6}$$

together with

$$\|u\|_{C\left([0,T];W_q^1(\Omega)\right)} \leqslant C\left(\|u\|_{W_q^{2,1}(\Omega\times(0,T))} + \|u_0\|_{W_q^{2-\frac{2}{q}}}\right),$$
$$\|u-v\|_{C([0,T];W_q^1(\Omega))} \leqslant C\|u-v\|_{W_q^{2,1}(\Omega\times(0,T))},$$

for $u, v \in W_q^{2,1}(\Omega \times (0,T))$ *with* $u|_{t=0} = v|_{t=0} = u_0$.

Lemma 2.4. *Let* Σ *be a compact sufficiently smooth hypersurface. For* $1 < q < \infty$, $\frac{1}{q} < \alpha \leqslant 1$ *and* $0 < T < \infty$, *define* $X_T := L^q(0,T; W_q^{2\alpha}(\Sigma)) \cap W_q^\alpha(0,T; L^q(\Sigma))$, *then*

$$X_T \hookrightarrow C([0,T]; X_\gamma),$$

where

$$X_\gamma = \left\{ u|_{t=0} : u \in X_T \right\} = W_q^{2\alpha-\frac{2}{q}}(\Sigma).$$

Moreover, if X_T *is endowed with the norm*

$$\|u\|_{X_T} := \|u\|_{L^q(0,T;X_1)} + \|u\|_{W_q^\alpha(0,T;X_0)} + \left\| u|_{t=0} \right\|_{X_\gamma},$$

then there is some $C > 0$ *independent of* T *such that for all* $u \in X_T$,

$$\|u\|_{C([0,T];X_\gamma)} \leqslant C\|u\|_{X_T}.$$

2.3. An equivalent system in Lagrangian reference configuration

In this section, we transform the free-boundary fluid–structure problem with growth from deformed Eulerian configuration to a fixed reference Lagrangian configuration and state the main result. For quantities in different configurations, we define

$$\begin{aligned} &\hat{\mathbf{v}}(X,t) = \mathbf{v}(x,t),\ \hat{\pi}(X,t) = \pi(x,t),\ \hat{\boldsymbol{\sigma}}(X,t) = \boldsymbol{\sigma}(x,t), \\ &\hat{\rho}(X,t) = \rho(x,t),\ \hat{\mu}(X,t) = \mu(x,t),\ \hat{\nu}(X,t) = \nu(x,t), \end{aligned} \tag{2.7}$$

for all $x = \varphi(X,t)$, $X \in \widehat{\Omega}$ and $t \geqslant 0$. Then one can easily deduce the relation of derivatives for quantities in different configurations as

$$\partial_t \hat{\mathbf{u}}(X,t) = (\partial_t + \mathbf{v}(x,t)\cdot\nabla)\,\mathbf{u}(x,t), \tag{2.8}$$

$$\nabla\phi = \hat{\mathbf{F}}^{-1}\hat{\nabla}\hat{\phi}, \quad \nabla\mathbf{u} = \hat{\mathbf{F}}^{-1}\hat{\nabla}\hat{\mathbf{u}}, \tag{2.9}$$

$$\mathrm{div}\mathbf{u} = \mathrm{tr}(\nabla\mathbf{u}) = \mathrm{tr}(\hat{\mathbf{F}}^{-1}\hat{\nabla}\hat{\mathbf{u}}) = \hat{\mathbf{F}}^{-\top} : \hat{\nabla}\hat{\mathbf{u}}, \tag{2.10}$$

where $\phi/\hat{\phi}$ is any scalar function in $\Omega^t/\widehat{\Omega}$ and $\mathbf{u}/\hat{\mathbf{u}}$ is any vector-valued function in $\Omega^t/\widehat{\Omega}$. From [14], we know that the Piola transform establishes a correspondence between tensor field defined in deformed and reference configurations, which is

$$\hat{\mathbf{T}}(X,t) = \hat{J}(X,t)\boldsymbol{\sigma}(x,t)\hat{\mathbf{F}}^{-\top}(X,t), \quad \text{for all } x = \varphi(X,t),\ X \in \widehat{\Omega}, \tag{2.11}$$

where $\hat{\mathbf{T}}$ is the first Piola–Kirchhoff stress tensor. Moreover, the following property of the Piola transformation will be useful:

Lemma 2.5 [14, theorem 1.7-1]. *For a stress tensor $\boldsymbol{\sigma}(x,t)$ in the deformed configuration Ω, and the corresponding first Piola–Kirchhoff stress tensor $\hat{\mathbf{T}}(X,t)$ in the reference configuration $\widehat{\Omega}$, we have:*

$$\widehat{\mathrm{div}}\hat{\mathbf{T}}(X,t) = \hat{J}(X,t)\mathrm{div}\boldsymbol{\sigma}(x,t), \quad \textit{for all } x = \varphi(X,t),\ X \in \widehat{\Omega},$$
$$\hat{\mathbf{T}}(X,t)\hat{\mathbf{n}}\mathrm{d}\hat{a} = \boldsymbol{\sigma}(x,t)\mathbf{n}\mathrm{d}a, \quad \textit{for all } x = \varphi(X,t),\ X \in \widehat{\Omega}.$$

For the fluid part, it follows from (2.1) that

$$\partial_t \hat{J}_\mathrm{f} = \mathrm{tr}\left(\hat{\mathbf{F}}^{-1}\partial_t\hat{\mathbf{F}}\right)\hat{J}_\mathrm{f} = \mathrm{tr}\left(\hat{\mathbf{F}}^{-1}\hat{\nabla}\hat{\mathbf{v}}\right)\hat{J}_\mathrm{f} = \mathrm{div}\mathbf{v}\hat{J}_\mathrm{f} = 0,$$

which implies

$$\hat{J}_\mathrm{f} = \hat{J}_\mathrm{f}\Big|_{t=0} = \det\mathbb{I} = 1, \quad \text{in } \Omega_\mathrm{f}. \tag{2.12}$$

For the solid part, since the deformation from natural configuration Ω_s^g to the deformed configuration Ω_s^t conserves mass, incompressibility yields $\hat{J}_\mathrm{s}^e = 1$ and hence

$$\hat{J}_\mathrm{s} = \hat{J}_\mathrm{s}^g = \hat{g}^n, \quad \text{in } \Omega_\mathrm{s}.$$

Now combining formulas (1.4), (2.7)–(2.12) and lemma 2.5, we rewrite the fluid–structure interaction problem (1.1) in the reference configuration $\widehat{\Omega}$.

$$\left.\begin{aligned} \hat{\rho}_\mathrm{f}\partial_t\hat{\mathbf{v}}_\mathrm{f} - \widehat{\mathrm{div}}\left(\hat{\boldsymbol{\sigma}}_\mathrm{f}\hat{\mathbf{F}}_\mathrm{f}^{-\top}\right) &= 0 \\ \hat{\mathbf{F}}_\mathrm{f}^{-\top} : \hat{\nabla}\hat{\mathbf{v}}_\mathrm{f} &= 0 \\ \partial_t\hat{c}_\mathrm{f} - \hat{D}_\mathrm{f}\widehat{\mathrm{div}}\left(\hat{\mathbf{F}}_\mathrm{f}^{-1}\hat{\mathbf{F}}_\mathrm{f}^{-\top}\hat{\nabla}\hat{c}_\mathrm{f}\right) &= 0 \end{aligned}\right\} \quad \text{in } \Omega_\mathrm{f}\times(0,T), \tag{2.13}$$

$$\left.\begin{aligned} \hat{\rho}_\mathrm{s}\partial_t\hat{\mathbf{v}}_\mathrm{s} - \hat{J}_\mathrm{s}^{-1}\widehat{\mathrm{div}}\left(\hat{J}_\mathrm{s}\hat{\boldsymbol{\sigma}}_\mathrm{s}\hat{\mathbf{F}}_\mathrm{s}^{-\top}\right) &= 0 \\ \hat{\mathbf{F}}_\mathrm{s}^{-\top} : \hat{\nabla}\hat{\mathbf{v}}_\mathrm{s} - \frac{\gamma\beta}{\hat{\rho}_\mathrm{s}}\hat{c}_\mathrm{s} &= 0 \\ \partial_t\hat{c}_\mathrm{s} - \hat{D}_\mathrm{s}\hat{J}_\mathrm{s}^{-1}\widehat{\mathrm{div}}\left(\hat{J}_\mathrm{s}\hat{\mathbf{F}}_\mathrm{s}^{-1}\hat{\mathbf{F}}_\mathrm{s}^{-\top}\hat{\nabla}\hat{c}_\mathrm{s}\right) + \beta\hat{c}_\mathrm{s}\left(1 + \frac{\gamma}{\hat{\rho}_\mathrm{s}}\hat{c}_\mathrm{s}\right) &= 0 \\ \partial_t\hat{c}_\mathrm{s}^* - \beta\hat{c}_\mathrm{s} + \frac{\gamma\beta}{\hat{\rho}_\mathrm{s}}\hat{c}_\mathrm{s}\hat{c}_\mathrm{s}^* = 0, \quad \partial_t\hat{g} - \frac{\gamma\beta}{n\hat{\rho}_\mathrm{s}}\hat{c}_\mathrm{s}\hat{g} &= 0 \end{aligned}\right\} \quad \text{in } \Omega_\mathrm{s}\times(0,T), \tag{2.14}$$

$$\left.\begin{aligned} [\hat{\mathbf{v}}] = 0, \quad \left[\!\left[\hat{\boldsymbol{\sigma}}\hat{\mathbf{F}}^{-\top}\right]\!\right]\hat{\mathbf{n}}_\Gamma = 0, \quad \left[\!\left[\hat{D}\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}^{-\top}\hat{\nabla}\hat{c}\right]\!\right]\hat{\mathbf{n}}_\Gamma &= 0 \\ \zeta\,[\hat{c}] - \hat{D}_\mathrm{s}\hat{\mathbf{F}}_\mathrm{s}^{-1}\hat{\mathbf{F}}_\mathrm{s}^{-\top}\hat{\nabla}\hat{c}_\mathrm{s}\cdot\hat{\mathbf{n}}_\Gamma &= 0 \end{aligned}\right\} \quad \text{on } \Gamma\times(0,T), \tag{2.15}$$

$$\hat{\boldsymbol{\sigma}}_\mathrm{s}\hat{\mathbf{F}}_\mathrm{s}^{-\top}\hat{\mathbf{n}}_{\Gamma_\mathrm{s}} = 0, \quad \hat{D}_\mathrm{s}\hat{\mathbf{F}}_\mathrm{s}^{-1}\hat{\mathbf{F}}_\mathrm{s}^{-\top}\hat{\nabla}\hat{c}_\mathrm{s}\cdot\hat{\mathbf{n}}_{\Gamma_\mathrm{s}} = 0 \quad \text{on } \Gamma_\mathrm{s}\times(0,T), \tag{2.16}$$

$$\hat{\mathbf{v}}|_{t=0} = \hat{\mathbf{v}}^0, \quad \hat{c}|_{t=0} = \hat{c}^0 \quad \text{in } \widetilde{\Omega}, \tag{2.17}$$

$$\hat{c}_\mathrm{s}^*|_{t=0} = 0, \quad \hat{g}|_{t=0} = 1 \quad \text{in } \Omega_\mathrm{s}, \tag{2.18}$$

where the corresponding stress tensors are

$$\hat{\boldsymbol{\sigma}}_\mathrm{f} = -\hat{\pi}_\mathrm{f}\mathbb{I} + \hat{\nu}_\mathrm{f}\left(\hat{\mathbf{F}}_\mathrm{f}^{-1}\hat{\nabla}\hat{\mathbf{v}}_\mathrm{f} + \hat{\nabla}^\top\hat{\mathbf{v}}_\mathrm{f}\hat{\mathbf{F}}_\mathrm{f}^{-\top}\right), \quad \hat{\boldsymbol{\sigma}}_\mathrm{s} = \hat{\boldsymbol{\sigma}}_\mathrm{s}^e + \hat{\boldsymbol{\sigma}}_\mathrm{s}^v,$$
$$\hat{\boldsymbol{\sigma}}_\mathrm{s}^e = -\hat{\pi}_\mathrm{s}\mathbb{I} + \hat{\mu}_\mathrm{s}\left(\hat{\mathbf{F}}_\mathrm{s}^e\hat{\mathbf{F}}_\mathrm{s}^{e\top} - \mathbb{I}\right) = -\hat{\pi}_\mathrm{s}\mathbb{I} + \hat{\mu}_\mathrm{s}\left(\frac{1}{(\hat{g})^2}\hat{\mathbf{F}}_\mathrm{s}\hat{\mathbf{F}}_\mathrm{s}^\top - \mathbb{I}\right),$$
$$\hat{\boldsymbol{\sigma}}_\mathrm{s}^v = \hat{\nu}_\mathrm{s}\left(\hat{\nabla}\hat{\mathbf{v}}_\mathrm{s} + \hat{\nabla}^\top\hat{\mathbf{v}}_\mathrm{s}\right)\hat{\mathbf{F}}_\mathrm{s}^\top.$$

For the maximal L^q-regularity setting, we assume

$$\hat{\mathbf{v}}^0 \in B^{1-1/q}_{q,q}(\widehat{\Omega})^n \cap B^{2(1-1/q)}_{q,q}(\widetilde{\Omega})^n, \quad \hat{c}^0 \in B^{2(1-1/q)}_{q,q}(\widetilde{\Omega}),$$

that is,

$$\hat{\mathbf{v}}^0 \in W^{1-1/q}_{q}(\widehat{\Omega})^n \cap W^{2(1-1/q)}_{q}(\widetilde{\Omega})^n =: \mathcal{D}^1_q, \quad \hat{c}^0 \in W^{2(1-1/q)}_{q}(\widetilde{\Omega}) =: \mathcal{D}^2_q,$$

where we define $\widetilde{\Omega} = \Omega_{\mathrm{f}} \cup \Omega_{\mathrm{s}}$. $\mathcal{D}_q := \mathcal{D}^1_q \times \mathcal{D}^2_q$ will be the initial space for velocities and concentrations. Moreover, we introduce the compatibility conditions for $q > n+2$, which were also used in e.g. Abels [1], Prüss and Simonett [37], Shibata and Shimizu [41], Shimizu [42]:

$$\begin{aligned} &\widehat{\mathrm{div}}\hat{\mathbf{v}}^0 = 0, \quad [\![\hat{\mathbf{v}}^0]\!]\big|_\Gamma = 0, \quad \left[\!\!\left[\left(\hat{\nu}\left(\hat{\nabla}\hat{\mathbf{v}}^0 + \hat{\nabla}^\top\hat{\mathbf{v}}^0\right)\hat{\mathbf{n}}_\Gamma\right)_\tau\right]\!\!\right]\Big|_\Gamma = 0, \\ &\left(\hat{\nu}\left(\hat{\nabla}\hat{\mathbf{v}}^0 + \hat{\nabla}^\top\hat{\mathbf{v}}^0\right)\hat{\mathbf{n}}_{\Gamma_{\mathrm{s}}}\right)_\tau\Big|_{\Gamma_{\mathrm{s}}} = 0, \end{aligned} \tag{2.19}$$

and

$$\left(\zeta\,[\![\hat{c}^0]\!] - \hat{D}_{\mathrm{s}}\hat{\nabla}\hat{c}^0_{\mathrm{s}} \cdot \hat{\mathbf{n}}_\Gamma\right)\Big|_\Gamma = 0, \quad \left[\!\!\left[\hat{D}\hat{\nabla}\hat{c}^0\right]\!\!\right] \cdot \hat{\mathbf{n}}_\Gamma\Big|_\Gamma = 0, \quad \hat{D}_{\mathrm{s}}\hat{\nabla}\hat{c}^0_{\mathrm{s}} \cdot \hat{\mathbf{n}}_{\Gamma_{\mathrm{s}}}\Big|_{\Gamma_{\mathrm{s}}} = 0, \tag{2.20}$$

where $(\cdot)_\tau$ denotes the tangential part on the surface, namely, $(\cdot)_\tau = (\mathbb{I} - \hat{\mathbf{n}} \otimes \hat{\mathbf{n}})\cdot$. Besides this, we define the solution space for $(\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}^*_{\mathrm{s}}, \hat{g})$ as $Y_T = Y^1_T \times Y^2_T \times Y^3_T \times Y^4_T \times Y^4_T$, where

$$\begin{aligned} Y^1_T &= L^q\left(0,T; W^2_q(\widetilde{\Omega}) \cap W^1_q(\widehat{\Omega})\right)^n \cap W^1_q\left(0,T; L^q(\widehat{\Omega})\right)^n, \\ Y^2_T &= \left\{ \begin{aligned} &\hat{\pi} \in L^q\left(0,T; W^1_q(\widehat{\Omega})\right) : [\hat{\pi}] \in W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma \times (0,T)) \\ &\qquad\qquad \hat{\pi}|_{\Gamma_{\mathrm{s}}} \in W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma_{\mathrm{s}} \times (0,T)) \end{aligned} \right\}, \\ Y^3_T &= L^q\left(0,T; W^2_q(\widetilde{\Omega})\right) \cap W^1_q\left(0,T; L^q(\widehat{\Omega})\right), \\ Y^4_T &= W^1_q\left(0,T; W^1_q(\Omega_{\mathrm{s}})\right), \end{aligned}$$

equipped with norms

$$\begin{aligned} \|\hat{\mathbf{v}}\|_{Y^1_T} &= \|\hat{\mathbf{v}}\|_{L^q\left(0,T;W^2_q(\widetilde{\Omega})\cap W^1_{q,0}(\widehat{\Omega})\right)^n} + \|\hat{\mathbf{v}}\|_{W^1_q\left(0,T;L^q(\widehat{\Omega})\right)^n}, \\ \|\hat{\pi}\|_{Y^2_T} &= \|\hat{\pi}\|_{L^q\left(0,T;W^1_q(\widehat{\Omega})\right)} + \|[\hat{\pi}]\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma\times(0,T))} \\ &\quad + \left\|\hat{\pi}|_{\Gamma_{\mathrm{s}}}\right\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma_{\mathrm{s}}\times(0,T))}, \\ \|\hat{c}\|_{Y^3_T} &= \|\hat{c}\|_{L^q\left(0,T;W^2_q(\widetilde{\Omega})\right)} + \|\hat{c}\|_{W^1_q\left(0,T;L^q(\widehat{\Omega})\right)}, \\ \|\hat{c}^*_{\mathrm{s}}\|_{Y^4_T} &= \|\hat{c}^*_{\mathrm{s}}\|_{W^1_q\left(0,T;W^1_q(\Omega_{\mathrm{s}})\right)}, \quad \|\hat{g}\|_{Y^4_T} = \|\hat{g}\|_{W^1_q\left(0,T;W^1_q(\Omega_{\mathrm{s}})\right)}. \end{aligned}$$

Moreover, we set $Y^v_T := Y^1_T \times Y^2_T$.

Remark 2.1. These spaces are constructed from the problem and the maximal regularity theory, endowed with the natural norms. In particular, $[\hat{\pi}]$ and $\hat{\pi}|_{\Gamma_{\mathrm{s}}}$ are determined by the regularities of the Neumann trace of $\hat{\mathbf{v}}$ on Γ and Γ_{s} respectively. Hence, we add the norm of $\|[\hat{\pi}]\|_{W_q^{1-1/q,(1-1/q)/2}(\Gamma\times(0,T))}$ and $\left\|\hat{\pi}|_{\Gamma_{\mathrm{s}}}\right\|_{W_q^{1-1/q,(1-1/q)/2}(\Gamma_{\mathrm{s}}\times(0,T))}$ in Y^2_T-norm correspondingly. One can easily verify that all spaces are Banach spaces.

Now the main theorem is given as follows.

Theorem 2.1 (Main theorem). *Let $q > n+2$. Assume that Γ, Γ_s are hypersurfaces of class C^3, $(\hat{\mathbf{v}}^0, \hat{c}^0) \in \mathcal{D}_q$ such that the compatibility conditions (2.19) and (2.20) hold, then there is a positive $T_0 = T_0(\|(\hat{\mathbf{v}}^0, \hat{c}^0)\|_{\mathcal{D}_q}) < \infty$ such that there exists a unique strong solution $(\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s^*, \hat{g}) \in Y_{T_0}$ to system (2.13)–(2.18). Moreover, $\hat{c} \geqslant 0$ and $\hat{c}_s^*, \hat{g} > 0$, if $\hat{c}^0 \geqslant 0$.*

Remark 2.2. In this work, the boundary of the domain is supposed to be C^3. We remark here that if the domain is not smooth enough, for example, with boundary contact, it is still an open problem. The authors considered a similar model with ninety degree contact angles in [3] recently.

The proof of theorem 2.1 relies on the Banach fixed-point theorem. To this end, we need to linearise the nonlinear system (2.13)–(2.18). Since we consider a nonzero initial reference configuration, a standard perturbation method is applied to (2.13)–(2.18), for which we linearise the system at the initial deformation and move all reminder terms to the right-hand side, namely,

$$\left.\begin{aligned} \hat{\rho}_f \partial_t \hat{\mathbf{v}}_f - \widehat{\operatorname{div}} \mathbf{S}(\hat{\mathbf{v}}_f, \hat{\pi}_f) = \mathbf{K}_f \\ \widehat{\operatorname{div}} \hat{\mathbf{v}}_f = G_f \end{aligned}\right\} \quad \text{in } \Omega_f \times (0,T), \tag{2.21}$$

$$\left.\begin{aligned} \hat{\rho}_s \partial_t \hat{\mathbf{v}}_s - \widehat{\operatorname{div}} \mathbf{S}(\hat{\mathbf{v}}_s, \hat{\pi}_s) = \bar{\mathbf{K}}_s + \mathbf{K}_s^g =: \mathbf{K}_s \\ \widehat{\operatorname{div}} \hat{\mathbf{v}}_s - \frac{\gamma\beta}{\hat{\rho}_s} \hat{c}_s = G_s \end{aligned}\right\} \quad \text{in } \Omega_s \times (0,T), \tag{2.22}$$

$$[\hat{\mathbf{v}}] = 0, \quad [\mathbf{S}(\hat{\mathbf{v}}, \hat{\pi})]\, \hat{\mathbf{n}}_\Gamma = \mathbf{H}^1 \quad \text{on } \Gamma \times (0,T), \tag{2.23}$$

$$\mathbf{S}(\hat{\mathbf{v}}_s, \hat{\pi}_s) \hat{\mathbf{n}}_{\Gamma_s} = \mathbf{H}^2 \quad \text{on } \Gamma_s \times (0,T), \tag{2.24}$$

$$\hat{\mathbf{v}}|_{t=0} = \hat{\mathbf{v}}^0 \quad \text{in } \widetilde{\Omega}, \tag{2.25}$$

$$\partial_t \hat{c}_f - \hat{D}_f \widehat{\Delta} \hat{c}_f = F_f^1 \quad \text{in } \Omega_f \times (0,T), \tag{2.26}$$

$$\partial_t \hat{c}_s - \hat{D}_f \widehat{\Delta} \hat{c}_s = \bar{F}_s^1 + F_s^g =: F_s^1 \quad \text{in } \Omega_s \times (0,T), \tag{2.27}$$

$$\left.\begin{aligned} \hat{D}_f \hat{\nabla} \hat{c}_f \cdot \hat{\mathbf{n}}_\Gamma = \hat{D}_s \nabla \hat{c}_s \cdot \hat{\mathbf{n}}_\Gamma + \bar{F}_f^2 =: F_f^2 \\ \hat{D}_s \hat{\nabla} \hat{c}_s \cdot \hat{\mathbf{n}}_\Gamma = \zeta [\hat{c}] + \bar{F}_s^2 =: F_s^2 \end{aligned}\right\} \quad \text{on } \Gamma \times (0,T), \tag{2.28}$$

$$\hat{D}_s \hat{\nabla} \hat{c}_s \cdot \hat{\mathbf{n}}_{\Gamma_s} = F^3 \quad \text{on } \Gamma_s \times (0,T), \tag{2.29}$$

$$\hat{c}|_{t=0} = \hat{c}^0 \quad \text{in } \widetilde{\Omega}, \tag{2.30}$$

$$\partial_t \hat{c}_s^* - \beta \hat{c}_s = F^4 \quad \text{in } \Omega_s \times (0,T), \tag{2.31}$$

$$\hat{c}_s^*|_{t=0} = 0 \quad \text{in } \Omega_s, \tag{2.32}$$

$$\partial_t \hat{g} - \frac{\gamma\beta}{n\hat{\rho}_s} \hat{c}_s = F^5 \quad \text{in } \Omega_s \times (0,T), \tag{2.33}$$

$$\hat{g}|_{t=0} = 1 \quad \text{in } \Omega_s, \tag{2.34}$$

where $\mathbf{S}(\hat{\mathbf{v}},\hat{\pi}) = -\hat{\pi}\mathbb{I} + \hat{\nu}\left(\hat{\nabla}\hat{\mathbf{v}} + \hat{\nabla}^\top\hat{\mathbf{v}}\right)$ in $\widetilde{\Omega}$ and

$$\begin{aligned}
&\mathbf{K}_\mathrm{f} = \widehat{\operatorname{div}}\tilde{\mathbf{K}}_\mathrm{f}, \quad \bar{\mathbf{K}}_\mathrm{s} = \widehat{\operatorname{div}}\tilde{\mathbf{K}}_\mathrm{s}, \quad \mathbf{K}_\mathrm{s}^g = -\left(\hat{\boldsymbol{\sigma}}_\mathrm{s}\hat{\mathbf{F}}_\mathrm{s}^{-\top}\right)\frac{n\hat{\nabla}\hat{g}}{\hat{g}},\\
&G = -\left(\hat{\mathbf{F}}^{-\top} - \mathbb{I}\right) : \hat{\nabla}\hat{\mathbf{v}}, \quad \mathbf{H}^1 = -[\![\tilde{\mathbf{K}}]\!]\cdot\hat{\mathbf{n}}_\Gamma, \quad \mathbf{H}^2 = -\tilde{\mathbf{K}}_\mathrm{s}\cdot\hat{\mathbf{n}}_{\Gamma_\mathrm{s}},\\
&F_\mathrm{f}^1 = \widehat{\operatorname{div}}\tilde{F}_\mathrm{f}, \quad \bar{F}_\mathrm{s}^1 = \widehat{\operatorname{div}}\tilde{F}_\mathrm{s},\\
&F_\mathrm{s}^g = -\beta\hat{c}_\mathrm{s}\left(1 + \frac{\gamma}{\hat{\rho}_\mathrm{s}}\hat{c}_\mathrm{s}\right) - \frac{n\hat{\nabla}\hat{g}}{\hat{g}}\cdot\left(\hat{D}_\mathrm{s}\hat{\mathbf{F}}_\mathrm{s}^{-1}\hat{\mathbf{F}}_\mathrm{s}^{-\top}\hat{\nabla}\hat{c}_\mathrm{s}\right),\\
&\bar{F}_\mathrm{f}^2 = -[\![\tilde{F}]\!]\cdot\hat{\mathbf{n}}_\Gamma, \quad \bar{F}_\mathrm{s}^2 = -\tilde{F}_\mathrm{s}\cdot\hat{\mathbf{n}}_\Gamma, \quad F^3 = -\tilde{F}_\mathrm{s}\cdot\hat{\mathbf{n}}_{\Gamma_\mathrm{s}},\\
&F^4 = -\frac{\gamma\beta}{\hat{\rho}_\mathrm{s}}\hat{c}_\mathrm{s}\hat{c}_\mathrm{s}^*, \quad F^5 = -\frac{\gamma\beta}{n\hat{\rho}_\mathrm{s}}\hat{c}_\mathrm{s}(\hat{g}-1),
\end{aligned} \tag{2.35}$$

with

$$\begin{aligned}
\tilde{\mathbf{K}}_\mathrm{f} &= -\hat{\pi}_\mathrm{f}\left(\hat{\mathbf{F}}_\mathrm{f}^{-\top} - \mathbb{I}\right) + \nu_\mathrm{f}\left(\hat{\mathbf{F}}_\mathrm{f}^{-1}\hat{\nabla}\hat{\mathbf{v}}_\mathrm{f} + \hat{\nabla}^\top\hat{\mathbf{v}}_\mathrm{f}\hat{\mathbf{F}}_\mathrm{f}^{-\top}\right)\left(\hat{\mathbf{F}}_\mathrm{f}^{-\top} - \mathbb{I}\right)\\
&\quad + \nu_\mathrm{f}\left(\left(\hat{\mathbf{F}}_\mathrm{f}^{-1} - \mathbb{I}\right)\hat{\nabla}\hat{\mathbf{v}}_\mathrm{f} + \hat{\nabla}^\top\hat{\mathbf{v}}_\mathrm{f}\left(\hat{\mathbf{F}}_\mathrm{f}^{-\top} - \mathbb{I}\right)\right),\\
\tilde{\mathbf{K}}_\mathrm{s} &= -\hat{\pi}_\mathrm{s}\left(\hat{\mathbf{F}}_\mathrm{s}^{-\top} - \mathbb{I}\right) + \mu_\mathrm{s}\left(\frac{1}{\hat{g}^2}\left(\hat{\mathbf{F}}_\mathrm{s} - \mathbb{I}\right) + \left(\frac{1}{\hat{g}^2} - 1\right)\mathbb{I} - \left(\hat{\mathbf{F}}_\mathrm{s}^{-\top} - \mathbb{I}\right)\right),\\
\tilde{F} &= \hat{D}\left(\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}^{-\top} - \mathbb{I}\right)\hat{\nabla}\hat{c}.
\end{aligned}$$

Then we analyse system (2.21)–(2.34), which is exactly (2.13)–(2.18).

Remark 2.3. It follows from the Piola identity, which can be found in [14, page 39], that

$$\widehat{\operatorname{div}}\left(\hat{J}\hat{\mathbf{F}}^{-\top}\right) = 0.$$

Then from (2.3),

$$\hat{J}\hat{\mathbf{F}}^{-\top} : \hat{\nabla}\hat{\mathbf{v}} = \widehat{\operatorname{div}}\left(\hat{J}\hat{\mathbf{F}}^{-1}\hat{\mathbf{v}}\right).$$

Hence, G possesses the form

$$G_\mathrm{f} = -\widehat{\operatorname{div}}\left(\left(\hat{\mathbf{F}}_\mathrm{f}^{-1} - \mathbb{I}\right)\hat{\mathbf{v}}_\mathrm{f}\right), \quad G_\mathrm{s} = -\widehat{\operatorname{div}}\left(\left(\hat{\mathbf{F}}_\mathrm{s}^{-1} - \mathbb{I}\right)\hat{\mathbf{v}}_\mathrm{s}\right) + \hat{\mathbf{v}}_\mathrm{s}\cdot\widehat{\operatorname{div}}\hat{\mathbf{F}}_\mathrm{s}^{-\top}. \tag{2.36}$$

Remark 2.4. The system (2.26)–(2.30) for the concentrations of monocytes and macrophages can be considered as a transmission problem in Ω_f and Ω_s with a common boundary Γ. However, if we use the concentration and stress jump condition as boundary condition on Γ, we will meet a regularity problem due to the high order term $D_\mathrm{s}\hat{\nabla}\hat{c}_\mathrm{s}\cdot\hat{\mathbf{n}}_\Gamma$ in $(2.28)_2$. More precisely, in our further perturbation argument, all perturbated or unrelated terms will be moved to the right-hand side of the equation and the regularities of both sides should coincide. The point is that in such argument, the right-hand side of $(2.28)_2$ contains $D_\mathrm{s}\hat{\nabla}\hat{c}_\mathrm{s}\cdot\hat{\mathbf{n}}_\Gamma$, which leads to a lower regularity, provided the same regularity of $\hat{c}$ on the both side.

Therefore, to avoid such awkward situation, we rewrite the transmission conditions as two Neumann type boundary conditions. Then the transmission problem can be decoupled into two separate parabolic system, which are both imposed with Neumann boundary and defined in Ω_f and Ω_s respectively. This is why we treat the boundary conditions on Γ as the form shown in (2.28).

Consequently, given data $(\mathbf{K}, G, \mathbf{H}^1, \mathbf{H}^2, F^1, F^2, F^3, F^4, F^5)$ with suitable regularities, existence and uniqueness of $(\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s^*, \hat{g})$ in the associated spaces will be obtained by the well-posedness of linear systems in the next section.

3. Analysis of the linear systems

As seen in (2.21)–(2.34), the linearised system can be seen as a two-phase type Stokes problem (2.21)–(2.25), two separate reaction–diffusion systems (2.26)–(2.30) and two ordinary differential equations (2.31)–(2.34) (equation for foam cells and growth, respectively). In this section, thanks to the maximal L^q-regularity theory, we establish the existence for strong solutions to these systems with prescribed initial data and source terms in appropriate spaces.

3.1. Two-phase Stokes problems with Neumann boundary condition

Observing that $(\mathbf{K}, G, \mathbf{H}^1, \mathbf{H}^2)\big|_{t=0} = 0$, one replaces $(\mathbf{K}, G, \mathbf{H}^1, \mathbf{H}^2)$ in (2.21)–(2.25) by known functions $(\mathbf{k}, g, \mathbf{h}^1, \mathbf{h}^2)$ with $(\mathbf{k}, g, \mathbf{h}^1, \mathbf{h}^2)\big|_{t=0} = 0$ in (2.22). Then we get the problem addressed in this subsection.

$$
\begin{aligned}
\hat{\rho}\partial_t \hat{\mathbf{v}} - \widehat{\operatorname{div}} \mathbf{S}(\hat{\mathbf{v}}, \hat{\pi}) &= \mathbf{k} && \text{in } \widetilde{\Omega} \times (0, T), \\
\widehat{\operatorname{div}} \hat{\mathbf{v}} &= g && \text{in } \widetilde{\Omega} \times (0, T), \\
[\![\hat{\mathbf{v}}]\!] &= 0 && \text{on } \Gamma \times (0, T), \\
[\![\mathbf{S}(\hat{\mathbf{v}}, \hat{\pi})]\!]\, \hat{\mathbf{n}}_\Gamma &= \mathbf{h}^1 && \text{on } \Gamma \times (0, T), \\
\mathbf{S}(\hat{\mathbf{v}}_s, \hat{\pi}_s) \hat{\mathbf{n}}_{\Gamma_s} &= \mathbf{h}^2 && \text{on } \Gamma_s \times (0, T), \\
\hat{\mathbf{v}}|_{t=0} &= \hat{\mathbf{v}}^0 && \text{in } \widetilde{\Omega}.
\end{aligned}
\tag{3.1}
$$

Now, we will prove the following theorem, namely, existence of unique solution to a two-phase Stokes problem with outer Neumann boundary condition.

Theorem 3.1. *Let $q > n+2$, $T > 0$, $\widehat{\Omega}$ a bounded domain as before with $\Gamma_s \in C^3$, Γ a closed hypersurface of class C^3. Assume that $(\mathbf{k}, g, \mathbf{h}^1, \mathbf{h}^2)$ are known functions contained in Z_T^v with initial value zero and $\hat{\mathbf{v}}^0 \in \mathcal{D}_q^1$ with compatibility conditions*

$$
\widehat{\operatorname{div}} \hat{\mathbf{v}}^0 = g|_{t=0}, \quad [\![\hat{\mathbf{v}}^0]\!]\,|_\Gamma = 0, \quad \left[\!\!\left[\left(\hat{\nu}\left(\hat{\nabla}\hat{\mathbf{v}}^0 + \hat{\nabla}^\top \hat{\mathbf{v}}^0\right)\hat{\mathbf{n}}_\Gamma\right)_\tau\right]\!\!\right]\Big|_\Gamma = 0,
$$
$$
\left(\hat{\nu}\left(\hat{\nabla}\hat{\mathbf{v}}^0 + \hat{\nabla}^\top \hat{\mathbf{v}}^0\right)\hat{\mathbf{n}}_{\Gamma_s}\right)_\tau\Big|_{\Gamma_s} = 0.
$$

Then the Stokes problem (3.1) admits a unique strong solution $(\hat{\mathbf{v}}, \hat{\pi})$ in Y_T^v. Moreover, there exist a time $T_0 > 0$ and a constant $C = C(T_0) > 0$ such that for $0 < T \leqslant T_0$,

$$
\|(\hat{\mathbf{v}}, \hat{\pi})\|_{Y_T^v} \leqslant C \left\|(\mathbf{k}, g, \mathbf{h}^1, \mathbf{h}^2, \hat{\mathbf{v}}^0)\right\|_{Z_T^v \times \mathcal{D}_q^1}, \tag{3.2}
$$

where $Z_T^v := Z_T^1 \times Z_T^2 \times Z_T^3 \times Z_T^4$ with

$$
Z_T^1 := L^q\left(0, T; L^q(\widetilde{\Omega})\right)^n, \tag{3.3}
$$

$$
Z_T^2 := \left\{
\begin{aligned}
& g \in L^q\left(0, T; W_q^1(\widetilde{\Omega})\right) \cap W_q^1\left(0, T; W_q^{-1}(\widehat{\Omega})\right) : \\
& \qquad \operatorname{tr}_\Gamma(g) \in W_q^{1-\frac{1}{q}, \frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma \times (0, T)), \\
& \qquad \operatorname{tr}_{\Gamma_s}(g) \in W_q^{1-\frac{1}{q}, \frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma_s \times (0, T))
\end{aligned}
\right\}, \tag{3.4}
$$

$$Z_T^3 := W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma \times (0,T))^n, \quad Z_T^4 := W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma_s \times (0,T))^n, \tag{3.5}$$

endowed with the norms

$$\begin{aligned}
\|\mathbf{k}\|_{Z_T^1} &= \|\mathbf{k}\|_{L^q\left(0,T;L^q(\widetilde{\Omega})\right)^n},\\
\|g\|_{Z_T^2} &= \|g\|_{L^q\left(0,T;W_q^1(\widetilde{\Omega})\right)} + \|g\|_{W_q^1\left(0,T;W_q^{-1}(\widehat{\Omega})\right)}\\
&\quad + \|\mathrm{tr}_\Gamma(g)\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma\times(0,T))} + \|\mathrm{tr}_{\Gamma_s}(g)\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma_s\times(0,T))},\\
\left\|\mathbf{h}^1\right\|_{Z_T^3} &= \|\mathbf{h}\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma\times(0,T))^n}, \quad \left\|\mathbf{h}^2\right\|_{Z_T^4} = \|\mathbf{h}\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma_s\times(0,T))^n}.
\end{aligned}$$

3.1.1. Reductions. To simplify the proof of theorem 3.1, we reduce (3.1) to the case $(\mathbf{k}, g, \hat{\mathbf{v}}^0) = 0$.

First of all, we define $\bar{\mathbf{v}}$ as the solution of the parabolic transmission problem

$$\begin{aligned}
\hat{\rho}_f\partial_t\bar{\mathbf{v}} - \widehat{\mathrm{div}}\mathbf{S}(\bar{\mathbf{v}},0) &= \mathbf{k} && \text{in } \widetilde{\Omega}\times(0,T),\\
[\bar{\mathbf{v}}] &= 0 && \text{on } \Gamma\times(0,T),\\
[\mathbf{S}(\bar{\mathbf{v}},0)]\,\hat{\mathbf{n}}_\Gamma &= 0 && \text{on } \Gamma\times(0,T),\\
\mathbf{S}(\bar{\mathbf{v}}_s,0)\hat{\mathbf{n}}_{\Gamma_s} &= 0 && \text{on } \Gamma_s\times(0,T),\\
\bar{\mathbf{v}}|_{t=0} &= \hat{\mathbf{v}}^0 && \text{in } \widetilde{\Omega},
\end{aligned} \tag{3.6}$$

with $\mathbf{k} \in L^q(\widetilde{\Omega}\times(0,T))$ and $\hat{\mathbf{v}}^0 \in \mathcal{D}_q^1$. Since the Lopatinskii–Shapiro conditions are satisfied, (3.6) is uniquely solvable in $W_q^{2,1}(\widehat{\Omega}\times(0,T))$, thanks to [37, theorem 6.5.1].

Now, we are in the position to reduce g to zero. To this end, we introduce a elliptic transmission problem with Dirichlet boundary

$$\begin{aligned}
\widehat{\Delta}\phi = g - \widehat{\mathrm{div}}\bar{\mathbf{v}} &=: \tilde{g} && \text{in } \widetilde{\Omega},\\
[\hat{\rho}\phi] &= 0 && \text{on } \Gamma,\\
\left[\!\left[\hat{\nabla}\phi\right]\!\right]\cdot\hat{\mathbf{n}}_\Gamma &= 0 && \text{on } \Gamma,\\
\hat{\rho}_s\phi_s &= 0 && \text{on } \Gamma_s,
\end{aligned} \tag{3.7}$$

with $\tilde{g} \in L^q(\widetilde{\Omega})$. Then (3.7) is uniquely solvable by proposition A.3. In addition, with the regularity of g and $\mathbf{v}$, the solution satisfies $\hat{\nabla}\phi \in Y_T^1$. Employing the decomposition

$$(\hat{\mathbf{v}}, \hat{\pi}) = (\bar{\mathbf{v}} + \hat{\nabla}\phi + \tilde{\mathbf{v}}, -\hat{\rho}\partial_t\phi + \hat{\nu}\widehat{\Delta}\phi + \tilde{\pi}), \tag{3.8}$$

we know that $(\tilde{\mathbf{v}}, \tilde{\pi})$ solves system (3.1) with $(\mathbf{k}, g, \hat{\mathbf{v}}^0) = 0$ and modified nonvanishing data $(\mathbf{h}^1, \mathbf{h}^2)$ (not to be relabeled) in the right regularity classes having a vanishing trace at $t = 0$. Thus, we will focus on the reduced system in the case $(\mathbf{k}, g, \hat{\mathbf{v}}^0) = 0$.

Remark 3.1. Because of the decomposition (3.8), the regularity of $\hat{\pi}$ given in Y_T^2 indicates that $\partial_t\phi$ and $\widehat{\Delta}\phi$ must be contained in Y_T^2. Since $\hat{\nabla}\phi \in Y_T^1 = L^q(0,T;W_q^2(\widetilde{\Omega})\cap W_q^1(\widehat{\Omega}))^n \cap W_q^1(0,T;L^q(\widehat{\Omega}))^n$, it is clear that $\partial_t\phi, \widehat{\Delta}\phi \in L^q(0,T;W_q^1(\widetilde{\Omega}))$. Moreover:

(a) The Vanishing Dirichlet boundary conditions of ϕ on Γ and Γ_s lead to $[\partial_t\phi]|_\Gamma = \partial_t\phi|_{\Gamma_s} = 0$, which naturally satisfy the boundary regularity $W_q^{1-1/q,(1-1/q)/2}(\Gamma\times(0,T))$ and $W_q^{1-1/q,(1-1/q)/2}(\Gamma_s\times(0,T))$. Hence $\partial_t\phi \in Y_T^2$.

(b) For $\widehat{\Delta}\phi = \tilde{g} = g - \widehat{\mathrm{div}}\bar{\mathbf{v}}$, the boundary regularity for $\widehat{\mathrm{div}}\bar{\mathbf{v}}$ is not a problem due to the zero Neumann boundary of $\bar{\mathbf{v}}$. Thus, to ensure the validation of the regularity for $\hat{\pi}$, we add trace regularities on Γ and Γ_s for g in Z_T^2. Namely,

$$\mathrm{tr}_\Gamma(g) \in W_q^{1-1/q,(1-1/q)/2}(\Gamma \times (0,T)), \quad \mathrm{tr}_{\Gamma_s}(g) \in W_q^{1-1/q,(1-1/q)/2}(\Gamma_s \times (0,T)).$$

Consequently, $\widehat{\Delta}\phi \in Y_T^2$.

3.1.2. Proof of theorem 3.1. As stated in the last section, we analyse the reduced system of (3.1) with $(\mathbf{k}, g, \hat{\mathbf{v}}^0) = 0$. Due to the outer Neumann boundary condition, the proof is proceeded by a truncation (localization) argument, based on the results given in appendix A. More precisely, with a suitable cutoff function, we decompose the system into a two-phase Stokes problem with Dirichlet boundary conditions and a one-phase nonstationary Stokes problem, which are uniquely solvable as in section A.1 and Abels [2, theorem 1.1] respectively.

Proof (Proof of theorem 3.1). ***Step 1.*** The first step is finding $(\hat{\mathbf{v}}^1, \hat{\pi}^1)$ to solve

$$\begin{aligned}
\hat{\rho}\partial_t\hat{\mathbf{v}}^1 - \widehat{\mathrm{div}}\mathbf{S}(\hat{\mathbf{v}}^1, \hat{\pi}^1) &= 0 && \text{in } \widetilde{\Omega} \times (0,T),\\
\widehat{\mathrm{div}}\hat{\mathbf{v}}^1 &= 0 && \text{in } \widetilde{\Omega} \times (0,T),\\
[\![\hat{\mathbf{v}}^1]\!] &= 0 && \text{on } \Gamma \times (0,T),\\
[\![\mathbf{S}(\hat{\mathbf{v}}^1, \hat{\pi}^1)]\!]\, \hat{\mathbf{n}}_\Gamma &= \mathbf{h}^1 && \text{on } \Gamma \times (0,T),\\
\hat{\mathbf{v}}^1 &= 0 && \text{on } \Gamma_s \times (0,T),\\
\hat{\mathbf{v}}^1|_{t=0} &= 0 && \text{in } \widetilde{\Omega},
\end{aligned} \tag{3.9}$$

where $\mathbf{h}^1 \in Z_T^3$ with $\mathbf{h}^1|_{t=0} = 0$. Since $\hat{\mathbf{v}}^1|_{t=0} = 0$, the compatibility conditions (A.2) hold true and then (3.9) admits a unique solution $(\hat{\mathbf{v}}^1, \hat{\pi})$ in Y_T^v, thanks to proposition A.1. In addition, we have the estimate

$$\|(\hat{\mathbf{v}}^1, \hat{\pi}^1)\|_{Y_T^v} \leqslant C\|\mathbf{h}^1\|_{Z_T^3}, \tag{3.10}$$

for some $C>0$ independent of $\hat{\mathbf{v}}^1, \hat{\pi}^1, \mathbf{h}^1$.

Step 2. Now, we construct $(\hat{\mathbf{v}}_s^2, \hat{\pi}_s^2)$ to solve the Stokes problem with Neumann boundary condition, which reads

$$\begin{aligned}
\hat{\rho}_s\partial_t\hat{\mathbf{v}}_s^2 - \widehat{\mathrm{div}}\mathbf{S}(\hat{\mathbf{v}}_s^2, \hat{\pi}_s^2) &= 0 && \text{in } \Omega_s \times (0,T),\\
\widehat{\mathrm{div}}\hat{\mathbf{v}}_s^2 &= 0 && \text{in } \Omega_s \times (0,T),\\
\mathbf{S}(\hat{\mathbf{v}}_s^2, \hat{\pi}_s^2)\hat{\mathbf{n}}_\Gamma &= 0 && \text{on } \Gamma \times (0,T),\\
\mathbf{S}(\hat{\mathbf{v}}_s^2, \hat{\pi}_s^2)\hat{\mathbf{n}}_{\Gamma_s} &= \mathbf{h}^2 && \text{on } \Gamma_s \times (0,T),\\
\hat{\mathbf{v}}^2|_{t=0} &= 0 && \text{in } \Omega_s,
\end{aligned} \tag{3.11}$$

where $\mathbf{h}^2 \in Z_T^4$ with $\mathbf{h}^2|_{t=0} = 0$. Thanks to theorem 1.1 in Abels [2] with $\Gamma_1 = \emptyset$, (3.11) admits a unique solution $(\hat{\mathbf{v}}^2, \hat{\pi}^2)$ in $W_q^{2,1}(\widetilde{\Omega}) \times L^q(0,T; W_q^1(\widetilde{\Omega}))$. Due to $\hat{\mathbf{v}}_s^2|_{t=0} = 0$, all the compatibility conditions are satisfied. Moreover,

$$\|(\hat{\mathbf{v}}^2, \hat{\pi}^2)\|_{W_q^{2,1}(\widetilde{\Omega}) \times L^q(0,T;W_q^1(\widetilde{\Omega}))} \leqslant C\|\mathbf{h}^2\|_{Z_T^4}, \tag{3.12}$$

for some $C>0$ independent of $\hat{\mathbf{v}}^2, \hat{\pi}^2, \mathbf{h}^2$.

Step 3. Finally, we combine the regularity results above by truncation. Specifically, let $\psi \in C_0^\infty(\widehat{\Omega})$ be a cutoff function over $\widehat{\Omega}$ such that

$$\psi(x) = \begin{cases} 1, & \text{in a neighborhood of } \Omega_{\mathrm{f}}, \\ 0, & \text{in a neighborhood of } \Gamma_{\mathrm{s}}. \end{cases} \tag{3.13}$$

We define

$$\tilde{\mathbf{v}} := \psi\hat{\mathbf{v}}^1 + (1-\psi)\hat{\mathbf{v}}^2, \quad \tilde{\pi} := \psi\hat{\pi}^1 + (1-\psi)\hat{\pi}^2.$$

Then $(\tilde{\mathbf{v}}, \tilde{\pi}) \in Y_T^\nu$ solves

$$\begin{aligned} \hat{\rho}\partial_t\tilde{\mathbf{v}} - \widehat{\operatorname{div}}\mathbf{S}(\tilde{\mathbf{v}}, \tilde{\pi}) &= \mathbf{R}^1 && \text{in } \widetilde{\Omega} \times (0,T), \\ \widehat{\operatorname{div}}\tilde{\mathbf{v}} &= R^2 && \text{in } \widetilde{\Omega} \times (0,T), \\ [\tilde{\mathbf{v}}] &= 0 && \text{on } \Gamma \times (0,T), \\ [\mathbf{S}(\tilde{\mathbf{v}}, \tilde{\pi})]\,\hat{\mathbf{n}}_\Gamma &= \mathbf{h}^1 && \text{on } \Gamma \times (0,T), \\ \mathbf{S}(\tilde{\mathbf{v}}_{\mathrm{s}}, \tilde{\pi}_{\mathrm{s}})\hat{\mathbf{n}}_{\Gamma_{\mathrm{s}}} &= \mathbf{h}^2 && \text{on } \Gamma_{\mathrm{s}} \times (0,T), \\ \tilde{\mathbf{v}}|_{t=0} &= 0 && \text{in } \widetilde{\Omega}, \end{aligned} \tag{3.14}$$

where $\mathbf{R}^1$ and R^2 vanish in Ω_{f}, while in Ω_{s},

$$\begin{aligned} \mathbf{R}^1 &= -\mathbf{S}(\hat{\mathbf{v}}_{\mathrm{s}}^1 - \hat{\mathbf{v}}_{\mathrm{s}}^2, \hat{\pi}_{\mathrm{s}}^1 - \hat{\pi}_{\mathrm{s}}^2)\hat{\nabla}\psi \\ &\quad - 2\hat{\nu}_{\mathrm{s}}\left(\widehat{\Delta}\psi\left(\hat{\mathbf{v}}_{\mathrm{s}}^1 - \hat{\mathbf{v}}_{\mathrm{s}}^2\right) + \left(\hat{\nabla}\hat{\mathbf{v}}_{\mathrm{s}}^1 - \hat{\nabla}\hat{\mathbf{v}}_{\mathrm{s}}^2\right)\hat{\nabla}\psi + \hat{\nabla}^2\psi\left(\hat{\mathbf{v}}_{\mathrm{s}}^1 - \hat{\mathbf{v}}_{\mathrm{s}}^2\right)\right), \\ R^2 &= \hat{\nabla}\psi \cdot \left(\hat{\mathbf{v}}_{\mathrm{s}}^1 - \hat{\mathbf{v}}_{\mathrm{s}}^2\right). \end{aligned}$$

Since the embedding

$${}_0W_q^{2,1}\left(\Omega_{\mathrm{s}} \times (0,T)\right) \hookrightarrow {}_0W_q^{\frac{1}{2}}\left(0,T; W_q^1(\Omega_{\mathrm{s}})\right)$$

holds, we know $\hat{\mathbf{v}}^i \in {}_0W_q^{\frac{1}{2}}(0,T; W_q^1(\Omega_{\mathrm{s}}))$, $i = 1,2$. For the reduced system, propositions 8.2.1 and 7.3.5 in Prüss and Simonett [37] imply that $\hat{\pi}^1$ and $\hat{\pi}_{\mathrm{s}}^2$ enjoys extra time regularities $\hat{\pi}^1 \in {}_0W_q^\alpha(0,T; L^q(\widehat{\Omega}))$ and $\hat{\pi}_{\mathrm{s}}^2 \in {}_0W_q^\alpha(0,T; L^q(\Omega_{\mathrm{s}}))$ respectively for $0 < \alpha < \frac{1}{2}\left(1 - \frac{1}{q}\right)$. Hence

$$\mathbf{R}^1 \in {}_0W_q^\alpha\left(0,T; L^q(\Omega_{\mathrm{s}})\right) \cap L^q\left(0,T; W_q^1(\Omega_{\mathrm{s}})\right),$$

for some fixed $0 < \alpha < \frac{1}{2}\left(1 - \frac{1}{q}\right)$.

To complete the proof, we still need to prove that the right-hand side terms of (3.14) can be in fact substituted by the right-hand side terms of (3.1) in appropriate spaces. Since the regularity of $\hat{\mathbf{v}}_{\mathrm{s}}^i$ and $\hat{\pi}_{\mathrm{s}}^i$, $i = 1,2$, are not enough to control $\mathbf{R}^1$ and R^2 for small times, we are going to remove the inhomogeneities $\mathbf{R}^1$ and R^2. For $\mathbf{R}^1$, we construct a $\bar{\phi}$ solving the problem

$$\begin{aligned} \bar{\phi}_{\mathrm{f}} &= 0 && \text{in } \Omega_{\mathrm{f}}, \\ \widehat{\Delta}\bar{\phi}_{\mathrm{s}} &= \widehat{\operatorname{div}}\mathbf{R}^1 && \text{in } \Omega_{\mathrm{s}}, \\ \bar{\phi}_{\mathrm{s}} &= 0 && \text{on } \Gamma, \\ \bar{\phi}_{\mathrm{s}} &= 0 && \text{on } \Gamma_{\mathrm{s}}. \end{aligned} \tag{3.15}$$

Then we obtain $\hat{\nabla}\bar{\phi}\Big|_{t=0} = \mathbf{R}^1\big|_{t=0} = 0$. By elliptic theory and regularity of $\mathbf{R}^1$, (3.15) admits a unique solution $\bar{\phi}$ satisfying ${}_0W_q^\alpha(0,T;W_q^1(\Omega_\mathrm{s}))\cap L^q(0,T;W_q^2(\Omega_\mathrm{s}))$. For R^2, we find a ϕ solving the elliptic transmission problem

$$\begin{aligned} \widehat{\Delta}\phi_\mathrm{f} &= 0 && \text{in } \Omega_\mathrm{f},\\ \widehat{\Delta}\phi_\mathrm{s} &= R^2 && \text{in } \Omega_\mathrm{s},\\ [\hat{\rho}\phi] &= 0 && \text{on } \Gamma,\\ \left[\!\left[\hat{\nabla}\phi\right]\!\right]\cdot\hat{\mathbf{n}}_\Gamma &= 0 && \text{on } \Gamma,\\ \hat{\rho}_\mathrm{s}\phi_\mathrm{s} &= 0 && \text{on } \Gamma_\mathrm{s}. \end{aligned} \tag{3.16}$$

Then we have $\phi|_{t=0} = 0$. Since $\hat{\mathbf{v}}_\mathrm{s}^1 - \hat{\mathbf{v}}_\mathrm{s}^2 \in {}_0W_q^{2,1}(\Omega_\mathrm{s}\times(0,T))^n$, $R^2 \in {}_0W_q^{2,1}(\Omega_\mathrm{s}\times(0,T)) \hookrightarrow {}_0W_q^{\frac{1}{2}}(0,T;W_q^1(\Omega_\mathrm{s}))$. Together with proposition A.3, one concludes that (3.16) admits a solution such that $\hat{\nabla}\phi$ is unique, with regularity

$$\hat{\nabla}\phi \in E_0 := {}_0W_q^1(0,T;W_q^1(\widetilde{\Omega}))^n \cap {}_0W_q^{\frac{1}{4}}(0,T;W_q^2(\widetilde{\Omega}))^n.$$

For its traces on Γ and Γ_s, we have

$$\begin{aligned} \left[\!\left[\hat{\nabla}\phi\right]\!\right] &\in E_1 := {}_0W_q^1\left(0,T;W_q^{1-\frac{1}{q}}(\Gamma)\right)^n \cap {}_0W_q^{\frac{1}{4}}\left(0,T;W_q^{2-\frac{1}{q}}(\Gamma)\right)^n,\\ \hat{\nabla}\phi_\mathrm{s} &\in E_1^s := {}_0W_q^1\left(0,T;W_q^{1-\frac{1}{q}}(\Gamma)\right)^n \cap {}_0W_q^{\frac{1}{4}}\left(0,T;W_q^{2-\frac{1}{q}}(\Gamma_\mathrm{s})\right)^n. \end{aligned}$$

Besides,

$$\begin{aligned} \left[\!\left[\hat{\nu}\hat{\nabla}^2\phi\right]\!\right] &\in E_2 := {}_0W_q^{1-\frac{1}{2q}}(0,T;L^q(\Gamma))^{n\times n} \cap {}_0W_q^{\frac{1}{4}}\left(0,T;W_q^{1-\frac{1}{q}}(\Gamma)\right)^{n\times n},\\ \hat{\nu}_\mathrm{s}\hat{\nabla}^2\phi_\mathrm{s} &\in E_2^s := {}_0W_q^{1-\frac{1}{2q}}(0,T;L^q(\Gamma_\mathrm{s}))^{n\times n} \cap {}_0W_q^{\frac{1}{4}}\left(0,T;W_q^{1-\frac{1}{q}}(\Gamma)\right)^{n\times n}. \end{aligned}$$

Moreover, the following estimate holds for a constant C, independent of $0 < T < T_0$,

$$\begin{aligned} &\left\|\hat{\nabla}\phi\right\|_{E_0} + \left\|\left[\!\left[\hat{\nabla}\phi\right]\!\right]\right\|_{E_1} + \left\|\hat{\nabla}\phi_\mathrm{s}\right\|_{E_1^s} + \left\|\left[\!\left[\hat{\nu}\hat{\nabla}^2\phi\right]\!\right]\right\|_{E_2} + \left\|\hat{\nu}_\mathrm{s}\hat{\nabla}^2\phi\right\|_{E_2^s}\\ &\quad\leqslant C\left\|\hat{\mathbf{v}}_\mathrm{s}^1 - \hat{\mathbf{v}}_\mathrm{s}^2\right\|_{W_q^{2,1}(\Omega_\mathrm{s}\times(0,T))^n}. \end{aligned}$$

Finally, define

$$\mathbf{v}^\sharp := \tilde{\mathbf{v}} - \hat{\nabla}\phi, \quad \pi^\sharp := \tilde{\pi} + \hat{\rho}\partial_t\phi - \bar{\phi} - 2\hat{\nu}\widehat{\Delta}\phi.$$

Since $[\hat{\rho}\phi]|_\Gamma$, $\hat{\rho}_\mathrm{s}\phi|_{\Gamma_\mathrm{s}} = 0$, we have $[\hat{\rho}\partial_t\phi]|_\Gamma$, $\hat{\rho}_\mathrm{s}\partial_t\phi|_{\Gamma_\mathrm{s}} = 0$. Then $(\mathbf{v}^\sharp, \pi^\sharp)$ solves

$$\begin{aligned} \hat{\rho}\partial_t\mathbf{v}^\sharp - \widehat{\operatorname{div}}\mathbf{S}(\mathbf{v}^\sharp,\pi^\sharp) &= \mathbf{R}^1 - \hat{\nabla}\bar{\phi} =: \mathbf{R}^0 && \text{in } \widetilde{\Omega}\times(0,T),\\ \widehat{\operatorname{div}}\mathbf{v}^\sharp &= 0 && \text{in } \widetilde{\Omega}\times(0,T),\\ \left[\!\left[\mathbf{v}^\sharp\right]\!\right] &= \mathbf{R}' && \text{on } \Gamma\times(0,T),\\ \left[\!\left[\mathbf{S}(\mathbf{v}^\sharp,\pi^\sharp)\right]\!\right]\hat{\mathbf{n}}_\Gamma &= \mathbf{h}^1 + \mathbf{R}^3 && \text{on } \Gamma\times(0,T),\\ \mathbf{S}(\mathbf{v}_\mathrm{s}^\sharp,\pi_\mathrm{s}^\sharp)\hat{\mathbf{n}}_{\Gamma_\mathrm{s}} &= \mathbf{h}^2 + \mathbf{R}^4 && \text{on } \Gamma_\mathrm{s}\times(0,T),\\ \mathbf{v}^\sharp\big|_{t=0} &= 0 && \text{in } \widetilde{\Omega}, \end{aligned} \tag{3.17}$$

where

$$\widehat{\operatorname{div}}\mathbf{R}^0 = 0, \quad \mathbf{R}' = -\left[\!\left[\hat{\nabla}\phi\right]\!\right]$$
$$\mathbf{R}^3 = \left[\!\left[2\hat{\nu}\hat{\nabla}^2\phi\right]\!\right]\hat{\mathbf{n}}_\Gamma - \left[\!\left[2\hat{\nu}_s\widehat{\Delta}\phi\right]\!\right]\hat{\mathbf{n}}_\Gamma \quad \mathbf{R}^4 = 2\hat{\nu}_s\hat{\nabla}^2\phi_s\hat{\mathbf{n}}_{\Gamma_s} - 2\hat{\nu}_s\widehat{\Delta}\phi_s\hat{\mathbf{n}}_{\Gamma_s}.$$

$\mathbf{R}^0$ can be seen as a Helmholtz projection of $\mathbf{R}^1$ and

$$\mathbf{R}^0 \in {}_0W_q^\alpha(0,T;L^q(\Omega_s))^n \cap L^q(0,T;W_q^{2\alpha}(\Omega_s))^n, \quad \text{for all } 0<\alpha<\frac{1}{2}-\frac{1}{2q}.$$

By lemma 2.4,

$$\mathbf{R}^0 \in C\left([0,T];W_q^{2\alpha-\frac{2}{q}}(\Omega_s)\right)^n \hookrightarrow C([0,T];L^q(\Omega_s))^n$$

holds for $\frac{1}{q}<\alpha<\frac{1}{2}-\frac{1}{2q}$. Hence, for $\mathbf{R}^0|_{t=0} = (\hat{\nabla}\bar{\phi}-\mathbf{R}^1)\Big|_{t=0} = 0$,

$$\begin{aligned}\|\mathbf{R}^0\|_{Z_T^1} &\leqslant CT^{\frac{1}{q}}\|\mathbf{R}^0\|_{C([0,T];L^q(\Omega_s))^n} \leqslant CT^{\frac{1}{q}}\|\mathbf{R}^0\|_{{}_0W_q^\alpha(0,T;L^q(\Omega_s))^n\cap L^q(0,T;W_q^{2\alpha}(\Omega_s))^n}\\ &\leqslant CT^{\frac{1}{q}}\left(\max_{i=1,2}\|(\hat{\mathbf{v}}^i,\hat{\pi}^i)\|_{Y_T^v}\right) \leqslant CT^{\frac{1}{q}}\|(\mathbf{k},g,\mathbf{h}^1,\mathbf{h}^2,\hat{\mathbf{v}}^0)\|_{Z_T^v\times\mathcal{D}_q^1},\end{aligned}$$

for $0<T<T_0$. According to appendix A, the regularity space of $\mathbf{R}'$ is defined as $Z_T' := W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\Gamma\times(0,T))$. Then with lemma 2.2 and $W_q^s(0,T;X)\hookrightarrow C([0,T];X)$ for $sq>1$,

$$\begin{aligned}\|\mathbf{R}'\|_{Z_T'} &\leqslant C\Bigg(\left\|\left[\!\left[\hat{\nabla}\phi\right]\!\right]\right\|_{L^q\left(0,T;W_q^{2-\frac{1}{q}}(\Gamma)\right)^n}\\ &\quad+\left\|\left[\!\left[\hat{\nabla}\phi\right]\!\right]\right\|_{L^q(0,T;L^q(\Gamma))^n} + \left[\left[\!\left[\hat{\nabla}\phi\right]\!\right]\right]_{W_q^{1-\frac{1}{2q}}(0,T;L^q(\Gamma))^n}\Bigg)\\ &\leqslant CT^{\frac{1}{q}}\left\|\left[\!\left[\hat{\nabla}\phi\right]\!\right]\right\|_{{}_0W_q^{\frac{1}{4}}\left(0,T;W_q^{2-\frac{1}{q}}(\Gamma)\right)^n} + CT^{\frac{1}{2q}}\left[\left[\!\left[\hat{\nabla}\phi\right]\!\right]\right]_{{}_0W_q^1\left(0,T;W_q^{1-\frac{1}{q}}(\Gamma)\right)^n}\\ &\leqslant CT^{\frac{1}{2q}}\left(\max_{i=1,2}\left\|(\hat{\mathbf{v}}^i,\hat{\pi}^i)\right\|_{Y_T^v}\right) \leqslant CT^{\frac{1}{2q}}\left\|(\mathbf{k},g,\mathbf{h}^1,\mathbf{h}^2,\hat{\mathbf{v}}^0)\right\|_{Z_T^v\times\mathcal{D}_q^1}.\end{aligned}$$

Since Γ and Γ_s are of class C^3, $\hat{\mathbf{n}}_\Gamma$ and $\hat{\mathbf{n}}_{\Gamma_s}$ are contained in C^2. Then we obtain

$$\begin{aligned}\left\|\mathbf{R}^3\right\|_{Z_T^3} &\leqslant C\Bigg(\left\|\left[\!\left[\hat{\nabla}^2\phi\right]\!\right]\right\|_{L^q\left(0,T;W_q^{1-\frac{1}{q}}(\Gamma)\right)^{n\times n}}\\ &\quad+\left\|\left[\!\left[\hat{\nabla}^2\phi\right]\!\right]\right\|_{L^q(0,T;L^q(\Gamma))^{n\times n}} + \left[\left[\!\left[\hat{\nabla}^2\phi\right]\!\right]\right]_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}(0,T;L^q(\Gamma))^{n\times n}}\Bigg)\\ &\leqslant CT^{\frac{1}{q}}\left\|\left[\!\left[\hat{\nabla}^2\phi\right]\!\right]\right\|_{{}_0W_q^{\frac{1}{4}}\left(0,T;W_q^{1-\frac{1}{q}}(\Gamma)\right)^{n\times n}} + CT^{\frac{1}{2}}\left[\left[\!\left[\hat{\nabla}^2\phi\right]\!\right]\right]_{{}_0W_q^{1-\frac{1}{2q}}(0,T;L^q(\Gamma))^{n\times n}}\\ &\leqslant CT^{\frac{1}{q}}\left(\max_{i=1,2}\left\|(\hat{\mathbf{v}}^i,\hat{\pi}^i)\right\|_{Y_T^v}\right) \leqslant CT^{\frac{1}{q}}\left\|(\mathbf{k},g,\mathbf{h}^1,\mathbf{h}^2,\hat{\mathbf{v}}^0)\right\|_{Z_T^v\times\mathcal{D}_q^1},\end{aligned}$$

with the help of lemma 2.2. Similarly,

$$\|\mathbf{R}^4\|_{Z_T^4} \leqslant CT^{\frac{1}{q}}\left(\max_{i=1,2}\|(\hat{\mathbf{v}}^i,\hat{\pi}^i)\|_{Y_T^v}\right) \leqslant CT^{\frac{1}{q}}\|(\mathbf{k},g,\mathbf{h}^1,\mathbf{h}^2,\hat{\mathbf{v}}^0)\|_{Z_T^v\times\mathcal{D}_q^1}.$$

Taking T_0 sufficiently small such that $CT_0^{\frac{1}{2q}} \leqslant \frac{1}{2}$, we have

$$\left\|\mathbf{R}^0(y)\right\|_{Z_T^1} + \|\mathbf{R}'(y)\|_{Z_T'} + \left\|\mathbf{R}^3(y)\right\|_{Z_T^3} + \left\|\mathbf{R}^4(y)\right\|_{Z_T^4} \leqslant \frac{1}{2}\|y\|_{Z_T^v \times \mathcal{D}_q^1},$$

for $y = (\mathbf{k}, g, \mathbf{h}^1, \mathbf{h}^2, \hat{\mathbf{v}}^0)^\top$. By a Neumann series argument,

$$\Phi : \tilde{y} \mapsto \tilde{y} + (\mathbf{R}^0, 0, \mathbf{R}', \mathbf{R}^3, \mathbf{R}^4, 0)^\top(\tilde{y})$$

is invertible for $\tilde{y} = (\mathbf{k}, g, 0, \mathbf{h}^1, \mathbf{h}^2, \hat{\mathbf{v}}^0)^\top$. Consequently, replacing $\tilde{y}$ by $\Phi^{-1}(\tilde{y})$ in (3.14) yields the solvability of (3.17) for $0 < T < T_0 \leqslant 1/(2C)^{2q}$. Solving (3.1) iteratively on $[0, T_0]$, $[T_0, 2T_0]$, ..., with initial values $\hat{\mathbf{v}}^0$, $\hat{\mathbf{v}}^0|_{t=T_0}$, ..., one obtains the solvability for any $T_0 > 0$. Additionally, estimate (3.2) is a result of (3.10) and (3.12). This completes the proof. □

Remark 3.2. Since $\hat{c}$ is contained in $Y_T^3 = L^q(0,T; W_q^2(\widetilde{\Omega})) \cap W_q^1(0,T; L^q(\widehat{\Omega}))$, which will be given in section 3.2, one can easily verify that $\hat{c} \in Z_T^2$. Hence, we replace g in (3.1) by $g + \frac{\gamma\beta}{\hat{\rho}_s}\hat{c}_s$ with the same existence and regularity results to the original linear system. To be more precise, we find $(\bar{\mathbf{v}}, \bar{\pi}) \in Y_T^v$ to solve

$$\begin{aligned}
\hat{\rho}\partial_t \bar{\mathbf{v}} - \widehat{\operatorname{div}}\mathbf{S}(\bar{\mathbf{v}}, \bar{\pi}) &= 0 && \text{in } \widetilde{\Omega} \times (0,T),\\
\widehat{\operatorname{div}}\bar{\mathbf{v}} &= \frac{\gamma\beta}{\hat{\rho}_s}\hat{c}_s && \text{in } \widetilde{\Omega} \times (0,T),\\
[\![\bar{\mathbf{v}}]\!] &= 0 && \text{on } \Gamma \times (0,T),\\
[\![\mathbf{S}(\bar{\mathbf{v}}, \bar{\pi})]\!] \cdot \hat{\mathbf{n}}_\Gamma &= 0 && \text{on } \Gamma \times (0,T),\\
\mathbf{S}(\bar{\mathbf{v}}_s, \bar{\pi}_s) \cdot \hat{\mathbf{n}}_{\Gamma_s} &= 0 && \text{on } \Gamma_s \times (0,T),\\
\bar{\mathbf{v}}|_{t=0} &= 0 && \text{in } \widetilde{\Omega},
\end{aligned}$$

with $\hat{c} \in Z_T^2$, thanks to theorem 3.1. Then $(\hat{\mathbf{v}} + \bar{\mathbf{v}}, \hat{\pi} + \bar{\pi})$ solves the original linear system.

3.2. Heat equations with Neumann boundary condition

From (2.26)–(2.30), we have two decoupled systems with given functions (f^1, f^2, f^3), that is,

$$\begin{cases}
\partial_t \hat{c}_f - \hat{D}_f \widehat{\Delta} \hat{c}_f = f_f^1 & \text{in } \Omega_f \times (0,T),\\
\hat{D}_f \widehat{\nabla} \hat{c}_f \cdot \hat{\mathbf{n}}_\Gamma = f_f^2 & \text{on } \Gamma \times (0,T),\\
\hat{c}_f|_{t=0} = \hat{c}_f^0 & \text{in } \Omega_f,
\end{cases} \tag{3.18}$$

and

$$\begin{cases}
\partial_t \hat{c}_s - \hat{D}_s \widehat{\Delta} \hat{c}_s = f_s^1 & \text{in } \Omega_s \times (0,T),\\
\hat{D}_s \widehat{\nabla} \hat{c}_s \cdot \hat{\mathbf{n}}_\Gamma = f_s^2 & \text{on } \Gamma \times (0,T),\\
\hat{D}_s \widehat{\nabla} \hat{c}_s \cdot \hat{\mathbf{n}}_\Gamma = f^3 & \text{on } \Gamma_s \times (0,T),\\
\hat{c}_s|_{t=0} = \hat{c}_s^0 & \text{in } \Omega_s,
\end{cases} \tag{3.19}$$

According to the maximal L^q-regularity results we introduced in appendix A.2, we immediately have following theorem.

Theorem 3.2. *Let $q > n+2$, $\widehat{\Omega}$ be a bounded domain as before with $\Gamma_s \in C^3$, Γ a closed hypersurface of class C^3. Assume that (f^1, f^2, f^3) are known functions contained in Z_T^c and $\hat{c}^0 \in \mathcal{D}_q^2$ with compatibility conditions*

$$\hat{D}_f \widehat{\nabla} \hat{c}_f^0 \cdot \hat{\mathbf{n}}_\Gamma\Big|_\Gamma = f_f^2\big|_{t=0}, \quad \hat{D}_s \widehat{\nabla} \hat{c}_s^0 \cdot \hat{\mathbf{n}}_\Gamma\Big|_\Gamma = f_s^2\big|_{t=0}, \quad \hat{D}_s \widehat{\nabla} \hat{c}_s^0 \cdot \hat{\mathbf{n}}_{\Gamma_s}\Big|_{\Gamma_s} = f^3\big|_{t=0}.$$

Then the parabolic problems (3.18) and (3.19) admit unique strong solutions $\hat{c}_{\mathrm{f}}$ and $\hat{c}_{\mathrm{s}}$ in Y_T^3 respectively. Moreover, there exist a constant $C>0$ and a time $T_0>0$ such that for $0<T<T_0$,

$$\|\hat{c}\|_{Y_T^3} \leqslant C\left\|(f^1, f^2, f^3, \hat{c}^0)\right\|_{Z_T^c\times\mathcal{D}_q^2}, \tag{3.20}$$

where $Z_T^c := Z_T^5\times Z_T^6\times Z_T^7$ with

$$Z_T^5 := L^q(0,T;L^q(\widetilde{\Omega})),$$
$$Z_T^6 := W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma\times(0,T)),\quad Z_T^7 := W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma_{\mathrm{s}}\times(0,T)).$$

3.3. Ordinary differential equations for foam cells and growth

Given functions (f^4, f^5) in Ω_{s}, we have

$$\begin{aligned}\partial_t\hat{c}_{\mathrm{s}}^* - \beta\hat{c}_{\mathrm{s}} &= f^4, \quad \text{in } \Omega_{\mathrm{s}}\times(0,T),\\ \hat{c}_{\mathrm{s}}^*|_{t=0} &= 0, \quad \text{in } \Omega_{\mathrm{s}},\end{aligned} \tag{3.21}$$

$$\begin{aligned}\partial_t\hat{g} - \frac{\gamma\beta}{n\hat{\rho}_{\mathrm{s}}}\hat{c}_{\mathrm{s}} &= f^5, \quad \text{in } \Omega_{\mathrm{s}}\times(0,T),\\ \hat{g}|_{t=0} &= 1, \quad \text{in } \Omega_{\mathrm{s}}.\end{aligned} \tag{3.22}$$

Since (3.21) and (3.22) are linear ordinary differential equations, it follows easily from $\hat{c}_{\mathrm{s}}\in Y_T^3$ in theorem 3.2 that system (3.21) and (3.22) admits a unique solution $\hat{c}_{\mathrm{s}}^*$ and $\hat{g}$, respectively, both in $Z_T^8 := L^q(0,T;W_q^1(\Omega_{\mathrm{s}}))$. Moreover, there exists a constant C independent of T such that

$$\|\hat{c}_{\mathrm{s}}^*\|_{Y_T^4} + \|\hat{g}\|_{Y_T^4} \leqslant C\left\|(f^4, f^5, \hat{c})\right\|_{Z_T^8\times Z_T^8\times Y_T^3}. \tag{3.23}$$

4. Local in time existence

This section is intended to prove theorem 2.1.

4.1. Some key estimates

Before showing theorem 2.1, let us give some useful estimates with regard to the deformation gradient $\hat{\mathbf{F}}^{-1}$and vector-valued Sobolev–Slobodeckij space $W_q^{\frac{1}{2}-\varepsilon}(0,T;L^q(\Omega))$.

Lemma 4.1 (Estimates on deformation gradient). *Let $q>n$, $n\geqslant 2$ and $\hat{\mathbf{F}}(\hat{\mathbf{v}})$ be the deformation gradient defined in (1.2) corresponding to a function $\hat{\mathbf{v}}\in Y_T^1$. Then for every $R>0$, there are a constant $C=C(R)>0$ and a finite time $0<T_R<1$ depending on R such that for all $0<T<T_R$, $\hat{\mathbf{F}}^{-1}$ exists and*

(a) $\left\|\hat{\mathbf{F}}^{-1}\right\|_{L^\infty\left(0,T;W_q^1(\widetilde{\Omega})\right)^{n\times n}} \leqslant C, \quad \left\|\partial_t\hat{\mathbf{F}}^{-1}\right\|_{L^q\left(0,T;W_q^1(\widetilde{\Omega})\right)^{n\times n}} \leqslant C\|\hat{\mathbf{v}}\|_{Y_T^1}$;

(b) $\left\|\hat{\mathbf{F}}^{-1}-\mathbb{I}\right\|_{L^\infty(0,T;W_q^1(\widetilde{\Omega}))^{n\times n}} \leqslant CT^{\frac{1}{q'}}\|\hat{\mathbf{v}}\|_{Y_T^1}$;

(c) $\sup\limits_{0\leqslant t\leqslant T}\left(\int_0^t \frac{\left\|\Delta_h\left(\hat{\mathbf{F}}^{-1}-\mathbb{I}\right)(\cdot,t)\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}^q}{h^{1+\frac{q}{2q'}}}\mathrm{d}h\right)^{\frac{1}{q}} \leqslant CT^{\frac{1}{2q'}}\|\hat{\mathbf{v}}\|_{Y_T^1}$;

(d) $\left[\hat{\mathbf{F}}^{-1}-\mathbb{I}\right]_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}\left(0,T;W_q^1(\widetilde{\Omega})\right)^{n\times n}} \leqslant CT^{\frac{1}{q}+\frac{1}{2q'}}\|\hat{\mathbf{v}}\|_{Y_T^1}$, *for all* $\|\hat{\mathbf{v}}\|_{Y_T^1}\leqslant R$, *where* $\Delta_h f(t):= f(t)-f(t-h)$ *is a difference of the time shift for a function f. Moreover, for another* $\hat{\mathbf{u}}\in Y_T^1$ *with* $\|\hat{\mathbf{u}}\|_{Y_T^1}\leqslant R$ *and* $\hat{\mathbf{v}}|_{t=0}=\hat{\mathbf{u}}|_{t=0}$, *we have*

(e) $\left\|\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{L^\infty\left(0,T;W_q^1(\widetilde{\Omega})\right)^{n\times n}} \leqslant CT^{\frac{1}{q'}}\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y_T^1}$;

$\left\|\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{L^q\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}} \leqslant CT^{\frac{1}{q}-\frac{1}{r}}\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y_T^1}$;

(f) $\sup\limits_{0\leqslant t\leqslant T}\left(\int_0^t \frac{\left\|\Delta_h\left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right)(\cdot,t)\right\|^q_{W_q^1(\widetilde{\Omega})^{n\times n}}}{h^{1+\frac{q}{2q'}}}\mathrm{d}h\right)^{\frac{1}{q}} \leqslant CT^{\frac{1}{2q'}}\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y_T^1}$;

(g) $\left[\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right]_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}\left(0,T;W_q^1(\widetilde{\Omega})\right)^{n\times n}} \leqslant CT^{\frac{1}{q}+\frac{1}{2q'}}\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y_T^1}$,

where $r=\frac{q^2}{n}$.

Proof. Recall from (1.2) the definition of $\hat{\mathbf{F}}$ that

$$\hat{\mathbf{F}}(X,t)=\mathbb{I}+\int_0^t\hat{\nabla}\hat{\mathbf{v}}(X,\tau)\mathrm{d}\tau,\quad \forall X\in\widehat{\Omega}.$$

Then we have

$$\sup_{0\leqslant t\leqslant T}\left\|\hat{\mathbf{F}}-\mathbb{I}\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}=\sup_{0\leqslant t\leqslant T}\left\|\int_0^t\hat{\nabla}\hat{\mathbf{v}}(X,\tau)\mathrm{d}\tau\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}\leqslant CT^{\frac{1}{q'}}R,$$

for all $\|\hat{\mathbf{v}}\|_{Y_T^1}\leqslant R$. Choosing $T_R>T$ so small that $CT_R^{\frac{1}{q'}}R\leqslant\frac{1}{2M_q}$, we know

$$\sup_{0\leqslant t\leqslant T}\left\|\hat{\mathbf{F}}-\mathbb{I}\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}\leqslant\frac{1}{2M_q},$$

where M_q is the constant of multiplication in $W_q^1(\widetilde{\Omega})$, see lemma 2.1. According to the Neumann series (see [5, section 5.7]), $\hat{\mathbf{F}}^{-1}$ does exist and

$$\hat{\mathbf{F}}^{-1}=\left(\hat{\mathbf{F}}-\mathbb{I}+\mathbb{I}\right)^{-1}=\left(\mathbb{I}-\left(\mathbb{I}-\hat{\mathbf{F}}\right)\right)^{-1}=\sum_{k=0}^\infty\left(\mathbb{I}-\hat{\mathbf{F}}\right)^k.$$

Then from lemma 2.1, one obtains

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\left\|\hat{\mathbf{F}}^{-1}\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}&\leqslant\sup_{0\leqslant t\leqslant T}\sum_{k=0}^\infty\left\|\left(\mathbb{I}-\hat{\mathbf{F}}\right)^k\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}\\&\leqslant\frac{1}{M_q}\sum_{k=0}^\infty\left(M_q\sup_{0\leqslant t\leqslant T}\left\|\mathbb{I}-\hat{\mathbf{F}}\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}\right)^k\leqslant\frac{1}{M_q}\sum_{k=0}^\infty\left(\frac{1}{2}\right)^k=\frac{2}{M_q},\end{aligned}$$

Consequently, if follows from (2.2) and lemma 2.1 that

$$\begin{aligned}&\left\|\partial_t\hat{\mathbf{F}}^{-1}\right\|_{L^q\left(0,T;W_q^1(\widetilde{\Omega})\right)^{n\times n}}\\&\leqslant M_q^2\left\|\hat{\mathbf{F}}^{-1}\right\|^2_{L^\infty\left(0,T;W_q^q(\widetilde{\Omega})\right)^{n\times n}}\left\|\hat{\nabla}\hat{\mathbf{v}}\right\|_{L^q\left(0,T:W_q^1(\widetilde{\Omega})\right)^{n\times n}}\leqslant C\|\hat{\mathbf{v}}\|_{Y_T^1},\end{aligned}$$

for all $0 < T < T_R$ and

$$\left\|\hat{\mathbf{F}}^{-1} - \mathbb{I}\right\|_{L^\infty\left(0,T;W^1_q(\widetilde{\Omega})\right)^{n\times n}} \leqslant \int_0^T \left\|\partial_t \hat{\mathbf{F}}^{-1}(\cdot,\tau)\right\|_{W^1_q(\widetilde{\Omega})^{n\times n}} \mathrm{d}\tau \leqslant CT^{\frac{1}{q'}} \left\|\hat{\mathbf{v}}\right\|_{Y^1_T},$$

where $C = C(R)$ depends on R. These estimates prove the first two statements.

For the third and fourth statements, we have

$$\left\|\Delta_h\left(\hat{\mathbf{F}}^{-1} - \mathbb{I}\right)(\cdot,t)\right\|_{W^1_q(\widetilde{\Omega})^{n\times n}} \leqslant \int_{t-h}^t \left\|\partial_t \hat{\mathbf{F}}^{-1}(\cdot,\tau)\right\|_{W^1_q(\widetilde{\Omega})^{n\times n}} \mathrm{d}\tau \leqslant Ch^{\frac{1}{q'}} \left\|\hat{\mathbf{v}}\right\|_{Y^1_T},$$

which can be used to deduce

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\left(\int_0^t \frac{\left\|\Delta_h\left(\hat{\mathbf{F}}^{-1} - \mathbb{I}\right)(\cdot,t)\right\|^q_{W^1_q(\widetilde{\Omega})^{n\times n}}}{h^{1+\frac{q}{2q'}}}\mathrm{d}h\right)^{\frac{1}{q}} \\
&\quad\leqslant C\sup_{0\leqslant t\leqslant T}\left(\int_0^t h^{-1+\frac{q}{2q'}}\,\mathrm{d}h\right)^{\frac{1}{q}} \left\|\hat{\mathbf{v}}\right\|_{Y^1_T} = C2q' \sup_{0\leqslant t\leqslant T} t^{\frac{1}{2q'}} \left\|\hat{\mathbf{v}}\right\|_{Y^1_T} \leqslant CT^{\frac{1}{2q'}} \left\|\hat{\mathbf{v}}\right\|_{Y^1_T},
\end{aligned}$$

and therefore from (2.4) and the definition of Sobolev–Slobodeckij space,

$$\left[\hat{\mathbf{F}}^{-1} - \mathbb{I}\right]_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}\left(0,T;W^1_q(\widetilde{\Omega})\right)^{n\times n}} \leqslant CT^{\frac{1}{q}+\frac{1}{2q'}} \left\|\hat{\mathbf{v}}\right\|_{Y^1_T}.$$

For the rest statements, we notice from (1.2) that

$$\hat{\mathbf{F}}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}(\hat{\mathbf{v}}) = \int_0^t \left(\hat{\nabla}\hat{\mathbf{u}} - \hat{\nabla}\hat{\mathbf{v}}\right)(X,\tau)\mathrm{d}\tau.$$

Then for all $0 < T < T_R$,

$$\sup_{0\leqslant t\leqslant T}\left\|\hat{\mathbf{F}}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}(\hat{\mathbf{v}})\right\|_{W^1_q(\widetilde{\Omega})^{n\times n}} \leqslant CT^{\frac{1}{q'}} \left\|\hat{\mathbf{u}} - \hat{\mathbf{v}}\right\|_{Y^1_T}.$$

Since

$$\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}) = -\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\left(\hat{\mathbf{F}}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}(\hat{\mathbf{v}})\right)\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}),$$

it follows from the multiplication property of $W^1_q(\widetilde{\Omega})$ again that for all $0 < T < T_R$,

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\left\|\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{W^1_q(\widetilde{\Omega})^{n\times n}} \\
&\quad\leqslant M_q^2 \sup_{0\leqslant t\leqslant T}\left\|\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\right\|_{W^1_q(\widetilde{\Omega})^{n\times n}}\left\|\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{W^1_q(\widetilde{\Omega})^{n\times n}}\left\|\hat{\mathbf{F}}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}(\hat{\mathbf{v}})\right\|_{W^1_q(\widetilde{\Omega})^{n\times n}} \\
&\quad\leqslant CT^{\frac{1}{q'}} \left\|\hat{\mathbf{u}} - \hat{\mathbf{v}}\right\|_{Y^1_T}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}}) - \partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}) = &-\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\left(\hat{\mathbf{F}}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}(\hat{\mathbf{v}})\right)\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}) - \hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\partial_t\left(\hat{\mathbf{F}}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}(\hat{\mathbf{v}})\right)\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}) \\
&- \hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\left(\hat{\mathbf{F}}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}(\hat{\mathbf{v}})\right)\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}). \qquad (4.1)
\end{aligned}$$

Hence

$$\begin{aligned}
&\left\|\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{L^q\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}}\\
&\quad\leqslant\left\|\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\left(\hat{\mathbf{F}}(\hat{\mathbf{u}})-\hat{\mathbf{F}}(\hat{\mathbf{v}})\right)\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{L^q\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}}\\
&\qquad+\left\|\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\left(\hat{\nabla}\hat{\mathbf{u}}-\hat{\nabla}\hat{\mathbf{v}}\right)\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{L^q\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}}\\
&\qquad+\left\|\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\left(\hat{\mathbf{F}}(\hat{\mathbf{u}})-\hat{\mathbf{F}}(\hat{\mathbf{v}})\right)\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{L^q\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}}=:F_1+F_2+F_3.
\end{aligned}$$

From the embedding (2.6), we know that for $\hat{\mathbf{v}}\in Y_T^1$,

$$\sup_{0\leqslant t\leqslant T}\left\|\hat{\nabla}\hat{\mathbf{v}}\right\|_{L^q(\widetilde{\Omega})^{n\times n}}\leqslant C\left(\|\hat{\mathbf{v}}\|_{Y_T^1}+\left\|\hat{\mathbf{v}}|_{t=0}\right\|_{W_q^1(\widetilde{\Omega})}\right).$$

The Gagliardo–Nirenberg inequality tells us

$$\left\|\hat{\nabla}\hat{\mathbf{v}}\right\|_{L^\infty(\widetilde{\Omega})^{n\times n}}\leqslant C\left\|\hat{\nabla}\hat{\mathbf{v}}\right\|_{L^q(\widetilde{\Omega})^{n\times n}}^{1-\frac{n}{q}}\left\|\hat{\nabla}\hat{\mathbf{v}}\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}^{\frac{n}{q}}.$$

For $r=\frac{q^2}{n}>q$, we obtain

$$\left\|\hat{\nabla}\hat{\mathbf{v}}\right\|_{L^r\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}}\leqslant C\left\|\hat{\nabla}\hat{\mathbf{v}}\right\|_{L^\infty\left(0,T;L^q(\widetilde{\Omega})\right)^{n\times n}}^{1-\frac{n}{q}}\left\|\hat{\nabla}\hat{\mathbf{v}}\right\|_{L^q\left(0,T;W_q^1(\widetilde{\Omega})\right)^{n\times n}}^{\frac{n}{q}}\leqslant C(R).$$

Then,

$$\left\|\hat{\nabla}\hat{\mathbf{v}}\right\|_{L^q\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}}\leqslant T^{\frac{1}{q}-\frac{1}{r}}\left\|\hat{\nabla}\hat{\mathbf{v}}\right\|_{L^r\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}}\leqslant C(R)T^{\frac{1}{q}-\frac{1}{r}},$$

and also, for $\hat{\mathbf{u}}\in Y_T^1$, $\|\hat{\mathbf{u}}\|_{Y_T^1}\leqslant R$,

$$\left\|\hat{\nabla}\hat{\mathbf{v}}-\hat{\nabla}\hat{\mathbf{u}}\right\|_{L^q\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}}\leqslant C(R)T^{\frac{1}{q}-\frac{1}{r}}\|\hat{\mathbf{v}}-\hat{\mathbf{u}}\|_{Y_T^1}. \tag{4.2}$$

Consequently, with $W_q^1(\widetilde{\Omega})\hookrightarrow L^\infty(\widetilde{\Omega})$ for $q>n$,

$$\begin{aligned}
F_1&\leqslant\left\|\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\right\|_{L^q\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}}\\
&\quad\times\left\|\hat{\mathbf{F}}(\hat{\mathbf{u}})-\hat{\mathbf{F}}(\hat{\mathbf{v}})\right\|_{L^\infty\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}}\left\|\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{L^\infty\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}}\\
&\leqslant\left\|\hat{\nabla}\hat{\mathbf{u}}\right\|_{L^q\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}}\left\|\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\right\|_{L^\infty\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}}^2\\
&\quad\times\left\|\hat{\mathbf{F}}(\hat{\mathbf{u}})-\hat{\mathbf{F}}(\hat{\mathbf{v}})\right\|_{L^\infty\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}}\left\|\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{L^\infty\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}}\\
&\leqslant CT^{\frac{1}{q}-\frac{1}{r}}\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y_T^1}.
\end{aligned}$$

Similarly,

$$F_2\leqslant CT^{\frac{1}{q}-\frac{1}{r}}\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y_T^1},\quad F_3\leqslant CT^{\frac{1}{q}-\frac{1}{r}}\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y_T^1}.$$

Thus,

$$\left\|\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{L^q\left(0,T;L^\infty(\widetilde{\Omega})\right)^{n\times n}}\leqslant CT^{\frac{1}{q}-\frac{1}{r}}\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y_T^1}.$$

Moreover, we can also conclude from (4.1) that

$$\left\|\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{L^q\left(0,T;W^1_q(\widetilde{\Omega})\right)^{n\times n}}\leqslant C\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y^1_T}.$$

Using that $\left(\hat{\mathbf{F}}_0^{-1}(\hat{\mathbf{u}})-\hat{\mathbf{F}}_0^{-1}(\hat{\mathbf{v}})\right)=0$,

$$\begin{aligned}&\left\|\Delta_h\left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right)(\cdot,t)\right\|_{W^1_q(\widetilde{\Omega})^{n\times n}}\\&\leqslant\int_{t-h}^t\left\|\partial_t\left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right)(\cdot,\tau)\right\|_{W^1_q(\widetilde{\Omega})^{n\times n}}\mathrm{d}\tau\leqslant Ch^{\frac{1}{q'}}\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y^1_T}.\end{aligned}$$

Therefore, for all $0<T<T_R$,

$$\begin{aligned}&\sup_{0\leqslant t\leqslant T}\left(\int_0^h\frac{\left\|\Delta_h\left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right)(\cdot,t)\right\|^q_{W^1_q(\widetilde{\Omega})^{n\times n}}}{h^{1+\frac{q}{2q'}}}\mathrm{d}h\right)^{\frac{1}{q}}\\&\leqslant C\sup_{0\leqslant t\leqslant T}t^{\frac{1}{2q'}}\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y^1_T}=CT^{\frac{1}{2q'}}\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y^1_T}.\end{aligned}$$

Again with the help of (2.4) and the definition of Sobolev–Slobodeckij space, one obtains the last statement. This completes the proof. □

Lemma 4.2. *Under the assumption of lemma 4.1, there exist a constant $C=C(R)>0$ and a finite time $T_R>0$ depending on R such that for all $0<T<T_R$ and for two arbitrary functions $f\in L^q(0,T;W^1_q(\widetilde{\Omega}))$ and $\mathbf{f}\in L^q(0,T;W^2_q(\widetilde{\Omega}))^n$,*

(a) $\left\|\left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})-\mathbb{I}\right)f\right\|_{L^q\left(0,T;W^1_q(\widetilde{\Omega})\right)^n}\leqslant CT^{\frac{1}{q'}}\|f\|_{L^q\left(0,T;W^1_q(\widetilde{\Omega})\right)}\|\hat{\mathbf{v}}\|_{Y^1_T}$;
$\left\|\left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})-\mathbb{I}\right)\left(\hat{\nabla}\mathbf{f}\right)\right\|_{L^q\left(0,T;W^1_q(\widetilde{\Omega})\right)^{n\times n}}\leqslant CT^{\frac{1}{q'}}\|\mathbf{f}\|_{L^q\left(0,T;W^2_q(\widetilde{\Omega})\right)^n}\|\hat{\mathbf{v}}\|_{Y^1_T}$;

(b) $\left\|\left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right)f\right\|_{L^q\left(0,T;W^1_q(\widetilde{\Omega})\right)^n}\leqslant CT^{\frac{1}{q'}}\|f\|_{L^q\left(0,T;W^1_q(\widetilde{\Omega})\right)}\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y^1_T}$;
$\left\|\left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right)\left(\hat{\nabla}\mathbf{f}\right)\right\|_{L^q\left(0,T;W^1_q(\widetilde{\Omega})\right)^{n\times n}}\leqslant CT^{\frac{1}{q'}}\|\mathbf{f}\|_{L^q\left(0,T;W^2_q(\widetilde{\Omega})\right)^n}\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y^1_T}$;

(c) $\left\|\left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right)\left(\hat{\nabla}\mathbf{f}\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\right)\right\|_{L^q\left(0,T;W^1_q(\widetilde{\Omega})\right)^{n\times n}}\leqslant CT^{\frac{1}{q'}}\|\mathbf{f}\|_{L^q\left(0,T;W^2_q(\widetilde{\Omega})\right)^n}\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y^1_T}$.

Proof. The key point to deduce these estimates is to use the multiplication property of $W^1_q(\widetilde{\Omega})$ with $q>n$, which was given in lemma 2.1. Then lemma 4.1 implies these results. □

Lemma 4.3. *Let $1<q<\infty$, $T_0>0$ and $\Omega\subseteq\mathbb{R}^n$, $n\geqslant 2$, be a bounded domain with $C^{1,1}$ boundary. Then*

$$\left[\hat{\nabla}\hat{\mathbf{v}}\right]_{W_q^{\frac{1}{2}-\varepsilon}(0,T;L^q(\Omega))^{n\times n}}\leqslant CT_0^{\varepsilon}\left[\hat{\mathbf{v}}\right]_{W_q^{2,1}(\Omega\times(0,T))^n},$$

for every $\hat{\mathbf{v}}\in W^{2,1}_q(\Omega\times(0,T))^n$, $\varepsilon\in(0,\frac{1}{2})$ and $0<T<T_0$. Here C depends on ε.

Proof. The lemma can be easily proved by using the argument in [1, lemma 4.2], where a layer-like domain with $C^{1,1}$ boundary is considered. Besides, it can be seen as a corollary of lemma 2.2. □

4.2. Proof of theorem 2.1

In this subsection, we prove theorem 2.1 by applying the strategy of a fixed-point procedure.

For the proof, we set $w = (\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s^*, \hat{g})$, $w_0 := (\hat{\mathbf{v}}^0, \hat{c}^0, 0, 1)$ and reformulate the initial and boundary value problem (2.21)–(2.34) as an abstract equation:

$$\mathscr{L}(w) = \mathscr{N}(w, w_0), \quad \text{for all } w \in Y_T,\ (\hat{\mathbf{v}}^0, \hat{c}^0) \in \mathcal{D}_q, \tag{4.3}$$

where

$$\mathscr{L}(w) := \begin{pmatrix} \partial_t \hat{\mathbf{v}} - \widehat{\operatorname{div}} \mathbf{S}(\hat{\mathbf{v}}, \hat{\pi}) \\ \widehat{\operatorname{div}}\,(\hat{\mathbf{v}}) - \dfrac{\gamma\beta}{\hat{\rho}_s} \hat{c}_s \\ [\mathbf{S}(\hat{\mathbf{v}}, \hat{\pi})] \cdot \hat{\mathbf{n}}_\Gamma \\ \mathbf{S}(\hat{\mathbf{v}}_s, \hat{\pi}_s) \cdot \hat{\mathbf{n}}_{\Gamma_s} \\ \partial_t \hat{c} - \hat{D} \hat{\Delta} \hat{c} \\ \hat{D} \hat{\nabla} \hat{c} \cdot \hat{\mathbf{n}}_\Gamma \\ \hat{D}_s \hat{\nabla} \hat{c}_s \cdot \hat{\mathbf{n}}_{\Gamma_s} \\ \partial_t \hat{c}_s^* - \beta \hat{c}_s \\ \partial_t \hat{g} - \dfrac{\gamma\beta}{n\hat{\rho}_s} \hat{c}_s \\ (\hat{\mathbf{v}}, \hat{c}, \hat{c}_s^*, \hat{g})^\top \big|_{t=0} \end{pmatrix}, \quad \mathscr{N}(w, w_0) := \begin{pmatrix} \mathbf{K}(w) \\ G(w) \\ \mathbf{H}^1(w) \\ \mathbf{H}^2(w) \\ F^1(w) \\ F^2(w) \\ F^3(w) \\ F^4(w) \\ F^5(w) \\ w_0 \end{pmatrix}.$$

In the sequel, we focus on (4.3). For $\mathscr{L}$, we have the following proposition.

Proposition 4.1. *Let $\mathscr{L}$ be defined as in (4.3). Then $\mathscr{L}$ is an isomorphism from Y_T to $Z_T \times \mathcal{D}_q$.*

Proof. As $\mathscr{L} \in \mathcal{L}(Y_T, Z_T \times \mathcal{D}_q)$, it suffices to show that $\mathscr{L}$ is bijective, thanks to the bounded inverse theorem.

Injective. Take any $w^1, w^2 \in Y_T$. Then, from (3.2), (3.20) and (3.23),

$$\left\| \mathscr{L}(w^1) - \mathscr{L}(w^2) \right\|_{Z_T \times \mathcal{D}_q} \leqslant C \left\| w^1 - w^2 \right\|_{Y_T},$$

which implies the injectivity of $\mathscr{L}$.

Surjective. The existence of (3.1), (3.18)–(3.19) and (3.21)–(3.22) immediately yields the surjectivity of $\mathscr{L}$.

□

To employ the contraction mapping principle to (4.3), we investigate the dependence and contraction property of $(\mathbf{K}, G, \mathbf{H}^1, \mathbf{H}^2, F^1, F^2, F^3, F^4, F^5)$ on $(\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s^*, \hat{g})$. To this end, we define

$$\mathscr{M}(w) := \left(\mathbf{K}(w), G(w), \mathbf{H}^1(w), \mathbf{H}^2(w), F^1(w), F^2(w), F^3(w), F^4(w), F^5(w)\right)^\top,$$

where the elements are given by (2.35). Then it is still needed to show that $\mathscr{M} : Y_T \to Z_T$ is well-defined for $w = (\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s^*, \hat{g}) \in Y_T$ and to verify that $\mathscr{M}$ possesses the contraction property.

Proposition 4.2. *Let $q > n$ and $R > 0$. Assume $w = (\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s^*, \hat{g}) \in Y_T$ with $\hat{g}|_{t=0} = 1$ and $\|w\|_{Y_T} \leqslant R$, then there exist a constant $C = C(R) > 0$, a finite time $T_R > 0$ depending on R and $\delta > 0$ such that for $0 < T < T_R$, $\mathscr{M} : Y_T \to Z_T$ is well-defined and bounded along with the estimates:*

$$\|\mathscr{M}(w)\|_{Z_T} \leqslant C(R) T^\delta \left(\|w\|_{Y_T} + 1\right). \tag{4.4}$$

Moreover, for $w^1 = (\hat{\mathbf{v}}^1, \hat{\pi}^1, \hat{c}^1, \hat{c}_s^{*1}, \hat{g}^1), w^2 = (\hat{\mathbf{v}}^2, \hat{\pi}^2, \hat{c}^2, \hat{c}_s^{*2}, \hat{g}^2) \in Y_T$ *with* $w^1 \neq w^2$, $\hat{c}^i\big|_{t=0} = \hat{c}^0$, $\hat{c}_s^*|_{t=0} = 0$, $\hat{g}^i\big|_{t=0} = 1$ *and* $\left\|w^i\right\|_{Y_T} \leqslant R$ $(i = 1, 2)$*, there exist a constant* $C = C(R) > 0$*, a finite time* $T_R > 0$ *depending on* R *and* $\delta > 0$ *such that for* $0 < T < T_R$*,*

$$\left\|\mathscr{M}(w^1) - \mathscr{M}(w^2)\right\|_{Z_T} \leqslant C(R)T^\delta \left\|w^1 - w^2\right\|_{Y_T}. \tag{4.5}$$

Proof. First of all, we prove the second part. To this end, for $\left\|w^i\right\|_{Y_T} \leqslant R$, $i = 1, 2$ we estimate the following terms respectively

$$\begin{aligned}
&\left\|\mathbf{K}(w^1) - \mathbf{K}(w^2)\right\|_{Z_T^1}, \quad \left\|G(w^1) - G(w^2)\right\|_{Z_T^2}, \\
&\left\|\mathbf{H}^j(w^1) - \mathbf{H}^j(w^2)\right\|_{Z_T^{j+2}}, \quad \left\|F^k(w^1) - F^k(w^2)\right\|_{Z_T^{k+4}}, \quad \left\|F^5(w^1) - F^5(w^2)\right\|_{Z_T^8},
\end{aligned}$$

where $j \in \{1, 2\}$, $k \in \{1, 2, 3, 4\}$. If $0 < T \leqslant 1$, we have $T^s < T^{s'}$ for $s > s' > 0$. In the sequel, we set a universal constant $\delta = \min\{\frac{1}{2q'}, \frac{1}{q} - \frac{1}{r}\}$, where $q' = \frac{q}{q-1}$, $r = \frac{q^2}{n}$.

Estimate of $\left\|\mathbf{K}(w^1) - \mathbf{K}(w^2)\right\|_{Z_T^1}$. For $\mathbf{K}_f = \widehat{\operatorname{div}}\tilde{\mathbf{K}}_f$ from (2.35), with the help of lemmas 2.1, 4.1 and 4.2, we derive that

$$\begin{aligned}
&\left\|\tilde{\mathbf{K}}_f(w^1) - \tilde{\mathbf{K}}_f(w^2)\right\|_{L^q\left(0,T;W_q^1(\Omega_f)\right)^{n\times n}} \\
&\leqslant \left\|\hat{\pi}_f^1 \left(\hat{\mathbf{F}}_f^{-\top}(\hat{\mathbf{v}}_f^1) - \hat{\mathbf{F}}_f^{-\top}(\hat{\mathbf{v}}_f^2)\right) + \left(\hat{\pi}_f^1 - \hat{\pi}_f^2\right)\left(\hat{\mathbf{F}}_f^{-\top}(\hat{\mathbf{v}}_f^2) - \mathbb{I}\right)\right\|_{L^q\left(0,T;W_q^1(\Omega_f)\right)^{n\times n}} \\
&\quad + \nu_f \left\|\left(\hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}_f^1)\hat{\nabla}\hat{\mathbf{v}}_f^1 + \hat{\nabla}^\top\hat{\mathbf{v}}_f^1\hat{\mathbf{F}}_f^{-\top}(\hat{\mathbf{v}}_f^1)\right)\left(\hat{\mathbf{F}}_f^{-\top}(\hat{\mathbf{v}}_f^1) - \hat{\mathbf{F}}_f^{-\top}(\hat{\mathbf{v}}_f^2)\right)\right\|_{L^q\left(0,T;W_q^1(\Omega_f)\right)^{n\times n}} \\
&\quad + 2\nu_f \left\|\left(\left(\hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}_f^1) - \hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}_f^2)\right)\hat{\nabla}\hat{\mathbf{v}}_f^1 + \hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}_f^2)\left(\hat{\nabla}\hat{\mathbf{v}}_f^1 - \hat{\nabla}\hat{\mathbf{v}}_f^2\right)\right)\right. \\
&\quad \left.\times\left(\hat{\mathbf{F}}_f^{-\top}(\hat{\mathbf{v}}_f^2) - \mathbb{I}\right)\right\|_{L^q\left(0,T;W_q^1(\Omega_f)\right)^{n\times n}} \\
&\quad + 2\nu_f \left\|\left(\hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}_f^1) - \hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}_f^2)\hat{\nabla}\hat{\mathbf{v}}_f^1\right)\left(\hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}_f^2) - \mathbb{I}\right)\left(\hat{\nabla}\hat{\mathbf{v}}_f^1 - \hat{\nabla}\hat{\mathbf{v}}_f^2\right)\right\|_{L^q\left(0,T;W_q^1(\Omega_f)\right)^{n\times n}} \\
&\leqslant CT^{\frac{1}{q'}}\left(\left\|\hat{\pi}_f^1\right\|_{Y_T^2}\left\|\hat{\mathbf{v}}_f^1 - \hat{\mathbf{v}}_f^2\right\|_{Y_T^1} + \left\|\hat{\pi}_f^1 - \hat{\pi}_f^2\right\|_{Y_T^2}\left\|\hat{\mathbf{v}}_f^2\right\|_{Y_T^1}\right) + CT^{\frac{1}{q'}}\left\|\hat{\mathbf{v}}_f^1\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}_f^1 - \hat{\mathbf{v}}_f^2\right\|_{Y_T^1} \\
&\quad + CT^{\frac{2}{q'}}\left\|\hat{\mathbf{v}}_f^1\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}_f^1 - \hat{\mathbf{v}}_f^2\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}_f^2\right\|_{Y_T^1} + CT^{\frac{1}{q'}}\left\|\hat{\mathbf{v}}_f^1 - \hat{\mathbf{v}}_f^2\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}_f^2\right\|_{Y_T^1} \\
&\quad + CT^{\frac{1}{q'}}\left(\left\|\hat{\mathbf{v}}_f^1\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}_f^1 - \hat{\mathbf{v}}_f^2\right\|_{Y_T^1} + \left\|\hat{\mathbf{v}}_f^1 - \hat{\mathbf{v}}_f^2\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}_f^2\right\|_{Y_T^1}\right) \leqslant C(R)T^\delta\left\|w^1 - w^2\right\|_{Y_T}.
\end{aligned}$$

Let $\hat{g} \in W_q^1(0, T; W_q^1(\Omega_s))$ with $\hat{g}|_{t=0} = 1$. Now we claim that there exists a time $T_R > 0$ such that for $0 < T < T_R$, $\hat{g} \geqslant \frac{1}{2} > 0$. Let $\hat{g}$ be such a function with $\|\hat{g}\|_{W_q^1(0,T;W_q^1(\Omega_s))} \leqslant R$ for some $R > 0$. Then for $0 < t < T$,

$$\|\hat{g}(t) - 1\|_{L^\infty(\Omega_s)} \leqslant C\left\|\int_0^t \partial_t\hat{g}(X,\tau)\mathrm{d}\tau\right\|_{W_q^1(\Omega_s)} \leqslant CT^{\frac{1}{q'}}R \leqslant \frac{1}{2},$$

where we choose $T_R > 0$ small enough such that $T_R^{\frac{1}{q'}} \leqslant \frac{1}{2CR}$. Hence,

$$\hat{g} \geqslant \frac{1}{2} > 0.$$

For $\mathbf{K}_s = \widehat{\operatorname{div}}\tilde{\mathbf{K}}_s + \bar{\mathbf{K}}_s^g$, the first part can be estimated similarly using

$$\left\|\tilde{\mathbf{K}}_s(w^1) - \tilde{\mathbf{K}}_s(w^2)\right\|_{L^q\left(0,T;W^1_q(\Omega_s)\right)^{n\times n}} \leqslant C(R)T^\delta \left\|w^1 - w^2\right\|_{Y_T}.$$

For the second part it follows from (2.4), lemma 4.1 and 4.2 that

$$\begin{aligned}
&\left\|\bar{\mathbf{K}}_s^g(w^1) - \bar{\mathbf{K}}_s^g(w^2)\right\|_{L^q(0,T;L^q(\Omega_s))^{n\times n}} \\
&\leqslant \left\|\left(\hat{\boldsymbol{\sigma}}_s(\hat{\mathbf{v}}_s^1,\hat{\pi}_s^1,\hat{g}^1)\hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}_s^1) - \hat{\boldsymbol{\sigma}}_s(\hat{\mathbf{v}}_s^2,\hat{\pi}_s^2,\hat{g}^2)\hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}_s^2)\right)\frac{n\hat{\nabla}\hat{g}^1}{\hat{g}^1}\right\|_{L^q(0,T;L^q(\Omega_s))^{n\times n}} \\
&\quad + \left\|\hat{\boldsymbol{\sigma}}_s(\hat{\mathbf{v}}_s^2,\hat{\pi}_s^2,\hat{g}^2)\hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}_s^2)\left(\frac{n\hat{\nabla}\hat{g}^1}{\hat{g}^1} - \frac{n\hat{\nabla}\hat{g}^2}{\hat{g}^2}\right)\right\|_{L^q(0,T;L^q(\Omega_s))^{n\times n}} =: N_1 + N_2.
\end{aligned}$$

From the definition of $\hat{\boldsymbol{\sigma}}_s$ and $\hat{g} \geqslant 1/2$,

$$N_1 \leqslant C\left\|\hat{\nabla}\hat{g}^1\right\|_{L^\infty(0,T;L^q(\Omega_s))^n} N_1^1 \leqslant C(R)T^{\frac{1}{q'}}N_1^1,$$

where

$$\begin{aligned}
N_1^1 :=& \left\|\hat{\pi}_s^1\left(\hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}_s^1) - \hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}_s^2)\right)\right\|_{L^q(0,T;L^\infty(\Omega_s))^{n\times n}} \\
&+ \left\|\left(\hat{\pi}_s^1 - \hat{\pi}_s^2\right)\hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}_s^2)\right\|_{L^q(0,T;L^\infty(\Omega_s))^{n\times n}} + \hat{\nu}_s\left\|\hat{\nabla}\hat{\mathbf{v}}_s^1 - \hat{\nabla}\hat{\mathbf{v}}_s^2\right\|_{L^q(0,T;L^\infty(\Omega_s))^{n\times n}} \\
&+ \hat{\mu}_s\left(\left\|\frac{1}{(\hat{g}^1)^2}\left(\hat{\mathbf{F}}_s(\hat{\mathbf{v}}_s^1) - \hat{\mathbf{F}}_s(\hat{\mathbf{v}}_s^2)\right)\right\|_{L^q(0,T;L^\infty(\Omega_s))^{n\times n}}\right. \\
&+ \left\|\left(\frac{1}{(\hat{g}^1)^2} - \frac{1}{(\hat{g}^2)^2}\right)\hat{\mathbf{F}}_s(\hat{\mathbf{v}}_s^2)\right\|_{L^q(0,T;L^\infty(\Omega_s))^{n\times n}} \\
&+ \left.\left\|\hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}_s^1) - \hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}_s^2)\right\|_{L^q(0,T;L^\infty(\Omega_s))^{n\times n}}\right) \\
\leqslant& CT^{\frac{1}{q'}}\left\|\hat{\pi}_s^1\right\|_{Y_T^2}\left\|\hat{\mathbf{v}}_s^1 - \hat{\mathbf{v}}_s^2\right\|_{Y_T^1} + C\left\|\hat{\pi}_s^1 - \hat{\pi}_s^2\right\|_{Y_T^2} + C\left\|\hat{\mathbf{v}}_s^1 - \hat{\mathbf{v}}_s^2\right\|_{Y_T^1} \\
&+ \hat{\mu}_s\left(CT^{\frac{1}{q'}}\left\|\hat{\mathbf{v}}_s^1 - \hat{\mathbf{v}}_s^2\right\|_{Y_T^1} + CT^{\frac{1}{q'}}\left\|\hat{g}^1 - \hat{g}^2\right\|_{Y_T^4}\left\|\hat{\mathbf{v}}_s^2\right\|_{Y_T^1}\left(\left\|\hat{g}^1\right\|_{Y_T^4} + \left\|\hat{g}^2\right\|_{Y_T^4}\right)\right. \\
&\left.+CT^{\frac{1}{q'}}\left\|\hat{\mathbf{v}}_s^1 - \hat{\mathbf{v}}_s^2\right\|_{Y_T^1}\right) \leqslant C(R)\left\|w^1 - w^2\right\|_{Y_T}.
\end{aligned}$$

Then we get

$$N_1 + N_2 \leqslant C(R)T^{\frac{1}{q'}}\left\|w^1 - w^2\right\|_{Y_T}.$$

Consequently,

$$\left\|\mathbf{K}(w^1) - \mathbf{K}(w^2)\right\|_{Z_T^1} \leqslant C(R)T^\delta\left\|w^1 - w^2\right\|_{Y_T}. \tag{4.6}$$

Estimate of $\left\|G(w^1) - G(w^2)\right\|_{Z_T^2}$. From the definition of Z_T^2 given by (3.4), we need to verify that $G(w^1) - G(w^2)$ is contained both in $L^q(0,T;W^1_q(\widetilde{\Omega}))$ and $W^1_q(0,T;W^{-1}_q(\widehat{\Omega}))$, as well as the trace regularity

$$\begin{aligned}
&\operatorname{tr}_\Gamma(G(w^1) - G(w^2)) \in W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma\times(0,T)), \\
&\operatorname{tr}_{\Gamma_s}(G(w^1) - G(w^2)) \in W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma_s\times(0,T)).
\end{aligned}$$

For the first regularity it follows easily from (2.35), lemmas 4.1 and 4.2 that

$$\begin{aligned}&\left\|G(w^1)-G(w^2)\right\|_{L^q\left(0,T;W^1_q(\widehat{\Omega})\right)}\\&\quad\leqslant CT^{\frac{1}{q'}}\left\|\hat{\mathbf{v}}^1\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}^1-\hat{\mathbf{v}}^2\right\|_{Y_T^1}+CT^{\frac{1}{q'}}\left\|\hat{\mathbf{v}}^2\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}^1-\hat{\mathbf{v}}^2\right\|_{Y_T^1}\leqslant CT^{\delta}R\left\|w^1-w^2\right\|_{Y_T}.\end{aligned}$$

From the approximation argument in [4, page 15], we know that a weak derivative with respect to time does exist for G. Hence, substituting G by the form (2.36) and using integration by parts, we have

$$\begin{aligned}\langle\partial_t G(\cdot,t),\phi\rangle_{W_q^{-1}\times W^1_{q',0}}&=\frac{\mathrm{d}}{\mathrm{d}t}\langle G(\cdot,t),\phi\rangle_{W_q^{-1}\times W^1_{q',0}}\\&=\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\langle\left(\hat{\mathbf{F}}^{-1}-\mathbb{I}\right)\hat{\mathbf{v}},\hat{\nabla}\phi\right\rangle_{L^q\times L^{q'}}-\left\langle\hat{\mathbf{v}}_{\mathrm{s}}\cdot\widehat{\operatorname{div}}\hat{\mathbf{F}}_{\mathrm{s}}^{-\top},\phi\right\rangle_{L^q\times L^{q'}}\right)\\&=\int_{\widehat{\Omega}}\left(\left(\partial_t\hat{\mathbf{F}}^{-1}\right)\hat{\mathbf{v}}+\left(\hat{\mathbf{F}}^{-1}-\mathbb{I}\right)\partial_t\hat{\mathbf{v}}\right)\cdot\hat{\nabla}\phi\mathrm{d}X\\&\quad+\int_{\Omega_{\mathrm{s}}}\left(\partial_t\hat{\mathbf{v}}_{\mathrm{s}}\cdot\widehat{\operatorname{div}}\hat{\mathbf{F}}_{\mathrm{s}}^{-\top}+\hat{\mathbf{v}}_{\mathrm{s}}\cdot\widehat{\operatorname{div}}\partial_t\hat{\mathbf{F}}_{\mathrm{s}}^{-\top}\right)\cdot\phi\mathrm{d}X\\&=\int_{\Omega_{\mathrm{f}}}\left(\partial_t\hat{\mathbf{F}}_{\mathrm{f}}^{-1}\right)\hat{\mathbf{v}}_{\mathrm{f}}\cdot\hat{\nabla}\phi\mathrm{d}X+\int_{\widehat{\Omega}}\left(\left(\hat{\mathbf{F}}^{-1}-\mathbb{I}\right)\partial_t\hat{\mathbf{v}}\right)\cdot\hat{\nabla}\phi\mathrm{d}X\\&\quad+\int_{\Omega_{\mathrm{s}}}\left(\partial_t\hat{\mathbf{v}}_{\mathrm{s}}\cdot\widehat{\operatorname{div}}\hat{\mathbf{F}}_{\mathrm{s}}^{-\top}+\partial_t\hat{\mathbf{F}}_{\mathrm{s}}^{-\top}:\hat{\nabla}\hat{\mathbf{v}}_{\mathrm{s}}\right)\cdot\phi\mathrm{d}X,\end{aligned}$$

for every $\phi\in W^1_{q',0}(\widehat{\Omega})$, where $\langle\cdot,\cdot\rangle_{X\times X'}$ denotes the duality product between a dual pair of spaces X and X'. Then according to (2.2), the Sobolev embedding $W^1_q(\widehat{\Omega})\hookrightarrow C^{0,1-n/q}(\widehat{\Omega})\hookrightarrow L^\infty(\widehat{\Omega})$ and lemma 4.1, one obtains

$$\begin{aligned}&\left\|\partial_t G(w^1)-\partial_t G(w^2)\right\|_{L^q\left(0,T;W_q^{-1}(\widehat{\Omega})\right)}\\&\leqslant\left\|\left(\partial_t\hat{\mathbf{F}}_{\mathrm{f}}^{-1}(\hat{\mathbf{v}}^1)-\partial_t\hat{\mathbf{F}}_{\mathrm{f}}^{-1}(\hat{\mathbf{v}}_{\mathrm{f}}^2)\right)\hat{\mathbf{v}}_{\mathrm{f}}^1+\partial_t\hat{\mathbf{F}}_{\mathrm{f}}^{-1}(\hat{\mathbf{v}}_{\mathrm{f}}^2)\left(\hat{\mathbf{v}}_{\mathrm{f}}^1-\hat{\mathbf{v}}_{\mathrm{f}}^2\right)\right\|_{L^q\left(0,T;L^q(\widehat{\Omega})\right)^n}\\&\quad+\left\|\left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}^1)-\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}^2)\right)\partial_t\hat{\mathbf{v}}^1+\left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}^2)-\mathbb{I}\right)\left(\partial_t\hat{\mathbf{v}}^1-\partial_t\hat{\mathbf{v}}^2\right)\right\|_{L^q\left(0,T;L^q(\widehat{\Omega})\right)^n}\\&\quad+\left\|\partial_t\hat{\mathbf{v}}_{\mathrm{s}}^1\cdot\left(\widehat{\operatorname{div}}\hat{\mathbf{F}}_{\mathrm{s}}^{-\top}(\hat{\mathbf{v}}^1)-\widehat{\operatorname{div}}\hat{\mathbf{F}}_{\mathrm{s}}^{-\top}(\hat{\mathbf{v}}_{\mathrm{s}}^2)\right)\right\|_{L^q(0,T;L^q(\Omega_{\mathrm{s}}))}\\&\quad+\left\|\left(\partial_t\hat{\mathbf{v}}_{\mathrm{s}}^1-\partial_t\hat{\mathbf{v}}_{\mathrm{s}}^2\right)\cdot\widehat{\operatorname{div}}\hat{\mathbf{F}}_{\mathrm{s}}^{-\top}(\hat{\mathbf{v}}_{\mathrm{s}}^2)\right\|_{L^q(0,T;L^q(\Omega_{\mathrm{s}}))}\\&\quad+\left\|\left(\partial_t\hat{\mathbf{F}}_{\mathrm{s}}^{-\top}(\hat{\mathbf{v}}_{\mathrm{s}}^1)-\partial_t\hat{\mathbf{F}}_{\mathrm{s}}^{-\top}(\hat{\mathbf{v}}_{\mathrm{s}}^2)\right):\hat{\nabla}\hat{\mathbf{v}}_{\mathrm{s}}^1\right\|_{L^q(0,T;L^q(\Omega_{\mathrm{s}}))}\\&\quad+\left\|\partial_t\hat{\mathbf{F}}_{\mathrm{s}}^{-\top}(\hat{\mathbf{v}}_{\mathrm{s}}^2):\left(\hat{\nabla}\hat{\mathbf{v}}_{\mathrm{s}}^1-\hat{\nabla}\hat{\mathbf{v}}_{\mathrm{s}}^2\right)\right\|_{L^q(0,T;L^q(\Omega_{\mathrm{s}}))}\\&\leqslant CT^{\frac{1}{q}-\frac{1}{r}}\left\|\hat{\mathbf{v}}^1-\hat{\mathbf{v}}^2\right\|_{Y_T^1}\left(1+T^{\frac{1}{q'}}\left\|\hat{\mathbf{v}}^1\right\|_{Y_T^1}\right)+CT^{\frac{1}{q'}}\left\|\hat{\mathbf{v}}^2\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}^1-\hat{\mathbf{v}}^2\right\|_{Y_T^1}\\&\quad+CT^{\frac{1}{q'}}\left\|\hat{\mathbf{v}}^1-\hat{\mathbf{v}}^2\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}^1\right\|_{Y_T^1}+CT^{\frac{1}{q'}}\left\|\hat{\mathbf{v}}^2\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}^1-\hat{\mathbf{v}}^2\right\|_{Y_T^1}\\&\quad+CT^{\frac{1}{q'}}\left\|\hat{\mathbf{v}}^1\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}^1-\hat{\mathbf{v}}^2\right\|_{Y_T^1}+CT^{\frac{1}{q'}}\left\|\hat{\mathbf{v}}^1-\hat{\mathbf{v}}^2\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}^2\right\|_{Y_T^1}\\&\quad+CT^{\frac{1}{q}-\frac{1}{r}}\left\|\hat{\mathbf{v}}^1-\hat{\mathbf{v}}^2\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}^1\right\|_{Y_T^1}+CT^{\frac{1}{q'}}\left\|\hat{\mathbf{v}}^2\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}^1-\hat{\mathbf{v}}^2\right\|_{Y_T^1}\\&\leqslant C(R)T^{\delta}\left\|w^1-w^2\right\|_{Y_T}.\end{aligned}$$

Then we are in the position to prove $\mathrm{tr}_\Gamma(G(w^1) - G(w^2)) \in W_q^{1-1/q,(1-1/q)/2}(\Gamma \times (0,T))$. Recalling the definition of such mixed space (2.5), we first write the norm explicitly:

$$\begin{aligned}
&\left\|\mathrm{tr}_\Gamma\left(G(w^1) - G(w^2)\right)\right\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma\times(0,T))} \\
&= \left\|\left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2)\right) : \hat{\nabla}\hat{\mathbf{v}}^1\right\|_{L^q\left(0,T;W_q^{1-\frac{1}{q}}(\Gamma)\right)} \\
&\quad + \left\|\left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2) - \mathbb{I}\right) : \left(\hat{\nabla}\hat{\mathbf{v}}^1 - \hat{\nabla}\hat{\mathbf{v}}^2\right)\right\|_{L^q\left(0,T;W_q^{1-\frac{1}{q}}(\Gamma)\right)} \\
&\quad + \left\|\left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2)\right) : \hat{\nabla}\hat{\mathbf{v}}^1\right\|_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}(0,T;L^q(\Gamma))} \\
&\quad + \left\|\left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2) - \mathbb{I}\right) : \left(\hat{\nabla}\hat{\mathbf{v}}^1 - \hat{\nabla}\hat{\mathbf{v}}^2\right)\right\|_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}(0,T;L^q(\Gamma))} =: \sum_{i=1}^4 I_i.
\end{aligned}$$

According to the trace theorem from $W_q^1(\widetilde{\Omega})$ into $W_q^{1-\frac{1}{q}}(\Gamma)$, lemmas 2.1, 4.1 and 4.2,

$$\begin{aligned}
I_1 &\leqslant C\left\|\left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2)\right) : \hat{\nabla}\hat{\mathbf{v}}^1\right\|_{L^q\left(0,T;W_q^1(\widetilde{\Omega})\right)} \leqslant C(R)T^\delta\left\|w^1 - w^2\right\|_{Y_T}, \\
I_2 &\leqslant C\left\|\left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2) - \mathbb{I}\right) : \left(\hat{\nabla}\hat{\mathbf{v}}^1 - \hat{\nabla}\hat{\mathbf{v}}^2\right)\right\|_{L^q\left(0,T;W_q^1(\widetilde{\Omega})\right)} \leqslant C(R)T^\delta\left\|w^1 - w^2\right\|_{Y_T}.
\end{aligned}$$

It follows from the definition of vector valued Sobolev–Slobodeckij spaces, lemmas 4.1 and 4.3 that

$$\begin{aligned}
I_3 &\leqslant \left(\int_0^T\int_0^t \frac{\left\|\Delta_h\left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2)\right)(t) : \hat{\nabla}\hat{\mathbf{v}}^1(t-h)\right\|_{L^q(\Gamma)}^q}{h^{1+\frac{q}{2}\left(1-\frac{1}{q}\right)}}\mathrm{d}h\mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^T\int_0^t \frac{\left\|\left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2)\right)(t) : \Delta_h\left(\hat{\nabla}\hat{\mathbf{v}}^1\right)(t)\right\|_{L^q(\Gamma)}^q}{h^{1+\frac{q}{2}\left(1-\frac{1}{q}\right)}}\mathrm{d}h\mathrm{d}t\right)^{\frac{1}{q}} \\
&\leqslant \sup_{0\leqslant t\leqslant T}\left(\int_0^t \frac{\left\|\Delta_h\left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2)\right)\right\|_{L^\infty(\Gamma)^{n\times n}}^q}{h^{1+\frac{q}{2}\left(1-\frac{1}{q}\right)}}\mathrm{d}h\right)^{\frac{1}{q}}\left\|\hat{\nabla}\hat{\mathbf{v}}^1\right\|_{L^q(0,T;L^q(\Gamma))^{n\times n}} \\
&\quad + \sup_{0\leqslant t\leqslant T}\left\|\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2)\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}\left[\hat{\nabla}\hat{\mathbf{v}}^1\right]_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}(0,T;L^q(\Gamma))^{n\times n}} \\
&\leqslant C\left(T^{\frac{1}{2q'}}\left\|\hat{\mathbf{v}}^1\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}^1 - \hat{\mathbf{v}}^2\right\|_{Y_T^1} + T^{\frac{1}{q'}}\left\|\hat{\mathbf{v}}^1 - \hat{\mathbf{v}}^2\right\|_{Y_T^1}\left\|\hat{\mathbf{v}}^1\right\|_{Y_T^1}\right) \\
&\leqslant C(R)T^\delta\left\|w^1 - w^2\right\|_{Y_T},
\end{aligned}$$

where we used the property of Δ_h that $\Delta_h(fg)(t) = \Delta_h f(t) g(t-h) + f(t)\Delta_h g(t)$. Similarly,

$$I_4 \leqslant C(R)T^\delta \left\|w^1 - w^2\right\|_{Y_T}.$$

Collecting I_i, $i = 1, \ldots, 4$, we get

$$\left\|\mathrm{tr}_\Gamma \left(G(w^1) - G(w^2)\right)\right\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma\times(0,T))} \leqslant C(R)T^\delta \left\|w^1 - w^2\right\|_{Y_T}.$$

Since the trace regularities for G on Γ and Γ_s are same, one also obtains

$$\left\|\mathrm{tr}_{\Gamma_\mathrm{s}} \left(G(w^1) - G(w^2)\right)\right\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma_\mathrm{s}\times(0,T))} \leqslant C(R)T^\delta \left\|w^1 - w^2\right\|_{Y_T}.$$

Then

$$\left\|G(w^1) - G(w^2)\right\|_{Z_T^2} \leqslant C(R)T^\delta \left\|w^1 - w^2\right\|_{Y_T},$$

Estimate of $\left\|\mathbf{H}^1(w^1) - \mathbf{H}^1(w^2)\right\|_{Z_T^3}$, $\left\|\mathbf{H}^2(w^1) - \mathbf{H}^2(w^2)\right\|_{Z_T^4}$. Since Γ is of class C^3, $\hat{\mathbf{n}}_\Gamma \in C^2(\partial\Omega_\mathrm{f})$. Then by similar estimates as for $\mathrm{tr}_\Gamma(G(w^1) - G(w^2))$, the norm of $\mathbf{H}^1(w^1) - \mathbf{H}^1(w^2)$ in Z_T^3 can be estimated as

$$\begin{aligned}
\left\|\mathbf{H}^1(w^1) - \mathbf{H}^1(w^2)\right\|_{Z_T^3} &= \left\| [\![\tilde{\mathbf{K}}(w^1) - \tilde{\mathbf{K}}(w^2)]\!]\, \hat{\mathbf{n}}_\Gamma \right\|_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}(0,T;L^q(\Gamma))^n} \\
&\quad + \left\| [\![\tilde{\mathbf{K}}(w^1) - \tilde{\mathbf{K}}(w^2)]\!]\, \hat{\mathbf{n}}_\Gamma \right\|_{L^q(0,T;W_q^{1-\frac{1}{q}}(\Gamma))^n} \\
&\leqslant C \left\|\tilde{\mathbf{K}}_\mathrm{f}(w^1) - \tilde{\mathbf{K}}_\mathrm{f}(w^2) + \tilde{\mathbf{K}}_\mathrm{s}(w^1) - \tilde{\mathbf{K}}_\mathrm{s}(w^2)\right\|_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}(0,T;L^q(\Gamma))^{n\times n}} \\
&\quad + C\left\|\left(\tilde{\mathbf{K}}(w^1) - \tilde{\mathbf{K}}(w^2)\right)\right\|_{L^q\left(0,T;W_q^1(\tilde{\Omega})\right)^{n\times n}} \leqslant C(R)T^\delta \left\|w^1 - w^2\right\|_{Y_T}.
\end{aligned}$$

As the similar situation, we can easily derive

$$\left\|\mathbf{H}^2(w^1) - \mathbf{H}^2(w^2)\right\|_{Z_T^4} \leqslant CT^\delta (1+R)^2 \left\|w^1 - w^2\right\|_{Y_T}.$$

Estimate of $\left\|F^1(w^1) - F^1(w^2)\right\|_{Z_T^5}$. For $F_\mathrm{f}^1 = \widehat{\mathrm{div}}\tilde{F}_\mathrm{f}$, we have

$$\begin{aligned}
&\left\|F_\mathrm{f}^1(w^1) - F_\mathrm{f}^1(w^2)\right\|_{Z_T^5} \\
&\quad \leqslant \left\|\tilde{F}_\mathrm{f}(w^1) - \tilde{F}_\mathrm{f}(w^2)\right\|_{L^q(0,T;W_q^1(\Omega_\mathrm{f}))^n} \\
&\quad \leqslant \hat{D}_\mathrm{f} \left\|\left(\hat{\mathbf{F}}_\mathrm{f}^{-1}(\hat{\mathbf{v}}_\mathrm{f}^1)\hat{\mathbf{F}}_\mathrm{f}^{-\top}(\hat{\mathbf{v}}_\mathrm{f}^1) - \hat{\mathbf{F}}_\mathrm{f}^{-1}(\hat{\mathbf{v}}_\mathrm{f}^2)\hat{\mathbf{F}}_\mathrm{f}^{-\top}(\hat{\mathbf{v}}_\mathrm{f}^2)\right)\hat{\nabla}\hat{c}_\mathrm{f}^1\right\|_{L^q(0,T;W_q^1(\Omega_\mathrm{f}))^n} \\
&\qquad + \hat{D}_\mathrm{f} \left\|\left(\hat{\mathbf{F}}_\mathrm{f}^{-1}(\hat{\mathbf{v}}_\mathrm{f}^2)\hat{\mathbf{F}}_\mathrm{f}^{-\top}(\hat{\mathbf{v}}_\mathrm{f}^2) - \mathbb{I}\right)\left(\hat{\nabla}\hat{c}_\mathrm{f}^1 - \hat{\nabla}\hat{c}_\mathrm{f}^2\right)\right\|_{L^q(0,T;W_q^1(\Omega_\mathrm{f}))^n} =: \mathfrak{F}_1 + \mathfrak{F}_2.
\end{aligned}$$

Lemma 4.1 and the multiplication property of $W_q^1(\Omega_\mathrm{f})$ in lemma 2.1 imply that

$$\begin{aligned}
\mathfrak{F}_1 &\leqslant C\left(\left\|\hat{\mathbf{F}}_\mathrm{f}^{-1}(\hat{\mathbf{v}}_\mathrm{f}^1)\left(\hat{\mathbf{F}}_\mathrm{f}^{-\top}(\hat{\mathbf{v}}_\mathrm{f}^1) - \hat{\mathbf{F}}_\mathrm{f}^{-\top}(\hat{\mathbf{v}}_\mathrm{f}^2)\right)\right\|_{L^\infty(0,T;W_q^1(\Omega_\mathrm{f}))^{n\times n}}\right. \\
&\quad \left. + \left\|\left(\hat{\mathbf{F}}_\mathrm{f}^{-1}(\hat{\mathbf{v}}_\mathrm{f}^1) - \hat{\mathbf{F}}_\mathrm{f}^{-1}(\hat{\mathbf{v}}_\mathrm{f}^2)\right)\hat{\mathbf{F}}_\mathrm{f}^{-\top}(\hat{\mathbf{v}}_\mathrm{f}^2)\right\|_{L^\infty(0,T;W_q^1(\Omega_\mathrm{f}))^{n\times n}}\right)\left\|\hat{\nabla}\hat{c}_\mathrm{f}^1\right\|_{L^q(0,T;W_q^1(\Omega_\mathrm{f}))^n} \\
&\leqslant C(R)T^{\frac{1}{q'}} \left\|w^1 - w^2\right\|_{Y_T},
\end{aligned}$$

and

$$\begin{aligned}\mathfrak{F}_2 &\leqslant C\Bigg(\left\|\hat{\mathbf{F}}_{\mathrm{f}}^{-1}(\hat{\mathbf{v}}_{\mathrm{f}}^2)\left(\hat{\mathbf{F}}_{\mathrm{f}}^{-\top}(\hat{\mathbf{v}}_{\mathrm{f}}^2)-\mathbb{I}\right)\right\|_{L^\infty(0,T;W^1_q(\Omega_{\mathrm{f}}))^{n\times n}} \\ &\quad+\left\|\hat{\mathbf{F}}_{\mathrm{f}}^{-1}(\hat{\mathbf{v}}_{\mathrm{f}}^2)-\mathbb{I}\right\|_{L^\infty(0,T;W^1_q(\Omega_{\mathrm{f}}))^{n\times n}}\Bigg)\left\|\hat{\nabla}\hat{c}_{\mathrm{f}}^1-\hat{\nabla}\hat{c}_{\mathrm{f}}^2\right\|_{L^q(0,T;W^1_q(\Omega_{\mathrm{f}}))^n} \\ &\leqslant C(R)T^{\frac{1}{q'}}\left\|w^1-w^2\right\|_{Y_T}.\end{aligned}$$

Then

$$\left\|F_{\mathrm{f}}^1(w^1)-F_{\mathrm{f}}^1(w^2)\right\|_{Z_T^5}\leqslant C(R)T^{\frac{1}{q'}}\left\|w^1-w^2\right\|_{Y_T}.$$

For $F_{\mathrm{s}}^1=\bar{F}_{\mathrm{s}}^1+F_{\mathrm{s}}^g=\widehat{\mathrm{div}}\tilde{F}_{\mathrm{s}}+F_{\mathrm{s}}^g$, it can be deduced similarly as for F_{f}^1 that

$$\left\|\bar{F}_{\mathrm{s}}^1(w^1)-\bar{F}_{\mathrm{s}}^1(w^2)\right\|_{Z_T^5}\leqslant C(R)T^{\frac{1}{q'}}\left\|w^1-w^2\right\|_{Y_T}.$$

Moreover,

$$\begin{aligned}&\left\|F_{\mathrm{s}}^g(w^1)-F_{\mathrm{s}}^g(w^2)\right\|_{Z_T^5} \\ &\quad\leqslant\beta\left\|\left(\hat{c}_{\mathrm{s}}^1-\hat{c}_{\mathrm{s}}^2\right)\left(1+\frac{\gamma}{\hat{\rho}_{\mathrm{s}}}\hat{c}_{\mathrm{s}}^1\right)\right\|_{L^q(\Omega_{\mathrm{s}}\times(0,T))}+\beta\left\|\hat{c}_{\mathrm{s}}^2\left(\hat{c}_{\mathrm{s}}^1-\hat{c}_{\mathrm{s}}^2\right)\right\|_{L^q(\Omega_{\mathrm{s}}\times(0,T))} \\ &\qquad+n\left\|\frac{\hat{\nabla}\hat{g}^1}{\hat{g}^1}\left(\tilde{F}_{\mathrm{s}}(w^1)-\tilde{F}_{\mathrm{s}}(w^2)+\left(\hat{\nabla}\hat{c}_{\mathrm{s}}^1-\hat{\nabla}\hat{c}_{\mathrm{s}}^2\right)\right)\right\|_{L^q(\Omega_{\mathrm{s}}\times(0,T))} \\ &\qquad+n\left\|\left(\frac{\hat{\nabla}\hat{g}^1}{\hat{g}^1}-\frac{\hat{\nabla}\hat{g}^2}{\hat{g}^2}\right)\hat{\mathbf{F}}_{\mathrm{s}}^{-1}(\hat{\mathbf{v}}_{\mathrm{s}}^2)\hat{\mathbf{F}}_{\mathrm{s}}^{-\top}(\hat{\mathbf{v}}_{\mathrm{s}}^2)\hat{\nabla}\hat{c}_{\mathrm{s}}^2\right\|_{L^q(\Omega_{\mathrm{s}}\times(0,T))}=:\sum_{i=1}^4\mathfrak{F}_i^g.\end{aligned}$$

Apparently, with $\hat{c}^i\big|_{t=0}=\hat{c}^0$, $i=1,2$,

$$\begin{aligned}\mathfrak{F}_1^g+\mathfrak{F}_2^g&\leqslant C\left\|\hat{c}_{\mathrm{s}}^1-\hat{c}_{\mathrm{s}}^2\right\|_{L^\infty(0,T;L^q(\Omega_{\mathrm{s}}))}\left\|1+\frac{\gamma}{\hat{\rho}_{\mathrm{s}}}\hat{c}_{\mathrm{s}}^1\right\|_{L^q(0,T;L^\infty(\Omega_{\mathrm{s}}))} \\ &\quad+C\left\|\hat{c}_{\mathrm{s}}^1-\hat{c}_{\mathrm{s}}^2\right\|_{L^\infty(0,T;L^q(\Omega_{\mathrm{s}}))}\left\|\hat{c}_{\mathrm{s}}^2\right\|_{L^q(0,T;L^\infty(\Omega_{\mathrm{s}}))}\leqslant C(R)T^{\frac{1}{q'}}\left\|w^1-w^2\right\|_{Y_T}.\end{aligned}$$

Proceeding the same estimates as $\tilde{F}_{\mathrm{f}}$ above, we have

$$\mathfrak{F}_3^g+\mathfrak{F}_4^g\leqslant C(R)T^{\frac{1}{q'}}\left\|w^1-w^2\right\|_{Y_T},$$

by $\hat{g}\geqslant\frac{1}{2}$ and lemma 4.1. Collecting $\mathfrak{F}_i^g$, $i=1,\ldots,4$ together, one concludes

$$\left\|F_{\mathrm{s}}^1(w^1)-F_{\mathrm{s}}^1(w^2)\right\|_{Z_T^5}\leqslant C(R)T^{\frac{1}{q'}}\left\|w^1-w^2\right\|_{Y_T}.$$

Estimate of $\left\|F^2(w^1)-F^2(w^2)\right\|_{Z_T^6}$, $\left\|F^3(w^1)-F^3(w^2)\right\|_{Z_T^7}$. Since the key ingredient here is to estimate $\tilde{F}(w^1)-\tilde{F}(w^2)$ in the space $W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma\times(0,T))$, we only give the details to handle this term. By definition,

$$\begin{aligned}&\left\|\tilde{F}(w^1)-\tilde{F}(w^2)\right\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma\times(0,T))^n} \\ &\quad=\left\|\tilde{F}(w^1)-\tilde{F}(w^2)\right\|_{L^q\left(0,T;W_q^{1-\frac{1}{q}}(\Gamma)\right)^n}+\left\|\tilde{F}(w^1)-\tilde{F}(w^2)\right\|_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}(0,T;L^q(\Gamma))^n}.\end{aligned}$$

The first term can be controlled easily by the trace theorem for $q > n$ and the estimates of $\tilde{F}$ in $\widetilde{\Omega}$ above. Namely,

$$\begin{aligned}&\left\|\tilde{F}(w^1)-\tilde{F}(w^2)\right\|_{L^q\left(0,T;W_q^{1-\frac{1}{q}}(\Gamma)\right)^n}\\&\quad\leqslant C\left\|\tilde{F}(w^1)-\tilde{F}(w^2)\right\|_{L^q(0,T;W_q^1(\widetilde{\Omega}))^n}\leqslant C(R)T^{\frac{1}{q'}}\left\|w^1-w^2\right\|_{Y_T}.\end{aligned}$$

For the second term, again by the definition of vector-valued Sobolev–Slobodeckij space, we have

$$\left\|\tilde{F}(w^1)-\tilde{F}(w^2)\right\|_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}(0,T;L^q(\Gamma))^n}\leqslant C(R)T^{\frac{1}{2q'}}\left\|w^1-w^2\right\|_{Y_T},$$

following the argument of estimating $\mathrm{tr}_\Gamma(G(w^1)-G(w^2))$. Then,

$$\left\|F^2(w^1)-F^2(w^2)\right\|_{Z_T^6}+\left\|F^3(w^1)-F^3(w^2)\right\|_{Z_T^7}\leqslant C(R)T^\delta\left\|w^1-w^2\right\|_{Y_T}.$$

Estimate of $\left\|F^4(w^1)-F^4(w^2)\right\|_{Z_T^8}$, $\left\|F^5(w^1)-F^5(w^2)\right\|_{Z_T^8}$. Observing that the nonlinearities in F^4 and F^5 are $\hat{c}_s\hat{c}_s^*$ and $\hat{c}_s\hat{g}$, which are all quadratic, we control them under the assumptions $\hat{c}^i|_{t=0}=\hat{c}^0$, $\hat{c}_s^*|_{t=0}=0$, $\hat{g}^i|_{t=0}=1$, $i=1,2$, and by

$$\|uv\|_{L^q(0,T;W_q^1(\Omega_s))}\leqslant M_q\|u\|_{L^\infty(0,T;W_q^1(\Omega_s))}\|v\|_{L^q(0,T;W_q^1(\Omega_s))},$$

for $u,v\in W_q^1(0,T;W_q^1(\Omega_s))$. Hence,

$$\left\|F^4(w^1)-F^4(w^2)\right\|_{Z_T^8}+\left\|F^5(w^1)-F^5(w^2)\right\|_{Z_T^8}\leqslant C(R)T^{\frac{1}{q'}}\left\|w^1-w^2\right\|_{Y_T}.$$

Consequently, we derive (4.5). Now, choosing $w^1=w$ and $w^2=(0,0,0,0,1)$ in (4.5), (4.4) follows immediately from the fact that $\mathscr{M}(0,0,0,0,1)=0$.

□

Proof (Proof of theorem 2.1). Since $\mathscr{L}: Y_T\to Z_T\times\mathcal{D}_q$ is an isomorphism as showed in proposition 4.1, and because of the estimates in theorem 3.1, we can set a well-defined constant

$$C_{\mathscr{L}}:=\sup_{0\leqslant T\leqslant 1}\left\|\mathscr{L}^{-1}\right\|_{\mathcal{L}(Z_T\times\mathcal{D}_q,Y_T)}.$$

We choose $R>0$ so large that $R\geqslant 2C_{\mathscr{L}}\left\|(\hat{\mathbf{v}}^0,\hat{c}^0)\right\|_{\mathcal{D}_q}$. Then

$$\left\|\mathscr{L}^{-1}\mathscr{N}(0,w_0)\right\|_{Y_T}\leqslant C_{\mathscr{L}}\left\|(\hat{\mathbf{v}}^0,\hat{c}^0)\right\|_{\mathcal{D}_q}\leqslant\frac{R}{2}.\tag{4.7}$$

For $\left\|w^i\right\|_{Y_T}\leqslant R$, $i=1,2$, we take $T_R>0$ small enough such that

$$C_{\mathscr{L}}C(R)T_R^\delta\leqslant\frac{1}{2},$$

where $C(R)$ is the constant in (4.5). Then for $0<T<T_R$, we infer from theorem 4.2 that

$$\begin{aligned}&\left\|\mathscr{L}^{-1}\mathscr{N}(w^1,w_0)-\mathscr{L}^{-1}\mathscr{N}(w^2,w_0)\right\|_{Y_T}\\&\quad\leqslant C_{\mathscr{L}}C(R)T^\delta\left\|w^1-w^2\right\|_{Y_T}\leqslant\frac{1}{2}\left\|w^1-w^2\right\|_{Y_T},\end{aligned}\tag{4.8}$$

which implies the contraction property. From (4.7) and (4.8), we have

$$\left\|\mathscr{L}^{-1}\mathscr{N}(w,w_0)\right\|_{Y_T}\leqslant\left\|\mathscr{L}^{-1}\mathscr{N}(0,w_0)\right\|_{Y_T}+\left\|\mathscr{L}^{-1}\mathscr{N}(w,w_0)-\mathscr{L}^{-1}\mathscr{N}(0,w_0)\right\|_{Y_T}\leqslant R.$$

We define $\mathcal{M}_{R,T}$ by

$$\mathcal{M}_{R,T} := \left\{ w \in \overline{B_{Y_T}(0,R)} : w = (\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s^*, \hat{g}), \quad \hat{g}|_{t=0} = 1, \quad \hat{c}|_{t=0} = \hat{c}^0 \right\},$$

which is a closed subset of Y_T. Hence, $\mathscr{L}^{-1}\mathscr{N} : \mathcal{M}_{R,T} \to \mathcal{M}_{R,T}$ is well-defined for all $0 < T < T_R$ and a strict contraction. Since Y_T is a Banach space, the Banach fixed-point theorem implies the existence of a unique fixed-point of $\mathscr{L}^{-1}\mathscr{N}$ in $\mathcal{M}_{R,T}$, i.e. (2.21)–(2.34) admits a unique strong solution in $\mathcal{M}_{R,T}$ for small time $0 < T < T_R$.

In the following, we prove the uniqueness of solutions in Y_T by a continuity argument. Let $w^1, w^2 \in Y_T$ be two different solutions of (2.21)–(2.34) and $\widetilde{R} := \max\{\|w^1\|_{Y_T}, \|w^2\|_{Y_T}\}$, then there is a time $T_{\widetilde{R}} \leqslant T$ such that $\mathscr{L}^{-1}\mathscr{N} : \mathcal{M}_{\widetilde{R},T_{\widetilde{R}}} \to \mathcal{M}_{\widetilde{R},T_{\widetilde{R}}}$ is a contraction and therefore $w^1\big|_{[0,T_{\widetilde{R}}]} = w^2\big|_{[0,T_{\widetilde{R}}]}$. Now we argue by contradiction. We define $\widetilde{T}$ as

$$\widetilde{T} := \sup\left\{ T' \in (0,T] : w^1\big|_{[0,T']} = w^2\big|_{[0,T']} \right\},$$

and assume $\widetilde{T} < T$. Since $w^1\big|_{[0,\widetilde{T}]} = w^2\big|_{[0,\widetilde{T}]}$, we consider $w^1\big|_{t=\widetilde{T}} = w^2\big|_{t=\widetilde{T}}$ as the initial value for (2.21)–(2.34). Repeating the argument above, we see that there is a time $\widehat{T} \in (\widetilde{T}, T)$ such that $w^1\big|_{[\widetilde{T},\widehat{T}]} = w^2\big|_{[\widetilde{T},\widehat{T}]}$, which contradicts the definition of $\widetilde{T}$.

In conclusion, (2.21)–(2.34) admits a unique solution in Y_T.

For the nonnegativity of $\hat{c}$, we show it in Eulerian coordinates. Let $U_T = (\Omega^t \backslash \Gamma^t) \times (0,T)$, $U_{f,T} = \Omega_f^t \times (0,T)$, $U_{s,T} = \Omega_s^t \times (0,T)$, and define the parabolic boundary $\partial_P U_{f,T} := (\overline{\Omega}_f^0 \times \{0\}) \cup (\Gamma^t \times [0,T])$, $\partial_P U_{s,T} := (\overline{\Omega}_s^0 \times \{0\}) \cup ((\Gamma^t \cup \Gamma_s^t) \times [0,T])$ and $\partial_P U_T := \partial_P U_{f,T} \cup \partial_P U_{s,T}$. First of all, we claim that $c \in C^{2,1}_{loc}(U_T) \cap C(\overline{U_T})$, where

$$C^{2s,s}(U_T) := \left\{ c(\cdot,t) \in C^{2s}(\Omega^t \backslash \Gamma^t),\ c(x,\cdot) \in C^s(0,T),\ \forall x \in \Omega^t \backslash \Gamma^t, t \in (0,T) \right\},$$

for $s > 0$. As shown above, we assume that $c \in Y_T^3$ is the solution of

$$\partial_t c - D\Delta c = -(\mathbf{v} \cdot \nabla c + (\operatorname{div} \mathbf{v} + \beta)c) =: f. \tag{4.9}$$

With the regularity of $\mathbf{v}, c$ and embedding theorems, we know that $f \in C^{\alpha,\alpha/2}_{\mathrm{loc}}(U_T)$ for some $0 < \alpha < 1$. By the local regularity theory for parabolic equations, one obtains

$$c \in C^{2+\alpha, 1+\frac{\alpha}{2}}_{\mathrm{loc}}(U_T) \hookrightarrow C^{2,1}_{\mathrm{loc}}(U_T).$$

The continuity of c can be derived directly from the lemma 2.3, especially (2.6) with

$$W_q^1 \hookrightarrow C^{1-\frac{n}{q}} \hookrightarrow C^0, \text{for } q > n.$$

Now, given a nonnegative initial value $c^0(x) \geqslant 0$, $x \in \Omega^0$. Define $c_\lambda := e^{-\lambda t} c$ where $\lambda > 0$ is a constant, which will be assigned later. Adding cc_λ to the both sides of (4.9), we have the equation for c_λ

$$\partial_t c_\lambda - D\Delta c_\lambda + \mathbf{v} \cdot \nabla c_\lambda + (\operatorname{div} \mathbf{v} + c + \beta + \lambda)c_\lambda = c^2 e^{-\lambda t} \geqslant 0.$$

Taking λ sufficiently large such that

$$\beta + \lambda \geqslant \sup_{0 \leqslant t \leqslant T, x \in \Omega^t \backslash \Sigma^t} |\operatorname{div} \mathbf{v}| + |c|,$$

one obtains

$$\mathrm{div}\mathbf{v} + c + \beta + \lambda \geqslant 0.$$

By the weak maximum principle for parabolic equations, we have

$$\min_{\overline{U}_{f,T}} c_{\mathrm{f}}(x,t) \geqslant -\max_{\partial_P U_{f,T}} c_{\mathrm{f}}^{-}(x,t), \quad \min_{\overline{U}_{s,T}} c_{\mathrm{s}}(x,t) \geqslant -\max_{\partial_P U_{s,T}} c_{\mathrm{s}}^{-}(x,\tau),$$

namely,

$$\min_{\overline{U}_T} c(x,t) \geqslant -\max_{\partial_P U_T} c^{-}(x,t),$$

where $c^{-}(x,t) := -\min\{c(x,t),0\}$.

Since $c^0(x) \geqslant 0$, now we claim that $c(x,t) \geqslant 0$ for all $(x,t) \in (\Gamma^t \cup \Gamma_{\mathrm{s}}^t) \times [0,T]$. To this end, we argue by contradiction. Assume that for some $t_0 \in (0,T]$, there exists a point $x_0 \in \Gamma^{t_0} \cup \Gamma_{\mathrm{s}}^{t_0}$, such that

$$c(x_0,t_0) = -\max_{x\in\Gamma^{t_0}\cup\Gamma_{\mathrm{s}}^{t_0}} c^{-}(x,t_0) < 0,$$

that is,

$$\min_{x\in\Gamma^{t_0}\cup\Gamma_{\mathrm{s}}^{t_0}} \min\{c(x,t_0),0\} < 0.$$

This implies that $x \mapsto \min\{c(x,t_0),0\}$ attains a negative minimum at x_0, i.e. $x \mapsto c(x,t_0)$ attains a negative minimum at x_0.

Case 1: $x_0 \in \Gamma^{t_0}$. For both $\Omega_{\mathrm{f}}^{t_0}$ and $\Omega_{\mathrm{s}}^{t_0}$, since Γ^{t_0} is assumed to be a C^{3-} interface, we infer from Hopf's Lemma that

$$D_{\mathrm{f}}\nabla c_{\mathrm{f}} \cdot \mathbf{n}_{\Gamma^{t_0}}(x_0) < 0, \quad D_{\mathrm{s}}\nabla c_{\mathrm{s}} \cdot \mathbf{n}_{\Gamma^{t_0}}(x_0) > 0, \mathrm{on}\Gamma^{t_0}.$$

Hence,

$$[D\nabla c] \cdot \mathbf{n}_{\Gamma^{t_0}}(x_0) < 0,$$

which contradicts (1.1*i*).

Case 2: $x_0 \in \Gamma_{\mathrm{s}}^{t_0}$. Again by Hopf's Lemma, one obtains

$$D\nabla c \cdot \mathbf{n}_{\Gamma_{\mathrm{s}}^{t_0}}(x_0) < 0, \mathrm{on}\Gamma_{\mathrm{s}}^{t_0},$$

which contradicts to (1.1*j*).

In summary, $c(x,t) \geqslant 0$ for all $(x,t) \in \overline{\Omega}^t \times [0,T]$.

For $\hat{c}_{\mathrm{s}}^*$ and $\hat{g}$, we note that the equations for them in Lagrangian coordinates are ordinary differential equations with suitable $\hat{c}_{\mathrm{s}} \geqslant 0$. Then

$$\hat{c}_{\mathrm{s}}^* = \int_0^t e^{\int_t^\sigma \frac{\gamma\beta}{\hat{\rho}_{\mathrm{s}}}\hat{c}_{\mathrm{s}}(x,\tau)\mathrm{d}\tau}\beta\hat{c}_{\mathrm{s}}(x,\sigma)\mathrm{d}\sigma > 0, \quad \hat{g} = e^{\int_0^t \frac{\gamma\beta}{n\hat{\rho}_{\mathrm{s}}}\hat{c}_{\mathrm{s}}(x,\tau)\mathrm{d}\tau} > 0,$$

which completes the proof. □

Acknowledgments

Y Liu was supported by the RTG 2339 'Interfaces, Complex Structures, and Singular Limits' of the German Science Foundation (DFG, Deutsche Forschungsgemeinschaft—321821685). The support is gratefully acknowledged. The authors would like to thank the referees for their valuable comments on the previous version of the manuscript.

Appendix A. Some results on linear systems

In this section, we give several maximal L^q-regularity results of different problems, which are needed for the whole system.

A.1. Two-phase Stokes problems with Dirichlet boundary condition

In this section, we focus on the following nonstationary two-phase Stokes problem.

$$\begin{aligned} \varrho\partial_t v - \operatorname{div}(2\mu Dv) + \nabla p &= \varrho f_u, && \text{in } \Omega\backslash\Sigma \times (0,T), \\ \operatorname{div} v &= g_d, && \text{in } \Omega\backslash\Sigma \times (0,T), \\ v &= g_b, && \text{on } \partial\Omega \times (0,T), \\ [\![v]\!] &= g_u, && \text{on } \Sigma \times (0,T), \\ [\![-2\mu Dv + p\mathbb{I}]\!]\nu_\Sigma &= g, && \text{on } \Sigma \times (0,T), \\ v|_{t=0} &= v_0, && \text{in } \Omega\backslash\Sigma, \end{aligned} \tag{A.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, is a bounded domain with $\partial\Omega \in C^3$, $\Sigma \subset \Omega$ a closed hypersurface of class C^3. ϱ_j are positive constants, $j = 1,2$. $v : \Omega \times (0,T) \to \mathbb{R}^n$ is the velocity of the fluid, $p : \Omega \times (0,T) \to \mathbb{R}$ denotes the pressure. $\mu > 0$ is the constant viscosity and $Dv = \frac{1}{2}\left(\nabla v + \nabla v^\top\right)$. ν_Σ represents the unit outer normal vector on Σ. f_u, g_d, g_b, g_u, g are given functions and v_0 is the prescribed initial value. System (A.1) has been investigated by many scholars in various aspects. We refer for the maximal L_q regularity results of such kind of two-phase Stokes problem to Prüss and Simonett [37]. Readers can also find similar results in Abels and Moser [4] for $(g_b, g_u) = 0$.

Proposition A.1. *Let $q > n+2$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^3$, $\Sigma \subset \Omega$ a closed hypersurface of class C^3. Assume that $(f_u, g_d, g_b, g_u, g) \in Z_T$ where*

$$Z_T := \left\{ \begin{aligned} & f_u \in L^q\left(0,T; L^q(\Omega)\right)^n, \quad g_d \in L^q\left(0,T; W_q^1(\Omega\backslash\Sigma)\right), \\ & g_b \in W_q^{2-\frac{1}{q}, 1-\frac{1}{2q}}(\partial\Omega \times (0,T))^n, \quad g_u \in W_q^{2-\frac{1}{q}, 1-\frac{1}{2q}}(\Sigma \times (0,T))^n, \\ & g \in W_q^{1-\frac{1}{q}, \frac{1}{2}\left(1-\frac{1}{q}\right)}(\Sigma \times (0,T))^n : (g_d, g_b \cdot \nu_{\partial\Omega}, g_u \cdot \nu_\Sigma) \in W_q^1\left(0,T; \widehat{W}_q^{-1}(\Omega)\right) \end{aligned} \right\}$$

and $v_0 \in W_q^{2-\frac{2}{q}}(\Omega\backslash\Sigma)^n$ satisfying the compatibility conditions

$$\operatorname{div} v_0 = g_d|_{t=0}, \ v_0|_{\partial\Omega} = g_b|_{t=0}, \ [\![v_0]\!]|_\Sigma = g_u|_{t=0}, \ [\![(2\mu D v_0 \nu_\Sigma)_\tau]\!]|_\Sigma = g_\tau|_{t=0}. \tag{A.2}$$

Then two-phase Stokes problem (A.1) admits a unique solution (v, p) with regularity

$$\begin{aligned} & v \in L^q\left(0,T; W_q^2(\Omega\backslash\Sigma)\right)^n \cap W_q^1\left(0,T; L^q(\Omega)\right)^n, \\ & p \in L^q\left(0,T; W_{q,(0)}^1(\Omega\backslash\Sigma)\right), \quad [\![p]\!] \in W_q^{1-\frac{1}{q}, \frac{1}{2}\left(1-\frac{1}{q}\right)}(\Sigma \times (0,T)). \end{aligned}$$

Moreover, for any fixed $0 < T_0 < \infty$, there is a constant C, independent of $T \in (0, T_0]$, such that

$$\begin{aligned}&\|v\|_{L^q\left(0,T;W_q^2(\Omega\backslash\Sigma)\right)^n}+\|v\|_{W_q^1(0,T;L^q(\Omega))^n}+\|p\|_{L^q\left(0,T;W^1_{q,(0)}(\Omega\backslash\Sigma)\right)}+\|[\![p]\!]\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Sigma\times(0,T))}\\ &\leqslant C\Bigg(\|f_u\|_{L^q(0,T;L^q(\Omega))^n}+\|g_d\|_{L^q\left(0,T;W_q^1(\Omega\backslash\Sigma)\right)}+\|g_b\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(0,T))^n}\\ &\quad+\|g_u\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\Sigma\times(0,T))^n}+\|\partial_t(g_d,g_b\cdot\nu_{\partial\Omega},g_u\cdot\nu_\Sigma)\|_{L^q\left(0,T;\widehat{W}_q^{-1}(\Omega)\right)}\\ &\quad+\|g\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Sigma\times(0,T))^n}+\|v_0\|_{W_q^{2-\frac{2}{q}}(\Omega\backslash\Sigma)^n}\Bigg).\end{aligned}\tag{A.3}$$

Here, $\widehat{W}_q^{-1}(\Omega)$ is the space of all triples $(\varphi,\psi,\chi)\in L^q(\Omega)\times W_q^{2-1/q}(\partial\Omega)^n\times W_q^{2-1/q}(\Sigma)^n$, which enjoy the regularity property $(\varphi,\psi\cdot\nu_{\partial\Omega},\chi\cdot\nu_\Sigma)\in\dot{W}_q^{-1}(\Omega)=(\dot{W}^1_{q'}(\Omega))'$, where

$$\langle(\varphi,\psi\cdot\nu_{\partial\Omega},\chi\cdot\nu_\Sigma),\phi\rangle:=-\langle\varphi,\phi\rangle_\Omega+\langle\psi\cdot\nu_{\partial\Omega},\phi\rangle_{\partial\Omega}+\langle\chi\cdot\nu_\Sigma,\phi\rangle_\Sigma,\tag{A.4}$$

for all $\phi\in\dot{W}^1_{q'}(\Omega)$.

Proof. We proceed to prove this theorem with theorem 8.1.4 in [37], by which we need some special treatments for (A.1). The first one is to extend the quintuple (f_u,g_d,g_b,g_u,g) from Z_T to Z_∞. Since $f_u\in L^q(0,T;L^q(\Omega))^n$ is without time derivatives, we simply extend it by zero to a new function $\bar{f}_u=\chi_{[0,T]}f_u\in L^q(0,\infty;L^q(\Omega))^n$. Since $g_d\in L^q(0,T;W_q^1(\Omega\backslash\Sigma))\cap W_q^1(0,T;W_q^{-1}(\Omega))$, by theorem B.2 with $X_1=W_q^1(\Omega\backslash\Sigma)$, $X_0=W_q^{-1}(\Omega)$, we obtain a new function $\bar{g}_d:=\mathcal{E}(g_d)\in L^q(0,\infty;W_q^1(\Omega\backslash\Sigma))\cap W_q^1(0,\infty;W_q^{-1}(\Omega))$, which is uniformly bounded for $T\leqslant T_0$. For $(g_b,g_u,g)\in W_q^{2-1/q,1-1/2q}(\partial\Omega\times(0,T))^n\times W_q^{2-1/q,1-1/2q}(\Sigma\times(0,T))^n\times W_q^{1-1/q,(1-1/q)/2}(\Sigma\times(0,T))^n$, theorem B.3 with $\alpha=1-1/2q>1/q$ and $(1-1/q)/2>1/q$ respectively imply that they can be extended as $(\bar{g}_b,\bar{g}_u,\bar{g}):=\mathcal{E}(g_b,g_u,g)\in W_q^{2-1/q,1-1/2q}(\partial\Omega\times(0,\infty))^n\times W_q^{2-1/q,1-1/2q}(\Sigma\times(0,\infty))^n\times W_q^{1-1/q,(1-1/q)/2}(\Sigma\times(0,\infty))^n$, which are uniformly bounded for $T\leqslant T_0$. In summary,

$$(\bar{f}_u,\bar{g}_d,\bar{g}_b,\bar{g}_u,\bar{g})\big|_{[0,T]}=(f_u,g_d,g_b,g_u,g)$$

and

$$(\bar{f}_u,\bar{g}_d,\bar{g}_b,\bar{g}_u,\bar{g})\in Z_\infty.$$

Now, for a constant $\omega>\omega_0\geqslant 0$, define

$$(\tilde{f}_u,\tilde{g}_d,\tilde{g}_b,\tilde{g}_u,\tilde{g})(t)=e^{-\omega t}(\bar{f}_u,\bar{g}_d,\bar{g}_b,\bar{g}_u,\bar{g})(t).$$

Then it is easy to verify that $(\tilde{f}_u,\tilde{g}_d,\tilde{g}_b,\tilde{g}_u,\tilde{g})$ is also contained in Z_∞, since $e^{-\omega t}$ is smooth and bounded with respect to time t.

Let (u,π) be the solution of (8.4) in [37] with $(f_u,g_d,g_b,g_u,g)=(\tilde{f}_u,\tilde{g}_d,\tilde{g}_b,\tilde{g}_u,\tilde{g})$ given above, as well as the constant viscosity $\mu>0$ in (A.1). For all $t\in\mathbb{R}_+$, we define

$$v(t)=e^{\omega t}u(t),\quad p(t)=e^{\omega t}\pi(t),$$

then (v,p) solves (A.1) for $t\in[0,T]$. Consequently, existence and regularity of (u,π), which are given by theorem 8.1.4 in [37], imply those of (v,p). Additionally, (A.3) holds under our construction of (v,p).

Finally, we need to show that our solution is unique. To this end, let $(v_1,p_1)\neq(v_2,p_2)$ be two solutions of (A.1) in $(0,T)$ with same source terms and initial value. Define $(v,p)=(v_1-v_2,p_1-p_2)$. Since (A.1) is linear, (v,p) satisfies

$$
\begin{aligned}
\varrho\partial_t v - \operatorname{div}(2\mu D v) + \nabla p &= 0, && \text{in } \Omega\backslash\Sigma \times (0,T), \\
\operatorname{div} v &= 0, && \text{in } \Omega\backslash\Sigma \times (0,T), \\
v &= 0, && \text{on } \partial\Omega \times (0,T), \\
[v] &= 0, && \text{on } \Sigma \times (0,T), \\
[-2\mu D v + p\mathbb{I}]\,\nu_\Sigma &= 0, && \text{on } \Sigma \times (0,T), \\
v|_{t=0} &= 0, && \text{in } \Omega\backslash\Sigma.
\end{aligned}
\tag{A.5}
$$

Multiplying the first equation of (A.5) by v and integrating by parts over $\Omega\backslash\Sigma \times (0,t)$, one obtains

$$
\int_{\Omega\backslash\Sigma} \varrho |v(t)|^2 \,\mathrm{d}x + \int_0^t \int_{\Omega\backslash\Sigma} 2\mu |Dv(x,t)|^2 \,\mathrm{d}x\mathrm{d}t = 0, \quad \text{for a.e.} t \in (0,T),
$$

which implies the uniqueness and completes the proof. □

Remark A.1. For $(g_d, g_b \cdot \nu_{\partial\Omega}, g_u \cdot \nu_\Sigma) \in W_q^1(0,T;\widehat{W}_q^{-1}(\Omega))$, we notice that

$$
\int_\Omega g_d \mathrm{d}x = \int_{\partial\Omega} g_b \cdot \nu_{\partial\Omega} \mathrm{d}(\partial\Omega) - \int_\Sigma g_u \cdot \nu_\Sigma \mathrm{d}\Sigma,
$$

when $\phi = 1$ in (A.4), the regularity property of $\widehat{W}_q^{-1}(\Omega)$. Thus, for the zero-Dirichlet problem, which means $g_b = g_u = 0$ in (A.1), one has an hidden compatibility condition

$$
\int_\Omega g_d \mathrm{d}x = 0.
$$

This is an important condition when we solve the Stokes type problems with homogeneous Dirichlet boundary conditions.

A.2. Parabolic equations with Neumann boundary conditions

Thanks to the general maximal regularity theory for parabolic problems, for example, Prüss and Simonett [37, section 6.3], we obtain the solvability of parabolic systems with Neumann boundary conditions. Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, be a domain with compact boundary $\partial\Omega$ of class C^2, we consider the following system

$$
\begin{aligned}
\partial_t u - D\Delta u &= f, && \text{in } \Omega \times (0,T), \\
D\nabla u \cdot \nu &= g, && \text{on } \partial\Omega \times (0,T), \\
u|_{t=0} &= u_0, && \text{in } \Omega,
\end{aligned}
\tag{A.6}
$$

where u represents some physical property, for example, temperature or concentration. D is the diffusion coefficient. ν denotes the unit outer normal vector on $\partial\Omega$. f and g are give functions standing for the source or reaction term. Now we state the proposition for (A.6).

Proposition A.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, be a domain with compact boundary $\partial\Omega$ of class C^2, $q > 3$. Assume that*

$$
f \in L^q(0,T;L^q(\Omega)), \quad g \in W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Omega \times (0,T)),
$$

and $u_0 \in W_q^{2-\frac{2}{q}}(\Omega)$ satisfying the compatibility condition

$$
D\nabla u_0 \cdot \nu|_{\partial\Omega} = g|_{t=0}.
$$

Then there exists a unique solution $u \in W_q^{2,1}(\Omega \times (0,T))$ *of (A.6). Moreover,*

$$\|u\|_{W_q^{2,1}(\Omega\times(0,T))} \leqslant C\left(\|f\|_{L^q(0,T;L^q(\Omega))} + \|g\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Omega\times(0,T))} + \|u_0\|_{W_q^{2-\frac{2}{q}}(\Omega)}\right),$$

where C *does not depend on* $T \in (0,T_0]$ *for any fixed* $0 < T_0 < \infty$.

Proof. This proposition can be easily shown by means of Prüss and Simonett [37, theorem 6.3.2], for which we need to extend the right-hand sides just as in the proof of proposition A.1 and construct a solution solving (6.45) in [37]. This can be done since we established general extension theorems in appendix B. □

A.3. Laplacian transmission problems with Dirichlet boundary

In this section, we investigate a transmission problem for the Laplacian equation with Dirichlet boundary condition, which reads

$$\begin{aligned} -\Delta\psi &= f && \text{in } \Omega\backslash\Sigma, \\ [\partial_\nu \psi] &= g && \text{on } \Sigma, \\ [\psi] &= h && \text{on } \Sigma, \\ \psi &= g_b && \text{on } \partial\Omega. \end{aligned} \tag{A.7}$$

Here, we denote the inner domain by Ω^-, resp. outer domain by Ω^+ and the unit normal vector on $\Sigma = \partial\Omega^-$ by ν.

The second result concerns strong solutions.

Proposition A.3. *Let* $1 < q < \infty$, $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, *with boundary* $\partial\Omega$ *of class* C^{3-}, *and let* $\Sigma \subset \Omega$ *be a closed hypersurface of class* C^{3-}, $s \in \{0,1\}$. *For all* $f \in W_q^s(\Omega\backslash\Sigma)$, $g \in W_q^{1+s-1/q}(\Sigma)$, $h \in W_q^{2+s-1/q}(\Sigma)$, $g_b \in W_q^{2+s-1/q}(\partial\Omega)$, *the problem (A.7) admits a unique solution* $\psi \in W_q^{2+s}(\Omega\backslash\Sigma)$. *Moreover, there is a constant* $C > 0$ *such that*

$$\|\psi\|_{W_q^{2+s}} \leqslant C\left(\|f\|_{W_q^s} + \|g\|_{W_q^{1+s-\frac{1}{q}}} + \|h\|_{W_q^{2+s-\frac{1}{q}}} + \|g_b\|_{W_q^{2+s-\frac{1}{q}}}\right).$$

Proof. ***Step 1: Reduction.*** We first reduce to the case $(h, g_b) = 0$. To this end, we find a φ solving

$$\begin{aligned} -\Delta\varphi &= 0 && \text{in } \Omega^-, \\ \varphi &= h && \text{on } \Sigma, \end{aligned}$$

and

$$\begin{aligned} -\Delta\varphi &= 0 && \text{in } \Omega^+, \\ \varphi &= 0 && \text{on } \Sigma, \\ \varphi &= g_b && \text{on } \partial\Omega. \end{aligned}$$

The existence and uniqueness of these two systems are clear due to elliptic theory. Thanks to the trace theorem, the extra outer normal derivatives terms on Σ enjoys the same regularities as g. Substracting φ from ψ, we can investigate the reduced system (A.7) with $(h, g_b) = 0$.

Step 2: Weak solution with L^2-setting. Now, let $H^k = W_2^k$ and $H_0^k = W_{2,0}^k$ for $k \in \mathbb{N}$. Testing (A.7) by a function $\phi \in H_0^1(\Omega)$ and integrating by parts, one obtains

$$\int_{\Omega\backslash\Sigma} \nabla\psi \cdot \nabla\phi \mathrm{d}x = \int_{\Omega\backslash\Sigma} f\phi \mathrm{d}x - \int_\Sigma g\phi \mathrm{d}\Sigma =: \langle F, \phi\rangle_{H^{-1}\times H_0^1},$$

as a result of the regularities of f and g. The Lax–Milgram Lemma implies existence of a unique weak solution $\psi \in H_0^1(\Omega)$ to (A.7) with $(h, g_b) = 0$.

Step 3: Truncation. Since the problem (A.7) with Neumann boundary conditions on $\partial\Omega$ has been uniquely solved, see e.g. Prüss and Simonett [37, proposition 8.6.1], we show the proposition by a truncation method. More specifically, we choose a cutoff function $\eta \in C_0^\infty(\Omega)$ such that

$$\eta(x) = \begin{cases} 1, & \text{in a neighborhood of } \Omega^-, \\ 0, & \text{in a neighborhood of } \Omega^+. \end{cases}$$

We decompose $\psi = \eta\psi + (1-\eta)\psi =: u_1 + u_2$, where u_1 solves

$$\begin{aligned} -\Delta u_1 &= \eta f - 2\nabla\eta\cdot\nabla\psi + \psi\Delta\eta =: f^1 && \text{in } \Omega\backslash\Sigma, \\ [\partial_\nu u_1] &= [\partial_\nu \psi] = g && \text{on } \Sigma, \\ [u] &= [\psi] = 0 && \text{on } \Sigma, \\ \partial_\nu u_1 &= 0 && \text{on } \partial\Omega, \end{aligned}$$

weakly and u_2 solves

$$\begin{aligned} -\Delta u_2 &= (1-\eta)f + 2\nabla\eta\cdot\nabla\psi - \psi\Delta\eta =: f^2 && \text{in } \Omega, \\ u_2 &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Step 4: Improving the regularity. From step 2, we already know that (A.7) admits a unique weak solution ψ enjoying the regularity $\nabla\psi \in L^2(\Omega)$, which means $f^i \in L^2(\Omega)$ in step 3. By classical elliptic theory and [37], one obtains $u_1 \in H^2(\Omega\backslash\Sigma)$, $u_2 \in H_0^1(\Omega)\cap H^2(\Omega)$. Then $\psi \in H_0^1(\Omega)\cap H^2(\Omega\backslash\Sigma)$. Moreover,

$$\nabla\psi \in H^1(\Omega\backslash\Sigma) \hookrightarrow \begin{cases} L^p(\Omega\backslash\Sigma), & \text{if } 1 \leqslant p < \infty, & n = 2, \\ L^p(\Omega\backslash\Sigma), & \text{if } 1 \leqslant p \leqslant p^* := \dfrac{2n}{n-2}, & n > 2, \end{cases}$$

due to the Sobolev Embedding Theorem. For $n=2$, the right-hand side terms f^1 and f^2 in Step 3 are contained in $L^p(\Omega\backslash\Sigma)$, $1 \leqslant p < \infty$. Consequently with $p = q$, proposition 8.6.1 and corollary 7.4.5 in Prüss and Simonett [37] indicate that $u_1 \in W_q^2(\Omega\backslash\Sigma)$ and $u_2 \in W_q^2(\Omega)$, which implies $\psi \in W_q^2(\Omega\backslash\Sigma)$. For $n>2$, we have $f^i \in L^{p^*}$, $i = 1, 2$. Again by regularity results in [37], we have $u_1 \in W_{p^*}^2(\Omega\backslash\Sigma)$ and $u_2 \in W_{p^*}^2(\Omega)$ and hence

$$\nabla\psi \in W_{p^*}^1(\Omega\backslash\Sigma) \hookrightarrow \begin{cases} L^p(\Omega\backslash\Sigma), & 1 \leqslant p < \infty, & n = q^*, \\ L^p(\Omega\backslash\Sigma), & 1 \leqslant p \leqslant p^{**} := \dfrac{np^*}{n-p^*}, & n > p^*, \\ C^\alpha(\overline{\Omega\backslash\Sigma}), & 0 < \alpha \leqslant 1 - \dfrac{n}{p^*} & 2 < n < p^*. \end{cases}$$

For the first and third cases, we find $f^i \in L^p(\Omega\backslash\Sigma)$, $i = 1, 2$, $1 \leqslant p < \infty$, and then get the regularity of ψ. For the second case, we know $p^{**} = \frac{np^*}{n-p^*} > p^*$. Therefore, by a bootstrapping argument, we can always increase the space index until we obtain L^q. Thus, by proposition 8.6.1

and corollary 7.4.5 in Prüss and Simonett [37], one obtains $u_1 \in W_q^2(\Omega\backslash\Sigma)$ and $u_2 \in W_q^2(\Omega)$, i.e. $\psi \in W_q^2(\Omega\backslash\Sigma)$ with the estimate

$$\|\psi\|_{W_q^2(\Omega\backslash\Sigma)} \leqslant C\left(\|f\|_{L^q(\Omega\backslash\Sigma)} + \|g\|_{W_q^{1-\frac{1}{q}}(\Sigma)} + \|h\|_{W_q^{2-\frac{1}{q}}(\Sigma)} + \|g_b\|_{W_q^{2-\frac{1}{q}}(\partial\Omega)}\right),$$

for some constant $C>0$. Then as above, one gets $f^i \in W_q^1(\Omega\backslash\Sigma)$, $i = 1,2$. With the help of proposition 8.6.1 and corollary 7.4.5 in Prüss and Simonett [37], we have the desired regularity and estimate with $s = 1$.

□

Appendix B. Extension of Sobolev–Slobodeckij space

In this section, we are intended to construct an extension operator from $W_q^s(0,T;X)$ to $W_q^s(0,\infty;X)$, where $s \in (\frac{1}{q},1]$ and X is a Banach space. The main feature is that the operator norms can be bounded independent of $T>0$, compared to the extension theorem for general Sobolev–Slobodeckij spaces. The reason we made such modification here is that if the constant depends on T, then the extended norm may blow up for small T, which is the case we addressed in this paper. For example, in the proof of theorem 5.4 in [18], the extension from $W_q^s(\Omega)$ to $W_q^s(\mathbb{R}^n)$ with $0<s<1$, several smooth functions ψ_j satisfying $0 \leqslant \psi_j \leqslant 1$ and $\sum_{j=0}^k \psi_j = 1$ are chosen to construct the extension operator. In the case $|\Omega| \to 0$, we have $\nabla\psi_j \to \infty$, which means that the extension is not valid. To avoid such problem, we employ an *even* extension and make use of the embedding results in Simon [43]. Now, we give the extension theorem.

Theorem B.1. *Let $q \geqslant 1$, $s=0$, or $s \in (\frac{1}{q},1]$, $T>0$ and X be a Banach space. Then there exists an extension operator $E_T : {}_0W_q^s(0,T;X) \to W_q^s(0,\infty;X)$, where ${}_0W_q^s(0,T;X) = \{u \in W_q^s(0,T;X) : u|_{t=0} = 0, \text{if} s > \frac{1}{q}\}$, such that $E_T(u)|_{[0,T]} = u$ and*

$$\|E_T(u)\|_{W_q^s(0,\infty;X)} \leqslant C\|u\|_{{}_0W_q^s(0,T;X)},$$

where $C>0$ depends on s, q and does not depend on T.

Proof. The proof is divided into three cases, namely, $s=0$, $\frac{1}{q}<s<1$ and $s=1$.

Case 1: $s = 0$. In this situation, $W_q^s(0,T;X)$ is just the Lebesgue space $L^q(0,T;X)$, which does not contain any time regularity. Hence for any function $u \in L^q(0,T;X)$, we can take the zero extension.

Case 2: $s = 1$. With $u|_{t=0} = 0$, we apply an even extension to u in $[0,T]$ around T to $[0,2T]$ and zero extension for $T>2T$ such that the extended function $\bar{u}$ is weakly differentiable with

$$\partial_t\bar{u}(t) = \begin{cases} \partial_t u(t), & \text{if} \quad 0 \leqslant t \leqslant T, \\ -\partial_t u(2T-t), & \text{if} \quad T<t\leqslant 2T, \\ 0, & \text{if} \quad t>2T. \end{cases}$$

Then we have

$$\|\bar{u}\|_{W_q^1(0,\infty;X)} = 2^{\frac{1}{q}}\|u\|_{W_q^1(0,T;X)}.$$

Case 3: $\frac{1}{q} < s < 1$. With the same extension as in Case 2, we define the same function $\tilde{u}$. Now we are in the position to show $\tilde{u} \in W_q^s(0,\infty;X)$, for which we only need to prove

$[\tilde{u}]_{W_q^s(0,\infty;X)} \leqslant C[u]_{W_q^s(0,T;X)}$, where C is independent of T. From the definition of Sobolev–Slobodeckij space,

$$[\tilde{u}]^q_{W_q^s(0,\infty;X)} = \int_0^T\int_0^T \frac{\|u(t)-u(\tau)\|_X^q}{|t-\tau|^{1+sq}}\mathrm{d}t\mathrm{d}\tau + \int_T^{2T}\int_T^{2T}\frac{\|u(2T-t)-u(2T-\tau)\|_X^q}{|t-\tau|^{1+sq}}\mathrm{d}t\mathrm{d}\tau$$
$$+2\int_0^T\int_T^{2T}\frac{\|u(t)-u(2T-\tau)\|_X^q}{|t-\tau|^{1+sq}}\mathrm{d}\tau\mathrm{d}t + 2\int_0^{2T}\int_{2T}^{\infty}\frac{\|\tilde{u}(t)\|_X^q}{|t-\tau|^{1+sq}}\mathrm{d}\tau\mathrm{d}t =: \sum_{i=1}^4 Q_i.$$

It is clear that

$$Q_1+Q_2 = 2\,[u]^q_{W_q^s(0,T;X)}.$$

Since $|t-\tau| \geqslant |t-(2T-\tau)|$ with $t\in[0,T]$ and $\tau\in[T,2T]$, we have

$$Q_3 \leqslant 2\int_0^T\int_0^T\frac{\|u(t)-u(h)\|_X^q}{|t-h|^{1+sq}}\mathrm{d}h\mathrm{d}t = 2\,[u]^q_{W_q^s(0,T;X)}.$$

Noticing that $\tilde{u}|_{t=2T}=0$ due to the even extension, we get

$$Q_4 = \frac{2}{sq}\int_0^{2T}\frac{\|\tilde{u}(2T-h)-\tilde{u}(2T)\|_X^q}{h^{sq}}\mathrm{d}h$$
$$\leqslant \frac{2}{sq}\int_0^{2T}\left(\frac{\|\tilde{u}(\cdot-h)-\tilde{u}(\cdot)\|_{L^\infty(h,2T;X)}}{h^{s-\frac{1}{q}}}\right)^q\frac{\mathrm{d}h}{h} = \frac{2}{sq}\,[\tilde{u}]^q_{B^{s-\frac{1}{q}}_{\infty,q}(0,2T;X)},$$

where the seminorm of $B^s_{p,q}(0,T;X)$ is given by

$$[f]_{B^s_{p,q}(0,T;X)} = \left(\int_0^T\left(\frac{\|\Delta_h f(t)\|_{L^p(h,T;X)}}{h^s}\right)^q\frac{\mathrm{d}h}{h}\right)^{\frac{1}{q}}$$

for $0<s<1$ and $1\leqslant p,q\leqslant\infty$. From theorem 10 in Simon [43], we know that for $\frac{1}{q}<s<1$ and $q\geqslant 1$,

$$[f]_{B^{s-\frac{1}{q}}_{\infty,q}(0,T;X)} \leqslant \frac{3\theta}{s-\frac{1}{q}}[f]_{B^s_{q,q}(0,T;X)} = \frac{3\theta}{s-\frac{1}{q}}[f]_{W_q^s(0,T;X)},\quad \forall f\in W_q^s(0,T;X),$$

where $\theta = 3^{1-(s-1/q)}$. Hence,

$$Q_4 \leqslant \frac{6\theta}{sq(sq-1)}[\tilde{u}]^q_{W_q^s(0,2T;X)} \leqslant \frac{24\theta}{sq(sq-1)}[u]^q_{W_q^s(0,T;X)}.$$

Combining the estimates of Q_i, $i=1,\ldots,4$, one obtains

$$[\tilde{u}]_{W_q^s(0,\infty;X)} \leqslant C[u]_{W_q^s(0,T;X)},$$

where $C = \left(4+\frac{24\theta}{sq(sq-1)}\right)^{1/q}$.

Now, let $E_T(u)=\tilde{u}$. Then $E_T(u)$ is well-defined from ${}_0W_q^s(0,T;X)$ to $W_q^s(0,T;X)$ as well as $E_T(u)|_{[0,T]}=u$ and

$$\|E_T(u)\|_{W_q^s(0,\infty;X)} \leqslant C\|u\|_{{}_0W_q^s(0,T;X)},$$

where $C>0$ depends on s, q and does not depend on T.

□

Next, we give an extension theorem for general functions.

Theorem B.2. *Let X_1, X_0 be two Banach spaces and $X_1 \hookrightarrow X_0$. For $1 < q < \infty$ and $0 < T < \infty$, define $X_T := L^q(0,T;X_1) \cap W_q^1(0,T;X_0)$ endowed with the norm*

$$\|u\|_{X_T} := \|u\|_{L^q(0,T;X_1)} + \|u\|_{W_q^1(0,T;X_0)} + \left\| u|_{t=0} \right\|_{X_\gamma},$$

where $X_\gamma = (X_0, X_1)_{1-1/q,q}$. Then there exists an extension operator $\mathcal{E} \in \mathcal{L}(X_T, X_\infty)$ satisfying $\mathcal{E}(u)|_{[0,T]} = u$, for all $u \in X_T$. Moreover, there is a constant $C > 0$, independent of $0 < T < \infty$, such that

$$\|\mathcal{E}(u)\|_{X_\infty} \leqslant C\|u\|_{X_T}, \tag{B.1}$$

for all $u \in X_T$.

Proof. First of all, we consider the case $u|_{t=0} = 0$. Let E be the extension operator as in theorem B.1. Define $\tilde{u} = E(u)$. Then we have $\tilde{u}|_{[0,T]} = u$ and

$$\|\tilde{u}\|_{X_\infty} \leqslant C\|u\|_{X_T},$$

where C does not depend on T.

Let $u_0 := u|_{t=0} \in X_\gamma$. Since $X_\gamma = (X_0, X_1)_{1-1/q,q}$, the trace method of interpolation implies that there exists a function $v \in X_\infty$ such that $v|_{t=0} = u_0$, see e.g. [32, proposition 1.13]. Moreover, it follows from the norm of X_T that there is a constant $C > 0$ such that

$$\|v\|_{X_\infty} \leqslant C\left\| u|_{t=0} \right\|_{X_\gamma} \leqslant C\|u\|_{X_T}.$$

Now for general $u \in X_T$, we define $w := u - v$. Then w is reduced to the case $w|_{t=0} = 0$ and can be extended to $E(w)$ in X_∞ like $\tilde{u}$. Now we define the extension operator as $\mathcal{E}(u) := w + v$. Then one obtains $\mathcal{E}(u)|_{[0,T]} = u$ and there is a constant, independent of T, such that

$$\|\mathcal{E}(u)\|_{X_\infty} \leqslant C\|w\|_{X_\infty} + C\|v\|_{X_\infty} \leqslant C\|u\|_{X_T},$$

for all $u \in X_T$, which completes the proof. □

With a similar argument, we have the following extension theorem for functions in $W_q^{2\alpha,\alpha}$.

Theorem B.3. *Let Σ be a compact sufficiently smooth hypersurface. For $1 < q < \infty$, $1/q < \alpha \leqslant 1$ and $0 < T < \infty$, let $W_q^{2\alpha,\alpha}(\Sigma \times (0,T)) := L^q(0,T;W_q^{2\alpha}(\Sigma)) \cap W_q^\alpha(0,T;L^q(\Sigma))$ be endowed with norm*

$$\|g\|_{W_q^{2\alpha,\alpha}(\Sigma\times(0,T))} := \|g\|_{L^q(0,T;W_q^{2\alpha}(\Sigma))} + \|g\|_{W_q^\alpha(0,T;L^q(\Sigma))} + \left\| g|_{t=0} \right\|_{W_q^{2\alpha-\frac{2}{q}}(\Sigma)}.$$

Then for $g \in W_q^{2\alpha,\alpha}(\Sigma \times (0,T))$, there exists an extension operator $\mathcal{E} \in \mathcal{L}(W_q^{2\alpha,\alpha}(\Sigma \times (0,T)), W_q^{2\alpha,\alpha}(\Sigma \times (0,\infty)))$ satisfying $\mathcal{E}(g)|_{[0,T]} = g$. Moreover, there is a constant $C > 0$, independent of $0 < T < \infty$, such that

$$\|\mathcal{E}(g)\|_{W_q^{2\alpha,\alpha}(\Sigma\times(0,\infty))} \leqslant C\|g\|_{W_q^{2\alpha,\alpha}(\Sigma\times(0,T))}. \tag{B.2}$$

Remark B.1. The proof is similar to what in theorem B.2, for which it relies on theorem B.1 for $1/q < \alpha < 1$ and the trace method interpolation, namely,

$$W_q^{2\alpha-\frac{2}{q}}(\Sigma) = \left\{ g(0) : g \in L^q(0,T;W_q^{2\alpha}(\Sigma)) \cap W_q^\alpha(0,T;L^q(\Sigma)) \right\},$$

see e.g. lemma 2.4 or [37, example 3.4.9(i)]. These results can also be extended to more general anisotropic Sobolev–Slobodeckij spaces with general trace theorem, see e.g. [37, theorem 3.4.8].

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