Correlators of heavy–light quark currents in HQET: OPE at three loops

K.G. Chetyrkin a,b,c, A.G. Grozin d,e,*

a Institut für Theoretische Teilchenphysik, Karlsruher Institut für Technologie, Karlsruhe, Germany
b Institut für Theoretische Physik, Universität Regensburg, 93040 Regensburg, Germany
c II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany
d Budker Institute of Nuclear Physics, Novosibirsk, Russia
e Novosibirsk State University, Novosibirsk, Russia

Received 23 December 2021; received in revised form 4 February 2022; accepted 5 February 2022
Available online 11 February 2022
Editor: Tommy Ohlsson

Abstract

Coefficient functions of the operator product expansion of correlators of HQET heavy–light quark currents are calculated up to operators of dimension 4 up to 3 loops.

© 2022 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP3.

1. Heavy–light currents and their correlators in HQET

QCD problems with a single heavy quark $Q$ having momentum $P = Mv + p$ ($M$ is its on-shell mass, $v^2 = 1$) can be described by heavy quark effective theory [1] (HQET, see, e.g., [2–4]) if its characteristic residual momentum is small ($p \ll M$), and characteristic momenta of light quarks and gluons are also small. QCD operators are expanded in $1/M$, the coefficients are HQET operators of corresponding dimensionalities. For example, QCD heavy–light quark currents at the leading order in $1/M$ are equal to the matching coefficients times the HQET heavy–light currents. These matching coefficients are known at 2 [5,6] and 3 loops [7]. Anomalous dimensions of all HQET heavy–light currents are the same and known at 2 [8–10] and 3 loops [11]. Cor-

* Corresponding author.
E-mail addresses: konstantin.chetyrkin@partner.kit.edu (K.G. Chetyrkin), A.G.Grozin@inp.nsk.su (A.G. Grozin).
relators of such currents at small distances can be calculated using operator product expansion (OPE); coefficient functions of operators up to dimension 3 are known up to 2 loops [12–14]. The \( \langle G^2 \rangle \) contribution vanishes at 1 loop; the \( \langle G^2 \rangle \) one is known at 1 loop [12]. Contributions of quark condensates up to dimension 8 are known at tree level [12,15]. Here we calculate the perturbative contribution expanded up to \( m^4 \) (\( m \) is the light-quark mass) and condensate contributions up to dimension 4 at 3 loops. The perturbative spectral densities of correlators of some QCD heavy–light currents with \( m = 0 \) in the threshold region at 3 loops were calculated [16]; they are related to the HQET spectral density by the corresponding matching coefficients.

If our heavy quark is \( b \), there are 2 different HQETs: with \( c \) quark and without it. The heavy–light HQET currents in these 2 theories are related by the decoupling coefficient, which is known up to 3 loops [17]. The HQET current in HQET with \( c \) is related to the QCD currents by the matching coefficients. In this paper we shall work in HQET without \( c \) quarks. There are \( n_f = 3 \) dynamic flavors (\( u, d, s \)) and the static \( b \) quark.

At the leading order in \( 1/M \) the heavy-quark spin does not interact with gluon field. We may rotate it at will without affecting physics (heavy-quark spin symmetry [18]). We may even switch it off (superflavor symmetry [19]). We shall use the effective theory of a scalar static antiquark. This particle has no antiparticle; its field \( \phi^* \) contains only annihilation operators. Its coordinate-space free propagator in the \( v \) rest frame is \( \delta(\vec{x})S_0(x^0) \) where \( S_0(t) = -i\theta(t) \). The momentum-space propagator \( S_0(p) = 1/(p^0 + i0) \) does not depend on \( \vec{p} \). Static-quark lines cannot form loops.

We consider the current

\[
j_0 = \phi^*_0 q_0 = Z_j(\alpha_s(\mu)) j(\mu)
\]

where \( q_0 \) is the bare light-quark field. The correlator of 2 currents in the \( v \) rest frame

\[
\langle T j_0(x) \bar{j}_0(0) \rangle = \delta(\vec{x}) \Pi_0(x^0)
\]

is non-zero only for \( x^0 \geq 0 \) (the symbol \( T \) is superfluous: the product \( j_0(0) j_0(x) = 0 \)). The momentum-space correlator

\[
\int d^4x \langle T j_0(x) \bar{j}_0(0) \rangle e^{ip\cdot x} = \Pi_0(p^0)
\]

(3) does not depend on \( \vec{p} \). They are related by the 1-dimensional Fourier transform

\[
\Pi_0(\omega) = \int_0^\infty d\tau \Pi_0(t)e^{i\omega t}, \quad \Pi_0(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \Pi_0(\omega)e^{-i\omega t}.
\]

(4) The correlator \( \Pi_0(\omega) \) has a cut from 0 to \( +\infty \), the discontinuity gives the spectral density

\[
\rho_0(\omega) = \frac{1}{2\pi} [\Pi_0(\omega + i0) - \Pi_0(\omega - i0)].
\]

(5) The correlator is expressed via the spectral density by the dispersion representation:

\[
\Pi_0(\omega) = i \int_0^\infty \frac{dv \rho_0(v)}{\omega - v + i0}, \quad \Pi_0(t) = \theta(t) \int_0^\infty d\omega \rho_0(\omega)e^{-i\omega t}.
\]

(6) We can analytically continue \( \Pi_0(t) \) from \( t > 0 \) to \( t = -i\tau, \tau > 0 \) and obtain the Euclidean correlator
The spectral density can be reconstructed from it by the inverse Mellin transform
\[
\rho_0(\omega) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\tau \Pi_0(\tau)e^{\omega\tau} ,
\]
where \( a \) is to the right from all singularities of \( \Pi_0(\tau) \).

Borel transform of the correlator \( \Pi(\omega) \) is often used in sum rules. In HQET it is defined by
\[
\hat{B}_E F(\omega) = \lim_{k \to \infty} \left( \frac{-\omega}{k} \right)^{k+1} \frac{k!}{(d/d\omega)^k} F(\omega) \biggr|_{\omega=-E_k} .
\]
It is equivalent to the correlator in imaginary time \( \Pi(\tau) \). For example, for the function
\[
F(\omega) = \frac{1}{(v-\omega-i0)^n} \quad \text{it is} \quad \hat{B}_E F(\omega) = \frac{e^{-v/E}}{\Gamma(n)E^{n-1}}.
\]
The Fourier transform (4) of \( F(\omega) \) is
\[
F(t) = i\theta(t) \frac{(it)^{n-1}}{\Gamma(n)} e^{-i(v-i0)t} ;
\]
its analytical continuation from the half-axis \( t > 0 \) to the half-axis \( t = -i\tau, \tau > 0 \) is
\[
F(\tau) = i\frac{\tau^{n-1}}{\Gamma(n)} e^{-v\tau} .
\]
Therefore,
\[
\hat{B}_E \Pi(\omega) = -i \Pi(\tau = 1/E) .
\]

The static-antiquark propagator in a gluon field is \( \delta(\vec{x}_1 - \vec{x}_0)S_0(x_1^0 - x_0^0)[x_1, x_0] \), where
\[
[x_1, x_0] = P \exp \left[ -ig_0 \int_{x_0}^{x_1} dx_\mu A_\mu^\alpha(x) \right]
\]
is the Wilson line in the antiquark representation, the integral is taken along the straight line from \( x_0 \) to \( x_1 \). Therefore, the correlator can be written as
\[
\Pi_0(t) = \langle q_0(\vec{v}t)[\vec{v}t, 0]\bar{q}_0(0) \rangle .
\]
We can consider a more general object [20]
\[
F_0(x) = \langle q_0(x)[x, 0]\bar{q}_0(0) \rangle ,
\]
where \( x \) is not necessarily timelike. The bilocal vacuum average can be expressed via \( F_0(x) \):
\[
\langle \bar{q}_0(0)[0, x]\Gamma q(x) \rangle = -\text{Tr} \Gamma F_0(x) ,
\]
where [0, x] is the Wilson line in the quark (fundamental) representation, and \( \Gamma \) is a Dirac matrix.
The correlator of the $\overline{\text{MS}}$ renormalized currents $j(\mu)$ still contains ultraviolet (UV) divergences when $t = 0$. Subtracting these divergences we obtain the renormalized correlator $\Pi(t; \mu)$. The dispersion representation should contain 3 subtractions:

\[
\Pi(\omega; \mu) = -i\omega^3 \int_0^\infty \frac{d\varepsilon}{\varepsilon^3(\varepsilon - \omega - i0)} + \sum_{n=0}^{2} c_n\omega^n,
\]

\[
\Pi(t; \mu) = \theta(t) \int_0^\infty d\omega \rho(\omega; \mu)e^{-i\omega t} + \sum_{n=0}^{2} c_n t^n \delta(t).
\] (15)

Divergences of the correlator of renormalized currents are in subtraction terms in (15): in coordinate space they are at $t = 0$, in momentum space they are polynomial in $\omega$. More exactly, in dimensional regularization only $c_2$ contains $1/\varepsilon^n$ divergences, whereas power divergences in $c_{0,1}$ are not seen in this scheme. The renormalized spectral density is simply given by $\rho_0(\omega) = Z_j^2(\alpha_s(\mu))\rho(\omega; \mu)$.

The correlator has 2 Dirac structures

\[
\Pi = A + B\gamma.
\] (16)

It is convenient to introduce the currents with definite parities $P = \pm 1$:

\[
j_P = \frac{1 + P\gamma}{2} j.
\] (17)

Their correlators are

\[
P \Pi P \frac{1 + P\gamma}{2} \quad \text{where} \quad \Pi_P = P(A + PB) = \frac{P}{4} \text{Tr}(1 + P\gamma)\Pi
\] (18)

(we shall see how it is convenient to introduce the factor $P$ here).

For sufficiently large $-\omega$ the operator product expansion (OPE) is valid

\[
\Pi(\omega; \mu) = \sum_i C_i(\omega; \mu)\langle O_i(\mu) \rangle,
\] (19)

where $O_i$ are all possible operators. If the $q$ mass is small, we can include operators with powers of $m(\mu)$ in the set $O_i$ and calculate the Wilson coefficients $C_i$ treating $q$ as massless. Then the terms with even-dimensional $O_i$ have Dirac structure $\gamma$, and those with odd-dimensional $O_i$ have the structure $1$.

Currently we are in the world where the antiquark $\bar{Q}$ has quantum numbers $0^+$. Then $S$-wave $\bar{Q}q$ mesons have $j^P = \frac{1}{2}^+$, and $P$-wave ones $\frac{1}{2}^-$ and $\frac{3}{2}^-$. The currents $j_{\pm}$ have quantum numbers of $\frac{1}{2}^\pm$ mesons (currents with quantum numbers of mesons with $j > \frac{1}{2}$ necessarily involve derivatives, we don’t consider them). The matrix elements of our currents are

\[
\langle 0 \mid j_P(\mu) \mid M \rangle = F(\mu)u,
\] (20)

where the meson states are normalized as

\[
\langle M, \bar{p}' \mid \bar{M}, \bar{p} \rangle = (2\pi)^3 \delta(\bar{p}' - \bar{p})
\] (21)

in the rest frame, and $u$ is the Dirac wave function of the $\frac{1}{2}^P$ meson $M$ satisfying $\bar{y}u = Pu$ and normalized as $u^+u = 1$. The contribution of the meson $M$ to the correlator $\Pi_P$ and its spectral density $\rho_P$ is
where \( \tilde{\Lambda} \) is the residual energy of this meson. If there are several mesons with given quantum numbers, we get sums of contributions (22); sums become integrals in the continuum spectrum.

Now let’s switch on the spin (and parity) \( \frac{3}{2}^- \) of the static antiquark \( \bar{Q} \) (still at \( M = \infty \)). The static antiquark is now described by the field \( \bar{h} \) satisfying \( \bar{h} \gamma = -\bar{h} \). The free propagator of this field contains the extra factor \( (1 - \gamma_5)/2 \) as compared to the scalar case. The currents are

\[
j_{\gamma 0} = \bar{h}_0 \gamma q_0 ;
\]

in the \( v \) rest frame the set of independent Dirac structures \( \Gamma \) is \( 1, \gamma_i, \gamma^{i[l} \gamma^{j]} , \gamma^{i[l} \gamma^{j] \gamma^k} \), where square brackets mean antisymmetrization. Instead of this set, we can use \( 1, \gamma_5^{HV}, \gamma^i, \gamma_5^{HV} \gamma^i \), where \( \gamma_5^{HV} \) is the ’t Hooft–Veltman \( \gamma_5 \). In HQET renormalized currents with the anticommuting \( \gamma_5^{AC} \) coincide with the corresponding currents with \( \gamma_5^{HV} \), because their anomalous dimensions are the same [5] (contrary to the QCD case). Therefore, in the following we shall just use \( \gamma_5 \). The correlator of \( j_1^+ \) and \( j_2^- \) is

\[
\Pi_{12} = - \text{Tr} \tilde{\Gamma}_1 \frac{1 - \gamma^0}{2} \Gamma_2 \Pi ,
\]

where \( \Pi \) is the correlator with the scalar static antiquark, and minus comes from the fermion loop. The Dirac matrices \( \Gamma \) either commute or anticommute with \( \gamma^0 : \Gamma \gamma^0 = -P \gamma^0 \Gamma \). Both \( \Gamma_{1,2} \) must have the same \( P \) (otherwise the correlator vanishes), and

\[
\Pi_{12} = -\Pi \frac{1}{2} \text{Tr} \tilde{\Gamma}_1 \Gamma_2 .
\]

The same formula works for the spectral densities.

\( S \)-wave mesons with light-fields quantum numbers \( j^P = \frac{1^+}{2} \) become degenerate doublets \( 0^- , 1^- ; P^- \)-wave ones with \( j^P = \frac{1^-}{2} , \frac{3^-}{2} \) form degenerate doublets \( 0^+ , 1^+ + 1^+ , 2^+ \). The currents with \( \Gamma \) anticommuting with \( \gamma^0 (\gamma_5, \gamma^i : P = +1) \) have quantum numbers of the \( S \)-wave \( 0^- , 1^- \) mesons \( (j^P = \frac{1^+}{2}) \); those with \( \Gamma \) commuting with \( \gamma^0 (1, \gamma_5 \gamma^i : P = -1) \) have quantum numbers of the \( P^- \)-wave \( 0^+ , 1^+ \) mesons \( (j^P = \frac{1^-}{2}) \). The spectral density of correlator of the currents with quantum numbers of \( 0^\pm \) mesons is \( 2 \rho_{\pm} \), and for \( 1^\pm \) mesons it is \( 2 \rho_{\pm \delta ij} \) (Table 1, eq. (25); this is the reason why we introduced the factor \( P \) in (18)). A \( 0^- \) meson contribution to the spectral density is \( |F_0^-|^2 \delta (\omega - \tilde{\Lambda}) \); for \( 1^- \) one it is \( |F_1^-|^2 \delta \delta ij \delta (\omega - \tilde{\Lambda}) \), where

\[
\langle 0| \bar{h} \gamma q |0^- \rangle = F_0^- (\mu) , \quad \langle 0| \bar{h} \gamma q |1^- \rangle = F_1^- (\mu) \tilde{e}
\]

\[
\langle 0| \bar{h} \gamma q |0^- \rangle = F_0^- (\mu) , \quad \langle 0| \bar{h} \gamma q |1^- \rangle = F_1^- (\mu) \tilde{e}
\]
(\vec{e} \text{ is the polarization vector of the } 1^- \text{ meson}). Therefore

$$F_0^-(\mu) = F_1^-(\mu) = \sqrt{2} F(\mu)$$  \hspace{1cm} (27)$$

(this is an example of the heavy-quark spin symmetry). The case of 0^+, 1^+ \text{ mesons is similar.}

Usually the relativistic normalization of one-particle states is used:

$$r \langle M, P | M, P \rangle = (2\pi)^3 2 P^0 \delta (\vec{P}' - \vec{P})$$  \hspace{1cm} (28)$$

(it becomes meaningless when the meson mass \( M \to \infty \), and thus is not usable in HQET; in the meson rest frame \( |M\rangle_r = \sqrt{2M|M\rangle} \)). The spin-symmetry result (27) can be written in a completely Lorentz-invariant way:

$$\langle 0|\vec{h} \Gamma q |M\rangle_r = \sqrt{M} F(\mu) \text{Tr} \Gamma M, \quad M = \frac{1 + \gamma^\tau}{2} \times \begin{cases} \gamma^5 - \gamma^\tau \gamma^5 & \text{for } 0^- , \\ \gamma^\tau & \text{for } 1^- . \end{cases}$$

For example \( P^\mu = M v^\mu \) is the meson momentum

$$\langle 0|\vec{h} \gamma^5 q^\mu |0^- \rangle_r = \frac{2 F(\mu)}{\sqrt{M}} M^\mu , \quad \langle 0|\vec{h} \gamma^5 q |0^- \rangle_r = \frac{2 F(\mu)}{\sqrt{M}} M ,$$

$$\langle 0|\vec{h} \gamma^\mu q^\tau |1^- \rangle_r = \frac{2 F(\mu)}{\sqrt{M}} Me^\mu , \quad \langle 0|\vec{h} \frac{1}{2}[\gamma^\mu, \gamma^\tau] q |1^- \rangle_r = \frac{2 F(\mu)}{\sqrt{M}} (e^\mu P^\tau - e^\tau P^\mu) .$$

Similarly, for \( 0^+, 1^+ \) mesons \( j^P = \frac{1}{2}^- \)

$$M = \frac{1 - \gamma^\tau}{2} \times \begin{cases} 1 & \text{for } 0^+ , \\ \gamma^5 \gamma^\tau & \text{for } 1^+ . \end{cases} \quad \psi M = \mathcal{M} = - \mathcal{M}.$$  \hspace{1cm} (30)$$

Of course, phases of \( |M\rangle \) states (and hence of \( \mathcal{M} \) can be redefined.

The vacuum average (13) for a general (timelike or spacelike) \( x \) has 2 Dirac structures

$$F_0(x) = -\frac{1}{4} \left[ F_S(x^2) - i \sqrt{x} F_V(x^2) \right] ,$$

$$F_S(x^2) = \langle \bar{q}(0)[0,x]q(x) \rangle , \quad F_V(x^2) = \frac{i}{x^2} \langle \bar{q}(0)[0,x]\gamma_5 q(x) \rangle .$$

(31)

In HQET \( x = ut \), and the scalar functions \( F_{S,V} \) have positive argument \( x^2 = t^2 \). If \( x \) is spacelike, the argument \( x^2 \) is negative. When we analytically continue HQET results to \( t = -i\tau \), we obtain \( F_{S,V} \) of negative argument \( -\tau^2 \).

2. Perturbative contribution

Perturbative contributions to the correlator are shown in Fig. 1: the one-loop diagram; one of three two-loop ones; and two examples of three-loop diagrams.

We use integration by parts (IBP) to reduce three-loop diagrams to master integrals with the C++ program\(^1\) FIRE6 [24]. Generation of Feynman diagram was done with QGRAF [25] and evaluation of color factors with the FORM [26] package COLOR [27].

\(^1\) We have also used the Mathematica program LiteRed 1.4 [21,22] and the REDUCE package Grinder [23] for testing purposes and the identification of the master integrals.
There are three non-trivial master integrals. Two of them are known exactly as hypergeometric functions with $\varepsilon$: [28] and [23,29]; for the last one, only a few terms of the $\varepsilon$ expansion are known [16], but this is sufficient for our purpose. This IBP procedure and the master integrals are reviewed in [30].

The renormalized perturbative correlator is

$$\Pi_P(\tau; \mu) = \frac{N_c}{\pi^2} \left( \frac{1}{2\tau^3} \left[ 1 + C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left( 6L_\tau + 4 \left( \frac{\pi^2}{3} + 2 \right) \right) \right] + C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[ 18L_\tau^2 + \left( \frac{40}{3}\pi^2 + 43 \right) L_\tau - 8\xi_3 + \frac{8}{45}\pi^4 + \frac{52}{3}\pi^2 + \frac{153}{8} \right] \right. \right.$$ 

$$+ C_A \left[ 22L_\tau^2 + \left( \frac{76}{9}\pi^2 + 75 \right) L_\tau - 104\xi_3 - \frac{8}{45}\pi^4 - \frac{5}{27}\pi^2 + \frac{6413}{72} \right] \left. \right.$$ 

$$- T_{F nl} \left[ 8L_\tau^2 + 4 \left( \frac{8}{9}\pi^2 + 7 \right) L_\tau - 32\xi_3 - \frac{16}{27}\pi^2 + \frac{589}{18} \right] \right\} \right] + P \frac{m^2}{4\tau^2} \left[ 1 + C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left( 12L_\tau + 4 \left( \frac{\pi^2}{3} + 3 \right) \right) \right]$$

$$+ C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[ 72L_\tau^2 + 2 \left( \frac{32}{3}\pi^2 + 71 \right) L_\tau - 20\xi_3 + \frac{8}{45}\pi^4 + \frac{52}{3}\pi^2 + \frac{233}{4} \right] \right. \right.$$ 

$$+ C_A \left[ 44L_\tau^2 + \frac{2}{3} \left( \frac{38}{3}\pi^2 + 205 \right) L_\tau - 116\xi_3 - \frac{8}{45}\pi^4 + \frac{31}{27}\pi^2 + \frac{4981}{36} \right] \left. \right.$$ 

$$- T_{F nl} \left[ 16L_\tau^2 + \frac{8}{3} \left( \frac{4}{3}\pi^2 + 17 \right) L_\tau - 32\xi_3 - \frac{16}{27}\pi^2 + \frac{401}{9} \right] \right\} \right]$$

$$- \frac{m^2}{8\tau} \left[ 1 + 6C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left( 3L_\tau + 1 \right) \right] + C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[ 162L_\tau^2 + \left( \frac{16}{3}\pi^2 + 109 \right) L_\tau + 40\xi_3 - \frac{32}{45}\pi^4 - 12\pi^2 + \frac{507}{8} \right] \right. \right.$$ 

$$+ C_A \left[ 66L_\tau^2 - \left( \frac{4}{3}\pi^2 - 125 \right) L_\tau - 22\xi_3 - \frac{4}{15}\pi^4 - \pi^2 + \frac{2789}{24} \right] \left. \right.$$ 

$$- T_{F nl} \left[ 24L_\tau^2 + 36L_\tau + \frac{229}{6} \right] \right\} \right]$$

$$- 2 \sum \frac{m_i^2}{\tau} C_F T_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left( \frac{\pi^2}{3} - 2 \right)$$
\[ + P \frac{m^3}{8} \left[ L_\tau + C_F \frac{\alpha_s}{4\pi} \left[ 12L^2_\tau + 10L_\tau - 2 \left( \frac{2}{3}\pi^2 - 3 \right) \right] \right. \\
+ C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[ 96L^3_\tau + 2 \left( \frac{8}{3}\pi^2 + 59 \right) L^2_\tau + \left( 16\zeta_3 - \frac{76}{3}\pi^2 + \frac{249}{2} \right) L_\tau \right. \\
- 140\zeta_5 + \frac{64}{3}\pi^2\zeta_3 + 134\zeta_3 - \frac{26}{15}\pi^4 - \frac{40}{3}\pi^2 + \frac{91}{24} \right] \\
+ C_A \left[ \frac{88}{3}L^3_\tau - \frac{2}{3}(2\pi^2 - 161) L^2_\tau - \left( 16\zeta_3 + \frac{97}{9}\pi^2 - \frac{889}{6} \right) L_\tau \right. \\
+ 60\zeta_5 - \frac{20}{3}\pi^2\zeta_3 + 89\zeta_3 - \frac{\pi^4}{45} - \frac{277}{27}\pi^2 + \frac{4357}{72} \right] \\
- T_F n_l \left[ \frac{32}{3}L^3_\tau + \frac{104}{3} L^2_\tau - \frac{2}{3} \left( \frac{16}{3}\pi^2 - 71 \right) L_\tau + 32\zeta_3 - \frac{80}{27}\pi^2 + \frac{245}{18} \right] \right\} \\
- P m \left( \sum m_i^2 \right) C_F T_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left( 6L_\tau + \frac{\pi^2}{3} - \frac{2}{3} \right) \\
- \frac{m^4\tau}{32} \left[ L_\tau - \frac{1}{4} + C_F \frac{\alpha_s}{4\pi} \left[ 18L^2_\tau - \frac{17}{2} L_\tau - \frac{8}{3}\pi^2 + \frac{45}{2} \right] \right. \\
+ C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[ 186L^3_\tau + \left( \frac{16}{3}\pi^2 - \frac{259}{2} \right) L^2_\tau + \left( 16\zeta_3 - \frac{212}{3}\pi^2 + \frac{4155}{8} \right) L_\tau \right. \\
- 420\zeta_5 + 64\pi^2\zeta_3 + 351\zeta_3 - \frac{20}{9}\pi^4 - \frac{113}{9}\pi^2 - \frac{10609}{32} \right] \\
+ C_A \left[ \frac{154}{3}L^3_\tau - \left( \frac{4}{3}\pi^2 - \frac{63}{2} \right) L^2_\tau - \left( 16\zeta_3 + \frac{200}{9}\pi^2 - \frac{2603}{8} \right) L_\tau \right. \\
+ 180\zeta_5 - 20\pi^2\zeta_3 + 179\zeta_3 - \frac{\pi^4}{15} + \frac{835}{108}\pi^2 - \frac{64801}{288} \right] \\
- T_F n_l \left[ \frac{56}{3}L^3_\tau + 2L^2 - \left( \frac{64}{9}\pi^2 - \frac{235}{2} \right) L + 64\zeta_3 + \frac{32}{27}\pi^2 - \frac{6137}{72} \right] \right\} \\
+ m^2 \frac{\sum m_i^2}{4} \tau C_F T_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left( 6L_\tau + \frac{17}{6} \right) \\
- \sum m_i^2 \tau C_F T_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left[ \frac{\pi^2}{3} - \frac{11}{4} \right] L_\tau - 3\zeta_3 - \frac{\pi^2}{2} + \frac{65}{8} \right] \\
+ \mathcal{O}(m^5, \alpha_s^3) \right\}, \] (32)

where \( \alpha_s = \alpha_s^{(n_i)}(\mu) \), \( m = m^{(n_i)}(\mu) \) is the mass of the quark \( q \) in our current (1), \( m_i = m_i^{(n_i)}(\mu) \) are all light-flavor masses (see the last diagram in Fig. 1), and

\[ L_\tau = \log \frac{\mu \tau e^{\gamma_E}}{2}. \] (33)

The coefficient functions \( C_{m^n}(\mu) \) with \( n = 0, 1, 2 \) satisfy simple renormalization group (RG) equations (while \( m^3 \) mixes with \( \tilde{q}\tilde{q} \), and \( m^4 \) mixes with \( m\tilde{q} \) and \( G^2 \), Sect. 3). Its solution is
\[ C_m \sim \exp \left\{ \frac{\alpha_s}{4\pi} \left( -2\gamma_0 L_\tau + c_1 \right) + \left( \frac{\alpha_s}{4\pi} \right)^2 \left[ -2\beta_0 \gamma_0 L_\tau^2 - 2(\gamma_1 - \beta_0 c_1) L_\tau + c_2 \right] \right\} + \mathcal{O}(\alpha_s^3) \],
\[ \gamma_k = 2\gamma_{jk} - n\gamma_{mk}, \quad \gamma_0 = -6C_F(n+1). \] (34)

Here
\[ \gamma_a(\alpha_s) = \frac{d\log Z_a}{d\log \mu} = \sum_{n=0}^{\infty} \gamma_{an} \left( \frac{\alpha_s}{4\pi} \right)^{n+1} \quad (a = j, m), \]
\[ \beta(\alpha_s) = \frac{1}{2} \frac{d\log Z_a}{d\log \mu} = \sum_{n=0}^{\infty} \beta_n \left( \frac{\alpha_s}{4\pi} \right)^{n+1}. \]

Our results (32) for \( n = 0, 1, 2 \) satisfy this condition.

The renormalized spectral density of the OPE terms having dimensionalities \( \leq 2 \) is
\[ \rho_p^{d\leq 2}(\omega; \mu) = \frac{N_c}{\pi^2} \left\{ \frac{\omega^2}{4} - 1 - C_F \frac{\alpha_s}{4\pi} \left( 6\omega \right) - \frac{3}{4} \pi^2 - 17 \right\} + C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[ 18L_\omega^2 - \left( \frac{40}{3} \pi^2 + 97 \right) L_\omega - 8\xi_3 + \frac{8}{45} \pi^4 + \frac{103}{3} \pi^2 + \frac{1173}{8} \right] \right. \\
+ C_A \left[ 22L_\omega^2 - \left( \frac{76}{9} \pi^2 + 141 \right) L_\omega - 104\xi_3 - \frac{8}{45} \pi^4 + \frac{238}{27} \pi^2 + \frac{20057}{72} \right] \\
- T_{FN I} \left[ 8L_\omega^2 - 4 \left( \frac{8}{9} \pi^2 + 13 \right) L_\omega - 32\xi_3 + \frac{92}{27} \pi^2 + \frac{1849}{18} \right] \right\} \\
+ \frac{P}{4} \left[ 1 - 4C_F \frac{\alpha_s}{4\pi} \left( 3L_\omega - \frac{\pi^2}{3} - 6 \right) \right] \\
+ 4C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[ 18L_\omega^2 - \left( \frac{16}{3} \pi^2 + \frac{143}{2} \right) L_\omega - 5\xi_3 + \frac{2}{45} \pi^4 + \frac{20}{3} \pi^2 + \frac{1377}{16} \right] \right. \\
+ C_A \left[ 11L_\omega^2 - \frac{1}{3} \left( \frac{19}{3} \pi^2 + \frac{337}{2} \right) L_\omega - 29\xi_3 - \frac{2}{45} \pi^4 + \frac{61}{108} \pi^2 + \frac{13069}{144} \right] \\
- T_{FN I} \left[ 4L_\omega^2 - \frac{2}{3} \left( \frac{4}{3} \pi^2 + 29 \right) L_\omega - 8\xi_3 + \frac{2}{27} \pi^2 + \frac{1097}{36} \right] \right\} \\
- \frac{m^2}{8} \left[ 1 - 6C_F \frac{\alpha_s}{4\pi} (3L_\omega - 1) \right] \\
+ 2C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[ 81L_\omega^2 - \left( \frac{8}{3} \pi^2 + \frac{109}{2} \right) L_\omega + 20\xi_3 - \frac{16}{45} \pi^4 - \frac{39}{2} \pi^2 + \frac{507}{16} \right] \right. \\
+ C_A \left[ 33L_\omega^2 + \left( \frac{2}{3} \pi^2 - \frac{125}{2} \right) L_\omega - 11\xi_3 - \frac{2}{15} \pi^4 - 6\pi^2 + \frac{2789}{48} \right] \\
- T_{FN I} \left[ 12L_\omega^2 - 18L_\omega - 2\pi^2 + \frac{229}{12} \right] \right\} \\
- \frac{2}{3} \left( \sum m_i^2 \right) C_F T_F \left( \frac{\alpha_s}{4\pi} \right)^2 (\pi^2 - 6) + \mathcal{O}(\alpha_s^3) \right\} \right. \]
\[ (35) \]

where
Terms up to two loops agree with [12]; the remaining ones are new. Multiplying the leading $m^0$ term in the HQET spectral density (35) by the corresponding matching coefficients [5,6], we reproduce the leading $\delta^0$ terms in the 3-loop QCD spectral densities (10), (14) in [16].

3. Quark and gluon condensates (dimensions 3 and 4)

Some 0-, 1-, and 2-loop diagrams for the quark condensate contribution are shown in Fig. 2.

\[
L_\omega = \log \frac{2\omega}{\mu}. \tag{36}
\]

Starting from 2 loops (the last diagram in the figure) contributions proportional to the singlet sum $\sum m_i \langle \bar{q}_i q_i \rangle$ appear. Our result for the coordinate-space correlator\(^2\) is

\[
\Pi^0_F(\tau; \mu) = -P \frac{(\bar{q}q)}{4} \left\{ 1 + 6C_F \frac{\alpha_s}{4\pi} \right. \\
+ C_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ 2C_F \left[ 4 \left( \frac{2}{3} \pi^2 - 1 \right) \right] + \frac{1}{3} \pi^2 - 7 \right\} L_\tau - \frac{16\zeta_3 + 10\pi^2 + 8}{3} \right. \\
- C_A \left[ 4 \left( \frac{\pi^2}{3} - 7 \right) \right] L_\tau - \frac{8\zeta_3 + \pi^2 - \frac{149}{3}}{2} - 16T_F n_l \left( L_\tau + \frac{4}{3} \right) \right. \\
+ C_F \left( \frac{\alpha_s}{4\pi} \right)^3 \left\{ C_F^2 \left[ 4 \left( 18\zeta_3 + \frac{4}{9} \pi^4 + \frac{8}{3} \pi^2 - 35 \right) \right] L_\tau \right. \\
+ \frac{1600}{3} \zeta_3 - \frac{928}{9} \pi^2 \zeta_3 - \frac{140}{3} \zeta_3 + \frac{479}{135} \pi^4 - \frac{8}{9} \pi^2 + 157 \right. \\
+ C_F C_A \left[ \frac{176}{3} \left( \frac{2}{3} \pi^2 - 1 \right) L_\tau^2 - 4 \left( 141\zeta_3 - \frac{4}{45} \pi^4 - \frac{902}{27} \pi^2 - \frac{737}{9} \right) L_\tau \right. \\
- \frac{1}{3} \left( 1216\zeta_5 - \frac{424}{3} \pi^2 \zeta_3 + \frac{23654}{9} \zeta_3 - \frac{3799}{270} \pi^4 - \frac{27122}{81} \pi^2 - \frac{23669}{27} \right) \right. \\
- C_A^2 \left[ \frac{88}{3} \left( \frac{2}{3} - 7 \right) \right] L_\tau^2 - 4 \left( 33\zeta_3 + \frac{2}{15} \pi^4 - \frac{164}{27} \pi^2 + \frac{1409}{9} \right) L_\tau \right. \\
- 72\zeta_5 + 12\pi^2 \zeta_3 - \frac{4856}{27} \zeta_3 + \frac{199}{810} \pi^4 + \frac{4094}{243} \pi^2 - \frac{69583}{81} \right. \\
\left. \right\}.
\]

\(^2\) We have used the well-known method of projectors [31,32] for computation of various condensate contributions (at 1 loop a similar method was used in [33,34]).
\[-2C_FT_Fn_l\left[\frac{32}{3} \left(\frac{2}{3} \pi^2 - 1\right) L_\tau^2 - 32 \left(3\zeta_3 - \frac{22}{27} \pi^2 - \frac{25}{9}\right) L_\tau\right.
\quad\left. - \frac{1}{27} \left(5936\zeta_3 - \frac{100}{3} \pi^4 - \frac{3536}{9} \pi^2 - \frac{10253}{3}\right)\right] \]
\[+ 2C_AT_Fn_l\left[16 \left(\frac{\pi^2}{9} - 6\right) L_\tau^2 - 4 \left(6\zeta_3 - \frac{32}{27} \pi^2 + \frac{217}{3}\right) L_\tau\right.
\quad\left. - \frac{1}{27} \left(2258\zeta_3 - \frac{89}{15} \pi^4 - \frac{830}{9} \pi^2 + \frac{27736}{3}\right)\right] \]
\[+ \frac{32}{3} \left(T_Fn_l\right)^2 \left(4L_\tau^2 + \frac{32}{3} L_\tau + \frac{109}{9}\right)\}
\[+ \frac{m(\bar{q}q)\tau}{16} \left\{1 + 2C_F \frac{\alpha_s}{4\pi} \left(3L_\tau - \frac{5}{2}\right)\right. \]
\[+ 4C_F \left(\frac{\alpha_s}{4\pi}\right)^2 \left[C_F \left[9L_\tau^3 + \frac{4}{3} \pi^2 + \frac{35}{4}\right] L_\tau - 8\zeta_3 + \frac{8}{3} \pi^2 - \frac{547}{32}\right] \]
\[+ C_A \left[\frac{11}{2} L_\tau^3 - \frac{1}{3} \left(\pi^2 + \frac{61}{4}\right) L_\tau + 2\zeta_3 - \frac{3}{4} \pi^2 + \frac{3415}{96}\right] \]
\[+ T_Fn_l \left(2L_\tau^2 - \frac{5}{3} L_\tau + \frac{335}{24}\right) \}
\[+ 4C_F \left(\frac{\alpha_s}{4\pi}\right)^3 \left[C_F^2 \left[9L_\tau^3 + 2(4\pi^2 - 15) L_\tau - \left(30\zeta_3 + \frac{4}{9} \pi^4 - 4\pi^2 + \frac{1393}{16}\right) L_\tau\right.ight.
\quad\left. + \frac{1}{3} \left(\frac{1450\zeta_3}{3} - \frac{712}{3} \pi^2 \zeta_3 + \frac{145\zeta_3}{3} + \frac{431}{180} \pi^4 + \frac{461}{6} \pi^2 - \frac{20141}{32}\right)\right] \]
\[+ C_F C_A \left[33L_\tau^3 + \frac{1}{3} \left(\frac{70}{3} \pi^2 - \frac{403}{2}\right) L_\tau^2\right.
\quad\left. - \left(129\zeta_3 - \frac{4}{45} \pi^4 - \frac{2551}{54} \pi^2 + \frac{6997}{144}\right) L_\tau\right.
\quad\left. - \frac{1}{3} \left(754\zeta_3 - \frac{256}{3} \pi^2 \zeta_3 + \frac{10715}{9} \zeta_3 - \frac{8119}{1080} \pi^4 - \frac{16439}{324} \pi^2 - \frac{207275}{432}\right)\right] \]
\[+ C_A^2 \left[\frac{242}{9} L_\tau^3 - \frac{1}{9} \left(22\pi^2 + \frac{365}{2}\right) L_\tau^2 + \left(33\zeta_3 + \frac{2}{15} \pi^4 - \frac{362}{27} \pi^2 + \frac{26281}{54}\right) L_\tau\right.
\quad\left. + \frac{67}{4} \zeta_5 - \frac{5}{2} \pi^2 \zeta_3 + \frac{2951}{27} \zeta_3 - \frac{419}{1620} \pi^4 - \frac{10103}{3888} \pi^2 - \frac{675449}{7776}\right]\]
\[- C_F T_Fn_l \left[12L_\tau^2 + \frac{4}{3} \left(\frac{8}{3} \pi^2 - 13\right) L_\tau^2 - \left(72\zeta_3 - \frac{496}{27} \pi^2 - \frac{13}{36}\right) L_\tau\right.
\quad\left. - \frac{1}{27} \left(3058\zeta_3 - \frac{223}{15} \pi^4 - \frac{1264}{9} \pi^2 - \frac{42839}{24}\right)\right] \]
\[- C_A T_Fn_l \left[\frac{176}{9} L_\tau^3 - \frac{2}{9} (4\pi^2 + 71) L_\tau^2 + 4 \left(9\zeta_3 - \frac{34}{27} \pi^2 + \frac{2486}{27}\right) L_\tau\right.
\quad\left. + \frac{1}{27} \left(436\zeta_3 - \frac{53}{30} \pi^4 - \frac{353}{18} \pi^2 - \frac{26521}{36}\right)\right] \]
\[
\begin{align*}
&+ \frac{(T_F n_l)^2}{3}\left(\frac{32}{3} L_3^2 - \frac{40}{3} L_2^2 + \frac{1940}{9} L_\tau + 8\zeta_3 - \frac{6395}{162}\right)
\end{align*}
\]
\[
+ \left(\sum m_i \langle \bar{q} i q_i \rangle\right) \frac{\tau}{4} C_F T_F \left(\frac{\alpha_s}{4\pi}\right)^2 \left\{\frac{1}{2} \left(\frac{4}{3} \pi^2 - 11\right)\right\}
\]
\[
+ \frac{\alpha_s}{4\pi} C_F \left[\frac{5}{4}\left(\frac{4}{3} \pi^2 - 11\right) L_\tau + 12\zeta_3 + \frac{8}{45} \pi^4 - \frac{44}{9} \pi^2 + \frac{76}{3}\right]
\]
\[
+ \frac{C_A}{3} \left[\left(\frac{100}{3} \pi^2 - 311\right) L_\tau - 256\zeta_3 - \frac{7}{15} \pi^4 - \frac{211}{9} \pi^2 + \frac{3559}{6}\right]
\]
\[
- 2T_F n_l \left[\frac{4}{3} \left(\frac{4}{3} \pi^2 - 11\right) L_\tau - 16\zeta_3 - \frac{44}{27} \pi^2 + \frac{313}{9}\right]\right]\right],
\end{align*}
\]

where \(\langle \bar{q} q \rangle\) is renormalized at \(\mu\). The terms up to 2 loops in the dimension-3 contribution agree with [12].

Finiteness of the renormalized coefficient function \(C_{\bar{q}q}\) provides an independent confirmation of \(2\gamma_j - \gamma_{\bar{q}q}\) at 3 loops [11]. This anomalous dimension vanishes at 1 loop; \(\gamma_{\bar{q}q} = -\gamma_m\), hence \(C_{\bar{q}q}\) has the structure (34) with \(n = -1\). We need one more term:

\[
\begin{align*}
C_{\bar{q}q} &\sim \exp\left[\frac{\alpha_s}{4\pi} \left\{\frac{\alpha_s}{4\pi} - \left(-2\left(\gamma_1 - \beta_0 c_1\right) L_\tau + c_2\right)\right\}
\end{align*}
\]

(38)

\[
\begin{align*}
+ \left(\frac{\alpha_s}{4\pi}\right)^3 \left[3\left(-\beta_0 (\gamma_1 - \beta_0 c_1) L_\tau^2 - 2(\gamma_2 - 2\beta_0 c_2 - \beta_1 c_1) L_\tau + c_3\right) + \mathcal{O}(\alpha_s^3)\right]
\end{align*}
\]

where \(\gamma_k = 2\gamma_{jk} + \gamma_{mk}, \gamma_0 = 0\). Hence the \(\alpha_s\) term contains no \(L_\tau\), the \(\alpha_s^2\) one contains \(L_\tau^2\), etc.

The dimension-3 operators \(O_3 = \left(m^3, m \sum m_i^2, \bar{q} q \right)^T\) satisfy the renormalization group equation [35–38]

\[
\begin{align*}
\frac{dO_3}{d\log \mu} + \gamma_3 O_3 = 0, \quad \gamma_3 = \begin{pmatrix} 3\gamma_m & 0 & 0 \\ 0 & 3\gamma_m & 0 \\ \gamma & \gamma' & -\gamma_m \end{pmatrix},
\end{align*}
\]

(39)

\[
\gamma = -\frac{N_c}{4\pi^2} \left[2 + 8C_F \frac{\alpha_s}{4\pi}\right]
\]

\[
+ C_F \left[C_F (96\zeta_3 - 131) - C_A (48\zeta_3 - 109) - 20T_F n_l\right] \left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3)\right],
\]

\[
\gamma' = 24 \frac{N_c}{\pi^2} C_F T_F \left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3).
\]

Therefore, the coefficient functions \(C_3 = (C_m^3, C_m \sum m_i^2, C_{\bar{q}q})^T\) satisfy the renormalization group equation

\[
\begin{align*}
\frac{dC_3}{d\log \mu} = \frac{\partial C_3}{\partial L_\tau} - 2\beta \frac{\partial C_3}{\partial \log \alpha_s} = (\gamma_3^T - 2\gamma_j) C_3.
\end{align*}
\]

(40)

The dimension-4 operators \(m O_3\) satisfy the renormalization group equation similar to (39) but with the anomalous dimension \(\gamma_3 + \gamma_m\). Hence we obtain

\[
\begin{align*}
\left[\frac{\partial}{\partial L_\tau} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_j - 3\gamma_m\right] C_m^3 = \gamma C_{\bar{q}q}.
\end{align*}
\]
Fig. 3. One-loop gluon condensate contribution.

\[
\frac{\partial}{\partial L_{\tau}} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_j - 3\gamma_m \right] C_m \sum m_i^2 = \gamma' C_{\bar{q}q}, \\
\frac{\partial}{\partial L_{\tau}} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_j - 4\gamma_m \right] C_{m^4} = \gamma C_{m\bar{q}q}, \\
\frac{\partial}{\partial L_{\tau}} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_j - 4\gamma_m \right] C_{m^2 \sum m_i^2} = \gamma' C_{m\bar{q}q}.
\]

Our results (32), (37) satisfy these equations.

It is well known that the gluon condensate contribution vanishes at 1 loop. In the fixed-point gauge the static quark does not interact with gluons, and the only remaining diagram is shown in Fig. 3. But the \(G^2\) correction to the massless quark propagator \(S(x, 0)\) vanishes after vacuum averaging [39]. The 2- and 3-loop contributions are

\[
\Pi^G_{\bar{q}q}(\tau; \mu) = \frac{\langle G^2 \rangle}{48} T_F \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ \left[ C_F \left( \frac{4}{3} \pi^2 - 11 \right) + \frac{C_A}{2} \left( \frac{4}{3} \pi^2 - 23 \right) \right] \\
+ \frac{3\alpha_s}{4\pi} \left[ C_F^2 \left( \frac{4}{3} \pi^2 - 11 \right) L_{\tau} + 56\zeta_3 + \frac{136}{135} \pi^4 - \frac{140}{9} \pi^2 - \frac{23}{2} \right] \right. \\
+ \frac{C_F C_A}{9} \left[ \left( \frac{124}{3} \pi^2 - 449 \right) L_{\tau} - 524\zeta_3 - \frac{20}{3} \pi^4 + 100\pi^2 + 167 \right] \left[ \right. \\
+ \frac{\alpha_s}{3} \left[ \left( \frac{4}{3} \pi^2 - 23 \right) L_{\tau} - 46\zeta_3 + \frac{4}{5} \pi^4 + \frac{65}{6} \pi^2 - \frac{93}{2} \right] \\
- \frac{2}{3} C_F T_F n_f \left[ \left( \frac{4}{3} \pi^2 - 11 \right) L_{\tau} - 16\zeta_3 - \frac{44}{27} \pi^2 + \frac{259}{9} \right] \\
- \frac{1}{3} C_A T_F n_f \left[ \left( \frac{4}{3} \pi^2 - 23 \right) L_{\tau} - 16\zeta_3 - \frac{20}{27} \pi^2 + \frac{253}{9} \right] \right\},
\]

where \(G^2 = G^{\mu\nu}_a G_{a\mu\nu}\). The anomalous dimension of this operator is [40–42]

\[
\gamma_{G^2} = -2 \frac{d^2}{d log \alpha_s},
\]

and hence the coefficient function must have the structure

\[
C_{G^2} \sim \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ 1 + \frac{\alpha_s}{4\pi} \left[ 2(\beta_0 - \gamma_{j0})L_{\tau} + c \right] + O(\alpha_s^2) \right\}.
\]

Our result (41) satisfies this condition.

The flavor-singlet dimension-4 operators \(O_4 = (\sum m_i^4, (\sum m_i^2)^2, \sum m_i \bar{q}_i q_i, G^2)^T\) satisfy the renormalization group equation [35–38]
\[
\frac{dO_4}{d\log \mu} + \gamma_4 O_4 = 0, \quad (44)
\]

\[
\gamma_4 = \begin{pmatrix}
4\gamma_m & 0 & 0 & 0 \\
0 & 4\gamma_m & 0 & 0 \\
\gamma & \gamma' & 0 & 0 \\
-\frac{d\gamma}{d\log \alpha_s} & -\frac{d\gamma'}{d\log \alpha_s} & 4\frac{d\gamma_m}{d\log \alpha_s} & -2\frac{d\beta}{d\log \alpha_s}
\end{pmatrix}
\]

and the corresponding coefficient functions \( C_4 = (C_{\sum m_i^4}, C_{(\sum m_i^2)^2}, C_{\sum m_i\bar{q}_i}, C_{G^2})^T \) — the equation

\[
\frac{dC_4}{d\log \mu} = (\gamma_4^T - 2\gamma_j)C_4; \quad (45)
\]

hence,

\[
\begin{align*}
\frac{\partial}{\partial L_\tau} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_j - 4\gamma_m \left[ C_{\sum m_i^4} = \gamma C_{\sum m_i\bar{q}_i} - \frac{d\gamma}{d\log \alpha_s} C_{G^2}, \right. \\
\frac{\partial}{\partial L_\tau} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_j - 4\gamma_m \left. \left[ C_{(\sum m_i^2)^2} = \gamma' C_{\sum m_i\bar{q}_i} - \frac{d\gamma'}{d\log \alpha_s} C_{G^2}, \right. \\
\frac{\partial}{\partial L_\tau} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_j \right] C_{\sum m_i\bar{q}_i} = 4\frac{d\gamma_m}{d\log \alpha_s} C_{G^2}.
\end{align*}
\]

The second equation here is satisfied trivially, because \( C_{(\sum m_i^2)^2} = O(\alpha_s^3) \). Our results (32), (37), (41) satisfy these equations.

4. Higher-dimensional condensates

The tree diagram in Fig. 2 can be written exactly in \( x = vt \):

\[
\Pi^T(t) = i\theta(t) \langle q(vt)|v_0\bar{q}(0) \rangle. \quad (46)
\]

It is expressed via the bilocal quark condensate [43] which has 2 Dirac structures:

\[
\langle q(x)|v_0\bar{q}(0) \rangle = -\frac{\langle \bar{q}q \rangle}{4} \left[ f_S(x^2) - \frac{i\gamma_4}{d} f_V(x^2) \right]. \quad (47)
\]

Its expansion in \( x \) via local quark condensates is known up to dimension 8 [15]. We use the bases of local condensates [44]

\[
\begin{align*}
Q_3 &= \langle \bar{q}q \rangle, & Q_5 &= i\langle \bar{q} G_{\mu\nu} \sigma_{\mu\nu} q \rangle, & Q_6 &= \langle \bar{q} f q \rangle, \\
Q_1^2 &= \langle \bar{q} G_{\mu\nu} G^{\mu\nu} q \rangle, & Q_2^2 &= i\langle \bar{q} G_{\mu\nu} \tilde{G}^{\mu\nu} \gamma q \rangle, \\
Q_3^2 &= \langle \bar{q} G_{\mu\nu} \tilde{G}_{\nu\sigma}^{\mu} q \rangle, & Q_4^2 &= i\langle \bar{q} D_{\mu} \gamma_{\nu}^{\sigma} q \rangle, \\
A &= i\langle \bar{q} D_{\alpha} D_{\beta} D_{\gamma} D_{\delta} \gamma_{\alpha}^{\beta} \gamma_{\gamma}^{\delta} q \rangle, \\
Q_5^3 &= \langle \bar{q} D_{\mu} J_{\nu}^{\gamma} q \rangle, & Q_6^2 &= \langle \bar{q} D^2 f q \rangle, \\
Q_7^3 &= i\langle \bar{q} G_{\mu\nu} D_{\nu}^{\gamma} q \rangle, & Q_8^2 &= \langle \bar{q} \bar{G}_{\mu\nu} J_{\nu}^{\gamma} q \rangle, \\
Q_9^3 &= \langle \bar{q} G_{\mu\nu} J_{\nu}^{\gamma} q \rangle, & Q_6^3 &= \langle \bar{q} [\tilde{G}_{\mu\nu}, J_{\nu}^{\gamma}] q \rangle,
\end{align*}
\]

where \( G_{\mu\nu} = g G_{\mu\nu}^{a}, J_{\mu} = g J_{\mu}^{a}, J_{\mu}^{a} = D_{\nu} G_{\mu\nu}^{a} = g \sum \bar{q}_i \gamma_\mu t^a q_i; \sigma_{\mu\nu} = \gamma_{[\mu} \gamma_{\nu]} \); operators containing \( \tilde{G}_{\mu\nu} = \frac{1}{2} \varepsilon_{\nu\rho\sigma\tau} G^{\rho\sigma} \) and \( \gamma_5 = \frac{i}{2} \varepsilon_{\mu\nu\rho\sigma} \gamma^\rho \gamma^\sigma \gamma^\delta \) are understood as short notations.
for the expressions from which both \( \varepsilon \) tensors are eliminated using \( \varepsilon^{\mu
u\rho\sigma} \varepsilon_{\alpha\beta\gamma\delta} = -4! \delta^{[\alpha}_{\mu} \delta^{\nu}_{\beta} \delta^{\rho}_{\gamma} \delta^{\sigma}_{\delta]} \). The anomalous condensate \( A \) does not vanish in the \( \overline{\text{MS}} \) scheme; it is a finite combination of dimension-8 gluon condensates \([44]\). We obtain the contribution of bilinear quark condensates up to dimension 8 to the correlator at the tree level

\[
\Pi^\prime_B(\tau) = -\frac{1}{4} \left\{ P Q^3 - \frac{\tau}{d} m Q^3 - P \frac{\tau^2}{21 d} \left[ \frac{1}{2} Q^5 - m^2 Q^3 \right] \right. \\
+ \frac{\tau^3}{3! d(d+2)} \left[ \frac{1}{2} Q^6 + \frac{3}{2} m Q^5 - 3 m^3 Q^3 \right] \\
+ P \frac{\tau^4}{4! d(d+2)} \left[ 3 Q_1^7 - \frac{3}{2} Q_2^7 - 3 Q_3^7 + Q_4^7 - 2 m Q^6 - 3 m^2 Q^5 + 3 m^4 Q^3 \right] \\
- \frac{\tau^5}{5! d(d+2)(d+4)} \left[ 5 A - \frac{5}{2} Q_2^8 + \frac{1}{4} Q_3^8 - \frac{1}{2} Q_4^8 - 3 Q_5^8 + 5 Q_6^8 \\
+ 5 m (3 Q_1^7 - Q_2^7 - 3 Q_3^7 + Q_4^7) - 15 m^2 (Q^6 + m Q^5 - m^3 Q^3) \right] \\
+ \mathcal{O}(\tau^6) \right\}.
\]

(49)
The terms up to dimension 7 at \( m = 0 \) agree with \([12]\).

5. Conclusion

The results obtained here can be used for extracting numerical values of \( F_P \) (and hence \( f_B = f_{B^*} \), \( f_{B_s} / f_B \) and similar quantities for \( 0^+, 1^+ \) mesons) and \( \Lambda_P \) (and hence \( m_{B_s} - m_B \), \( m_{B(0^+)} - m_B, m_{B_s(0^+)} - m_{B(0^+)} \), \( m_B \)) from HQET sum rules \((1 / m_b \) corrections should be calculated separately). For sufficiently small \( \tau \) the correlator \( \Pi_P(\tau; \mu) \) is given by the truncated OPE series

\[
\Pi_P(\tau; \mu) = \int_0^\infty d\omega \rho_P^{d\geq 2} (\omega; \mu) e^{-\omega \tau} + \Pi_P^{d\geq 3}(\tau; \mu),
\]

(50)

where the coefficient functions are known as truncated series in \( \alpha_s \). On the other hand, we can represent it as

\[
\Pi_P(\tau; \mu) = \int_0^\infty d\omega \rho_P(\omega; \mu) e^{-\omega \tau},
\]

(51)

where the spectral density is given by the ground-state meson contribution \((22)\) plus the continuum of excited states. We can use the rough model of the continuum contribution \([45]\)

\[
\rho_P(\omega; \mu) = |F_P(\mu)|^2 \delta(\omega - \Lambda_P) + \rho_P^{d\geq 2}(\omega; \mu) \theta(\omega - \omega_{eP}),
\]

(52)

where \( \omega_{eP} \) is the effective continuum threshold. Equating these two expressions, we obtain the sum rule
\[ |F_p(\mu)|^2 e^{-\Lambda_p \tau} = \int_0^{\omega_{c}^P} d\omega \, \rho_P^{d \leq 2}(\omega; \mu) e^{-\omega \tau} + \Gamma_P^{d \geq 3}(\tau; \mu). \tag{53} \]

It is approximately valid at sufficiently large \( \tau \), where the continuum contribution is small, and the uncertainty introduced by its rough model is not essential. If there is a window of \( \tau \) where both conditions are satisfied, we can use this sum rule to extract an approximate value of \( F_p(\mu) \).

Differentiating (53) in \( \tau \) and dividing by (53) we obtain the sum rule for the ground-state residual energy

\[ \tilde{\Lambda}_p = \frac{\int_0^{\omega_{c}^P} d\omega \, \rho_P^{d \leq 2}(\omega; \mu) \omega e^{-\omega \tau} - d\Gamma_P^{d \geq 3}(\tau; \mu)/d\tau}{\int_0^{\omega_{c}^P} d\omega \, \rho_P^{d \leq 2}(\omega; \mu) e^{-\omega \tau} + \Gamma_P^{d \geq 3}(\tau; \mu)}. \tag{54} \]

The continuum thresholds \( \omega_{c}^P \) are tuned in such a way that the resulting \( \tilde{\Lambda}_p \) do not depend on \( \tau \) in the region of applicability.

**CRediT authorship contribution statement**

**K.G. Chetyrkin:** Investigation, Methodology, Software, Validation, Writing – review & editing. **A.G. Grozin:** Investigation, Validation, Writing – original draft.

**Declaration of competing interest**

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

A.G. Grozin reports financial support was provided by Russian Science Foundation.

**Acknowledgements**

We are grateful to S.V. Mikhailov for useful comments. The work of K.Ch. was supported in part by DFG grant CH 1479/2-1 and the Research Unit FOR 2926, project number 40824754. The work of A.G. has been supported by Russian Science Foundation under grant 20-12-00205.

**References**


