



Correlators of heavy–light quark currents in HQET: OPE at three loops

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Abstract

Coefficient functions of the operator product expansion of correlators of HQET heavy–light quark currents are calculated up to operators of dimension 4 up to 3 loops.

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1. Heavy–light currents and their correlators in HQET

QCD problems with a single heavy quark Q having momentum $P = Mv + p$ (M is its on-shell mass, $v^2 = 1$) can be described by heavy quark effective theory [1] (HQET, see, e.g., [2–4]) if its characteristic residual momentum is small ($p \ll M$), and characteristic momenta of light quarks and gluons are also small. QCD operators are expanded in $1/M$, the coefficients are HQET operators of corresponding dimensionalities. For example, QCD heavy–light quark currents at the leading order in $1/M$ are equal to the matching coefficients times the HQET heavy–light currents. These matching coefficients are known at 2 [5,6] and 3 loops [7]. Anomalous dimensions of all HQET heavy–light currents are the same and known at 2 [8–10] and 3 loops [11]. Cor-

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relators of such currents at small distances can be calculated using operator product expansion (OPE); coefficient functions of operators up to dimension 3 are known up to 2 loops [12–14]. The $\langle G^2 \rangle$ contribution vanishes at 1 loop; the $\langle G^3 \rangle$ one is known at 1 loop [12]. Contributions of quark condensates up to dimension 8 are known at tree level [12,15]. Here we calculate the perturbative contribution expanded up to m^4 (m is the light-quark mass) and condensate contributions up to dimension 4 at 3 loops. The perturbative spectral densities of correlators of some QCD heavy–light currents with $m = 0$ in the threshold region at 3 loops were calculated [16]; they are related to the HQET spectral density by the corresponding matching coefficients.

If our heavy quark is b , there are 2 different HQETs: with c quark and without it. The heavy–light HQET currents in these 2 theories are related by the decoupling coefficient, which is known up to 3 loops [17]. The HQET current in HQET with c is related to the QCD currents by the matching coefficients. In this paper we shall work in HQET without c quarks. There are $n_l = 3$ dynamic flavors (u, d, s) and the static b quark.

At the leading order in $1/M$ the heavy-quark spin does not interact with gluon field. We may rotate it at will without affecting physics (heavy-quark spin symmetry [18]). We may even switch it off (superflavor symmetry [19]). We shall use the effective theory of a scalar static antiquark. This particle has no antiparticle; its field φ^* contains only annihilation operators. Its coordinate-space free propagator in the v rest frame is $\delta(\vec{x})S_0(x^0)$ where $S_0(t) = -i\theta(t)$. The momentum-space propagator $S_0(p) = 1/(p^0 + i0)$ does not depend on \vec{p} . Static-quark lines cannot form loops.

We consider the current

$$j_0 = \varphi_0^* q_0 = Z_j(\alpha_s(\mu)) j(\mu) \tag{1}$$

where q_0 is the bare light-quark field. The correlator of 2 currents in the v rest frame

$$\langle T j_0(x) \bar{j}_0(0) \rangle = \delta(\vec{x}) \Pi_0(x^0) \tag{2}$$

is non-zero only for $x^0 \geq 0$ (the symbol T is superfluous: the product $\bar{j}_0(0)j_0(x) = 0$). The momentum-space correlator

$$\int d^d x \langle T j_0(x) \bar{j}_0(0) \rangle e^{ip \cdot x} = \Pi_0(p^0) \tag{3}$$

does not depend on \vec{p} . They are related by the 1-dimensional Fourier transform

$$\Pi_0(\omega) = \int_0^\infty dt \Pi_0(t) e^{i\omega t}, \quad \Pi_0(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \Pi_0(\omega) e^{-i\omega t}. \tag{4}$$

The correlator $\Pi_0(\omega)$ has a cut from 0 to $+\infty$, the discontinuity gives the spectral density

$$\rho_0(\omega) = \frac{1}{2\pi} [\Pi_0(\omega + i0) - \Pi_0(\omega - i0)]. \tag{5}$$

The correlator is expressed via the spectral density by the dispersion representation:

$$\Pi_0(\omega) = i \int_0^\infty \frac{dv \rho_0(v)}{\omega - v + i0}, \quad \Pi_0(t) = \theta(t) \int_0^\infty d\omega \rho_0(\omega) e^{-i\omega t}. \tag{6}$$

We can analytically continue $\Pi_0(t)$ from $t > 0$ to $t = -i\tau$, $\tau > 0$ and obtain the Euclidean correlator

$$\Pi_0(\tau) = \int_0^\infty d\omega \rho_0(\omega) e^{-\omega\tau}. \tag{7}$$

The spectral density can be reconstructed from it by the inverse Mellin transform

$$\rho_0(\omega) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\tau \Pi_0(\tau) e^{\omega\tau}, \tag{8}$$

where a is to the right from all singularities of $\Pi_0(\tau)$.

Borel transform of the correlator $\Pi(\omega)$ is often used in sum rules. In HQET it is defined by

$$\hat{B}_E F(\omega) = \lim_{k \rightarrow \infty} \frac{(-\omega)^{k+1}}{k!} \left(\frac{d}{d\omega} \right)^k F(\omega) \Big|_{\omega=-Ek}. \tag{9}$$

It is equivalent to the correlator in imaginary time $\Pi(\tau)$. For example, for the function

$$F(\omega) = \frac{1}{(v - \omega - i0)^n} \quad \text{it is} \quad \hat{B}_E F(\omega) = \frac{e^{-v/E}}{\Gamma(n) E^{n-1}}.$$

The Fourier transform (4) of $F(\omega)$ is

$$F(t) = i\theta(t) \frac{(it)^{n-1}}{\Gamma(n)} e^{-i(v-i0)t};$$

its analytical continuation from the half-axis $t > 0$ to the half-axis $t = -i\tau$, $\tau > 0$ is

$$F(\tau) = i \frac{\tau^{n-1}}{\Gamma(n)} e^{-v\tau}.$$

Therefore,

$$\hat{B}_E \Pi(\omega) = -i \Pi(\tau = 1/E). \tag{10}$$

The static-antiquark propagator in a gluon field is $\delta(\vec{x}_1 - \vec{x}_0) S_0(x_1^0 - x_0^0) \overline{[x_1, x_0]}$, where

$$\overline{[x_1, x_0]} = P \exp \left[-ig_0 \int_{x_0}^{x_1} dx_\mu A_0^\mu(x) \right] \tag{11}$$

is the Wilson line in the antiquark representation, the integral is taken along the straight line from x_0 to x_1 . Therefore, the correlator can be written as

$$\Pi_0(t) = \langle q_0(vt) \overline{[vt, 0]} \bar{q}_0(0) \rangle. \tag{12}$$

We can consider a more general object [20]

$$F_0(x) = \langle q_0(x) \overline{[x, 0]} \bar{q}_0(0) \rangle, \tag{13}$$

where x is not necessarily timelike. The bilocal vacuum average can be expressed via $F_0(x)$:

$$\langle \bar{q}_0(0) [0, x] \Gamma q(x) \rangle = -\text{Tr} \Gamma F_0(x), \tag{14}$$

where $[0, x]$ is the Wilson line in the quark (fundamental) representation, and Γ is a Dirac matrix.

The correlator of the $\overline{\text{MS}}$ renormalized currents $j(\mu)$ still contains ultraviolet (UV) divergences when $t = 0$. Subtracting these divergences we obtain the renormalized correlator $\Pi(t; \mu)$. The dispersion representation should contain 3 subtractions:

$$\begin{aligned} \Pi(\omega; \mu) &= -i\omega^3 \int_0^\infty \frac{d\varepsilon \rho(\varepsilon; \mu)}{\varepsilon^3(\varepsilon - \omega - i0)} + \sum_{n=0}^2 c_n \omega^n, \\ \Pi(t; \mu) &= \theta(t) \int_0^\infty d\omega \rho(\omega; \mu) e^{-i\omega t} + \sum_{n=0}^2 c_n i^n \delta^{(n)}(t). \end{aligned} \tag{15}$$

Divergences of the correlator of renormalized currents are in subtraction terms in (15): in coordinate space they are at $t = 0$, in momentum space they are polynomial in ω . More exactly, in dimensional regularization only c_2 contains $1/\varepsilon^n$ divergences, whereas power divergences in $c_{0,1}$ are not seen in this scheme. The renormalized spectral density is simply given by $\rho_0(\omega) = Z_j^2(\alpha_s(\mu))\rho(\omega; \mu)$.

The correlator has 2 Dirac structures

$$\Pi = A + B\not{p}. \tag{16}$$

It is convenient to introduce the currents with definite parities $P = \pm 1$:

$$j_P = \frac{1 + P\not{p}}{2} j. \tag{17}$$

Their correlators are

$$P\Pi_P \frac{1 + P\not{p}}{2} \quad \text{where} \quad \Pi_P = P(A + PB) = \frac{P}{4} \text{Tr}(1 + P\not{p})\Pi \tag{18}$$

(we shall see soon why it is convenient to introduce the factor P here).

For sufficiently large $-\omega$ the operator product expansion (OPE) is valid

$$\Pi(\omega; \mu) = \sum_i C_i(\omega; \mu) \langle O_i(\mu) \rangle, \tag{19}$$

where O_i are all possible operators. If the q mass is small, we can include operators with powers of $m(\mu)$ in the set O_i and calculate the Wilson coefficients C_i treating q as massless. Then the terms with even-dimensional O_i have Dirac structure \not{p} , and those with odd-dimensional O_i have the structure 1.

Currently we are in the world where the antiquark \bar{Q} has quantum numbers 0^+ . Then S -wave $\bar{Q}q$ mesons have $j^P = \frac{1}{2}^+$, and P -wave ones $\frac{1}{2}^-$ and $\frac{3}{2}^-$. The currents j_\pm have quantum numbers of $\frac{1}{2}^\pm$ mesons (currents with quantum numbers of mesons with $j > \frac{1}{2}$ necessarily involve derivatives, we don't consider them). The matrix elements of our currents are

$$\langle 0 | j_P(\mu) | M \rangle = F(\mu)u, \tag{20}$$

where the meson states are normalized as

$$\langle M, \vec{p}' | M, \vec{p} \rangle = (2\pi)^3 \delta(\vec{p}' - \vec{p}) \tag{21}$$

in the v rest frame, and u is the Dirac wave function of the $\frac{1}{2}^P$ meson M satisfying $\not{p}u = Pu$ and normalized as $u^\dagger u = 1$. The contribution of the meson M to the correlator Π_P and its spectral density ρ_P is

Table 1
Correlators of currents with spin $\frac{1}{2}$ heavy antiquark.

P	Γ_1	Γ_2	Π_{12}
+1	γ_5	γ_5	$2\Pi_+$
	γ^i	γ^j	$2\Pi_+\delta^{ij}$
-1	1	1	$2\Pi_-$
	$\gamma_5\gamma^i$	$\gamma_5\gamma^j$	$2\Pi_-\delta^{ij}$

$$\begin{aligned} \Pi_M(t) &= |F|^2 e^{-i\bar{\Lambda}t} \theta(t), & \Pi_M(\tau) &= |F|^2 e^{-\bar{\Lambda}\tau}, \\ \Pi_M(\omega) &= \frac{i|F|^2}{\omega - \bar{\Lambda} + i0}, & \rho_M(\omega) &= |F|^2 \delta(\omega - \bar{\Lambda}), \end{aligned} \tag{22}$$

where $\bar{\Lambda}$ is the residual energy of this meson. If there are several mesons with given quantum numbers, we get sums of contributions (22); sums become integrals in the continuum spectrum.

Now let's switch on the spin (and parity) $\frac{1}{2}^-$ of the static antiquark \bar{Q} (still at $M = \infty$). The static antiquark is now described by the field \bar{h} satisfying $\bar{h}\not{v} = -\bar{h}$. The free propagator of this field contains the extra factor $(1 - \not{v})/2$ as compared to the scalar case. The currents are

$$j_{\Gamma 0} = \bar{h}_0 \Gamma q_0; \tag{23}$$

in the v rest frame the set of independent Dirac structures Γ is $1, \gamma^i, \gamma^{[i} \gamma^{j]}$, $\gamma^{[i} \gamma^j \gamma^{k]}$, where square brackets mean antisymmetrization. Instead of this set, we can use $1, \gamma_5^{\text{HV}}, \gamma^i, \gamma_5^{\text{HV}} \gamma^i$, where γ_5^{HV} is the 't Hooft–Veltman γ_5 . In HQET renormalized currents with the anticommuting γ_5^{AC} coincide with the corresponding currents with γ_5^{HV} , because their anomalous dimensions are the same [5] (contrary to the QCD case). Therefore, in the following we shall just use γ_5 . The correlator of j_1^+ and j_2 is

$$\Pi_{12} = -\text{Tr} \bar{\Gamma}_1 \frac{1 - \gamma^0}{2} \Gamma_2 \Pi, \tag{24}$$

where Π is the correlator with the scalar static antiquark, and minus comes from the fermion loop. The Dirac matrices Γ either commute or anticommute with γ^0 : $\Gamma\gamma^0 = -P\gamma^0\Gamma$. Both $\Gamma_{1,2}$ must have the same P (otherwise the correlator vanishes), and

$$\Pi_{12} = -\Pi_P \frac{1}{2} \text{Tr} \bar{\Gamma}_1 \Gamma_2. \tag{25}$$

The same formula works for the spectral densities.

S -wave mesons with light-fields quantum numbers $j^P = \frac{1}{2}^+$ become degenerate doublets $0^-, 1^-$; P -wave ones with $j^P = \frac{1}{2}^-, \frac{3}{2}^-$ form degenerate doublets $0^+, 1^+$ and $1^+, 2^+$. The currents with Γ anticommuting with γ^0 (γ_5, γ^i : $P = +1$) have quantum numbers of the S -wave $0^-, 1^-$ mesons ($j^P = \frac{1}{2}^+$); those with Γ commuting with γ^0 ($1, \gamma_5\gamma^i$: $P = -1$) have quantum numbers of the P -wave $0^+, 1^+$ mesons ($j^P = \frac{1}{2}^-$). The spectral density of correlator of the currents with quantum numbers of 0^\mp mesons is $2\rho_\pm$, and for 1^\mp mesons it is $2\rho_\pm\delta^{ij}$ (Table 1, eq. (25); this is the reason why we introduced the factor P in (18)). A 0^- meson contribution to the spectral density is $|F_{0^-}|^2\delta(\omega - \bar{\Lambda})$; for 1^- one it is $|F_{1^-}|^2\delta^{ij}\delta(\omega - \bar{\Lambda})$, where

$$\langle 0|\bar{h}\gamma_5 q|0^- \rangle = F_{0^-}(\mu), \quad \langle 0|\bar{h}\vec{\gamma} q|1^- \rangle = F_{1^-}(\mu)\vec{e} \tag{26}$$

(\vec{e} is the polarization vector of the 1^- meson). Therefore

$$F_{0^-}(\mu) = F_{1^-}(\mu) = \sqrt{2}F(\mu) \quad (27)$$

(this is an example of the heavy-quark spin symmetry). The case of 0^+ , 1^+ mesons is similar.

Usually the relativistic normalization of one-particle states is used:

$${}_r\langle M, P' | M, P \rangle_r = (2\pi)^3 2P^0 \delta(\vec{P}' - \vec{P}) \quad (28)$$

(it becomes meaningless when the meson mass $M \rightarrow \infty$, and thus is not usable in HQET; in the meson rest frame $|M\rangle_r = \sqrt{2M}|M\rangle$). The spin-symmetry result (27) can be written in a completely Lorentz-invariant way:

$$\begin{aligned} \langle 0 | \bar{h} \Gamma q | M \rangle_r &= \sqrt{M} F(\mu) \text{Tr} \Gamma \mathcal{M}, \quad \mathcal{M} = \frac{1 + \not{v}}{2} \times \begin{cases} \gamma_5 & \text{for } 0^- \\ \not{e} & \text{for } 1^- \end{cases}, \\ \not{v} \mathcal{M} &= -\mathcal{M} \not{v} = \mathcal{M}. \end{aligned} \quad (29)$$

For example ($P^\mu = Mv^\mu$ is the meson momentum)

$$\begin{aligned} \langle 0 | \bar{h} \gamma_5 \gamma^\mu q | 0^- \rangle_r &= \frac{2F(\mu)}{\sqrt{M}} P^\mu, \quad \langle 0 | \bar{h} \gamma_5 q | 0^- \rangle_r = \frac{2F(\mu)}{\sqrt{M}} M, \\ \langle 0 | \bar{h} \gamma^\mu q | 1^- \rangle_r &= \frac{2F(\mu)}{\sqrt{M}} M e^\mu, \quad \langle 0 | \bar{h} \frac{1}{2} [\gamma^\mu, \gamma^\nu] q | 1^- \rangle_r = \frac{2F(\mu)}{\sqrt{M}} (e^\mu P^\nu - e^\nu P^\mu). \end{aligned}$$

Similarly, for 0^+ , 1^+ mesons ($j^P = \frac{1}{2}^-$)

$$\mathcal{M} = \frac{1 - \not{v}}{2} \times \begin{cases} 1 & \text{for } 0^+ \\ \gamma_5 \not{e} & \text{for } 1^+ \end{cases}, \quad \not{v} \mathcal{M} = \mathcal{M} \not{v} = -\mathcal{M}. \quad (30)$$

Of course, phases of $|M\rangle$ states (and hence of \mathcal{M}) can be redefined.

The vacuum average (13) for a general (timelike or spacelike) x has 2 Dirac structures

$$\begin{aligned} F_0(x) &= -\frac{1}{4} \left[F_S(x^2) - i \not{x} F_V(x^2) \right], \\ F_S(x^2) &= \langle \bar{q}(0) [0, x] q(x) \rangle, \quad F_V(x^2) = \frac{i}{x^2} \langle \bar{q}(0) [0, x] \not{x} q(x) \rangle. \end{aligned} \quad (31)$$

In HQET $x = vt$, and the scalar functions $F_{S,V}$ have positive argument $x^2 = t^2$. If x is spacelike, the argument x^2 is negative. When we analytically continue HQET results to $t = -i\tau$, we obtain $F_{S,V}$ of negative argument $-\tau^2$.

2. Perturbative contribution

Perturbative contributions to the correlator are shown in Fig. 1: the one-loop diagram; one of three two-loop ones; and two examples of three-loop diagrams.

We use integration by parts (IBP) to reduce three-loop diagrams to master integrals with the C++ program¹ FIRE6 [24]. Generation of Feynman diagram was done with QGRAF [25] and evaluation of color factors with the FORM [26] package COLOR [27].

¹ We have also used the Mathematica program LiteRed 1.4 [21,22] and the REDUCE package Grinder [23] for testing purposes and the identification of the master integrals.

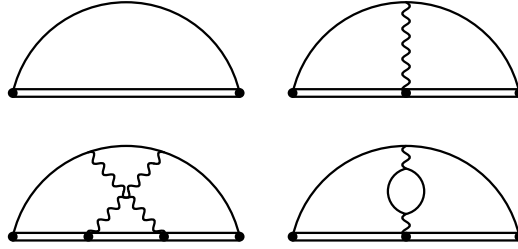


Fig. 1. Perturbative contributions to the correlator.

There are three non-trivial master integrals. Two of them are known exactly as hypergeometric functions with ε : [28] and [23,29]; for the last one, only a few terms of the ε expansion are known [16], but this is sufficient for our purpose. This IBP procedure and the master integrals are reviewed in [30].

The renormalized perturbative correlator is

$$\begin{aligned}
 \Pi_P(\tau; \mu) = & \frac{N_c}{\pi^2} \left\{ \frac{1}{2\tau^3} \left[1 + C_F \frac{\alpha_s}{4\pi} \left[6L_\tau + 4 \left(\frac{\pi^2}{3} + 2 \right) \right] \right. \right. \\
 & + C_F \left(\frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[18L_\tau^2 + \left(\frac{40}{3} \pi^2 + 43 \right) L_\tau - 8\zeta_3 + \frac{8}{45} \pi^4 + \frac{52}{3} \pi^2 + \frac{153}{8} \right] \right. \\
 & + C_A \left[22L_\tau^2 + \left(\frac{76}{9} \pi^2 + 75 \right) L_\tau - 104\zeta_3 - \frac{8}{45} \pi^4 - \frac{5}{27} \pi^2 + \frac{6413}{72} \right] \\
 & \left. \left. - T_{Fnl} \left[8L_\tau^2 + 4 \left(\frac{8}{9} \pi^2 + 7 \right) L_\tau - 32\zeta_3 - \frac{16}{27} \pi^2 + \frac{589}{18} \right] \right\} \right\} \\
 & + P \frac{m}{4\tau^2} \left[1 + C_F \frac{\alpha_s}{4\pi} \left[12L_\tau + 4 \left(\frac{\pi^2}{3} + 3 \right) \right] \right. \\
 & + C_F \left(\frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[72L_\tau^2 + 2 \left(\frac{32}{3} \pi^2 + 71 \right) L_\tau - 20\zeta_3 + \frac{8}{45} \pi^4 + \frac{52}{3} \pi^2 + \frac{233}{4} \right] \right. \\
 & + C_A \left[44L_\tau^2 + \frac{2}{3} \left(\frac{38}{3} \pi^2 + 205 \right) L_\tau - 116\zeta_3 - \frac{8}{45} \pi^4 + \frac{31}{27} \pi^2 + \frac{4981}{36} \right] \\
 & \left. \left. - T_{Fnl} \left[16L_\tau^2 + \frac{8}{3} \left(\frac{4}{3} \pi^2 + 17 \right) L_\tau - 32\zeta_3 - \frac{16}{27} \pi^2 + \frac{401}{9} \right] \right\} \right\} \\
 & - \frac{m^2}{8\tau} \left[1 + 6C_F \frac{\alpha_s}{4\pi} (3L_\tau + 1) \right. \\
 & + C_F \left(\frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[162L_\tau^2 + \left(\frac{16}{3} \pi^2 + 109 \right) L_\tau + 40\zeta_3 - \frac{32}{45} \pi^4 - 12\pi^2 + \frac{507}{8} \right] \right. \\
 & + C_A \left[66L_\tau^2 - \left(\frac{4}{3} \pi^2 - 125 \right) L_\tau - 22\zeta_3 - \frac{4}{15} \pi^4 - \pi^2 + \frac{2789}{24} \right] \\
 & \left. \left. - T_{Fnl} \left[24L_\tau^2 + 36L_\tau + \frac{229}{6} \right] \right\} \right\} \\
 & - 2 \frac{\sum_i m_i^2}{\tau} C_F T_F \left(\frac{\alpha_s}{4\pi} \right)^2 \left(\frac{\pi^2}{3} - 2 \right)
 \end{aligned}$$

$$\begin{aligned}
 &+ P \frac{m^3}{8} \left[L_\tau + C_F \frac{\alpha_s}{4\pi} \left[12L_\tau^2 + 10L_\tau - 2 \left(\frac{2}{3} \pi^2 - 3 \right) \right] \right. \\
 &+ C_F \left(\frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[96L_\tau^3 + 2 \left(\frac{8}{3} \pi^2 + 59 \right) L_\tau^2 + \left(16\xi_3 - \frac{76}{3} \pi^2 + \frac{249}{2} \right) L_\tau \right. \right. \\
 &\quad \left. \left. - 140\xi_5 + \frac{64}{3} \pi^2 \xi_3 + 134\xi_3 - \frac{26}{15} \pi^4 - \frac{40}{3} \pi^2 + \frac{91}{24} \right] \right. \\
 &+ C_A \left[\frac{88}{3} L_\tau^3 - \frac{2}{3} (2\pi^2 - 161) L_\tau^2 - \left(16\xi_3 + \frac{97}{9} \pi^2 - \frac{889}{6} \right) L_\tau \right. \\
 &\quad \left. + 60\xi_5 - \frac{20}{3} \pi^2 \xi_3 + 89\xi_3 - \frac{\pi^4}{45} - \frac{277}{27} \pi^2 + \frac{4357}{72} \right] \\
 &\quad \left. \left. - T_{Fnl} \left[\frac{32}{3} L_\tau^3 + \frac{104}{3} L_\tau^2 - \frac{2}{3} \left(\frac{16}{3} \pi^2 - 71 \right) L_\tau + 32\xi_3 - \frac{80}{27} \pi^2 + \frac{245}{18} \right] \right\} \right] \\
 &- Pm \left(\sum m_i^2 \right) C_F T_F \left(\frac{\alpha_s}{4\pi} \right)^2 \left(6L_\tau + \frac{\pi^2}{3} - \frac{2}{3} \right) \\
 &- \frac{m^4 \tau}{32} \left[L_\tau - \frac{1}{4} + C_F \frac{\alpha_s}{4\pi} \left[18L_\tau^2 - \frac{17}{2} L_\tau - \frac{8}{3} \pi^2 + \frac{45}{2} \right] \right. \\
 &+ C_F \left(\frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[186L_\tau^3 + \left(\frac{16}{3} \pi^2 - \frac{259}{2} \right) L_\tau^2 + \left(16\xi_3 - \frac{212}{3} \pi^2 + \frac{4155}{8} \right) L_\tau \right. \right. \\
 &\quad \left. \left. - 420\xi_5 + 64\pi^2 \xi_3 + 351\xi_3 - \frac{20}{9} \pi^4 - \frac{113}{9} \pi^2 - \frac{10609}{32} \right] \right. \\
 &+ C_A \left[\frac{154}{3} L_\tau^3 - \left(\frac{4}{3} \pi^2 - \frac{63}{2} \right) L_\tau^2 - \left(16\xi_3 + \frac{200}{9} \pi^2 - \frac{2603}{8} \right) L_\tau \right. \\
 &\quad \left. + 180\xi_5 - 20\pi^2 \xi_3 + 179\xi_3 - \frac{\pi^4}{15} + \frac{835}{108} \pi^2 - \frac{64801}{288} \right] \\
 &\quad \left. \left. - T_{Fnl} \left[\frac{56}{3} L_\tau^3 + 2L_\tau^2 - \left(\frac{64}{9} \pi^2 - \frac{235}{2} \right) L_\tau + 64\xi_3 + \frac{32}{27} \pi^2 - \frac{6137}{72} \right] \right\} \right] \\
 &+ m^2 \frac{\sum m_i^2}{4} \tau C_F T_F \left(\frac{\alpha_s}{4\pi} \right)^2 \left(6L_\tau + \frac{17}{6} \right) \\
 &- \frac{\sum m_i^4}{4} \tau C_F T_F \left(\frac{\alpha_s}{4\pi} \right)^2 \left[\left(\frac{\pi^2}{3} - \frac{11}{4} \right) L_\tau - 3\xi_3 - \frac{\pi^2}{2} + \frac{65}{8} \right] \\
 &+ \mathcal{O}(m^5, \alpha_s^3) \Big\}, \tag{32}
 \end{aligned}$$

where $\alpha_s = \alpha_s^{(n_l)}(\mu)$, $m = m^{(n_l)}(\mu)$ is the mass of the quark q in our current (1), $m_i = m_i^{(n_l)}(\mu)$ are all light-flavor masses (see the last diagram in Fig. 1), and

$$L_\tau = \log \frac{\mu \tau e^{\gamma_E}}{2}. \tag{33}$$

The coefficient functions $C_{m^n}(\mu)$ with $n = 0, 1, 2$ satisfy simple renormalization group (RG) equations (while m^3 mixes with $\bar{q}q$, and m^4 mixes with $m\bar{q}q$ and G^2 , Sect. 3). Its solution is

$$C_{m^n} \sim \exp \left\{ \frac{\alpha_s}{4\pi} (-2\gamma_0 L_\tau + c_1) + \left(\frac{\alpha_s}{4\pi} \right)^2 [-2\beta_0 \gamma_0 L_\tau^2 - 2(\gamma_1 - \beta_0 c_1) L_\tau + c_2] + \mathcal{O}(\alpha_s^3) \right\}, \quad \gamma_k = 2\gamma_{jk} - n\gamma_{mk}, \quad \gamma_0 = -6C_F(n+1). \tag{34}$$

Here

$$\gamma_a(\alpha_s) = \frac{d \log Z_a}{d \log \mu} = \sum_{n=0}^{\infty} \gamma_{an} \left(\frac{\alpha_s}{4\pi} \right)^{n+1} \quad (a = j, m),$$

$$\beta(\alpha_s) = \frac{1}{2} \frac{d \log Z_\alpha}{d \log \mu} = \sum_{n=0}^{\infty} \beta_n \left(\frac{\alpha_s}{4\pi} \right)^{n+1}.$$

Our results (32) for $n = 0, 1, 2$ satisfy this condition.

The renormalized spectral density of the OPE terms having dimensionalities ≤ 2 is

$$\begin{aligned} \rho_P^{d \leq 2}(\omega; \mu) = & \frac{N_c}{\pi^2} \left\{ \frac{\omega^2}{4} \left[1 - C_F \frac{\alpha_s}{4\pi} \left(6L_\omega - \frac{4}{3}\pi^2 - 17 \right) \right. \right. \\ & + C_F \left(\frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[18L_\omega^2 - \left(\frac{40}{3}\pi^2 + 97 \right) L_\omega - 8\zeta_3 + \frac{8}{45}\pi^4 + \frac{103}{3}\pi^2 + \frac{1173}{8} \right] \right. \\ & + C_A \left[22L_\omega^2 - \left(\frac{76}{9}\pi^2 + 141 \right) L_\omega - 104\zeta_3 - \frac{8}{45}\pi^4 + \frac{238}{27}\pi^2 + \frac{20057}{72} \right] \\ & \left. \left. - T_{Fnl} \left[8L_\omega^2 - 4 \left(\frac{8}{9}\pi^2 + 13 \right) L_\omega - 32\zeta_3 + \frac{92}{27}\pi^2 + \frac{1849}{18} \right] \right\} \right\} \\ & + P \frac{m\omega}{4} \left[1 - 4C_F \frac{\alpha_s}{4\pi} \left(3L_\omega - \frac{\pi^2}{3} - 6 \right) \right. \\ & + 4C_F \left(\frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[18L_\omega^2 - \left(\frac{16}{3}\pi^2 + \frac{143}{2} \right) L_\omega - 5\zeta_3 + \frac{2}{45}\pi^4 + \frac{20}{3}\pi^2 + \frac{1377}{16} \right] \right. \\ & + C_A \left[11L_\omega^2 - \frac{1}{3} \left(\frac{19}{3}\pi^2 + \frac{337}{2} \right) L_\omega - 29\zeta_3 - \frac{2}{45}\pi^4 + \frac{61}{108}\pi^2 + \frac{13069}{144} \right] \\ & \left. \left. - T_{Fnl} \left[4L_\omega^2 - \frac{2}{3} \left(\frac{4}{3}\pi^2 + 29 \right) L_\omega - 8\zeta_3 + \frac{2}{27}\pi^2 + \frac{1097}{36} \right] \right\} \right\} \\ & - \frac{m^2}{8} \left[1 - 6C_F \frac{\alpha_s}{4\pi} (3L_\omega - 1) \right. \\ & + 2C_F \left(\frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[81L_\omega^2 - \left(\frac{8}{3}\pi^2 + \frac{109}{2} \right) L_\omega + 20\zeta_3 - \frac{16}{45}\pi^4 - \frac{39}{2}\pi^2 + \frac{507}{16} \right] \right. \\ & + C_A \left[33L_\omega^2 + \left(\frac{2}{3}\pi^2 - \frac{125}{2} \right) L_\omega - 11\zeta_3 - \frac{2}{15}\pi^4 - 6\pi^2 + \frac{2789}{48} \right] \\ & \left. \left. - T_{Fnl} \left[12L_\omega^2 - 18L_\omega - 2\pi^2 + \frac{229}{12} \right] \right\} \right\} \\ & - \frac{2}{3} \left(\sum m_i^2 \right) C_F T_F \left(\frac{\alpha_s}{4\pi} \right)^2 (\pi^2 - 6) + \mathcal{O}(\alpha_s^3) \Big\}, \tag{35} \end{aligned}$$

where

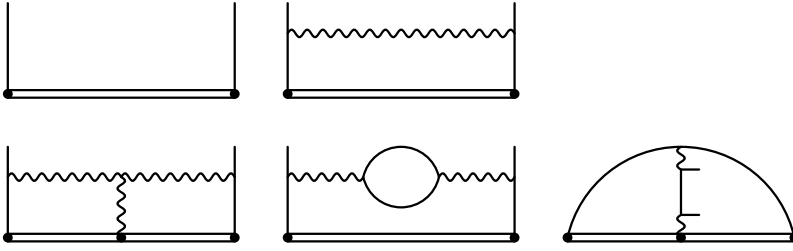


Fig. 2. Quark-condensate contributions to the correlator.

$$L_\omega = \log \frac{2\omega}{\mu}. \tag{36}$$

Terms up to two loops agree with [12]; the remaining ones are new. Multiplying the leading m^0 term in the HQET spectral density (35) by the corresponding matching coefficients [5,6], we reproduce the leading δ^0 terms in the 3-loop QCD spectral densities (10), (14) in [16].

3. Quark and gluon condensates (dimensions 3 and 4)

Some 0-, 1-, and 2-loop diagrams for the quark condensate contribution are shown in Fig. 2. Starting from 2 loops (the last diagram in the figure) contributions proportional to the singlet sum $\sum m_i \langle \bar{q}_i q_i \rangle$ appear. Our result for the coordinate-space correlator² is

$$\begin{aligned} \Pi_p^q(\tau; \mu) = & -P \frac{\langle \bar{q}q \rangle}{4} \left\{ 1 + 6C_F \frac{\alpha_s}{4\pi} \right. \\ & + C_F \left(\frac{\alpha_s}{4\pi} \right)^2 \left\{ 2C_F \left[4 \left(\frac{2}{3}\pi^2 - 1 \right) L_\tau - 16\zeta_3 + \frac{10}{3}\pi^2 + 11 \right] \right. \\ & \left. - C_A \left[4 \left(\frac{\pi^2}{3} - 7 \right) L_\tau - 8\zeta_3 + \pi^2 - \frac{149}{3} \right] - 16T_F n_l \left(L_\tau + \frac{4}{3} \right) \right\} \\ & + C_F \left(\frac{\alpha_s}{4\pi} \right)^3 \left\{ C_F^2 \left[4 \left(18\zeta_3 + \frac{4}{9}\pi^4 + \frac{8}{3}\pi^2 - 35 \right) L_\tau \right. \right. \\ & \left. \left. + \frac{1600}{3}\zeta_5 - \frac{928}{9}\pi^2\zeta_3 - \frac{140}{3}\zeta_3 + \frac{479}{135}\pi^4 - \frac{8}{9}\pi^2 + 157 \right] \right. \\ & + C_F C_A \left[\frac{176}{3} \left(\frac{2}{3}\pi^2 - 1 \right) L_\tau^2 - 4 \left(141\zeta_3 - \frac{4}{45}\pi^4 - \frac{902}{27}\pi^2 - \frac{737}{9} \right) L_\tau \right. \\ & \left. \left. - \frac{1}{3} \left(1216\zeta_5 - \frac{424}{3}\pi^2\zeta_3 + \frac{23654}{9}\zeta_3 - \frac{3799}{270}\pi^4 - \frac{27122}{81}\pi^2 - \frac{23669}{27} \right) \right] \right. \\ & \left. - C_A^2 \left[\frac{88}{3} \left(\frac{\pi^2}{3} - 7 \right) L_\tau^2 - 4 \left(33\zeta_3 + \frac{2}{15}\pi^4 - \frac{164}{27}\pi^2 + \frac{1409}{9} \right) L_\tau \right. \right. \\ & \left. \left. - 72\zeta_5 + 12\pi^2\zeta_3 - \frac{4856}{27}\zeta_3 + \frac{199}{810}\pi^4 + \frac{4094}{243}\pi^2 - \frac{69583}{81} \right] \right\} \end{aligned}$$

² We have used the well-known method of projectors [31,32] for computation of various condensate contributions (at 1 loop a similar method was used in [33,34]).

$$\begin{aligned}
 & -2C_F T_{Fnl} \left[\frac{32}{3} \left(\frac{2}{3} \pi^2 - 1 \right) L_\tau^2 - 32 \left(3\xi_3 - \frac{22}{27} \pi^2 - \frac{25}{9} \right) L_\tau \right. \\
 & \quad \left. - \frac{1}{27} \left(5936\xi_3 - \frac{100}{3} \pi^4 - \frac{3536}{9} \pi^2 - \frac{10253}{3} \right) \right] \\
 & + 2C_A T_{Fnl} \left[16 \left(\frac{\pi^2}{9} - 6 \right) L_\tau^2 - 4 \left(6\xi_3 - \frac{32}{27} \pi^2 + \frac{217}{3} \right) L_\tau \right. \\
 & \quad \left. - \frac{1}{27} \left(2258\xi_3 - \frac{89}{15} \pi^4 - \frac{830}{9} \pi^2 + \frac{27736}{3} \right) \right] \\
 & \quad \left. + \frac{32}{3} (T_{Fnl})^2 \left(4L_\tau^2 + \frac{32}{3} L_\tau + \frac{109}{9} \right) \right\} \\
 & + \frac{m(\bar{q}q)\tau}{16} \left\{ 1 + 2C_F \frac{\alpha_s}{4\pi} \left(3L_\tau - \frac{5}{2} \right) \right. \\
 & \quad + 4C_F \left(\frac{\alpha_s}{4\pi} \right)^2 \left\{ C_F \left[\frac{9}{2} L_\tau^2 + \left(\frac{4}{3} \pi^2 - \frac{35}{4} \right) L_\tau - 8\xi_3 + \frac{8}{3} \pi^2 - \frac{547}{32} \right] \right. \\
 & \quad + C_A \left[\frac{11}{2} L_\tau^2 - \frac{1}{3} \left(\pi^2 + \frac{61}{4} \right) L_\tau + 2\xi_3 - \frac{3}{4} \pi^2 + \frac{3415}{96} \right] \\
 & \quad \left. \left. - T_{Fnl} \left(2L_\tau^2 - \frac{5}{3} L_\tau + \frac{335}{24} \right) \right\} \right. \\
 & \quad + 4C_F \left(\frac{\alpha_s}{4\pi} \right)^3 \left[C_F^2 \left[9L_\tau^3 + 2(4\pi^2 - 15)L_\tau^2 - \left(30\xi_3 - \frac{4}{9} \pi^4 - 4\pi^2 + \frac{1393}{16} \right) L_\tau \right. \right. \\
 & \quad \left. \left. + \frac{1}{3} \left(1450\xi_5 - \frac{712}{3} \pi^2 \xi_3 + 145\xi_3 + \frac{431}{180} \pi^4 + \frac{461}{6} \pi^2 - \frac{20141}{32} \right) \right] \right. \\
 & \quad + C_F C_A \left[33L_\tau^3 + \frac{1}{3} \left(\frac{70}{3} \pi^2 - \frac{403}{2} \right) L_\tau^2 \right. \\
 & \quad \left. - \left(129\xi_3 - \frac{4}{45} \pi^4 - \frac{2551}{54} \pi^2 + \frac{6997}{144} \right) L_\tau \right. \\
 & \quad \left. \left. - \frac{1}{3} \left(754\xi_5 - \frac{256}{3} \pi^2 \xi_3 + \frac{10715}{9} \xi_3 - \frac{8119}{1080} \pi^4 - \frac{16439}{324} \pi^2 - \frac{207275}{432} \right) \right] \right. \\
 & \quad + C_A^2 \left[\frac{242}{9} L_\tau^3 - \frac{1}{9} \left(22\pi^2 + \frac{365}{2} \right) L_\tau^2 + \left(33\xi_3 + \frac{2}{15} \pi^4 - \frac{362}{27} \pi^2 + \frac{26281}{54} \right) L_\tau \right. \\
 & \quad \left. + \frac{67}{4} \xi_5 - \frac{5}{2} \pi^2 \xi_3 + \frac{2951}{27} \xi_3 - \frac{419}{1620} \pi^4 - \frac{10103}{3888} \pi^2 - \frac{675449}{7776} \right] \\
 & \quad - C_F T_{Fnl} \left[12L_\tau^3 + \frac{4}{3} \left(\frac{8}{3} \pi^2 - 13 \right) L_\tau^2 - \left(72\xi_3 - \frac{496}{27} \pi^2 - \frac{13}{36} \right) L_\tau \right. \\
 & \quad \left. - \frac{1}{27} \left(3058\xi_3 - \frac{223}{15} \pi^4 - \frac{1264}{9} \pi^2 - \frac{42839}{24} \right) \right] \\
 & \quad - C_A T_{Fnl} \left[\frac{176}{9} L_\tau^3 - \frac{2}{9} (4\pi^2 + 71) L_\tau^2 + 4 \left(9\xi_3 - \frac{34}{27} \pi^2 + \frac{2486}{27} \right) L_\tau \right. \\
 & \quad \left. + \frac{1}{27} \left(436\xi_3 - \frac{53}{30} \pi^4 - \frac{353}{18} \pi^2 - \frac{26521}{36} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{(T_{Fnl})^2}{3} \left(\frac{32}{3} L_\tau^3 - \frac{40}{3} L_\tau^2 + \frac{1940}{9} L_\tau + 8\zeta_3 - \frac{6395}{162} \right) \Bigg\} \\
 & + \left(\sum m_i \langle \bar{q}_i q_i \rangle \right) \frac{\tau}{4} C_F T_F \left(\frac{\alpha_s}{4\pi} \right)^2 \left\{ \frac{1}{2} \left(\frac{4}{3} \pi^2 - 11 \right) \right. \\
 & + \frac{\alpha_s}{4\pi} \left[C_F \left[5 \left(\frac{4}{3} \pi^2 - 11 \right) L_\tau + 12\zeta_3 + \frac{8}{45} \pi^4 - \frac{44}{9} \pi^2 + \frac{76}{3} \right] \right. \\
 & + \frac{C_A}{3} \left[\left(\frac{100}{3} \pi^2 - 311 \right) L_\tau - 256\zeta_3 - \frac{7}{15} \pi^4 - \frac{211}{9} \pi^2 + \frac{3559}{6} \right] \\
 & \left. \left. - 2T_{Fnl} \left[\frac{4}{3} \left(\frac{4}{3} \pi^2 - 11 \right) L_\tau - 16\zeta_3 - \frac{44}{27} \pi^2 + \frac{313}{9} \right] \right] \right\}, \tag{37}
 \end{aligned}$$

where $\langle \bar{q}q \rangle$ is renormalized at μ . The terms up to 2 loops in the dimension-3 contribution agree with [12].

Finiteness of the renormalized coefficient function $C_{\bar{q}q}$ provides an independent confirmation of $2\gamma_j - \gamma_{\bar{q}q}$ at 3 loops [11]. This anomalous dimension vanishes at 1 loop; $\gamma_{\bar{q}q} = -\gamma_m$, hence $C_{\bar{q}q}$ has the structure (34) with $n = -1$. We need one more term:

$$\begin{aligned}
 C_{\bar{q}q} \sim \exp \left\{ c_1 \frac{\alpha_s}{4\pi} + \left(\frac{\alpha_s}{4\pi} \right)^2 \left[-2(\gamma_1 - \beta_0 c_1) L_\tau + c_2 \right] \right. \\
 \left. + \left(\frac{\alpha_s}{4\pi} \right)^3 \left[-4\beta_0(\gamma_1 - \beta_0 c_1) L_\tau^2 - 2(\gamma_2 - 2\beta_0 c_2 - \beta_1 c_1) L_\tau + c_3 \right] + \mathcal{O}(\alpha_s^4) \right\}, \tag{38}
 \end{aligned}$$

where $\gamma_k = 2\gamma_{jk} + \gamma_{mk}$, $\gamma_0 = 0$. Hence the α_s term contains no L_τ , the α_s^2 one contains L_τ^1 , etc.

The dimension-3 operators $O_3 = (m^3, m \sum m_i^2, \bar{q}q)^T$ satisfy the renormalization group equation [35–38]

$$\frac{dO_3}{d \log \mu} + \gamma_3 O_3 = 0, \quad \gamma_3 = \begin{pmatrix} 3\gamma_m & 0 & 0 \\ 0 & 3\gamma_m & 0 \\ \gamma & \gamma' & -\gamma_m \end{pmatrix}, \tag{39}$$

$$\begin{aligned}
 \gamma = -\frac{N_c}{4\pi^2} \left\{ 2 + 8C_F \frac{\alpha_s}{4\pi} \right. \\
 \left. + C_F \left[C_F(96\zeta_3 - 131) - C_A(48\zeta_3 - 109) - 20T_{Fnl} \right] \left(\frac{\alpha_s}{4\pi} \right)^2 + \mathcal{O}(\alpha_s^3) \right\},
 \end{aligned}$$

$$\gamma' = 24 \frac{N_c}{\pi^2} C_F T_F \left(\frac{\alpha_s}{4\pi} \right)^2 + \mathcal{O}(\alpha_s^3).$$

Therefore, the coefficient functions $C_3 = (C_{m^3}, C_{m \sum m_i^2}, C_{\bar{q}q})^T$ satisfy the renormalization group equation

$$\frac{dC_3}{d \log \mu} = \frac{\partial C_3}{\partial L_\tau} - 2\beta \frac{\partial C_3}{\partial \log \alpha_s} = (\gamma_3^T - 2\gamma_j) C_3. \tag{40}$$

The dimension-4 operators mO_3 satisfy the renormalization group equation similar to (39) but with the anomalous dimension $\gamma_3 + \gamma_m$. Hence we obtain

$$\left[\frac{\partial}{\partial L_\tau} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_j - 3\gamma_m \right] C_{m^3} = \gamma C_{\bar{q}q},$$

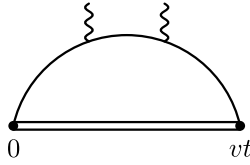


Fig. 3. One-loop gluon condensate contribution.

$$\begin{aligned} \left[\frac{\partial}{\partial L_\tau} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_j - 3\gamma_m \right] C_{m \sum m_i^2} &= \gamma' C_{\bar{q}q}, \\ \left[\frac{\partial}{\partial L_\tau} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_j - 4\gamma_m \right] C_{m^4} &= \gamma C_{m\bar{q}q}, \\ \left[\frac{\partial}{\partial L_\tau} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_j - 4\gamma_m \right] C_{m^2 \sum m_i^2} &= \gamma' C_{m\bar{q}q}. \end{aligned}$$

Our results (32), (37) satisfy these equations.

It is well known that the gluon condensate contribution vanishes at 1 loop. In the fixed-point gauge the static quark does not interact with gluons, and the only remaining diagram is shown in Fig. 3. But the G^2 correction to the massless quark propagator $S(x, 0)$ vanishes after vacuum averaging [39]. The 2- and 3-loop contributions are

$$\begin{aligned} \Pi_P^G(\tau; \mu) &= \frac{\langle G^2 \rangle \tau}{48} T_F \left(\frac{\alpha_s}{4\pi} \right)^2 \left\{ \left[C_F \left(\frac{4}{3} \pi^2 - 11 \right) + \frac{C_A}{2} \left(\frac{4}{3} \pi^2 - 23 \right) \right] \right. \\ &+ 3 \frac{\alpha_s}{4\pi} \left\{ C_F^2 \left[2 \left(\frac{4}{3} \pi^2 - 11 \right) L_\tau + 56\zeta_3 + \frac{136}{135} \pi^4 - \frac{140}{9} \pi^2 - \frac{23}{2} \right] \right. \\ &+ \frac{C_F C_A}{9} \left[\left(\frac{124}{3} \pi^2 - 449 \right) L_\tau - 524\zeta_3 - \frac{20}{3} \pi^4 + 100\pi^2 + 167 \right] \\ &+ \frac{C_A^2}{9} \left[11 \left(\frac{4}{3} \pi^2 - 23 \right) L_\tau - 46\zeta_3 + \frac{4}{5} \pi^4 + \frac{65}{6} \pi^2 - \frac{93}{2} \right] \\ &- \frac{2}{3} C_F T_F n_l \left[\frac{4}{3} \left(\frac{4}{3} \pi^2 - 11 \right) L_\tau - 16\zeta_3 - \frac{44}{27} \pi^2 + \frac{259}{9} \right] \\ &\left. \left. - \frac{1}{3} C_A T_F n_l \left[\frac{4}{3} \left(\frac{4}{3} \pi^2 - 23 \right) L_\tau - 16\zeta_3 - \frac{20}{27} \pi^2 + \frac{253}{9} \right] \right\} \right\}, \end{aligned} \tag{41}$$

where $G^2 = G_{\mu\nu}^a G^{a\mu\nu}$. The anomalous dimension of this operator is [40–42]

$$\gamma_{G^2} = -2 \frac{d\beta}{d \log \alpha_s}, \tag{42}$$

and hence the coefficient function must have the structure

$$C_{G^2} \sim \left(\frac{\alpha_s}{4\pi} \right)^2 \left\{ 1 + \frac{\alpha_s}{4\pi} [2(\beta_0 - \gamma_{j0})L_\tau + c] + \mathcal{O}(\alpha_s^2) \right\}. \tag{43}$$

Our result (41) satisfies this condition.

The flavor-singlet dimension-4 operators $O_4 = (\sum m_i^4, (\sum m_i^2)^2, \sum m_i \bar{q}_i q_i, G^2)^T$ satisfy the renormalization group equation [35–38]

$$\frac{dO_4}{d \log \mu} + \gamma_4 O_4 = 0, \tag{44}$$

$$\gamma_4 = \begin{pmatrix} 4\gamma_m & 0 & 0 & 0 \\ 0 & 4\gamma_m & 0 & 0 \\ \gamma & \gamma' & 0 & 0 \\ -\frac{d\gamma}{d \log \alpha_s} & -\frac{d\gamma'}{d \log \alpha_s} & 4\frac{d\gamma_m}{d \log \alpha_s} & -2\frac{d\beta}{d \log \alpha_s} \end{pmatrix}$$

and the corresponding coefficient functions $C_4 = (C_{\Sigma m_i^4}, C_{(\Sigma m_i^2)^2}, C_{\Sigma m_i \bar{q}_i q_i}, C_{G^2})^T$ — the equation

$$\frac{dC_4}{d \log \mu} = (\gamma_4^T - 2\gamma_j)C_4; \tag{45}$$

hence,

$$\begin{aligned} \left[\frac{\partial}{\partial L_\tau} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_j - 4\gamma_m \right] C_{\Sigma m_i^4} &= \gamma C_{\Sigma m_i \bar{q}_i q_i} - \frac{d\gamma}{d \log \alpha_s} C_{G^2}, \\ \left[\frac{\partial}{\partial L_\tau} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_j - 4\gamma_m \right] C_{(\Sigma m_i^2)^2} &= \gamma' C_{\Sigma m_i \bar{q}_i q_i} - \frac{d\gamma'}{d \log \alpha_s} C_{G^2}, \\ \left[\frac{\partial}{\partial L_\tau} - 2\beta \frac{\partial}{\partial \log \alpha_s} + 2\gamma_j \right] C_{\Sigma m_i \bar{q}_i q_i} &= 4\frac{d\gamma_m}{d \log \alpha_s} C_{G^2}. \end{aligned}$$

The second equation here is satisfied trivially, because $C_{(\Sigma m_i^2)^2} = \mathcal{O}(\alpha_s^3)$. Our results (32), (37), (41) satisfy these equations.

4. Higher-dimensional condensates

The tree diagram in Fig. 2 can be written exactly in $x = vt$:

$$\Pi^q(t) = i\theta(t)\langle q(vt)\overline{[vt, 0]}\bar{q}(0)\rangle. \tag{46}$$

It is expressed via the bilocal quark condensate [43] which has 2 Dirac structures:

$$\langle q(x)\overline{[x, 0]}\bar{q}(0)\rangle = -\frac{\langle \bar{q}q \rangle}{4} \left[f_S(x^2) - \frac{i\not{x}}{d} f_V(x^2) \right]. \tag{47}$$

Its expansion in x via local quark condensates is known up to dimension 8 [15]. We use the bases of local condensates [44]

$$\begin{aligned} Q^3 &= \langle \bar{q}q \rangle, \quad Q^5 = i\langle \bar{q}G_{\mu\nu}\sigma^{\mu\nu}q \rangle, \quad Q^6 = \langle \bar{q}\not{x}q \rangle, \\ Q_1^7 &= \langle \bar{q}G_{\mu\nu}G^{\mu\nu}q \rangle, \quad Q_2^7 = i\langle \bar{q}G_{\mu\nu}\tilde{G}^{\mu\nu}\gamma_5q \rangle, \\ Q_3^7 &= \langle \bar{q}G_{\mu\lambda}G^\lambda_\nu\sigma^{\mu\nu}q \rangle, \quad Q_4^7 = i\langle \bar{q}D_\mu J_\nu\sigma^{\mu\nu}q \rangle, \\ A &= i\langle \bar{q}D_\alpha D_\beta D_\gamma D_\delta D_\varepsilon \gamma^{[\alpha}\gamma^\beta\gamma^\gamma\gamma^\delta\gamma^{\varepsilon]}q \rangle, \\ Q_1^8 &= i\langle \bar{q}[[G_{\mu\lambda}, G^\lambda_\nu]_+, D^\mu]_+\gamma^nuq \rangle, \quad Q_2^8 = -\langle \bar{q}[[G_{\mu\lambda}, \tilde{G}^\lambda_\nu], D^\mu]_+\gamma^nuq \rangle, \\ Q_3^8 &= i\langle \bar{q}[\not{x}G_{\mu\nu}, G^{\mu\nu}]q \rangle, \quad Q_4^8 = \langle \bar{q}D^2\not{x}q \rangle, \\ Q_5^8 &= i\langle \bar{q}[G_{\mu\nu}, J^\mu]\gamma^nuq \rangle, \quad Q_6^8 = \langle \bar{q}[\tilde{G}_{\mu\nu}, J^\mu]_+\gamma^nuq \rangle, \end{aligned} \tag{48}$$

where $G_{\mu\nu} = gG_{\mu\nu}^a t^a$, $J_\mu = gJ_\mu^a t^a$, $J_\mu^a = D^\nu G_{\mu\nu}^a = g\sum \bar{q}_i \gamma_\mu t^a q_i$; $\sigma_{\mu\nu} = \gamma_{[\mu}\gamma_{\nu]}$; operators containing $\tilde{G}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}$ and $\gamma_5 = \frac{i}{4!}\varepsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta$ are understood as short notations

for the expressions from which both ε tensors are eliminated using $\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\delta} = -4!\delta_{[\alpha}^{\mu}\delta_{\beta}^{\nu}\delta_{\gamma}^{\rho}\delta_{\delta]}^{\sigma}$. The anomalous condensate A does not vanish in the $\overline{\text{MS}}$ scheme; it is a finite combination of dimension-8 gluon condensates [44].

We obtain the contribution of bilinear quark condensates up to dimension 8 to the correlator at the tree level

$$\begin{aligned} \Pi_P^q(\tau) = & -\frac{1}{4}\left\{PQ^3 - \frac{\tau}{d}mQ^3 - P\frac{\tau^2}{2!d}\left[\frac{1}{2}Q^5 - m^2Q^3\right] \right. \\ & + \frac{\tau^3}{3!d(d+2)}\left[\frac{1}{2}Q^6 + \frac{3}{2}mQ^5 - 3m^3Q^3\right] \\ & + P\frac{\tau^4}{4!d(d+2)}\left[3Q_1^7 - \frac{3}{2}Q_2^7 - 3Q_3^7 + Q_4^7 - 2mQ^6 - 3m^2Q^5 + 3m^4Q^3\right] \\ & - \frac{\tau^5}{5!d(d+2)(d+4)}\left[5A - \frac{5}{2}Q_2^8 + \frac{1}{4}Q_3^8 - \frac{1}{2}Q_4^8 - 3Q_5^8 + 5Q_6^8 \right. \\ & \left. + 5m(3Q_1^7 - Q_2^7 - 3Q_3^7 + Q_4^7) - 15m^2(Q^6 + mQ^5 - m^3Q^3)\right] \\ & \left. + \mathcal{O}(\tau^6)\right\}. \end{aligned} \tag{49}$$

The terms up to dimension 7 at $m = 0$ agree with [12].

5. Conclusion

The results obtained here can be used for extracting numerical values of F_P (and hence $f_B = f_{B^*}$, f_{B_s}/f_B and similar quantities for 0^+ , 1^+ mesons) and $\bar{\Lambda}_P$ (and hence $m_{B_s} - m_B$, $m_{B(0^+)} - m_B$, $m_{B_s(0^+)} - m_{B(0^+)}$, m_b) from HQET sum rules ($1/m_b$ corrections should be calculated separately).

For sufficiently small τ the correlator $\Pi_P(\tau; \mu)$ is given by the truncated OPE series

$$\Pi_P(\tau; \mu) = \int_0^\infty d\omega \rho_P^{d \leq 2}(\omega; \mu)e^{-\omega\tau} + \Pi_P^{d \geq 3}(\tau; \mu), \tag{50}$$

where the coefficient functions are known as truncated series in α_s . On the other hand, we can represent it as

$$\Pi_P(\tau; \mu) = \int_0^\infty d\omega \rho_P(\omega; \mu)e^{-\omega\tau}, \tag{51}$$

where the spectral density is given by the ground-state meson contribution (22) plus the continuum of excited states. We can use the rough model of the continuum contribution [45]

$$\rho_P(\omega; \mu) = |F_P(\mu)|^2\delta(\omega - \bar{\Lambda}_P) + \rho_P^{d \leq 2}(\omega; \mu)\theta(\omega - \omega_{cP}), \tag{52}$$

where ω_{cP} is the effective continuum threshold. Equating these two expressions, we obtain the sum rule

$$|F_P(\mu)|^2 e^{-\bar{\Lambda}_P \tau} = \int_0^{\omega_{cP}} d\omega \rho_P^{d \leq 2}(\omega; \mu) e^{-\omega \tau} + \Pi_P^{d \geq 3}(\tau; \mu). \quad (53)$$

It is approximately valid at sufficiently large τ , where the continuum contribution is small, and the uncertainty introduced by its rough model is not essential. If there is a window of τ where both conditions are satisfied, we can use this sum rule to extract an approximate value of $F_P(\mu)$.

Differentiating (53) in τ and dividing by (53) we obtain the sum rule for the ground-state residual energy

$$\bar{\Lambda}_P = \frac{\int_0^{\omega_{cP}} d\omega \rho_P^{d \leq 2}(\omega; \mu) \omega e^{-\omega \tau} - d \Pi_P^{d \geq 3}(\tau; \mu) / d\tau}{\int_0^{\omega_{cP}} d\omega \rho_P^{d \leq 2}(\omega; \mu) e^{-\omega \tau} + \Pi_P^{d \geq 3}(\tau; \mu)}. \quad (54)$$

The continuum thresholds ω_{cP} are tuned in such a way that the resulting $\bar{\Lambda}_P$ do not depend on τ in the region of applicability.

CRedit authorship contribution statement

K.G. Chetyrkin: Investigation, Methodology, Software, Validation, Writing – review & editing. **A.G. Grozin:** Investigation, Validation, Writing – original draft.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

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