Higher and Quantum Invariants of Knots

Höhere und Quanteninvarianten von Knoten

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Contents

Chapter 1

Thesis Overview

The aim of the present thesis is to contribute towards the theory of higher and quantum invariants of knots and links. The thesis is structured as follows.

Chapter 2 is devoted to a brief historical account of knot theory in mathematics. We start in Section 2.1 by describing how knots found their way from the real world into mathematics, and continue in Sections 2.2 to 2.4 with a description of the developments of a selection of the most common and important knot invariants, divided into classical, higher, and quantum invariants, respectively.[1] Along the way, we pay careful attention to provide many references to original papers, textbooks, or survey articles with additional sources.

The remainder of the thesis is divided into two parts. In Part I we turn our attention to higher invariants, more precisely the study of knot concordance and a particular invariant defined by C. Livingston in 2010 called the *stable* 4-*genus* of knots [Liv10]. In the main result we derive a lower bound on the stable 4-genus of a knot in terms of Casson-Gordon τ-signatures. As an application, we compute the lower bounder for an infinite family of knots, the *twist knots*, and complete the classification of their order in the knot concordance group. Part I is based on the author's paper *"A lower bound on the stable* 4*-genus of knots"* from 2020 [Ilt20].

Part II is concerned with contributions towards quantum invariants. We use a variation of Khovanov homology called $\mathbb{Z}[G]$*-homology* in order to define a new knot invariant λ that takes non-negative integer values. The highlight of this invariant is that it provides a lower bound on the proper rational unknotting number of a knot, which in turn is a lower bound for the ordinary unknotting number. We further show that the invariant λ can be arbitrarily big by constructing concrete examples. Part II is based on the collaborative work of the author with L. Lewark and L. Marino in 2021, which resulted in the paper *"Khovanov homology and rational unknotting"* [ILM21].

[1] This classification of knot invariants is neither standard nor precisely defined, but to the author's best knowledge widely accepted.

An extensive introduction to our results with background information is given at the beginning of Part I and Part II, respectively, where we also provide a more detailed description on how the individual parts are structured.

While the new contributions towards the theory of higher and quantum invariants are certainly the main attraction of the present text, the reader will notice that a large part of the thesis is formed by the preliminaries in Part I and Part II. This is due to the fact that we aimed to provide a mostly self-contained introduction to the mathematical theory needed in order to understand our own results. Moreover, we tried to put emphasis on proper notation and explanations in hope to achieve a presentation of the material that might not be found elsewhere in the literature.

1.1 Acknowledgments

I would like to sincerely thank my advisor Lukas Lewark for his extraordinary support and advice during the past years. His tireless will to pass knowledge and the countless insightful discussions formed the pillar of this thesis and my mathematical understanding.

I also thank the research groups of Stefan Friedl and Clara Löh in Regensburg for their support and camaraderie. In particular, discussions with Stefan Friedl and his astonishing lecture notes in topology played a fundamental role in my understanding of the theory of homology with twisted coefficients, and ultimately Casson-Gordon invariants.

I would further like to thank Anthony Conway, Stefan Friedl, Charles Livingston and Allison Miller for reading a preliminary version of the paper *“A lower bound on the stable 4-genus of knots”* [Ilt20] and giving final comments and advice. Likewise, I would like to thank Eaman Eftekhary, Duncan McCoy, Dirk Schütz and Claudius Zibrowius for comments on drafts of the paper *“Khovanov homology and rational unknotting”* [ILM21]. Thanks in particular to Claudius Zibrowius who pointed the authors of the paper to the work of Naot and Thompson which are essential to our results, Dirk Schütz who kindly shared his calculations of Rasmussen invariants [Sch21b], and Daniel Schäppi who led us towards Remark 10.20. Also thanks to all participants of the *Khovanov homology reading course* in Regensburg in 2020, Felix Eberhart, Marco Moraschini, Lars Munser, Paula Truöl, and José Pedro Quintanilha for the exciting times.

I am eternally grateful for the continued and endless support of my parents Michael and Nicole, and my sister Fabienne during my mathematical studies. Without them, this work would have never been possible.

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Chapter 2

Historical Note

The journey of knots into the abstract spheres of modern day mathematics is an astonishing one. Starting off as a primary example for Leibniz' *geometria situs*, the abstract study of knots has produced highly sophisticated tools in mathematics and revealed deep connections with the geometrical model of our real world. Let's take a closer look.

2.1 The Beginnings

The craft of tying knots has been used by mankind since prehistoric times and is still essential in many areas such as sailing, climbing, or art. Regarding knots in science, one of the earliest records goes back to ancient Greece. Heraklas, a Greek physicist, described in an essay in the first century A.D. sixteen knots and nooses for surgical and orthopedic use [Hag08].[1] Knots didn't enter mathematics however until 1771, when A.-T. Vandermonde (1735–1796) published his paper "*Remarques sur les problèmes de situation*" [Van71]. The newly emerging branch of mathematics "geometry of position" (*geometria situs*, the ancestor of modern topology), first mentioned by G. W. Leibniz (1646–1716), called for a geometry that deals with position directly, but convincing examples had been lacking so far [Fre72, Prz98]. In his paper, Vandermonde observed that it is the "question of position", i.e. the way "...in which [...] threads are interlaced" that is of main interest for knots, and not any quantitative measures such as length or magnitude, making knots a primary example for the geometry of position.

C. F. Gauss (1777–1855) and his student J. B. Listing (1808–1882) also showed interest in knots and, more generally, links (i.e. a union of several, possibly intertwined knots) [Prz98, RLR11]. The linking number, introduced by Gauss in 1833, was the first mathematical object that provides information about knotted strings in space. Intuitively, the linking number

[1] As noted by J. J. Hage [Hag08], seven of Heraklas' knots are still in use and four more have been rediscovered recently.

counts how many times a curve winds around a second curve, and provides therefore a first characterization of links. Indeed, the linking number can be used for instance to tell the 2-component unlink apart from the Hopf link, see Figure 2.1.

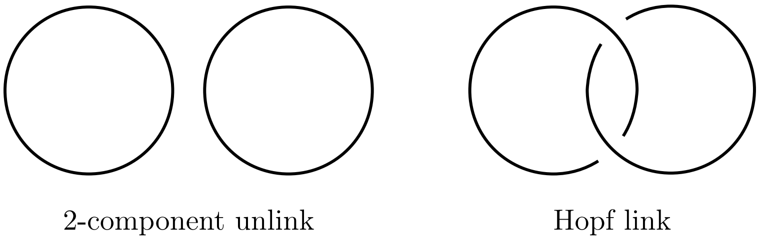


Figure 2.1: The 2-component unlink on the left with linking number 0, and the Hopf link on the right with linking number +1 or −1 depending on the chosen orientation.

There is evidence that Gauss' derivation of the linking number was in fact motivated by his studies of electromagnetism [RLR11]. Gauss' discovery therefore sparked the interest of physicists such as J. C. Maxwell (1831–1879) or H. Von Helmholtz (1821–1894) [Prz98]. It is interesting to note that Maxwell provided a first example of two intertwined curves that cannot be separated but have the same linking number as the 2-component unlink, giving a first glimpse of what should become one of the major problems in mathematical knot theory.

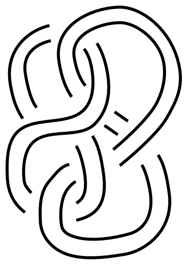


Figure 2.2: The link of Maxwell (thickened, redrawn from [Prz98, Figure 7])

The journey of knots into mathematics continues as Scottish physicist P. G. Tait (1831–1901) showed W. Thomson, also known as Lord Kelvin (1824–1907), how to experiment with vortex smoke rings in 1867 [Sil06]. Thomson, at that time trying to understand the fundamental particles of matter, deduced from his observations the theory of the vortex atom. The theory suggests that atoms are knotted vortices in the luminiferous aether and that different knots represent different chemical elements [Tho67].

Motivated by his friend's theory and anticipating a characterization of all periodic elements, Tait started in 1878 his extensive study of knots [Sil06]. Tait attempted to tabulate knots by depicting them with diagrams, but he was confronted with a major difficulty: how does one prove that two knot diagrams represent different or equal knots? There were no rigorous techniques for this task available, so Tait had to work with intuition and common sense [Sil06]. He made several observations which later became known as the *Tait conjectures*, and achieved a tabulation of knots up to 7 crossings [Sto08, Prz98].[2]

While Thomson's theory of the vortex atom eventually turned out to be wrong [Sil06], Tait's work on knots could withstand time: his tabulation was proven to be essentially correct by the results of subsequent knot tabulists (see Footnote 2 on p. 5), and his conjectures were solved by the works of L. Kauffman, K. Murasugi, M. B. Thistlethwaite and W. Menasco between 1987 and 1993 [Kau87b, Mur87a, Mur87b, Thi87, Thi88, MT93]. Because of his contributions, Tait can be considered as the first knot theorist in mathematics.

Despite its influence, Tait's work wasn't the sole reason for the formation of knot theory as its own mathematical discipline. In the 19th-century, the theory of algebraic functions and more generally algebraic geometry experienced great popularity within mathematics. As noted in [Epp95], W. Wirtinger (1865–1945) was the first to recognize a connection between knots and singular points of algebraic curves, even though he never directly published his findings. Many of the immediate knot theoretical inventions and discoveries after Tait stem in fact from this connection of knots with problems in algebraic geometry. However, this relation is barely noticeable in the early papers of modern knot theory by mathematicians such as H. Tietze, M. Dehn, K. Reidemeister or J. W. Alexander; a consequence of forgetting contexts [Epp95] (also, see [DH07] for a historical survey of knot theory in the 19th-century). According to M. Epple [Epp95, Introduction], it is the result of a "... process of *differentiation* and ... a subsequent *elimination of contexts*" [Epp95, Thesis] that fully established knot theory as an independent discipline within mathematics.

Whether one's motivation stems from Tait's work on knot tabulation or the theory of algebraic functions, the fundamental problem about knots still remained: how does one distinguish non-equivalent knots in general? In order to treat this problem a proper mathematical framework was required. The development of such a framework was initiated by H. Poincaré (1854–1912), who laid with his paper "Analysis Situs" [Poi95] the foundations of algebraic topology in 1895. Looking at the bigger picture, Poincaré's work was part of an ongoing revolution throughout mathematics at the start of the 19th-century: the change of mathematics as a tool to treat questions

[2] T. P. Kirkman (1806–1895) and C. N. Little (1858–1923) contributed to a further tabulation of knots so that by 1900 a list of knots with up to 10 crossings was available (see [Prz98] and references therein). The tabulation was continually extended by many mathematicians such as M. G. Haseman (1889–1960(?)), J. H. Conway (1937–2020), C. H. Dowker (1912–1982), J. Hoste, J. Weeks and M. B. Thistlethwaite [Has17, Con70, DT83, Thi85, HTW98]. The latest result is to the best knowledge of the author by B. A. Burton [Bur20], achieving a tabulation of topologically distinct prime knots up to 19 crossings. The tabulation consists of more than 350 million entries!

motivated by exact sciences into an autonomous discipline with axiomatic foundations, rigorous formalism and problems arising from mathematical research itself.[3] This change is reflected in knot theory as knots became more and more independent objects studied in three- and even higher-dimensional spaces, formalized in the language of topology.[4] It was only this abstraction which made it possible to tackle the evolving questions within knot theory.

Equipped with the evolving tools of (algebraic) topology, the early 20th-century marks the beginning of modern knot theory. The developments from now on are incredibly vast and rapid, and we have to make a choice on the topics we wish to discuss. We do so by choosing subjects that benefit us the most for the rest of this thesis. In particular, we have to omit important contemporary topics such as Vassiliev (or finite-type) invariants, or contact geometry and Legendrian knots, for instance. We would also like to take the opportunity to apologize for all intended and non-intended omissions of results and references in what follows. For more extensive accounts we refer the interested reader to one of the many excellent textbooks such as [Rol76, Lic97, Kau87a, Ada04, Gei08, Mur08, Tur10, LN16, JM19, AFH$^+$21].

2.2 Classical invariants

Let us now switch to a more modern language in order to continue our historical note. A *knot* K is a smooth embedding of the unit circle S^1 into the Euclidean 3-space $\mathbb{R}^3$ or the unit 3-sphere S^3, and two such knots are called *equivalent* if there is an ambient isotopy between them.[5] An n-*component link* is a disjoint union of $n \in \mathbb{N}$ knots (hence a one-component link is just a regular knot). It is clear that certain tools are needed in order to prove theorems about knots; heuristic or empirical arguments would not lead to satisfying proofs. These tools are known as *knot invariants*, i.e. mathematical objects that are identical for equivalent knots. Let us refer to the knot invariants that were predominantly studied in roughly the first half of the 20th-century as *classical knot invariants*[6]

The introduction of the fundamental group of a topological space by H. Poincaré in 1895 [Poi95] led to one of the first and simultaneously most famous classical invariants, the so-called *knot group* $\pi_1(S^3 \backslash K)$. In 1905, W. Wirtinger (1865–1945) described an algorithm to obtain a presentation for the knot group, the *Wirtinger presentation* [Wir05]. Three years later in 1908, H. Tietze (1880–1964) used the knot group to show that the trefoil knot is not equivalent to the unknot [Tie08], providing the first formal

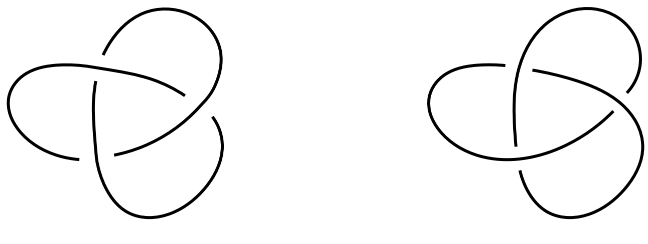
[3] For an excellent account of this process within mathematics with focus on knot theory see [Epp99].

[4] Throughout this text we focus on the classical dimension, i.e. 1-dimensional knots in 3-dimensional space. For knot theory in higher dimensions see for instance [Rol76] or [Ran98].

[5] For reasons of compactness, knots are frequently considered in S^3 rather than $\mathbb{R}^3$.

[6] This terminology is chosen by the author and neither standard nor precisely defined.

proof that non-trivial knots exist [Epp95]. Without relying on previous results, M. Dehn (1878–1952) applied his own techniques to the knot group and published a series of papers between 1910 and 1914, with the latest containing the remarkable result that the left- and right-handed trefoil are not equivalent (in other words, the trefoil is not *amphicheiral*) [Deh10, Deh12, Deh14].



3_1 $\overline{3}_1$

Figure 2.3: The right-handed trefoil 3_1 and its mirror image, the left-handed trefoil $\overline{3}_1$. M. Dehn showed in 1914 that the two trefoils are not equivalent [Deh14]. Note that it is possible to obtain $\overline{3}_1$ from 3_1 by an orientation-reversing homeomorphism of S^3; such a homeomorphism induces an isomorphism of the corresponding knot groups, showing that the isomorphism type of the knot group alone cannot distinguish the two trefoils.

Covering spaces also play an important role in the study of classical invariants. The abelianization of the knot group is always infinite cyclic, hence the commutator subgroup gives rise to a covering of the knot exterior $S^3\backslash K$ with automorphism group $\mathbb{Z}$, the *infinite cyclic covering* of K. If one composes the Abelianization homomorphism with an epimorphism from $\mathbb{Z}$ to $\mathbb{Z}/k\mathbb{Z}$ for some $k \in \mathbb{N}$, $k \geq 2$, one obtains the *k-fold finite cyclic covering* of K. These give rise to a set of invariants, the *k^{th} torsion numbers* of K (see [Rol76, Chapter 6.A]). If one adds back the knot K to the exterior, the finite cyclic covering extends to a *branched* cyclic covering of S^3 with branch locus K.[7]

In his 1898 dissertation [Hee98], P. Heegaard (1871–1948) constructed the 2-fold branched cyclic covering of the trefoil knot and the unknot, and showed that the coverings are different from each other; interestingly, he did not explicitly state that this distinguishes the trefoil from the unknot [Sti93, Prz98].

Arguably one of the most important invariants that arises from a cyclic covering of the knot exterior is the *Alexander polynomial* Δ_K, due to J. W. Alexander (1888–1971) in 1923 [Ale28]. It is defined as the order ideal of the *Alexander invariant*, the first homology of the infinite cyclic covering viewed as a module over the ring of integer Laurent polynomials $\mathbb{Z}[t^{\pm 1}]$ (see

[7]Invariants that are determined by the (co-)homology of abelian coverings of the knot exterior are known as *abelian knot invariants*.

[Rol76, Chapter 8.C]). The Alexander polynomial forms the first and, for a long time only instance of a new class of invariants, the *knot polynomials*, and has been studied extensively ever since its discovery in 1923 [Sei35, Tor53, Fox53, Bro60, Kon79, Tur86, Mur87a, Mur87b, Lin01].

There is another way to obtain the Alexander polynomial of a knot. F. Frankl (1905–1961) and L. Pontrjagin (1908–1988) showed in 1930 that every knot bounds an orientable two-dimensional surface in S^3 [FP30]. Five years later in 1935, H. Seifert (1907–1996) described an algorithm that produces such a bounding surface for any given knot K [Sei35]. This algorithm is now known as *Seifert algorithm*, and an orientable bounding surface is called *Seifert surface.* Associated to a Seifert surface is a bilinear form on first homology called *Seifert pairing*, and a matrix for this pairing is called *Seifert matrix* (see [LN16, Chapter 2]).

If A is a Seifert matrix for K, then it can be shown that $A - tA^\top$ is a square presentation matrix for the Alexander invariant[8], hence $\det(A - tA^\top)$ is equal to the Alexander polynomial Δ_K (see [Rol76, Chapter 8.C]). Seifert surfaces and matrices give rise to many new invariants, such as the *genus* of a knot [Sei35], the *knot signature* [Tro62, Mur65], the *knot determinant* [Goe33], or the *Arf invariant* [Rob65]. The knot signature was further generalized by J. Levine and A. Tristram in the 1960s, yielding the *Levine-Tristram signatures* [Lev69a, Lev69b, Tri69]. Around the same time a deep connection of links with 3-dimensional topology was revealed when W. B. R. Lickorish and independently A. H. Wallace showed that every closed, oriented and connected 3-manifold can be obtained by performing surgery on a (framed) link in S^3 [Lic62, Wal61].

In 1927, K. Reidemeister (1893–1971) showed that any two diagrams of equivalent knots are related by a sequence of three local moves, the so-called *Reidemeister moves* [Rei26] (see Figure 2.4).[9] While not being directly a knot invariant, the Reidemeister moves are still of great importance up to the present day; a diagrammatically defined object is a knot invariant if and only if it is invariant under the three Reidemeister moves. As a note beside, it was also Reidemeister that published the first textbook on knot theory in 1932 [Rei32].

One of the simplest, yet most puzzling invariants is the *unknotting number* of a knot K. It is defined as the minimal number of crossing changes needed to turn a diagram of K into a diagram the unknot. It is a simple exercise to show that the unknotting number is always less than half the *crossing number* of the knot. However, it is in general surprisingly difficult to determine the *exact* value of the unknotting number. One of the main reasons is that the unknotting number can not always be determined from a *minimal* diagram of K, i.e. a diagram that realizes the crossing number. The first examples of such knots were found by Y. Nakanishi and inde-

[8] A presentation matrix for the Alexander invariant is called *Alexander matrix.*

[9] This was independently discovered by J. W. Alexander and G. B. Briggs in 1927 [AB27].

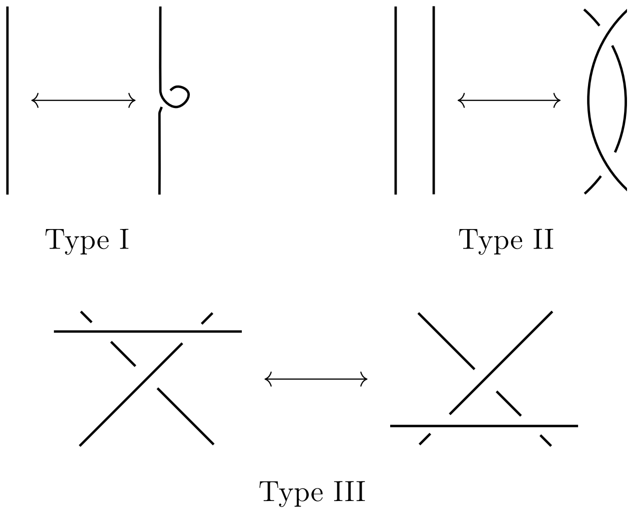


Figure 2.4: The three Reidemeister moves

pendently S. A. Bleiler in 1983 and 1984, respectively [Nak83, Ble84] (see Figure 2.5).

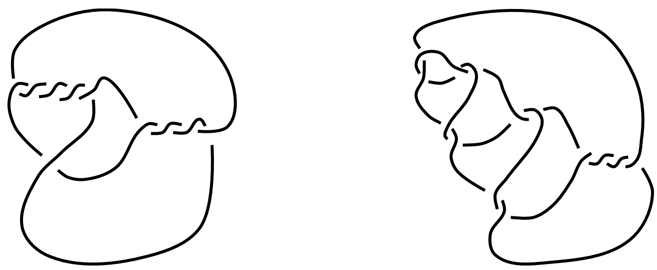


Figure 2.5: Two diagrams of the knot 10_8 (redrawn from [Ble84, Figures 1 and 2]). On the left, a minimal diagram of 10_8, and on the right, the (non-minimal) diagram used by S. A. Bleiler to obtain the unknotting number $u(10_8) = 2$. Note that Bleiler also showed that no two crossing changes in the left diagram are sufficient to produce the unknot!

The difficulty is also reflected in the fact that many results about the unknotting number have only been appearing since the 1980s. Outstanding results are that knots of unknotting number one are prime, shown by M. G. Scharlemann in 1985 [Sch85], and the solution of the Milnor conjecture by P. Kronheimer and T. Mrowka in 1993 [KM93], which states that the unknotting number of the (p, q)-torus knot equals $(p-1)(p-q)/2$. An excellent survey of results on the unknotting number with focus on classical invariants is given in [BF15].

E. Artin (1898–1962) described in 1925 the construction of certain knotted 2-spheres in Euclidean 4-space $\mathbb{R}^4$ [Art25]. The intersection of such a knotted 2-sphere with $\mathbb{R}^3 \subset \mathbb{R}^4$ is a non-trivial knot bounding a 2-dimensional disk in the upper half-space (see [Rol76, Chapter 3.J] or [LN16, Chapter 3]). Knots with this property are called *slice* [Fox62, p.135], and it was an open question until the '60s whether every knot arises in this way or not. The answer to the question is negative, with first examples of non-slice knots given by K. Murasugi [Mur65] and R.H. Fox and J. W. Milnor [Fox62, FM66]. This paved the way for an entire new subfield within knot theory: the study of *knot concordance*, which is an equivalence relation on the set of isotopy classes of knots, and the connection of knots with 4-dimensional topology. Two knots K_1 and K_2 are called *concordant* if the connected sum $K_1\# - K_2$ is slice, where $-K_2$ denotes the mirror image of K_2 with reversed orientation. It can be shown that the concordance classes form an abelian group under the operation of connected sum, the *knot concordance group* $\mathcal{C}$. This group was first introduced by Fox and Milnor in 1966 during their work on surface singularities in 4-manifolds [FM66]. Related to knot concordance is the problem of determining the *slice genus* (also called *4-genus*) $g_4(K)$ of a knot K, which is defined as the minimal genus of a smoothly and properly embedded surface in B^4 bounding the knot in $S^3 = \partial B^4 \subset B^4$. Clearly, K is slice if and only if $g_4(K) = 0$.

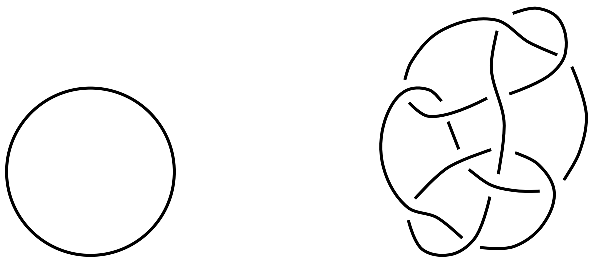
Knot concordance can be studied in two settings: topological and smooth. The above description assumes that everything is smooth. However, if one replaces smooth with *locally flat*, then one arrives at *topologically slice knots* and the *topological concordance group* $\mathcal{C}_{\text{top}}$. It is a non-trivial result that the topological and smooth theory are in fact not equivalent, but we shall come back to this at a later point. For now, we focus on the smooth concordance group and use the adjective "topological" to emphasize the locally flat setting.

The first results about knot concordance go back to Fox, who used the Alexander polynomial to show that the figure-eight knot is of order two in the knot concordance group [Fox62], and Murasugi who used the knot signature to obtain an obstruction to the sliceness of a knot, showing that the trefoil has infinite order in $\mathcal{C}$ [Mur65]. In 1969, J. Levine introduced an algebraic counterpart to slice knots and knot concordance based on the Seifert form and Seifert matrices: so-called *algebraically slice knots* and the corresponding *algebraic concordance group* $\mathcal{G}$ [Lev69b].[10] It can be shown that algebraic sliceness is a necessary condition for a knot to be slice, but it was unknown at this point if it is also sufficient [Lev69b]. The application of classical invariants found a culminating point when Levine showed that the algebraic concordance group $\mathcal{G}$ is isomorphic to $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$ [Lev69a, Lev69b]. This isomorphism also reveals more structure of the knot concordance group as it induces an epimorphism $\mathcal{C} \twoheadrightarrow \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$.[11]

[10] In his work, Levine speaks of *knot cobordism*, rather than *knot concordance*.

[11] A great survey on the knot concordance group and concordance invariants is given by C.

Despite the diversity of classical invariants, none of them can fully solve the problem of distinguishing all knots; for instance, the Alexander polynomial cannot detect the unknot (see Figure 2.6 for an example), and there are non-equivalent knots with isomorphic knot groups (such as the left- and right-handed trefoil in Figure 2.3, for instance).



unknot $\qquad 11_{34}$

Figure 2.6: The unknot and the knot 11_{34} are not equivalent, but have both Alexander polynomial 1 (information of and picture about 11_{34} retrieved from [LM]).

The study of the knot complement reached its high when C. McA. Gordon and J. Luecke showed in 1989 that knots are determined by their complement [GL89], but the practical use of this significant theoretical result is limited. However, explicit algorithms that can decide the equivalence of any pair of knots exist, with the first one developed by W. Haken in 1961 using the theory of normal surfaces [Hak61]. Nonetheless, the available algorithms can be extremely time-consuming, and it is an open question to determine the computational complexity of this so-called *recognition problem* [Has98, HLP99].[12]

2.3 Higher invariants

In the second half of the 20th-century several breakthroughs in mathematics and knot theory led to entirely new classes of invariants, forming knot theory as we know it today in the 21st-century.

Despite Levine's work, many questions about the knot concordance group $\mathcal{C}$ remained open; for instance, as noted at the end of [Lev69a] it is unknown whether there is torsion of order greater than 2 in $\mathcal{C}$. Surprisingly, this question is still unsolved as of 2022! Nevertheless, remarkable progress has been made ever since Levine's results by applying advances in mathematics to the problem of knot concordance. Central to this is the work of M. H. Freedman and S. Donaldson on topological respectively

Livingston in [Liv05].

[12]Recently in February 2021, Marc Lackenby has announced a new unknot recognition algorithm that runs in quasi-polynomial time [Lac21].

smooth 4-dimensional manifolds in the 1980s, and the work of M. F. Atiyah, I. M. Singer and V. K. Patodi on global analysis between 1960 and 1980. Let us give a brief overview.

The G-signature theorem by Atiyah and Singer in 1968 [AS68] led A. J. Casson and C. M. Gordon in the 1970s to the definition of certain new invariants, nowadays known as *Casson-Gordon invariants*, and used them to show that Levine's epimorphism $\mathcal{C} \twoheadrightarrow \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$ has non-trivial kernel [CG75, CG78]. They found the first examples of algebraically slice knots that are not slice. This result was extended by B. Jiang [Jia81] and C. Livingston [Liv99], who showed that the kernel contains a subgroup isomorphic to $\mathbb{Z}^\infty$ and $\mathbb{Z}_2^\infty$, respectively. An obstruction to 4-torsion in the knot concordance group was given by C. Livingston and S. Naik in 1999 using once more Casson-Gordon invariants [LN99, LN01].

In [APS75], Atiyah-Patodi-Singer introduced the *η-invariant* for an odd-dimensional, oriented, compact Riemannian manifold Y and a unitary representation $\alpha\colon \pi_1(Y) \to U(n)$. η-invariants were applied to knot theory by Levine to obtain new link invariants [Lev94], and by C. F. Letsche who found new approaches that yield sliceness obstructions [Let00]. In 2004, S. Friedl built on Letsche's work and furthermore related η-invariants to Casson-Gordon invariants [Fri04].

In 1985, J. Cheeger and M. Gromov refined η and defined the *von Neumann η-invariant* $\eta^{(2)}$ (also known as *L^2-eta invariant*) associated to Y as above and a homomorphism $\varphi\colon \pi_1(Y) \to \Gamma$, where Γ is some group [CG85, Section 4].[13] They further showed that the difference $\rho := \eta^{(2)} - \eta$ does not depend on the particular choice of Riemannian metric on Y, leading to yet another invariant, the *von Neumann ρ-invariant.* Atiyah's L^2-signature [Ati76] is closely related to the von Neumann η-invariant: it can be shown that if the pair (Y, φ) is the boundary of some $4k$-dimensional (W, ψ), then $\eta^{(2)}(Y, \varphi)$ equals the difference of the L^2-signature and the ordinary signature of W, the so-called *reduced* L^2-signature of (W, ψ) [COT03, Lemma 5.9 and Remark 5.10].[14] As one might guess, it was just a matter of time until these L^2-invariants found their way into knot theory as well.

In the groundbreaking work of T. D. Cochran, K. E. Orr, and P. Teichner in 2001, a filtration $\mathcal{F}_n$ indexed by $\frac{1}{2}\mathbb{N}_0$ was exhibited on the knot concordance group $\mathcal{C}$, revealing more of its structure [COT03, COT04]. This so-called *solvable filtration* is formed in terms of the newly introduced *(n)-solvability* of knots for $n \in \frac{1}{2}\mathbb{N}_0$, a property defined using the theory of intersection forms on 4-manifolds [COT03, Definition 1.2]. The theory of L^2-invariants found an impressive application when Cochran-Orr-Teichner used von Neumann ρ-invariants to show the non-triviality of all filtrations

[13] Cheeger and Gromov work with the trivial unitary representation, but everything generalizes to arbitrary unitary representations as well.

[14] The fascinating theory of L^2-invariants finds a great exposition in [COT03, Section 5], and in more condensed form in [CT07, Section 2]. For the reader who wishes to learn the theory more thoroughly, the textbook [Lü02] by W. Lück is an excellent start.

steps [COT04, CT07]. Since slice knots are (n)-solvable for all $n \in \frac{1}{2}\mathbb{N}_0$ [COT03, Remark 1.3], this yields an infinite amount of new sliceness obstructions. In particular, L^2-eta invariants were used to find the first examples of non-slice knots with vanishing Casson-Gordon invariants [COT03, Section 6]. Another remarkable result about the solvable filtration is that the first few filtration steps are related to all previously known concordance invariants [CT07]: $\mathcal{F}_0$ corresponds to knots with vanishing Arf invariant, $\mathcal{F}_{1/2}$ represents algebraically slice knots, and knots in $\mathcal{F}_{3/2}$ have trivial Casson-Gordon invariants. Regarding $\mathcal{F}_1$, it is an open question as of 2022 whether $\mathcal{F}_{1/2} = \mathcal{F}_1$, but there is evidence that the answer is affirmative [DMOP19].

In the 1980s, topology experienced a revolution when M. H. Freedman and F. S. Quinn presented their work on the structure of topological 4-manifolds [Fre82, FQ90]. A major consequence for knot theory was that the results by Levine and Casson-Gordon on knot concordance also apply in the topological setting, rather than just in the smooth. On the other hand, Freedman also showed that all knots with trivial Alexander polynomial are topologically slice. Around the same time, the pioneering work of S. Donaldson on the application of gauge theory in 4-dimensional topology provided deep insight into smooth 4-manifolds [Don96, DK90]. The impacts were far-reaching: it showed that the smooth h-cobordism theorem fails in dimension 4, and provided in combination with Freedman's results the first examples of exotic $\mathbb{R}^4$ [Kir89, FQ90]. In knot theory, gauge theoretic methods were used by R. E. Gompf to find the first examples of topologically slice knots that are not *smoothly* slice [Gom86], and P. B. Kronheimer and T. S. Mrowka succeeded in proving the Milnor conjecture [KM93], which states that the slice genus of a (p, q)-torus knot equals $(p-1)(p-q)/2$. In 2005, Friedl and Teichner discovered new examples of topologically slice knots with *non-trivial* Alexander polynomial [FT05].

The introduction of Seiberg-Witten invariants of smooth 4-manifolds by E. Witten [Wit94] marks yet another breakthrough in topology, and later knot theory. Just as the 4-manifold invariants discovered by Donaldson, Seiberg-Witten invariants not only reveal the structure of smooth 4-manifolds, but also allow for simpler and more general proofs of results similar to those of Donaldson [Don96, Mor96]. These significant advances in 4-dimensional topology also sparked developments one dimension lower, in the study of 3-manifolds. In 1988, Andreas Floer constructed a three-dimensional analogue to Donaldson's theory, the *instanton Floer homology* [Flo88], and Seiberg-Witten type invariants were obtained for 3-manifolds with *Seiberg-Witten Floer homology*, which was constructed rigorously for the first time by Kronheimer and Mrowka in 2007 in terms of their *monopole Floer homology* [KM07]. In an attempt to understand the geometric foundations of the Seiberg-Witten theory [OS18], P. Ozsváth and S. Szabó defined in the early 2000s the *Heegaard Floer homology* of a closed smooth

3-manifold [OS04d, OS04c].[15] In 2003, Ozsváth-Szabó [OS04b] and independently J. Rasmussen [Ras03] discovered that a knot inside a 3-manifold induces a filtration of the corresponding Heegaard Floer homology, leading to the so called *knot Floer homology*.

Heegaard Floer and knot Floer homology has been proven as extremely fruitful in the study of 3-manifolds and knots within them, producing many insights and strong invariants – too many to be discussed here in detail. For an overview of results and further references we refer the reader to [OS18] and [Hom17]. Nevertheless, we would like to mention some of the most beautiful properties of knot Floer homology: it detects the unknot and the 3-genus [OS04a], its Euler characteristic equals the Alexander polynomial (that is, knot Floer homology is a *categorification* of the Alexander polynomial) [OS04b], and it recognizes fibered knots [Ghi08, Ni07]. The invariants τ [OS03a, OS03b, Ras03], ν [OS11] and Υ [OSS17] coming from knot Floer homology provide strong concordance homomorphisms and yield lower bounds on the (smooth) slice genus. As an application, one obtains another proof of the Milnor conjecture [OS03b], and it was shown by J. Hom [Hom15] and independently Ozsváth-Stipsicz-Szabó [OSS17] that the subgroup of the smooth concordance group generated by all topologically slice knots contains a summand isomorphic to $\mathbb{Z}^\infty$!

A new type of invariants appeared in the 1990s when X. S. Lin defined the *twisted Alexander polynomial* of a knot K [Lin01]. Given the knot exterior X_K and a representation $\alpha\colon \pi_1(X_K) \to \mathrm{GL}(k, R[F])$, where R is a Noetherian unique factorization domain and F a free abelian group, the twisted Alexander polynomial $\Delta^\alpha_{X_K}$ is defined as the order of the *twisted Alexander module* $H_1(X_K; R[F]^k)$ (see [FV11, Section 2.3 and 2.4]). This construction was soon generalized to yield *twisted invariants* of arbitrary 3-manifolds, most notably twisted Reidemeister torsion, twisted Blanchfield pairings, and twisted signatures; see [FV11, BCP20] for a survey and many more references. As explained in [FV11], the general idea of a twisted invariant is to extract more information about a 3-manifold (e.g. the knot complement) by combining an algebraic object or invariant with a choice of representation of the fundamental group of the manifold under consideration. One advantage of this approach is that the resulting invariants are not only stronger but also remain computable in many cases, e.g. for satellite constructions [KL99]. As a result, twisted invariants yield sliceness and concordance obstructions [KL99, BCP20], but also an explicit formula for the 3-genus of a knot has been discovered [FV15]. Furthermore, the twisted Alexander polynomial is known to detect the unknot [SW06], as opposed to the ordinary Alexander polynomial. In similar spirit, the theory of L^2-invariants was applied by W. Li and W. Zhang in 2006 to obtain yet another generalization of the Alexander polynomial, the *L^2-Alexander*

[15] The previously conjectured equivalence of Heegaard Floer and Seiberg-Witten Floer homology [OS18] was proven to be true by the works of Ç. Kutluhan, Y.-J. Lee, C. Taubes [KLT20] and V. Colin, P. Ghiggini, K. Honda [CGH11, CGH20].

invariant [LZ06]. In 2014, F. Ben Aribi showed that the L^2-Alexander invariant also serves as an unknot detector [Ben16].

2.4 Quantum invariants

For nearly 60 years, the Alexander polynomial was the only one of its kind: a polynomial knot invariant. This changed suddenly when V. F. R. Jones discovered in 1984 a new polynomial invariant which now bears his name, the celebrated *Jones polynomial* V_K [Jon85, Jon87]. Its discovery was certainly surprising to both knot theorists and the general mathematician, as it resulted from Jones' study on the index of subfactors of certain von Neumann algebras [Jon83, Jon85]. This stands in strong contrast to all previously discovered knot invariants which are underpinned by geometric observations. The origin of the Jones polynomial and the lack of an obvious geometric interpretation sparked the interest of mathematicians and physicists alike. It marks the start of a beautiful interaction between knot theory and physics, as well as the emergence of a new theory: the theory of quantum invariants.

Let us first emphasize some of the properties of the Jones polynomial V_K, which is contained in $\mathbb{Z}[\sqrt{t^{\pm 1}}]$. Since its discovery, several ways to define the polynomial have appeared, most notably Kauffman's definition in terms of the *bracket polynomial* (also known as the *Kauffman bracket*) [Kau87b].[16]

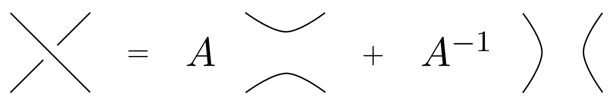


Figure 2.7: The main equation appearing in the computation of the Kauffman bracket. We will encounter this equation again in Part II, Section 8.1.

The Jones polynomial of the mirror image of a knot K equals $V_K(t^{-1})$, giving a simple criterion for checking the amphicheirality of a knot. M. Thistlethwaite has shown that the Jones polynomial of an alternating link is alternating as well [Thi87]. V_K has also played a major role in the solution of the Tait conjectures (see Section 2.1). Last but not least, the Jones polynomial takes the value 1 on any diagram of the unknot. However, as of 2022 it is surprisingly still unknown whether it detects the unknot or not. The Jones polynomial was generalized shortly after its discovery with the *HOMFLY-PT polynomial* [FYH$^+$85, PT87] in 1985,[17] and the *Kauffman*

[16] In [Jon91, Lecture 8], Jones refers to Kauffman's definition as "fourth and best definition" of the Jones polynomial.

[17] The name stands for the initials of its founders: J. Hoste, A. Ocneanu, K. Millett, P. Freyd, W. B. R. Lickorish, D. Yetter, J. Przytycki and P. Traczyk. In the literature, the HOMFLY-PT polynomial is often referred to as just the *HOMFLY polynomial*. The addition "PT" is to

polynomial in 1986 [Kau90].

Originally, the Jones polynomial arose as the trace of a representation from the braid group on n strands into a certain finite-dimensional von Neumann algebra, also known as the Temperley-Lieb algebra [Jon85, Jon91]. This algebra also appears in the context of statistical mechanics, where it is used as a tool to calculate the partition function of the Potts model [Jon85, Kau88]. Consequently, a relationship between knot theory and statistical mechanics started to unfold [Kau88, YG89]. The Yang-Baxter equation, fundamental to quantum integrable systems in statistical mechanics [Jim89, Tur88, Bax82], relates to the Jones polynomial since solutions of this equation give rise to Jones-type representations of the braid group [Res87, YG89]. In 1988, V. G. Turaev showed that both the HOMFLY-PT and the Kauffman polynomial can be obtained from certain solutions of the Yang-Baxter equation [Tur88], strengthening the relation between knot theory and theoretical physics.

The Yang-Baxter equation and quantum integrable systems were predominantly studied in the '80s at the Leningrad school of mathematical physics directed by L. Faddeev [Jim89, Tur10]. Their work led to a procedure to obtain explicit solutions to quantum integrable systems, the *quantum inverse scattering method* [STF80, Fad84, KS82b, KS82a, Fad95]. Algebraic structures appearing within this theory [Tji92] found a proper mathematical formalization when V. G. Drinfel'd and M. Jimbo introduced certain deformations of Lie algebras, the so-called *quantum groups* [Dri85, Dri86, Jim85]. Roughly speaking, a quantum group is a non-trivial deformation (or *quantization*) of the enveloping Hopf algebra of a semisimple Lie algebra, giving it the structure of a quasitriangular Hopf algebra [Kas95]. Basic, yet important examples of quantum groups are quantized special linear Lie algebras, the *quantum* $\mathfrak{sl}_n$ denoted by $U_q(\mathfrak{sl}_n)$.

Connections of quantum groups to knot theory were established quickly: as outlined in [Saw96], the deformation process that leads to a quantum group can be understood as a mean to obtain *non-commutative*, *non-cocommutative* ribbon Hopf algebras (i.e. a quasitriangular Hopf algebra with a certain invertible central element, see [RT90, Section 3.3]), whose representation theory gives rise to interesting knot invariants. Indeed, N. Yu. Reshetikhin and V. G. Turaev used in 1990 the representation theory of quantum $\mathfrak{sl}_\mathfrak{n}$ to obtain new polynomial invariants of knots for all $n \in \mathbb{N}$ [RT90]. Polynomials that correspond to the fundamental representation are referred to as *the quantum* $\mathfrak{sl}_n$ *polynomials*, with $n = 2$ corresponding to the original Jones polynomial. Additionally, it was observed in [RT90] that the quantum $\mathfrak{sl}_n$ polynomials can also be obtained from the HOMFLY-PT polynomial through a suitable substitution of variables.

Overall, the relation between topology and physics remained puzzling in the early '80s. On one hand, there is the discovery of the Jones polynomial,

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emphasize the independent work that was done by J. Przytycki and P. Traczyk.

an invariant of a 3-dimensional geometric situation and closely related to physical models, but lacking a topological description. On the other hand, there is Donaldson's gauge theoretic work on 4-manifolds with a clear topological picture, yet without a physical interpretation despite its usage of the Yang-Mills equations which emerged from physics [YM54, Wit88]. As noted by Atiyah [Ati88a, Ati88b], this situation called for a unifying approach towards physics and low-dimensional topology. First successes in this direction were achieved in 1988 with Witten's introduction of *topological quantum field theories* (TQFTs for short) [Wit88], based on his earlier work on the geometric description of super-symmetry [Wit82]. Witten's theory encompasses Donaldson's 4-dimensional invariants as well as Floer's homology groups of 3-manifolds. Shortly after, TQFTs were given an axiomatic description by Atiyah [Ati88b]. Only one year later in 1989, Witten constructed a 3-dimensional TQFT using Chern-Simons theory that yields invariants of knots and 3-manifolds as observables, in particular the Jones polynomial, giving it the first description ever in a 3-dimensional framework [Wit89].[18]

Naturally, TQFTs were quickly picked up by a wide mathematical and physical audience, and in particular by knot theorists. Speaking loosely in mathematical terms, an $(n+1)$-dimensional TQFT is defined as a symmetric monoidal functor from the category of $(n+1)$-cobordisms to the category of vector spaces over a field, or more generally, modules over a ring [Koc04, Section 1.3]. In 1991, Reshetikhin and Turaev introduced the notion of a *modular Hopf algebra* and observed that each such algebra gives rise to a 3-dimensional TQFT, producing invariants of 3-manifolds and links within them [RT91]. In particular, they show that given a root of unity q, the associated quantum group $U_q(\mathfrak{sl}_2)$ carries the structure of a modular Hopf algebra and has a corresponding 3-dimensional TQFT. They further show that this relates to the construction of invariants of more general 3-manifolds and links inside them using surgery theory as well as their earlier work on the representation theory of quantum $\mathfrak{sl}_2$ and the associated link invariants in S^3 [Res87, RT90, RT91]. This gives rise to a large new class of invariants, the *Reshetikhin-Turaev invariants*, and form far-reaching generalizations of the Jones polynomial. As pointed out by Reshetikhin-Turaev [RT91, Introduction], their work was partly inspired by Witten's 3-dimensional constructions using TQFTs, and hence their invariants can be understood as a "mathematical realization of Witten's program".

The next breakthrough in the theory of quantum invariants appeared when M. Khovanov presented a categorification of the Jones polynomial in 1999 [Kho00]. Khovanov's construction associates to a link diagram and a

[18] A short heuristic description of Witten's argument to compute the Jones polynomial can be found in the introduction of [Saw96]. A general exposition of topological quantum field theories with applications to topology can be found in the book of J. Labastida and M. Marino [LM05]. The book by J. Kock [Koc04] treats TQFTs from an axiomatic and categorical point of view and provides many historical remarks.

2-dimensional TQFT coming from an algebraic structure called *Frobenius algebra* a bigraded chain complex whose homology, the *Khovanov homology*, is a link invariant and whose graded Euler characteristic is the Jones polynomial of the link. It can be argued that Khovanov homology is at least as mysterious as the Jones polynomial: the unnatural combinatorial construction combined with an enormous amount of topological information made Khovanov homology seem to appear out of nowhere. This mystery was only strengthened when E. S. Lee defined a deformation of Khovanov homology in 2002 [Lee05]. Lee showed that her theory, also known as *Lee homology*, is not bigraded but filtered, and that it relates to Khovanov homology by means of a spectral sequence. She also showed that the direct sum over all her homology groups – which are in fact vector spaces over $\mathbb{Q}$ – always has dimension 2^n, where n is the number of components of the underlying link. This makes her deformation seem rather rigid. However, in a striking paper by J. Rasmussen [Ras10] it was observed that for knots the filtration degree of the generators of Lee homology give rise to a knot invariant: the *Rasmussen s-invariant*. The big surprise now is that the s-invariant carries lots of topological information, as it defines a homomorphism from the knot concordance group to the integers and provides a lower bound on the slice genus of a knot [Ras10]. Moreover, Rasmussen gave the first purely combinatorial proof of the Milnor conjecture using his invariant [Ras10]. Quite remarkable having its origin in mind!

Just as for the Jones polynomial, Khovanov homology sparked the interest of researchers throughout mathematics and physics. On the physical side, the fact that the Jones polynomial arises as an observable in a 3-dimensional TQFT led physicists to believe that Khovanov homology should correspond to an observable of a theory in dimension 4. A first interpretation of Khovanov homology in this direction was given by S. Gukov, A. Schwarz and C. Vafa in 2005 in terms of topological strings [GSV05]. A reinterpretation using gauge theory was given by Witten in 2011 [Wit12]. In particular, Witten observed that Khovanov homology can be computed by counting solutions of certain elliptic partial differential equations in $(4 + 1)$ dimensions. This method of computation was further studied by Gaiotto and Witten in 2012, giving the physical understanding of Khovanov homology a "good foundation" [GW12].

On the mathematical side, generalizations of Khovanov homology were soon established. In [Kho06], Khovanov investigated different variations of Khovanov homology by specifying alternative Frobenius algebras, leading to the discovery of a "universal" Khovanov homology from which other variations can be obtained. An extension of Khovanov homology to *tangles* (i.e. a proper embedding of a disjoint union of arcs and circles into the 3-ball B^3) was given in [Kho00, Kho02], with further work done in this direction by D. Bar-Natan in 2005 [Bar05]. In 2003, Khovanov described a categorification of the quantum $\mathfrak{sl}_3$ polynomial [Kho02]. One year later this result was extended by M. Khovanov and L. Rozansky who constructed a categorification of the quantum $\mathfrak{sl}_n$ polynomials for arbitrary $n > 0$, the so-called *Khovanov-Rozansky homologies* [KR08a]. In their sequel [KR08b], Khovanov and Rozansky gave a categorification of the HOMFLY-PT polynomial in terms of a triply graded homology theory.

Together with the categorification of the Alexander polynomial through knot Floer homology (see Section 2.3), a categorification of all major knot polynomials was achieved. However, the sheer number of knot homology theories left one wonder whether a unifying theory exists. Since the HOMFLY-PT polynomial specializes to all quantum polynomials, Dunfield, Gukov and Rasmussen conjectured in [DGR06] that such a theory should take the form of a triply graded homology theory that categorifies the HOMFLY-PT polynomial together with an additional set of differentials from which the Khovanov-Rozansky homologies and knot Floer homology can be obtained (see [DGR06, Conjecture 3.1]).

While Dunfield, Gukov and Rasmussen didn't give a precise definition, they made many observations that strongly support the existence of such a triply graded theory. A first candidate is Khovanov and Rozansky's categorification of the HOMFLY-PT polynomial which is indeed a triply graded homology theory, but lacks the additional set of differentials as noted in [DGR06, Section 1.9]. A preliminary connection between Khovanov homology and knot Floer homology was established when Ozsváth and Szabó constructed in 2005 a spectral sequence from Khovanov homology to Heegaard Floer homology of the double branched cover of the underlying link [OS05]. Several years later, Kronheimer and Mrowka proved in 2011 the existence of a spectral sequence from Khovanov homology to instanton Floer homology [KM11].

In 2006, Rasmussen showed that for each $N > 0$ there is a spectral sequence that starts at HOMFLY-PT homology and converges to the Khovanov-Rozansky $\mathfrak{sl}_N$ homology [Ras15]. As noted by Manolescu in [Man14, Introduction], after Rasmussen's result "it became more natural" to expect the existence of a spectral sequence from HOMFLY-PT homology to knot Floer homology, rather than the existence of a triply graded theory with additional differentials as proposed in [DGR06]. A first step towards such a spectral sequence was given by Ozsváth and Szabó in 2007 with their combinatorial description of knot Floer homology [OS09]. In particular, they observed that their construction specializes to a theory that resembles the HOMFLY-PT homology of Khovanov and Rozansky [OS09, Introduction]. This specialization was further developed by Manolescu in 2011, who showed that an associated spectral sequence converges to knot Floer homology, and conjectured that the first term of this spectral sequence is isomorphic to Khovanov and Rozansky's HOMFLY-PT chain complex [Man14]. Recently, N. Dowlin showed that the second page of this spectral sequence is a triply graded link invariant that categorifies the HOMFLY-PT polynomial and converges to knot Floer homology [Dow17, Dow18]. He also showed that if this second term is torsion-free, then it would be in fact isomorphic to Khovanov and Rozansky's HOMFLY-PT homology [Dow17, Theorem 1.4], giving a strong foundation for the correctness of the conjecture stated by Manolescu in [Man14]. At the time of writing, the spectral sequence coming from Ozsváth and Szabó's construction appears to be the most promising candidate for a theory that links Khovanov-Rozansky, HOMFLY-PT, and knot Floer homology.

Ever since their discovery, knot homologies and the associated invariants have become one of the major and strongest tools of 21st-century knot theory with deep connections to 3- and 4-dimensional topology and physics. For instance, it has been shown that Khovanov homology serves as a detector for many knots and links, such as the unknot [KM11], the unlink [BS15, Xie18], the trefoil and its mirror image [BS21], the Hopf links [BSX19], and the figure-eight knot [BDL+21]. The Rasmussen s-invariant, a powerful concordance invariant and provider for a lower bound on the slice genus, was recently used by L. Piccirillo to give a negative answer to the long lasting question whether the Conway knot is slice or not [Pic20]. On the other hand, knot Floer homology is known to detect many properties of knots such as the 3-genus and fiberedness, and serves as a source for a plethora of link invariants (see Section 2.3). The intricate relationship of knot homologies with theoretical physics that is being unfolded by the work of Witten et al. demonstrates the significance outside of mathematical knot theory. It remains thrilling to see what discoveries are yet to be made in the domain of knots, mathematics, physics, and science in general.

Part I

Higher Invariants: *A lower bound on the stable 4-genus of knots*

Chapter 3

Introduction and Results

In 2010, Charles Livingston [Liv10] introduced a new knot invariant called the *stable 4-genus* g_{st} of a knot K, which is defined as

$$g_{st}(K) = \lim_{n \to \infty} \frac{g_4(nK)}{n},$$

where nK denotes the n-fold connected sum $K \# \cdots \# K$ and g_4 is the (topological) 4-genus.[1] Recall that the 4-genus of a knot K is defined as the minimal genus over all properly embedded and locally flat surfaces $\Sigma \subset B^4$ with $\partial \Sigma = K$.

In general, it is rather difficult to compute the stable 4-genus of a knot. Most of the knot invariants that give bounds on the 4-genus, such as the Levine-Tristram signatures [Lev69a, Lev69b, Tri69], are additive under connected sum, hence they cannot yield bounds that are different for g_{st}.

More promising are Casson-Gordon invariants [CG75, CG78]. For instance, Livingston used them in [Liv10] to show that a specific satellite construction yields knots whose stable 4-genus is close to but not greater than 1/2. Note that it is an open question whether there exists a knot K such that $0 < g_{st}(K) < 1/2$. In what follows we use Casson-Gordon invariants once more to construct a lower bound on g_{st}. Our results show that already a simple family of knots, the twist knots, contains an infinite subfamily with stable 4-genus close to but not greater than 1/2. The main results are as follows.

Theorem 5.5 (Main Theorem). Let K be a knot with d-fold branched cover X_d where d is a prime-power, and let p be any prime. If the rational numbers $L_1, L_2, \ldots, L_m$ (defined below) have the same sign, and if $\sum_{s=1}^{d-1} \sigma_{s/d}(K) = 0$, where $\sigma_{s/d}(K)$ is the Levine-Tristram signature of K associated to $e^{2\pi i s/d}$, then

$$g_{st}(K) \geq \frac{t \cdot L}{4d(p-1) + 2(d-1)L},$$

[1] Throughout this part of the thesis, we will work in the topological category.

where $t \coloneqq \dim H^1(X_d;\mathbb{F}_p)$ and $L \coloneqq \min_{j=1,\dots,m} |L_j|$.

Here, the numbers L_j for $j = 1,\dots,m$ are defined as

$$L_j \coloneqq \sum_{\chi\in A_j} \operatorname{sign}_1^{\mathrm{av}}(\tau(K,\chi)) \in \mathbb{Q},$$

where $A_1,\dots,A_m$ are the one-dimensional subspaces of $H^1(X_d;\mathbb{F}_p)$, and $\operatorname{sign}_1^{\mathrm{av}}(\tau(K,\chi))$ is the Casson-Gordon τ-signature corresponding to χ (see Section 4.3 and Definition 5.1).

The strength of the bound in Theorem 5.5 depends on the choice of d and p. However, $t \cdot L/(4d(p-1) + 2(d-1)L)$ is bounded from above by $t/(2(d-1))$, so a priori the best bounds are obtained in the case $d = 2$, i.e. when working with the double branched cover X_2. Note that if one of the numbers L_j is zero, then also $L = 0$ and Theorem 5.5 will yield the trivial bound $g_{st}(K) \geq 0$.

Note that in Theorem 5.5, we make the assumption that the sum of Levine-Tristram signatures $\sum_{s=1}^{d-1}\sigma_{s/d}(K)$ of K vanishes. This is needed so that one of our main tools in the proof – Gilmer's lower bound on the 4-genus given by Casson-Gordon τ-signatures (see Theorem 4.93 and Corollary 4.94 in Subsection 4.3.9) – admits a more simplified application. The given proof of Theorem 5.5 doesn't hold without this assumption, and we currently do not know if of our methods generalize to the case where $\sum_{s=1}^{d-1}\sigma_{s/d}(K)$ doesn't vanish. However, if any of the Levine-Tristram signatures $\sigma_{s/d}(K)$ is non-zero, then there is the Murasugi-Tristram bound $g_4(nK) \geq \frac{n}{2}|\sigma_{s/d}(K)|$ [Mur65, Tri69] which implies $g_{st}(K) \geq \frac{1}{2}|\sigma_{s/d}(K)|$, so we still obtain a non-trivial lower bound on g_{st} when $\sum_{s=1}^{d-1}\sigma_{s/d}(K)$ doesn't vanish.

The main theorem is an immediate consequence of the following proposition.

Proposition 5.4. With the same assumptions as in Theorem 5.5,

$$g_4(nK) \geq \frac{nt\cdot L}{4d(p-1)+2(d-1)L}$$

for any $n \in \mathbb{N}$.

For the reader who would like to skip the (quite lengthy) proof of the main theorem without missing out on the key proof technique, we will provide a quick proof of the following introductory result.

Proposition 5.2. Let K be a knot and p a prime such that $H^1(X_d;\mathbb{F}_p)$ is one-dimensional. If $L = |L_1| > 0$, then $g_4(nK) \neq 0$ for all $n \in \mathbb{N}$.

As a sample application, we compute the lower bound given by Theorem 5.5 with $d = 2$ for an infinite family of knots, the twist knots K_n (see Figure 3.1). The result is as follows.

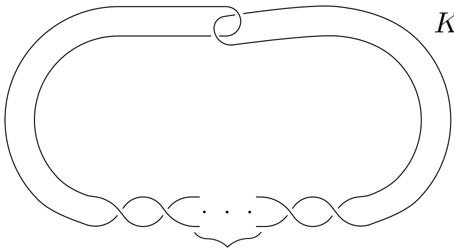


Figure 3.1: The twist knot K_n (picture taken from [Ilt19])

Corollary 6.3. Let K_n be the twist knot with $n \in \mathbb{N}\backslash\{0,2\}$ full right hand twists and p a prime dividing $4n+1$. Then

$$g_{st}(K_n) \geq \begin{cases} \frac{(pq+q-6)}{2(pq+q+18)}, & (p-1)/2 \text{ even} \\ \frac{p^2q-6p-q-6}{2(p^2q+18p-q-30)}, & (p-1)/2 \text{ odd}, \end{cases}$$

where $q = (4n+1)/p$.

While Corollary 6.3 is directly obtained from the main theorem, the bounds are not particularly easy to grasp. By estimating further from below, we obtain a single bound which is weaker but easier to grasp and holds for all twist knots K_n simultaneously.

Corollary 6.7 (Corollary 6.3, weakened). Let K_n with $n \in \mathbb{N}$ be any twist knot. Then

$$g_{st}(K_n) \geq \frac{1}{2} - \frac{6}{2n+7}.$$

It is straightforward to see that for growing n, the bound given in Corollary 6.7, and in fact also the stronger bounds in Corollary 6.3, tend towards 1/2. This means in particular that our bounds for the twist knots lie in the interval $[0, \frac{1}{2})$. Since the Levine-Tristram signatures of all twist knots vanish, these are the best bounds that we currently know.

Note that the first three twist knots K_0, K_1 and K_2 form a special case. Casson and Gordon proved [CG75, CG78] that the unknot K_0 and the so-called Stevedore knot K_2 are the only slice knots among the twist knots, and so the stable 4-genus of K_0 and K_2 vanishes. The knot K_1 represents the figure-eight knot, and it is well known that the figure-eight has order 2 in the knot concordance group $\mathcal{C}$. It follows that the stable 4-genus of K_1 vanishes as well. This coincides with the trivial bound obtained from Corollary 6.3 in those cases. In fact, K_0, K_1 and K_2 are the only twist knots with $L = 0$ (see Chapter 6), which means that for any other twist knot, a non-trivial lower bound can be obtained from Corollary 6.3 (resp. Theorem 5.5). This establishes the following corollaries.

Corollary 6.5. Let K_n be any twist knot. Then

$$K_n \text{ is torsion in } \mathcal{C} \iff g_{st}(K_n) = 0.$$

Corollary 6.6. K_n is of infinite order in $\mathcal{C}$ except for $n = 0, 1, 2$.

It is not known whether Corollary 6.5 holds for arbitrary knots K.

Problem (Livingston [Liv10]). Given a knot K, does $g_{st}(K) = 0$ imply that K is torsion in the knot concordance group $\mathcal{C}$?

Corollary 6.6 gives a full description of the concordance order of twist knots in the topological category. This result was to the best knowledge of the author not completely known before, despite the vast work that has been done on the topological concordance order of twist knots in the past. We would like to mention at this point some of the previously obtained results. First and foremost, Casson and Gordon [CG75, CG78] showed that if K_n is algebraically slice and $n \neq 0, 2$, then K_n has infinite order in the knot concordance group $\mathcal{C}$. Livingston and Naik [LN99, LN01] obtained the same result for all twist knots of algebraic concordance order 4. The remaining case of algebraic order 2 was partially solved by Davis [Dav12] and Tamulis [Tam02], who obtained infinite concordance order of K_n if $n = x^2 + x + 1$ with $x \geq 2$ (see [Dav12, Corollary 6]), or if $4n + 1$ is prime with $n \geq 3$ (see [Tam02, Theorem 1.1]). With our Corollary 6.6, all other cases are now solved.

A recent result by Baader and Lewark [BL17] shows that $g_{st}(K_n) \leq 2/3$ for every $n \in \mathbb{N}$. In fact, this bound can be improved for certain n as will be shown in Section 6.4.

Proposition 6.11. Let $n \in \mathbb{N}$ be such that the negative Pell equation $x^2 - (4n + 1)y^2 = -1$ has a solution $x, y \in \mathbb{Z}$. Then

$$g_{st}(K_n) \leq \frac{1}{2}.$$

As shown by Rippon and Taylor in [RT04], the negative Pell equation $x^2 - (4n + 1)y^2 = -1$ has a solution if and only if the continued fraction of $\sqrt{4n + 1}$ has odd period length. This is the case, for example, if $4n+1 = p^k$, where p is a prime such that $p \equiv 1 \mod 4$ and $k \in \mathbb{N}$ is odd [RT04]. This yields the infinite family of twist knots whose stable 4-genus is close to but not greater than 1/2.

3.1 Smooth vs. topological setting

We would like to make a short remark about the situation in the smooth setting. Since the smooth stable 4-genus is always greater than or equal to the topological stable 4-genus, our result also applies in the smooth setting. When it comes to potential lower bounds obtained by smooth knot invariants such as the Rasmussen s- and τ-invariant [Ras03, Ras10, OS03a], or the Ozsváth-Stipsicz-Szabó Υ-invariant [OSS17], we face the same problem as with the Levine-Tristram signatures in the topological setting: all these invariants are additive under connected sum. In particular, no better lower bounds in the smooth setting are known.

Regarding the results about twist knots, Corollary 6.3 still holds in the smooth setting. This is no longer true for Proposition 6.11: the upper bound is obtained by using machinery that is exclusive to the topological setting and is therefore no longer valid. However, a result by Baader and Lewark [BL17, Lemma 6] implies that the smooth stable 4-genus of twist knots is strictly smaller than 1, so we can still say that it is contained in the interval $[0, 1)$. Since our lower bounds also hold in the smooth setting, we can further say that K_0, K_1 and K_2 are the only twist knots with smooth stable 4-genus equal to zero. Other information is not known. In particular, we cannot tell whether the topological and smooth stable 4-genus of twist knots coincide or not, except in the case of K_0, K_1 and K_2 where they are the same. This raises the following open question.

Problem. Compute the smooth and topological stable 4-genus for any twist knot K_n with $n \geq 3$.

Twist knots form one of the simplest family of knots. The fact that we don't know the exact value of the stable 4-genus for any twist knot (other than the first three) shows how difficult it is to determine g_{st} in general. Moreover, to the best knowledge of the author all known exact values of the stable 4-genus are integers. This raises further the following question.

Problem. Does there exist either in the smooth or topological setting a knot K such that $g_{st}(K) \notin \mathbb{Z}$ and for which the exact value of the stable 4-genus can be computed?

We would like to conclude this discussion with a remark regarding Corollary 6.6. In the smooth category, Lisca [Lis07] gave a complete description of the concordance order of 2-bridge knots, a class to which the twist knots belong. It follows from his result that the only twist knots of finite order in the smooth concordance group are K_0, K_1, and K_2. This coincides with the results obtained in the present work and shows that the concordance order of the twist knots is the same in the topological and smooth category.

3.2 Organization

The remainder of the current Part I is structured as follows. In Chapter 4, we state the tools and definitions needed to prove our results. Chapter 5 forms the heart of Part I and is occupied with the proof of Proposition 5.4 and Theorem 5.5. Here we also prove Proposition 5.2. In Chapter 6, we compute the lower bound for g_{st} for the twist knots K_n and complete as a corollary the classification of the concordance order of the twist knots. We further show that there is an infinite subfamily of twist knots with stable 4-genus close to but not greater than $1/2$, and we provide a partial answer to their linear independency in the knot concordance group $\mathcal{C}$.

This introduction, Section 4.1 and Subsection 4.3.9, as well as the contents of Chapter 5 and Chapter 6, have appeared previously in the paper *"A lower bound on the stable* 4*-genus of knots"* by the author [Ilt20]. New is the discussion on how our results behave with respect to connected sum of knots at the end of Chapter 5, as well as the partial result about the linear independency of the twist knots in the knot concordance group $\mathcal{C}$, see Corollary 6.9.

Chapter 4

Preliminaries

The purpose of this chapter is to review the definitions and notions of the mathematical objects that are used throughout Part I. The chapter is organized as follows. In Section 4.1 we introduce the abstract notion of linking forms on finite abelian groups and metabolizers thereof. In Section 4.2, we recall some of the basic definitions and constructions from knot theory relevant to us. Here, we also introduce our main examples for computations in Chapter 6, the twist knots.

Section 4.3 forms the main part of Chapter 4 and is concerned with Casson-Gordon invariants. In Subsections 4.3.1 to 4.3.3, we start with an introduction to the concepts and objects involved in the definition of these invariants, such as homology with twisted coefficients and twisted intersection forms. In Subsection 4.3.4, we describe a specific situation where twisted homology is formed using finite cyclic coverings, and discuss how in this case the twisted intersection form relates to a certain intersection form on untwisted homology. Also, in this scenario a twisted signature arises that relates to the Atiyah-Singer G-signature, which we describe in Subsection 4.3.5. Note that this setting is of particular interest as it forms the starting point for the definition and computation of the Casson-Gordon invariants. The last ingredient needed are Witt groups of non-singular hermitian forms on free modules, which we introduce in Subsection 4.3.6. In Subsections 4.3.7 and 4.3.8 we finally arrive at the definition of the Casson-Gordon invariants and provide a discussion of the main theorems and results around them, and show how they relate to each other. In Subsection 4.3.9, we conclude our discussion with a description of results obtained by P. Gilmer about Casson-Gordon invariants, which will be our main tools in Chapter 5.

Our discussion about Casson-Gordon invariants in Section 4.3 is quite detailed and strives to provide a mostly self-contained account for this at times confusing subject. However, not all of Section 4.3 is needed in order to understand our main results in the next chapter. The reader who wishes to proceed to Chapter 5 as fast as possible is advised to have a look at Section 4.2 and Subsection 4.3.9 for the objects and tools used in our main theorems and proofs. In particular, the explicit definition of the Casson-Gordon invariants is not needed and may be taken as a blackbox.

Unless otherwise mentioned, we work in the topological category. References for Chapter 4 include [CG75, CG78, Gil82, Gil83, LN16, Con17, Fri22], with further references provided throughout the individual sections.

4.1 Linking forms and metabolizers

We follow the definitions of [Gil82].

Definition 4.1. Let G be a finite abelian group. A *linking form* on G is a symmetric bilinear map

$$\alpha\colon G \times G \to \mathbb{Q}/\mathbb{Z}$$

which is non-singular, i.e. the adjoint map $c\colon G \to \operatorname{Hom}(G, \mathbb{Q}/\mathbb{Z})$ is an isomorphism.

If α is a linking form on a finite abelian group G and $H \subseteq G$ a subgroup, we define

$$H^\perp := \{g \in G \mid \alpha(g,h) = 0 \text{ for all } h \in H\}.$$

Definition 4.2. Let α be a linking form on a finite abelian group G. If there is a subgroup $H \subseteq G$ such that $H = H^\perp$, then α is called *metabolic* and H is called a *metabolizer*.

A useful property of metabolizers is the following.

Proposition 4.3. If H is a metabolizer for α, then

$$|H|^2 = |G|.$$

Proof. Write $G^* := \operatorname{Hom}(G, \mathbb{Q}/\mathbb{Z})$ for short. Observe that there is a short exact sequence

$$0 \to H^\perp \xrightarrow{c} G^* \to H^* \to 0,$$

hence

$$G^*/\mathrm{im}\, c \cong H^*.$$

Using Lagrange's Theorem and that $\mathrm{im}\, c \cong H^\perp$, we get

$$|G^*|/|H^\perp| = |G^* : \mathrm{im}\, c| = |H^*| \iff |G^*| = |H^\perp||H^*|$$

Since G is a finite abelian group, we have $|G| = |G^*|$ and in particular $|H| = |H^*|$. Moreover we have $H = H^\perp$ since H is a metabolizer for α, hence we obtain

$$|G| = |H|^2$$

as desired. □

It will be convenient for our purposes to consider the above in a slightly different setting. Given a prime p, let V be a finite-dimensional vector space over $\mathbb{F}_p$, the finite field with p elements. If V can be embedded in G as a finite abelian group, then α induces a linking form $\overline{\alpha}$ on V in the following way. After choosing an embedding $\varphi\colon V \hookrightarrow G$, we define $\overline{\alpha}$ to be the restriction of α to the subgroup $\varphi(V) \subseteq G$. In other words,

$$\overline{\alpha} := \alpha \circ (\varphi \times \varphi).$$

Note that the subgroup $\varphi(V)$ consists only of p-torsion. In general, the induced form $\overline{\alpha}$ depends on the chosen embedding φ and will no longer be non-degenerate. However, any two forms obtained in this way are isometric provided that their domains are the same: if $\varphi_1, \varphi_2\colon V \hookrightarrow G$ are two embeddings with $\varphi_1(V) = \varphi_2(V)$ inducing the forms $\overline{\alpha}_1$ and $\overline{\alpha}_2$, then

$$f\colon (V, \overline{\alpha}_1) \to (V, \overline{\alpha}_2), \quad f = \varphi_2^{-1} \circ \varphi_1$$

defines an isometry between them. Given such an induced form $\overline{\alpha}$, we can similarly define $F^\perp$ for a given subspace $F \subseteq V$. If F is such that $F = F^\perp$, then F will be called a *generalized metabolizer* for $\overline{\alpha}$. Similar to Proposition 4.3, generalized metabolizers satisfy the following useful property.

Proposition 4.4. If F is a generalized metabolizer for $\overline{\alpha}$, then

$$2 \dim F \geq \dim V.$$

In particular, a generalized metabolizer F can consist of the entire space V.

Proof. Write $V^* := \operatorname{Hom}(V, \mathbb{Q}/\mathbb{Z})$ for short. Similar to the proof of Proposition 4.3, there is an exact sequence

$$F^\perp \xrightarrow{c} V^* \xrightarrow{r} F^* \to 0,$$

where $c\colon V \to V^*$ is the adjoint map and r is restriction of maps. Note that this sequence is in general not split since $\overline{\alpha}$ may be degenerate. However, by the rank-nullity theorem we have

$$\dim V^* = \dim \operatorname{im}\, r + \dim \ker r. \tag{4.1}$$

Since V is finite-dimensional we have $\dim V = \dim V^*$. Moreover, using that c and r sit in an exact sequence and that by assumption $F = F^\perp$, we get

$$\begin{aligned} \dim \operatorname{im}\, r &= \dim F^* = \dim F \\ \dim \ker r &= \dim \operatorname{im}\, c \leq \dim F^\perp = \dim F. \end{aligned}$$

Applying this to (4.1) we therefore obtain

$$\dim V \leq 2 \dim F. \qquad \square$$

The following example demonstrates that a generalized metabolizer may indeed consist of the entire space V itself (thus realizing strict inequality in Proposition 4.4).

Example 4.5. Let $G = \mathbb{Z}_{75} \cong \mathbb{Z}_{5^2} \times \mathbb{Z}_3$ with

$$\alpha\colon \mathbb{Z}_{75} \times \mathbb{Z}_{75} \to \mathbb{Q}/\mathbb{Z}, \quad (x, y) \mapsto \frac{x \cdot y}{75}.$$

Clearly, α is a symmetric pairing on $\mathbb{Z}_{75}$, and the adjoint map

$$y \mapsto \alpha(\cdot, y) \in \mathrm{Hom}(\mathbb{Z}_{75}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}_{75}$$

is an isomorphism since any element in $\mathrm{Hom}(\mathbb{Z}_{75}, \mathbb{Q}/\mathbb{Z})$ is determined by its image of $1 \in \mathbb{Z}_{75}$, which is necessarily a multiple of $1/75$. Hence, α defines a linking form on $\mathbb{Z}_{75}$. Now let $p = 5$ and consider $V = \mathbb{F}_5$ as a vector space over itself. We embed $\mathbb{F}_5$ into $\mathbb{Z}_{75}$ via

$$\varphi\colon \mathbb{F}_5 \to \mathbb{Z}_{75}, \quad \varphi(1) = 15.$$

As described above, this induces a linking form

$$\overline{\alpha}\colon \mathbb{F}_5 \times \mathbb{F}_5 \to \mathbb{Z}_{75}, \quad (x, y) \mapsto \frac{x \cdot y \cdot 15^2}{75}.$$

Since $15^2 = 225 = 0 \mod 75$, we see that $\overline{\alpha}$ vanishes on the entire vector space $\mathbb{F}_5$, making it a generalized metabolizer for $\overline{\alpha}$.

4.2 Notions from knot theory

The aim of this section is to recall some notions from knot theory that will be used throughout this part of the thesis. Here, we also introduce our main examples of knots, the twist knots K_n. We assume familiarity with the upcoming objects and constructions and refer the interested reader for more details to one of the standard textbooks about knot theory such as [Rol76] or [BZH14]. We work in the topological setting unless otherwise mentioned.

We start by recalling the following definitions, which will be needed shortly when talking about the results involving Casson-Gordon invariants.

Definition 4.6. A *knot* K is a smooth embedding $S^1 \hookrightarrow S^3$ (sometimes also denoted by $K \subset S^3$). A knot K is called:

1. *(topologically) slice* if there exists a properly embedded locally flat 2-disc $D \subset B^4$ with $\partial D = K$, called a *(topological) slice disc* for K;

2. *algebraically slice* if the Seifert pairing of K is metabolic in the sense of Definition 4.61;

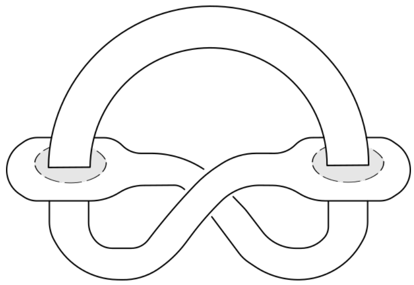


Figure 4.1: A ribbon knot with a ribbon disc. An open neighborhood of the ribbon singularities is highlighted in grey (figure taken from [Ilt19]).

3. *ribbon* if there exists a smooth immersion of a 2-disc $D^2 \to S^3$ such that all singularities are of the type shown in Figure 4.1. The immersed disc is called a *ribbon disc* for K.

Unless otherwise mentioned, our knots will always be assumed to be oriented.

Remark 4.7. If K is a ribbon knot with ribbon disc D, then one can push the interior of D into B^4 and deform afterwards a small open neighborhood of the ribbon singularities to obtain a *smoothly and properly embedded* slice disc for K, meaning that K is *smoothly slice* (cf. [LN16, Theorem 3.1.6]). Hence every ribbon knot is smoothly slice, but the converse, the famous *Slice-Ribbon Conjecture*, has been an open question ever since it was first posed by Fox in the 60's [Fox62]. Note that since there exist topologically slice knots which are not smoothly slice [Fre82], ribbon knots are not relevant in the topological setting.

Definition 4.8. Let $K \subset S^3$ be a knot.

1. The 3-*genus* of K is defined as

$$g(K) \coloneqq \min\{g(F) \mid F \text{ is a Seifert surface for } K\}.$$

2. The *(topological) slice genus* or *(topological)* 4-*genus* of K is defined as

$$g_4(K) \coloneqq \min\left\{g(F) \;\middle|\; \begin{array}{l} F \subset B^4 \text{ properly embedded, locally} \\ \text{flat surface with } \partial F = K \end{array}\right\}.$$

3. The *stable* 4-*genus* of K is defined as

$$g_{st}(K) \coloneqq \lim_{n\to\infty} \frac{g_4(nK)}{n},$$

where $nK \coloneqq K\#\cdots\#K$ denotes the n-fold connected sum of K.

Remark 4.9. Clearly, if $g_4(K) = 0$ then K is slice. Pushing a Seifert surface of K into the 4-ball while keeping its boundary fixed, we obtain the following inequality:

$$g_4(K) \leq g(K).$$

Of course, one can also define the *smooth 4-genus* g_4^{smooth} of a knot K by replacing locally flat with smooth in Definition 4.8. Since smooth implies locally flat, we see that

$$g_4(K) \leq g_4^{\text{smooth}}(K) \leq g(K),$$

where the last inequality is obtained by noting that a pushed-in Seifert surface produces in fact a smoothly and properly embedded surface $F \subset B^4$ with $\partial F = K$. None of these inequalities are in fact equalities, the first because it is known since the work of Freedman [Fre82] that there are topologically slice knots which are not smoothly slice, and the second because there are many smoothly slice knots which are not trivial, such as the connected sum of the figure-eight with itself, or in general any ribbon knot.

As mentioned earlier, the stable 4-genus was introduced by Charles Livingston in 2010 [Liv10] as a mean to provide a new insights into the 4-genus and knot concordance. For instance, g_{st} induces a semi-norm on the rationalized knot concordance group $\mathcal{C}_{\mathbb{Q}} = \mathcal{C} \otimes_{\mathbb{Z}} \mathbb{Q}$, see [Liv10, Theorem 2]. In Chapter 5, we will obtain a new lower bound for g_{st} using Casson-Gordon τ-signatures, which will allow us to derive new concordance information about the twist knots in Chapter 6.

The set of isotopy classes of knots together with the operation of connected sum forms a monoid. Using the notion of sliceness however, one is able to turn this monoid into a group, the knot concordance group $\mathcal{C}$.

Definition 4.10. Two knots K_1 and K_2 are called *concordant* if the connected sum $K_1 \# -K_2$ is slice, where $-K_2$ denotes the mirror image of K_2 with orientation reversed.

Theorem 4.11. Knot concordance is a well-defined equivalence relation on the set of isotopy classes of knots. The resulting equivalence classes form a group with operation the connected sum of knots. The neutral element is given by the class of slice knots, and the inverse of a knot K is given by $-K$.

For a proof, see for instance [LN16, Theorem 3.3.3].

Definition 4.12. The group described in Theorem 4.11 is called the *knot concordance group* $\mathcal{C}$.

Remark 4.13. As mentioned in Chapter 3, the knot concordance group was first introduced by Fox and Milnor in the 60's [FM66]. Originally, it was formulated in the smooth setting, in which also one of the most important results, namely that there is an epimorphism

$$\mathcal{C} \to \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty,$$

was obtained by Levine [Lev69a, Lev69b]. Moreover, Casson and Gordon showed that the kernel of this epimorphism is non-trivial [CG75, CG78], thus proving that this map is not an isomorphism (see also Subsections 4.3.7 and 4.3.8). However, the work of Freedman [Fre82, FQ90] implies that these results also apply in the topological setting (cf. [Liv05, Section 6]). Despite being extensively studied since its introduction, many questions about the knot concordance group remain open. For instance, it is still unknown whether there is torsion of order greater than 2 in $\mathcal{C}$. In Chapter 6, we will apply our results from Chapter 5 to complete the classification of the concordance order of the twist knots K_n (see Corollary 6.6). For a general survey of the knot concordance group $\mathcal{C}$, see for instance [Liv05].

We recall the following definitions of manifolds and coverings arising from the ambient space of a knot K. Details are given for instance in [Con17, Section 2.1 and 2.7].

Definition 4.14. Let $K \subset S^3$ be a knot.

1. The *knot exterior of* K is defined as

$$X_K \coloneqq S^3 \backslash \nu K,$$

   where νK is a tubular neighborhood of K in S^3.

2. By Alexander duality $H_1(X_K;\mathbb{Z}) \cong \mathbb{Z}$, with generator given by a meridian of K (unique up to sign). Hence for each $d \in \mathbb{N}$, we define the *d-fold cyclic coverings of the knot exterior* as the regular coverings $X'_d \to X_K$ corresponding to the kernel of the composition

$$\pi_1(X_K) \to H_1(X_K;\mathbb{Z}) \cong \mathbb{Z} \stackrel{\text{proj}}{\to} \mathbb{Z}_d.$$

   If $d = 0$, we write X'_∞ and refer to it as *the infinite cyclic covering of* X_K.

3. The boundary of X_K is a torus $S^1 \times S^1$, and so is the boundary of X'_d since the Euler characteristic is multiplicative under finite coverings. Let μ_K be a meridian of ∂X_K and $d \in \mathbb{N}$. Then μ_K^d lifts to a closed loop $(\mu_K^d)'$ in X'_d, and we define the *d-fold cyclic branched covering of* K as

$$X_d \coloneqq X'_d \cup_\partial T \to S^3,$$

   where $T \coloneqq S^1 \times D^2$ is a solid torus glued to X'_d via an orientation-reversing homeomorphism that identifies the meridian of T with $(\mu_K^d)'$.

4. The *zero framed surgery of* K is defined as

$$M_K := X_K \cup_\partial T,$$

where $T := S^1 \times D^2$ is a solid torus glued to X_K via an orientation-reversing homeomorphism that identifies the meridian of T with the longitude of ∂X_K.

5. A Mayer-Vietoris argument applied to the decomposition $M_K = X_K \cup T$ shows that $H_1(M_K;\mathbb{Z}) \cong \mathbb{Z}$, with generator given by a meridian of K. Hence for each $d \in \mathbb{N}$, we define the *d-fold cyclic coverings of the zero-framed surgery of* K as the regular coverings $M_d \to M_K$ corresponding to the kernel of the composition

$$\pi_1(M_K) \to H_1(M_K;\mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\text{proj}} \mathbb{Z}_d.$$

If $d = 0$, we write M_∞ and refer to it as *the infinite cyclic covering of* M_K.

Our main examples for computations are going to be the twist knots K_n, defined as follows.

Definition 4.15. The twist knot K_n is obtained by adding $n \in \mathbb{N}$ full right-hand twists into an unknot and adding a clasp; see Figure 4.2.

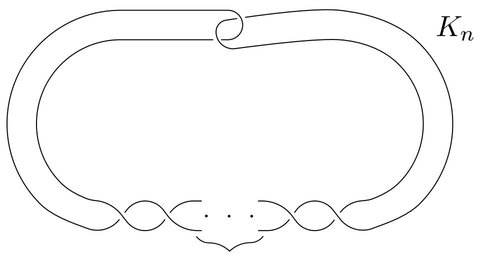


Figure 4.2: The twist knot K_n (picture taken from [Ilt19]).

The following properties of twist knots are well-known.

Proposition 4.16. Let K_n be a twist knot.

1. If $n \neq 0$, then $g(K_n) = 1$.
2. The signature of K_n vanishes, i.e. $\sigma(K_n) = 0$.
3. K_n is algebraically slice if and only if $4n + 1$ is a square.

Proof. If $n = 0$, then K_0 is the unknot and the statements 2. and 3. are automatically true. So assume without loss of generality that $n \neq 0$. Consider Figure 4.3. The surface F_n forms a genus one Seifert surface for K_n, and since K_n is not trivial it follows that

$$g(K_n) = 1.$$

The curves a and b shown in Figure 4.3 form a basis for $H_1(F_n;\mathbb{Z}) \cong \mathbb{Z}^2$. In this basis, the Seifert matrix of K_n takes the form

$$A_n := \begin{pmatrix} -1 & 1 \\ 0 & n \end{pmatrix},$$

and the Seifert pairing is represented by the matrix

$$A_n + A_n^T = \begin{pmatrix} -2 & 1 \\ 1 & 2n \end{pmatrix}.$$

The signature of $A_n + A_n^T$ is easily seen to vanish, hence $\sigma(K_n) = 0$. In order to determine when the Seifert pairing is metabolic, let $\begin{pmatrix} x & y \end{pmatrix} \in H_1(F;\mathbb{Z})$. We have:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2(-x^2 + xy + ny^2). \tag{4.2}$$

We wish to find $x, y \in \mathbb{Z}$ such that the right-hand side of (4.2) vanishes, for then the element $\begin{pmatrix} x & y \end{pmatrix} \in H_1(F;\mathbb{Z})$ generates a submodule of half-rank that equals its orthogonal complement with respect to the Seifert pairing. To find a solution, assume $y = 1$. Solving over the reals $\mathbb{R}$, we obtain

$$-x^2 + x + n = 0 \iff x_{1,2} = \frac{-1 \pm \sqrt{4n+1}}{-2}.$$

Observe that the solutions x_1 and x_2 are in $\mathbb{Z}$ if and only if $4n+1$ is a square. It follows that if this is the case, then the Seifert pairing of K_n is metabolic. A similar computation can also be applied to obtain the other direction of the equivalence. However, a more elegant argument is the following: if K_n is algebraically slice, then the existence of a metabolizer for the symmetric Seifert pairing implies that the absolute value of the determinant of K_n is a square of a non-zero integer (cf. [LN16, Exercise 4.3.1]). Since

$$|\det K_n| = |\det(A_n + A_n^T)| = 4n + 1,$$

it follows that $4n + 1$ has to be a square. □

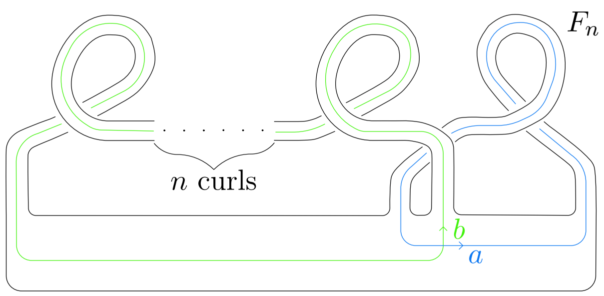


Figure 4.3: The Seifert surface F_n for K_n with a basis for $H_1(F_n;\mathbb{Z})$ (picture taken from [Ilt19]).

Remark 4.17. As mentioned in Section 2.2, Levine used the notion of algebraic sliceness to introduce the *algebraic concordance group* $\mathcal{G}$, defined similar to the knot concordance group $\mathcal{C}$ [Lev69a, Lev69b]. He showed that there is an epimorphism $\mathcal{C} \twoheadrightarrow \mathcal{G}$ and that $\mathcal{G} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$, giving in fact the epimorphism mentioned in Remark 4.13. In [LN01, 4.4 Corollary], Livingston and Naik provided a complete classification of the order of the twist knots K_n in the algebraic concordance group $\mathcal{G}$:[1]

1. If $4n + 1$ is a perfect square, then K_n is algebraically slice.
2. If $4n+1$ is not a perfect square, and every prime congruent to 3 mod 4 has even exponent in the prime power factorization of $4n + 1$, then K_n is of order 2 in $\mathcal{G}$.
3. If there is some prime congruent to 3 mod 4 with odd exponent in the prime power factorization of $4n + 1$, then K_n is of order 4 in $\mathcal{G}$.

4.3 Casson-Gordon invariants and τ-signatures

In the 1970s, A. J. Casson and C. M. Gordon defined several invariants of knots and 3-manifolds and applied them successfully to the study of knot concordance. These invariants are:

1. The closely related $\sigma(M, \chi) \in \mathbb{Q}$ and $\sigma_r(M, \chi) \in \mathbb{Q}$, associated to a closed oriented topological 3-manifold M and an epimorphism $\chi\colon H_1(M; \mathbb{Z}) \to \mathbb{Z}_m$ for some $m \in \mathbb{N}$ and $0 < r < m$ [CG75, CG78];
2. $\sigma(K, \chi) \in \mathbb{Q}$, a special instance of $\sigma(M, \chi)$ associated to a knot K with $M = X_d$ being the d-fold cyclic branched cover of K [CG75];
3. $\tau(K, \chi) \in W(\mathbb{C}(t)) \otimes \mathbb{Q}$, associated to a knot K and an epimorphism $\chi\colon H_1(X_d; \mathbb{Z}) \to \mathbb{Z}_m$ with $m \in \mathbb{N}$ being a prime-power. Here, X_d denotes again the d-fold cyclic branched cover of K and $W(\mathbb{C}(t))$ is the Witt group of non-singular hermitian forms on finite-dimensional vector spaces over $\mathbb{C}(t)$ [CG75].

In the upcoming Chapter 5, our main theorems will be obtained by working exclusively with $\tau(K, \chi)$ and a certain signature function that evaluates this invariant. However, we would like to take the opportunity and describe all of the above Casson-Gordon invariants in hope to resolve some of the confusion that might be encountered when learning this subject.[2] Along the way, we provide an introduction to several concepts that will be needed in order to define the Casson-Gordon invariants, such as homology with twisted coefficients, twisted intersection forms,

[1] Their result holds more generally for the n-twisted double of a knot K; if $n < 0$ then the n-twisted double of K is of infinite order in $\mathcal{G}$, and if $n > 0$ then the same results as for the twist knots apply. Note that if K is the unknot and $n > 0$, one obtains the twist knots K_n.

[2] Such confusion was actually encountered by the author himself when first learning the subject.

and Witt groups. The reader who is already familiar with these concepts and/or Casson-Gordon invariants, or who would simply like to take this bit of mathematics as a blackbox is advised to continue with Subsection 4.3.9, which contains the description of one of the main tools needed throughout Chapter 5. Unless otherwise mentioned, we will work in the topological locally flat category. Main references for this section are [CG75, CG78, Kau87a, LN16, Con17, Fri22].

4.3.1 Twisted homology

Twisted homology, also known as *homology with twisted coefficients*, is an extension of ordinary homology and cohomology that takes (roughly speaking) coverings into account in order to obtain more information about the space under consideration. Arguably its most famous application is the extension of Poincaré duality to non-orientable manifolds (see for instance [DK01, Theorem 5.7]). In knot theory, a major tool are covering spaces of the knot exterior, so homology with twisted coefficients provides a suitable framework in this situation (see for instance [Hil12]).

Twisted homology is usually formed using modules over (not necessarily commutative) rings with unity. A similar and in most settings equivalent theory (see [Fri22, Part XIX] or [Hat02, Chapter 3.H]) is constructed by using bundles of groups, leading to so-called *homology with local coefficients*.[3] While we will only describe the former in the following, it is worth to note that both theories have their strengths and weaknesses depending on the context and the results one wishes to prove (cf. [Fri22]).

We wish to remark that we do not intend to develop the theory in its full generality at this point; our description stays within the scope of an introduction that is only as general as needed in later applications. We refer the interested reader to [Fri22, Part XIX] for a full account instead (see also [DK01, Hat02, Her19]). Our discussion follows mainly [Fri22, Part XIX], with additional resources being [DK01, Hat02, LN16, Con17, FLNP17, Her19]. Proofs will mostly be omitted, but references to proofs of the stated results will be provided.

Let X be a path-connected and locally contractible topological space with a base point $x_0 \in X$, and let $Y \subset X$ be a (possibly empty) subspace. Further, let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be the universal cover of X and write $\widetilde{Y} := p^{-1}(Y)$.[4] Using the identification of $\pi_1(X, x_0)$ with the group of deck transformations $\mathrm{Aut}_p(\widetilde{X})$ (see [Fri22, Proposition 167.1]), the univer-

[3] The distinction between twisted homology and homology with local coefficients is not always made in the literature. For instance, [Hat02] refers to both theories as homology with local coefficients.

[4] In order to develop the theory of (co-)homology with twisted coefficients, one usually works with an explicit description of the universal covering in terms of certain equivalence classes of paths in X based at x_0 (as for instance in [Fri22, Part XIX]). However, [Fri22, Proposition 168.6] implies that the theory also holds for an arbitrary description of the universal cover of X with any fixed base point $\widetilde{x}_0$ in the fibre of x_0.

sal cover $\widetilde{X}$ admits a left π-action. The boundary map of the singular chain complex $C_*(\widetilde{X},\widetilde{Y};\mathbb{Z})$ is natural and thus commutes with the $\pi_1(X,x_0)$-action on $\widetilde{X}$, hence $C_*(\widetilde{X},\widetilde{Y};\mathbb{Z})$ inherits the structure of a chain complex of free left $\mathbb{Z}[\pi_1(X,x_0)]$-modules, where $\mathbb{Z}[\pi_1(X,x_0)]$ is the group ring of $\pi_1(X,x_0)$ over the integers.

Let R be a (not necessarily commutative) unitary ring together with an involution $r \mapsto \overline{r}$ that reverses the order of multiplication (a so-called *involutive anti-automorphism*). We assume that ring homomorphisms are involution preserving, that is $f(\overline{r}) = \overline{f(r)}$ for all $r \in R$ and ring homomorphisms $f\colon R \to S$. Our primary examples are group rings $R = \mathbb{Z}[G]$ with involution defined by $\sum \alpha_i g_i \mapsto \sum \alpha_i g_i^{-1}$. If M is a left R-module, we will denote by $\overline{M}$ the right R-module obtained by inverting the left R-multiplication on M using the involution, i.e. $m \cdot r := \overline{r}m$, and vice-versa if M is a right R-module.

Now let G be a group and let M be a left R-module. Given a representation $\alpha\colon G \to \mathrm{Aut}_{\text{left}-R}(M)$, we can turn M into a left $\mathbb{Z}[G]$-module by defining $g \cdot m := \alpha(g)(m)$ for $g \in G$ and $m \in M$. This structure is compatible with the left R-structure on M in the sense that

$$(g \cdot (r \cdot m)) = \alpha(g)(r \cdot m) = r \cdot \alpha(g)(m) = (r \cdot (g \cdot m)),$$

thus turning M into a $(R,\mathbb{Z}[G])$-left-left module. Conversely, if M is a $(R,\mathbb{Z}[G])$-left-left module, we obtain a representation $\widetilde{\alpha}\colon G \to \mathrm{Aut}_{\text{left}-R}(M)$ by mapping $g \mapsto L_g$, where L_g is the left multiplication $L_g(m) := g \cdot m$. Hence a left $\mathbb{Z}[G]$-structure on a left R-module M is the same as specifying a representation $\alpha\colon G \to \mathrm{Aut}_{\text{left}-R}(M)$. In particular, if $R = \mathbb{Z}$ then a $(\mathbb{Z},\mathbb{Z}[G])$-left-left module M is simply a left $\mathbb{Z}[G]$-module, and a left $\mathbb{Z}[G]$-structure on (the abelian group) M is the same as specifying a representation $\alpha\colon G \to \mathrm{Aut}_{\mathbb{Z}}(M)$.

Notation. Let us fix some conventions and notation.

1. If we wish to emphasize the representation α that defines a left $\mathbb{Z}[G]$-structure on a left R-module M, we will write M_α.

2. If M carries the trivial left $\mathbb{Z}[G]$-structure corresponding to the representation defined by $\alpha(g)(m) := m$ for $g \in G$ and $m \in M$, we use the special notation M_{triv}.

3. Whenever we speak of a left $\mathbb{Z}[G]$-module M, we will implicitly assume that M is in fact a $(\mathbb{Z},\mathbb{Z}[G])$-left-left module unless otherwise mentioned or clear from the context.

4. We will use the multiplication symbol $\cdot$ to indicate group actions or specifically defined module structures. Ordinary ring multiplication will be denoted by juxtaposition.

5. Throughout this section we will abbreviate $\pi := \pi_1(X,x_0)$.

Definition 4.18. Let X be a locally contractible and path-connected topological space with base point $x_0 \in X$ and $Y \subset X$ a (possibly empty) subspace. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be the universal cover and write $\widetilde{Y} := p^{-1}(Y)$. Further, let M be a $(R, \mathbb{Z}[\pi])$-left-left module. The *twisted (co-)chain complex* of the pair (X, Y) with coefficients in M is defined as[5]

$$C^t_*(X, Y; M) := \overline{M} \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Z}),$$
$$C^*_t(X, Y; M) := \operatorname{Hom}_{\text{left}-\mathbb{Z}[\pi]}(C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Z}), M),$$

with boundary operator $\mathrm{id}_{\overline{M}} \otimes \partial$ and ∂^*, respectively. We denote the corresponding *twisted (co-)homology groups* by $H^t_*(X, Y; M)$ and $H^*_t(X, Y; M)$, respectively.

Remark 4.19. The conventions used in Definition 4.18 differ throughout the literature; some authors define twisted (co-)homology using a *right* instead of a *left* action of π on the universal cover $\widetilde{X}$ (e.g. [CG75, DK01, FK06]), others assume that M is a *right* instead of a *left* $\mathbb{Z}[\pi]$-module (e.g. [Con17]). Although the different conventions lead to slightly altered definitions of twisted (co-)homology, the resulting theories however are essentially the same. Each convention has their strengths and weaknesses, and we opted for the one that will lead to the least difficulties in our later applications (note that we use the same conventions as in our main reference [Fri22, Part XIX]).

Remark 4.20.

1.) Observe that since M is a $(R, \mathbb{Z}[\pi])$-left-left module, both $C^t_*(X, Y; M)$ and $C^*_t(X, Y; M)$ are chain complexes of left R-modules. Consequently, both twisted homology $H^t_*(X, Y; M)$ and twisted cohomology $H^*_t(X, Y; M)$ carry the structure of a left R-module as well.

2.) The choice of base point is essential for the definition of twisted (co-)homology. However, one can show that a different choice of base point results in isomorphic twisted (co-)chain complexes and (co-)homology groups, see [Fri22, Proposition 167.6].

3.) If X is a connected CW-complex, then it is possible to define twisted (co-)homology starting from cellular instead of singular homology (see [Fri22, Chapter 170]). Just as in the ordinary case, the resulting homology groups are naturally isomorphic, see [Fri22, Theorem 170.2].

The following example shows that one can extract ordinary (co-)homology with coefficients from twisted (co-)homology.

Example 4.21. Let (X, x_0) and $Y \subseteq X$ be as above and let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be the universal cover with $\widetilde{Y} := p^{-1}(Y)$. Given a left R-module

[5] Here, "t" stands for "twisted".

M, let M_{triv} be the corresponding $(R, \mathbb{Z}[\pi])$-left-left module with trivial left $\mathbb{Z}[\pi]$-structure. Then the maps

$$\begin{aligned} C_*^t(X, Y; M_{\mathrm{triv}}) &\to M \otimes_{\mathbb{Z}} C_*(X, Y; \mathbb{Z}) \\ \sum_{i=0}^k m_i \otimes \sigma_i &\mapsto \sum_{i=0}^k m_i \otimes p_*(\sigma_i) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Hom}_{\mathbb{Z}}(C_*(X, Y; \mathbb{Z}), M) &\to C_t^*(X, Y; M_{\mathrm{triv}}) \\ \varphi &\mapsto \varphi \circ p_* \end{aligned}$$

are easily verified to be natural isomorphisms of chain complexes of left R-modules by using the fact that the trivial $\mathbb{Z}[\pi]$-structure on M_{triv} "cancels" the $\mathbb{Z}[\pi]$-action on $C_*(\widetilde{X}; \mathbb{Z})$ in $C_*^t(X; M_{\mathrm{triv}})$ and $C_t^*(X; M_{\mathrm{triv}})$, respectively (see [Fri22, Lemma 167.3]). Hence the maps above induce natural isomorphisms of left R-modules

$$\begin{aligned} H_*^t(X, Y; M_{\mathrm{triv}}) &\cong H_*(X, Y; M) \\ H_t^*(X, Y; M_{\mathrm{triv}}) &\cong H^*(X, Y; M) \end{aligned}$$

In other words, twisted (co-)homology with coefficients in M_{triv} is isomorphic to ordinary (co-)homology with coefficients in the left R-module M.

When defining a new (co-)homology theory it is always interesting to see what happens in degree zero. The following proposition gives the desired answer (see [Fri22, Proposition 168.1] for a proof).

Proposition 4.22. Let (X, x_0) be as above with universal cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$, and let M be a $(R, \mathbb{Z}[\pi])$-left-left module. Define the left R-modules

$$\begin{aligned} M_\pi &:= M / \Big\{ \sum_{i=0}^k (g_i \cdot m_i - m_i) \;\Big|\; g_i \in \pi,\ m_i \in M,\ k \in \mathbb{N} \Big\} \\ M^\pi &:= \{ m \in M \mid g \cdot m = m \text{ for all } g \in \pi \}. \end{aligned}$$

Then the *twisted augmentation map*

$$\begin{aligned} \epsilon_M \colon H_0^t(X; M) &\to M_\pi \\ \Big[\sum_{i=0}^k m_i \otimes \sigma_i \Big] &\mapsto \Big[\sum_{i=0}^k m_i \Big] \end{aligned}$$

and *twisted evaluation map*

$$\begin{aligned} \mathrm{ev}_M \colon H_t^0(X; M) &\to M^\pi \\ \varphi &\mapsto \varphi(\sigma_{\widetilde{x}_0}), \end{aligned}$$

where $\sigma_{\widetilde{x}_0} \colon \Delta^0 \to \widetilde{X}$ is the unique singular 0-simplex mapping Δ^0 to $\widetilde{x}_0$, are both natural isomorphisms of left R-modules.

Remark 4.23. If π acts trivially on M then obviously $M_\pi \cong M$ and $M^\pi = M$, so in this case Proposition 4.22 shows that both $H_0^t(X;M)$ and $H_t^0(X;M)$ are isomorphic to M as left R-modules.

In the upcoming sections we will encounter change of coefficients on twisted (co-)homology, so let us state the corresponding definition.

Definition 4.24. Let M and N be $(R, \mathbb{Z}[\pi])$-left-left modules and suppose we are given a homomorphism

$$\Theta \colon M \to N$$

of $(R, \mathbb{Z}[\pi])$-left-left modules. For each $k \in \mathbb{N}$, Θ induces chain maps of (co-)chain complexes of left R-modules[6]

$$\Theta_* \colon C_k(X,Y;M) \to C_k(X,Y;N), \quad m \otimes \sigma \mapsto \Theta(m) \otimes \sigma$$
$$\Theta_* \colon C^k(X,Y;M) \to C_k(X,Y;N), \quad \varphi \mapsto \Theta \circ \varphi$$

Consequently, Θ_* induces homomorphisms of left R-modules on twisted (co-)homology, which we denote by Θ_* as well. Θ_* is called a *change of coefficients*.

Twisted homology can also be formed using other coverings of X instead of the universal cover. Let $\Gamma \subseteq \pi$ be a normal subgroup and let $p_\Gamma \colon (X_\Gamma, x_0') \to (X, x_0)$ be the corresponding covering with group of deck transformations isomorphic to π/Γ. Let $Y \subset X$ and set $Y_\Gamma \coloneqq p_\Gamma^{-1}(Y)$. As for the universal cover, this induces a left $\mathbb{Z}[\pi/\Gamma]$-action on $C_*(X_\Gamma, Y_\Gamma; \mathbb{Z})$. Given the universal cover $p \colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$, there exists a unique covering $q \colon (\widetilde{X}, \widetilde{x}_0) \to (X_\Gamma, x_0')$ with $p_\Gamma \circ q = p$.

Let M be a $(R, \mathbb{Z}[\pi/\Gamma])$-left-left module with left $\mathbb{Z}[\pi/\Gamma]$-module structure given by a representation $\alpha_\Gamma \colon \pi/\Gamma \to \mathrm{Aut}_{\mathbb{Z}}(M)$. Observe that α_Γ induces $\alpha \colon \pi \to \mathrm{Aut}_{\mathbb{Z}}(M)$ by setting $\alpha \coloneqq \alpha_\Gamma \circ \mathrm{proj}$, where $\mathrm{proj} \colon \pi \to \pi/\Gamma$ is the canonical projection, making M also into a $(R, \mathbb{Z}[\pi])$-left-left module. (Note that this left $\mathbb{Z}[\pi]$-structure is in general *not* compatible with the left $\mathbb{Z}[\pi/\Gamma]$-structure on M).

Proposition 4.25. In the situation above, for each $k \in \mathbb{N}$ the following maps are isomorphisms of (co-)chain complexes of left R-modules:

$$C_k^t(X,Y;M) \xrightarrow{\cong} \overline{M} \otimes_{\mathbb{Z}[\pi/\Gamma]} C_k(X_\Gamma, Y_\Gamma; \mathbb{Z})$$
$$m \otimes \sigma \mapsto m \otimes (q \circ \sigma)$$

$$\mathrm{Hom}_{\text{left}-\mathbb{Z}[\pi/\Gamma]}(C_k(X_\Gamma, Y_\Gamma; \mathbb{Z}), M) \xrightarrow{\cong} C_t^k(X,Y;M)$$
$$\varphi \mapsto (\sigma \mapsto \varphi(q \circ \sigma)),$$

where $\sigma \colon \Delta^k \to \widetilde{X}$ is a singular simplex.

[6] Here we abuse notation and denote the change of coefficients on both the twisted chain and cochain complex by Θ_*. That Θ induces chain maps is a direct generalization of the fact that singular (co-)homology is covariantly functorial in the coefficients, see [Fri22, Lemma 89.7 and 108.9].

Proposition 4.25 follows essentially from the fact that the chain complexes under consideration are generated by the same “basis” due to the specific left $\mathbb{Z}[\pi]$-module structure on M; see for instance [Fri22, Proposition 168.6] for more details and a proof.

Corollary 4.26. Let $M = \mathbb{Z}[\pi/\Gamma]$ with the left $\mathbb{Z}[\pi/\Gamma]$-module structure given by $h\Gamma \cdot g\Gamma := gh^{-1}\Gamma$. Then there is an isomorphism of chain complexes

$$C_*^t(X; \mathbb{Z}[\pi/\Gamma]) \cong C_*(X_\Gamma; \mathbb{Z}).$$

If we further equip $\mathbb{Z}[\pi/\Gamma]$ with the left $\mathbb{Z}[\pi/\Gamma]$-structure given by ordinary left multiplication, then $\mathbb{Z}[\pi/\Gamma]$ is a $(\mathbb{Z}[\pi/\Gamma], \mathbb{Z}[\pi/\Gamma])$-left-left module and the above is an isomorphism of left $\mathbb{Z}[\pi/\Gamma]$-modules.

Proof. Using Proposition 4.25 and properties of the tensor product, we obtain

$$\begin{aligned} C_*^t(X, Y; \mathbb{Z}[\pi/\Gamma]) &\cong \overline{\mathbb{Z}[\pi/\Gamma]} \otimes_{\mathbb{Z}[\pi/\Gamma]} C_*(X_\Gamma, Y_\Gamma; \mathbb{Z}) \\ &\cong C_*(X_\Gamma, Y_\Gamma; \mathbb{Z}). \end{aligned}$$

The second left $\mathbb{Z}[\pi/\Gamma]$-structure on $\mathbb{Z}[\pi/\Gamma]$ given by left multiplication turns $C_*^t(X; \mathbb{Z}[\pi/\Gamma])$ into a left $\mathbb{Z}[\pi/\Gamma]$-module, making the above isomorphisms left $\mathbb{Z}[\pi/\Gamma]$-linear. □

Example 4.27. Proposition 4.25 shows that one can extract the ordinary homology of any regular covering from twisted homology, which is in particular true for the trivial and the universal covering of X. Indeed, taking $\Gamma = \pi$ and $M = \mathbb{Z}_{\text{triv}}$ with α_Γ trivial yields an isomorphism

$$C_*^t(X; \mathbb{Z}_{\text{triv}}) \cong C_*(X; \mathbb{Z}),$$

Similarly, taking the trivial subgroup $\Gamma = \{e\}$ and $M = \mathbb{Z}[\pi]$ with $\alpha_\Gamma(g) = \mathrm{id}_M$ for all $g \in \pi$ gives us

$$C_*^t(X; \mathbb{Z}[\pi]) \cong C_*(\widetilde{X}; \mathbb{Z}).$$

Note that Corollary 4.26 is in general *not* true for twisted cohomology (cf. [LN16, Example 7.12.5]). For instance, let $X = S^1$, $\Gamma = \{e\}$ and $M = \mathbb{Z}[\pi] \cong \mathbb{Z}[t^{\pm 1}]$ with $\alpha_\Gamma(g) = \mathrm{id}_M$ for all $g \in \pi$. Then $C_t^*(S^1; \mathbb{Z}[\mathbb{Z}])$ takes the form

$$0 \to \mathrm{Hom}_{\text{left}-\mathbb{Z}[t^{\pm 1}]}(\mathbb{Z}[\mathbb{R}], \mathbb{Z}[t^{\pm 1}]) \xrightarrow{\partial^*} \mathrm{Hom}_{\text{left}-\mathbb{Z}[t^{\pm 1}]}(C_1(\mathbb{R}; \mathbb{Z}), \mathbb{Z}[t^{\pm 1}]) \to \cdots,$$

where we used $C_0(\mathbb{R}; \mathbb{Z}) \cong \mathbb{Z}[\mathbb{R}]$. In the following we identify all real numbers $x \in \mathbb{R}$ with t^x, meaning in particular that $\mathbb{Z}[t^{\pm 1}]$ acts on $\mathbb{Z}[\mathbb{R}]$ via $t^{\pm 1} \cdot t^x := t^{x \pm 1}$. Let us show that $H_t^0(X; \mathbb{Z}[t^{\pm 1}])$ is trivial, in other words that ∂^* is injective. Given $f \in \mathrm{Hom}_{\text{left}-\mathbb{Z}[t^{\pm 1}]}(\mathbb{Z}[\mathbb{R}], \mathbb{Z}[t^{\pm 1}])$, observe that by the $\mathbb{Z}[t^{\pm 1}]$-linearity of f we have

$$f(t^{x+1}) = t f(t^x) \tag{4.3}$$

for all $x \in \mathbb{R}$. Let $\sigma\colon [0,1] \to \mathbb{R}$ be a singular 1-simplex. Then

$$\partial^*(f)(\sigma) = f(\partial\sigma) = f(t^{\sigma(1)} - t^{\sigma(0)}) = f(t^{\sigma(1)}) - f(t^{\sigma(0)}).$$

Assume that $\partial^*(f)$ is trivial, so that in particular $\partial^*(f)(\sigma) = 0$ for all singular 1-simplices σ. Then

$$\begin{aligned} \partial^*(f)(\sigma) = 0 &\implies f(t^{\sigma(1)}) = f(t^{\sigma(0)}) \\ &\implies f(t^x) = f(t^y) \quad \forall x, y \in \mathbb{R} \\ &\overset{(4.3)}{\implies} f(t^x) = 0 \quad \forall x \in \mathbb{R} \\ &\implies f = 0, \end{aligned}$$

proving that ∂^* is injective, and hence $H^0_t(S^1; \mathbb{Z}[t^{\pm 1}]) \cong 0$ as desired. However $\mathbb{R}$ is contractible, so the untwisted cohomology group $H^0(\mathbb{R}; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$. Therefore the corresponding chain complexes can't be isomorphic:

$$C^*_t(S^1; \mathbb{Z}[t^{\pm 1}]) \not\cong C^*(\mathbb{R}; \mathbb{Z}).$$

While Corollary 4.26 does in general not hold for twisted cohomology, there is an analogue provided that the group $\Gamma \subseteq \pi$ has finite index. The result is quite technical, but it will be needed later in Subsection 4.3.4. So let $\Gamma \subseteq \pi$ be a normal subgroup of finite index, and let $p_\Gamma\colon (X_\Gamma, x_0') \to (X, x_0)$ be the corresponding covering. Given the universal cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$, there exists a unique covering $q\colon (\widetilde{X}, \widetilde{x}_0) \to (X_\Gamma, x_0')$ with $p_\Gamma \circ q = p$. As before, we equip $M = \mathbb{Z}[\pi/\Gamma]$ with a left $\mathbb{Z}[\pi/\Gamma]$-module structure via $h\Gamma \cdot g\Gamma := gh^{-1}\Gamma$, which also induces a left $\mathbb{Z}[\pi]$-structure by composing with the canonical projection $\mathrm{proj}\colon \pi \to \pi/\Gamma$.

Given $k \in \mathbb{N}$, observe that by definition of the group ring $\mathbb{Z}[\pi/\Gamma]$ every $f \in \mathrm{Hom}_{\mathrm{left}-\mathbb{Z}[\pi]}(C_k(\widetilde{X}; \mathbb{Z}), \mathbb{Z}[\pi/\Gamma])$ is of the form

$$f(x) = \sum_{g\Gamma \in \pi/\Gamma} \psi_{g\Gamma}(x) g\Gamma, \quad x \in C_k(\widetilde{X}; \mathbb{Z})$$

with unique coefficient functions $\psi_{g\Gamma} \in \mathrm{Hom}_{\mathbb{Z}}(C_k(\widetilde{X}; \mathbb{Z}), \mathbb{Z})$. In fact more is true: we can write

$$f(x) = \sum_{g\Gamma \in \pi/\Gamma} \psi_{e\Gamma}(g \cdot x) g\Gamma, \quad x \in C_k(\widetilde{X}; \mathbb{Z}),$$

where $\psi_{e\Gamma}$ is the unique coefficient function corresponding to the neutral element $e\Gamma \in \pi/\Gamma$. Indeed, let $h \in \pi$ and $x \in C_k(\widetilde{X}; \mathbb{Z})$ and consider

$$f(h \cdot x) = \sum_{g\Gamma \in \pi/\Gamma} \psi_{g\Gamma}(h \cdot x) g\Gamma \tag{4.4}$$

$$h \cdot f(x) = \sum_{g\Gamma \in \pi/\Gamma} \psi_{g\Gamma}(x) g h^{-1}\Gamma \tag{4.5}$$

Observe that the coefficient of some given $g\Gamma \in \pi/\Gamma$ is $\psi_{g\Gamma}(h \cdot x)$ in (4.4) and $\psi_{gh\Gamma}(x)$ in (4.5). Since $f(h \cdot x) = h \cdot f(x)$ by the left $\mathbb{Z}[\pi]$-linearity of f, we have by comparing coefficients that

$$\psi_{g\Gamma}(h \cdot x) = \psi_{gh\Gamma}(x).$$

Taking $h = g^{-1}$ and substituting $x' = h \cdot x$ yields

$$\psi_{h^{-1}\Gamma}(x') = \psi_{e\Gamma}(h^{-1} \cdot x').$$

Hence we obtain

$$f(x) = \sum_{g\Gamma \in \pi/\Gamma} \psi_{g\Gamma}(x) g\Gamma = \sum_{g\Gamma \in \pi/\Gamma} \psi_{e\Gamma}(g \cdot x) g\Gamma$$

as claimed.

Now, for each $k \in \mathbb{N}$ define the maps

$$\begin{aligned}\Phi\colon \operatorname{Hom}_{\text{left}-\mathbb{Z}[\pi]}(C_k(\widetilde{X};\mathbb{Z}), \mathbb{Z}[\pi/\Gamma]) &\to \operatorname{Hom}_{\mathbb{Z}}(C_k(X_\Gamma;\mathbb{Z}), \mathbb{Z}) \\ f &\mapsto \big(\sigma \mapsto \psi_{e\Gamma}(\widetilde{\sigma})\big),\end{aligned}$$

where $\widetilde{\sigma}\colon \Delta^k \to \widetilde{X}$ is a lift of $\sigma\colon \Delta^k \to X_\Gamma$, and

$$\begin{aligned}\Psi\colon \operatorname{Hom}_{\mathbb{Z}}(C_k(X_\Gamma;\mathbb{Z}), \mathbb{Z}) &\to \operatorname{Hom}_{\text{left}-\mathbb{Z}[\pi]}(C_k(\widetilde{X};\mathbb{Z}), \mathbb{Z}[\pi/\Gamma]) \\ f &\mapsto \Big(\sigma \mapsto \sum_{g\Gamma \in \pi/\Gamma} f(g \cdot q(\sigma)) g\Gamma\Big)\end{aligned}$$

(here we use that Γ has finite index). We are now finally ready to state the cohomological analogue to Corollary 4.26.

Proposition 4.28. The maps Φ and Ψ defined above are well-defined cochain maps and form inverses of each other. In particular, if we equip $\mathbb{Z}[\pi/\Gamma]$ with a second left $\mathbb{Z}[\pi/\Gamma]$-structure given by ordinary left multiplication, and $\operatorname{Hom}_{\mathbb{Z}}(C_k(X_\Gamma;\mathbb{Z}),\mathbb{Z})$ with the left $\mathbb{Z}[\pi/\Gamma]$-structure given by $g\Gamma \cdot f \coloneqq f \circ (T_{g\Gamma}^{-1})_*$ where $T_{g\Gamma}$ is the deck transformation corresponding to $g\Gamma \in \pi/\Gamma$, then Φ and Ψ establish for each $k \in \mathbb{N}$ isomorphisms of left $\mathbb{Z}[\pi/\Gamma]$-modules

$$\operatorname{Hom}_{\text{left}-\mathbb{Z}[\pi]}(C_k(\widetilde{X};\mathbb{Z}), \mathbb{Z}[\pi/\Gamma]) \cong \operatorname{Hom}_{\mathbb{Z}}(C_k(X_\Gamma;\mathbb{Z}), \mathbb{Z})$$

and

$$H_t^k(X;\mathbb{Z}[\pi/\Gamma] \cong H^k(X_\Gamma;\mathbb{Z}).$$

A proof of Proposition 4.28 is given for instance in [Fri22, Proposition 168.4].

4.3.2 Twisted cup and cap product

Twisted (co-)homology shares many of the familiar properties of ordinary singular homology, such as long exact sequences of triples (see [Fri22, Proposition 167.11] or excision (see [Fri22, Theorem 167.12] or [Her19, Theorem B.1]). Of most interest to us are cup and cap products on twisted homology, which will allow us to state a version of Poincaré duality for twisted homology which will ultimately lead us to twisted intersection forms. While it is relatively simple to define cup and cap products on *absolute* twisted homology, our applications are going to need cup and cap products on *relative* twisted homology. The subject is quite delicate, so we are only going to state the necessary definitions and refer the interested reader for a full treatment to [Fri22, Chapters 169 and 171].

As before, we abbreviate in the following $\pi := \pi_1(X, x_0)$. We start by recalling the definition of an excisive triad of topological spaces.

Definition 4.29.

1.) A *triad of topological spaces* is a triple (X, A_1, A_2) where X is a topological space and $A_1, A_2 \subseteq X$ are subspaces.

2.) Let (X, A_1, A_2) be a triad of topological spaces. Define for each $n \in \mathbb{N}$ the subgroup

$$C_n^{\{A_1,A_2\}}(A_1 \cup A_2; \mathbb{Z}) := \left\{ \sum_{i=0}^{k} \alpha_i \sigma_i \;\middle|\; \begin{array}{l} \operatorname{im} \sigma_i \subset A_1 \text{ or } \operatorname{im} \sigma_i \subset A_2, \\ i = 0, \dots, k \end{array} \right\} \subseteq C_n(A_1 \cup A_2; \mathbb{Z}).$$

The boundary of $C_*(A_1 \cup A_2; \mathbb{Z})$ restricts to a boundary on $C_*^{\{A_1,A_2\}}(A_1 \cup A_2; \mathbb{Z})$, hence $C_n^{\{A_1,A_2\}}(A_1 \cup A_2; \mathbb{Z})$ is a chain complex whose homology groups we denote by $H_*^{\{A_1,A_2\}}(A_1 \cup A_2; \mathbb{Z})$.

3.) A triad (X, A_1, A_2) is called excisive if the inclusion

$$\iota\colon C_*^{\{A_1,A_2\}}(A_1 \cup A_2; \mathbb{Z}) \to C_*(A_1 \cup A_2; \mathbb{Z})$$

is a chain homotopy equivalence.

Excisive triads are used in ordinary homology theory to obtain Mayer-Vietoris sequences or relative cup and cap products. The following definition now adapts this notion to the twisted setting.

Definition 4.30. Let X be a locally contractible and path-connected space with base point x_0 and universal cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. A triad (X, A_1, A_2) is called *universally excisive* if the following two conditions are satisfied:

1. $(\widetilde{X}, p^{-1}(A_1), p^{-1}(A_2))$ is an excisive triad;

2. The subspaces A_1, A_2 and $A_1 \cap A_2$ are locally contractible.

In [Fri22, Theorems 169.7-169.9] it is shown that a universally excisive triad as defined in Definition 4.30 is the correct notion to obtain a Mayer-Vietoris sequence on twisted homology. For the relative cup and cap product however we are going to need one more definition.

Definition 4.31. Let X be a locally contractible and path-connected space with base point x_0 and universal cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. Given a triad (X, A, B), write $\widetilde{A} := p^{-1}(A)$, $\widetilde{B} := p^{-1}(B)$. Let M be a $(R, \mathbb{Z}[\pi])$-left-left module. Consider the obvious maps

$$\begin{aligned} \varphi\colon \overline{M} \otimes_{\mathbb{Z}[\pi]} C_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B}; \mathbb{Z}) &\to C_*^t(X; M) \\ \psi\colon C_t^*(X; M) &\to \operatorname{Hom}_{\text{left}-\mathbb{Z}[\pi]}(C_*^{\{\widetilde{A},\widetilde{B}\}}(\widetilde{A} \cup \widetilde{B}; \mathbb{Z}), M) \end{aligned}$$

and define

$$\begin{aligned} C_*^t(X, \{A, B\}; M) &:= \operatorname{coker} \varphi \\ C_t^*(X, \{A, B\}; M) &:= \ker \psi. \end{aligned}$$

The boundary on $C_*^t(X; M)$ induces a boundary on $C_*^t(X, \{A, B\}; M)$, turning it into a chain complex of left R-modules. Similarly, the boundary on $C_t^*(X; M)$ induces a boundary on $C_t^*(X, \{A, B\}; M)$, turning it into a cochain complex of left R-modules. We write $H_*^t(X, \{A, B\}; M)$ and $H_t^*(X, \{A, B\}; M)$ for the corresponding homology and cohomology groups.

As we will see shortly, the "natural" definition of a relative cup and cap product on twisted homology has as target the quite unhandy groups $H_*^t(X, \{A, B\}; M)$ and $H_t^*(X, \{A, B\}; M)$. However, Lemma 4.32 shows that these are in fact isomorphic to some familiar relative twisted homology groups (see [Fri22, Lemma 171.4 and 171.10] for a proof).

Lemma 4.32. Let X be a locally contractible and path-connected space with base point x_0 and suppose that (X, A, B) is a universally excisive triad. Then the obvious maps

$$\begin{aligned} C_*^t(X, A \cup B; M) &\to C_*^t(X, \{A, B\}; M) \\ C_t^*(X, A \cup B; M) &\to C_t^*(X, \{A, B\}; M) \end{aligned}$$

induce isomorphisms

$$\begin{aligned} H_*^t(X, \{A, B\}; M) &\cong H_*^t(X, A \cup B; M) \\ H_t^*(X, \{A, B\}; M) &\cong H_t^*(X, A \cup B; M). \end{aligned}$$

The last ingredient we need is a remark on the tensor product of left-left modules over $\mathbb{Z}$.

Remark 4.33. Let R and S be unital rings with involution and let M be a $(R, \mathbb{Z}[\pi])$-left-left module and N a $(S, \mathbb{Z}[\pi])$-left-left module. Observe that the tensor product $M \otimes_{\mathbb{Z}} N$ inherits a natural left $\mathbb{Z}[\pi]$-module structure via the diagonal action

$$g \cdot (m \otimes n) \coloneqq g \cdot m \otimes g \cdot n,$$

where $g \in \pi$ and $m \in M$, $n \in N$. Furthermore, $M \otimes_{\mathbb{Z}} N$ is a left $R \otimes_{\mathbb{Z}} S$-module via

$$(r \otimes s) \cdot (m \otimes n) \coloneqq r \cdot m \otimes s \cdot n$$

where $g \in \pi$ and $m \in M$, $n \in N$. Those two structures are compatible with each other since M is a $(R, \mathbb{Z}[\pi])$-left-left module and N is a $(S, \mathbb{Z}[\pi])$-left-left module. Thus $M \otimes_{\mathbb{Z}} N$ turns into a $(R \otimes_{\mathbb{Z}} S, \mathbb{Z}[\pi])$-left-left module, and we will always consider $M \otimes_{\mathbb{Z}} N$ with this specific left-left module structure in the following.

We are now finally ready to state the definition of the relative twisted cup and cap product.

Definition 4.34. Let X be a locally contractible and path-connected space with base point x_0 and let (X, A, B) be a universally excisive triad. Let R and S be unital rings with involution. Further, let M be a $(R, \mathbb{Z}[\pi])$-left-left module and N a $(S, \mathbb{Z}[\pi])$-left-left module and $p, q \in \mathbb{N}$. Then the *(relative) twisted cup product* is the map

$$\begin{aligned} \smile_t \colon C_t^p(X, A; M) \times C_t^q(X, B; N) &\to C_t^{p+q}(X, \{A, B\}; M \otimes_{\mathbb{Z}} N) \\ &\xrightarrow{\cong} C_t^{p+q}(X, A \cup B; M \otimes_{\mathbb{Z}} N) \end{aligned}$$

defined by the formula

$$(\varphi \smile_t \psi)(\sigma) \coloneqq \varphi(\sigma|_{[0,\ldots,p]}) \otimes \psi(\sigma|_{[p,\ldots,p+q]}),$$

where $\varphi \in C_t^p(X, A; M)$, $\psi \in C_t^q(X, B; N)$, and $\sigma \colon \Delta^{p+q} \to \widetilde{X}$ is a singular simplex with $\sigma|_{[i,\ldots,j]}$ being the restriction of σ to the face spanned by the standard basis vectors indexed i through j.

Definition 4.35. Let X be a locally contractible and path-connected space with base point x_0 and let (X, A, B) be a universally excisive triad. Let R and S be unital rings with involution. Further, let M be a $(R, \mathbb{Z}[\pi])$-left-left module and N a $(S, \mathbb{Z}[\pi])$-left-left module and $p, q \in \mathbb{N}$. Then the *(relative) twisted cap product* is the map

$$\begin{aligned} \frown_t \colon C_t^p(X, A; M) \times C^t_q(X, A \cup B; N) &\xrightarrow{\cong} C_t^p(X, A; M) \times C^t_q(X, \{A, B\}; N) \\ &\to C^t_{q-p}(X, B; M \otimes_{\mathbb{Z}} N) \end{aligned}$$

defined by the formula

$$\varphi \frown_t (n \otimes \sigma) \coloneqq \big(\varphi(\sigma|_{[0,\ldots,p]}) \otimes n\big) \otimes \sigma|_{[p,\ldots,q]},$$

where $\varphi \in C_t^p(X, A; M)$, $m \otimes \sigma \in C_q^t(X, A \cup B; N)$, and $\sigma\colon \Delta^q \to X$ is a singular simplex with $\sigma|_{[i,\dots,j]}$ being the restriction of σ to the face spanned by the standard basis vectors indexed i through j. If $p > q$, the (relative) twisted cap product is defined to be zero.

Remark 4.36. In [Fri22, Lemma 171.5 and 171.11], it is shown that both the (relative) twisted cup and cap product descend to well-defined products on twisted homology:

$$\smile_t\colon H_t^p(X, A; M) \times H_t^q(X, B; N) \to H_t^{p+q}(X, A \cup B; M \otimes_{\mathbb{Z}} N)$$
$$\frown_t\colon H_t^p(X, A; M) \times H_q^t(X, A \cup B; N) \to H_{q-p}^t(X, B; M \otimes_{\mathbb{Z}} N).$$

The left $R \otimes_{\mathbb{Z}} S$-structure on $M \otimes_{\mathbb{Z}} N$ yields a (R, S)-left-left module structure on $M \otimes_{\mathbb{Z}} N$ via

$$r \cdot m \otimes n := (r \otimes 1) \cdot (m \otimes n), \quad s \cdot m \otimes n := (1 \otimes s) \cdot (m \otimes n),$$

where $r \in R$, $s \in S$ and $m \otimes n \in M \otimes_{\mathbb{Z}} N$. This turns $H_t^{p+q}(X, A \cup B; M \otimes_{\mathbb{Z}} N)$ and $H_{q-p}^t(X, B; M \otimes_{\mathbb{Z}} N)$ into (R, S)-left-left modules, respectively, and then both $\smile_t$ and $\frown_t$ are R-linear in the first and S-linear in the second argument.[7] If $A = B = \emptyset$, then $(X, \emptyset, \emptyset)$ is trivially a universally excisive triad and we obtain by the same formulas as in Definition 4.34 and 4.35 a cup and cap product on absolute twisted homology.

Remark 4.37. The twisted cup product shares symmetry properties similar to the ordinary cup product provided $A = B$. Indeed, given $\varphi \in H_t^p(X, A; M)$ and $\psi \in H_t^q(X, A; N)$, we have

$$\tau(\varphi \smile_t \psi) = (-1)^{pq}(\psi \smile_t \varphi) \in H_t^{p+q}(X, A; N \otimes_{\mathbb{Z}} M),$$

where $\tau\colon M \otimes_{\mathbb{Z}} N \to N \otimes_{\mathbb{Z}} M$ is the flip isomorphism $m \otimes n \mapsto n \otimes m$ (see [Fri22, Proposition 171.7]). Moreover, if $A \subset X$ is a subset, $C, D \in \{\emptyset, A\}$, $\varphi \in H_t^p(X, C; L)$, $\psi \in H_t^q(X, D; M)$, and $x \in H_n^t(X, C \cup D; N)$, then we have the following relation between the twisted cup and cap product (see [Fri22, Lemma 171.14]):

$$\varphi \frown_t (\psi \frown_t x) = (\psi \smile_t \varphi) \frown_t x \in H_{n-p-q}^t(X; L \otimes_{\mathbb{Z}} M \otimes_{\mathbb{Z}} N).$$

This is again similar to the relation between the cup and cap product in the ordinary case.

There is an important special case of the twisted cap product that makes it possible to cap a twisted cohomology class with an untwisted homology class. To state this special case we need the following observation. Let

[7] Alternatively, one can use the tensor-hom adjunction (see Proposition 4.44) to obtain induced maps $\smile_t\colon H_t^p(X, A; M) \otimes_{\mathbb{Z}} H_t^q(X, B; N) \to H_t^{p+q}(X, A \cup B; M \otimes_{\mathbb{Z}} N)$ and $\frown_t\colon H_t^p(X, A; M) \otimes_{\mathbb{Z}} H_q^t(X, A \cup B; N) \to H_{q-p}^t(X, B; M \otimes_{\mathbb{Z}} N)$, which then are homomorphisms of left $R \otimes_{\mathbb{Z}} S$-modules.

M be a $(R, \mathbb{Z}[\pi])$-left-left module and consider $\mathbb{Z}_{\mathrm{triv}}$, the integers equipped with the trivial left $\mathbb{Z}[\pi]$-module structure. Then the map

$$\Theta \colon M \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathrm{triv}} \to M, \quad m \otimes n \mapsto n \cdot m$$

is a natural isomorphism of $(R \otimes_{\mathbb{Z}} \mathbb{Z}, \mathbb{Z}[\pi])$-left-left modules.

Definition 4.38. Let X be a locally contractible and path-connected space with base point x_0 and let (X, A, B) be a universally excisive triad. Let

$$\psi \colon C_*^t(X, A; \mathbb{Z}_{\mathrm{triv}}) \xrightarrow{\cong} C_*(X, A; \mathbb{Z})$$

be the isomorphism described in Example 4.27. Finally, let M be a $(R, \mathbb{Z}[\pi])$-left-left module and $p, q \in \mathbb{N}$. Then we extend Definition 4.35 of the relative twisted cap product to

$$\begin{aligned} \frown_{\mathbb{Z}} \colon C_t^p(X, A; M) \times C_q(X, A \cup B; \mathbb{Z}) &\to C_{q-p}^t(X, B; M) \\ (\varphi, \sigma) &\mapsto \Theta_*(\varphi \frown_t \psi^{-1}(\sigma)), \end{aligned}$$

where Θ_* is the change of coefficients that is induced by Θ as described in Definition 4.24.

Remark 4.39. As for the relative twisted cap product, $\frown_{\mathbb{Z}}$ descends to a well-defined product on homology

$$\frown_{\mathbb{Z}} \colon H_t^p(X, A; M) \times H_q(X, A \cup B; \mathbb{Z}) \to H_{q-p}^t(X, B; M),$$

which is R-linear in the first and $\mathbb{Z}$-linear in the second argument.

We are now equipped with enough machinery to state a version of Poincaré-Lefschetz duality for homology with twisted coefficients as in [Fri22, Theorem 172.1].

Theorem 4.40 (Twisted Poincaré-Lefschetz Duality). Let W be a compact oriented path-connected topological manifold of dimension $n \in \mathbb{N}$, possibly with boundary, and with a base point $w_0 \in W$. Suppose that we are given a decomposition $\partial W = A \cup B$, where A and B are compact $(n-1)$-dimensional submanifolds such that $A \cap B = \partial A = \partial B$.[8] Further, let M be a left $(R, \mathbb{Z}[\pi_1(W, w_0)])$-left-left module. Then for any $k \in \mathbb{N}$ there is an isomorphism of left R-modules

$$\begin{aligned} \mathrm{PD}_t \colon H_t^k(W, A; M) &\to H_{n-k}^t(W, B; M) \\ x &\mapsto x \frown_{\mathbb{Z}} [W], \end{aligned}$$

where $[W] \in H_n(W, \partial W; \mathbb{Z})$ denotes the fundamental class of W.

The proof is similar to the classical, untwisted case. We refer the interested reader to [Fri22, Chapter 172].

Notation. In order to simplify notation, we will denote both twisted Poincaré-Lefschetz duality and its inverse with PD_t. Similarly, we will use the notation PD for Poincaré-Lefschetz duality and its inverse in the classical untwisted case.

[8] Most important to us are the cases $A = \emptyset$ or $A = \partial W$.

4.3.3 Twisted intersection forms

One of the central invariants that arises from Poincaré-Lefschetz duality is the intersection form of an even-dimensional manifold together with its signature, which finds a natural generalization to homology with twisted coefficients. Let us quickly recall the definition of and some facts about the ordinary intersection form, focusing on the dimension most important to us, dimension 4.[9] Our discussion follows essentially [Fri22, Chapter 173].

We start by recalling a basic fact from algebraic topology. Let X be a topological space and consider the *augmentation map*

$$\epsilon\colon H_0(X;\mathbb{Z}) \to \mathbb{Z}, \quad \Big[\sum_{i=0}^{k} \alpha_i \sigma_i\Big] \mapsto \sum_{i=0}^{k} \alpha_i,$$

which is frequently used to form reduced homology (recall that we also encountered a twisted augmentation map in Proposition 4.22). If X is path-connected then the augmentation map is an isomorphism, thus giving $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$. We will use this in the upcoming definition of the ordinary intersection form. But first, let us fix some notions about pairings between modules.

Definition 4.41. Let R be a unitary ring with involution and let M, N be left R-modules. A *pairing* is a $\mathbb{Z}$-bilinear map $B\colon M \times N \to R$. Let $m \in M, n \in N$ and $\alpha, \beta \in R$. A pairing B is called:

1. *sesquilinear* if $B(\alpha \cdot m, \beta \cdot n) = \alpha B(m,n)\overline{\beta}$;

2. *hermitian* if B is sesquilinear and satisfies $B(m,n) = \overline{B(n,m)}$;

3. *non-degenerate* if B is sesquilinear and the *adjoint maps*

$$\begin{aligned} M &\to \mathrm{Hom}_{\mathrm{right}-R}(\overline{N}, R) \\ m &\mapsto \big(n \mapsto B(m,n)\big) \end{aligned}$$

and

$$\begin{aligned} \overline{N} &\to \mathrm{Hom}_{\mathrm{left}-R}(M, R) \\ n &\mapsto \big(m \mapsto B(m,n)\big) \end{aligned}$$

are injective;

4. *non-singular* if B is sesquilinear and both adjoints are isomorphisms.

[9] The (twisted) intersection form that we describe is a special case of the more general *(twisted) intersection pairing*, which can be defined in arbitrary dimensions, but we restrict our considerations to the case most important to us. A full account can be found in [Fri22, Chapters 132-135 and 173].

Definition 4.42 (Intersection Form). Let W be a compact oriented topological 4-manifold, possibly with non-empty boundary ∂W. The *intersection form* of W is defined as

$$\begin{aligned} Q_{\mathbb{Z}}\colon H_2(W;\mathbb{Z}) \times H_2(W;\mathbb{Z}) &\to \mathbb{Z} \\ (x,y) &\mapsto \epsilon\Big(\big(\mathrm{PD}(y) \smile \mathrm{PD}(x)\big) \frown [W]\Big), \end{aligned}$$

where $[W] \in H_4(W,\partial W;\mathbb{Z})$ denotes the fundamental class of W and ϵ is the augmentation map. The *signature* of W is defined as the signature of $Q_{\mathbb{Z}}$ and denoted by $\mathrm{sign}_{\mathbb{Z}}(W)$.

Remark 4.43. It is evident from the definition that $Q_{\mathbb{Z}}$ is a bilinear pairing. Moreover, since we are working in even dimensions the cup product is symmetric, and it follows immediately that $Q_{\mathbb{Z}}$ is symmetric as well. However, the intersection form is in general not non-singular. Indeed, using properties of the cup and cap product we have that

$$\begin{aligned} Q_{\mathbb{Z}}(x,y) &= \epsilon\Big(\big(\mathrm{PD}(y) \smile \mathrm{PD}(x)\big) \frown [W]\Big) \\ &= \epsilon\Big(\mathrm{PD}(x) \frown \big(\mathrm{PD}(y) \frown [W]\big)\Big) \\ &= \epsilon\Big(\mathrm{PD}(x) \frown i_*(y)\Big), \end{aligned}$$

where $i_*\colon H_2(W;\mathbb{Z}) \to H_2(W,\partial W;\mathbb{Z})$ is induced by inclusion. If we now set

$$\Phi\colon H_2(W;\mathbb{Z}) \overset{\mathrm{PD}}{\to} H^2(W,\partial W;\mathbb{Z}) \overset{\mathrm{ev}}{\to} \mathrm{Hom}_{\mathbb{Z}}(H_2(W,\partial W;\mathbb{Z}),\mathbb{Z}),$$

where ev is evaluation, then we see that

$$Q_{\mathbb{Z}}(x,y) = \epsilon\Big(\mathrm{PD}(x) \frown i_*(y)\Big) = \Phi(x)(i_*(y)),$$

and it becomes evident that the intersection form $Q_{\mathbb{Z}}$ is non-singular when both i_* and ev are isomorphisms. Whether i_* is an isomorphism can be determined by considering the long exact sequence of the pair $(W,\partial W)$. On the other hand, by universal coefficients ev is always an isomorphism on the free part of $H_2(W;\mathbb{Z})$. Since $Q_{\mathbb{Z}}$ is always zero on the torsion part of $H_2(W;\mathbb{Z})$, one frequently mods out torsion and considers the form that is induced by $Q_{\mathbb{Z}}$ on the free part of $H_2(W;\mathbb{Z})$ so that evaluation is an isomorphism.

Let us now discuss how to define the intersection form on twisted homology. As in the previous sections, let R be a unitary ring with an involution $r \mapsto \overline{r}$. Let X be locally contractible and path-connected with base point $x_0 \in X$, write as usual $\pi := \pi_1(X,x_0)$, and consider the trivial left $\mathbb{Z}[\pi]$-module R_{triv} (note that R_{triv} is a $(R,\mathbb{Z}[\pi])$-left-left module). Recall from

Proposition 4.22 that there is a *twisted augmentation map*

$$\epsilon_{R_{\mathrm{triv}}}\colon H_0^t(X;R_{\mathrm{triv}})\to R,\quad \Big[\sum_{i=0}^k r_i\otimes\sigma_i\Big]\mapsto\sum_{i=0}^k r_i,$$

which in this case is an isomorphism of left R-modules.

Similar to the ordinary intersection form, the definition of the twisted intersection form involves the twisted cup product. However, as we have seen in Definition 4.34, the target space of the twisted cup product is twisted homology with coefficients in a tensor product, which in most cases is not very desirable. So let us show how to obtain a suitable change of coefficients. For this, let us recall the tensor-hom adjunction in our setting.

Proposition 4.44. Let R, S, T unitary rings with involution, and

- M an (R,T)-left left module;
- N an (T,S)-left left module;
- L an (R,S)-left left module.

Let $\overline{M}$ denote the (R,T)-left-right module obtained from M by turning the left T-module structure into a right T-module structure using the involution on T. Then there is a natural isomorphism of $\mathbb{Z}$-modules

$$\mathrm{Hom}_{(R,T)\text{-left-right}}(\overline{M},\mathrm{Hom}_{(\mathbb{Z},S)\text{-left-left}}(N,L))\cong\mathrm{Hom}_{(R,S)\text{-left-left}}(\overline{M}\otimes_T N,L).$$

The proof is a routine check and left to the reader.

Remark 4.45. Observe that $\mathrm{Hom}_{(R,T)\text{-left-right}}(\overline{M},\mathrm{Hom}_{(\mathbb{Z},S)\text{-left-left}}(N,L))$ consists precisely of all maps $B\colon\overline{M}\times N\to L$ that are R-linear in the first and S-linear in the second argument and T-balanced. Indeed, if

$$f\in\mathrm{Hom}_{(R,T)\text{-left-right}}(\overline{M},\mathrm{Hom}_{(\mathbb{Z},S)\text{-left-left}}(N,L)),$$

then we obtain a map

$$B_f\colon\overline{M}\times N\to L,\quad B_f(m,n):=f(m)(n)$$

that is obviously R-linear in the first and S-linear in the second argument, and moreover satisfies

$$B_f(m\cdot t,n)=f(m\cdot t)(n)\overset{f\text{ is }T\text{-linear}}{=}(f(m)\cdot t)(n)\overset{(*)}{=}f(m)(t\cdot n)=B_f(m,t\cdot n)$$

for all $t\in T$, meaning that B_f is T-balanced. Here, the equality $(*)$ follows since $f(m)\in\mathrm{Hom}_{(\mathbb{Z},S)\text{-left-left}}(N,L)$ and $\mathrm{Hom}_{(\mathbb{Z},S)\text{-left-left}}(N,L)$ is a (R,T)-left-right module via

$$(r\cdot h)(n):=r\cdot h(n),\quad (h\cdot t)(n):=h(t\cdot n).$$

Conversely, a T-balanced map $B\colon \overline{M} \times N \to L$ that is R-linear in the first and S-linear in the second argument defines an element f_B that is contained in $\operatorname{Hom}_{(R,T)\text{-left-right}}(\overline{M}, \operatorname{Hom}_{(\mathbb{Z},S)\text{-left-left}}(N, L))$ via

$$f_B(m) \coloneqq B(m, \cdot) \in \operatorname{Hom}_{(\mathbb{Z},S)\text{-left left}}(N, L).$$

As a special case, observe that if $T = \mathbb{Z}$, then M can be identified with $\overline{M}$, and any map $B\colon M \times N \to L$ that is R-linear in the first and S-linear in the second argument is automatically $\mathbb{Z}$-balanced.

Now, let M and N be two $(R, \mathbb{Z}[\pi])$-left-left modules and suppose we are given an R-sesquilinear pairing

$$\Theta\colon M \times N \to R_{\text{triv}}$$

that is preserved by the action of π, that is $\Theta(g \cdot m, g \cdot n) = \Theta(m, n)$ for all $g \in \pi$ and $m \in M$, $n \in N$. We consider R_{triv} as a $(R, R, \mathbb{Z}[\pi])$-left-left-left module as follows:

1. the first left R-structure is given by $r \cdot x \coloneqq rx$;
2. the second left R-structure is given by $r \cdot x \coloneqq x\overline{r}$;
3. the left $\mathbb{Z}[\pi]$-structure is given by the trivial action of π on R.

Then Θ can be understood as an R-bilinear pairing since

$$\Theta(r_1 \cdot m_1, r_2 \cdot m_2) = r_1 \Theta(m_1, m_2)\overline{r_2} \overset{(*)}{=} r_1 \cdot (r_2 \cdot \Theta(m_1, m_2)),$$

where the equality $(*)$ follows from our specifically chosen module structures on R. Now Θ is $\mathbb{Z}$-balanced, hence we obtain by Proposition 4.44 a homomorphism of (R, R)-left-left modules

$$\Theta^{\otimes}\colon M \otimes_{\mathbb{Z}} N \to R_{\text{triv}}.$$

that is moreover $\mathbb{Z}[\pi]$-left linear. This map now induces a change of coefficients on the relative twisted cup product by post-composing $\smile_t$ with the induced map $\Theta^{\otimes}_*$:

$$\begin{aligned} H^p_t(X, A; M) \times H^q_t(X, B; N) &\to H^{p+q}_t(X, A \cup B; R_{\text{triv}}) \\ (\varphi, \psi) &\mapsto \Theta^{\otimes}_*(\varphi \smile_t \psi) \end{aligned}$$

In particular, observe that the resulting map is now R-sesquilinear:

$$\begin{aligned} \Theta^{\otimes}_*(r_1\varphi \smile_t r_2\psi) &= \Theta^{\otimes}_*(r_1 \cdot r_2 \cdot (\varphi \smile_t \psi)) \\ &= r_1 \cdot (r_2 \cdot \Theta^{\otimes}_*(\varphi \smile_t \psi)) \\ &= r_1 \Theta^{\otimes}_*(\varphi \smile_t \psi)\overline{r_2}. \end{aligned}$$

We are now ready to state the definition of the twisted intersection form.

Definition 4.46 (Twisted Intersection Form). Let W be a compact oriented path-connected 4-dimensional topological manifold, possibly with boundary, and with a base point $w_0 \in W$. Let M be a $(R, \mathbb{Z}[\pi_1(W, w_0)])$-left-left module, and suppose that we are given an R-sesquilinear pairing

$$\Theta\colon M \times M \to R_{\mathrm{triv}}$$

that preserves the $\pi_1(W, w_0)$-action, so that $\Theta(g \cdot m_1, g \cdot m_2) = \Theta(m_1, m_2)$ for all $g \in \pi_1(W, w_0)$ and $m_1, m_2 \in M$. Then the *M-twisted intersection form* of W is defined as the map

$$\begin{aligned} Q^t_M\colon H^t_2(W; M) \times H^t_2(W; M) &\to R \\ (x, y) &\mapsto \epsilon_{R_{\mathrm{triv}}}\Big(\Theta^{\otimes}_*\big(\mathrm{PD}_t(y) \smile_t \mathrm{PD}_t(x)\big) \frown_{\mathbb{Z}} [W]\Big), \end{aligned}$$

where $[W] \in H_4(W, \partial W; \mathbb{Z})$ is the fundamental class of W and $\epsilon_{R_{\mathrm{triv}}}$ is the twisted augmentation map.

Remark 4.47. Observe that since PD_t and $\epsilon_{R_{\mathrm{triv}}}$ are R-linear, $\smile_t$ and $\frown_{\mathbb{Z}}$ are R-bilinear, and Θ is R-sesquilinear, it follows immediately that the twisted intersection form Q^t_M is R-sesquilinear. Moreover, if Θ is hermitian then Q^t_M is hermitian as well (this uses symmetry properties of the twisted cup product described in Remark 4.37, see also [Fri22, Proposition 173.1]). Just as the ordinary intersection form $Q_{\mathbb{Z}}$, the twisted intersection form Q^t_M is in general not non-singular, and it is in general difficult to determine the non-singularity of Q^t_M. As a result, one can show that if $R = \mathbb{F}$ is a skew-field and $\Theta\colon M \times M \to \mathbb{F}$ is a non-singular $\mathbb{F}$-sesquilinear pairing, then Q^t_M is indeed non-singular (see [Fri22, Proposition 173.3]).

Note that in Definition 4.46, we did not define any form of signature for Q^t_M. This is simply because the usual notion of a signature is in general not defined for pairings on modules over arbitrary rings. In the next section however, we will describe a scenario where a twisted intersection form arises that indeed does have a well-defined signature.

4.3.4 Finite cyclic coverings, twisted intersection forms, and eigenspace decompositions

It is evident from the definition that computing twisted intersection forms is in general not a simple task. However, there is a specific situation in which computations are quite possible, a situation that we will also encounter in the discussion about Casson-Gordon invariants. This subsection is devoted to this scenario, with main references being [Con17], [LN16, Section 7.8.1] and [Fri22, Chapter 173]. Note that in the following we are going to encounter exclusively rings (and fields) that are commutative. However, in order to stay consistent with Subsections 4.3.1 to 4.3.3, we continue to use the language and notation of the general, non-commutative case.

Throughout this section we denote by $\mathbb{Z}_m$ the cyclic group of order $m \in \mathbb{N}$, generated by some formal ξ and written multiplicatively:

$$\mathbb{Z}_m = \{1, \xi, \xi^2, \dots, \xi^{m-1}\}.$$

We equip the group ring $\mathbb{Z}[\mathbb{Z}_m]$ with a left $\mathbb{Z}[\mathbb{Z}_m]$-module structure via $\xi^i \cdot \xi^j := \xi^{j-i}$. We will use this structure in the following to form twisted homology. We further turn $\mathbb{Z}[\mathbb{Z}_m]$ into a $(\mathbb{Z}[\mathbb{Z}_m], \mathbb{Z}[\mathbb{Z}_m])$-left-left module with second left $\mathbb{Z}[\mathbb{Z}_m]$-structure given by ordinary left multiplication.

Let W be a compact oriented path-connected topological 4-manifold, possibly with non-empty boundary, and with a base point $w_0 \in W$. In the following we write $\pi := \pi_1(W, w_0)$, and we denote the corresponding universal cover as usual by $p\colon (\widetilde{W}, \widetilde{w}_0) \to (W, w_0)$. Given an epimorphism $\varphi\colon \pi \to \mathbb{Z}_m$ for some $m \in \mathbb{N}$, let $p_m\colon (W_m, w_0') \to (W, w_0)$ denote the m-fold cyclic covering associated to $\ker\varphi \subseteq \pi$. The group of deck transformations $\mathrm{Aut}_{p_m}(W_m) \cong \mathbb{Z}_m$ induces a left $\mathbb{Z}[\mathbb{Z}_m]$-action on $H_*(W_m; \mathbb{Z})$. We write T_ξ for the deck transformation corresponding to $\xi \in \mathbb{Z}_m$.

Set $\omega = e^{\frac{2\pi i}{m}}$ and consider the cyclotomic field $\mathbb{Q}(\omega)$ with involution defined by complex conjugation $\omega \mapsto \overline{\omega}$. We endow $\mathbb{Q}(\omega)$ with the structure of a left $\mathbb{Z}[\mathbb{Z}_m]$-module via $\xi \cdot x := x\overline{\omega}$, and we will use this structure in the following to form twisted homology. Further, using ordinary left multiplication we turn $\mathbb{Q}(\omega)$ into a $(\mathbb{Q}(\omega), \mathbb{Z}[\mathbb{Z}_m])$-left-left module. Now by Maschke's Theorem, $\mathbb{Q}[\mathbb{Z}_m]$ is semisimple, hence $\mathbb{Q}(\omega)$ is projective over $\mathbb{Q}[\mathbb{Z}_m]$ and thus flat over $\mathbb{Z}[\mathbb{Z}_m]$. This gives us an isomorphism of left $\mathbb{Q}(\omega)$-modules

$$H_*(\overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} C_*(W_m; \mathbb{Z})) \cong \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_*(W_m; \mathbb{Z}).$$

We have the following definition.

Definition 4.48. In the situation above, we define the homology groups

$$H_*^\varphi(W; \mathbb{Q}(\omega)) := \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_*(W_m; \mathbb{Z}).$$

Note that $H_*^\varphi(W; \mathbb{Q}(\omega))$ arises in fact as a certain twisted homology of W (cf. [Con17, Example 2.24]): as described in Corollary 4.26, there is an isomorphism of left $\mathbb{Z}[\mathbb{Z}_m]$-modules

$$C_*(W_m; \mathbb{Z}) \cong \overline{\mathbb{Z}[\mathbb{Z}_m]} \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{W}; \mathbb{Z}).$$

The left $\mathbb{Z}[\mathbb{Z}_m]$-structure on $\mathbb{Q}(\omega)$ extends to a left $\mathbb{Z}[\pi]$-structure in the obvious way by using the projection $\mathrm{proj}\colon \pi \to \pi/\ker\varphi \cong \mathbb{Z}_m$. Applying properties of the tensor product, we obtain isomorphisms of chain complexes of left $\mathbb{Q}(\omega)$-modules

$$\begin{aligned}\overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} C_*(W_m; \mathbb{Z}) &\cong \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} \left(\mathbb{Z}[\mathbb{Z}_m] \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{W}; \mathbb{Z})\right) \\ &\cong \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{W}; \mathbb{Z}).\end{aligned}$$

Taking homology on both sides then yields

$$H_*^\varphi(W;\mathbb{Q}(\omega)) \cong H_*^t(W;\mathbb{Q}(\omega)). \tag{4.6}$$

Remark 4.49. The reason why we are working here with $H_*^\varphi(W;\mathbb{Q}(\omega))$ instead of $H_*^t(W;\mathbb{Q}(\omega))$ directly will become apparent shortly when talking about eigenspace decompositions.

Using the $\mathbb{Q}(\omega)$-sesquilinear pairing of left $\mathbb{Q}(\omega)$-modules defined by

$$\Theta\colon \mathbb{Q}(\omega)\times\mathbb{Q}(\omega)\to\mathbb{Q}(\omega)_{\mathrm{triv}}, \quad (x,y)\mapsto x\overline{y},$$

we obtain a $\mathbb{Q}(\omega)$-twisted intersection form $Q^t_{\mathbb{Q}(\omega)}$ on $H_2^t(W;\mathbb{Q}(\omega))$ as in Definition 4.46, and thus also one on $H_2^\varphi(W;\mathbb{Q}(\omega))$ using the isomorphism (4.6). For sake of simplicity we will denote this induced form by $Q^t_{\mathbb{Q}(\omega)}$ as well. Note that by Remark 4.47, $Q^t_{\mathbb{Q}(\omega)}$ is hermitian and non-singular. This form can also be obtained in a different way, as follows.

Definition 4.50. The $\mathbb{Z}[\mathbb{Z}_m]$-*equivariant intersection form* of W_m is defined as the pairing

$$\begin{aligned} Q^{\mathrm{eq}}_{\mathbb{Z}[\mathbb{Z}_m]}\colon H_2(W_m;\mathbb{Z})\times H_2(W_m;\mathbb{Z}) &\to \mathbb{Z}[\mathbb{Z}_m] \\ (x,y) &\mapsto \sum_{i=0}^{m-1} Q_{\mathbb{Z}}(x,\xi^i\cdot y)\xi^i. \end{aligned}$$

Remark 4.51. Observe that $Q^{\mathrm{eq}}_{\mathbb{Z}[\mathbb{Z}_m]}$ is $\mathbb{Z}[\mathbb{Z}_m]$-sesquilinear. Indeed, using that $Q_{\mathbb{Z}}$ is invariant under the action of $\mathbb{Z}_m$, we have that

$$\begin{aligned} Q^{\mathrm{eq}}_{\mathbb{Z}[\mathbb{Z}_m]}(\xi^j\cdot x,y) &= \sum_{i=0}^{m-1} Q_{\mathbb{Z}}(\xi^j\cdot x,\xi^i\cdot y)\xi^i \\ &= \sum_{i=0}^{m-1} Q_{\mathbb{Z}}(x,\xi^{i-j}\cdot y)\xi^i \\ &= \sum_{i=0}^{m-1} Q_{\mathbb{Z}}(x,\xi^{i}\cdot y)\xi^{i+j} \\ &= \sum_{i=0}^{m-1} Q_{\mathbb{Z}}(x,\xi^{i}\cdot y)\xi^{i}\xi^j \\ &= \xi^j Q^{\mathrm{eq}}_{\mathbb{Z}[\mathbb{Z}_m]}(x,y). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} Q^{\mathrm{eq}}_{\mathbb{Z}[\mathbb{Z}_m]}(x, \xi^j \cdot y) &= \sum_{i=0}^{m-1} Q_{\mathbb{Z}}(x, \xi^i \cdot (\xi^j \cdot y))\xi^i \\ &= \sum_{i=0}^{m-1} Q_{\mathbb{Z}}(x, \xi^{i+j} \cdot y)\xi^i \\ &= \sum_{i=0}^{m-1} Q_{\mathbb{Z}}(x, \xi^i \cdot y)\xi^{i-j} \\ &= \sum_{i=0}^{m-1} Q_{\mathbb{Z}}(x, \xi^i \cdot y)\xi^i\xi^{-j} \\ &= Q^{\mathrm{eq}}_{\mathbb{Z}[\mathbb{Z}_m]}(x, y)\xi^{-j}. \end{aligned}$$

Moreover, the symmetry and the $\mathbb{Z}_m$-invariance of $Q_{\mathbb{Z}}$ implies that $Q^{\mathrm{eq}}_{\mathbb{Z}[\mathbb{Z}_m]}$ is hermitian:

$$\begin{aligned} Q^{\mathrm{eq}}_{\mathbb{Z}[\mathbb{Z}_m]}(x, y) &= \sum_{i=0}^{m-1} Q_{\mathbb{Z}}(x, \xi^i \cdot y)\xi^i \\ &= \sum_{i=0}^{m-1} Q_{\mathbb{Z}}(\xi^i \cdot y, x)\xi^i \\ &= \sum_{i=0}^{m-1} Q_{\mathbb{Z}}(y, \xi^{-i} \cdot x)\xi^i \\ &= \sum_{i=0}^{m-1} Q_{\mathbb{Z}}(y, \xi^i \cdot x)\xi^{-i} \\ &= \sum_{i=0}^{m-1} \overline{Q_{\mathbb{Z}}(y, \xi^i \cdot x)\xi^i} \\ &= \overline{Q^{\mathrm{eq}}_{\mathbb{Z}[\mathbb{Z}_m]}(y, x)}. \end{aligned}$$

However, the form $Q^{\mathrm{eq}}_{\mathbb{Z}[\mathbb{Z}_m]}$ is in general not non-singular.

The $\mathbb{Z}[\mathbb{Z}_m]$-equivariant intersection form is in fact a twisted intersection form in disguise: using the $\mathbb{Z}[\mathbb{Z}_m]$-sesquilinear pairing of left $\mathbb{Z}[\mathbb{Z}_m]$-modules

$$\Theta\colon \mathbb{Z}[\mathbb{Z}_m] \times \mathbb{Z}[\mathbb{Z}_m] \to \mathbb{Z}[\mathbb{Z}_m], \quad (x, y) \mapsto x\overline{y},$$

we obtain a $\mathbb{Z}[\mathbb{Z}_m]$-twisted intersection form $Q^t_{\mathbb{Z}[\mathbb{Z}_m]}$ on $H^t_2(W; \mathbb{Z}[\mathbb{Z}_m])$, and by Corollary 4.26 we know that $H^t_*(W; \mathbb{Z}[\mathbb{Z}_m]) \cong H_*(W_m; \mathbb{Z})$. We have the following proposition.

Proposition 4.52. The following spaces are isometric as left $\mathbb{Z}[\mathbb{Z}_m]$-modules:

$$(H_2(W_m; \mathbb{Z}), Q^{\mathrm{eq}}_{\mathbb{Z}[\mathbb{Z}_m]}) \cong (H^t_2(W; \mathbb{Z}[\mathbb{Z}_m]), Q^t_{\mathbb{Z}[\mathbb{Z}_m]}).$$

Proof. Let $q\colon (\widetilde{W}, \widetilde{w}_0) \to (W_m, w_0')$ be the unique covering such that $p_m \circ q = p$. By Proposition 4.25 and Corollary 4.26, there is an isomorphism of left $\mathbb{Z}[\mathbb{Z}_m]$-modules

$$\eta\colon C_2^t(W;\mathbb{Z}[\mathbb{Z}_m]) \xrightarrow{\cong} \overline{\mathbb{Z}[\mathbb{Z}_m]} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} C_2(W_m;\mathbb{Z}) \xrightarrow{\cong} C_2(W_m;\mathbb{Z})$$
$$n \otimes \sigma \mapsto n \otimes (q \circ \sigma) \mapsto n \cdot (q \circ \sigma),$$

where $n \in \mathbb{Z}[\mathbb{Z}_m]$ and $\sigma\colon \Delta^2 \to \widetilde{W}$ is a singular 2-simplex, which descends to an isomorphism η_* on homology. Our goal is to show that η_* provides the desired isometry. Recall that by Proposition 4.28,[10] there are cochain maps of left $\mathbb{Z}[\mathbb{Z}_m]$-modules

$$\Phi\colon \mathrm{Hom}_{\mathrm{left}-\mathbb{Z}[\pi]}(C_*(\widetilde{W}, \partial\widetilde{W};\mathbb{Z}), \mathbb{Z}[\mathbb{Z}_m]) \to \mathrm{Hom}_{\mathbb{Z}}(C_*(W_m, \partial W_m;\mathbb{Z}), \mathbb{Z})$$
$$f \mapsto \big(\sigma \mapsto \psi_e(\widetilde{\sigma})\big),$$

and

$$\Psi\colon \mathrm{Hom}_{\mathbb{Z}}(C_*(W_m, \partial W_m;\mathbb{Z}), \mathbb{Z}) \to \mathrm{Hom}_{\mathrm{left}-\mathbb{Z}[\pi]}(C_*(\widetilde{W}, \partial\widetilde{W}_m;\mathbb{Z}), \mathbb{Z}[\mathbb{Z}_m])$$
$$f \mapsto \left(\sigma \mapsto \sum_{i=0}^{m-1} f(\xi^i \cdot q(\sigma))\xi^i\right)$$

that are inverse to each other and thus establish an isomorphism

$$H_t^2(W, \partial W;\mathbb{Z}[\mathbb{Z}_m]) \cong H^2(W_m, \partial W_m;\mathbb{Z}).$$

Further, observe the following: suppose that $\xi \in \mathbb{Z}_m$ corresponds to $g\Gamma \in \pi/\Gamma$, where $\Gamma = \ker\varphi$. Given $f \in \mathrm{Hom}_{\mathrm{left}-\mathbb{Z}[\pi]}(C_*(\widetilde{W}, \partial\widetilde{W};\mathbb{Z}), \mathbb{Z}[\mathbb{Z}_m])$ and any representative $h \in g\Gamma$, the left $\mathbb{Z}[\pi]$-linearity implies

$$f(h \cdot x) = h \cdot f(x) = f(x)h^{-1}\Gamma = f(x)g^{-1}\Gamma = f(x)\xi^{-1}.$$

This means that given any $n \in \mathbb{Z}[\mathbb{Z}_m]$, we have a well-defined equality

$$f(\widetilde{n} \cdot x) = f(x)\overline{n},$$

where $\widetilde{n}$ is any representative of n under the isomorphism $\mathbb{Z}[\mathbb{Z}_m] \cong \mathbb{Z}[\pi/\Gamma]$. In order to continue we need the following claim.

Claim:

$$\Phi_*(\mathrm{PD}_t(x)) = \mathrm{PD}(\eta_*(x)) \tag{4.7}$$

For a proof of (4.7), see [FL19] or [Fri22, Proposition 173.4]. Now, let $x, \big[\sum_{j=1}^k n_i \otimes \mu_i\big] \in H_2^t(W;\mathbb{Z}[\mathbb{Z}_m])$ with $n_j \in \mathbb{Z}[\mathbb{Z}_m]$ and $\mu_j\colon \Delta^2 \to \widetilde{W}$ a

[10]Proposition 4.28 is stated in terms of absolute twisted cohomology, but also holds for relative twisted cohomology.

singular 2-simplex for each j. We have:

$$
\begin{aligned}
&Q^t_{\mathbb{Z}[\mathbb{Z}_m]}(x, [\Sigma_{j=1}^k n_j \otimes \mu_j]) \\
&\qquad = \epsilon_{\mathbb{Z}[\mathbb{Z}_m]_{\mathrm{triv}}}\Big(\Theta_*^{\otimes}\big(\mathrm{PD}_t([\Sigma_{j=1}^k n_j \otimes \mu_j]) \smile_t \mathrm{PD}_t(x)\big) \frown_{\mathbb{Z}} [W]\Big) \\
&\qquad \overset{*}{=} \epsilon_{\mathbb{Z}[\mathbb{Z}_m]_{\mathrm{triv}}}\Big(\Theta_*^{\otimes}\big(\mathrm{PD}_t(x) \frown_t (\mathrm{PD}_t([\Sigma_{j=1}^k n_j \otimes \mu_j]) \frown_{\mathbb{Z}} [W])\big)\Big) \\
&\qquad = \epsilon_{\mathbb{Z}[\mathbb{Z}_m]_{\mathrm{triv}}}\Big(\Theta_*^{\otimes}\big(\mathrm{PD}_t(x) \frown_t ([\Sigma_{j=1}^k n_j \otimes \iota(\mu_j)])\big)\Big) \\
&\qquad = \sum_{j=1}^k \rho(\iota(\mu_j))\overline{n}_j \\
&\qquad = \sum_{j=1}^k \rho(\widetilde{n}_j \cdot \iota(\mu_j))
\end{aligned}
$$

Here, $\iota\colon C_2^t(W;\mathbb{Z}[\mathbb{Z}_m]) \to C_2^t(W,\partial W;\mathbb{Z}[\mathbb{Z}_m])$ denotes the map induced by inclusion and ρ is a representative of $\mathrm{PD}_t(x) \in H^2_t(W,\partial W;\mathbb{Z}[\mathbb{Z}_m])$. Note that the equality $*$ is obtained by using the commutativity of the diagram (here we write β for the pairing Θ in Definition 4.38 of $\frown_{\mathbb{Z}}$)

$$
\begin{array}{ccc}
\mathbb{Z}[\mathbb{Z}_m] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}_m] \otimes_{\mathbb{Z}} \mathbb{Z} & \xrightarrow{\Theta^{\otimes}\otimes \mathrm{id}} & \mathbb{Z}[\mathbb{Z}_m] \otimes_{\mathbb{Z}} \mathbb{Z} \\
\big\downarrow{\scriptstyle \mathrm{id}\otimes\beta} & & \big\downarrow{\scriptstyle \beta} \\
\mathbb{Z}[\mathbb{Z}_m] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}_m] & \xrightarrow{\quad\Theta^{\otimes}\quad} & \mathbb{Z}[\mathbb{Z}_m]
\end{array}
$$

together with Remark 4.37:

$$
\begin{aligned}
&\Theta_*^{\otimes}\big(\mathrm{PD}_t([\Sigma_{j=1}^k n_j \otimes \mu_j]) \smile_t \mathrm{PD}_t(x)\big) \frown_{\mathbb{Z}} [W] \\
&\qquad = \Theta_*^{\otimes}\big((\mathrm{PD}_t([\Sigma_{j=1}^k n_j \otimes \mu_j]) \smile_t \mathrm{PD}_t(x)) \frown_{\mathbb{Z}} [W]\big) \\
&\qquad = \Theta_*^{\otimes}\big(\mathrm{PD}_t(x) \frown_t (\mathrm{PD}_t([\Sigma_{j=1}^k n_j \otimes \mu_j]) \frown_{\mathbb{Z}} [W])\big).
\end{aligned}
$$

Now we know from Subsection 4.3.1 that any representative ρ of $\mathrm{PD}_t(x)$ is of the form[11]

$$\rho(y) = \sum_{i=0}^{m-1} \psi_e(\xi^i \cdot y)\xi^i.$$

[11] Here we abuse notation and denote by $\xi^i \cdot y$ the action of a representative of $\xi^i \in \mathbb{Z}_m \cong \pi/\ker(\varphi)$ on y. We have seen in Subsection 4.3.1 that the left $\mathbb{Z}[\pi]$-linearity of $\mathrm{PD}_t(x)$ implies that this is independent of the choice of representative for ξ^i.

Then:

$$\begin{aligned}
\sum_{j=1}^{k} \rho(\widetilde{n}_j \cdot \iota(\mu_j)) \\
&= \sum_{i=0}^{m-1} \psi_e\big(\Sigma_{j=1}^{k} \xi^i \cdot \widetilde{n}_j \cdot \iota(\mu_j)\big)\xi^i \\
&\overset{1.}{=} \sum_{i=0}^{m-1} \Phi\big(\rho\big)\big(\Sigma_{j=1}^{k} q_*(\xi^i \cdot \widetilde{n}_j \cdot \iota(\mu_j))\big)\xi^i \\
&\overset{2.}{=} \sum_{i=0}^{m-1} \Phi\big(\rho\big)\big(\Sigma_{j=1}^{k} n_j \cdot q_*(\xi^i \cdot \iota(\mu_j))\big)\xi^i \\
&\overset{3.}{=} \sum_{i=0}^{m-1} \gamma\big(\Sigma_{j=1}^{k} \xi^i \cdot \iota(\eta(n_j \otimes \mu_j))\big)\xi^i \\
&\overset{4.}{=} \sum_{i=0}^{m-1} \epsilon\Big(\mathrm{PD}(\eta_*(x)) \frown \big(\big[\Sigma_{j=1}^{k} \xi^i \cdot \iota(\eta(n_j \otimes \mu_j))\big]\big)\Big)\xi^i \\
&\overset{5.}{=} \sum_{i=0}^{m-1} \epsilon\Big(\mathrm{PD}(\eta_*(x)) \frown \big(\mathrm{PD}\big(\big[\Sigma_{j=1}^{k} \xi^i \cdot \eta(n_j \otimes \mu_j)\big]\big) \frown [W]\big)\Big)\xi^i \\
&= \sum_{i=0}^{m-1} Q_{\mathbb{Z}}\Big(\eta_*(x), \big[\Sigma_{j=1}^{k} \xi^i \cdot \eta(n_j \otimes \mu_j)\big]\Big)\xi^i \\
&= Q^{\mathrm{eq}}_{\mathbb{Z}[\mathbb{Z}_m]}\Big(\eta_*(x), \eta_*\big(\big[\Sigma_{j=1}^{k} n_j \otimes \mu_j\big]\big)\Big)
\end{aligned}$$

Here, γ is a representative of $\mathrm{PD}(\eta_*(x)) \in H^2(W_m, \partial W_m; \mathbb{Z})$. Let us explain the equalities above:

1. Definition of Φ.
2. Holds since q is the unique cover $(\widetilde{W}, \widetilde{w}_0) \to (W_m, w_0')$ such that $p_m \circ q = p$ (see [Fri22, Proposition 167.1] for details).
3. Follows from (4.7) and definition and $\mathbb{Z}[\mathbb{Z}_m]$-linearity of η.
4. Application of Remark 4.43.
5. Another application of Remark 4.43.

Overall, we have shown that

$$Q^{t}_{\mathbb{Z}[\mathbb{Z}_m]}\Big(x, \big[\Sigma_{j=1}^{k} n_j \otimes \mu_j\big]\Big) = Q^{\mathrm{eq}}_{\mathbb{Z}[\mathbb{Z}_m]}\Big(\eta_*(x), \eta_*\big(\big[\Sigma_{j=1}^{k} n_j \otimes \mu_j\big]\big)\Big)$$

as desired. □

A more general version of Proposition 4.52 is proven in [Fri22, Proposition 173.4]. The form $Q^t_{\mathbb{Q}(\omega)}$ can now be obtained from $Q^{\mathrm{eq}}_{\mathbb{Z}[\mathbb{Z}_m]}$ by changing coefficients appropriately. Consider $H_2(W_m;\mathbb{Z})$ and switch to $\mathbb{Q}(\omega)$-coefficients. By universal coefficients, we have

$$H_2(W_m;\mathbb{Q}(\omega)) \cong \mathbb{Q}(\omega) \otimes_{\mathbb{Z}} H_2(W_m;\mathbb{Z}), \tag{4.8}$$

and we identify $H_2(W_m;\mathbb{Q}(\omega))$ with the right-hand side of (4.8) in the following. Let $Q_{\mathbb{Q}(\omega)}$ be the hermitian extension of the ordinary intersection form $Q_{\mathbb{Z}}$ to $H_2(W_m;\mathbb{Q}(\omega))$, i.e.

$$Q_{\mathbb{Q}(\omega)}(\alpha \otimes x, \beta \otimes y) \coloneqq \alpha\overline{\beta} Q_{\mathbb{Z}}(x,y), \tag{4.9}$$

where $x, y \in H_2(W_m;\mathbb{Z})$ and $\alpha, \beta \in \mathbb{Q}(\omega)$. Now recall from Definition 4.48 that we defined $H_2^{\varphi}(W;\mathbb{Q}(\omega))$ as

$$H_2^{\varphi}(W;\mathbb{Q}(\omega)) \coloneqq \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_2(W_m;\mathbb{Z}).$$

Similar to $Q_{\mathbb{Q}(\omega)}$, $Q_{\mathbb{Z}}$ induces a hermitian form $\widetilde{Q}_{\mathbb{Q}(\omega)}$ on $H_2^{\varphi}(W;\mathbb{Q}(\omega))$ by the same formula as in (4.9). Then we may imitate Definition 4.50 in order to obtain a *twisted* $\mathbb{Q}(\omega)$-equivariant intersection form $Q^{\mathrm{eq},t}_{\mathbb{Q}(\omega)}$ on $H_2^{\varphi}(W;\mathbb{Q}(\omega))$ as follows:[12]

$$\begin{aligned} Q^{\mathrm{eq},t}_{\mathbb{Q}(\omega)}\colon H_2^{\varphi}(W;\mathbb{Q}(\omega)) \times H_2^{\varphi}(W;\mathbb{Q}(\omega)) &\to \mathbb{Q}(\omega) \\ (x,y) &\mapsto \sum_{i=0}^{m-1} \widetilde{Q}_{\mathbb{Q}(\omega)}(x, \xi^i \cdot y)\omega^i. \end{aligned}$$

The next proposition shows that this form is in fact the $\mathbb{Q}(\omega)$-twisted intersection form $Q^t_{\mathbb{Q}(\omega)}$ on $H_2^{\varphi}(W;\mathbb{Q}(\omega))$ in disguise.

Proposition 4.53. The forms $Q^{\mathrm{eq},t}_{\mathbb{Q}(\omega)}$ and $Q^t_{\mathbb{Q}(\omega)}$ on $H_2^{\varphi}(W;\mathbb{Q}(\omega))$ are equal.

Proof. Recall from Proposition 4.25 and Corollary 4.26 that there is an isomorphism of left $\mathbb{Z}[\mathbb{Z}_m]$-modules

$$H_2^t(W;\mathbb{Z}[\mathbb{Z}_m]) \cong H_2(W_m;\mathbb{Z}).$$

Moreover, we have seen at the beginning of this section that $H_2^{\varphi}(W;\mathbb{Q}(\omega))$ is isomorphic to $H_2^t(W;\mathbb{Q}(\omega))$, see (4.6). Therefore

$$\begin{aligned} H_2^t(W;\mathbb{Q}(\omega)) &\cong H_2^{\varphi}(W;\mathbb{Q}(\omega)) \\ &= \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_2(W_m;\mathbb{Z}) \\ &\cong \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_2^t(W;\mathbb{Z}[\mathbb{Z}_m]). \end{aligned} \tag{4.10}$$

[12] Intuitively speaking, we "tensor Definition 4.50 with $\mathbb{Q}(\omega)$ over $\mathbb{Z}[\mathbb{Z}_m]$". Alternatively, one may first "tensor Definition 4.50 with $\mathbb{Q}(\omega)$ over $\mathbb{Z}$" in order to obtain a $\mathbb{Q}(\omega)[\mathbb{Z}_m]$-equivariant intersection form $Q^{\mathrm{eq}}_{\mathbb{Q}(\omega)}$ on untwisted homology $H_2(W_n;\mathbb{Q}(\omega))$, and then consider the induced $\mathbb{Q}(\omega)$-form on $H_2^{\varphi}(W_m;\mathbb{Q}(\omega))$ which is equal to $Q^{\mathrm{eq},t}_{\mathbb{Q}(\omega)}$.

As in (4.9), one may extend $Q^t_{\mathbb{Z}[\mathbb{Z}_m]}$ on $H^t_2(W;\mathbb{Z}[\mathbb{Z}_m])$ to the tensor product $\overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H^t_2(W;\mathbb{Z}[\mathbb{Z}_m])$ via

$$(\alpha \otimes x, \beta \otimes y) \mapsto \alpha\overline{\beta} Q^t_{\mathbb{Z}[\mathbb{Z}_m]}(x,y). \tag{4.11}$$

Using (4.10) and the naturality of twisted cup and cap product (see [Fri22, Proposition 171.8 and 171.13]), one sees that the form (4.11) on $\overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H^t_2(W;\mathbb{Z}[\mathbb{Z}_m])$ is isometric to $Q^t_{\mathbb{Q}(\omega)}$ on $H^t_2(W;\mathbb{Q}(\omega))$. Similarly, extending $Q^{\text{eq}}_{\mathbb{Z}[\mathbb{Z}_m]}$ on $H_2(W_m;\mathbb{Z})$ to $\overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_2(W_m;\mathbb{Z})$ yields by definition the form $Q^{\text{eq},t}_{\mathbb{Q}(\omega)}$ on $H^\varphi_2(W;\mathbb{Q}(\omega))$. Therefore, (4.10) and Proposition 4.52 imply that the following spaces are isometric:

$$(H^\varphi_2(W;\mathbb{Q}(\omega)), Q^{\text{eq},t}_{\mathbb{Q}(\omega)}) \cong (H^t_2(W;\mathbb{Q}(\omega)), Q^t_{\mathbb{Q}(\omega)}).$$

Since we defined the $\mathbb{Q}(\omega)$-twisted intersection form on $H^\varphi_2(W;\mathbb{Q}(\omega))$ as the form induced by $Q^t_{\mathbb{Q}(\omega)}$ via the isomorphism $H^\varphi_2(W;\mathbb{Q}(\omega)) \cong H^t_2(W;\mathbb{Q}(\omega))$ (using the same notation), the result follows. □

There is a third description of the form $Q^{\text{eq},t}_{\mathbb{Q}(\omega)} = Q^t_{\mathbb{Q}(\omega)}$ in terms of untwisted homology which will be useful when studying signatures. Consider again $H_2(W_m;\mathbb{Q}(\omega))$ (untwisted) with the hermitian form $Q_{\mathbb{Q}(\omega)}$. Observe that $Q_{\mathbb{Q}(\omega)}$ (and in fact also $Q_{\mathbb{Z}}$) is invariant under the action of $\mathbb{Z}_m$ on $H_2(W_m;\mathbb{Q}(\omega))$:

$$Q_{\mathbb{Q}(\omega)}(\xi\cdot x, \xi\cdot y) = Q_{\mathbb{Q}(\omega)}(x,y).$$

In other words, the deck transformation T_ξ corresponding to ξ defines an isometry of $(H_2(W_m;\mathbb{Q}(\omega)), Q_{\mathbb{Q}(\omega)})$, which we will denote by $(T_\xi)_*$. The eigenvalues of $(T_\xi)_*$ are ω^i for $i=0,\dots,m-1$, and $H_2(W_m;\mathbb{Q}(\omega))$ admits a $\mathbb{Z}_m$-invariant orthogonal decomposition into the eigenspaces $E(\omega^i)$ of $(T_\xi)_*$ (cf. [LN16, Section 7.8.1]):

$$H_2(W_m;\mathbb{Q}(\omega)) \cong \bigoplus_{i=0}^{m-1} E(\omega^i).$$

The connection to twisted homology is given with the next proposition.

Proposition 4.54. There is an isomorphism of $\mathbb{Q}(\omega)$-vector spaces

$$E(\omega) \cong H^\varphi_2(W;\mathbb{Q}(\omega)),$$

and $Q^{\text{eq},t}_{\mathbb{Q}(\omega)} = m\cdot (Q_{\mathbb{Q}(\omega)})|_{E(\omega)}$, where $(Q_{\mathbb{Q}(\omega)})|_{E(\omega)}$ is the restriction of $Q_{\mathbb{Q}(\omega)}$ to the ω-eigenspace $E(\omega)$.

Proof. Recall that by definition, we have

$$H^\varphi_2(W;\mathbb{Q}(\omega)) = \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_2(W_m;\mathbb{Z}).$$

Observe that the right-hand side carries a $\mathbb{Q}(\omega)[\mathbb{Z}_m]$-module structure that is defined on elementary tensors via

$$\Big(\sum_{i=0}^{m-1} \alpha_i \xi^i\Big) \cdot (\beta \otimes x) := \sum_{i=0}^{m-1} \beta\alpha_i \otimes \xi^i \cdot x$$

and extended linearly, and there is an isomorphism of $\mathbb{Q}(\omega)[\mathbb{Z}_m]$-modules

$$\overline{\mathbb{Q}(\omega)[\mathbb{Z}_m]} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_2(W_m;\mathbb{Z}) \cong \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}} H_2(W_m;\mathbb{Z}).$$

Using this isomorphism as well as properties of the tensor product, the universal coefficient theorem, and the decomposition of $H_2(W_m;\mathbb{Q}(\omega))$ into eigenspaces, we obtain a sequence of isomorphisms of $\mathbb{Q}(\omega)$-vector spaces

$$\begin{aligned} H_2^{\varphi}(W;\mathbb{Q}(\omega)) &= \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_2(W_m;\mathbb{Z}) \\ &\cong \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Q}(\omega)[\mathbb{Z}_m]} (\overline{\mathbb{Q}(\omega)[\mathbb{Z}_m]} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_2(W_m;\mathbb{Z})) \\ &\cong \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Q}(\omega)[\mathbb{Z}_m]} (\overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}} H_2(W_m;\mathbb{Z})) \\ &\cong \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Q}(\omega)[\mathbb{Z}_m]} H_2(W_m;\mathbb{Q}(\omega)) \\ &\cong \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Q}(\omega)[\mathbb{Z}_m]} \bigoplus_{i=0}^{m-1} E(\omega^i) \\ &\cong E(\omega). \end{aligned}$$

In order to see the last isomorphism, note that

$$\overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Q}(\omega)[\mathbb{Z}_m]} E(\omega^i) \cong \begin{cases} E(\omega), & i = 1 \\ 0, & i \neq 1. \end{cases}$$

Indeed, if $\sum_j \alpha_j \otimes x_j \in \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Q}(\omega)[\mathbb{Z}_m]} E(\omega^i)$, then

$$\begin{aligned} \omega \cdot \Big(\sum_j \alpha_j \otimes x_j\Big) &= \sum_j \alpha_j\omega \otimes x_j \\ &= \sum_j \alpha_j \otimes (T_\xi)_* x_j \\ &= \sum_j \alpha_j \otimes \omega^i x_j \\ &= \omega^i \cdot \Big(\sum_j \alpha_j \otimes x_j\Big), \end{aligned}$$

so

$$(\omega - \omega^i) \cdot \Big(\sum_j \alpha_j \otimes x_j\Big) = 0.$$

Hence if $i \neq 1$, then necessarily $\sum_j \alpha_j \otimes x_j = 0$, and if $i = 1$ the map defined by $\sum_j \alpha_j \otimes x_j \mapsto \sum_j \alpha_j x_j$ yields the desired isomorphism $\overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Q}(\omega)[\mathbb{Z}_m]} E(\omega) \cong E(\omega)$.

For the second statement, let $x, y \in E(\omega) \cong H_2^{\varphi}(W; \mathbb{Q}(\omega))$. We have:

$$\begin{aligned} Q^{\mathrm{eq},t}_{\mathbb{Q}(\omega)}(x,y) &= \sum_{i=0}^{m-1} Q_{\mathbb{Q}(\omega)}(x, \xi^i \cdot y)\omega^i \\ &= \sum_{i=0}^{m-1} Q_{\mathbb{Q}(\omega)}(x, (T_\xi)_*^i y)\omega^i \\ &= \sum_{i=0}^{m-1} Q_{\mathbb{Q}(\omega)}(x, \omega^i \cdot y)\omega^i \\ &= \sum_{i=0}^{m-1} Q_{\mathbb{Q}(\omega)}(x, y) \\ &= m \cdot Q_{\mathbb{Q}(\omega)}(x,y). \end{aligned}$$

□

Let us briefly summarize the previous discussion. Our main actor is the homology theory

$$H_*^{\varphi}(W; \mathbb{Q}(\omega)) = \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_*(W_m; \mathbb{Z}),$$

where $\omega = e^{\frac{2\pi i}{m}}$, which is isomorphic to the homology of W with twisted $\mathbb{Q}(\omega)$-coefficients:

$$H_*^t(W; \mathbb{Q}(\omega)) \cong H_*^t(W; \mathbb{Q}(\omega)).$$

On $H_*^t(W; \mathbb{Q}(\omega))$ there is the twisted intersection form $Q^t_{\mathbb{Q}(\omega)}$, which induces via the isomorphism above a twisted intersection form on $H_*^t(W; \mathbb{Q}(\omega))$ that we denote by $Q^t_{\mathbb{Q}(\omega)}$ as well. This form can also be obtained from the equivariant intersection form $Q^{\mathrm{eq}}_{\mathbb{Z}[\mathbb{Z}_m]}$ by switching coefficients appropriately, giving the form $Q^{\mathrm{eq},t}_{\mathbb{Q}(\omega)} = Q^t_{\mathbb{Q}(\omega)}$. Moreover, there is an isomorphism

$$H_2^{\varphi}(W; \mathbb{Q}(\omega)) \cong E(\omega),$$

where $E(\omega)$ is the ω-eigenspace of $(T_\xi)_*$ on $H_2(W_m; \mathbb{Q}(\omega))$ (untwisted). It turns out that $Q^{\mathrm{eq},t}_{\mathbb{Q}(\omega)}$ is equal to m times the hermitian extension of the ordinary intersection form $Q_{\mathbb{Z}}$ to $H_2(W_m; \mathbb{Q}(\omega))$ restricted to the ω-eigenspace, in symbols

$$Q^{\mathrm{eq},t}_{\mathbb{Q}(\omega)} = m \cdot (Q_{\mathbb{Q}(\omega)})|_{E(\omega)}.$$

4.3.5 $\mathbb{Q}(\omega)$-twisted signatures

We continue with the setup from the previous Subsection 4.3.4. There we have described two equal hermitian forms $Q^t_{\mathbb{Q}(\omega)}$ and $Q^{\mathrm{eq},t}_{\mathbb{Q}(\omega)}$ on $H_2^{\varphi}(W; \mathbb{Q}(\omega))$, and we have shown how they are related to the untwisted $\mathbb{Q}(\omega)$-intersection form $Q_{\mathbb{Q}(\omega)}$. Since a hermitian form on a $\mathbb{Q}(\omega)$-vector space has a well-defined signature, we can make the following definition.

Definition 4.55. The $\mathbb{Q}(\omega)$-*twisted signature* of W is defined as

$$\operatorname{sign}^t_{\mathbb{Q}(\omega)}(W) := \operatorname{sign}(Q^t_{\mathbb{Q}(\omega)}).$$

Now on $H_2(W_m;\mathbb{Q}(\omega))$ we have the hermitian extension $Q_{\mathbb{Q}(\omega)}$ of the ordinary intersection form, with signature $\operatorname{sign}(Q_{\mathbb{Q}(\omega)})$. Using the decomposition of $H_2(W_m;\mathbb{Q}(\omega))$ into the eigenspaces of T_ξ, we obtain

$$\operatorname{sign}(Q_{\mathbb{Q}(\omega)}) = \sum_{i=0}^{m-1} \operatorname{sign}((Q_{\mathbb{Q}(\omega)})|_{E(\omega^i)}),$$

and by Proposition 4.54 we have

$$\operatorname{sign}^t_{\mathbb{Q}(\omega)}(W) = \operatorname{sign}((Q_{\mathbb{Q}(\omega)})|_{E(\omega)}).$$

Thus the $\mathbb{Q}(\omega)$-twisted signature of W is equal to the signature of $Q_{\mathbb{Q}(\omega)}$ restricted to the ω-eigenspace $E(\omega)$. It is in general not an easy task to compute any signature that arises from a twisted intersection form. However in our situation, one can apply the G-signature theorem by Atiyah and Singer [AS68, Theorem 6.12] to obtain information about the signatures $\operatorname{sign}((Q_{\mathbb{Q}(\omega)})|_{E(\omega^i)})$, and consequently about the twisted signature $\operatorname{sign}^t_{\mathbb{Q}(\omega)}(W_m)$. Let us briefly state the definition of the G-signature in our scenario.

Let X be a compact oriented topological 4-dimensional manifold, and suppose that a finite group G acts on X via orientation-preserving homeomorphisms. On $H_2(X;\mathbb{Z})$ we have the ordinary intersection form $Q_\mathbb{Z}$, which extends to a hermitian form $Q_\mathbb{C}$ on $H := H_2(X;\mathbb{C})$ via

$$Q_\mathbb{C}(\alpha\otimes x, \beta\otimes y) = \alpha\overline{\beta}Q_\mathbb{Z}(x,y),$$

where $\alpha,\beta\in\mathbb{C}$ and $x,y\in H_2(X;\mathbb{Z})$. Note that the form $Q_\mathbb{C}$ is invariant under the induced action of G on H. We make the following general observation

Lemma 4.56. Let G be a finite group acting on a finite-dimensional complex vector space V. Further, let $B\colon V\times V\to\mathbb{C}$ be a hermitian form that is invariant under the action of G, i.e. $B(g\cdot x, g\cdot y) = B(x,y)$ for all $g\in G$ and $x,y\in V$. Then there exists a G-invariant orthogonal decomposition

$$V = V^+\oplus V^-\oplus V^0,$$

where B is $\pm$-definite on $V^\pm$ and totally isotropic (zero) on V^0.

Proof. The action of G on H defines a representation

$$\rho\colon G\to \operatorname{Aut}_\mathbb{C}(V).$$

Since any representation of a finite group decomposes as a sum of irreducible subrepresentations, it suffices to show the following claim.

Claim: If ρ is irreducible then B is either positive definite, negative definite, or totally isotropic.

Indeed, the lemma then follows from the claim by noting that the decomposition of ρ into irreducible subrepresentations yields G-invariant subspaces of V on which B is either positive definite, negative definite, or totally isotropic, and one can form V^+, V^- and V^0 from these subspaces by taking direct sums accordingly.

To see the claim, choose any inner product $\langle \cdot, \cdot \rangle$ on V and define a new form

$$\widetilde{B}\colon V \times V \to \mathbb{C}, \quad \widetilde{B}(x, y) = \sum_{g \in G} \langle g \cdot x, g \cdot y \rangle.$$

Since $\langle \cdot, \cdot \rangle$ is an inner product, it follows that $\widetilde{B}$ is hermitian, positive definite and non-singular. Moreover, $\widetilde{B}$ is by construction G-invariant. Let A_B and $A_{\widetilde{B}}$ be matrices representing B and $\widetilde{B}$, respectively, and set $A := (A_B A_{\widetilde{B}}^{-1})^*$. Then

$$\widetilde{B}(Ax, y) = x^* A^* A_{\widetilde{B}} y = x^* A_B y = B(x, y)$$

for all $x, y \in V$. It follows that the matrix A commutes with the action of G, i.e. $Ag \cdot x = g \cdot Ax$ for all $g \in G$ and $x \in V$. Indeed, observe that

$$\begin{aligned}
\widetilde{B}(Ax, y) &= B(x, y) \\
&= B(g \cdot x, g \cdot y) \\
&= \widetilde{B}(Ag \cdot x, g \cdot y)) \\
&= B(g^{-1} \cdot (Ag \cdot x), y)
\end{aligned}$$

Since $\widetilde{B}$ is non-singular, it follows that $Ax = g^{-1}(Ag \cdot x)$ and thus $g \cdot Ax = Ag \cdot x$. Since ρ is irreducible, we can now apply Schur's lemma to obtain $A = \lambda I$ for some $\lambda \in \mathbb{C}$, where I is the identity matrix. Therefore

$$B(x, y) = \lambda \widetilde{B}(x, y)$$

for all $x, y \in V$. Since both B and $\widetilde{B}$ are hermitian, it follows that $\lambda \in \mathbb{R}$, and the sign of λ determines if B is positive definite, negative definite, or totally isotropic on V. □

Returning to our setting, Lemma 4.56 shows that there exists a G-invariant orthogonal decomposition

$$H = H^+ \oplus H^- \oplus H^0,$$

where $Q_{\mathbb{C}}$ is $\pm$-definite on $H^{\pm}$ and totally isotropic (zero) on H^0. Hence we obtain representations

$$\rho^{\pm}\colon G \to \mathrm{Aut}_{\mathbb{C}}(H^{\pm}), \quad \rho^{\pm}(g) := g_*|_{H^{\pm}},$$

where g_* is the automorphism of H that is induced by $g \in G$. Atiyah and Singer make the following definition [AS68, Section 6].

Definition 4.57 (G-Signature). The *G-signature* of the pair (G, X) is defined as

$$\operatorname{sign}(G, M) \coloneqq \rho^+ - \rho^- \in R(G),$$

where $R(G)$ denotes the complex representation ring of G. If $g \in G$, we define the *g-signature* as

$$\operatorname{sign}(g, X) \coloneqq \operatorname{tr}(g_*|_{H^+}) - \operatorname{tr}(g_*|_{H^-}),$$

where tr denotes the trace.

The Atiyah-Singer G-signature theorem [AS68, Theorem 6.12] states that one can express the g-signatures $\operatorname{sign}(g, X)$ using the action of g on the normal bundle of the fixed-point set $\operatorname{Fix}(g) \subseteq X$. However, the precise statement of the theorem is quite technical and requires some preparation. Since the exact statement is of little use to us, we omit a formulation and continue with a remark about the G-signature, which also contains further references to the G-signature theorem and its proof.

Remark 4.58. As mentioned before, the notion of the G-signature is due to Atiyah and Singer in [AS68, Section 6]. It is originally and more generally defined in the smooth setting, where a compact Lie group G acts on a compact, oriented, smooth $2n$-dimensional manifold X for some $n \in \mathbb{N}$ via orientation-preserving diffeomorphisms (the definition is identical to our Definition 4.57, however). In this scenario, the original proof of the G-signature theorem given by Atiyah and Singer [AS68, Theorem 6.12] uses analysis on Riemannian manifolds. A proof that uses only topological methods was given by Gilmer [Gil81], provided that G is finite. In [Gor86], Gordon gives an accessible proof of the G-signature theorem in dimension 4 for G finite (which is closest to our scenario), that uses only little analysis.

The transition to the topological setting was made by Wall, who noticed that the formula in the G-signature theorem also holds when G acts via orientation-preserving homeomorphisms on a topological manifold, provided that the action is *semi-free* and *tame* [Wal99, Theorem 14B.2]. Here, the action is called

- *semi-free* if the stabilizer of each $x \in X$ is either trivial or the entire group G; and
- *tame* if the action is semi-free and for each $g \in G$,
  1. $\operatorname{Fix}(g)$ is a manifold;
  2. there is a G-vector bundle N over F equivariantly homeomorphic to a neighborhood of F in X;
  3. $(X \backslash \operatorname{Fix}(g))/G$ is a manifold.

(see [Wal99, Chapter 14B]). In particular, the action of the group of deck transformations on the total space of a covering is free and hence also semi-free and tame, so the G-signature theorem holds in this scenario in the topological setting.

Turning back to our original situation, one can see immediately that we find ourselves in the setup of the G-signature. Indeed, the $\mathbb{Z}_m$-covering W_m takes the role of X, and $\mathbb{Z}_m$ takes the role of G, which acts via deck transformations $\mathrm{Aut}_p(W_m) \cong \mathbb{Z}_m$ on W_m. Furthermore, if we consider $\mathbb{Q}(\omega)$ as a subfield of $\mathbb{C}$, we can replace $Q_{\mathbb{C}}$ with $Q_{\mathbb{Q}(\omega)}$ in the situation above. Then $H = H_2(W_m; \mathbb{Q}(\omega))$, and $H_2(W_m; \mathbb{Q}(\omega))$ decomposes into the ω^i-eigenspaces $E(\omega^i)$ of $(T_\xi)_*$, where T_ξ is the canonical generating deck transformation, and this decomposition is invariant under the $\mathbb{Z}_m$-action on $H_2(W_m; \mathbb{Q}(\omega))$. If we now choose and fix subspaces $E(\omega^i)^\pm \subseteq E(\omega^i)$ where $Q_{\mathbb{Q}(\omega)}$ is $\pm$-definite, then we obtain a $\mathbb{Z}_m$-invariant orthogonal decomposition of $H_2(W_m; \mathbb{Q}(\omega))$ as in the situation preceding Definition 4.57 of the G-signature:

$$H_2(W_m; \mathbb{Q}(\omega)) = H^+ \oplus H^-,$$

where $H^\pm = \sum_{i=0}^{m-1} E(\omega^i)^\pm$ (note that H^0 is trivial since $Q_{\mathbb{Q}}(\omega)$ is non-singular). Thus we have the $\mathbb{Z}_m$-signature $\mathrm{sign}(\mathbb{Z}_m, W_m)$, and for each $\xi^s \in \mathbb{Z}_m$ we have the ξ^s-signatures

$$\mathrm{sign}(\xi^s, W_m) = \mathrm{tr}((T_{\xi^s})_*|_{H^+}) - \mathrm{tr}((T_{\xi^s})_*|_{H^-})$$

where $s = 0, \ldots, m-1$. Observe that it follows directly from our chosen decomposition of $H_2(W_m; \mathbb{Q}(\omega))$ that for each $s = 0, \ldots, m-1$,

$$\mathrm{sign}(\xi^s, W_m) = \sum_{r=0}^{m-1} \omega^{rs} \mathrm{sign}((Q_{\mathbb{Q}(\omega)})|_{E(\omega^r)}).$$

Now, the G-signature theorem is used by Rohklin [Rok71] (see also [CG78, Lemma 2.1]) to prove the following key identity, which can be seen as one motivation for the definition of the Casson-Gordon invariant $\sigma_r(M, \chi)$.

Lemma 4.59. Let $p\colon W_m \to W$ be as above, and assume additionally that both W_m and W are closed and that the covering is branched over a (possibly empty) surface $F \subset W$. Let $[F]$ denote the self-intersection number of the branching surface F with respect to $Q_{\mathbb{Q}(\omega)}$. Then for each $r = 0, \cdots, m-1$ we have

$$\mathrm{sign}((Q_{\mathbb{Q}(\omega)})|_{E(\omega^r)}) = \mathrm{sign}_{\mathbb{Z}}(W) - \frac{2[F]^2 r(m-r)}{m^2}.$$

Since we know that the $\mathbb{Q}(\omega)$-twisted signature $\mathrm{sign}^t_{\mathbb{Q}(\omega)}(W_m)$ is equal to $\mathrm{sign}((Q_{\mathbb{Q}(\omega)})|_{E(\omega)})$, Lemma 4.59 tells us in particular that

$$\mathrm{sign}^t_{\mathbb{Q}(\omega)}(W_m) = \mathrm{sign}_{\mathbb{Z}}(W) - \frac{2[F]^2(m-1)}{m^2}.$$

This makes the $\mathbb{Q}(\omega)$-twisted signature $\operatorname{sign}^t_{\mathbb{Q}(\omega)}(W_m)$ much more treatable, and as we will see shortly it is also this relation that Casson and Gordon use to define their invariant $\sigma_r(M,\chi)$.

4.3.6 Witt groups

In many cases it is desirable to have a group structure on the set of non-singular hermitian forms on modules over a ring with involution R. Such a group structure was first introduced by Ernst Witt in 1936 [Wit36], and is now known as the *Witt group* $W(R)$ of R.[13] In topology, they arise naturally in the study of (algebraic) knot concordance (see for instance [LN16, Chapter 5]), or in surgery theory where they arise in generalized form as the so-called *L-groups* (see for instance [Ran02]). In fact, we have already dealt indirectly with elements of a Witt group in Section 4.1 on linking forms and metabolizers.

The Witt group $W(\mathbb{C}(t))$ will be the most interesting to us, as it appears in the definition of the Casson-Gordon τ-invariant. The purpose of this section is thus to introduce the notion of a Witt group and a certain signature function in the special case of $W(\mathbb{C}(t))$. Main references are [Con17, Section 2.6] and [MH73, Chapter 1].

As in Subsection 4.3.1, let R be a (not necessarily commutative) unitary ring together with an involution $r \mapsto \overline{r}$ that reverses the order of multiplication. We further assume that R is not of characteristic 2.[14] We start by recalling Definition 4.41 from Subsection 4.3.3 about pairings on modules over R.[15]

Definition 4.41. Let R be a unital ring with involution and let M, N be left R-modules. A *pairing* is a $\mathbb{Z}$-bilinear map $B\colon M \times N \to R$. Let $m \in M, n \in N$ and $\alpha, \beta \in R$. A pairing B is called:

1. *sesquilinear* if $B(\alpha \cdot m, \beta \cdot n) = \alpha B(m,n)\overline{\beta}$;
2. *hermitian* if B is sesquilinear and satisfies $B(m,n) = \overline{B(n,m)}$;
3. *non-degenerate* if B is sesquilinear and the *adjoint maps*

$$\begin{aligned} M &\to \operatorname{Hom}_{\text{right}-R}(\overline{N}, R) \\ m &\mapsto \big(n \mapsto B(m,n)\big) \end{aligned}$$

and

$$\begin{aligned} \overline{N} &\to \operatorname{Hom}_{\text{left}-R}(M, R) \\ n &\mapsto \big(m \mapsto B(m,n)\big) \end{aligned}$$

[13] Witt's original definition was on the set of isometry classes of anisotropic quadratic forms over an arbitrary field k, but this was soon adapted to more general settings.

[14] There is also a theory of Witt groups in the case of characteristic 2, see [MH73].

[15] As in Subsection 4.3.1, we will use the multiplication symbol '$\cdot$' to indicate scalar multiplication on modules. Ordinary ring multiplication will be denoted by juxtaposition.

are injective;

4. *non-singular* if B is sesquilinear and both adjoints are isomorphisms.

Definition 4.60. Let M be a free left R-module, and let $B\colon M \times M \to R$ be a non-singular and hermitian pairing on M. Then B is called a *non-singular hermitian form* on M and denoted by (M, B).[16]

If the underlying module is unimportant or clear from the context, we will simply write B instead of (M, B) for a non-singular hermitian form on M. Given two non-singular hermitian forms $(M_1, B_1), (M_2, B_2)$, we can form their direct sum

$$(M_1, B_1) \oplus (M_2, B_2) := (M_1 \oplus M_2, B_1 \oplus B_2)$$

where $B_1 \oplus B_2$ is defined in the obvious way. Given a submodule $N \subseteq M$, we define the *orthogonal complement* $N^\perp$ as

$$N^\perp := \{x \in M \mid B(x, y) = 0 \; \forall \; y \in N\}.$$

We have the following definitions.

Definition 4.61.

1.) A non-singular hermitian form (M, B) is called *metabolic* if there exists a direct summand $P \subseteq M$ such that $P = P^\perp$. In this case P is called a *metabolizer* for (M, B).

2.) Two non-singular hermitian forms $(M_1, B_1), (M_2, B_2)$ are called *Witt equivalent* if the sum $(M_1, B_1) \oplus -(M_2, B_2)$ is metabolic, where $-(M_2, B_2) := (M_2, -B_2)$.

Remark 4.62. Note that the proof of Proposition 4.4 carries over to our setting and shows that if a non-singular hermitian form (M, B) is metabolic with metabolizer $P \subset M$, then

$$\operatorname{rank}(M) = 2 \operatorname{rank}(P).$$

As the name indicates, Witt equivalence is in fact an equivalence relation on the set of hermitian forms on M.

Theorem 4.63. Let R be a unitary ring with involution and suppose further that R is an integral domain and not of characteristic 2. Then Witt equivalence is an equivalence relation on the set of isomorphism classes of non-singular hermitian forms on free modules over R, and the equivalence classes form a group with respect to direct sum. The neutral element is given by the class of metabolic non-singular hermitian forms, and the additive inverse of a class $[(M, B)]$ is given by $[-(M, B)]$.

[16] One can work more generally with projective instead of free modules (see [MH73]). However, we won't need this generality in the present text.

We refer the reader for a proof of Theorem 4.63 to [Con17, Theorem 2.27].

Definition 4.64. Let R be a unitary ring with involution and suppose further that R is an integral domain and not of characteristic 2. Then the set of Witt equivalence classes of non-singular hermitian forms on free modules over R is called the *Witt group* of R and denoted by $W(R)$.

Example 4.65. Let $R = \mathbb{F}$ with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, where $\mathbb{R}$ is equipped with the trivial involution and $\mathbb{C}$ is equipped with the involution given by complex conjugation. Then a hermitian form on a vector space over $\mathbb{F}$ can be represented by a hermitian matrix, and has therefore a well-defined signature. Observe that over $\mathbb{F}$, a non-singular hermitian form is metabolic if and only if its signature vanishes (see [MH73, Chapter III, §2]). Indeed, suppose that (V, B) is a non-singular hermitian form. By Sylvester's law of inertia, there exists an orthogonal basis of V with respect to B such that $V = V_+ \oplus V_-$, where B is positive definite on V_+ and negative definite on V_-. Then the signature of B is given by $\dim V_+ - \dim V_-$. Suppose that (V, b) is metabolic with metabolizer P. Since $P = P^\perp$, we have that $P \cap V_+ = P \cap V_- = \{0\}$. Therefore

$$\dim P \leq \dim V - \dim V_\pm = \dim V_\mp.$$

Since $2 \dim P = \dim V$, it follows that

$$\dim P = \dim V_+ = \dim V_-,$$

hence the signature of B vanishes. On the other hand, if B has zero signature, then there exists again by Sylvester's law of inertia an orthogonal basis of V such that B is represented by the matrix

$$\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix},$$

where I is the identity matrix of size $n = \frac{1}{2} \dim V$. Then the subspace spanned by all elements that have coordinate vectors whose first n entries equal the second n entries (respecting orders) obviously forms a metabolizer for (V, B), showing that the non-singular hermitian form is metabolic.

The previous observation shows that the signature induces a well-defined map

$$\operatorname{sign}\colon W(\mathbb{F}) \to \mathbb{Z},$$

which is linear since the signature respects the direct sum of hermitian forms. In fact, sign is an isomorphism: surjectivity is clear, and injectivity follows again from the observation that over $\mathbb{F}$, a non-singular hermitian form is metabolic if and only if its signature vanishes. Therefore

$$W(\mathbb{F}) \cong \mathbb{Z}.$$

With some additional work, one can also show that the signature induces an isomorphism $W(\mathbb{Z}) \cong \mathbb{Z}$, where $\mathbb{Z}$ is equipped with the trivial involution (see for instance [MH73, Chapter IV, Corollary 2.7]).

As noted earlier, the case $R = \mathbb{C}(t)$, the field of complex rational functions with involution defined by complex conjugation and $t \mapsto t^{-1}$, will be the most interesting to us as it appears in the definition of the Casson-Gordon τ-invariant. While it is not incredibly hard to show that $W(\mathbb{C}[t^{\pm 1}]) \cong W(\mathbb{C})$ (see [Con17, Example 2.30]), $W(\mathbb{C}(t))$ is more intricate (see [Lit84, Appendix A7]). The computation uses a certain signature function defined on $W(\mathbb{C}(t))$, which we are going to describe now.

Let (V, B) be a non-singular hermitian form over $\mathbb{C}(t)$ represented by a matrix $A(t)$. Each entry $A_{ij}(t)$ of $A(t)$ is a complex rational function $f(t)/g(t) \in \mathbb{C}(t)$, and both $f(t)$ and $g(t)$ have only finitely many zeros (provided that f is not zero). Let

$$\begin{aligned} S &:= \bigcup_{i,j} \text{zeros of the denominator of } A_{ij}(t) \\ D &:= \text{zeros of } \det A(t) \end{aligned}$$

and set

$$Z := S^1 \backslash (S^1 \cap (S \cup D)) \subset \mathbb{C}.$$

Since S and D are finite, Z contains all but finitely many points of S^1. Let $\omega \in Z$ and consider

$$A(\omega) := A(t)|_{t=\omega}.$$

This is a well-defined non-singular hermitian matrix over $\mathbb{C}$, and thus has a signature

$$\text{sign}_\omega(A(t)) := \text{sign}(A(\omega)).$$

As a hermitian matrix, $A(\omega)$ has real eigenvalues. These eigenvalues depend on complex rational functions, hence they vary continuously in a neighborhood of ω, meaning that their signs remain unchanged. Thus the function

$$Z \to \mathbb{Z}, \quad \omega \mapsto \text{sign}_\omega(A(t)) \tag{4.12}$$

is locally constant. We wish to extend this function to all of S^1. Let us make the following definition.

Definition 4.66. Let (V, B) be a non-singular hermitian form over $\mathbb{C}(t)$ represented by a matrix $A(t)$, and let $\omega \in S^1$. Then we define the *averaged signature* of $A(t)$ as the average of the one-sided limits as $\eta \in S^1$ approaches ω, in symbols

$$\text{sign}^{\text{av}}_\omega(A(t)) := \frac{1}{2}\Big(\lim_{\eta \nearrow \omega} \text{sign}(A(\eta)) + \lim_{\eta \searrow \omega} \text{sign}(A(\eta))\Big).$$

Lemma 4.67. There exists a well-defined map

$$S^1 \to \mathbb{Z}, \quad \omega \mapsto \operatorname{sign}^{\mathrm{av}}_{\omega}(A(t)).$$

Proof. Consider again the set Z from (4.12). If $\omega \in Z$, then clearly

$$\lim_{\eta \nearrow \omega} \operatorname{sign}(A(\eta)) = \lim_{\eta \searrow \omega} \operatorname{sign}(A(\eta)) = \operatorname{sign}(A(\omega))$$

and thus we simply have

$$\operatorname{sign}^{\mathrm{av}}_{\omega}(A(t)) = \operatorname{sign}_{\omega}(A(t)) = \operatorname{sign}(A(\omega)) \in \mathbb{Z}$$

as described above. Now let $\omega \in S^1 \backslash Z$ and recall that $S^1 \backslash Z$ is a finite set of points. As noted earlier, the assignment $Z \to \mathbb{Z}$, $\eta \mapsto \operatorname{sign}(A(\eta))$ is locally constant, hence it follows that the limits in the definition of $\operatorname{sign}^{\mathrm{av}}_{\omega}(A(t))$ exist and are finite. In order to show that $\operatorname{sign}^{\mathrm{av}}_{\omega}(A(t)) \in \mathbb{Z}$, we recall the following fact: given a hermitian matrix $A \in \mathbb{C}^{n \times n}$ with $\det A \neq 0$, we have

$$\begin{aligned}
\operatorname{sign}(A) &= \#\ \text{pos. eigenvalues} - \#\ \text{neg. eigenvalues} \\
&\equiv \#\ \text{pos. eigenvalues} + \#\ \text{neg. eigenvalues} \mod 2 \\
&\equiv n \mod 2.
\end{aligned}$$

□

Lemma 4.67 shows that the averaged signature is a suitable notion to obtain information about non-singular hermitian forms over $\mathbb{C}(t)$, in particular because it also takes discontinuities of complex rational functions into account. Furthermore, $\operatorname{sign}^{\mathrm{av}}_{\omega}$ behaves well under Witt equivalence, as the next lemma shows.

Lemma 4.68. Let $\omega \in S^1$. Then the averaged signature $\operatorname{sign}^{\mathrm{av}}_{\omega}$ induces a well-defined homomorphism of groups

$$\operatorname{sign}^{\mathrm{av}}_{\omega} \colon W(\mathbb{C}(t)) \to \mathbb{Z}, \quad [(V,B)] \mapsto \operatorname{sign}^{\mathrm{av}}_{\omega}(A(t)),$$

where $A(t)$ is a matrix representative of the Witt class $[(V,B)]$.

Proof. Let $(V_1, B_1), (V_2, B_2)$ be two Witt equivalent, non-singular hermitian forms over $\mathbb{C}(t)$, represented by matrices $A_1(t)$ and $A_2(t)$, respectively. We wish to show that

$$\operatorname{sign}^{\mathrm{av}}_{\omega}(A_1(t)) = \operatorname{sign}^{\mathrm{av}}_{\omega}(A_2(t)). \tag{4.13}$$

First, note that the notion of Witt equivalence descends to matrix representatives of non-singular hermitian forms in the sense that if B_1 and B_2 are Witt equivalent, then there exists a basis of $V_1 \oplus -V_2$ such that

$$A_1(t) \oplus -A_2(t) \sim \begin{pmatrix} 0 & U(t) \\ U(t)^* & P(t) \end{pmatrix}$$

for some square matrices $U(t)$ and $P(t)$ (note that $P(t)$ is necessarily hermitian). If this is the case, we will call the corresponding matrices *Witt equivalent*.

In order to show (4.13), we start with a special case. Let (V, B) be a non-singular hermitian form over $\mathbb{C}(t)$, represented by a matrix $A(t) \in \mathbb{C}(t)^{n\times n}$, and let $T(t) \in \mathrm{GL}(n, \mathbb{C}(t))$. Then $T(t)$ defines a change of basis of V, and the form in this new basis is represented by the matrix

$$\widetilde{A}(t) := T(t)^* A(t) T(t),$$

where $T(t)^*$ denotes the conjugate transpose. Since Witt equivalence is basis independent, we moreover have that $\widetilde{A}(t)$ and $A(t)$ are Witt equivalent. We wish to show that $\mathrm{sign}^{\mathrm{av}}_{\omega}$ remains unaffected under change of basis, i.e. $\mathrm{sign}^{\mathrm{av}}_{\omega}(\widetilde{A}(t)) = \mathrm{sign}^{\mathrm{av}}_{\omega}(A(t))$. For this we proceed with a similar approach that led to the definition of the averaged signature. Let

$$\begin{aligned} S &:= \bigcup_{i,j} \text{zeros of the denominator of } A_{ij}(t) \text{ and } T_{ij}(t) \\ D &:= \bigcup_{i,j} \text{zeros of } \det A(t) \text{ and } \det T(t) \end{aligned}$$

and set

$$Z := S^1 \backslash (S^1 \cap (S \cup D)) \subset \mathbb{C}.$$

Note that since S and D are finite, Z contains all but finitely many points in S^1. Let $\omega \in Z$. Then $\widetilde{A}(\omega)$ and $A(\omega)$ are two non-singular hermitian matrices over $\mathbb{C}$ that are related by a change of basis $T(\omega)$. Since the ordinary signature is invariant under change of basis, we have that in this case

$$\mathrm{sign}^{\mathrm{av}}_{\omega}(\widetilde{A}(t)) = \mathrm{sign}^{\mathrm{av}}_{\omega}(A(t)). \tag{4.14}$$

As before, an eigenvalue argument shows that the averaged signature is locally constant on Z. Hence it follows from (4.14) that if $\omega \in S^1 \backslash Z$, then

$$\frac{1}{2}\Big(\lim_{\eta \nearrow \omega} \mathrm{sign}(A(\eta)) + \lim_{\eta \searrow \omega} \mathrm{sign}(A(\eta))\Big) = \frac{1}{2}\Big(\lim_{\eta \nearrow \omega} \mathrm{sign}(\widetilde{A}(\eta)) + \lim_{\eta \searrow \omega} \mathrm{sign}(\widetilde{A}(\eta))\Big)$$

and so $\mathrm{sign}^{\mathrm{av}}_{\omega}(\widetilde{A}(t)) = \mathrm{sign}^{\mathrm{av}}_{\omega}(A(t)$ for all $\omega \in S^1$. Turning back to the general case, note that since $A_1(t)$ and $A_2(t)$ are Witt equivalent, we can find a sequence of basis changes of $V_1 \oplus -V_2$ such that

$$\begin{aligned} A_1(t) \oplus -A_2(t) &\sim \begin{pmatrix} 0 & U(t) \\ U(t)^* & P(t) \end{pmatrix} \\ &\sim \begin{pmatrix} 0 & U(t) \\ U(t)^* & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 0 & -U(t) \\ -U(t)^* & 0 \end{pmatrix} =: Q(t). \end{aligned}$$

Note that the second similarity is obtained using the basis change

$$\begin{pmatrix} \mathbf{1} & 0 \\ -\frac{1}{2}P(t)U(t)^{-1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & U(t) \\ U(t)^* & P(t) \end{pmatrix} \begin{pmatrix} \mathbf{1} & -\frac{1}{2}(U(t)^*)^{-1}P(t) \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} 0 & U(t) \\ U(t)^* & 0 \end{pmatrix}.$$

Clearly $\operatorname{sign}^{\mathrm{av}}_\omega(Q(t)) = 0$. Now, we have shown that the averaged signature is invariant under change of basis. Moreover, the additivity of the ordinary signature over $\mathbb{C}$ with respect to block sum of matrices implies that $\operatorname{sign}^{\mathrm{av}}_\omega$ is additive under block sum as well. Overall, we find that

$$\begin{aligned} \operatorname{sign}^{\mathrm{av}}_\omega(A_1(t)) - \operatorname{sign}^{\mathrm{av}}_\omega(A_2(t)) &= \operatorname{sign}^{\mathrm{av}}_\omega(A_1(t) \oplus -A_2(t)) \\ &= \operatorname{sign}^{\mathrm{av}}_\omega(Q(t)) \\ &= 0. \end{aligned}$$

Therefore

$$\operatorname{sign}^{\mathrm{av}}_\omega(A_1(t)) = \operatorname{sign}^{\mathrm{av}}_\omega(A_2(t))$$

as desired. We have proven that $\operatorname{sign}^{\mathrm{av}}_\omega$ is a well-defined map on the Witt group $W(\mathbb{C}(t))$. The linearity follows again from the fact that the averaged signature is additive under block sum of matrices. □

It can be shown that the averaged signature contains in fact all the information about $W(\mathbb{C}(t))$. In [Lit84], Litherland defines the notion of a *balanced function* as a map $f\colon S^1 \to \mathbb{Z}$ with finitely many discontinuities such that

$$f(\omega) = \frac{1}{2}\Big(\lim_{\eta \nearrow \omega} f(\eta) + \lim_{\eta \searrow \omega} f(\eta)\Big)$$

for all $\omega \in S^1$. Clearly, $\operatorname{sign}^{\mathrm{av}}_\omega$ is a balanced function in Litherland's sense. The set of balanced functions forms a group B with respect to addition of maps. The averaged signature can now be used to obtain the following result.

Theorem 4.69 ([Lit84, Corollary A1]). There is a well-defined isomorphism from $W(\mathbb{C}(t))$ to the group of balanced functions B

$$\Phi\colon W(\mathbb{C}(t)) \to B, \quad [(V,B)] \to \big(\omega \mapsto \operatorname{sign}^{\mathrm{av}}_\omega(A(t))\big),$$

where $A(t)$ is a matrix of a representative of the Witt class $[(V,B)]$.

4.3.7 The Casson-Gordon invariants $\sigma_r(M,\chi)$ and $\sigma(K,\chi)$

Equipped with the theory from the previous sections, we are now finally ready to state the definition of the Casson-Gordon invariants. We start with the definition of $\sigma(M,\chi)$ following the original source [CG75], and proceed to describe the closely related $\sigma_r(M,\chi)$ following [CG78]. The invariant $\tau(K,\chi)$ will be described in the upcoming Subsection 4.3.8.

Remark 4.70. Casson and Gordon describe their invariants and results in [CG75, CG78] in the smooth category. However, the work of Freedman [Fre82] implies that the methods used by Casson and Gordon also apply in the topological setting, meaning that their results hold without change (see also [Liv05]). This is in particular true for the G-signature theorem that is applied by Casson and Gordon, as we have already noted in Subsection 4.3.5. Hence in the following, we won't restrict ourselves to the smooth category and work in the topological setting as well. Unless otherwise stated, all manifolds will be assumed to be compact and oriented. In contrast to previous subsections, we will now omit base points whenever possible for better readability.

Let us define the Casson-Gordon invariant $\sigma(M, \chi)$. Let M be a closed oriented topological 3-manifold, and let $\chi\colon H_1(M;\mathbb{Z}) \to \mathbb{Z}_m$ be an epimorphism for some $m \in \mathbb{N}$, $m > 1$, where $\mathbb{Z}_m$ denotes as usual the cyclic group of order m. The bordism group $\Omega_3^{\mathrm{top}}(K(\mathbb{Z}_m, 1))$ is finite (see [Con17, Proposition 2.12]),[17] so there exists a compact oriented 4-manifold W and an epimorphism $\psi\colon H_1(W;\mathbb{Z}) \to \mathbb{Z}_m$ such that:

1. the boundary of W consists of s disjoint copies of M for some $s \in \mathbb{N}$, i.e. $\partial W = \bigsqcup_s M$;
2. ψ restricts on each boundary component to χ.

In short, $\partial(W, \psi) = s(M, \chi)$. Note that since the number of boundary components s may vary, one can always assume that W is path-connected by taking a path-connected component with non-empty boundary from the 4-manifold obtained through bordism theory.

Let $p\colon W_m \to W$ denote the m-fold cyclic covering associated to ψ. Set $\omega = e^{\frac{2\pi i}{m}}$, and consider the cyclotomic field $\mathbb{Q}(\omega)$. We find ourselves in a situation similar to Subsection 4.3.4: we equip $\mathbb{Q}(\omega)$ with a $(\mathbb{Q}(\omega), \mathbb{Z}[\mathbb{Z}_m])$-left-left module structure where $\mathbb{Q}(\omega)$ acts via left multiplication and $\mathbb{Z}[\mathbb{Z}_m]$ acts via $\xi \cdot x := x\overline{\omega}$, and we can form the homology groups

$$H_*^{\psi}(W;\mathbb{Q}(\omega)) := \overline{\mathbb{Q}(\omega)} \otimes_{\mathbb{Z}[\mathbb{Z}_m]} H_*(W_m;\mathbb{Z}),$$

which carry the structure of left $\mathbb{Q}(\omega)$-modules. Recall that $H_*^{\psi}(W;\mathbb{Q}(\omega))$ is isomorphic to the twisted homology $H_*^t(W;\mathbb{Q}(\omega))$ (see Subsection 4.3.4), giving us the $\mathbb{Q}(\omega)$-twisted intersection form $Q^t_{\mathbb{Q}(\omega)}$ on $H_2^{\psi}(W;\mathbb{Q}(\omega))$. The form $Q^t_{\mathbb{Q}(\omega)}$ is hermitian (see Remark 4.47), and we defined in Subsection 4.3.5 the $\mathbb{Q}(\omega)$-twisted signature $\operatorname{sign}^t_{\mathbb{Q}(\omega)}(W)$ as the signature of $Q^t_{\mathbb{Q}(\omega)}$. We have the following definition.

Definition 4.71. Let M be a closed oriented topological 3-manifold and $\chi\colon H_1(M;\mathbb{Z}) \to \mathbb{Z}_m$ an epimorphism. Let W be a compact path-connected

[17] We refrain from introducing bordism groups at this point and refer the reader for more details to [Con17, Section 2.3].

oriented 4-manifold and $\psi\colon H_1(W;\mathbb{Z}) \to \mathbb{Z}_m$ an epimorphism such that $\partial(W,\psi) = s(M,\chi)$ for some $s \in \mathbb{N}$. Then we define

$$\sigma(M,\chi) \coloneqq \frac{1}{s}\big(\operatorname{sign}^t_{\mathbb{Q}(\omega)}(W) - \operatorname{sign}_{\mathbb{Z}}(W)\big) \in \mathbb{Q}$$

If $M = X_d$ is the d-fold cyclic branched cover of a knot K, then we set

$$\sigma(K,\chi) \coloneqq \sigma(X_d,\chi).$$

Remark 4.72.

1.) It is not obvious that the Casson-Gordon invariant $\sigma(M,\chi)$ is well-defined, i.e. does not depend on the particular choice of (W,ψ). The proof uses Novikov additivity (i.e. that the (twisted) signature is additive with respect to boundary connected sum) and bordism theory, in particular that the 4-dimensional oriented topological bordism group is generated by simply connected manifolds ($\mathbb{C}P^2$ and E_8). A proof in the smooth setting is given in [Con17, Lemma 2.15], which carries over to the topological setting using [Hsu87, Corollary 2.4] (see also [BKK$^+$21, Section 21.6.6]).

2.) There is a slight ambiguity in the notation $\sigma(K,\chi)$ as it of course depends on the order $d \in \mathbb{N}$ of the finite cyclic branched cover used to construct the invariant. Originally, Casson and Gordon [CG75] define $\sigma(K,\chi)$ for the double branched cover X_2, but use later in their paper the same notation for the invariant obtained by starting with other d-fold cyclic branched coverings (see for instance [CG75, Theorem 3]). To not complicate our notation further, we chose to tolerate this ambiguity at the potential cost of minor confusion.

3.) Note that Casson and Gordon only work with $\sigma(K,\chi)$ in [CG75] but mention on p. 182 that their construction also works for arbitrary closed 3-manifolds, giving $\sigma(M,\chi)$.

The main result regarding $\sigma(M,\chi)$ is the following.

Theorem 4.73 (Casson-Gordon). Let K be a slice knot with d-fold cyclic branched cover X_d and an epimorphism $\chi\colon H_1(X_d;\mathbb{Z}) \to \mathbb{Z}_m$ where both $d, m \in \mathbb{N}$ are prime-powers. If the cover $\widehat{X}_d \to X_d$ that is induced by χ satisfies $H_1(\widehat{X}_d;\mathbb{Q}) = 0$, then

$$|\sigma(K,\chi)| \leq 1.$$

Remark 4.74. Theorem 4.73 is in fact a combination of [CG75, Theorem 2 and 3] (see also [Con17, Theorem 2.16]), and uses results regarding the Casson-Gordon invariant $\tau(K,\chi)$ which we will describe in Subsection 4.3.8. Without $\tau(K,\chi)$, the invariant $\sigma(K,\chi)$ only yields an obstruction for the ribbonness of a knot K whose double branched cover X_2 is a lens space (see [CG75, Theorem 1]).

Let us now discuss how the Casson-Gordon invariant $\sigma(M,\chi)$ relates to Subsection 4.3.5, in particular the Atiyah-Singer G-signature. The definition of $\sigma(M,\chi)$ involves the signature of the $\mathbb{Q}(\omega)$-twisted intersection form $Q^t_{\mathbb{Q}(\omega)}$ on $H_2^{\psi}(W;\mathbb{Q}(\omega))$. Consider $H_2(W_m;\mathbb{Q}(\omega))$ (untwisted) with the hermitian extension $Q_{\mathbb{Q}(\omega)}$ of the ordinary intersection form. As described in Subsection 4.3.4, $(H_2(W_m;\mathbb{Q}(\omega)), Q_{\mathbb{Q}(\omega)})$ admits a $\mathbb{Z}_m$-invariant orthogonal decomposition into the eigenspaces $E(\omega^i)$ of $(T_\xi)_*$, $i=0,\dots,m-1$, where T_ξ is the canonical generating deck transformation of $\mathrm{Aut}_p(W_m)\cong\mathbb{Z}_m$. By Proposition 4.54 we know that $H_2^{\psi}(W;\mathbb{Q}(\omega))\cong E(\omega)$ and $Q^t_{\mathbb{Q}(\omega)} = m\cdot(Q_{\mathbb{Q}(\omega)})|_{E(\omega)}$, so in particular

$$s\sigma(M,\chi)+\mathrm{sign}_{\mathbb{Z}}(W)=\mathrm{sign}^t_{\mathbb{Q}(\omega)}(W)=\mathrm{sign}((Q_{\mathbb{Q}(\omega)})|_{E(\omega)}).$$

Similarly, one obtains

$$s\sigma(M,\chi^i)+\mathrm{sign}_{\mathbb{Z}}(W)=\mathrm{sign}((Q_{\mathbb{Q}(\omega)})|_{E(\omega^i)}) \tag{4.15}$$

for $i=1,\dots,m-1$ (to see this, use the fact that the group of deck transformations of the m-fold covering of 4-manifolds used to construct $\sigma(M,\chi^i)$ has as canonical generator T^i_ξ and replicate the proof of Proposition 4.54). On the other hand, in this context there is also the ξ-signature $\mathrm{sign}(\xi,W)$ (see Definition 4.57), and as described at the end of Subsection 4.3.5 we know that

$$\mathrm{sign}(\xi,W_m)=\sum_{i=0}^{m-1}\omega^i\mathrm{sign}((Q_{\mathbb{Q}(\omega)})|_{E(\omega^i)}).$$

Thus, the Casson-Gordon invariant $\sigma(M,\chi)$ relates to the Atiyah-Singer G-signature via

$$\mathrm{sign}(\xi,W_m)=\sum_{i=0}^{m-1}\omega^i\big(s\sigma(M,\chi^i)+\mathrm{sign}_{\mathbb{Z}}(W)\big).$$

This explains why Casson and Gordon refer to $\sigma(M,\chi)$ as a "disguised form of a standard Atiyah-Singer invariant of 3-manifolds", see [CG75, p. 183]. Moreover, it is also the starting point for the computations made by Casson and Gordon, which led to an explicit formula for $\sigma(K,\chi)=\sigma(X_2,\chi)$ in the case where the double branched cover X_2 is a lens space (see [CG75, p. 186-188]).

Let us now turn our attention to the invariant $\sigma_r(M,\chi)$, following the original source [CG78]. Let M be a closed oriented topological 3-manifold together with an epimorphism $\chi\colon H_1(M;\mathbb{Z})\to\mathbb{Z}_m$ for some $m\in\mathbb{N}$, $m>1$. The kernel of χ induces an m-fold cyclic covering $p\colon M_m\to M$, with group of deck transformations $\mathrm{Aut}_p(M_m)$ isomorphic to $\mathbb{Z}_m$. There is a canonical generator of $\mathrm{Aut}_p(M_m)$ corresponding to $\xi\in\mathbb{Z}_m$ which we will denote by T_ξ.

Suppose that for some $n \in \mathbb{N}$, there exists an mn-fold cyclic branched covering of 4-manifolds $q\colon W_{mn} \to W$, branched over a surface $F \subset \operatorname{int} W$, such that:

1. $\partial W_{mn} = \bigsqcup_n M_m$;
2. $\partial W = M$;
3. the deck transformation $T'_\xi \in \operatorname{Aut}_q(W_{mn}) \cong \mathbb{Z}_{mn}$ that induces rotation through $2\pi/m$ on the fibres of the normal bundle of $F' := q^{-1}(F)$ restricts on each boundary component of W_{mn} to T_ξ.

Consider $H_2(W_{mn};\mathbb{C}) \cong H_2(W_{mn};\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$, and let $(T'_\xi)_*$ be induced by T'_ξ on this homology group. We find ourselves in a situation similar to that of Subsection 4.3.4: we have the hermitian extension $Q_{\mathbb{C}}$ of the ordinary intersection form $Q_{\mathbb{Z}}$ of W_{mn}, invariant under the action of $\mathbb{Z}_m$ on $H_2(W_{mn};\mathbb{C})$, and $H_2(W_{mn};\mathbb{C})$ admits a $\mathbb{Z}_m$-invariant orthogonal decomposition into the eigenspaces $E(\omega^r)$ of T'_ξ, where $\omega = e^{\frac{2\pi i}{m}}$ and $r = 0, \dots, m-1$. We have the following definition.

Definition 4.75 ([CG78]). Let $0 < r < m$. In the situation above, define

$$\sigma_r(M,\chi) := \frac{1}{n}\left(\operatorname{sign}_{\mathbb{Z}}(W) - \operatorname{sign}((Q_{\mathbb{C}})|_{E(\omega^r)}) - \frac{2[F]^2 r(m-r)}{m^2}\right),$$

where $[F]$ denotes the self-intersection number of F.

Remark 4.76. As the notation indicates, $\sigma_r(M,\chi)$ only depends on the pair (M,χ) and r. The proof uses Lemma 4.59 and Novikov additivity and proceeds similar to the argument that shows that $\sigma(M,\chi)$ is well-defined (see [Con17, Section 2.3]).

Given (M,χ) as above, it is not obvious that there exists an mn-fold cyclic branched covering $q\colon W_{mn} \to W$ such that $\sigma_r(M,\chi)$ is defined. However, Casson and Gordon show that $\sigma_r(M,\chi)$ does indeed always exist.

Lemma 4.77 ([CG78, Lemma 2.2]). Let (M,χ) as above and assume that there exists a 4-manifold W such that $\partial W = M$ and $H_1(W;\mathbb{Z}_m) = 0$. Then the m-fold cyclic covering $p\colon M_m \to M$ extends to an m-fold cyclic branched covering $q\colon W_m \to W$, branched over a surface $F \subset \operatorname{int} W$, such that the canonical deck transformation $T_\xi \in \operatorname{Aut}_p(M_m)$ extends to $T'_\xi \in \operatorname{Aut}_q(W_m)$ that induces rotation through $2\pi/m$ on each fibre of the normal bundle of $F' := q^{-1}(F) \subset \operatorname{int} W_m$.

Note that Lemma 4.77 does not only show that $\sigma_r(M,\chi)$ always exists, it also shows that it is always possible to take $n = 1$.

The main application of $\sigma_r(M,\chi)$ given by Casson and Gordon is concerned with slice knots. Fix a prime q, and let X_{q^s} denote the q^s-fold cyclic branched covering of K for some $s \in \mathbb{N}$, $s \geq 1$. Note that for each s, there exists a canonical cyclic branched covering $\psi\colon X_{q^s} \to X_q$ that induces a surjection on the fundamental groups, and hence also on first homology. Thus, given an epimorphism $\chi\colon H_1(X_q;\mathbb{Z}) \to \mathbb{Z}_m$, composition with ψ_* defines epimorphisms $\chi_s\colon H_1(X_{q^s};\mathbb{Z}) \to \mathbb{Z}_m$ for all $s \in \mathbb{N}$, $s \geq 0$. We have the following theorem.

Theorem 4.78 ([CG78, Theorem 4.1]). Let K be a slice knot. Then there exists a constant c, and a subgroup $G \subseteq H_1(X_q;\mathbb{Z})$ with $|G|^2 = |H_1(X_q;\mathbb{Z})|$ such that if m is a prime-power and $\chi\colon H_1(X_q;\mathbb{Z}) \to \mathbb{Z}_m$ an epimorphism satisfying $\chi(G) = 0$, then

$$|\sigma_r(X_{q^s}, \chi_s)| < c$$

for all $0 < r < m$ and $s \in \mathbb{N}$.

As an application, Casson and Gordon use $\sigma_r(M,\chi)$ in order to determine which twist knots K_n are slice, a problem that was previously open.

Theorem 4.79 ([CG78, Theorem 5.1]). The only twist knots K_n that are slice are the unknot K_0 and the Stevedore's knot K_2.

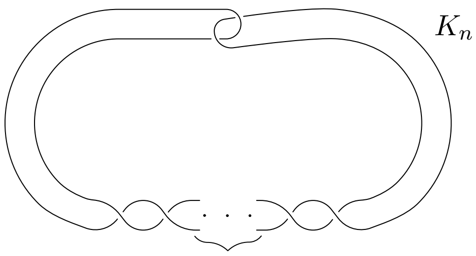


Figure 4.4: The twist knot K_n (picture taken from [Ilt19]).

Since K_n is algebraically slice if and only if $4n+1$ is a square (see Proposition 4.16), Theorem 4.79 is of particular significance as it shows that there are infinitely many algebraically slice knots which are not slice. Note that Theorem 4.79 was obtained previously by Casson and Gordon in [CG75] using their invariants $\sigma(K,\chi)$ and $\tau(K,\chi)$ (see Corollary 4.91 below); Theorem 4.79 provides the same result using the invariant $\sigma_r(M,\chi)$.

Let us now discuss how $\sigma(M,\chi)$ relates to $\sigma_r(M,\chi)$.

Proposition 4.80. Let M be a closed oriented topological 3-manifold and $\chi\colon H_1(M;\mathbb{Z}) \to \mathbb{Z}_m$ an epimorphism. Then

$$\sigma(M,\chi) = -\sigma_1(M,\chi).$$

Proof. Let us first consider a special case. Suppose that W is a 4-manifold and $\psi\colon H_1(W;\mathbb{Z}) \to \mathbb{Z}_m$ an epimorphism such that $\partial(W,\psi) = (M,\chi)$, so that $\sigma(M,\chi)$ can be formed with $s = 1$. In this case, the m-fold cyclic (unbranched) covering $W_m \to W$ induced by ψ can be used to obtain $\sigma_r(M,\chi)$.

Observe that the hermitian extensions $Q_{\mathbb{Q}(\omega)}$ (defined on $H_2(W_m;\mathbb{Q}(\omega))$) and $Q_{\mathbb{C}}$ (defined on $H_2(W_m;\mathbb{C})$) of the ordinary intersection form have the same signature. Thus using Proposition 4.54 we obtain

$$\operatorname{sign}^t_{\mathbb{Q}(\omega)}(W) = \operatorname{sign}((Q_{\mathbb{Q}(\omega)})|_{E(\omega)}) = \operatorname{sign}((Q_{\mathbb{C}})|_{E(\omega)}),$$

and therefore

$$\begin{aligned}\sigma(M,\chi) &= \operatorname{sign}^t_{\mathbb{Q}(\omega)}(W) - \operatorname{sign}_{\mathbb{Z}}(W)\\ &= \operatorname{sign}((Q_{\mathbb{C}})|_{E(\omega)}) - \operatorname{sign}_{\mathbb{Z}}(W)\\ &= -\sigma_1(M,\chi).\end{aligned}$$

Now, assume for the general case that $\partial(W,\psi) = s(M,\chi)$ for some $s \in \mathbb{N}$. Consider the disjoint union $\sqcup_s M$ with epimorphism

$$\overline{\chi} = (\chi,\dots,\chi)\colon H_1(\sqcup_s M;\mathbb{Z}) \cong \bigoplus_s H_1(M;\mathbb{Z}) \to \mathbb{Z}_m.$$

Then $\partial(W,\psi) = (\sqcup_s M, \overline{\chi})$, so by the above

$$\sigma(\sqcup_s M, \overline{\chi}) = -\sigma_1(\sqcup_s M, \overline{\chi}).$$

On the other hand, signatures are additive with respect to disjoint union, so in fact

$$s\sigma(M,\chi) = \sigma(\sqcup_s M, \overline{\chi}) = -\sigma_1(\sqcup_s M, \overline{\chi}) = -s\sigma_1(M,\chi).$$

Therefore

$$\sigma(M,\chi) = -\sigma_1(M,\chi)$$

as claimed. □

4.3.8 The Casson-Gordon invariant $\tau(K,\chi)$

As mentioned after the statement of Theorem 4.73, the invariant $\sigma(K,\chi)$ alone only yields an obstruction for the ribbonness of a knot K whose double branched cover is a lens space (see [CG75, Theorem 1]). The main ingredient used by Casson and Gordon is that if a ribbon knot has a lens space L as double branched cover, then L is bounded by a compact rational homology 4-ball with cyclic fundamental group (see [CG75, Lemmas 1-3]). In order to obtain an obstruction for sliceness, Casson and Gordon extend their construction of the invariant $\sigma(K,\chi)$ and take infinite cyclic coverings into account, leading to the invariant $\tau(K,\chi)$. We outline the definition of this invariant below, following [CG75] and [Con17].

Let $K \subset S^3$ be a knot with knot exterior X_K and zero-framed surgery M_K. Given $d \in \mathbb{N}$, there are d-fold cyclic (branched) coverings

$$p_d\colon M_d \to M_K, \quad q'_d\colon X'_d \to X_K, \quad q_d\colon X_d \to S^3$$

as described in Definition 4.14. The following lemma relates the first homology of these coverings.

Lemma 4.81. Let $K \subset S^3$ be a knot with meridian μ_K and $d \in \mathbb{N}$. Then there are isomorphisms of $\mathbb{Z}$-modules

1. $H_1(M_d;\mathbb{Z}) \cong H_1(X'_d;\mathbb{Z})$, induced by the inclusion $X'_d \hookrightarrow M_d$;
2. $H_1(X'_d;\mathbb{Z}) \cong H_1(X_d;\mathbb{Z}) \oplus \mathbb{Z}$, where the $\mathbb{Z}$-summand is generated by a lift of μ_K^d to X'_d.

Proof. Our proof follows essentially [Con17, Lemma 2.18 and 2.31]. Let μ_K and ℓ_k be a meridian and longitude of K, respectively. Recall that $H_1(X_K;\mathbb{Z})$ is freely generated by μ_K and that $\partial X_K = S^1 \times S^1$. The zero-framed surgery M_K is obtained from X_K by gluing in a solid torus $T \coloneqq S^1 \times D^2$ to X_K, identifying a meridian m of T with the longitude ℓ_K and a longitude λ of T with μ_K. However, ℓ_K is null-homologous in X_K (since it bounds a Seifert surface), and it follows that $H_1(M_K;\mathbb{Z})$ is freely generated by μ_K as well. In other words, the inclusion $i\colon X_K \hookrightarrow M_K$ induces an isomorphism $i_*\colon H_1(X_K;\mathbb{Z}) \to H_1(M_K;\mathbb{Z})$.

Now the d-fold cyclic coverings X'_d and M_d are obtained by definition from the kernel of the maps

$$\pi_1(X_K) \xrightarrow{\text{ab}} H_1(X_K;\mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\text{proj}} \mathbb{Z}_d$$
$$\pi_1(M_K) \xrightarrow{\text{ab}} H_1(M_K;\mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\text{proj}} \mathbb{Z}_d,$$

where in both cases the map proj reduces the class of the meridian $[\mu_K]$ mod d. Hence $X'_d \subset M_d$. We make the following observations:

1. The meridian μ_K does not lift to a closed loop in X'_d (reps. M_d) but μ_K^d does. Let $\widetilde{\mu}_K^d$ denote the corresponding lift.
2. The longitude λ of T is identified with μ_K in M_K, hence λ doesn't lift to a closed loop in M_d either, but λ^d does. Considering $X'_d \subset M_d$, $\widetilde{\mu}_K^d$ corresponds to a lift of λ^d to M_d.
3. The longitude ℓ_K however lifts to a closed loop in both X'_d and M_d since it is already null-homologous in X_K. Let $\widetilde{\ell}_K$ denote the corresponding lift. Note that $\widetilde{\ell}_K$ is null-homologous in X'_d (resp. M_d) (simply lift a Seifert surface, for instance).

It follows from these observations that T lifts to $T_d \coloneqq S^1_d \times D^2 \cong S^1 \times D^2$ in M_d, where S^1_d is the d-fold covering of S^1. Now, applying Mayer-Vietoris to the decomposition $M_d = X'_d \cup T_d$, we obtain the following exact sequence:

$$\underbrace{H_1(\partial X'_d;\mathbb{Z})}_{\cong\langle[\widetilde{\mu}_K^d]\rangle\times\langle[\widetilde{\ell}_K]\rangle\cong\mathbb{Z}\times\mathbb{Z}} \xrightarrow{(\iota_*,\nu_*)} H_1(X'_d;\mathbb{Z}) \oplus \underbrace{H_1(T_d;\mathbb{Z})}_{\cong\mathbb{Z}} \xrightarrow{k_*-j_*} H_1(M_d;\mathbb{Z}) \longrightarrow 0 \quad (4.16)$$

where ι_*, ν_*, k_*, j_* are the maps induced on homology from the corresponding inclusions. We have:

$$\iota_*([\widetilde{\mu}_K^d]) = [\widetilde{\mu}_K^d], \quad \iota_*([\widetilde{\ell}_K]) = [\widetilde{\ell}_K] = 0 \quad (4.17)$$

$$\nu_*([\widetilde{\mu}_K^d]) = 1, \quad \nu_*([\widetilde{\ell}_K]) = 0 \tag{4.18}$$

Note that (4.18) holds because the meridian of T_d gets identified with the longitude $\widetilde{\ell}_K$ in X_d', and the longitude of T_d gets identified with $\widetilde{\mu}_K^d$. Overall, we see that ν_* is surjective and so

$$\operatorname{im}(i_*, \nu_*) = \operatorname{im} i_* \oplus H_1(T_d;\mathbb{Z}) = \ker k_* - j_*.$$

Therefore (4.16) reduces to

$$\ker \nu_* \xrightarrow{\iota_*} H_1(X_d';\mathbb{Z}) \xrightarrow{k_*} H_1(M_d;\mathbb{Z}) \longrightarrow 0.$$

By (4.17) and (4.18) we know that the image of ι_* restricted to $\ker \nu_*$ is generated by $\widetilde{\ell}_K$, which is trivial in $H_1(X_d';\mathbb{Z})$. Therefore k_* is an isomorphism and so

$$H_1(X_d';\mathbb{Z}) \cong H_1(M_d;\mathbb{Z}).$$

For the second isomorphism, we proceed identical. Let $\widehat{T}_d := S^1 \times D_d^2$, where D_d^2 denotes the d-fold branched cover of D^2 obtained by extending the d-fold cover of S^1 to D^2. Then $X_d = X_d' \cup_\partial \widehat{T}_d$, where the meridian of $\widehat{T}_d$ gets identified with $\widetilde{\mu}_K^d$ and the longitude of $\widehat{T}_d$ gets identified with $\widetilde{\ell}_K$. Applying Mayer-Vietoris to $X_d = X_d' \cup \widehat{T}_d$ we obtain

$$\underbrace{H_1(\partial X_d';\mathbb{Z})}_{\cong\langle[\widetilde{\mu}_K^d]\rangle\times\langle[\widetilde{\ell}_K]\rangle\cong\mathbb{Z}\times\mathbb{Z}} \xrightarrow{(\iota_*,\nu_*)} H_1(X_d';\mathbb{Z}) \oplus \underbrace{H_1(\widehat{T}_d;\mathbb{Z})}_{\cong\mathbb{Z}} \xrightarrow{k_*-j_*} H_1(X_d;\mathbb{Z}) \longrightarrow 0, \tag{4.19}$$

where ι_*, ν_*, k_*, j_* denote again the maps on homology induced by the corresponding inclusions. We have:

$$\iota_*([\widetilde{\mu}_K^d]) = [\widetilde{\mu}_K^d], \quad \iota_*([\widetilde{\ell}_K]) = [\widetilde{\ell}_K] = 0 \tag{4.20}$$

$$\nu_*([\widetilde{\mu}_K^d]) = 0, \quad \nu_*([\widetilde{\ell}_K]) = 1 \tag{4.21}$$

Hence ν_* is surjective and so

$$\operatorname{im}(i_*, \nu_*) = \operatorname{im} i_* \oplus H_1(\widehat{T}_d;\mathbb{Z}) = \ker k_* - j_*.$$

Therefore (4.19) reduces to

$$\ker \nu_* \xrightarrow{\iota_*} H_1(X_d';\mathbb{Z}) \xrightarrow{k_*} H_1(X_d;\mathbb{Z}) \longrightarrow 0.$$

By (4.20) and (4.21) we know that the image of ι_* restricted to $\ker \nu_*$ is generated by $[\widetilde{\mu}_K^d]$, so

$$H_1(X_d;\mathbb{Z}) \cong H_1(X_d')/([\widetilde{\mu}_K^d])$$

Therefore

$$H_1(X_d';\mathbb{Z}) \cong H_1(X_d;\mathbb{Z}) \oplus \mathbb{Z}$$

as desired. □

Suppose we are given an epimorphism $\chi\colon H_1(X_d;\mathbb{Z}) \to \mathbb{Z}_m$ for some $m \in \mathbb{N}$. The image of the composition

$$\pi_1(M_d) \xhookrightarrow{(p_d)_*} \pi_1(M_K) \xrightarrow{\text{ab}} H_1(M_K;\mathbb{Z}) \cong \mathbb{Z},$$

is isomorphic to $d\mathbb{Z}$ generated by $[\mu_K^d]$, producing a surjection $\varphi\colon \pi_1(M_d) \to d\mathbb{Z} \cong \mathbb{Z}$. On the other hand, by Lemma 4.81 χ defines an epimorphism

$$\psi\colon \pi_1(M_d) \xrightarrow{\text{ab}} H_1(M_d;\mathbb{Z}) \cong H_1(X_d;\mathbb{Z}) \oplus \mathbb{Z} \xrightarrow{\chi\times 0} \mathbb{Z}_m,$$

that combines with φ to an epimorphism

$$\psi \times \varphi\colon \pi_1(M_d) \to \mathbb{Z}_m \times \mathbb{Z}.$$

Let $\widehat{M_d}$ denote the $\mathbb{Z}_m \times \mathbb{Z}$-covering corresponding to the kernel of $\psi \times \varphi$.

Remark 4.82. The covering $\widehat{M_d}$ can also be obtained differently, as follows. The kernel of the surjection $\varphi\colon \pi_1(M_d) \to d\mathbb{Z} \cong \mathbb{Z}$ yields an infinite cyclic covering $r_\infty\colon (M_d)_\infty \to M_d$. Observe that this covering also corresponds to the kernel of the composition

$$\widetilde{\varphi}\colon \pi_1(M_d) \xrightarrow{\text{ab}} H_1(M_d;\mathbb{Z}) \cong H_1(X_d;\mathbb{Z}) \oplus \mathbb{Z} \xrightarrow{0\times \text{id}} \mathbb{Z}.$$

Indeed, this follows directly from the commutativity of the following diagram:

$$\begin{array}{ccc}
\pi_1(M_d) & \xrightarrow[{[\widetilde{\mu}_K^d]\mapsto 1}]{\widetilde{\varphi}} & \mathbb{Z} \\
{\scriptstyle (p_d)_*}\big\downarrow & \searrow^{\varphi} & \big\downarrow{\scriptstyle \cdot d} \\
\pi_1(M_K) & \xrightarrow{\text{ab}} & H_1(M_K;\mathbb{Z}) \cong \mathbb{Z}
\end{array}$$

So $\ker\varphi = \ker\widetilde{\varphi}$, and therefore the kernel of the epimorphism

$$\rho\colon \pi_1(M_d) \xrightarrow{\text{ab}} H_1(M_d;\mathbb{Z}) \cong H_1(X_d;\mathbb{Z}) \oplus \mathbb{Z} \xrightarrow{\chi\times\text{id}} \mathbb{Z}_m \oplus \mathbb{Z} \tag{4.22}$$

yields precisely the $\mathbb{Z}_m \times \mathbb{Z}$-covering $\widehat{M_d}$.

There is yet another description of $\widehat{M_d}$ that appears in the original source [CG75] by Casson and Gordon. Observe that $(M_d)_\infty$ gives a cover of M_K via

$$r\colon (M_d)_\infty \xrightarrow{r_\infty} M_d \xrightarrow{p_d} M_K$$

and this is in fact the same as the infinite cyclic covering $p_\infty\colon M_\infty \to M_K$ since

$$(r_\infty)_*(\pi_1((M_d)_\infty)) = \ker\varphi = (p_d)_*^{-1}(\ker \text{ab})$$

and therefore

$$r_*(\pi_1((M_d)_\infty)) = \ker \text{ab} = [\pi_1(M_K), \pi_1(M_K)] \subset \pi_1(M_K).$$

Using again Lemma 4.81, we now obtain an epimorphism

$$\pi_1(M_\infty) \xhookrightarrow{(r_\infty)_*} \pi_1(M_d) \xrightarrow{\psi} \mathbb{Z}_m \tag{4.23}$$

where ψ is the surjection

$$\psi\colon \pi_1(M_d) \to H_1(M_d;\mathbb{Z}) \cong H_1(X_d;\mathbb{Z}) \oplus \mathbb{Z} \overset{\chi\times 0}{\to} \mathbb{Z}_m$$

$$\widetilde{\varphi}\colon \pi_1(M_d) \xrightarrow{\text{ab}} H_1(M_d;\mathbb{Z}) \cong H_1(X_d;\mathbb{Z}) \oplus \mathbb{Z} \xrightarrow{0\times\text{id}} \mathbb{Z}.$$

The kernel of (4.23) gives an m-fold cyclic covering $(M_\infty)_m$ of M_∞, which combines to a covering

$$\ell\colon (M_\infty)_m \longrightarrow M_\infty \xrightarrow{r_\infty} M_d$$

But this is precisely the $\mathbb{Z}_m \times \mathbb{Z}$-covering $\widehat{M_d}$ since

$$\ell_*(\pi_1((M_\infty)_m)) = \ker\rho,$$

where ρ is the map in (4.22).

Similar to $\Omega_3^{\text{top}}(K(\mathbb{Z}_m,1))$ that appeared in the construction leading to $\sigma(K,\chi)$, the bordism group $\Omega_3^{\text{top}}(K(\mathbb{Z}\times\mathbb{Z}_m,1))$ is finite, hence there exists a compact oriented 4-manifold V and an epimorphism $\phi\colon \pi_1(V) \to \mathbb{Z}_m \times \mathbb{Z}$ such that

$$\partial(V,\phi) = s(M_d, \psi\times\varphi)$$

for some $s \in \mathbb{N}$. Let $\widehat{V}$ denote the $\mathbb{Z}_m\times\mathbb{Z}$-covering associated the $\ker\phi$. Set $\omega := e^{\frac{2\pi i}{m}}$ and consider the field of rational functions over $\mathbb{Q}(\omega)$, denoted by $\mathbb{Q}(\omega)(t)$. We equip $\mathbb{Q}(\omega)(t)$ with an involution defined by $\omega \mapsto \overline{\omega}$ and $t \mapsto t^{-1}$. We are again in a situation similar to Subsection 4.3.4: we endow $\mathbb{Q}(\omega)(t)$ with a $(\mathbb{Q}(\omega)(t), \mathbb{Z}[\mathbb{Z}_m\times\mathbb{Z}])$-left-left module structure, where $\mathbb{Q}(\omega)(t)$ acts via left multiplication and $\mathbb{Z}[\mathbb{Z}_m\times\mathbb{Z}] \cong \mathbb{Z}[\mathbb{Z}_m][t^{\pm1}]$ acts via

$$\xi\cdot x := x\overline{\omega}, \quad t^{\pm1}\cdot x := xt^{\mp1}.$$

Then we can form the homology groups

$$H_*^\phi(V;\mathbb{Q}(\omega)(t)) := \overline{\mathbb{Q}(\omega)(t)} \otimes_{\mathbb{Z}[\mathbb{Z}_m\times\mathbb{Z}]} H_*(\widehat{V};\mathbb{Z}),$$

which carry the structure of left $\mathbb{Q}(\omega)(t)$-modules. Using the same argument as in Subsection 4.3.4, we see that $H_*^\phi(V;\mathbb{Q}(\omega)(t))$ is isomorphic to the twisted homology $H_*^t(V;\mathbb{Q}(\omega)(t))$, which yields the $\mathbb{Q}(\omega)(t)$-twisted intersection form $Q^t_{\mathbb{Q}(\omega)(t)}$ on $H_2^\phi(V;\mathbb{Q}(\omega)(t))$. This form is hermitian, but in general not non-singular. However, we have the following result.

Lemma 4.83. If the epimorphism $\chi\colon H_1(X_d;\mathbb{Z}) \to \mathbb{Z}_m$ has prime-power order, then $Q^t_{\mathbb{Q}(\omega)(t)}$ is non-singular.

A proof is given in [CG75, Lemma 4 and the Corollary afterwards]. Now, the standard $\mathbb{Q}$-valued intersection form $Q_{\mathbb{Q}}$ on $H_2(V;\mathbb{Q})$ (untwisted) might be singular (see Remark 4.43), but we can consider the induced non-singular form on $\widehat{H}_2(V;\mathbb{Q}) := H_2(V;\mathbb{Q})/(\mathrm{im}(H_2(\partial V;\mathbb{Q}) \to H_2(V;\mathbb{Q})))$ which we denote by $Q_{\mathbb{Q}}$ as well. The canonical inclusions $\iota\colon \mathbb{Q} \to \mathbb{C}(t)$ and $\iota'\colon \mathbb{Q}(\omega)(t) \to \mathbb{C}(t)$ induce homomorphisms

$$\iota_*\colon W(\mathbb{Q}) \to W(\mathbb{C}(t)), \quad \iota'_*\colon W(\mathbb{Q}(\omega)(t)) \to W(\mathbb{C}(t)),$$

giving us the elements

$$[\widetilde{Q}_{\mathbb{Q}}] := \iota_*\big((\widehat{H}_2(V;\mathbb{Q}), Q_{\mathbb{Q}})\big)$$
$$[\widetilde{Q}_{\mathbb{Q}(\omega)(t)}] := \iota'_*\big((H_2^{\phi}(V;\mathbb{Q}(\omega)(t)), Q^t_{\mathbb{Q}(\omega)(t)})\big)$$

in $W(\mathbb{C}(t))$, provided that m is a prime-power. We have the following definition.

Definition 4.84 ([CG75]). Let $K \subset S^3$ be a knot and $d \in \mathbb{N}$. Further, let X_d be the d-fold cyclic branched cover of S^3, and let $\chi\colon H_1(X_d;\mathbb{Z}) \to \mathbb{Z}_m$ be an epimorphism of prime-power order. Define

$$\tau(K,\chi) := \big([\widetilde{Q}_{\mathbb{Q}(\omega)(t)}] - [\widetilde{Q}_{\mathbb{Q}}]\big) \otimes \frac{1}{r} \in W(\mathbb{C}(t)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Remark 4.85. Similar to the invariant $\sigma(K,\chi)$, the Casson-Gordon invariant $\tau(K,\chi)$ only depends on the knot K and epimorphism of prime-power order $\chi\colon H_1(X_d;\mathbb{Z}) \to \mathbb{Z}_m$. The argument is similar to the one that shows that $\sigma(K,\chi)$ (resp. $\sigma(M,\chi)$) is well-defined, and a detailed proof can be found for instance in [GL92, Theorem 1]. Also, note that there is again the same ambiguity in the notation of $\tau(K,\chi)$ as with $\sigma(K,\chi)$, whose construction depends on the order $d \in \mathbb{N}$ of the finite cyclic coverings used.

In order to state the main results regarding $\tau(K,\chi)$, we quickly recall the definition of the geometric linking form on $H_1(X_d;\mathbb{Z})$. Let $d \in \mathbb{N}$ be a prime-power, and let $\eta\colon H^1(X_d;\mathbb{Z}) \to H^2(X_d;\mathbb{Q}/\mathbb{Z})$ be the Bockstein homomorphism associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ of coefficients in cohomology. Since d is a prime-power, X_d is a rational homology sphere and thus η is in fact an isomorphism. Consider the composition

$$\Phi\colon H_1(X_d;\mathbb{Z}) \overset{\mathrm{PD}}{\to} H^2(X_d;\mathbb{Z}) \overset{\eta^{-1}}{\to} H^1(X_d;\mathbb{Q}/\mathbb{Z}) \overset{\mathrm{ev}}{\to} \mathrm{Hom}(H_1(X_d;\mathbb{Z}),\mathbb{Q}/\mathbb{Z}),$$

where PD is Poincaré-Lefschetz duality, η^{-1} is the inverse of the Bockstein map, and ev is the Kronecker evaluation map (see [CFH16] for more details).

Definition 4.86. Let K be a knot with d-fold cyclic branched cover X_d, where $d \in \mathbb{N}$ is a prime-power. The *geometric linking form* l on $H_1(X_d;\mathbb{Z})$ is defined as

$$l\colon H_1(X_d;\mathbb{Z}) \times H_1(X_d;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}, \quad l(x,y) := \Phi(x)(y).$$

Remark 4.87. It is evident from the definition that the geometric linking form is in fact non-singular. The form is also symmetric, but this is less obvious and requires some additional work (see [CFH16]). Of course, the definition of l is not restrained to finite cyclic branched coverings of knots and works in a more general setting, see [CFH16]. Note that l is a linking form in the sense of Definition 4.1.

We are now ready to state the main theorem regarding $\tau(K, \chi)$.

Theorem 4.88 (Casson-Gordon). Let K be a knot with d-fold cyclic branched cover X_d with $d \in \mathbb{N}$ a prime-power. If K is slice, then there is a subgroup $G \subseteq H_1(X_d; \mathbb{Z})$ such that

1. $l(G \times G) = 0$;
2. $\tau(K, \chi) = 0$ for every $\chi \colon H_1(X_d; \mathbb{Z}) \to \mathbb{Z}_m$ of prime-power order with $\chi(G) = 0$.

A proof of Theorem 4.88 is given in [CG75, Theorem 2]. The proof shows that one can take G that satisfies $G = G^\perp$ with respect to l, so G is in fact a metabolizer for the geometric linking form in the sense of Definition 4.2.

The next main result relates the invariant $\tau(K, \chi)$ with $\sigma(K, \chi)$. For this, let $\omega \in S^1$ and recall from Lemma 4.68 that the averaged signature $\operatorname{sign}^{\mathrm{av}}_\omega$ induces a homomorphism

$$\operatorname{sign}^{\mathrm{av}}_\omega \colon W(\mathbb{C}(t)) \to \mathbb{Z}$$

which by Theorem 4.69 contains the entire information about $W(\mathbb{C}(t))$. This signature homomorphism extends to

$$\operatorname{sign}^{\mathrm{av}}_\omega \colon W(\mathbb{C}(t)) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$$

in the obvious way,[18] allowing us to consider the signatures $\operatorname{sign}^{\mathrm{av}}_\omega(\tau(K, \chi)) \in \mathbb{Q}$. Let us give the special case of $\omega = 1$ a specific name.

Definition 4.89. The *Casson-Gordon τ-signature* is defined as

$$\operatorname{sign}^{\mathrm{av}}_1(\tau(K, \chi)) \in \mathbb{Q}.$$

We wish to remark at this point that the terminology of Casson-Gordon τ-signature is not standard and chosen by the author.[19] The next theorem relates $\sigma(K, \chi)$ with the Casson-Gordon τ-signature.

[18] As there is already plenty of notation involved, we keep things simple and use $\operatorname{sign}^{\mathrm{av}}_\omega$ to denote both the homomorphism defined on $W(\mathbb{C}(t))$ and the one on $W(\mathbb{C}(t)) \otimes_{\mathbb{Z}} \mathbb{Q}$.

[19] The Casson-Gordon τ-signature is not be confused with the notion of *Casson-Gordon signature*, which is usually reserved for the invariant $\sigma_r(M, \chi)$ in [CG78].

Theorem 4.90 (Casson-Gordon). Let K be a knot with d-fold cyclic branched covering X_d, where $d \in \mathbb{N}$ is a prime-power. Further, let $\chi\colon H_1(X_d;\mathbb{Z}) \to \mathbb{Z}_m$ be an epimorphism of prime-power order. If the covering $\widehat{X}_d \to X$ induced by χ satisfies $H_1(\widehat{X}_d;\mathbb{Q}) = 0$, then

$$|\sigma(K,\chi) - \operatorname{sign}_1^{\mathrm{av}}(\tau(K,\chi))| \leq 1.$$

A proof of Theorem 4.90 is given in [CG75, Theorem 3]. As noted earlier, if K is slice, then by Theorem 4.88 $\operatorname{sign}_1^{\mathrm{av}}(\tau(K,\chi)) = 0$, and we see that in this case Theorem 4.90 yields the earlier mentioned Theorem 4.73.

The main application of $\tau(K,\chi)$ given by Casson and Gordon determines which twist knots K_n are slice, a problem that was previously open.

Corollary 4.91 (Casson-Gordon). The only twist knots which are slice are the unknot K_0 and the Stevedore's knot K_2.

Since K_n is algebraically slice if and only if $4n+1$ is a square (see Proposition 4.16), Corollary 4.91 is of particular relevance as it shows that there are infinitely many non-slice knots which are algebraically slice. Corollary 4.91 is derived in [CG75, Computations after Theorem 3], and was obtained again in Casson and Gordon's second paper [CG78] using the invariant $\sigma_r(M,\chi)$ (see Theorem 4.79).

This concludes our first discussion about the invariants $\sigma(K,\chi)$, $\sigma_r(M,\chi)$, and $\tau(K,\chi)$. We proceed with a short overview of some results by Patrick Gilmer about $\tau(K,\chi)$, which will be needed later to prove our main theorems in Chapter 5.

4.3.9 Gilmer's work on $\tau(K,\chi)$

This subsection is devoted to some results by Patrick Gilmer regarding the Casson-Gordon invariant $\tau(K,\chi)$ as found in [Gil82, Gil83]. Our main interest lies in Theorem 4.93 and Corollary 4.94 below, which will be one of our main ingredients of the proofs in the upcoming Chapter 5.

Let K be a knot with d-fold cyclic branched cover X_d (see Definition 4.14). As described in Subsection 4.3.8, given a homomorphism $\chi\colon H_1(X_d;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ of prime-power order,[20] there is the associated Casson-Gordon invariant $\tau(K,\chi)$ (see Definition 4.84), and for each $\omega \in S^1$ we have the signatures $\operatorname{sign}_\omega^{\mathrm{av}}(\tau(K,\chi))$. If $\omega = 1$, we further defined $\operatorname{sign}_1^{\mathrm{av}}(\tau(K,\chi))$ as the Casson-Gordon τ-signature (see Definition 4.89).

From now on we always assume that $d \in \mathbb{N}$ is a prime-power. In order to state one of our main tools, we need to define a certain linking form β on $H^1(X_d;\mathbb{Q}/\mathbb{Z})$ which is closely related to the geometric linking form l on $H_1(X_d;\mathbb{Z})$ described in Definition 4.86. The definition is as follows. Recall

[20] We adapt here the conventions of Gilmer and consider homomorphisms $\chi\colon H_1(X_d;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ of prime-power order instead of $\chi\colon H_1(X_d;\mathbb{Z}) \to \mathbb{Z}_m$.

that we defined l using the composition

$$\Phi\colon H_1(X_d;\mathbb{Z}) \overset{\mathrm{PD}}{\to} H^2(X_d;\mathbb{Z}) \overset{\eta^{-1}}{\to} H^1(X_d;\mathbb{Q}/\mathbb{Z}) \overset{\mathrm{ev}}{\to} \mathrm{Hom}(H_1(X_d;\mathbb{Z}),\mathbb{Q}/\mathbb{Z}),$$

where PD is Poincaré-Lefschetz duality, η^{-1} is the inverse of the Bockstein map, and ev is the Kronecker evaluation map, and then set

$$l\colon H_1(X_d;\mathbb{Z}) \times H_1(X_d;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}, \quad l(x,y) := \Phi(x)(y).$$

Let us define β as follows.

Definition 4.92. In the situation above, define

$$\beta\colon H^1(X_d;\mathbb{Q}/\mathbb{Z}) \times H^1(X_d;\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}, \quad \beta(x,y) := -l(\nu(x),\nu(y)),$$

where $\nu := \mathrm{PD} \circ \eta$.

The symmetry of l implies that β is symmetric as well, so β is a linking form in the sense of Definition 4.1. Since $H^1(X_d;\mathbb{Q}/\mathbb{Z})$ is isomorphic to $\mathrm{Hom}(H_1(X_d;\mathbb{Z}),\mathbb{Q}/\mathbb{Z})$ by universal coefficients, β can be seen as the dual of the geometric linking form l on $(H_1(X_d;\mathbb{Z}))^* = \mathrm{Hom}(H_1(X_d;\mathbb{Z}),\mathbb{Q}/\mathbb{Z})$. The following theorem is due to Gilmer [Gil82].

Theorem 4.93 ([Gil82, Theorem 1]). Let K be a knot with $g_4(K) = g$. Then $(H^1(X_d;\mathbb{Q}/\mathbb{Z}),\beta)$ splits as direct sum $(B_1 \oplus B_2, \beta_1 \oplus \beta_2)$ such that:

1. β_1 has an even presentation with rank $2(d-1)g$ and signature $\sum_{s=1}^{d-1} \sigma_{s/d}(K)$;
2. β_2 has a metabolizer H such that for every $\chi \in H$ of prime-power order,

$$|\mathrm{sign}_1^{\mathrm{av}}(\tau(K,\chi)) + \sum_{s=1}^{d-1} \sigma_{s/d}(K)| \leq 2dg.$$

Here, $\sigma_{s/d}(K)$ denotes the Levine-Tristram signature of K associated to $e^{2\pi i s/d} \in S^1$.

It will be convenient for our purposes to consider only elements in $H^1(X_d;\mathbb{Q}/\mathbb{Z})$ (resp. H) that have prime order. For this, let p be any prime (for example one that divides the order of $H^1(X_d;\mathbb{Q}/\mathbb{Z})$), and consider the vector space $H^1(X_d;\mathbb{F}_p)$. Using the canonical embedding $\psi\colon \mathbb{F}_p \to \mathbb{Q}/\mathbb{Z}$ defined by $1 \mapsto 1/p$, we obtain a diagram that defines an embedding φ of $H^1(X_d;\mathbb{F}_p)$ into $H^1(X_d;\mathbb{Q}/\mathbb{Z})$:

$$\begin{array}{ccc} H^1(X_d;\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\cong} & \mathrm{Hom}(H_1(X_d;\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) \\ {\scriptstyle\varphi}\big\uparrow & & \big\uparrow{\scriptstyle\widetilde{\psi}} \\ H^1(X_d;\mathbb{F}_p) & \xrightarrow{\cong} & \mathrm{Hom}(H_1(X_d;\mathbb{Z}),\mathbb{F}_p). \end{array}$$

Here, the two horizontal isomorphisms are given by the universal coefficient theorem and $\widetilde{\psi}$ is the map induced by postcomposition with ψ. As described in Section 4.1, the linking form β now induces a linking form $\overline{\beta}$ on $H^1(X_d;\mathbb{F}_p)$ by setting $\overline{\beta} := \beta\circ(\varphi\times\varphi)$. Note that this form will in general no longer be non-singular, and that $\varphi(H^1(X_d;\mathbb{F}_p))$ forms the entire p-torsion subgroup of $H^1(X_d;\mathbb{Q}/\mathbb{Z})$. Since φ is injective, we can uniquely identify the elements of $H^1(X_d;\mathbb{F}_p)$ with the elements of $H^1(X_d;\mathbb{Q}/\mathbb{Z})$ of order p. In particular, we obtain a well-defined Casson-Gordon τ-signature for the elements in $H^1(X_d;\mathbb{F}_p)$ by setting $\operatorname{sign}_1^{\mathrm{av}}(\tau(K,\chi)) := \operatorname{sign}_1^{\mathrm{av}}(\tau(K,\varphi(\chi))$ for $\chi\in H^1(X_d;\mathbb{F}_p)$. This allows us to translate Theorem 4.93 into this setting as follows.

Corollary 4.94. Let K be a knot with $g_4(K) = g$ and p a prime. Then $(H^1(X_d;\mathbb{F}_p),\overline{\beta})$ splits as a direct sum $(G_1\oplus G_2,\gamma_1\oplus\gamma_2)$ such that:

1. the dimension of G_1 over $\mathbb{F}_p$ is at most $2(d-1)g$;
2. γ_2 has a generalized metabolizer F such that for every $\chi\in F$,

$$|\operatorname{sign}_1^{\mathrm{av}}(\tau(K,\chi)) + \sum_{s=1}^{d-1}\sigma_{s/d}(K)| \leq 2dg. \tag{4.24}$$

The most important difference is that the inequality (4.24) now holds for every $\chi\in F$, meaning that we no longer have to make the distinction between elements of arbitrary order and prime-power order.

We conclude this section with a short remark about connected sums. As before, let X_d be the d-fold branched cover of K where d is a prime-power and consider the n-fold connected sum nK. The d-fold branched cover of nK is a connected sum of n copies of X_d, and the first cohomology splits accordingly. Litherland [Lit84] showed that if $\chi = (\chi_1,\dots,\chi_n)\in H^1(nX_d;\mathbb{Q}/\mathbb{Z})$, then

$$\tau(nK,\chi) = \bigoplus_{i=1}^{n}\tau(K,\chi_i).$$

In particular,

$$\operatorname{sign}_\omega^{\mathrm{av}}(\tau(nK,\chi)) = \sum_{i=1}^{n}\operatorname{sign}_\omega^{\mathrm{av}}(\tau(K,\chi_i))$$

since $\operatorname{sign}_\omega^{\mathrm{av}}$ is a homomorphism. In short, Casson-Gordon invariants and τ-signatures behave well under connected sum. Note that the above also holds with $\mathbb{F}_p$-coefficients instead of $\mathbb{Q}/\mathbb{Z}$-coefficients.

Chapter 5

Main Result

Let $K \subset S^3$ be a knot with d-fold branched cover X_d, where $d \in \mathbb{N}$ is a prime-power, and $p \in \mathbb{N}$ a prime. Consider the vector space $H^1(X_d; \mathbb{F}_p)$. Every element $\chi \in H^1(X_d; \mathbb{F}_p)$ can be considered as a map $\chi\colon H_1(X_d; \mathbb{Z}) \to \mathbb{F}_p$ via universal coefficients, and to each such element there is the Casson-Gordon τ-signature $\operatorname{sign}_1^{\text{av}}(\tau(K, \chi))$ (see Subsections 4.3.8 and 4.3.9). Since $H^1(X_d; \mathbb{F}_p)$ is finite, there are only finitely many Casson-Gordon τ-signatures. We make the following definition.

Definition 5.1. Assume that $H^1(X_d; \mathbb{F}_p)$ is non-trivial. Let $A_1, \ldots, A_m$ be the one-dimensional subspaces of $H^1(X_d; \mathbb{F}_p)$. Define for $j = 1, \ldots, m$,

$$L_j := \sum_{\chi \in A_j} \operatorname{sign}_1^{\text{av}}(\tau(K, \chi)) \in \mathbb{Q}.$$

Moreover, we set $L := \min_{j=1,\ldots,m} |L_j|$. If $H^1(X_d; \mathbb{F}_p)$ is trivial we define $L_1 := 0$, $L := 0$.[1]

Note that $m = (p^t - 1)/(p - 1)$ with $t := \dim H^1(X_d; \mathbb{F}_p)$ whenever $H^1(X_d; \mathbb{F}_p)$ is non-trivial.

Before we move on, let us quickly prove an introductory result as a warm-up exercise. This will not only show one of the key ideas used later on in a simplified context, but also allow the reader who would like to skip the (quite lengthy) proof of the main result to not miss out on the main proof technique.

Proposition 5.2. Let $K \subset S^3$ be a knot and $p \in \mathbb{N}$ a prime such that $H^1(X_d; \mathbb{F}_p)$ is one-dimensional. If $L = |L_1| > 0$, then $g_4(nK) \neq 0$ for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$ be a natural number and consider the n-fold connected sum nK. We would like to show with the given assumptions that nK is not slice, i.e. $g_4(nK) \neq 0$.

[1] The somewhat artificial definition of L_1 and L in the case of trivial $H^1(X_d; \mathbb{F}_p)$ is needed in order to obtain concise statements in Proposition 5.4 and 5.7, and Theorem 5.5.

If any of the Levine-Tristram signatures $\sigma_{s/d}(K)$ is non-zero, then the Murasugi-Tristram bound $g_4(nK) \geq \frac{n}{2}|\sigma_{s/d}(K)|$ holds [Mur65, Tri69], so we may assume in the following that $\sum_{s=1}^{d-1} \sigma_{s/d}(K) = 0$. Let $\chi \in H^1(nX_d; \mathbb{F}_p)$ be a non-zero element. We claim that there exists a $k \in \mathbb{Z}$ such that $\mathrm{sign}_1^{\mathrm{av}}(\tau(nK, k \cdot \chi)) \neq 0$. Indeed, take the canonical basis on $H^1(nX_d; \mathbb{F}_p)$ given by the decomposition

$$H^1(nX_d; \mathbb{F}_p) = \underbrace{H^1(X_d; \mathbb{F}_p) \oplus \cdots \oplus H^1(X_d; \mathbb{F}_p)}_{n \text{ times}}$$

and let R be the number of non-zero components of χ with respect to this basis. Then

$$\left| \sum_{\ell=1}^{p-1} \mathrm{sign}_1^{\mathrm{av}}(\tau(nK, \ell \cdot \chi)) \right| = R \cdot L > 0.$$

Now by Corollary 4.94, if nK was slice there would exist a subspace $F \subset H^1(nX_d; \mathbb{F}_p)$ consisting only of elements with vanishing Casson-Gordon τ-signature. However, we have just shown that every non-zero element in $H^1(nX_d; \mathbb{F}_p)$ has a multiple with non-zero τ-signature. Thus nK is not slice, and the result follows. □

Let us now get back to the proof of our main result. We start with the following technical proposition.

Proposition 5.3. Let K be a knot. If the rational numbers $L_1, L_2, \ldots, L_m$ have the same sign and $L \neq 0$, then for any given $g \in \mathbb{N}$, there exists some $N \in \mathbb{N}$ such that $g_4(nK) > g$ for all $n \geq N$.

Proof. Fix some $g \in \mathbb{N}$ and consider the connected sum nK for some $n \in \mathbb{N}$. If one of the Levine-Tristram signatures $\sigma_{s/d}(K)$ is non-zero, then the Murasugi-Tristram bound $g_4(nK) \geq \frac{n}{2}|\sigma_{s/d}(K)|$ holds [Mur65, Tri69], so we may assume in the following that $\sum_{s=1}^{d-1} \sigma_{s/d}(K) = 0$.

Recall Corollary 4.94: If $g_4(nK) = g$, then $(H^1(nX_d; \mathbb{F}_p), \overline{\beta})$ splits as a direct sum $(G_1 \oplus G_2, \gamma_1 \oplus \gamma_2)$, where the dimension of G_1 over $\mathbb{F}_p$ is at most $2(d-1)g$, and G_2 has a generalized metabolizer F such that for all $\chi = (\chi_1, \ldots, \chi_n) \in F$,

$$|\mathrm{sign}_1^{\mathrm{av}}(\tau(nK, \chi))| = \left| \sum_{i=1}^{n} \mathrm{sign}_1^{\mathrm{av}}(\tau(K, \chi_i)) \right| \leq 2dg.$$

Our goal is to show that by choosing n appropriately, we can find an element $\chi \in F$ such that this inequality does not hold.

Observe the following. Since $\dim G_1 \leq 2(d-1)g$ and g is fixed, the dimension of G_1 is bounded when increasing n. On the other hand, increasing n increases the dimension of G_2 since $\dim G_1 + \dim G_2 = \dim H^1(nX_d; \mathbb{F}_p)$, and with it the dimension of F.

The Gauss-Jordan algorithm shows that if $\dim F = r$, then there is at least one element $\widetilde{\chi} = (\widetilde{\chi}_1, \ldots, \widetilde{\chi}_n) \in F$ with $R \geq r$ non-zero entries. We may assume without loss of generality that the first R entries are non-zero, i.e.

$$\widetilde{\chi} = (\widetilde{\chi}_1, \ldots, \widetilde{\chi}_R, \underbrace{0, \ldots, 0}_{n-R}).$$

From now on, we fix this element $\widetilde{\chi}$. Note that $k \cdot \widetilde{\chi} \in F$ for any $k \in \mathbb{N}$ since F is a subspace. We are now going to show that there exists a multiple $k \cdot \widetilde{\chi}$ whose Casson-Gordon τ-signature can be bounded from below by a value that depends on n.

As in Definition 5.1, let $A_1, \ldots, A_m$ be the one-dimensional subspaces of $H^1(X_d; \mathbb{F}_p)$. Write

$$A_j = \{0, \phi_1^j, \ldots, \phi_{p-1}^j\} \subseteq H^1(X_d; \mathbb{F}_p)$$

for $j = 1, \ldots, m$. For every ϕ_i^j there is an associated Casson-Gordon τ-signature $s_i^j := \operatorname{sign}_1^{\mathrm{av}}(\tau(K, \phi_i^j))$. In particular, for every component $\widetilde{\chi}_h$ for $h = 1, \ldots, R$, there is some $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, p-1\}$ such that

$$\widetilde{\chi}_h = \phi_i^j \in A_j, \quad \operatorname{sign}_1^{\mathrm{av}}(\tau(K, \widetilde{\chi}_h)) = s_i^j.$$

Let a_i^j denote the number of components in $\widetilde{\chi}$ with τ-signature s_i^j and set $r_j := \sum_{i=1}^{p-1} a_i^j$. Observe that

$$a_i^j \geq 0, \qquad \sum_{i=1}^{p-1} s_i^j = L_j, \qquad \sum_{j=1}^{m} r_j = R$$

for all $i = 1, \ldots, p-1$ and $j = 1, \ldots, m$. Recall from Definition 5.1 that we defined $L := \min_{j=1,\ldots,m} |L_j|$. We claim the following.

Claim. There exists some $1 \leq k \leq p-1$ such that

$$\left| \sum_{i=1}^{n} \operatorname{sign}_1^{\mathrm{av}}(\tau(K, k \cdot \widetilde{\chi}_i)) \right| \geq \frac{R}{p-1} \cdot L.$$

Proof. Consider the elements

$$\begin{aligned} \widetilde{\chi} &= \ (\widetilde{\chi}_1, \ldots, \widetilde{\chi}_R, 0 \ldots, 0) \\ 2\widetilde{\chi} &= 2 \cdot (\widetilde{\chi}_1, \ldots, \widetilde{\chi}_R, 0, \ldots, 0) \\ &\ \vdots \\ (p-1)\widetilde{\chi} &= (p-1) \cdot (\widetilde{\chi}_1, \ldots, \widetilde{\chi}_R, 0, \ldots, 0). \end{aligned}$$

For $\ell = 1, \ldots, p-1$, let $a_{i,\ell}^j$ denote the number of components in $\ell \cdot \widetilde{\chi}$ with τ-signature s_i^j. Since every A_j is one-dimensional, the numbers $a_{1,\ell}^j, \ldots, a_{p-1,\ell}^j$

are just a permutation of $a_1^j, \dots, a_{p-1}^j$ for every j. In fact, by looking at the multiples $\ell \cdot \widetilde{\chi}$ for $\ell = 1, \dots, p-1$, we have that for any given a_i^j and s_h^j, there is an ℓ such that a_i^j is the number of components in $\ell \cdot \widetilde{\chi}$ corresponding to the τ-signature s_h^j. Therefore,

$$\sum_{\ell=1}^{p-1}\sum_{j=1}^{m}\sum_{i=1}^{p-1} a_{i,\ell}^j s_i^j = \sum_{j=1}^{m}\sum_{i=1}^{p-1}\left(\sum_{\ell=1}^{p-1} a_{i,\ell}^j\right) s_i^j = \sum_{j=1}^{m}\sum_{i=1}^{p-1} r_j s_i^j = \sum_{j=1}^{m} r_j L_j$$

Since also

$$(p-1)\sum_{j=1}^{m}\sum_{i=1}^{p-1}\frac{r_j}{p-1}s_i^j = \sum_{j=1}^{m} r_j L_j,$$

we have that

$$\sum_{\ell=1}^{p-1}\left(\sum_{j=1}^{m}\sum_{i=1}^{p-1}\left(a_{i,\ell}^j s_i^j - \frac{r_j}{p-1}s_i\right)\right) = 0.$$

Therefore, there have to be some $\ell_1, \ell_2 \in \{1, \dots, p-1\}$ such that

$$\sum_{j=1}^{m}\sum_{i=1}^{p-1}\left(a_{i,\ell_1}^j s_i^j - \frac{r_j}{p-1}s_i^j\right) \geq 0 \quad \text{and} \quad \sum_{j=1}^{m}\sum_{i=1}^{p-1}\left(a_{i,\ell_2}^j s_i^j - \frac{r_j}{p-1}s_i^j\right) \leq 0.$$

Choose $k \in \{\ell_1, \ell_2\}$ such that

$$\left|\sum_{j=1}^{m}\sum_{i=1}^{p-1} a_{i,k}^j s_i^j\right| \geq \left|\sum_{j=1}^{m}\sum_{i=1}^{p-1}\frac{r_j}{p-1}s_i^j\right|.$$

Unraveling the notations and using that all L_j have the same sign, we find

$$\begin{aligned}\left|\sum_{i=1}^{n}\operatorname{sign}_1^{\mathrm{av}}(\tau(K, k\cdot\widetilde{\chi}_i))\right| &= \left|\sum_{j=1}^{m}\sum_{i=1}^{p-1} a_{i,k}^j s_i^j\right| \\ &\geq \left|\sum_{j=1}^{m}\sum_{i=1}^{p-1}\frac{r_j}{p-1}s_i^j\right| \\ &= \left|\sum_{j=1}^{m}\frac{r_j}{p-1}L_j\right| \\ &\geq \left(\sum_{j=1}^{m}\frac{r_j}{p-1}\right)\cdot L \\ &= \frac{R}{p-1}\cdot L.\end{aligned}$$

□

The claim shows that there exists an element $k \cdot \widetilde{\chi} \in F$ such that

$$|\operatorname{sign}_1^{\mathrm{av}}(\tau(nK, k\cdot\widetilde{\chi}))| \geq \frac{R}{p-1}\cdot L.$$

As mentioned earlier, the total number R of non-zero components in $\widetilde{\chi}$ (or any multiple of it) depends on the dimension of F, which in turn depends on n, i.e. the number of summands in the connected sum nK. Since g is fixed, adding more and more knots to the connected sum increases the dimension of F. Thus, choose $N \in \mathbb{N}$ such that the connected sum NK admits a generalized metabolizer F with dimension $r \leq R$ satisfying

$$\frac{r}{p-1} \cdot L > 2dg.$$

Then for any $n \geq N$, the connected sum nK admits an element $k \cdot \widetilde{\chi} \in F$ such that

$$|\operatorname{sign}_1^{\mathrm{av}}(\tau(nK, k \cdot \widetilde{\chi}))| \geq \frac{R}{p-1} \cdot L > 2dg,$$

proving that $g_4(nK) > g$ by Corollary 4.94. □

Before we continue, let us recall again Definition 5.1: if $A_1, \dots, A_m$ are the one-dimensional subspaces of $H^1(X_d; \mathbb{F}_p)$, then we define

$$L_j \coloneqq \sum_{\chi \in A_j} \operatorname{sign}_1^{\mathrm{av}}(\tau(K, \chi)) \in \mathbb{Q}, \quad j = 1, \dots, m.$$

Proposition 5.4. Let K be a knot with d-fold branched cover X_d where d is a prime-power, and let p be any prime. If the rational numbers $L_1, L_2, \dots, L_m$ have the same sign, and if $\sum_{s=1}^{d-1} \sigma_{s/d}(K) = 0$, where $\sigma_{s/d}(K)$ is the Levine-Tristram signature of K associated to $e^{2\pi i s/d}$, then

$$g_4(nK) \geq \frac{nt \cdot L}{4d(p-1) + 2(d-1)L}$$

for any $n \in \mathbb{N}$, where $t \coloneqq \dim H^1(X_d; \mathbb{F}_p)$ and $L \coloneqq \min_{j=1,\dots,m} |L_j|$.

Proof. We wish to determine the maximal number $g \in \mathbb{N}$ in terms of n for which the proof of Proposition 5.3 applies. So suppose that $H^1(nX_d; \mathbb{F}_p)$ splits as in Corollary 4.94 for some $g \in \mathbb{N}$, and let $r = \dim F$. If $L = 0$, then $nt \cdot L/(4d(p-1) + 2(d-1)L) = 0$ and we obtain the trivial bound $g_4(nK) \geq 0$. So suppose that $L \neq 0$. The proof of Proposition 5.3 showed that if

$$\frac{r}{p-1} \cdot L > 2dg,$$

then $g_4(nK) \neq g$. Since $\dim G_2 \geq nt - 2(d-1)g$, we know that $r \geq (nt - 2(d-1)g)/2$, so

$$\frac{r}{p-1} \cdot L \geq \frac{nt - 2(d-1)g}{2(p-1)} \cdot L.$$

The right-hand side of the last inequality is strictly greater than $2dg$ if and only if

$$g < \frac{nt \cdot L}{4d(p-1) + 2(d-1)L}.$$

Thus, if

$$g \leq \left\lceil \frac{nt \cdot L}{4d(p-1) + 2(d-1)L} \right\rceil - 1,$$

then $g_4(nK) \neq g$ by applying the argument in the proof of Proposition 5.3. It follows that

$$g_4(nK) \geq \frac{nt \cdot L}{4d(p-1) + 2(d-1)L}$$

as claimed. □

Theorem 5.5 (Main Theorem). Let K be a knot with d-fold branched cover X_d where d is a prime-power, and let p be any prime. If the rational numbers $L_1, L_2, \ldots, L_m$ have the same sign, and if $\sum_{s=1}^{d-1} \sigma_{s/d}(K) = 0$, where $\sigma_{s/d}(K)$ is the Levine-Tristram signature of K associated to $e^{2\pi i s/d}$, then

$$g_{st}(K) \geq \frac{t \cdot L}{4d(p-1) + 2(d-1)L},$$

where $t := \dim H^1(X_d; \mathbb{F}_p)$ and $L := \min_{j=1,\ldots,m} |L_j|$.

Proof. By Proposition 5.4,

$$g_4(nK) \geq \frac{nt \cdot L}{4d(p-1) + 2(d-1)L}$$

for any $n \in \mathbb{N}$. Therefore

$$\begin{aligned} g_{st}(K) &= \lim_{n\to\infty} \frac{g_4(nK)}{n} \\ &\geq \lim_{n\to\infty} \frac{1}{n} \cdot \frac{nt \cdot L}{4d(p-1) + 2(d-1)L} \\ &= \frac{t \cdot L}{4d(p-1) + 2(d-1)L}. \end{aligned}$$

□

In the remainder of this chapter we will consider without further mention only knots for which the sum of Levine-Tristram signatures $\sum_{s=1}^{d-1} \sigma_{s/d}(K)$ vanishes.

Let us now discuss how the lower bounds from Theorem 5.5 resp. Proposition 5.4 behave when K itself is a connected sum. For this we introduce some new notation.

Notation. Let K be a knot with d a prime-power and p a prime. Then we write:

1. $X_d(K)$ for the d-fold branched cover of K;
2. $A_1(K), \ldots, A_{m(K)}(K) \subseteq H^1(X_d(K); \mathbb{F}_p)$ for the one-dimensional subspaces of $H^1(X_d(K); \mathbb{F}_p)$;

3. $L_1(K), \ldots, L_{m(K)}(K)$ for the rational numbers in Definition 5.1;
4. $L(K)$ for L as in Definition 5.1;
5. $t(K)$ for the dimension of $H^1(X_d(K); \mathbb{F}_p)$.

Let's consider a simple example as a warm-up exercise.

Example 5.6. Let J_1, J_2 be knots. Take $d = 2$ and $p = 5$, and suppose that both $H^1(X_2(J_1); \mathbb{F}_5)$ and $H^1(X_2(J_2); \mathbb{F}_5)$ are one-dimensional, so that

$$H^1(X_2(J_1); \mathbb{F}_5) \cong H^1(X_2(J_2); \mathbb{F}_5) \cong \mathbb{F}_5,$$

and in particular $L(J_1) = |L_1(J_1)|$, $L(J_2) = |L_1(J_2)|$, $t(J_1) = t(J_2) = 1$. Now consider the connected sum

$$J \coloneqq J_1 \# J_2.$$

We have

$$H^1(X_2(J); \mathbb{F}_5) \cong \mathbb{F}_5 \oplus \mathbb{F}_5$$

with $t(J) = t(J_1) + t(J_2) = 2$, and we see that the one-dimensional subspaces of $H^1(X_2(J); \mathbb{F}_5)$ are given by

$$\begin{aligned} A_1(J) &= \operatorname{span}_{\mathbb{F}_5} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A_2(J) = \operatorname{span}_{\mathbb{F}_5} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ A_3(J) &= \operatorname{span}_{\mathbb{F}_5} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad A_4(J) = \operatorname{span}_{\mathbb{F}_5} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ A_5(J) &= \operatorname{span}_{\mathbb{F}_5} \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad A_6(J) = \operatorname{span}_{\mathbb{F}_5} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \end{aligned}$$

with $m(J) = 6$. It follows that

$$L_1(J) = L_1(J_1), \quad L_2(J) = L_1(J_2), \quad L_i(J) = L_1(J_1) + L_1(J_2)$$

for $i = 3, \ldots, 6$. If all $L_1(J), \ldots, L_6(J)$ have the same sign, then we can apply Theorem 5.5 to J and obtain

$$g_{st}(J) \geq \frac{(t(J_1) + t(J_2)) \cdot L(J)}{4d(p-1) + 2(d-1)L(J)} = \frac{L(J)}{16 + L(J)}$$

However, since $L_1(J) = L_1(J_1)$ and $L_2(J) = L_1(J_2)$, the condition that all $L_1(J), \ldots, L_6(J)$ have the same sign also implies that $L_1(J_1)$ and $L_1(J_2)$ have the same sign, thus posing a condition on the rational numbers of the knots in the connected sum J.

Generalizing Example 5.6, we obtain the following result.

Proposition 5.7. Let $J_1, J_2, \dots, J_k$ be a family of knots, and consider the connected sum

$$J := n_1 J_1 \# n_2 J_2 \# \cdots \# n_k J_k,$$

where $n_i \in \mathbb{N}\backslash\{0\}$ and $n_i J_i$ denotes the n_i-fold connected sum of J_i with itself for $i = 1, \dots, k$. Further, let d be a prime-power and p a prime. Then:

1. The rational numbers $L_1(J), L_2(J), \dots, L_{m(J)}(J)$ have the same sign if and only if the $L_1(J_i), L_2(J_i), \dots, L_{m(J_i)}(J_i)$ obtained from non-trivial $H^1(X_d(J_i); \mathbb{F}_p)$ have the same sign simultaneously for all $i = 1, \dots, k$.

2. If the rational numbers $L_1(J), L_2(J), \dots, L_{m(J)}(J)$ have the same sign, then

$$g_{st}(J) \geq \sum_{i=1}^{k} \frac{n_i t(J_i) L(J)}{4d(p-1) + 2(d-1)L(J)},$$

   where $t(J_i) := \dim H^1(X_d(J_i); \mathbb{F}_p)$ and $L(J) := \min_{j=1,\dots,m(J)} |L_j(J)|$. In particular, if $L(J) \neq 0$ then J is not slice.

Proof. The observation made in Example 5.6 generalizes to prove the first part of the proposition. We know that there is a decomposition

$$H^1(X_d(J); \mathbb{F}_p) \cong \bigoplus_{j=1}^{k} \left(\bigoplus_{i=1}^{n_j} H^1(X_d(J_i); \mathbb{F}_p) \right).$$

Now depending on the prime p some of the $H^1(X_d(J_i); \mathbb{F}_p)$ may be trivial and are thus not part of one-dimensional subspaces of $H^1(X_d(J); \mathbb{F}_p)$. In other words, the rational numbers $L_1(J), L_2(J), \dots, L_{m(J)}(J)$ consist of sums of the $L_1(J_i), L_2(J_i), \dots, L_{m(J_i)}(J_i)$ obtained from the non-trivial $H^1(X_d(J_i); \mathbb{F}_p)$, $i = 1, \dots, k$. In particular, for any such $L_j(J_i)$, there is some $h \in \{1, \dots, m(J)\}$ such that $L_h(J) = L_j(J_i)$, and the first part of the proposition immediately follows. For the second part, we apply Theorem 5.5 to J to obtain the inequality

$$g_{st}(J) \geq \sum_{i=1}^{k} \frac{n_i t(J_i) L(J)}{4d(p-1) + 2(d-1)L(J)},$$

which is strictly greater than zero if $L(J) \neq 0$. □

The reader might have noticed that Proposition 5.7 points towards the problem of linear independency in the knot concordance group $\mathcal{C}$. The rational numbers $L_1(J), L_2(J), \dots, L_{m(J)}(J)$ play a crucial role in Proposition 5.7, and we know that they are made of sums of the $L_j(J_i)$. However, the condition that all these numbers have the same sign is quite restrictive, and the next step would be to find an improvement on this condition. What makes this difficult is that our main results rely on the fact that when considering a connected sum nK with $g_4(nK) = g$ and $g \in \mathbb{N}$ fixed, then there exists a generalized metabolizer $F \subset H^1(X_d(nK);\mathbb{F}_p)$, and we can manually increase the dimension of F by adding more copies of K to the connected sum with itself, that is by increasing n (see the proof of Proposition 5.3). However, we do in general not know how the elements in F (and with them the appearing Casson-Gordon τ-signatures) change under this process, which is needed in order to obtain lower bounds for expressions that involve the numbers $L_i(K)$. Nonetheless, when the Casson-Gordon τ-signatures are explicitly known the results may be improved, and this will allow us in the upcoming Chapter 6 to obtain a partial result for the linear independency of the twist knots in the knot concordance group $\mathcal{C}$.

Chapter 6

Example: Twist Knots

The results from Chapter 5 are in particular applicable for (classical) genus one knots, for example the twist knots. This is because for genus one knots, there is an explicit formula, due to Gilmer, for computing the Casson-Gordon τ-signature. We proceed by describing this formula in Section 6.1, following the original source [Gil83]. In the subsequent Sections 6.2 and 6.3, we use this formula and the results from Chapter 5 to obtain the lower bound for the stable 4-genus of twist knots. Moreover, this will allow us to complete the classification of the concordance order of the twist knots, and we will be able to provide a partial answer to their linear independency in the knot concordance group $\mathcal{C}$. In Section 6.4, we apply a different technique recently used by Baader and Lewark [BL17] to obtain an upper bound for the stable 4-genus of twist knots, yielding the subfamily with g_{st} close to but not greater than 1/2.

6.1 Gilmer's formula for Casson-Gordon τ-signatures

The results in the following section are due to Gilmer [Gil83]. From now on, we will work exclusively with the double branched cover X_2 (see the remark at the end of Section 6.1). Let K be a knot with Seifert surface F, Seifert pairing $\theta\colon H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \to \mathbb{Z}$, and double branched cover X_2. Define

$$\varepsilon\colon H_1(F;\mathbb{Z}) \to H^1(F;\mathbb{Z}), \quad x \mapsto \varepsilon_x(\cdot) = \theta(x,\cdot) + \theta(\cdot,x).$$

There is an isomorphism

$$H^1(X_2;\mathbb{Q}/\mathbb{Z}) \cong \ker(\varepsilon \otimes \mathrm{id}_{\mathbb{Q}/\mathbb{Z}}),$$

which is natural up to sign. Given $\chi \in H^1(X_2;\mathbb{Q}/\mathbb{Z})$, this allows for the identification $\chi = x \otimes s/m$ for some $x \in H_1(F;\mathbb{Z})$ and $0 \leq s < m$. Given $\omega = e^{2\pi i s/m} \in S^1$, let $\sigma_{s/m}(K)$ denote the Levine-Tristram signature of K associated to ω.

Suppose now that $g(F) = 1$, i.e. K is of genus one. If $x \in H_1(F;\mathbb{Z})$ is primitive, let J_x be the knot in S^3 obtained by representing x by a simple closed curve γ on F and then viewing γ as a knot in S^3. Note that J_x is unique up to isotopy since $g(F) = 1$. We have the following theorem.

Theorem 6.1 (Gilmer [Gil83]). Let K be a genus one knot with genus-minimal Seifert surface F and ordinary signature $\sigma(K)$. If $\chi = x \otimes s/m \in H_1(F;\mathbb{Q}/\mathbb{Z})$, where $0 < s < m$, m is a prime-power and x is primitive, then

$$\operatorname{sign}_1^{\mathrm{av}}(\tau(K,\chi)) = 2\sigma_{s/m}(J_x) + \frac{4(m-s)s}{m^2}\theta(x,x) + \sigma(K).$$

Using Theorem 6.1, the computation of Casson-Gordon τ-signatures for genus one knots boils down to the computation of generators of $\ker(\varepsilon \otimes \mathrm{id}_{\mathbb{Q}/\mathbb{Z}})$, and then identifying the corresponding knots J_x and their Levine-Tristram signature.[1]

Remark 6.2. Gilmer's formula in [Gil83] for the Casson-Gordon τ-signatures of genus one knots is stated in terms of the double branched cover X_2. It is worth to note that generalizations of this formula to higher branched covers exist [Gil93, Nai96, Kim05]. However, since the double branched cover is the most accessible and since our formula for the stable 4-genus yields a priori the best result for $d = 2$, we continue our computations for the twist knots with the double branched cover X_2.

6.2 Casson-Gordon τ-signatures of twist knots

The main actors in the remaining sections are the twist knots. Given $n \in \mathbb{N}$, we will denote by K_n the twist knot with n full right hand twists, as depicted in Figure 6.1 (see also Definition 4.15).

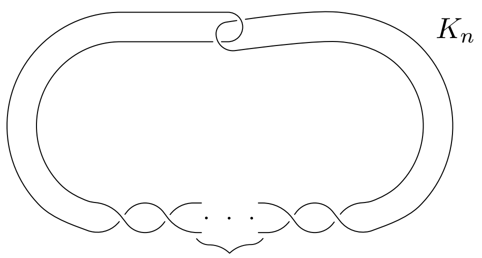


Figure 6.1: The twist knot K_n (picture taken from [Ilt19]).

The double branched cover of K_n is the lens space $L(4n+1, 2)$, with $\mathbb{Q}/\mathbb{Z}$-(co-)homology

$$H_1(X_2;\mathbb{Q}/\mathbb{Z}) \cong H^1(X_2;\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}_{4n+1}.$$

[1] There is also a result about Casson-Gordon τ-signatures for knots with higher genus, see [Gil83, Theorem 3.4]. However, this result yields in general only an inequality for τ-signatures.

Given $\chi \in H^1(X_2;\mathbb{Q}/\mathbb{Z})$ of prime-power order, there is the Casson-Gordon invariant $\tau(K_n,\chi)$ and the Casson-Gordon τ-signature $\operatorname{sign}_1^{\mathrm{av}}(\tau(K_n,\chi))$. Since $g(K_n) = 1$ for all $n \in \mathbb{N}$, we can use Gilmer's formula (see Theorem 6.1) to compute the τ-signatures of the twist knots. We would like to note at this point that the computations in the current section have already appeared in the literature previously in greater generality [Kim05]. Moreover, they were also already made by the author in [Ilt19]. However, for the sake of completeness and because we use slightly different conventions, we chose to perform the computations once more.

Let F_n be the genus-one Seifert surface for K_n with a and b as a basis for the first homology as shown in Figure 6.2.

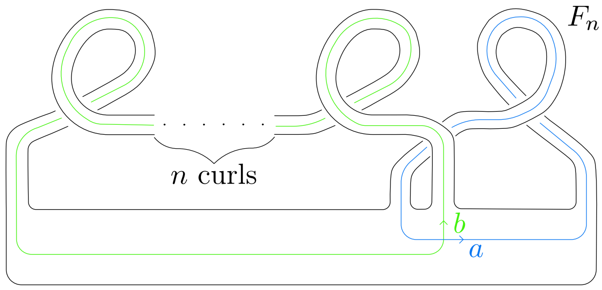


Figure 6.2: The Seifert surface F_n for K_n with a basis for $H_1(F_n;\mathbb{Z})$ (picture taken from [Ilt19]).

In this setting, the Seifert matrix of K_n takes the form

$$A_n = \begin{pmatrix} -1 & 1 \\ 0 & n \end{pmatrix}$$

We are interested in finding generators for $\ker(\varepsilon \otimes \mathrm{id}_{\mathbb{Q}/\mathbb{Z}}) \cong H^1(X_2;\mathbb{Q}/\mathbb{Z})$. Since $H^1(X_2;\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}_{4n+1}$ is a finite cyclic group of order $4n+1$, any element of order $4n+1$ in this kernel will form a generating set. Let $x = (1,2)^\top \in H_1(F_n;\mathbb{Z})$. We claim that the element

$$x \otimes \frac{1}{4n+1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \frac{1}{4n+1} \in H_1(F;\mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z}$$

is of order $4n+1$ and contained in $\ker(\varepsilon \otimes \mathrm{id}_{\mathbb{Q}/\mathbb{Z}})$. The former assertion is clear. To see the latter, note that given any $y = (y_1,y_2)^\top \in H_1(F_n;\mathbb{Z})$,

$$\varepsilon_x(y) = x^\top A_n y + y^\top A_n x = (4n+1)y_2,$$

showing that $(\varepsilon \otimes \mathrm{id}_{\mathbb{Q}/\mathbb{Z}})(x \otimes \frac{1}{4n+1})$ is zero in $H^1(F_n;\mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z}$.

The next step consists of representing $x = (1,2)^\top$ as a simple closed curve on F_n and determining what knot J_x this curve represents in S^3. Figure 6.3 below shows this process. It turns out that the element x represents a $(2,2n+1)$-torus knot in S^3, i.e. $J_x = T(2,2n+1)$.

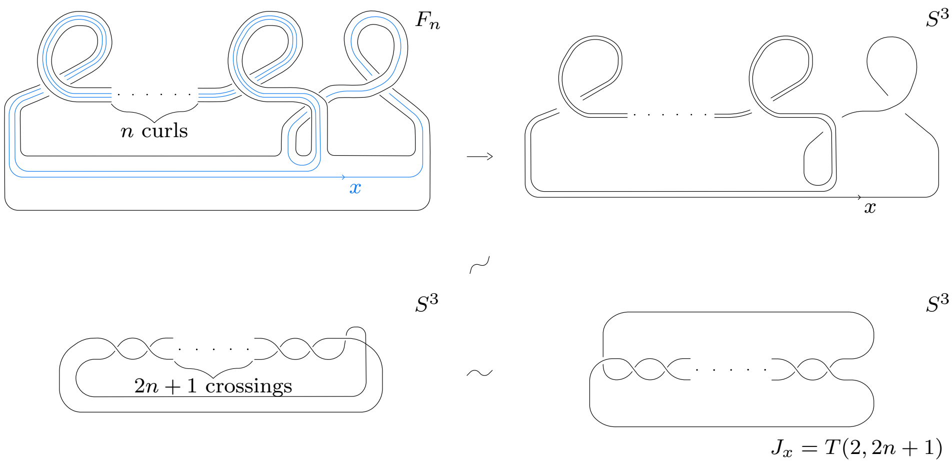


Figure 6.3: The element $x = (1, 2)^\top \in H_1(F_n; \mathbb{Z})$ represents a $(2, 2n+1)$-torus knot in S^3 (picture taken from [Ilt19]).

We now have all ingredients to compute the Casson-Gordon τ-signatures from Theorem 6.1. Assume for the moment that $4n + 1$ is a power of a prime, so that every element in $H^1(X_2; \mathbb{Q}/\mathbb{Z})$ has prime-power order. Given

$$\chi = x \otimes \frac{s}{4n+1} \in H^1(X_2; \mathbb{Q}/\mathbb{Z}), \quad 0 < s < 4n + 1,$$

we get

$$\begin{aligned}\operatorname{sign}_1^{\mathrm{av}}(\tau(K_n, \chi)) &= \sigma_{s/(4n+1)}(T(2, 2n+1)) + \frac{4((4n+1)-s)s}{(4n+1)^2}\theta(x, x) + \sigma(K_n) \\ &= \sigma_{s/(4n+1)}(T(2, 2n+1)) + \frac{4((4n+1)-s)s}{4n+1}\end{aligned}$$

since $\theta(x, x) = 4n + 1$ and $\sigma(K_n) = 0$ for all $n \in \mathbb{N}$. The Levine-Tristram signatures of $(2, 2n + 1)$-torus knots are well-known and readily computed (see for instance [Lit79]). Thus overall,

$$\operatorname{sign}_1^{\mathrm{av}}(\tau(K_n, \chi)) = \begin{cases} -4\left\lceil \frac{s}{2} \right\rceil + \frac{4((4n+1)-s)s}{4n+1}, & s = 1, \dots, 2n \\ -4\left\lceil \frac{4n+1-s}{2} \right\rceil + \frac{4((4n+1)-s)s}{4n+1}, & s = 2n+1, \dots, 4n. \end{cases} \tag{6.1}$$

Table 6.1 shows the Casson-Gordon τ-signatures for the twist knot K_6. Notice the symmetry of the values about 12.

Suppose now that $4n + 1$ is a general number, not necessarily a prime-power. It is still true that the right-hand side of the equation for the Casson-Gordon τ-signature given by Theorem 6.1 is equal to the right-hand side of (6.1) for twist knots. However, the whole equation (6.1) only holds if the corresponding character χ has prime-power order. So the best way to get information about the Casson-Gordon τ-signature of an arbitrary twist knot is to study the right-hand side of (6.1), and keep in mind that it

Element	Signature	Element	Signature
0	0	13	0.96
1	−0.16	14	0.64
2	3.36	15	4
3	2.56	16	3.04
4	5.44	17	5.76
5	4	18	4.16
6	6.24	19	6.24
7	4.16	20	4
8	5.76	21	5.44
9	3.04	22	2.56
10	4	23	3.36
11	0.64	24	−0.16
12	0.96		

Table 6.1: Elements in $H^1(X_2;\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}_{25}$ and their corresponding Casson-Gordon τ-signatures for the twist knot K_6.

is equal to $\operatorname{sign}_1^{\mathrm{av}}(\tau(K_n,\chi))$ if and only if the corresponding character has prime-power order.

In general, the values of the two formulas for $\operatorname{sign}_1^{\mathrm{av}}(\tau(K_n,\chi))$ in (6.1) are symmetric about $2n$. Thus, it suffices to consider only the first formula and take symmetry into account. We will do so in the following section.

6.3 Lower bound for the stable 4-genus of twist knots

We proceed by computing the lower bound for $g_{st}(K_n)$ from our results in Chapter 5. Let p be a prime dividing $4n+1$. Since $H^1(X_2;\mathbb{Q}/\mathbb{Z})$ is finite cyclic, $H^1(X_2;\mathbb{F}_p)$ is one-dimensional, hence there is only one number

$$L_1 = \sum_{\chi \in H^1(X_2;\mathbb{F}_p)} \operatorname{sign}_1^{\mathrm{av}}(\tau(K,\chi)).$$

In particular, $|L_1| = L$. Observe that if p appears with exponent k in the prime-decomposition of $4n+1$, then the elements of $H^1(X_2;\mathbb{F}_p)$, considered as elements in $\mathbb{Z}_{4n+1}$, are $0, q, 2q, \ldots, (p-1)q$ with $q = (4n+1)/p$. Then, using the symmetry of the Casson-Gordon τ-signatures for the twist knots,

$$\begin{aligned}
L_1 &= 2\sum_{s=1}^{(p-1)/2} -4\left\lceil \frac{sq}{2} \right\rceil + \frac{4((4n+1)-sq)sq}{4n+1} \\
&= q(p-1)(p+1) - \left(\frac{pq^2(p-1)(p+1)}{3(4n+1)}\right) - \sum_{s=1}^{(p-1)/2} 8\left\lceil \frac{sq}{2} \right\rceil \qquad (6.2) \\
&= \frac{2}{3}q(p^2-1) - \sum_{s=1}^{(p-1)/2} 8\left\lceil \frac{sq}{2} \right\rceil .
\end{aligned}$$

To evaluate the remaining sum, we distinguish two cases. If $(p-1)/2$ is even, then

$$\begin{aligned}\sum_{s=1}^{(p-1)/2} 8\left\lceil\frac{sq}{2}\right\rceil &= \sum_{i=1}^{(p-1)/4} 8\Big(iq + \frac{(2i-1)q+1}{2}\Big)\\ &= (1-q)(p-1) + \frac{q(p-1)(p+3)}{2}\\ &= \frac{1}{2}(p-1)(pq+q+2).\end{aligned}$$

On the other hand, if $(p-1)/2$ is odd, then

$$\begin{aligned}\sum_{s=1}^{(p-1)/2} 8\left\lceil\frac{sq}{2}\right\rceil &= \Big(\sum_{i=1}^{(p-3)/4} 8\Big(iq + \frac{(2i-1)q+1}{2}\Big)\Big) + 4q\Big(\frac{p-1}{2}+1\Big)\\ &= \frac{q(p-3)(p+1)}{2} + (p-3)(1-q) + 2q(p-1) + 8\\ &= \frac{1}{2}(p^2q + 2p - q + 2).\end{aligned}$$

Putting this into the equation (6.2) above and simplifying further, we overall obtain

$$L_1 = \begin{cases} \frac{1}{6}(p-1)(pq+q-6), & (p-1)/2 \text{ even}\\ \frac{1}{6}(p^2q-6p-q-6), & (p-1)/2 \text{ odd}\ . \end{cases}$$

Note that $p=2$ cannot occur since $4n+1$ is always odd, so we covered all cases above. It is not hard to see that $L_1 \geq 0$ for all primes $p \geq 3$. Moreover, all Levine-Tristram signatures of twist knots vanish, hence we can apply Theorem 5.5 and obtain the following.

Corollary 6.3. Let K_n be the twist knot with $n \in \mathbb{N}\backslash\{0,2\}$ full right hand twists and p a prime dividing $4n+1$. Then

$$g_{st}(K_n) \geq \begin{cases} \frac{(pq+q-6)}{2(pq+q+18)}, & (p-1)/2 \text{ even}\\ \frac{p^2q-6p-q-6}{2(p^2q+18p-q-30)}, & (p-1)/2 \text{ odd}\ , \end{cases}$$

where $q = (4n+1)/p$.

Remark 6.4. One can check from the previous computations that K_0, K_1 and K_2 are the only twist knots with $L=0$. This means that a non-trivial lower bound for g_{st} can be obtained from Corollary 6.3 (resp. Theorem 5.5) for any twist knot K_n with $n \geq 3$. The unknot K_0 and the Stevedore knot K_2 are slice, and the figure-eight K_1 represents torsion in the knot concordance group $\mathcal{C}$. Thus we obtain the following corollaries.

Corollary 6.5. Let K_n with $n \in \mathbb{N}$ be any twist knot. Then

$$K_n \text{ is torsion in } \mathcal{C} \iff g_{st}(K_n) = 0.$$

Corollary 6.6. K_n is of infinite order in $\mathcal{C}$ except for $n = 0, 1, 2$.

For arbitrary knots K, it is an open question whether $g_{st}(K) = 0$ implies that K is torsion in the knot concordance group $\mathcal{C}$.

It is important to note that the strength of the lower bound in Corollary 6.3 depends on the choice of prime factor p; a priori, there is no preferred choice of p to obtain the strongest bound. In order to obtain the best result, one has to compute the lower bound for all primes p in the prime decomposition of $4n + 1$ and then compare. In Table 6.2 below we have computed the lower bounds given by Corollary 6.3 for the twist knots K_5, K_{11}, K_{16}, K_{21} and K_{400}.

$K_n \setminus p$	3	5	7	13	17	89	$4n+1$
K_5	1/5	0	1/5	0	0	0	$21 = 3 \cdot 7$
K_{11}	1/3	1/3	0	0	0	0	$45 = 5 \cdot 9$
K_{16}	0	3/8	0	4/11	0	0	$65 = 5 \cdot 13$
K_{21}	0	2/5	0	0	7/18	0	$85 = 5 \cdot 17$
K_{400}	22/45	0	0	0	0	67/138	$801 = 9 \cdot 89$

Table 6.2: Examples of lower bounds for the stable 4-genus of twist knots obtained from Theorem 5.5.

While the bounds in Corollary 6.3 are directly obtained from the main theorem and are the strongest that we currently know, they are not particularly easy to grasp. By estimating further we obtain a weaker result that holds for all twist knots simultaneously and is easier to grasp.

Corollary 6.7 (Corollary 6.3, weakened). Let K_n with $n \in \mathbb{N}$ be any twist knot. Then

$$g_{st}(K_n) \geq \frac{1}{2} - \frac{6}{2n+7}.$$

Proof. The result is obtained by estimating the two bounds given in Corollary 6.3. Recall that p is a prime dividing $4n+1$ and $q = (4n+1)/p$. Observe that $3 \leq p \leq 4n+1$, $1 \leq q \leq (4n+1)/3$, and $pq = 4n+1$.

1.) In the first case of Corollary 6.3,

$$\begin{aligned} g_{st}(K_n) &\geq \frac{pq+q-6}{2(pq+q+18)} \\ &= \frac{pq+q+18}{2(pq+q+18)} - \frac{24}{2(pq+q+18)} \\ &= \frac{1}{2} - \frac{12}{4n+q+19} \\ &\geq \frac{1}{2} - \frac{3}{n+5}. \end{aligned} \tag{6.3}$$

The last inequality is obtained by estimating q from below with 1.

2.) In the second case of Corollary 6.3,

$$
\begin{aligned}
g_{st}(K_n) &\geq \frac{p^2q - 6p - q - 6}{2(p^2q + 18p - q - 30)} \\
&= \frac{1}{2} - \frac{12(p-1)}{p^2q + 18p - q - 30} \\
&= \frac{1}{2} - \frac{12(p-1)}{(4n+1)p + 18p - q - 30} \\
&= \frac{1}{2} - \frac{12}{4n + q + 19 - \frac{12}{p-1}} \\
&\geq \frac{1}{2} - \frac{6}{2n+7}.
\end{aligned} \tag{6.4}
$$

The third equality is obtained by using

$$
\begin{aligned}
(4n+1)p + 18p - q - 30 &= (4n+1)p + 18p - 4n + 1 - 18 \\
&\quad + pq - q - 12 \\
&= (p-1)\Big(4n + 1 + q + 18 - \frac{12}{p-1}\Big),
\end{aligned}
$$

while the last inequality is obtained by estimating q from below with 1 and $12/(p-1)$ from above with $12/2$ by using $p \geq 3$.

Comparing the lower bounds obtained in (6.3) and (6.4), we see that

$$\frac{1}{2} - \frac{3}{n+5} \geq \frac{1}{2} - \frac{6}{2n+7}$$

for all $n \in \mathbb{N}$, and the result follows. □

It is immediate from Corollary 6.7 that for growing n, the bound tends towards $1/2$. This implies that the stronger bounds in Corollary 6.3 also tend towards $1/2$ as n grows, since $1/2$ is an upper bound for the lower bound in the main theorem. We will use this fact in the next section to show that there exists an infinite subfamily of twist knots with stable 4-genus close to but not greater than $1/2$.

Our next result is concerned with the linear independency of the twist knots in the knot concordance group $\mathcal{C}$. We provide a partial answer by applying the discussion leading to Proposition 5.7 at the end of Chapter 5. For this we make the following definition.

Definition 6.8. Let $\{J_i\}_{i\in I}$ be a finite family of knots for some finite index set $I \subset \mathbb{N}$. Then we say that $\{J_i\}_{i\in I}$ is *positively linearly independent in* $\mathcal{C}$ if the sliceness of any connected sum

$$m_1 J_{i_1} \# m_2 J_{i_2} \# \cdots \# m_j J_{i_k},$$

where $i_1, i_2, \ldots, i_k \in I$ and $m_1, m_2, \ldots, m_k \in \mathbb{N}$, implies that $m_1 = m_2 = \cdots = m_k = 0$.

Corollary 6.9. Let $\{K_n\}_{n\in I}$ be a finite family of twist knots for some finite index set $I \subset \mathbb{N}\backslash\{0,1,2\}$. Then $\{K_n\}_{n\in I}$ is positively linearly independent in the knot concordance group $\mathcal{C}$.

Proof. Throughout the proof, we use the notation from Proposition 5.7. Consider the connected sum

$$K \coloneqq m_1 K_{n_1} \# m_2 K_{n_2} \# \cdots \# m_k K_{n_k},$$

where $n_1, n_2, \ldots, n_k \in I$ and $m_1, m_2, \ldots, m_k \in \mathbb{N}$. Fix some n_i and let p be a prime dividing $4n_i + 1$. Then we know that $H^1(X_2(K_{n_i}), \mathbb{F}_p)$ is one-dimensional, so there exists only one rational number $L_1(K_{n_i})$. Our computations at the beginning of Section 6.3 showed that

$$L_1(K_{n_i}) = \begin{cases} \frac{1}{6}(p-1)(pq+q-6), & (p-1)/2 \text{ even} \\ \frac{1}{6}(p^2q-6p-q-6), & (p-1)/2 \text{ odd }, \end{cases}$$

where $q = (4n_i + 1)/p$, and in particular $L_1(K_{n_i}) > 0$. Moreover, if K_{n_j} is any other twist knot in the connected sum K for which $H^1(X_2(K_{j_l}); \mathbb{F}_p)$ is non-trivial, the same argument applies to show that $L_1(K_{n_j}) > 0$.

Let $K_{j_1}, K_{j_2}, \ldots, K_{j_h}$ be the twist knots in the sum K for which $H^1(X_2(K_{j_l}); \mathbb{F}_p)$ is non-trivial. Then we know that the rational numbers $L_1(K), \ldots, L_{m(K)}(K)$ consist of sums of the numbers $L_1(K_{j_1}), \ldots, L_1(K_{j_h})$ which by the above are strictly positive. Hence $L(K) > 0$ and we can apply Proposition 5.7 to obtain

$$g_{st}(K) \geq \frac{L(K)}{8(p-1) + 2L(K)} \sum_{i=1}^{h} m_{j_i} \tag{6.5}$$

If K is slice then the lower bound in (6.5) is necessarily zero, and this is the case if and only if $m_{j_1} = m_{j_2} = \cdots = m_{j_h} = 0$. Replicating the above argument (if necessary) for other choices of p then shows that K is slice if and only if $m_1 = m_2 = \cdots = m_k = 0$, proving that $\{K_n\}_{n\in I}$ is positively linearly independent in the knot concordance group $\mathcal{C}$.

□

Remark 6.10. Of course, the exact value of the bound in (6.5) depends on the choice of fixed n_i and prime p dividing $4n_i + 1$. This means that the bound may be improved with a different choice of n_i or p.

6.4 Twists knots with stable 4-genus close to but not greater than 1/2

A recent result by Baader and Lewark [BL17] implies that $g_{st}(K_n) \leq 2/3$ for any $n \in \mathbb{N}$ (see [BL17, Lemma 5]). The idea is to take the three-fold connected sum $3K_n$ with Seifert surface 3Σ, where Σ is a genus-minimal Seifert surface for K_n, and find a subgroup of rank two of $H_1(3\Sigma;\mathbb{Z})$ on which the Seifert form has the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & c \end{pmatrix} \tag{6.6}$$

for some $c \in \mathbb{Z}$. As the proof in [BL17, Lemma 5] shows, one can always find such a subgroup when taking at least three copies of K_n. In this setting, Baader and Lewark show that one can achieve a situation in which Freedman's disc theorem [Fre84, FQ90] can be applied to obtain $g_4(3K_n) \leq g(3K_n) - 1 = 2$, implying that $g_{st}(K_n) \leq 2/3$.

A natural question that arises is under what conditions one can find such a subgroup of rank two starting with two copies of the knot instead of three. Thus, let K_n be any twist knot with its standard genus-one Seifert surface Σ and Seifert matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -n \end{pmatrix}.$$

The two-fold connected sum $2K_n$ has then 2Σ as a Seifert surface with Seifert matrix the block sum $\overline{A} := A \oplus A$. Consider the vectors

$$v = (1,0,x,y)^\top,\ w = (0,1,0,0)^\top \in H_1(2\Sigma;\mathbb{Z}),$$

where $x, y \in \mathbb{Z}$ are yet to be found. Independent of the choice of x and y, we have

$$v^\top \overline{A} w = 1,\ w^\top \overline{A} v = 0,\ w^\top \overline{A} w = -n.$$

Thus, it remains to find $x, y \in \mathbb{Z}$ such that $v^\top \overline{A} v = 0$; that is

$$v^\top \overline{A} v = x^2 + xy - ny^2 + 1 = 0.$$

Similar to the proof of [BL17, Lemma 4], we complete the square to obtain

$$\begin{aligned} x^2 + xy - ny^2 &= \left(x + \frac{y}{2}\right)^2 - (4n+1)\left(\frac{y}{2}\right)^2 \\ &= \overline{x}^2 - (4n+1)\overline{y}^2, \end{aligned}$$

where in the last equation we substituted $\overline{y} = y/2$ and $\overline{x} = x + \overline{y}$. We obtain

$$v^\top \overline{A} v = 0 \iff \overline{x}^2 - (4n+1)\overline{y}^2 = -1.$$

The equation $\overline{x}^2 - (4n+1)\overline{y}^2 = -1$ is generally known as a negative Pell equation. If this equation has a solution $\overline{x}, \overline{y} \in \mathbb{Z}$, then setting $x = \overline{x} - \overline{y}$ and $y = 2\overline{y}$ in v gives us a vector such that $v^\top \overline{A} v = 0$. It follows that the Seifert form of $2K_n$ restricted to the rank-two subgroup spanned by v and w is of the form

$$\begin{pmatrix} 0 & 1 \\ 0 & c \end{pmatrix}$$

for some $c \in \mathbb{Z}$. Proceeding as in the proof of [BL17, Lemma 5], we obtain

$$g_4(2K_n) \leq 1 \implies g_{st}(K_n) \leq \frac{1}{2},$$

provided that for $4n+1$, the negative Pell equation has an integer valued solution. We summarize the above observations in the following proposition.

Proposition 6.11. Let $n \in \mathbb{N}$ be such that the negative Pell equation $x^2 - (4n+1)y^2 = -1$ has a solution $x, y \in \mathbb{Z}$. Then

$$g_{st}(K_n) \leq \frac{1}{2}. \qquad \square$$

As mentioned in the introduction, a necessary and sufficient condition for the existence of a solution of the negative Pell equation is that the continued fraction of $\sqrt{4n+1}$ has odd period length [RT04]. This is the case, for example, if $4n+1 = p^k$, where p is a prime such that $p \equiv 1 \mod 4$ and $k \in \mathbb{N}$ [RT04]. This yields, together with the lower bound given in Corollary 6.3, the infinite subfamily of twist knots with g_{st} close to but not greater than 1/2.

Note that in the approach above, we specified two explicit vectors v and w and derived a sufficient condition for them to span a rank-two subgroup on which the Seifert matrix has the desired form as in (6.6). A natural question to ask is under what circumstances such a subgroup exists in general. Indeed, although the solvability of the negative Pell equation is a sufficient condition, it is not necessary. For example, if $n = 51$, then $4n+1 = 205$, but $x^2 - 205y^2 = -1$ has no solution. Yet, the Seifert form of $2K_{51}$ restricted to the rank-two subgroup spanned by the vectors $(13, 2, 3, 0)^\top$ and $(14, 2, -2, 1)^\top$ gives the desired matrix (6.6), so $g_{st}(K_{51}) \leq 1/2$.

More generally, there might also be other, different methods to obtain the upper bound given by 1/2. We do not know of a full characterization of twist knots for which $g_{st} \leq 1/2$ holds.

Part II

Quantum Invariants: *Khovanov homology and the λ-invariant*

Chapter 7

Introduction and Results

In 1999, M. Khovanov introduced a categorification of the Jones polynomial that takes the form of a homology theory and is a link invariant [Kho00]. This theory is now known as Khovanov homology, and it soon became apparent that it is a strictly stronger invariant than the Jones polynomial [BN02]. More so, ever since its discovery Khovanov homology and its variations has served as a fruitful source to obtain geometric information about the underlying knot or link.

Arguably the most famous piece of information obtained from Khovanov homology is J. Rasmussen's s-invariant [Ras10]. It induces a homomorphism from the knot concordance group $\mathcal{C}$ to the integers and provides a lower bound for the slice genus g_4 of a knot. The Rasmussen s-invariant can be computed as follows: there is a variation of Khovanov homology, known as Lee homology [Lee05], that is the limit of the so-called Lee spectral sequence which starts at Khovanov homology. The Lee homology of a knot is always of rank 2, and the filtration degree of the two generators may be used to compute the Rasmussen s-invariant.

Recently, Alishahi and Dowlin [AD19] used Lee homology in a new and different way to obtain geometric information. They discovered a knot invariant u_X taking non-negative integer values that yields a lower bound for the unknotting number u of a knot K. Interestingly, this number behaves rather differently than the s-invariant: it does not induce a concordance homomorphism, and it is not additive under connected sum of knots. Inspired by their methods, the author defined in joint work with L. Lewark and L. Marino in [ILM21] a new invariant λ on yet another variation of Khovanov homology: $\mathbb{Z}[G]$-homology. Let's take a closer look.

7.1 A simple universal Khovanov homology

A *Frobenius system* is a tuple $\mathcal{F} = (R, A, \Delta, \varepsilon)$ consisting of a commutative ring R, a commutative algebra A, a cocommutative comultiplication $\Delta\colon A \to A \otimes_R A$, and a counit $\epsilon\colon A \to R$ such that $\Delta \circ m =$

$(\mathrm{Id} \otimes m) \circ (\Delta \otimes \mathrm{Id})$. The algebra A of a Frobenius system is called a *Frobenius algebra*. In the present work we will only consider so-called *rank two Frobenius systems*, i.e. Frobenius systems $\mathcal{F}$ with an $X \in A$ such that A is freely generated by 1 and X as an R-module. Moreover, all our Frobenius algebras will be equipped with a filtration or a grading, such that 1 and X are homogeneous elements of degree 0 and -2, respectively. We call this the *quantum grading*.

It is well known that every rank two Frobenius system $\mathcal{F}$ yields a variation of Khovanov homology, i.e. a way to associate to all diagrams D of a link L a chain complex $C_{\mathcal{F}}(D)$ of free R-modules, well-defined up to homotopy equivalence for different diagrams of L, thus giving $C_{\mathcal{F}}(L)$ [Kho06]. For links with a marked component or for knots, there is an action of A on $C_{\mathcal{F}}(D)$ which is well-defined up to homotopy, so we may consider $C_{\mathcal{F}}(D)$ as a chain complex of free A-modules.

Khovanov's original homology theory corresponds to the Frobenius algebra $\mathbb{Z}[X]/(X^2)$ over $\mathbb{Z}$. On the other hand, the theory coming from the Frobenius algebra $A_{\mathrm{univ}} = R_{\mathrm{univ}}[X]/(X^2 - hX - t)$ over $R_{\mathrm{univ}} = \mathbb{Z}[h, t]$ is called *universal* since for all rank two Frobenius algebras $\mathcal{F}$, the chain complex $C_{\mathcal{F}}(D)$ is determined by $C_{\mathrm{univ}}(D)$ [Kho06]. Recently, Khovanov and Robert defined another theory called α-homology, which is also universal in the sense above [KR22]. To define λ, we will use a third universal theory, which we call $\mathbb{Z}[G]$-homology. The universality of this theory is due to Naot [Nao06, Nao07]. This $\mathbb{Z}[G]$-theory associates to a diagram D of a knot K the *reduced* Khovanov chain complex coming from the Frobenius algebra $R[X]/(X^2+GX)$ with $R = \mathbb{Z}[G]$. We denote this chain complex by $[\![D]\!]$ (well-defined up to isomorphism) or $[\![K]\!]$ (well-defined up to homotopy equivalence). Our reason to use $\mathbb{Z}[G]$-homology is that it is the simplest of the three mentioned universal theories, in the sense that the ground ring is a polynomial ring in only one, instead of two variables. Let us explicitly state how $\mathbb{Z}[G]$-homology determines $\mathcal{F}_{\mathrm{univ}}$-homology (this is implicit in the work of Naot [Nao06, Nao07]).

Theorem 7.1. Endow $A_{\mathrm{univ}} = \mathbb{Z}[h, t][X]/(X^2 - hX - t)$ with the structure of a $\mathbb{Z}[G]$-module by letting G act as $2X - h$. Then for every oriented link with base point,

$$C_{\mathrm{univ}}(L) \simeq [\![L]\!] \otimes_{\mathbb{Z}[G]} A_{\mathrm{univ}}\{1\}.$$

Here, $\{\cdot\}$ denotes a shift in quantum degree.

Corollary 7.2. For every knot K, $C_{\mathrm{univ}}(K)$ is homotopy equivalent to a chain complex of free shifted A_{univ}-modules, with differentials consisting only of integer multiples of powers of $2X - h$.

Theorem 7.1 and Corollary 7.2 can be understood to say that $\mathbb{Z}[G]$-homology encodes the same amount of information present in $\mathcal{F}_{\mathrm{univ}}$- and α-homology in a more compact way. In particular, the original reduced Khovanov homology over $\mathbb{Z}$ of K as defined in [Kho03] may be obtained from $[\![K]\!]$ simply by setting $G = 0$, i.e. by tensoring with $\mathbb{Z}[G]/(G) \cong \mathbb{Z}$. The original unreduced Khovanov homology over $\mathbb{Z}$ is also determined by $\mathbb{Z}[G]$-homology, see Corollary 9.10.

Let us give three examples of $\mathbb{Z}[G]$-complexes $[\![\cdot]\!]$ of knots. For the unknot U, $[\![U]\!]$ is simply homotopy equivalent to one copy of $\mathbb{Z}[G]$ supported in homological degree 0. For the trefoil $T_{2,3}$, we have a homotopy equivalence

$$[\![T_{2,3}]\!] \simeq {}_0\mathbb{Z}[G]\{2\} \longrightarrow 0 \longrightarrow \mathbb{Z}[G]\{6\} \xrightarrow{G} \mathbb{Z}[G]\{8\},$$

where the subscript to the left denotes homological degree. Finally, for the $T_{3,4}$ torus knot we have

$$[\![T_{3,4}]\!] \simeq$$

$${}_0\mathbb{Z}[G]\{6\} \to 0 \to \mathbb{Z}[G]\{10\} \xrightarrow{G} \mathbb{Z}[G]\{12\} \xrightarrow{0} \mathbb{Z}[G]\{12\} \xrightarrow{G^2} \mathbb{Z}[G]\{16\}.$$

We will show how to compute $[\![T_{2,3}]\!]$ in Example 9.3. The complex $[\![T_{3,4}]\!]$ may be computed using the same methods, or by using the computer programs `khoca` [LL18] and `homca` [Ilt21a], see Section 7.7 and Subsection 11.3.1.

7.2 The definition of λ

Having an understanding of $\mathbb{Z}[G]$-homology, we are ready to introduce the previously mentioned new knot invariant λ.

Definition 7.3. For a knot K, let $\lambda(K)$ be the minimal integer $k \geq 0$ such that there exist ungraded chain maps (i.e. chain maps that do not need to respect the homological or the quantum degree, see Definition 10.1)

$$[\![K]\!] \underset{g}{\overset{f}{\rightleftarrows}} [\![U]\!]$$

such that $g \circ f$ and $f \circ g$ are homotopic to multiplication with G^k:

$$g \circ f \simeq G^k \cdot \mathrm{id}_{[\![K]\!]}, \quad f \circ g \simeq G^k \cdot \mathrm{id}_{[\![U]\!]}.$$

It is not obvious that for a given knot, f, g and k as in Definition 7.3 exist at all. So for the time being we simply set $\lambda(K) = \infty$ if they do not, but it will be a consequence of our Theorem 7.4 that this case does in fact not occur. To get acquainted with calculating λ, the reader is invited to convince themself that $\lambda(U) = 0$, $\lambda(T_{2,3}) = 1$, and $\lambda(T_{3,4}) = 2$.

As mentioned previously, the definition of λ is based on the work of Alishahi and Dowlin, who use analogous maps f, g in the proof that their invariant $\mathfrak{u}_X$ is a lower bound for the unknotting number u. The invariant

$\mathfrak{u}_X(K)$ is defined as the maximal X-torsion order of the homology of K coming from the Frobenius algebra $\mathcal{F}_{\text{AD}} = \mathbb{Q}[X,t]/(X^2 - t)$ over $\mathbb{Q}[t]$, i.e. the minimal n such that $X^n H_{\mathcal{F}_{\text{AD}}}(K)$ is torsion-free. Note that setting $t = 1$ in $\mathcal{F}_{\text{AD}}$ yields the system $\mathcal{F}_{\text{Lee}} = \mathbb{Q}[X]/(X^2 - 1)$ over $\mathbb{Q}$, which is used to obtain Lee homology.

At first glance the definition of $\mathfrak{u}_X$ and λ appear to be rather different; but on a closer look, one finds that $\mathfrak{u}_X(K) = \lambda_X(K)$, where $\lambda_X(K)$ is the minimal $k \geq 0$ such that there exist ungraded chain maps

$$C_{\mathcal{F}_{\text{AD}}}(K) \underset{g}{\overset{f}{\rightleftarrows}} C_{\mathcal{F}_{\text{AD}}}(U)$$

and homotopies

$$g \circ f \simeq (2X)^k \cdot \text{id}_{C_{\mathcal{F}_{\text{AD}}}(K)}, \quad f \circ g \simeq (2X)^k \cdot \text{id}_{C_{\mathcal{F}_{\text{AD}}}(U)}.$$

In this sense, λ is a direct generalization of $\mathfrak{u}_X$, obtained from the reduced homology coming from the Frobenius algebra $\mathcal{F}_{\mathbb{Z}[G]}$ instead of from the unreduced homology coming from $\mathcal{F}_{\text{AD}}$. But why don't we instead of λ consider $\mathfrak{u}_G(K)$, defined as the maximal G-torsion order of $\mathbb{Z}[G]$-homology of K (see Definition 10.23)? There are two reasons. First, λ is not equal to $\mathfrak{u}_G$ for all knots; the proof of the equality $\lambda_X = \mathfrak{u}_X$ does not carry over from $\mathbb{Q}[X,t]/(X^2-t)$ to $\mathbb{Z}[G]$, because it relies on $\mathbb{Q}[X,t]/(X^2-t) \cong \mathbb{Q}[X]$ being a PID, which $\mathbb{Z}[G]$ is not. In fact, $\lambda(K) \geq \mathfrak{u}_G(K)$ holds for all knots K (cf. Lemma 10.28). Second, $\mathfrak{u}_G$ displays some unfavorable behavior; for example, the value of $\mathfrak{u}_G(-K)$ is not determined by the value of $\mathfrak{u}_G(K)$, where $-K$ denotes the mirror image of K. Again, the ring $\mathbb{Z}[G]$ not being a PID is to blame for this.

7.3 Main results

Our main results about λ are the following.

Theorem 7.4. For all knots K, one has $\lambda(K) \leq u_q(K)$.

Theorem 7.5. For every $n \in \mathbb{N}$ there exists a knot K such that $\lambda(K) = n$.

Here, $u_q(K)$ denotes the proper rational unknotting number, which is defined as follows.

Definition 7.6. Two knots K and K' are related by a *rational replacement* if K' may be obtained from K by replacing a rational tangle T in K with another rational tangle T'. If the arcs of T and T' connect the same tangle end points, we say that the rational replacement is *proper*. Now, $u_q(K)$ is defined as the minimal number of proper rational replacements relating K to the unknot.

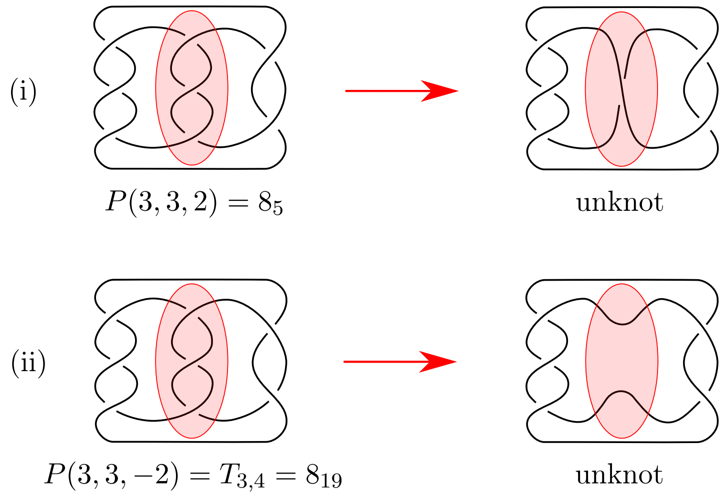


Figure 7.1: In (i), an example of a proper rational replacement (1/3 by −1 in the language of Definition 12.15), showing that the $P(3,3,2)$ pretzel knot has proper rational unknotting number 1. In (ii), an example of a non-proper rational replacement (1/3 by 0), showing that the $P(3,3,-2)$ pretzel knot, which is also the $T_{3,4}$ torus knot, has rational unknotting number 1. Since $\lambda(T_{3,4}) = 2$, it follows from Theorem 7.4 that there is no *proper* rational replacement transforming the $T_{3,4}$ pretzel knot into the unknot, i.e. $T_{3,4}$ has proper rational unknotting number at least 2 (and in fact equal to 2).

A more refined definition of rational replacement will be given in Definition 12.15. Figure 7.1 shows examples of rational replacements, and an application of Theorem 7.4. Since a crossing change is merely a special case of a proper rational replacement, we find that $u_q(K) \leq u(K)$ holds for all knots K. So Theorem 7.4 can be seen as a strengthening of the inequality $\mathfrak{u}_X(K) \leq u(K)$ obtained by Alishahi and Dowlin in [AD19]. Indeed, we have

$$\mathfrak{u}_X(K) \leq \lambda(K) \leq u_q(K) \leq u(K). \tag{7.1}$$

We shall see that none of these inequalities are equalities, and in fact, the gaps between the four involved invariants can be arbitrarily large. Observe that Theorem 7.4 and (7.1) show that $\lambda(K)$ always exists and is finite.

In contrast to Theorem 7.5, let us note that currently $\mathfrak{u}_X(K) \leq 3$ holds for all knots K for which $\mathfrak{u}_X$ has been computed. The essentially only known knot with $\mathfrak{u}_X(K) = 3$ was recently found by Manolescu and Marengon [MM20]. Still, it seems a reasonable conjecture that $\mathfrak{u}_X$ is unbounded, though the proof of that conjecture might require more complicated knots and methods of computation than our proof of the unboundedness of λ in Theorem 7.5.

7.4 Rational replacements and rational unknotting

Rational unknotting has previously been considered by Lines [Lin96] and McCoy [McC15], and the recent work by McCoy and Zentner [MZ21] deals with proper rational unknotting. In those papers, rational unknotting is obstructed via the double branched cover, relying on the so-called Montesinos trick: if two knots K and J are related by a rational replacement, then their double branched covers M_K, M_J are related by a surgery.

As one consequence of this observation, the minimal number of generators of $H_1(M_K;\mathbb{Z})$ is a lower bound for the rational unknotting number of K. For example, this implies that the connected sum of n trefoil knots has (proper and non-proper) rational unknotting number equal to n. On the other hand, one may easily compute that λ of the connected sum of $n \geq 1$ trefoil knots equals 1. This may be taken as a first sign that our lower bound λ is quite different from the lower bounds for u_q obtainable from the double branched cover.

Note that the gap between the proper rational unknotting number u_q and the (classical) unknotting number u may also be arbitrarily high. For example, $u_q(K) = 1$ clearly holds for all two-bridge knots K; but $u(K)$ of two-bridge knots can take any value, which can e.g. be shown using the signature bound $|\sigma(K)/2| \leq u(K)$ [Mur65]. This also demonstrates that $|\sigma(K)/2|$ is not a lower bound for the proper rational unknotting number.

As an aside, let us also remark that in the definition of the proper rational unknotting number $u_q(K)$, the proper rational replacements relating K and the unknot are sequential: happening one after the other. However, by a standard transversality argument (see e.g. [Sch85]) one can show that for every knot K, there exist $u_q(K)$ many simultaneous rational replacements, i.e. rational replacements taking place in pairwise disjoint balls.

7.5 Further properties and generalizations of λ

Let us now state further properties of our invariant λ. First and foremost, λ is a unknot detector, which is a direct consequence of Kronheimer and Mrowka's result that Khovanov homology detects the unknot [KM11].

Proposition 7.7. The λ-invariant detects the unknot, i.e. $\lambda(K) = 0$ holds if and only if K is the unknot.

Given a connected sum of knots $K\#J$, we will see that the value of $\lambda(K\#J)$ is not determined by the values of $\lambda(K)$ and $\lambda(J)$. However, we can say the following.

Proposition 7.8.

1. $\lambda(K\#J) \leq \lambda(K) + \lambda(J)$ for all knots K, J.
2. λ does not change under taking mirror images, or orientation reversal.

Let us call a knot K *thin* if its reduced integral Khovanov homology consists of free modules supported in a single δ-degree (see Section 8.11 for the definition of δ and further details).

Proposition 7.9. For all non-trivial thin knots K, we have $\lambda(K) = 1$.

In particular $\lambda(K) = 1$ holds for all non-trivial quasi-alternating knots, since those knots are thin in the above sense [MO08]. This leads to applications such as the following.

Example 7.10. In Example 10.19, we will compute $\lambda(T_{5,6}) = 3$. It follows that there is no proper rational replacement relating $T_{5,6}$ to a quasi-alternating knot (compare this to [CGL$^+$20, Example 10]).

In the definition of λ, replacing U by an arbitrary second knot J yields the definition of a function $\lambda(K, J) \geq 0$ that is symmetric and obeys the triangle inequality: $\lambda(\,\cdot\,,\,\cdot\,)$ is a pseudometric on the set of isotopy classes of knots. In fact, we can even further extend the definition of λ and define it for pairs of tangles. This leads to a pseudometric on the set of equivalence classes of tangles in a fixed ball, with fixed base point and connectivity, see Proposition 10.14. Details will be provided in Chapter 10.

7.6 A comparison of λ with previously known invariants

Alishahi and Dowlin's article [AD19] appeared at the same time as an article by Alishahi [Ali19], in which similar to $\mathfrak{u}_X$ a lower bound $\mathfrak{u}_h$ for the unknotting number was obtained using the Frobenius algebra $\mathbb{F}_2[X,h]/(X^2 + hX)$ over $\mathbb{F}_2[h]$. Then, further papers followed: Caprau-González-Lee-Lowrance-Sazdanović-Zhang generalized Alishahi and Dowlin's work for $\mathbb{Q}$ to the fields $\mathbb{F}_p$ for all odd primes p [CGL$^+$20]. Using the previously mentioned α-homology, Gujral [Guj20] defined an invariant ν which can be seen to equal our invariant $\mathfrak{u}_G$, and showed that it provides a lower bound for the ribbon distance between knots; this was a generalization of earlier work by Sarkar [Sar20]. Here, the *ribbon distance* between two smoothly concordant knots K and J is the minimal k such that there is a sequence $K = K_1, \dots, K_n = J$ of knots, such that each consecutive K_i, K_{i+1} are related by a ribbon concordance in either direction with at most k saddles. This leads to the following question (see Section 7.5 or Definition 10.2 for the definition of $\lambda(K, J)$).

Question. Is $\lambda(K, J)$ less than or equal to the ribbon distance of K and J for all pairs of knots K, J?

The previously defined invariants mentioned above will be discussed in more detail in Section 10.4. By construction, λ is greater than or equal to all of them (the price to pay is that λ is generally slightly harder to compute). Let us explicitly emphasize that this observation combined with Theorem 7.4 implies that all of those previously defined invariants are also lower bounds for the proper rational unknotting number.

Alishahi and Eftekhary applied the same construction principle that underlies λ to knot Floer homology [AE20], obtaining a lower bound for the unknotting number as well as lower bounds for other quantities, such as the minimal number of negative-to-positive crossing changes in any unknotting sequence of a knot. Further knot Floer torsion order invariants were defined by Juhász, Miller and Zemke [JMZ20], who find lower bounds for even more topological quantities, such as the bridge index, the band-unlinking number, etc. Still, the following question remains open.

Question. Is one of the knot Floer torsion order invariants a lower bound for the proper rational unknotting number?

7.7 Computations

Computations of $\mathbb{Z}[G]$-homology are theoretically possible by hand using Bar-Natan's divide-and-conquer approach [BN07]. Nevertheless, to proceed efficiently, we use the program `khoca` [LL18] (originally written for [LL16]) to compute $\mathbb{Z}[G]$-complexes of knots.[1] As input, `khoca` accepts diagrams of a knot K, e.g. in PD notation. From `khoca`'s output, one may read off a chain complex of $\mathbb{Z}[G]$-modules in the homotopy class of $[\![K]\!]$. For further simplification, `khoca`'s output may be fed into the new program `homca` [Ilt21a], which attempts to decompose $[\![K]\!]$ as a direct sum of simpler chain complexes. From these simpler pieces, one may typically calculate λ by hand. See Example 10.19 for an application of this strategy to the $T_{5,6}$ torus knot. For small knots, we find the following.

Proposition 7.11. For all knots up to 10 crossings we have $\lambda = 1$, except for the knots 8_{19}, 10_{124}, 10_{128}, 10_{139}, 10_{152}, 10_{154}, 10_{161}, where $\lambda = 2$.

7.8 Organization and overview of Part II

The remainder of Part II is organized as follows. In Chapter 8 we provide a mostly self-contained introduction to the mathematical topics and notions needed in subsequent chapters, and in particular to those already present in Chapter 7. We treat subjects such as Khovanov homology and Bar-Natan's generalization to tangles, Frobenius systems and topological quantum field theories, or the Rasmussen s-invariant. In particular, we lay the categorical framework for $\mathbb{Z}[G]$-homology and fix conventions regarding

[1] Note that `javakh` [GM05], while very fast, apparently only calculates Morrison's 'universal homology', which corresponds to $\mathbb{Q}[G]$-homology. Currently, the program `kht++` [Zib21] also only simplifies complexes over fields, not over the integers.

tangles and tangle diagrams. A detailed overview of the contents is given at the beginning of Chapter 8.

Chapter 9 is devoted to the theoretical foundations of $\mathbb{Z}[G]$-homology and contains the proof of Theorem 7.1. In Chapter 10 we provide a detailed introduction to the λ-invariant. Here, we prove Proposition 7.7 to 7.9 and 7.11, and we compute λ of the $(5, 6)$-torus knot in Example 10.19. Chapter 11 deals with calculations surrounding $\mathbb{Z}[G]$-homology and the λ-invariant. In particular, we prove Theorem 7.5 that λ can be arbitrarily big, and we provide descriptions of the author's computer programs `homca` [Ilt21a] and `tenpro` [Ilt21b] which were used throughout our work. Finally, Chapter 12 is concerned with our main Theorem 7.4 which states that λ yields a lower bound for the proper rational unknotting number of a knot.

A more detailed overview of the contents is given at the beginning of each chapter. Most of the results in Part II have appeared previously in the paper *"Khovanov homology and rational unknotting"* by the author in joint work with L. Lewark and L. Marino [ILM21]. New is Chapter 8 (aside from parts in Sections 8.3 to 8.5), Section 11.3 on the computer programs `homca` and `tenpro`, Remark 11.9 on a knot whose $\mathbb{Z}[G]$-complex potentially splits off interesting new pieces, and Lemma 12.12 and Proposition 12.13 which are generalizations of [ILM21, Lemma 5.12] and [ILM21, Lemma 5.13], respectively. In particular, Lemma 12.12 fixes a gap in the proof of [ILM21, Lemma 5.13], which is crucial to the proof of Theorem 7.4 (see also Remark 12.14). Proposition 12.13 extends [ILM21, Lemma 5.13] to positive rational tangles of arbitrary connectivity.

Chapter 8

Preliminaries

The aim of this chapter is to introduce the mathematical notions and concepts needed in later chapters. We start in Section 8.1 by recalling how to obtain the Jones polynomial of an oriented link using the (scaled) Kauffman bracket. In Section 8.2, we will introduce Khovanov homology by categorifying step-by-step the ingredients from the previous section for computing the Jones polynomial, following Bar-Natan's exposition in [BN02]. This intuitive approach is well-suited for the reader unfamiliar with Khovanov homology, as it conveys Khovanov's idea without introducing unnecessary formalism. Nonetheless, precise formalism is necessary and will be needed for our discussions about $\mathbb{Z}[G]$-homology and the invariant λ. In particular, we will work with Bar-Natan's generalization of Khovanov homology to tangles and cobordisms [BN02], and Sections 8.3 to 8.6 are devoted to the introduction thereof. More precisely, in Section 8.3 we will introduce tangles and tangle diagrams, and explain the 1:1-correspondence between oriented links with base point in S^3 and 2-ended tangles in a fixed 3-ball. Here, we also describe rational tangles which play a major role in proving that λ forms a lower bound on the proper rational unknotting number. In Section 8.4, we will introduce the necessary categorical framework in which Bar-Natan's theory is constructed, and we define the specific categories that will be used for $\mathbb{Z}[G]$-homology in Chapter 9.

A key ingredient in the construction of Khovanov homology is the specification of a *Frobenius system* (also called *Frobenius algebra*) and the corresponding *topological quantum field theory* (TQFT for short). Any such Frobenius system yields a Khovanov-type homology theory for oriented links in S^3, and we will take a closer look at these systems in Section 8.5. Moreover, they will also be needed in order to obtain a homology theory from Bar-Natan's complex of tangles, which we will finally introduce in Section 8.6. Before doing so however, we describe in Section 8.7 important composability and compatibility properties of tangles and their corresponding Bar-Natan complexes in the language of Bar-Natan's planar arc diagrams and planar algebras, which will ultimately lead to tools that simplify computations. These tools will be described in Section 8.8, and we will demonstrate them in action by computing the Bar-Natan complex of the right-handed trefoil in the same section. In Section 8.9 we describe how to obtain a homology theory from Bar-Natan's complex of tangles using TQFTs. In Section 8.10 we discuss Lee's deformation of Khovanov homology and the famous Rasmussen s-invariant, and show how MacKaay-Turner-Vaz and Liphshitz-Sarkar generalized the s-invariant to arbitrary fields. Last but not least, we return to ordinary Khovanov homology in Section 8.11 and discuss more about its structure, such as the *Knight Move Conjecture*, and introduce a reduced version of Khovanov homology as well as the notion of *homologically thin* knots.

In this chapter we tried to put emphasis on proper descriptions and formalism, with additional explanations that may not be found or may be implicit in the literature. However, not all of Chapter 8 is needed in detail in order to understand our own results. The reader who is already familiar with Khovanov homology resp. Bar-Natan's theory and wishes to proceed as fast as possible to Chapter 9 and subsequent chapters is advised to have a look at Sections 8.3 to 8.5 for the main definitions that we use. Main references for this chapter are [Kho00, BN02, Bar05, Kho06, Ras10, MTV07].

8.1 The scaled Kauffman bracket and the Jones polynomial

We start by recalling the (scaled) Kauffman bracket and its relation to the Jones polynomial (for motivation and historical background of these invariants, see Section 2.4). Our main references are [Kau87b] and [Kho00, Section 2.4].

Definition 8.1. Let $n \in \mathbb{N}$. An *n-component link L* in S^3 is a smooth embedding $L\colon S^1 \sqcup \cdots \sqcup S^1 \to S^3$ of the disjoint union of n circles into S^3. Each connected component of the image of L is a knot $K \subset S^3$, and we write $L = K_1 \sqcup \cdots \sqcup K_n$. A *regular plane projection* of a link L is a smooth immersion $p\colon S^3 \to P$ of S^3 to a plane P, such that all self-intersections in the image of $p \circ L$ are transverse double points (i.e. no more than two points intersect transversally and no points intersect tangentially), endowed with over- and undercrossing information. The image of $p \circ L$ is called a *link diagram* of L, and will be used to represent links. See Figure 8.1 for an example.

Convention. Unless otherwise mentioned or clear from the context, all links and link diagrams are understood to be oriented.

In [Kau87b], Kauffman constructs a state model for the Jones polynomial. In this model, the Jones polynomial results from a normalization of an invariant of unoriented links, the so-called *bracket polynomial.* The

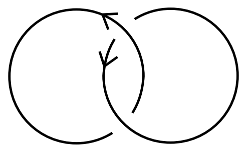


Figure 8.1: A regular plane projection of the (positive) Hopf link.

bracket polynomial is defined in a combinatorial way based on a diagram of the link. The main ingredient is the *resolution* (also called *smoothing*) of a crossing in a link diagram. There are two types of resolutions, the 0- and 1-resolution, as shown in Figure 8.2.

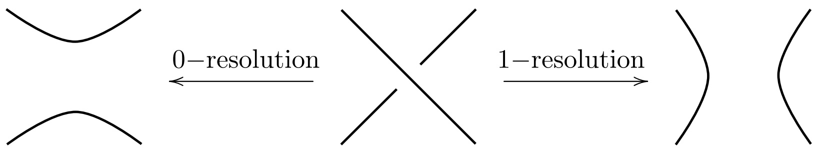


Figure 8.2: The two possible ways of resolving a crossing in a link diagram.

Observe that orientations are irrelevant for smoothings. Resolving all crossings in a link diagram with any sort of resolutions always results in a diagram that consists of a disjoint union of planar circles (a so-called *complete smoothing*). This is the main observation for the upcoming recursive definition of the bracket polynomial (and in fact also the starting point for the construction of Khovanov homology as we shall see in Section 8.2).[1]

Definition 8.2. Let $L \subset S^3$ be a link with diagram D. Then the *bracket polynomial* of D is the Laurent polynomial $\langle D \rangle \in \mathbb{Z}[q^{\pm 1}]$ defined by the following rules:

1. $\langle \bigcirc \rangle = q + q^{-1}$;
2. $\langle \times \rangle = \langle \asymp \rangle - q\langle)\,(\rangle$;
3. $\langle D_1 \sqcup D_2 \rangle = \langle D_1 \rangle \langle D_2 \rangle$.

Here, $D_1 \sqcup D_2$ denotes the disjoint union of two link diagrams.

The rules in Definition 8.2 are to be understood *locally*, that is, in order to compute $\langle D \rangle$, one starts to resolve crossings, disjoint unions, and circles within D in a recursive fashion according to the rules in Definition 8.2. Note that the bracket polynomial is *not* an invariant of oriented links; it is not invariant under the first and second Reidemeister move (cf. [Kho00, Section

[1]Note that our definition of the bracket polynomial follows the (up to normalization equivalent) convention of [Kho00], instead of the original source [Kau87b]. See Remark 8.4 below for further details.

2.4]). In order to obtain an invariant, one has to take the orientation of L and the induced orientation on a diagram of L into account. Figure 8.3 shows our convention regarding the sign of an oriented crossing.

Figure 8.3: A positive and negative crossing on the left and right, respectively.

Definition 8.3. Let $L \subset S^3$ be a link with diagram D and set

$$n_+(D) := \text{number of positive } (+1) \text{ crossings in } D$$
$$n_-(D) := \text{number of negative } (-1) \text{ crossings in } D.$$

Then the *scaled Kauffman bracket* of L is defined as

$$K(L) := (-1)^{n_-(D)} q^{n_+(D)-2n_-(D)} \langle D \rangle \in \mathbb{Z}[q^{\pm 1}].$$

One can check that $K(L)$ does not depend on the choice of the diagram D, making it an invariant of L (see [Kau87b]), and hence justifying the notation in Definition 8.3.

Remark 8.4. The conventions regarding the notions of the bracket polynomial and the Kauffman bracket differ throughout the literature, but are equivalent up to normalization. Our conventions are closest to [Kho00] and [BN02]. However in [Kho00], there is no name for the bracket polynomial $\langle D \rangle$. The invariant $K(L)$ is called *scaled* Kauffman bracket because in Kauffman's original work [Kau87b], the *Kauffman bracket* is defined as

$$f[L] = (-A)^{-3(n_+(D)-n_-(D))} \langle D \rangle_{\mathrm{Ka}} \in \mathbb{Z}[A^{\pm 1}],$$

where the polynomial $\langle D \rangle_{\mathrm{Ka}} \in \mathbb{Z}[A^{\pm 1}]$ is determined by the rules

1. $\langle \bigcirc \rangle_{\mathrm{Ka}} = 1$;
2. $\langle \times \rangle_{\mathrm{Ka}} = A \langle \asymp \rangle_{\mathrm{Ka}} + A^{-1} \langle)\,(\rangle_{\mathrm{Ka}}$;
3. $\langle \bigcirc \sqcup D' \rangle_{\mathrm{Ka}} = (-A^2 - A^{-2}) \langle D' \rangle_{\mathrm{Ka}}$,

where D' is a non-empty link diagram. It is easy to see that $K(L)$ and $f[L]$ are related via

$$K(L)|_{q=-A^{-2}} = (-A^2 - A^{-2}) f[L].$$

Yet another naming convention appears in [BN02], where the bracket polynomial $\langle D \rangle$ is called the Kauffman bracket and $K(L)$ is referred to as the unnormalized Jones polynomial.

Let us now recall the skein relation of the Jones polynomial. Let L_1, L_2, and L_3 be oriented links with diagrams D_1, D_2, and D_3 that differ at a single crossing as shown in Figure 8.4.

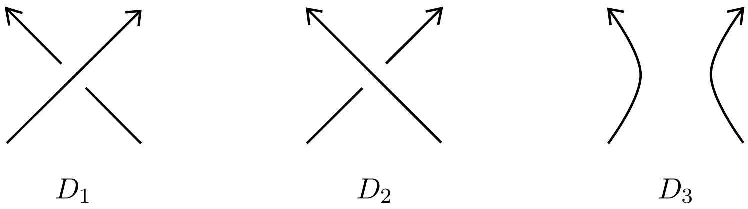


Figure 8.4: The link diagrams D_1, D_2, and D_3 that differ at a single crossing.

Proposition 8.5 ([Jon85]). Let $L \subset S^3$ be an oriented link with diagram D. Then there exists a unique polynomial $V(L) \in \mathbb{Z}[\sqrt{t^{\pm 1}}]$ such that:

1. $V(\bigcirc) = 1$;
2. $t^{-1}V(L_1) - tV(L_2) = (\sqrt{t} - \frac{1}{\sqrt{t}})V(L_3)$.

Here, L_1, L_2, L_3 are the links obtained by changing L (respectively D) in a single crossing as in Figure 8.4. Moreover, $V(L)$ is equal to the Jones polynomial of L.

We wish to relate the scaled Kauffman bracket $K(L)$ to the Jones polynomial $V(L)$. For this, let $L \subset S^3$ be an oriented link with diagram D, fix a crossing in D, and consider the corresponding links L_1, L_2, L_3 and diagrams D_1, D_2, D_3 as in Figure 8.4. The second rule in Definition 8.2 of the bracket polynomial implies that

$$\langle D_1 \rangle = \langle D_3 \rangle - q\langle \asymp \rangle, \quad \langle D_2 \rangle = \langle \asymp \rangle - q\langle D_3 \rangle.$$

Hence

$$\langle D_1 \rangle + q\langle D_2 \rangle = (1 - q^2)\langle D_3 \rangle. \tag{8.1}$$

Observe that

$$\begin{aligned} n_+(D_1) &= n_+(D_2) + 1 = n_+(D_3) + 1 \\ n_-(D_1) &= n_-(D_2) - 1 = n_-(D_3). \end{aligned}$$

Therefore

$$\begin{aligned} K(L_1) &= (-1)^{n_-(D_1)} q^{n_+(D_1) - 2n_-(D_1)} \langle D_1 \rangle = c\langle D_1 \rangle \\ K(L_2) &= (-1)^{n_-(D_1)+1} q^{n_+(D_1) - 1 - 2n_-(D_1) - 2} \langle D_2 \rangle = -q^{-3} c\langle D_2 \rangle \\ K(L_3) &= (-1)^{n_-(D_1)} q^{n_+(D_1) - 1 - 2n_-(D_1)} \langle D_3 \rangle = q^{-1} c\langle D_3 \rangle, \end{aligned}$$

where $c := (-1)^{n_-(D)} q^{n_+(D)-2n_-(D)}$. Applying these observations to (8.1), we get

$$\begin{aligned}\langle D_1\rangle + q\langle D_2\rangle = (1-q^2)\langle D_3\rangle &\Leftrightarrow c\langle D_1\rangle + qc\langle D_2\rangle = (1-q^2)c\langle D_3\rangle\\ &\Leftrightarrow K(L_1) - q^4 K(L_2) = (q-q^3)K(L_3)\\ &\Leftrightarrow q^{-2}K(L_1) - q^2 K(L_2) = (q^{-1}-q)K(L_3).\end{aligned}$$

Thus we have the following skein relation for the scaled Kauffman bracket:

$$q^{-2}K(L_1) - q^2K(L_2) = (q^{-1}-q)K(L_3).$$

Noting that $K(\text{unknot}) = q+q^{-1}$ and $V(\text{unknot}) = 1$, we now immediately obtain that the scaled Kauffman bracket and the Jones polynomial are related via

$$V(L)|_{\sqrt{t}=-q} = \frac{K(L)}{q+q^{-1}}.$$

Before moving on to Khovanov homology, let us quickly emphasize again a certain point of view in computing the Jones polynomial using the scaled Kauffman bracket, as explained in [BN02, Section 2].

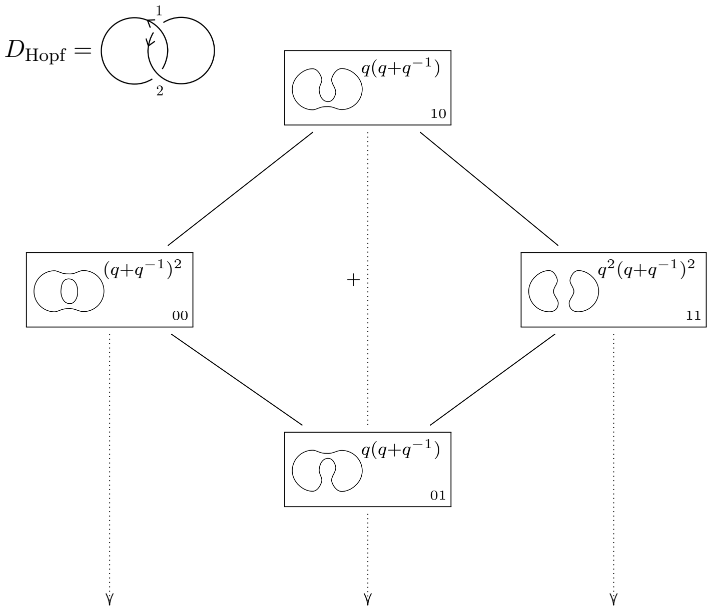
Let $L \subset S^3$ be an oriented link with diagram D, and let $\mathcal{X} \subset D$ denote the set of crossings in D, i.e. the transverse double points. Fix an enumeration of the crossings in $\mathcal{X}$ and set $n = |\mathcal{X}|$. As mentioned at the beginning of this section, each crossing $x \in \mathcal{X}$ admits a 0- and a 1-resolution. Thus any tuple $\alpha \in \{0,1\}^{\mathcal{X}}$ defines a unique way to resolve all crossings in D, so that we end up with a diagram that consists of a disjoint union of planar circles (a so-called *complete smoothing* of D). In the process of computing the scaled Kauffman bracket $K(L)$ we eventually arrive at every possible complete smoothing of D, which then gets replaced by a suitable polynomial in $\mathbb{Z}[q^{\pm 1}]$.

Let us be more specific. Each vertex $\alpha \in \{0,1\}^{\mathcal{X}}$ of the n-dimensional cube $[0,1]^{\mathcal{X}}$ defines a complete smoothing S_α of D. Looking at Definition 8.2 and Definition 8.3, we see that in order to compute the scaled Kauffman bracket $K(L)$, we replace each such union S_α with a term of the form

$$(-1)^r q^r (q+q^{-1})^{k_\alpha}, \tag{8.2}$$

where k_α denotes the number of planar circles in S_v, and r is the *height* of a smoothing, i.e. the number of 1-resolutions used in order to obtain S_α. We then sum (8.2) over all possible $\alpha \in \{0,1\}^{\mathcal{X}}$, and multiply the result by the final term $(-1)^{n_-}q^{n_+-2n_-}$, where as before n_+ and n_- denote the number of positive and negative crossings in D, respectively. This process can be depicted nicely in a diagram as shown in Figure 8.5, which also explains why we speak of vertices and cubes in this scenario.[2]

[2]Bar-Natan's diagrams (1)-(3) for the (right-handed) trefoil in [BN02] served as templates for our Figures 8.5, 8.7 and 8.9 for the (positive) Hopf link.



$$(q+q^{-1})^2 \quad - \quad 2q(q+q^{-1}) \quad + \quad q^2(q+q^{-1})^2$$

$$= \; q^{-2}+1+q^2+q^4 \xrightarrow[\text{(with } (n_+,n_-)=(2,0))]{\cdot(-1)^{n_-}q^{n_+-2n_-}} 1+q^2+q^4+q^6 \xrightarrow{\cdot(q+q^{-1})^{-1}} V(\text{Hopf link})|_{\sqrt{t}=-q} = q+q^5.$$

Figure 8.5: The cube $[0,1]^{\mathcal{X}}$ for a diagram D_{Hopf} of the (positive) Hopf link. Each vertex $\alpha \in \{0,1\}^{\mathcal{X}}$ is decorated with a box containing the vertices' coordinates in the lower right corner, and the corresponding complete smoothing S_α and polynomial $(-1)^r q^r (q+q^{-1})^{k_\alpha}$ in the center and upper right corner, respectively. The bottom line shows how to obtain the Jones polynomial of the (positive) Hopf link from the depicted cube.

Convention. It will be convenient in later sections and for computations to have a convention giving an enumeration of the circles in a complete smoothing S_α of D. We adapt the convention described in [BN02, Section 3.3]. First, we label each edge of D individually by integers in an ascending order starting from 1. Then, given a complete smoothing S_α, we label every circle by the minimal integer of the edges appearing in it. This gives us an ascending enumeration of circles, although not necessarily by consecutive integers. So last but not least, we relabel the circles according to their enumeration starting at 1. See Figure 8.6 for an example. However, for better readability we will keep the enumeration implicit and most of the time omit them in our figures, and only refer to them when needed.

The 1-skeleton of the cube $[0,1]^{\mathcal{X}}$ as depicted in Figure 8.5 will play an important role in the upcoming sections, so let's give it a proper name.

Definition 8.6 (Cube of Resolutions). Let $L \subset S^3$ be an oriented link with diagram D. Let $\mathcal{X} \subset D$ be the subset of crossings of D. Then the 1-skeleton of the cube $[0,1]^{\mathcal{X}}$ with vertices $\alpha \in \{0,1\}^{\mathcal{X}}$ identified with the corresponding complete smoothings S_α with circles enumerated according

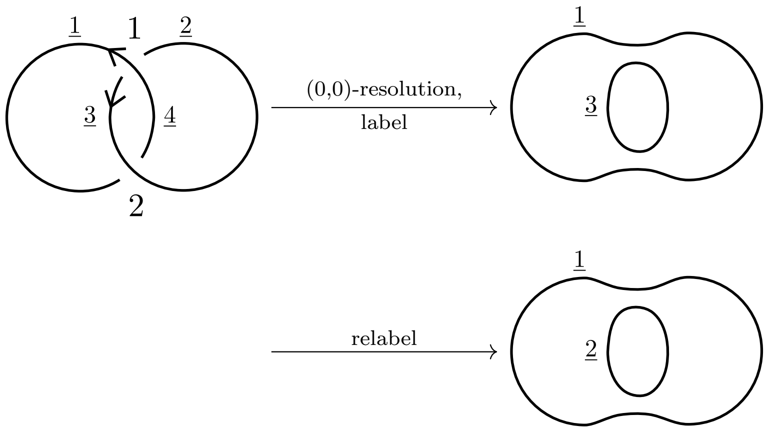


Figure 8.6: Our convention for enumerating circles in complete smoothings of link diagrams.

to our convention is called the *cube of resolutions* of D, and denoted by $Q(D)$.

8.2 Khovanov homology I

Khovanov homology was introduced by M. Khovanov in 1999 as a "categorification of the Jones polynomial" [Kho00]. Two decades after its discovery however we know that Khovanov homology (and all its variations) bears much more information. For instance, one may extract a lower bound on the slice genus of a knot using the Rasmussen s-invariant [Ras10], and it is know that Khovanov homology detects the unknot [KM11], a question that is still open for the Jones polynomial at the time of writing. In this sense, one could say that the categorification of the Jones polynomial is just a mere property. Nonetheless, approaching Khovanov homology via categorification is still a suitable way to get a first understanding of its spirit. Thus, we will pick up where we stopped in Section 8.1 and show how to obtain Khovanov homology by categorifying the Jones polynomial, following Bar-Natan's exposition in [BN02].[3]

Khovanov's categorification can intuitively be understood as the process of encoding the Jones polynomial into a homological object. The basic idea is to replace the polynomials of the form $(-1)^r q^r (q+q^{-1})^{k_\alpha}$ that pop up in the computation of the Jones polynomial by graded vector spaces such that their graded dimension equals said polynomials. These vector spaces then form the cornerstones of a bigraded chain complex $\mathcal{C}$, with a *homological* and a so-called *quantum* grading,[4] such that the graded Euler characteristic

[3]A more formal approach is given by Khovanov in his original paper [Kho00].

[4]The gradings of Khovanov homology, in particular the quantum grading, will be discusses

of this complex equals the Jones polynomial. Let us be more precise.

Definition 8.7. Let $V = \bigoplus_{m\in\mathbb{Z}} V_m$ be a graded finite-dimensional vector space with homogeneous components V_m.

1. The *graded dimension* of V is defined as
$$\operatorname{qdim} V := \sum_{m\in\mathbb{Z}} q^m \dim V_m \in \mathbb{Z}[q^{\pm 1}].$$

2. Let $l \in \mathbb{Z}$. Then the *degree* (or *grading*) *shift* operation $\cdot\{l\}$ on V is defined as
$$V\{l\} := \bigoplus_{m\in\mathbb{Z}} V\{l\}_m, \quad V\{l\}_m := V_{m-l}$$

3. The *degree* of a non-zero homogeneous element $x \in V_m$, $m \in \ Z$, is defined as
$$\deg x := m.$$
The degree of 0 is set to be indefinite.

Remark 8.8. Observe that the degree shift has the following effect on the graded dimension of V:
$$\operatorname{qdim} V\{l\} = q^l(\operatorname{qdim} V).$$

Definition 8.9. Let
$$\mathcal{C} = \cdots \to C^r \xrightarrow{d^r} C^{r+1} \to \cdots$$
be a chain complex of (possibly graded) finite-dimensional vector spaces.

1. Let $s \in \mathbb{Z}$. Then the *height shift* operation $\cdot[s]$ on C is defined as
$$\mathcal{C}[s] := \cdots \to C^r[s] \xrightarrow{d^r[s]} C^{r+1}[s] \to \cdots, \quad C^r[s] := C^{r-s},$$
with differentials shifted accordingly. Here, r is called the *height* of a chain group C^r.

2. The *graded Euler characteristic* of $\mathcal{C}$ is defined as
$$\chi_q(\mathcal{C}) := \sum_{r\in\mathbb{Z}} (-1)^r \operatorname{qdim} H_r(\mathcal{C}).$$

Remark 8.10. Let
$$\mathcal{C} = \cdots \to C^r \xrightarrow{d^r} C^{r+1} \to \cdots$$

more precisely in Section 8.10.

be a chain complex of *graded* finite-dimensional vector spaces. Note that $\mathcal{C}$ can be considered as what is sometimes called a *differential graded module* (e.g. in [McC85]): a $\mathbb{Z}$-graded module

$$\mathcal{C} = \bigoplus_{r\in\mathbb{Z}} C^r$$

with a differential $d = \bigoplus_{r\in\mathbb{Z}} d^r$, i.e. an endomorphism $d\colon \mathcal{C} \to \mathcal{C}$ that squares to the identity. From this point of view, (C, d) becomes a *bigraded* chain complex, where the $\mathbb{Z}$-grading of $\mathcal{C}$ corresponds to the first, homological grading, and the second grading is given by the individual graded chain groups C^r. Note that the degree and height shift operators act on $(\mathcal{C}, d)$ as follows:

$$\mathcal{C}[s]\{l\} = \bigoplus_{r\in\mathbb{Z}} \bigoplus_{m\in\mathbb{Z}} C^{r-s,m-l}.$$

Convention. From now on we will work with the usual notion of a graded chain complex and its corresponding understanding as a differential graded module as explained in Remark 8.10 interchangeably.

Let us now move towards the categorification of the Jones polynomial. Let $L \subset S^3$ be an oriented link with diagram D, let $\mathcal{X} \subset D$ be the subset of crossings, and set $n \coloneqq |\mathcal{X}|$. As before, we write $n = n_+ + n_-$, where n_+ and n_- denote the number of positive and negative crossings in D, respectively. Let A be the graded 2-dimensional $\mathbb{Q}$-vector space

$$A \coloneqq \mathbb{Q}[X]/(X^2) \cong \langle 1\rangle_{\mathbb{Q}} \oplus \langle X\rangle_{\mathbb{Q}}$$

with

$$\deg 1 = +1, \quad \deg X = -1,$$

so that $\operatorname{qdim} A = (q + q^{-1})$. Given a vertex $\alpha \in \{0,1\}^{\mathcal{X}}$ of the cube of resolutions $Q(D)$, we define the vector space

$$A_\alpha(D) \coloneqq A^{\otimes k_\alpha}\{r\},$$

where k_α denotes the number of planar circles in the complete smoothing S_α of D, and r is the height $r = |\alpha| = \sum_i \alpha_i$, i.e. the number of 1-resolutions used in order to obtain S_α. Note that $A_\alpha(D)$ inherits a grading via

$$\deg(a_1 \otimes a_2 \otimes \cdots \otimes a_{k_\alpha}) \coloneqq \deg a_1 + \deg a_2 + \cdots \deg a_{k_\alpha},$$

where $a_i \in A$, $i = 1, 2, \ldots, k_\alpha$, are homogeneous.

Observe that $A_\alpha(D)$ is defined precisely such that $\operatorname{qdim} A_\alpha(D)$ equals the polynomial that appears at the vertex α of the cube of resolutions when computing the Jones polynomial (see the end of Section 8.1 and Figure 8.5). Now, define for $0 \leq r \leq n$

$$C^r(D) \coloneqq \bigoplus_{\substack{\alpha\in\{0,1\}^{\mathcal{X}},\\ |\alpha|=r}} A_\alpha(D),$$

where we order the summands in reversed lexicographical order with respect to α (so that we "flatten" the cube of resolutions top-down),[5] and set

$$\mathcal{C}(D) \coloneqq \bigoplus_{r=0}^{n} C^r(D).$$

Finally, we define

$$\mathcal{C}_{\mathrm{Kh}}(D) \coloneqq \mathcal{C}(D)[-n_-]\{n_+ - 2n_-\}.$$

As an example, Figure 8.7 shows how the cube of resolutions for the Hopf link in Figure 8.5 translates to this new setting.

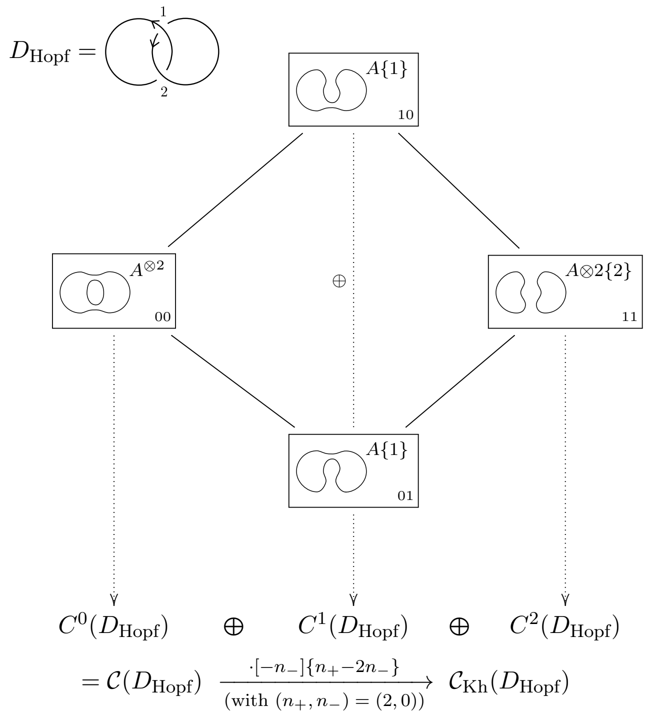


Figure 8.7: The cube of resolutions $Q(D_{\mathrm{Hopf}})$ for a diagram D_{Hopf} of the (positive) Hopf link. Each vertex $\alpha \in \{0,1\}^{\mathcal{X}}$ is decorated with a box containing the vertices' coordinates in the lower right corner, and the corresponding complete smoothing S_α and vector spaces $A_\alpha(D_{\mathrm{Hopf}})$ in the center and upper right corner, respectively. The bottom lines show how to obtain the (soon to be) chain complex $\mathcal{C}_{\mathrm{Kh}}(D_{\mathrm{Hopf}})$ from the depicted cube.

The notation indicates that $\mathcal{C}_{\mathrm{Kh}}(D)$ is going to be the chain complex corresponding to Khovanov homology. However, there is still a suitable differential missing, which we are going to construct now. The plan is

[5]This is our convention for ordering summands in $C^r(D)$, but the result does in fact not depend on the choice of ordering, see [BN02].

to define a linear map d_σ for every edge of the cube of resolutions $Q(D)$, and use those to construct a differential. Thus, let $\sigma \subset Q(D)$ be an edge connecting two vertices $\alpha_1, \alpha_2 \in \{0,1\}^{\mathcal{X}}$ with $|\alpha_1| < |\alpha_2|$. Observe that the coordinates of α_1 and α_2 differ at exactly *one* entry by a 0 and a 1 (so that in fact $|\alpha_1| = |\alpha_2| - 1$). We use this to encode an edge $\sigma \subset Q(D)$ by a sequence $\sigma \in \{0,1,\star\}^{\mathcal{X}}$ such that σ contains exactly one entry with a $\star$ at the position where the coordinates of α_1 and α_2 differ. We define the *height* of σ as

$$|\sigma| \coloneqq \sum_{i=0}^{n} \widetilde{\sigma}_i, \quad \widetilde{\sigma}_i \coloneqq \begin{cases} \widetilde{\sigma}_i = \sigma_i & \text{if } \sigma_i \neq \star, \\ \widetilde{\sigma}_i = 0 & \text{if } \sigma_i = \star. \end{cases} \tag{8.3}$$

In other words, $|\sigma|$ equals the height of α_1.

Now, let $\sigma \in \{0,1,\star\}^{\mathcal{X}}$ be an edge connecting two vertices $\alpha_1, \alpha_2 \in \{0,1\}^{\mathcal{X}}$ as above with $|\alpha_1| = |\alpha_2| - 1$. The two corresponding complete smoothings S_{α_1} and S_{α_2} differ at exactly one crossing $x \in \mathcal{X}$ that was 0-resolved in S_{α_1} and 1-resolved in S_{α_2}. This means that by the nature of resolving crossings either two planar circles merge, or one planar circle splits while moving from S_{α_1} to S_{α_2} along the edge σ. This splitting respectively merging along an edge of the cube of resolutions can abstractly be visualized by a $(1+1)$-dimensional cobordism as shown in Figure 8.8 below (in fact, we can consider a cobordism from S_{α_1} to S_{α_2} that is a product except in a neighborhood of the crossing that was resolved differently, where the cobordism is the obvious saddle between the 0- and 1-resolution, see Remark 8.63).

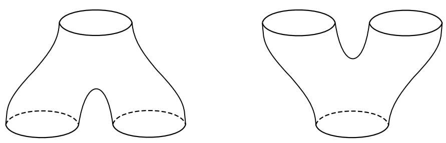
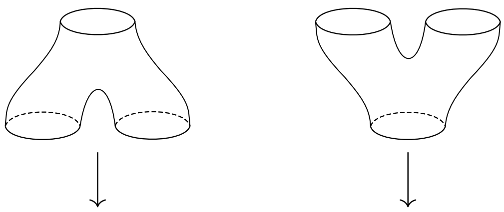


Figure 8.8: Two circles merging into one and one circle splitting into two via a $(1+1)$-dimensional cobordism

Now, the circles in a complete smoothing of D correspond to tensor factors in $A_\alpha(D)$, and so both splitting and merging of circles can be translated into maps as follows:



$m\colon A \otimes A \to A \qquad \Delta\colon A \to A \otimes A$

Let us define m and Δ on basis elements as follows:

$$m\colon \begin{cases} 1\otimes X \mapsto X & 1\otimes 1\mapsto 1 \\ X\otimes 1\mapsto X & X\otimes X\mapsto 0 \end{cases} \qquad \Delta\colon \begin{cases} 1\mapsto 1\otimes X + X\otimes 1 \\ X\mapsto X\otimes X \end{cases}$$

Clearly, both m and Δ are linear maps. Define

$$d_\sigma\colon A_{\alpha_1}(D)\to A_{\alpha_2}(D),$$

where d_σ acts as the identity on all tensor factors except on those factors that are involved in the merging or splitting process, where d_σ acts as either m or Δ.

Remark 8.11. As explained in [BN02], the definition of the maps m and Δ are not chosen arbitrarily and are in fact forced up to scalars by several requirements. First, recall that our goal is categorification; the graded Euler characteristic (see Definition 8.9) of our chain complex $\mathcal{C}_{\mathrm{Kh}}$ should equal the Jones polynomial (up to normalization). If the degree of the differential of an arbitrary graded chain complex equals zero and all chain groups are finite-dimensional, then the graded Euler characteristic equals the alternating sum of the chain groups. Thus it is favorable that the maps d_σ are of degree zero. This means that m and Δ have to be of degree -1 due to the degree shift in the spaces $A_\alpha(D)$. Moreover, since there is no canonical order of the circles in S_α and the corresponding tensor factors of $A_\alpha(D)$, m and Δ must be commutative and co-commutative, respectively (see Section 8.5). These requirements lead to the definition of m and Δ above.

We have now defined a linear map d_σ for every edge of the cube of resolutions $Q(D)$, and just as we did to obtain the spaces $C^r(D)$, we now take appropriate sums of these maps:

$$d^r := \sum_{|\sigma|=r} (-1)^\sigma d_\sigma, \quad 0\le r\le n$$

with

$$(-1)^\sigma := (-1)^{\Sigma_{i<j}\sigma_i},$$

where j is the coordinate of the single $\star$ in σ. The reader is invited to check that the signs are chosen in a way such that all square-faces in the cube of resolutions anti-commute, which guarantees that $d^{r+1}\circ d^r = 0$, thus making d^r a differential. Figure 8.9 extends the diagram from Figure 8.7 by the maps leading to the differentials d^r. If we now set

$$d_{\mathrm{Kh}} := \bigoplus_{r=0}^{n} d^r,$$

then $(\mathcal{C}_{\mathrm{Kh}}(D), d_{\mathrm{Kh}})$ becomes a bigraded chain complex in the sense of Remark 8.10. We are now ready for the following definition.

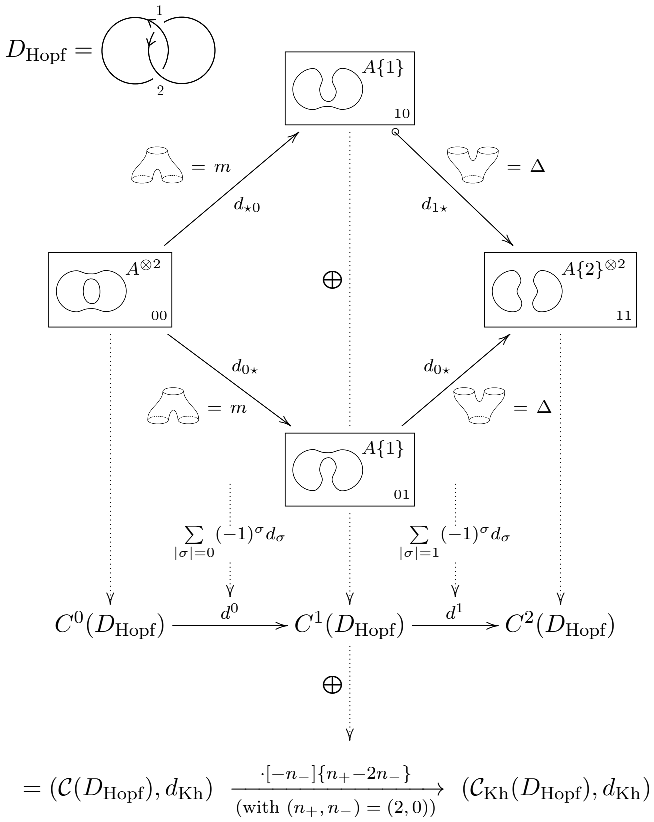


Figure 8.9: The cube of resolutions $Q(D_{\text{Hopf}})$ for a diagram D_{Hopf} of the (positive) Hopf link as in Figure 8.7, but with edges decorated by the maps d_σ that lead to the differential d_{Kh}. A little circle at the tail of an edge indicates that the map appears with a minus sign in the sum d^r. The bottom lines show how to obtain the chain complex $(\mathcal{C}_{\text{Kh}}(D_{\text{Hopf}}), d_{\text{Kh}})$ from the depicted cube.

Definition 8.12 (Khovanov homology). Let $L \subset S^3$ be an oriented link with diagram D. Then $(\mathcal{C}_{\text{Kh}}(D), d_{\text{Kh}})$ is called *Khovanov chain complex* of D. It is by construction a bigraded chain complex, with a *homological* and a *quantum grading*. The gradings may be read off via

$$h(x) \coloneqq |\alpha| - n_- \qquad (\textit{homological grading})$$
$$q(x) \coloneqq \deg x + h(x) + n_+ - n_- \qquad (\textit{quantum grading})$$

where $x \in A_\alpha(D)$ is homogeneous. The homology of the chain complex $(\mathcal{C}_{\text{Kh}}(D), d_{\text{Kh}})$ is bigraded as well and called *Khovanov homology* of L and denoted by $\text{Kh}_{\mathbb{Q}}(L)$.

The notation $\text{Kh}_{\mathbb{Q}}(L)$ will be justified shortly. Comparing Figure 8.5 and Figure 8.9, it is easy to see that $(\mathcal{C}_{\text{Kh}}(D), d_{\text{Kh}})$ does indeed categorify the Jones polynomial of L by construction of the chain complex. The next theorem captures this statement, and a short proof is given for instance in [BN02, Theorem 1].

Theorem 8.13. The graded Euler characteristic of $\mathcal{C}_{\mathrm{Kh}}(D)$ is equal to the scaled Kauffman bracket of L which in turn equals the Jones polynomial of L up to normalization. In symbols:

$$\chi_q(\mathcal{C}_{\mathrm{Kh}}(D)) = K(L) = (q+q^{-1})V(L)|_{\sqrt{t}=q}.$$

In the notation $\mathrm{Kh}_{\mathbb{Q}}(L)$ of Khovanov homology we have secretly hidden the fact that while the entire construction of the chain complex $\mathcal{C}_{\mathrm{Kh}}(D)$ depends heavily on the chosen diagram D of L, the homology does surprisingly not! In fact, different choices of diagrams for L yield homotopy equivalent Khovanov chain complexes. Before stating the corresponding theorem, let us recall that the *Poincaré polynomial* of a bigraded chain complex $\mathcal{C}$ is defined as

$$P_{\mathcal{C}}(t) \coloneqq \sum_{r\in\mathbb{Z}} t^r \operatorname{qdim} H^r(\mathcal{C}) \in \mathbb{Z}[t^{\pm 1}, q^{\pm 1}],$$

where $H^r(\mathcal{C})$ denotes the r-th homology group of $\mathcal{C}$.

Theorem 8.14 ([Kho00]). The graded dimensions of the homology groups $\mathrm{Kh}^r_{\mathbb{Q}}(L)$ and the Poincaré polynomial $P_{\mathcal{C}_{\mathrm{Kh}(D)}}(t)$ of the complex $\mathcal{C}_{\mathrm{Kh}}(D)$ are link invariants. In particular, $P_{\mathcal{C}_{\mathrm{Kh}(D)}}(-1)$ equals the Jones polynomial of L up to normalization.

A proof of Theorem 8.14 is given for instance in [Kho00] or [BN02, Theorem 3]. Let us now compute the Khovanov homology of the Hopf link.

Example 8.15. Consider the (positive) Hopf link L_{Hopf} with diagram D_{Hopf} and cube of resolutions $Q(D_{\mathrm{Hopf}})$ as in Figure 8.9. Then $\mathcal{C}_{\mathrm{Kh}}(D_{\mathrm{Hopf}})$ takes the form

$$0 \longrightarrow A\{2\}^{\otimes 2} \xrightarrow{d^0} A\{3\} \oplus A\{3\} \xrightarrow{d^1} A\{4\}^{\otimes 2} \longrightarrow 0$$

We use the ordered basis $(1,0), (X,0), (0,1), (0,X)$ for $A^{\oplus 2}$, and $1\otimes 1, X\otimes X, 1\otimes X, X\otimes 1$ for $A^{\otimes 2}$. Then the matrices of d^0 and d^1 respectively take the form

$$M_{d^0} \coloneqq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad M_{d^1} \coloneqq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

Putting M_{d^0} and M_{d^1} into Smith normal form[6], we obtain

$$M_{d^0} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{d^1} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From this, we can immediately read off the isomorphism types of kernels and images, and obtain

$$\begin{aligned} \mathrm{Kh}^0_{\mathbb{Q}}(L_{\mathrm{Hopf}}) &\cong \mathbb{Q}_0 \oplus \mathbb{Q}_2 \\ \mathrm{Kh}^1_{\mathbb{Q}}(L_{\mathrm{Hopf}}) &\cong 0 \\ \mathrm{Kh}^2_{\mathbb{Q}}(L_{\mathrm{Hopf}}) &\cong \mathbb{Q}_4 \oplus \mathbb{Q}_6. \end{aligned}$$

Here, the subscript denotes the quantum degree of the individual generators.

This concludes our introduction to Khovanov homology from the point of view of categorification. However, the maps m and Δ, as well as the mentioning of $(1+1)$-dimensional cobordisms already gave a hint of what's secretly hidden in the intuitive construction of Khovanov homology that we followed above. Namely, the $\mathbb{Q}$-vector space $A = \mathbb{Q}[X]/(X^2)$ and the maps m and Δ are part of a more general algebraic structure, a so-called *Frobenius system*. These structures exist more generally over (graded) rings R instead of fields, and any Frobenius system of rank two gives rise to a functor $\mathcal{F}$ from the category of $(1+1)$-dimensional cobordisms to the category of modules over R, a so-called *topological quantum field theory* or *TQFT* for short. Without introducing precise definitions at this point, it should be clear that any such TQFT can be used to translate the cube of resolutions of a link into a chain complex just as we did above, giving rise to many possible Khovanov-type link homology theories. The upcoming sections are concerned with making these notions precise in the more general context of Bar-Natan's theory for tangles. We will return however to Khovanov homology in Section 8.11, where we discuss more about its structure.

8.3 Tangles and tangle diagrams

Tangles can be understood in two ways; either as the building blocks of knots and links, or as a generalization of them, see (8.4) and (8.5) below. In the context of Khovanov homology, tangles first appeared in Khovanov's papers [Kho00, Kho02], with one result being the discovery of a suitable

[6] Note that since column operations change the basis of the kernel and row operations change the basis of the image, we have to do the corresponding inverse operations on the preceding and succeeding differentials, respectively.

functoriality property for his homology theory of links.[7] However, Khovanov's approach towards this property is rather involved. In his paper [Bar05], Bar-Natan picked up Khovanov's ideas for tangles and simplified as well as extended his approach using a new language, leading to *Bar-Natan's theory for tangles and cobordisms.*

In the upcoming chapters we will exclusively work with Bar-Natan's complex for tangles, so let us pick up Bar-Natan's more general theory and introduce the necessary formalism. We start with a treatment of tangles and tangle diagrams in the present section, and move on to the introduction of a suitable categorical framework in Section 8.4. In Section 8.5 we discuss Frobenius algebras and TQFTs, before introducing the Bar-Natan complex in Section 8.6.

Definition 8.16.

1.) A *tangle* T is a proper smooth 1-submanifold of a closed oriented 3-ball B.

2.) Every tangle is $2n$-ended for some $n \geq 0$, and the $2n$ points in $T \cap \partial B$ are called *end points* of T.

3.) Two tangles in the same 3-ball B with the same set of $2n$ end points in ∂B are called *equivalent* if there is an orientation-preserving homeomorphism of B, fixing the boundary pointwise, mapping one tangle to the other, and preserving the orientation of the tangles if they are oriented.

Remark 8.17. Throughout this text we will consider oriented tangles, unless explicitly mentioned otherwise. Note that a tangle with no end points (i.e. $n = 0$) is just an ordinary link in S^3, and thus of no further interest beyond that.

Observe that a $2n$-ended tangle consists of n arcs and a finite number of circles.

Definition 8.18. Let $n > 0$. The *connectivity* of a $2n$-ended tangle T with arcs $\alpha_1, \alpha_2, \ldots, \alpha_n \subseteq T$ is defined as the set $\{\partial\alpha_1, \partial\alpha_2, \ldots, \partial\alpha_n\}$. If $n = 0$, we define the connectivity of T as $\emptyset$.

It is not hard to see that a tangle with $2n > 0$ end points has $(2n-1)!!$ possible connectivities, where !! denotes the double factorial. Hence, if a tangle has $0, 2, 4, 6, \ldots$ end points, then there are $1, 1, 3, 15, \ldots$ possible connectivities. Bleiler [Ble85] called this notion "string attachments", but for its brevity we prefer the term connectivity, which is also used in [SW21, KWZ21].

[7] Roughly speaking, this functoriality property says that a 4-dimensional cobordism C between two links L_1 and L_2 induces a well-defined homomorphism $\mathrm{Kh}(C)\colon \mathrm{Kh}(L_1) \to \mathrm{Kh}(L_2)$ on Khovanov homology. A similar property was also discovered independently by Jacobsson [Jac04].

Definition 8.19.

1.) A *tangle diagram* D is an immersed proper smooth 1-submanifold of a closed 2-disk E, such that all self-intersections are transverse double points, endowed with over-under information at each such double point.

2.) Similar to tangles, a tangle diagram has an even number of *end points* in $D \cap \partial E$.

3.) Two tangle diagrams in the same disk E with the same set of end points are called *equivalent* if there is an orientation-preserving homeomorphism of E, fixing the boundary pointwise, mapping one diagram to the other while preserving over-under information, and preserving orientation if the tangle diagram is oriented.

Remark 8.20. All tangles in a ball that is embedded into the 3-sphere arise as intersections of that ball with a link that is transverse to the ball's boundary sphere. Similarly, all tangle diagrams in a disk embedded into the plane arise as intersection of that disk with a link diagram that is transverse to the disk's boundary circle.

A natural question that arises is how tangle diagrams with $2n$ end points in two different disks E_1, E_2 are related to each other. In order to answer this question, consider orientation-preserving homeomorphisms $\varphi\colon E_1 \to E_2$ that preserve end points, i.e. that map end points to end points. If two such homeomorphisms φ, φ' are isotopic along end point preserving maps, then they send a tangle diagram $D \subset E_1$ to two equivalent tangle diagrams $\varphi(D), \varphi'(D) \subset E_2$. By Alexander's trick (see e.g. [FM11, discussion after Lemma 2.1]), the isotopy class of a homeomorphism $E_1 \to E_2$ is determined by the isotopy class of its restriction to the boundary. So there are $2n$ end point preserving isotopy classes of homeomorphisms $E_1 \to E_2$, each giving a way to identify equivalence classes of $2n$-ended tangle diagrams in two different disks. If one considers tangle diagrams with *base points*, i.e. one distinguished end point, then requiring that φ sends base point to base point determines φ uniquely up to isotopy.

The situation is more complicated for tangles in different balls B_1, B_2, however. As before, the end point-preserving isotopy classes of homeomorphisms $\varphi\colon B_1 \to B_2$ are determined by the end point-preserving isotopy classes of homeomorphisms $\partial B_1 \to \partial B_2$. Those are in (non-canonical) one-to-one correspondence with the elements of the mapping class group of the $2n$-punctured 2-sphere (see e.g. [FM11] for an introduction to mapping class groups). For $2n \geq 4$, there are non-trivial mapping classes fixing some boundary point; so, in contrast to the situation for tangle diagrams, base pointed tangles with four or more end points in different balls cannot be identified in a canonical fashion.

This also has consequences for the tangle diagrams of a tangle, which one may obtain by projection.

Definition 8.21. Let $T \subset B$ be a $2n$-ended tangle. Let φ be a homeomorphism from B to the unit ball $B_0 \subset \mathbb{R}^3$, mapping the end points of T on ∂B to $\{(\cos(2k\pi/n), \sin(2k\pi/n), 0) \mid 0 \leq k \leq n\}$. If the projection $\mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x, y)$ sends $\varphi(T)$ to a tangle diagram D_T in the unit disk in the xy-plane, we call D_T a *tangle diagram of* T.

A fixed homeomorphism $\varphi\colon B \to B_0$ sends equivalent tangles T, T' to tangle diagrams D_T, D'_T related by Reidemeister moves and tangle diagram equivalence. But this is no longer true if one does not specify φ, and the equivalence class of T is no longer determined by D_T.

Let us now focus on the case $n = 1$, i.e. tangles with 2 end points and the corresponding tangle diagrams. They will serve as the building blocks for the construction of $\mathbb{Z}[G]$-homology in Chapter 9. We have the following one-to-one correspondences:

isotopy classes of base-pointed oriented links $L \subset S^3$	$\overset{1:1}{\longleftrightarrow}$	equivalence classes of oriented 2-ended tangles T in a fixed ball with fixed end points x, y, with the arc of T oriented from x to y.	(8.4)
isotopy classes of base-pointed oriented link diagrams	$\overset{1:1}{\longleftrightarrow}$	equivalence classes of oriented 2-ended tangle diagrams D in a fixed disk with fixed end points x, y, with the arc of D oriented from x to y.	(8.5)

Here, *'base-pointed'* simply means a fixed distinguished point on a link $L \subset S^3$ or on a link diagram away from crossings. Figure 8.10 below shows an example of the correspondence (8.5).

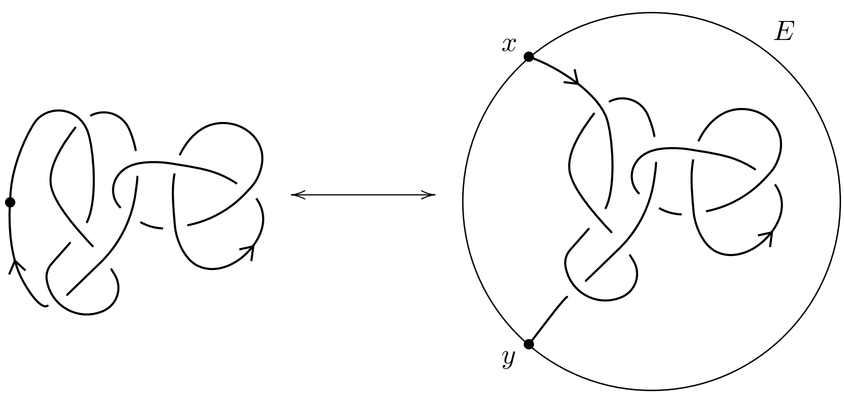


Figure 8.10: A base-pointed oriented link diagram on the left and the corresponding 2-ended tangle diagram in a fixed disk E with fixed end points x, y and with arc oriented from x to y on the right.

Let us describe how to get from L to T and vice versa in (8.4). The complement of an open ball neighborhood of the base point of L is a closed ball B containing a 2-ended tangle $B \cap L$. There are two non-isotopic homeomorphisms sending end points to end points between B and another fixed ball; these two correspond to the two elements of the mapping class group of the twice-punctured sphere. By specifying the orientation of the arc on the right-hand side of (8.4), we eliminate this ambiguity. In the other direction, a fixed ball containing a 2-ended tangle T may be embedded into S^3, and the two end points of T may be joined by an arc outside of the embedded ball, producing a link $L \subset S^3$. The correspondence (8.5) can be shown in a similar way.

So, from now on, we will work with the notions of base-pointed link (diagrams) and 2-ended tangle (diagrams) interchangeably. Moreover, we may associate tangle diagrams to given 2-ended tangles without the ambiguities arising for tangles with more end points.

In Chapter 12 we are going to work with a special class of tangles, so-called *rational tangles.*

Definition 8.22. A 4-ended (oriented or unoriented) tangle T is called *rational* if (B, T) is homeomorphic to $(D^2 \times [0,1], \{(-\frac{1}{2}, 0), (\frac{1}{2}, 0)\} \times [0,1])$, drawn in Figure 8.11.

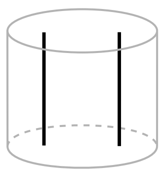


Figure 8.11: The rational tangle $(D^2 \times [0,1], \{(-\frac{1}{2}, 0), (\frac{1}{2}, 0)\} \times [0,1])$.

Let us briefly summarize Conway's famous one-to-one correspondence [Con70, GK97]

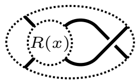
$$R\colon \mathbb{Q} \cup \{\infty\} \xrightarrow{1:1} \{\text{unoriented rational tangles}\}/\text{equivalence} \tag{8.6}$$

(for a general introduction to this topic see e.g. [Cro04]). Let us work with unoriented tangles in the unit ball $B_0 \subset \mathbb{R}^3 \subset S^3$ with the four end points $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}, 0)$, and base point $(-1/\sqrt{2}, -1/\sqrt{2}, 0)$. Generically, the projection to $D^2 \times \{0\}$ yields tangle diagrams; these are the tangle diagrams we consider in what follows. Then, R may be defined by the rules in Figure 8.12 (where we set $1/\infty = 0$ and $1/0 = \infty = \infty + 1 = -\infty$). By a slight abuse of notation, we denote by $R(x)$ both the tangle and the tangle diagram (both well-defined up to equivalence).

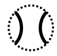
As stated, these rules are consistent and determine the correspondence R completely, but they are somewhat redundant: for example, (8.11) to (8.13) can be derived from the other rules. For simplicity, we will focus

 $= R(0)$ (8.7)

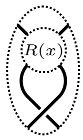
 $= R(1)$ (8.8)

 $= R(x+1)$ (8.9)

$R(x)$ mirrored at plane $\langle e_1 - e_2, e_3\rangle$ $= R\left(\frac{1}{x}\right)$ (8.10)

 $= R(\infty)$ (8.11)

 $= R(-1)$ (8.12)

 $= R\left(\frac{x}{x+1}\right)$ (8.13)

$R(x)$ mirrored at plane $\langle e_1, e_2\rangle$ $= R(-x)$ (8.14)

Figure 8.12: The recursive definition of the bijection R between $\mathbb{Q} \cup \{\infty\}$ and equivalence classes of unoriented rational tangles. In (5.4) and (5.8), e_1, e_2, e_3 denote the standard basis vectors of $\mathbb{R}^3$.

in Chapter 12 only on rational tangles T such that $R^{-1}(T) \in \mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$ (in particular excluding $R(0)$ and $R(\infty)$). The one-to-one correspondence between such rational tangles and $\mathbb{Q}^+$ is completely determined by (8.8),(8.9),(8.10), or by (8.8),(8.9),(8.13).

8.4 Categorical framework for Bar-Natan's theory of tangles

The aim of this section is to introduce the necessary categorical notions for Bar-Natan's theory of tangles and cobordisms, as well as for our further discussions regarding $\mathbb{Z}[G]$-homology and the invariant λ. The main reference for this section is Bar-Natan [Bar05, Sections 3 and 4.1.1-4.1.2].

Convention. Whenever we refer to "category" in this thesis, we assume that the category is small, i.e. its classes of objects and morphisms are actually sets.

Definition 8.23. A category $\mathcal{C}$ is called *pre-additive* if it has the following additional structure: for any two given objects $\mathcal{O}, \mathcal{O}' \in \mathrm{ob}(\mathcal{C})$, the set $\hom_{\mathcal{C}}(\mathcal{O}, \mathcal{O}')$ is an abelian group and the composition of morphisms is bilinear.

Remark 8.24. An arbitrary category $\mathcal{C}$ can be made pre-additive by allowing formal $\mathbb{Z}$-linear combinations in every set of morphisms $\hom_{\mathcal{C}}(\mathcal{O}, \mathcal{O}')$ and by extending composition of morphisms in the natural bilinear way.

Definition 8.25. Let $\mathcal{C}$ be a pre-additive category. The *additive closure* $\mathrm{Mat}(\mathcal{C})$ of $\mathcal{C}$ is defined as follows:

- The objects of $\mathrm{Mat}(\mathcal{C})$ are (possibly empty) formal direct sums $\oplus_{i=1}^n \mathcal{O}_i$ where $\mathcal{O}_i \in \mathrm{ob}(\mathcal{C})$, and thought of as column vectors $(\mathcal{O}_1\ \mathcal{O}_2\ \cdots\ \mathcal{O}_n)^\top$;

- If $\mathcal{O} = \oplus_{i=1}^n \mathcal{O}_i, \mathcal{O}' = \oplus_{i=1}^m \mathcal{O}'_i \in \mathrm{ob}(\mathrm{Mat}(\mathcal{C}))$, then a morphism $F\colon \mathcal{O} \to \mathcal{O}'$ in $\mathrm{Mat}(\mathcal{C})$ is an $m \times n$-matrix $F = (F_{ij})$ of morphisms $F_{ij} \in \hom_{\mathcal{C}}(\mathcal{O}_i, \mathcal{O}'_j)$;
- The addition of morphisms in $\mathrm{Mat}(\mathcal{C})$ is given by matrix addition;
- The composition of morphisms in $\mathrm{Mat}(\mathcal{C})$ is defined using the rules of matrix multiplication using the composition of morphisms in $\mathcal{C}$, i.e. if $F = (F_{ij})\colon \mathcal{O} \to \mathcal{O}'$ and $G = (G_{jk})\colon \mathcal{O}' \to \mathcal{O}''$, then

$$G \circ F = ((G \circ F)_{ik}) := \Big(\Big(\sum_j G_{jk} \circ F_{ij}\Big)_{ik}\Big).$$

Definition 8.26. Let $\mathcal{C}$ be a pre-additive category. The *category of complexes* $\mathrm{Kom}(\mathcal{C})$ *over* $\mathcal{C}$ is defined as follows:

- The objects of $\mathrm{Kom}(\mathcal{C})$ are chains of finite length

$$(C, d) = \cdots \longrightarrow C^{r-1} \xrightarrow{d^{r-1}} C^r \xrightarrow{d^r} C^{r+1} \longrightarrow \cdots,$$

  with $C^r \in \mathrm{ob}(\mathcal{C})$ and $d^r \in \hom_{\mathcal{C}}(C^r, C^{r+1})$ such that $d^r \circ d^{r-1} = 0$ for all r;
- The morphisms of $\mathrm{Kom}(\mathcal{C})$ are chain maps $F\colon (C_1, d_1) \to (C_2, d_2)$ defined just as in ordinary homological algebra;
- Composition of morphisms in $\mathrm{Kom}(\mathcal{C})$ is given as well by composition of chain maps just as in ordinary homological algebra.

Definition 8.27. Let $\mathcal{C}$ be a pre-additive category. Then two morphisms

$$F, G\colon (C_1, d_1) \to (C_2, d_2)$$

are called homotopic, in symbols $F \sim G$, if for all $r \in \mathbb{Z}$ there exist morphisms $h\colon C_1^r \to C_2^{r-1}$ such that

$$F^r - G^r = h^{r+1} \circ d_1^r + d_2^{r-1} \circ h^r.$$

If there are morphisms $F\colon (C_1, d_1) \to (C_2, d_2)$ and $G\colon (C_2, d_2) \to (C_1, d_1)$ such that $G \circ F$ and $F \circ G$ are homotopic to the identity morphisms, then the complexes (C_1, d_1) and (C_2, d_2) are called *homotopy equivalent* and F, G form a *homotopy equivalence.*

Definition 8.26 and 8.27 mimics the notions of ordinary chain complexes and chain maps of, say, abelian groups, for a pre-additive category $\mathcal{C}$, and many properties of the ordinary setting are preserved. In particular, homotopy equivalence defines an equivalence relation on the morphisms of $\mathrm{Kom}(\mathcal{C})$, leading to the following definition.

Definition 8.28. Let $\mathcal{C}$ be a pre-additive category. Then we define the category $\mathrm{Kom}_{/h}(\mathcal{C})$ as $\mathrm{Kom}(\mathcal{C})$ with morphisms considered up to homotopy equivalence.

We continue with the notion of a graded category.

Definition 8.29. We call a pre-additive category $\mathcal{C}$ *graded* if it carries the following additional structure:

1. For any two objects $\mathcal{O}, \mathcal{O}' \in \mathrm{ob}(\mathcal{C})$, $\hom_{\mathcal{C}}(\mathcal{O}, \mathcal{O}')$ forms a graded abelian group such that composition of morphisms respects the grading and such that all identity maps are of degree zero.

2. There is a $\mathbb{Z}$-action $(m, \mathcal{O}) \mapsto \mathcal{O}\{m\}$ on the objects of $\mathcal{C}$, called *grading shift by m*.

If $\mathcal{C}$ only satisfies 1., we define the *graded closure* $\mathrm{cl}(\mathcal{C})$ of $\mathcal{C}$ by adding "artificial" objects $\mathcal{O}\{m\}$ for any $\mathcal{O} \in \mathrm{ob}(\mathcal{C})$ and $m \in \mathbb{Z}$, and define the grading shift in the obvious way (so that $\mathrm{cl}(\mathcal{C})$ satisfies 1. and 2. and becomes graded).

Remark 8.30.

1.) Note that the grading shift changes gradings of morphisms, but not the set of morphisms itself: $\hom_{\mathcal{C}}(\mathcal{O}, \mathcal{O}') = \hom_{\mathcal{C}}(\mathcal{O}\{m\}, \mathcal{O}'\{n\})$, but if $f \in \hom_{\mathcal{C}}(\mathcal{O}, \mathcal{O}')$ has degree d, then $f \in \hom_{\mathcal{C}}(\mathcal{O}\{m\}, \mathcal{O}'\{n\})$ has degree $d - m + n$.

2.) If $\mathcal{C}$ is a graded category, the additive closure $\mathrm{Mat}(\mathcal{C})$ and the category of complexes $\mathrm{Kom}(\mathcal{C})$ can be considered as graded categories as well.

Next, let us describe the categories we are going to work with in order to obtain Bar-Natan's theory for tangles.

Definition 8.31. We define the following categories:

1. The category $\mathrm{Cob}^3(2n)$:

   - The objects of $\mathrm{Cob}^3(2n)$ are crossingless unoriented $2n$-ended tangle diagrams D_T (possibly empty if $n = 0$) inside the unit disk $D^2 \subset \mathbb{R}^2$ with fixed end points, together with an enumeration of every circle appearing in D_T (see Figure 8.13 below).

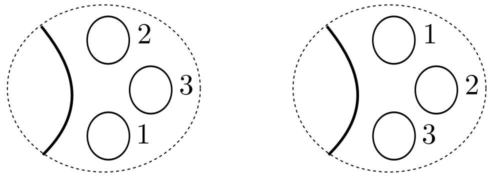
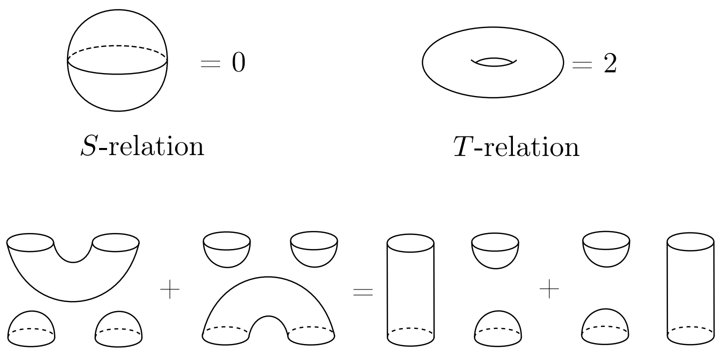


Figure 8.13: Two non-equal but isomorphic objects in $\mathrm{Cob}^3(2n)$.

- Morphisms are 2-dimensional cobordisms (orientable, possibly disconnected surfaces) between two such diagrams D_T, $D_{T'}$, considered up to boundary-fixing isotopy. We assume that cobordisms are properly embedded in the cylinder $D^2 \times [0,1]$ such that if C is a cobordism from D_T to $D_{T'}$, then:
  (a) $C \cap D^2 \times \{0\} = D_T$;
  (b) $C \cap D^2 \times \{1\} = D_{T'}$;
  (c) $C \cap S^1 \times [0,1] =$ the straight boundary arcs connecting the end points of D_T and $D_{T'}$ in the cylinder $D^2 \times [0,1]$.
- The identity is given by the product cobordism, and composition is done by concatenating cobordisms.
- We turn $\mathrm{Cob}^3(2n)$ into a pre-additive category as described in Remark 8.24. For better readability we will frequently keep the enumeration implicit and omit it in our diagrams.

2. The category $\mathrm{Cob}^3_{/l}(2n)$:

- The objects of $\mathrm{Cob}^3_{/l}(2n)$ are the same as the objects of $\mathrm{Cob}^3(2n)$.
- The morphisms of $\mathrm{Cob}^3_{/l}(2n)$ are those of $\mathrm{Cob}^3(2n)$, modulo the *local relations* S, T, and $4Tu$ (see Figure 8.14).
- The identity morphism is the product cobordism, and composition is done by concatenating cobordisms.
- The local nature of the S-, T-, and $4Tu$-relation preserve pre-additivity of $\mathrm{Cob}^3(2n)$ when taking the quotient on morphisms, thus making $\mathrm{Cob}^3_{/l}(2n)$ pre-additive as well.



$4Tu$-relation

Figure 8.14: The defining relations for $\mathrm{Cob}^3_{/l}(2n)$.

The local relations are to be understood as follows. Whenever a cobordism contains a connected component that is a closed sphere or a closed torus (i.e. with no boundary), the S- and T-relation say that we may set these components to 0 and 2, respectively. The $4Tu$-relation says that if a cobordism C contains four embedded 2-disks D_1, D_2, D_3, D_4 (possibly in different connected components), then

$$C_{12} + C_{34} = C_{13} + C_{24},$$

where C_{ij}, $i, j \in \{1,2,3,4\}$, denotes the cobordism C with disks D_i and D_j replaced by a tube connecting the boundary of the two disks. The local relations are needed in order to obtain invariance under the three Reidemeister moves for the Bar-Natan complex of a tangle, see Section 8.6.

The category $\mathrm{Cob}^3_{/l}(2)$ will play an important role in subsequent sections. The tangle diagrams that form the objects in this category consist of the disjoint union of a single arc and a certain number of closed circles. Let's give this arc a special name.

Definition 8.32. Let $\mathcal{O}$ be an object in $\mathrm{Cob}^3_{/l}(2)$. Then the single arc in $\mathcal{O}$ is called *special strand.*

In order to construct $\mathbb{Z}[G]$-homology in Chapter 9 we are further going to need the following categories.

Definition 8.33. Let R be a graded ring. Similar to Definition 8.7, we write $R\{m\}$ for R with grading shifted by $m \in \mathbb{Z}$, i.e. $R\{m\}_n = R_{n-m}$. We define the following categories:

1. The category $\mathcal{E}$:
   - There is only one object in $\mathcal{E}$, namely the crossingless diagram D_{T_0} of the trivial 2-ended tangle T_0 in the unit disk in $\mathbb{R}^2$ with the same fixed end points as for the objects in $\mathrm{Cob}^3(2)$.
   - Morphisms are connected cobordisms up to boundary-fixing isotopy.
   - The identity is given by the product cobordism (a "curtain" of genus zero, see Definition 8.35), and composition is done by concatenating cobordisms.
   - We turn $\mathcal{E}$ into a pre-additive category using Remark 8.24.
2. The category $\mathcal{M}_R$:
   - The objects of $\mathcal{M}_R$ are graded R-modules isomorphic to a direct sum $\oplus_{i=1}^n R\{m_i\}$.
   - Morphisms are graded homomorphisms between R-modules.
   - We turn $\mathcal{M}_R$ into a graded category by introducing the shift operation

$$\Big(\bigoplus_{i=1}^n R\{m_i\}\Big)\{n\} := \bigoplus_{i=1}^n R\{m_i + n\}, \quad n \in \mathbb{Z}.$$

Remark 8.34. Note that our definition of the category $\mathrm{Cob}^3(2n)$ (resp. $\mathrm{Cob}^3_{/l}(2n)$) differs from Bar-Natan's definition in [Bar05]: we require that the objects in $\mathrm{Cob}^3(2n)$, i.e. crossingless tangle diagrams, come with an enumeration of the circles in the diagram. This enumeration will be needed in subsequent sections in order to obtain well-defined TQFTs. It is worthwhile to note that while the enumeration enlarges the set of objects in $\mathrm{Cob}^3(2n)$, it does *not* introduce any new isomorphism classes of objects compared to Bar-Natan's definition of $\mathrm{Cob}^3(2n)$. Moreover, morphisms remain unaffected by the enumeration. In fact, the functor that forgets the enumeration of the circles is an equivalence of categories. In particular, all results obtained by Bar-Natan in [Bar05] remain true for our definition of $\mathrm{Cob}^3(2n)$.

Connected cobordisms between the trivial 2-ended tangle diagram D_{T_0} and itself will have a special role throughout in the upcoming sections and chapters, so let's give them a proper name.

Definition 8.35. A connected cobordism of genus k between the trivial 2-ended tangle diagram and itself will be called a *curtain of genus k*.

Figure 8.15 shows a curtain of genus one.

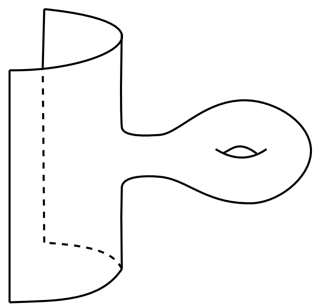


Figure 8.15: A curtain of genus one.

Remark 8.36. Let G be a formal variable. Observe that there is an isomorphism of $\mathbb{Z}$-modules

$$\hom_{\mathrm{Cob}^3(2))}(D_{T_0}, D_{T_0}) \cong \mathbb{Z}[G]$$

that is generated by mapping the product cobordism to 1 and the curtain of genus one to G in $\mathbb{Z}[G]$, respectively.

We introduce a notion of degree on cobordisms as follows.

Definition 8.37. Let $C \in \hom_{\mathrm{Cob}^3(2n)}(D_{T_1}, D_{T_2})$ be a morphism between two tangle diagrams D_{T_1} and D_{T_2}. Then we define the *degree of C* as

$$\deg C := \chi(C) - n,$$

where $\chi(C)$ is the Euler characteristic of C

Using Definition 8.37, we can turn $\hom_{\mathrm{Cob}^3(2n)}(D_{T_1}, D_{T_2})$ into a graded abelian group. Consequently, we can extend $\mathrm{Cob}^3(2n)$ to become a graded category by taking the graded closure, see Definition 8.29. Since the three local relations S, T, and $4Tu$ are degree-homogeneous, $\mathrm{Cob}^3_{/l}(2n)$ is graded too. Last but not least, we use the same notion of degree to turn $\mathcal{E}$ into a graded category as well.

Notation. For the sake of simplicity, we will use the same notation for the graded versions of $\mathrm{Cob}^3(2n)$, $\mathrm{Cob}^3_{/l}(2n)$ and $\mathcal{E}$.

8.5 Frobenius systems and TQFTs

As mentioned at the end of Section 8.2, what's secretly hidden in our construction of Khovanov homology is that the chain complex $\mathcal{C}_{\mathrm{Kh}}(D)$ results from applying a $(1+1)$-dimensional topological quantum field theory (TQFT for short) defined by a Frobenius algebra to the cube of resolutions $Q(D)$. In particular, given a cube of resolutions $Q(D)$, any TQFT coming from a rank two Frobenius algebra yields a Khovanov-type homology theory, and this principle extends easily to Bar-Natan's generalization in the case of 2-ended tangles. We start by recalling some basic definitions.

Definition 8.38. Let R be a commutative unitary ring. An *algebra* over R (or *R-algebra* for short) is a unitary ring A together with a ring homomorphism

$$\iota\colon R \to Z(A)$$

called *unit*, where $Z(A)$ denotes the center of A. An R-algebra A is called *commutative* if the ring multiplication of A is commutative.

Remark 8.39. The unit ι turns an R-algebra A into an R-module via

$$r \cdot a := \iota(r)a, \tag{8.15}$$

turning the ring multiplication $m\colon A \times A \to A$ into an R-bilinear map. Using the universal property of tensor products, we can identify the multiplication m with a map

$$m_\otimes\colon A \otimes_R A \to A.$$

Note that our definition of algebra implies that the algebra multiplication is associative.

Convention. From now on, we will always consider the multiplication of an R-algebra as a map $m\colon A \otimes_R A \to A$ (and dropping the subscript $\otimes$ from $m_\otimes$). Furthermore, we will consider A both as a ring and R-module interchangeably (using the scalar multiplication (8.15)).

Definition 8.40. Let R be a commutative unitary ring. A *coalgebra* over R is an R-module A with R-linear maps

$$\begin{aligned} \Delta\colon A &\to A \otimes_R A && \text{(comultiplication)} \\ \varepsilon\colon A &\to R && \text{(counit)} \end{aligned}$$

such that:

$$\begin{aligned} (\mathrm{id}_A \otimes \Delta) \circ \Delta &= (\Delta \otimes \mathrm{id}_A) \circ \Delta && \text{(coassociativity)} \\ (\mathrm{id}_A \otimes \varepsilon) \circ \Delta &= \mathrm{id}_A && \text{(counit identity).} \end{aligned}$$

A coalgebra A is called *cocommutative* if

$$\tau \circ \Delta = \Delta,$$

where $\tau\colon A \otimes_R A \to A \otimes_R A$ denotes the flip map $\tau(a \otimes b) := b \otimes a$.

We assume that the reader is familiar with the notion of a graded ring and algebra. However, we will eventually also encounter filtered algebras, so let's recall the definition of a filtration.

Definition 8.41. Let R be a commutative unitary ring and A an algebra over R. An *ascending filtration* on A is a sequence of subalgebras $F = (F_i)_{i \in \mathcal{I}}$, indexed by a set $\mathcal{I} \subseteq \mathbb{Z}$ that is bounded from *below* and closed under addition, such that:

1. $F_j \subseteq F_i$ for $i, j \in \mathcal{I}$ with $j \leq i$;
2. $F_i \cdot F_j \subseteq F_{i+j}$ for all $i, j \in \mathcal{I}$;
3. $A = \bigcup_{i \in \mathcal{I}} F_i$.

Similarly, a *descending* filtration on A is a sequence of subalgebras $F = (F_i)_{i \in \mathcal{I}}$, indexed by a set $\mathcal{I} \subseteq \mathbb{Z}$ that is bounded from *above* and closed under addition, such that 2. and 3. hold, and $F_i \subseteq F_j$ for $i, j \in \mathcal{I}$ with $j \leq i$. The algebra A together with a filtration F is called *filtered*. If $\mathcal{I}$ is finite, then a filtration F is said to be of *finite-length*. If $\varphi\colon A \to A'$ is a homomorphism of algebras A, A' with (both ascending or descending) filtrations $F = (F_i)_{i \in \mathcal{I}}, F' = (F'_i)_{i \in \mathcal{I}}$ that are indexed by the same set $\mathcal{I}$ respectively, we say that φ is *filtered of degree k* if $i+k \in \mathcal{I}$ and $\varphi(F_i) \subseteq F'_{i+k}$ for all $i \in \mathcal{I}$.

Convention. If unimportant or clear from the context, we will refer to an ascending or descending filtration simply as filtration.

Remark 8.42. Let $A \cong \bigoplus_{i \in \mathcal{I}} A_i$ be an $\mathcal{I}$-graded algebra over an $\mathcal{I}$-graded commutative unitary ring R, where $\mathcal{I} \subseteq \mathbb{Z}$. If $\mathcal{I}$ is bounded from below, then the grading on A induces an *ascending* filtration $F^a = (F_i^a)_{i\in\mathcal{I}}$ by setting

$$F_i^a := \bigoplus_{\substack{j\in\mathcal{I},\\ j\leq i}} A_j, \quad i \in \mathcal{I}.$$

Similarly, if $\mathcal{I}$ is bounded from above then the grading induces a *descending* filtration $F^d = (F_i^d)_{i\in\mathcal{I}}$ via

$$F_i^d := \bigoplus_{\substack{j\in\mathcal{I},\\ j\geq i}} A_j, \quad i \in \mathcal{I}.$$

Definition 8.43. Let A be a filtered algebra over a commutative unitary ring R with filtration $F = (F_i)_{i\in\mathcal{I}}$. Given a non-zero element $a \in A$ such that

$$a \in F_i,\ a \notin F_j \begin{cases} \text{for all } j < i \text{ if } F \text{ is ascending,} \\ \text{for all } j > i \text{ if } F \text{ is descending,} \end{cases}$$

for some $i \in \mathcal{I}$, then we define the *filtration degree* of a as

$$\deg_F a := i.$$

The filtration degree of 0 is set to be indefinite (i.e. of no particular value).

Definition 8.44. Let A be a filtered algebra over a commutative unitary ring R with filtration $F = (F_i)_{i\in\mathcal{I}}$. Further, let $l \in \mathbb{Z}$ and set

$$\mathcal{I} + l := \{i + l \mid i \in \mathcal{I}\} \subseteq \mathbb{Z}.$$

Then the *(filtration) degree shift* operation $\cdot\{l\}$ on A is defined as returning the algebra A with new filtration $F' = (F_i')_{i\in\mathcal{I}+l}$ given by

$$F_i' := F_{i-l}, \quad i \in \mathcal{I} + l.$$

We write $A\{l\}$ for the algebra A with filtration F'.

Remark 8.45. Let $A_1, \dots, A_n$ be filtered algebras over a commutative unitary ring R with ascending filtrations $F^j = (F_i^j)_{i\in\mathcal{I}}$ for $j = 1, \dots, n$ respectively (so that all F^j are indexed by the same set $\mathcal{I} \subseteq \mathbb{Z}$). Then there is an induced ascending filtration $F^{\otimes_R} = (F_i^{\otimes_R})_{i\in\mathcal{I}}$ on the tensor product $\bigotimes_{j=1}^n A_j$ by setting

$$F_i^{\otimes_R} := \operatorname{span}_R \{a_1 \otimes \cdots \otimes a_n \in \bigotimes_{j=1}^n A_j \mid \sum_{j=1}^n \deg_{F^j} a_i \leq i\}$$

for all $i \in \mathcal{I}$. The filtration degree of an element $\underline{a} = \sum_{i=1}^k a_{i,1} \otimes \cdots \otimes a_{i,n} \in \bigotimes_{j=1}^n A_j$ is then given as

$$\deg_{F^{\otimes_R}}(\underline{a}) = \max_{i=1,\dots,k} \Big(\sum_{j=1}^n \deg_{F^j} a_{i,j} \Big). \tag{8.16}$$

Similarly, there is an induced ascending filtration $F^{\oplus} = (F_i^{\oplus})_{i\in\mathcal{I}}$ on the direct sum $\bigoplus_{j=1}^n A_j$ by simply setting

$$F_i^{\oplus} := \bigoplus_{j=1}^{n} F_i^j. \tag{8.17}$$

The filtration degree of an element $(a_1, \dots, a_n) \in \bigoplus_{j=1}^n A_j$ is then given as

$$\deg_{F^{\oplus}}((a_1, \dots, a_n)) = \max_{j=1,\dots,n} \deg_{F^j} a_j$$

The same holds if the filtrations F^j are descending by replacing max with min in (8.16) and (8.17), respectively.

We will not only encounter filtered algebras, but also filtered chain complexes. Let's quickly translate the above definitions into the language of chain complexes.

Definition 8.46. Let R be a commutative unitary ring and R-Mod the category of modules over R. Further, let $(\Omega, d) \in \mathrm{Kom}(R\text{-Mod})$ be a chain complex. An *ascending filtration* of Ω is a sequence of subcomplexes $C = ((C_i, d_i))_{i\in\mathcal{I}}$, indexed by a set $\mathcal{I} \subseteq \mathbb{Z}$ that is bounded from *below* and closed under addition, such that:

1. $C_j \subseteq C_i$ for all $i, j \in \mathcal{I}$ with $j \leq i$;
2. $d_i(C_i) \subseteq C_i$ for all $i \in \mathcal{I}$;
3. $\bigcup_{i\in\mathcal{I}} C_i = \Omega$.

Similarly, a *descending* filtration on (Ω, d) is a sequence of subcomplexes $C = ((C_i, d_i))_{i\in\mathcal{I}}$, indexed by a set $\mathcal{I} \subseteq \mathbb{Z}$ that is bounded from *above* and closed under addition, such that 2. and 3. hold, and $C_i \subseteq C_j$ for all $i, j \in \mathcal{I}$ with $j \leq i$. A chain complex together with a filtration is called *filtered*, and the filtration is of *finite length* if the set $\mathcal{I}$ is finite. If $\varphi\colon (\Omega, d) \to (\Omega', d')$ is a chain map of chain complexes (Ω, d) and (Ω', d') with filtrations $C = ((C_i, d_i))_{i\in\mathcal{I}}$ and $C' = ((C'_i, d'_i))_{i\in\mathcal{I}}$ that are indexed by the same set $\mathcal{I}$ respectively, we say that φ is *filtered of degree k* if $i + k \in \mathcal{I}$ and $\varphi(C_i) \subseteq C'_{i+k}$ for all $i \in \mathcal{I}$. The *filtration degree* $\deg_C$ and *(filtration) degree shift* operation $\cdot\{l\}$ on Ω for $l \in \mathbb{Z}$ are defined in analogy with the corresponding notions for filtered algebras (see Definition 8.43 and 8.44).

A filtration on a chain complex Ω induces one on homology, as follows.

Definition 8.47. Let R be a commutative unitary ring and R-Mod the category of modules over R. Further, let $(\Omega, d) \in \mathrm{Kom}(R\text{-Mod})$ a chain complex with ascending (resp. descending) filtration $C = ((C_i, d_i))_{i\in\mathcal{I}}$. Then C

induces an ascending (resp. descending) filtration $S = (S_i)_{i\in\mathcal{I}}$ on homology $H_*(\Omega)$ by setting[8]

$$S_i := \{[x] \in H_*(\Omega) \mid \text{there exists some } y \in [x] \text{ such that } y \in C_i\}.$$

If $\varphi\colon H_*(\Omega) \to H_*(\Omega')$ is a morphism on homology of filtered chain complexes with induced filtrations $S = (S_i)_{i\in\mathcal{I}}$ and $S' = (S'_i)_{i\in\mathcal{I}}$ that are indexed by the same set $\mathcal{I}$ respectively, we say that φ is *filtered degree* k if $i + k \in \mathcal{I}$ and $\varphi(S_i) \subseteq S'_{i+k}$ for all $i \in \mathcal{I}$. The *filtration degree* $\deg_S$ and *(filtration) degree shift* operation $\cdot\{l\}$ on $H_*(\Omega)$ for $l \in \mathbb{Z}$ are defined in analogy with the corresponding notions for filtered algebras (see Definition 8.43 and 8.44).

Remark 8.48. We make the following observations.

1. The differential of a filtered chain complex is by definition required to be filtered of degree zero.

2. If $\varphi\colon \Omega \to \Omega'$ is a chain map filtered of degree k, then $\varphi_*\colon H_*(\Omega) \to H_*(\Omega')$ is filtered of degree k as well.

3. Let Ω be a filtered chain complex with ascending filtration C and induced ascending filtration S on homology $H_*(\Omega)$. Then given an element $[x] \in H_*(\Omega)$, observe that

$$\deg_S([x]) = \min_{y\in[x]} \deg_C y.$$

   Replacing max with min yields the same formula in the descending case.

4. Let $\Omega \cong \bigoplus_{i\in\mathcal{I}} \Omega_i$ be an $\mathcal{I}$-graded chain complex (so that each Ω_i is a subcomplex), where $\mathcal{I} \subseteq \mathbb{Z}$. Similar to Remark 8.42, if $\mathcal{I}$ is bounded from below, then the grading on Ω induces an *ascending* filtration $C^a = (C^a_i)_{i\in\mathcal{I}}$ by setting

$$C^a_i := \bigoplus_{\substack{j\in\mathcal{I},\\ j\le i}} \Omega_j, \quad i \in \mathcal{I}.$$

   Similarly, if $\mathcal{I}$ is bounded from above then the grading induces a *descending* filtration $C^d = (C^d_i)_{i\in\mathcal{I}}$ via

$$C^d_i := \bigoplus_{\substack{j\in\mathcal{I},\\ j\ge i}} \Omega_j, \quad i \in \mathcal{I}.$$

[8] We refrain from the introduction of a filtration on homology as the definition should be clear to the reader at this point.

5. An ascending finite length filtration C on a chain complex Ω induces a spectral sequence that converges to the so-called *associated graded group* $\mathcal{G}(H_*(\Omega))$ of $H_*(\Omega)$ with induced filtration S. Here, $\mathcal{G}(H_*(\Omega))$ is defined as
$$\mathcal{G}(H_*(\Omega)) \coloneqq \bigoplus_{i\in\mathcal{I}} G_i, \quad G_i \coloneqq S_i/S_j$$
with $i > j$ such that $i > k \geq j$ implies $k = j$. A similar statement is true for descending filtrations. See [McC85] for details.

We are now ready to introduce the notion of a Frobenius system.

Definition 8.49. A *Frobenius system* (or *Frobenius algebra*) is a 4-tuple $\mathcal{F} = (R, A, \Delta, \varepsilon)$ consisting of a graded commutative unitary ring R and a graded or filtered free R-module A equipped with

1. a commutative algebra structure (multiplication m, unit ι); and
2. a cocommutative coalgebra structure (comultiplication Δ, counit ε),

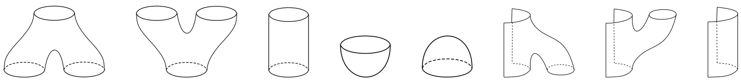
such that the so-called *Frobenius identity* holds:
$$\Delta \circ m = (\mathrm{Id} \otimes m) \circ (\Delta \otimes \mathrm{Id}). \tag{8.18}$$
The maps defining the (co-)algebra structure are required to be homogeneous respectively filtered of a certain degree. The Frobenius system $\mathcal{F}$ is called *graded* (resp. *filtered*) if A is graded (resp. filtered).

As one might guess from the definition there exist many different Frobenius algebras, and we have already encountered one in Section 8.2, namely $A = \mathbb{Q}[X]/(X^2)$, with more examples following shortly. However, we will only be interested in rank 2 Frobenius systems, i.e. Frobenius systems where $A \cong R1_A \oplus RX$ as R-modules for some $X \in A$.

We already mentioned several times that a Frobenius system gives rise to a topological quantum field theory, i.e. a functor from the category of $(1 + 1)$-dimensional cobordisms to a certain category of modules. Before making this precise, let us make two observations.

First, given a rank 2 Frobenius algebra $A \cong R1_A \oplus RX$, we have an isomorphisms of R-modules
$$\underbrace{A \otimes_R \cdots \otimes_R A}_{n \text{ times}} \cong R^{\oplus 2^n} \cong \underbrace{A \oplus \cdots \oplus A}_{2^{n-1} \text{ times}}.$$
In particular, this allows us to consider tensor products $A^{\otimes_R n}$ as objects in the category $\mathcal{M}_A$ (see Definition 8.33).

Second, the morphisms of $\mathrm{Cob}^3_{/l}(2)$ can be expressed as sums (recall that $\mathrm{Cob}^3_{/l}(2)$ is pre-additive) of compositions of disjoint unions of the following elementary cobordisms (details can be found in [Kho00]):

 (8.19)

Thus in order to define a functor on $\mathrm{Cob}^3_{/l}(2)$, it is enough to specify how it acts on objects and the elementary cobordisms in (8.19).

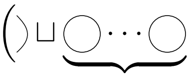
We make the following definition.

Definition 8.50. Let $\mathcal{F} = (R, A, \Delta, \varepsilon)$ be a rank 2 Frobenius system. A *topological quantum field theory* (or *TQFT* for short) is a functor

$$\mathcal{F}\colon \mathrm{Cob}^3_{/l}(2) \to \mathcal{M}_A$$

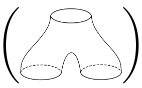
defined as follows:

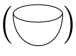
1. On objects, $\mathcal{F}$ acts in the following way:

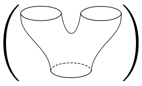
   $\mathcal{F}\Big($  $\Big) = \underline{A}\{1\} \otimes_R \underbrace{A\{1\} \otimes_R \cdots \otimes_R A\{1\}}_{n \text{ times}}.$

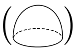
   Here, the special strand corresponds to the first tensor factor while the other factors are ordered according to the enumeration of the circles. The underline indicates the action of A on the tensor product $\underline{A}\{1\}\otimes_R A\{1\}^{\otimes n}$, turning it into an A-module. If an object in $\mathrm{Cob}^3_{/l}(2)$ is shifted in grading, say k, then the resulting tensor factors are shifted by $k+1$, respectively.

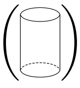
2. On morphisms, $\mathcal{F}$ is defined via

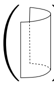
   $\mathcal{F}\Big($  $\Big) = m\colon A\{1\} \otimes A\{1\} \to A\{1\}$

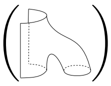
   $\mathcal{F}\Big($  $\Big) = \iota\colon R\{1\} \to A\{1\}$

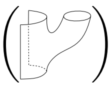
   $\mathcal{F}\Big($  $\Big) = \Delta\colon A\{1\} \to A\{1\} \otimes A\{1\}$

   $\mathcal{F}\Big($  $\Big) = \varepsilon\colon A\{1\} \to R\{1\}$

   $\mathcal{F}\Big($  $\Big) = \mathrm{Id}\colon A\{1\} \to A\{1\}$

   $\mathcal{F}\Big($  $\Big) = \mathrm{Id}\colon \underline{A}\{1\} \to \underline{A}\{1\}$

   $\mathcal{F}\Big($  $\Big) = m\colon \underline{A}\{1\} \otimes A\{1\} \to \underline{A}\{1\}$

   $\mathcal{F}\Big($  $\Big) = \Delta\colon \underline{A}\{1\} \to \underline{A}\{1\} \otimes A\{1\}$

If A is graded, then $\mathcal{F}$ is called a *graded* TQFT, and if A is filtered, then $\mathcal{F}$ is called a *filtered* TQFT. We require that $\mathcal{F}$ respects the grading on $\mathrm{Cob}^3_{/l}(2)$, so that on objects $\mathcal{F}(\mathcal{O}\{k\}) = \mathcal{F}(\mathcal{O})\{k\}$ for $k \in \mathbb{Z}$ and similar for morphisms.

Notation. As the symbols in Definition 8.49 and 8.50 already indicate, we abuse notation and use the letter $\mathcal{F}$ to denote the Frobenius system, the corresponding algebra A, and the corresponding TQFT. Furthermore, we abuse language and refer to $\mathcal{F}$ as a Frobenius system (resp. Frobenius algebra) and TQFT interchangeably. We will sometimes also use the algebra A to denote the entire Frobenius system (resp. Frobenius algebra).

Let us make several remarks regarding TQFTs.

Remark 8.51.

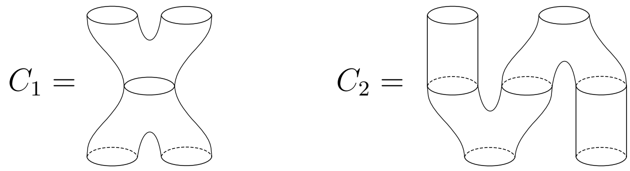
1.) If $\mathcal{F}$ is a filtered TQFT, then we use Remark 8.45 to endow tensor products $\underline{A}\{1\} \otimes_R A\{1\} \otimes_R \cdots \otimes_R A\{1\}$ with the induced filtration so that $\mathcal{F}$ returns when applied to objects in $\mathrm{Cob}^3_{/l}(2)$ a filtered space.

2.) The grading shift on A by 1 in the definition of a TQFT are to be explained as follows: we will only work with rank 2 Frobenius system that are either graded, or whose filtration is induced by a grading. In this sense, if $A \cong R1_A \oplus RX$ in specific examples, then we will always use the grading
$$\deg 1 = 0, \quad \deg X = -2, \tag{8.20}$$
and endow A if necessary with the filtration induced by that grading (as for instance in Section 8.10). Now, the grading shift is needed in order to obtain a graded (resp. filtered) chain complex whose differential is of graded (resp. filtered) degree zero when applying $\mathcal{F}$ to the cube of resolutions of a link or tangle, see Section 8.6. The advantage of (8.20) is however that in our examples, the algebra multiplication will always be of graded or filtered degree zero.

3.) Let D_T be an object in $\mathrm{Cob}^3_{/l}(2)$, i.e. an unoriented crossingless tangle diagram in the unit disk. Endowing D_T with an orientation, the 1:1-correspondence between 2-ended tangle diagrams and base pointed link diagrams (see (8.5)) assigns to the equivalence class of D_T a unique isotopy class of a base-pointed oriented crossingless link diagram. If we forget about the base point and orientation, and endow circles with an enumeration, then each link diagram in this isotopy class is an object in $\mathrm{Cob}^3_{/l}(0)$. Since a TQFT $\mathcal{F}$ assigns by definition isomorphic A-modules to isomorphic objects in $\mathrm{Cob}^3_{/l}(2)$, $\mathcal{F}$ induces a well-defined functor
$$\mathcal{F}' \colon \mathrm{Cob}^3_{/l}(0) \to \mathcal{M}_R$$
by similar rules as in Definition 8.50. Observe that the target category of $\mathcal{F}'$ is $\mathcal{M}_R$ and not $\mathcal{M}_A$; this is due to the fact that after translating an

object in $\mathrm{Cob}^3_{/l}(0)$ into a tensor product $A\{1\}^{\otimes n}$, there is no canonical way to obtain an A-module structure on said tensor product without introducing further conventions.

4.) Let $\mathcal{F}$ be a TQFT corresponding to a rank 2 Frobenius system. Consider the following two cobordisms in $\mathrm{Cob}^3_{/l}(2)$:



Clearly, C_1 and C_2 are isotopic preserving the boundary, so we better have $\mathcal{F}(C_1) = \mathcal{F}(C_2)$. Now observe that

$$\mathcal{F}(C_1) = \Delta \circ m$$
$$\mathcal{F}(C_2) = (\mathrm{Id} \otimes m) \circ (\Delta \otimes \mathrm{Id}).$$

The equality $\Delta \circ m = (\mathrm{Id} \otimes m) \circ (\Delta \otimes \mathrm{Id})$ is precisely the Frobenius identity of a Frobenius system (see Definition 8.49), and so $\mathcal{F}(C_1) = \mathcal{F}(C_2)$ does indeed hold.

5.) The term *topological quantum field theory* has a broad use throughout physics and mathematics (see Section 2.4), and definitions might differ throughout the literature. In mathematics, a TQFT is usually understood in the sense of Atiyah's axiomatic formulation in [Ati88b], with the basic idea being that a TQFT should be a functor from a certain category of cobordisms to a certain category of vector spaces or modules. Precise definitions are then made depending on the context and individual use, such as we did with Definition 8.50.

Before looking at examples, let us introduce as in [Kho06] two operations that deform Frobenius systems.

Definition 8.52. Let $\mathcal{F} = (R, A, \Delta, \varepsilon)$ be a Frobenius system, R' a graded commutative unitary ring, and $\varphi\colon R \to R'$ a ring homomorphism. Then we define a Frobenius system $\mathcal{F}' = (R', A', \Delta', \varepsilon')$ as follows:

$$A' := A \otimes_R R'$$

$$m' := m \otimes \mathrm{Id}_{R'}, \quad \iota' := \iota \otimes \mathrm{Id}_{R'}$$
$$\Delta' := \Delta \otimes \mathrm{Id}_{R'}, \quad \varepsilon' := \varepsilon \otimes \mathrm{Id}_{R'}$$

(here, we used that $A' \otimes_{R'} A' \cong (A \otimes_R A) \otimes_R R'$ in order to define m'). The Frobenius system $\mathcal{F}'$ is said to be obtained from $\mathcal{F}$ by a *base change*.

Remark 8.53. If the Frobenius system $\mathcal{F}$ is graded, then a base change might collapse the grading, depending on whether $\varphi\colon R \to R'$ is homogeneous of a certain degree. Thus the resulting Frobenius system $\mathcal{F}'$ might no longer be graded, but can filtered instead (for an example, see Definition 8.59).

Definition 8.54. Let $\mathcal{F} = (R, A, \Delta, \varepsilon)$ be a Frobenius system and $y \in A$ invertible. Then we define a Frobenius system $\mathcal{F}^t = (R, A, \Delta', \varepsilon')$ by setting

$$\Delta'(x) := \Delta(y^{-1}x), \quad \varepsilon'(x) := \varepsilon(yx).$$

The Frobenius system $\mathcal{F}^t$ is said to be obtained from $\mathcal{F}$ by a *twist*.

In order to motivate the following examples, we will take for the moment for granted that a Frobenius system and the corresponding TQFT yields a Khovanov-type homology theory. We will explain this process in the upcoming Section 8.6.

Let us now introduce the explicit Frobenius systems that we will encounter throughout this text. In the following, we will accept for motivational purposes that a Frobenius system induces a homology theory, and will explain this process in the upcoming Section 8.9.

Definition 8.55 (Khovanov System). The Frobenius system $\mathcal{F}_{\mathbb{Z}} = (R_{\mathbb{Z}}, A_{\mathbb{Z}}, \Delta, \varepsilon)$ is defined as

$$R_{\mathbb{Z}} = \mathbb{Z}, \quad A_{\mathbb{Z}} = \mathbb{Z}[X]/(X^2), \quad \iota(1) = 1,$$

$$\begin{aligned} m(1 \otimes 1) &= 1 & m(X \otimes X) &= 0 \\ m(1 \otimes X) &= X & m(X \otimes 1) &= X \\ \Delta(1) &= 1 \otimes X + X \otimes 1 & \varepsilon(1) &= 0 \\ \Delta(X) &= X \otimes X & \varepsilon(X) &= 1 \end{aligned}$$

We equip $\mathcal{F}_{\mathbb{Z}}$ with a grading by setting

$$\deg 1 = 0, \quad \deg X = -2.$$

The maps $m, \iota, \Delta, \varepsilon$ are then homogeneous of degree $0, 0, -2, 2$, respectively. If $\mathbb{F}$ is a field (e.g. $\mathbb{Q}$ or $\mathbb{F}_p$ for a prime p), we define $\mathcal{F}_{\mathbb{F}}$ as $\mathcal{F}_{\mathbb{Z}}$ with coefficients switched to $\mathbb{F}$ (more formally, the unique map $\varphi\colon \mathbb{Z} \to \mathbb{F}$ given by $1 \mapsto 1$ defines a base change from $\mathcal{F}_{\mathbb{Z}}$ to $\mathcal{F}_{\mathbb{F}}$).

The system $\mathcal{F}_{\mathbb{Z}}$ is the Frobenius system originally used by Khovanov [Kho00] in order to construct his homology theory (up to a shift in grading).[9] The system $\mathcal{F}_{\mathbb{Q}}$ is the one that we encountered in Section 8.2 (up to a shift in grading).

[9] In fact, Khovanov worked over the even more general ground ring $\mathbb{Z}[c]$, but noticed later in [Kho06] that adding c does not provide any new information on the resulting homology theories.

Definition 8.56 (Universal System). Let h, t be formal variables. The Frobenius system $\mathcal{F}_{\text{univ}} = (R_{\text{univ}}, A_{\text{univ}}, \Delta, \varepsilon)$ is defined as

$$R_{\text{univ}} = \mathbb{Z}[h,t], \quad A_{\text{univ}} = R_{\text{univ}}[X]/(X^2 - hX - t), \quad \iota\colon R_{\text{univ}} \hookrightarrow A_{\text{univ}},$$

$$\begin{aligned} m(1 \otimes 1) &= 1 & m(X \otimes X) &= hX + t \\ m(1 \otimes X) &= X & m(X \otimes 1) &= X \\ \Delta(1) &= 1 \otimes X + X \otimes 1 - h1 \otimes 1 & \varepsilon(1) &= 0 \\ \Delta(X) &= X \otimes X + t1 \otimes 1 & \varepsilon(X) &= 1 \end{aligned}$$

We equip $\mathcal{F}_{\text{univ}}$ with a grading by setting

$$\deg 1 = 0, \quad \deg X = \deg h = -2, \quad \deg t = -4.$$

The maps $m, \iota, \Delta, \varepsilon$ are then homogeneous of degree $0, 0, -2, 2$, respectively.

This Frobenius system was first described by Khovanov in [Kho06]. As the subscript indicates, $\mathcal{F}_{\text{univ}}$ is a *universal* Frobenius system in the sense that any other rank 2 Frobenius system can be obtained from $\mathcal{F}_{\text{univ}}$ by a composition of base change and twist, see [Kho06, Proposition 5]. As an example, the ring homomorphism $\varphi\colon R_{\text{univ}} \to \mathbb{Z}$ defined by $\varphi(h) = \varphi(t) = 0$ and $\varphi(1) = 1$ defines a base change giving back the previously described Frobenius system $\mathcal{F}_{\mathbb{Z}}$.

Definition 8.57 ($\mathbb{Z}[G]$-System). Let G be a formal variable. The Frobenius system $\mathcal{F}_{\mathbb{Z}[G]} = (R_{\mathbb{Z}[G]}, A_{\mathbb{Z}[G]}, \Delta, \varepsilon)$ is defined as

$$R_{\mathbb{Z}[G]} = \mathbb{Z}[G], \quad A_{\mathbb{Z}[G]} = R_{\mathbb{Z}[G]}[X]/(X^2 + GX), \quad \iota\colon R_{\mathbb{Z}[G]} \hookrightarrow A_{\mathbb{Z}[G]},$$

$$\begin{aligned} m(1 \otimes 1) &= 1 & m(X \otimes X) &= -GX \\ m(1 \otimes X) &= X & m(X \otimes 1) &= X \\ \Delta(1) &= 1 \otimes X + X \otimes 1 + G1 \otimes 1 & \varepsilon(1) &= 0 \\ \Delta(X) &= X \otimes X & \varepsilon(X) &= 1 \end{aligned}$$

We equip $\mathcal{F}_{\mathbb{Z}[G]}$ with a grading by setting

$$\deg 1 = 0, \quad \deg X = \deg G = -2.$$

The maps $m, \iota, \Delta, \varepsilon$ are then homogeneous of degree $0, 0, -2, 2$, respectively.

As the subscript indicates, $\mathcal{F}_{\mathbb{Z}[G]}$ will be the Frobenius system that we use in order to construct $\mathbb{Z}[G]$-homology in Chapter 9. Observe that $\mathcal{F}_{\mathbb{Z}[G]}$ can be obtained from the universal system $\mathcal{F}_{\text{univ}}$ by a base change that sends $1 \mapsto 1$, $h \mapsto -G$, $t \mapsto 0$. We will see in Section 9.2 that $\mathcal{F}_{\mathbb{Z}[G]}$ yields a homology theory that is as strong as the one induced by $\mathcal{F}_{\text{univ}}$, despite having less formal variables. Note that if we switch coefficients from $\mathbb{Z}$ to $\mathbb{F}_2$ and replace G with the formal variable h, then we obtain Bar-Natan's system in [Bar05, Section 9].

Definition 8.58 (Alishahi-Dowlin System). Let t be a formal variable. The Frobenius system $\mathcal{F}_{\mathrm{AD}} = (R_{\mathrm{AD}}, A_{\mathrm{AD}}, \Delta, \varepsilon)$ is defined as

$$R_{\mathrm{AD}} = \mathbb{Q}[t], \quad A_{\mathrm{AD}} = R_{\mathrm{AD}}[X]/(X^2 - t), \quad \iota\colon R_{\mathrm{AD}} \hookrightarrow A_{\mathrm{AD}},$$

$$\begin{aligned} &m(1 \otimes 1) = 1 && m(X \otimes X) = t \\ &m(1 \otimes X) = X && m(X \otimes 1) = X \\ &\Delta(1) = 1 \otimes X + X \otimes 1 && \varepsilon(1) = 0 \\ &\Delta(X) = X \otimes X + t1 \otimes 1 && \varepsilon(X) = 1 \end{aligned}$$

We equip $\mathcal{F}_{\mathrm{AD}}$ with a grading by setting

$$\deg 1 = 0, \quad \deg X = -2, \quad \deg t = -4.$$

The maps $m, \iota, \Delta, \varepsilon$ are then homogeneous of degree $0, 0, -2, 2$, respectively.

The system $\mathcal{F}_{\mathrm{AD}}$ was used recently by Alishahi-Dowlin [AD19] to construct their invariant $\mathfrak{u}_X(K)$ (resp. $\mathfrak{u}_t(K)$) that yields a lower bound on the unknotting number of a knot K (see also Section 10.4). $\mathcal{F}_{\mathrm{AD}}$ is closely related to the following famous system.

Definition 8.59 (Lee System). The Frobenius system $\mathcal{F}_{\mathrm{Lee}} = (R_{\mathrm{Lee}}, A_{\mathrm{Lee}}, \Delta, \varepsilon)$ is defined as

$$R_{\mathrm{Lee}} = \mathbb{Q}, \quad A_{\mathrm{Lee}} = \mathbb{Q}[X]/(X^2 - 1), \quad \iota\colon \mathbb{Q} \hookrightarrow A_{\mathrm{Lee}},$$

$$\begin{aligned} &m(1 \otimes 1) = 1 && m(X \otimes X) = 1 \\ &m(1 \otimes X) = X && m(X \otimes 1) = X \\ &\Delta(1) = 1 \otimes X + X \otimes 1 && \varepsilon(1) = 0 \\ &\Delta(X) = X \otimes X + 1 \otimes 1 && \varepsilon(X) = 1 \end{aligned}$$

We equip $\mathcal{F}_{\mathrm{Lee}}$ with a descending filtration given by

$$\{0\} \subset F_0 \subset F_{-2} = A_{\mathrm{Lee}},$$

where F_0 is generated by 1 and F_{-2} is generated by 1 and X. The maps $m, \iota, \Delta, \varepsilon$ are then filtered of degree $0, 0, -2, 0$ respectively.

The system $\mathcal{F}_{\mathrm{Lee}}$ was first introduced by Lee [Lee05] as a deformation of Khovanov homology. It is of particular interest because it induces a filtered homology theory that gives rise to a spectral sequence with E_2 page Khovanov homology that converges to the associated graded of the filtration on homology. Moreover, the system was used by Rasmussen [Ras10] in order to obtain his famous s-invariant, which provides a lower bound on the slice genus of a knot. More details will be given in Section 8.10. Note that $\mathcal{F}_{\mathrm{Lee}}$ results from $\mathcal{F}_{\mathrm{AD}}$ by setting $t = 1$. This collapses the grading in $\mathcal{F}_{\mathrm{AD}}$ (i.e. the (co-)algebra maps are no longer homogeneous), but the descending filtration on A_{AD} that is induced by the grading yields precisely the filtration on A_{Lee} after setting $t = 1$.

Definition 8.60 (Bar-Natan System). Let h be a formal variable. The Frobenius system $\mathcal{F}_{\mathrm{BN}} = (R_{\mathrm{BN}}, A_{\mathrm{BN}}, \Delta, \varepsilon)$ is defined as

$$R_{\mathrm{BN}} = \mathbb{F}_2[h], \quad A_{\mathrm{BN}} = R_{\mathrm{BN}}[X]/(X^2 - hX), \quad \iota\colon R_{\mathrm{BN}} \hookrightarrow A_{\mathrm{BN}},$$

$$\begin{aligned} m(1\otimes 1) &= 1 & m(X\otimes X) &= hX \\ m(1\otimes X) &= X & m(X\otimes 1) &= X \end{aligned}$$

$$\begin{aligned} \Delta(1) &= 1\otimes X + X\otimes 1 - h1\otimes 1 & \varepsilon(1) &= 0 \\ \Delta(X) &= X\otimes X & \varepsilon(X) &= 1 \end{aligned}$$

We equip $\mathcal{F}_{\mathrm{BN}}$ with a grading by setting

$$\deg 1 = 0, \quad \deg X = \deg h = -2.$$

The maps $m, \iota, \Delta, \varepsilon$ are then homogeneous of degree $0, 0, -2, 2$, respectively.

The system $\mathcal{F}_{\mathrm{BN}}$ was first introduced by Bar-Natan in [Bar05]. Turner [Tur06] showed that setting $h = 1$ yields a filtered Frobenius system similar to the Lee system with ground field $\mathbb{F}_2$. This filtered system also induces a spectral sequence with E_1 page Khovanov homology over $\mathbb{F}_2$ (i.e. the homology resulting from $\mathcal{F}_2$) that converges to the associated graded, and it can be used to define the Rasmussen s-invariant over $\mathbb{F}_2$. Again, more details will be provided in Section 8.10.

Definition 8.61 ((α, β)-System). Let $\mathbb{F}$ be a field (e.g. $\mathbb{Q}$ or $\mathbb{F}_p$ with p a prime), and $\alpha, \beta \in \mathbb{K}$. The Frobenius system $\mathcal{F}_{\alpha,\beta} = (R_{\alpha,\beta}, A_{\alpha,\beta}, \Delta, \varepsilon)$ is defined as

$$R_{\alpha,\beta} = \mathbb{F}, \quad A_{\alpha,\beta} = \mathbb{F}[X]/(X^2 - \alpha X - \beta), \quad \iota\colon \mathbb{F} \hookrightarrow A_{\alpha,\beta},$$

$$\begin{aligned} m(1\otimes 1) &= 1 & m(X\otimes X) &= \alpha X + \beta \\ m(1\otimes X) &= X & m(X\otimes 1) &= X \end{aligned}$$

$$\begin{aligned} \Delta(1) &= 1\otimes X + X\otimes 1 - \alpha 1\otimes 1 & \varepsilon(1) &= 0 \\ \Delta(X) &= X\otimes X + \beta 1\otimes 1 & \varepsilon(X) &= 1 \end{aligned}$$

We equip $\mathcal{F}_{\alpha,\beta}$ with a descending filtration given by

$$\{0\} \subset F_0 \subset F_{-2} = A_{\alpha,\beta},$$

where F_0 is generated by 1 and F_{-2} is generated by 1 and X. The maps $m, \iota, \Delta, \varepsilon$ are then filtered of degree $0, 0, -2, 0$ respectively.

The system $\mathcal{F}_{\alpha,\beta}$ was used by MacKaay-Turner-Vaz in [MTV07] in order to define the Rasmussen invariant over the field $\mathbb{K}$. Note that $\mathcal{F}_{\alpha,\beta}$ results from $\mathcal{F}_{\mathrm{univ}}$ by switching coefficients form $\mathbb{Z}$ to $\mathbb{K}$ and specializing h and t to elements of the field. Observe that choosing $\mathbb{F} = \mathbb{Q}$ and $\alpha = 0, \beta = 1$ recovers the system $\mathcal{F}_{\mathrm{Lee}}$. As for the Lee system, this collapses the grading but the induced descending filtration on A_{univ} yields the filtration on $\mathcal{F}_{\alpha,\beta}$ after switching coefficients and specializing h and t.

8.6 The Bar-Natan complex of tangles

Armed with the definitions and notions from Sections 8.3 to 8.5, we are ready to introduce Bar-Natan's theory of tangles and cobordisms with ease. In fact, the construction of the Bar-Natan chain complex is very similar to the construction of the Khovanov complex $\mathcal{C}_{\mathrm{Kh}}$ in Section 8.4. Let's start.

Let T be a $2n$-ended (oriented) tangle with (oriented) tangle diagram D_T (as defined in Definition 8.21) and with crossings enumerated. Let $\mathcal{X} \subset D_T$ denote the subset of crossings, set $n := |\mathcal{X}|$ and write $n = n_+ + n_-$, where n_+ and n_- denote the number of positive and negative crossings in D_T, respectively.

Just as for an ordinary link diagram, we can form the cube of resolutions of D_T as follows. An n-tuple $\alpha \in \{0,1\}^{\mathcal{X}}$ defines a complete smoothing S_α of D_T by 0- and 1-resolving the crossings according to α with respect to the enumeration of crossings. In order to obtain an ordering on the circles in S_α, we introduce the following convention.

Convention. Given a $2n$-ended tangle T with diagram D_T, we label every edge of D_T by integers in an ascending fashion starting at 1. Given a complete smoothing S_α of D_T, we label each component with the minimal integer of the edge appearing in it. In particular, circles are now ordered in an ascending fashion with not necessarily consecutive integers. Then we forget the labelling of the arcs, and reenumerate the circles according to their order starting at 1. See Figure 8.16 for an example.

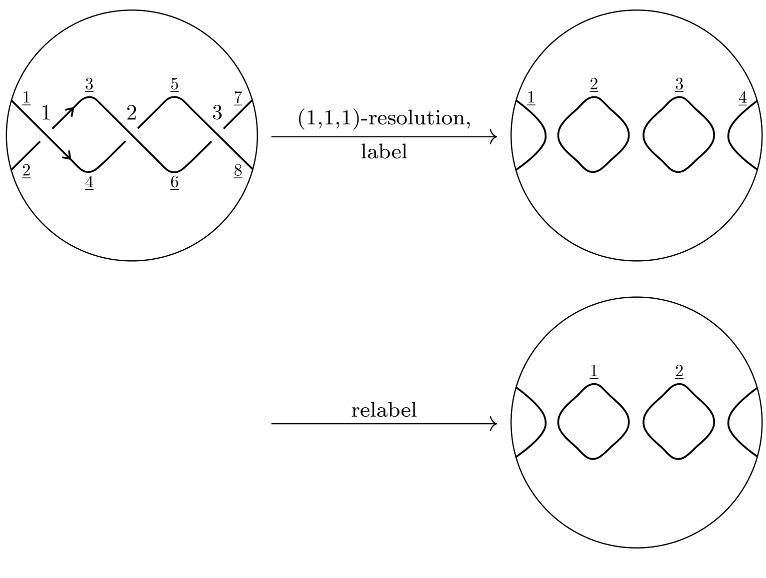


Figure 8.16: Our convention for enumerating circles in complete smoothings of tangle diagrams.

With the above convention, each complete smoothing S_α defines an object in the category $\mathrm{Cob}^3_{/l}(2n)$.

Given an edge σ in the 1-skeleton of the n-dimensional cube $[0,1]^{\mathcal{X}}$ connecting vertices α_1 and α_2 with $|\alpha_1| = |\alpha_2| - 1$, we encode σ again using an n-tuple $\sigma \in \{0, 1, \star\}$ that contains exactly one $\star$ at the single position where the coordinates of α_1 and α_2 change.

Just as for links diagrams, the complete smoothings S_{α_1} and S_{α_2} are obtained by resolving crossings in D_T in the same way, except at a single crossing, say the i-th, that gets 0-resolved in S_{α_1} and 1-resolved in S_{α_2}. Geometrically, S_{α_1} and S_{α_2} differ by a merging or splitting of a combination of circles or arcs. Now we can consider again a cobordism C_σ from S_{α_1} to S_{α_2} that is a product except in a neighborhood of the former i-th crossing, where the cobordism is made up of the obvious saddle between the 0- and 1-resolution.

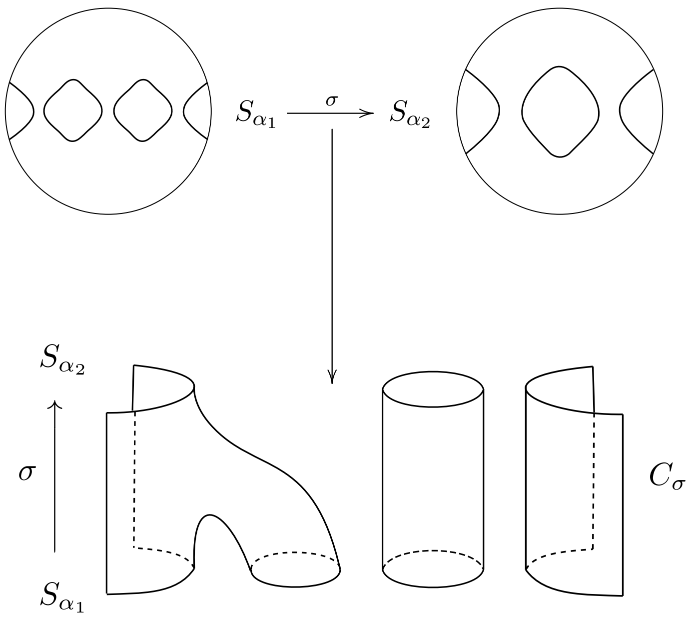


Figure 8.17: The cobordism C_σ assigned to an edge σ connecting the complete smoothings S_{α_1} and S_{α_2}.

This gives us a way to identify the edge σ with the cobordism

$$C_\sigma \in \hom_{\mathrm{Cob}^3_{/l}(2n)}(S_{\alpha_1}, S_{\alpha_2}),$$

well-defined up to boundary-fixing isotopy, and we are now ready to define the cube of resolutions for a tangle diagram D_T.

Definition 8.62. Let T be a $2n$-ended oriented tangle with oriented tangle diagram D_T with crossings enumerated. Let $\mathcal{X} \subset D_T$ be the subset of crossings of D_T. Then the 1-skeleton of the cube $[0,1]^{\mathcal{X}}$ with

1. vertices $\alpha \in \{0,1\}^{\mathcal{X}}$ identified with the corresponding enumerated complete smoothings $S_\alpha \in \mathrm{ob}(\mathrm{Cob}^3_{/l}(2n))$; and
2. edges σ identified with the corresponding cobordism C_σ that is contained in $\hom_{\mathrm{Cob}^3_{/l}(2n)}(S_{\alpha_1}, S_{\alpha_2})$

is called the *cube of resolutions* of D_T and denoted by $Q(D_T)$.

Remark 8.63. Let L be a 0-ended tangle, that is a link in S^3, with diagram D_L. Then $Q(D_L)$ is nothing but the cube of resolutions of a link diagram as described in Definition 8.6, with vertices and edges described in the language of $\mathrm{Cob}^3_{/l}(0)$.

Now, when constructing the Khovanov chain complex $\mathcal{C}_{Kh}$ in Section 8.2, we translated the cobordisms C_σ into linear maps m and Δ in order to obtain a differential d_{Kh}. However, for the Bar-Natan chain complex we won't translate anything into algebra yet and instead form a chain complex in $\mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2n)))$ from the cube of resolutions $Q(D_T)$. For $0 \leq r \leq n$, define

$$C^r(D_T) \coloneqq \bigoplus_{\substack{\alpha\in\{0,1\},\\ |\alpha|=r}} S_\alpha\{r\} \in \mathrm{ob}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2n))),$$

where we order the summands in reversed lexicographical order with respect to α (so that we "flatten" the cube of resolutions top-down),[10] and set[11]

$$[D_T] \coloneqq \bigoplus_{r=0}^{n} C^r(D_T)[-n_-]\{n_+ - 2n_-\}.$$

For the differentials, let $\sigma \in \{0,1,\star\}$ be an edge of the cube of resolutions identified with the cobordism C_σ. Then we define

$$d_\sigma \coloneqq C_\sigma \in \hom_{\mathrm{Cob}^3_{/l}(2n)}(S_{\alpha_1}, S_{\alpha_2}).$$

and set for $0 \leq r \leq n$

$$d^r \coloneqq \sum_{|\sigma|=r} (-1)^\sigma d_\sigma \in \hom_{\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2n))}(C^r(D_T), C^{r+1}(D_T))$$

with $C^{n+1}(D_T) \coloneqq 0$. Here, $|\sigma|$ denotes the height of σ as defined in (8.3), and $(-1)^\sigma$ is defined as

$$(-1)^\sigma \coloneqq (-1)^{\sum_{i<j}\sigma_i},$$

where j is the coordinate of the single $\star$ in σ. Let us make two observations:

[10] This is our convention for ordering summands in $C^r(D_T)$, but the result does in fact not depend on the choice of ordering, see [BN02].

[11] The notation $[D_T]$ is chosen to be consistent with [ILM21].

1. The linear map d^r is homogeneous of degree zero: this follows from our grading shifts in the definition of $C^r(D_T)$ and the fact that $\deg C_\sigma = -1$, which in turn follows since the degree of a saddle (which is a morphism in $\mathrm{Cob}^3_{/l}(4)$) is -1 (see Definition 8.37).

2. We have that $d^{r+1} \circ d^r = 0$. Indeed, in order to see this it is enough to understand that all square faces in $Q(D_T)$ anti-commute. Just as for the Khovanov complex, our choice of signs guarantees anti-commutativity of square faces provided that they positively commute (i.e. commute with signs ignored). But this is clear, because we can isotope saddles in a composition $C_{\sigma_2} \circ C_{\sigma_1}$ so they are arranged in reverse order (cf. [Bar05]).

Setting

$$d_{\mathrm{BN}} := \bigoplus_{r=0}^{n} d^r,$$

it follows from the previous two observations that we have constructed an honest chain complex $([D_T], d_{\mathrm{BN}})$. We are ready for the following definition.

Definition 8.64 (Bar-Natan chain complex). Let T be a $2n$-ended oriented tangle with oriented tangle diagram D_T. Then the complex

$$([D_T], d_{\mathrm{BN}}) \in \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2n)))$$

is called the *Bar-Natan chain complex* of D_T. Likewise, the complex

$$[T] := ([D_T], d_{\mathrm{BN}}) \in \mathrm{Kom}_{/h}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2n)))$$

is called *Bar-Natan chain complex* of T.

An example of $([D_T], d_{\mathrm{BN}})$ is given in Figure 8.18. The notation $[T]$ for the Bar-Natan chain complex of a tangle will be justified shortly. In contrast to Khovanov homology, the main actor in Bar-Natan's theory is the complex itself. Even more, we are not yet able to obtain any sort of homology theory from $[D_T]$ because $\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2n))$ is not an abelian category. Nonetheless, $[D_T]$ is in fact an invariant of tangles when considered up to chain homotopy equivalence, and a strong tool for applications as subsequent sections will show. The following is a main theorem of Bar-Natan.

Theorem 8.65 ([Bar05, Theorem 1]). The isomorphism class of the complex $([D_T], d_{\mathrm{BN}})$ considered in $\mathrm{Kom}_{/h}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2n)))$ is an invariant of the tangle T. That is, it does not depend on the ordering of the crossings of D_T and on the ordering of the complete smoothings in $C^r(D_T)$, and is invariant under the three Reidemeister moves.

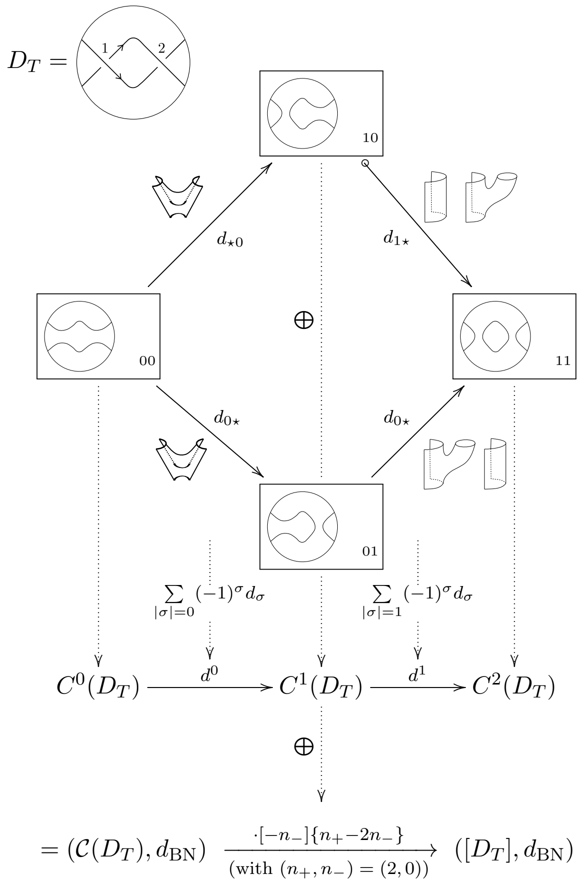


Figure 8.18: The cube of resolutions $Q(D_T)$ for the tangle diagram D_T with edges decorated by the maps d_σ that lead to the differential d_{BN}. A little circle at the tail of an edge indicates that the map appears with a minus sign in the sum d^r. The bottom lines show how to obtain the chain complex $([D_T], d_{\mathrm{BN}})$ from the depicted cube.

A proof of Theorem 8.65 is given in [BN02, Section 4.3]. Note that this is the point where the local relations S, T and $4Tu$ are needed in $\mathrm{Cob}^3_{/l}(2n)$ in order to obtain invariance under the three Reidemeister moves.

The Bar-Natan complex of a tangle has a very geometrical and combinatorial nature, which is very advantageous for computations as we shall see shortly in Sections 8.7 and 8.8. Of course, one may also obtain a homology theory from the Bar-Natan complex using a TQFT, and we will discuss how this is done in Section 8.9.

8.7 Planar arc diagrams and compatibility results

One of the great advantages of tangles and the complex $[T]$ is their composition properties, formalized in terms of *planar arc diagrams*. This composability in combination with a certain form of Gaussian elimination and an isomorphism in $\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))$ called *delooping* yields an algorithm known as *divide-and-conquer* for computing and simplifying $[T]$, which further simplifies the calculation of homology theories obtained from the Bar-Natan complex. This will be the subject of upcoming sections, but for now let's take a closer look at planar arc diagrams. Main reference is [Bar05, Section 5].

As mentioned at the beginning of Section 8.3, tangles can be considered as the building blocks of links, and it should be intuitively clear how to build a link diagram from a finite collection of tangle diagrams ("place tangle diagrams in a big fixed disk and close up strands in any fashion"). This process of gluing tangle diagrams together was formalized by Bar-Natan [Bar05], using so-called planar arc diagrams.

Convention. For reasons of well-definedness, we will consider in the following without further mention all disks up to orientation preserving homeomorphisms mapping end points to end points, base points to base points, and keeping enumeration whenever it makes sense.

Definition 8.66. Let $d \in \mathbb{N}$. An *oriented d-input planar arc diagram $\mathcal{D}$* consists of

1. a disk $\mathcal{D}$ (called *output disk*) with d enumerated open disjoint so-called *input disks E_i* removed from its interior; and
2. a proper smooth oriented 1-submanifold of $\mathcal{D}$ with end points on the boundary $\partial\mathcal{D}$.

Here, we have that

$$\partial\mathcal{D} = \bigcup_{D \in \{E, E_1, \dots, E_d\}} \partial D$$

where E denotes the output disk with the input disks placed back in. The number of end points on each ∂D is required to be even; if it is non-zero then one of the end points is required to be chosen and fixed as base point of D. Forgetting about the orientation of the proper smooth 1-submanifolds, $\mathcal{D}$ is called an *unoriented planar arc diagram.*

Let $\mathcal{D}$ be an oriented d-input planar arc diagram and E_i an input disk with $2n_{E_i} > 0$ end points on ∂E_i. The orientation of the proper smooth 1-submanifold yields an ordered sequence $s_{E_i} \in \{\mathrm{i}, \mathrm{o}\}^{2n_{E_i}}$ of "in" (i) and "out" (o) that describes the direction of the arcs connected to the end points of ∂E_i, with ordering given by starting at the base point and moving counterclockwise (see Figure 8.19 for an example). Given a $2n_{E_i}$-ended oriented tangle diagram D in a disk H, let $D^\bullet$ denote the same diagram with one of the end points fixed as base point. Then the orientation of $D^\bullet$ induces an ordered sequence $s_{D^\bullet} \in \{\mathrm{i}, \mathrm{o}\}^{2n_{E_i}}$ in the exact same way as s_{E_i} is defined. Let $\varphi\colon H \to E_i$ be an orientation-preserving homeomorphism mapping end points to end points and base point to base point (note that φ is

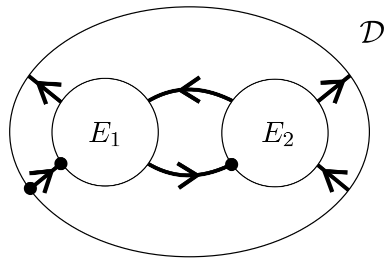


Figure 8.19: An oriented 2-input planar arc diagram $\mathcal{D}$. Here, $2n_{E_1} = 2n_{E_2} = 4$ and $s_{E_1} = (\mathrm{i}, \mathrm{o}, \mathrm{i}, \mathrm{o})$, $s_{E_2} = (\mathrm{i}, \mathrm{i}, \mathrm{o}, \mathrm{o})$.

unique up to isotopy, see Section 8.3). If $s_{D^\bullet} = s_{E_i}$, then we can use the homeomorphism φ in order to place the tangle diagram $D^\bullet$ inside the input disk E_i, which in turn can be put into the planar arc diagram $\mathcal{D}$. Let's make the following definition.

Definition 8.67. Let $n \in \mathbb{N}_{>0}$ and $s \in \{\mathrm{i}, \mathrm{o}\}^{2n}$. Then we define $\mathcal{T}^0(s)$ as the collection of all $2n$-ended oriented base-pointed tangle diagrams $D^\bullet$ with $s_{D^\bullet} = s$, and $\mathcal{T}(s)$ for the corresponding quotient of $\mathcal{T}^0(s)$ modulo the three Reidemeister moves. We further write $\mathcal{T}^0(0)$ for the collection of all 0-ended oriented tangle diagrams and $\mathcal{T}(0)$ for the quotient modulo the three Reidemeister moves.

Definition 8.68. Let $\mathcal{D}$ be an oriented d-input planar arc diagram. Then $\mathcal{D}$ defines an operator

$$\mathcal{D}_{\mathcal{T}} \colon \mathcal{T}(s_{E_1}) \times \cdots \times \mathcal{T}(s_{E_d}) \to \mathcal{T}(s_E) \tag{8.21}$$

that places d oriented base-pointed tangle diagrams inside the d holes of the planar arc diagram $\mathcal{D}$ using the construction above. If either E or one of the input disks E_i has no end-points, then we set $s_E = 0$ or $s_{E_i} = 0$ in (8.21), respectively. The operator $\mathcal{D}_{\mathcal{T}}$ is called *planar arc diagram operator*.

The procedure that we described above in order to place a tangle diagram inside an oriented planar arc diagram also works in the unoriented case. Indeed, let $\mathcal{D}$ be an unoriented d-input planar arc diagram and E_i an input disk with $2n_{E_i} > 0$ end points. Given an *unoriented* $2n$-ended tangle diagram D in some fixed disk H, let $D^\bullet$ denote the same tangle diagram with one of the end points fixed as base point. As before, any orientation-preserving homeomorphism $\varphi \colon H \to E_i$ that maps end points to end points and base point to base point (unique up to isotopy) can be used to place the diagram $D^\bullet$ inside the i-th hole of $\mathcal{D}$. Let us imitate Definition 8.67 and 8.68 in the unoriented scenario.

Definition 8.69. Let $n \in \mathbb{N}$. Then we define $\mathcal{S}^0(2n)$ as the collection of all $2n$-ended unoriented base-pointed (without base point if $n = 0$) tangle diagrams $D^\bullet$, and $\mathcal{S}(2n)$ as the corresponding quotient modulo the three Reidemeister moves.

Definition 8.70. Let $\mathcal{D}$ be an unoriented d-input planar arc diagram. Then $\mathcal{D}$ defines an operator

$$\mathcal{D}_{\mathcal{S}}\colon \mathcal{S}(2n_{E_1}) \times \cdots \times \mathcal{S}(2n_{E_d}) \to \mathcal{S}(2n_E) \tag{8.22}$$

that places d oriented base-pointed tangle diagrams inside the d holes of the planar arc diagram $\mathcal{D}$ using the construction above. The operator $\mathcal{D}_{\mathcal{S}}$ is called *unoriented planar arc diagram operator*.

Let us make the following observations.

Remark 8.71.

1.) Of course, an oriented (resp. unoriented) d-input planar arc diagram $\mathcal{D}$ yields also an operator $\mathcal{D}_{\mathcal{T}^0}$ (resp. $\mathcal{D}_{\mathcal{S}^0}$) in the sense of Definition 8.68 and 8.70.

2.) For each $s \in \{\mathrm{i}, \mathrm{o}\}^{2n}$ with $n > 0$ or $s = 0$, there exists an oriented $|s|$-input "*radial*" planar arc diagram $\mathcal{I}$ such that $\mathcal{I}_{\mathcal{T}}\colon \mathcal{T}(s) \to \mathcal{T}(s)$ is the identity operator. The same is true in the unoriented case, giving an identity operator $\mathcal{I}_{\mathcal{S}}$.

3.) Oriented planar arc diagram operators are "*associative*", that is compatible with each other in a natural way, as follows. Suppose we are given oriented planar arc diagrams $\mathcal{D}$ and $\mathcal{D}'$ such that $\mathcal{D}'$ can be placed into the i-th hole of $\mathcal{D}$ (i.e. so that s_E of $\mathcal{D}'$ matches s_{E_i} of $\mathcal{D}$), and call the resulting oriented planar arc diagram $\widetilde{\mathcal{D}}$. Then on the level of operators, if

$$\mathcal{D}_{\mathcal{T}}\colon \mathcal{T}(s_{E_1}) \times \cdots \times \mathcal{T}(s_{E_d}) \to \mathcal{T}(s_E)$$
$$\mathcal{D}'_{\mathcal{T}}\colon \mathcal{T}(t_{E_1}) \times \cdots \times \mathcal{T}(t_{E_{d'}}) \to \mathcal{T}(s_{E_i})$$

then we have

$$\widetilde{\mathcal{D}}_{\mathcal{T}} = \mathcal{D}_{\mathcal{T}} \circ (\mathcal{I}_{\mathcal{T}} \times \cdots \times \mathcal{D}'_{\mathcal{T}} \times \cdots \times \mathcal{I}_{\mathcal{T}}).$$

As before, the same compatibility holds in the unoriented case and without Reidemeister moves modded out.

4.) The identity and associativity properties also hold for $\mathcal{T}^0(s)$ and $\mathcal{S}^0(2n)$, respectively.

Generalizing the above discussion, we make the following definition.

Definition 8.72. Let S be the collection of all possible finite sequences of symbols i and o. An *oriented planar algebra* is a collection of (possibly empty) sets $\mathcal{P} = \{\mathcal{P}(s)\}_{s\in S}$ such that each oriented d-input planar arc diagram $\mathcal{D}$ defines operators

$$\mathcal{D}_{\mathcal{P}}\colon \mathcal{P}(s_{E_1}) \times \cdots \times \mathcal{P}(s_{E_d}) \to \mathcal{P}(s_E)$$

satisfying identity and associativity properties similar as in Remark 8.71. Likewise, we define an *unoriented planar algebra* as a collection of sets $\mathcal{Q} = \{\mathcal{Q}(k)\}_{k\in\mathbb{N}}$ such that each unoriented d-input planar arc diagram defines operators with the corresponding identity and associativity properties.

Remark 8.73. Every unoriented planar algebra $\mathcal{Q} = \{\mathcal{Q}(k)\}_{k\in\mathbb{N}}$ can trivially be considered as an oriented planar algebra $\mathcal{Q}_{\mathrm{or}} = \{\mathcal{Q}_{\mathrm{or}}(s)\}_{s\in S}$ by setting $\mathcal{Q}_{\mathrm{or}}(s) := \mathcal{Q}(|s|)$ and simply ignoring all orientations on planar arc diagrams $\mathcal{D}$.

Definition 8.74. A *morphism* of oriented planar algebras $\mathcal{P}_1, \mathcal{P}_2$ is a collection of maps $\Phi = \{\Phi_s\colon \mathcal{P}_1(s) \to \mathcal{P}_2(s)\}_{s\in S}$ such that for each oriented d-input planar arc diagram $\mathcal{D}$

$$\Phi_{s_E} \circ \mathcal{D}_{\mathcal{P}_1} = \mathcal{D}_{\mathcal{P}_2} \circ (\Phi_{s_{E_1}} \times \cdots \times \Phi_{s_{E_d}}).$$

A morphism of unoriented planar algebras is defined similarly.

Of course, the collections $\mathcal{T}^0 = \{\mathcal{T}^0(s)\}_{s\in S}$, $\mathcal{T} = \{\mathcal{T}(s)\}_{s\in S}$ and $\mathcal{S}^0 = \{\mathcal{S}^0(2n)\}_{n\in\mathbb{N}}$, $\mathcal{S} = \{\mathcal{S}(2n)\}_{n\in\mathbb{N}}$ form examples of oriented and unoriented planar algebras, respectively.

Now, recall that the objects of $\mathrm{Cob}^3(2n)$ are crossingless unoriented $2n$-ended tangle diagrams in a fixed disk with fixed end points and circles enumerated. We would like to see the objects $\mathrm{ob}(\mathrm{Cob}^3(2n))$ and in fact also the morphisms $\mathrm{mor}(\mathrm{Cob}^3(2n))$ as part of a planar algebra as well. For this, we need to adapt $\mathrm{Cob}^3(2n)$ as follows.

Definition 8.75. Let $\mathrm{Cob}^{3,\bullet}(2n)$ denote the category $\mathrm{Cob}^3(2n)$ but with one end point of the fixed disk containing the unoriented crossingless $2n$-ended tangle diagrams marked as base point. The quotient $\mathrm{Cob}^{3,\bullet}_{/l}(2n)$ is defined similarly.

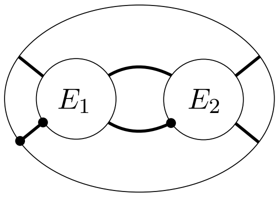
Considered as a stand-alone object, $\mathrm{Cob}^{3,\bullet}(2n)$ (resp. $\mathrm{Cob}^{3,\bullet}_{/l}(2n)$) provides no new information; the base point is just decoration. In particular, morphisms are still composed in the same way as in $\mathrm{Cob}^3(2n)$ (resp. $\mathrm{Cob}^3_{/l}(2n)$). However, forgetting about the enumeration of circles we have $\mathrm{ob}(\mathrm{Cob}^{3,\bullet}(2n)), \mathrm{ob}(\mathrm{Cob}^{3,\bullet}_{/l}(2n)) \subset \mathcal{S}^0(2n)$, and are thus part of the unoriented planar algebra $\mathcal{S}^0$. In fact,

$$\mathrm{ob} := \{\mathrm{ob}(\mathrm{Cob}^{3,\bullet}(2n))\}_{n\in\mathbb{N}}, \quad \mathrm{ob}_{/l} := \{\mathrm{ob}(\mathrm{Cob}^{3,\bullet}_{/l}(2n))\}_{n\in\mathbb{N}}$$

form unoriented sub-planar algebras of $\mathcal{S}^0$.

Convention. In practice we want to consider ob and $\mathrm{ob}_{/l}$ as individual planar algebras, that is with enumeration of circles taken into account. For this, we have to choose a convention how to enumerate circles in the image of the corresponding unoriented planar arc diagram operators. We will do so by starting with the given enumeration of the circles of the diagram that gets placed in the first input disk, and then continue this enumeration in subsequent placed-in diagrams while keeping the individual absolute ordering of the circle within a diagram; see Example 8.76 for an example.

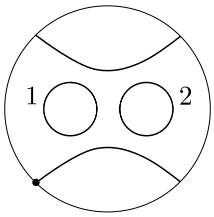
Example 8.76. Consider the unoriented 2-input planar arc diagram

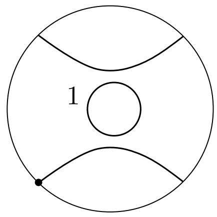
$\mathcal{D} =$ 

Then $\mathcal{D}$ defines an operator

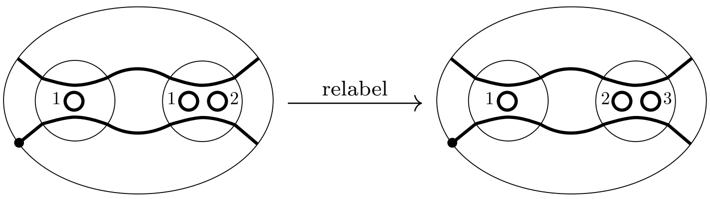
$$\mathcal{D}_{\mathrm{ob}}\colon\ \mathrm{ob}(\mathrm{Cob}^{3,\bullet}(4)) \times \mathrm{ob}(\mathrm{Cob}^{3,\bullet}(4)) \longrightarrow \mathrm{ob}(\mathrm{Cob}^{3,\bullet}(4)).$$

As an example and to demonstrate our convention for enumerating circles, let

$D_1 =$ 

$D_2 =$ 

Then



$= \mathcal{D}_{\mathrm{ob}}(D_1, D_2)$

Similar to the objects, the collection of sets of morphisms

$$\mathrm{mor} \coloneqq \{\mathrm{mor}(\mathrm{Cob}^{3,\bullet}(2n))\}_{n\in\mathbb{N}}, \quad \mathrm{mor}_{/l} \coloneqq \{\mathrm{mor}(\mathrm{Cob}^{3,\bullet}_{/l}(2n))\}_{n\in\mathbb{N}}$$

form also unoriented planar algebras, respectively. Indeed, let $\mathcal{D}$ be an unoriented d-input planar arc diagram. Then the thickening $\mathcal{D} \times [0,1]$ yields an operator

$$\mathcal{D}_{\mathrm{mor}}\colon \prod_{i=1}^{d} \mathrm{mor}(\mathrm{Cob}^{3,\bullet}(2n_{E_i})) \longrightarrow \mathrm{mor}(\mathrm{Cob}^{3,\bullet}(2n_E))$$

that places a cobordism $C \in \mathrm{mor}(\mathrm{Cob}^{3,\bullet}(2n_{E_i}))$ inside the i-th cylinder of $\mathcal{D}\times[0,1]$ via a orientation-preserving homeomorphism $\varphi\colon D^2\times[0,1] \to E_i\times[0,1]$ that preserves the arcs on the boundary and maps the arc connecting the base points in $D^2 \times [0,1]$ to the arc connecting the base points in $E_i \times [0,1]$ (note that φ is unique up to isotopy). Here, D^2 denotes the unit disk in $\mathbb{R}^2$ containing the unoriented crossingless tangle diagrams in $\mathrm{Cob}^{3,\bullet}(2n_{E_i})$. The same construction applies also to $\mathrm{ob}_{/l}$ and $\mathrm{mor}_{/l}$. See [Bar05, Section 5] for further details.

The unoriented planar algebra structures on ob and mor (resp. $\mathrm{ob}_{/l}$ and $\mathrm{mor}_{/l}$) define an unoriented planar algebra structure on

$$\mathrm{Cob} := \{\mathrm{Cob}^{3,\bullet}(2n)\}_{n\in\mathbb{N}}, \quad \mathrm{Cob}_{/l} := \{\mathrm{Cob}^{3,\bullet}_{/l}(2n)\}_{n\in\mathbb{N}},$$

where an unoriented d-input planar arc diagram $\mathcal{D}$ yields an operator as a functor

$$\mathcal{D}_{\mathrm{Cob}}\colon \prod_{i=1}^{d} \mathrm{Cob}^{3,\bullet}(2n_{E_i}) \to \mathrm{Cob}^{3,\bullet}(2n_E)$$

and

$$\mathcal{D}_{\mathrm{Cob}_{/l}}\colon \prod_{i=1}^{d} \mathrm{Cob}^{3,\bullet}_{/l}(2n_{E_i}) \to \mathrm{Cob}^{3,\bullet}_{/l}(2n_E),$$

respectively, that acts on objects and morphisms using the operator that $\mathcal{D}$ defines in the planar algebras ob, mor and $\mathrm{ob}_{/l}$, $\mathrm{mor}_{/l}$, respectively. The following theorem by Bar-Natan shows that the unoriented planar algebra structures of Cob and $\mathrm{Cob}_{/l}$ descend in fact to additive closures and complexes over additive closures, respectively.

Theorem 8.77 ([Bar05, Theorem 2]).

1.) The collection

$$\mathcal{K} = \{\mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^{3,\bullet}_{/l}(2n)))\}_{n\in\mathbb{N}}$$

has a natural structure of an unoriented planar algebra.

2.) The operations $\mathcal{D}_{\mathcal{K}}$ on $\mathcal{K}$ send homotopy equivalent chain complexes to homotopy equivalent chain complexes, hence

$$\mathcal{K}_{/h} = \{\mathrm{Kom}_{/h}(\mathrm{Mat}(\mathrm{Cob}^{3,\bullet}_{/l}(2n)))\}_{n\in\mathbb{N}}$$

also has a natural structure of an unoriented planar algebra.

3.) The Bar-Natan chain complex $[T]$ of a tangle T descends to an oriented planar algebra morphism

$$[\,\cdot\,]^{\bullet}\colon \mathcal{T} \to (\mathcal{K}_{/h})_{\mathrm{or}},$$

where $(\mathcal{K}_{/h})_{\mathrm{or}}$ is $\mathcal{K}_{/h}$ considered as an oriented planar algebra as described in Remark 8.73.

The basic idea is that an unoriented d-input planar arc diagram $\mathcal{D}$ yields an operator that is a functor

$$\mathcal{D}_{\mathcal{K}} \colon \prod_{i=1}^{d} \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^{3;\bullet}_{/l}(2n_{E_i}))) \to \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^{3;\bullet}_{/l}(2n_E)))$$

that takes as input a d-tuple of chain complexes $(\mathcal{C}_1, \ldots, \mathcal{C}_d)$ and returns their tensor product $\mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_d$ that is formed using $\mathcal{D}$ (this is best understood by looking at an example, so we refer the reader to Example 8.80 instead of giving a precise definition). Then, 3.) of Theorem 8.77 says that the following complexes are homotopy equivalent:

$$\mathcal{D}_{\mathcal{K}}([D_1]^\bullet, \ldots, [D_d]^\bullet) \simeq [\mathcal{D}_{\mathcal{T}}(D_1, \ldots, D_d)]^\bullet, \tag{8.23}$$

where T_i is a $2n_{E_i}$-ended oriented tangle diagram with base point for all $i = 1, \ldots, d$. We refer the reader to [Bar05] for a proof of Theorem 8.77.

8.8 Delooping, Gaussian elimination, and divide-and-conquer

The compatibility result (8.23) from the previous Section 8.7 gives us an effective way of computing the Bar-Natan chain complex $[T]^\bullet$ of any tangle T: we break T down into "smaller" tangles T_i from which we successively compute $[T]^\bullet$ using the simpler $[T_i]^\bullet$ and suitable planar arc diagrams. Along the way, one simplifies the intermediate complexes using two tools called *delooping* and Gaussian elimination. This strategy of computation is known as *divide-and-conquer*, a term coined by Bar-Natan [BN07]. We will see how this works in detail in Example 8.80 below where we compute the Bar-Natan complex of the right-handed trefoil, but first let us introduce the two aforementioned tools that will greatly simplify our computations.

The first tool is an isomorphism in the category $\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))$ known as *delooping*. It is described in Figure 8.20.

Put differently, we have an isomorphism of objects in $\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))$

 $\cong$  $\{-1\} \oplus$  $\{+1\}.$

We invite the reader to check that the morphisms in Figure 8.20 are indeed mutually inverse to each other. Thus given a 2-ended tangle with diagram D_T, we can use delooping to successively resolve every circle appearing in the complex $\mathcal{C}_{BN}(D_T)$ which reduces the computational complexity of $[T]$ in the sense that there are less different objects and morphisms involved.

Remark 8.78. Delooping was first described by Bar-Natan [BN07], with a different version given by Naot [Nao06]. Our version is closest to Naot's, with the exception that we don't use dots.

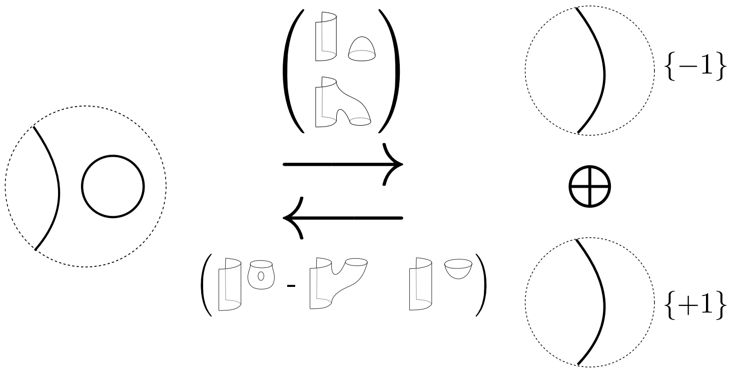
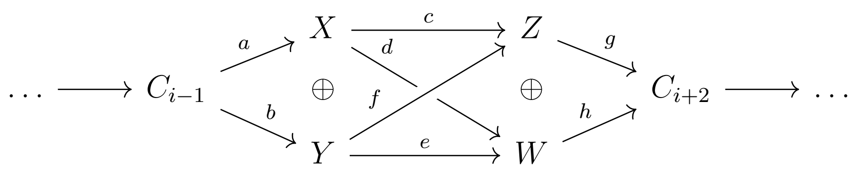


Figure 8.20: The depicted morphisms in $\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))$ are mutually inverse to each other, thus yielding an isomorphism of objects known as *delooping*.

The second tool is *Gaussian elimination*, which is described in the following lemma.

Lemma 8.79. Assume (C, d) is a chain complex in some additive category taking the following form:

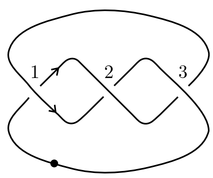


where e is an isomorphism. Then C is homotopy equivalent to

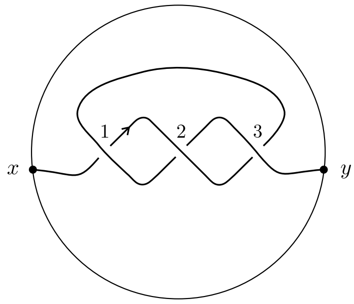
$$\cdots \longrightarrow C_{i-1} \xrightarrow{a} X \xrightarrow{c-fe^{-1}d} Z \xrightarrow{g} C_{i+2} \longrightarrow \cdots.$$

For a proof, see [BN07, Lemma 4.2]. Using Gaussian elimination as stated in Lemma 8.79, one may eliminate the domain and target of an isomorphism in a chain complex, by paying the price of introducing a new differential $f \circ e^{-1} \circ d$. The compatibility property as described in (8.23) in combination with delooping and Gaussian elimination yields an algorithm known as *divide-and-conquer* for computing the Bar-Natan chain complex of a tangle [BN07], which is best understood by looking at a hands-on example.

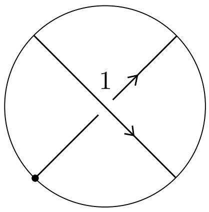
Example 8.80. Let K be the right-handed trefoil with diagram D_K as depicted below:

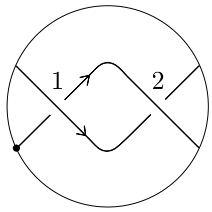
$D_K =$ 

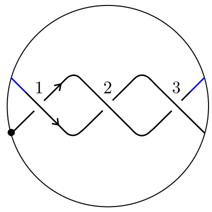
We wish to compute the Bar-Natan complex $[K]^\bullet$ using the diagram D_K. For this, we apply the correspondences (8.4) and (8.5) in order to obtain a 2-ended tangle T with diagram D_T corresponding to K and D_K, respectively:

$D_T =$ 

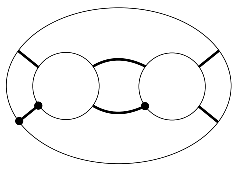
Then, computing $[K]^\bullet$ is equivalent to compute $[T]^\bullet$ (resp. $[D_T]^\bullet$ up to homotopy equivalence). We will do so by applying Bar-Natan's divide-and-conquer strategy: we compute $[T]^\bullet$ by computing the Bar-Natan complexes of the tangles T_1, T_2, T_3 corresponding to the diagrams $D_{T_1}, D_{T_2}, D_{T_3}$ respectively:[12]

$D_{T_1} =$ 

$D_{T_2} =$ 

$D_{T_3} =$ 

Observe that if we close up the blue marked strands in D_{T_3} in the most obvious way, we obtain a 2-ended tangle diagram equivalent to D_T. In order to compute $[T_2]^\bullet$ and $[T_3]^\bullet$, we will use the following unoriented 2-input planar arc diagram:

$\mathcal{D} =$ 

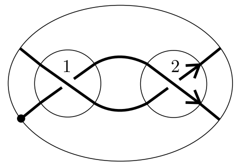
(here, the leftmost inner input disk is the first, and the rightmost inner input disk is the second). Before starting the computation, let us give an overview of the process for better understanding:

1. The first step consists of computing $[T_1]^\bullet$ using D_{T_1} which, as we shall see shortly, will be a very easy task.

2. Next, we compute $[T_2]^\bullet$. We do so by considering $\mathcal{D}_\mathcal{T}(D_{T_1}, D_{T_1})$; if we equip this diagram with an orientation and an enumeration of

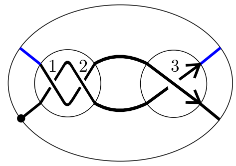
[12]For illustrative reasons, the base point doesn't appear at the same position in D_{T_1}, D_{T_2}, D_{T_3}, but the pictures are to be understood with a fixed disk and fixed base point.

crossings, and forget about the inner input disks, we see that

$$[\mathcal{D}_{\mathcal{T}}(D_{T_1}, D_{T_1})]^{\bullet} = \Bigg[$$  $$\Bigg]^{\bullet} = [D_{T_2}]^{\bullet}$$

We then obtain $[T_2]^{\bullet}$ by considering $[D_{T_2}]^{\bullet}$ up to homotopy equivalence. In order to compute $[\mathcal{D}_{\mathcal{T}}(D_{T_1}, D_{T_1})]^{\bullet}$, we use $[T_1]^{\bullet}$ from the previous step in combination with (8.23).

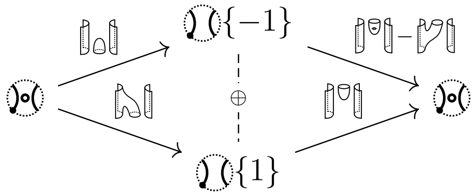
3. The third step consists of computing $[T_3]^{\bullet}$ which proceeds similar to the second step. We consider $\mathcal{D}_{\mathcal{T}}(D_{T_2}, D_{T_1})$ and equip this diagram with an orientation and enumeration of crossings, and forget about the inner input disks. Then

$$[\mathcal{D}_{\mathcal{T}}(D_{T_2}, D_{T_1})]^{\bullet} = \Bigg[$$  $$\Bigg]^{\bullet} = [D_{T_3}]^{\bullet}$$

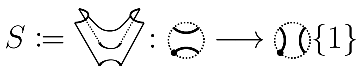
We then obtain $[T_3]^{\bullet}$ by considering $[D_{T_3}]^{\bullet}$ up to homotopy equivalence. In order to compute $[\mathcal{D}_{\mathcal{T}}(D_{T_2}, D_{T_1})]^{\bullet}$, we use $[T_1]^{\bullet}, [T_2]^{\bullet}$ from the previous step in combination with (8.23).

4. The last step consists of closing up the blue marked strands in D_{T_3} and adjusting the complex $[D_{T_3}]^{\bullet}$ accordingly. Up to homotopy equivalence, we then obtain the complex of the right-handed trefoil $[T]^{\bullet}$ (resp. $[K]^{\bullet}$).

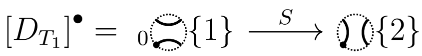
Let us now start with the computations. In what follows we omit the enumeration of circles (in objects of $\mathrm{Cob}^{3;\bullet}_{/l}$) since there will always be at most one circle appearing in diagrams. A left subscript next to diagrams indicates homological degree, and a number in curly brackets denotes as usual the shift in quantum grading. We call the strand that connects the base point the *marked strand*. The main tools in our computations are delooping (see Figure 8.20) and Gaussian elimination (see Lemma 8.79). In the upcoming computations we will always deloop with respect to the marked strand as shown below:



Finally, let us introduce the following abbreviation for the saddle morphism:

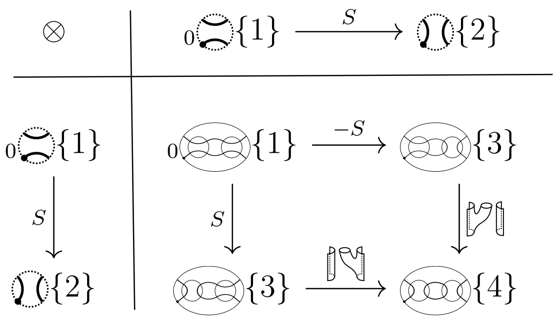


Let us now start with the first step by computing $[D_{T_1}]^\bullet$. We have:

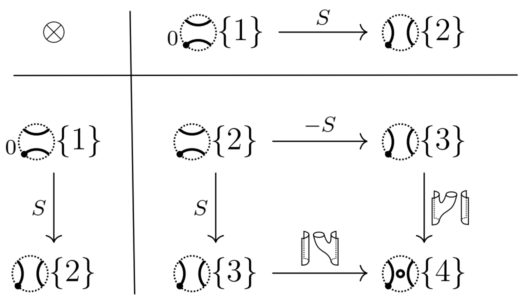


Up to homotopy equivalence, this is the Bar-Natan complex of $[T_1]^\bullet$.

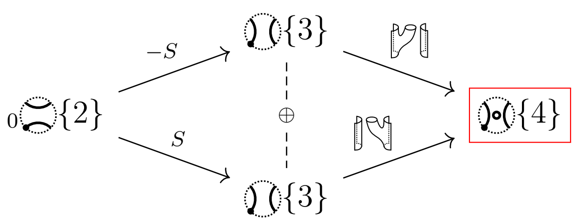
For the second step we compute $[\mathcal{D}_\mathcal{T}(D_{T_1}, D_{T_1})]^\bullet$ using $[D_{T_1}]^\bullet$ and (8.23). Hence we form the tensor product of $[D_{T_1}]^\bullet$ with itself, where we tensor objects in $\mathrm{Cob}^{3,\bullet}_{/l}(4)$ using the planar arc diagram $\mathcal{D}$:[13]



Forgetting about the inner input disks, the diagram above simply becomes



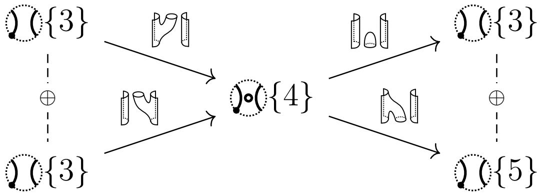
From this, we obtain the following chain complex:



(8.24)

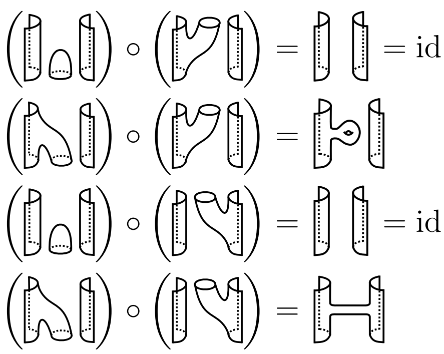
[13] We use here the convention that the column gets placed in the first, and the row in the second inner input disk.

Next, we deloop the diagram in the red box. This means that we have to compute the following compositions of maps:

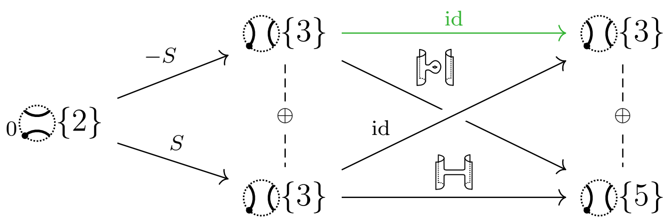


(8.25)

We get:

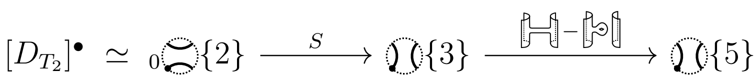


Hence, (8.24) is overall isomorphic to the following isomorphic complex:

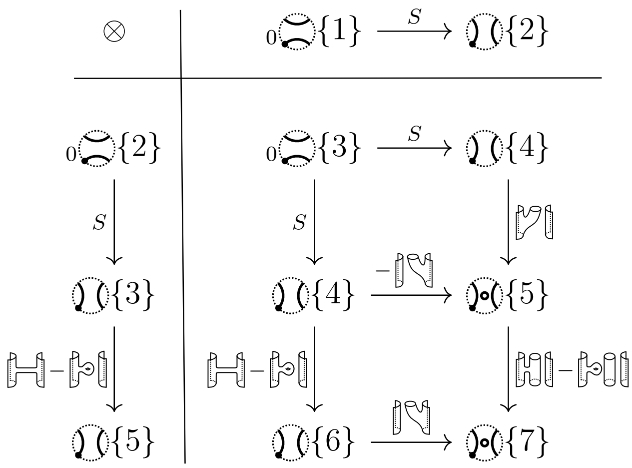


(8.26)

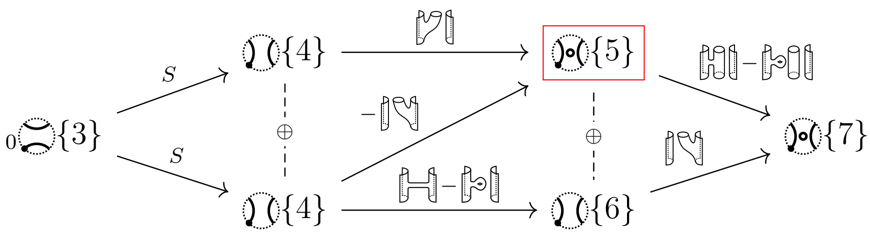
Last but not least, we apply Gaussian elimination to the green edge and finally obtain



Again, up to homotopy equivalence this is the complex of $[T_2]^\bullet$. For the third step, we proceed exactly in the same way as before. We compute $[\mathcal{D}_{\mathcal{T}}(D_{T_2}, D_{T_1})]^\bullet$ by considering

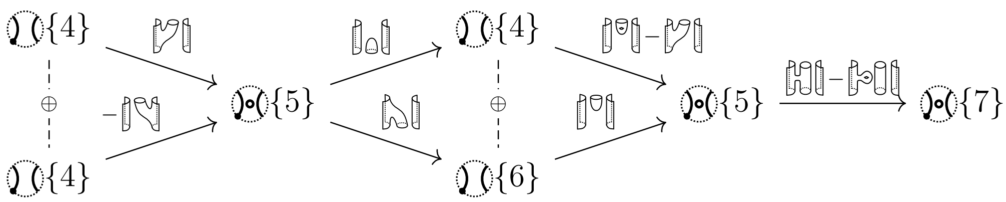


This gives us the complex



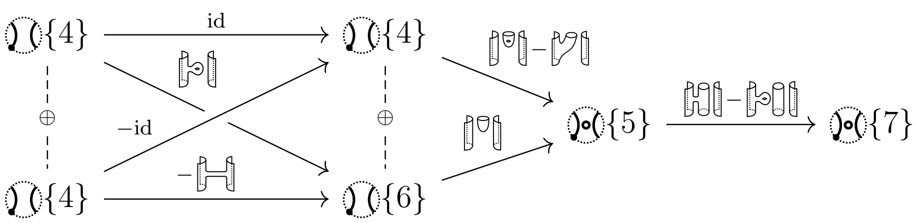
(8.27)

Let us deloop the diagram in the red box. Note that in contrast to (8.24), we now have a non-zero outgoing arrow from the red box. So we effectively have to compute the following compositions:



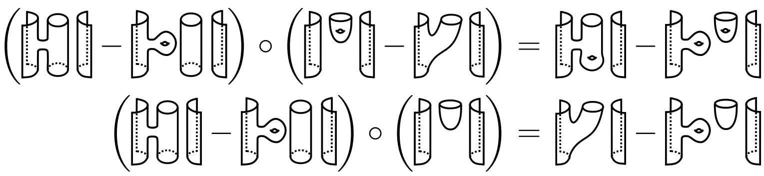
(8.28)

Observe that the left rectangular half of (8.28) is simply (8.25), which we computed in (8.26), with an additional minus sign at one edge. Hence (8.28) becomes

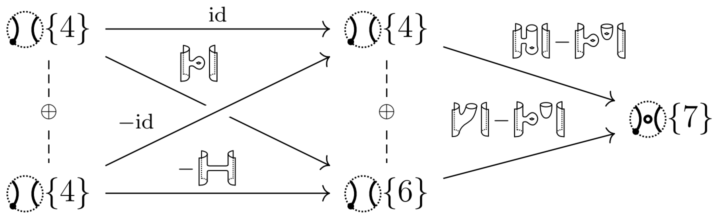


(8.29)

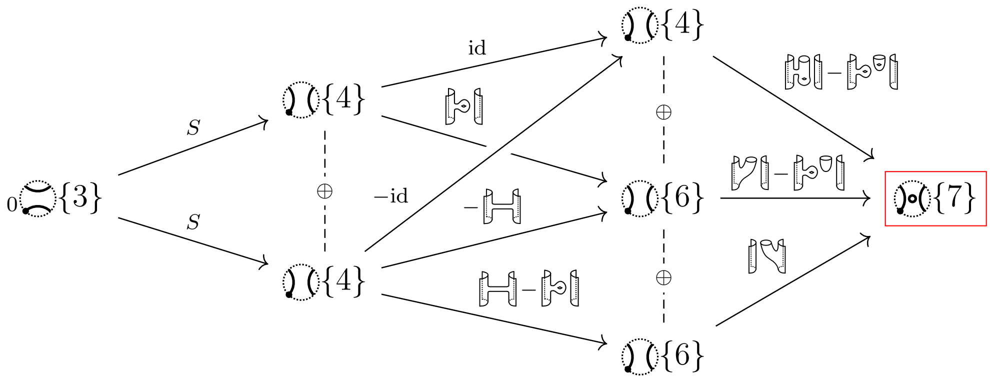
Now let us compute the compositions in the right half of (8.29):



Hence (8.29) becomes

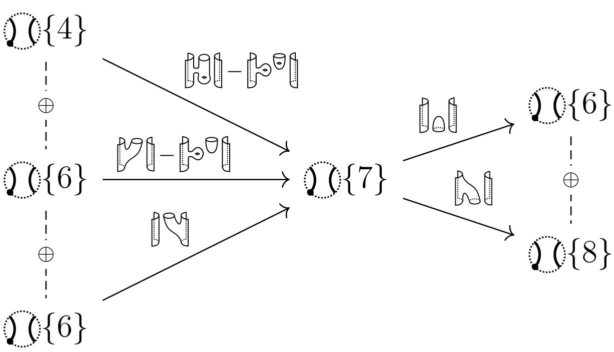


Plugging this into (8.27), we overall obtain the following isomorphic complex:



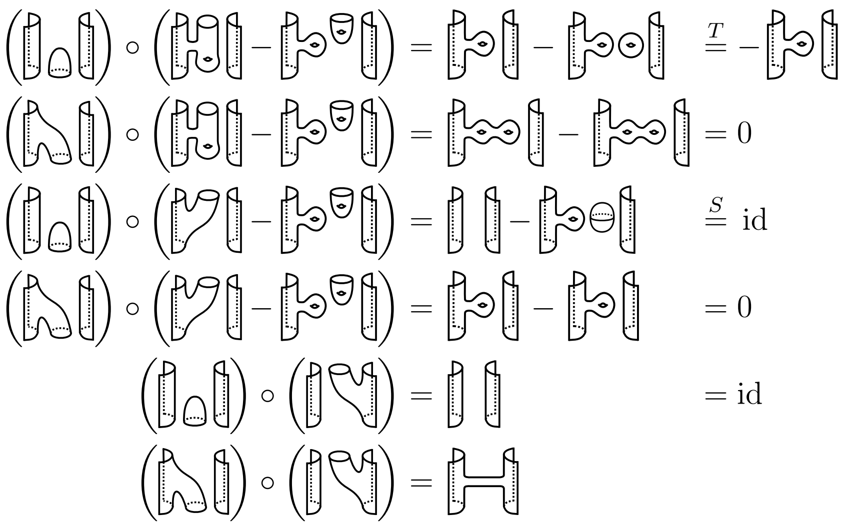
(8.30)

Let us deloop once more the diagram in the red box. We have to compute the following compositions:

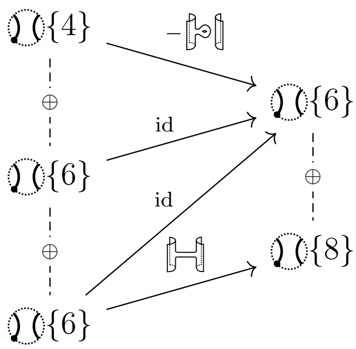


(8.31)

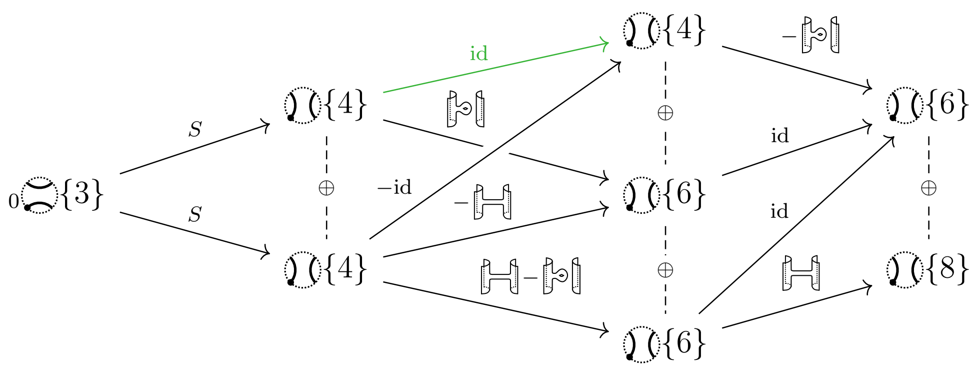
Using the local relations S and T (see Figure 8.14, not to be confused with the saddle morphism S)



Hence (8.31) becomes

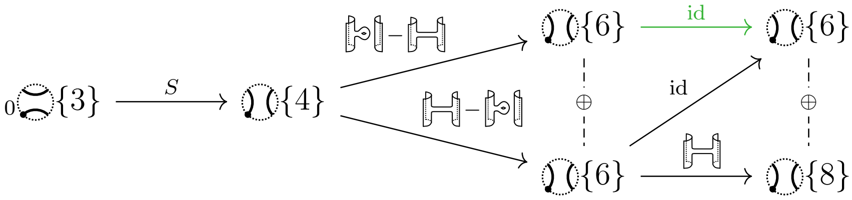


Plugging this into (8.30), we overall obtain the following complex isomorphic to $[D_{T_3}]^\bullet$:



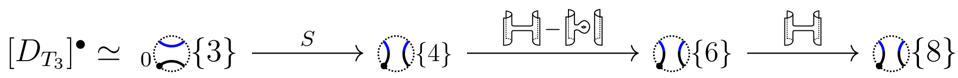
(8.32)

Simplifying further, we apply Gaussian elimination to the green edge in (8.32) and get



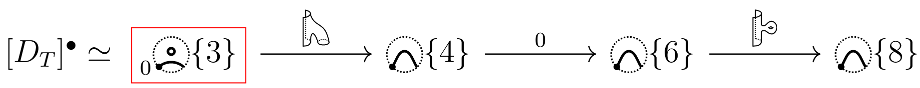
(8.33)

Finally, we apply once more Gaussian elimination to the green edge in (8.33) and obtain

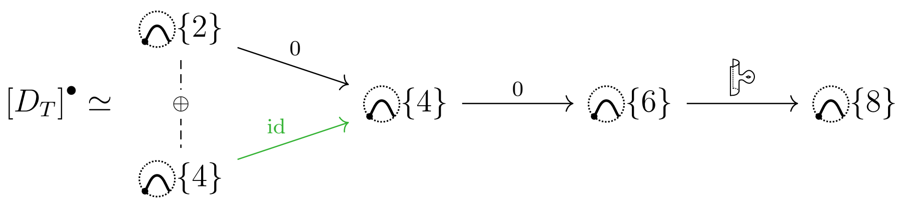


(8.34)

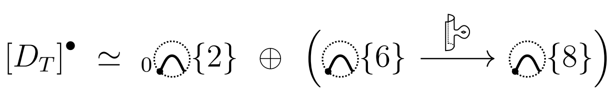
Note that (8.34) is equal to $[T_3]^\bullet$ up to homotopy equivalence. Observe that we have indicated blue strands in (8.34); this leads us to the final step in our computation of $[T]^\bullet$. Indeed, closing up the blue strands in (8.34) in the most obvious way and doing the same for the morphisms, we obtain



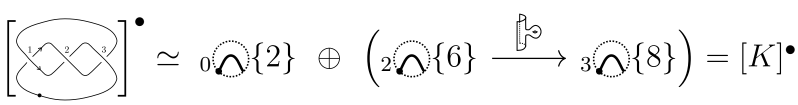
Delooping the diagram in the red box gives



Finally, applying Gaussian elimination to the edge in green we obtain



This complex is equal to $[T]^\bullet$ up to homotopy equivalence. This concludes our computation of the Bar-Natan complex of the right-handed trefoil with the final result:



(8.35)

8.9 Obtaining homology from the Bar-Natan complex

The goal of this section is to show how to obtain a homology theory by applying a TQFT to the Bar-Natan complex of a 2-ended tangle (or link). The basic idea is very similar to the construction of Khovanov homology in Section 8.2, where we translated vertices and edges of the cube of resolutions into algebraic objects. A TQFT yields precisely the rules to do the same with more general algebraic systems. It goes as follows.

Let $\mathcal{F} = (R, A, \Delta, \varepsilon)$ be a rank 2 Frobenius system with corresponding TQFT $\mathcal{F}\colon \operatorname{Cob}^3_{/l}(2) \to \mathcal{M}_A$ as described in Section 8.5. Note that the category $\mathcal{M}_A$ is by definition not an abelian category (see Definition 8.33). However, there exists an inclusion functor

$$I\colon \mathcal{M}_A \hookrightarrow A\text{-Mod}$$

where A-Mod denotes the usual category of graded A-modules with graded A-module homomorphisms. Post-composing $\mathcal{F}$ with I gives us a new functor

$$I \circ \mathcal{F}\colon \operatorname{Cob}^3_{/l}(2) \to A\text{-Mod}$$

which we denote by $\mathcal{F}$ as well at the cost of minor confusion. By definition of the TQFT $\mathcal{F}$, it should be clear that $\mathcal{F}$ induces a functor

$$\widehat{\mathcal{F}}\colon \operatorname{Kom}(\operatorname{Mat}(\operatorname{Cob}^3_{/l}(2))) \to \operatorname{Kom}(A\text{-Mod}).$$

Indeed, $\mathcal{F}$ descends to $\widetilde{\mathcal{F}}\colon \operatorname{Mat}(\operatorname{Cob}^3_{/l}(2)) \to A$-Mod by applying $\mathcal{F}$ to summands in a direct sum of objects of $\operatorname{Cob}^3_{/l}(2)$ and entries of matrices individually, and the step from $\widetilde{\mathcal{F}}$ to $\widehat{\mathcal{F}}$ is then immediate. Note that $\widehat{F}$ respects homotopy equivalence of complexes and thus descend to a functor on $\operatorname{Kom}_{/h}$, for which we use the same notation.

Now, given a 2-ended tangle T with diagram D_T, we have the Bar-Natan complexes $[D_T] \in \operatorname{Kom}(\operatorname{Mat}(\operatorname{Cob}^3_{/l}(2)))$ and $[T] \in \operatorname{Kom}_{/h}(\operatorname{Mat}(\operatorname{Cob}^3_{/l}(2)))$ respectively, and applying $\widehat{\mathcal{F}}$ gives us a well-defined chain complex over A, from which we can take homology.

Definition 8.81. Let T be a 2-ended tangle with diagram D_T and $\mathcal{F} = (R, A, \Delta, \varepsilon)$ a rank 2 Frobenius system giving a TQFT $\mathcal{F}\colon \operatorname{Cob}^3_{/l}(2) \to \mathcal{M}_A$. Then we define the *$\mathcal{F}$-complex* of D_T as

$$C_{\mathcal{F}}(D_T) \coloneqq \widehat{\mathcal{F}}([D_T]) \in \operatorname{Kom}(A\text{-Mod}),$$

and the *$\mathcal{F}$-complex* of T as

$$C_{\mathcal{F}}(T) \coloneqq \widehat{\mathcal{F}}([T]) \in \operatorname{Kom}_{/h}(A\text{-Mod}).$$

The homology

$$H_{\mathcal{F}}(T) \coloneqq H(C_{\mathcal{F}}(T)) \in A\text{-Mod}$$

is called the $\mathcal{F}$-*homology* of T. If $L \subseteq S^3$ is a base-pointed link with base-pointed diagram D_L and T_L the corresponding 2-ended tangle with diagram D_{T_L} (see (8.4)), we define the $\mathcal{F}$-*complex* of D_L and L as

$$\begin{aligned} C_{\mathcal{F}}(D_L) &:= C_{\mathcal{F}}(D_{T_L}) \in \mathrm{Kom}(A\text{-Mod}) \\ C_{\mathcal{F}}(L) &:= C_{\mathcal{F}}(T_L) \in \mathrm{Kom}_{/h}(A\text{-Mod}), \end{aligned}$$

respectively, and the $\mathcal{F}$-*homology* of L as

$$H_{\mathcal{F}}(L) := H_{\mathcal{F}}(T_L) \in A\text{-Mod}$$

Let us make several remarks regarding Definition 8.81.

Remark 8.82.

1.) By Theorem 8.65, $C_{\mathcal{F}}(T)$ does not depend on the choice of diagram D_T, thus making the $\mathcal{F}$-complex and $\mathcal{F}$-homology of T an invariant of tangles. The same is true for the $\mathcal{F}$-complex and $\mathcal{F}$-homology of a base-pointed link L, respectively.

2.) If two rank 2 Frobenius systems $\mathcal{F}_1$ and $\mathcal{F}_2$ are related by a twist Definition 8.54, then the corresponding complexes $C_{\mathcal{F}_1}(T)$ and $C_{\mathcal{F}_2}(T)$, as well as the homologies $H_{\mathcal{F}_1}(T)$ and $H_{\mathcal{F}_2}(T)$ are isomorphic (see [Kho06, Proposition 3 and Corollary 1]).

3.) Similar to the Khovanov complex from Section 8.2, the $\mathcal{F}$-complex $C_{\mathcal{F}}(T)$ carries by construction a *homological grading* given by the height in the cube of resolutions and the final shift by $-n_-$. Moreover, if $\mathcal{F}$ is graded, then the $\mathcal{F}$-complex $C_{\mathcal{F}}(T)$ carries a second grading which we call *quantum grading*, which makes $C_{\mathcal{F}}(T)$ and therefore also the $\mathcal{F}$-homology $H_{\mathcal{F}}(T)$ bigraded. Likewise, if $\mathcal{F}$ is filtered, then $C_{\mathcal{F}}(T)$ is a filtered complex (use Remark 8.45 in order to obtain filtrations induced by $\mathcal{F}$ on tensor products and direct sums in $C_{\mathcal{F}}(T)$) with a single grading given by the homological grading. The filtration on $C_{\mathcal{F}}(T)$ then induces one on homology as described in Definition 8.47, thus making $H_{\mathcal{F}}(T)$ filtered.

4.) Recall Remark 8.51: the TQFT $\mathcal{F}\colon \mathrm{Cob}^3_{/l}(2) \to \mathcal{M}_A$ induces a new TQFT $\mathcal{F}'\colon \mathrm{Cob}^3_{/l}(0) \to \mathcal{M}_R$, and Definition 8.81 translates mutatis mutandis to 0-ended tangles, i.e. oriented links in S^3. This gives us a way to obtain a chain complex and homology theory from the cube of resolutions $Q(D_L)$ of a link diagram D_L to which we previously referred to as a *Khovanov-type homology theory*. Indeed, if we take the Frobenius system $\mathcal{F}_{\mathbb{Q}}$ described in Definition 8.55, then we see that $C_{\mathcal{F}'_{\mathbb{Q}}}(L)$ and $H_{\mathcal{F}'_{\mathbb{Q}}}(L)$ are precisely the Khovanov chain complex (up to homotopy equivalence) and homology of a link L that we defined in Section 8.2.

5.) More generally, if $J\colon A\text{-Mod} \to R\text{-Mod}$ is the functor that forgets the A-module structure and considers them as R-modules, then $J(H_{\mathcal{F}}(L))$ is isomorphic to $H_{\mathcal{F}'}(L)$ over R (cf. [Bar05, Section 9]).

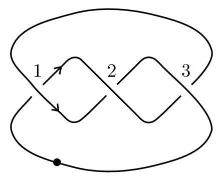
6.) It should be clear that Definition 8.81 generalizes to arbitrary $2n$-ended tangles by specifying a suitable functor $\mathcal{F}\colon \mathrm{Cob}^3_{/l}(2n) \to A\text{-Mod}$. However, since we will work exclusively with 2-ended tangles in upcoming sections, we decided to restrict our considerations and descriptions to this special case. For more information, see [Bar05, Section 7].

In Section 8.5, we have defined several explicit Frobenius systems. Let's give the corresponding homology theories proper names.

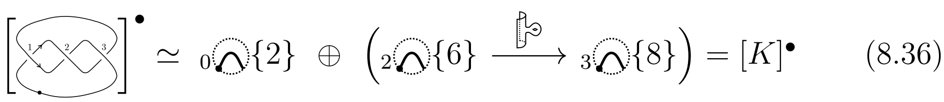
Definition 8.83. Let $L \subset S^3$ be a link with base point. Then:

1. $\mathrm{Kh}_{\mathbb{Z}}(L) \coloneqq H_{\mathcal{F}_{\mathbb{Z}}}(L)$ and $\mathrm{Kh}_{\mathbb{Q}}(L) \coloneqq H_{\mathcal{F}_{\mathbb{Q}}}(L)$ are called *integral* and *rational Khovanov homology* of L, respectively.
2. More generally, if $\mathbb{F}$ is a field then $H_{\mathcal{F}_{\mathbb{F}}}(L)$ is called *Khovanov homology of L over* $\mathbb{F}$.
3. $H_{\mathrm{univ}}(L) \coloneqq H_{\mathcal{F}_{\mathrm{BN}}}(L)$ is called *universal Khovanov homology* of L.
4. $H_{\mathrm{Lee}}(L) \coloneqq H_{\mathcal{F}_{\mathrm{Lee}}}(L)$ is called *Lee homology* of L.
5. $H_{\mathrm{BN}}(L) \coloneqq H_{\mathcal{F}_{\mathrm{BN}}}(L)$ is called *Bar-Natan homology* of L.

Example 8.84. Let K be the right-handed trefoil as shown below.



In Example 8.80, we computed the Bar-Natan complex of K using divide-and-conquer and obtained the result (see (8.35))



Here, the left subscript denotes as usual the homological degree. From (8.36) we can easily compute $\mathcal{F}$-complexes and $\mathcal{F}$-homology of K, so let's look at some examples. Consider the Khovanov system $\mathcal{F}_{\mathbb{Z}}$ as described in Definition 8.55. Then the $\mathcal{F}_{\mathbb{Z}}$-complex of K is given as

$$\mathcal{C}_{\mathcal{F}_{\mathbb{Z}}}(K) = {}_0\mathbb{Z}[X]/(X^2)\{3\} \;\oplus\; \Big({}_2\mathbb{Z}[X]/(X^2)\{7\} \xrightarrow{m\circ\Delta} {}_3\mathbb{Z}[X]/(X^2)\{9\}\Big)$$

In the usual basis $1, X$ of $\mathbb{Z}[X]/(X^2)$, the matrix of $m \circ \Delta$ takes the form

$$m \circ \Delta = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

Hence the integral Khovanov homology of the right-handed trefoil is

$$\begin{aligned}
\mathrm{Kh}^0_{\mathbb{Z}}(K) &\cong \mathbb{Z}_1 \oplus \mathbb{Z}_3 \\
\mathrm{Kh}^1_{\mathbb{Z}}(K) &= 0 \\
\mathrm{Kh}^2_{\mathbb{Z}}(K) &\cong \mathbb{Z}_5 \\
\mathrm{Kh}^3_{\mathbb{Z}}(K) &\cong \mathbb{Z}_7 \oplus \mathbb{Z}/2\mathbb{Z}_9,
\end{aligned}$$

where the right subscript denotes the quantum degree of the individual generator. Switching from $\mathbb{Z}$ to $\mathbb{Q}$ and noting that 2 is invertible over $\mathbb{Q}$, we can immediately read off the rational Khovanov homology of the right-handed trefoil from the above:

$$\begin{aligned}
\mathrm{Kh}^0_{\mathbb{Q}}(K) &\cong \mathbb{Q}_1 \oplus \mathbb{Q}_3 \\
\mathrm{Kh}^1_{\mathbb{Q}}(K) &= 0 \\
\mathrm{Kh}^2_{\mathbb{Q}}(K) &\cong \mathbb{Q}_5 \\
\mathrm{Kh}^3_{\mathbb{Q}}(K) &\cong \mathbb{Q}_7.
\end{aligned}$$

Consider the universal Frobenius system $\mathcal{F}_{\text{univ}}$ from Definition 8.56. We write $A_{\text{univ}} = \mathbb{Z}[h, t, X](X^2 - hX - t)$. Then the corresponding complex for the right-handed trefoil takes the form

$$\mathcal{C}_{\mathcal{F}_{\text{univ}}}(K) = {}_0A_{\text{univ}}\{3\} \;\oplus\; \Big({}_2A_{\text{univ}}\{7\} \xrightarrow{m\circ\Delta} {}_3A_{\text{univ}}\{9\}\Big)$$

In the basis $1, X$ of A_{univ}, the matrix of $m \circ \Delta$ is given as

$$m \circ \Delta = \begin{pmatrix} -h & 2t \\ 2 & h \end{pmatrix}$$

Hence the universal Khovanov homology of the right-handed trefoil is

$$\begin{aligned}
H^0_{\text{univ}}(K) &\cong \mathbb{Z}[h,t]_1 \oplus \mathbb{Z}[h,t]_3 \\
H^1_{\text{univ}}(K) &= 0 \\
H^2_{\text{univ}}(K) &= 0 \\
H^3_{\text{univ}}(K) &\cong \operatorname{coker} \begin{pmatrix} -h & 2t \\ 2 & h \end{pmatrix}
\end{aligned}$$

Now let's look at the $\mathbb{Z}[G]$-system $\mathcal{F}_{\mathbb{Z}[G]}$ as described in Definition 8.57. We write $A_{\mathbb{Z}[G]} = \mathbb{Z}[X, G]/(X^2 + GX)$. Then the corresponding complex for the right-handed trefoil takes the form

$$\mathcal{C}_{\mathcal{F}_{\mathbb{Z}[G]}}(K) = {}_0A_{\mathbb{Z}[G]}\{3\} \;\oplus\; \left({}_2A_{\mathbb{Z}[G]}\{7\} \xrightarrow{m\circ\Delta} {}_3A_{\mathbb{Z}[G]}\{9\}\right)$$

In the basis $1, X$ of $A_{\mathbb{Z}[G]}$, the matrix of $m \circ \Delta$ is given as

$$m \circ \Delta = \begin{pmatrix} G & 0 \\ 2 & -G \end{pmatrix}$$

Hence the $\mathcal{F}_{\mathbb{Z}[G]}$-homology of the right-handed trefoil is

$$\begin{aligned} H^0_{\mathcal{F}_{\mathbb{Z}[G]}}(K) &\cong \mathbb{Z}[G]_1 \oplus \mathbb{Z}[G]_3 \\ H^1_{\mathcal{F}_{\mathbb{Z}[G]}}(K) &= 0 \\ H^2_{\mathcal{F}_{\mathbb{Z}[G]}}(K) &= 0 \\ H^3_{\mathcal{F}_{\mathbb{Z}[G]}}(K) &\cong \operatorname{coker} \begin{pmatrix} G & 0 \\ 2 & -G \end{pmatrix} \end{aligned}$$

Finally, let's look at the Lee system $\mathcal{F}_{\text{Lee}}$. We write $A_{\text{Lee}} = \mathbb{Q}[X](X^2 - 1)$. Then the corresponding complex for the right-handed trefoil takes the form

$$\mathcal{C}_{\mathcal{F}_{\text{Lee}}}(K) = {}_0A_{\text{Lee}}\{3\} \;\oplus\; \left({}_2A_{\text{Lee}}\{7\} \xrightarrow{m\circ\Delta} {}_3A_{\text{Lee}}\{9\}\right)$$

In the basis $1, X$ of A_{Lee}, the matrix of $m \circ \Delta$ is given as

$$m \circ \Delta = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence the Lee homology of the right-handed trefoil is

$$\begin{aligned} H_{\text{Lee}}(K)^0(K) &\cong \mathbb{Q}_1 \oplus \mathbb{Q}_3 \\ H_{\text{Lee}}(K)^1(K) &= 0 \\ H_{\text{Lee}}(K)^2(K) &= 0 \\ H_{\text{Lee}}(K)^3(K) &= 0 \end{aligned}$$

It is not a coincidence that $H_{\text{Lee}}(K)$ is concentrated in homological degree 0 with dimension 2; this is in fact true for the Lee homology of any knot. Also, note that the quantum degrees of the two generators differ by 2; this is a key observation used for the definition of the Rasmussen s-invariant, which will be the topic of the upcoming section.

8.10 Lee homology and the Rasmussen s-invariant

In 2002, Lee [Lee05] defined a deformation of Khovanov homology by specifying a Frobenius system that applied to the cube of resolutions of a link diagram yields a *filtered* homology theory. A consequence of this is the existence of a spectral sequence that starts at Khovanov homology and ends at the associated graded space corresponding to the filtration of the Lee homology of a link L. This spectral sequence was later used by Rasmussen [Ras10] in order to extract an invariant of knots that provides much geometrical information, such as a lower bound on the slice genus of the knot. Let's take a closer look.

In Definition 8.59 we already encountered the Frobenius system that was specified by Lee: it is defined as the system $\mathcal{F}_{\text{Lee}} = (R_{\text{Lee}}, A_{\text{Lee}}, \Delta, \varepsilon)$, where

$$R_{\text{Lee}} = \mathbb{Q}, \quad A_{\text{Lee}} = \mathbb{Q}[X]/(X^2-1), \quad \iota\colon \mathbb{Q} \hookrightarrow A_{\text{Lee}},$$

$$\begin{aligned} &m(1\otimes 1) = 1 && m(X\otimes X) = 1 \\ &m(1\otimes X) = X && m(X\otimes 1) = X \\ &\Delta(1) = 1\otimes X + X\otimes 1 && \varepsilon(1) = 0 \\ &\Delta(X) = X\otimes X + 1\otimes 1 && \varepsilon(X) = 1 \end{aligned}$$

We equip $\mathcal{F}_{\text{Lee}}$ with a descending filtration

$$\{0\} \subset F_0 \subset F_{-2} = A_{\text{Lee}},$$

where F_0 is generated by 1 and F_{-2} is generated by 1 and X. The maps $m, \iota, \Delta, \varepsilon$ are then filtered of degree $0, 0, -2, 0$ respectively. This filtration is induced by the grading on the system $\mathcal{F}_{\text{AD}}$ with algebra $Q[t,X]/(X^2-t)$ after setting $t = 1$, see Section 8.5.

Now, let $L \subset S^3$ be a link with diagram D. Applying the TQFT given by $\mathcal{F}_{\text{Lee}}$ to the cube of resolutions $Q(D)$, we obtain a filtered chain complex $(\mathcal{C}_{\mathcal{F}_{\text{Lee}}}(D), d_{\mathcal{F}_{\text{Lee}}})$ as described at the end of Section 8.6.

Remark 8.85. In the literature, the filtration on $\mathcal{C}_{\mathcal{F}_{\text{Lee}}}(D)$ is frequently described as "induced by the quantum grading" (as for instance in [Ras10]). This is to be explained as follows. Recall from Section 8.2 that the Khovanov chain complex $\mathcal{C}_{\text{Kh}}(D)$ is bigraded with a homological and a quantum grading. The gradings may be read off via

$$h(x) \coloneqq |\alpha| - n_- \qquad (\textit{homological grading}) \tag{8.37}$$

$$q(x) \coloneqq \deg x + h(x) + n_+ - n_- \qquad (\textit{quantum grading}), \tag{8.38}$$

where $x \in A_\alpha(D)$ is homogeneous (see Definition 8.12). The quantum grading can now be used to obtain the filtration on the Lee complex in the following way. Keep the homological grading on $\mathcal{C}_{\mathcal{F}_{\text{Lee}}}(D)$, but other than that think of the Lee complex as an ordinary vector space, so that each homologically graded component of $\mathcal{C}_{\mathcal{F}_{\text{Lee}}}(D)$ is simply a direct sum of tensor products

$$\mathcal{C}^i_{\mathcal{F}_{\text{Lee}}}(D) = \bigoplus_{j=1}^{n} \bigotimes_{1}^{k_j} A_{\text{Lee}}\{1\}$$

We grade $A_{\text{Lee}}\{1\}$ once more internally as a vector space with $\deg 1 = +1$ and $\deg X = -1$. Now for each $j = 1, \dots, n$, define a quantum grading on $\bigotimes_1^{k_j} A_{\text{Lee}}\{1\}$ as in (8.38), and form the corresponding induced descending filtration. Finally, form the descending filtration on the direct sum using the filtrations on the tensor products (see Remark 8.45). This defines a filtration on $\mathcal{C}^i_{\mathcal{F}_{\text{Lee}}}(D)$, and one may check that d^i_{Lee} is filtered. Thus $\mathcal{C}_{\mathcal{F}_{\text{Lee}}}(D)$ is a filtered complex whose filtration coincides with the one induced by $\mathcal{F}_{\text{Lee}}$ (which can be seen by noting that the quantum grading contains precisely all additional degree shifts that are made while forming the Bar-Natan complex of D). With this description, the filtration degree of an element in $\mathcal{C}_{\mathcal{F}_{\text{Lee}}}(D)$ may now be read off as follows (see [MTV07, Section 3]): let $x = (x_1, \dots, x_n) \in \mathcal{C}^i_{\mathcal{F}_{\text{Lee}}}(D)$ be an element in the i-th homologically graded component. Then each x_j is contained in a tensor product of $A_{\text{Lee}}\{1\}$ and in general not homogeneous, but we can write $x_j = x_j^1 + \cdots + x_j^k$ with x_j^l homogeneous and of different degree for all $l = 1, \dots, k$ and $j = 1, \dots, n$. We set

$$r(x_j) := \min_{l=1,\dots,k} \deg x_j^l,$$

where $\deg x_j^l$ denotes the degree of x_j^l in the tensor product with respect to the grading $\deg 1 = +1$, $\deg X = -1$ on $A_{\text{Lee}}\{1\}$ considered as a vector space. Then

$$\deg_C(x) = \min_{j=1,\dots n} r(x_j) + i + n_+ - n_-,$$

where C denotes the filtration induced by the quantum grading on $\mathcal{C}^i_{\mathcal{F}_{\text{Lee}}}(D)$.

As described in Definition 8.47, the filtration on $\mathcal{C}_{\mathcal{F}_{\text{Lee}}}(D)$ induces a filtration S on homology $H_{\text{Lee}}(L)$. The main results of Lee are now the following (see also [Ras10]).

Theorem 8.86 ([Lee05]). Let $L \subset S^3$ be an oriented link. Then there is a spectral sequence with E_2 term the Khovanov homology $\text{Kh}_{\mathbb{Q}}(L)$ of L that converges to the associated graded space $\mathcal{G}_S(H_{\text{Lee}}(L))$ of the induced filtration S on $H_{\text{Lee}}(L)$. The terms E_i with $i \geq 2$ form invariants of the link L.

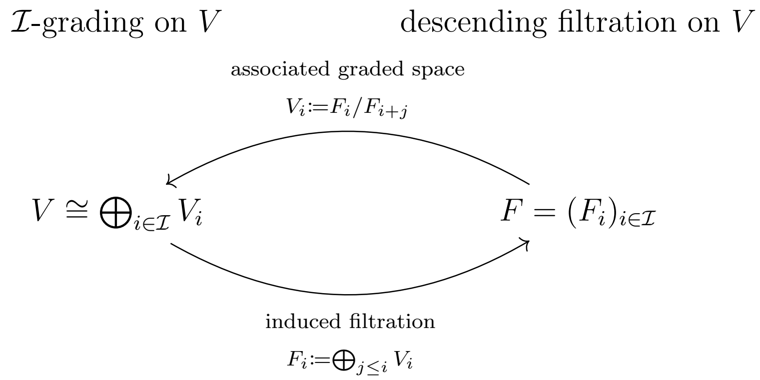
Theorem 8.87 ([Lee05]). Let $L \subset S^3$ be an oriented link. Then

$$\dim_{\mathbb{Q}} H_{\text{Lee}}(L) = 2^n,$$

where n denotes the number of components of L. If L is a knot (i.e. $n = 1$), then both generators lie in homological degree 0.

The proof of Theorem 8.87 is very explicit and obtained by describing a bijection between the set of possible orientations of L and a generating set of $H_{\text{Lee}}(L)$. Details regarding Theorem 8.86 and 8.87 can be found in [Lee05] and [Ras10]. Before we continue, let us make a remark regarding the relation between a grading and filtration on a finite-dimensional vector space.

Remark 8.88. Let $\mathbb{F}$ be a field and V a finite-dimensional $\mathbb{F}$-vector space. Further, let $\mathcal{I} \subseteq \mathbb{Z}$ be an index set that is bounded from above. Then we have the following relation between $\mathcal{I}$-gradings and descending filtrations $F = (F_i)_{i \in \mathcal{I}}$ on V:



Here, $j \in \mathcal{I}$ is chosen such that $i < k \leq j \implies k = j$ (i.e. the next greater number after i in $\mathcal{I}$). Note that in the case of finite-dimensional vector spaces, both arrows are mutually inverse to each other: starting with an $\mathcal{I}$-grading on V, then passing to the induced filtration and going back via the associated graded (which in this case if isomorphic to V) yields the same grading on V. Vice-versa, the associated graded of a descending filtration on V induces again the original filtration on V. As a consequence, if $v \in V \cong \bigoplus_{i \in \mathcal{I}} V_i$ is homogeneous and F the induced filtration on V, then the degree and filtration degree coincide:

$$\deg v = \deg_F v.$$

However, note that the above is in general *not* true: consider for instance the abelian group $A := \mathbb{Z}$ with filtration given by

$$\{0\} \subset F_1 := 2A \subset F_0 := A. \tag{8.39}$$

Then the associated graded $A_0 \oplus A_1$ is given by

$$A_0 = \mathbb{Z}/2\mathbb{Z}, \quad A_1 = 2\mathbb{Z},$$

and it is clear that the corresponding induced descending filtration is no longer the one in (8.39). In fact, the associated graded $A_0 \oplus A_1$ isn't even isomorphic to the original group $A = \mathbb{Z}$.

Let $K \subset S^3$ be an oriented knot with diagram D. By Theorem 8.87,

$$H_{\text{Lee}}(K) \cong \mathbb{Q} \oplus \mathbb{Q}.$$

Hence by Theorem 8.86, there exist two generators in $\text{Kh}_{\mathbb{Q}}(K)$ that survive the spectral sequence and end up in the associated graded space corresponding to the induced filtration on $H_{\text{Lee}}(K)$. By Remark 8.88, the quantum degree of these two elements equals the filtration degree of the two generators of $H_{\text{Lee}}(K)$. Rasmussen makes the following definition.

Definition 8.89 ([Ras10, Definition 3.1]). Let $K \subset S^3$ be an oriented knot. Define

$$s_{\min}(K) \coloneqq \min\{\deg_S([x]) \mid [x] \in H_{\mathrm{Lee}}(K), [x] \neq 0\}$$
$$s_{\max}(K) \coloneqq \max\{\deg_S([x]) \mid [x] \in H_{\mathrm{Lee}}(K), [x] \neq 0\}$$

Note that $s_{\min}(K)$ and $s_{\max}(K)$ correspond precisely to the quantum degree of the two surviving elements in the associated graded space, respectively. Since the the spectral sequence in Theorem 8.86 is up to isomorphism an invariant of K, both $s_{\min}(K)$ and $s_{\max}(K)$ are invariants of K as well.

Proposition 8.90 ([Ras10, Proposition 3.3]). Let $K \subset S^3$ be an oriented knot. Then

$$s_{\max}(K) = s_{\min}(K) + 2.$$

The previous proposition justifies the following definition.

Definition 8.91 ([Ras10, Definition 3.4]). Let $K \subset S^3$ be an oriented knot. Define

$$s(K) \coloneqq s_{\max}(K) - 1 = s_{\min}(K) + 1.$$

Since $s_{\min}(K)$ and $s_{\max}(K)$ are invariants of K, $s(K)$ is an invariant of K as well.

This is the famous *Rasmussen s-invariant* of a knot. The invariant behaves nicely with respect to mirror images and connected sum. Indeed, [Ras10, Proposition 3.9 and 3.11] show that

$$s(\overline{K}) = -s(K), \quad s(K_1 \# K_2) = s(K_1) + s(K_2)$$

for knots K, K_1, K_2, where $\overline{K}$ denotes the mirror image of K and $K_1 \# K_2$ is the connected sum of K_1 and K_2. Note that the additivity of s with respect to connected sum implies that s induces a homomorphism from the knot concordance group $\mathcal{C}$ (see Definition 4.12) to the integers $\mathbb{Z}$. However, the probably strongest and most surprising result is that the s-invariant yields a lower bound on the smooth 4-genus g_4^{smooth} of a knot K (see Definition 4.8 and Remark 4.9 for the definition of smooth 4-genus).

Theorem 8.92 ([Ras10, Theorem 1]). Let $K \subset S^3$ be an oriented knot. Then

$$|s(K)| \leq 2g_4^{\mathrm{smooth}}(K).$$

The proof proceeds by showing that an oriented connected cobordism S between two knots K_1 and K_2 induces a well-defined isomorphism $\phi_S \colon H_{\mathrm{Lee}}(K_1) \to H_{\mathrm{Lee}}(K_2)$ that is filtered of degree $\chi(S)$, the Euler characteristic of S. Then, if a knot K bounds a smoothly and properly embedded genus g surface in B^4, there exists an orientable connected cobordism S

of Euler characteristic $-2g$ between K and the unknot U, so the induced map ϕ_S is filtered of degree $-2g$. One then obtains inequalities

$$1 \geq \deg_S(\phi([x])) \geq \deg_S([x]) - 2g$$

for $[x] \in H_{\text{Lee}}(K)$, where the first inequality follows from the fact that $s_{\max}(U) = 1$, which in turn follows since $\text{Kh}(U)$ has dimension 2 and is supported in quantum degrees $+1$ and -1. It follows that $\deg_S([x]) \leq 2g + 1$, hence $s_{\max}(K) \leq 2g + 1$ and thus by definition $s(K) \leq 2g$. The other inequality $s(K) \geq -2g$ is shown by applying the same argument to the mirror image $\overline{K}$ and using $s(\overline{K}) = -s(K)$. Details can be found in [Ras10, Section 4].

The Rasmussen s-invariant can be defined more generally over any given field. Indeed, using the Bar-Natan system $\mathcal{F}_{\text{BN}}$ with h set equal to 1, Turner [Tur06] showed that the resulting filtered homology theory yield analogues of Theorem 8.86 and 8.87 over $\mathbb{F}_2$, thus giving a definition of the s-invariant over $\mathbb{F}_2$. This was later generalized to prime fields $\mathbb{F}_p$ by MacKaay-Turner-Vaz in [MTV07], using the filtered (α, β)-system $\mathcal{F}_{\alpha,\beta}$ (see Definition 8.61). In fact, their results hold for arbitrary fields $\mathbb{F}$ as shown by Lipshitz and Sarkar [LS14, Section 2]. In any case, the resulting s-invariant only depends on the characteristic of the field, see Schütz [Sch22]. Thus, let us state the results of [MTV07] but for arbitrary fields.

Proposition 8.93 ([MTV07, Proposition 2.2], [LS14, Section 2]). Let $L \subset S^3$ be a link and $\mathbb{F}$ a field. Further, let $\alpha, \beta, \widetilde{\alpha}, \widetilde{\beta} \in \mathbb{F}$.

1. If $\alpha^2 + 4t = 0$ then there is an isomorphism

$$H_{\mathcal{F}_{\alpha,\beta}}(L) \cong H_{\mathcal{F}_{\mathbb{F}}}(L),$$

   where $H_{\mathcal{F}_{\mathbb{F}}}(L)$ denotes the Khovanov homology of L over $\mathbb{F}$.

2. If $\alpha^2 + \beta \neq 0$ and $(\widetilde{\alpha}^2 + 4\widetilde{\beta}^2)/(\alpha^2 + 4\beta) = \gamma^2$ for some non-zero $\gamma \in \mathbb{F}$, then there is an isomorphism

$$H_{\mathcal{F}_{\alpha,\beta}}(L) \cong H_{\mathcal{F}_{\widetilde{\alpha},\widetilde{\beta}}}(L)$$

   that is induced by a base change and twist on $\mathcal{F}_{\alpha,\beta}$.

They proceed to show that in many cases, the homology $H_{\mathcal{F}_{\alpha,\beta}}(L)$ is as simple as $H_{\text{Lee}}(L)$.

Proposition 8.94 ([MTV07, Proposition 2.3]). Let $L \subset S^3$ be an oriented link with n components and $\mathbb{F}$ a field. Further, let $\alpha, \beta \in \mathbb{F}$. If $\alpha^2 + 4\beta = \gamma^2$ for some non-zero $\gamma \in \mathbb{F}$, then

$$\dim_{\mathbb{F}} H_{\mathcal{F}_{\alpha,\beta}}(L) = 2^n.$$

All generators of $H_{\mathcal{F}_{\alpha,\beta}}(L)$ have even homological degree, except in the case of a knot (i.e. $n = 1$) where both generators lie in homological degree zero.

We are now in a similar situation that led to the definition of the Rasmussen s-invariant. Indeed, let $\mathbb{F}$ be a field, and let $\alpha, \beta \in \mathbb{F}$ such that $\alpha^2 + 4\beta = \gamma^2$ for some non-zero $\gamma \in \mathbb{F}$. Let S be the filtration on $H_{\mathcal{F}_{\alpha,\beta}}(L)$ that is induced by the filtration on the corresponding chain complex. Let $K \subset S^3$ be a knot and set

$$s_{\min}(K, \mathbb{F})_{\alpha,\beta} := \min\{\deg_S([x]) \mid [x] \in H_{\mathcal{F}_{\alpha,\beta}}(L),\ [x] \neq 0\}$$
$$s_{\max}(K, \mathbb{F})_{\alpha,\beta} := \max\{\deg_S([x]) \mid [x] \in H_{\mathcal{F}_{\alpha,\beta}}(L),\ [x] \neq 0\}.$$

Then we have the analogue of Proposition 8.90.

Proposition 8.95 ([LS14, Proposition 2.6]). We have

$$s_{\max}(K, \mathbb{F})_{\alpha,\beta} = s_{\min}(K, \mathbb{F})_{\alpha,\beta} + 2$$

Definition 8.96 ([MTV07, Definition 4.1]). Let $K \subset S^3$ be a knot and $\mathbb{F}$ a field. Further, let $\alpha, \beta \in \mathbb{F}$ such that $\alpha^2 + 4\beta = \gamma^2$ for some non-zero $\gamma \in \mathbb{F}$. Define

$$s(K, \mathbb{F})_{\alpha,\beta} := \frac{s_{\min}(K, \mathbb{K})_{\alpha,\beta} + s_{\max}(K, \mathbb{K})_{\alpha,\beta}}{2}.$$

In the case of $\alpha = 1$, $\beta = 1$, we set

$$s_{\mathbb{F}}(K) := s(K, \mathbb{F})_{1,0}.$$

$s_{\mathbb{F}}(K)$ is called the *Rasmussen s-invariant over* $\mathbb{F}$.

Remark 8.97.

1.) $s(K, \mathbb{K})_{\alpha,\beta}$ is an invariant of K for the same reasons as for the Rasmussen s-invariant.

2.) By Proposition 8.93 2., if $\mathbb{F} = \mathbb{Q}$ then $s_{\mathbb{Q}}(K) = s(K)$.

3.) For a long time it was an open question whether $s(K, \mathbb{F})_{\alpha,\beta}$ is the same over all fields. In fact, [MTV07, Theorem 4.2] provided a seemingly affirmative answer to this question, but later the proof turned out to contain an error, see [MTV13]. Then, a first counter-example was given by Cotton Seed [See13, LS14], who showed that

$$s_{\mathbb{F}_2}(14n19265) \neq s_{\mathbb{F}_3}(14n19265).$$

Few years later, Dirk Schütz [Sch21a, Sch21b] found all knots with up to 15 crossings for which $s_{\mathbb{F}_2} \neq s_{\mathbb{F}_3}$. They are listed in [Sch21a, Table 1] together with the following additional knots [Sch21b]:

$$15n154386, \quad 15n165952, \quad 15n165966, \quad 15n166064, \quad 15n166244.$$

Recently, Lewark and Zibrowius [LZ21] found the first example of a knot for which $s_{\mathbb{Q}} \neq s_{\mathbb{F}_3}$. In the upcoming Chapter 9, we will show that one can extract the Rasmussen s-invariant over any finite field $\mathbb{F}_p$ or $\mathbb{Q}$ from the $\mathbb{Z}[G]$-homology of a knot, see Remark 9.16.

4.) One may also define an s-invariant over $\mathbb{Z}$. Indeed, let $\mathcal{F}^{\mathbb{Z}}_{\alpha,\beta}$ denote the Frobenius system $\mathcal{F}_{\alpha,\beta}$ with coefficients changed from $\mathbb{F}$ to $\mathbb{Z}$. MacKaay, Turner, and Vaz show in [MTV07, Proposition 2.4] that if $L \subset S^3$ is a link with n components and $\alpha^2 + 4\beta = \gamma^2$ for some non-zero $\gamma \in \mathbb{Z}$, then
$$H_{\mathcal{F}^{\mathbb{Z}}_{\alpha,\beta}}(L) \cong \mathbb{Z}^{\oplus n} \oplus T^*,$$
where T^* is torsion (they further show that if $\alpha, \beta < p$ and $\gamma \neq 0 \mod p$ for p a prime, then $H_{\mathcal{F}^{\mathbb{Z}}_{\alpha,\beta}}(L)$ has no p-torsion). For a knot $K \subset S^3$, one may then define $s(K, \mathbb{Z})_{\alpha,\beta}$ as in Definition 8.96 but with $s_{\min}$ and $s_{\max}$ restricted to the torsion-free part of $H_{\mathcal{F}^{\mathbb{Z}}_{\alpha,\beta}}(L)$.

8.11 Khovanov homology II

Let us return once more to ordinary Khovanov homology. Let $K \subset S^3$ be an oriented knot with diagram D, and consider the (rational) Khovanov homology $\mathrm{KH}(K)$ of K obtained from the Khovanov chain complex $\mathcal{C}_{\mathrm{Kh}}(D)$, or equivalently from the $\mathcal{F}_{\mathbb{Q}}$-complex $C_{\mathcal{F}_{\mathbb{Q}}}(D)$ corresponding to the Frobenius system $\mathcal{F}_{\mathbb{Q}}$ with algebra $\mathbb{Q}[X]/(X)^2$ (see Definition 8.55). Based on early computations of Khovanov homology by Bar-Natan, Khovanov, and Garoufalidis, there have been several phenomenological conjectures about the structure of $\mathrm{Kh}(K)$ [BN02, Section 3.6]. The most famous one is the so-called *Knight Move Conjecture.*

Conjecture 8.98 ([BN02, Conjecture 1]). Let $K \subset S^3$ be an oriented prime knot. Then the rational Khovanov homology of K consists of a single *pawn move* piece
$$\mathbb{Q}_{(0,s(K)-1)} \oplus \mathbb{Q}_{(0,s(K)+1)},$$
and several *knight move* pieces
$$\mathbb{Q}_{(i,j)} \oplus \mathbb{Q}_{(i+1,j+4)}$$
where $i, j \in \mathbb{Z}$. Here, $\mathbb{Q}_{(i,j)}$ stands for a copy of $\mathbb{Q}$ generated by a single element in homological degree i and quantum degree j, and $s(K)$ stands for the Rasmussen s-invariant of K.

The terms "pawn move" and "knight move" were coined by Bar-Natan [BN02, Section 3.6] and origin from the observation that if one arranges the Khovanov homology in a two dimensional table where the horizontal axis corresponds to the homological grading and the vertical axis corresponds to the quantum grading, then the pawn and knight move pieces look like
$$\begin{array}{|c|}\hline \mathbb{Q} \\ \hline \mathbb{Q} \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline & \mathbb{Q} \\ \hline & \\ \hline \mathbb{Q} & \\ \hline \end{array} \tag{8.40}$$

respectively. We will give a more formal definitions of "pawns" and "knights" in Section 10.3, where they appear in the decomposition of the $\mathbb{Z}[G]$-complex of a knot K.

The Knight Move Conjecture was supported by many examples for a long time. In particular, Bar-Natan [BN02, Section 3.6] showed by computations that the conjecture is true for all prime knots with up to 11 crossings, Lee [Lee02] proved that the Knight Move Conjecture is true for all alternating knots, and Manolescu-Ozsváth [MO08] showed it for all quasi-alternating knots. Since the Lee spectral sequence (see Theorem 8.86) has on page E_n a differential of bidegree $(1, 4n)$, it follows that if the sequence for a knot K collapses on the second page, then the Knight Move Conjecture is true for K. Alishahi and Dowlin [AD19] linked the Knight Move Conjecture to the unknotting number of knots and showed that it is true for all knots with unknotting number not bigger than 2 (see Corollary 10.25). However, a counterexample to the Knight Move Conjecture was found very recently by Manolescu-Marengon [MM20], proving that the conjecture is in general *wrong* (their counterexample is a knot with more than 30 crossings!).

Another structural observation about Khovanov homology was made by Khovanov in [Kho03]. Let $L \subset S^3$ be an oriented link with diagram D and consider the rational or integral Khovanov complex $C_{\mathcal{F}_\mathbb{Q}}(D)$ or $C_{\mathcal{F}_\mathbb{Z}}(D)$, respectively. By our definition of a TQFT, $C_{\mathcal{F}_\mathbb{Q}}(D)$ is equipped with an $A_\mathbb{Q} := \mathbb{Q}[X]/(X^2)$-module structure, where $A_\mathbb{Q}$ acts on the first factor of tensor products in $C_{\mathcal{F}_\mathbb{Q}}(D)$. Similarly, $C_{\mathcal{F}_\mathbb{Z}}(D)$ has an $A_\mathbb{Z} := \mathbb{Z}[X]/(X^2)$-module structure. Khovanov [Kho03] makes the following definition.[14]

Definition 8.99. Let $L \subset S^3$ be an oriented link with diagram D. Then the homology of the *reduced rational Khovanov complex*

$$C^{\mathrm{red}}_{\mathcal{F}_\mathbb{Q}}(D) := C_{\mathcal{F}_\mathbb{Q}}(D) \otimes_{A_\mathbb{Q}} A_\mathbb{Q}/(X)\{-1\}$$

is called the *reduced rational Khovanov homology* of L and denoted by $\mathrm{Kh}^{\mathrm{red}}(L)$. Similarly, the homology of the *reduced integral Khovanov complex*

$$C^{\mathrm{red}}_{\mathcal{F}_\mathbb{Z}}(D) := C_{\mathcal{F}_\mathbb{Z}}(D) \otimes_{A_\mathbb{Z}} A_\mathbb{Z}/(X)\{-1\}$$

is called the *reduced integral Khovanov homology* of L and denoted by $\mathrm{Kh}^{\mathrm{red}}_\mathbb{Z}(L)$.

As for ordinary Khovanov homology, the notation $\mathrm{Kh}^{\mathrm{red}}(L)$ respectively $\mathrm{Kh}^{\mathrm{red}}_\mathbb{Z}(L)$ is justified since a different choice of diagram for L yields homotopy equivalent reduced Khovanov complexes. Further, note that reducing preserves the homological as well as the quantum grading on complexes, making reduced Khovanov homology bigraded as well.

[14]Khovanov's definition of reduced complexes and homology in [Kho03] is for knots and only in the rational case, but it works equally well for links and the integral case.

Before discussing results about reduced Khovanov homology, let us quickly note that one may also reduce other $\mathcal{F}$-complexes in the sense of Definition 8.99; in fact, we will do so with the $\mathbb{Z}[G]$-complex in Section 9.1. In order to distinguish between ordinary and reduced complexes, we introduce a new term.

Definition 8.100. Let $L \subset S^3$ be an oriented link with diagram D and $\mathcal{F}$ a rank 2 Frobenius system. Then the $\mathcal{F}$-complex $C_{\mathcal{F}}(L)$ (resp. $C_{\mathcal{F}}(D)$) and $\mathcal{F}$-homology $H_{\mathcal{F}}(L)$ is called *unreduced.*

Convention. If clear from the context, we will refer to an unreduced $\mathcal{F}$-complex (resp. $\mathcal{F}$-homology) simply as $\mathcal{F}$-complex (resp. $\mathcal{F}$-homology) as we did before.

Representing Khovanov homology in a table as in (8.40), the computations of Bar-Natan in [BN02] have shown that the rational Khovanov homology of almost all knots with at most 10 crossings is supported on two adjacent diagonals; there are only twelve exceptions. Here, if h denotes the homological and q the quantum grading, then a "diagonal" refers to the line $q - 2h = b$ in the table for some $b \in \mathbb{Z}$. This observation has led to several notions of *homological thinness* and *width* of knots, see [Shu14, Shu21]. Before introducing one of them, let us state how this observation translates to reduced rational Khovanov homology.

Proposition 8.101 ([Kho03, Proposition 4]). Let $K \subset S^3$ be an oriented knot. Then the reduced rational Khovanov homology $\mathrm{Kh}^{\mathrm{red}}(K)$ is supported in a single diagonal (i.e. $q - 2h$ is constant) if and only if the rational Khovanov homology $\mathrm{Kh}(K)$ is supported in two adjacent diagonals.

This motivates the following definition.

Definition 8.102. Let $L \subset S^3$ be an oriented link with diagram D. Define on the rational and integral Khovanov complex the *δ-grading* by setting

$$\delta(x) := h(x) - 2q(x),$$

where $h(x)$ denotes the homological degree and $q(x)$ denotes the quantum degree of a homogeneous element $x \in C_{\mathcal{F}_{\mathbb{Z}}}(D)$ or $x \in C_{\mathcal{F}_{\mathbb{Q}}}(D)$, respectively. The δ-grading descends to homology, and L is called

1. *homologically thin* (or *h-thin*) if the reduced rational Khovanov homology of L is supported in a single δ-degree; and
2. *δ-thin* (or simply *thin*) if the reduced integral Khovanov homology of L consists of free modules and is supported in a single δ-degree.

If K is not homologically thin or δ-thin, then L is called called *homologically thick* (or *h-thick*) or *δ-thick* (or simply *thick*), respectively.

Remark 8.103. Observe that our notion of homological thinness agrees with Khovanov's notion in [Kho03], whereas the notion of δ-thinness follows [MO08], where it is referred to as "Khovanov homological thinness". Note that we have the following implication:

$$\delta\text{-thin} \implies h\text{-thin}.$$

However, the converse is in general *not* true: we require that the reduced integral Khovanov homology of a δ-thin link is *free*.

As mentioned above, all but twelve knots with up to 10 crossings are h-thin. More generally, Lee has shown that all alternating links are h-thin [Lee05]. In the upcoming Proposition 7.9, we will see that λ of all non-trivial δ-thin knots is equal to one.

Chapter 9

$\mathbb{Z}[G]$-Homology

In 2006, Khovanov [Kho06] described a rank 2 Frobenius system $\mathcal{F}_{\text{univ}}$ from which any other rank 2 Frobenius system can be obtain by a base change and twist (see [Kho06, Proposition 5]). As a consequence, the $\mathcal{F}_{\text{univ}}$-complex determines the $\mathcal{F}$-complex of any other rank 2 Frobenius system $\mathcal{F}$. In this sense, $\mathcal{F}_{\text{univ}}$ yields a *universal* Khovanov homology theory.

As mentioned in Chapter 7, other Frobenius systems that yield universal Khovanov-type chain complexes were found later on, one of them being the system $\mathcal{F}_{\mathbb{Z}[G]}$, which gives rise to $\mathbb{Z}[G]$-homology of a link. This system was first described by Naot [Nao06, Nao07], and will form the basis for the definition of our knot invariant λ. The $\mathbb{Z}[G]$-system is of particular interest as it is a simpler theory than $\mathcal{F}_{\text{univ}}$ in the sense that there are less indeterminates involved. The aim of this chapter is to provide the theoretical foundations of the $\mathbb{Z}[G]$-theory and show that it is equivalent to Khovanov's $\mathcal{F}_{\text{univ}}$-theory.

Chapter 9 is organized as follows. In Section 9.1, we give the definition of the $\mathbb{Z}[G]$-complex of a knot K using the Frobenius algebra $\mathbb{Z}[G, X]/(X^2+GX)$ (see Definition 9.1). We then describe an alternative way to obtain the $\mathbb{Z}[G]$-complex using the Bar-Natan complex of the 2-ended tangle obtained by cutting K open at some point (see Definition 9.5). In Proposition 9.8 we show that both these ways to obtain the $\mathbb{Z}[G]$-complex are equivalent, which forms the starting point for the proof of Theorem 7.1 in Section 9.2 about its universality. In Section 9.3, we describe a reduced version of $\mathbb{Z}[G]$-homology obtained by setting $G = 1$. This gives a particularly simple homology theory (see Proposition 9.15), from which one may read off the Rasmussen s-invariant over any field, for instance (see Remark 9.16). Lastly, in Section 9.4 we introduce a $\mathbb{Z}[G]$-action on a certain category of cobordisms, which is a necessary technicality for later chapters. We assume that the reader is familiar with the contents of Chapter 8, in particular Sections 8.3 to 8.5 and 8.7.

9.1 The $\mathbb{Z}[G]$-complex and homology

In Section 8.5, we already encountered the definition of the Frobenius system $\mathcal{F}_{\text{univ}}$ and the $\mathbb{Z}[G]$-system $\mathcal{F}_{\mathbb{Z}[G]}$. For the readers convenience, we recall the details in formal definitions.

Definition 8.56. Let h and t be formal variables. The Frobenius system $\mathcal{F}_{\text{univ}} = (R_{\text{univ}}, A_{\text{univ}}, \Delta, \varepsilon)$ is defined as

$$R_{\text{univ}} = \mathbb{Z}[h,t], \quad A_{\text{univ}} = R_{\text{univ}}[X]/(X^2 - hX - t), \quad \iota\colon R_{\text{univ}} \hookrightarrow A_{\text{univ}},$$

$$\begin{aligned} &m(1 \otimes 1) = 1 && m(X \otimes X) = hX + t \\ &m(1 \otimes X) = X && m(X \otimes 1) = X \\ &\Delta(1) = 1 \otimes X + X \otimes 1 - h1 \otimes 1 && \varepsilon(1) = 0 \\ &\Delta(X) = X \otimes X + t1 \otimes 1 && \varepsilon(X) = 1 \end{aligned}$$

We equip $\mathcal{F}_{\text{univ}}$ with a grading by setting

$$\deg 1 = 0, \quad \deg X = \deg h = -2, \quad \deg t = -4.$$

The maps $m, \iota, \Delta, \varepsilon$ are then homogeneous of degree $0, 0, -2, 2$, respectively.

Definition 8.57. Let G be a formal variable. The Frobenius system $\mathcal{F}_{\mathbb{Z}[G]} = (R_{\mathbb{Z}[G]}, A_{\mathbb{Z}[G]}, \Delta, \varepsilon)$ is defined as

$$R_{\mathbb{Z}[G]} = \mathbb{Z}[G], \quad A_{\mathbb{Z}[G]} = R_{\mathbb{Z}[G]}[X]/(X^2 + GX), \quad \iota\colon R_{\mathbb{Z}[G]} \hookrightarrow A_{\mathbb{Z}[G]},$$

$$\begin{aligned} &m(1 \otimes 1) = 1 && m(X \otimes X) = -GX \\ &m(1 \otimes X) = X && m(X \otimes 1) = X \\ &\Delta(1) = 1 \otimes X + X \otimes 1 + G1 \otimes 1 && \varepsilon(1) = 0 \\ &\Delta(X) = X \otimes X && \varepsilon(X) = 1 \end{aligned}$$

We equip $\mathcal{F}_{\mathbb{Z}[G]}$ with a grading by setting

$$\deg 1 = 0, \quad \deg X = \deg G = -2.$$

The maps $m, \iota, \Delta, \varepsilon$ are then homogeneous of degree $0, 0, -2, 2$, respectively.

As explained in Section 8.5, a rank 2 Frobenius system induces a functor $\mathcal{F}\colon \operatorname{Cob}^3_{/l}(2) \to \mathcal{M}_A$ which is called topological quantum field theory (TQFT for short). Let T be a 2-ended tangle with diagram D_T. Then a TQFT applied to the cube of resolutions of D_T yields a chain complex $C_{\mathcal{F}}(T)$ as described in Section 8.6. The chain complex depends on the chosen diagram D_T, but any two diagrams of T yield homotopy equivalent chain complexes (see Theorem 8.65).

Notation. To simplify notation, we will write $C_{\mathrm{univ}}(T)$ and $C_{\mathbb{Z}[G]}(T)$ for the complexes coming from the TQFTs $\mathcal{F}_{\mathrm{univ}}$ and $\mathcal{F}_{\mathbb{Z}[G]}$, respectively. Further, we abuse notation as in Section 8.5 and denote a Frobenius system as well as the corresponding TQFT with the same symbol $\mathcal{F}$.

As we will see in Section 9.2, $C_{\mathrm{univ}}(T)$ is determined by a *reduced* version of $C_{\mathbb{Z}[G]}(T)$ and vice-versa (see Section 8.11 for a discussion of reduced homology). Let us give the definition of the reduced $\mathcal{F}_{\mathbb{Z}[G]}$-complex.

Definition 9.1. Let T be a 2-ended tangle with diagram D_T and $C_{\mathbb{Z}[G]}(T)$, $C_{\mathbb{Z}[G]}(D_T)$ the corresponding (unreduced) $\mathcal{F}_{\mathbb{Z}[G]}$-complexes, respectively. The *reduced* $\mathcal{F}_{\mathbb{Z}[G]}$-complex of D_T is defined as

$$[\![D_T]\!] := C_{\mathbb{Z}[G]}(D_T) \otimes_{A_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}/(X)\{-1\} \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]}).$$

Similarly, the *reduced* $\mathcal{F}_{\mathbb{Z}[G]}$-complex of T is defined as

$$[\![T]\!] := C_{\mathbb{Z}[G]}(T) \otimes_{A_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}/(X)\{-1\} \in \mathrm{Kom}_{/h}(\mathcal{M}_{\mathbb{Z}[G]}).$$

Using the inclusion functor $\mathcal{M}_{\mathbb{Z}[G]} \hookrightarrow \mathbb{Z}[G]$-Mod as described in Section 8.9, we define the $\mathbb{Z}[G]$-*homology* of T as the homology of $[\![T]\!]$ considered in $\mathrm{Kom}_{/h}(\mathbb{Z}[G]\text{-Mod})$ and is denoted by $H_{\mathbb{Z}[G]}(T)$. If $L \subset S^3$ is a base-pointed link with base-pointed diagram D_L and T_L the corresponding 2-ended tangle T_L with diagram D_{T_L} (see (8.4)), we define the same notions for L by setting

$$[\![D_L]\!] := [\![D_{T_L}]\!], \quad [\![L]\!] := [\![T_L]\!], \quad H_{\mathbb{Z}[G]}(L) := H_{\mathbb{Z}[G]}(T_L).$$

Remark 9.2.

1.) Since $C_{\mathbb{Z}[G]}(T)$ is an invariant of 2-ended tangles, so is the reduced version $[\![T]\!]$ and does in particular not depend on the choice of diagram D_T.

2.) The grading on $\mathcal{F}_{\mathbb{Z}[G]}$ induces a grading on $C_{\mathbb{Z}[G]}(T)$ and thus also on $[\![T]\!]$, making the reduced $\mathbb{Z}[G]$-complex and $\mathbb{Z}[G]$-homology bigraded. We call the gradings as usual *homological* and *quantum* grading.

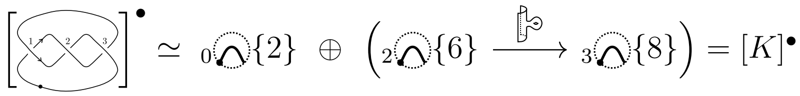
3.) Observe that reducing has the following effect on summands in $C_{\mathbb{Z}[G]}$:

$$\underline{A_{\mathbb{Z}[G]}}\{1\} \otimes_{R_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}\{1\}^{\otimes n} \xrightarrow{\text{reduce}} R_{\mathbb{Z}[G]} \otimes_{R_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}\{1\}^{\otimes n}.$$

In particular, the reduced complex has no longer an $A_{\mathbb{Z}[G]}$-module structure. Also note that the first factor is no longer affected by a shift in grading.[1]

[1] This is needed in order that the dual of the reduced complex corresponds to the reduced complex of the mirror image. Here, we use the usual convention that the signs of the homological and quantum grading in the dual are switched.

Example 9.3. Let K be once more the right-handed trefoil. Recall from Example 8.80 that the Bar-Natan complex of the right-handed trefoil is given as



and the unreduced $\mathcal{F}_{\mathbb{Z}[G]}$-complex of K takes the form

$$\mathcal{C}_{\mathcal{F}_{\mathbb{Z}[G]}}(K) = {}_0A_{\mathbb{Z}[G]}\{3\} \ \oplus \ \Big({}_2A_{\mathbb{Z}[G]}\{7\} \xrightarrow{m\circ\Delta} {}_3A_{\mathbb{Z}[G]}\{9\}\Big)$$

where $A_{\mathbb{Z}[G]} = \mathbb{Z}[X,G]/(X^2+GX)$ and the left subscript denotes the homological degree. In the basis $1, X$ of $A_{\mathbb{Z}[G]}$, the matrix of $m \circ \Delta$ is

$$m \circ \Delta = \begin{pmatrix} G & 0 \\ 2 & -G \end{pmatrix}$$

Then the reduced $\mathcal{F}_{\mathbb{Z}}[G]$-complex $[\![K]\!]$ is given as

$$\begin{aligned} [\![K]\!] &\stackrel{\text{def.}}{=} C_{\mathbb{Z}[G]}(K) \otimes_{A_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}/(X)\{-1\} \\ &= {}_0\mathbb{Z}[G]\{2\} \ \oplus \ \Big({}_2\mathbb{Z}[G]\{6\} \xrightarrow{G} {}_3\mathbb{Z}[G]\{8\}\Big) \end{aligned}$$

It will be convenient to give an alternative description of $[\![T]\!]$, using the category $\mathcal{E}$ (see Definition 8.33). Let T be a 2-ended tangle with diagram D_T. Using delooping (see Figure 8.20), we can successively resolve every circle that appears in the Bar-Natan complex $[D_T]$. This yields an isomorphic complex whose chain objects consist solely of sums of grading shifted copies of D_{T_0} (where D_{T_0} is the diagram of the trivial 2-ended tangle in the unit disk), giving us a connection between the categories $\mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))$ and $\mathrm{Kom}(\mathrm{Mat}(\mathcal{E}))$. In fact:

Proposition 9.4. The functor $B\colon \mathcal{E} \to \mathrm{Cob}^3_{/l}(2)$ given by inclusion is an equivalence of categories.

Proof. We need to check that B is faithful, full and dense. As explained before, density of B follows directly from delooping. To show that B is faithful and full, we are going to look at the morphism spaces

$$\begin{aligned} &\hom_{\mathcal{E}}(D_{T_0}, D_{T_0}), \\ &\hom_{\mathrm{Cob}^3(2)}(B(D_{T_0}), B(D_{T_0})), \\ &\hom_{\mathrm{Cob}^3_{/l}(2)}(B(D_{T_0}), B(D_{T_0})), \end{aligned}$$

where D_{T_0} is the diagram of the trivial 2-ended tangle T_0.

Let G, Σ_i, $i \in \mathbb{Z}_{\geq 0}$ be formal variables. We introduce a grading on $\mathbb{Z}[G]$ and $\mathbb{Z}[G, \Sigma_0, \Sigma_1, \Sigma_2, \dots]$ by setting $\deg G = -2$ and $\deg \Sigma_i = 2 - 2i$. Then there is an isomorphism of graded abelian groups[2]

$$\hom_{\mathcal{E}}(D_{T_0}, D_{T_0}) \cong \mathbb{Z}[G]$$

given by mapping a connected cobordism of genus k to G^k. Similarly, there is an isomorphism of graded Abelian groups[2]

$$\hom_{\mathrm{Cob}^3(2)}(B(D_{T_0}), B(D_{T_0})) \cong \mathbb{Z}[G, \Sigma_0, \Sigma_1, \Sigma_2, \dots],$$

given by mapping a cobordism, which consists of the marked component with genus k and a disjoint union of n_i many closed surfaces of genus i, to the product $G^k \prod_{i=0}^{\infty} \Sigma_i^{n_i}$. In order to determine the morphism space $\hom_{\mathrm{Cob}^3_{/l}(2)}(B(D_{T_0}), B(D_{T_0}))$, we need to understand how the local relations S, T, and $4Tu$ in $\mathrm{Cob}^3_{/l}(2)$ affect the ring $\mathbb{Z}[G, \Sigma_0, \Sigma_1, \Sigma_2, \dots]$. Introducing S and T translates to $\Sigma_0 = 0$ and $\Sigma_1 = 2$. Next, we replace $4Tu$ with the equivalent $3S_2$ relation (cf. [Bar05, Section 11.4], and Figure 9.1 below) which is easier to handle as there are at most three surfaces involved. We name the surfaces in the relation A, B, C in a clockwise manner starting top left. Suppose that $g(A) = a$, $g(B) = b$, $g(C) = c$. Let M_n be the curtain with genus $g(M_n) = n$.

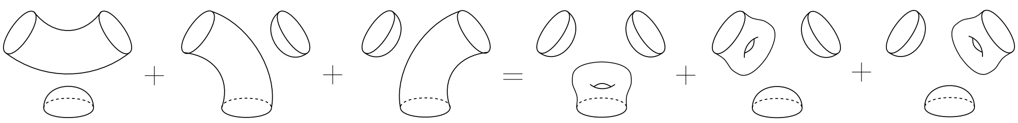


Figure 9.1: The $3S_2$-relation.

Suppose first that $A = B = M_0 \neq C$. In this case, the $3S_2$ relation translates to

$$G\Sigma_c + G^c + G^c = \Sigma_{c+1} + G\Sigma_c + G\Sigma_c \iff \Sigma_{c+1} = 2G^c - G\Sigma_c.$$

By the S relation, we have $\Sigma_0 = 0$ and thus $\Sigma_1 = 2G^0 - G \cdot 0 = 2$, which coincides with the T relation. By induction, we therefore obtain the relation

$$\Sigma_c = \begin{cases} 2G^{c-1}, & c \text{ odd}, \\ 0, & c \text{ even}, \end{cases} \tag{9.1}$$

giving us a surjection $\hom_{\mathrm{Cob}^3_{/l}(2)}(B(D_{T_0}), B(D_{T_0})) \twoheadrightarrow \mathbb{Z}[G]$. We claim that there are no other relations introduced, i.e. that this surjection is an isomorphism. For this, we check all possible general cases of the $3S_2$ relation.

[2]This is in fact an isomorphism of graded rings if we declare multiplication in $\hom_{\mathcal{E}}$ (resp. $\hom_{\mathrm{Cob}^3(2)}$) as composition of cobordisms.

Case 1: A, B, C are three different closed surfaces (i.e. none of them is a curtain M_n). In this case, $3S_2$ translates to

$$\Sigma_{a+b}\Sigma_c + \Sigma_{a+c}\Sigma_b + \Sigma_{b+c}\Sigma_a = \Sigma_{a+1}\Sigma_b\Sigma_c + \Sigma_a\Sigma_{b+1}\Sigma_c + \Sigma_a\Sigma_b\Sigma_{c+1}.$$

If all $a \equiv b \equiv c \mod 2$ or if a is odd and b, c even, both sides of the equation vanish after applying (9.1). On the other hand, if a is even and b, c odd, then (9.1) gives us

$$4G^{a+b+c-2} + 4G^{a+b+c-2} + 0 = 8G^{a+b+c-2} + 0 + 0,$$

showing that there is no new relation introduced. Since $3S_2$ is symmetric in A, B, C, no other parities of a, b, c need to be checked in this case.

Case 2: $A = B \neq C$, none of them is a curtain M_n. In this case we have

$$\Sigma_{a+1}\Sigma_c + \Sigma_{a+c} + \Sigma_{a+c} = \Sigma_a\Sigma_{c+1} + \Sigma_{a+1}\Sigma_c + \Sigma_{a+1}\Sigma_c.$$

If $a \equiv c \mod 2$ both sides of the equation vanish after applying (9.1). If a is even and c odd, we obtain

$$4G^{a+c-1} + 0 + 0 = 0 + 2G^{a+c-1} + 2G^{a+c-1},$$

showing that there is no new relation introduced. Again by symmetry of the $3S_2$ relation, no other cases a and c need to be checked.

Case 3: $A = B = C$, none of them is a curtain M_n. In this case we have

$$3\Sigma_{a+1} = 3\Sigma_{a+1},$$

showing immediately that there are no new relations introduced.

Case 4: $A = M_a$, B, C are three different surfaces. The $3S_2$ relations translates to

$$G^{a+b}\Sigma_c + G^{a+c}\Sigma_b + G^a\Sigma b + c = G^{a+1}\Sigma_b\Sigma_c + G^a\Sigma_{b+1}\Sigma_c + G^a\Sigma_b\Sigma_{c+1}.$$

If b, c are even, both sides vanish after applying (9.1). If b is even and c is odd, we obtain

$$2G^{a+b+c-1} + 0 + 2G^{a+b+c-1} = 0 + 4G^{a+b+c-1} + 0,$$

and if b is odd and c even, we get

$$0 + 2G^{a+b+c-1} + 2G^{a+b+c-1} = 0 + 0 + 4G^{a+b+c-1}.$$

In both cases, no new relations are introduced.

Case 5: $A = B = M_a \neq C$. In this case, we get

$$G^{a+1}\Sigma_c + G^{a+c} + G^{a+c} = G^{a+1}\Sigma_c + G^{a+1}\Sigma_c + G^a\Sigma_{c+1}.$$

If c is odd, applying (9.1) yields

$$2G^{a+c} + G^{a+c} + G^{a+c} = 2G^{a+c} + 2G^{a+c} + 0,$$

and if c is even,

$$0 + G^{a+c} + G^{a+c} = 0 + 0 + 2G^{a+c}.$$

In both cases, no new relations are introduced.

Case 6: $A = M_a$, $B = C = M_c$, $M_a \neq M_c$. We have

$$G^{a+c} + G^{a+c} + G^a\Sigma_{c+1} = G^{a+1}\Sigma_c + G^a\Sigma_{c+1} + G^a\Sigma_{c+1}.$$

If c is odd, we obtain after applying (9.1)

$$G^{a+c} + G^{a+c} + 0 = 2G^{a+c} + 0 + 0,$$

and if c is even,

$$G^{a+c} + G^{a+c} + 2G^{a+c} = 0 + 2G^{a+c} + 2G^{a+c}.$$

In both cases, no new relations are introduced.

Case 7: $A = B = C = M_a$. As in the third case, we have

$$3G^a = 3G^a,$$

showing immediately that there are no new relations introduced.

The above shows that there are isomorphisms

$$\varphi : \hom_{\mathcal{E}}(D_{T_0}, D_{T_0}) \xrightarrow{\cong} \mathbb{Z}[G]$$
$$\psi : \hom_{\mathrm{Cob}^3_{/l}(2)}(B(D_{T_0}), B(D_{T_0})) \xrightarrow{\cong} \mathbb{Z}[G].$$

Consider the diagram

$$\begin{array}{ccc} & \mathbb{Z}[G] & \\ {}^{\varphi}\swarrow & & \searrow^{\psi} \\ \hom_{\mathcal{E}}(D_{T_0}, D_{T_0}) & \xrightarrow{\quad B \quad} & \hom_{\mathrm{Cob}^3_{/l}(2)}(B(D_{T_0}), B(D_{T_0})). \end{array}$$

By construction this diagram commutes, i.e. $B \circ \varphi = \psi$. Since both φ and ψ are isomorphisms, B has to be an isomorphism as well. □

The functor B induces an equivalence of categories on the additive closure $\mathrm{Mat}(\mathcal{E}) \to \mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))$ which we denote by the same symbol. This means in particular that there is a functor $I\colon \mathrm{Mat}(\mathrm{Cob}^3_{/l}(2)) \to \mathrm{Mat}(\mathcal{E})$ such that $I \circ B$ and $B \circ I$ are naturally isomorphic to the identity functors $\mathrm{Id}_{\mathrm{Mat}(\mathcal{E})}$ and $\mathrm{id}_{\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))}$, respectively. The functor I can be constructed by using delooping as well, though not in a natural way: there is an ambiguity in the order of which circles get delooped.[3] Observe that if I is constructed in this way, then $I \circ B = \mathrm{Id}_{\mathrm{Mat}(\mathcal{E})}$ while $B \circ I$ is still only naturally isomorphic to $\mathrm{Id}_{\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))}$. The functor I induces an equivalence of categories

$$\widehat{I}\colon \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))) \to \mathrm{Kom}(\mathrm{Mat}(\mathcal{E})).$$

Now as before let G be a formal variable and consider the ring $\mathbb{Z}[G]$. We equip $\mathbb{Z}[G]$ with a grading by setting $\deg 1 = 0$ and $\deg G = -2$, and we consider the category $\mathcal{M}_{\mathbb{Z}[G]}$. There is a functor $F\colon \mathcal{E} \to \mathcal{M}_{\mathbb{Z}[G]}$ sending the object $D_{T_0}\{m\} \in \mathrm{ob}(\mathcal{E})$ to the $\mathbb{Z}[G]$-module $\mathbb{Z}[G]\{m\}$ and a cobordism of genus k to the linear map given by multiplication with G^k. This functor extends by linearity to a functor $F\colon \mathrm{Mat}(\mathcal{E}) \to \mathcal{M}_{\mathbb{Z}[G]}$, which is in fact an isomorphism. Moreover, it induces yet another functor

$$\widehat{F}\colon \mathrm{Kom}(\mathrm{Mat}(\mathcal{E})) \to \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]}).$$

Note that the functors $\widehat{I}$ and $\widehat{F}$ respect homotopy equivalence of complexes and thus descend to functors on $\mathrm{Kom}_{/h}$, for which we use the same notation. Let us make the following definition.

Definition 9.5. Let T be a 2-ended tangle with diagram D_T. The $\mathbb{Z}[G]$-*complex* of D_T is defined as the chain complex

$$\Omega(D_T) \coloneqq \widehat{F}(\widehat{I}([D_T])) \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]}),$$

where $[D_T]$ denotes the Bar-Natan complex of D_T. Similarly, the $\mathbb{Z}[G]$-*complex* of T is defined as the chain complex

$$\Omega(T) \coloneqq \widehat{F}(\widehat{I}([T])) \in \mathrm{Kom}_{/h}(\mathcal{M}_{\mathbb{Z}[G]}),$$

where $[T]$ is the Bar-Natan complex of T. If L is a link with base point and diagram D_L, and T_L its corresponding 2-ended tangle with diagram D_{T_L} (see (8.4) and (8.5)), we define the $\mathbb{Z}[G]$-*complex of* D_L and the $\mathbb{Z}[G]$-*complex of* L as

$$\begin{aligned} \Omega(D_L) &\coloneqq \widehat{F}(\widehat{I}([D_{T_L}])) \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]}) \\ \Omega(L) &\coloneqq F(\widehat{I}([T_L])) \in \mathrm{Kom}_{/h}(\mathcal{M}_{\mathbb{Z}[G]}), \end{aligned}$$

respectively.

Since the Bar-Natan complex is an invariant of tangles, $\Omega(T)$ (resp. the homotopy class of $\Omega(D_T)$) is an invariant for 2-ended tangles as well. The construction is summarized in the schematic below.

[3] This problem can be resolved by introducing the convention of always delooping the *last* circle with respect to the enumeration. However, since we don't need a natural inverse we aren't going to introduce such a convention.

$$
\begin{array}{c}
\text{2-ended tangles } T \\
\big\downarrow \\
\text{cube of resolutions of a diagram of } T \\
\big\downarrow{\scriptstyle \text{Bar-Natan}} \\
[T] \in \mathrm{Kom}_{/h}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))) \\
\big\downarrow{\scriptstyle \widehat{I},\text{Proposition 9.4}} \\
\widehat{I}([T]) \in \mathrm{Kom}_{/h}(\mathrm{Mat}(\mathcal{E})) \\
\big\downarrow{\scriptstyle \widehat{F}} \\
\Omega(T) \coloneqq \widehat{F}(\widehat{I}([T])) \in \mathrm{Kom}_{/h}(\mathcal{M}_{\mathbb{Z}[G]}).
\end{array}
$$

Remark 9.6. Observe that by construction, the differentials in $\Omega(D_T)$ (resp. $\Omega(T)$) are given by multiplication with nG^k for some $n \in \mathbb{Z}$ and $k \in \mathbb{N}$.

Let us now show that the $\mathbb{Z}[G]$-complex $\Omega(T)$ is isomorphic to the reduced $\mathcal{F}_{\mathbb{Z}[G]}$-complex $[\![T]\!]$. For this, we need the following lemma.

Lemma 9.7. Let D_T be a 2-ended tangle diagram and let $\mathcal{F} = (R, A, \Delta, \varepsilon)$ be a rank 2 Frobenius system. Let α be the functor

$$\alpha\colon \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2))) \to \mathrm{Kom}(\mathcal{M}_A)$$

that is induced by the TQFT $\mathcal{F}$. We can see A as a $\mathbb{Z}[G]$-module by letting G act as $\mathcal{F}($$)$. Then the functor $\gamma\colon \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]}) \to \mathrm{Kom}(\mathcal{M}_A)$ defined as

$$\gamma(Y) \coloneqq Y \otimes_{\mathbb{Z}[G]} A\{1\}, \quad Y \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$$

satisfies

$$\alpha([D_T]) \cong \gamma(\Omega(D_T))$$

(see Figure 9.2 for a diagram of the situation).

Proof. We know that the functor $\widehat{I}$ is an equivalence of categories (with "inverse" $\widehat{B}$, see Proposition 9.4) and that $\widehat{F}$ is an isomorphism of categories, thus if $\beta = \widehat{F} \circ \widehat{I}$ and $\zeta = \widehat{B} \circ \widehat{F}^{-1}$ we have that $\zeta(\beta(C)) \cong C$ for all $C \in \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^3_{/l}(2)))$ (see Figure 9.3 below). In order to show that $\alpha([D_T]) \cong \gamma(\widehat{F}(\widehat{I}([D_T])))$, it is enough to prove that $\alpha(\zeta(Y)) \cong \gamma(Y)$ for all $Y \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$. This is done by introducing a new functor $J\colon \mathcal{E} \to \mathcal{M}_A$ sending the trivial 2-ended tangle diagram D_{T_0} to the A-module A and the curtain with genus $k \geq 0$ to the linear map given by $\mathcal{F}($$)^k$. The functor J induces a functor

$$\widehat{J}\colon \mathrm{Kom}(\mathrm{Mat}(\mathcal{E})) \to \mathrm{Kom}(\mathcal{M}_A).$$

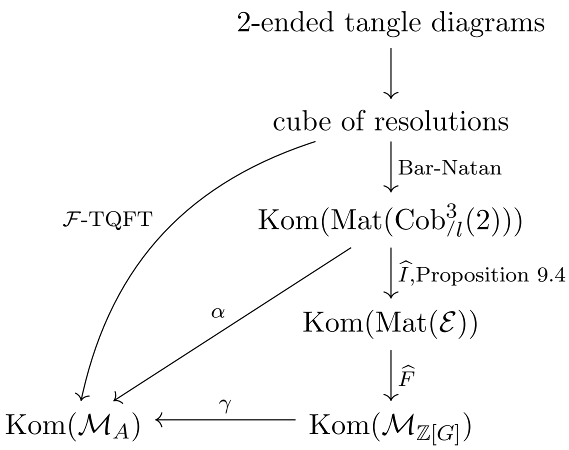


Figure 9.2: The functors and constructions in the statement of Lemma 9.7.

It is easy to see that $\widehat{J} = \alpha \circ \widehat{B}$, so α is naturally isomorphic to $\widehat{J} \circ \widehat{I}$. Thus it only remains to check that $\widehat{J} \cong \gamma \circ \widehat{F}$. This follows immediately from the definitions. $\square$

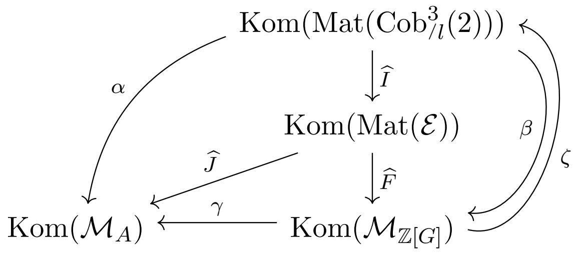


Figure 9.3: Functors used in the proof of Lemma 9.7.

Proposition 9.8. Let T be a 2-ended tangle. Then the reduced $\mathcal{F}_{\mathbb{Z}[G]}$-complex $[\![T]\!]$ is isomorphic to the $\mathbb{Z}[G]$-complex $\Omega(T)$.

Proof. Let D_T be a diagram for T. Using Lemma 9.7 with $\mathcal{F} = \mathcal{F}_{\mathbb{Z}[G]}$ and that $A_{\mathbb{Z}[G]}/(X) \cong \mathbb{Z}[G]$, we obtain

$$\begin{aligned}
[\![D_T]\!] &= C_{\mathbb{Z}[G]}(D_T) \otimes_{A_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}/(X)\{-1\} \\
&= \alpha([D_T]) \otimes_{A_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}/(X)\{-1\} \\
&\cong \gamma(\Omega(D_T)) \otimes_{A_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}/(X)\{-1\} \\
&\cong \Omega(D_T) \otimes_{\mathbb{Z}[G]} A_{\mathbb{Z}[G]}/(X)\{-1\} \\
&\cong \Omega(D_T).
\end{aligned}$$

Considering the above up to homotopy equivalence of chain complexes then yields the isomorphism $[\![T]\!] \cong \Omega(T)$. $\square$

Proposition 9.8 tells us that the reduced $\mathcal{F}_{\mathbb{Z}[G]}$-complex and the $\mathbb{Z}[G]$-complex are isomorphic, so let's consider them as the same.

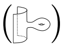
Notation. We will from now on denote both the reduced $\mathcal{F}_{\mathbb{Z}[G]}$-complex and the $\mathbb{Z}[G]$-complex with $[\![\cdot]\!]$, and no longer distinguish between them.

9.2 Equivalence of the $\mathcal{F}_{\text{univ}}$- and $\mathbb{Z}[G]$-theory

The aim of this section is to prove the previously claimed statement that the $\mathcal{F}_{\mathbb{Z}[G]}$-theory is universal in the sense of Khovanov's $\mathcal{F}_{\text{univ}}$-theory. Thus our goal is to show that the $\mathbb{Z}[G]$-complex determines the $\mathcal{F}_{\text{univ}}$-complex and vice-versa. We start by showing that the $\mathbb{Z}[G]$-complex determines any $\mathcal{F}$-complex obtained from a rank 2 Frobenius system.

Theorem 9.9. Let T be a 2-ended tangle with diagram D_T and let $\mathcal{F} = (R, A, \Delta, \varepsilon)$ be a rank 2 Frobenius system. The $\mathcal{F}$-complex $C_{\mathcal{F}}(D_T)$ is determined by the $\mathbb{Z}[G]$-complex $[\![D_T]\!]$ in the following way:

$$C_{\mathcal{F}}(D_T) \cong [\![D_T]\!] \otimes_{\mathbb{Z}[G]} A\{1\} \in \operatorname{Kom}(\mathcal{M}_A),$$

where A is a $\mathbb{Z}[G]$-module via G acting as $\mathcal{F}\Big($$\Big)$. Considered up to homotopy equivalence of chain complexes, the same is true for $\mathcal{C}_{\mathcal{F}}(T)$ and $[\![T]\!]$.

Proof. The statement of the theorem follows immediately from Lemma 9.7 and Proposition 9.8:

$$C_{\mathcal{F}}(D_T) = \alpha([D_T]) \cong \gamma(\Omega(D_T)) \cong [\![D_T]\!] \otimes_{\mathbb{Z}[G]} A\{1\}. \qquad \square$$

Observe that Theorem 9.9 specializes to Theorem 7.1 that we already mentioned in the introduction:

Theorem 7.1. Endow $A_{\text{univ}} = \mathbb{Z}[h,t][X]/(X^2 - hX - t)$ with the structure of a $\mathbb{Z}[G]$-module by letting G act as $2X - h$. Then for every oriented link L with base point,

$$C_{\text{univ}}(L) \simeq [\![L]\!] \otimes_{\mathbb{Z}[G]} A_{\text{univ}}\{1\}.$$

Proof. Apply Theorem 9.9 with $\mathcal{F} = \mathcal{F}_{\text{univ}}$. $\square$

Let us make explicit how $\mathbb{Z}[G]$-homology determines the original Khovanov homology (Naot mentions this statement in [Nao06, Section 6.6]).

Corollary 9.10. For all knots K, the unreduced integral Khovanov chain complex may be obtained from $[\![K]\!]$ by tensoring with the $\mathbb{Z}[G]$-module $\mathbb{Z}\{-1\} \oplus \mathbb{Z}\{1\}$, where G acts as $\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right)$. More sloppily said, replace every copy of $\mathbb{Z}[G]\{m\}$ by $\mathbb{Z}\{m-1\} \oplus \mathbb{Z}\{m+1\}$, and every differential nG^k with $n, k \in \mathbb{Z}, k \geq 0$ by $\left(\begin{smallmatrix} n & 0 \\ 0 & n \end{smallmatrix}\right)$ for $k = 0$, by $\left(\begin{smallmatrix} 0 & 2n \\ 0 & 0 \end{smallmatrix}\right)$ for $k = 1$, and by the zero matrix for $k \geq 2$.

Proof. Apply Theorem 9.9 to the Frobenius system $\mathbb{Z}[X]/(X^2)$ over $\mathbb{Z}$, and forget the action of the algebra. $\square$

Theorem 9.9 shows us how to obtain the $\mathcal{F}$-complex $C_{\mathcal{F}}(D_T)$ from the $\mathbb{Z}[G]$-complex $[\![D_T]\!]$ for any rank 2 Frobenius system $\mathcal{F}$, which is in particular true for the universal system $\mathcal{F}_{\text{univ}}$. In order to show that the $\mathcal{F}_{\text{univ}}$- and the $\mathbb{Z}[G]$-theory are in fact equivalent, it remains to prove that $[\![D_T]\!]$ is also determined by C_{univ}.

Theorem 9.11. Let T be a 2-ended tangle with diagram D_T. The $\mathbb{Z}[G]$-complex $[\![D_T]\!]$ is determined by the $\mathcal{F}_{\text{univ}}$-complex $C_{\text{univ}}(D_T)$ in the following way:

$$[\![D_T]\!] \cong C_{\text{univ}}(D_T) \otimes_{A_{\text{univ}}} \mathbb{Z}[G]\{-1\} \in \text{Kom}(\mathcal{M}_{\mathbb{Z}[G]}),$$

where $\mathbb{Z}[G]$ is an A_{univ}-module by X and t acting as 0 and h as $-G$. Considered up to homotopy equivalence of chain complexes, the same is true for $[\![T]\!]$ and $C_{\text{univ}}(D_T)$.

Proof. By Theorem 9.9,

$$C_{\mathcal{F}_{\text{univ}}}(D_T) \cong [\![D_T]\!] \otimes_{\mathbb{Z}[G]} A_{\text{univ}}\{1\}. \tag{9.2}$$

Consider $A_{\mathbb{Z}[G]}$ as an A_{univ}-module by letting X and t act as 0 and h as $-G$. Tensoring (9.2) with $A_{\mathbb{Z}[G]}$ over A_{univ} yields

$$\begin{aligned} C_{\text{univ}}(D_T) \otimes_{A_{\text{univ}}} A_{\mathbb{Z}[G]} &\cong \big([\![D_T]\!] \otimes_{\mathbb{Z}[G]} A_{\text{univ}}\{1\}\big) \otimes_{A_{\text{univ}}} A_{\mathbb{Z}[G]} \\ &\cong [\![D_T]\!] \otimes_{\mathbb{Z}[G]} A_{\mathbb{Z}[G]}\{1\} \\ &\cong C_{\mathbb{Z}[G]}(D_T). \end{aligned}$$

Therefore

$$\begin{aligned} [\![D_T]\!] &= C_{\mathbb{Z}[G]}(D_T) \otimes_{A_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}/(X)\{-1\} \\ &\cong (C_{\text{univ}}(D_T) \otimes_{A_{\text{univ}}} A_{\mathbb{Z}[G]}) \otimes_{A_{\mathbb{Z}[G]}} A_{\mathbb{Z}[G]}/(X)\{-1\} \\ &\cong C_{\text{univ}}(D_T) \otimes_{A_{\text{univ}}} A_{\mathbb{Z}[G]}/(X)\{-1\} \\ &\cong C_{\text{univ}}(D_T) \otimes_{A_{\text{univ}}} \mathbb{Z}[G]\{-1\}. \end{aligned}$$ $\square$

The discussion in this section can be summarized by the commutative diagram in Figure 9.4, where $\xi\colon \text{Kom}(\mathcal{M}_{A_{\text{univ}}}) \to \text{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$ is the functor given by

$$\xi(C) := C \otimes_{A_{\text{univ}}} \mathbb{Z}[G]\{-1\}$$

for $C \in \text{Kom}(\mathcal{M}_{A_{\text{univ}}})$.

Remark 9.12. For the reader's convenience, let us quickly summarize the current state of matters from Sections 9.1 and 9.2. We have the following $\mathbb{Z}[G]$-related objects:

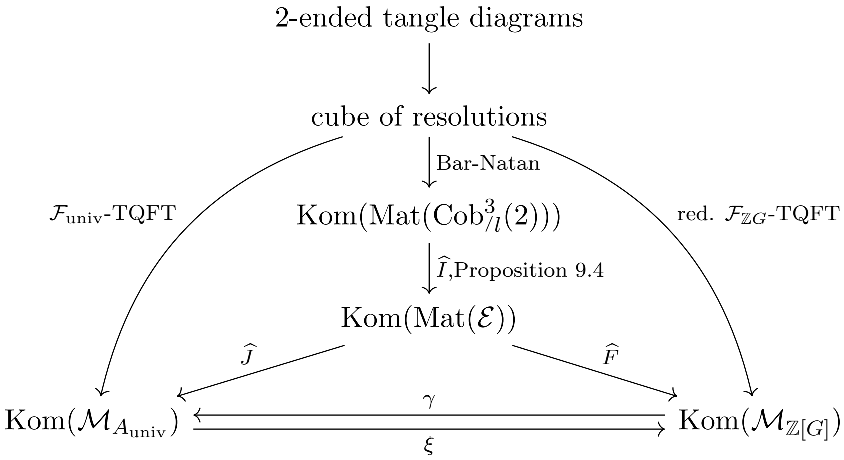


Figure 9.4: A summary of the relationships discussed in Section 9.2.

1. The *unreduced* $\mathcal{F}_{\mathbb{Z}[G]}$-complex $C_{\mathbb{Z}[G]}(T)$ obtained by applying the TQFT $\mathcal{F}_{\mathbb{Z}[G]}$ to the Bar-Natan complex of a diagram of T;
2. The *reduced* $\mathcal{F}_{\mathbb{Z}[G]}$-complex $[\![T]\!]$ obtained by "setting X equal to 0" in $C_{\mathbb{Z}[G]}(T)$ (see Definition 9.1). It's homology is called the $\mathbb{Z}[G]$-homology of T;
3. The $\mathbb{Z}[G]$-complex $\Omega(T)$ obtained from the equivalence of categories $\text{Kom}_{/h}(\text{Mat}(\text{Cob}^3_{/l}(2)))$ and $\text{Kom}_{/h}(\text{Mat}(\mathcal{E}))$ in combination with the isomorphism $\widehat{F}\colon \text{Kom}_{/h}(\text{Mat}(\mathcal{E})) \to \text{Kom}_{/h}(\mathcal{M}_{\mathbb{Z}[G]})$.

By Proposition 9.8 we have $[\![T]\!] \cong \Omega(T)$, which is why we no longer distinguish between them and denote both by $[\![T]\!]$. By Theorem 9.9, the $\mathbb{Z}[G]$-complex $[\![T]\!]$ determines any $\mathcal{F}$-complex $C_{\mathcal{F}}(T)$ obtained from a TQFT corresponding to a rank 2 Frobenius system $\mathcal{F}$, so in particular $C_{\text{univ}}(T)$. On the other hand, $C_{\text{univ}}(T)$ determines the $\mathbb{Z}[G]$-complex $[\![T]\!]$ by Theorem 9.11, showing that both theories are "universal".

9.3 Reduced $\mathbb{Z}[G]$-homology

Let T be a 2-ended tangle. We have seen in the previous subsection that the reduced $\mathbb{Z}[G]$-complex $[\![T]\!]$ is determined by the $\mathcal{F}_{\text{univ}}$-complex $C_{\text{univ}}(T)$ and vice-versa. One advantage of the $\mathbb{Z}[G]$-complex is that setting $G = 1$ yields a particularly simple homology theory.

Definition 9.13. Let T be a 2-ended tangle with a single component, so that T corresponds to a knot K with base point. Then the homology $H([\![T]\!]_{G=1})$ of the complex

$$[\![T]\!]_{G=1} \coloneqq [\![T]\!] \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/(G-1)$$

is called the *reduced* $\mathbb{Z}[G]$*-homology* of T.

Remark 9.14.

1.) Observe that setting $G = 1$ collapses the quantum grading of $[\![T]\!]$. However, forming on $[\![T]\!]$ the descending filtration that is induced by the quantum grading, we obtain that $[\![T]\!]_{G=1}$ is filtered (compare Remark 8.85).

2.) To avoid the reader's confusion, let us notice the slight ambiguity in our usage of the word "reduced": the homology of the $\mathbb{Z}[G]$-complex, which by Proposition 9.8 we identified with the *reduced* $\mathcal{F}_{\mathbb{Z}[G]}$-complex, is called the $\mathbb{Z}[G]$-*homology* of an arbitrary 2-ended tangle T. On the other hand, the *reduced* $\mathbb{Z}[G]$-*homology* of a 2-ended tangle T with a single component is obtained from $[\![T]\!]$ by "setting $G = 1$" as in Definition 9.13.

Proposition 9.15. Let T be a 2-ended tangle with a single component, so that T corresponds to a knot K with base point. Then

$$H([\![T]\!]_{G=1}) \cong \mathbb{Z}.$$

Proof. Let us first look at the unreduced situation over the rationals with G set equal to 1, i.e. the complex

$$C(T) := (C_{\mathbb{Z}[G]}(T) \otimes_{A_{\mathbb{Z}[G]}} A_{\mathbb{Q}}) \otimes_{A_{\mathbb{Q}[G]}} \mathbb{Q}[G]/(G-1),$$

where $A = \mathbb{Z}[X, G]/(X^2+GX)$ and $A_{\mathbb{Q}[G]} = \mathbb{Q}[X, G]/(X^2+GX)$. Note that $C(T)$ can be equivalently obtained from the Frobenius system $\mathcal{F}_{\mathbb{Z}[G]}$ in the usual way after switching coefficients to $\mathbb{Q}$ and setting $G = 1$ (the algebra of this system is then given by $\mathbb{Q}[X]/(X^2 + X)$). Then Proposition 8.94 implies that

$$H(C(T)) \cong \mathbb{Q} \oplus \mathbb{Q}.$$

In fact, using Wehrli's edge-coloring technique [Weh08, Section 2.1], one obtains a decomposition

$$C(T) = XC(T) \oplus (X + 1)C(T),$$

where $XC(T)$ and $(X + 1)C(T)$ are the subcomplexes generated by all elements having X and $X + 1$ as the first tensor factor (i.e. at the factor corresponding to the special strand), respectively. Similar to [Weh08, Theorem 5], one can show that both $XC(T)$ and $(X+1)C(T)$ have homology of dimension one. Now let's look at the reduced situation over the rationals, i.e. the complex

$$[\![T]\!]_{\mathbb{Q},G=1} := [\![T]\!]_{G=1} \otimes_{\mathbb{Z}} \mathbb{Q},$$

By construction, $[\![T]\!]_{\mathbb{Q},G=1}$ is equivalent to the complex $C(T)$ with X set to 0 in the first tensor factor of every summand in $C(T)$, which means that the summand $XC(T)$ becomes trivial after reducing. Hence

$$H([\![T]\!]_{\mathbb{Q},G=1}) \cong \mathbb{Q}.$$

The above tells us that $\dim_{\mathbb{Q}}([\![T]\!]_{G=1}) = 1$. Thus switching back to the integers, it remains to show that $[\![T]\!]_{G=1}$ has no torsion. This is done in the same way as in the proof of [MTV07, Proposition 2.4, ii.)]. □

Remark 9.16. Let T be a 2-ended tangle corresponding to a knot K. It is interesting to note that one can extract the Rasmussen $s_{\mathbb{F}}$-invariant of K over any field $\mathbb{F}$ (see Definition 8.96) from the $\mathbb{Z}[G]$-homology of K. Indeed, consider the $\mathbb{Z}[G]$-complex with coefficients switched to $\mathbb{F}$, i.e.

$$[\![K]\!]_{\mathbb{F}[G]} \coloneqq [\![K]\!] \otimes_{\mathbb{Z}[G]} \mathbb{F}[G].$$

By Proposition 9.15 and since $\mathbb{F}[G]$ is a PID, $[\![K]\!]_{\mathbb{F}[G]}$ decomposes into a single grading-shifted copy of the base ring $\mathbb{F}[G]\{n\}$ and some summands of the form $\mathbb{F}[G]\{m\} \xrightarrow{G^k} \mathbb{F}[G]\{2k+m\}$ for $k, m, n \in \mathbb{Z}$ (a so-called *pawn* and several G^k*-knights*, cf. Definition 10.17). Therefore, setting $G = 1$ in $[\![K]\!]_{\mathbb{F}[G]}$ yields

$$[\![K]\!]_{\mathbb{F},G=1} \coloneqq [\![K]\!]_{\mathbb{F}[G]} \otimes_{\mathbb{F}[G]} \mathbb{F}[G]/(G-1) \simeq \mathbb{F}[G]\{n\} \otimes_{\mathbb{F}[G]} \mathbb{F}[G]/(G-1).$$

As in the case of $\mathbb{F} = \mathbb{Q}$, $[\![K]\!]_{\mathbb{F},G=1}$ is a filtered complex. We claim that n, i.e. the filtered degree of the generator of $\mathbb{F}[G]\{n\}$ in homology, is equal to $s_{\mathbb{F}}(K)$. Indeed, let $\mathcal{F}_{\mathbb{F}[G]}$ denote the Frobenius system $\mathcal{F}_{\mathbb{Z}[G]}$ with coefficients switched from the integers to $\mathbb{F}$, so that the algebra of this system is $\mathbb{F}[G,X](X^2+GX)$. Setting $G = 1$ in $\mathcal{F}_{\mathbb{F}[G]}$ defines a filtered Frobenius system in the usual way that yields the unreduced complex $C_{\mathcal{F}_{\mathbb{F}[G]}}(K)$ whose homology can be used to obtain $s_{\mathbb{F}}(K)$ (see Section 8.10). Now, using Theorem 9.9 and the decomposition of $[\![K]\!]_{\mathbb{F}[G]}$ described above, we can write $C_{\mathcal{F}_{\mathbb{F}[G]}}(K)$ as

$$\begin{aligned} C_{\mathcal{F}_{\mathbb{F}[G]}}(K) &\cong [\![K]\!]_{\mathbb{F}[G]} \otimes_{\mathbb{F}[G]} \mathbb{F}[G,X]/(X^2+GX)\{1\} \\ &\cong (\mathbb{F}[G]\{n\} \oplus \mathcal{R}) \otimes_{\mathbb{F}[G]} \mathbb{F}[G,X]/(X^2+GX)\{1\} \\ &\cong \mathbb{F}[G,X]/(X^2+GX)\{n+1\} \oplus \\ &\qquad\qquad (\mathcal{R} \otimes_{\mathbb{F}[G]} \mathbb{F}[G,X]/(X^2+GX)\{1\}), \end{aligned}$$

where $\mathcal{R}$ consists solely of summands $\mathbb{F}[G]\{m\} \xrightarrow{G^k} \mathbb{F}[G]\{2k+m\}$. If we now set $G = 1$ and take homology, we obtain

$$H(C_{\mathcal{F}_{\mathbb{F}[G]}}(K) \otimes_{\mathbb{F}[G]} \mathbb{F}[G]/(G-1)) \cong \mathbb{F}[X]/(X^2+X)\{n+1\}.$$

Now $\mathbb{F}[X]/(X^2+X)\{n+1\}$ is generated by 1 and X in filtered degrees $n+1$ and $n-1$ respectively (see Proposition 8.95), hence $s_{\mathbb{F}}(K) = n$ as claimed.

Example 9.17. In Example 9.3, we computed the $\mathbb{Z}[G]$-complex of the right-handed trefoil and obtained

$$[\![K]\!] = {}_0\mathbb{Z}[G]\{2\} \;\oplus\; \Big({}_2\mathbb{Z}[G]\{6\} \xrightarrow{G} {}_3\mathbb{Z}[G]\{8\}\Big).$$

Setting $G = 1$ yields

$$\begin{aligned} [\![K]\!]_{G=1} &\stackrel{\text{def.}}{=} [\![K]\!] \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/(G-1) \\ &\cong {}_0\mathbb{Z}\{2\} \ \oplus \ \left({}_2\mathbb{Z}\{6\} \xrightarrow{1} {}_3\mathbb{Z}]\{8\}\right) \\ &\simeq {}_0\mathbb{Z}\{2\} \end{aligned}$$

Hence the reduced $\mathbb{Z}[G]$-homology of the right-handed trefoil is given by a single copy of $\mathbb{Z}$ in homological degree 0 and quantum degree 2, confirming Proposition 9.15. Moreover, we can also immediately read off the Rasmussen invariant of the right-handed trefoil over any field $\mathbb{F}$, which is given by 2.

9.4 The $\mathbb{Z}[G]$-enriched category $\mathrm{Cob}^{3,\bullet}_{/l}(2n)$

In preparation for upcoming chapters about the λ-invariant, we need to be able to speak about "multiplication by G" on the Bar-Natan complex of a tangle diagram $[D]$. Thus, let's introduce the necessary formalism.

As in Section 8.7, let $\mathrm{Cob}^{3,\bullet}_{/l}(2n)$ denote the category $\mathrm{Cob}^{3}_{/l}(2n)$ with one of the fixed end points of the unoriented crossingless $2n$-ended tangle diagrams marked as base point. We wish to obtain a $\mathbb{Z}[G]$-action on the morphism groups of this category. For this, let $C \in \hom_{\mathrm{Cob}^{3,\bullet}(2)}(D_{T_0}, D_{T_0})$ be a cobordism between the trivial 2-ended tangle diagram D_{T_0} and itself, and let $T \in \hom_{\mathrm{Cob}^{3,\bullet}(2n)}(D, D')$ be any cobordism between two unoriented crossingless $2n$-ended tangle diagrams D, D'. Then one may glue the cobordisms C and T together, so that the base point of D_{T_0} gets attached to the base point of D and D', respectively. This gives a bilinear map

$$\hom_{\mathrm{Cob}^{3,\bullet}(2)}(D_{T_0}, D_{T_0}) \times \hom_{\mathrm{Cob}^{3,\bullet}(2n)}(D, D') \to \hom_{\mathrm{Cob}^{3,\bullet}(2n)}(D, D').$$

Modding out the local relations l and using that $\hom_{\mathrm{Cob}^{3,\bullet}(2)}(D_{T_0}, D_{T_0})$ is isomorphic to $\mathbb{Z}[G]$ (see Remark 8.36), we obtain a $\mathbb{Z}[G]$-action on each of the morphism $\mathbb{Z}$-modules of $\mathrm{Cob}^{3}_{/l}(2n)$, thus turning them into $\mathbb{Z}[G]$-modules. Let us redefine $\mathrm{Cob}^{3,\bullet}_{/l}(2n)$ as follows.

Definition 9.18. Let $\mathrm{Cob}^{3,\bullet}_{/l}(2n)$ denote the $\mathbb{Z}[G]$-enriched category obtained from $\mathrm{Cob}^{3}_{/l}(2n)$ by marking one of the fixed tangle diagram end points as base point and letting $\mathbb{Z}[G]$-act on the morphism groups as described above. Given a $2n$-ended tangle diagram D with base point, we denote by $[D]^\bullet$ the Bar-Natan chain complex of D in $\mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^{3,\bullet}_{/l}(2n)))$. Here, we identify equivalence classes of tangle diagrams in the disk that contains D with equivalence classes of tangle diagrams in the fixed unit disk in $\mathbb{R}^2$ for $\mathrm{Cob}^{3,\bullet}_{/l}(2n)$, using a homeomorphism (which is unique up to isotopy) between these disks that sends end points to end points and base point to base point.

Remark 9.19.

1.) The $\mathbb{Z}[G]$-action on $\mathrm{Cob}^{3;\bullet}_{/l}(2n)$ induces a $\mathbb{Z}[G]$-action on $[D]^{\bullet}$ in the obvious way.

2.) Note that one may recover $\mathrm{Cob}^{3}_{/l}(2n)$ from $\mathrm{Cob}^{3;\bullet}_{/l}(2n)$ and $[D]$ from $[D]^{\bullet}$ by simply forgetting the $\mathbb{Z}[G]$-action and the base point. In other words, Definition 9.18 only introduces the action of G, but does not introduce any new objects and morphisms.

3.) For $n = 1$, the $\mathbb{Z}[G]$-action on $\mathrm{Cob}^{3;\bullet}_{/l}(2)$ is by construction the same as the one obtained via the equivalence of $\mathrm{Cob}^{3}_{/l}(2)$ and $\mathcal{M}_{\mathbb{Z}[G]}$ (see Proposition 9.4 and the discussion after). In particular, for 2-ended tangle diagrams D, both choices of base point result in the same $\mathbb{Z}[G]$-action on $[D]^{\bullet}$.

Notation. In order to distinguish the two versions of $\mathrm{Cob}^{3;\bullet}_{/l}(2n)$ as in Definition 9.18 (i.e. with $\mathbb{Z}[G]$-action and base point) and Definition 8.75 (i.e. only with base point), we drop from now on the $\bullet$ from the latter and keep in mind that in the context of planar arc diagrams there is always a fixed base point.

In Section 8.7 we have described how an unoriented d-input planar arc diagram $\mathcal{D}$ (see Definition 8.66) yields a functor

$$\mathcal{D}_{\mathrm{Cob}/l}\colon \prod_{i=1}^{d} \mathrm{Cob}^{3}_{/l}(2n_{E_i}) \to \mathrm{Cob}^{3}_{/l}(2n_E)$$

that intuitively "glues" d suitable tangle diagrams into the planar arc diagram $\mathcal{D}$. Here, $2n_{E_i}$ is the number of end points on the boundary of the i-th removed input disk from $\mathcal{D}$, and $2n_E$ is the number of end points on the boundary of $\mathcal{D}$ with the input disks placed back in. According to Theorem 8.77, $\mathcal{D}_{\mathrm{Cob}/l}$ extends by taking tensor products to a functor

$$\mathcal{D}_{\mathcal{K}}\colon \prod_{i=1}^{d} \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^{3}_{/l}(2n_{E_i}))) \to \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^{3}_{/l}(2n_E))),$$

which again by Theorem 8.77 is compatible with homotopy equivalence of chain complexes (see Section 8.7 for more details). Thus an unoriented d-input planar arc diagram establishes the following compatibility result:

$$\mathcal{D}_{\mathcal{K}}([D_1], \ldots, [D_d]) \simeq [\mathcal{D}_{\mathcal{T}}(D_1, \ldots, D_d)], \tag{9.3}$$

where D_i is a $2n_{E_i}$-ended tangle diagram D_i for $i = 1, \ldots, d$ and $D_{\mathcal{T}}$ is the planar arc diagram operator defined by $\mathcal{D}$ (see Definition 8.68). Note that with this tool set, one could give a more formal definition of the $\mathbb{Z}[G]$-action on $[D]^{\bullet}$ given in Definition 9.18 by using an unoriented 2-input planar arc diagram whose two input disks have 2 and $2n$ end points, respectively.

To adapt to $\mathbb{Z}[G]$-complexes, let us consider unoriented d-input planar arc diagrams $\mathcal{D}$ that contain an arc connecting the base point of E to the base point of the first input disk E_1. Then $\mathcal{D}_{(\mathcal{K}_{/h})_{\text{or}}}$ induces a functor

$$\mathrm{Kom}_{/h}(\mathrm{Mat}(\mathrm{Cob}^{3;\bullet}_{/l}(2n_{E_1}))) \times \prod_{i=2}^{d} \mathrm{Kom}_{/h}(\mathrm{Mat}(\mathrm{Cob}^{3}_{/l}(2n_{E_i}))) \\ \longrightarrow \mathrm{Kom}_{/h}(\mathrm{Mat}(\mathrm{Cob}^{3;\bullet}_{/l}(2n_E))).$$

For this functor, it is straightforward to obtain the following analogue of (9.3):

$$\mathcal{D}_{\mathcal{K}}([D_1]^{\bullet}, [D_2] \dots, [D_d]) \simeq [\mathcal{D}(D_1, \dots, D_d)]^{\bullet}.$$

Chapter 10

The λ-Invariant

The aim of this chapter is to provide a detailed introduction to the λ-invariant. It is organized as follows. We start in Section 10.1 with the general definition of λ as described in Section 7.5 (see Definition 10.2 and 10.3), and prove first properties, such as Proposition 7.8. In Section 10.2, we take a closer look at λ for tangles, and prove Proposition 10.14 which states that λ induces a pseudometric on the set of equivalence classes of tangles in a fixed ball with fixed base point and connectivity. In Section 10.3 we describe how to decompose the $\mathbb{Z}[G]$-complex of a knot into a direct sum of simpler complexes (so-called *pieces*), which leads to the computation of λ for the $(5,6)$-torus knot $T_{5,6}$ in Example 10.19. In Section 10.4, we return to the relation of λ with torsion order invariants. We start with a discussion of $\mathfrak{u}_G$ (see Definition 10.23) and related invariants, and proceed to show that $\mathfrak{u}_G$ detects the unknot (see Lemma 10.27) and that $\mathfrak{u}_G(K) \leq \lambda(K)$ for all knots K (see Lemma 10.28). It then follows directly from these results that λ is an unknot detector (see Proposition 7.7). The remaining Sections 10.5 and 10.6 are concerned with λ of thin and small knots respectively, and contain proofs of Proposition 7.9 and 7.11

10.1 Definition and basic properties

Let us start by making the preliminary definition of the λ-invariant for a knot K in Chapter 7 more formal in a general context. However, before doing so let us introduce the notion of an *ungraded* chain map.

Definition 10.1. For chain complexes $(C, d), (C', d')$ in some additive category, an *ungraded* chain map $f\colon C \to C'$ is a morphism

$$f\colon \bigoplus_{i=-\infty}^{\infty} C_i \to \bigoplus_{i=-\infty}^{\infty} C'_i$$

that need *not* respect homological degree, such that $d' \circ f = f \circ d$. Whenever we want to highlight the difference, we call a chain map in the usual sense *graded*. If the underlying category is Abelian (so that one may take homology), then the ungraded chain map f induces a morphism

$$f_*\colon H(C) = \bigoplus_{i=-\infty}^{\infty} H_i(C) \to \bigoplus_{i=-\infty}^{\infty} H_i(C') = H(C').$$

Definition 10.2. Let $A, B \in \mathrm{Kom}(\mathcal{C})$ with $\mathcal{C} = \mathrm{Mat}(\mathrm{Cob}^{3;\bullet}_{/l}(2n))$ or $\mathbb{Z}[G]$-Mod. Then we define $\lambda(A, B)$ as the minimal integer $k \geq 0$ such that there exists ungraded chain maps

$$A \underset{g}{\overset{f}{\rightleftarrows}} B$$

and chain homotopies $g \circ f \simeq G^k \cdot \mathrm{id}_A$, $f \circ g \simeq G^k \cdot \mathrm{id}_B$. If such a k does not exist, then we set $\lambda(A, B) = \infty$.

Based on Definition 10.2, we introduce the following abbreviations for $\lambda(A, B)$.

Definition 10.3. Let K and J be knots, let U be the unknot, and let $A \in \mathrm{Kom}(\mathbb{Z}[G]\text{-Mod})$. Further, let D, D' be two $2n$-ended base-pointed tangle diagrams in a fixed disk with the same endpoints and the same base point. Then we set:

1. $\lambda(A) \coloneqq \lambda(A, [\![U]\!])$;
2. $\lambda(D, D') \coloneqq \lambda([D]^\bullet, [D']^\bullet)$;
3. $\lambda(K, J) \coloneqq \lambda([\![K]\!], [\![J]\!])$;
4. $\lambda(K) \coloneqq \lambda([\![K]\!], [\![U]\!]) = \lambda([\![K]\!])$.

Remark 10.4.

1.) One can naturally extend the definition of λ from knots to links with base point, by setting $\lambda(L, L')$ to be $\lambda(T, T')$, where T, T' are the 2-ended tangles corresponding to the links L, L' via (8.4). In this sense, most of the results regarding λ will generalize from knots to links. For simplicity's sake, however, we are sticking with knots.

2.) Strictly speaking, the $\mathbb{Z}[G]$-complex of a knot K is by definition considered up to homotopy equivalence and not contained in $\mathrm{Kom}(\mathbb{Z}[G]\text{-Mod})$. However, λ as in Definition 10.2 is invariant under homotopy equivalence of complexes, hence we accept the slight inaccuracy in formalism by plugging $\mathbb{Z}[G]$-complexes of knots into λ as in Definition 10.3.

Next, let us prove some useful basic properties of λ.

Lemma 10.5. For some $k \geq 1$, let $A_1, \ldots, A_k, B_1, \ldots, B_k$ be chain complexes over $\mathbb{Z}[G]$-Mod or $\mathrm{Mat}(\mathrm{Cob}^{3;\bullet}_{/l}(2n))$. Then, for $A = A_1 \oplus \cdots \oplus A_k$ and $B = B_1 \oplus \cdots \oplus B_k$ one has

$$\lambda(A, B) \;\leq\; \max(\lambda(A_1, B_1), \ldots, \lambda(A_k, B_k)).$$

Proof. W.l.o.g. we can assume that $k = 2$. If either $\lambda(A_1, B_1)$ or $\lambda(A_2, B_2)$ are equal to ∞ the statement of the lemma is trivial, so let us assume that they are both finite. We pick chain maps f_1, g_1 such that $f_1 \circ g_1 \simeq G^{\lambda(A_1,B_1)} \cdot \mathrm{id}_{B_1}$ and $g_1 \circ f_1 \simeq G^{\lambda(A_1,B_1)} \cdot \mathrm{id}_{A_1}$, and choose maps f_2, g_2 similarly for $\lambda(A_2, B_2)$. Let $m = \max(\lambda(A_1, B_1), \lambda(A_2, B_2))$ and define $f\colon A_1 \oplus A_2 \to B_1 \oplus B_2$, $g\colon B_1 \oplus B_2 \to A_1 \oplus A_2$ as follows:

$$f = \begin{pmatrix} G^{m-\lambda(A_1,B_1)} \cdot f_1 & 0 \\ 0 & G^{m-\lambda(A_2,B_2)} \cdot f_2 \end{pmatrix}, \quad g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$$

We leave it to the reader to check that $f \circ g \simeq G^m \cdot \mathrm{id}_{B_1 \oplus B_2}$ and $g \circ f \simeq G^m \cdot \mathrm{id}_{A_1 \oplus A_2}$. □

Taking one of the B_i as $[\![U]\!]$, and all the others as 0, we obtain the following special case of Lemma 10.5, which gives a useful upper bound for λ of a direct sum.

Corollary 10.6. Let $C^1, \ldots, C^n$ be chain complexes over $\mathbb{Z}[G]$-Mod, fix a $j \in \{1, \ldots, n\}$ and let $l_i = \lambda(C^i, 0)$, for all $i \neq j$, and $l_j = \lambda(C^j)$. Then:

$$\lambda\Big(\bigoplus_{i=1}^{n} C^i\Big) \leq \max_{i=1,\ldots,n} l_i.$$

Lemma 10.7. Let A, A_1 and A_2 be chain complexes over $\mathbb{Z}[G]$-Mod. Then:

1. $\lambda(A_1 \otimes A_2) \leq \lambda(A_1) + \lambda(A_2)$;
2. $\lambda(\overline{A}) = \lambda(A)$, where $\overline{A}$ is the dual of A.

Proof. For the first statement, let us assume that $\lambda(A_1), \lambda(A_2)$ are both finite (if either one is ∞ the statement is trivial). Let $f_i\colon A_i \to [\![U]\!]$, $g_i\colon [\![U]\!] \to A_i$ be chain maps such that $g_i \circ f_i \simeq G^{\lambda(A_i)} \cdot \mathrm{id}_{A_i}$ and $f_i \circ g_i \simeq G^{\lambda(A_i)} \cdot \mathrm{id}_{[\![U]\!]}$, for $i = 1, 2$. Define $f\colon A_1 \otimes A_2 \to [\![U]\!] \otimes [\![U]\!] \cong [\![U]\!]$ and $g\colon [\![U]\!] \otimes [\![U]\!] \cong [\![U]\!] \to A_1 \otimes A_2$ as

$$f = f_1 \otimes f_2, \qquad g = g_1 \otimes g_2.$$

Then $g \circ f \simeq G^{\lambda(A_1)+\lambda(A_2)} \cdot \mathrm{id}_{A_1 \otimes A_2}$ and $f \circ g \simeq G^{\lambda(A_1)+\lambda(A_2)} \cdot \mathrm{id}_{[\![U]\!]}$, so $\lambda(A_1 \otimes A_2) \leq \lambda(A_1) + \lambda(A_2)$ as desired.

As for the second statement, it follows from the fact that if $f\colon A \to [\![U]\!]$, $g\colon [\![U]\!] \to A$ are chain maps such that $g \circ f \simeq G^k \cdot \mathrm{id}_A$ and $f \circ g \simeq G^k \cdot \mathrm{id}_{[\![U]\!]}$, then the induced dual chain maps $\overline{g}\colon \overline{A} \to \overline{[\![U]\!]} \cong [\![U]\!]$ and $\overline{f}\colon \overline{[\![U]\!]} \cong [\![U]\!] \to \overline{A}$ satisfy $\overline{f} \circ \overline{g} \simeq G^k \cdot \mathrm{id}_{\overline{A}}$ and $\overline{g} \circ \overline{f} \simeq G^k \cdot \mathrm{id}_{[\![U]\!]}$. □

The in the introduction mentioned Proposition 7.8 now follows directly from Lemma 10.7, since $[\![K\#J]\!] \cong [\![K]\!] \otimes [\![J]\!]$ and $[\![-K]\!] \cong \overline{[\![K]\!]}$.

Proposition 7.8.

1. $\lambda(K\#J) \leq \lambda(K) + \lambda(J)$ for all knots K, J.

2. λ does not change under taking mirror images, or orientation reversal.

10.2 A closer look at λ for tangles

The aim of this section is to study closer the behaviour of λ for tangles. Here, we will again make use of planar arc diagrams and the corresponding operators, as described in Sections 8.7 and 9.4. In the following, all planar arc diagrams $\mathcal{D}$ are to be understood as oriented, and in order to simplify notation we will denote the various induced operators with $\mathcal{D}$ as well.

Lemma 10.8. Let $\mathcal{D}$ be a 2-input planar arc diagram containing an arc connecting the base points of the output disk and the first input disk. Let D_1 and D_1' be two tangle diagrams fitting into the first input disk, and let D_2 be a tangle diagram fitting into the second input disk. Then

$$\lambda(\mathcal{D}(D_1, D_2), \mathcal{D}(D_1', D_2)) \;\leq\; \lambda(D_1, D_1').$$

Proof. Let $f\colon [D_1]^\bullet \to [D_1']^\bullet$ and $g\colon [D_1']^\bullet \to [D_1]^\bullet$ be chain maps satisfying $f \circ g \simeq G^n \cdot \mathrm{id}_{[D_1']^\bullet}$ and $g \circ f \simeq G^n \cdot \mathrm{id}_{[D_1]^\bullet}$ for $n = \lambda(D_1, D_1')$. Using the functor induced by $\mathcal{D}$, we may define maps $\tilde{f}$ and $\tilde{g}$ as

$$\mathcal{D}([D_1]^\bullet, [D_2]) \underset{\tilde{g}=\mathcal{D}(g,\mathrm{id}_{[D_2]})}{\overset{\tilde{f}=\mathcal{D}(f,\mathrm{id}_{[D_2]})}{\rightleftarrows}} \mathcal{D}([D_1']^\bullet, [D_2]).$$

These maps satisfy

$$\tilde{g} \circ \tilde{f} = \mathcal{D}(g \circ f, \mathrm{id}_{D_2}) \simeq \mathcal{D}(G^n \cdot \mathrm{id}_{D_1}, \mathrm{id}_{D_2}) = G^n \cdot \mathrm{id}_{\mathcal{D}([D_1']^\bullet, [D_2])},$$

and the analog equality for $\tilde{g} \circ \tilde{f}$. This shows the desired statement. $\square$

See Figure 10.1 for examples of the following definitions.

Definition 10.9. Let $\mathcal{D}_{2n}$ be the following 2-input planar arc diagram: the two input disks are $2n$-ended and $(4n-2)$-ended, respectively; $\mathcal{D}_{2n}$ consists of one arc connecting the base point of the output disk to the base point of the first input disk, $2n-1$ arcs connecting end points of the two input disks, and $2n-1$ arcs connecting end points of the second input disk to end points of the output disk. We say that a tangle diagram Q with $2m$ end points is *braid-like*, if it may be isotoped such that m end points are

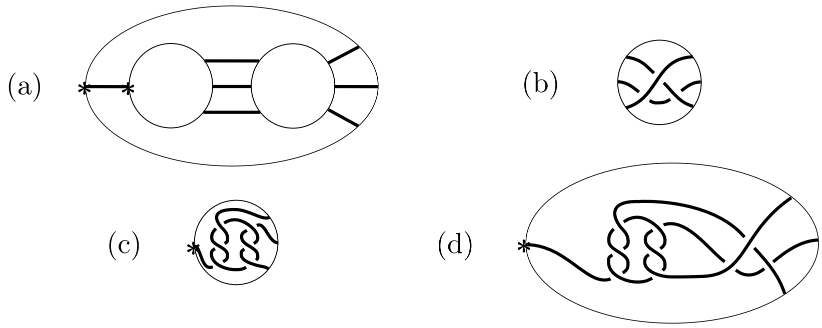


Figure 10.1: Examples of the concepts introduced in Definition 10.9. An asterisk marks the base point. (a) $\mathcal{D}_4$, (b) a braid-like 6-ended tangle diagram Q, (c) a 4-ended tangle diagram T, (d) $\mathcal{D}_4(T, Q)$, called a braiding of T using Q.

on the left, m end points are on the right, and Q consists of m arcs that at no point have a vertical tangent. For $\mathcal{D}_{2n}$ as above, Q a $(4n-2)$-ended braid-like tangle diagram, and D a $2n$-ended tangle diagram, we say that $\mathcal{D}(D, Q)$ is a *braiding* of D.

Recall from Definition 8.21 that to obtain a tangle diagram of a given tangle in a ball B, one must choose a homeomorphism between B and the unit ball B_0. We will now show that two tangle diagrams of a fixed tangle are related by a finite sequence of Reidemeister moves and a braiding. In fact, the braiding only depends on the homeomorphisms between the balls, and not on the tangles. Let us make this precise.

Lemma 10.10. Let B be a ball, and $P = \{p_1, \ldots, p_{2n}\} \subset \partial B$ for some $n \geq 1$. Let φ_1, φ_2 be homeomorphisms from B to the unit ball B_0 with $\varphi_1(P) = \varphi_2(P)$ and $\varphi_1(p_1) = \varphi_2(p_1)$. Let $\mathcal{D}_{2n}$ be the 2-input planar arc diagram from Definition 10.9. Then there is an unoriented braid-like $(4n-2)$-ended tangle diagram Q, such that for all tangles T in B with end points P and base point p_1 the following holds: if D_1 and D_2 are the tangle diagrams of T coming from φ_1 and φ_2, respectively, then $\mathcal{D}_{2n}(D_1, Q)$ and D_2 are related by a finite sequence of Reidemeister moves and tangle diagram equivalences.

Proof. Let $f\colon S^2 \to S^2$ be the restriction of $\varphi_2 \circ \varphi_1^{-1}$ to $S^2 = \partial B_0$. Let us write $\tilde{P} = \varphi_1(P) = \varphi_2(P)$ and $\tilde{p}_i = \varphi_1(p_i)$. Note $f(\tilde{P}) = \tilde{P}$. In case that f is isotopic to id_{S^2} along homeomorphisms fixing $\tilde{P}$ pointwise, it follows that $\varphi_1(T)$ and $\varphi_2(T)$ are equivalent tangles, and thus D_1 and D_2 are related by a finite sequence of Reidemeister moves and tangle diagram equivalences. In order to deal with general f, let us consider the mapping class group of homeomorphisms $f\colon S^2 \to S^2$ with $f(\tilde{P}) = \tilde{P}$ and $f(\tilde{p}_1) = \tilde{p}_1$, up to isotopy along such maps. Every such f is isotopic to a homeomorphism fixing a neighborhood of $\tilde{p}_1$ pointwise, and so this mapping class group is isomorphic to the mapping class group of the $(2n-1)$-punctured disk, which is isomorphic to the braid group on $2n-1$ strands. More explicitly, it is generated by $\sigma_1, \ldots, \sigma_{2n-2}$, where σ_i is a so-called *half-twist*, switching the positions of the punctures $\tilde{p}_{i+1}$ and $\tilde{p}_{i+2}$ [FM11, Section 9.1.3]. So f is isotopic to a product β of the generators $\sigma_1, \ldots, \sigma_{2n-2}$. Let Q be a braid-like $(4n-2)$-ended tangle diagram corresponding to β. Then one sees that $\mathcal{D}_{2n}(D_1, Q)$ is a tangle diagram of T coming from the homeomorphism $(\varphi_2 \circ \varphi_1^{-1}) \circ \varphi_1 = \varphi_2$. Therefore, $\mathcal{D}_{2n}(D_1, Q)$ and D_2 are related by a finite sequence of Reidemeister moves and tangle diagram equivalences, as desired. □

Proposition 10.11. Let S and T be tangles with the same end points and the same base point in a ball B. Let φ_1 and φ_2 be homeomorphisms from B to the unit ball B_0, leading to tangle diagrams D_{S1}, D_{S2} for S and D_{T1}, D_{T2} for T, respectively. Then

$$\lambda(D_{S1}, D_{T1}) = \lambda(D_{S2}, D_{T2})$$

Proof. By Lemma 10.10, there is a 2-input planar arc diagram $\mathcal{D}$ and a tangle Q such that $\mathcal{D}(D_{S1}, Q)$ and D_{S2} are related by a finite sequence of Reidemeister moves, and so are $\mathcal{D}(D_{T1}, Q)$ and D_{T2}. By Lemma 10.8, it follows that $\lambda(D_{S2}, D_{T2}) \leq \lambda(D_{S1}, D_{T1})$. Switching the roles of φ_1 and φ_2, the opposite inequality also follows. □

As a consequence, the following is well-defined, since it does not depend on the choice of homeomorphism.

Definition 10.12. Let S and T be tangles with the same end points and the same base point in a ball B. Then let $\lambda(S, T)$ be defined as $\lambda(D_S, D_T)$, where D_S and D_T are tangle diagrams of S and T, respectively, obtained via the same homeomorphism from B to the unit ball B_0.

Proposition 10.13. Let S and T be two tangles in a ball B with the same connectivity, base point and end points. Let R be a tangle in another ball B', and $\varphi\colon \partial B \to \partial B'$ an orientation-reversing homeomorphism sending end points to end points, such that $S \cup R$ and $T \cup R$ are knots in $B \cup_\varphi B' \cong S^3$. Then

$$\lambda(S \cup R, T \cup R) \leq \lambda(S, T).$$

Proof. One may pick tangle diagrams D_S, D_T and D_R for S, T and R, respectively, such that: D_S and D_T come from the same homeomorphism from B to B_0; gluing D_S and D_R (using a 2-input planar arc diagram $\mathcal{D}$) results in a knot diagram of $S \cup R$; and similarly, $\mathcal{D}(D_T, D_R)$ is a diagram of $T \cup R$. Then, we have

$$\begin{aligned}\lambda(S \cup R, T \cup R) &= \lambda(\mathcal{D}(D_S, D_R), \mathcal{D}(D_T, D_R)) \\ &\leq \lambda(D_S, D_T) \\ &= \lambda(S, T)\end{aligned}$$

by the definition of λ for knots, Lemma 10.8, and Definition 10.12, respectively. □

Proposition 10.14. Let us fix a ball with $2n$ end points, and consider unoriented tangles T with a fixed connectivity and fixed base point in that ball. On the set of equivalence classes of such tangles T, λ is a pseudometric.

Proof. It is straight-forward to see that λ is symmetric, satisfies the triangle inequality, and $\lambda(T,T) = 0$. Since any two tangles S,T with same connectivity are related by crossing changes, Theorem 7.4 implies that $\lambda(S,T) < \infty$. □

Note that $\lambda(S,T) = 0$ if and only if $[S]^\bullet$ and $[T]^\bullet$ are ungradedly homotopy equivalent. So the existence of non-equivalent tangles with homotopy equivalent Bar-Natan homology prevents λ from being a metric. Still, this pseudometric allows for a nice formulation of the main step of the proof of Theorem 7.4.

Proposition 10.15. Fix a ball with four end points on it, one of them distinguished as base point. On the set of equivalence classes of unoriented rational tangles in that ball with fixed connectivity, the pseudometric given by λ is in fact equal to the discrete metric. That is to say, $\lambda(S,T) = 1$ for inequivalent rational tangles S and T.

The proof of Proposition 10.15 will be given in Chapter 12.

10.3 Decomposing $\mathbb{Z}[G]$-chain complexes into pieces

To analyze the $\mathbb{Z}[G]$-chain complex $[\![K]\!]$ of a knot K and compute $\lambda(K)$, one may follow the same divide-and-conquer strategy as described in Section 8.8 for the Bar-Natan complex and decompose $[\![K]\!]$ as a direct sum of simpler complexes. This motivates the following definition.

Definition 10.16. For a graded ring R, a graded chain complex C of free shifted R-modules of finite total rank, i.e. $C \in \mathrm{Kom}(\mathcal{M}_R)$, is called a *piece* if it satisfies the following: C is not contractible (i.e. not homotopy equivalent to the trivial complex), and if C is homotopy equivalent to $C' \oplus C''$ with $C', C'' \in \mathrm{Kom}(\mathcal{M}_R)$, then either C' or C'' is contractible. In other words, a piece is an indecomposable object in the category $\mathrm{Kom}_{/h}(\mathcal{M}_R)$ of chain complexes of finite total rank up to homotopy equivalence.

In Section 8.11 and Remark 9.16 we have already implicitly encountered pieces called pawns and knights. Those are the two most common kinds of pieces, and we introduce them now formally.

Definition 10.17. For a graded integral domain R, let the *pawn piece*, denoted by ♙, be the chain complex consisting of just one copy R in homological degree 0 and quantum degree 0. Given a non-trivial prime power $z \in R$, we define the *z-knight piece*, denoted by ♘(z), to be the chain complex

$${}_0R \xrightarrow{z} R\{\deg z\},$$

where the left subscript denotes the homological degree.

As mentioned in Section 8.11, the terminology of these pieces are coined by Bar-Natan [BN02] and come from the fact that when obtaining unreduced rational Khovanov homology from $[\![K]\!]$ using Theorem 9.9, a ♙ and ♘(G) piece in $[\![K]\!]$ result in the patterns

$$\begin{array}{|c|}\hline \mathbb{Q} \\ \hline \mathbb{Q} \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline & \mathbb{Q} \\ \hline & \\ \hline \mathbb{Q} & \\ \hline \end{array},$$

respectively in the unreduced rational Khovanov homology. This can be seen using Corollary 9.10.

Remark 10.18. A complex P is a piece if and only if the ring of endomorphisms of P up to homotopy has precisely two distinct idempotents, namely the zero map and the identity map. Let us use this to check that pawns and knights actually are pieces. For $P =$ ♙, the endomorphism ring of P is isomorphic to R, and there are no non-trivial homotopies. Since R is assumed to be an integral domain, the only idempotents are 0 and 1, and $0 \neq 1$. So ♙ is indeed a piece.

Now consider $P =$ ♘(z). Ignoring the chain complex structure, R-module endomorphisms $P \to P$ are given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Which among those maps are chain maps? To respect homological degree, we must have $b = c = 0$. To commute with the differential, we must have $az = dz$. Since $z \neq 0$ and R is an integral domain, this implies $a = d$. So the endomorphism ring of P consists (as for ♙) just of multiples of id_P, i.e. this ring is isomorphic to R. All homotopies are multiples of $h\colon P_1 \to P_0$, $h(1) = 1$. We have $h \circ d + d \circ h = z \cdot \mathrm{id}_P$, and so the endomorphism ring of P modulo homotopy is isomorphic to $R/(z)$. Since z is by assumption a non-trivial prime power, $R/(z)$ has no non-trivial idempotents. Thus ♘(z) is indeed a piece.

In Example 10.19 and Chapter 11 below, we will claim that various chain complexes are pieces. This may be checked by similar arguments as above; but since we don't actually make use of the fact that those complexes are pieces, we omit these arguments from the text.

If R is a graded PID, then it is not hard to see that pawns and knights are the only pieces. This fact has been used previously to analyze homology theories coming from Frobenius algebras over fields, e.g. by Khovanov [Kho06], or by Morrison [Mor07].[1] In the introduction, we have seen $[\![K]\!]$ for $K = U$, $T_{2,3}$, $T_{3,4}$, and for those examples, $[\![K]\!]$ also decomposes as sum

[1] Morrison's "universal Khovanov homology" is equivalent to $[\![\,\cdot\,]\!] \otimes \mathbb{Q}$, i.e. the reduced theory coming from the Frobenius algebra $\mathbb{Q}[X]/(X^2 - tX)$ over $\mathbb{Q}[t]$. Since $\mathbb{Q}[t]$ is a PID, the chain complexes coming from that theory are homotopy equivalent to a sum of ♙ and ♘(t^n) pieces, which Morrison calls E and C_n (or $KhC[n]$), respectively. This homology theory can be calculated with `JavaKh` [GM05].

of pawns and knights. Let us consider a further example, which demonstrates that the pieces of $\mathbb{Z}[G]$-chain complexes can be significantly more complicated (in fact, we do not know a classification of those pieces).

Example 10.19. As one may compute with `khoca` and `homca`, the chain complex $[\![T_{5,6}]\!]$ is homotopy equivalent to the sum of

$$ {}_0\text{♙}\{20\} \;\oplus\; {}_2\text{♘}(G)\{24\} \;\oplus\; {}_4\text{♘}(G^2)\{26\} $$

and the following four more complicated pieces (where we write $R = \mathbb{Z}[G]$):

$$ P_1 = \begin{array}{ccc} {}_6R\{28\} & \xrightarrow{G} & R\{30\} \\ \oplus & \nearrow^{2} & \oplus \\ R\{30\} & \xrightarrow[G]{} & R\{32\} \end{array} $$

$$ P_2 = \; {}_8R\{30\} \begin{array}{c} \xrightarrow{2G^2} \\ \\ \xrightarrow[G^3]{} \end{array} \begin{array}{c} R\{34\} \\ \oplus \\ R\{36\} \end{array} \begin{array}{c} \xrightarrow{G} \\ \\ \xrightarrow[-2]{} \end{array} R\{36\}, $$

$$ P_3 = \; {}_{10}R\{34\} \begin{array}{c} \xrightarrow{5G} \\ \\ \xrightarrow[G^2]{} \end{array} \begin{array}{c} R\{36\} \\ \oplus \\ R\{38\} \end{array} \begin{array}{c} \xrightarrow{G^2} \\ \\ \xrightarrow[-5G]{} \end{array} R\{40\}, $$

$$ P_4 = \; {}_{12}R\{36\} \begin{array}{c} \xrightarrow{3G^2} \\ \\ \xrightarrow[G^3]{} \end{array} \begin{array}{c} R\{40\} \\ \oplus \\ R\{42\} \end{array} \begin{array}{c} \xrightarrow{G} \\ \\ \xrightarrow[-3]{} \end{array} R\{42\}. $$

Note that P_3 is isomorphic to ${}_{10}\text{♘}(G^2) \otimes \text{♘}(5G)\{34\}$. Let us now compute λ of $T_{5,6}$. We have $\lambda(\text{♘}(G^k)) = k$ for $k \in \{1,2\}$ (in fact, for all $k \geq 1$) and leave it to the reader to check that $\lambda(P_i, 0) \leq 3$ for $i \in \{1,2,3,4\}$. Using Corollary 10.6, this implies $\lambda(T_{5,6}) \leq 3$. To show $\lambda(T_{5,6}) \geq 3$, we rely on the maximal G-torsion order of $\mathbb{Z}[G]$-homology, denoted by $\mathfrak{u}_G$. This invariant is discussed in detail in the next Section 10.4. It gives a lower bound $\mathfrak{u}_G \leq \lambda$ (see Lemma 10.28). Consider the homology of $\overline{P_4}$, the dual of P_4. The annihilator of the class of a generator of ${}_{-12}R\{-36\}$ is the ideal $(3G^2, G^3) \subset \mathbb{Z}[G]$, and so the G-torsion order of that homology class is equal to 3. Hence $\lambda(T_{5,6}) = \lambda(-T_{5,6}) \geq \mathfrak{u}_G(-T_{5,6}) \geq 3$, and thus $\lambda(T_{5,6}) = 3$.

Remark 10.20. If R is Noetherian, then every chain complex in $\mathrm{Kom}_{/h}(\mathcal{M}_R)$ can be written as a sum of finitely many pieces. If R is a graded PID, then this decomposition is essentially unique, i.e. unique up to the order of the summands. This is not true for $R = \mathbb{Z}[G]$, as the following Example 10.21 demonstrates. As a consequence, in this text we will often decompose chain complexes $[\![K]\!]$ as sums of pieces, but we will never rely on this decomposition being unique.

Example 10.21. Let us give an example of a chain complex that admits two essentially different decompositions as sums of pieces. Let $R = \mathbb{Z}[G]$, and for any integer $n > 0$ let Q_n be the complex

$$
{}_0R\{0\} \begin{array}{c} \xrightarrow{nG} \\ \\ \xrightarrow[G^2]{} \end{array} \begin{array}{c} R\{2\} \\ \oplus \\ R\{4\} \end{array} \begin{array}{c} \xrightarrow{-G} \\ \\ \xrightarrow[n]{} \end{array} R\{4\}.
$$

One computes that the endomorphism ring of Q_n modulo homotopy is isomorphic to $R/(G^2, nG)$. This ring does not admit non-trivial idempotents, and so Q_n is a piece. Now, the Smith normal form gives us invertible 2×2 integer matrices S, T such that $S \left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right) T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 6 \end{smallmatrix}\right)$. This leads to the following change of basis, which demonstrates $Q_2 \oplus Q_3 \cong Q_1 \oplus Q_6$, giving us the desired example. Note that $Q_1 \simeq$ ♘(G).

$$
{}_0R\{0\}^{\oplus 2} \begin{array}{c} \xrightarrow{G\left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)} \\ \\ \xrightarrow[G^2]{} \end{array} \begin{array}{c} R\{2\}^{\oplus 2} \\ \oplus \\ R\{4\}^{\oplus 2} \end{array} \begin{array}{c} \xrightarrow{-G} \\ \\ \xrightarrow[\left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)]{} \end{array} R\{4\}^{\oplus 2} \quad \cong
$$

$$
{}_0R\{0\}^{\oplus 2} \begin{array}{c} \xrightarrow{GS\left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)T} \\ \\ \xrightarrow[G^2]{} \end{array} \begin{array}{c} R\{2\}^{\oplus 2} \\ \oplus \\ R\{4\}^{\oplus 2} \end{array} \begin{array}{c} \xrightarrow{-G} \\ \\ \xrightarrow[S\left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)T]{} \end{array} R\{4\}^{\oplus 2}
$$

10.4 Torsion orders

When computing λ of a knot K it is fairly simple to find an upper bound $k \geq \lambda(K)$ by defining ungraded chain maps $f\colon [\![K]\!] \to [\![U]\!]$, $g\colon [\![U]\!] \to [\![K]\!]$ such that $g \circ f$ and $f \circ g$ are homotopic to multiplication with G^k. In order to compute the exact value of λ however, one has to find the minimal such k, which can be a hard task. The invariants described in this subsection give lower bounds for λ in terms of the maximal torsion order in homology.

In 2017, Alishahi and Dowlin [Ali19, AD19] introduced the following knot invariants which are lower bounds for the unknotting number: $\mathfrak{u}_h$ is defined as the maximal order of h-torsion in Bar-Natan homology (with Frobenius algebra $\mathbb{F}_2[h, X]/(X^2 + hX)$, see Definition 8.60), while $\mathfrak{u}_t$ is the maximal t-torsion order of the homology corresponding to the Alishahi-Dowlin system $\mathcal{F}_{\text{AD}}$ with Frobenius algebra $\mathbb{Q}[t, X]/(X^2 - t)$ (cf. Definition 8.58). Since $X^2 = t$ in this setting, one has $\lceil \mathfrak{u}_X/2 \rceil = \mathfrak{u}_t$, where $\mathfrak{u}_X$ is the maximal X-torsion order. It was then remarked in [CGL+20] that for the latter invariant one can replace $\mathbb{Q}$ with $\mathbb{F}_p$ for any odd prime p in order to obtain new bounds $\mathfrak{u}_{t,p}$.

The following invariant is the analog of those bounds in the $\mathbb{Z}[G]$-setting.

Definition 10.22. Let K be a knot with $\mathbb{Z}[G]$-homology $H_{\mathbb{Z}[G]}(K)$. Given $a \in H_{\mathbb{Z}[G]}(K)$, we say that a is *G-torsion* if there is some $n \in \mathbb{Z}_{\geq 0}$ such that $G^n \cdot a = 0$. We further define the *order* of a G-torsion element a as

$$\operatorname{ord}_G(a) \coloneqq \min\{n \in \mathbb{Z}_{\geq 0} \mid G^n \cdot a = 0\}$$

and write $T\left(H_{\mathbb{Z}[G]}(K)\right)$ for the $\mathbb{Z}[G]$-submodule of G-torsion elements.

Definition 10.23. Let K be a knot. Then we define $\mathfrak{u}_G(K)$ to be the maximal order of a G-torsion element:

$$\mathfrak{u}_G(K) \coloneqq \max_{a \in T\left(H_{\mathbb{Z}[G]}(K)\right)} \operatorname{ord}_G(a)$$

In a recent paper, Gujral [Guj20] introduced a lower bound ν for the ribbon distance[2] in terms of the maximal order of $(2X - (\alpha_1 + \alpha_2))$-torsion in the α-homology of a knot. α-homology, first described in [KR22], is universal as well and thus equivalent to our $\mathbb{Z}[G]$-homology. Hence the invariant ν is equal to $\mathfrak{u}_G$. The following is a natural consequence of the universality of $\mathbb{Z}[G]$-homology:

Proposition 10.24. We have

$$\mathfrak{u}_G = \nu \geq \mathfrak{u}_t, \mathfrak{u}_{t,p}, \mathfrak{u}_h.$$

As a side note it is worthwhile to observe, although we will not make use of it, that $\mathfrak{u}_h, \mathfrak{u}_t$ are also linked to the convergence of the Bar-Natan [Tur06] and the Lee [Lee05] spectral sequences respectively (see Section 8.10). For all knots K, these sequences start at Khovanov homology of K (with coefficients in $\mathbb{F}_2$ and $\mathbb{Q}$ respectively) and, letting n_{BN} and n_{Lee} be the pages at which they collapse, we have $\mathfrak{u}_h(K) = n_{\text{BN}} - 1$ and $\mathfrak{u}_t(K) = n_{\text{Lee}} - 1$. As a consequence, the following interesting result holds which we already mentioned in Section 8.11:

Corollary 10.25 ([AD19]). The Knight Move Conjecture is true for all knots K with $u(K) \leq 2$.

In light of Theorem 7.4, we even have:

[2]Here, the *ribbon distance* of two knots K and J is defined as the minimal $k \geq 0$ such that there exists a sequence of knots $K = K_1, K_2 \ldots, K_n = J$ where each consecutive pair K_i, K_{i+1} is connected by ribbon concordance with at most k saddles, see [Sar20].

Corollary 10.26. The Knight Move Conjecture is true for all knots K with $u_q(K) \leq 2$.

The connection discussed above between invariants related to λ and spectral sequences brings us to the following natural question:

Question. Is there a spectral sequence E_G such that for any knot K, $E_G(K)$ starts at Khovanov homology (with coefficients in $\mathbb{Z}$) and collapses at page $\mathfrak{u}_G(K) + c$, for some constant c?

We suspect this question has a positive answer. Namely, consider the chain complex obtained from $[\![K]\!]$ by setting $G = 1$. The resulting complex is filtered, and gives rise to a spectral sequence $E_G(K)$ starting at (reduced) Khovanov homology with integer coefficients. It seems likely that $E_G(K)$ converges at page $\mathfrak{u}_G(K) - 1$.

Let us continue by showing that $\mathfrak{u}_G$ is an unknot detector.

Lemma 10.27. The invariant $\mathfrak{u}_G$ detects the unknot, i.e. $\mathfrak{u}_G(K) = 0$ holds if and only if K is trivial.

Proof. We start by noticing that $\mathfrak{u}_G(U) = 0$: this is clear since $H_{\mathbb{Z}[G]}(U) = \mathbb{Z}[G]$ is torsion free. It is also clear that $\lambda(U) = 0$, using the ungraded chain maps $f = g = \mathrm{id}_{[\![U]\!]}$.

Let $H_{\mathcal{F}_{\mathrm{AD}}}$ be the unreduced Khovanov homology theory coming from the TQFT $\mathcal{F}_{\mathrm{AD}}$ with algebra $\mathbb{Q}[t][X]/(X^2 - t)$ (see Definition 8.58). Then

$$H_{\mathcal{F}_{\mathrm{AD}}}(K) \cong \mathbb{Q}[t] \oplus \mathbb{Q}[t] \oplus T\left(H_{\mathcal{F}_{\mathrm{AD}}}(K)\right),$$

where $T\left(H_{\mathcal{F}_{\mathrm{AD}}}(K)\right)$ is the t-torsion part. Since Khovanov homology detects the unknot we have $T\left(H_{\mathcal{F}_{\mathrm{AD}}}(K)\right) = \{0\}$ only if $K = U$. This implies that if K is not the unknot $\mathfrak{u}_G(K) \geq \mathfrak{u}_t(K) > 0$. More details on $\mathfrak{u}_t$ can be found in [AD19] (the homology $H_{\mathcal{F}_{\mathrm{AD}}}$ is called H_{Lee} there). □

Lemma 10.28. Let K be a knot. Then $\mathfrak{u}_G(K) \leq \lambda(K)$.

Proof. Let $n = \lambda(K)$ and let $f\colon [\![K]\!] \to [\![U]\!]$, $g\colon [\![U]\!] \to [\![K]\!]$ be ungraded chain maps such that $g \circ f \simeq G^n \cdot \mathrm{id}_{[\![K]\!]}$ and $f \circ g \simeq G^n \cdot \mathrm{id}_{[\![U]\!]}$. Then, for every $a \in T(H_{\mathbb{Z}[G]}(K))$:

$$\mathrm{ord}_G\left(f_*(a)\right) \geq \mathrm{ord}_G\left(g_* \circ f_*(a)\right) = \mathrm{ord}_G\left(G^n \cdot a\right) \geq \mathrm{ord}_G\left(a\right) - n.$$

Taking the maximum over $T\left(H_{\mathbb{Z}[G]}(K)\right)$ we get:

$$\begin{aligned} 0 = \mathfrak{u}_G(U) &= \max_{a \in T\left(H_{\mathbb{Z}[G]}(K)\right)} \mathrm{ord}_G\left(f_*(a)\right) \\ &\geq \max_{a \in T\left(H_{\mathbb{Z}[G]}(K)\right)} \mathrm{ord}_G\left(a\right) - n \\ &= \mathfrak{u}_G(K) - n. \end{aligned}$$

This shows that $\mathfrak{u}_G(K) \leq n = \lambda(K)$. □

The two previous lemmas combined show that λ detects the unknot, as claimed in the introduction.

Proposition 7.7. The λ-invariant detects the unknot, i.e. $\lambda(K) = 0$ holds if and only if K is the unknot.

A more direct proof of Proposition 7.7 may also be given as follows:

Proof of Proposition 7.7. Let K be a knot. Khovanov homology detects the unknot [KM11], so

$$\begin{aligned} \lambda(K) = 0 &\iff [\![K]\!] \simeq [\![U]\!] \\ &\iff K = U. \end{aligned}$$

□

In Definition 10.3 we saw how to define λ for the $\mathbb{Z}[G]$-complex of any knot. Similarly, one can define the invariants $\mathfrak{u}_G, \mathfrak{u}_h, \mathfrak{u}_t$, and $\mathfrak{u}_{t,p}$ on chain complexes over $\mathcal{M}_R$, where R is equal to $\mathbb{Z}[G], \mathbb{F}_2[h], \mathbb{Q}[t]$ or $\mathbb{F}_p[t]$, respectively. Indeed, if $C \in \mathrm{Kom}(\mathcal{M}_R)$ and $\eta = G, h, t$ or (t, p), then $\mathfrak{u}_\eta(C)$ is the maximal order of η-torsion in the homology $H(C) \in R$-Mod of C. Note that Lemma 10.28 also holds for arbitrary complexes over $\mathcal{M}_{\mathbb{Z}[G]}$.

Let us now state a few properties of $\mathfrak{u}_G, \mathfrak{u}_h, \mathfrak{u}_t$ and $\mathfrak{u}_{t,p}$.

Lemma 10.29. Let $R_G = \mathbb{Z}[G]$, $R_h = \mathbb{F}_2[h]$, $R_t = \mathbb{Q}[t]$, $R_{t,p} = \mathbb{F}_p[t]$, and let A, B be chain complexes over $\mathcal{M}_{R_\eta}$, where $\eta = G, h, t$ or (t, p). We have:

1. $\mathfrak{u}_\eta(A \oplus B) = \max\{\mathfrak{u}_\eta(A), \mathfrak{u}_\eta(B)\}$.
2. $\mathfrak{u}_\eta(K \# J) = \max\{\mathfrak{u}_\eta(K), \mathfrak{u}_\eta(J)\}$ for $\eta = h, t$ or (t, p) and knots K, J.

Proof. The first statement is clear. For the second one, we use the fact that R_α is a PID. As noted earlier, this implies that

$$C_\eta(K) \cong \text{♙} \oplus \text{♘}(a_1) \oplus \ldots \oplus \text{♘}(a_n),$$

where

$$a_i = \begin{cases} t^{k_i} & \text{if } \eta = t \text{ or } (t,p) \\ h^{k_i} & \text{if } \eta = h \end{cases}$$

and $k_i \le k_{i+1}$ for all $i = 1, \ldots, n$. Similarly,

$$C_\eta(J) \cong \text{♙} \oplus \text{♘}(b_1) \oplus \ldots \oplus \text{♘}(b_m),$$

where $b_j = t^{l_j}$ or $b_j = h^{l_j}$ and $l_j \le l_{j+1}$ for all $j = 1, \ldots, m$. One checks that $\mathfrak{u}_\eta(\text{♙}) = 0$ and $\mathfrak{u}_\eta(\text{♘}(a_i)) = k_i$, therefore

$$\mathfrak{u}_\eta(K) = \max\{\mathfrak{u}_\eta(\text{♙}), \mathfrak{u}_\eta(\text{♘}(a_1)), \ldots, \mathfrak{u}_\eta(\text{♘}(a_n))\} = k_n$$

Similarly, $\mathfrak{u}_\eta(J) = l_m$. Now

$$C_\eta(K\#J) \cong C_\eta(K) \otimes C_\eta(K) \cong \bigoplus_{\substack{i\in\{0,\dots,n\}\\ j\in\{0,\dots,m\}}} A_i \otimes B_j,$$

with $A_0 = B_0 = \text{♙}$, $A_i = \text{♘}(a_i)$ and $B_j = \text{♘}(b_j)$. It is a simple exercise to check that $\mathfrak{u}_\eta(\text{♘}(a_i) \otimes \text{♘}(b_j)) = \min\{k_i, l_j\}$. It follows that

$$\begin{aligned}
\mathfrak{u}_\eta(K\#J) &= \max_{i,j}\{\mathfrak{u}_\eta(A_i \otimes B_j)\} \\
&= \max\{\mathfrak{u}_\eta(A_0 \otimes B_m), \mathfrak{u}_\eta(A_n \otimes B_0)\} \\
&= \max\{\mathfrak{u}_\eta(\text{♙} \otimes \text{♘}(b_m)), \mathfrak{u}_\eta(\text{♘}(a_n) \otimes \text{♙})\} \\
&= \max\{\mathfrak{u}_\eta(K), \mathfrak{u}_\eta(J)\}
\end{aligned}$$

as desired. □

Remark 10.30. Statement 2. of Lemma 10.29 does not hold in general for $\mathfrak{u}_G$. This is due to the fact that for $\eta \neq G$, the $\mathfrak{u}_\eta$ are defined over PIDs, while $\mathfrak{u}_G$ is not (cf. Section 10.3). Later on in this article, Remark 11.7 and 11.8 will provide us with examples of knots K_1, K_2 such that

$$\mathfrak{u}_G(K_1\#K_2) < \max\{\mathfrak{u}_G(K_1), \mathfrak{u}_G(K_2)\},$$

as well as examples of knots J_1, J_2 where

$$\mathfrak{u}_G(J_1\#J_2) > \max\{\mathfrak{u}_G(J_1), \mathfrak{u}_G(J_2)\}.$$

In particular, if K_1, K_2 satisfy 1. of Proposition 11.4 and J satisfies 2. then

$$\mathfrak{u}_G(K_1) = \mathfrak{u}_G(K_2) = \mathfrak{u}_G(J) = 1,$$

but

$$\begin{aligned}
\mathfrak{u}_G(K_1\#K_2) &= 2 = \mathfrak{u}_G(K_1) + \mathfrak{u}_G(K_2) \\
\mathfrak{u}_G((K_1\#K_2)\#J) &= 1 = \mathfrak{u}_G(K_1\#K_2) - \mathfrak{u}_G(J)
\end{aligned}$$

Therefore, the best that we can hope for is that $\mathfrak{u}_G(K\#J) \leq \mathfrak{u}_G(K) + \mathfrak{u}_G(J)$.

For generic complexes A, B over $\mathcal{M}_{R_\eta}$ things get also more intrigued for $\mathfrak{u}_\eta(A \otimes B)$ for general $\eta = G, h, t$ or (t, p), as the pieces that appear in the decompositions of a generic chain complex are often much more complicated than those we saw appear in $C_\eta(K)$ of a knot K. In this case, the only result we can hope for is again $\mathfrak{u}_\eta(A \otimes B) \leq \mathfrak{u}_\eta(A) + \mathfrak{u}_\eta(B)$, for $\eta = G, h, t$ or (t, p).

10.5 λ of thin knots

In Section 8.11, we introduced the notion of δ-thin knots: a knot K is called *δ-thin* (or simply *thin*) if its reduced integral Khovanov homology consists of free modules supported in a single δ-degree, see Definition 8.102. The aim of this section is to prove that λ of non-trivial thin knots is 1.

Lemma 10.31. If a chain complex $C \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$ decomposes (ignoring gradings) as a sum of one ♙ and finitely many ♘(G) pieces, then $\lambda(C) \leq 1$.

Proof. Since $\lambda(\text{♙}) = 0$ and $\lambda(\text{♘}(G), 0) = 1$, this follows from Corollary 10.6. □

Lemma 10.32. Let K be a knot whose reduced integral Khovanov homology is torsion free. Then $[\![K]\!]$ is homotopy equivalent to a chain complex $C \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$ of free shifted $\mathbb{Z}[G]$-modules, such that the Poincaré polynomial of C is equal to the Poincaré polynomial of reduced integral Khovanov homology of K.

Proof. Start by picking an arbitrary chain complex $C' \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$ that is homotopy equivalent to $[\![K]\!]$. Consider the chain complex $C' \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/(G)$. This is a chain complex over the integers, whose homology is isomorphic to reduced integral Khovanov homology of K. In particular, it has torsion free homology by assumption. One may select bases for the chain groups of the complex $C' \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/(G)$, with respect to which the matrices of the differentials are in Smith normal form. Because homology is torsion free, all the entries of these matrices are 0 or 1. Gaussian elimination (see e.g. Lemma 8.79) of all the entries equal to 1 yields a homotopy equivalence between $C' \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/(G)$ and a complex Z with trivial differentials. So, Z is isomorphic to the reduced integral Khovanov homology of K.

Now, one may lift the bases of $C' \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/(G)$ to homogeneous bases of C'. Since the matrices of the differentials of C' have homogeneous entries, it follows that if a matrix entry of a differential of $C' \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/(G)$ equals 1, then the corresponding matrix entry of the corresponding differential of C' also equals 1. Therefore, one may lift the homotopy equivalence constructed above, obtaining a homotopy equivalence between C' and a complex $C \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$, such that $C \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/(G)$ is isomorphic to Z. It follows that C and the reduced integral Khovanov homology of K have the same Poincaré polynomial, as desired. □

Lemma 10.33. For all thin knots K, $[\![K]\!]$ is up to degree shifts homotopy equivalent to a sum of one ♙ piece and finitely many ♘(G) pieces.

Proof. By Lemma 10.32, we may pick a chain complex $C \in \mathrm{Kom}(\mathcal{M}_{\mathbb{Z}[G]})$ that is homotopy equivalent to $[\![K]\!]$, and has the same Poincaré polynomial as reduced integral Khovanov homology of K. Since the latter is supported on a single δ-degree, so is C. Choosing arbitrary bases for the chain modules of C, it follows that every entry of the matrices of the differentials is an integer multiple of G. Similarly as in the proof of Lemma 10.32, one may choose new bases for the modules of C, such that the matrices of the differentials equal G times a matrix in Smith normal form. Consequently, ignoring gradings C decomposes as a direct sum of ♙ and ♘(aG) pieces, with a priori varying $a \in \mathbb{Z}_{>0}$. By Proposition 9.15, there is exactly one ♙ piece, and all other pieces are ♘(G) pieces. □

Proposition 7.9. For all non-trivial thin knots K, we have $\lambda(K) = 1$.

Proof. Lemma 10.33 and Lemma 10.31 imply $\lambda(K) \leq 1$, whereas Proposition 7.7 implies $\lambda(K) \geq 1$. □

Remark 10.34. Note that Lemma 10.33 also provides a proof (at least for knots) for Bar-Natan's "structural conjecture" that all alternating links are "Khovanov basic" [Bar05, Conjecture 1].

In [CGL$^+$20], upper bounds for $\mathfrak{u}_X, \mathfrak{u}_h, \mathfrak{u}_t$ and $\mathfrak{u}_{t,p}$ are given in terms of the homological width of Khovanov homologies. This motivates the following question.

Question. Let K be a knot such that $[\![K]\!]$ is homotopy equivalent to a complex supported in n adjacent δ-degrees. Does then $\lambda(K) \leq n$ follow?

10.6 λ of small knots

We start this section by computing λ for all knots with up to 10 crossings.

Proposition 7.11. For all knots up to 10 crossings we have $\lambda = 1$, except for the knots 8_{19}, 10_{124}, 10_{128}, 10_{139}, 10_{152}, 10_{154}, 10_{161}, where $\lambda = 2$.

Proof. By Proposition 7.9, if a knot is thin then $\lambda = 1$, so it suffices to look at knots which are not thin. Among the knots with up to 10 crossings, there are twelve knots that are thick:

8_{19}, 9_{42}, 10_{124}, 10_{128}, 10_{132}, 10_{136}, 10_{139}, 10_{145}, 10_{152}, 10_{153}, 10_{154}, 10_{161}.

Using `khoca` [LL18] and `homca` [Ilt21a], one can compute that the $\mathbb{Z}[G]$-complex of the knots 9_{42}, 10_{132}, 10_{136}, 10_{145}, 10_{153} decomposes into a sum of a ♙ and several ♘(G) pieces, hence $\lambda = 1$ by Lemma 10.31. The $\mathbb{Z}[G]$-complex of the remaining knots 8_{19}, 10_{124}, 10_{128}, 10_{139}, 10_{152}, 10_{154}, 10_{161} decomposes into a sum of a ♙, several ♘(G) pieces and a single ♘(G^2) piece. Using Corollary 10.6 and Lemma 10.29, one obtains $\lambda = 2$ for these knots. □

A natural question to ask when introducing a new invariant is how it compares to other already existing invariants. For example, how does λ compare to the classical 3-genus g of a knot K? We know that λ is a lower bound for the unknotting number u, while g can be a lower or upper bound for u depending on the knot. For instance, Lee-Lee [LL13] showed that for all knots with braid index ≤ 3, the inequality $u(K) \leq g(K)$ holds. However, this is no longer true for knots with braid index ≥ 4: as pointed out in their work, there are six braid-index 4 knots with up to 9 crossings for which $u < g$ holds. How does λ fit into this scheme? For knots up to 12 crossings, we can provide the following answer.

K	$u(K)$	$g(K)$
9_{46}	2	1
$11n_{139}$	1 or 2	1
$12n_{203}$	3 or 4	3
$12n_{260}$	2 or 3	2
$12n_{404}$	2 or 3	2
$12n_{432}$	2 or 3	2
$12n_{554}$	3	2
$12n_{642}$	3 or 4	2
$12n_{764}$	3 or 4	3
$12n_{809}$	1 or 3	2
$12n_{851}$	3 or 4	3

Table 10.1: Non-quasi-alternating prime knots with up to 12 crossings for which (possibly) $g < u$ holds.

Proposition 10.35. For all knots up to 12 crossings, the 3-genus g is an upper bound for λ.

Proof. Since λ is a lower bound for the unknotting number u, it is sufficient to consider knots with up to 12 crossings for which (possibly) $g < u$ holds. Using that quasi-alternating knots are thin and that for thin knots $\lambda = 1$ (cf. Proposition 7.9) there are 11 non-quasi-alternating knots with at most 12 crossings with (possibly) $g < u$. They were found using Livingston's wonderful KnotInfo [LM] and Jablan's table of quasi-alternating knots for up to 12 crossings [Jab14]. The knots are listed in Table 10.1.

A computation using `khoca` [LL18] and its extension `homca` [Ilt21a] showed that the $\mathbb{Z}[G]$-complex of all knots in Table 10.1 decomposes into ♙ and ♘(G) summands. By Lemma 10.31, this implies that $\lambda = 1$ for all knots in Table 10.1. □

Proposition 10.35 raises the following question.

Question. Does $\lambda(K) \leq g(K)$ hold for all knots K?

Chapter 11

Calculations of $\mathbb{Z}[G]$-Homology and the λ-Invariant

In the previous Chapter 10, we have seen how to define the λ-invariant for chain complexes over $\mathbb{Z}[G]$ resp. $\mathrm{Mat}(\mathrm{Cob}^{3;\bullet}_{/l}(2n))$, and in particular for the $\mathbb{Z}[G]$-complex of a knot. Moreover, we have seen basic properties and results regarding λ and related torsion order invariants. In this chapter we now provide further results that involve computations around $\mathbb{Z}[G]$-homology and the λ-invariant.

Chapter 11 is organized as follows. Section 11.1 is occupied with the proof that λ can be arbitrarily big, see Theorem 7.5. In Section 11.2 we study the behavior of λ under connected sums of knots. Section 11.3 is occupied with a description of the author's computer programs used throughout our work. Subsection 11.3.1 starts with `homca` [Ilt21a] and gives an intuitive description of the algorithm used to simplify $\mathbb{Z}[G]$-complexes. In Subsection 11.3.2 we introduce the program `tenpro` [Ilt21b], which is able to compute tensor products of $\mathbb{Z}[G]$-complexes.

11.1 λ can be arbitrarily big

The aim of this section is to prove our main Theorem 7.5 which states that λ can be arbitrarily big.

Theorem 7.5. For every $n \in \mathbb{N}$ there exists a knot K such that $\lambda(K) = n$.

The proof proceeds by constructing an explicit knot K such that $\lambda(K) = n$ for any given $n \in \mathbb{N}$. Since λ does not depend on the quantum grading of the $\mathbb{Z}[G]$-complex, we will in the following omit quantum degree shifts for better readability. For the proof we are going to need a special type of chain complex.

Definition 11.1. For every $n \in \mathbb{Z}_{>0}$ the *staircase of rank* $2n+1$, denoted by S_n, is defined as the chain complex

$$0 \longrightarrow C_0 \xrightarrow{d_{S_n}} C_1 \longrightarrow 0,$$

where $C_0 = \mathbb{Z}[G]^{\oplus n+1}, C_1 = \mathbb{Z}[G]^{\oplus n}$ and

$$d_{S_n} = \begin{pmatrix} 2 & G & & 0 \\ & \ddots & \ddots & \\ 0 & & 2 & G \end{pmatrix}.$$

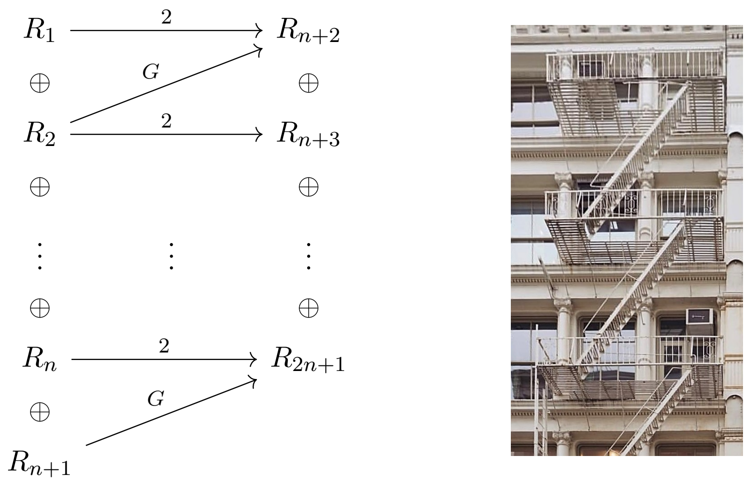


Figure 11.1: The staircase S_n of rank $2n+1$. Here, $R_i = \mathbb{Z}[G]$ for all $i = 1, \ldots, 2n+1$ so that the left-hand column corresponds to C_0 and the right-hand column to C_1. The diagonal arrows indicate a shift in quantum degree.

Lemma 11.2. Let S_n be a staircase of rank $2n+1$. Then

$$\lambda(\mathrm{S}_n) = \mathfrak{u}_G(S_n) = n.$$

The proof of Lemma 11.2 proceeds by finding chain maps $f\colon S_n \to [\![U]\!]$ and $g\colon [\![U]\!] \to S_n$ such that $f \circ g \simeq G^n \cdot \mathrm{id}_{[\![U]\!]}$ and $g \circ f \simeq G^n \cdot \mathrm{id}_{S_n}$, showing that $\lambda(S_n) \leq n$. In order to show the reversed inequality, one finds an element $x \in H_{\mathbb{Z}[G]}(S_n)$ with G-torsion order n. Using that $\mathfrak{u}_G \leq \lambda$, one then obtains

$$n \leq \mathfrak{u}_G(S_n) \leq \lambda(S_n) \leq n.$$

Details can be found in [ILM21, Lemma 4.2]

Lemma 11.3. Forgetting about shifts in quantum degree we have

$$\mathrm{S}_1 \otimes \mathrm{S}_n \cong \mathrm{S}_{n+1} \oplus (♘(G) \otimes ♘(2))^{\oplus n}.$$

Lemma 11.3 is a straightforward computation that involves manipulation of large matrices. We refer the interested reader for the proof to [ILM21, Lemma 4.3].

Proof of Theorem 7.5. Lemma 11.2 and Lemma 11.3, together with the fact that $[\![K_1 \# K_2]\!] \cong [\![K_1]\!] \otimes [\![K_2]\!]$ for any two knots K_1, K_2, are enough to construct knots with arbitrarily big λ. Indeed, let us consider the knot $K = 14n19265$. This knot was used by Seed to show that $s(K) \neq s_{\mathbb{F}_2}(K)$ [See13, LS14], where s is the classical Rasmussen invariant over $\mathbb{Q}$ and $s_{\mathbb{F}_2}$ is the invariant computed over $\mathbb{F}_2$. We observe using `khoca` and `homca` that the $\mathbb{Z}[G]$-complex $[\![K]\!]$ decomposes as a sum of a staircase S_1 and finitely many $\text{♘}(G)$ and $\text{♘}(G) \otimes \text{♘}(2)$. Therefore, by Corollary 10.6:

$$\lambda(K) \leq \max\{\lambda(S_1),\ \lambda(\text{♘}(G), 0),\ \lambda(\text{♘}(G) \otimes \text{♘}(2), 0)\} = 1$$

Since $K \neq U$ it follows that $\lambda(K) = 1$. By Proposition 7.8, given $n \in \mathbb{Z}_{>0}$, we then have:

$$\lambda(K^{\# n}) \leq n \cdot \lambda(K) = n.$$

On the other hand, it follows from Lemma 11.3 that $[\![K^{\# n}]\!] \cong S_n \oplus C$ for some chain complex C. We know that $\mathfrak{u}_G(S_n \oplus C) = \max(\mathfrak{u}_G(S_n), \mathfrak{u}_G(C))$ (cf. Lemma 10.29), so

$$\lambda(K^{\# n}) = \lambda(S_n \oplus C) \geq \mathfrak{u}_G(S_n \oplus C) \geq \mathfrak{u}_G(S_n) = n.$$

This proves that $\lambda(K^{\# n}) = n$ for all $n \geq 0$. □

The fact that $\lambda(K^{\# n}) = n$ will also follow from the upcoming Proposition 11.4.

11.2 Further calculations

Proposition 11.4. Let $K_1, \ldots, K_n$ and $J_1, \ldots, J_m$ be knots such that:

1. for all $i = 1, \ldots, n$ the complex $[\![K_i]\!]$ splits as a sum of one staircase S_1 and finitely many $\text{♘}(G)$ and $\text{♘}(G) \otimes \text{♘}(2)$ pieces,
2. for all $j = 1, \ldots, m$ the complex $[\![J_j]\!]$ decomposes as a sum of one dual staircase $\overline{S_1}$ and finitely many $\text{♘}(G)$ and $\text{♘}(G) \otimes \text{♘}(2)$.

Then

$$\lambda\big((\mathop{\#}_{i \leq n} K_i) \# (\mathop{\#}_{j \leq m} J_j)\big) = \begin{cases} |n - m| & \text{if } n \neq m \text{ and } m, n \geq 0 \\ 1 & \text{if } n = m \neq 0 \\ 0 & \text{if } n = m = 0 \end{cases}$$

where we set the empty $\#$ to be equal to the unknot.

For the proof of Proposition 11.4 we will need the following lemmas.

Lemma 11.5. Ignoring shifts in quantum degree we have

$$\text{♘}(G) \otimes S_n \cong \text{♘}(G) \otimes \overline{S_n} \cong (\text{♘}(G) \otimes \text{♘}(2))^{\oplus n} \oplus \text{♘}(G)$$

and

$$\text{♘}(2) \otimes S_n \cong \text{♘}(2) \otimes \overline{S_n} \cong (\text{♘}(G) \otimes \text{♘}(2))^{\oplus n} \oplus \text{♘}(2).$$

Lemma 11.6. Let $z \in \mathbb{Z}[G]$ and $a, b \in \mathbb{Z}_{\geq 0}$ with $a \leq b$. Then

$$♘(z^a) \otimes ♘(z^b) \cong ♘(z^a) \oplus ♘(z^a)$$

(shifts in quantum degree are omitted).

We refer the reader for a proof of Lemma 11.5 and 11.6 to [ILM21, Lemma 4.6] and [ILM21, Lemma 4.7], respectively.

Proof of Proposition 11.4. Let us start by stating a few facts that we will use throughout this proof.

$$(♘(G) \otimes ♘(2)) \otimes (♘(G) \otimes ♘(2)) \cong (♘(G) \otimes ♘(2))^{\oplus 4} \tag{11.1}$$

$$♘(G) \otimes (♘(G) \otimes ♘(2)) \cong (♘(G) \otimes ♘(2))^{\oplus 2} \tag{11.2}$$

$$(♘(G) \otimes ♘(2)) \otimes S_1 \cong (♘(G) \otimes ♘(2)) \otimes \overline{S_1} \cong (♘(G) \otimes ♘(2))^{\oplus 6} \tag{11.3}$$

$$S_1 \otimes \overline{S_1} \cong (♘(G) \otimes ♘(2))^{\oplus 2} \oplus ♙. \tag{11.4}$$

The equations (11.1) to (11.3) follow easily from Lemma 11.6. The equation (11.4) requires some more work and may be checked by hand or using the program `tenpro`. Details are left to the reader. We also remind the reader that for two knots K_1 and K_2 we have $[\![K_1 \# K_2]\!] \cong [\![K_1]\!] \otimes [\![K_2]\!]$.

Let $L = (\mathop{\#}_{i \leq n} K_i) \# (\mathop{\#}_{j \leq m} J_j)$. If $n = m = 0$ then $\lambda(L) = \lambda(U) = 0$, so assume $n, m \neq 0$. Using equations (11.1) to (11.4), Lemma 11.5 and Lemma 11.6 we find that for all $i, j \geq 1$ the complex $[\![K_i \# J_j]\!]$ splits as a sum of the following pieces:

$$♙, \quad ♘(G), \quad ♘(G) \otimes ♘(2).$$

The same pieces also give a decomposition of $[\![\mathop{\#}_{i,j \geq 1}(K_i \# J_j)]\!]$. If $n = m \neq 0$ then $L = \mathop{\#}_{1 \leq i \leq m}(K_i \# J_i)$. Using Corollary 10.6 and the fact that

$$\lambda(♘(G), 0) = \lambda(♘(G) \otimes ♘(2), 0) = 1$$

one obtains:

$$\begin{aligned} \lambda(L) &= \lambda(\mathop{\#}_{1 \leq i \leq m}(K_i \# J_i)) \\ &\leq \max\{\lambda(♙),\ \lambda(♘(G), 0),\ \lambda(♘(G) \otimes ♘(2), 0)\} \\ &= 1 \end{aligned}$$

Since $L \neq U$, we also have $\lambda(L) \geq 1$ by Proposition 7.7. This shows that $\lambda(L) = 1$.

Suppose now that $n > m \geq 0$. Then

$$L = \mathop{\#}_{j \leq m}(K_j \# J_j) \# (\mathop{\#}_{m+1 \leq i \leq n} K_i).$$

Using the equations (11.1) to (11.3), Lemma 11.3 as well as Lemma 11.5 and Lemma 11.6, it is easy to see that $[\![\#_{m+1 \leq i \leq n} K_i]\!]$ splits as a sum of

$$\knight(G), \quad \knight(G) \otimes \knight(2), \quad S_{n-m}.$$

Now

$$[\![L]\!] \cong [\![\#_{j \leq m} (K_j \# J_j)]\!] \otimes [\![\#_{m+1 \leq i \leq n} K_i]\!],$$

therefore (11.1) to (11.3) together with Lemma 11.5 and Lemma 11.6 show that the same pieces also give a decomposition of $[\![L]\!]$. Thus, in order to prove that $\lambda(L) \leq n - m$ all we have to do is apply Corollary 10.6, which yields:

$$\lambda(L) \leq \max\{\lambda(S_{n-m}),\ \lambda(\knight(G), 0),\ \lambda(\knight(G) \otimes \knight(2), 0)\} = n - m.$$

The inequality $\lambda(L) \geq n - m$ also holds: the complex $[\![\#_{j \leq m} (K_j \# J_j)]\!]$ has a $\pawn$ piece, and $[\![\#_{m+1 \leq i \leq n} K_i]\!]$ has a S_{n-m} piece, so there is a piece $S_{n-m} \cong \pawn \otimes S_{n-m}$ in $[\![L]\!]$. Using Lemma 10.29 and that $\mathfrak{u}_G(S_{n-m}) = n - m$ by Lemma 11.2, one finds

$$\lambda(L) \geq \mathfrak{u}_G(L) = n - m.$$

Overall, it follows that $\lambda(L) = n - m$.

Lastly, let $m > n \geq 0$. Then

$$L = \#_{i \leq n} (K_i \# J_i) \# (\#_{n+1 \leq j \leq m} J_j),$$

and the only pieces appearing in $[\![L]\!]$ are

$$\knight(G), \quad \knight(G) \otimes \knight(2), \quad \overline{S_{m-n}}.$$

It follows that the pieces appearing in $[\![-L]\!] = \overline{[\![L]\!]}$ are

$$\knight(G), \quad \knight(G) \otimes \knight(2), \quad S_{m-n}.$$

Hence, by Proposition 7.8 and looking at the proof of the case $n > m$ just above, one finds $\lambda(L) = \lambda(-L) = m - n$. □

Remark 11.7. Using `khoca` and `homca` one finds that there are many knots satisfying requirements 1. or 2. of Proposition 11.4. For instance, one can take any knot with up to 15 crossings such that $s_{\mathbb{F}_2} \neq s_{\mathbb{F}_3}$. One of those is the above-mentioned knot $14n19265$, and a complete list is given in Table 1 of Schütz's paper [Sch21a], supplemented by the following knots [Sch21b]:

$$15n154386, \quad 15n165952, \quad 15n165966, \quad 15n166064, \quad 15n166244.$$

We also note that if a knot K satisfies condition 1. of Proposition 11.4 then its mirror image $-K$ will satisfy condition 2. and vice-versa.

Remark 11.8. It's easy to see from the above proof that a similar result as Proposition 11.4 also holds for $\mathfrak{u}_G$. Namely, if $K_1, \dots, K_n$, $J_1, \dots, J_m$ satisfy conditions 1. and 2. of Proposition 11.4, we have:

$$\mathfrak{u}_G(\underset{i\leq n}{\#} K_i \# \underset{j\leq m}{\#} J_j) = \begin{cases} n-m & \text{if } n > m \geq 0 \\ 1 & \text{if } n = m \neq 0 \text{ or } m > n \geq 0 \\ 0 & \text{if } n = m = 0 \end{cases}$$

The partial difference is due to the fact that $\mathfrak{u}_G(\overline{S_k}) = 0$, while $\lambda(\overline{S_k}) = \lambda(S_k) = k$ for all $k \geq 1$.

Remark 11.9. So far we have only encountered knots whose $\mathbb{Z}[G]$-complex decomposes into either a staircase or a pawn, and (tensor products of) knights. In particular, we have used knots that split off a staircase to exhibit special behavior of λ. However, it is very likely that there exist knots whose $\mathbb{Z}[G]$-complex decomposes into summands other than the ones mentioned above. A potential candidate is the knot 18nh9772775, used by Schütz in [Sch22]:[1] using `khoca` and `homca`, one finds that the $\mathbb{Z}[G]$-complex of this knots splits off the following summand of rank 9 (left subscript denotes homological degree):

$$_{-1}R\{0\} \oplus R\{2\} \xrightarrow{d^{-1}} R\{-2\} \oplus R\{-2\} \oplus R\{-2\} \oplus R\{0\} \oplus R\{0\} \xrightarrow{d^0} R\{-4\} \oplus R\{-2\}$$

where

$$d^{-1} = \begin{pmatrix} -2G & 0 \\ 3G & 0 \\ 2G & 0 \\ -4 & 0 \\ 2 & -G \end{pmatrix}, \quad d^0 = \begin{pmatrix} 0 & 2G & -3G & 0 & 0 \\ -2 & 0 & 0 & G & 0 \end{pmatrix}.$$

11.3 The computer programs `homca` and `tenpro`

Many of the results in the current chapter are obtained or supported by computations using the program `khoca` by Lewark [LL18] and the extensions `homca`, `tenpro` by the author [Ilt21a, Ilt21b]. The purpose of this section is to give an introduction to the author's programs `homca` and `tenpro` and provide a description of the algorithms and working mechanisms that are implement. A documentation of `khoca` is available at `https://github.com/LLewark/khoca`.

[1] Here, we use the notation provided by the knot tables of `https://regina-normal.github.io/data.html`.

11.3.1 `homca`

The $\mathbb{Z}[G]$-complex of a knot K can be difficult to handle; without further simplification, the size of $[\![K]\!]$ grows exponentially in terms of rank with increasing crossing number of K. This is problematic in hands-on examples, as special behaviour seems to often appear for knots with large crossing number (such as non-equal values of the Rasmussen invariant over different fields (see Remark 11.7) or the counterexample to the Knight Move Conjecture by Manolescu and Marengon with over 30 crossings [MM20], for instance). While Lewark's program `khoca` [LL18] is able to produce the $\mathbb{Z}[G]$-complex $[\![K]\!]$ of any knot, it doesn't particularly attempt to decompose the complex into pieces in the sense of Section 10.3. This is where `homca` [Ilt21a] comes into play.

The program `homca`, written in Python and SageMath, was designed with the purpose to simplify the $\mathbb{Z}[G]$-complex $[\![K]\!]$ of a knot K produced by `khoca`. It does so by performing basis changes on the free chain modules in $[\![K]\!]$ in order to produce as many zeros as possible in matrix representatives of the differentials, so that one may more easily read off indecomposable summands. For instance, given the following complex of free $\mathbb{Z}[G]$-modules

$$C = 0 \longrightarrow \mathbb{Z}[G] \oplus \mathbb{Z}[G] \xrightarrow{\left(\begin{smallmatrix} G & 0 \\ 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)} \mathbb{Z}[G] \oplus \mathbb{Z}[G] \oplus \mathbb{Z}[G] \longrightarrow 0 \ ,$$

we may read off a decomposition into a staircase of rank 1 and a 3-knight:

$$C \cong S_0 \oplus \text{♘}(3) = \Big(\mathbb{Z}[G] \xrightarrow{\left(\begin{smallmatrix} G \\ 2 \end{smallmatrix}\right)} \mathbb{Z}[G] \oplus \mathbb{Z}[G]\Big) \bigoplus \Big(\mathbb{Z}[G] \xrightarrow{(3)} \mathbb{Z}[G]\Big).$$

For a general chain complex of free $\mathbb{Z}[G]$-modules, there are two main difficulties one has to deal with when trying to implement a simplification algorithm.

1.) $\mathbb{Z}[G]$ is not a PID, hence there is no Smith normal form. In particular, there is no known algorithm that transforms any given matrix (of a differential) over $\mathbb{Z}[G]$ into a form from which one may read off only indecomposable summands of the corresponding chain complex.

2.) Basis changes correspond to elementary row and column operations on the matrix representatives of the differentials. However, these operations change the basis of the image and kernel of the differential, respectively, which means that one has to do the corresponding inverse operations on preceding and succeeding differentials. More precisely, if we are given a complex of free modules

$$\cdots \longrightarrow C^{i-1} \xrightarrow{d^{i-1}} C^{i} \xrightarrow{d^{i}} C^{i+1} \xrightarrow{d^{i+1}} C^{i+2} \longrightarrow \cdots$$

then:

elementary row operation changes basis of im d^i	$\longrightarrow$	perform inverse column operation on d^{i+1}
elementary column operation changes basis of $\ker d^i$	$\rightarrow$	perform inverse row operation on d^{i-1}

However, these inverse operations may introduce new non-zero entries, again because $\mathbb{Z}[G]$ is not a PID and we have no Smith normal form.

`homca` tries to solve problems 1.) and 2.) by performing a reduction algorithm that runs twice through every differential in the complex. It starts at the lowest homological degree moving to the highest, and performs at each differential row and column operations that introduce strictly more 0's in the matrix. When it has reached the highest homological degree, the algorithm runs backwards through every differential again in order to undo non-zero entries that were introduce because of 2.). However, this time we only perform elementary column operations so that differentials of higher homological degree don't get changed again; we consider them as "done". Again, because $\mathbb{Z}[G]$ is not a PID this algorithm may not yield the best result in terms of producing most zero entries in every matrix, but in our sample computations it has almost always produced optimal results.

Let us now take a closer look at the steps involved when executing `homca`. The start is always given by parsing an output file of `khoca`, from which we initialize the chain complex.

Step 1: Initialize *zgcomplex* from *khocaoutput.txt* including homological and quantum degrees.

Step 2: Reduction Algorithm First Iteration
Let $A^m, \dots, A^n$ be the matrices of the differentials of *zgcomplex* in ascending order with respect to homological degree. Starting at A^m and moving towards A^n, search in each A^i for elementary row and column operations that introduce strictly more zeros. If a row operation is performed, do inverse column operation on A^{i+1}, and if a column operation is performed, do inverse row operation on A^{i-1} (with $A^{m-1} := 0$ and $A^{n+1} := 0$).

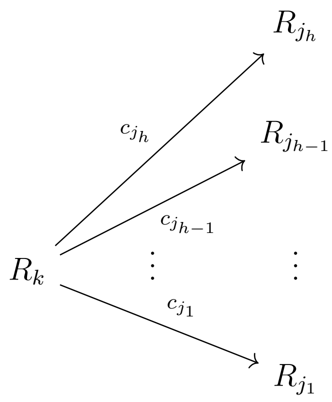
Step 3: Reduction Algorithm Second Iteration
Perform **Step 2** in *reversed order*: start at A^n, move towards Am, and *only* search for column operations that introduce strictly more zeros. If a column operation is performed, do the corresponding inverse row operation on the preceding differential.

Step 4: Search for direct summands in *zgcomplex*

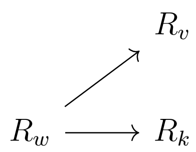
This is done by `homca` as follows: suppose we are given

$$\cdots \longrightarrow \bigoplus_{j=1}^{a} R_j \xrightarrow{A^{i-1}} \bigoplus_{j=1}^{b} R_j \xrightarrow{A^{i}} \bigoplus_{j=1}^{m} R_j \xrightarrow{A^{i+1}} \bigoplus_{j=1}^{n} R_j \longrightarrow \cdots$$

where each $R_j = \mathbb{Z}[G]$. For each k, let $c_{j_1}, \ldots, c_{j_h}$ be the non-zero entries in the k-th column of A^i. Then we store an object of the form



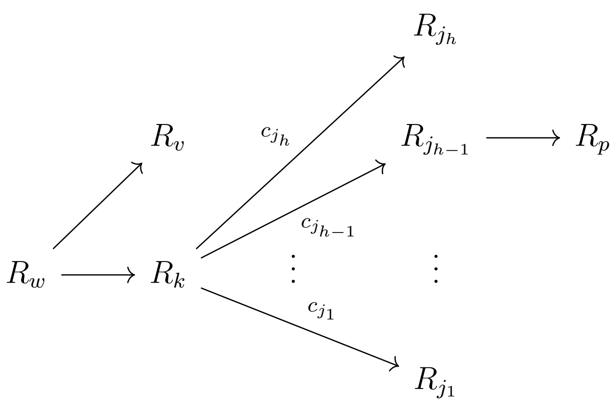
where a tower of R_j's is to be understood as a direct sum. Call R_k the *source* and the R_{j_l} the *targets* of the k-th column of A^i. Check if R_k appears as a target of some column in A^{i-1}, check if any of the R_{j_l} form the source of a non-trivial column in A^{i+1}, and then merge the corresponding objects together. So for instance, if A^{i-1} contains (labels of the arrows ignored)



and A^{i+1} contains

$$R_{j_{h-1}} \longrightarrow R_p$$

then we merge the objects to



This process causes backwards and forwards reactions through all preceding and succeeding differentials of A^i, where we have to search for matching sources and targets of columns that we need to append to the current object as above. If there's nothing left to be merged, we have found a direct summand and finish by constructing the differentials of the summand from the given arrows.

We refer the reader for more details to the source code of `homca` found at `https://github.com/dilt1337/homca`, which contains detailed comments about the algorithms and methods implemented. One may also find an installation guide for `homca` under the same link.

As an example, Figure 11.2 and Figure 11.3 at the end of this chapter show the `khoca` output of the Cotton Seed knot $14n288160$ and the corresponding `homca` output, respectively.

11.3.2 `tenpro`

When defining a new invariant, it is always desirable to find good examples that exhibit special behaviour of the invariant. Moreover, such examples may lead to theorems, as it was the case with our Theorem 7.5 that λ can grow arbitrarily big. When search for a knot K such that $\lambda(K) = n$ for a given $n\mathbb{N}$, we knew for two reasons that connected sums of the Cotton Seed knot $CS = 14n268810$ might be a good candidate: first, because in general $[\![K_1 \# K_2]\!] = [\![K_1]\!] \otimes [\![K_2]\!]$, and second because the piece of odd rank in $[\![CS]\!]$ is different from a pawn since $s_{\mathbb{F}_2}(CS) \neq s_{\mathbb{F}_3}(CS)$. Using `khoca` and `homca`, we figured that the piece of odd rank in $[\![CS]\!]$ is a staircase S_1 for which one can easily compute $\lambda(S_1) = 1$. Expecting that in general $\lambda(S_n) = n$, all we had to do is to find a way to effectively compute tensor products $[\![CS]\!]^{\otimes n}$ to check that this $\mathbb{Z}[G]$-complex has the staircase S_n as piece of odd rank and then prove that $\lambda([\![CS]\!]^{\otimes n}) = n$.[2] This is where the program `tenpro` by the author comes into play [Ilt21b]. `tenpro` takes as input two chain complexes over $\mathcal{M}_{\mathbb{Z}[G]}$ (specified by the user by hand or via two `khoca` output files containing $\mathbb{Z}[G]$-complexes), computes their tensor product and then decomposes it into summands using `homca`. This way, we verified for several n that $[\![CS]\!]^{\otimes n}$ has S_n as piece of odd rank, and in fact many of the results in Chapter 11 were verified using the combination of `khoca`, `homca`, and `tenpro`.

The aim of this section is to explain the algorithm implemented by `tenpro` that computes the tensor product of two complexes over $\mathcal{M}_{\mathbb{Z}[G]}$. Before we start, let us mention that `tenpro` is written in Python and SageMath and available at `https://github.com/dilt1337/tenpro`, where also an installation and usage manual can be found. We assume that the reader is familiar with forming the tensor product of two general chain complexes.

[2] Of course, one can also simply compute the case $n = 2$ by hand giving $S_1 \otimes S_1 = S_2$ and then start working out the proof for general n. But it's always desirable to be able to verify the hypothesis on more than one example.

Let $(A, d_A), (B, d_B)$ be chain complexes over $\mathcal{M}_{\mathbb{Z}[G]}$, so that each chain module of A respectively B is a direct sum of (grading shifted) copies of $\mathbb{Z}[G]$. Write

$$A = A^m \xrightarrow{d_A^m} A^{m+1} \xrightarrow{d_A^{m+1}} \cdots \xrightarrow{d_A^{n-1}} A^{n-1} \xrightarrow{d_A^n} A^n$$
$$B = B^p \xrightarrow{d_B^p} B^{p+1} \xrightarrow{d_B^{p+1}} \cdots \xrightarrow{d_B^{q-1}} B^{q-1} \xrightarrow{d_B^q} B^q$$

for some $m, n, p, q \in \mathbb{Z}$, where

$$A^i \cong \bigoplus_{k=1}^{l_i} R_k, \qquad B^j \cong \bigoplus_{f=1}^{z_j} R_f$$

with $R_k \cong R_f \cong \mathbb{Z}[G]$ for each k and f. Here, we omit quantum shifts for better readability but they are of course included in `tenpro`. We further identify the differentials with their matrix corresponding to the standard basis of a direct sum of $\mathbb{Z}[G]$'s. Now, the important part when forming $A \otimes B$ is to define how we choose a basis on the resulting chain modules. So, we start by ordering summands in tensor products $A^i \otimes B^j$ as follows:

$$A^i \otimes B^j = \Big(\bigoplus_{k=1}^{l_i} R_k\Big) \otimes \Big(\bigoplus_{f=1}^{z_j} R_f\Big) \cong \bigoplus_{k=1}^{l_i} \bigoplus_{f=1}^{z_j} R_k \otimes R_f,$$

and each $R_k \otimes R_f$ is generated by $1 \otimes 1$ over $\mathbb{Z}[G]$ for every k and f. In `tenpro`, the basis element $1 \otimes 1$ of $R_k \otimes R_f$ is encoded as array $[k_i, f_j]$, with subscript being the homological degree of A^i and B^j, respectively. We order the corresponding basis on $A^i \otimes B^j$ in lexicographical order. Overall, `tenpro` stores the basis of $A^i \otimes B^j$ as an ordered list of arrays:

$$[1_i, 1_j], [1_i, 2_j], \ldots, [1_i, (z_j)_j], [2_i, 1_j], \ldots, [(l_i)_i, (z_j)_j]. \tag{11.5}$$

For $A \otimes B$, we start with a diagram as follows.

$$\begin{array}{c|ccccccc}
\otimes & A^m & \xrightarrow{d_A^m} & A^{m+1} & \xrightarrow{d_A^{m+1}} & \cdots & \xrightarrow{d_A^{n-1}} & A^n \\
\hline
B^p & A^m \otimes B^p & \longrightarrow & A^{m+1} \otimes B^p & \longrightarrow & \cdots & \longrightarrow & A^n \otimes B^p \\
{\scriptstyle d_B^p}\downarrow & \downarrow & & \downarrow & & & & \\
B^{p+1} & A^m \otimes B^{p+1} & \longrightarrow & A^{m+1} \otimes B^{p+1} & \longrightarrow & \cdots & \longrightarrow & A^n \otimes B^{p+1} \\
{\scriptstyle d_B^{p+1}}\downarrow & \downarrow & & \downarrow & & & & \downarrow \\
\vdots & \vdots & & \vdots & & \ddots & & \vdots \\
{\scriptstyle d_B^{q-1}}\downarrow & \downarrow & & \downarrow & & & & \downarrow \\
B^q & A^m \otimes B^q & \longrightarrow & A^{m+1} \otimes B^q & \longrightarrow & \cdots & \longrightarrow & A^n \otimes B^q
\end{array}$$

From this, we form the chain complex $A \otimes B$ as follows:

$$A^m \otimes B^p \longrightarrow (A^{m+1} \otimes B^p) \oplus (A^m \otimes B^{p+1}) \longrightarrow \\ (A^{m+2} \otimes B^p) \oplus (A^{m+1} \otimes B^{p+1}) \oplus (A^m \otimes B^{p+2}) \longrightarrow \cdots \\ \cdots \longrightarrow \bigoplus_{i=0,\dots,h} A^{m+h-i} \otimes B^{p+i} \longrightarrow \cdots \longrightarrow A^n \otimes B^q$$

As one can see, we order the summands in a descending fashion with respect to the homological degree of A. On $\bigoplus_{i=0}^h A^{m+h-i} \otimes B^{p+i}$ we use the standard ordered basis induced by summands, so that using (11.5) it reads

$$[1_{m+h}, 1_p]_h, [1_{m+h}, 2_p]_h, \dots, [(l_{m+h})_{m+h}, (z_p)_p]_h, \\ [1_{m+h-1}, 1_{p+1}]_h, \dots, [(l_m)_m, (z_{p+h})_{p+h}]_h. \tag{11.6}$$

The subscript h at each bracket tracks the homological degree. We write L_h for the ordered set containing (11.6). Let's look at the differentials. Consider

$$\bigoplus_{i=0,\dots,h} A^{m+h-i} \otimes B^{p+i} \overset{d^h_{A\otimes B}}{\longrightarrow} \bigoplus_{i=0,\dots,h+1} A^{m+h+1-i} \otimes B^{p+i}.$$

Both domain and range of $d^h_{A\otimes B}$ have an ordered basis as in (11.6), stored in L_h and L_{h+1}, respectively. `tenpro` builds the matrix of the differential $d^h_{A\otimes B}$ now as follows. Pick elements $X = [k_{m+h-i}, f_{p+i}]_h \in L_h$ and $Y = [e_{m+h+1-j}, g_{p+j}]_{h+1} \in L_{h+1}$ where $k \in \{1, \dots, l_{m+h-i}\}$, $f \in \{1, \dots, z_{p+i}\}$, $e \in \{1, \dots, l_{m+h+1-j}\}$ and $g \in \{1, \dots, z_{m+j}\}$. Suppose that X is at position v in L_h and Y is at position w in L_{h+1}, respectively. Then we determine the matrix entry $(d^h_{A\otimes B})_{v,w}$ by comparing the numerical values of the entries of X and Y, including subscripts. For this, we write

$$a_b \doteq c_d :\Longleftrightarrow a = c,\ b = d \qquad a, b, c, d \in \mathbb{Z}$$

Then we have the following cases:

1. $k_{m+h-i} \doteq e_{m+h+1-j}$: then $p + i = p + j - 1$ and we set
$$(d^h_{A\otimes B})_{v,w} = (d^{p+i}_B)_{g,f}$$
2. $f_{p+i} \doteq g_{p+j}$: then $m + h - i = m + h + 1 - j - 1$ and we set
$$(d^h_{A\otimes B})_{v,w} = (-1)^{c+1} (d^{m+h-i}_A)_{e,k},$$
where $c := (p + i) \mod 2$.[3]
3. Neither $k_{m+h-i} \doteq e_{m+h+1-j}$ nor $f_{p+i} \doteq g_{p+j}$: then we set
$$(d^h_{A\otimes B})_{v,w} = 0$$

[3] In our sign convention, each horizontal arrow of even homological degree of B gets a minus sign in the commutative diagram of $A \otimes B$.

Doing this for each pair of entries in L_h and L_{h+1} builds the matrix $d^h_{A\otimes B}$. Observe that if one fixes an element in L_h and compares it with all other elements in L_{h+1} as above, one builds a column of $d^h_{A\otimes B}$ (this is how it's implemented in `tenpro`).

Example 11.10. In the following, let $R_i = \mathbb{Z}[G]$ for $i = 1, 2$. Consider the complexes

$$A = \underbrace{R_1}_{=A^0} \xrightarrow{d_A^0} \underbrace{R_1 \oplus R_2}_{=A^1} \xrightarrow{d_A^1} \underbrace{R_1}_{=A^2}$$

$$B = \underbrace{R_1 \oplus R_2}_{=B^0} \xrightarrow{d_B^0} \underbrace{R_1}_{=B^1}$$

where

$$d_A^0 = \begin{pmatrix} G \\ 2 \end{pmatrix}, \quad d_A^1 = \begin{pmatrix} -2 & G \end{pmatrix}, \quad d_B^0 = \begin{pmatrix} G \\ 2 \end{pmatrix}$$

In the notation above, we have

$$l_0 = 1,\ l_1 = 2,\ l_2 = 1,\ z_1 = 2,\ z_2 = 1.$$

For the tensor product $A \otimes B$, consider the diagram below.

$$\begin{array}{c|ccccc}
\otimes & R_1 & \xrightarrow{d_A^0} & R_1 \oplus R_2 & \xrightarrow{d_A^1} & R_1 \\
\hline
R_1 \oplus R_2 & R_1 \otimes (R_1 \oplus R_2) & \longrightarrow & (R_1 \oplus R_2)^{\otimes 2} & \longrightarrow & R_1 \otimes (R_1 \oplus R_2) \\
{\scriptstyle d_B^0}\big\downarrow & \big\downarrow & & \big\downarrow & & \\
R_1 & R_1 \otimes R_1 & \longrightarrow & (R_1 \oplus R_2) \otimes R_1 & \longrightarrow & R_1 \otimes R_1
\end{array}$$

This yields

$$\begin{aligned}
A \otimes B = & R_1 \otimes (R_1 \oplus R_2) \xrightarrow{d^0_{A\otimes B}} ((R_1 \oplus R_2) \otimes (R_1 \oplus R_2)) \oplus (R_1 \otimes R_1) \\
& \xrightarrow{d^1_{A\otimes B}} (R_1 \otimes (R_1 \oplus R_2)) \oplus ((R_1 \oplus R_2) \otimes R_1) \\
& \xrightarrow{d^2_{A\otimes B}} R_1 \otimes R_1
\end{aligned}$$

For $h = 0, 1, 2, 3$, the ordered sets L_h containing the basis of the chain modules in $A \otimes B$ (as in (11.6)) are given by

$$\begin{aligned}
L_0 &: \ [1_0, 1_0]_0, [1_0, 2_0]_0 \\
L_1 &: \ [1_1, 1_0]_1, [1_1, 2_0]_1, [2_1, 1_0]_1, [2_1, 2_0]_1, [1_0, 1_1]_1 \\
L_2 &: \ [1_2, 1_0]_2, [1_2, 2_0]_2, [1_1, 1_1]_2, [2_1, 1_1]_2 \\
L_3 &: \ [1_2, 1_1]_3
\end{aligned}$$

Let us now step-by-step show how to construct the first differential $d^0_{A\otimes B}$. We follow the convention of `tenpro` and build column by column. So, for the first column, fix $[1_0, 1_0]_0 \in L_0$, compare with elements in L_1 respecting the order, and distinguish the cases 1. – 3. above:

- $[1_0, 1_0]_0 \leftrightarrow [1_1, 1_0]_1$: This is case 2., so we obtain

$$d^0_{A\otimes B} = \begin{pmatrix} -G & * \\ * & * \\ * & * \\ * & * \\ * & * \end{pmatrix}.$$

- $[1_0, 1_0]_0 \leftrightarrow [2_1, 1_0]_1$: This is case 3., so we obtain

$$d^0_{A\otimes B} = \begin{pmatrix} -G & * \\ 0 & * \\ * & * \\ * & * \\ * & * \end{pmatrix}.$$

- $[1_0, 1_0]_0 \leftrightarrow [2_1, 1_0]_1$: This is case 2., so we obtain

$$d^0_{A\otimes B} = \begin{pmatrix} -G & * \\ 0 & * \\ -2 & * \\ * & * \\ * & * \end{pmatrix}.$$

- $[1_0, 1_0]_0 \leftrightarrow [2_1, 2_0]_1$: This is case 3., so we obtain

$$d^0_{A\otimes B} = \begin{pmatrix} -G & * \\ 0 & * \\ -2 & * \\ 0 & * \\ * & * \end{pmatrix}.$$

- $[1_0, 1_0]_0 \leftrightarrow [1_0, 1_1]_1$: This is case 1., so we obtain

$$d^0_{A\otimes B} = \begin{pmatrix} -G & * \\ 0 & * \\ -2 & * \\ 0 & * \\ G & * \end{pmatrix}.$$

For the second column, we proceed exactly the same. Fix $[1_0, 2_0]_0 \in L_0$ and compare:

- $[1_0, 2_0]_0 \leftrightarrow [1_1, 1_0]_1$: This is case 3., so we obtain

$$d^0_{A\otimes B} = \begin{pmatrix} -G & 0 \\ 0 & * \\ -2 & * \\ 0 & * \\ G & * \end{pmatrix}.$$

- $[1_0, 2_0]_0 \leftrightarrow [1_1, 2_0]_1$: This is case 2., so we obtain

$$d^0_{A\otimes B} = \begin{pmatrix} -G & 0 \\ 0 & -G \\ -2 & * \\ 0 & * \\ G & * \end{pmatrix}.$$

- $[1_0, 2_0]_0 \leftrightarrow [2_1, 1_0]_1$: This is case 3., so we obtain

$$d^0_{A\otimes B} = \begin{pmatrix} -G & 0 \\ 0 & -G \\ -2 & 0 \\ 0 & * \\ G & * \end{pmatrix}.$$

- $[1_0, 2_0]_0 \leftrightarrow [2_1, 2_0]_1$: This is case 2., so we obtain

$$d^0_{A\otimes B} = \begin{pmatrix} -G & 0 \\ 0 & -G \\ -2 & 0 \\ 0 & -2 \\ G & * \end{pmatrix}.$$

- $[1_0, 2_0]_0 \leftrightarrow [1_0, 1_1]_1$: This is case 1., so we obtain

$$d^0_{A\otimes B} = \begin{pmatrix} -G & 0 \\ 0 & -G \\ -2 & 0 \\ 0 & -2 \\ G & 2 \end{pmatrix}.$$

We hope that by now the algorithm is clear to the reader. As an exercise, we invite the reader to write down the remaining matrices using the algorithm and compare with our results:

$$d^1_{A\otimes B} = \begin{pmatrix} 2 & 0 & -G & 0 & 0 \\ 0 & 2 & 0 & -G & 0 \\ G & 2 & 0 & 0 & G \\ 0 & 0 & G & 2 & 2 \end{pmatrix}, \quad d^2_{A\otimes B} = \begin{pmatrix} G & 2 & -2 & G \end{pmatrix}.$$

```
Frobenius algebra:  Z[a, b][X] / (1*X^2 + b*X + a).

[[0]; [(-b)]]

[[0, 0]; [(-b), 0]]

[[0, 0]; [0, 0]; [(b), 0]; [(-2), 0]]

[[(-b), 0, 0, (2*a)]; [0, 0, 0, 0]; [0, (-b), 0, 0];
[(-2), (-2), 0, 0]; [(2), 0, (2), (b)]]

[[0, 0, 0, 0, 0]; [0, (-b), 0, 0, 0]; [0, 0, 0, 0, 0];
[(2), (2), (2), (-b), 0]]

[[0, 0, (b^2 + -4*a), 0]; [(-b), 0, (b^2 + -4*a), 0];
[0, 0, (b), 0]; [(2), 0, 0, 0]]

[[(b), 0, (-b^2 + 4*a), 0]; [(-2), 0, (2*b), 0];
[(2), (2), (-4*b), (b)]; [0, 0, 0, 0]]

[[0, 0, 0, 0]; [0, 0, 0, (-b)]]

[[(-b), 0]; [0, 0]]

[[0, (b)]; [0, 0]]

[[0, (-b)]]

[[0]]

[[(-b)]]

[[3, 17, 22, 16, 21], [3, 1, 10, 0, 9], [3, 21, 14, 20, 13], [2, 22, 7,
23, 8], [2, 14, 3, 15, 4], [2, 11, 18, 12, 19], [3, 13, 26, 12, 25], [3,
6, 11, 5, 10], [2, 19, 26, 20, 28], [3, 0, 5, 27, 4], [3, 25, 18, 24,
17], [2, 6, 23, 7, 24], [2, 2, 15, 3, 16], [3, 9, 2, 8, 1]]

Result:
Reduced Homology:
Equivariant homology:
t^-7q^14 + t^-6q^12 + t^-6q^12 + t^-5q^10 + t^-5q^10 + t^-4q^8 + t^-4q^8
+ t^-4q^8 + t^-4q^10 + t^-3q^6 + t^-3q^6 + t^-3q^6 + t^-3q^8 + t^-3q^8
+ t^-2q^4 + t^-2q^4 + t^-2q^6 + t^-2q^6 + t^-1q^2 + t^-1q^2 + t^-1q^4 +
t^-1q^4 + t^0q^0 + t^0q^2 + t^0q^2 + t^0q^4 + t^1q^0 + t^1q^2 + t^2q^-2
+ t^2q^0 + t^3q^-2 + t^3q^-2 + t^4q^-4 + t^5q^-6 + t^6q^-8
```

Figure 11.2: The `khoca` output for the Cotton Seed knot $14n22180$, giving its $\mathbb{Z}[G]$-complex. It contains (from top to bottom) the Frobenius system used (with $b = -G$ and $a = 0$), the matrices of the differentials, the PD-notation of the knot, and the Poincaré polynomial of the $\mathbb{Z}[G]$-complex.

`Summands of rank 2:`

`hdeg start:  -7, hdeg stop:  -6, length:  2`
$R_1\{14\} \longrightarrow R_2\{12\}$
$(-G)$

`hdeg start:  -6, hdeg stop:  -5, length:  2`
$R_1\{12\} \longrightarrow R_2\{10\}$
$(-G)$

`hdeg start:  -4, hdeg stop:  -3, length:  2`
$R_1\{8\} \longrightarrow R_1\{6\}$
$(-G)$

`hdeg start:  -3, hdeg stop:  -2, length:  2`
$R_2\{6\} \longrightarrow R_2\{4\}$
$(-G)$

`hdeg start:  -2, hdeg stop:  -1, length:  2`
$R_3\{6\} \longrightarrow R_3\{4\}$
(G)

`hdeg start:  0, hdeg stop:  1, length:  2`
$R_4\{4\} \longrightarrow R_2\{2\}$
$(-G)$

`hdeg start:  0, hdeg stop:  1, length:  2`
$R_4\{4\} \longrightarrow R_2\{2\}$
$(-G)$

`hdeg start:  1, hdeg stop:  2, length:  2`
$R_1\{0\} \longrightarrow R_1\{-2\}$
$(-G)$

`hdeg start:  2, hdeg stop:  3, length:  2`
$R_2\{0\} \longrightarrow R_1\{-2\}$
(G)

`hdeg start:  3, hdeg stop:  4, length:  2`
$R_2\{-2\} \longrightarrow R_1\{-4\}$
$(-G)$

`hdeg start:  5, hdeg stop:  6, length:  2`
$R_1\{-g\} \longrightarrow R_2\{-8\}$
$(-G)$

```
Summands of rank 3:

hdeg start:  -1, hdeg stop:  0, length:  2
```
$R_1\{2\} \longrightarrow R_1\{0\} \oplus R_2\{2\}$

$\begin{pmatrix} G \\ -2 \end{pmatrix}$

```
Summands of rank 4:

hdeg start:  -5, hdeg stop:  -3, length 3
```
$R_1\{10\} \longrightarrow R_3\{8\} \oplus R_4\{10\} \longrightarrow R_5\{8\}$

$\begin{pmatrix} G \\ -2 \end{pmatrix}$

$\begin{pmatrix} G & -2 \end{pmatrix}$

```
hdeg start:  -4, hdeg stop:  -2, length 3
```
$R_2\{8\} \longrightarrow R_3\{6\} \oplus R_4\{8\} \longrightarrow R_4\{6\}$

$\begin{pmatrix} -G \\ -2 \end{pmatrix}$

$\begin{pmatrix} 2 & -G \end{pmatrix}$

```
hdeg start:  -2, hdeg stop:  -0, length 3
```
$R_1\{4\} \longrightarrow R_2\{2\} \oplus R_4\{4\} \longrightarrow R_3\{2\}$

$\begin{pmatrix} -G \\ 2 \end{pmatrix}$

$\begin{pmatrix} 2 & G \end{pmatrix}$

```
Piece of odd rank:

hdeg start:  -1, hdeg stop:  0, length:  2
```
$R_1\{2\} \longrightarrow R_1\{0\} \oplus R_2\{2\}$

$\begin{pmatrix} G \\ -2 \end{pmatrix}$

Figure 11.3: The `homca` output for the Cotton Seed knot $14n22180$. Shown are direct summands of a certain rank. For each summand, it shows the beginning (*hdeg start*) and ending (*hdeg end*) homological degree, and the length of the summand (i.e. number of appearing homological degrees). Within a summand, $R_j\{q\}$ stands for the j-th copy of $\mathbb{Z}[G]$ in the corresponding chain module of $[\![14n22180]\!]$, with quantum degree q. The differentials are ordered according to the arrows from left to right. At the bottom it shows the single piece of odd rank, from which one may read off the non-equal Rasmussen invariants $s_{\mathbb{F}_2} = 2$ and $s_{\mathbb{F}_3} = 0$.

Chapter 12

λ and Rational Tangles

One of our main results states that the λ-invariant yields a lower bound on the proper rational unknotting number u_q of a knot, see Theorem 7.4. The key step in the proof is to show that if two knots K and J are related by a proper rational replacement, then $\lambda(K, J) \leq 1$. For this we have to relate the Bar-Natan complex of different rational tangles, as follows.

First, we need to compute the Bar-Natan complexes of rational tangles. For this we heavily rely on a result by Thompson [Tho17], who showed that the Bar-Natan complex of a rational tangle T over the category of *dotted* cobordisms is homotopy equivalent to a so-called *zigzag complex*. However, we work over categories of cobordisms without dots, and so we require the analogue of Thompson's result in this more general setting. We also use Kotelskiy-Watson-Zibrowius' theorem [KWZ19, Theorem 1.1] that Bar-Natan's category of 4-ended tangles and cobordisms is equivalent to a category coming from a quiver with two vertices and four edges, which yields a quite simple calculus for chain complexes of 4-ended tangles. This forms the content of Section 12.1.

Second, we need to study zigzag complexes themselves in order to prove Proposition 10.15, which states that for rational tangles in a fixed ball with fixed base point and connectivity (up to equivalence), the pseudometric given by λ (see Proposition 10.14) is in fact equal to the discrete metric. This implies $\lambda(S, T) \leq 1$ for two rational tangles S, T, from which $\lambda(K, J) \leq 1$ and ultimately $\lambda(K) \leq u_q(K)$ will follow. The analysis of zigzag complexes is the subject of Section 12.2, which also contains the proof of Theorem 7.4.

For simplicity we will in the following only consider positive rational tangles, i.e. rational tangles $R(x)$ with $x \in \mathbb{Q}^+$ (see Section 8.3). However, since $R(-x)$ is the mirror image of $R(x)$ and the dual of the Bar-Natan complex of a tangle corresponds to the Bar-Natan complex of its mirror image, everything admits a straightforward generalization.

12.1 The Bar-Natan complex of rational tangles

The Bar-Natan complex $[T]^\bullet$ of an oriented rational tangle T is by our definition a complex over $\mathrm{Mat}(\mathrm{Cob}^{3;\bullet}_{/l}(4))$ (considered up to homotopy equivalence), where $\mathrm{Cob}^{3;\bullet}_{/l}(4)$ is the $\mathbb{Z}[G]$-enriched category of 4-ended unoriented crossingless tangle diagrams with base point, and cobordisms between them (see Section 9.4). Recall that in Section 9.1, we have found by using delooping and simplifying cobordisms that $\mathrm{Cob}^{3}_{/l}(2)$, and hence also $\mathrm{Cob}^{3;\bullet}_{/l}(2)$, is equivalent to the category $\mathcal{M}_{\mathbb{Z}[G]}$. A similar strategy gives the following analogue for $\mathrm{Cob}^{3;\bullet}_{/l}(4)$, which is a reformulation of a theorem by Kotelskiy, Watson and Zibrowius.

Theorem 12.1 ([KWZ19, Theorem 1.1]). Consider the $\mathbb{Z}[G]$-enriched category with the two objects  and , and $\mathbb{Z}[G]$-morphism modules consisting of compositions of the identity (product) cobordisms

and the obvious saddle cobordisms

modulo the relation

$$S^3 = GS.$$

We grade the $\mathbb{Z}[G]$-morphism modules according to Definition 8.37. Then the inclusion of the additive graded closure of this category into $\mathrm{Cob}^{3;\bullet}_{/l}(4)$ is an equivalence of categories.

Notation. As already present in Theorem 12.1, we abuse notation and denote both identity and saddle cobordisms with I and S, respectively.

Theorem 12.1 gives us a compact notation for $\mathrm{Cob}^{3;\bullet}_{/l}(4)$: objects are isomorphic to (quantum) grading shifted sums of  and , and morphisms are equal to $\mathbb{Z}[G]$-linear combinations of I, S and S^2.

Convention. Following [KWZ19], we write $D := S^2 - G$ for convenience. Also, we will in the following mostly omit homological and quantum gradings without further mention.

Remark 12.2. Observe that

1. $SD = DS = 0$;
2. $D^2 = -GD$;
3. $\deg I = 0$, $\deg S = 1$, and $\deg D = 2$.

Let us now move towards the analogue of Thompson's result mentioned at the beginning of Chapter 12 by introducing the notion of a zigzag complex.

Definition 12.3. Let

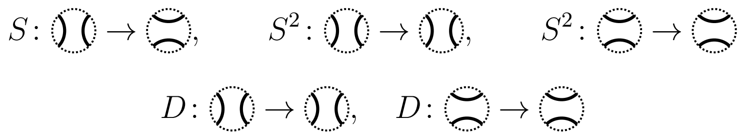
$$\mathcal{C} = C^p \xrightarrow{d^p} C^{p+1} \xrightarrow{d^{p+1}} \cdots \xrightarrow{d^{q-1}} C^q$$

be a graded chain complex over $\mathrm{Mat}(\mathrm{Cob}^{3;\bullet}_{/l}(4))$, so that each C^k is a formal direct sum of grading shifted objects in $\mathrm{Cob}^{3;\bullet}_{/l}(4)$, i.e.

$$C^k = \bigoplus_{j=0}^{l_k} \mathcal{O}^k_j\{r^k_j\} \in \mathrm{ob}(\mathrm{Mat}(\mathrm{Cob}^{3;\bullet}_{/l}(4))),$$

and each differential d^k is a matrix consisting of morphisms in $\mathrm{Cob}^{3;\bullet}_{/l}(4)$. $\mathcal{C}$ is called a *zigzag complex* if it satisfies the following:

1. Each $\mathcal{O}^k_j\{r^k_j\}$ is either $\{r^k_j\}$ or $\{r^k_j\}$.
2. If $\sum_{i=p}^{q} l_i = n + 1$, then there are in total n non-zero entries in all differentials combined.
3. Each non-zero entry in a differential is one of the following five maps (ignoring grading shifts):



4. There is at least one entry in some differential that is a saddle S.
5. If m is a non-zero entry at position (i, j) in d^k, then there is one of the following cases:
   (a) There is exactly one more non-zero entry m' in either the i-th row or the j-th column of d^k, and if $m = D$ then $m' = S$ or S^2, and if $m = S$ or S^2 then $m' = D$;
   (b) m is the only non-zero entry in the i-th row and j-th column of d^k, and if $k \neq 0$, then there is precisely one non-zero entry m' in the j-th row of d^{k-1}, and if $m = D$ then $m' = S$ or S^2, and if $m = S$ or S^2 then $m' = D$.

The definition of a zigzag complex certainly looks confusing, but has in fact a very simple intuition behind it. The following lemma explains this and follows directly from Definition 12.3.

Lemma 12.4. Let $\mathcal{C}$ be a zigzag complex. Then there exists an enumeration $A_0, \ldots, A_n$ of all appearing objects $\mathcal{O}^k_j\{r^k_j\}$ in $\mathcal{C}$ and an enumeration $m_1, \ldots, m_n$ of all non-zero entries in the differentials of $\mathcal{C}$, such that:

1. Each m_i is a morphism $A_{i-1} \to A_i$ or $A_i \to A_{i-1}$ increasing homological degree by 1; and

2. Two consecutive morphisms m_i, m_{i+1} are either S and D (or vice-versa), or S^2 and D (or vice-versa).

Moreover, this enumeration is unique up to *reindexing* given by $A'_i := A_{n-i}$ for $i = 0, \ldots, n$ and $m'_i := m_{n+1-j}$ for $j = 1, \ldots, n$.

Notation. Given a zigzag complex $\mathcal{C}$ with an enumeration as in Lemma 12.4, we write $(\mathcal{C}, \bigoplus_{i=0}^n A_i, \sum_{i=1}^n m_i)$. For the reindexed complex, we write $(\mathcal{C}, \bigoplus_{i=0}^n A_{n-i}, \sum_{i=1}^n m_{n+1-i})$.

Note that reindexing does not change the isomorphism type of the zigzag complex.

Convention. From now on, we will always consider an enumeration part of the data of a zigzag complex.

The enumeration as in Lemma 12.4 allows for a very simple depiction of the chain complex, which also explains why it's called a zigzag complex; see Figure 12.2 for an example. Moreover, using this depiction one may easily write down chain maps and homotopies between zigzag complexes by simply specifying directed arrows with non-zero labels between the objects, which correspond to entries in the matrices of the actual chain maps and homotopies over $\mathrm{Mat}(\mathrm{Cob}^{3;\bullet}_{/l}(4))$ (a missing arrow is interpreted as zero in the matrix). See Figure 12.3 for an example.

Now, let us encode zigzag complexes further using the following type of graphs.

Definition 12.5. Let us consider a directed finite graph with two *types* of vertices, $\bullet$ and $\circ$. Let us call an edge connecting a $\bullet$ and a $\circ$ vertex a *saddle edge*. Such a graph is called a *zigzag graph* if it satisfies the following conditions.

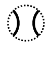
1. The graph has the shape of a line, i.e. there are exactly two vertices of valency 1 (which we call the *ends*), and all other vertices have valency 2.

2. There is a partition of edges into *odd* and *even* edges, such that all saddle edges are odd, and if two edges are adjacent, then one of them is odd and the other one even.

3. All saddle edges are directed like this: $\circ \to \bullet$.

4. There is at least one saddle edge.

Note that because there is at least one saddle edge, the partition of edges into odd and even edges is unique.

Definition 12.6. The *graph of a zigzag complex* $(\mathcal{C}, \bigoplus_{i=0}^n A_i, \sum_{i=1}^n m_i)$ is defined as the zigzag graph with a vertex corresponding to each A_i, where we assign a $\bullet$ if $A_i =$  and a $\circ$ if $A_i =$  (shifts ignored), and one directed edge corresponding to each m_i.

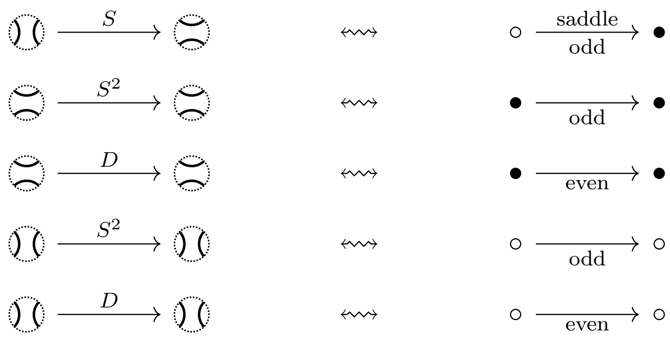


Figure 12.1: Summary of the correspondence between objects and differentials of zigzag complexes (left column) and vertices and edges of zigzag graphs (right columns).

One easily checks that the graph of a zigzag complex really is a zigzag graph. Moreover, every zigzag graph is the graph of a zigzag complex; and the graph of a zigzag complex determines the zigzag complex up to reindexing, and up to global shifts in homological and quantum degree. The correspondence between zigzag complexes and zigzag graphs is summarized in Figure 12.1.

Let us now recursively define a zigzag graph $zz(x)$ for all positive rational numbers $x \in \mathbb{Q}^+$ by the following rules:

$$zz(1) = \circ \longrightarrow \bullet \tag{12.1}$$

$zz(1/x)$ is obtained from $zz(x)$ by switching $\circ$ and $\bullet$, and reversing the directions of all edges. (12.2)

$zz(x+1)$ is obtained from $zz(x)$ by replacing each edge as shown in Table 12.1. (12.3)

Note that these cases are exhaustive, since two adjacent $\bullet$-vertices cannot both be ends (because there is at least one saddle edge), and since a saddle edge is always directed from $\circ$ to $\bullet$. Let us check that zz is well-defined. Indeed, $\circ \to \bullet$ is a zigzag graph, and one may verify that (12.2) and (12.3) map zigzag graphs to zigzag graphs. Every positive rational number can be obtained from 1 by a sequence of $x \mapsto 1/x$ and $x \mapsto x+1$. Moreover, that sequence is unique up to inserting or removing two consecutive $x \mapsto 1/x$. Since applying (12.2) twice has no effect, (12.1) to (12.3) indeed define $zz(x)$ for every positive rational x.

We are now ready to state our generalization of Thompson's theorem.

Theorem 12.7. Let $R(x)$ be the unoriented rational tangle corresponding to a positive rational number x. Let T be the tangle $R(x)$ equipped with some orientation o. Then, the Bar-Natan complex $[T]^\bullet$ is homotopy equivalent to a zigzag complex with graph $zz(x)$.

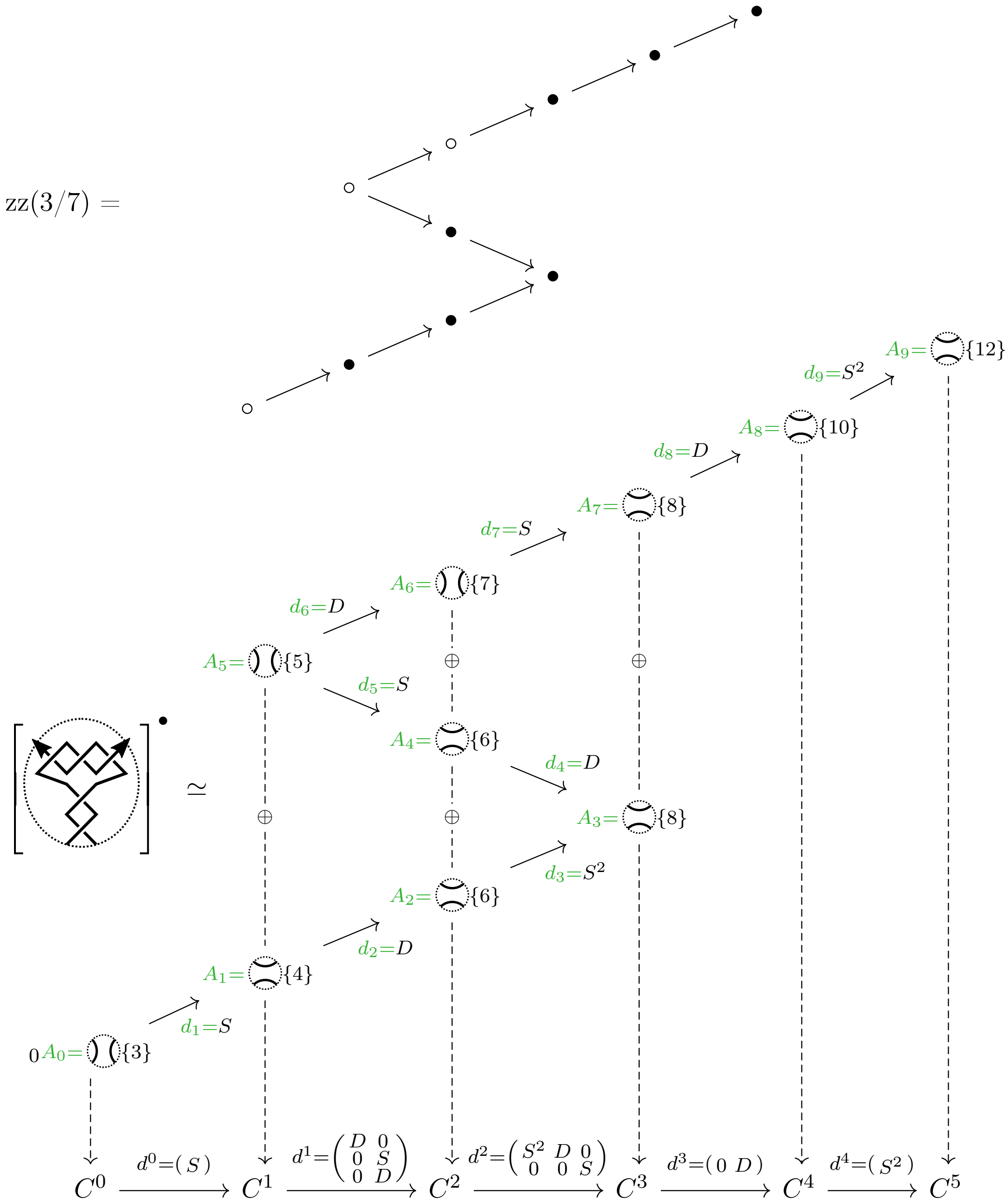


Figure 12.2: As an illustration of the correspondence between zigzag graphs and zigzag complexes, and of Theorem 12.7: on the top, the zigzag graph $zz(3/7)$; on the bottom, a zigzag complex (with $\mathcal{C}$ omitted) that is homotopy equivalent to the Bar-Natan complex of the rational tangle $R(3/7)$ endowed with some orientation. The left subscript gives the homological degree.

Replace	by
$\bullet \xrightarrow{\text{odd}} \bullet$	$\bullet \longleftarrow \circ \longrightarrow \circ \longrightarrow \bullet$
not an end $\bullet \xrightarrow{\text{even}} \bullet$ not an end	$\bullet \longrightarrow \bullet$
not an end $\bullet \xrightarrow{\text{even}} \bullet$ end	$\bullet \longrightarrow \bullet \longleftarrow \circ$
$\circ \longrightarrow \bullet$	$\circ \longrightarrow \circ \longrightarrow \bullet$
$\circ \longrightarrow \circ$	$\circ \longrightarrow \circ$
$\bullet \xleftarrow{\text{odd}} \bullet$	$\bullet \longleftarrow \circ \longleftarrow \circ \longrightarrow \bullet$
not an end $\bullet \xleftarrow{\text{even}} \bullet$ not an end	$\bullet \longleftarrow \bullet$
end $\bullet \xleftarrow{\text{even}} \bullet$ not an end	$\circ \longrightarrow \bullet \longleftarrow \bullet$
$\bullet \longleftarrow \circ$	$\bullet \longleftarrow \circ \longleftarrow \circ$
$\circ \longleftarrow \circ$	$\circ \longleftarrow \circ$

Table 12.1: How to obtain $zz(x+1)$ from $zz(x)$: each edge e in $zz(x)$ falls into a unique one of the six cases shown in the left column of the table. Apply the rule, i.e. replace e by the graph Γ_e in the right column in the same row. In this way, each of the two vertices v, w adjacent to e in $zz(x)$ are replaced by vertices v_e, w_e in $zz(x+1)$, namely the leftmost and the rightmost vertex in Γ_e. If two edges e and f of $zz(x)$ are adjacent to a common vertex v, identify the vertices v_e and v_f in $zz(x+1)$. Note that this is possible since v_e and v_f always have the same type: in fact, v_e has the same type as v if v (or equivalently, v_e) is not an end.

The proof follows essentially [Tho17] with the necessary modifications. On proceeds by induction over the number of transformations $y \mapsto 1/y$ and $y \mapsto y+1$ necessary to reach $R(x)$ from $R(1)$. The main ingredient in the induction steps is then Bar-Natan's divide-and-conquer strategy for simplifying complexes, which consists of a combination of delooping (see Figure 8.20) and Gaussian elimination (see Lemma 8.79). We refer the reader for details of the proof to [ILM21, Theorem 5.6].

Remark 12.8. Since the Bar-Natan complex of the mirror image of a tangle is isomorphic to the dual of the Bar-Natan complex of that tangle, Theorem 12.7 yields a rather simple representative of the homotopy equivalence class of the Bar-Natan complex of any rational tangle, up to global shifts in homological and quantum degree. These shifts depend on the orientation of the tangle; since they do not matter for our work on λ, we will neglect them. Thompson computes the shifts in [Tho17, Theorem 5.1].

12.2 The λ-distance between rational tangles

We start with an analysis on the zigzag graph $zz(x)$ for $x \in \mathbb{Q}^+$.

Lemma 12.9. Let $x \in \mathbb{Q}^+$. For every non-saddle edge e of the zigzag graph $zz(x)$, there is a subgraph Γ_e of $zz(x)$ as follows for some $n \geq 1$ (in the following, a missing arrowhead means that the edge's direction is unknown):

$$\Gamma_e = \begin{array}{ccccccc} A_1 & \text{——} & A_2 & \text{——} & \dots & \text{——} & A_n \\ {\scriptstyle e}\downarrow & & & & & & \\ B_1 & \text{——} & B_2 & \text{——} & \dots & \text{——} & B_n, \end{array}$$

such that for each i with $1 \leq i < n$, the vertices A_i and B_i are of the same type ($\circ$ or $\bullet$), and the edges between A_i and A_{i+1} and between B_i and B_{i+1} are either both directed to the right, or both to the left (in the above drawing of Γ_e); and such that moreover, one of the following statements is true:

1. A_n and B_n are of the same type, A_n has no outgoing *external* edge (i.e. an edge towards a vertex in $zz(x) \setminus \Gamma_e$), and B_n has no incoming external edge.

2. $n \geq 2$, and

$$\begin{array}{ccc} A_{n-1} \text{ —— } A_n & & \circ \longrightarrow \circ \\ & \text{looks like} & \\ B_{n-1} \text{ —— } B_n & & \circ \longrightarrow \bullet. \end{array}$$

3. $n \geq 2$, and

$$\begin{array}{ccc} A_{n-1} \text{ —— } A_n & & \bullet \longleftarrow \circ \\ & \text{looks like} & \\ B_{n-1} \text{ —— } B_n & & \bullet \longleftarrow \bullet. \end{array}$$

Similar to the proof of Theorem 12.7, the proof of Lemma 12.9 proceeds by induction over the number of transformations $y \mapsto 1/y$ and $y \mapsto y+1$ necessary to reach x from 1. One has to do an extensive case distinction, and we refer the reader to [ILM21, Lemma 5.9] for a proof.

The existence of a subgraph Γ_e for non-saddle edges as in Lemma 12.9 allows for a statement about multiplication on the individual A_i's in a zigzag complex $\mathcal{C}$.

Lemma 12.10. Let $x \in \mathbb{Q}^+$, and let $(C, d) = (\bigoplus_{i=0}^n A_i, \sum_{i=1}^n d_i)$ be a zigzag complex corresponding to $zz(x)$. Then, for every $i \in \{1, \dots, n\}$, there exists a homotopy $h\colon C \to C$ such that $h\circ d + d\circ h = f\cdot(\mathrm{id}_{A_i} + \mathrm{id}_{A_{i-1}})$ with $f = S^2$ if d_i is odd (i.e. $d_i = S$ or $d_i = S^2$), and $f = D$ if d_i is even, i.e. $d_i = D$.

Proof. By reindexing C if necessary, we assume w.l.o.g. that d_i is a map $A_{i-1} \to A_i$. If d_i is S, then let h be given by $S\colon A_i \to A_{i-1}$. Then, for $h \circ d_j$ to be non-zero, the target of d_j and the domain of h must match; this happens only if $j = i$, or $j = i+1 \leq n$ and d_{i+1} is a map $A_{i+1} \to A_i$. In the latter case, we nevertheless have $h \circ d_{i+1} = 0$, since h is S and d_{i+1} is D. Similarly, one sees that $d_j \circ h = 0$ unless $j = i$. Overall, we find

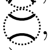
$$h \circ d + d \circ h = \sum_{j=1}^{n} h \circ d_j + d_j \circ h = h \circ d_i + d_i \circ h = S^2 \cdot (\mathrm{id}_{A_i} + \mathrm{id}_{A_{i-1}})$$

as desired. If d_i is not S, denote by e the edge in $zz(x)$ corresponding to d_i. Since e is not a saddle edge, by the previous Lemma 12.9 there is a subgraph Γ satisfying (i), (ii) or (iii). The part of C corresponding to Γ is the following (drawn in black):

$$\begin{array}{ccccccc} A_{i-1} & \overset{d_{i-1}}{\text{——}} & A_{i-2} & \text{——} & \cdots & \overset{d_{i-k+1}}{\text{——}} & A_{i-k} \\ d_i \downarrow \;\; \uparrow h_0 & & \uparrow h_1 & & & & \uparrow h_{k-1} \\ A_i & \underset{d_{i+1}}{\text{——}} & A_{i+1} & \text{——} & \cdots & \underset{d_{i+k-1}}{\text{——}} & A_{i+k-1}. \end{array} \tag{12.4}$$

Let the homotopy h, drawn in (12.4) in red and dashed, be defined as the sum

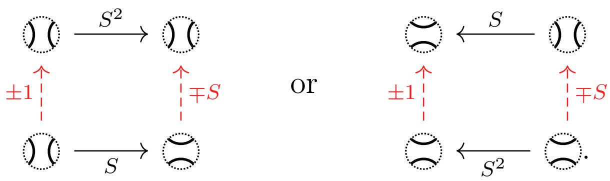
$$h = \sum_{j=0}^{k-1} h_j\colon A_{i+j} \to A_{i-j-1}$$

with h_j equal to $(-1)^{i-j}$ times the identity cobordism if the domain and target of h_j are both  or both ; and h_j equal to $(-1)^{i-j}$ times S if one of the domain and target of h_j is , and the other . Note that the latter case only happens if Γ satisfies (ii) or (iii) and $j = k-1$. Now, $h\circ d + d\circ h$ is equal to the sum of the following terms $\alpha, \beta_j, \gamma_j, \delta$ (all other compositions of h_j and d_k vanish because target and domain do not match):

$$\underbrace{d_i \circ h_0 + h_0 \circ d_i}_{\alpha} \quad + \quad \sum_{j=1}^{k-1} \underbrace{d_{i-j} \circ h_{j-1} + h_j \circ d_{i+j}}_{\beta_j} \quad +$$
$$\sum_{j=1}^{k-1} \underbrace{d_{i-j} \circ h_j + h_{j-1} \circ d_{i+j}}_{\gamma_j} \quad + \quad \underbrace{d_{i-k} \circ h_{k-1} + h_{k-1} \circ d_{i+k}}_{\delta}.$$

Now, observe that α equals $f \cdot (\mathrm{id}_{A_i} + \mathrm{id}_{A_{i-1}})$ with $f = S^2$ if d_i is S^2 and $f = D$ if d_i is D. So it just remains to show that the terms β_j, γ_j and δ are

0. For each j, one of β_j and γ_j is 0 because targets and domains do not match; and the other term is 0 because the squares in (12.4) anticommute (remember that d_{i-j} and d_{i+j} both point to the left, or both to the right). This is also true for the last square in case that Γ satisfies (ii) or (iii), in which case that square respectively looks like



Finally, in case Γ satisfies (i), δ is 0 because targets and domains mismatch. If Γ satisfies (ii) or (iii), δ is 0 either for the same reason, or because h_{k-1} is S and d_{i-j} and d_{i+j} are D. ☐

The following lemma is well-known (see e.g. [Ble85, KWZ21]), and can easily be checked inductively.

Lemma 12.11. For $i \in \{1, 2\}$, let p_i and q_i be coprime integers. Then $R(p_1/q_1)$ and $R(p_2/q_2)$ have the same connectivity if and only if $p_1 \equiv p_2 \pmod 2$ and $q_1 \equiv q_2 \pmod 2$.

Let us call an end of a zigzag graph *even* or *odd* depending on whether the unique edge adjacent to it is even or odd.

Lemma 12.12. Let $p, q \in \mathbb{Q}^+$ be coprime and let $zz(p/q)$ be the corresponding zigzag graph.

1. If both p and q are odd, then the ends of $zz(p/q)$ are given by one of the following configurations:

$$\begin{array}{lllll}
\text{end } \circ \xrightarrow{\text{odd}} \bullet & \cdots & \circ \xrightarrow{\text{odd}} \bullet \text{ end} \\
\text{end } \circ \xrightarrow{\text{odd}} \circ & \cdots & \circ \xrightarrow{\text{odd}} \bullet \text{ end} \\
\text{end } \circ \xrightarrow{\text{odd}} \bullet & \cdots & \bullet \xrightarrow{\text{odd}} \bullet \text{ end} \\
\text{end } \bullet \xleftarrow{\text{odd}} \circ & \cdots & \bullet \xleftarrow{\text{odd}} \circ \text{ end} \\
\text{end } \bullet \xleftarrow{\text{odd}} \circ & \cdots & \circ \xleftarrow{\text{odd}} \circ \text{ end} \\
\text{end } \bullet \xleftarrow{\text{odd}} \bullet & \cdots & \bullet \xleftarrow{\text{odd}} \circ \text{ end}
\end{array} \tag{12.5}$$

2. If p is odd and q even, then the ends of $zz(p/q)$ are given by one of

the following configurations:

$$\begin{array}{ccccccccc}
\text{end} & \circ & \xrightarrow{\text{odd}} & \circ & \cdots & \circ & \xleftarrow{\text{even}} & \circ & \text{end} \\
\text{end} & \circ & \xrightarrow{\text{odd}} & \bullet & \cdots & \circ & \xleftarrow{\text{even}} & \circ & \text{end} \\
\text{end} & \circ & \xrightarrow{\text{odd}} & \bullet & \cdots & \bullet & \xrightarrow{\text{even}} & \bullet & \text{end} \\
\text{end} & \circ & \xrightarrow{\text{even}} & \circ & \cdots & \circ & \xleftarrow{\text{odd}} & \circ & \text{end} \\
\text{end} & \circ & \xrightarrow{\text{even}} & \circ & \cdots & \bullet & \xleftarrow{\text{odd}} & \circ & \text{end} \\
\text{end} & \bullet & \xleftarrow{\text{even}} & \bullet & \cdots & \bullet & \xleftarrow{\text{odd}} & \circ & \text{end}
\end{array} \tag{12.6}$$

3. If p is even and q odd, then the ends of $zz(p/q)$ are given by one of the following configurations:

$$\begin{array}{ccccccccc}
\text{end} & \circ & \xrightarrow{\text{even}} & \circ & \cdots & \circ & \xrightarrow{\text{odd}} & \bullet & \text{end} \\
\text{end} & \bullet & \xleftarrow{\text{odd}} & \bullet & \cdots & \bullet & \xrightarrow{\text{even}} & \bullet & \text{end} \\
\text{end} & \bullet & \xleftarrow{\text{odd}} & \circ & \cdots & \bullet & \xrightarrow{\text{even}} & \bullet & \text{end} \\
\text{end} & \bullet & \xleftarrow{\text{odd}} & \circ & \cdots & \circ & \xleftarrow{\text{even}} & \circ & \text{end} \\
\text{end} & \bullet & \xleftarrow{\text{even}} & \bullet & \cdots & \bullet & \xrightarrow{\text{odd}} & \bullet & \text{end} \\
\text{end} & \bullet & \xleftarrow{\text{even}} & \bullet & \cdots & \circ & \xrightarrow{\text{odd}} & \bullet & \text{end}
\end{array} \tag{12.7}$$

Proof. We advise the reader who wishes to go through the proof in detail to have a printed copy of Table 12.1 at hand. Let us start with the following claim.

Claim: Given $x \in \mathbb{Q}^+$ with $x \neq 1$, the only edges appearing in $zz(x)$ that connect an end are among the following:

$$\begin{array}{ccccccccc}
\text{end} & \circ & \xrightarrow{\text{odd}} & \bullet & \qquad & \circ & \xrightarrow{\text{odd}} & \bullet & \text{end} \\
\text{end} & \bullet & \xleftarrow{\text{odd}} & \circ & \qquad & \bullet & \xleftarrow{\text{odd}} & \circ & \text{end}
\end{array} \tag{12.8}$$

$$\begin{array}{ccccccccc}
\text{end} & \circ & \xrightarrow{\text{even}} & \circ & \qquad & \circ & \xleftarrow{\text{even}} & \circ & \text{end} \\
\text{end} & \circ & \xrightarrow{\text{odd}} & \circ & \qquad & \circ & \xleftarrow{\text{odd}} & \circ & \text{end}
\end{array} \tag{12.9}$$

$$\begin{array}{ccccccccc}
\text{end} & \bullet & \xleftarrow{\text{even}} & \bullet & \qquad & \bullet & \xrightarrow{\text{even}} & \bullet & \text{end} \\
\text{end} & \bullet & \xleftarrow{\text{odd}} & \bullet & \qquad & \bullet & \xrightarrow{\text{odd}} & \bullet & \text{end}
\end{array} \tag{12.10}$$

We prove the claim by induction on the steps necessary to reach x from 1 using the transformations $y \mapsto 1/y$ and $y \mapsto y+1$. The claim is clearly true for $x = 1$, since $zz(1)$ only consists of an odd saddle edge. So suppose that the statement is true for $zz(x)$. In order to show that it also holds for $zz(x+1)$, we have to examine Table 12.1. It is not difficult to see that the only new edges introduced in $zz(x+1)$ connecting an end are either saddle edges, or one of the following:

$$\text{end } \circ \xrightarrow{\text{even}} \circ \qquad \circ \xleftarrow{\text{even}} \circ \text{ end}$$

$$\text{end } \circ \xrightarrow{\text{odd}} \circ \qquad \circ \xleftarrow{\text{odd}} \circ \text{ end}$$

This shows that the claim also holds for $zz(x+1)$. Now recall that one obtains $zz(1/x)$ from $zz(x)$ by switching $\circ$ and $\bullet$, and reversing the direction of all edges. In particular, now new edges are introduced. This means that saddle edges connecting an end remain saddle edges connecting an end in $zz(1/x)$. Furthermore, the edges in (12.10) are precisely the edges in (12.9) obtained under the transformation $x \mapsto 1/x$, and vice-versa. Thus the claim also holds for $zz(1/x)$.

The statement is clearly true for $zz(1)$, so suppose it holds for $zz(x)$. The transformation $x \mapsto 1/x$ switches the roles and with it the parity of p and q. Now observe that the configurations in (12.5) are clearly closed under switching $\circ$ and $\bullet$ and reversing edges. On the other hand, doing the same on any case in (12.6) yields a case in (12.7) and vice-versa, which corresponds to the interchanged roles and parities of p and q under the transformation $x \mapsto 1/x$. Hence the statement holds for $zz(1/x)$ as well.

It remains to show the statement for $x \mapsto x+1$. Note that this will also show that (12.5) to (12.7) are the only possible configurations appearing. Let $x = p/q$ with $p, q \in \mathbb{Q}^+$ coprime. In order to show that the statement holds for $zz(x+1)$, we have to check depending on the parity of p and q what edges adjacent to ends can appear, and how they get replaced in Table 12.1. The possible configurations of parities and ends in $zz(x)$ and $zz(x+1)$ respectively are summarized in the table below, which is taken from the proof of [ILM21, Lemma 5.12]:

parities of p, q	ends of $zz(x)$	parities of $p+q, q$	ends of $zz(x+1)$
odd, odd	odd $\circ$, odd $\bullet$	even, odd	even $\circ$, odd $\bullet$
even, odd	even $\circ$, odd $\bullet$	odd, odd	odd $\circ$, odd $\bullet$
even, odd	even $\bullet$, odd $\bullet$	odd, odd	odd $\circ$, odd $\bullet$
odd, even	even $\bullet$, odd $\circ$	odd, even	even $\circ$, odd $\circ$
odd, even	even $\circ$, odd $\circ$	odd, even	even $\circ$, odd $\circ$

In what follows we will list zigzag graphs up to rotation by 180 degrees. Suppose first that $x+1 = (p+q)/q$ with both $p+q$ and q odd, so that p is even and q odd. By the claim we know the possible configurations of edges connecting these ends, and Table 12.1 tells us how they get replaced:

$zz(x)$ with p even, q odd	$zz(x+1)$ with $p+q$ odd, q odd
end $\circ \xrightarrow{\text{even}} \circ \cdots \circ \xrightarrow{\text{odd}} \bullet$ end	end $\circ \xrightarrow{\text{odd}} \circ \cdots \circ \xrightarrow{\text{odd}} \bullet$ end
end $\circ \xrightarrow{\text{even}} \circ \cdots \bullet \xrightarrow{\text{odd}} \bullet$ end	end $\circ \xrightarrow{\text{odd}} \circ \cdots \bullet \xleftarrow{\text{odd}} \circ$ end
end $\bullet \xleftarrow{\text{odd}} \circ \cdots \bullet \xrightarrow{\text{even}} \bullet$ end	end $\bullet \xleftarrow{\text{odd}} \circ \cdots \bullet \xleftarrow{\text{odd}} \circ$ end
end $\bullet \xleftarrow{\text{odd}} \bullet \cdots \bullet \xrightarrow{\text{even}} \bullet$ end	end $\bullet \xleftarrow{\text{odd}} \circ \cdots \bullet \xleftarrow{\text{odd}} \circ$ end

Observe that the second entry in the table cannot appear, as it results in two odd $\circ$-ends. All the above configurations of edges connecting ends in $zz(x+1)$ are listed in (12.5), proving the statement.

For the other cases we proceed similarly. Assume that $x+1 = (p+q)/q$ with $p+q$ even and q odd. Then both p and q are odd, and by the previous case we simply have to check how the edges in (12.5) get replaced:

$zz(x)$ with p odd, q odd	$zz(x+1)$ with $p+q$ even, q odd
end $\circ \xrightarrow{\text{odd}} \bullet \cdots \circ \xrightarrow{\text{odd}} \bullet$ end	end $\circ \xrightarrow{\text{even}} \circ \cdots \circ \xrightarrow{\text{odd}} \bullet$ end
end $\circ \xrightarrow{\text{odd}} \circ \cdots \circ \xrightarrow{\text{odd}} \bullet$ end	end $\circ \xrightarrow{\text{even}} \circ \cdots \circ \xrightarrow{\text{odd}} \bullet$ end
end $\circ \xrightarrow{\text{odd}} \bullet \cdots \bullet \xrightarrow{\text{odd}} \bullet$ end	end $\circ \xrightarrow{\text{even}} \circ \cdots \circ \xrightarrow{\text{odd}} \bullet$ end

Finally, suppose that $p+q$ is odd and q even. Then p odd and q even, and we have to check again two cases. Assume first that $zz(p/q)$ has an even $\bullet$-end and an odd $\circ$-end. Then using the claim and Table 12.1, we get the following configurations and replacements:

$zz(x)$ with p odd, q even	$zz(x+1)$ with $p+q$ odd, q even
end $\circ \xrightarrow{\text{odd}} \bullet \cdots \bullet \xrightarrow{\text{even}} \bullet$ end	end $\circ \xrightarrow{\text{even}} \circ \cdots \bullet \xleftarrow{\text{odd}} \circ$ end
end $\circ \xrightarrow{\text{odd}} \circ \cdots \bullet \xrightarrow{\text{even}} \bullet$ end	end $\circ \xrightarrow{\text{even}} \circ \cdots \bullet \xleftarrow{\text{odd}} \circ$ end
end $\circ \xrightarrow{\text{odd}} \bullet \cdots \circ \xleftarrow{\text{even}} \circ$ end	end $\circ \xrightarrow{\text{even}} \circ \cdots \circ \xleftarrow{\text{odd}} \circ$ end
end $\circ \xrightarrow{\text{odd}} \circ \cdots \circ \xleftarrow{\text{even}} \circ$ end	end $\circ \xrightarrow{\text{even}} \circ \cdots \circ \xleftarrow{\text{odd}} \circ$ end

All these configurations are listed in (12.6), thus completing the proof. $\square$

The upcoming Proposition 12.13 is the heart of the proof of Theorem 7.4. It is the analog of Lemmas 3.1 and 3.2 in [AD19], and a generalization of [ILM21, Lemma 5.13], which only dealt with the case p and q odd.

Proposition 12.13. Let $p, q \in \mathbb{Q}^+$ be coprime. Let $(C, \bigoplus_{i=0}^n A_n, \sum_{i=1}^n m_i)$ denote the zigzag complex corresponding to $zz(p/q)$. Further, let (C', d') be the complex

$C'_0 =$  $\xrightarrow{d'=S}$  $\{1\} = C'_1,$

which is (up to global shifts) the Bar-Natan complex of $R(-1)$ equipped with some orientation, coming from a negative crossing. Then there exist ungraded chain maps $f\colon C \to C'$ and $g\colon C' \to C$ such that:

1. If both p and q are odd, then

$$f \circ g \simeq G \cdot \mathrm{id}_{C'}, \quad g \circ f \simeq G \cdot \mathrm{id}_C$$

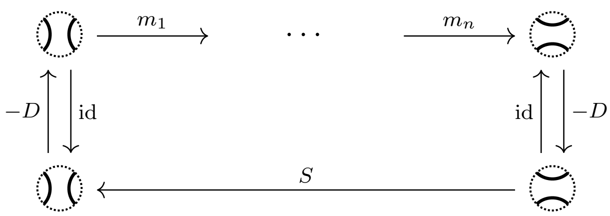
2. If p is odd and q is even, then

$$f \circ g \simeq \begin{pmatrix} S^2 & 0 \\ 0 & G \end{pmatrix}, \quad g \circ f \simeq D \cdot \mathrm{id}_C$$

3. If p is even and q odd, then

$$f \circ g \simeq \begin{pmatrix} G & 0 \\ 0 & S^2 \end{pmatrix}, \quad g \circ f \simeq D \cdot \mathrm{id}_C$$

Proof. Suppose first that both p and q are odd. Then by Lemma 12.12, $zz(p/q)$ has an odd ∘-end and an odd •-end, and one end has an incoming edge and the other has an outgoing edge. By reindexing if necessary, we may assume that A_0 corresponds to the odd ∘-end and A_n to the odd •-end in $zz(p/q)$. Then again by Lemma 12.12, A_0 has an outgoing and A_n an incoming edge, with $m_1, m_n \in \{S, S^2\}$. Let us define $f\colon C \to C'$ and $g\colon C' \to C$ as follows:



Here we define

$$\begin{aligned} f &:= (\mathrm{id}\colon A_0 \to C_1') \ + \ (-D\colon A_n \to C_0') \\ g &:= (\mathrm{id}\colon C_0' \to A_n) \ + \ (-D\colon C_1' \to A_0) \end{aligned}$$

The map f is set to be zero on summands other than A_0 and A_n. Clearly, f and g are ungraded chain maps. Moreover,

$$f \circ g = -D \cdot \mathrm{id}_{C'}, \quad g \circ f = -D \cdot (\mathrm{id}_{A_0} + \mathrm{id}_{A_n})$$

Let $h'\colon C' \to C'$ be the homotopy defined by $S\colon C_1' \to C_0'$. Then

$$d' \circ h' + h' \circ d' = G \cdot \mathrm{id}_{C'} - f \circ g,$$

so $f \circ g$ is homotopic to multiplication with G as desired. Regarding $g \circ f$, we apply Lemma 12.10 to obtain for every $i \in \{1, \dots, n\}$ a homotopy

$h_i \colon C \to C$ such that $d \circ h + h \circ d = u \cdot (\mathrm{id}_{A_i} + \mathrm{id}_{A_{i-1}})$ where $u = S^2$ if m_i is odd and $u = D$ if m_i is even. Define

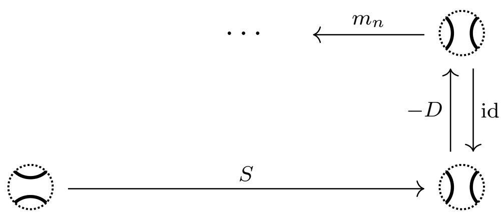
$$h = \sum_{i=1}^{n} (-1)^{i+1} h_i$$

Then it is easy to see that

$$d \circ h + h \circ d = G \cdot \mathrm{id}_C - g \circ f,$$

showing that $g \circ f$ is homotopic to multiplication with G as desired.

Next, suppose that p is odd and q is even. Then by Lemma 12.12, $zz(p/q)$ has an even end and an odd $\circ$-end. By reindexing C if necessary, we may assume that A_0 corresponds to the even end, and that A_n corresponds to the odd $\circ$-end. In order to define $f \colon C \to C'$ and $g \colon C' \to C$, we will only be interested in A_n, which by Lemma 12.12 has an outgoing edge with $m_n \in \{S, S^2\}$. Consider



Here we define

$$f \coloneqq \mathrm{id} \colon A_n \to C'_1 \quad g \coloneqq -D \colon C'_1 \to A_n$$

and set f and g to be zero on other summands. Clearly f and g are ungraded chain maps, and

$$f \circ g = -D \cdot \mathrm{id}_{C'_1}, \quad g \circ f = -D \cdot \mathrm{id}_{A_n}$$

Let $h' \colon C' \to C'$ be the homotopy defined by $S \colon C'_1 \to C'_0$. Then

$$d' \circ h' + h' \circ d' = S^2 \cdot \mathrm{id}_{C'_0} + G \cdot \mathrm{id}_{C'_1} - f \circ g$$

as desired. Now since $zz(p/q)$ has an even and an odd end, we have by Lemma 12.10 that for every $i \in \{1, \dots, n/2\}$ there exists a homotopy $h_{2i-1} \colon C \to C$ such that

$$h_{2i-1} \circ d + d \circ h_{2i-1} = D \cdot (\mathrm{id}_{A_{2i-1}} + \mathrm{id}_{A_{2i-2}}).$$

Define $h \colon C \to C$ as

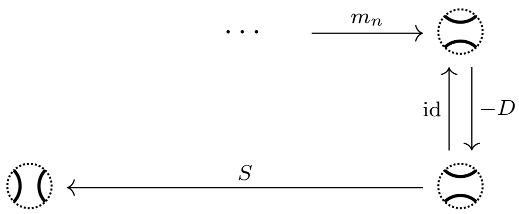
$$h = h_0 + h_2 + \cdots h_{n-2} = \sum_{i=1}^{n/2} h_{2i-2}.$$

Then

$$d \circ h + h \circ d = D \cdot \mathrm{id}_C - g \circ f$$

as desired.

Finally, suppose that p is even and q odd. Then by Lemma 12.12, $zz(p/q)$ has an even end and an odd $\bullet$-end. By reindexing C if necessary, we may assume that A_0 corresponds to the even end, and that A_n corresponds to the odd $\bullet$-end. In order to define $f\colon C \to C'$ and $g\colon C' \to C$, we will only be interested in A_n, which by Lemma 12.12 has an incoming edge with $m_n \in \{S, S^2\}$. Consider



Here we define

$$f \coloneqq -D\colon A_n \to C'_0 \quad g \coloneqq \mathrm{id}\colon C'_0 \to A_n$$

and set f and g to be zero on other summands. Clearly f and g are ungraded chain maps, and

$$f \circ g = -D \cdot \mathrm{id}_{C'_0}, \quad g \circ f = -D \cdot \mathrm{id}_{A_n}.$$

Let $h'\colon C' \to C'$ be the homotopy defined by $S\colon C'_1 \to C'_0$. Then

$$d' \circ h' + h' \circ d' = S^2 \cdot \mathrm{id}_{C'_1} + G \cdot \mathrm{id}_{C'_0} - f \circ g$$

as desired. Similar to the previous case, since $zz(p/q)$ has an even and an odd end, we have by Lemma 12.10 that for every $i \in \{1, \ldots, n/2\}$ there exists a homotopy $h_{2i-1}\colon C \to C$ such that

$$h_{2i-1} \circ d + d \circ h_{2i-1} = D \cdot (\mathrm{id}_{A_{2i-1}} + \mathrm{id}_{A_{2i-2}}).$$

Define $h\colon C \to C$ as

$$h = h_0 + h_2 + \cdots h_{n-2} = \sum_{i=1}^{n/2} h_{2i-2}.$$

Then

$$d \circ h + h \circ d = D \cdot \mathrm{id}_C - g \circ f$$

completing the proof. □

Remark 12.14. Proposition 12.13 is a generalization of [ILM21, Lemma 5.13] which dealt with the case p and q odd. However, the proof contained a gap: it assumed that the odd $\bullet$-end has *no* outgoing edge. This gap is now closed with the new Lemma 12.12, and the proof of Proposition 12.13 provides the correct arguments.

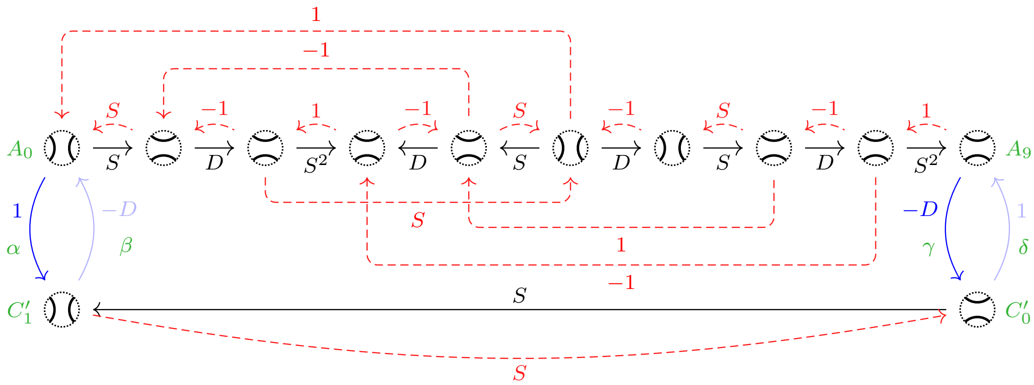


Figure 12.3: Illustration of the proof of Proposition 12.13. In the top row, a zigzag complex C with graph $zz(3/7)$ (compare Figure 12.2). On the bottom row, the complex C'. In light and dark blue, the ungraded chain maps $f\colon C \to C'$ (going down) and $g\colon C' \to C$ (going up). Red and dashed, the required homotopies. Homological and quantum degree shifts are omitted from the diagram.

We now need to examine rational replacements (first seen in Definition 7.6) more closely.

Definition 12.15. Two unoriented link $L, L' \subset S^3$ are related by a *rational replacement* if, after an isotopy, there exists a ball $B \subset S^3$ whose boundary sphere intersects L and L' transversely, such that $L \setminus B^\circ = L' \setminus B^\circ$, and the two tangles $T = L \cap B$ and $T' = L' \cap B$ are rational. If T and T' have the same connectivity, we say that the rational replacement is *proper*. If there is a homeomorphism between B and the unit ball that sends T to $R(x)$ and T' to $R(y)$ for some $x, y \in \mathbb{Q} \cup \{\infty\}$, we speak of an *$x$ by y rational replacement*.

It is a frequently used fact that a crossing change may be seen as -1 by 1 rational replacement, but also as 0 by 2 rational replacement. The following lemma generalizes this.

Lemma 12.16. Let S, T be rational tangles in a ball B, let $x, y \in \mathbb{Q} \cup \{\infty\}$, and let φ be a homeomorphism of B to the unit ball B_0 such that $\varphi(S) = R(x)$ and $\varphi(T) = R(y)$. Then there exists $z \in \{-1\} \cup [0, \infty)$, and a homeomorphism $\varphi'\colon B \to B_0$ such that $\varphi'(S) = R(-1)$ and $\varphi'(T) = R(z)$.

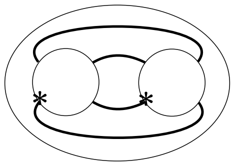
Proof. Let ψ_1 be a homeomorphism of B_0 with $\psi_1(R(x)) = R(\infty)$. Then $\psi_1(R(y)) = R(y')$ for some y'. Let ψ_2 be a homeomorphism of B_0 such that $\psi_2(R(\infty)) = R(\infty)$ and $\psi_2(R(y')) = R(y'')$ with $y'' \in (0, 1] \cup \{\infty\}$. Such a ψ_2 may be constructed by adding a certain number of twists to the right side of the ball. Finally, let ψ_3 be the homeomorphism of B_0 that sends $R(w)$ to $R(1/w - 1)$ for all $w \in \mathbb{Q} \cup \{\infty\}$. Then $\psi_3 \circ \psi_2 \circ \psi_1 \circ \varphi(S) = R(-1)$ and $\psi_3 \circ \psi_2 \circ \psi_1 \circ \varphi(T) = R(z)$ for $z \in \{-1\} \cup [0, \infty)$, as desired. □

Proposition 10.15. Fix a ball with four end points on it, one of them distinguished as base point. On the set of equivalence classes of unoriented rational tangles in that ball with fixed connectivity, the pseudometric given by λ is in fact equal to the discrete metric. That is to say, $\lambda(S, T) = 1$ for inequivalent rational tangles S and T.

Proof. By Lemma 12.16, there exists $x \in \{-1\} \cup [0, \infty)$ and a homeomorphism that sends S to $R(-1)$ and T to $R(x)$. Since S and T are not equivalent, we have $x \neq -1$. Since the connectivities of S and T are the same, Lemma 12.11 implies that $x = p/q$ with both p and q odd (in particular, $p/q \neq 0$). By Proposition 10.11, λ is equivariant under homeomorphisms, and so we have $\lambda(S, T) = \lambda(R(-1), R(x))$. So it will be sufficient to show that $\lambda(R(-1), R(x)) = 1$.

By Theorem 12.7, $[R(x)]^\bullet$ is homotopy equivalent to a zigzag complex C with graph $zz(x)$. By Proposition 12.13, there are ungraded chain maps $f\colon [R(-1)]^\bullet \to C$ and $g\colon C \to [R(-1)]^\bullet$ with $g \circ f \simeq G \cdot \mathrm{id}_{[R(-1)]^\bullet}$ and $f \circ g \simeq G \cdot \mathrm{id}_C$, showing $\lambda(R(-1), R(x)) \leq 1$.

Let $\mathcal{D}$ be the following 2-input planar arc diagram:



Then $\mathcal{D}(R(-1), R(2))$ is the unknot, and $\mathcal{D}(R(x), R(2))$ is the two-bridge knot K corresponding to $x + 2 = (p + 2q)/q$. Since $x + 2 > 1$, this is a non-trivial knot, and so we have $\lambda(K) > 0$ because λ detects the unknot (see Proposition 7.7).[1] Overall, using Lemma 10.8, we get

$$\lambda(R(-1), R(x)) \geq \lambda(\mathcal{D}(R(-1), R(2)), \mathcal{D}(R(x), R(2))) = \lambda(K) > 0.$$

This concludes the proof. □

Let us restate and prove our main theorem.

Theorem 7.4. For all knots K, one has $\lambda(K) \leq u_q(K)$.

Proof. To show $\lambda(K) \leq u_q(K)$, it is sufficient to show the following: if two knots K and J are related by a proper rational replacement, then $\lambda(K, J) \leq 1$. So let knots K, J related by a proper rational replacement be given. By definition, there exists a 4-ended tangle T, such that K is the union of T with a rational tangle S, and J is the union of T with a another rational tangle S'. Since the replacement is proper, S and S' have the same connectivity. So $\lambda(S, S') \leq 1$ by Proposition 10.15, and thus $\lambda(K, J) \leq 1$ by Proposition 10.13. □

[1]Here, we do not even need Kronheimer-Mrowka's theorem that Khovanov homology detects the unknot, but only the (much easier) theorem that Khovanov homology detects the unknot among two-bridge knots (in fact, already the Jones polynomial can be seen to accomplish that).

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