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Fractional-order operators on nonsmooth domains

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Abstract

The fractional Laplacian $(-\Delta)^a$, $a \in (0,1)$, and its generalizations to variable-coefficient $2a$-order pseudodifferential operators P, are studied in L_q-Sobolev spaces of Bessel-potential type H_q^s. For a bounded open set $\Omega \subset \mathbb{R}^n$, consider the homogeneous Dirichlet problem: $Pu = f$ in Ω, $u = 0$ in $\mathbb{R}^n \setminus \Omega$. We find the regularity of solutions and determine the exact Dirichlet domain $D_{a,s,q}$ (the space of solutions u with $f \in H_q^s(\overline{\Omega})$) in cases where Ω has limited smoothness $C^{1+\tau}$, for $2a < \tau < \infty$, $0 \leqslant s < \tau - 2a$. Earlier, the regularity and Dirichlet domains were determined for smooth Ω by the second author, and the regularity was found in low-order Hölder spaces for $\tau = 1$ by Ros-Oton and Serra. The H_q^s-results obtained now when $\tau < \infty$ are new, even for $(-\Delta)^a$. In detail, the spaces $D_{a,s,q}$ are identified as a-transmission spaces $H_q^{a(s+2a)}(\overline{\Omega})$, exhibiting estimates in terms of $\operatorname{dist}(x, \partial\Omega)^a$ near the boundary.

The result has required a new development of methods to handle nonsmooth coordinate changes for pseudodifferential operators, which have not been available before; this constitutes another main contribution of the paper.

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1 INTRODUCTION

The present work has two main purposes. One is to solve the regularity question and determine the domain for the Dirichlet problem for $(-\Delta)^a$ and its fractional-order generalizations, in L_q-Sobolev spaces over domains of finite smoothness degrees between $C^{1,\sigma}$ and C^∞. The other is to develop a tool that has been missing in the theory of pseudodifferential operators: nonsmooth coordinate changes. It plays an important role in the solution of the regularity question.

The fractional Laplacian $(-\Delta)^a$, $0 < a < 1$, and its generalizations P of the same order $2a$, have been much studied in recent years, with applications in probability, finance, differential geometry, and mathematical physics. To mention some of the studies through the times: Blumenthal and Getoor [13], Hoh and Jacob [37], Kulczycki [43], Chen and Song [19], Jakubowski [41], Bogdan, Burdzy, and Chen [14], Cont and Tankov [20], Caffarelli and Silvestre [18], Gonzales, Mazzeo, and Sire [25], Ros-Oton and Serra [50, 51], Abatangelo [1], Felsinger, Kassmann, and Voigt [23], Bonforte, Sire, and Vazquez [15], Dipierro, Ros-Oton, and Valdinoci [21], and Abatangelo, Jarohs, and Saldana [2]. They refer to many more works, also with applications to nonlinear problems. From its action on $\mathbb{R}^n$ one defines the *homogeneous Dirichlet problem:*

$$Pu = f \text{ on } \Omega, \quad \operatorname{supp} u \subset \overline{\Omega}, \tag{1.1}$$

on bounded open subset Ω of $\mathbb{R}^n$ (with some boundary regularity). For operators P with $\operatorname{Re} \int_\Omega Pu\, \bar{u}\, dx > 0$, there is unique solvability for $f \in L_2(\Omega)$ by a variational argument. One of the fundamental questions in then: How do the solutions look? The variational theory gives that the solution belongs to H^a-functions supported in $\overline{\Omega}$, but is u in fact more regular? And will higher regularity of f increase that of u? This is often called *the regularity question.* Early results of Vishik and Eskin (see [22]) imply, for example, for $a \geqslant \frac{1}{2}$ that u is $H^{a+\frac{1}{2}-\varepsilon}$. More precisely, one can ask: What is the *Dirichlet domain* for P; the space of functions u solving (1.1), when f runs through a Sobolev space $H^s(\overline{\Omega})$?

It was an important step forward when Ros-Oton and Serra [50] showed that for $f \in L_\infty(\Omega)$, u is Hölder-continuous with a singularity $d(x)^a$ at the boundary (where $d(x) \sim \operatorname{dist}(x, \partial\Omega)$ near $\partial\Omega$, extended smoothly to a positive function on Ω),

$$f \in L_\infty(\Omega) \Rightarrow u/d^a \in C^\alpha(\overline{\Omega}), \tag{1.2}$$

for small $\alpha > 0$. Ω was assumed to be $C^{1,1}$; a later study lifted α up to a. The methods were delicate potential-theoretic arguments, based on the representation of $(-\Delta)^a$ as a real singular integral operator; they were later extended to other real translation-invariant singular integral operators.

A very different method was introduced by one of the present authors [31]: Fourier analysis in the form of pseudodifferential operator (ψdo) theory. This theory (necessarily for complex functions) is designed to allow x-dependent operators (not translation-invariant), taking care of the composition rules that arise, which make the theory quite technical. Here it was shown when Ω is a C^∞-domain that if, say, $u \in \dot{H}^a_q(\overline{\Omega})$ and $s \geqslant 0$,

$$f \in C^\infty(\overline{\Omega}) \iff u \in d^a C^\infty(\overline{\Omega}), \tag{1.3}$$

$$f \in H^s_q(\overline{\Omega}) \iff u \in D_{a,s,q}, \tag{1.4}$$

where $D_{a,s,q}$ is a certain space contained in $\dot{H}_q^{s+2a}(\overline{\Omega}) + d^a H_q^{s+a}(\overline{\Omega})$ for $s + a - \frac{1}{q} > 0$ (and $\notin \mathbb{Z}$), $1 < q < \infty$; here H_q^s denotes the L_q-Sobolev space of Bessel-potential type. For $0 \leqslant s < \frac{1}{q} - a$, $D_{a,s,q} = \dot{H}_q^{s+2a}(\overline{\Omega})$ (the $H_q^{s+2a}(\mathbb{R}^n)$-functions supported in $\overline{\Omega}$).

In detail, the space $D_{a,s,q}$ equals the so-called a-transmission space $H_q^{a(s+2a)}(\overline{\Omega})$ introduced in [31]. (1.4) was in [30] extended to a wealth of other scales of function spaces, including Hölder–Zygmund spaces C_*^s; here $f \in C^s(\overline{\Omega})$ implies $u \in \dot{C}^{s+2a}(\overline{\Omega}) + d^a C^{s+a}(\overline{\Omega})$ (when $s, s+a, s+2a \notin \mathbb{N}$), the component in $d^a C^{s+a}(\overline{\Omega})$ described more precisely in [34].

There is a large gap between the results (1.2) and (1.4), in that the former allows low smoothness of the domain Ω and correspondingly shows smoothness of u in a low range, whereas the latter, under a very high smoothness assumption on Ω, shows results in the full scale $s \geqslant 0$.

This gap remained open until recently. Ros-Oton with coauthor Abatangelo presented a study [3] treating the regularity question for translation-invariant real singular integral operators, lifting (1.2) to $\alpha = a + s$ when $f \in C^s(\overline{\Omega})$, Ω is $C^{1+\tau}$ with $\tau > a$, for $0 < s \leqslant \tau - a$ with $s, a+s, 2a+s \notin \mathbb{N}$. This allows τ to be a step a lower than we assume below, but does not exhibit a more precise domain, and does not treat Sobolev spaces. Abatangelo and Ros-Oton[3] give important consequences for regularity questions for the obstacle problem.

One may also compare with the standard knowledge for second order strongly elliptic *differential* operators (cf. Gilbarg–Trudinger [24, Theorem 8.13]), where $a = 1$: When Ω is a C^{2+s}-domain, $s \in \mathbb{N}_0$, the Dirichlet solutions for $f \in \overline{H}^s(\Omega)$ lie in $\overline{H}^{s+2}(\Omega) \cap \dot{H}^1(\overline{\Omega})$; this allows a first step lower regularity of Ω than we do.

We shall in the present paper fill the gap in the category of L_q-Sobolev spaces, showing that the characterizations (1.4) can also be obtained when Ω has finite smoothness $C^{1+\tau}$, for s in a finite interval whose upper bound follows τ linearly. This is the first treatment of the fractional-order operators acting in L_q-Sobolev spaces over domains with limited but high smoothness, giving correspondingly high regularity of solutions.

In this connection there enters a condition on the behavior of the operator P at $\partial\Omega$, the a-transmission condition, known from [31] to be necessary for (1.3) on smooth domains. Part of our work consists in finding the appropriate generalization of this condition to nonsmooth domains, as well as generalizing the 0-transmission condition of Boutet de Monvel [16, 17] and of Grubb and Hörmander [36]. The a-transmission condition is satisfied in all directions when P of order $2a$ is *even*, that is, its symbol $p(x, \xi)$ has a graded symmetry property in ξ, cf. Section 2.2 below. This holds for $(-\Delta)^a$. We can now show a result that is new even for $(-\Delta)^a$:

Theorem 1.1. *Let $1 < q < \infty$, $0 < a < 1$, $\tau > 2a$, let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1+\tau}$-domain, and let P be an **even** pseudodifferential operator of order $2a$ with symbol depending C^τ on x (that is, $p \in C^\tau S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$, cf. Section 2.2 below).*

There is a family of spaces $D_{a,s,q}$, equal to the a-transmission spaces $H_q^{a(s+2a)}(\overline{\Omega})$ (cf. Definition 4.3 below), such that the following holds:

1° $r^+P : u \mapsto (Pu)|_\Omega$ maps $D_{a,s,q}$ continuously into $H_q^s(\overline{\Omega})$ for $0 \leqslant s < \tau - 2a$.

*2° Let P moreover be strongly elliptic. If $u \in \dot{H}_q^a(\overline{\Omega})$ is a solution of the homogeneous Dirichlet problem (1.1), then (1.4) holds for $0 \leqslant s < \tau - 2a$. This shows that $D_{a,s,q}$ is **the Dirichlet domain** for P with data in $H_q^s(\overline{\Omega})$:*

$$D_{a,s,q} = \{u \in \dot{H}_q^a(\overline{\Omega}) \mid (Pu)|_\Omega \in H_q^s(\overline{\Omega})\}. \tag{1.5}$$

The definition of the a-transmission spaces $H_q^{a(t)}(\overline{\Omega})$ over $C^{1+\tau}$-domains Ω ($t < \tau + 1$) involves the order-reducing operators Ξ_+^a connected with the a-transmission condition. Here $H_q^{a(s+2a)}(\overline{\Omega})$ equals $\dot{H}_q^{s+2a}(\overline{\Omega})$ if $s < \frac{1}{q} - a$, and is, if $\tau \geqslant 1$, included in $\dot{H}_q^{s+2a}(\overline{\Omega}) + d^a H_q^{s+a}(\overline{\Omega})$ for larger s, with $d \in C^{1+\tau}(\overline{\Omega}, \mathbb{R}_+)$ equal to the distance to $\partial\Omega$ in a neighborhood of $\partial\Omega$. The condition $\tau \geqslant 1$ assures that d is differentiable in a neighborhood of $\partial\Omega$ and normal coordinates exist. In cases $2a < \tau < 1$, which can only occur if $a < \frac{1}{2}$, there is a local interpretation of the inclusion, cf. Remark 4.7.

We show furthermore that for $u \in H_q^{a(s+2a)}(\overline{\Omega})$, the function u/d^a has boundary value in $B_q^{s+a-\frac{1}{q}}(\partial\Omega)$ when $s + a - \frac{1}{q} > 0$, cf. Theorem 4.6 below.

It is a new result that the exact Dirichlet domains $D_{a,s,q}$ have been found in cases where Ω has limited smoothness. A remarkable fact is that these Dirichlet domains are *universal*, depending on a, s, and q, but not on the symbol of P within the class of even, strongly elliptic ψdo's with C^τ-smooth symbols.

Our analysis has a number of other new applications for nonsmooth domains, for example, (1) solution of evolution problems by functional analysis as in [33], where the determination of the Dirichlet domain leads to precise results, and (2) solution of nonhomogeneous boundary value problems with local Dirichlet conditions as in [30]. They are worked out in [35].

At the end, we include some consequences in Hölder and Hölder–Zygmund spaces that follow by use of embedding theorems by letting $q \to \infty$; this expands the scope of [3] to our classes of pseudodifferential operators.

It would of course be interesting to extend the general principles to other scales of function spaces, namely, the Triebel–Lizorkin spaces $F_{p,q}^s$ and the Besov spaces $B_{p,q}^s$ (including Hölder–Zygmund spaces $B_{\infty,\infty}^s = C_*^s$), as it was done in smooth cases in [30] after the treatment in Bessel potential spaces H_q^s in [31]. But even for integer-order operators, the basic results [4] on pseudo-differential boundary problems with nonsmooth x-dependence have so far only been established in H_q^s-spaces (plus immediate consequences by interpolation), so a substantial additional work would be required.

The new results are based on a development of pseudodifferential theory that makes non-smooth coordinate changes possible beyond the principal symbol level; this is the other main contribution from the present work.

Recall that pseudodifferential operators were originally developed from singular integral operators as a systematic calculus (containing differential operators) that could handle compositions of x-dependent operators and constructions of inverses in elliptic cases, by use of the mechanisms of Fourier transformation $\mathcal{F}$ (Seeley [54], Kohn and Nirenberg [42], Hörmander [38], and others). From a symbol $p(x,\xi)$ depending smoothly on $x, \xi \in \mathbb{R}^n$ (except possibly at $\xi = 0$) one defines the operator $P = \mathrm{OP}(p(x,\xi))$ by

$$Pu(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\mathcal{F}u(\xi)\, d\xi, \tag{1.6}$$

for nice functions u, extended to general u as so-called oscillatory integrals.

Theories for boundary value problems for ψdo's, resembling those for differential operators, were soon set up in Boutet de Monvel [16, 17] and further developed in, for example, Rempel–Schulze [49], Grubb [26, 28], acting on C^∞-domains $\Omega \subset \mathbb{R}^n$. The definitions of appropriate symbol classes were focused at first on the behavior with respect to ξ, but after some years, non-smooth behavior in x was also introduced (cf., for example, Kumano-go and Nagase [45], Beals and Reed [12], Marschall [46, 47], Witt [58], or Taylor [56, 57]). Presently, we focus on symbols $p(x,\xi)$ in classes $C^\tau S^m_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$, with C^τ-smoothness in x and standard estimates in ξ. For boundary value problems, a nonsmooth generalization of the calculus of Boutet de Monvel for integer-order ψdo's was worked out in Abels [4].

In the study of boundary value problems, one needs not only the x-dependence in the symbol to be less smooth than C^∞; one also needs to be able to apply the theory to domains Ω with nonsmooth boundary. When localization techniques are used, this means that one needs to perform changes-of-variables with nonsmooth transition functions. For some questions it is sufficient to do this only at the principal symbol level, with estimates for the remainder operators (since the result is possibly sought in low-regularity spaces anyway); this has been done, for example, in applications to the Navier–Stokes problem (cf. Abels [5], with Terasawa [11]) and spectral theory (cf. Abels, Grubb, and Wood [7]). Invariance under smooth coordinate changes for nonsmooth pseudodifferential operators and boundary value problems were discussed by Jiménez and the first author in [8].

Full symbol rules for nonsmooth changes of variables have not to our knowledge been established anywhere (for example, [47] leaves out this point, [10] works with a restrictive symbol condition that avoids the issue). Changes of variables are of course also interesting for interior problems, for example, if one wants to let one coordinate play a special role.

Since the behavior of a symbol under a coordinate change has a nice exact expression when one allows symbols 'in (x,y)-form' (also called amplitude functions), which define operators by formulas

$$Au(x)=(2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)\,d\xi dy=\mathrm{OP}(a(x,y,\xi))u, \tag{1.7}$$

the question of how to reduce an operator $\mathrm{OP}(a(x,y,\xi))$ to the form $\mathrm{OP}(p(x,\xi))$ is intimately related to the change-of-variables question. For this question one has to establish the validity of (1.7), as well as set up the formula for the reduction with appropriate remainder terms, and to our knowledge neither of these points has been treated before when the y-dependence is nonsmooth. We shall show the following theorem.

Theorem 1.2. *Let $\tau>0$ and $m<\tau$.*

1° *Let $a\in C^\tau S^m_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$. Then (1.7) has a meaning as an oscillatory integral, cf. Theorem 3.1 below for the details. It defines an operator $\mathrm{OP}(a(x,y,\xi))$ mapping continuously*

$$\mathrm{OP}(a(x,y,\xi)):\ H^{s+m}_q(\mathbb{R}^n)\to H^s_q(\mathbb{R}^n),$$

when $|m|$, $|s|$, and $|s+m|$ are $<\tau$, and $1<q<\infty$. Moreover, for any nonnegative integer $l<\tau$,

$$\mathrm{OP}(a(x,y,\xi))u(x)=\sum_{|\alpha|\leqslant l}\mathrm{OP}(p_\alpha(x,\xi))u(x)+\mathrm{OP}(r(x,y,\xi))u(x), \tag{1.8}$$

where $p_\alpha(x,\xi)=\frac{1}{\alpha!}\partial_y^\alpha D_\xi^\alpha a(x,y,\xi)|_{y=x}\in C^{\tau-|\alpha|}S^{m-|\alpha|}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$ and $r(x,y,\xi)\in C^{\tau-l}S^{m-l}_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$, with $r(x,x,\xi)=0$. Here the operators map continuously

$$\mathrm{OP}(p_\alpha(x,\xi)):\ H^{s+m-|\alpha|}_q(\mathbb{R}^n)\to H^s_q(\mathbb{R}^n)\qquad \text{for } |s|<\tau-|\alpha|,$$

$$\mathrm{OP}(r(x,y,\xi)):\ H^{(s+m-l)_+}_q(\mathbb{R}^n)\to H^s_q(\mathbb{R}^n)\qquad \text{for } 0\leqslant s<\min\{\tau-l,\tau-m\}.$$

2° *Let* $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$*. Under a* $C^{1+\tau}$*-diffeomorphism* $F : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $c_0 \leqslant |\det(\nabla F(x))| \leqslant C_0$ *with* $c_0, C_0 > 0$*,* P *transforms to an operator* $\underline{P}$*,*

$$\underline{P} = F^* P F^{*,-1} = \mathrm{OP}(q(x, y, \xi)) + R_1,$$

where $q \in C^\tau S^m_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$*,* $R_1 : L_q(\mathbb{R}^n) \to H^s_q(\mathbb{R}^n)$ *for* $s < \min\{\tau, \tau - m\}$*, and* $\mathrm{OP}(q(x, y, \xi))$ *is as under* 1°*; in particular reducing to* x*-form as in (1.8).*

We address the problem on subsets of $\mathbb{R}^n$, because that is what is asked for in the probabilistic and financial applications. Based on the work of Jiménez and the first author [8] for nonsmooth operators on smooth manifolds, the various results are likely to carry over without trouble to suitable nonsmooth domains in smooth manifolds. The case where the basis manifold itself is nonsmooth would demand a larger effort because of various conditions between the regularities of the manifolds, the symbols, and the order of the function spaces.

The structure of the manuscript is as follows: In Section 2 we summarize necessary preliminaries on function spaces and (nonsmooth) pseudodifferential operators. The first main results are established in Section 3, where pseudodifferential operators with nonsmooth symbols in (x, y)-form, oscillatory integrals for them, and their reduction to x-form operators, are discussed. These results are applied to treat nonsmooth coordinate changes for nonsmooth pseudodifferential operators, and Theorem 1.2 is proved. In Section 4, the description of μ-transmission spaces is generalized from the smooth case to nonsmooth domains. In Section 5, we introduce general versions of a μ-transmission condition for nonsmooth pseudodifferential operators, and show their mapping properties in relation to the μ-transmission spaces, as a key to the regularity results. Finally, in Section 6 the results of the preceding sections are applied to show regularity results for the homogeneous Dirichlet problem for even, strongly elliptic pseudodifferential operators on nonsmooth domains, proving Theorem 1.1.

2 | PRELIMINARIES

2.1 | Function spaces

Recall that the standard Sobolev spaces $W^s_q(\mathbb{R}^n)$, $1 < q < \infty$ and $s \geqslant 0$, have a different character according to whether s is integer or not. Namely, for s integer, they consist of L_q-functions with derivatives in L_q up to order s, hence coincide with the Bessel-potential spaces $H^s_q(\mathbb{R}^n)$, defined for $s \in \mathbb{R}$ by

$$H^s_q(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u}) \in L_q(\mathbb{R}^n)\}. \tag{2.1}$$

Here $\mathcal{F}$ is the Fourier transform $\hat{u}(\xi) = \mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx$, and the function $\langle\xi\rangle$ equals $(|\xi|^2 + 1)^{\frac{1}{2}}$. For noninteger s, the W^s_q-spaces coincide with the Besov spaces $B^s_q(\mathbb{R}^n) = B^s_{q,q}(\mathbb{R}^n)$, defined, for example, as follows: For $0 < s < 2$ and measurable $f : \mathbb{R}^n \to \mathbb{C}$,

$$f \in B^s_q(\mathbb{R}^n) \iff \|f\|^q_{L_q} + \int_{\mathbb{R}^{2n}} \frac{|f(x) + f(y) - 2f((x + y)/2)|^q}{|x - y|^{n+qs}}\, dxdy < \infty; \tag{2.2}$$

and $B^{s+t}_q(\mathbb{R}^n) = (1 - \Delta)^{-t/2} B^s_q(\mathbb{R}^n)$ for all $t \in \mathbb{R}$. The Bessel-potential spaces are important because they are most directly related to $L_q(\mathbb{R}^n)$; the Besov spaces have other convenient properties, and are needed for boundary value problems in an H_q^s-context, because they are the correct range spaces for trace maps $\gamma_j u = (\partial_n^j u)|_{x_n=0}$:

$$\gamma_j : \overline{H}_q^s(\mathbb{R}_+^n), \overline{B}_q^s(\mathbb{R}_+^n) \to B_q^{s-j-\frac{1}{q}}(\mathbb{R}^{n-1}), \text{ for } s - j - \tfrac{1}{q} > 0, \tag{2.3}$$

surjectively and with a continuous right inverse; see, for example, the overview in the introduction to [27]. For $q = 2$, the two scales are identical, but for $q \neq 2$ they are related by strict inclusions: $H_q^s \subset B_q^s$ when $q > 2$, $H_q^s \supset B_q^s$ when $q < 2$. When $q = 2$, the index q is usually omitted. We will always use B_q^s as abbreviation of $B_{q,q}^s$.

We shall also use the spaces $C^k(\mathbb{R}^n) \equiv C_b^k(\mathbb{R}^n)$ of k-times differentiable functions with uniform norms $\|u\|_{C^k} = \sup_{|\alpha| \leqslant k, x \in \mathbb{R}^n} |D^\alpha u(x)|$ ($k \in \mathbb{N}_0$), and the Hölder spaces $C^\tau(\mathbb{R}^n)$, $\tau = k + \sigma$ with $k \in \mathbb{N}_0, 0 < \sigma < 1$, also denoted as $C^{k,\sigma}(\mathbb{R}^n)$, with norms $\|u\|_{C^\tau} = \|u\|_{C^k} + \sup_{|\alpha|=k, x\neq y} |D^\alpha u(x) - D^\alpha u(y)|/|x-y|^\sigma$. The latter definition extends to Lipschitz spaces $C^{k,1}(\mathbb{R}^n)$. There are similar spaces over subsets of $\mathbb{R}^n$. Finally, we denote $C_b^\infty(\mathbb{R}^n) = \bigcap_{k \in \mathbb{N}} C_b^k(\mathbb{R}^n)$.

The halfspaces $\mathbb{R}_\pm^n$ are defined by $\mathbb{R}_\pm^n = \{x \in \mathbb{R}^n \mid x_n \gtrless 0\}$, with points denoted as $x = (x', x_n)$, $x' = (x_1, \dots, x_{n-1})$. When $\gamma \in C^{1+\tau}(\mathbb{R}^{n-1})$ for some $\tau > 0$, we define the curved halfspace $\mathbb{R}_\gamma^n$ by $\mathbb{R}_\gamma^n = \{x \in \mathbb{R}^n \mid x_n > \gamma(x')\}$.

Also bounded $C^{1+\tau}$-domains Ω will be considered. By this we mean that $\Omega \subset \mathbb{R}^n$ is open and bounded, and every boundary point x_0 has an open neighborhood U such that, after a translation of x_0 to 0 and a suitable rotation, $U \cap \Omega = U \cap \mathbb{R}_\gamma^n$ for a function $\gamma \in C^{1+\tau}(\mathbb{R}^{n-1})$ with $\gamma(0) = 0$. (In some texts, a hypothesis on connectedness of Ω is included here, but we do not need this.)

Restriction from $\mathbb{R}^n$ to $\mathbb{R}_\pm^n$ (or from $\mathbb{R}^n$ to Ω, respectively, $\complement\overline{\Omega} = \mathbb{R}^n \setminus \overline{\Omega}$) is denoted as $r^\pm$, extension by zero from $\mathbb{R}_\pm^n$ to $\mathbb{R}^n$ (or from Ω, respectively, $\complement\overline{\Omega}$ to $\mathbb{R}^n$) is denoted as $e^\pm$. (The notation is also used for $\Omega = \mathbb{R}_\gamma^n$). Restriction from $\overline{\mathbb{R}}_+^n$ or $\overline{\Omega}$ to $\partial\mathbb{R}_+^n$, respectively, $\partial\Omega$ is denoted as γ_0.

By $d(x)$ we denote (as in [31, Definition 2.1] for the C^∞-case) a function that is $C^{1+\tau}$ on $\overline{\Omega}$, positive on Ω and vanishes only to the first order on $\partial\Omega$ (that is, $d(x) = 0$ and $\nabla d(x) \neq 0$ for $x \in \partial\Omega$). On bounded sets it satisfies near $\partial\Omega$:

$$C^{-1} d_0(x) \leqslant d(x) \leqslant C d_0(x) \tag{2.4}$$

with $C > 0$, where $d_0(x)$ equals $\operatorname{dist}(x, \partial\Omega)$ on a neighborhood of $\partial\Omega$ and is extended as a correspondingly smooth positive function on Ω.

When $\tau \geqslant 1$, d_0 itself can be taken $C^{1+\tau}$. This holds since there is then a tubular neighborhood of $\partial\Omega$ where $d_0(x)$ plays the role of a normal coordinate; its gradient equals the interior unit normal vector $\nu(x)$ (cf., for example, Prüss and Simonett [48, pp. 65–66]). Since ν is C^τ, d_0 is $C^{1+\tau}$. Then moreover, d/d_0 is a positive C^τ-function on $\overline{\Omega}$.

We take $d_0(x) = x_n$ in the case of $\mathbb{R}_+^n$. For $\mathbb{R}_\gamma^n$, the function $d(x) = x_n - \gamma(x')$ satisfies (2.4) near $\partial\mathbb{R}_\gamma^n$, and does so globally on $\mathbb{R}_\gamma^n$ if we choose the extension of $d_0(x)$ further away from $\partial\mathbb{R}_\gamma^n$ to equal $x_n + C_1$, where $C_1 > \sup_{x'} |\gamma(x')|$. Then when $\tau \geqslant 1$, d/d_0 is a positive function in $C^\tau(\overline{\mathbb{R}}_\gamma^n)$.

Along with the spaces $H_q^s(\mathbb{R}^n)$ defined in (2.1), there are the two scales of spaces associated with Ω for $s \in \mathbb{R}$:

$$\begin{aligned} &\overline{H}_q^s(\Omega) = \{u \in \mathcal{D}'(\Omega) \mid u = r^+ U \text{ for some } U \in H_q^s(\mathbb{R}^n)\}, \text{ the } \textit{restricted} \text{ space,} \\ &\dot{H}_q^s(\overline{\Omega}) = \{u \in H_q^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}, \text{ the } \textit{supported} \text{ space;} \end{aligned} \tag{2.5}$$

here supp u denotes the support of u. $\overline{H}^s_q(\Omega)$ is in other texts often denoted as $H^s_q(\Omega)$ or $H^s_q(\overline{\Omega})$, and $\dot{H}^s_q(\overline{\Omega})$ may be indicated with a ring, zero, or twiddle; the current notation stems from Hörmander [40, Appendix B2]. There are similar spaces with B^s_q.

Besides for the H^s_q and B^s_q-spaces, there are in [30] for C^∞-domains established the relevant results in many other scales of spaces, namely, Besov spaces $B^s_{p,q}$ for $1 \leqslant p, q \leqslant \infty$ and Triebel–Lizorkin spaces $F^s_{p,q}$ (for the same p, q but with $p < \infty$). We shall not pursue this in the present work, except that we want to refer to the Hölder–Zygmund scale $B^s_{\infty,\infty}$, also denoted as C^s_*. Here C^s_* identifies with the Hölder space C^s when $s \in \mathbb{R}_+ \setminus \mathbb{N}$, and for positive integer k satisfies $C^{k-\varepsilon} \supset C^k_* \supset C^{k-1,1} \supset C^k_b$ for small $\varepsilon > 0$; moreover, $C^0_* \supset L_\infty \supset C^0_b$ (with strict inclusions everywhere). Similarly to (2.5) we denote the spaces of restricted, respectively, supported distributions

$$\overline{C}^s_*(\Omega) = \{u \in \mathcal{D}'(\Omega) \mid u = r^+U \text{ for some } U \in C^s_*(\mathbb{R}^n)\},$$

$$\dot{C}^s_*(\overline{\Omega}) = \{u \in C^s_*(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\};$$

the star can be omitted when $s \in \mathbb{R}_+ \setminus \mathbb{N}$.

2.2 | Pseudodifferential operators

A *pseudodifferential operator* (ψdo) P on $\mathbb{R}^n$ is defined from a symbol $p(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$Pu = \mathrm{OP}(p(x,\xi))u = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, đ\xi = \mathcal{F}^{-1}_{\xi\to x}(p(x,\xi)\mathcal{F}u(\xi)), \tag{2.6}$$

using the Fourier transform $\mathcal{F}$ and the notation $đ\xi = (2\pi)^{-n}\, d\xi$. We refer to textbooks such as Kumano-go [44], Hörmander [40], Taylor [55], Grubb [29], and Abels [6] for the rules of calculus, in particular the definition by oscillatory integrals in [6, 40]. For precision, the notation $\mathrm{Os}-\int$ is often used when an integral is understood as an oscillatory integral.

The symbols p of order $m \in \mathbb{R}$ were originally taken to lie in the symbol space $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, consisting of complex C^∞-functions $p(x, \xi)$ such that $\partial^\beta_x \partial^\alpha_\xi p(x, \xi)$ is $O(\langle\xi\rangle^{m-|\alpha|})$ for all α, β, for some $m \in \mathbb{R}$, with global estimates for $x \in \mathbb{R}^n$ (as in [40, start of Section 18.1] and [28]). P (of order m) then maps $H^s_q(\mathbb{R}^n)$ continuously into $H^{s-m}_q(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, cf. (2.1). P is said to be *classical* when p has an asymptotic expansion $p(x, \xi) \sim \sum_{j\in\mathbb{N}_0} p_j(x, \xi)$ with p_j homogeneous in ξ of degree $m - j$ for all $|\xi| \geqslant 1$ and $j \in \mathbb{N}_0$, such that

$$\partial^\beta_x \partial^\alpha_\xi \big(p(x,\xi) - \textstyle\sum_{j<J} p_j(x,\xi)\big) \text{ is } O(\langle\xi\rangle^{m-\alpha-J}) \text{ for all } \alpha, \beta \in \mathbb{N}^n_0, J \in \mathbb{N}_0. \tag{2.7}$$

For a complete theory one adds to these operators the smoothing operators (mapping any $H^s_q(\mathbb{R}^n)$ into $\bigcap_t H^t_q(\mathbb{R}^n)$), regarded as operators of order $-\infty$. (For example, $(-\Delta)^a$ fits into the calculus when it is written as $\mathrm{OP}((1-\zeta(\xi))|\xi|^{2a}) + \mathrm{OP}(\zeta(\xi)|\xi|^{2a})$, where $\zeta(\xi)$ is a C^∞-function that equals 1 for $|\xi| \leqslant \frac{1}{2}$ and 0 for $|\xi| \geqslant 1$; the second term is smoothing.)

In the present study we moreover consider symbols with limited smoothness in x. For later purposes we here replace $x \in \mathbb{R}^n$ by $X \in \mathbb{R}^{n'}$, where n' will usually be n or $2n$.

The space $C^\tau S^m_{1,0}(\mathbb{R}^{n'} \times \mathbb{R}^n)$ for $\tau > 0$, $m \in \mathbb{R}$, $n', n \in \mathbb{N}$ consists of functions $p : \mathbb{R}^{n'} \times \mathbb{R}^n \to \mathbb{C}$ that are continuous w.r.t. $(X, \xi) \in \mathbb{R}^{n'} \times \mathbb{R}^n$ and smooth with respect to $\xi \in \mathbb{R}^n$, such that for every $\alpha \in \mathbb{N}_0^n$ we have: $\partial_\xi^\alpha p(X, \xi)$ is in $C^\tau(\mathbb{R}^{n'})$ with respect to X and satisfies for all $\xi \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_0^n$,

$$\|\partial_\xi^\alpha p(\cdot, \xi)\|_{C^\tau(\mathbb{R}^{n'})} \leqslant C_\alpha \langle \xi \rangle^{m-|\alpha|}, \tag{2.8}$$

with $C_\alpha > 0$. We equip the symbol space with the semi-norms

$$|p|_{k, C^\tau S^m_{1,0}(\mathbb{R}^{n'} \times \mathbb{R}^n)} := \max_{|\alpha| \leqslant k} \sup_{\xi \in \mathbb{R}^n} \langle \xi \rangle^{-m+|\alpha|} \|\partial_\xi^\alpha p(\cdot, \xi)\|_{C^\tau(\mathbb{R}^{n'})} \quad \text{for } k \in \mathbb{N}_0. \tag{2.9}$$

The following theorem is well known.

Theorem 2.1. *Let $\tau > 0$, $1 < q < \infty$, $m \in \mathbb{R}$ and $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Then $\mathrm{OP}(p)$ extends to a bounded linear operator*

$$\mathrm{OP}(p) : H_q^{s+m}(\mathbb{R}^n) \to H_q^s(\mathbb{R}^n) \quad \textit{for all } |s| < \tau.$$

Moreover, for every $s \in (-\tau, \tau)$ there is some $k \in \mathbb{N}$ and $C > 0$ such that

$$\| \mathrm{OP}(p)\|_{\mathcal{L}(H_q^{s+m}(\mathbb{R}^n), H_q^s(\mathbb{R}^n))} \leqslant C |p|_{k, C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)} \quad \textit{for all } p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n).$$

The mapping properties follow from [46, Theorem 2.7]. The boundedness of the operator norm by a symbol semi-norm is a consequence of the closed graph theorem or the Hahn–Banach theorem, cf., for example, [9, Theorem 3.7].

The subspace of classical symbols $C^\tau S^m(\mathbb{R}^{n'} \times \mathbb{R}^n)$ consists of those functions that moreover have expansions into terms p_j homogeneous in ξ of degree $m - j$ for $|\xi| \geqslant 1$, all j, such that for all $\xi \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_0^n$, $J \in \mathbb{N}_0$,

$$\|\partial_\xi^\alpha \big(p(\cdot, \xi) - \textstyle\sum_{j<J} p_j(\cdot, \xi)\big)\|_{C^\tau(\mathbb{R}^{n'})} \leqslant C_{\alpha, J} \langle \xi \rangle^{m-J-|\alpha|}. \tag{2.10}$$

A classical symbol $p(x, \xi)$ (and the associated operator P) is said to be *strongly elliptic* when $\mathrm{Re}\, p_0(x, \xi) \geqslant c|\xi|^m$ for $|\xi| \geqslant 1$, with $c > 0$. Moreover, a classical ψdo $P = \mathrm{OP}(p(x, \xi))$ of order $m \in \mathbb{R}$ is said to be *even*, when the terms in the symbol expansion $p \sim \sum_{j \in \mathbb{N}_0} p_j$ satisfy

$$p_j(x, -\xi) = (-1)^j p_j(x, \xi) \quad \text{for all } x \in \mathbb{R}^n, |\xi| \geqslant 1, j \in \mathbb{N}_0. \tag{2.11}$$

(The word 'even' is short for *even-to-even parity*, meaning that the terms with even j are even in ξ, and the terms with odd j are odd in ξ.) Similarly, p is *odd* (short for odd-to-even parity) if $p \sim \sum_{j \in \mathbb{N}_0} p_j$, where

$$p_j(x, -\xi) = (-1)^{j+1} p_j(x, \xi) \quad \text{for all } x \in \mathbb{R}^n, |\xi| \geqslant 1, j \in \mathbb{N}_0.$$

Note that when p is even of order m, $p - p_0$ is odd of order $m - 1$.

In part of the present paper, we consider symbols that are moreover assumed to satisfy a μ-transmission condition, as introduced in the smooth case by Hörmander [39, 40] and Grubb [31], see Section 4.1. To handle operators with such properties, we must introduce *order-reducing* *operators*. There is a simple definition of operators $\Xi^t_\pm$ on $\mathbb{R}^n$, $t \in \mathbb{R}$,

$$\Xi^t_\pm = \mathrm{OP}(\chi^t_\pm), \quad \chi^t_\pm(\xi) = (\langle \xi' \rangle \pm i\xi_n)^t; \tag{2.12}$$

they preserve support in $\overline{\mathbb{R}}^n_\pm$, respectively, because the symbols extend as holomorphic functions of ξ_n into $\mathbb{C}_\mp$, respectively; $\mathbb{C}_\pm = \{z \in \mathbb{C} : \operatorname{Im} z \gtrless 0\}$. (The functions $(\langle \xi' \rangle \pm i\xi_n)^t$ satisfy only part of the estimates (2.7) with $m = t$, but the ψdo definition can be applied anyway.) There is a more refined choice $\Lambda^t_\pm$, cf. [27, 31], with symbols $\lambda^t_\pm(\xi)$ that do satisfy all the required estimates, and where $\overline{\lambda^t_+} = \lambda^t_-$. These symbols likewise have holomorphic extensions in ξ_n to the complex half-spaces $\mathbb{C}_\mp$, so that the operators preserve support in $\overline{\mathbb{R}}^n_\pm$, respectively. Operators with that property are called 'plus', respectively, 'minus' operators.

There is also a pseudodifferential definition $\Lambda^{(t)}_\pm$ adapted to the situation of a smooth domain Ω, by [27, 31].

It follows from the Lizorkin multiplier theorem and the definition of the spaces $H^s_q(\mathbb{R}^n)$ in terms of Fourier transformation that the operators define homeomorphisms $\Xi^t_\pm : H^s_q(\mathbb{R}^n) \xrightarrow{\sim} H^{s-t}_q(\mathbb{R}^n)$, for all $s \in \mathbb{R}$. The special interest is that the 'plus'/'minus' operators also define homeomorphisms related to $\overline{\mathbb{R}}^n_+$ and $\overline{\Omega}$, for all $s \in \mathbb{R}$:

$$\Xi^t_+ : \dot{H}^s_q(\overline{\mathbb{R}}^n_+) \xrightarrow{\sim} \dot{H}^{s-t}_q(\overline{\mathbb{R}}^n_+), \quad r^+\Xi^t_- e^+ : \overline{H}^s_q(\mathbb{R}^n_+) \xrightarrow{\sim} \overline{H}^{s-t}_q(\mathbb{R}^n_+),$$

$$\Lambda^{(t)}_+ : \dot{H}^s_q(\overline{\Omega}) \xrightarrow{\sim} \dot{H}^{s-t}_q(\overline{\Omega}), \quad r^+\Lambda^{(t)}_- e^+ : \overline{H}^s_q(\Omega) \xrightarrow{\sim} \overline{H}^{s-t}_q(\Omega),$$

with similar rules for $\Lambda^t_\pm$. Moreover, the operators Ξ^t_+ and $r^+\Xi^t_- e^+$ identify with each other's adjoints over $\overline{\mathbb{R}}^n_+$, because of the support-preserving properties. There is a similar statement for Λ^t_+ and $r^+\Lambda^t_- e^+$, and for $\Lambda^{(t)}_+$ and $r^+\Lambda^{(t)}_- e^+$ relative to the set Ω.

3 | NONSMOOTH COORDINATE CHANGES

Since some of the proofs of the results in this chapter are quite technical, we have moved them to the Appendix, in order to keep the flow of the presentation leading to results on fractional-order boundary problems.

Basic rules of calculus for pseudodifferential operators with nonsmooth symbols were set up by Kumano-go and Nagase [45], Beals and Reed [12], Marschall [46, 47], and Witt [58], however, leaving out the question of nonsmooth coordinate changes. To our knowledge, this question has been open since then, and it is this that we want to work out now. A basic step is the reduction of operators defined by symbols in (x, y)-form to symbols in x-form, in particular to handle the remainders arising from this. The definition of operators from symbols $a(x, y, \xi)$ (also called amplitude functions) follows from known facts when a is C^τ in x and C^∞ in y, see, for example, [10], but a definition when a is merely C^τ in y has not to our knowledge been established in detail; this is the first thing we undertake here.

Symbol classes and their seminorms were defined in the preliminaries section. In the following let $\tau > 0$, $m \in \mathbb{R}$, $a \in C^\tau S^m_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$.

The first task is to show the existence of oscillatory integrals for nonsmooth (x, y)-form symbols, and to find derived x-form operators.

Assuming $\tau > m$, we have to give a meaning to,

$$\begin{aligned}\mathrm{OP}(a)u(x) &= \mathrm{Os}-\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a(x,y,\xi)u(y)\,dy\,đ\xi \\ &:= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^{2n}} \chi(\varepsilon y,\varepsilon\xi)e^{i(x-y)\cdot\xi} a(x,y,\xi)u(y)\,dy\,đ\xi\end{aligned}$$

for every $u \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, where $\chi \in \mathcal{S}(\mathbb{R}^{2n})$ with $\chi(0,0) = 1$. To this end (and for other purposes later on) we shall use a Taylor expansion

$$a(x,y,\xi) = \sum_{|\alpha|\leqslant l} \frac{1}{\alpha!}\partial_y^\alpha a(x,y,\xi)|_{y=x}(y-x)^\alpha + \sum_{|\alpha|=l}(y-x)^\alpha r_\alpha(x,y,\xi), \tag{3.1}$$

where $l \in \mathbb{N}_0$ with $l < \tau$ and

$$r_\alpha(x,y,\xi) = \frac{|\alpha|}{\alpha!}\int_0^1 (1-t)^{|\alpha|-1}(\partial_y^\alpha a)(x,(1-t)x+ty,\xi)\,dt - \frac{1}{\alpha!}\partial_y^\alpha a(x,y,\xi)|_{y=x}. \tag{3.2}$$

We have here subtracted the αth precise term from the usual remainder term, to achieve that

$$r_\alpha(x,x,\xi) = 0 \quad \text{for all } x,\xi \in \mathbb{R}^n.$$

Note that $r_\alpha \in C^{\tau-|\alpha|}S^m_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$. Hence for every $\beta \in \mathbb{N}_0^n$

$$|\partial_\xi^\beta r_\alpha(x,y,\xi)| \leqslant C_\beta |x-y|^{\min\{\tau-|\alpha|,1\}}\langle\xi\rangle^{m-|\beta|} \quad \text{for all } x,y,\xi \in \mathbb{R}^n. \tag{3.3}$$

We will use a dyadic partition of unity $(\varphi_j)_{j\in\mathbb{N}_0}$ of $\mathbb{R}^n$ in the proof. More precisely, let $\varphi_j \in C_0^\infty(\mathbb{R}^n)$, $j \in \mathbb{N}_0$, be a partition of unity such that

$$\operatorname{supp}\varphi_j \subset \{\xi \in \mathbb{R}^n \mid 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\} \tag{3.4}$$

and $\varphi_j(\xi) = \varphi_1(2^{1-j}\xi)$ for all $j \geqslant 1$, $\xi \in \mathbb{R}^n$.

For later purposes we show existence of the oscillatory integral in a more general form. We denote by $[\sigma]$ the largest integer $\leqslant \sigma$.

Theorem 3.1. *Let $a \in C^\tau S^m_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$, $\tau > 0$, $m \in \mathbb{R}$, $\gamma \in \mathbb{N}_0^n$, and assume that $m < \tau + |\gamma|$. Then for every $x \in \mathbb{R}^n$ and $u \in \mathcal{S}(\mathbb{R}^n)$ the limit*

$$\mathrm{OP}((y-x)^\gamma a(x,y,\xi))u(x) := \lim_{\varepsilon\to 0}\int_{\mathbb{R}^{2n}} \chi(\varepsilon y,\varepsilon\xi)e^{i(x-y)\cdot\xi}(y-x)^\gamma a(x,y,\xi)u(y)\,dy\,đ\xi$$

exists and coincides with

$$\sum_{|\alpha|\leqslant l}\frac{1}{\alpha!}\mathrm{OP}(\partial_y^\alpha D_\xi^{\alpha+\gamma} a(x,y,\xi)|_{y=x})u(x) + \sum_{|\alpha|=l}\mathrm{OP}((y-x)^{\alpha+\gamma} r_\alpha(x,y,\xi))u(x),$$

where $l = \max\{[m-|\gamma|],0\} < \tau$, and

$$\mathrm{OP}((y-x)^\alpha r_\alpha(x,y,\xi))u(x) = \int_{\mathbb{R}^n} k_{\alpha,\gamma}(x,y,x-y)u(y)\,dy,$$

$|k_{\alpha,\gamma}(x, y, x-y)| \leqslant g(x-y)$ *for some nonnegative* $g \in L_1(\mathbb{R}^n)$, *and*

$$k_{\alpha,\gamma}(x, y, z) = \sum\nolimits_{j\in\mathbb{N}_0} k_{\alpha,\gamma,j}(x, y, z),$$

$$k_{\alpha,\gamma,j}(x, y, z) = \int_{\mathbb{R}^n} e^{iz\cdot\xi}(x-y)^{\alpha+\gamma} r_\alpha(x, y, \xi)\varphi_j(\xi)\, đ\xi$$

for all $x, y, z \in \mathbb{R}^n$ *with* $z \neq 0$.

The proof of Theorem 3.1 is given in the Appendix.

It will be convenient to observe.

Lemma 3.2. *Let* $a \in C^\tau S^m_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$, $\tau > 0$, $m \in \mathbb{R}$, $\gamma \in \mathbb{N}_0^n$, *and assume that* $m < \tau + |\gamma|$. *Then for every* $x \in \mathbb{R}^n$ *and* $u \in \mathcal{S}(\mathbb{R}^n)$ *and* $\beta \in \mathbb{N}_0^n$ *with* $\beta \leqslant \gamma$:

$$\mathrm{OP}((y-x)^\gamma a(x, y, \xi))u(x) = \mathrm{OP}((y-x)^{\gamma-\beta} D_\xi^\beta a(x, y, \xi))u(x).$$

The proof of Lemma 3.2 is given in the Appendix.

We also have the following.

Corollary 3.3. *Let* $a \in C^\tau S^m_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$, $\tau > 0$, $m \in \mathbb{R}$, *and assume that* $m < \tau$. *Setting*

$$p_\alpha(x, \xi) = \frac{1}{\alpha!}\partial_y^\alpha D_\xi^\alpha a(x, y, \xi)|_{y=x},$$

we have for every $l \in \mathbb{N}_0$ *with* $l < \tau$:

$$\mathrm{OP}(a(x, y, \xi))u(x) = \sum_{|\alpha|\leqslant l} \mathrm{OP}(p_\alpha(x, \xi))u(x) + \sum_{|\alpha|=l} \mathrm{OP}(D_\xi^\alpha r_\alpha(x, y, \xi))u(x),$$

where $p_\alpha(x, \xi) \in C^{\tau-|\alpha|} S^{m-|\alpha|}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ *and* $D_\xi^\alpha r_\alpha \in C^{\tau-l} S^{m-l}_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$ *for all* $|\alpha| = l$ *(as defined in (3.2)), with* $D_\xi^\alpha r_\alpha(x, x, \xi) = 0$.

Proof. The equality follows directly from Theorem 3.1 with $\gamma = 0$ and Lemma 3.2 applied to $a = r_\alpha$ and $\gamma = \beta = \alpha$. The statements on $D_\xi^\alpha r_\alpha$ follow from (3.2)ff. □

The next task is to determine the mapping properties of (x, y)-form operators.

The following result will be the basis for all further results on mapping properties of (x, y)-form operators. It applies not only to remainders, but also to full operators, without a graded expansion that would reduce the regularity with respect to x.

Theorem 3.4. *Let* $a \in C^\tau S^m_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$ *with* $|m| < \tau$, *and let* $1 < q < \infty$. *Then*

$$\mathrm{OP}(a(x, y, \xi)) : H_q^{s+m}(\mathbb{R}^n) \to H_q^s(\mathbb{R}^n)$$

is a bounded linear operator, provided that $|s| < \tau$, $|s+m| < \tau$.

The proof of Theorem 3.4 is given in the Appendix.

We observe a particular conclusion for the remainder terms (3.2), in case $s \geqslant 0$.

Corollary 3.5. *Let r_α be as in (3.2) and Corollary 3.3, let $m < \tau$, $l \in \mathbb{N}_0$ with $l < \tau$, $1 < q < \infty$, and let $s \in \mathbb{R}$ such that $0 \leqslant s < \tau - l$, $s + m < \tau$. Then*

$$\mathrm{OP}(D_\xi^\alpha r_\alpha(x, y, \xi)) : H_q^{(s+m-l)_+}(\mathbb{R}^n) \to H_q^s(\mathbb{R}^n) \tag{3.5}$$

is bounded. Moreover, there is some $k \in \mathbb{N}$ and $C_{s,q} > 0$ such that

$$\| \mathrm{OP}(D_\xi^\alpha r_\alpha(x, y, \xi))\|_{\mathcal{L}(H_q^{(s+m-l)_+}(\mathbb{R}^n), H_q^s(\mathbb{R}^n))} \leqslant C_{s,q} |a|_{k, C^\tau S_{1,0}^m(\mathbb{R}^{2n} \times \mathbb{R}^n)}.$$

Proof. If $m - l > -(\tau - l)$, we can apply Theorem 3.4 directly with τ and m replaced by $\tau' = \tau - l$ and $m' = m - l$; this gives

$$\mathrm{OP}(D_\xi^\alpha r_\alpha(x, y, \xi)) : H_q^{s+m-l}(\mathbb{R}^n) \to H_q^s(\mathbb{R}^n),$$

and (3.5) holds à fortiori. If $m - l \leqslant -(\tau - l)$, we note that $-(\tau - l) < -s$, so that $D_\xi^\alpha r_\alpha \in C^{\tau - l} S_{1,0}^{m-l}(\mathbb{R}^{2n} \times \mathbb{R}^n) \subset C^{\tau - l} S_{1,0}^{-s}(\mathbb{R}^{2n} \times \mathbb{R}^n)$; here Theorem 3.4 can be applied with τ and m replaced by $\tau' = \tau - l$ and $m'' = -s$, giving

$$\mathrm{OP}(D_\xi^\alpha r_\alpha(x, y, \xi)) : H_q^0(\mathbb{R}^n) \to H_q^s(\mathbb{R}^n).$$

This is as desired since in this case $(s + m - l)_+ = 0$.

Finally, the boundedness of the operator norm by a symbol semi-norm is a consequence of the closed graph theorem. It can be shown in the same way as in the proof of Theorem 5.13 below. □

The next result will help improve remainder estimates; it is related to statements given in Taylor [57, Proposition 9.5].

Theorem 3.6. *Let $a \in C^\tau S_{1,0}^m(\mathbb{R}^{2n} \times \mathbb{R}^n)$ with $m < \tau$ and*

$$\partial_x^\alpha \partial_y^\beta a(x, y, \xi)|_{y=x} = 0 \quad \textit{for all } x, \xi \in \mathbb{R}^n \textit{ and } |\alpha| + |\beta| < \tau.$$

Moreover, let $1 < q < \infty$. Then

$$\mathrm{OP}(a(x, y, \xi)) : L_q(\mathbb{R}^n) \to H_q^s(\mathbb{R}^n)$$

is a bounded linear operator provided that $0 \leqslant s < \min(\tau - m, \tau)$.

The proof of Theorem 3.6 is given in the Appendix.

Remark 3.7. We note that, if $a \in C^\tau S_{1,0}^m(\mathbb{R}^{2n} \times \mathbb{R}^n)$ with $0 \leqslant m < \tau$ satisfies $a(x, y, \xi) = 0$ for all $x, y, \xi \in \mathbb{R}^n$ with $|x - y| < \delta$ for some $\delta > 0$, then the conditions of Theorem 3.6 are satisfied. In particular, if $p \in C^\tau S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n)$ with $0 \leqslant m < \tau$ and $\varphi, \psi \in C_b^\infty(\mathbb{R}^n)$ are such that $\operatorname{supp} \varphi \cap \operatorname{supp} \psi = \emptyset$, then

$$\psi\, \mathrm{OP}(p(x, \xi))(\varphi u) \in H_q^s(\mathbb{R}^n) \qquad \text{for all } u \in L_q(\mathbb{R}^n), 0 \leqslant s < \tau.$$

The third task is to establish rules for coordinate changes, by use of the results on (x, y)-form operators.

In the following, let $F: \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-diffeomorphism with $DF \in C^\tau(\mathbb{R}^n)^{n\times n}$, and $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ for some $m < \tau$ and $\tau > 0$. Moreover, let

$$(\underline{P}u)(x) = (P(u\circ F^{-1}))(F(x)) = (F^*PF^{*,-1}u)(x) \quad \text{for all } u \in \mathcal{S}(\mathbb{R}^n), \tag{3.6}$$

where $(F^*v)(y) := v(F(y))$ and $(F^{*,-1}u)(x) := u(F^{-1}(x))$. We will first consider the case that

$$\sup_{x\in\mathbb{R}^n} |\nabla F(x) - I| \leqslant \tfrac{1}{2}. \tag{3.7}$$

Then one obtains for all $u \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$:

$$\begin{aligned}\underline{P}u(x) &= \mathrm{Os}-\int_{\mathbb{R}^{2n}} e^{i(F(x)-z)\cdot\eta} p(F(x),\eta)u(F^{-1}(z))\,dz\,đ\eta \\ &= \mathrm{Os}-\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} q(x,y,\xi)u(y)\,dy\,đ\xi,\end{aligned} \tag{3.8}$$

where a well-known formula for coordinate changes (explained, for example, in [6, Proof of Theorem 3.48]) gives

$$\begin{aligned} q(x,y,\xi) &= p(F(x), A(x,y)^{-1,T}\xi)|\det A(x,y)|^{-1}|\det \nabla_y F(y)|, \\ A(x,y) &= \int_0^1 \nabla_x F(x + t(y-x))\,dt,\end{aligned} \tag{3.9}$$

for all $x, y, \xi \in \mathbb{R}^n$. Here $q(x,y,\xi) \in C^\tau S^m_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$ and $A(x,y)^{-1,T} = (A(x,y)^{-1})^T$. We note that (3.7) ensures $\det A(x,y) \neq 0$ for every $x, y \in \mathbb{R}^n$. In our nonsmooth case, the existence of the limit in the oscillatory integral in (3.8) follows from Theorem 3.1. The existence of this limit also implies the existence of the limit in the oscillatory integral preceding it. Moreover, using that for every $\tau' \in (0,\tau]$ there is some $C > 0$ such that

$$\|a\circ F - a\|_{C^{\tau'}(\mathbb{R}^n)} \leqslant C\|a\|_{C^\tau(\mathbb{R}^n)}\|F - \mathrm{id}\,\|^{\min(1,\tau-\tau')}_{C^{1+\tau}(\mathbb{R}^n)} \qquad \text{for all } a \in C^\tau(\mathbb{R}^n),$$

one verifies in a straightforward manner that for every $r_0 > 0$ and $k \in \mathbb{N}_0$ there is some C_k independent of F such that

$$|q-p|_{k,C^{\tau'}S^m_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)} \leqslant C_k\|F - \mathrm{id}\,\|^{\min(1,\tau-\tau')}_{C^{1+\tau}(\mathbb{R}^n)}|p|_{k+1,C^\tau S^m_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)} \tag{3.10}$$

for all $p \in C^\tau S^m_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$ and C^1-diffeomorphisms F such that $DF \in C^\tau(\mathbb{R}^n)^{n\times n}$, $\|F - \mathrm{id}\,\|_{C^{1+\tau}} \leqslant r_0$ and (3.7) holds since

$$\|A - I\|_{C^\tau(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant C\|F - \mathrm{id}\,\|_{C^{1+\tau}(\mathbb{R}^n)}.$$

We shall now apply the Taylor expansion in (3.1) and Corollary 3.3 to q. This gives that for any $N \in \mathbb{N}_0$ with $N < \tau$,

$$\underline{P}u(x) = \sum_{|\alpha|\leqslant N} \mathrm{OP}(q_\alpha(x,\xi))u(x) + \mathrm{OP}(\tilde{r}_N(x,y,\xi))u(x),$$

where

$$q_\alpha(x,\xi) = \frac{1}{\alpha!}\partial_y^\alpha D_\xi^\alpha q(x,y,\xi)\Big|_{y=x}, \tag{3.11}$$
$$\tilde{r}_N(x,y,\xi) = \sum_{|\alpha|=N}\frac{N}{\alpha!}\int_0^1 (1-t)^{N-1}\partial_y^\alpha D_\xi^\alpha q(x,x+t(y-x),\xi)\,dt - \sum_{|\alpha|=N} q_\alpha(x,\xi).$$

Theorem 3.8. *Let $p \in C^\tau S^m_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$ with $m<\tau$ and $\tau>0$, let F be a C^1-diffeomorphism with $DF \in C^\tau(\mathbb{R}^n)^{n\times n}$ such that (3.7) holds true, $|\alpha| \leqslant N < \tau$ and let q be defined as in (3.11). Then $q_\alpha \in C^{\tau-|\alpha|}S^{m-|\alpha|}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$ and $\tilde{r}_N \in C^{\tau-N}S^{m-N}_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$ with $\tilde{r}_N(x,x,\xi)=0$ for all $x,\xi\in\mathbb{R}^n$. Moreover, for every $r_0>0$, $\tau'\in(N,\tau]$, and $k\in\mathbb{N}_0$ there is some C_k independent of F such that*

$$|q_0 - p|_{k,C^{\tau'}S^m_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant C_k\|F-\mathrm{id}\|^{\min(1,\tau-\tau')}_{C^{1+\tau}(\mathbb{R}^n)}|p|_{k+1,C^\tau S^m_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)},$$
$$|q_\alpha|_{k,C^{\tau'-|\alpha|}S^{m-|\alpha|}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant C_k\|F-\mathrm{id}\|^{\min(1,\tau-\tau')}_{C^{1+\tau}(\mathbb{R}^n)}|p|_{k+1,C^\tau S^m_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)},$$
$$|\tilde{r}_N|_{k,C^{\tau'-N}S^{2a-N}_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)} \leqslant C_k\|F-\mathrm{id}\|^{\min(1,\tau-\tau')}_{C^{1+\tau}(\mathbb{R}^n)}|p|_{k+1,C^\tau S^m_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)}$$

for all $1\leqslant|\alpha|\leqslant N$, provided that $\|F-\mathrm{id}\|_{C^{1+\tau}}\leqslant r_0$.

Proof. The first statements follow easily from the definitions. Morever, as for (3.10) one can verify the stated estimates in a straightforward manner. □

For general $C^{1+\tau}$-diffeomorphisms we obtain the following.

Theorem 3.9. *Let $p\in C^\tau S^m_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$ with $m<\tau$ and $\tau>0$, let $F:\mathbb{R}^n\to\mathbb{R}^n$ be a C^1-diffeomorphism with $DF\in C^\tau(\mathbb{R}^n)^{n\times n}$ such that*

$$c_0 \leqslant |\det(\nabla F(x))| \leqslant C_0 \qquad \text{for all } x\in\mathbb{R}^n \tag{3.12}$$

for some $c_0, C_0>0$. Then there is some $q\in C^\tau S^m_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$ such that

$$\underline{P}u = F^*PF^{*,-1}u = \mathrm{OP}(q(x,y,\xi))u + Ru \quad \text{for all } u\in\mathcal{S}(\mathbb{R}^n), \tag{3.13}$$

where

$$R: L_q(\mathbb{R}^n)\to H^s_q(\mathbb{R}^n) \qquad \text{for all } s<\min(\tau-m,\tau).$$

Moreover, for any $N\in\mathbb{N}_0$ with $N<\tau$

$$\mathrm{OP}(q(x,y,\xi)) = \sum_{|\alpha|\leqslant N}\mathrm{OP}(q_\alpha(x,\xi)) + \mathrm{OP}(\tilde{r}_N(x,y,\xi)),$$

where the entries are defined by (3.11), with $A(x,y)=\int_0^1\nabla_x F(x+t(y-x))\,dt$ for every $y\in\mathbb{R}^n$ sufficiently close to x.

Proof. Since $\nabla F\in C^\tau(\mathbb{R}^n)$ and satisfies (3.12), there is some $\delta>0$ such that $A(x,y)$ is invertible for every $x,y\in\mathbb{R}^n$ with $|x-y|<\delta$. Now choose $\psi\in C_0^\infty(\mathbb{R}^n)$ with $\psi\equiv 1$ on $B_{\delta/2}(0)$ and

$\operatorname{supp}\psi \subset B_\delta(0)$. Then

$$\begin{aligned} Pu &= \mathrm{Os}-\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}\psi(x-y)p(x,\xi)u(y)\,dy\,đ\xi \\ &\quad + \mathrm{Os}-\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}(1-\psi(x-y))p(x,\xi)u(y)\,dy\,đ\xi \equiv P'u + \mathrm{OP}(a(x,y,\xi))u, \end{aligned}$$

where

$$\mathrm{OP}(a(x,y,\xi)) : L_q(\mathbb{R}^n) \to H_q^s(\mathbb{R}^n) \qquad \text{for all } s < \min(\tau - m, \tau)$$

by Theorem 3.6 since $a(x,y,\xi) = 0$ if $|x-y| < \delta/2$. Moreover,

$$\begin{aligned} \underline{P}'u(x) &= \mathrm{Os}-\int_{\mathbb{R}^{2n}} e^{i(F(x)-z)\cdot\eta}\psi(F(x)-z)p(F(x),\eta)u(F^{-1}(z))\,dz\,đ\eta \\ &= \mathrm{Os}-\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}q(x,y,\xi)u(y)\,dy\,đ\xi, \end{aligned}$$

where

$$q(x,y,\xi) = \psi(F(x)-F(y))p(F(x),A(x,y)^{-1,T}\xi)|\det A(x,y)|^{-1}|\det\nabla_y F(y)| \tag{3.14}$$

for all $x,y,\xi \in \mathbb{R}^n$. The rest follows in the same way as in the proof of Theorem 3.8, since $\eta(F(x)-F(y))|_{x=y} = 1$ and $\partial_x^\alpha(\eta(F(x)-F(y)))|_{x=y} = 0$ for every $|\alpha| \leqslant N$. □

Proof of Theorem 1.2. The first part is a consequence of Corollary 3.3, Theorem 2.1 applied to p_α, and Corollary 3.5 applied to r_α. Theorem 3.8 implies the second part. □

4 | PRELIMINARIES FOR OPERATORS ON DOMAINS

4.1 | The μ-transmission spaces for the halfspace and other smooth domains

For Ω equal to $\mathbb{R}^n_+$ or a bounded smooth domain, the special *μ-transmission spaces* were introduced for all $\mu \in \mathbb{C}$ by Hörmander [39] for $q = 2$, cf. the account in [31] where the spaces are redefined and extended to general $q \in (1,\infty)$. In the present paper we take μ real with $\mu > -1$. The spaces are defined by use of the order-reducing operators recalled in Section 2.2:

$$\begin{aligned} H_q^{\mu(s)}(\overline{\mathbb{R}}^n_+) &= \begin{cases} \Xi_+^{-\mu}e^+\overline{H}_q^{s-\mu}(\mathbb{R}^n_+) = \Lambda_+^{-\mu}e^+\overline{H}_q^{s-\mu}(\mathbb{R}^n_+), & \text{for } s > \mu - \frac{1}{q'}, \\ \dot{H}_q^s(\overline{\mathbb{R}}^n_+), & \text{for } s \leqslant \mu - \frac{1}{q'}, \end{cases} \\ H_q^{\mu(s)}(\overline{\Omega}) &= \begin{cases} \Lambda_+^{(-\mu)}e^+\overline{H}_q^{s-\mu}(\Omega), & \text{for } s > \mu - \frac{1}{q'}, \\ \dot{H}_q^s(\overline{\Omega}), & \text{for } s \leqslant \mu - \frac{1}{q'}. \end{cases} \end{aligned} \tag{4.1}$$

Here $\frac{1}{q'} = 1 - \frac{1}{q}$; for convenience of notation we have included the cases $s \leqslant \mu - \frac{1}{q'}$ (as mentioned in [31, Definition 1.5]), although they play a very small role in regularity studies. The spaces decrease with growing s: $H_q^{\mu(s)}(\overline{\Omega}) \subset H_q^{\mu(s')}(\overline{\Omega})$ when $s > s'$.

Remark 4.1. From the definition and the interpolation properties of Bessel potential spaces it follows that $H_q^{\mu(s)}(\overline{\Omega})$ is preserved under complex interpolation in s when $s > \mu - \frac{1}{q'}$.

We now list some further properties, formulated for Ω with the convention that $\Lambda_+^{(t)}$ is replaced by Ξ_+^t or Λ_+^t when $\mathbb{R}_+^n$ is considered. There holds (cf. [31, Definition 1.8])

$$\|u\|_{H_q^{\mu(s)}(\overline{\Omega})} \simeq \|r^+\Lambda_+^{(\mu)}u\|_{\overline{H}_q^{s-\mu}(\Omega)}, \text{ when } s > \mu - \tfrac{1}{q'}. \tag{4.2}$$

Observe in particular that

$$\begin{aligned} H_q^{\mu(s)}(\overline{\Omega}) &= \dot{H}_q^s(\overline{\Omega}) \text{ for } s - \mu \in (-\tfrac{1}{q'}, \tfrac{1}{q}), \\ \dot{H}_q^s(\overline{\Omega}) \subset H_q^{\mu(s)}(\overline{\Omega}) &\subset H_{q,\mathrm{loc}}^s(\Omega) \text{ for all } s \in \mathbb{R}, \end{aligned} \tag{4.3}$$

since $e^+\overline{H}_q^{s-\mu}(\Omega) \supset \dot{H}_q^{s-\mu}(\overline{\Omega})$ for all $s > \mu - \frac{1}{q'}$, with equality if $s - \mu \in (-\frac{1}{q'}, \frac{1}{q})$, and since $\Lambda_+^{(-\mu)}$ is elliptic. We have moreover, for $s \geqslant \mu$,

$$H_q^{\mu(s)}(\overline{\Omega}) \subset H_q^{\mu(\mu)}(\overline{\Omega}) = \dot{H}_q^{\mu}(\overline{\Omega}). \tag{4.4}$$

The great interest of the spaces $H_q^{\mu(s)}(\overline{\Omega})$ lies in the following facts.

- Pseudodifferential operators P of order m satisfying the μ-transmission condition map these spaces into standard spaces $\overline{H}_q^{s-m}(\Omega)$, by [31, Theorem 4.2],

$$\|r^+Pu\|_{\overline{H}_q^{s-m}(\Omega)} \leqslant C\|u\|_{H_q^{\mu(s)}(\overline{\Omega})} \quad \text{for } s > \mu - \tfrac{1}{q'}. \tag{4.5}$$

- When P furthermore is strongly elliptic, the spaces are the solution spaces for the homogeneous Dirichlet problem, cf. [31, Theorem 4.4].

Example 4.2. To demonstrate how (4.1) enters into the picture, we give a simple example: Let $P = \mathrm{OP}(p(\xi))$, where the C^∞-function $p(\xi)$ is homogeneous of degree $2a$ and even for $|\xi| \geqslant 1$, and strongly elliptic satisfying $\operatorname{Re} p(\xi) \geqslant c > 0$ for $\xi \in \mathbb{R}^n$. The Dirichlet problem on $\mathbb{R}_+^n$,

$$r^+Pu = f \text{ on } \mathbb{R}_+^n, \quad \operatorname{supp} u \subset \overline{\mathbb{R}}_+^n,$$

is uniquely solvable for $f \in L_2(\mathbb{R}_+^n)$, when u is sought in $\dot{H}^a(\overline{\mathbb{R}}_+^n)$; this follows by applying the Lax–Milgram lemma (cf., for example, [29, Section 12.4]) to the sesquilinear form $s(u, v) = \int_{\mathbb{R}_+^n} r^+Pu\,\bar{v}\,dx$ on $\dot{H}^a(\overline{\mathbb{R}}_+^n)$. A precise description of the domain $D(P_D) = \{u \in \dot{H}^a(\overline{\mathbb{R}}_+^n) \mid r^+Pu \in L_2(\mathbb{R}_+^n)\}$ can be found by letting $Q = \Lambda_-^{-a}P\Lambda_+^{-a}$; it defines a bijection $Q_+ = r^+Qe^+$ in $L_2(\mathbb{R}_+^n)$. Here Q is of order 0 and satisfies Boutet de Monvel's 0-transmission condition at $\partial\mathbb{R}_+^n$; hence, Q_+ is also a bijection in $\overline{H}_q^s(\mathbb{R}_+^n)$ for all $1 < q < \infty$, $s > -1/q'$. When this is carried back to P, we see that r^+P maps the a-transmission space $\Lambda_+^{-a}e^+\overline{H}_q^{s+a}(\mathbb{R}^n_+) = H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_+)$ bijectively to $\overline{H}_q^s(\mathbb{R}^n_+)$. In particular, $D(P_D) = H^{a(2a)}(\overline{\mathbb{R}}^n_+)$.

We note that the spaces do not depend on P. As a very special case of (4.5), since $\Lambda_+^{(\mu)}$ satisfies the μ-transmission condition, so does the composition $P = \Lambda_+^{(\mu)}\circ\varphi$ for any $\varphi \in C_b^\infty(\mathbb{R}^n)$; hence, for any $s > \mu - \frac{1}{q'}$ there is some $C > 0$ such that

$$\|\varphi u\|_{H_q^{\mu(s)}(\overline{\Omega})} \simeq \|r^+\Lambda_+^{(\mu)}\varphi u\|_{\overline{H}_q^{s-\mu}(\Omega)} \leqslant C\|u\|_{H_q^{\mu(s)}(\overline{\Omega})} \text{ for all } u \in H_q^{\mu(s)}(\overline{\Omega}),$$

cf. also (4.2). Thus the multiplication by φ maps $H_q^{\mu(s)}(\overline{\Omega})$ into itself (also for lower s, since the property is well known for the spaces $\dot{H}_q^s(\overline{\Omega})$).

With d defined as in (2.4) (in particular, d can equal d_0), there are local weighted boundary operators

$$\gamma_j^\mu u = \Gamma(\mu+1+j)\gamma_j(u/d^\mu) : H_q^{\mu(s)}(\overline{\Omega}) \to B_q^{s-\mu-j-\frac{1}{q}}(\partial\Omega) \tag{4.6}$$

defined for $s > \mu + j + \frac{1}{q}$, here γ_j denotes the jth normal derivative $\gamma_j v = (\partial_n^j v)|_{\partial\Omega}$, and Γ denotes the Gamma function. There is a hierarchy (cf. [31, Section 5]), $H_q^{\mu(s)}(\overline{\Omega}) \supset H_q^{(\mu+1)(s)}(\overline{\Omega}) \supset \cdots \supset H_q^{(\mu+j)(s)}(\overline{\Omega})$ for $s > \mu + j - \frac{1}{q'}$, and

$$u \in H_q^{(\mu+j)(s)}(\overline{\Omega}) \iff u \in H_q^{\mu(s)}(\overline{\Omega}) \text{ with } \gamma_0^\mu u = \cdots = \gamma_{j-1}^\mu u = 0; \tag{4.7}$$

this is of importance for the study of nonhomogeneous boundary conditions.

Defining $\mathcal{E}_\mu(\overline{\Omega}) \equiv e^+d^\mu\overline{C}^\infty(\Omega)$, we have for bounded smooth domains that $\bigcap_s H_q^{\mu(s)}(\overline{\Omega}) = \mathcal{E}_\mu(\overline{\Omega})$, which is dense in $H_q^{\mu(s)}(\overline{\Omega})$. For $\mathbb{R}^n_+$, $\bigcap_s H_q^{\mu(s)}(\overline{\mathbb{R}}^n_+) = \mathcal{E}_\mu(\overline{\mathbb{R}}^n_+) \cap \bigcap_s H_q^{\mu(s)}(\overline{\mathbb{R}}^n_+)$, and $\mathcal{E}_\mu(\overline{\mathbb{R}}^n_+) \cap \mathcal{E}'(\mathbb{R}^n)$ is dense in $H_q^{\mu(s)}(\overline{\mathbb{R}}^n_+)$.

It was shown in [31] that

$$H_q^{\mu(s)}(\overline{\Omega}) \subset \dot{H}_q^s(\overline{\Omega}) + e^+d^\mu\overline{H}_q^{s-\mu}(\Omega), \text{ for } s > \mu + \tfrac{1}{q}, s - \mu - \tfrac{1}{q} \notin \mathbb{N}, \tag{4.8}$$

and the inclusion holds with $\dot{H}_q^s(\overline{\Omega})$ replaced by $\dot{H}_q^{s-\varepsilon}(\overline{\Omega})$ if $s - \mu - \frac{1}{q} \in \mathbb{N}$. There is a similar statement on $\mathbb{R}^n_+$ with d replaced by x_n.

In a recent paper [34], the representations (4.8) were sharpened by an identification of which functions in $e^+d^\mu\overline{H}_q^{s-\mu}(\Omega)$ actually enter in $H_q^{\mu(s)}(\overline{\Omega})$.

For the Hölder–Zygmund scale C_*^s, the μ-transmission spaces are defined analogously by

$$C_*^{\mu(s)}(\overline{\Omega}) = \Lambda_+^{(-\mu)}e^+\overline{C}_*^{s-\mu}(\Omega), \qquad \text{for } s > \mu - 1,$$

$$C_*^{\mu(s)}(\overline{\Omega}) = \dot{C}_*^s(\overline{\Omega}), \qquad \text{for } s \leqslant \mu - 1,$$

and all the properties listed above for H_q^s-spaces carry over to the C_*^s-spaces. The formulas hold with H_q^s replaced by C_*^s and $\frac{1}{q}, \frac{1}{q'}$ replaced by 0, 1. In particular, there is the analog of (4.8):

$$C_*^{\mu(s)}(\overline{\Omega}) \subset \dot{C}_*^s(\overline{\Omega}) + e^+d^\mu\overline{C}_*^{s-\mu}(\Omega), \text{ for } s > \mu, s - \mu \notin \mathbb{N}, \tag{4.9}$$

and the inclusion holds with $\dot{C}^s_*(\overline{\Omega})$ replaced by $\dot{C}^{s-\varepsilon}_*(\overline{\Omega})$ if $s - \mu \in \mathbb{N}$. There is a similar statement on $\mathbb{R}^n_+$ with d replaced by x_n.

4.2 | Definitions in nonsmooth cases

Recall that for any C^1-diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$ with $DF \in C^\tau(\mathbb{R}^n)^{n\times n}$ and $0 \leqslant s < 1+\tau$ we have that

$$F^* : H^s_q(\mathbb{R}^n) \to H^s_q(\mathbb{R}^n) : u \mapsto u\circ F \tag{4.10}$$

is bounded. In the case $s \leqslant 1 + [\tau]$, this follows by interpolation from the corresponding statement for $H^m_q(\mathbb{R}^n)$, $m = 0, \dots, 1 + [\tau]$. In the case $1 + [\tau] < s < 1 + \tau$ one uses that

$$\partial_{x_j}(u\circ F)(x) = (\nabla u)(F(x)) \cdot \partial_{x_j} F(x) \quad \text{for } u \in H^s_q(\mathbb{R}^n),$$

where $\nabla u\circ F \in H^{s-1}_q(\mathbb{R}^n)$ since $s - 1 \leqslant 1 + [\tau]$, $\partial_{x_j} F \in C^\tau(\mathbb{R}^n)^n$, and one can apply well-known multiplication results for Bessel potential spaces, namely, that $fg \in H^t_q(\mathbb{R}^n)$ for any $f \in C^\tau(\mathbb{R}^n)$, $g \in H^t_q(\mathbb{R}^n)$ if $|t| < \tau$ and $1 < q < \infty$, cf., for example, the book by Runst and Sickel [52, Section 4.7.1]. This result also follows from Theorem 2.1 for $p(x, \xi) = f(x)$. The mapping is a bijection if also $D(F^{-1}) \in C^\tau(\mathbb{R}^n)^{n\times n}$.

In the following, let $\mathbb{R}^n_\gamma = \{x \in \mathbb{R}^n \mid x_n > \gamma(x')\}$ for some $\gamma \in C^{1+\tau}(\mathbb{R}^{n-1})$ with $\tau > 0$, and let $F_\gamma : \mathbb{R}^n \to \mathbb{R}^n$ be such that

$$F_\gamma(x) = (x', x_n - \gamma(x')) \quad \text{for all } x \in \mathbb{R}^n, \text{ where } x' = (x_1, \dots, x_{n-1}); \tag{4.11}$$

note that for both F_γ and F^{-1}_γ (defined by $F_\gamma(x)^{-1} = (x', x_n + \gamma(x'))$) the differential is in $C^\tau(\mathbb{R}^n)^{n\times n}$.

Moreover, we consider a bounded open set $\Omega \subset \mathbb{R}^n$ with $C^{1+\tau}$-boundary.

Definition 4.3. Let $\mu > -1$, $\tau > 0$, $1 < q < \infty$, and let $\mu - \frac{1}{q'} < s < 1 + \tau$.

1° For the set $\mathbb{R}^n_\gamma$ with $\gamma \in C^{1+\tau}(\mathbb{R}^{n-1})$, we define

$$u \in H^{\mu(s)}_q(\overline{\mathbb{R}}^n_\gamma) \iff u\circ F^{-1}_\gamma \in H^{\mu(s)}_q(\overline{\mathbb{R}}^n_+),$$

and provide $H^{\mu(s)}_q(\overline{\mathbb{R}}^n_\gamma)$ with the inherited norm. In other words,

$$H^{\mu(s)}_q(\overline{\mathbb{R}}^n_\gamma) = F^*_\gamma(H^{\mu(s)}_q(\overline{\mathbb{R}}^n_+)). \tag{4.12}$$

2° When Ω is a bounded $C^{1+\tau}$-domain, each point $x_0 \in \partial\Omega$ has a bounded open neighborhood $U \subset \mathbb{R}^n$ and a $\gamma \in C^{1+\tau}(\mathbb{R}^{n-1})$, such that (after a suitable rotation) $\Omega \cap U = \mathbb{R}^n_\gamma \cap U$. We denote by $H^{\mu(s)}_q(\overline{\Omega})$ the set of all $u \in H^s_{q,loc}(\Omega)$ such that for each x_0, with a $\varphi \in C^\infty_0(U)$ with $\varphi \equiv 1$ in a neighborhood of x_0, we have

$$F^{*,-1}_\gamma(\varphi u) := (\varphi u)\circ F^{-1}_\gamma \in H^{\mu(s)}_q(\overline{\mathbb{R}}^n_+)$$

in the rotated situation, with F_γ defined by (4.11).

3° There are similar spaces defined with H_q^s replaced by C_*^s, q, q' replaced by $\infty, 1$.

From the inclusions (4.8) and (4.9) in the halfspace case we obtain the following.

Proposition 4.4. *Let $\mu > -1$ and $\mu - \frac{1}{q'} < s < 1 + \tau$ with $s - \mu < 1 + \tau$, and let $\gamma \in C^{1+\tau}(\mathbb{R}^{n-1})$. There is a function d as in (2.4)ff. such that (with $\varepsilon > 0$):*

$$H_q^{\mu(s)}(\overline{\mathbb{R}}^n_\gamma) \begin{cases} = \dot{H}_q^s(\overline{\mathbb{R}}^n_\gamma), & \textit{for } s < \mu + \frac{1}{q}, \\ \subset \dot{H}_q^{s(-\varepsilon)}(\overline{\mathbb{R}}^n_\gamma) + d^\mu e^+ \overline{H}_q^{s-\mu}(\overline{\mathbb{R}}^n_\gamma), & \textit{for } s > \mu + \frac{1}{q}, \end{cases}$$

where $(-\varepsilon)$ is active if $s - \mu - \frac{1}{q} \in \mathbb{N}$.

Moreover, the mapping $\gamma_0^\mu : u \mapsto \Gamma(\mu+1)(u/d^\mu)|_{\partial\mathbb{R}^n_\gamma}$ is continuous:

$$\gamma_0^\mu : H_q^{\mu(s)}(\overline{\mathbb{R}}^n_\gamma) \to B_q^{s-\mu-\frac{1}{q}}(\partial\mathbb{R}^n_\gamma), \textit{ for } s - \mu - \tfrac{1}{q} > 0.$$

If $\tau \geqslant 1$, then one can replace d by d_0, chosen as a $C^{1+\tau}$-function coinciding with the distance to $\partial\mathbb{R}^n_\gamma$ near $\partial\mathbb{R}^n_\gamma$, as indicated after (2.4).

There are similar results in C_*^s-spaces with $\frac{1}{q}$ replaced by 0.

Proof. The properties of $H_q^{\mu(s)}(\mathbb{R}^n_\gamma)$ follow from (4.1) and (4.8)ff, since the function x_n^μ carries over to the function d^μ where $d(x) = x_n - \gamma(x')$ has the mentioned properties, and since $F^{*,-1}$ maps $H_q^s(\mathbb{R}^n)$ and $H_q^{s-\mu}(\mathbb{R}^n)$ to themselves. Similarly, the properties of $C_*^{\mu(s)}(\mathbb{R}^n_\gamma)$ are carried over from (4.9). The definition of the trace follows from (4.6) for $\Omega = \mathbb{R}^n_+$.

If $\tau \geqslant 1$, we can replace $d(x) = x_n - \gamma(x')$ by $d_0(x)$, since $d(x) = f(x)d_0(x)$, where $f \in C^\tau(\overline{\mathbb{R}}^n_\gamma)$ is strictly positive, and multiplication with a $C^\tau(\overline{\mathbb{R}}^n_\gamma)$-functions maps $H_q^{s-\mu}(\overline{\mathbb{R}}^n_\gamma)$ into itself. ☐

For the use of local coordinates, we need to know how the multiplication by regular functions acts in the transmission spaces.

Proposition 4.5. *Let $\mu > -1$ and $\sigma > 0$. If $\mu \geqslant 0$, assume that $\sigma > \mu$ and $\mu - \frac{1}{q'} < s < \sigma$. If $-1 < \mu < 0$, assume that $\sigma > \mu + 1$ and $\mu - \frac{1}{q'} < s < \sigma - 1$. Then multiplication by a function $\varphi \in C^\sigma(\mathbb{R}^n)$ maps $H_q^{\mu(s)}(\overline{\mathbb{R}}^n_+)$ into itself.*

Consequently, if $\gamma \in C^\sigma(\mathbb{R}^{n-1})$, then multiplication by a function $\varphi \in C^\sigma(\mathbb{R}^n)$ maps $H_q^{\mu(s)}(\overline{\mathbb{R}}^n_\gamma)$ into itself.

Proof. For $s - \mu \in (-\frac{1}{q'}, \frac{1}{q})$, $H_q^{\mu(s)}(\overline{\mathbb{R}}^n_+) = \dot{H}_q^s(\overline{\mathbb{R}}^n_+)$, where the property is well known since $|s| < \sigma$, so we can assume $s \geqslant \mu$. There holds in general:

$$v \in H_q^{\mu(s)}(\overline{\mathbb{R}}^n_+) \iff \Lambda_+^\mu v \in e^+ \overline{H}_q^{s-\mu}(\mathbb{R}^n_+). \tag{4.13}$$

Let $u \in H_q^{\mu(s)}(\overline{\mathbb{R}}_+^n)$, and let $\varphi \in C_b^\sigma(\mathbb{R}^n)$. To show that $\varphi u \in H_q^{\mu(s)}(\overline{\mathbb{R}}_+^n)$, we must show that $\Lambda_+^\mu(\varphi u) \in e^+\overline{H}_q^{s-\mu}(\mathbb{R}_+^n)$. Here

$$\Lambda_+^\mu(\varphi u) = \varphi\Lambda_+^\mu u + [\Lambda_+^\mu, \varphi]u.$$

For the first term, $\Lambda_+^\mu u \in e^+\overline{H}_q^{s-\mu}(\mathbb{R}_+^n)$ by hypothesis, and multiplication by φ is known to preserve this space since $|s-\mu| < \sigma$. It remains to show that

$$[\Lambda_+^\mu, \varphi]u \in e^+\overline{H}_q^{s-\mu}(\mathbb{R}_+^n). \tag{4.14}$$

Here we shall borrow some continuity properties shown later in Theorem 5.13 and Corollary 5.14. The operator $[\Lambda_+^\mu, \varphi]$ may be written as a ψdo in (x,y)-form,

$$[\Lambda_+^\mu, \varphi] = \mathrm{OP}(a(x,y,\xi)); \quad a(x,y,\xi) = \lambda_+^\mu(\xi)(\varphi(x) - \varphi(y)),$$

satisfying the global μ-transmission condition and with symbol in $C^\sigma S^\mu(\mathbb{R}^{2n} \times \mathbb{R}^n)$.

If $\mu \geqslant 0$, we apply Theorem 5.13 with $\tau = \sigma$, $m = \mu$, and s replaced by s', to conclude that

$$r^+[\Lambda_+^\mu, \varphi] \colon H_q^{\mu(\mu+s')}(\overline{\mathbb{R}}_+^n) \to \overline{H}_q^{s'}(\mathbb{R}_+^n), \tag{4.15}$$

when $\sigma > \mu$, $0 \leqslant s' < \sigma - \mu$. The operator preserves support in $\overline{\mathbb{R}}_+^n$ since Λ_+^μ does so. With $s' = s - \mu$ we conclude that when $\sigma > \mu$, $\mu \leqslant s < \sigma$,

$$[\Lambda_+^\mu, \varphi] \colon H_q^{\mu(s)}(\overline{\mathbb{R}}_+^n) \to e^+\overline{H}_q^{s-\mu}(\mathbb{R}_+^n). \tag{4.16}$$

If $\mu \in (-1, 0)$, we apply Corollary 5.14 below with $\tau = \sigma$, $m = \mu$, and s replaced by s', to conclude that (4.15) holds when $\sigma > \mu + 1$, $0 \leqslant s' < \sigma - \mu - 1$. With $s' = s - \mu$ we conclude that (4.16) holds when $\sigma > \mu + 1$, $\mu \leqslant s < \sigma - 1$.

For the last statement, we note that when $u \in H_q^{\mu(s)}(\overline{\mathbb{R}}_\gamma^n)$, then $\varphi u \in H_q^{\mu(s)}(\overline{\mathbb{R}}_\gamma^n)$ holds if $(\varphi u)\circ F_\gamma^{-1} \in H_q^{\mu(s)}(\overline{\mathbb{R}}_+^n)$, by Definition 4.3. Here $(\varphi u)\circ F_\gamma^{-1} = (\varphi\circ F_\gamma^{-1})(u\circ F_\gamma^{-1})$, where $\varphi\circ F_\gamma^{-1} \in C_b^\sigma(\mathbb{R}^n)$, so the statement follows from what was proved in the case of $\overline{\mathbb{R}}_+^n$. □

The inclusions (4.8)ff. are shown for bounded $C^{1+\tau}$-domains as follows.

Theorem 4.6. *Let $\mu > -1$ and $\mu - \frac{1}{q'} < s < \tau$ with $s - \mu < \tau$, and let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1+\tau}$-domain with $\tau \geqslant 1$. Then (with $\varepsilon > 0$)*

$$H_q^{\mu(s)}(\overline{\Omega}) \begin{cases} = \dot{H}_q^s(\overline{\Omega}), & \text{for } s < \mu + \frac{1}{q}, \\ \subset \dot{H}_q^{s-\varepsilon}(\overline{\Omega}), & \text{for } s = \mu + \frac{1}{q}, \\ \subset \dot{H}_q^s(\overline{\Omega}) + d_0^\mu e^+\overline{H}_q^{s-\mu}(\Omega), & \text{for } s - \mu - \frac{1}{q} \in \mathbb{R}_+ \setminus \mathbb{N}, \\ \subset \dot{H}_q^{s-\varepsilon}(\overline{\Omega}) + d_0^\mu e^+\overline{H}_q^{s-\mu}(\Omega) & \text{for } s - \mu - \frac{1}{q} \in \mathbb{N}. \end{cases} \tag{4.17}$$

Moreover, the mapping $\gamma_0^\mu : u \mapsto \Gamma(\mu+1)(u/d_0^\mu)|_{\partial\Omega}$ *is continuous:*

$$\gamma_0^\mu : H_q^{\mu(s)}(\overline{\Omega}) \to B_q^{s-\mu-\frac{1}{q}}(\partial\Omega), \quad \text{for } s-\mu-\tfrac{1}{q} > 0.$$

There are similar results in C_*^s*-spaces, with* q, q' *replaced by* $\infty, 1$.

Proof. We formulate the proof for H_q^s-spaces; the proof for C_*^s-spaces is similar. We can assume $s > \mu + \frac{1}{q}$, noting that the identity is known when $s \in (\mu - \frac{1}{q'}, \mu + \frac{1}{q})$, and the spaces decrease with increasing s.

Let $u \in H_q^{\mu(s)}(\overline{\Omega})$ and let x_0, U, γ, φ be as in Definition 4.3 2°. Let $\psi \in C_0^\infty(U)$ satisfy $\psi\varphi = \varphi$. Let U' be the interior of the set where $\varphi = 1$. By Proposition 4.4 with $d = d_0$,

$$\varphi u = w + d_0^\mu e^+ v = \psi w + d_0^\mu e^+ \psi v \text{ with } \psi w \in \dot{H}_q^s(\overline{\Omega}), \psi v \in \overline{H}_q^{s-\mu}(\Omega), \tag{4.18}$$

using that multiplication by ψ preserves the space, by Proposition 4.5.

There is a finite set of points $\{x_{0,i}\}_{i=1,\dots,I}$ such that $\bigcup_i U_i' \supset \partial\Omega$ holds for the associated data $\{U_i, \gamma_i, \varphi_i, U_i', \psi_i\}$. Supply these sets with an open set $U_0' \supset \overline{\Omega} \setminus \bigcup_i U_i'$ with $\overline{U}_0' \subset \Omega$, and let $\{\varrho_i\}_{i=0,\dots,I}$ be an associated partition of unity, $\varrho_i \in C_0^\infty(U_i')$. Then $u = \sum_i \varrho_i u$. Moreover, $\varrho_i u = \varrho_i \varphi_i u$ for $i \geqslant 1$, where the $\varphi_i u$ satisfy (4.18).

Summation over i gives the statement in (4.17) with d replaced by d_0. (Note that the functions d_0 in the different charts are consistent near $\partial\Omega$, and their extensions further away play no role since u is in $H_{q,\mathrm{loc}}^s(\Omega)$ anyway).

The statement on the trace operator follows from Proposition 4.4 in a similar way. □

Remark 4.7. If we replace the assumption $\tau \geqslant 1$ in Theorem 4.6 by $\tau > 0$, we still obtain the following local inclusion: If $u \in H_q^{\mu(s)}(\overline{\Omega})$ and x_0, U, γ, φ are as in Definition 4.3.2°, then by Proposition 4.4:

$$\varphi u = w + d^\mu e^+ v \qquad \text{with } w \in \dot{H}_q^s(\overline{\mathbb{R}}_\gamma^n),\ v \in \overline{H}_q^{s-\mu}(\overline{\mathbb{R}}_\gamma^n),$$

where d may depend on γ.

Also higher traces as in (4.6) can be defined; we intend to take up their applications in later works.

The concepts are applied to the fractional Laplacian $(-\Delta)^a$ and its generalizations primarily for $a \in (0,1)$, but also higher values of a are of interest. The a-transmission spaces enter as solution spaces for the homogeneous Dirichlet problem. The $(a-1)$-transmission spaces allow the definition of nonzero Dirichlet and Neumann traces; this is the reason that we have made an effort to include cases $\mu \in (-1, 0)$ in the treatment.

The following commutation result will be needed later.

Proposition 4.8. *Let* $p \in C^\tau S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n)$ *for some* $\tau > 0$, $\tau \notin \mathbb{N}$, $m \in \mathbb{R}$, *and* $\varphi \in C_b^\infty(\mathbb{R}^n)$. *Then there is some* $q \in C^\tau S_{1,0}^{m-1}(\mathbb{R}^n \times \mathbb{R}^n)$ *such that*

$$[\mathrm{OP}(p), \varphi]u = \mathrm{OP}(q)u \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n).$$

Moreover, if $p \in C^\tau S^m(\mathbb{R}^n \times \mathbb{R}^n)$, *then* $q \in C^\tau S^{m-1}(\mathbb{R}^n \times \mathbb{R}^n)$ *and, if p is even, then q is odd.*

Proof. First of all we have

$$
\begin{aligned}
[P,\varphi]u(x) &= \mathrm{Os}-\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} p(x,\xi)(\varphi(y)-\varphi(x))u(y)\,dy\,đ\xi \\
&= \mathrm{Os}-\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} p(x,\xi)(y-x)\cdot\Phi(x,y)u(y)\,dy\,đ\xi \\
&= \mathrm{Os}-\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} D_\xi p(x,\xi)\cdot\Phi(x,y)u(y)\,dy\,đ\xi
\end{aligned}
$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, where $\Phi \in C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)^n$ is defined by

$$
\Phi(x,y) = \int_0^1 (\nabla\varphi)(x+t(y-x))\,dt \qquad \text{for all } x,y \in \mathbb{R}^n.
$$

Hence

$$
a(x,y,\xi) = D_\xi p(x,\xi)\cdot\Phi(x,y) \qquad \text{for all } x,y,\xi \in \mathbb{R}^n \tag{4.19}
$$

defines a symbol in $C^\tau S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n;\infty)$ as defined in [10]. Therefore, because of [10, Theorem 4.15], there is some $q = a_L \in C^\tau S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$
[P,\varphi]u(x) = \mathrm{Os}-\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a(x,y,\xi)u(y)\,dy\,đ\xi = \mathrm{OP}(q)u(x)
$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$. More precisely,

$$
a_L(x,\xi) = \mathrm{Os}-\int_{\mathbb{R}^{2n}} e^{-iy\cdot\eta} a(x,x+y,\eta+\xi)dy\,đ\eta \quad \text{for all } x,\xi \in \mathbb{R}^n
$$

and it follows first from [10, Theorem 4.15] that $a_L \in C^\tau S^{m-1}_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$. In order to see that $a_L \in C^\tau S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ one uses that

$$
\partial_\xi^\alpha a_L(x,\xi) = \mathrm{Os}-\int_{\mathbb{R}^{2n}} e^{-iy\cdot\eta}(\partial_\xi^\alpha a)(x,x+y,\eta+\xi)dy\,đ\eta \quad \text{for all } x,\xi \in \mathbb{R}^n
$$

due to [10, Theorem 2.11], where $\partial_\xi^\alpha a \in C^\tau S^{m-|\alpha|-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n;\infty)$. Applying [10, Theorem 4.15] again, we obtain $\partial_\xi^\alpha a_L \in C^\tau S^{m-|\alpha|-1}_{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0$, that is, $q = a_L \in C^\tau S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$.

Finally, we note that, using a Taylor expansion of $a(x,x+y,\eta+\xi)$ with respect to η (around 0) in a standard manner, it is easy to show that we have the asymptotic expansion

$$
a_L(x,\xi) \sim \sum\nolimits_{\alpha\in\mathbb{N}_0^n} \tfrac{1}{\alpha!}\partial_\xi^\alpha D_y^\alpha a(x,y,\xi)|_{y=x}.
$$

Moreover, $a \in C^\tau S^{m-1}(\mathbb{R}^{2n} \times \mathbb{R}^n)$ if $p \in C^\tau S^m(\mathbb{R}^n \times \mathbb{R}^n)$. Using the asymptotic expansion one easily observes that $q = a_L \in C^\tau S^{m-1}(\mathbb{R}^n \times \mathbb{R}^n)$. Furthermore, one verifies in a straightforward manner that a and $q = a_L$ are odd if p is even. $\square$

There is a corollary to Proposition 4.8, showing that an operator sandwiched between smooth functions with disjoint supports acts like an operator of arbitrarily low order and the same Hölder smoothness.

Corollary 4.9. *Let $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ for some $\tau > 0$, $\tau \notin \mathbb{N}$, $m \in \mathbb{R}$, and let $\varphi, \psi \in C_b^\infty(\mathbb{R}^n)$ with disjoint supports. For any $N \in \mathbb{N}$ there is a $q_N \in C^\tau S^{m-N}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that*

$$\varphi\,\mathrm{OP}(p)\psi u = \varphi\,\mathrm{OP}(q_N)u \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n).$$

(If $p \in C^\tau S^m(\mathbb{R}^n \times \mathbb{R}^n)$, then $q_N \in C^\tau S^{m-N}(\mathbb{R}^n \times \mathbb{R}^n)$ and, if p is even, then q_N is even for even N, odd for odd N.)

Proof. Setting $\psi_0 = \psi$, we can for $N = 1, 2, \ldots$ choose a nested sequence of C_b^∞-functions ψ_N with supports disjoint from $\operatorname{supp}\varphi$, such that ψ_N is 1 on $\operatorname{supp}\psi_{N-1}$ for all N. By Proposition 4.8, $\varphi\,\mathrm{OP}(p)\psi_0 = \varphi[\mathrm{OP}(p), \psi_0] = \varphi\,\mathrm{OP}(q_1)$ with $q_1 \in C^\tau S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Since $\psi_1\psi_0 = \psi_0$, we can repeat the argument with

$$\varphi\,\mathrm{OP}(p)\psi_0 = \varphi\,\mathrm{OP}(p)\psi_0\psi_1 = \varphi\,\mathrm{OP}(q_1)\psi_1 = \varphi[\mathrm{OP}(q_1), \psi_1] = \varphi\,\mathrm{OP}(q_2),$$

where $q_2 \in C^\tau S^{m-2}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Continuing in this way, with the Nth step being

$$\varphi\,\mathrm{OP}(p)\psi_0 = \varphi\,\mathrm{OP}(p)\psi_0\psi_N = \varphi\,\mathrm{OP}(q_N)\psi_N = \varphi[\mathrm{OP}(q_N), \psi_N] = \varphi\,\mathrm{OP}(q_{N+1}),$$

shows the main assertion. The last statement follows from the corresponding statement in Proposition 4.8. □

5 | NONSMOOTH TRANSMISSION CONDITIONS

5.1 | Transmission conditions for nonsmooth symbols

For the consideration of ψdo's P on open subsets of $\mathbb{R}^n$ one needs conditions that govern their behavior at a boundary. There have been many contributions through the times, mainly for ψdo's with smooth x-dependence. The *transmission property* in case of a smooth open set Ω is the property that r^+Pe^+ maps $C^\infty(\overline{\Omega}) \cap \mathcal{E}'(\mathbb{R}^n)$ into $C^\infty(\overline{\Omega})$. Necessary and sufficient conditions for this property have been established in several works, and sufficient conditions have been introduced under additional requirements (for example, parameter-dependence) — called *transmission conditions*. (References are given in Example 5.2 below.)

For some ψdo's P, r^+Pe^+ does not preserve $C^\infty(\overline{\Omega})$, but maps another space $d^\mu C^\infty(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$; they satisfy the so-called *μ-transmission condition*, where the above-mentioned case is the case $\mu = 0$.

We now consider nonsmooth situations: $P = \mathrm{OP}(p(x,\xi))$ is a ψdo with symbol p in $C^\tau S^m(\mathbb{R}^n \times \mathbb{R}^n)$, and it is considered relative to an open subset $\Omega \subset \mathbb{R}^n$ with $C^{1+\tau}$-boundary. The definition of the μ-transmission condition from [39] and [31] (and [40, Section 18.2] with a different notation) will here be generalized to be the requirement that the difference between p and a certain symbol obtained by twisted reflections of the homogeneous symbol terms vanishes to the order τ at $\partial\Omega$. In details:

Definition 5.1. Let $m, \mu \in \mathbb{R}$ and $\tau \in \overline{\mathbb{R}}_+$. Let $p(X,\xi) \in C^\tau S^m(\mathbb{R}^{n'} \times \mathbb{R}^n)$ with the expansion in homogeneous terms $p(X,\xi) \sim \sum_{j\in\mathbb{N}_0} p_j(X,\xi)$.

1° $p(X,\xi)$ will be said to satisfy the μ-transmission condition with respect to $\overline{\mathbb{R}}^n_+$ at X, when there holds for all $j \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$,

$$\partial_\xi^\alpha p_j(X,0,-1) = e^{i\pi(m-2\mu-j-|\alpha|)}\partial_\xi^\alpha p_j(X,0,1). \tag{5.1}$$

For $n' = n$, $X = x$, $p(x,\xi)$ will be said to satisfy the μ-transmission condition with respect to $\overline{\mathbb{R}}^n_+$, when for all $x' \in \mathbb{R}^{n-1}$, $j \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$,

$$\partial_x^\beta\partial_\xi^\alpha p_j(x',0,0,-1) = e^{i\pi(m-2\mu-j-|\alpha|)}\partial_x^\beta\partial_\xi^\alpha p_j(x',0,0,1), \text{ for } |\beta| \leqslant \tau. \tag{5.2}$$

2° For $n' = n$ or $2n$, $X = x$ or (x,y), $p(X,\xi)$ will be said to satisfy the **extended** μ-transmission condition with respect to $\overline{\mathbb{R}}^n_+$, when there is an $\varepsilon > 0$ such that for the points X in the region $\{0 \leqslant x_n < \varepsilon\}$, respectively, $\{0 \leqslant x_n, y_n < \varepsilon\}$, (5.1) holds for $j \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$. The condition is said to be **global** if all $\varepsilon > 0$ can be used.

3° For an open set Ω with $C^{1+\tau}$-boundary, analogous conditions are formulated with $(x',0)$ replaced by $x_0 \in \partial\Omega$, $(0,1)$ replaced by $\nu(x_0)$, and $x = (x', x_n)$ replaced by $x = x_0 + t\nu(x_0)$, $x_0 \in \partial\Omega$ and $0 \leqslant t < \varepsilon$ (with t playing the role of x_n).

Note that in 2°, equalities for X-derivatives as in 1° follow simply by differentiating the identities (5.1) up to order $[\tau]$. Note also that addition of an integer to μ does not change the formulas; the conditions depend only on $\mu(\mathrm{mod}\, \mathbb{Z})$.

We need the extended 0-transmission condition in order to apply the results of Abels [4], where the mapping properties for truncated integer-order operators depend on an extended definition of Poisson operators.

Example 5.2.

(1) *The case* $\mu = 0$. The operators considered by Boutet de Monvel [17], Grubb [26, 28, 29], Rempel and Schulze [49] are of integer order and satisfy the *0*-transmission condition with $\tau = \infty$. In [16] also noninteger-order classical ψdo's were included. Grubb and Hörmander [36] treated operators of arbitrary orders with symbols in $S^m_{\varrho,\delta}$-spaces, giving general conditions that are necessary and sufficient for the transmission property. Abels [4] introduced a generalization to operators of integer order with finite C^τ-smoothness (defining a slightly different version of the global 0-transmission condition).

(2) *General* μ. Simple examples of symbols satisfying the global μ-transmission condition with respect to $\overline{\mathbb{R}}^n_+$ are $\chi_+^\mu(\xi) = (\langle\xi'\rangle + i\xi_n)^\mu$ and its truly ψdo variant λ_+^μ (see (2.10)ff.); here $\chi^\mu_{+,0}(0,\pm1) = \lambda^\mu_{+,0}(0,\pm1) = (\pm i)^\mu$. Note that λ^μ_- is of 0-transmission type.

(3) *Even symbols.* When P is of order $m = 2a$ and even, cf. (2.11), it satisfies the a-transmission condition with respect to any halfspace, so it in fact fulfills the global a-transmission condition relative to any $C^{1+\tau}$-domain. Examples of operators in this category are: *fractional powers of elliptic differential operators*, including $(-\Delta)^a$. (cf. [31, Lemma *2.9* and Example *3.2*].) Note that the symbol $p' = p - p_0$, considered as a symbol of order $m' = m - 1$, is *odd*. If $m = 2a$, hence $m' = 2a - 1$, p' satisfies the a-transmission condition.

The conditions are preserved under multiplication in the following way.

Proposition 5.3. *When $p(X,\xi)\in C^\tau S^m(\mathbb{R}^{n'}\times\mathbb{R}^n)$ and $p'(X,\xi)\in C^\tau S^{m'}(\mathbb{R}^{n'}\times\mathbb{R}^n)$, satisfying the (extended) μ-transmission, respectively, μ'-transmission condition, then $p(X,\xi)p'(X,\xi)\in C^\tau S^{m+m'}(\mathbb{R}^{n'}\times\mathbb{R}^n)$ satisfying the (extended) $(\mu+\mu')$-transmission condition.*

Proof. This follows straightforwardly from the definition, using that C^τ is preserved under multiplication. The jth term in pp' is $(pp')_j=\sum_{k+l=j}p_kp'_l$, homogeneous of order $m+m'-j$, where

$$p_k(X,0,-\xi_n)p'_l(X,0,-\xi_n)=e^{i\pi(m+m'-2(\mu+\mu')-j)}p_k(X,0,\xi_n)p'_l(X,0,\xi_n).\qquad\square$$

The 0-transmission condition for $p(X,\xi)$ can also be expressed by formulations in terms of $\tilde p(X,\xi',z_n)=\mathcal{F}^{-1}_{\xi_n\to z_n}p$, where it means smoothness from the right.

Theorem 5.4. *Let $m\in\mathbb{R}$ and $\tau\in\overline{\mathbb{R}}_+$, and let $p(X,\xi)\in C^\tau S^m(\mathbb{R}^{n'}\times\mathbb{R}^n)$. Then p satisfies the 0-transmission condition with respect to $\overline{\mathbb{R}}^n_+$ at X((5.1) with $\mu=0$) if and only if $r^+\tilde p(X,\xi',z_n)=r^+\mathcal{F}^{-1}_{\xi_n\to z_n}p(X,\xi)$ satisfies*

$$r^+z_n^{\alpha_n}\partial_{\xi'}^{\alpha'}\tilde p_j(X,0,z_n)\in C^\infty(\overline{\mathbb{R}}_+),\ \textit{all } j\in\mathbb{N}_0,\alpha\in\mathbb{N}_0^n.\tag{5.3}$$

When $p(X,\xi)$ satisfies the extended 0-transmission condition with respect to $\overline{\mathbb{R}}^n_+$, $\tilde p$ moreover satisfies the estimates, for all $j,k,l\in\mathbb{N}_0,\alpha\in\mathbb{N}_0^{n-1}$:

$$\|z_n^k\partial_{z_n}^l\partial_{\xi'}^\alpha r^+\tilde p_j(.,\xi',z_n)\|_{C^\tau(U;L_{2,z_n}(\mathbb{R}_+))}\leqslant C_{k,l,\alpha}\langle\xi'\rangle^{m+\frac12-k+l-j-|\alpha|},\tag{5.4}$$

$$\|z_n^k\partial_{z_n}^l\partial_{\xi'}^\alpha r^+\tilde p(.,\xi',z_n)\|_{C^\tau(U;L_{2,z_n}(\mathbb{R}_+))}\leqslant C_{k,l,\alpha}\langle\xi'\rangle^{m+\frac12-k+l-|\alpha|},\tag{5.5}$$

where $U=\{x\in\mathbb{R}^n\mid x_n\in[0,\varepsilon)\}$ if $n'=n$ and $U=\{(x,y)\in\mathbb{R}^{2n}\mid x_n,y_n\in[0,\varepsilon)\}$ if $n'=2n$. In the case that p satisfies the global 0-transmission condition with respect to $\overline{\mathbb{R}}^n_+$, the estimates (5.4)–(5.5) hold true with $U=\overline{\mathbb{R}}^n_+$, respectively, $\overline{\mathbb{R}}^n_+\times\overline{\mathbb{R}}^n_+$.

Proof. Many of the ingredients in the proof were already present in [16]. The details we give below make use of later developments.

First let $m\in\mathbb{Z}$. Here (5.2) takes the form

$$\partial_\xi^\alpha p_j(X,0,-\xi_n)=(-1)^{m-j-|\alpha|}\partial_\xi^\alpha p_j(X,0,\xi_n),\tag{5.6}$$

and clearly holds also with ξ_n and $-\xi_n$ interchanged (is in some texts in fact written as such), so it is *two-sided*, valid also with respect to $\overline{\mathbb{R}}^n_-$. In [17], and in many later works, for example, [4, 28, 29, 53], it is replaced by a condition where p as a function of ξ_n takes values in the space $\mathcal{H}=\mathcal{F}_{z_n\to\xi_n}(e^+\mathcal{S}(\overline{\mathbb{R}}_+)\oplus e^-\mathcal{S}(\overline{\mathbb{R}}_-))\oplus\mathbb{C}[\xi_n]$ with certain estimates; they imply (5.3)–(5.5), and vice versa. The equivalence of (5.6) with the $\mathcal{H}$-estimates is shown in [17], and in [28, Theorem 2.2.5] in a situation with a parameter. We shall not take up further space here with details.

Now let $m\in\mathbb{R}\setminus\mathbb{Z}$. First consider the case where $m<-1$. We study each homogeneous term in the symbol individually; take, for example, p_0. Consider $p_0(X,0,\xi_n)$ at a fixed X. It is homogeneous in ξ_n of degree m for $|\xi_n|\geqslant1$. By the rules of Fourier transformation of homogeneous functions of one variable we have, as shown in detail, for example, in [31, Lemma 2.7] (which takes the behavior for $|\xi_n| \leqslant 1$ into account), that the twisted parity property

$$p_0(X, 0, -\xi_n) = e^{i\pi m} p_0(X, 0, \xi_n) \text{ for } |\xi_n| \geqslant 1,$$

holds if and only if $r^+ \tilde{p}_0(X, 0, z_n)$ is zero on $\mathbb{R}_+$ modulo $C^\infty(\overline{\mathbb{R}}_+)$. Likewise, the ξ-derivatives have the corresponding twisted parity if and only if $r^+ z_n^{\alpha_n} \partial_{\xi'}^{\alpha'} \tilde{p}_0(X, 0, z_n)$ is zero on $\mathbb{R}_+$ modulo $C^\infty(\overline{\mathbb{R}}_+)$. This shows the equivalence with (5.3) for $j = 0$.

The consequences can be further analyzed: Since $p_0(X, \xi)$ is a symbol of order m, one has that

$$\|\partial_{\xi_n}^k \xi_n^l \partial_{\xi'}^\alpha p_0\|_{L_{2,\xi_n}(\mathbb{R})} \leqslant C_{k,l,\alpha} \langle \xi' \rangle^{m+\frac{1}{2}-k+l-|\alpha|}, \text{ for } k > m + l - |\alpha| + 2,$$

and hence $z_n^k \partial_{z_n}^l \partial_{\xi'}^\alpha \tilde{p}_0$ is in $L_{2,z_n}(\mathbb{R})$ and

$$\|z_n^k \partial_{z_n}^l \partial_{\xi'}^\alpha r^+ \tilde{p}_0\|_{L_{2,z_n}(\mathbb{R}_+)} \leqslant \|z_n^k \partial_{z_n}^l \partial_{\xi'}^\alpha \tilde{p}_0\|_{L_{2,z_n}(\mathbb{R})} \leqslant C_{k,l,\alpha} \langle \xi' \rangle^{m+\frac{1}{2}-k+l-|\alpha|}, \tag{5.7}$$

for such indices. When $\xi' = 0$, the left entry is moreover, a fortiori, bounded for all lower values of $k \in \mathbb{N}_0$, in view of the smoothness for $z_n \to 0+$ shown above. In particular,

$$r^+ \partial_{\xi'}^\alpha \tilde{p}_0(X, 0, z_n) \in \mathcal{S}(\overline{\mathbb{R}}_+) \tag{5.8}$$

as a function of z_n.

We can extend the estimates to $\xi' \neq 0$ by a Taylor expansion in ξ', using the estimates for $\xi' = 0$ (and handling remainders by use of symbol estimates), as in the detailed proof of [31, Theorem 2.6].

The estimates moreover hold with ∂_X^β inserted, for $|\beta| \leqslant [\tau]$. This shows that estimates (5.4) hold for p_0 with τ replaced by $[\tau]$. When $\tau = [\tau] + \sigma$, $\sigma \in (0, 1)$, we also apply the considerations to $(\partial^\beta p_0(X, \xi) - \partial_x^\beta p_0(Y, \xi))/|X - Y|^\sigma$ for $|\beta| = [\tau]$, $X \neq Y$; these functions likewise have the twisted parity property allowing to conclude the smoothness for $z_n \to 0+$ of the inverse Fourier transformed function when $\xi' = 0$, with estimates as above. This can, as above, be extended to estimates of the type (5.4) with τ replaced by 0. The validity with uniform bounds in X, Y then implies that (5.4) holds for p_0.

There are similar proofs for the other terms p_j.

Larger values of m are included as follows: When a positive integer r is chosen so large that $m - 2r < -1$, then the above analysis applies to $q(X, \xi) = p(X, \xi)(1 + \sum_{l=1}^n \xi_l^{2r})^{-1}$. Now $p = q + \sum_{l=1}^{n-1} q\xi_l^{2r} + q\xi_n^{2r}$. The above analysis carries over directly to the first two terms. The third term leads to the function $r^+ D_{z_n}^{2r} \tilde{q}$, which is also included in the analysis.

For the last estimate (5.5), we use that $p - \sum_{j<J} p_j$ satisfies estimates like (5.7) with $m \to -\infty$ when $J \to \infty$, so that when the term is added to the finite sum for $j < J$, the estimates cover more of the desired indices, the larger J is taken. □

Corollary 5.5. *Let $m, \mu \in \mathbb{R}$ and $\tau \in \overline{\mathbb{R}}_+$, and let $p(X, \xi) \in C^\tau S^m(\mathbb{R}^{n'} \times \mathbb{R}^n)$. Then p satisfies the (extended) μ-transmission condition with respect to $\overline{\mathbb{R}}^n_+$ if and only if $b(X, \xi) = p(X, \xi)\lambda_+^{-\mu}(\xi)$ satisfies the equivalent conditions in Theorem 5.4 with m replaced by $m - \mu$.*

One can here replace $\lambda_+^{-\mu}(\xi)$ by any other invertible symbol $l(X, \xi) \in C^\tau S^{-\mu}(\mathbb{R}^n \times \mathbb{R}^{n'})$ satisfying the (extended) $(-\mu)$-transmission condition, for which $1/l(X, \xi)$ is in $C^\tau S^\mu(\mathbb{R}^{n'} \times \mathbb{R}^n)$ satisfying the (extended) μ-transmission condition.

Proof. Note that $b = p\lambda_+^{-\mu}$ is of order $m' = m - \mu$ with homogeneous terms $b_j = p_j\lambda_+^{-\mu}$ of degree $m' - j$. The system of identities (5.2) for p is equivalent with the analogous system of identities for b with m, μ replaced by $m - \mu, 0$. To check this, it suffices in view of the homogeneity to evaluate the symbols for $\xi_n = \pm 1$. Since $\lambda_+^{-\mu}(0, \pm 1) = (\pm i)^{-\mu} = e^{\mp i\mu\pi/2}$, b_0 satisfies

$$b_0(X,0,-1) - e^{i\pi(m-\mu)}b_0(X,0,1) = p_0(X,0,-1)e^{i\mu\pi/2} - e^{i\pi(m-\mu)}p_0(X,0,1)e^{-i\mu\pi/2}$$
$$= e^{i\mu\pi/2}[p_0(X,0,-1) - e^{i\pi(m-2\mu)}p_0(X,0,1)] = 0,$$

by the hypothesis on p_0. The lower order terms and derivatives are checked similarly, and then Theorem 5.4 readily applies.

The last statement is likewise straightforward, in view of Proposition 5.3. □

As a special case, the even $2a$-order symbols fit into the setup as follows.

Example 5.6. Let $a \in \mathbb{R}$, and let $p(X,\xi) \in C^\tau S^{2a}(\mathbb{R}^{n'} \times \mathbb{R}^n)$. Assume that p is *even*, cf. Example 5.2. Then p satisfies the global a-transmission condition with respect to $\overline{\mathbb{R}}^n_+$ with $m = 2a$, and $b = p\lambda_+^{-a}$ is in $C^\tau S^a(\mathbb{R}^{n'} \times \mathbb{R}^n)$ satisfying the global 0-transmission condition with respect to $\overline{\mathbb{R}}^n_+$.

5.2 | Mapping properties over the halfspace and smooth sets

In preparation for showing mapping properties of ψdo's $P = \mathrm{OP}(p(x,\xi))$ truncated to the half-space $\mathbb{R}^n_+$, we shall consider the Poisson-type operators that arise in connection with the truncation. Recall that there holds (as a version of Green's formula):

$$D_n^k e^+ u = e^+ D_n^k u - i\sum\nolimits_{l=0}^{k-1} \gamma_l^c u(x') \otimes D_n^{k-1-l}\delta(x_n),$$

cf., for example, [28, (2.2.39)]. (We are here using the complex trace operator $\gamma_j^c u(x') = D_n^j u(x',0) = i^{-j}(\partial_n^j u)|_{x_n=0}$.) Then when r^+P is applied to the extension by zero of a normal derivative of a function $u \in \mathcal{S}(\overline{\mathbb{R}}^n_+)$ — which will usually have a jump at $x_n = 0$ — one finds

$$r^+PD_n^k e^+ u - r^+Pe^+D_n^k u = -ir^+P\sum\nolimits_{l=0}^{k-1}\gamma_l^c u \otimes D_n^{k-1-l}\delta(x_n) = -i\sum\nolimits_{l=0}^{k-1} K_{p,k-1-l}\gamma_l^c u,$$
$$\text{with } K_{p,r}v = r^+P(v(x') \otimes D_n^r\delta(x_n)). \tag{5.9}$$

The application of the ψdo $P = \mathrm{OP}(p(x,\xi))$ is understood as an application by Fourier transformation for each fixed x (considered as a parameter). If p is independent of x_n, satisfying the 0-transmission condition with respect to $\overline{\mathbb{R}}^n_+$, $K_{p,0}$ is the Poisson operator with symbol-kernel $\tilde{k}(x', z_n, \xi') = r^+\tilde{p}(x', \xi', z_n)$, as defined in [17, 28, 29]. If p depends on x_n, one can for smooth symbols use an expansion derived from the Taylor expansion of p in x_n to define the operator, but in case of limited smoothness in x, this is unsatisfactory. In [4], this point is solved for non-smooth symbols by allowing a more general definition of Poisson operators incorporating the x_n-dependence, and requiring the *global* 0-transmission condition for the involved ψdo's.

The estimates in Theorem 5.4 assure precisely that when $p(x,\xi) \in C^\tau S^m(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies the global 0-transmission condition introduced in Definition 5.1, the function $r^+\tilde{p}$ is a Poisson symbol-kernel as in [4, Definition 4.1] with $d = m$, lying in the space $C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$

defined there. The general definition of a Poisson operator from a symbol-kernel $\tilde{k}(x,\xi',z_n) \in C^\tau S^m_{1,0}(\mathbb{R}^n\times\mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$ is

$$\mathrm{OPK}(\tilde{k}(x,\xi',z_n))v = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}\tilde{k}(x,\xi',z_n)\hat{v}(\xi')\, đ\xi'\Big|_{z_n=x_n}. \tag{5.10}$$

For the operator in (5.9), the calculation is, when $v \in \mathcal{S}(\mathbb{R}^{n-1})$ and rules for Fourier transformation of distributions are applied:

$$\begin{aligned} K_{p,r}v &= r^+P(v(x')\otimes D^r_n\delta(x_n)) = r^+\mathcal{F}^{-1}_{\xi\to x}[p(x,\xi',\xi_n)\hat{v}(\xi')\xi_n^r] \\ &= r^+\mathcal{F}^{-1}_{\xi'\to x'}[D^r_{z_n}\tilde{p}(x,\xi',z_n)\hat{v}(\xi')]\big|_{z_n=x_n} \\ &= \mathcal{F}^{-1}_{\xi'\to x'}[\tilde{k}_{p,r}(x,\xi',z_n)\hat{v}(\xi')]\big|_{z_n=x_n} = \mathrm{OPK}(\tilde{k}_{p,r}(x,\xi',z_n))v, \end{aligned}$$

where

$$\tilde{k}_{p,r}(x,\xi',z_n) = r^+\mathcal{F}^{-1}_{\xi_n\to z_n}[p(x,\xi',\xi_n)\xi_n^r] = r^+D^r_{z_n}\tilde{p}(x,\xi',z_n), \tag{5.11}$$

a Poisson symbol-kernel in $C^\tau S^{m+r}_{1,0}(\mathbb{R}^n\times\mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$. We have shown the following lemma.

Lemma 5.7. *Let $p(x,\xi) \in C^\tau S^m(\mathbb{R}^n\times\mathbb{R}^n)$ satisfy the global 0-transmission condition, let $r \in \mathbb{N}_0$, and define $K_{p,r}$ by (5.9). Then $K_{p,r}$ is the Poisson operator $\mathrm{OPK}(\tilde{k}_{p,r})$, where $\tilde{k}_{p,r}(x,\xi',z_n) \in C^\tau S^{m+r}_{1,0}(\mathbb{R}^n\times\mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$ is defined by (5.11).*

For integer-order nonsmooth ψdo's there is a deduction of such Poisson operators in [4, Lemma 5.4].

We shall now show that when $P = \mathrm{OP}(p(x,\xi))$ is a C^τ-smooth pseudodifferential operator satisfying the extended 0-transmission condition w.r.t. $\overline{\mathbb{R}}^n_+$, then the truncated version $P_+ = r^+Pe^+$ preserves regularity in $\overline{\mathbb{R}}^n_+$ up to orders dominated by τ. There holds as follows.

Theorem 5.8. *Let $\tau > 0$, $1 < q < \infty$ and $m \in \mathbb{R}$. When $P = \mathrm{OP}(p(x,\xi))$ with $p \in C^\tau S^m(\mathbb{R}^n\times\mathbb{R}^n)$ satisfying the extended 0-transmission condition according to Definition 5.1, then the truncated version $P_+ = r^+Pe^+$ satisfies*

$$\begin{aligned} &P_+ : \overline{H}^{s+m}_q(\mathbb{R}^n_+) \to \overline{H}^s_q(\mathbb{R}^n_+), \quad \textit{for } |s| < \tau \textit{ with } s+m > -\tfrac{1}{q'}. \\ &P_+ : \dot{H}^{s+m}_q(\overline{\mathbb{R}}^n_+) \to \overline{H}^s_q(\mathbb{R}^n_+), \quad \textit{for } |s| < \tau. \end{aligned} \tag{5.12}$$

Proof. Assume in the following that $|s| < \tau$. The second statement is an immediate consequence of Theorem 2.1 since $\dot{H}^{s+m}_q(\overline{\mathbb{R}}^n_+)$ is a closed subspace of $H^{s+m}_q(\mathbb{R}^n)$. When $-\frac{1}{q'} < s+m < \frac{1}{q}$, the first statement also follows immediately, since $\overline{H}^{s+m}_q(\mathbb{R}^n_+)$ identifies with $\dot{H}^{s+m}_q(\overline{\mathbb{R}}^n_+)$ then.

To show the first statement for higher s, we use a method similar to that of [4, Lemma 5.6]. (The iterative proof in [36] does not adapt well, since commutation of P with D_j introduces a decrease in the Hölder smoothness.)

Assume to begin with that P satisfies the global 0-transmission condition. Let us show that the estimate holds for $s+m \in (k-\frac{1}{q'}, k+\frac{1}{q})$, $k = 1,2,\ldots$ with $|s| < \tau$. Fix k and write $s = s_0 + k$,

$s_0 + m \in (-\frac{1}{q'}, \frac{1}{q})$. Setting $r(\xi) = (\sum_{j=1}^n \xi_j^{2k} + 1)^{-1}$, we write p as a sum of three terms:

$$p(x,\xi) = p_1(x,\xi) + p_2(x,\xi) + p_3(x,\xi),$$
$$p_1(x,\xi) = \sum\nolimits_{j=1}^{n-1} p(x,\xi) r(\xi)\xi_j^{2k},\ p_2(x,\xi) = p(x,\xi)r(\xi)\xi_n^{2k},\ p_3(x,\xi) = p(x,\xi)r(\xi),$$

defining operators $P_i = \mathrm{OP}(p_i(x,\xi))$ satisfying the global 0-transmission condition.

We can write

$$r^+ P_1 e^+ = \sum\nolimits_{j=1}^{n-1} r^+ \mathrm{OP}(pr\xi_j^k) D_j^k e^+ = \sum\nolimits_{j=1}^{n-1} r^+ \mathrm{OP}(pr\xi_j^k) e^+ D_j^k,$$

since the tangential derivatives D_j commute with e^+. When $u \in \overline{H}_q^{m+s_0+k}(\mathbb{R}^n_+)$, then $D_j^k u \in \overline{H}_q^{m+s_0}(\mathbb{R}^n_+) \simeq \dot{H}_q^{m+s_0}(\overline{\mathbb{R}}^n_+)$, so since $\mathrm{OP}(pr\xi_j^k)$ is of order $m-k$, $r^+ \mathrm{OP}(pr\xi_j^k)e^+$ maps $\overline{H}_q^{m+s_0}(\mathbb{R}^n_+)$ to $\overline{H}_q^{s_0+k}(\mathbb{R}^n_+)$. Summing over j we see that $r^+P_1e^+$ has the desired mapping property.

For P_2, we have that

$$r^+ P_2 e^+ u = r^+ \mathrm{OP}(pr\xi_n^k) D_n^k e^+ u = r^+ \mathrm{OP}(pr\xi_n^k) e^+ D_n^k u + r^+ \mathrm{OP}(pr\xi_n^k)[D_n^k, e^+]u.$$

The term $r^+ \mathrm{OP}(pr\xi_n^k)e^+D_n^k u$ is treated like the terms in P_1, defining an operator with the desired mapping property. The other term satisfies, by (5.9) applied to $\mathrm{OP}(pr\xi_n^k)$,

$$\begin{aligned} r^+ \mathrm{OP}(pr\xi_n^k)[D_n^k, e^+]u &= -ir^+ \mathrm{OP}(pr\xi_n^k) \sum\nolimits_{l=0}^{k-1} \gamma_l^c u \otimes D_n^{k-1-l}\delta(x_n) \\ &= -i \sum\nolimits_{l=0}^{k-1} K_{pr\xi_n^k, k-1-l} \gamma_l^c u, \end{aligned}$$

with Poisson operators defined by Lemma 5.7. Here $K_{pr\xi_n^k, k-1-l}$ is a Poisson operator with symbol-kernel in $C^\tau S_{1,0}^{m-1-l}(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathcal{S}(\overline{\mathbb{R}}_+))$, hence continuous from $B_q^{s+m-l-\frac{1}{q}}(\mathbb{R}^{n-1})$ to $\overline{H}_q^s(\mathbb{R}^n_+)$ for $|s| < \tau$, by [4, Theorem 4.8]. The trace operator γ_l^c goes from $\overline{H}_q^{s+m}(\mathbb{R}^n_+)$ to $B_q^{s+m-l-\frac{1}{q}}(\mathbb{R}^{n-1})$ for any $s > -m + \frac{1}{q}$, so $K_{pr\xi_n^k, k-1-1}\gamma_l^c u \in \overline{H}_q^s(\mathbb{R}^n_+)$. Altogether, P_2 has the asserted mapping property.

The term P_3 is easily treated: Since P_3 is of order $m-2k$, it maps $H_q^{m-2k+s_0+k}(\mathbb{R}^n) = H_q^{m-k+s_0}(\mathbb{R}^n)$ to $H_q^{s_0+k}(\mathbb{R}^n)$. Here $\overline{H}_q^{m-k+s_0}(\mathbb{R}^n_+) \supset \overline{H}_q^{m+s_0}(\mathbb{R}^n_+) \simeq \dot{H}_q^{m+s_0}(\overline{\mathbb{R}}^n_+)$, so $r^+P_3e^+$ maps $\overline{H}_q^{m+s_0}(\mathbb{R}^n_+)$ to $\overline{H}_q^{s_0+k}(\mathbb{R}^n_+)$, and a fortiori $\overline{H}_q^{m+k+s_0}(\mathbb{R}^n_+)$ to $\overline{H}_q^{s_0+k}(\mathbb{R}^n_+)$.

We have then obtained (5.15) for all $|s| < \tau$ with $s + m + \frac{1}{q'} \in \mathbb{R}_+ \setminus \mathbb{N}$. The intermediate integer values are included by interpolation. This ends the proof in the case where P satisfies the global 0-transmission condition with respect to $\overline{\mathbb{R}}^n_+$.

Finally, consider the case where the 0-transmission condition is only satisfied for x with $0 \leqslant x_n < \varepsilon$, some $\varepsilon > 0$. Let $\eta(x_n), \zeta_0(x_n) \in C_0^\infty(\mathbb{R})$, supported in $(-\varepsilon, \varepsilon)$, equal to 1 on a neighborhood of 0, and such that $\eta = 1$ on a neighborhood of $\operatorname{supp}\zeta_0$. Then

$$P = P_1 + P_2 + P_3, \text{ where } P_1 = \eta P,\ P_2 = (1-\eta)P\zeta_0,\ P_3 = (1-\eta)P(1-\zeta_0).$$

The term P_1 satisfies the global 0-transmission condition, and hence has the asserted mapping properties. For the term P_3, $r^+P_3e^+$ acts on $\overline{H}_q^{s+m}(\mathbb{R}^n_+)$ as on $\dot{H}_q^{s+m}(\overline{\mathbb{R}}^n_+)$, so it likewise has them. For the middle term P_2, we note that since $1-\eta$ and ζ_0 have disjoint supports, we can by Corollary 4.9 for any large N write

$$P_2 = (1-\eta)P\zeta_0 = (1-\eta)Q_N,$$

where Q_N has symbol in $C^\tau S^{m-N}(\mathbb{R}^n \times \mathbb{R}^n)$. Now $e^+\overline{H}_q^{s+m}(\mathbb{R}^n_+) \subset H_q^{-M}(\mathbb{R}^n)$ for some large M for the considered values of s, and this will be mapped into $H_q^{\tau-\delta}(\mathbb{R}^n)$ (any $\delta > 0$) by P_2 when N is chosen large enough in the above representation, by Theorem 2.1. □

As a corollary we get the mapping property for operators satisfying the extended μ-transmission condition.

Corollary 5.9. *Let $\tau > 0$, $1 < q < \infty$, and $\mu > -1$. Let P have symbol $C^\tau S^m(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying the extended μ-transmission condition with respect to $\overline{\mathbb{R}}^n_+$. Then*

$$r^+P : H_q^{\mu(m+s)}(\overline{\mathbb{R}}^n_+) \to \overline{H}_q^s(\mathbb{R}^n_+), \tag{5.13}$$

holds for $|s| < \tau$.

Proof. By Corollary 5.5, the ψdo $B = P\Lambda_+^{-\mu}$, of order $m-\mu$, satisfies the extended 0-transmission condition. By Theorem 5.8,

$$\begin{aligned} r^+B : e^+\overline{H}_q^{s+m-\mu}(\mathbb{R}^n_+) &\to \overline{H}_q^s(\mathbb{R}^n_+), \text{ for } |s| < \tau \text{ with } s+m-\mu > -\tfrac{1}{q'}, \\ r^+B : \dot{H}_q^{s+m-\mu}(\overline{\mathbb{R}}^n_+) &\to \overline{H}_q^s(\mathbb{R}^n_+), \text{ for } |s| < \tau. \end{aligned}$$

Let $|s| < \tau$. Recalling that $H_q^{\mu(m+s)}(\overline{\mathbb{R}}^n_+) = \Lambda_+^{-\mu} e^+\overline{H}_q^{s+m-\mu}(\mathbb{R}^n_+)$ for $s+m-\mu > -\frac{1}{q'}$, we infer that

$$r^+P = r^+B\Lambda_+^\mu : \Lambda_+^{-\mu}e^+\overline{H}_q^{m-\mu+s}(\mathbb{R}^n_+) = H_q^{\mu(m+s)}(\overline{\mathbb{R}}^n_+) \to \overline{H}_q^s(\mathbb{R}^n_+),$$

for $|s| < \tau$ with $s+m-\mu > -\frac{1}{q'}$. For $s+m-\mu < \frac{1}{q}$, we use that $H_q^{\mu(m+s)}(\overline{\mathbb{R}}^n_+) = \dot{H}_q^{m+s}(\overline{\mathbb{R}}^n_+)$ by definition. □

There is in particular a consequence for operators as in Example 5.6.

Corollary 5.10. *Let $\tau > 0$ and $1 < q < \infty$. When P is even of order $2a > 0$ as in Example 5.6, then*

$$r^+P : H_q^{a(2a+s)}(\overline{\mathbb{R}}^n_+) \to \overline{H}_q^s(\mathbb{R}^n_+), \text{ for } |s| < \tau. \tag{5.14}$$

The result can be generalized to bounded smooth domains by tools that are already available in the literature, namely, the result of Abels and Jiménez [8] that $C^\tau S^m(\mathbb{R}^n \times \mathbb{R}^n)$ is preserved under C^∞-transformations, and the localization explained, for example, in [32].

Theorem 5.11. *Let $\tau > 0$, $1 < q < \infty$, and $\mu > -1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞-domain, and let $P = \mathrm{OP}(p(x,\xi))$ with $p \in C^\tau S^m(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying the extended μ-transmission condition with* *respect to $\overline{\Omega}$. Then the restricted operator r^+P has the mapping property:*

$$r^+P: H_q^{\mu(m+s)}(\overline{\Omega}) \to \overline{H}_q^s(\Omega), \text{ for } |s| < \tau. \tag{5.15}$$

Proof. For $s + m - \mu < \frac{1}{q}$, the statement follows immediately from Theorem 2.1. For $s + m - \mu > -\frac{1}{q'}$, we use local coordinates and a subordinated partition of unity, as in [32, Remark 4.3ff.]. It is described there how $\overline{\Omega}$ has a cover by bounded open sets U_i with diffeomorphisms $\kappa_i : U_i \to V_i$ such that $U_i \cap \Omega$ is mapped to $V_i \cap \mathbb{R}^n_+$, $i = 0, \dots, I_1$, and there is a *subordinated* partition of unity $\{\varrho_j\}_{j=0,\dots,J}$ where for each pair j, k there is an index $i = i(j,k)$ such that $\varrho_j, \varrho_k \in C_0^\infty(U_i)$. Choose also ζ_j in $C_0^\infty(U_i)$ satisfying $\zeta_j\varrho_j = \varrho_j$. Let $u \in H_q^{\mu(m+s)}(\overline{\Omega})$, and let $u_k = \varrho_k u = \zeta_k u_k$, then $Pu = \sum_{j,k} \varrho_j P\zeta_k u_k$. Here the operators $P_{jk} = \varrho_j P \zeta_k$ carry over via κ_i to operators $\underline{P}_{jk}$ acting over V_i with symbols in $C^\tau S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$ in view of Proposition 4.8 and [8], satisfying the μ-transmission condition with respect to $\overline{\mathbb{R}}^n_+$, and u_k carries over to $\underline{u}_k \in H_q^{\mu(m+s)}(\overline{\mathbb{R}}^n_+)$. Now we apply Corollary 5.9 to each $\underline{P}_{jk}\underline{u}_k$, carry the contributions back to $\overline{\Omega}$, and sum over j and k to end the proof. $\square$

A similar result holds for $\Omega = \mathbb{R}^n_\gamma$ when $\gamma \in C_b^\infty(\mathbb{R}^{n-1})$.

5.3 | Mapping properties of (x, y)-form operators over the halfspace

As a preparation for the treatment of operators on nonsmooth sets we consider operators with symbols in (x, y)-form on $\mathbb{R}^n_+$. We begin with an observation on remainders:

Corollary 5.12. *Let r_α and $m < \tau$ be as in Corollary 3.3, with $l < \tau$, and let $\mu \geqslant 0$. Then*

$$r^+ \operatorname{OP}(D_\xi^\alpha r_\alpha(x, y, \xi)) : H_q^{\mu((m-l+s)_+)}(\overline{\mathbb{R}}^n_+) \to \overline{H}_q^s(\mathbb{R}^n_+)$$

is bounded if $0 \leqslant s < \tau$ and $s + m < \tau$, and $s + m < \mu + l + \frac{1}{q}$. Moreover, there is some $k \in \mathbb{N}$ and $C_{s,m,\mu} > 0$ independent of a such that

$$\| \operatorname{OP}(D_\xi^\alpha r_\alpha(x, y, \xi)) \|_{\mathcal{L}(H_q^{\mu((m-l+s)_+)}(\overline{\mathbb{R}}^n_+), \overline{H}_q^s(\mathbb{R}^n_+))} \leqslant C_{s,m,\mu} |a|_{k, C^\tau S^m_{1,0}}.$$

Proof. We use that

$$H_q^{\mu((m-l+s)_+)}(\overline{\mathbb{R}}^n_+) = \dot{H}_q^{(s+m-l)_+}(\overline{\mathbb{R}}^n_+)$$

since $(m - l + s)_+ < \mu + \frac{1}{q}$. Hence

$$r^+ \operatorname{OP}(D_\xi^\alpha r_\alpha(x, y, \xi)) : H_q^{\mu((m-l+s)_+)}(\overline{\mathbb{R}}^n_+) \to \overline{H}_q^s(\mathbb{R}^n_+),$$

with the mentioned estimates, because of Corollary 3.5. $\square$

We now show a mapping property for restricted (x, y)-form operators, with limitations on both μ, m, and s.

Theorem 5.13. *Let $1 < q < \infty$, $\tau > 0$, $0 \leqslant \mu \leqslant m < \tau$, and $\mu - m \leqslant s < \tau - m$, and let $a \in C^\tau S^m(\mathbb{R}^{2n} \times \mathbb{R}^n)$, satisfying the global μ-transmission condition with respect to $\overline{\mathbb{R}}^n_+$. Then*

$$r^+ \operatorname{OP}(a(x,y,\xi)) : H_q^{\mu(m+s)}(\overline{\mathbb{R}}^n_+) \to \overline{H}^s_q(\mathbb{R}^n_+)$$

is bounded. Moreover, there is some $N \in \mathbb{N}$ and $C_{s,m,\mu,q} > 0$ such that

$$\|r^+ \operatorname{OP}(a(x,y,\xi))\|_{\mathcal{L}(H_q^{\mu(m+s)}(\overline{\mathbb{R}}^n_+), \overline{H}^s_q(\mathbb{R}^n_+))} \leqslant C_{s,m,\mu,q} |a|_{N, C^\tau S^m_{1,0}}. \tag{5.16}$$

Proof. We will prove the statement in the cases $s = \mu - m$ and $s = \tau - m - \varepsilon$ for $\varepsilon > 0$ sufficiently small. Then the general statement follows by complex interpolation since $s + m \geqslant \mu > \mu - \frac{1}{q'}$, cf. Remark 4.1.

Case $s = \mu - m$: In this case we have

$$H_q^{\mu(m+s)}(\overline{\mathbb{R}}^n_+) = H_q^{\mu(\mu)}(\overline{\mathbb{R}}^n_+) = \dot{H}^\mu_q(\overline{\mathbb{R}}^n_+) \subset H^\mu_q(\mathbb{R}^n)$$

and

$$r^+ \operatorname{OP}(p_\alpha(x,\xi)) : \dot{H}^\mu_q(\overline{\mathbb{R}}^n_+) \to \overline{H}^{\mu-m}_q(\mathbb{R}^n_+)$$

by Theorem 3.4, using that $|\mu - m| = m - \mu \leqslant m < \tau$ and $\mu < \tau$.

Case $s = \tau - m - \varepsilon$, $\varepsilon > 0$ **sufficiently small:** We can assume $\tau - m \notin \mathbb{N}$ without loss of generality. (Otherwise replace τ by some $\tau' \in (s + m, \tau)$.) Then $k := [s] = [\tau - m]$, if $\varepsilon > 0$ is sufficiently small.

First we consider the case $k = 0$. Then $0 \leqslant s = \tau - m - \varepsilon < \tau - m < 1$. We use again the expansion in Corollary 3.3 with $l = [m]$. Here Theorem 3.6 yields:

$$r^+ \operatorname{OP}(D^\alpha_\xi r_\alpha(x,y,\xi)) : H_q^{\mu(s+m)}(\overline{\mathbb{R}}^n_+) \hookrightarrow e^+ L_q(\mathbb{R}^n_+) \to \overline{H}^s_q(\mathbb{R}^n_+)$$

because of $D^\alpha_\xi r_\alpha \in C^{\tau - [m]} S^{m-[m]}_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$ and $s < \tau - [m] - (m - [m]) = \tau - m$. Moreover,

$$r^+ \operatorname{OP}(p_\alpha(x,\xi)) : H_q^{\mu(s+m-|\alpha|)}(\overline{\mathbb{R}}^n_+) \to \overline{H}^s_q(\mathbb{R}^n_+)$$

by Corollary 5.9. This shows the case $k = 0$.

Next we consider the case $k \geqslant 1$. We shall use that $s = s' + k$ with $s' \in [0,1)$ and

$$v \in \overline{H}^s_q(\mathbb{R}^n_+) \iff \partial^\beta_x v \in \overline{H}^{s'}_q(\mathbb{R}^n_+) \quad \text{for all } |\beta| = k \text{ and } \beta = 0,$$

with corresponding norm equivalences. Let $|\beta| = k$. The composition of the differential operator ∂^β_x with $r^+ \operatorname{OP}(a(x,y,\xi))$ is a finite sum

$$\partial^\beta_x r^+ \operatorname{OP}(a(x,y,\xi)) = \sum_{0 \leqslant \gamma \leqslant \beta} \binom{\beta}{\gamma} r^+ \operatorname{OP}(\partial^\gamma_x a(x,y,\xi)(i\xi)^{\beta-\gamma}) = \sum_{j=0}^k r^+ \operatorname{OP}(a_{\beta,j}(x,y,\xi)),$$

where $a_{\beta,j}(x,y,\xi) \in C^{\tau-j} S^{m+k-j}(\mathbb{R}^{2n} \times \mathbb{R}^n)$. Here the result for the case $k = 0$ yields that

$$r^+ \operatorname{OP}(a_{\beta,j}) : H_q^{\mu(m+k+s'-j)}(\overline{\mathbb{R}}^n_+) \to \overline{H}^{s'}_q(\mathbb{R}^n_+)$$

since $s' = s - k < \tau - j - (m + k - j) = \tau - m - k$ and $m + k - j < \tau - j$ due to $k = [\tau - m] < \tau - m$. This treats the case $|\beta| = k$. For the case $|\beta| = 0$ we apply the first case directly in a similar way.

Finally, the last statement is a consequence of the closed graph theorem. More precisely, let

$$C^\tau S^m_{\mu,tr}(\mathbb{R}^{2n} \times \mathbb{R}^n) := \{a \in C^\tau S^m(\mathbb{R}^{2n} \times \mathbb{R}^n) \mid a \text{ satisfies the global } \mu\text{-transmission condition}\}$$

and consider the mapping

$$\mathrm{OP}_+ : C^\tau S^m_{\mu,tr}(\mathbb{R}^{2n} \times \mathbb{R}^n) \to \mathcal{L}(H_q^{\mu(m+s)}(\overline{\mathbb{R}}^n_+), \overline{H}^s_q(\mathbb{R}^n_+)) : a \mapsto r^+ \mathrm{OP}(a(x, y, \xi)).$$

Note that $C^\tau S^m_{\mu,tr}(\mathbb{R}^{2n} \times \mathbb{R}^n)$ is a closed subspace of the Fréchet space $C^\tau S^m_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$ and therefore a Fréchet space. If $(a_k)_{k\in\mathbb{N}} \subset C^\tau S^m_{\mu,tr}(\mathbb{R}^{2n} \times \mathbb{R}^n)$ is such that

$$\begin{aligned} a_k &\to_{k\to\infty} a \quad &&\text{in } C^\tau S^m_{1,0;\mu,tr}(\mathbb{R}^{2n} \times \mathbb{R}^n), \\ r^+ \mathrm{OP}(a_k(x, y, \xi)) &\to_{k\to\infty} A \quad &&\text{in } \mathcal{L}(H_q^{\mu(m+s)}(\overline{\mathbb{R}}^n_+), \overline{H}^s_q(\mathbb{R}^n_+)), \end{aligned}$$

then for any $u \in \mathcal{E}_\mu \cap \mathcal{E}'$ and a suitable subsequence

$$r^+ \mathrm{OP}(a_k(x, y, \xi))u(x) \to_{k\to\infty} Au(x) \qquad \text{for almost every } x \in \mathbb{R}^n_+.$$

Moreover, using the representation in Theorem 3.1 of $\mathrm{OP}(a(x, y, \xi))u(x)$ with a replaced by a_k it is easy to observe that

$$r^+ \mathrm{OP}(a_k(x, y, \xi))u(x) \to_{k\to\infty} r^+ \mathrm{OP}(a(x, y, \xi))u(x) \qquad \text{for all } x \in \mathbb{R}^n_+.$$

Hence $Au(x) = r^+ \mathrm{OP}(a(x, y, \xi))u(x)$ for almost all $x \in \mathbb{R}^n_+$ and all $u \in \mathcal{E}_\mu \cap \mathcal{E}'$. This shows the closedness of OP_+ since $\mathcal{E}_\mu \cap \mathcal{E}'$ is dense in $H_q^{\mu(m+s)}(\overline{\mathbb{R}}^n_+)$. Hence OP_+ is continuous and therefore bounded, which yields the last statement. □

Also cases where μ and m are in $(-1, 0)$ can be included, with a loss of Hölder-regularity by one step.

Corollary 5.14. *Let $\tau > 0$, and $m \geqslant \mu > -1$, and let $a \in C^\tau S^m(\mathbb{R}^{2n} \times \mathbb{R}^n)$ satisfy the global μ-transmission condition with respect to $\overline{\mathbb{R}}^n_+$. If $m < \tau - 1$ and $\mu - m \leqslant s < \tau - m - 1$, then*

$$r^+ \mathrm{OP}(a(x, y, \xi)) : H_q^{\mu(m+s)}(\overline{\mathbb{R}}^n_+) \to \overline{H}^s_q(\mathbb{R}^n_+)$$

is bounded. Moreover, there is some $N \in \mathbb{N}$ and $C_{s,m,\mu} > 0$ such that

$$\|r^+ \mathrm{OP}(a(x, y, \xi))\|_{\mathcal{L}(H_q^{\mu(m+s)}(\overline{\mathbb{R}}^n_+), \overline{H}^s_q(\mathbb{R}^n_+))} \leqslant C_{s,m,\mu} |a|_{N, C^\tau S^m_{1,0}}. \tag{5.17}$$

Proof. We use that by definition,

$$H_q^{\mu(m+s)}(\overline{\mathbb{R}}^n_+) = \Xi^1_+ \Xi^{-(\mu+1)}_+ e^+ \overline{H}^{m+s-\mu}_q(\mathbb{R}^n_+) = \Xi^1_+ H_q^{(\mu+1)(m+s+1)}(\overline{\mathbb{R}}^n_+),$$

where $\Xi^1_+ = \partial_{x_n} + \mathrm{OP}(\langle \xi' \rangle)$. Here $\mathrm{OP}(\langle \xi' \rangle) : H_q^{(\mu+1)(m+s+1)}(\overline{\mathbb{R}}^n_+) \to H_q^{(\mu+1)(m+s)}(\overline{\mathbb{R}}^n_+)$ since the operator commutes with e^+. Thus for every $u \in H_q^{\mu(m+s)}(\overline{\mathbb{R}}^n_+)$ there are $v \in H_q^{(\mu+1)(m+s+1)}(\overline{\mathbb{R}}^n_+)$

and $w \in H_q^{(\mu+1)(m+s)}(\overline{\mathbb{R}}^n_+)$ (depending continuously on u) such that $u = \partial_{x_n} v + w$. Moreover,

$$r^+ \operatorname{OP}(a(x,y,\xi))\partial_{x_n} v = r^+ \operatorname{OP}(a(x,y,\xi)i\xi_n)v - r^+ \operatorname{OP}(\partial_{y_n} a(x,y,\xi))v,$$

where $a(x,y,\xi)i\xi_n \in C^\tau S^{m+1}(\mathbb{R}^{2n} \times \mathbb{R}^n)$ and $\partial_{y_n} a(x,y,\xi) \in C^{\tau-1} S^m(\mathbb{R}^{2n} \times \mathbb{R}^n)$ satisfy the global $(\mu+1)$-transmission condition. Considering $\partial_{y_n} a(x,y,\xi)$ and $a(x,y,\xi)$ as symbols of order $m+1$, we get from Theorem 5.13 that all three maps $r^+ \operatorname{OP}(a(x,y,\xi)i\xi_n)$, $r^+ \operatorname{OP}(\partial_{y_n} a(x,y,\xi))$, and $r^+ \operatorname{OP}(a(x,y,\xi))$ are bounded from $H_q^{(\mu+1)(m+1+s)}(\overline{\mathbb{R}}^n_+)$ to $\overline{H}_q^s(\mathbb{R}^n_+)$, when $\tau > 0$, $0 \leqslant \mu+1 \leqslant m+1 < \tau$, and $(\mu+1)-(m+1) \leqslant s < \tau-(m+1)$. In view of the assumption $-1 < \mu \leqslant m$, the latter conditions reduce to $m < \tau - 1$, $\mu - m \leqslant s < \tau - m - 1$. Then when they hold,

$$r^+ \operatorname{OP}(a(x,y,\xi))u = r^+ \operatorname{OP}(a(x,y,\xi)i\xi_n)v - r^+ \operatorname{OP}(\partial_{y_n} a(x,y,\xi))v + r^+ \operatorname{OP}(a(x,y,\xi))w$$

belongs to $\overline{H}_q^s(\mathbb{R}^n_+)$, and the corresponding map is bounded.

The last statement follows likewise from Theorem 5.13. □

The results can be generalized to symbols satisfying the extended μ-transmission condition.

6 | THE HOMOGENEOUS DIRICHLET PROBLEM ON NONSMOOTH DOMAINS

We shall now apply the analysis to the homogeneous Dirichlet problem for those ψdo's that are close generalizations of the fractional Laplacian, namely, operators P of order $2a$ with an *even* symbol. As already noted, they satisfy the global a-transmission condition with respect to any choice of normal coordinate. The homogeneous Dirichlet problem is, for strongly elliptic operators,

$$Pu = f \text{ on } \Omega, \quad \operatorname{supp} u \subset \overline{\Omega}, \tag{6.1}$$

where the solution u is sought in $H^a(\mathbb{R}^n)$, and it is known in the smooth case [31] that it is Fredholm solvable for $f \in \overline{H}_q^s(\Omega)$, with $u \in H_q^{a(s+2a)}(\overline{\Omega})$, when $s > -a - \frac{1}{q'}$. Our present aim is to extend the regularity result to symbols p and open sets Ω with $C^{1+\tau}$-boundary, for s as large as possible relative to the Hölder exponents.

6.1 | Coordinate changes at a boundary, boundedness over nonsmooth domains

As in Section 4.2, $\mathbb{R}^n_\gamma = \{x \in \mathbb{R}^n \mid x_n > \gamma(x')\}$ for some $\gamma \in C^{1+\tau}(\mathbb{R}^{n-1})$ with $\tau > 0$, and $F_\gamma : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^{1+\tau}$-diffeomorphism such that $F_\gamma(\mathbb{R}^n_\gamma) = \mathbb{R}^n_+$. We take $F_\gamma(x) = (x', x_n - \gamma(x'))$ for all $x \in \mathbb{R}^n$, where $x' = (x_1, \dots, x_{n-1})$. Moreover, let $p \in C^\tau S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$ be even and $\tau > 2a$ and let P_γ be the transformed operator:

$$(P_\gamma u)(x) = (P(u \circ F_\gamma^{-1}))(F_\gamma(x)) = (F_\gamma^* P F_\gamma^{*,-1} u)(x) \quad \text{for all } u \in \mathcal{S}(\mathbb{R}^n), \tag{6.2}$$

and let $\|\gamma\|_{C^1(\mathbb{R}^n)} \leqslant r_0$ for some $r_0 \in (0,1]$. We assume for simplicity that r_0 is so small that

$$\sup_{x\in\mathbb{R}^n} |\nabla F_\gamma(x) - I| \leqslant \tfrac{1}{2}. \tag{6.3}$$

Then one obtains by the results of Section 3 that for all $u \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$

$$P_\gamma u(x) = \mathrm{Os}-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} q_\gamma(x,y,\xi)u(y)\,dy\,đ\xi, \tag{6.4}$$

where

$$q_\gamma(x,y,\xi) = p(F_\gamma(x), A_\gamma(x,y)^{-1,T}\xi)|\det A_\gamma(x,y)|^{-1}|\det \nabla_y F_\gamma(y)|, \tag{6.5}$$

$$A_\gamma(x,y) = \int_0^1 \nabla_x F_\gamma(x + t(y-x))\,dt,$$

for all $x, y, \xi \in \mathbb{R}^n$. Here $q_\gamma(x,y,\xi) \in C^\tau S^m_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$. Moreover, for every $0 < \tau' < \tau$ and $k \in \mathbb{N}_0$ there is some C_k independent of γ and p such that

$$|q_\gamma - p|_{k, C^{\tau'}S^{2a}_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)} \leqslant C_k \|\gamma\|^{\min(\tau-\tau',1)}_{C^{1+\tau}(\mathbb{R}^{n-1})} |p|_{k+1, C^\tau S^{2a}_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)} \tag{6.6}$$

for all $\gamma \in C^{1+\tau}(\mathbb{R}^{n-1})$, $\|\gamma\|_{C^{1+\tau}(\mathbb{R}^{n-1})} \leqslant r_0$, since

$$\|F_\gamma - \mathrm{id}\,\|_{C^\tau(\mathbb{R}^n)} \leqslant C\|\gamma\|_{C^{1+\tau}(\mathbb{R}^{n-1})}, \quad \|A_\gamma - I\|_{C^\tau(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant C\|\gamma\|_{C^{1+\tau}(\mathbb{R}^{n-1})}.$$

In order to apply the results to the nonlocal equations on $\mathbb{R}^n_\gamma$, we have to extend (6.2) and (6.4) to $u \in H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_+)$. First of all, $\mathrm{OP}(q_\gamma)$ extends to a bounded linear operator from $H^a_q(\mathbb{R}^n)$ to $H^{-a}_q(\mathbb{R}^n)$ because of Theorem 3.4, due to $0 < a < 2a < \tau$. Moreover, F^*_γ and $F^{*,-1}_\gamma$ map $H^a_q(\mathbb{R}^n)$ to itself since $0 < a < \tau$. Because of $\det DF_\gamma(x) \equiv 1$,

$$\int_{\mathbb{R}^n} (F^*_\gamma u)(x)v(x)\,dx = \int_{\mathbb{R}^n} u(x)(F^{*,-1}_\gamma v)(x)\,dx \qquad \text{for all } u, v \in \mathcal{S}(\mathbb{R}^n).$$

Hence F^*_γ and $F^{*,-1}_\gamma$ map $H^{-a}_q(\mathbb{R}^n)$ to itself as well. Therefore we obtain

$$P_\gamma u = F^*_\gamma P F^{*,-1}_\gamma u = \mathrm{OP}(q_\gamma(x,y,\xi))u \tag{6.7}$$

for all $u \in H^a_q(\mathbb{R}^n)$. In particular, we obtain the identity for all $u \in H_q^{a(a)}(\overline{\mathbb{R}}^n_+) = \dot{H}^a_q(\overline{\mathbb{R}}^n_+)$, and conclude

$$r^+ F^*_\gamma P F^{*,-1}_\gamma u = r^+\,\mathrm{OP}(q_\gamma(x,y,\xi))u$$

for all $u \in H_q^{a(a)}(\overline{\mathbb{R}}^n_+)$. Note that $H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_+) \subset H_q^{a(a)}(\overline{\mathbb{R}}^n_+)$ for any $s \geqslant -a$.

In the case of a classical even symbol p this leads to following.

Theorem 6.1. *Let $p \in C^\tau S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$ be even, where $0 < a < 1$, let $\gamma \in C^{1+\tau}(\mathbb{R}^{n-1})$ and $N < \tau$, let q_γ be the transformed symbol (6.5), and let $P_\gamma = \mathrm{OP}(q_\gamma(x,y,\xi))$. Then $q_\gamma \in C^\tau S^{2a}(\mathbb{R}^{2n}\times\mathbb{R}^n)$ satisfies the global a-transmission condition. Moreover, with $1 < q < \infty$,*

$$r^+ P_\gamma \equiv r^+\,\mathrm{OP}(q_\gamma(x,y,\xi)) : H_q^{a(2a+s)}(\overline{\mathbb{R}}^n_+) \to \overline{H}^s_q(\mathbb{R}^n_+)$$

is bounded for any $s \in \mathbb{R}$ *such that* $-a \leqslant s < \tau - 2a$. *Furthermore, for any* $r_0 > 0$ *there is some* $C_{s,r_0,q} > 0$, $\theta > 0$ *and* $k \in \mathbb{N}_0$ *independent of* p *and* γ *such that*

$$\|r^+ P_\gamma - r^+ \operatorname{OP}(p(x,\xi))\|_{\mathcal{L}(H_p^{\mu(2a+s)}(\overline{\mathbb{R}}^n_+), \overline{H}^s_p(\mathbb{R}^n_+))} \leqslant C_{s,r_0,q} \|\gamma\|^{\theta}_{C^{1+\tau}(\mathbb{R}^{n-1})} |p|_{k, C^\tau S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)} \tag{6.8}$$

provided that $\|\gamma\|_{C^{1+\tau}} \leqslant r_0$.

Proof. Since p is even, it is easy to observe that q_γ is even as well. Therefore $q_\gamma \in C^\tau S^{2a}(\mathbb{R}^{2n} \times \mathbb{R}^n)$ satisfies the global a-transmission condition and we can apply Theorem 5.13 to q_γ. This implies the statement for the mapping properties of $r^+ \operatorname{OP}(q_\gamma(x,y,\xi))$. To show (6.8) one chooses some $\tau' \in (0,\tau)$ sufficiently close to τ, $\theta = \min(\tau - \tau', 1)$ and applies (5.17) for $r^+ P_\gamma - r^+ \operatorname{OP}(p(x,\xi)) = r^+ \operatorname{OP}(q_\gamma(x,y,\xi) - p(x,\xi))$ and with τ' instead of τ. Moreover, one uses (6.6). □

Corollary 6.2. *Let* $0 < a < 1$ *and* $\tau > 2a$, *and let* $p \in C^\tau S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$ *be even. Then* $P = \operatorname{OP}(p)$ *maps*

$$r^+ P : H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_\gamma) \to \overline{H}^s_q(\mathbb{R}^n_\gamma),$$

continuously for $-a \leqslant s < \tau - 2a$.

Proof. Follows directly from Theorem 6.1, since $F_\gamma^*(H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_+)) = H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_\gamma)$, using that $F_\gamma^{-1,*} = F^*_{-\gamma}$ maps $H^s_q(\mathbb{R}^n)$ to itself (since $|s| < 1 + \tau$). □

Corollary 6.3. *Let* $0 < a < 1$ *and* $\tau > 2a$, *and let* $p \in C^\tau S^{2a-1}(\mathbb{R}^n \times \mathbb{R}^n)$ *be odd. Then* $P = \operatorname{OP}(p)$ *maps*

$$r^+ P : H_q^{a(\max(a,s+2a-1))}(\overline{\mathbb{R}}^n_\gamma) \to \overline{H}^s_q(\mathbb{R}^n_\gamma),$$

continuously for $-a \leqslant s < \tau - 2a$.

Proof. We first consider the case that $\max(a, s+2a-1) = a$, that is, $s \leqslant 1 - a$. Then

$$H_q^{a(\max(a,s+2a-1))}(\overline{\mathbb{R}}^n_\gamma) = \dot{H}^a_q(\overline{\mathbb{R}}^n_\gamma)$$

and

$$r^+ P : \dot{H}^a_q(\overline{\mathbb{R}}^n_\gamma) \to \overline{H}^s_q(\mathbb{R}^n_\gamma)$$

because of Theorem 3.4, $-\tau < -a \leqslant s < \tau - 2a < \tau$, $p \in C^\tau S^{2a-1}(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\tau S^{a-s}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, and $|a - s| < \tau$.

Next we consider the case $s > 1 - a$. Then $\max(a, s + 2a - 1) = s + 2a - 1$ and $\tau > s + 2a > 1 + a$. We use that

$$f \in \overline{H}^s_q(\mathbb{R}^n_\gamma) \iff \partial_x^\alpha f \in \overline{H}^{s-1}_q(\mathbb{R}^n_\gamma) \quad \text{for all } |\alpha| \leqslant 1,$$

and write $\partial_x^\alpha P = \operatorname{OP}(a_\alpha(x,y,\xi))$ for some even $a_\alpha \in C^{\tau-1} S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$. In the case $\tau > 1 + 2a$, Corollary 6.2 implies that

$$r^+ \operatorname{OP}(a_\alpha(x,y,\xi)) : H_q^{a(s+2a-1)}(\overline{\mathbb{R}}^n_\gamma) \to \overline{H}^{s-1}_q(\mathbb{R}^n_\gamma),$$

due to $2a < \tau - 1$ and $-a < s - 1 < \tau - 1 - 2a$. In the case $\tau \leqslant 1 + 2a$ we use that $a_\alpha(x, y, \xi) = q_\gamma(x, y, \xi)(i\xi)^\alpha + \partial_x^\alpha q_\gamma(x, y, \xi)$ if $|\alpha| = 1$, where $q_\gamma(x, y, \xi)(i\xi)^\alpha \in C^\tau S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$ is even and $\partial_x^\alpha q_\gamma(x, y, \xi) \in C^{\tau-1} S^{2a-1}(\mathbb{R}^n \times \mathbb{R}^n)$. Again using Corollary 6.2 implies that

$$r^+ \operatorname{OP}(q_\gamma(x, y, \xi)(i\xi)^\alpha) : H_q^{a(s+2a-1)}(\overline{\mathbb{R}}_+^n) \to \overline{H}_q^{s-1}(\mathbb{R}_\gamma^n),$$

due to $2a < \tau$ and $-a < s - 1 < \tau - 2a$. Moreover,

$$r^+ \operatorname{OP}(\partial_x^\alpha q_\gamma(x, y, \xi)) : H_q^{a(a)}(\overline{\mathbb{R}}_+^n) = \dot{H}_q^a(\mathbb{R}_\gamma^n) \to \overline{H}_q^{s-1}(\mathbb{R}_\gamma^n)$$

because of $-a < s - 1 < \tau - 1$, $a < \tau - 1$ and $a - s + 1 \geqslant 2a - 1$ due to $s < \tau - 2a \leqslant 1$. Altogether this yields the desired mapping properties. □

For general domains we obtain the following theorem.

Theorem 6.4. *Let $0 < a < 1$ and $\tau > 2a$, let Ω be a bounded $C^{1+\tau}$-domain, and let $P = \operatorname{OP}(p(x, \xi))$ where $p \in C^\tau S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$ is even. Then*

$$r^+ P : H_q^{a(s+2a)}(\overline{\Omega}) \to \overline{H}_q^s(\Omega) \tag{6.9}$$

holds for $-a \leqslant s < \tau - 2a$, $1 < q < \infty$.

Proof. Let $u \in H_q^{a(s+2a)}(\overline{\Omega})$ and let x_0, U, γ, φ be as in Definition 4.3 2°. Let $\psi \in C_0^\infty(U)$ satisfy $\psi\varphi = \varphi$. Let U' be the interior of the set where $\varphi = 1$. In the translated and rotated situation, where the objects will be marked with an underline, we then have that $\underline{\varphi u} \in H_q^{a(s+2a)}(\overline{\mathbb{R}}_\gamma^n)$; and then by Corollary 6.2, $r^+ \underline{P}(\underline{\varphi u}) \in \overline{H}_q^s(\mathbb{R}_\gamma^n)$, and also $r^+ \underline{\psi} \underline{P}(\underline{\varphi u})$ lies there. Thus in the original position, $r^+ \psi P \varphi u \in \overline{H}_q^s(\Omega)$.

By Corollary 4.9, $(1 - \psi) P \varphi u = (1 - \psi) \operatorname{OP}(q_N) u$ with $q_N \in C^\tau S_{1,0}^{2a-N}(\mathbb{R}^n \times \mathbb{R}^n)$ for arbitrarily large N. Take $N \geqslant \tau + 2a$. By Theorem 2.1, $\operatorname{OP}(q_N)$ maps $H_q^{s+2a-N}(\mathbb{R}^n)$ into $H_q^s(\mathbb{R}^n)$ for $|s| < \tau$. When $s \in [-a, \tau - 2a)$, $H_q^{a(s+2a)}(\overline{\Omega}) \subset H_q^{a(a)}(\overline{\Omega}) = \dot{H}_q^a(\overline{\Omega})$ is thus mapped by $r^+(1 - \psi) \operatorname{OP}(q_N)$ into $\overline{H}_q^s(\Omega)$.

Altogether, we see that $r^+ P \varphi u \in \overline{H}_q^s(\Omega)$.

There is a finite set of points $\{x_{0,i}\}_{i=1,\dots,I}$ such that $\bigcup_i U_i' \supset \partial\Omega$ holds for the associated data $\{U_i, \gamma_i, \varphi_i, U_i', \psi_i\}$. Supply these sets with an open set $U_0' \supset \overline{\Omega} \setminus \bigcup_i U_i'$ with $\overline{U}_0' \subset \Omega$, and let $\{\varrho_i\}_{i=0,\dots,I}$ be an associated partition of unity, $\varrho_i \in C_0^\infty(U_i')$. Then $u = \sum_i \varrho_i u$, where also $\varrho_i u$ belongs to $H_q^{a(s+2a)}(\overline{\Omega})$ by Proposition 4.5. Moreover, $\varrho_i u = \varphi_i \varrho_i u$ for $i \geqslant 1$, where the initial considerations apply to $\varrho_i u$ to show that $r^+ P \varphi_i \varrho_i u \in \overline{H}_q^s(\Omega)$.

Summation over i gives the mapping property for u. □

6.2 | Elliptic regularity in an almost flat curved halfspace

We now turn to the study of regularity properties of solutions of elliptic problems in this context. Also here, we restrict the attention in the present paper to even operators; this suffices for the treatment of $(-\Delta)^a$ and its pseudodifferential generalizations.

In the following let $\overline{p} \in S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$ and $p \in C^\tau S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$ be strongly elliptic and even, and assume that for some $1 < q < \infty$ and $s \geqslant 0$

$$r^+ \operatorname{OP}(\overline{p}) : H_q^{a(t+2a)}(\overline{\mathbb{R}}^n_+) \to \overline{H}^t_q(\mathbb{R}^n_+) \quad \text{is invertible for } t = s, s', \tag{6.10}$$

where $s' := \min(s-1, -a)$. Moreover, let $\gamma \in C^{1+\tau}(\mathbb{R}^{n-1})$, $\mathbb{R}^n_\gamma = \{x \in \mathbb{R}^n \mid x_n > \gamma(x')\}$, and let $F_\gamma : \mathbb{R}^n \to \mathbb{R}^n$ be as in the preceding section. For the following it is assumed that $s + 2a < \tau$. Finally, let P_γ be defined as in (6.7).

Proposition 6.5. *Let $\overline{p} \in S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$ and $p \in C^\tau S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$ be strongly elliptic and even, with $\overline{p}$ invertible as in (6.10), and let $0 \leqslant s < \tau - 2a$. There are some $k \in \mathbb{N}$ and $\delta = \delta(\overline{p}, s, q) > 0$ such that*

$$r^+ P_\gamma : H_q^{a(t+2a)}(\overline{\mathbb{R}}^n_+) \to \overline{H}^t_q(\mathbb{R}^n_+)$$

is invertible for $t = s$ and $t = s' := \max(s-1, -a)$ if

$$|\overline{p} - p|_{k, C^\tau S^{2a}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)} \leqslant \delta \quad \text{and} \quad \|\gamma\|_{C^{1+\gamma}(\mathbb{R}^{n-1})} \leqslant \delta.$$

Proof. Because of (6.10), there is some $\varepsilon > 0$ such that $r^+ P_\gamma : H_q^{a(t+2a)}(\overline{\mathbb{R}}^n_+) \to \overline{H}^t_q(\mathbb{R}^n_+)$ is invertible for $t = s$ and $t = s'$, provided

$$\|r^+ \operatorname{OP}(\overline{p}) - r^+ P_\gamma\|_{\mathcal{L}(H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_+), \overline{H}^s_q(\mathbb{R}^n_+))} < \varepsilon.$$

Moreover, because of Theorems 5.13 and 6.1, there are some $k \in \mathbb{N}$, $\theta > 0$ and some $C > 0$ independent of p and γ with $|p|_{k, C^\tau S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)} \leqslant 1$, $\|\gamma\|_{C^{1+\tau}(\mathbb{R}^n)} \leqslant 1$ such that

$$\begin{aligned}
&\|r^+ \operatorname{OP}(\overline{p}) - r^+ P_\gamma\|_{\mathcal{L}(H_q^{a(t+2a)}(\overline{\mathbb{R}}^n_+), \overline{H}^t_q(\mathbb{R}^n_+))} \\
&\leqslant \|r^+ \operatorname{OP}(\overline{p}) - r^+ \operatorname{OP}(p)\|_{\mathcal{L}(H_q^{a(t+2a)}(\overline{\mathbb{R}}^n_+), \overline{H}^t_q(\mathbb{R}^n_+))} + \|r^+ \operatorname{OP}(p) - r^+ P_\gamma\|_{\mathcal{L}(H_q^{a(t+2a)}(\overline{\mathbb{R}}^n_+), \overline{H}^t_q(\mathbb{R}^n_+))} \\
&\leqslant C|\overline{p} - p|_{k, C^\tau S^{2a}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)} + C\|\gamma\|^\theta_{C^{1+\tau}(\mathbb{R}^n)}
\end{aligned}$$

for $t = s$ and $t = s'$. Hence there is some $\delta > 0$ such that the right-hand side is smaller than ε, provided $|\overline{p} - p|_{k, C^\tau S^{2a}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)} \leqslant \delta$ and $\|\gamma\|_{C^{1+\gamma}(\mathbb{R}^{n-1})} \leqslant \delta$. □

Next, we apply the preceding result to obtain a local regularity result in $\mathbb{R}^n_\gamma$. Denote $\{x \in \mathbb{R}^n \mid |x - x_0| < r\} = B_r(x_0)$.

Theorem 6.6. *Let $0 \leqslant s < \tau - 2a$. Assume that $\gamma \in C^{1+\tau}(\mathbb{R}^{n-1})$ satisfies $\gamma(0) = 0$, $\nabla\gamma(0) = 0$, and that $u \in H_q^{a(s'+2a)}(\overline{\mathbb{R}}^n_\gamma)$ is a solution of*

$$r^+ P u = f \qquad \text{in } \mathbb{R}^n_\gamma$$

for some $f \in L_q(\mathbb{R}^n_\gamma)$ *with* $f|_{B_{R_1}(0)} \in \overline{H}^{s}_q(\mathbb{R}^n_\gamma \cap B_{R_1}(0))$ *for some* $R_1 > 0$, *where* $s' = \max(s-1, -a)$. *Then there is some* $R > 0$ *such that*

$$u = v \qquad \text{in } \mathbb{R}^n_\gamma \cap B_R(0)$$

for some $v \in H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_\gamma)$.

We assume for simplicity that $R_1 = 1$. By a suitable scaling in space, one can always reduce to this case. The idea of the proof is to rescale in space and localize in order to apply the results for operators close to a constant coefficient pseudodifferential operator, that is, Proposition 6.5. For the rescaling we define for $R > 0$

$$\gamma_R(x') = R^{-1}\eta((x',0))\gamma(Rx'),$$

$$p_R(x,\xi) = \eta(x)p_0(Rx,\xi) + (1-\eta(x))p_0(0,\xi),$$

$$\overline{p}(x,\xi) = p_0(0,\xi),$$

for all $x, \xi \in \mathbb{R}^n$, $x' = (x_1, \dots, x_{n-1})$, where $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\eta \equiv 1$ on $B_1(0)$ and $\operatorname{supp}\eta \subset B_2(0)$. To assure that $\mathrm{OP}(\overline{p})$ is invertible, we can in view of the strong ellipticity assume that $p_0(0,\xi)$ has been modified for $|\xi| \leqslant 1$ such that $\operatorname{Re} p_0(0,\xi) \geqslant c > 0$ for all $\xi \in \mathbb{R}^n$ (also done for $p_0(x,\xi)$ with x in a small neighborhood of 0). Then $\overline{P} = \mathrm{OP}(p_0(0,\xi))$ satisfies (6.10), cf. Example 4.2. Furthermore, for $v : \mathbb{R}^n \to \mathbb{C}$ and $R > 0$ we define $\sigma_R v : \mathbb{R}^n \to \mathbb{C}$ by

$$(\sigma_R v)(x) = v(Rx) \qquad \text{for all } x \in \mathbb{R}^n.$$

Define moreover

$$q_R(x,\xi) = p_0(Rx, R^{-1}\xi) - R^{-2a}p_0(Rx,\xi),$$

and note that since $p_0(Rx, R^{-1}\xi) = R^{-2a}p_0(Rx,\xi)$ for all $|\xi| \geqslant 1$ and $R \in (0,1]$ by the homogeneity, $q_R(x,\xi) = 0$ for all $|\xi| \geqslant 1$, $R \in (0,1]$. Hence $q_R \in C^\tau S^{-\infty}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Now $P_0 = \mathrm{OP}(p_0)$ satisfies for all $x \in \mathbb{R}^n$ and suitable $v : \mathbb{R}^n \to \mathbb{C}$:

$$\sigma_R(P_0 v)(x) = \int_{\mathbb{R}^n} e^{iRx\cdot\xi} p_0(Rx,\xi)\hat{v}(\xi)\, đ\xi = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p_0(Rx, R^{-1}\xi)\widehat{\sigma_R(v)}(\xi)\, đ\xi.$$

Since $p_R(x,\xi) = p_0(Rx,\xi)$ if $|x| \leqslant 1$, this may by use of q_R be written:

$$\sigma_R(P_0 v)(x) = R^{-2a}(\mathrm{OP}(p_R)\sigma_R(v))(x) + (\mathrm{OP}(q_R)\sigma_R(v))(x) \text{ for } |x| \leqslant 1. \tag{6.11}$$

Moreover, we show a technical lemma in order to control remainder terms.

Lemma 6.7. *For any* $k \in \mathbb{N}$ *and* $R \in (0,1]$,

$$\|\gamma_R\|_{C^{1+\tau}(\mathbb{R}^{n-1})} \leqslant CR^{\min(1,\tau)}, \qquad |p_R - \overline{p}|_{k, C^\tau S^{2a}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant CR^{\min(1,\tau)}.$$

Proof. Using $\gamma(0) = 0$, $\nabla\gamma(0) = 0$ we have

$$\gamma(Rx') = \int_0^1 (\nabla\gamma(sRx') - \nabla\gamma(0))\, ds \cdot Rx',$$

and therefore

$$\sup_{|x'|\leqslant 2} |\gamma(Rx')| \leqslant C|R|^{1+\min(\tau,1)}.$$

since $\nabla\gamma \in C^\tau(\mathbb{R}^{n-1})$. Now let $\alpha \in \mathbb{N}_0^{n-1}$ with $|\alpha| < 1+\tau$. Using

$$\partial_{x'}^\alpha(\gamma(Rx')) = R^{|\alpha|}(\partial_{x'}^\alpha\gamma)(Rx'),$$

one obtains in the same way

$$\sup_{|x'|\leqslant 2} |\partial_{x'}^\alpha(\gamma(Rx'))| \leqslant \begin{cases} C|R|^{1+\min(\tau,1)} & \text{if } |\alpha| = 1, \\ C|R|^{|\alpha|} \leqslant C|R|^{1+\min(\tau,1)} & \text{if } |\alpha| \geqslant 2, \end{cases}$$

since $\partial_{x'}^\alpha\gamma \in C^{\tau+1-|\alpha|}(\mathbb{R}^{n-1})$, $\partial_{x'}^\alpha\gamma(0) = 0$ if $|\alpha| = 1$, and $R \leqslant 1$.

Now let $|\alpha| = 1 + [\tau]$. Then one obtains

$$\sup_{|x'|,|y'|\leqslant 2, x'\neq y'} \frac{|\partial_{x'}^\alpha(\gamma(Rx')) - \partial_{y'}^\alpha(\gamma(Ry'))|}{|x'-y'|^\tau}$$
$$\leqslant \sup_{x',y'\in\mathbb{R}^{n-1}, x'\neq y'} \frac{|(\partial_{x'}^\alpha\gamma)(Rx')) - (\partial_{x'}^\alpha\gamma)(Ry'))|}{|Rx'-Ry'|^\tau} R^{|\alpha|+\tau} \leqslant C|R|^{1+\tau}.$$

Therefore $\|R^{-1}\gamma(R\cdot)\|_{C^{1+\tau}(\overline{B_2(0)})} \leqslant CR^{\min(1,\tau)}$. This implies

$$\|\gamma_R\|_{C^{1+\tau}(\mathbb{R}^n)} = \|\eta R^{-1}\gamma(R\cdot)\|_{C^{1+\tau}(\overline{B_2(0)})} \leqslant C'\|R^{-1}\gamma(R\cdot)\|_{C^{1+\tau}(\overline{B_2(0)})} \leqslant CR^\tau.$$

In a similar manner one shows for every $\alpha, \beta \in \mathbb{N}_0$ with $|\beta| < \tau$

$$\sup_{|x|\leqslant 2,} |\partial_x^\beta\partial_\xi^\alpha(p_0(Rx,\xi) - p_0(0,\xi))| \leqslant \begin{cases} C_{\alpha,\beta}R^{\min(1,\tau)}\langle\xi\rangle^{m-|\alpha|} & \text{if } \beta = 0, \\ C_{\alpha,\beta}R^{|\beta|}\langle\xi\rangle^{m-|\alpha|} \leqslant C_{\alpha,\beta}R^{\min(1,\tau)}\langle\xi\rangle^{m-|\alpha|} & \text{if } \beta \neq 0 \end{cases}$$

and, if $|\beta| = [\tau]$,

$$\sup_{|x|,|y|\leqslant 2} \frac{|\partial_x^\beta\partial_\xi^\alpha(p_0(Rx,\xi) - p_0(Ry,\xi))|}{|x-y|^{\tau-[\tau]}} \leqslant C_{\alpha,\beta}R^\tau\langle\xi\rangle^{m-|\alpha|}$$

for all $\xi \in \mathbb{R}^n$ and $R \in (0,1]$. From this one derives the second statement in a straightforward manner with the aid of the product rule. □

Proof of Theorem 6.6. First of all, because of Proposition 6.5 and Lemma 6.7, there is some $R \in (0,1]$ such that

$$r^+P_{\gamma_R} : H_q^{a(t+2a)}(\overline{\mathbb{R}}_+^n) \to \overline{H}_q^t(\mathbb{R}_+^n)$$

is invertible for $t = s$ and $t = s'$, where $P_{\gamma_R} u = F^*_{\gamma_R} P_R F^{*,-1}_{\gamma_R} u$ for all $u \in H^a_q(\mathbb{R}^n)$. For the following we fix such an R. Then we have that

$$r^+ P_R : H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_{\gamma_R}) \to \overline{H}^s_q(\overline{\mathbb{R}}^n_{\gamma_R})$$

is invertible.

Now we localize the given solution $u \in H_q^{a(s'+2a)}(\overline{\mathbb{R}}^n_\gamma)$. To this end let $\psi \in C_0^\infty(B_1(0))$ with $\psi \equiv 1$ on $B_{1/2}(0)$ and set $\psi_R(x) = \psi(x/R) = (\sigma_{1/R}\psi)(x)$ for all $x \in \mathbb{R}^n$. Then

$$r^+ P_0(\psi_R u) = r^+ \psi_R P u + g = \psi_R r^+ P u + g = \psi_R f + g =: \tilde{f} \qquad \text{in } \mathbb{R}^n_\gamma, \tag{6.12}$$

where $P_0 = \mathrm{OP}(p_0)$, $g := \psi_R r^+(P_0 - P)u + r^+[P_0, \psi_R]u$, and $[P_0, \psi_R] = \mathrm{OP}(q)$ for some odd $q \in C^\tau S^{2a-1}(\mathbb{R}^n \times \mathbb{R}^n)$ due to Proposition 4.8 and $P_0 - P = \mathrm{OP}(p_0 - p)$, where $p_0 - p \in C^\tau S^{2a-1}(\mathbb{R}^n \times \mathbb{R}^n)$ is odd. Therefore q and $p_0 - p$ satisfy the a-transmission condition and we conclude that $g \in \overline{H}^s_q(\mathbb{R}^n_\gamma)$ because of $u \in H_q^{a(2a+s')}(\overline{\mathbb{R}}^n_\gamma)$, $2a + s' = \max(a, s + 2a - 1)$ and Corollary 6.3. Hence $\tilde{f} \in \overline{H}^s_q(\mathbb{R}^n_\gamma)$ since $f|_{B_1(0)} \in \overline{H}^s_q(\mathbb{R}^n_\gamma \cap B_1(0))$.

Moreover, by the definition of γ_R

$$Rx \in \mathbb{R}^n_\gamma \cap B_R(0) \quad \text{if and only if} \quad x \in \mathbb{R}^n_{\gamma_R} \cap B_1(0).$$

Hence (6.12) in $\mathbb{R}^n_\gamma \cap B_R(0)$ is equivalent to

$$r^+ P_R(\psi \sigma_R(u)) = R^{2a}(\sigma_R(\tilde{f}) - \mathrm{OP}(q_R)(\psi \sigma_R(u)) \qquad \text{in } \mathbb{R}^n_{\gamma_R} \cap B_1(0)$$

because of (6.11) and $\psi\sigma_R(u) = \sigma_R(\psi_R u)$. Moreover, since $\mathrm{supp}(\psi\sigma_R u)$ is compactly contained in $B_1(0)$, and thus contained in $B_\lambda(0)$ for some $\lambda < 1$, we have $P_R(\psi\sigma_R u) \in \overline{H}^s_q(\mathbb{R}^n \setminus B_\lambda(0)))$ because of Remark 3.7. Hence

$$r^+ P_R(\psi \sigma_R(u)) = h \qquad \text{in } \mathbb{R}^n_{\gamma_R}$$

for some $h \in \overline{H}^s_q(\mathbb{R}^n_{\gamma_R})$. Since this equation has a unique solution in $H_q^{a(2a+s)}(\overline{\mathbb{R}}^n_{\gamma_R})$ and in $H_q^{a(2a+s')}(\overline{\mathbb{R}}^n_{\gamma_R})$, we conclude $\psi\sigma_R(u) \in H_q^{a(2a+s)}(\overline{\mathbb{R}}^n_{\gamma_R})$. Scaling back implies $u = \tilde{v}$ in $B_{R/2}(0) \cap \mathbb{R}^n_\gamma$ for some $\tilde{v} \in H_q^{a(2a+s)}(\overline{\mathbb{R}}^n_\gamma)$. □

Corollary 6.8. *Let $0 \leqslant s < \tau - 2a$. Assume that $\gamma \in C^{1+\tau}(\mathbb{R}^{n-1})$ satisfies $\gamma(0) = 0$, $\nabla\gamma(0) = 0$ and that $u \in \dot{H}^a_q(\overline{\mathbb{R}}^n_\gamma)$ is a solution of*

$$r^+ P u = f \qquad \text{in } \mathbb{R}^n_\gamma$$

for some $f \in L_q(\mathbb{R}^n_\gamma)$ with $f|_{B_{R_1}(0)} \in \overline{H}^s_q(\mathbb{R}^n_\gamma \cap B_{R_1}(0))$ for some $R_1 > 0$. Then there is some $R > 0$ such that

$$u = v \qquad \text{in } \mathbb{R}^n_\gamma \cap B_R(0)$$

for some $v \in H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_\gamma)$.

Proof. If $s \leqslant 1-a$, then the statement follows directly from Theorem 6.6 since $s' = -a$ and $H_q^{a(s'+2a)}(\overline{\mathbb{R}}^n_\gamma) = \dot{H}_q^a(\overline{\mathbb{R}}^n_\gamma)$ in that case. Let us consider the case $s > 1-a$. Then we see from the proof of Theorem 6.6 that

$$r^+P(\psi_R u) = \psi_R f + r_{\mathbb{R}^n_\gamma}[P,\psi_R]u =: f' \qquad \text{in } \mathbb{R}^n_{\gamma_R},$$

where $\psi_R u \in H_q^{a(a+1)}(\overline{\mathbb{R}}^n_\gamma)$. Moreover, if $\eta \in C_0^\infty(B_{R/2}(0))$ such that $\eta \equiv 1$ on $B_{R/4}(0)$, then

$$\begin{aligned}\eta[P,\psi_R]u &= \mathrm{OP}(\eta(x)p(x,\xi)(\psi_R(x)-\psi_R(y))u = \mathrm{OP}(\eta(x)p(x,\xi)(1-\psi_R(y))u \\ &= \eta P((1-\psi_R)u) \in H_q^s(\mathbb{R}^n)\end{aligned}$$

because of Remark 3.7 and $\operatorname{supp}\eta \cap \operatorname{supp}(1-\psi_R) = \emptyset$. Hence $f' \in \overline{H}_q^s(\mathbb{R}^n_\gamma \cap B_{R/4}(0))$. Therefore we can apply Theorem 6.6 to $\psi_R u$ and f' again to conclude the statement of the corollary, provided that $s \leqslant 2-a$. Repeating this argument finitely many times with the help of Corollary 4.9, we obtain the statement in the general case. □

6.3 | Elliptic regularity in a bounded domain

Now we are in the position to prove the following.

Theorem 6.9. *Let $1 < q < \infty$, $a \in (0,1)$, $\tau > 2a$, and $0 \leqslant s < \tau - 2a$. Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1+\tau}$-domain, and let $p \in C^\tau S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$ be an even and strongly elliptic symbol, $P = \mathrm{OP}(p(x,\xi))$. If $u \in \dot{H}_q^a(\overline{\Omega})$ solves*

$$r^+Pu = f \qquad \text{in } \Omega \tag{6.13}$$

for some $f \in \overline{H}_q^s(\Omega)$, then $u \in H_q^{a(s+2a)}(\overline{\Omega})$.

Proof. Let $x_0 \in \partial\Omega$ be arbitrary. Moreover, let $\gamma \in C^{1+\tau}(\mathbb{R}^{n-1})$ and $R_0 > 0$ be such that

$$\Omega \cap B_{R_0}(x_0) = \mathbb{R}^n_\gamma \cap B_{R_0}(x_0)$$

(after a suitable rotation). By a simple translation and rotation we can always reduce to the case $x_0 = 0$, $\gamma(0) = 0$, and $\nabla\gamma(0) = 0$. It suffices to show that there is an $R \in (0, R_0]$ such that

$$u = v \qquad \text{in } \mathbb{R}^n_\gamma \cap B_R(x_0)$$

for some $v \in H_q^{a(2a+s)}(\overline{\mathbb{R}}^n_\gamma)$. Now let $\psi \in C_0^\infty(B_{R_0}(x_0))$ with $\psi \equiv 1$ on $B_{R_0/2}(x_0)$. Then

$$r^+P(\psi u) = \psi r_\Omega P u + g = \psi f + g \qquad \text{in } \mathbb{R}^n_\gamma,$$

where $g := r^+[P,\psi]u \in \overline{H}_q^{1-a}(\mathbb{R}^n_\gamma) \subset L_q(\mathbb{R}^n_\gamma)$ since $u \in \dot{H}_q^a(\overline{\Omega})$, and $q \in C^\tau S^{2a-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\tau S^a_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Moreover, $g|_{B_{R_0/4}(0)} \in \overline{H}_q^s(\mathbb{R}^n_\gamma \cap B_{R_0/4}(0))$ because of Remark 3.7 and the same observations as in the proof of Corollary 6.8. Hence Corollary 6.8 implies that there is an $R > 0$

such that

$$u = v \qquad \text{in } \mathbb{R}^n_\gamma \cap B_R(0)$$

for some $v \in H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_\gamma)$. Hence the statement of the theorem follows. □

Combining Theorem 6.9 with the forward mapping property shown in Theorem 6.4 we find the following.

Corollary 6.10. *Hypotheses as in Theorem 6.9. A function $u \in \dot{H}_q^a(\overline{\Omega})$ solves (6.13) for some $f \in \overline{H}_q^s(\Omega)$ if and only if $u \in H_q^{a(s+2a)}(\overline{\Omega})$.*

Hence the Dirichlet domain (1.5) for P with data in $\overline{H}_q^s(\Omega)$ equals $H_q^{a(s+2a)}(\overline{\Omega})$.

Proof of Theorem 1.1. The result 1° is a consequence of Theorem 6.4, and 2° is shown above in Corollary 6.10. □

The theorem applies to $(-\Delta)^a$ in the way that $(-\Delta)^a = P_1 + P_2$, where $P_1 = \mathrm{OP}((1-\psi(\xi))|\xi|^{2a})$ satisfies the hypotheses and P_2 is smoothing, so that $r^+(-\Delta)^a u = f \in \overline{H}_q^s(\Omega)$ is turned into $r^+ P_1 u = f_1$, with $f_1 = f - r^+ P_2 u \in \overline{H}_q^s(\Omega)$ since P_2 is smoothing.

There is a corollary on regularity in Hölder spaces.

Corollary 6.11. *Hypotheses as in Theorem 6.9.* 1° *If $f \in \overline{C}_*^s(\Omega)$ for some $s \in (0, \tau - 2a)$, then $u \in C_*^{a(s+2a-\varepsilon)}(\overline{\Omega})$ for every small $\varepsilon > 0$. When $\tau \geqslant 1$, it satisfies*

$$u \in \dot{C}^{s+2a-\varepsilon}(\overline{\Omega}) + d^a e^+ \overline{C}^{s+a-\varepsilon}(\Omega). \tag{6.14}$$

If $2a < 1$ and $2a < \tau < 1$, it satisfies a local version of (6.14), cf. Remark 4.7.

2° *If $f \in L_\infty(\Omega)$, then $u \in C_*^{a(2a-\varepsilon)}(\overline{\Omega})$ for every small $\varepsilon > 0$, satisfying (6.14) ff. with s replaced by* 0.

Proof. We use the Sobolev embedding property $H_q^t(\mathbb{R}^n) \subset C_*^{t-n/q-\varepsilon}(\mathbb{R}^n)$, which implies $H_q^{a(t)}(\overline{\Omega}) \subset C_*^{a(t-n/q-\varepsilon)}(\overline{\Omega})$ in view of Definition 4.3 (when $a \leqslant t - n/q - \varepsilon < t < 1 + \tau$). For 1°, $f \in \overline{C}_*^s(\Omega) \subset \overline{H}_q^{s-\varepsilon/2}(\Omega)$ implies when $n/q < \varepsilon/2$ that $u \in H_q^{a(s+2a-\varepsilon/2)}(\overline{\Omega}) \subset C_*^{a(s+2a-\varepsilon)}(\overline{\Omega})$, and (6.14)ff. follow from Theorem 4.6 and Remark 4.7. For 2°, we conclude similarly from $f \in L_\infty(\Omega) \subset L_q(\Omega) = \overline{H}_q^0(\Omega)$ that $u \in H_q^{a(2a)}(\overline{\Omega}) \subset C_*^{a(2a-\varepsilon)}(\overline{\Omega})$, when $n/q < \varepsilon$, with the ensuing descriptions. □

APPENDIX

Proof of Theorem 3.1. We use (3.1) with $l = \max\{[m - |\gamma|], 0\}$. Since $\partial_y^\alpha a(x,y,\xi)|_{y=x}(y-x)^\alpha u(y) = (\partial_y^\alpha a)(x,x,\xi)(y-x)^\alpha u(y)$ is smooth with respect to (y,ξ), the existence of

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^{2n}} \chi(\varepsilon y, \varepsilon\xi) e^{i(x-y)\cdot\xi} \left(\partial_y^\alpha a(x,y,\xi)|_{y=x}\right)(x-y)^\alpha u(y)\, dy\, đ\xi = \mathrm{OP}(p_\alpha)u(x)$$

follows from standard results on oscillatory integrals. Here

$$p_\alpha(x,\xi) = \partial_y^\alpha D_\xi^\alpha a(x,y,\xi)|_{y=x} \qquad \text{for all } x,\xi \in \mathbb{R}^n,$$

because of the calculation rules for oscillatory integrals. Therefore it only remains to show the existence of the oscillatory integrals for $(y-x)^{\alpha+\gamma} r_\alpha(x,y,\xi)$. Now we use that

$$\int_{\mathbb{R}^{2n}} \chi(\varepsilon y, \varepsilon\xi) e^{i(x-y)\cdot\xi}(y-x)^{\alpha+\gamma} r_\alpha(x,y,\xi) u(y)\,dy\,đ\xi$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\chi(\varepsilon y,\varepsilon\xi)(y-x)^{\alpha+\gamma} r_\alpha(x,y,\xi)\,đ\xi\, u(y)\,dy = \int_{\mathbb{R}^n} k_\varepsilon(x,y,x-y)u(y)\,dy,$$

where

$$k_\varepsilon(x,y,z) := (y-x)^{\alpha+\gamma}\int_{\mathbb{R}^n} e^{iz\cdot\xi}\chi(\varepsilon y,\varepsilon\xi) r_\alpha(x,y,\xi)\,đ\xi = \sum_{j\in\mathbb{N}_0} k_{\varepsilon,j}(x,y,z),$$

$$k_{\varepsilon,j}(x,y,z) := (y-x)^{\alpha+\gamma}\int_{\mathbb{R}^n} e^{iz\cdot\xi}\chi(\varepsilon y,\varepsilon\xi) r_\alpha(x,y,\xi)\varphi_j(\xi)\,đ\xi.$$

Using (3.3) one shows in the same manner as in the proof of [6, Lemma 5.14] that for every $N\in\mathbb{N}_0$ there is some $C_N > 0$ such that

$$|k_{\varepsilon,j}(x,y,z)| \leqslant C_N|x-y|^{l+|\gamma|+\theta}|z|^{-N}2^{j(n+m-N)}$$

for all $z\neq 0$, $j\in\mathbb{N}_0$, $\varepsilon\in(0,1)$, $x,y\in\mathbb{R}^n$, where $\theta = \min\{\tau - l, 1\}$. Using

$$\sum_{j\in\mathbb{N}_0} k_{\varepsilon,j}(x,y,z) = \sum_{2^j\leqslant|z|^{-1}} k_{\varepsilon,j}(x,y,z) + \sum_{2^j>|z|^{-1}} k_{\varepsilon,j}(x,y,z),$$

one can derive in the same way as in the proof of [6, Theorem 5.12] that the series $\sum_{j\in\mathbb{N}_0} k_{\varepsilon,j}(x,y,z)$ converges absolutely and uniformly in $|z|\geqslant\delta$, $x,y\in\mathbb{R}^n$ for any $\delta>0$ to a function $k_\varepsilon : \mathbb{R}^n\times\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})\to\mathbb{C}$ that satisfies for any $N\in\mathbb{N}_0$

$$|k_\varepsilon(x,y,z)| \leqslant \begin{cases} C_N|x-y|^{l+|\gamma|+\theta}|z|^{-m-n}\langle z\rangle^{-N} & \text{if } m+n>0,\\ C_N|x-y|^{l+|\gamma|+\theta}(1+\log|z|^{-1})\langle z\rangle^{-N} & \text{if } m+n=0,\\ C_N|x-y|^{l+|\gamma|+\theta}\langle z\rangle^{-N} & \text{if } m+n<0,\end{cases}$$

for all $x,y\in\mathbb{R}^n$, $z\neq0$, $\varepsilon\in(0,1)$, $j\in\mathbb{N}_0$. Now, if we choose $N\in\mathbb{N}_0$ sufficiently large and $z = x-y$, the right-hand side is in $L_1(\mathbb{R}^n)$ with respect to y since $\tau+|\gamma|>m$. Hence by the dominated convergence theorem the limit

$$\lim_{\varepsilon\to0}\int_{\mathbb{R}^{2n}}\chi(\varepsilon y,\varepsilon\xi)e^{i(x-y)\cdot\xi}(y-x)^{\alpha+\gamma}r_\alpha(x,y,\xi)u(y)\,dy\,đ\xi = \int_{\mathbb{R}^n} k_{\alpha,\gamma}(x,y,x-y)u(y)\,dy$$

exists, where $k_{\alpha,\gamma}$ and $k_{\alpha,\gamma,j}$ are as in the theorem. This concludes the proof. □

Proof of Lemma 3.2. First of all, note that both oscillatory integrals exist because of Theorem 3.1, $D_\xi^\beta a\in C^\tau S^{m-|\beta|}_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$, and $m-|\beta| < \tau+|\gamma|-|\beta|$. Moreover, it is sufficient to consider the case $\beta = e_k$ for some $k \in \{1, \dots, n\}$. The general case follows inductively. In view of Theorem 3.1, it only remains to show that

$$\mathrm{OP}((y-x)^{\alpha+\gamma} r_\alpha(x,y,\xi))u(x) = \mathrm{OP}((y-x)^{\alpha+\gamma-e_k} D_{\xi_k} r_\alpha(x,y,\xi))u(x).$$

By Theorem 3.1 applied to $D_{\xi_k} a$ we have

$$\mathrm{OP}((y-x)^{\alpha+\gamma-e_k} D_{\xi_k} r_\alpha(x,y,\xi))u(x) = \int_{\mathbb{R}^n} \tilde{k}_{\alpha,\gamma}(x,y,x-y)u(y)\,dy,$$

where $\tilde{k}_{\alpha,\gamma}(x,y,z) = \sum_{j\in\mathbb{N}_0} \tilde{k}_{\alpha,\gamma,j}(x,y,z)$ and

$$\tilde{k}_{\alpha,\gamma,j}(x,y,z) = \int_{\mathbb{R}^n} e^{iz\cdot\xi}(y-x)^{\alpha+\gamma-e_k} D_{\xi_k} r_\alpha(x,y,\xi)\varphi_j(\xi)\,đ\xi.$$

For $z = x - y \neq 0$, an integration by parts yields

$$\begin{aligned}\tilde{k}_{\alpha,\gamma,j}(x,y,x-y) &= \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}(y-x)^{\alpha+\gamma} r_\alpha(x,y,\xi)\varphi_j(\xi)\,đ\xi \\ &\quad - \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}(y-x)^{\alpha+\gamma-e_k} r_\alpha(x,y,\xi) D_{\xi_k}\varphi_j(\xi)\,đ\xi \\ &= k_{\alpha,\gamma,j}(x,y,x-y) - \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}(y-x)^{\alpha+\gamma-e_k} r_\alpha(x,y,\xi) D_{\xi_k}\varphi_j(\xi)\,đ\xi.\end{aligned}$$

By the results in the proof of Theorem 3.1

$$\sum_{j\in\mathbb{N}_0} \tilde{k}_{\alpha,\gamma,j}(x,y,x-y) \quad \text{and} \quad \sum_{j\in\mathbb{N}_0} k_{\alpha,\gamma,j}(x,y,x-y)$$

converge absolutely for every $x \neq y$. Hence the same is true for

$$\sum_{j\in\mathbb{N}_0} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}(y-x)^{\alpha+\gamma-e_k} r_\alpha(x,y,\xi) D_{\xi_k}\varphi_j(\xi)\,đ\xi.$$

Here for $x \neq y$ and $N \in \mathbb{N}$

$$\begin{aligned}&\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}(y-x)^{\alpha+\gamma-e_k} r_\alpha(x,y,\xi) D_{\xi_k}\varphi_j(\xi)\,đ\xi \\ &= |x-y|^{-2N} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}(y-x)^{\alpha+\gamma-e_k}(-\Delta_\xi)^N\Big(r_\alpha(x,y,\xi) D_{\xi_k}\varphi_j(\xi)\Big)\,đ\xi,\end{aligned}$$

where

$$\left|(-\Delta_\xi)^N\Big(r_\alpha(x,y,\xi) D_{\xi_k}\varphi_j(\xi)\Big)\right| \leqslant C_N\langle\xi\rangle^{m-2N} \quad \text{for all } \xi \in \mathbb{R}^n.$$

Hence choosing $N \in \mathbb{N}$ such that $m - 2N < -n$, we can apply the dominated convergence theorem to conclude

$$\sum_{j\in\mathbb{N}_0}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}(y-x)^{\alpha+\gamma-e_k} r_\alpha(x,y,\xi) D_{\xi_k}\varphi_j(\xi)\,đ\xi$$
$$=\sum_{j\in\mathbb{N}_0}|x-y|^{-2N}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}(y-x)^{\alpha+\gamma-e_k}(-\Delta_\xi)^N\Big(r_\alpha(x,y,\xi)\sum_{j\in\mathbb{N}_0} D_{\xi_k}\varphi_j(\xi)\Big)\,đ\xi = 0,$$

because of $0=\sum_{j=0}^\infty D_{\xi_k}\varphi_j(\xi)$ for all $\xi\in\mathbb{R}^n$. Thus $\tilde{k}_{\alpha,\gamma}(x,y,x-y)=k_{\alpha,\gamma}(x,y,x-y)$ for all $x\neq y$, and the statement of the lemma follows. □

Proof of Theorem 3.4. First we treat the case $s\geqslant 0$ by splitting it up in several cases. Afterward the case $s<0$ follows by duality.

Case $m\in[0,1]$ **and** $0\leqslant s<1$: Let us define $\sigma := m$ and $\varrho := \tau-m$ if $\tau-m<1$, and $\varrho\in(s,1)$ arbitrary if $\tau-m\geqslant 1$. Then $a\in C^{\varrho+\sigma}S^\sigma_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$, and [57, Proposition 9.8] yields the boundedness of

$$\mathrm{OP}(a(x,y,\xi))\colon H^{s+m}_q(\mathbb{R}^n)\to H^s_q(\mathbb{R}^n).$$

Case $m\in(-1,0)$ **and** $0\leqslant s<1$: We use the decomposition

$$\mathrm{OP}(a(x,y,\xi))=\mathrm{OP}(a(x,x,\xi))+\mathrm{OP}(b(x,y,\xi)),$$

where $b(x,y,\xi)=a(x,y,\xi)-a(x,x,\xi)$. Because of Theorem 2.1, it is sufficient to show the corresponding mapping property for $\mathrm{OP}(b(x,y,\xi))$. Since $b\in C^\tau S^m_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)\subset C^\tau S^0_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$ and $b(x,x,\xi)=0$ for all $x,\xi\in\mathbb{R}^n$, [57, Proposition 9.5] yields that

$$\mathrm{OP}(b(x,y,\xi))\colon H^\sigma_q(\mathbb{R}^n)\to H^{\sigma+t}_q(\mathbb{R}^n)\quad\text{for all }-\tau<\sigma\leqslant 0,\ 0\leqslant t<\tau. \tag{A.1}$$

If $s+m\leqslant 0$, we can choose $\sigma=s+m\in(-\tau,0]$ and $t=-m\in[0,\tau)$ and obtain the desired mapping property. If $s+m>0$, we use that $H^{s+m}_q(\mathbb{R}^n)\hookrightarrow L_q(\mathbb{R}^n)$ and (A.1) with $\sigma=0$ and $t=s$ to conclude that $\mathrm{OP}(b(x,y,\xi))\colon H^{s+m}_q(\mathbb{R}^n)\to H^s_q(\mathbb{R}^n)$ is bounded.

Case $m\in(-1,1]$ **and** $s\geqslant 1$: Let $k=[s]$, $s'=s-k\in[0,1)$. We use that

$$u\in H^s_q(\mathbb{R}^n)\quad\text{if and only if}\quad \partial^\alpha_x u\in H^{s'}_q(\mathbb{R}^n)\text{ for all }|\alpha|\leqslant k$$

for $u=\mathrm{OP}(a(x,y,\xi))f$ for some $f\in H^{s+m}_q(\mathbb{R}^n)$.

First let $m\in[0,1]$. Using that

$$[\partial_{x_j},\mathrm{OP}(a(x,y,\xi)]=\mathrm{OP}(\partial_{x_j}a(x,y,\xi)+\partial_{y_j}a(x,y,\xi))$$

for all $j=1,\dots,n$ one obtains

$$\partial^\alpha_x\,\mathrm{OP}(a(x,y,\xi))f=\sum_{0\leqslant\beta\leqslant\alpha}\mathrm{OP}(a_\beta(x,y,\xi))\partial^{\alpha-\beta}_x f$$

for some $a_\beta\in C^{\tau-|\beta|}S^m_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)\subset C^{\tau-k}S^m_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$, where $0\leqslant s'=s-k<\tau' := \tau-k$ and $0\leqslant s'+m=s+m-k<\tau'$ because of $|s|,|s+m|<\tau$. Moreover, $\partial^{\alpha-\beta}_x f\in H^{s'+m}_q(\mathbb{R}^n)$. By the preceding cases,

$$\mathrm{OP}(a_\beta(x,y,\xi))\colon H^{s'+m}_q(\mathbb{R}^n)\to H^{s'}_q(\mathbb{R}^n)$$

is bounded. Altogether this yields the boundedness of $\mathrm{OP}(a(x,y,\xi))$ in this case if $m\in[0,1]$.

If $m \in (-1,0)$ and $1 \leqslant |\alpha| \leqslant k$, then $\alpha = \alpha' + e_j$ for some $j \in \{1, \dots, n\}$. Using

$$\partial_{x_j} \mathrm{OP}(a(x,y,\xi))f = \mathrm{OP}(a(x,y,\xi)i\xi_j) + \mathrm{OP}(\partial_{x_j} a(x,y,\xi)),$$

one obtains similarly as before

$$\partial_x^\alpha \mathrm{OP}(a(x,y,\xi))f = \sum_{0\leqslant\beta\leqslant\alpha'} \mathrm{OP}(a'_\beta(x,y,\xi) + a''_\beta(x,y,\xi))\partial_x^{\alpha'-\beta} f$$

for some $a'_\beta \in C^{\tau-k+1}S^{m+1}_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$ and $a''_\beta \in C^{\tau-k}S^m_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n) \subset C^{\tau-k}S^0_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$. Since $m+1 \in (0,1)$, one obtains by the preceding cases that

$$\mathrm{OP}(a'_\beta(x,y,\xi)) : H_q^{s'+m+1}(\mathbb{R}^n) \to H_q^{s'}(\mathbb{R}^n), \quad \mathrm{OP}(a''_\beta(x,y,\xi)) : H_q^{s'}(\mathbb{R}^n) \to H_q^{s'}(\mathbb{R}^n)$$

are bounded due to $0 \leqslant s'+m+1 = s+m-k+1 < \tau-k+1$ and $0 \leqslant s' < \tau-k$. This yields the statement in this case since $\partial_x^{\alpha'-\beta} f$ is in $H_q^{s'+m+1}(\mathbb{R}^n)$ for all $f \in H_q^{s+m}(\mathbb{R}^n)$ and $0 \leqslant \beta \leqslant \alpha'$, where $|\alpha'| \leqslant k-1$, and the case $\alpha = 0$ is easy.

Case $m \in [0,\tau)$ **and** $s \geqslant 0$: Now let $0 \leqslant m < \tau$ and $s \geqslant 0$ and set $m' = [m]$. We use that there are polynomials $p_k : \mathbb{R}^n \to \mathbb{R}$ of order at most m' and $q_k \in S^0_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$, independent of x, $k = 0, \dots, n$ such that

$$\langle\xi\rangle^{m'} = \sum\nolimits_{k=0}^n q_k(\xi)p_k(\xi),$$

cf., for example, [6, Proof of Theorem 6.8]. Hence

$$a(x,y,\xi) = \sum\nolimits_{k=0}^n a_k(x,y,\xi)p_k(\xi),$$

where $a_k \in C^\tau S^{m-m'}_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$. Combining this with the general relation

$$\mathrm{OP}(b(x,y,\xi)i\xi_j) = \mathrm{OP}(b(x,y,\xi))\partial_{x_j} + \mathrm{OP}(\partial_{y_j} b(x,y,\xi)), \tag{A.2}$$

we have the representation

$$\mathrm{OP}(a(x,y,\xi)) = \sum\nolimits_{|\alpha|\leqslant m'} \mathrm{OP}(a_\alpha(x,y,\xi))\partial_x^\alpha$$

for some $a_\alpha \in C^{\tau-m'}S^{m-m'}_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$. Because of the case "$m \in [0,1]$ and $s \geqslant 0$", we conclude that

$$\mathrm{OP}(a_\alpha(x,y,\xi)) : H_q^{s+m-m'}(\mathbb{R}^n) \to H_q^s(\mathbb{R}^n)$$

is bounded for every $|\alpha| \leqslant m'$. This implies the statement in this case.

Case: $m \in (-\tau,0)$ **and** $s \geqslant 0$: Let $m' \in \mathbb{N}_0$ be such that $m+m' \in (-1,0]$, that is, $m' = [-m]$. First we consider the case $s = 0$. As noted above, $\langle\xi\rangle^{m'} = \sum_{k=0}^n q_k(\xi)p_k(\xi)$, for some $q_k \in S^0_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$ independent of x and polynomials p_k of order at most $m' < \tau$. Hence

$$\mathrm{OP}(a(x,y,\xi)) = \mathrm{OP}(a(x,y,\xi))\langle D_x\rangle^{m'}\langle D_x\rangle^{-m'} = \sum_{k=0}^n \mathrm{OP}(a(x,y,\xi))p_k(D_x)\tilde{q}_k(D_x),$$

where $\tilde{q}_k \in S^{-m'}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is independent of x and $\tilde{q}_k(D_x) : H^m_q(\mathbb{R}^n) \to H^{m+m'}_q(\mathbb{R}^n)$. Therefore it remains to show that

$$\mathrm{OP}(a(x,y,\xi))p_k(D_x) : H^{s+m+m'}_q(\mathbb{R}^n) \to H^s_q(\mathbb{R}^n)$$

is bounded. Using that $p_k(D_x)$ is a differential operator of order m' and (A.2) one obtains the representation

$$\mathrm{OP}(a(x,y,\xi))p_k(D_x) = \sum\nolimits_{|\alpha|\leqslant m'} \mathrm{OP}(a_{\alpha,k}(x,y,\xi))$$

for some $a_{\alpha,k} \in C^{\tau-m'}S^{m+m'}_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$. Hence the case "$m \in (-1,0]$ and $s \geqslant 0$" implies that

$$\mathrm{OP}(a_{\alpha,k}(x,y,\xi)) : H^{m+m'}_q(\mathbb{R}^n) \to L_q(\mathbb{R}^n).$$

Altogether we conclude that

$$\mathrm{OP}(a(x,y,\xi)) : H^m_q(\mathbb{R}^n) \to L_q(\mathbb{R}^n)$$

is bounded. Next let $s \in [-m,\tau)$ and $s' = s - m'$. Using

$$u \in H^s_q(\mathbb{R}^n) \quad \text{if and only if} \quad \partial^\alpha_x u \in H^{s'}_q(\mathbb{R}^n) \text{ for all } |\alpha| \leqslant m'$$

for $u = \mathrm{OP}(a(x,y,\xi))f$ for some $f \in H^{s+m}_q(\mathbb{R}^n)$, it is sufficient to show that

$$\partial^\alpha_x \mathrm{OP}(a(x,y,\xi)) : H^{s+m}_q(\mathbb{R}^n) \to H^{s'}_q(\mathbb{R}^n)$$

is bounded for all $|\alpha| \leqslant m'$. Similarly as before

$$\partial^\alpha_x \mathrm{OP}(a(x,y,\xi)) = \mathrm{OP}(a_\alpha(x,y,\xi))$$

for some $a_\alpha \in C^{\tau-|\alpha|}S^{m+|\alpha|}_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n) \subset C^{\tau-m'}S^{m+m'}_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$, where $0 \leqslant s+m \leqslant s' = s - m' < \tau' := \tau - m'$. By the preceding cases,

$$\mathrm{OP}(a_\alpha(x,y,\xi)) : H^{s+m}_q(\mathbb{R}^n) = H^{s'+m+m'}_q(\mathbb{R}^n) \to H^{s'}_q(\mathbb{R}^n)$$

is bounded. Now the mapping properties for general $s \in [0,\tau)$ follow by interpolation between the case $s = 0$ and $s \in [-m,\tau)$.

Case $s < 0$: By the assumptions of the theorem, $|s| < \tau$ and $|s+m| < \tau$. First we consider the case that additionally $s \in (-\tau,-m)$. Since $|m| < \tau$, there are some s, which satisfy all these assumptions. Note that in the case $m < 0$ this condition is trivial. By the case "$s \geqslant 0$" we obtain that

$$\mathrm{OP}(a(x,y,\xi))^* = \mathrm{OP}(\overline{a(y,x,\xi)}) : H^{-s}_{q'}(\mathbb{R}^n) \to H^{-s-m}_{q'}(\mathbb{R}^n)$$

since $s' := -s-m \in (0,\tau)$ and $-\tau < -s = s' + m < \tau$. Hence we conclude by duality that

$$\mathrm{OP}(a(x,y,\xi)) : H^{s+m}_q(\mathbb{R}^n) \to H^s_q(\mathbb{R}^n)$$

if additionally $s \in (-\tau,-m)$. Interpolation with the case $s \geqslant 0$ yields the statement for all $s \in (-\tau, \tau - m)$. □

Proof of Theorem 3.6. It is sufficient to consider the case $m \geqslant 0$ since $C^\tau S^m_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n) \subset C^\tau S^0_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$ and $\min(\tau - m, \tau) = \tau$ if $m < 0$.

Let us first consider the case $0 \leqslant s < 1$. Since $0 \leqslant s < \tau - m$ is arbitrary and $B^{s+\varepsilon}_{q,\infty}(\mathbb{R}^n) \hookrightarrow H^s_q(\mathbb{R}^n)$ for any $\varepsilon > 0$, it is sufficient to prove that

$$\mathrm{OP}(a(x, y, \xi)) \colon L_q(\mathbb{R}^n) \to B^s_{q,\infty}(\mathbb{R}^n)$$

is a bounded linear operator for any $0 \leqslant s < \min(\tau - m, \tau)$. To this end we use that

$$\mathrm{OP}(a(x, y, \xi))u(x) = \int_{\mathbb{R}^n} k(x, y, x - y)u(y)\, dy \qquad \text{for all } u \in \mathcal{S}(\mathbb{R}^n), x \in \mathbb{R}^n, \tag{A.3}$$

where $k \colon \mathbb{R}^n \times \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{C}$ is smooth with respect to the third variable and satisfies for any $\alpha \in \mathbb{N}^n_0$ and $N \in \mathbb{N}$

$$\|\partial^\alpha_z k(.,.,z)\|_{C^\tau(\mathbb{R}^{2n})} \leqslant C_{\alpha,N}|z|^{-n-m-|\alpha|}(1 + |z|)^{-N} \qquad \text{for all } z \neq 0.$$

Here k can be defined as

$$\sum\nolimits_{j\in\mathbb{N}_0} k_j(x, y, z) \qquad \text{for all } x, y, z \in \mathbb{R}^n, z \neq 0,$$

where $k_j(x, y, z) = \mathcal{F}^{-1}_{\xi\mapsto z}(a(x, y, \xi)\varphi_j(\xi))$ and $\varphi_j \in C^\infty_0(\mathbb{R}^n)$, $j \in \mathbb{N}_0$, is a smooth dyadic partition of unity as in (3.4)ff. The proofs in [6, Section 5.4] carry directly over to the present situation.

Moreover, $(\partial^\alpha_y k)(x, y, z)|_{y=x} = 0$ since $\partial^\alpha_y a(x, y, \xi)|_{y=x} = 0$ for all $x, \xi \in \mathbb{R}^n$ and $|\alpha| < \tau$, and we have for any $N \in \mathbb{N}$:

$$\begin{aligned} |k(x, y, z)| &= \left| k(x, y, z) - \sum\nolimits_{|\alpha|<\tau} \frac{1}{\alpha!}(\partial^\alpha_y k)(x, x, z)z^\alpha \right| \\ &\leqslant C_N|x - y|^\tau |z|^{-n-m}(1 + |z|)^{-N}, \end{aligned} \tag{A.4}$$

and in particular

$$|k(x, y, x - y)| \leqslant C_N|x - y|^{-n-m+\tau}(1 + |x - y|)^{-N},$$

where $|z|^{-n-m+\tau}(1 + |z|)^{-N}$ is in $L_1(\mathbb{R}^n)$ with respect to z for sufficiently large $N \in \mathbb{N}$ since $m < \tau$. Now let $(\Delta_h f)(x) := f(x + h) - f(x)$ for any $x, h \in \mathbb{R}^n$ and $f \colon \mathbb{R}^n \to \mathbb{C}$. Then

$$\begin{aligned} (\Delta_h \mathrm{OP}(a(x, y, \xi))u)(x) = &\int_{|x-y|<2|h|} (k(x + h, y, x + h - y) - k(x, y, x - y))u(y)\, dy \\ &+ \int_{|x-y|\geqslant 2|h|} (k(x + h, y, x - y) - k(x, y, x - y))u(y)\, dy \equiv I_1 + I_2. \end{aligned}$$

In order to estimate I_1 we use that

$$\begin{aligned} |k(x + h, y, x + h - y) - k(x, y, x - y)| &\leqslant |k(x + h, y, x + h - y)| + |k(x, y, x - y)| \\ &\leqslant C\big(|x + h - y|^{-n-m+\tau} + |x - y|^{-m-m+\tau}\big). \end{aligned}$$

Hence Young's inequality implies

$$\begin{aligned}\|I_1\|_{L_q(\mathbb{R}^n)} &\leqslant C\int_{|z|\leqslant 2|h|}(|z+h|^{-n-m+\tau}+|z|^{-n-m+\tau})\|u\|_{L_q(\mathbb{R}^n)}\\ &\leqslant C'|h|^{\tau-m}\|u\|_{L_q(\mathbb{R}^n)}\leqslant C'|h|^{s}\|u\|_{L_q(\mathbb{R}^n)}\end{aligned}$$

for any $s<\tau-m$ and $|h|\leqslant 1$. In order to estimate I_2 we use

$$\begin{aligned}I_2 &= \int_{|x-y|\geqslant 2|h|}(k(x+h,y,x+h-y)-k(x+h,y,x-y))u(y)\,dy\\ &\quad+\int_{|x-y|\geqslant 2|h|}(k(x+h,y,x-y)-k(x,y,x-y))u(y)\,dy\equiv J_1+J_2.\end{aligned}$$

For the second integral we use that for any $|x-y|\geqslant 2|h|$

$$\begin{aligned}|k(x+h,y,x-y)-k(x,y,x-y)| &\leqslant C_N|h|^{\tau}|x-y|^{-n-m}(1+|x-y|)^{-N}\\ &\leqslant C_N'|h|^{s}|x-y|^{-n-m+\tau-s}(1+|x-y|)^{-N}.\end{aligned}$$

Therefore

$$\|J_2\|_{L_q(\mathbb{R}^n)}\leqslant C_N|h|^s\int_{\mathbb{R}^n}|z|^{-n-m+\tau-s}(1+|z|)^{-N}\,dz\|u\|_{L_q(\mathbb{R}^n)}\leqslant C_s'|h|^s\|u\|_{L_q(\mathbb{R}^n)}$$

for any $s<\tau-m$ and $|h|\leqslant 1$ and sufficiently large N. Furthermore, for any $|x-y|\geqslant 2|h|$

$$\begin{aligned}&|k(x+h,y,x+h-y)-k(x+h,y,x-y)|\\ &=\left|\int_0^1 D_zk(x+h,y,x+sh-y)\,ds\,h\right|\leqslant C_N|x+h-y|^{\tau}|x-y|^{-n-m-1}(1+|x-y|)^{-N}|h|\\ &\leqslant C_N'|h|^s|x-y|^{-n-m+\tau-s}(1+|x-y|)^{-N}.\end{aligned}$$

by (A.4) with k replaced by D_zk and since

$$\tfrac{1}{2}|x-y|\leqslant|x-y|-|h|\leqslant|x+sh-y|\leqslant|x-y|+|h|\leqslant\tfrac{3}{2}|x-y|$$

for every $s\in[0,1]$. Thus we obtain as before

$$\|J_1\|_{L_q(\mathbb{R}^n)}\leqslant C_N|h|^s\int_{\mathbb{R}^n}|z|^{-n-m+\tau-s}(1+|z|)^{-N}\,dz\|u\|_{L_q(\mathbb{R}^n)}\leqslant C_s'|h|^s\|u\|_{L_q(\mathbb{R}^n)}$$

for any $s<\tau-m$, $|h|\leqslant 1$, and suitable $N\in\mathbb{N}$. Altogether we obtain

$$\|\Delta_h\,\mathrm{OP}(a(x,y,\xi))u\|_{L_q(\mathbb{R}^n)}\leqslant C|h|^s\|u\|_{L_q(\mathbb{R}^n)}$$

uniformly in $|h|\leqslant 1$ for any $s<\tau-m$. Moreover, one obtains by similar, but simpler estimates

$$\|\,\mathrm{OP}(a(x,y,\xi))u\|_{L_q(\mathbb{R}^n)}\leqslant C\|u\|_{L_q(\mathbb{R}^n)}.$$

This implies the boundedness of $\mathrm{OP}(a(x,y,\xi)):\ L_q(\mathbb{R}^n)\to B^s_{q,\infty}(\mathbb{R}^n)$ for any $0\leqslant s<\tau-m$.

Finally, let $s \geqslant 1$. Now let $k \in \mathbb{N}$ such that $s = k + s'$ with $s' \in [0,1)$. We use that $u \in H_q^s(\mathbb{R}^n)$ if and only if $\partial_x^\alpha u \in H_q^{s'}(\mathbb{R}^n)$ for every $|\alpha| \leqslant k$. For $|\alpha| \leqslant k$ we have

$$\partial_x^\alpha \operatorname{OP}(a(x,y,\xi)) = \sum_{0\leqslant\beta\leqslant\alpha} \binom{\alpha}{\beta} \operatorname{OP}(\partial_x^\beta a(x,y,\xi)(i\xi)^{\alpha-\beta}),$$

where $\partial_x^\beta a(x,y,\xi)(i\xi)^{\alpha-\beta} \in C^{\tau-|\beta|}S_{1,0}^{m+|\alpha|-|\beta|}(\mathbb{R}^{2n} \times \mathbb{R}^n)$. Because of

$$0 \leqslant s' = s - k < s - |\alpha| \leqslant \tau - m - |\alpha| = (\tau - |\beta|) - (m + |\alpha| - |\beta|),$$

we can apply the case "$0 \leqslant s < 1$" to $\partial_x^\beta a(x,y,\xi)(i\xi)^{\alpha-\beta}$ (with s' instead of s) and obtain the result. □

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