Dolbeault cohomology of weakly smooth forms on non-archimedean abelian varieties

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Chapter 1

Introduction

For a complex manifold, we can define a sheaf $\mathcal{A}^k$ of complex-valued differential forms. There is a canonical decomposition $\mathcal{A}^k = \bigoplus_{p+q=k} \mathcal{A}^{p,q}$ equipping this sheaf with a bigrading. Furthermore, there are differential operators ∂ and $\overline{\partial}$ on differential forms, yielding a bicomplex $\left(\mathcal{A}^{\bullet,\bullet}, \partial, \overline{\partial}\right)$. The cohomology of the complex $\left(\mathcal{A}^{p,\bullet}, \overline{\partial}\right)$ is called the Dolbeault cohomology and it is canonically isomorphic to the sheaf cohomology of the sheaf of holomorphic differential p-forms. This result is known as Dolbeault's Theorem (from about 1950), and it is a complex analogue of de Rham's Theorem.

In non-archimedean analytic geometry, complex manifolds are replaced by analytic spaces over non-archimedean fields. Non-archimedean fields were discovered in the beginning of the last century, and they are defined as fields k together with an absolute value $|.|$ that satisfies the ultrametric triangle inequality, meaning that $|x+y| \leq \max\{|x|, |y|\}$. As a result, the topology of a non-archimedean field $(k, |.|)$ is totally disconnected. For this reason, the naive approach defining analytic spaces over a non-archimedean field k analogously to the complex analytic case does not work.

This lead to different approaches towards the definition of non-archimedean analytic spaces together with a suitable sheaf of analytic functions. Tate developed the theory of rigid geometry around 1960. His work gave rise to a good theory of sheaves of analytic functions, but the underlying spaces only come with a Grothendieck topology. More recently, Raynaud's formal models and Huber's adic spaces lead to other approaches to non-archimedean geometry.

In this thesis we are going to build upon the theory of analytic spaces developed by Berkovich in 1990. His analytic spaces are actual, well-behaved topological spaces. For example, they are locally compact and locally path-connected. An algebraic variety X over a non-archimedean field k gives rise in a natural way to a Berkovich analytic space X^{an}, the so-called Berkovich analytification of X. Locally, on affine open subsets $\mathrm{Spec}(R)$ of X, the Berkovich analytification is given by the collection of all multiplicative seminorms on R extending the given absolute value on k, together with the coarsest topology such that for all $f \in R$ the map $\mathrm{Spec}(R)^{\mathrm{an}} \to \mathbb{R}, |.|_x \mapsto |f|_x$ is continuous.

There are results – known as GAGA-results – which connect algebraic properties of the variety X with properties of the topological space X^{an}. To name a few, X is separated (resp. proper, connected) if and only if X^{an} is Hausdorff (resp. compact, arcwise connected). All analytic spaces we consider in this thesis are meant to be good strictly k-analytic spaces in the sense of [Ber93].

There is a theory of smooth real-valued differential forms on Berkovich analytic spaces together with differential operators and a so-called Dolbeault cohomology, which is due to Chambert-Loir and Ducros [CD12] and can be seen as an analogue of the complex analytic case in the non-archimedean world. This theory was generalized by Gubler, Jell and Rabinoff in their recent paper [GJR21], giving rise to a larger bigraded sheaf of so-called weakly smooth differential forms with essentially the same properties, but with a better cohomological behaviour. Furthermore – in contrast to the space of smooth forms – they contain certain forms that arise in a natural way on the analytfication of abelian varieties.

In this thesis, our main goal is to study the Dolbeault cohomology of weakly smooth forms of a particular class of Berkovich analytic spaces. Namely, we consider the Berkovich analytic space A^{an} associated to an abelian variety A over k, where $(k, |.|)$ is a field which is complete with respect to a non-trivial non-archimedean absolute value. The motivation comes from the fact that in the complex analytic case, this class of cohomology groups can be described very precisely: For an abelian variety A over $\mathbb{C}$ of dimension n, the complex analytic space $A(\mathbb{C})$ is a connected compact Lie group, and hence a complex torus. Denoting by V the tangent space of $A(\mathbb{C})$ at zero, the exponential map $\exp : V \to A(\mathbb{C})$ is a surjective morphism of Lie groups and a topological covering map whose kernel Λ is a complete lattice in V. The morphism exp and the description of $A(\mathbb{C})$ as the quotient of V by its kernel then allow us to construct a canonical isomorphism of complex vector spaces

$$H^{p,q}_{\overline{\partial}}(A(\mathbb{C})) \cong \bigwedge^{p} \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \otimes_{\mathbb{C}} \bigwedge^{q} \mathrm{Hom}_{\mathbb{C}\text{-antilinear}}(V, \mathbb{C}),$$

where $H^{p,q}_{\overline{\partial}}(A(\mathbb{C}))$ denotes the q-th $\overline{\partial}$-Dolbeault cohomology group of $A(\mathbb{C})$.

The goal is now to study the Dolbeault cohomology of abelian varieties in the non-archimedean setting. The key ingredient here is the uniformization theory by Raynaud, Bosch and Lütkebohmert and its analytic consequences investigated by Berkovich. Given an abelian variety A over a non-archimedean field k, their results provide us (after possibly passing to a finite separable extension of k) with a short exact sequence

$$1 \longrightarrow \mathbb{T} \longrightarrow E \longrightarrow B \longrightarrow 0 \tag{1.1}$$

of algebraic groups over k. In the sequence above, $\mathbb{T}$ is a split algebraic torus over k, the so-called *algebraic torus associated with* A, and B is an abelian variety over k of good reduction. This gives rise to a similar setup as in the complex case. Now the connection with our original Berkovich analytic space A^{an} is the following: The short exact sequence (1.1) comes with a morphism $\mathfrak{p} : E^{\mathrm{an}} \to A^{\mathrm{an}}$ of k-analytic groups which is a covering map in the topological sense, and whose kernel $\Lambda := \ker(\mathfrak{p})$ is a discrete subgroup of the k-rational points $E(k) \subseteq E^{\mathrm{an}}$ such that the map

$$E^{\mathrm{an}}/\Lambda \simeq A^{\mathrm{an}}$$

induced by $\mathfrak{p}$ is an isomorphism of k-analytic groups. As a result, we obtain the so-called Raynaud uniformization cross

$$\begin{tikzcd} & \Lambda \arrow[d, hook] & \\ \mathbb{T}^{\mathrm{an}} \arrow[r, hook] & E^{\mathrm{an}} \arrow[r, two heads] \arrow[d, two heads, "\mathfrak{p}"] & B^{\mathrm{an}} \\ & A^{\mathrm{an}} & \end{tikzcd},$$

a diagram of k-analytic groups. In his thesis [Sto21], Stoffel considered the case where A has totally degenerate reduction, meaning that $B = 0$ and $E = \mathbb{T}$ (which always holds in the complex case). He showed that in this case, there is a canonical injective morphism of real vector spaces

$$\Phi_A^{p,q} : \bigwedge^{p,q} M_{\mathbb{R}} := \bigwedge^{p} M_{\mathbb{R}} \otimes_{\mathbb{R}} \bigwedge^{q} M_{\mathbb{R}} \hookrightarrow H^{p,q}(A^{\mathrm{an}}),$$

where $M_{\mathbb{R}}$ denotes the real vector space associated to the character group $M = \mathrm{Hom}\left(\mathbb{T}, \mathrm{Spec}\left(k\left[T^{\pm 1}\right]\right)\right)$ of the torus $\mathbb{T}$, and $H^{p,q}(A^{\mathrm{an}})$ denotes the Dolbeault cohomology of (smooth) forms on A^{an}. The aim of this thesis is to generalise this result by dropping the assumption that A has totally degenerate reduction. In this more general setting, we will on the one hand recover Stoffel's result, but also specify the contribution of the abelian variety B of good reduction to the Dolbeault cohomology of A. More precisely, we will define another injective morphism

$$\bigwedge^{p,q} M_{\mathbb{R}} \hookrightarrow H_{\mathcal{D}}^{p+\dim B, q+\dim B}(A^{\mathrm{an}}),$$

this time to the cohomology of so-called strong currents, which are continuous linear functionals on the space of compactly supported weakly smooth forms on A^{an}.

Let us consider the construction of the map $\Phi_A^{p,q}$ and the strategy of proof for its injectivity in more detail. First of all, let us consider the construction of the sheaf of weakly smooth forms on a Berkovich analytic space X over k. An essential ingredient are Lagerberg forms. Lagerberg forms on $\mathbb{R}^n$ were introduced by Lagerberg in [Lag12] as analogues of real-valued (p, q)-differential forms on complex manifolds. More precisely, for an open subset U of a finite dimensional real vector space $N_{\mathbb{R}}$, a Lagerberg form of bidegree (p, q) on U is an element of

$$\mathcal{A}^{p,q}(U) := \mathcal{A}^p(U, \mathbb{R}) \otimes_{C^\infty(U)} \mathcal{A}^q(U, \mathbb{R}),$$

where $\mathcal{A}^k(U, \mathbb{R})$ denotes the space of smooth real differential forms of degree k on U. The resulting *sheaf of Lagerberg forms* on $N_{\mathbb{R}}$ is then given by $\mathcal{A} := \bigoplus_{p,q} \mathcal{A}^{p,q}$. There is a $\wedge$-product and natural differential operators d' and d'' on Lagerberg forms turning $(\mathcal{A}^{\bullet,\bullet}, d', d'')$ into a bicomplex of real vector spaces. Furthermore, by a shrinking procedure, we obtain a space of Lagerberg forms $\mathcal{A}(\sigma)$ on each polyhedron $\sigma \subseteq N_{\mathbb{R}}$, and also on each polyhedral set.

Now we can use so-called tropicalization maps to pass from the analytic space X to some finite dimensional real vector space, and thus build the bridge from Lagerberg forms to the sheaf of (weakly) smooth forms $\mathcal{A}_X = \bigoplus_{p,q} \mathcal{A}_X^{p,q}$ on X. More precisely, a *smooth differential form on X* is locally given by pull-backs of Lagerberg forms via smooth tropicalization maps. These maps are given by analytic morphisms to the Berkovich analytification of some split torus $\mathbb{G}_m^r = \operatorname{Spec}(k\left[T_1^{\pm 1}, \dots, T_r^{\pm 1}\right])$ over k composed with the *tropicalization map* $\operatorname{trop}\colon \mathbb{G}_m^{r,\mathrm{an}} \to \mathbb{R}^r$ which maps a multiplicative seminorm $|.|_x \in \mathbb{G}_m^{r,\mathrm{an}}$ to the r-tuple $(-\log|T_1|_x, \dots, -\log|T_r|_x) \in \mathbb{R}^r$.

In [GJR21], Gubler, Jell and Rabinoff extend this theory of differential forms by allowing pull-backs via *harmonic tropicalization maps*, which are G-locally given by smooth tropicalization maps. Here we mean the G-topology on Berkovich analytic spaces, this is a Grothendieck topology generated by analytic subdomains, closely related to rigid geometry. Both the smooth as well as the harmonic tropicalization maps share the important property that their image is a polyhedral set which admits in a natural way a polyhedral complex structure together with weights turning them into a tropical cycle away from the image of the boundary.

Coming back to our abelian variety A, there is the so-called *canonical tropicalization map* $\operatorname{trop}_E : E^{\mathrm{an}} \to N_{\mathbb{R}}$ on the Raynaud extension E^{an}, where $N_{\mathbb{R}}$ is now defined as the real vector space associated to the cocharacter lattice of the algebraic torus $\mathbb{T}$ associated to A from (1.1). In general, this tropicalization map fails to be smooth, but in [GJR21, §16] the authors show that it is a harmonic tropicalization map. This was also one of the motivations for Gubler, Jell and Rabinoff to introduce their theory of weakly smooth forms based on harmonic tropicalizations. In Chapter 3, we are going to use the fact that trop_E locally induces a harmonic tropicalization map on A^{an} to show that each Λ-invariant Lagerberg form $\alpha' \in \mathcal{A}^{p,q}(N_{\mathbb{R}})^{\Lambda}$ defines a weakly smooth form $\phi_A^{p,q}(\alpha') = \alpha \in \mathcal{A}^{p,q}_{A^{\mathrm{an}}}(A^{\mathrm{an}})$ on $A^{\mathrm{an}} \simeq E^{\mathrm{an}}/\Lambda$ via trop_E. Denoting by $M_{\mathbb{R}}$ the dual vector space of $N_{\mathbb{R}}$, the canonical map

$$\Phi_A^{p,q} : \bigwedge^{p,q} M_{\mathbb{R}} \to H^{p,q}(A^{\mathrm{an}}), \alpha' \mapsto \Phi_A^{p,q}(\alpha') := \left[\phi_A^{p,q}(\alpha')\right]$$

obtained by this construction will help us to study the contribution of the torus part to the Dolbeault cohomology. The map $\Phi_A^{p,q}$ can be defined on all d''-closed Λ-invariant (p,q)-Lagerberg forms on $N_{\mathbb{R}}$, but the result by Stoffel that the natural morphism $\bigwedge^{p,q} M_{\mathbb{R}} \to H^q(\mathcal{A}^{p,\bullet}(N_{\mathbb{R}})^{\Lambda}, d'')$ given by mapping a Lagerberg form with constant coefficients onto its class in the cohomology group is an isomorphism (see [Sto21, 3.4.26]) shows that with regards to cohomology, it is enough to consider only the forms with constant coefficients. We note here also that, while in the case of totally degenerate reduction, the injectivity of $\Phi_A^{p,q}$ showed that the Dolbeault cohomology $H^{p,q}(A^{\mathrm{an}})$ is non-trivial for all $p, q \in \mathbb{N}$ with $\max\{p,q\} \leq \dim A$, in our more general setting we obtain that $H^{p,q}(A^{\mathrm{an}})$ is non-trivial for all $p, q \in \mathbb{N}$ with $\max\{p,q\} \leq \dim \mathbb{T}$ if $\dim \mathbb{T} > 0$. So already here the abelian variety B of good reduction appears implicitly.

The construction of the map $\Phi_A^{p,q}$ is based on the torus part $\mathbb{T}$ of the Raynaud extension (1.1), and the proof of its injectivity is where the abelian variety B

of good reduction comes in. The overall idea of the proof is to find for every weakly smooth form of the form $\phi_A^{p,q}(\alpha')$ induced by $\alpha' \in \bigwedge^{p,q} M_{\mathbb{R}}$ and trop_E another d''-closed form β – now coming from the good reduction part B^{an} – such that pairing it with $\phi_A^{p,q}(\alpha')$ yields a non-zero integral. The requirements on β are hence to yield a non-zero integral, and to come from B^{an} but descend to A^{an}. Passing to an algebraic closure of k, Gubler's and Künnemann's results in [GK17] and [GK19] suggest a canonical candidate for the choice of β: the *Chern current* $[c_1(L, \| \,.\, \|_{\mathcal{L}})]$ associated to an ample line bundle L on B together with the canonical metric $\| \,.\, \|_{\mathcal{L}}$ on L induced by a cubic model $\mathcal{L}$ for L over the valuation ring of k. They proved that the pull-back of this element to E^{an} can be shown to be translation invariant under Λ, hence it descends to $A^{\mathrm{an}} \simeq E^{\mathrm{an}}/\Lambda$. Furthermore, its integral can be computed using the degree of the line bundle L on B.

The current $[c_1(L, \| \,.\, \|_{\mathcal{L}})]$ is not induced by a weakly smooth form, but it is a so-called δ-form in the sense of [GK17]. Hence, in order to pair a weakly smooth form $\phi_A^{p,q}(\alpha')$ as above with a δ-form, we have to pass to a larger space containing both types of forms. This is where the space of δ-forms introduced by Mihatsch in [Mih21] comes in. It contains both the weakly smooth forms as well as the δ-forms in the sense of [GK17]. Furthermore, the sheaf of δ-forms in the sense of [Mih21] comes with a $\wedge$-product. Then, letting β denote the δ-form on A^{an} which is induced by the δ-form $[c_1(L, \| \,.\, \|_{\mathcal{L}})]^{\wedge(\dim B)}$ on B^{an}, our main Theorem 5.2.8 states that the formula

$$\int_{A^{\mathrm{an}}} \phi_A^{\dim \mathbb{T}, \dim \mathbb{T}}(\alpha') \wedge \beta = \int_{F_\Lambda^\circ} \alpha' \cdot \deg_{L,\ldots,L}(B)$$

holds for all $\alpha' \in \bigwedge^{\dim \mathbb{T}, \dim \mathbb{T}} M_{\mathbb{R}}$, where F_Λ denotes a fundamental domain for the lattice $\mathrm{trop}_E(\Lambda)$ in $N_{\mathbb{R}}$. The injectivity of the natural map $\Phi_A^{p,q}$ can be derived from this product formula, and furthermore we can use it to show that the δ-form $[c_1(L, \| \,.\, \|_{\mathcal{L}})]$ gives rise to an injective map

$$\bigwedge^{p,q} M_{\mathbb{R}} \hookrightarrow H_{\mathcal{D}}^{p+\dim B, q+\dim B}(A^{\mathrm{an}})$$

to the cohomology of strong currents, i.e. of continuous functionals on compactly supported weakly smooth forms on A^{an}. It is given by mapping $\alpha' \in \bigwedge^{p,q} M_{\mathbb{R}}$ to the class of the strong current

$$\mathcal{A}_c^{(\dim \mathbb{T})-p, (\dim \mathbb{T})-q}(A^{\mathrm{an}}) \to \mathbb{R}, \eta \mapsto \int_{A^{\mathrm{an}}} \eta \wedge \phi_A^{p,q}(\alpha') \wedge \beta$$

in the cohomology of strong currents. In particular, the cohomology groups $H_{\mathcal{D}}^{r,s}(A^{\mathrm{an}})$ are non-trivial for all $r, s \in \mathbb{N}$ with $r, s \geq \dim B$ and $\max\{r, s\} \leq \dim A$.

Outline of the thesis. In Chapter 2 we define the sheaf of weakly smooth differential forms on Berkovich spaces based on harmonic tropicalizations after [GJR21]. As a preparation, in Section 2.1 we give basic definitions from polyhedral geometry and then introduce Lagerberg forms on real vector spaces and polyhedra. In the second part of Chapter 2, we define smooth and harmonic tropicalization maps giving rise to the sheaf of weakly smooth forms. Then we can define the corresponding Dolbeault cohomology.

In Chapter 3 we deal with the analytification of abelian varieties over non-archimedean fields. First, we recall results from the uniformization theory of Bosch, Lütkebohmert and Raynaud, and analytic consequences thereof outlined by Berkovich. This then allows us to define the canonical tropicalization map on the Raynaud extension of an abelian variety. It is a harmonic tropicalization map and gives rise to the definition of the canonical map describing the contribution of the torus part of an abelian variety to its Dolbeault cohomology.

Chapter 4 has two parts: We define δ-forms on Berkovich analytic spaces, and furthermore we prove a product formula for their integration theory. In order to define δ-forms on Berkovich spaces, we first need the notion of δ-forms on finite dimensional real vector spaces after [Mih23]. This is the content of Section 4.1. In Section 4.2, we first introduce the notion of tropical spaces in order to be able to give a definition of δ-forms first on tropical spaces, and then on Berkovich spaces. Furthermore, we investigate the integration theory of δ-forms in order to prove the product formula. This formula is stated in much more generality than it will be used in the proof of the main theorem. Its proof uses Mihatsch's intersection theory as well as analytic results from [Ber90] and [CD12], and it does not rely on any properties of abelian varieties.

The fifth and last chapter contains the proof of our main theorem. In Section 5.1, we introduce the δ-form on the good reduction part B^{an} of the Raynaud extension, which is crucial for our study of the cohomology. For this, we collect some results by Gubler and Künnemann in [GK17] and [GK19]. The aim of the second part is to use the short exact sequence (1.1) of the Raynaud extension to apply the product formula for integration of δ-forms in our specific setting, in order to obtain our main theorem. For this, some technical observations and constructions are needed. In Section 5.3 we finally use the product formula from our main Theorem 5.2.8 to describe the (non-trivial) contributions of the torus part as well as the good reduction part of the abelian variety to its cohomology.

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Chapter 2

Weakly smooth forms on Berkovich spaces and their cohomology

2.1 Lagerberg forms

The aim of this section is to recall the construction of Lagerberg forms and Lagerberg currents introduced by Lagerberg in [Lag12, §1], which are real analogues of complex (p, q)-forms and currents on $\mathbb{C}^r$. They are defined on real vector spaces, as well as on polyhedral subsets, and together with their integration theory they build the basis for the construction of the later considered weakly smooth forms and δ-forms on non-archimedean analytic spaces. For all this we follow [Gub16, §2, §3] and use the language of polyhedral geometry from [GJR21, §2.4] and [Mih23].

2.1.1 Definitions from polyhedral geometry

In this first part we introduce some fundamental definitions from polyhedral geometry following [GJR21, §2.4] and [Mih23, §2.2].

In the subsequent, let N always be a *lattice* of rank r, i.e. a free $\mathbb{Z}$-module of finite rank r, and we denote by $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ the ambient real vector space. We denote by $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ the *dual abelian group of* of N, and by $M_{\mathbb{R}}$ the *dual vector space of* $N_{\mathbb{R}}$.

Convention 2.1.1. We fix a subring R of $\mathbb{R}$ which is commutative and with 1, and an R-submodule G of $\mathbb{R}$. Let $M_R := \mathrm{Hom}_{\mathbb{Z}}(N, R)$.

Remark 2.1.2. In applications R is often taken to be the integers, and G is taken to be the value group $\Gamma := v(k^{\times})$ of a non-archimedean valued field (k, v).

Definition 2.1.3.
i) An *affine function on* $N_{\mathbb{R}}$ is a map $f : N_{\mathbb{R}} \to \mathbb{R}$ of the form $f = u + c$ for some $u \in M_{\mathbb{R}}$ and $c \in \mathbb{R}$.

ii) An affine function $f = u + c$ ($u \in M_{\mathbb{R}}, c \in \mathbb{R}$) on $N_{\mathbb{R}}$ is called (R, G)-*linear* if $u \in M_R$ and $c \in G$.

iii) Let N' be another lattice. Then a map $L : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ is called *(R,G)-linear* if $f \circ L : N'_{\mathbb{R}} \to \mathbb{R}$ is (R,G)-linear on $N'_{\mathbb{R}}$ for each (R,G)-linear function $f : N_{\mathbb{R}} \to \mathbb{R}$.

Definition 2.1.4. i) A *polyhedron σ in $N_{\mathbb{R}}$* is a finite intersection $\sigma = \bigcap_{i=1}^k H_i$ of halfspaces

$$H_i := \{w \in N_{\mathbb{R}} \mid f_i(w) \geq 0\},$$

where $f_i = u_i + c_i$ ($u_i \in M_{\mathbb{R}}, c_i \in \mathbb{R}$) is an affine function on $N_{\mathbb{R}}$ for all $i \in \{1, \dots, k\}$.

ii) We call a polyhedron σ as in i) an *(R,G)-polyhedron* if all affine functions f_i, $i = 1, \dots, k$, can be chosen (R,G)-linear.

iii) A *face* of a polyhedron σ is the intersection of σ with the boundary of a closed halfspace in $N_{\mathbb{R}}$ containing σ. Here, also σ and the empty set are allowed as faces of σ. We write $\rho \preceq \sigma$ if ρ is a face of σ.

iv) The *relative interior* of a polyhedron σ in $N_{\mathbb{R}}$ is defined by

$$\operatorname{relint}(\sigma) := \sigma^\circ := \sigma \setminus \bigcup_{\tau \preceq \sigma \text{ proper}} \tau,$$

where the union is taken over all faces τ of σ with $\tau \neq \sigma$.

Definition 2.1.5. For an (R,G)-polyhedron σ in $N_{\mathbb{R}}$, let N_σ denote the linear space spanned by all differences $x - y$ for $x, y \in \sigma$ and let $M_\sigma := N_\sigma^\vee$ denote its $\mathbb{R}$-dual. Furthermore, as in [GK17, A.2], let $\mathbb{A}_\sigma$ be the smallest affine space in $N_{\mathbb{R}}$ containing σ. Then $\mathbb{A}_\sigma = x + N_\sigma$ for some $x \in \sigma$. The *dimension* of σ is defined as the dimension of N_σ.

Definition 2.1.6. i) An *(R,G)-polyhedral complex* in $N_{\mathbb{R}}$ is a locally finite collection Π of (R,G)-polyhedra such that for any $\sigma \in \Pi$, all faces of σ are contained in Π and such that for $\sigma, \rho \in \Pi$, the intersection $\sigma \cap \rho$ is a face of both σ and ρ. The property of being locally finite is meant to be the following: There is an open covering $N_{\mathbb{R}} = \bigcup_{i \in I} U_i$ of $N_{\mathbb{R}}$ such that for every $i \in I$, the set U_i is disjoint from almost all polyhedra $\sigma \in \Pi$. If Π is finite, then Π is called a *finite (R,G)-polyhedral complex.*

ii) The *support* of a polyhedral complex Π is the union $|\Pi| := \bigcup_{\sigma \in \Pi} \sigma$ of all polyhedra in Π.

iii) For a polyhedral complex Π and some $n \in \mathbb{N}$, we denote by Π_n the collection of all polyhedra of Π of dimension n.

iv) For $n \in \mathbb{N}$ a polyhedral complex Π in $N_{\mathbb{R}}$ is called *of pure dimension n* if all its maximal polyhedra (with respect to inclusion) have dimension n.

v) A *subcomplex* of a polyhedral complex Π is a polyhedral complex $\Sigma \subseteq \Pi$.

vi) A *subdivision* of a polyhedral complex Π is a polyhedral complex Π' with $|\Pi| = |\Pi'|$ and such that each polyhedron $\sigma \in \Pi$ is a union of polyhedra in Π'. If both Π and Π' are (R, G)-polyhedral complexes, we call Π' an (R, G)-*subdivision* of Π.

vii) A subset $S \subseteq N_{\mathbb{R}}$ is called a *polyhedral set* if S is the support of a polyhedral complex in $N_{\mathbb{R}}$. It is called *pure dimensional* if it is the support of a pure dimensional polyhedral complex.

Remark 2.1.7. We note that the definition of a polyhedral complex we use here is the one from [Mih23]. A polyhedral complex in the sense of [GJR21] and [GK17] is the same as a finite polyhedral complex in our sense.

2.1.2 Lagerberg forms on real vector spaces

The aim of this section is to recall the construction of Lagerberg forms and Lagerberg currents introduced by Lagerberg in [Lag12, §1]. Lagerberg forms on $\mathbb{R}^r$ are bigraded real-valued differential forms, which come with a wedge product and differential operators d', d'' and d analogous to the differential operators $\partial, \overline{\partial}$ and $\partial + \overline{\partial}$ on complex differential forms. For all this we follow [Gub16, §2] and [Mih23, §2].

Convention 2.1.8. We choose a $\mathbb{Z}$-basis of M leading to coordinates $x_1, \ldots, x_r$ on $N_{\mathbb{R}}$. Furthermore, we fix an open subset U of $N_{\mathbb{R}}$.

Definition 2.1.9. Let $\mathcal{A}^k(U, \mathbb{R})$ denote the *space of smooth real differential forms of degree k*. A *Lagerberg form of bidegree (p, q) on U* is an element of

$$\mathcal{A}^{p,q}(U) := \mathcal{A}^p(U, \mathbb{R}) \otimes_{C^\infty(U)} \mathcal{A}^q(U, \mathbb{R}) = C^\infty(U) \otimes_{\mathbb{Z}} \bigwedge^p M \otimes_{\mathbb{Z}} \bigwedge^q M.$$

Remark 2.1.10. Formally, a Lagerberg form $\alpha \in \mathcal{A}^{p,q}(U)$ may be written as

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{IJ} d'x_I \wedge d''x_J, \text{ where } I = \{i_1 < \ldots < i_p\} \subseteq \{1, \ldots, r\},$$
$$J = \{j_1 < \ldots j_q\} \subseteq \{1, \ldots, r\}, \alpha_{IJ} \in C^\infty(U) \text{ and}$$
$$d'x_I \wedge d''x_J := (dx_{i_1} \wedge \ldots \wedge dx_{i_p}) \otimes_{\mathbb{R}} (dx_{j_1} \wedge \ldots \wedge dx_{j_q}).$$

Definition 2.1.11. We denote by $\mathcal{A}(U) := \bigoplus_{p,q \leq r} \mathcal{A}^{p,q}(U)$ the *space of Lagerberg forms on U*.

Remark 2.1.12. There is a *wedge product* on $\mathcal{A}(U)$, defined in the usual way. In coordinates, the wedge product on Lagerberg forms of bigedree (p, q) and (p', q') is given by

$$\wedge : \mathcal{A}^{p,q}(U) \times \mathcal{A}^{p',q'}(U) \to \mathcal{A}^{p+p',q+q'}(U)$$

$$\left(\alpha_{IJ} d'x_I \wedge d''x_J, \alpha'_{I'J'} d'x_{I'} \wedge d''x_{J'}\right) \mapsto \alpha_{IJ}\alpha'_{I'J'} d'x_I \wedge d''x_J \wedge d'x_{I'} \wedge d''x_{J'} :=$$
$$(-1)^{p'q} \alpha_{IJ}\alpha'_{I'J'} d'x_I \wedge d'x_{I'} \wedge d''x_J \wedge d''x_{J'},$$

where $\alpha = \sum_{|I|=p, |J|=q} \alpha_{IJ} d'x_I \wedge d''x_J \in \mathcal{A}^{p,q}(U)$ and $\alpha' = \sum_{|I'|=p', |J'|=q'} \alpha'_{I'J'} d'x_{I'} \wedge d''x_{J'} \in \mathcal{A}^{p',q'}(U)$.

Remark/Definition 2.1.13. There is a canonical $C^\infty(U)$-linear isomorphism $J^{p,q} : \mathcal{A}^{p,q}(U) \to \mathcal{A}^{q,p}(U)$ obtained by switching factors in the tensor product $\mathcal{A}^{p,q}(U) = \mathcal{A}^p(U,\mathbb{R}) \otimes_{C^\infty(U)} \mathcal{A}^q(U,\mathbb{R})$, i.e. $J^{p,q}$ is given by

$$\begin{aligned} J^{p,q} : \mathcal{A}^{p,q}(U) &\to \mathcal{A}^{q,p}(U) \\ \alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J \quad &\mapsto \quad \sum_{|I|=p,|J|=q} \alpha_{IJ} d''x_I \wedge d'x_J = \\ &= (-1)^{pq} \sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_J \wedge d''x_I. \end{aligned}$$

The maps $J^{p,q}$ and $J^{q,p}$ are inverse to each other. We call a Lagerberg form $\alpha \in \mathcal{A}^{p,p}(U)$ of bidegree (p,p) *symmetric* if $J^{p,p}(\alpha) = \alpha$.

Remark 2.1.14. There are *differential operators*

$$d' : \mathcal{A}^{p,q}(U) \to \mathcal{A}^{p+1,q}(U) \text{ resp. } d'' : \mathcal{A}^{p,q}(U) \to \mathcal{A}^{p,q+1}(U) \text{ given by}$$

$$d'\left(\sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J\right) := \sum_{|I|=p,|J|=q} \sum_{i=1}^{r} \frac{\partial \alpha_{IJ}}{\partial x_i} d'x_i \wedge d'x_I \wedge d''x_J \text{ resp.}$$

$$d''\left(\sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J\right) := \sum_{|I|=p,|J|=q} \sum_{j=1}^{r} \frac{\partial \alpha_{IJ}}{\partial x_j} d''x_j \wedge d'x_I \wedge d''x_J$$

which anti-commute and thus induce a differential operator $d := d' + d''$. The definitions above do not depend on the choice of coordinates as d' (resp. d'') is given by $d' = D \otimes \mathrm{id}$ using $\mathcal{A}^{p,q}(U) = \mathcal{A}^p(U,\mathbb{R}) \otimes_{\mathbb{Z}} \bigwedge^q M$ (resp. $d'' = (-1)^p \mathrm{id} \otimes D$ using $\mathcal{A}^{p,q}(U) = \bigwedge^p M \otimes_{\mathbb{Z}} \mathcal{A}^q(U,\mathbb{R})$), where D denotes the classical differential operator on forms. By linearity, we extend these differential operators to the space of Lagerberg forms $\mathcal{A}(U)$.

Remark 2.1.15. As in the classical theory, the *Leibniz formulae*

$$\begin{aligned} d'(\alpha \wedge \alpha') &= d'\alpha \wedge \alpha' + (-1)^{p+q} \alpha \wedge d'\alpha' \text{ and} \\ d''(\alpha \wedge \alpha') &= d''\alpha \wedge \alpha' + (-1)^{p+q} \alpha \wedge d''\alpha' \end{aligned}$$

hold for all $\alpha \in \mathcal{A}^{p,q}(U)$, see [CD12, (1.2.11)].

Remark 2.1.16. For all $p, q \in \mathbb{N}$ the functor

$$U \mapsto \mathcal{A}^{p,q}(U)$$

defines a sheaf on $N_\mathbb{R}$. We obtain a sheaf of bigraded differential algebras $(\mathcal{A}^{\bullet,\bullet}, d', d'')$ on $N_\mathbb{R}$, see [CD12, (1.2.6)].

Remark 2.1.17. Let N' be a lattice of rank r', $N'_\mathbb{R}$ the ambient real vector space and $F : N'_\mathbb{R} \to N_\mathbb{R}$ an affine map with $F(V) \subseteq U$ for an open subset V of $N'_\mathbb{R}$. Then there is a well-defined *pullback morphism* $F^* : \mathcal{A}^{p,q}(U) \to \mathcal{A}^{p,q}(V)$ that commutes with the differentials d', d'', d, the operator $J^{p,q}$ and the wedge product, see [Gub16, 2.3], [Jel16, Remark 2.1.5].

Definition 2.1.18. The *support* $\mathrm{supp}(\alpha)$ *of a Lagerberg form* $\alpha \in \mathcal{A}^{p,q}(U)$ is its support in the sense of sheaves, thus it is the set of points $x \in U$ which do not have a neighbourhood U_x in U such that $\alpha|_{U_x} = 0$. A Lagerberg form is said to have *compact support* if its support is a compact set. We denote by $\mathcal{A}_c(U)$ the *space of Lagerberg forms on* U *with compact support in* U.

Remark 2.1.19. If $\alpha \in \mathcal{A}^{p,q}(U)$ and $\alpha' \in \mathcal{A}^{p',q'}(U)$ such that α or α' has compact support on U, then $\alpha \wedge \alpha'$ has compact support on U by [JSS19, Remark 2.2].

Definition 2.1.20. For a Lagerberg form $\alpha = \sum_{|I|=p,|J|=q} \alpha_{IJ} d'x_I \wedge d''x_J \in \mathcal{A}_c^{p,q}(U)$ as in Remark 2.1.10 we define

$$\int_U \alpha := (-1)^{\frac{r(r-1)}{2}} \int_U \alpha_{LL} dx_1 \wedge \ldots \wedge dx_r \text{ with } L := \{1, \ldots, r\},$$

where the right hand side denotes the usual integration of r-forms with respect to the orientation induced by the choice of coordinates.

Definition 2.1.21. A *Lagerberg current on* U is a continuous linear functional on the locally convex vector space $\mathcal{A}_c^{p,q}(U)$, where continuity is with respect to the usual topology from the theory of distributions as explained in [GGJ+20, §3.2]. The space of such Lagerberg currents, i.e. the topological dual space of $\mathcal{A}_c^{p,q}(U)$, is denoted by $\mathcal{D}_{p,q}(U)$, and furthermore we set $\mathcal{D}(U) := \bigoplus_{p,q \le r} \mathcal{D}_{p,q}(U)$. We denote the evaluation of a Lagerberg current $T \in \mathcal{D}_{p,q}(U)$ at a Lagerberg form $\alpha \in \mathcal{A}_c^{p,q}(U)$ by $\langle T, \alpha \rangle$ or by $T(\alpha)$.

Remark 2.1.22. There is a canonical embedding

$$\mathcal{A}^{p,q}(U) \to \mathcal{D}_{r-p,r-q}(U), \alpha \mapsto [\alpha],$$

where $[\alpha]$ is given by $\langle [\alpha], \beta \rangle := [\alpha](\beta) := \int_U \alpha \wedge \beta$ for any $\beta \in \mathcal{A}_c^{r-p,r-q}(U)$. We note that this is well-defined by Remark 2.1.19. The linear differential operators d, d' and d'' extend to $\mathcal{D}(U)$. More precisely, for a Lagerberg current $T \in \mathcal{D}_{p,q}(U)$, the evaluation of the differentials of T at a Lagerberg form $\alpha \in \mathcal{A}_c^{p-1,q}(U)$ (resp. $\alpha \in \mathcal{A}_c^{p,q-1}(U)$) is given by

$$\langle d'T, \alpha \rangle = (-1)^{p+q+1} \langle T, d'\alpha \rangle \text{ (resp. } \langle d''T, \alpha \rangle = (-1)^{p+q+1} \langle T, d''\alpha \rangle).$$

For a Lagerberg form $\alpha \in \mathcal{A}^{p,q}(U)$, the formation of the corresponding Lagerberg current $[\alpha] \in \mathcal{D}_{r-p,r-q}(U)$ is compatible with the operators d, d' and d''.

2.1.3 Lagerberg forms on polyhedra and integration

The aim of this section is to give a notion of Lagerberg forms on polyhedra and define their integral with respect to weights. We follow [Mih21, §2.2] here.

Remark/Definition 2.1.23. Let $n \in \mathbb{N}$, and let σ be an arbitrary polyhedron in $\mathbb{R}^n$.

i) A *smooth function on* σ is a map $\varphi : \sigma \to \mathbb{R}$ such that there exists a smooth function $\tilde{\varphi} \in C^\infty(\mathbb{R}^n)$ with $\tilde{\varphi}|_\sigma = \varphi$. We denote the ring of smooth functions on σ by $C^\infty(\sigma)$.

ii) A *Lagerberg form of bidegree* (p, q) *on* σ is defined to be an element of

$$\mathcal{A}^{p,q}(\sigma) := C^\infty(\sigma) \otimes_{C^\infty(\mathbb{A}_\sigma)} \mathcal{A}^{p,q}(\mathbb{A}_\sigma),$$

and a *Lagerberg form on* σ is defined as an element of the direct sum $\bigoplus_{p,q\in\mathbb{N}} \mathcal{A}^{p,q}(\sigma)$. Following [Mih23, §2.2], the space of Lagerberg forms of bidegree (p, q) on σ consists of the (p, q)-Lagerberg forms on the open subset $\mathrm{relint}(\sigma) \subseteq \mathbb{A}_\sigma$ that come by restriction of elements in $\mathcal{A}^{p,q}(\mathbb{A}_\sigma)$. There is a restriction map $\mathcal{A}^{p,q}(\sigma) \to \mathcal{A}^{p,q}(\tau)$ for each face $\tau \preceq \sigma$ which commutes with the $\wedge$-product and the differentials d' and d''.

Definition 2.1.24. Let $n \in \mathbb{N}$.

i) Let σ be a polyhedron in $\mathbb{R}^n$. A *weight on* σ is a generator μ of $\det(N_\sigma)$ up to sign. We use the following convention on 0-dimensional polyhedra: The determinant of the 0-space is $\mathbb{R}$, and a weight is a positive scalar.

ii) A *weighted polyhedron* in $\mathbb{R}^n$ is a pair $[\sigma, \mu]$, where σ is a polyhedron in $\mathbb{R}^n$ and μ is any weight on σ.

iii) A *weighted polyhedral complex* is the datum of a polyhedral complex $\mathcal{T}$ together with weights μ_ρ for all its polyhedra $\rho \in \mathcal{T}$.

Remark 2.1.25. i) Following [Mih23, §2.3], a *weight* on a polyhedron σ in $\mathbb{R}^n$ can equivalently be defined as the datum of a Haar measure for M_σ. In the following, if we consider a weighted polyhedron $[\sigma, \mu]$ with $\mu \in \det(N_\sigma)$, we will always write λ_μ for the associated Haar measure on M_σ.

ii) In the situation of i), more precisely the mapping $\mu \mapsto \lambda_\mu$ is given as follows ([Mih23, §2.3]): If $\dim(N_\sigma) = 0$, then λ_μ is given by the Dirac measure of volume $\mu \in \mathbb{R}_{>0}$. In the case where $\dim(N_\sigma) > 0$, we pick a basis $e_1, \dots, e_n \in N_\sigma$ of N_σ such that $\mu = \pm e_1 \wedge \dots e_n$. Denoting by $x_1, \dots, x_n \in M_\sigma$ the associated dual basis of M_σ, the Haar measure λ_μ is defined to be the Haar measure for M_σ such that the parallelepiped spanned by $x_1, \dots, x_n$ has measure 1.

iii) For every $(\mathbb{Q}, \mathbb{R})$-polyhedron $\sigma \subseteq \mathbb{R}^n$ there is a natural weight on σ, namely the - up to sign - unique generator μ_0 of $\det(N_\sigma \cap \mathbb{Z}^n)$. We note here that $N_\sigma \cap \mathbb{Z}^n$ is a lattice in N_σ, see [GK17, A.2]. Any other weight on σ is then in a unique way of the form $c\mu_0$ for $c \in \mathbb{R}_{>0}$.

Remark/Definition 2.1.26. Let $\sigma \subseteq \mathbb{R}^n$ be a polyhedron of dimension d and let $\mu \in \det(N_\sigma)$ be a weight for σ. Let $\eta \in \mathcal{A}_c^{d,d}(\sigma)$ be a Lagerberg form of top degree on σ. Let $e_1, \dots, e_d \in N_\sigma$ be a basis of N_σ such that $\mu = e_1 \wedge \dots \wedge e_d$ and let $x_1, \dots, x_d$ denote the associated dual basis of M_σ. Let $\varphi \in C^\infty(\sigma)$ such that $\eta = \varphi d'x_1 \wedge d''x_1 \wedge \ldots \wedge d'x_d \wedge d''x_d$. Then the *integral of* η *along* $[\sigma, \mu]$ is defined as (see [Mih23, (2.11)])

$$\int_{[\sigma,\mu]} \eta := \int_\sigma \varphi d\lambda_\mu,$$

where the right hand side is the Lebesgue integral with respect to the measure defined by the choice of the isomorphism $(x_1, \ldots, x_d) : N_\sigma \cong \mathbb{R}^d$.

Definition 2.1.27. For $n \in \mathbb{N}$, the *standard weight on* $\mathbb{R}^n$ is defined as the Lebesgue measure on the dual space of $\mathbb{R}^n$, where the identification $(\mathbb{R}^n)^\vee = \mathbb{R}^n$ is with respect to the dual of the standard basis of $\mathbb{R}^n$. This measure on $(\mathbb{R}^n)^\vee$ is denoted by $\lambda_{\mu_{\mathbb{R}^n}}$. Equivalently, it is defined as $\mu_{\mathbb{R}^n} := e_1 \wedge \ldots \wedge e_n \in \det(\mathbb{R}^n)$ for the standard basis vectors $e_1, \ldots, e_n$ of $\mathbb{R}^n$, see [Mih21, Definition 2.8].

Remark 2.1.28.

i) The transformation formula, stating that for an affine map $f : \mathbb{R}^n \to \mathbb{R}^n$ and a Lagerberg form $\eta \in \mathcal{A}_c^{n,n}(\mathbb{R}^n)$, the equality

$$\int_{\mathbb{R}^n} f^*\eta = |\det f| \int_{\mathbb{R}^n} \eta$$

holds, ensures that the integral in Definition 2.1.26 is well-defined. Here the determinant of f is meant to be the determinant of the linear part of f.

ii) A weighted d-dimensional polyhedron $[\sigma, \mu]$ in $\mathbb{R}^n$ may be viewed as an $(n-d, n-d)$- Lagerberg current

$$[\sigma, \mu] : \mathcal{A}_c^{d,d}(\sigma) \to \mathbb{R}, \eta \mapsto \int_{[\sigma,\mu]} \eta.$$

iii) Every weighted d-dimensional polyhedron $[\sigma, \mu]$ in $\mathbb{R}^n$ together with a Lagerberg form $\alpha \in \mathcal{A}^{p,q}(\sigma)$ defines an $(n-d+p, n-d+q)$-Lagerberg current

$$\alpha \wedge [\sigma, \mu] : \mathcal{A}_c^{d-p,d-q}(\sigma) \to \mathbb{R}, \eta \mapsto \int_{[\sigma,\mu]} \alpha \wedge \eta.$$

Remark/Definition 2.1.29. Let S be polyhedral set of pure dimension n in $N_\mathbb{R}$, and let Π be a polyhedral complex in $N_\mathbb{R}$ with support $|\Pi| = S$. Following [GK17, 3.1], we define the *sheaf* $\mathcal{A}_{|\Pi|}^{p,q}$ *of* (p,q)*- Lagerberg forms on* $|\Pi|$ as follows: For any open subset U of $|\Pi|$, a Lagerberg form $\alpha \in \mathcal{A}_{|\Pi|}^{p,q}(U)$ on U is represented by a Lagerberg form $\tilde{\alpha} \in \mathcal{A}^{p,q}(\tilde{U})$ for any open subset $\tilde{U} \subseteq N_\mathbb{R}$ with $\tilde{U} \cap |\Pi| = U$, where two such elements are identified if they induce the same element in $\mathcal{A}^{p,q}(\sigma)$ for every maximal polyhedron $\sigma \in \Pi_n$. We denote the corresponding *sheaf of Lagerberg forms on* $|\Pi| = S$ by $\mathcal{A}_{|\Pi|}$.

Definition 2.1.30. Let (Π, μ) be a finite weighted polyhedral complex in $N_\mathbb{R}$ of pure dimension d. Let $S := |\Pi|$ denote the support of Π. Let $\eta \in \mathcal{A}_{S,c}^{d,d}(S)$ be a Lagerberg form of top bidegree on S with compact support. Then the *integral of* η *along* (Π, μ) is defined as

$$\int_{(\Pi,\mu)} \eta := \sum_{\sigma \in \Pi_d} \int_{[\sigma,\mu]} \eta|_\sigma.$$

2.2 Weakly smooth forms on Berkovich spaces based on harmonic tropicalizations

In their paper [GJR21], Gubler, Jell and Rabinoff enhance the construction of the bigraded sheaf of smooth real-valued differential forms on non-archimedean analytic spaces by Chambert-Loir and Ducros in [CD12]. Their approach is to allow pulling back Lagerberg forms via more general harmonic tropicalization maps instead of smooth tropicalization maps. This gives rise to a larger sheaf of differential forms with essentially the same properties, but with a better cohomological behaviour. The main focus of this thesis is to investigate this cohomological behaviour in the case of the Berkovich analytification of abelian varieties - where the so-called canonical tropicalization map fails to be smooth, but not to be harmonic. To do so, at first a detailed study of the theory of the so-called weakly smooth forms from [GJR21] is needed. This is the content of this section.

For the rest of this chapter, we fix a field k that is complete with respect to a non-trivial non-archimedean absolute value $|.| : k \to \mathbb{R}$. We denote by $v := -\log|.|$ the corresponding valuation and by $\Gamma := v(k^\times)$ its value group. An *analytic space* is meant to be a good strictly k-analytic space in the sense of [Ber93]. For a field extension l of k which is complete with respect to a non-archimedean absolute value extending the absolute value on k (called a non-archimedean field extension), there is a *ground field extension functor* from the category of k-analytic spaces to the category of l-analytic spaces. Once we pass to some extension field, we also consider our analytic spaces as objects of the larger category of analytic spaces over k (see [Ber93, 1.4]), whose objects are l-analytic spaces for any field extension l of k as above. This allows us in particular to consider the structure morphism $\pi : X\hat{\otimes}_k l \to X$ for a k-analytic space X. Furthermore, for an algebraic variety Y over k we denote by Y^{an} its *Berkovich analytification.* As a set, Y^{an} consists of pairs of the form $(p, |.|_p)$, where $p \in Y$ and $|.|_p$ is an absolute value on the residue field $\kappa(p)$ which extends the absolute value $|.|$ on the field k.

2.2.1 Tori and (smooth) tropicalization maps

In this chapter, we introduce smooth tropicalization maps, which we will also call tropical coordinates. They form - together with the Lagerberg forms - the building blocks for the smooth differential forms from [CD12], but also for the so-called δ-forms in the sense of [GK17] as well as in the sense of [Mih21]. Furthermore, they are needed to define skeletons which are of great importance for both the weakly smooth forms as well as the δ-forms. In this chapter we follow [GJR21, §3, §4] and [Mih21, §3].

Remark/Definition 2.2.1. Let $\mathbb{T}$ be an n-dimensional split torus over k with character lattice M and cocharacter lattice N. For $u \in M$ let $\chi^u : \mathbb{T} \to \mathrm{Spec}\left(k[T^{\pm 1}]\right) = \mathbb{G}_m^1$ be the corresponding character of $\mathbb{T} = \mathrm{Spec}(k\,[M])$. Let $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, and choose a $\mathbb{Z}$-basis of M leading to coordinates $x_1, \ldots, x_n$ on $\mathbb{T}$ identifying N with $\mathbb{Z}^n$.

i) We define the *tropicalization map* $\mathrm{trop}: \mathbb{T}^{\mathrm{an}} \to N_{\mathbb{R}}$ as follows: The image $\mathrm{trop}(x) \in N_{\mathbb{R}}$ of $x \in \mathbb{T}^{\mathrm{an}}$ is characterized by

$$< u, \mathrm{trop}(x) >= -\log|\chi^u(x)| \text{ for all } u \in M.$$

Under the identification $N_{\mathbb{R}} \simeq \mathbb{R}^n$ given by the x_i, $\mathrm{trop}(x)$ is given by the tuple $(-\log|x_i(x)|)_{i=1,\dots,n}$ for all $x \in \mathbb{T}^{\mathrm{an}}$.

ii) There is a canonical section $\iota_{\mathbb{T}} : N_{\mathbb{R}} \to \mathbb{T}^{\mathrm{an}}$ of the tropicalization map which is defined by

$$w \mapsto \begin{cases} \iota_{\mathbb{T}}(w) : k\,[M] \to \mathbb{R} \\ \sum_{u\in M} a_u \xi^u \mapsto \|\sum_{u\in M} a_u \xi^u\|_{\iota_{\mathbb{T}}(w)} := \\ \qquad\qquad \exp\left(-\min\{-\log|a_u| + < u, w >|\ a_u \neq 0\}\right) \end{cases}$$

for all $w \in N_{\mathbb{R}}$.

iii) The image of $\iota_{\mathbb{T}}$ in $\mathbb{T}^{\mathrm{an}}$ is called the *canonical skeleton* $\Sigma(\mathbb{T})$ *of* $\mathbb{T}$.

Convention 2.2.2. For an n-dimensional split torus $\mathbb{T}$ over k, we always use the notations from Remark/Definition 2.2.1.

Remark 2.2.3. The tropicalization map trop on a split torus $\mathbb{T}$ is a continuous proper map of topological spaces.

Definition 2.2.4. A *moment map* is a morphism $\varphi : X \to \mathbb{T}^{\mathrm{an}}$ of an analytic space X to the analytification of some split torus $\mathbb{T}$. The map φ is also called *tropical coordinates on* X. The corresponding map

$$t_\varphi := \mathrm{trop} \circ \varphi : X \to N_{\mathbb{R}}$$

is called *smooth tropicalization map.*

Example 2.2.5. For $n \in \mathbb{N}$ the multiplicative split torus

$$\mathbb{G}_m^n := \mathrm{Spec}\left(k\left[T_1^{\pm 1}, \dots, T_n^{\pm 1}\right]\right)$$

over k is an n-dimensional split torus over k with character lattice $\mathbb{Z}^n$.

Definition 2.2.6. For $r, s \in \mathbb{N}$, a *morphism of tori* $\mathbb{G}_m^r \to \mathbb{G}_m^s$ is defined to be an algebraic group homomorphism composed with a multiplicative translation.

Convention 2.2.7. For $r, s \in \mathbb{N}$, by a morphism $\mathbb{G}_m^r \to \mathbb{G}_m^s$ (resp. $\mathbb{G}_m^{r,\mathrm{an}} \to \mathbb{G}_m^{s,\mathrm{an}}$) we always mean a morphism of tori (resp. the map on analytifications induced by a morphism of tori).

Definition 2.2.8. Let X be a compact analytic space over k of pure dimension $d \in \mathbb{N}$, and let $\varphi : X \to \mathbb{T}^{\mathrm{an}}$ be a moment map. Then the *tropical variety of* φ is by definition the image $t_\varphi(X) \subseteq N_{\mathbb{R}}$. We call this image also the *tropicalization of* X *with respect to* φ and denote it by $T'(X, \varphi)$.

Remark 2.2.9. In the situation of Definition 2.2.8, by [Duc12, Théorème 3.2] the following holds: The tropical variety $t_\varphi(X)$ is the support of a finite $(\mathbb{Z}, \Gamma)$-polytopal complex Π in $N_\mathbb{R}$ of dimension at most d. Furthermore, the polyhedral complex Π might be chosen such that $t_\varphi(\partial X)$ is contained in a subcomplex of dimension at most $d-1$. If X is chosen strictly affinoid, then $t_\varphi(\partial X)$ equals the support of such a subcomplex.

Definition 2.2.10. Let X be an analytic space and $f : X \to \mathbb{G}_m^{r,\mathrm{an}}$ a moment map on X. Then a *refinement of* f is a pair (g, p) consisting of a moment map $g : X \to \mathbb{G}_m^{s,\mathrm{an}}$ and a map $p : \mathbb{G}_m^{s,\mathrm{an}} \to \mathbb{G}_m^{r,\mathrm{an}}$ which is induced by a morphism $\mathbb{G}_m^s \to \mathbb{G}_m^r$ of tori, such that $f = p \circ g$. This means that the diagram

$$\begin{tikzcd}
 & \mathbb{G}_m^{r,\mathrm{an}} \arrow[r, "\mathrm{trop}"] & \mathbb{R}^r \\
X \arrow[ur, "f"] \arrow[dr, "g"'] & & \\
 & \mathbb{G}_m^{s,\mathrm{an}} \arrow[uu, "p"] \arrow[r, "\mathrm{trop}"] & \mathbb{R}^s \arrow[uu, "p"]
\end{tikzcd}$$

commutes. Here and also in the following, the map $\mathbb{R}^s \to \mathbb{R}^r$ which is induced by p, trop and $\iota_{\mathbb{G}_m^s}$ is also denoted by p.

2.2.2 Harmonic tropicalization maps

The aim of this part is to introduce harmonic tropicalization maps as described in [GJR21, §8]. They can G-locally be described in terms of smooth tropicalization maps as introduced in the previous chapter, and allow to define weakly smooth forms on non-archimedean analytic spaces. A crucial ingredient for this is that the skeleta or tropical varieties arising from harmonic tropicalization maps are - as in the case of smooth tropicalization maps before - balanced, as we will explain in Chapter 4.

Definition 2.2.11. Let X be an analytic space and $\Lambda \subseteq \mathbb{R}$ an additive subgroup. Then a continuous map $h : X \to \mathbb{R}$ is called *piecewise* Λ*-linear* if there exists a G-covering $\{X_i\}_{i \in I}$ of X such that for each $i \in I$, there exists a finite index set J_i and $\lambda_{ij} \in \Lambda$, $f_{ij} \in \mathcal{O}(X_i)^\times$ for all $j \in J_i$ such that

$$h|_{X_i} = \sum_{j \in J_i} \lambda_{ij} \log|f_{ij}|$$

for all $i \in I$. A piecewise $\mathbb{Z}$-linear function is called *piecewise linear map.* The *group* of piecewise Λ-linear functions on X is denoted by $\mathrm{PL}_\Lambda(X)$.

Remark 2.2.12. Let X be an analytic space and $\Lambda \subseteq \mathbb{R}$ an additive subgroup. Then by [GJR21, 5.2], the following holds:

i) A function $h : X \to \mathbb{R}$ is piecewise linear if and only if there exists a G-covering $\{X_i\}_{i \in I}$ of X and $f_i \in \mathcal{O}(X_i)^\times$ for all $i \in I$ such that $h|_{X_i} = \log|f_i|$ for all $i \in I$.

ii) The restriction of a piecewise Λ-linear function to a strictly analytic domain is again piecewise Λ-linear.

iii) PL_Λ defines a sheaf in the G-topology and in the analytic topology.

Definition 2.2.13. Let X be an analytic space and $\Lambda \subseteq \mathbb{R}$ an additive subgroup. We assume that $\Lambda = \mathbb{Z}$ or that Λ is divisible. Let $h : X \to \mathbb{R}$ be a piecewise Λ-linear function. The map h is called *harmonic* at $x \in X$ if there exists a paracompact neighbourhood U of x in X, a formal k°-model $\mathcal{U}$ of U and a numerically trivial element $\mathcal{L} \in M(\mathcal{U})_\Lambda := M(\mathcal{U}) \otimes_{\mathbb{Z}} \Lambda$ such that $h|_U = -\log\|1\|_{\mathcal{L}}$, where $\|.\|_{\mathcal{L}}$ denotes the canonical metric induced by $\mathcal{L}$ on $\mathcal{O}_U$. Here $M(\mathcal{U})$ denotes the abelian group of isomorphism classes of line bundles $\mathcal{O}$ on $\mathcal{U}$ equipped with an identification $\mathcal{O}_\eta = \mathcal{O}_U$ of the generic fibre, and a line bundle $\mathcal{L} \in M(\mathcal{U})$ is said to be *numerically trivial* if its restriction to the special fibre of $\mathcal{U}$ has this property. The map h is then called *harmonic* if it is harmonic at all points $x \in X$.

Remark 2.2.14. More details and equivalent definitions for a map to be harmonic can be found in [GJR21, §7]. The definition chosen above is due to [GJR21, 7.3].

Convention 2.2.15. For the rest of this chapter, let X be a compact analytic space of pure dimension $d \in \mathbb{N}$.

Definition 2.2.16. Let

$$h = (h_1, \dots, h_n) : X \to \mathbb{R}^n$$

be a map. Then h is called *piecewise linear tropicalization map* if $h_1, \dots, h_n$ are piecewise linear functions, h is called *harmonic tropicalization map* if in addition $h_1, \dots, h_n$ are harmonic, and h is called *smooth tropicalization map* if for all $i \in \{1, \dots, n\}$ there is some $f_i \in \mathcal{O}_X^\times(X)$ such that $h_i = \log|f_i|$.

Remark 2.2.17. A smooth tropicalization map $h : X \to \mathbb{R}^n$ has the form t_φ for a moment map $\varphi : X \to \mathbb{G}_m^{n,\mathrm{an}}$, i.e. the definition above is consistent with the one from Definition 2.2.4. Furthermore, a piecewise linear tropicalization map on X is G-locally a smooth tropicalization map. This important property will be used to define a tropical space structure on the image $h(X)$ of X under a harmonic tropicalization map h using the tropical space structure on the tropical variety coming from smooth tropicalization maps (G-locally).

Remark 2.2.18. By [GJR21, 8.2 (4)], harmonic tropicalization maps pull back under morphisms of analytic spaces and arbitrary base change.

Proposition 2.2.19. *A map $h : X \to \mathbb{R}$ is piecewise linear if and only if each $x \in X$ admits a strictly affinoid neighbourhood U together with a smooth tropicalization map $h' : U \to \mathbb{R}^n$ and a piecewise $(\mathbb{Z}, \Gamma)$-linear function $F : h'(U) \to \mathbb{R}$ such that $h|_U = F \circ h'$.*

Proof. [GJR21, Proposition 8.3] □

Definition 2.2.20. Let $h : X \to \mathbb{R}^n$ be a piecewise linear tropicalization. Then h is said to be *covered by a smooth tropicalization map* $h' : U \to \mathbb{R}^m$ on a compact strictly analytic subdomain $U \subseteq X$ if there is a pieceweise $(\mathbb{Z}, \Gamma)$-linear function $F : h'(U) \to \mathbb{R}^n$ such that $h|_U = F \circ h'$.

Remark 2.2.21. Proposition 2.2.19 yields that locally on X, a pieceweise linear tropicalization is always covered by a smooth tropicalization map.

Remark 2.2.22. Let $h : X \to \mathbb{R}^n$ be a piecewise linear tropicalization map. Then $h(X)$ is the support of an at most d-dimensional finite $(\mathbb{Z}, \Gamma)$-polyhedral complex. In the following, we denote by $h(X)_d$ always its d-dimensional locus.

Proposition 2.2.23. *Let l/k be a non-archimedean field extension, let $X_l := X \hat{\otimes}_k l$ and let $\pi : X_l \to X$ be the structure morphism. Let h be a piecewise linear function on X and let $h_l := h \circ \pi$. Then h is harmonic if and only if h_l is harmonic.*

Proof. [GJR21, Proposition 7.7 (2')] □

2.2.3 Weakly smooth forms on Berkovich spaces

In this section, we introduce weakly smooth forms on non-archimedean analytic spaces. All the constructions are due to [GJR21, §10, §11].

Definition 2.2.24. [GJR21, Definition 10.3] Let X be an analytic space and $U \subseteq X$ an open subset with compact strictly analytic closure $\overline{U}$. A *weakly smooth preform of type* (p, q) on U is given by a tuple (h, α), where $h : \overline{U} \to \mathbb{R}^n$ is a harmonic tropicalization map and $\alpha \in \mathcal{A}^{p,q}_{h(\overline{U})}(h(\overline{U}))$ a Lagerberg form on $h(\overline{U}) \subseteq \mathbb{R}^n$. Two such tuples (h, α) and (h', α') define the same preform if and only if $p_1^*\alpha = p_2^*\alpha'$ on $(h, h')(\overline{U})$, where p_1 resp. p_2 denote the projection maps $p_1 : \mathbb{R}^n \times \mathbb{R}^{n'} \to \mathbb{R}^n$ resp. $p_2 : \mathbb{R}^n \times \mathbb{R}^{n'} \to \mathbb{R}^{n'}$ and $h' : \overline{U} \to \mathbb{R}^{n'}, \alpha' \in \mathcal{A}^{p,q}_{h'(\overline{U})}(h'(\overline{U}))$.

Lemma 2.2.25. *In the situation of Definition 2.2.24, the equivalence $(h, \alpha) = 0 \Leftrightarrow \alpha = 0$ holds.*

Proof. [GJR21, Lemma 10.4] □

Definition 2.2.26. [GJR21, 10.6] Let X be an analytic space and $U \subseteq X$ an open subset with compact strictly analytic closure $\overline{U}$. Let (h, α) and (h', α') be two weakly smooth preforms on U of type (p, q) and (p', q'), respectively. Then

$$g := (h, h') : \overline{U} \to \mathbb{R}^{n+n'}, x \mapsto (h(x), h'(x))$$

is a harmonic tropicalization map and the equalities $(g, p_1^*\alpha) = (h, \alpha)$ and $(g, p_2^*\alpha') = (h', \alpha')$ of weakly smooth preforms hold, where p_1 and p_2 denote the canonical projection maps on $\mathbb{R}^n \times \mathbb{R}^{n'}$. Then we define the *wedge product* of (h, α) and (h', α') as

$$(h, \alpha) \wedge (h', \alpha') := (g, p_1^*\alpha \wedge p_2^*\alpha').$$

It is a weakly smooth preform of type $(p+p', q+q')$ on U and satisfies

$$(h', \alpha') \wedge (h, \alpha) = (-1)^{(p+p')(q+q')}(h, \alpha) \wedge (h', \alpha').$$

Furthermore, we define *differentials of preforms* by

$$d'(h, \alpha) := (h, d'\alpha) \text{ and } d''(h, \alpha) := (h, d''\alpha).$$

They satisfy the *Leibniz rule*

$$d'\left((h, \alpha) \wedge (h', \alpha')\right) = d'(h, \alpha) \wedge (h', \alpha') + (-1)^{p+q}(h, \alpha) \wedge d'(h', \alpha')$$

and likewise for d''.

Definition 2.2.27. Let X be an analytic space. Following [GJR21, 10.5, 10.7], the weakly smooth preforms define a presheaf of bigraded differential $\mathbb{R}$-algebras on the category of open subets of X whose closure is a compact strictly analytic domain. Such open subsets form a basis for the analytic topology of the analytic space X. We denote the associated sheaf on the underlying topological space of X by $\mathcal{A}_X$ or simply by $\mathcal{A}$. It is a sheaf of bigraded differential $\mathbb{R}$-algebras $\mathcal{A} = \bigoplus_{p,q} \mathcal{A}^{p,q}$. The elements of $\mathcal{A}$ are called *weakly smooth forms.*

Remark 2.2.28. Following [GJR21, 10.8], for a morphism $f : X' \to X$ of analytic spaces over k, there is a *pull-back* homomorphism

$$f^* : \mathcal{A}_X \to f_* \mathcal{A}_{X'}$$

of sheaves of bigraded differential $\mathbb{R}$-algebras, defined on preforms as follows: For open subsets $U \subseteq X$ and $U' \subseteq X'$ with $U' \subseteq f^{-1}(U)$ and such that the closures of U and U' are compact strictly analytic domains, the pull-back $f^*(h, \alpha)$ of a preform (h, α) on U is given by $f^*(h, \alpha) := (h \circ f, \alpha)$.

Convention 2.2.29. For a weakly smooth preform (h, α) on a compact analytic space X over k we denote the associated weakly smooth form by $h^*(\alpha) \in \mathcal{A}(X)$.

Proposition 2.2.30. *Let X be a compact analytic space.*

i) The natural map from weakly smooth preforms to weakly smooth forms on X is injective.

ii) If $h : X \to \mathbb{R}^n$ is a harmonic tropicalization map, then the corresponding pull-back morphism $h^* : \mathcal{A}^{\bullet,\bullet}_{h(X)}(h(X)) \to \mathcal{A}^{\bullet,\bullet}_X(X)$ is an injective homomorphism of bigraded differential $\mathbb{R}$-algebras.

iii) If $f : X' \to X$ is a morphism of analytic spaces, then $(h \circ f)^* = f^* \circ h^*$.

Proof. [GJR21, Proposition 10.9] □

Definition 2.2.31. Let X be an analytic space. The *support* of a weakly smooth form $\omega \in \mathcal{A}^{p,q}(X)$ is defined in the sense of sheaves: It is the set of points x of X such that the germ of ω in x is non-zero. This is a closed subset denoted by $\mathrm{supp}(\omega)$. Furthermore we denote the cosheaf of weakly smooth (p, q)-forms with compact support on X by $\mathcal{A}_c^{p,q}$.

Proposition 2.2.32. *Let X be an analytic space of pure dimension d. Then there is a unique linear functional*

$$\int_X : \mathcal{A}_c^{d,d} \to \mathbb{R}$$

such that the following holds for all $\omega \in \mathcal{A}_c^{d,d}(X)$:

i) If $\operatorname{supp}(\omega) \subseteq W$ for a strictly analytic domain W of X, then $\int_W \omega = \int_X \omega$.

ii) For closed analytic domains V, W of X we have $\int_{V\cup W} \omega = \int_V \omega + \int_W \omega - \int_{V\cap W} \omega$.

iii) If X is compact and ω is given by $h^*(\alpha)$ for a harmonic tropicalization map $h : X \to \mathbb{R}^n$ and a Lagerberg form $\alpha \in \mathcal{A}^{d,d}(h(X))$, then $\int_X \omega = \int_{h(X)_d} \alpha$, where the d-dimensional locus $h(X)_d$ of $h(X)$ is considered as a weighted polyhedral complex via Remark 4.2.43.

Proof. [GJR21, Proposition 11.1] □

2.2.4 Strong currents

The aim of this section is to introduce the notion of continuous linear functionals on the space of compactly supported weakly smooth forms on non-archimedean analytic spaces: This space is equipped with a topology that coincides with the topology on smooth differential forms in the Schwartz sense introduced in [CD12, (4.1.1)] if one restricts to the smooth case. This gives rise to the notion of the so-called strong currents which will be of use to study the Dolbeault cohomology of the analytification of abelian varieties later. All this is object of [GJR21, §12].

Throughout this section, let X denote a Hausdorff analytic space of some pure dimension d with empty boundary.

Definition 2.2.33. Following [GJR21, 12.1], we define a (locally convex) topology on the space of weakly smooth (p, q)-forms on X with compact support exactly as in [CD12, (4.1.1)], replacing smooth tropicalization maps by harmonic tropicalization maps. For details, see [CD12, (4.1.1)].

Remark 2.2.34. Following [GJR21, 12.1], roughly speaking, convergence in $\mathcal{A}_c^{p,q}(X)$ means that the supports of the weakly smooth forms are covered by finitely many compact strictly analytic subdomains V in X and the strictly analytic subdomains tropicalize the weakly smooth forms such that all the derivatives of the corresponding Lagerberg forms converge uniformly.

Definition 2.2.35. A *strong current on X of bidegree* (p, q) is defined to be a continuous linear functional $\mathcal{A}_c^{p,q}(X) \to \mathbb{R}$. The space of strong currents on X of bidegree (p, q) is denoted by $\mathcal{D}_{p,q}(X)$ or $\mathcal{D}^{d-p,d-q}(X)$, and a *strong current on X* is an element of $\mathcal{D}(X) := \bigoplus_{p,q\in\mathbb{N}} \mathcal{D}_{p,q}(X)$.

Remark 2.2.36. Mapping open subsets U of X to the space $\mathcal{D}(U)$ of strong currents on X defines a sheaf on X by [GJR21, 12.2].

Example 2.2.37. The *current of integration* $\delta_X : \mathcal{A}_c^{d,d}(X) \to \mathbb{R}, \eta \mapsto \int_X \eta$ defines a strong current $\delta_X \in \mathcal{D}_{d,d}(X)$ on X.

Remark 2.2.38. Following [GJR21, 12.4], there is an injective linear map

$$\mathcal{A}^{p,q}(X) \to \mathcal{D}_{d-p,d-q}(X), \omega \mapsto (\eta \mapsto \delta_X(\omega \wedge \eta))$$

for $\eta \in \mathcal{A}_c^{d-p,d-q}(X)$.

Remark/Definition 2.2.39. We define differentials $d' : \mathcal{D}_{p,q}(X) \to \mathcal{D}_{p-1,q}(X)$ and $d'' : \mathcal{D}_{p,q}(X) \to \mathcal{D}_{p,q-1}(X)$ on strong currents by

$$d'T(\omega) := (-1)^{p+q+1}T(d'\omega) \text{ and } d''T(\eta) := (-1)^{p+q+1}T(d''\eta)$$

for all $T \in \mathcal{D}_{p,q}(X), \omega \in \mathcal{A}_c^{p-1,q}(X)$ and $\eta \in \mathcal{A}_c^{p,q-1}(X)$.

Remark 2.2.40. Following [GJR21, 12.5], the homomorphism $\mathcal{A} \to \mathcal{D}$ induced by mapping a weakly smooth form onto its associated strong current respects the differentials d' and d''.

Remark 2.2.41. As in [CD12, (4.3.5)], there is a generalization of Remark 2.2.38: For every weakly smooth form $\omega \in \mathcal{A}^{p,q}(X)$ and every strong current $T \in \mathcal{D}_{r,s}(X)$ we obtain a strong current $\omega \wedge T \in \mathcal{D}_{r-p,s-q}(X)$ which is defined by

$$\omega \wedge T : \mathcal{A}_c^{r-p,s-q}(X) \to \mathbb{R}, \eta \mapsto T(\omega \wedge \eta).$$

Lemma 2.2.42. *Let $\omega \in \mathcal{A}^{p,q}(X)$ be a d''-closed weakly smooth form and $T \in \mathcal{D}_{r,s}(X)$ a d''-closed strong current on X. Then the strong current $\omega \wedge T \in \mathcal{D}_{r-p,s-q}(X)$ is again d''-closed.*

Proof. For d''-closed elements $\omega \in \mathcal{A}^{p,q}(X)$ and $T \in \mathcal{D}_{r,s}(X)$, the definition of the differentials on strong currents together with the Leibniz formula gives that

$$\begin{aligned} d''(\omega \wedge T)(\eta) &= \pm(\omega \wedge T)\left(d''\eta\right) \\ &= \pm T\left(\omega \wedge d''\eta\right) \\ &= \pm T\left(\pm d''(\omega \wedge \eta) \pm (d''\omega \wedge \eta)\right) \\ &= \pm T\left(d''(\omega \wedge \eta)\right) \pm T\left(d''\omega \wedge \eta\right) \\ &= \pm d''T(\omega \wedge \eta) = 0 \end{aligned}$$

for all $\eta \in \mathcal{A}_c^{r-p,s-q}(X)$, where in the last two steps we use that ω and T are d''-closed. This yields that $\omega \wedge T$ is d''-closed. □

2.2.5 Dolbeault cohomology of weakly smooth forms

In this section, we define the Dolbeault cohomology of weakly smooth forms for general Berkovich analytic spaces following [GJR21, §13].

We fix an analytic space X of dimension d. We note that by applying the Lagerberg involution $J^{p,q}$, every result for d'' has a natural counterpart for d' after switching the roles of p and q. So we only consider d'' here.

Theorem 2.2.43. *For all $p \in \mathbb{N}_{\geq 0}$, the differential operator d'' on the sheaf $\mathcal{A}$ of weakly smooth forms induces a complex*

$$0 \to \mathcal{A}^{p,0} \to \ldots \to \mathcal{A}^{p,d} \to 0$$

of sheaves on X which is exact at $\mathcal{A}^{p,q}$ for all $q \in \mathbb{N}_{\geq 0}$.

Proof. [GJR21, Theorem 13.1] □

Definition 2.2.44. Applying the global section functor to the complex in Theorem 2.2.43, we get the so-called *Dolbeault complex*

$$0 \to \mathcal{A}^{p,0}(X) \to \ldots \to \mathcal{A}^{p,d}(X) \to 0.$$

The resulting cohomology is called the *q-th Dolbeault cohomology* and the cohomology groups are denoted by $H^{p,q}(X)$ for all $p, q \in \mathbb{N}$.

Remark 2.2.45. Let l/k be a non-archimedean field extension, let $X_l := X \hat{\otimes}_k l$ and let $\pi : X_l \to X$ be the corresponding structure morphism. Then for all $p, q \in \mathbb{N}$, the base change from k to l induces a natural map on the Dolbeault cohomology groups

$$H^{p,q}(X) \to H^{p,q}(X_l)$$

which is obtained as follows: Every weakly smooth form $\alpha \in \mathcal{A}^{p,q}(X)$ is given by an open cover $(U_i)_{i \in I}$ of X such that the closure $\overline{U}_i$ of U_i is a compact strictly analytic domain for all $i \in I$, together with a compatible family $(h_i, \alpha_i)_{i \in I}$, where $h_i : \overline{U}_i \to \mathbb{R}^{n_i}$ is a harmonic tropicalization map and $\alpha_i \in \mathcal{A}^{p,q}(\mathbb{R}^{n_i})$ a Lagerberg form for all $i \in I$. Using Proposition 2.2.23, the assignment

$$(h_i, \alpha_i)_{i \in I} \mapsto (h_i \circ \pi, \alpha_i)_{i \in I}$$

defines a natural map $\varphi^{p,q} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q}(X_l)$ which is compatible with d'' by definition of d''. More precisely, the definition of d'' gives

$$d''\varphi^{p,q}((h_i, \alpha_i)) = d''(h_i \circ \pi, \alpha_i) = (h_i \circ \pi, d''\alpha_i) = \varphi^{p,q+1}(h_i, d''\alpha_i)$$

for all $i \in I$. The d''-compatibility of $\varphi^{p,q}$ shows that $\varphi^{p,q}$ induces a natural map

$$H^{p,q}(X) \to H^{p,q}(X_l), [\alpha] \mapsto [\varphi^{p,q}(\alpha)]$$

on the Dolbeault cohomology groups.

Chapter 3

Weakly smooth forms and Dolbeault cohomology on the analytification of abelian varieties

In this section, we recall results from the uniformization theory of Bosch, Lütkebohmert and Raynaud and their analytic consideration by Berkovich, which give rise to the building blocks for the main result: Roughly speaking, every abelian variety over a suitable field admits a covering space, the so-called Raynaud extension, which can locally be divided in two parts. One of them is a multiplicative split torus which gives in a natural way rise to a harmonic tropicalization map. The other part is an abelian variety of good reduction. This is explained in [GJR21, §16], [Gub10, §4], [BL93] and [Ber90, 6.5]. We follow those papers here.

3.1 Canonical tropicalization of abelian varieties

The aim of this first part of the chapter is to explain the construction of the canonical tropicalization of abelian varieties. It is based on the torus part of the so-called Raynaud extension of the abelian variety, and gives in a natural way rise to weakly smooth forms on the Berkovich analytification of abelian varieties. These weakly smooth forms will be the ones occurring in our main theorem, and the ones which we will use to describe the Dolbeault cohomology.

In this section, we fix a field k that is complete with respect to a non-trivial non-archimedean absolute value $|.| : k \to \mathbb{R}$. We denote by k° the corresponding valuation ring.

Remark/Definition 3.1.1. Let A be an abelian variety over k. Following [Ber90, 6.5], there exists a unique compact subgroup G of A^{an} such that

i) G is an analytic domain;

ii) G is a formal k-analytic group;

iii) for some finite separable extension l of k, $G\hat{\otimes}_k l$ has *semiabelian reduction* which means that the reduction of $G\hat{\otimes}_k l$ is an extension of an abelian variety by a torus.

Furthermore, the group G contains a unique closed k-analytic subgroup $\mathbb{T}_1^{\mathrm{an}}$ that is a k-affinoid torus. This affinoid torus $\mathbb{T}_1^{\mathrm{an}}$ is isomorphic to the locus $\{p \in \mathbb{T}^{\mathrm{an}} \mid |T_1(p)| = \ldots = |T_d(p)| = 1\}$ for some algebraic torus $\mathbb{T}$ over k, where $T_1, \ldots, T_d$ denotes a $\mathbb{Z}$-basis of the character lattice of $\mathbb{T}$. The algebraic torus $\mathbb{T}$ is called the *algebraic torus associated with A*.

Definition 3.1.2. Let A be an abelian variety over k. Furthermore, let G be the compact subgroup of A^{an} from Remark/Definition 3.1.1, and let $\mathbb{T}$ be the algebraic torus associated with A. Then A is called *split over k* if G has semiabelian reduction and $\mathbb{T}$ is split over k.

Remark 3.1.3. Following [Ber90, 6.5], for any abelian variety A over k there is some finite Galois extension l of k such that $A \otimes_k l$ is split over l.

Remark/Definition 3.1.4. Let A be an abelian variety over k that is split over k. Following [BL93], [Ber90, 6.5], [GJR21, §16] and [Gub10, §4], the uniformization theory of Bosch, Lütkebohmert and Raynaud yields the following: Let G be the the compact subgroup of A^{an} from Remark/Definition 3.1.1, and let $\mathbb{T}$ be the algebraic torus associated with A. We identify the unique closed k-analytic subgroup $\mathbb{T}_1^{\mathrm{an}}$ of G that is a k-affinoid torus with $\{p \in \mathbb{T}^{\mathrm{an}} \mid |T_1(p)| = \ldots = |T_d(p)| = 1\}$, where $T_1, \ldots, T_d$ denotes a $\mathbb{Z}$-basis of the character lattice of $\mathbb{T}$. There is a unique abelian scheme $\mathcal{B}$ over $\mathrm{Spec}(k^\circ)$ with generic fiber B yielding an exact sequence

$$1 \longrightarrow \mathbb{T}_1^{\mathrm{an}} \longrightarrow G \xrightarrow{\mathfrak{q}_1} B^{\mathrm{an}} \longrightarrow 0$$

of formal analytic groups. We embed $\mathbb{T}_1^{\mathrm{an}}$ into $G \times_k \mathbb{T}^{\mathrm{an}}$ by mapping $x \in \mathbb{T}_1^{\mathrm{an}}$ to (x, x^{-1}), and call the analytic group $E^{\mathrm{an}} := (G \times_k \mathbb{T}^{\mathrm{an}}) / \mathbb{T}_1^{\mathrm{an}}$ obtained via the resulting group action of $\mathbb{T}_1^{\mathrm{an}}$ on $G \times_k \mathbb{T}^{\mathrm{an}}$ the *uniformization* of A^{an}. Using the canonical maps $\iota_G : G \to E^{\mathrm{an}}$ and $\mathfrak{q} : E^{\mathrm{an}} \to B^{\mathrm{an}}$, following [BL93, §1] we get a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{T}^{\mathrm{an}} \arrow[r] & E^{\mathrm{an}} \arrow[r, "\mathfrak{q}"] & B^{\mathrm{an}} \arrow[r] & 0 \\
1 \arrow[r] & \mathbb{T}_1^{\mathrm{an}} \arrow[r] \arrow[u, hook] & G \arrow[r, "\mathfrak{q}_1"] \arrow[u, hook, "\iota_G"] & B^{\mathrm{an}} \arrow[r] \arrow[u, equal] & 0,
\end{tikzcd}$$

where the upper row is an exact sequence of analytic groups, called the *Raynaud extension* of the abelian variety A. Furthermore, the Raynaud extension is algebraizable, i.e. there is an algebraic variety E over k with Berkovich analytification E^{an} for the analytic group E^{an} constructed above, and a canonical short exact sequence

$$1 \longrightarrow \mathbb{T} \longrightarrow E \xrightarrow{\mathfrak{q}} B \longrightarrow 0 \tag{3.1}$$

of algebraic groups which induces the exact sequence on the Berkovich analytifications above. The closed immersion $\mathbb{T}_1^{\mathrm{an}} \hookrightarrow G$ extends uniquely to a morphism $\mathbb{T}^{\mathrm{an}} \to A^{\mathrm{an}}$ of analytic groups, and the morphism $G \hookrightarrow A^{\mathrm{an}}$ extends uniquely to a morphism $\mathfrak{p} : E^{\mathrm{an}} \to A^{\mathrm{an}}$ of analytic groups. The kernel $\Lambda := \ker(\mathfrak{p})$ of $\mathfrak{p} : E^{\mathrm{an}} \to A^{\mathrm{an}}$ is a discrete subgroup of the k-rational points $E(k)$, and the homomorphism $E^{\mathrm{an}}/\Lambda \to A^{\mathrm{an}}$ induced by $\mathfrak{p}$ is an isomorphism. Hence we may use the identification

$$A^{\mathrm{an}} \cong E^{\mathrm{an}}/\Lambda. \tag{3.2}$$

Furthermore, in (3.1), B is of good reduction, and we denote the unique Shilov point of B^{an} by ξ_B. The homomorphism $\mathfrak{q} : E \to B$ is an algebraic $\mathbb{T}$-torsor for the Zariski topology, whereas the quotient homomorphism $\mathfrak{p} : E^{\mathrm{an}} \to A^{\mathrm{an}}$ is only an analytic morphism. It is a covering map in the sense of topology, meaning that $\mathfrak{p}$ is surjective and for all $x \in A^{\mathrm{an}}$ there is an open neighborhood U of x such that $\mathfrak{p}^{-1}(U) = \bigsqcup_{i\in\Lambda} U_i$ with $U_i \subseteq E^{\mathrm{an}}$ open for all $i \in \Lambda$ and such that $\mathfrak{p}$ maps U_i homeomorphically onto U for all $i \in \Lambda$.

Convention 3.1.5. For the rest of the section let A be an abelian variety which is split over k. We fix the notations from Remark/ Definition 3.1.4. Furthermore, we denote the character lattice of the split torus $\mathbb{T}$ by M and let $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ be the cocharacter lattice. Let $d := \dim N_{\mathbb{R}}, b := \dim B$ and $n := \dim A$.

Remark/Definition 3.1.6. Let $u \in M$ and consider the corresponding character $\chi^u : \mathbb{T} \to \mathbb{G}_m$, i.e. the canonical map from the torus $\mathbb{T}$ to the multiplicative split torus $\mathbb{G}_m$ in one variable over K. Then the pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{T} \arrow[r] \arrow[d, "\chi^u"] & E \arrow[r, "\mathfrak{q}"] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{G}_m \arrow[r] & E_u \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

of the Raynaud extension with respect to χ^u exists and gives rise to a translation invariant extension of B by $\mathbb{G}_m$, and to a rigidified translation invariant line bundle E_u on B by [GJR21, §16] and [BL93, §3]. We note that $\mathfrak{q}^*(E_u)$ is trivial over E and the pushout construction gives a canonical frame $e_u : E \to \mathfrak{q}^*(E_u)$. Then there is a unique map

$$\mathrm{trop}_E : E^{\mathrm{an}} \to N_{\mathbb{R}}, \tag{3.3}$$

called *canonical tropicalization map*, satisfying

$$< \mathrm{trop}_E(x), u >= -\log \mathfrak{q}^* \parallel e_u(x) \parallel_{E_u} \tag{3.4}$$

for all $x \in E^{\mathrm{an}}$, where $\parallel . \parallel_{E_u}$ denotes the canonical metric of the rigidified line bundle E_u. This is the formal metric associated to the - up to isomorphism - unique formal k°-model of E_u on the formal completion of $\mathcal{B}$. For more details we refer to [Gub10, §3]. For $x \in \mathbb{T}^{\mathrm{an}}$ considered as an element of E^{an} via the map $\mathbb{T}^{\mathrm{an}} \to E^{\mathrm{an}}$ induced by the Raynaud extension, we obtain $<\mathrm{trop}(x), u> = -\log|.| \circ \chi^u(x)$, i.e. the canonical tropicalization map trop agrees with the usual tropicalization map on the split torus $\mathbb{T}^{\mathrm{an}}$. Another important fact is that trop maps Λ isomorphically onto a complete lattice in $N_{\mathbb{R}}$ by [Gub10, 4.2] which we also denote by $\Lambda \subseteq N_{\mathbb{R}}$ in the following.

Remark 3.1.7. Following [Gub10, 4.2], an alternative description of the canonical tropicalization map $\mathrm{trop}_E : E^{\mathrm{an}} \to N_{\mathbb{R}}$ is given as follows: There is an open affine cover $\{\mathcal{U}_j\}_{j\in J}$ of $\mathcal{B}$ such that $\mathfrak{q}_1^{-1}(\mathcal{U}_j^{\beth}) \simeq \mathcal{U}_j^{\beth} \times_k \mathbb{T}_1^{\mathrm{an}}$ for all $j \in J$, where

$$\mathcal{U}_j^{\beth} := \{x \in U_j^{\mathrm{an}} \mid |f(x)| \leq 1 \, \forall f \in \mathcal{O}_{\mathcal{U}_j}(\mathcal{U}_j)\} \subseteq U_j^{\mathrm{an}} \subseteq B^{\mathrm{an}},$$

and where U_j denotes the generic fiber of $\mathcal{U}_j$. For every $j \in J$, we fix such a trivialization $\mathfrak{q}_1^{-1}(\mathcal{U}_j^{\beth}) \simeq \mathcal{U}_j^{\beth} \times_k \mathbb{T}_1^{\mathrm{an}}$ given by a section $s_j^{\beth} : \mathcal{U}_j^{\beth} \to G$. The image of the transition functions $g_{jl} := s_j^{\beth} - s_l^{\beth}$ are maps from $\mathcal{U}_j^{\beth} \cap \mathcal{U}_l^{\beth}$ to $\mathbb{T}_1^{\mathrm{an}}$. We choose a $\mathbb{Z}$-basis $T_1, \ldots, T_d$ of the character lattice M of $\mathbb{T}$. Now using that the image of the transition functions g_{jl} lies in $\mathbb{T}_1^{\mathrm{an}} = \{x \in \mathbb{T}^{\mathrm{an}} \mid |T_1(x)| = \ldots = |T_d(x)| = 1\}$, the pull-backs of $|T_i(.)| : \mathbb{T}^{\mathrm{an}} \to \mathbb{R}$ to the trivializations $\mathcal{U}_j^{\beth} \times_k \mathbb{T}^{\mathrm{an}}$ via the projections $\mathcal{U}_j^{\beth} \times_k \mathbb{T}^{\mathrm{an}} \to \mathbb{T}^{\mathrm{an}}$ yield well-defined maps

$$|T_i(.)| : E^{\mathrm{an}} \to \mathbb{R}$$

for all $i \in \{1, \ldots, d\}$. Furthermore, they are independent of the choice of the open affine cover $\{\mathcal{U}_j\}_{j\in J}$ of $\mathcal{B}$. Identifying $N_{\mathbb{R}}$ with $\mathbb{R}^d$ via the $\mathbb{Z}$-basis $T_1, \ldots, T_d$ of the character lattice M of $\mathbb{T}$, the canonical tropicalization map trop_E on E^{an} is then given by

$$\mathrm{trop}_E : E^{\mathrm{an}} \to \mathbb{R}^d \simeq N_{\mathbb{R}}, x \mapsto (-\log|T_1(x)|, \ldots, -\log|T_d(x)|) \in \mathbb{R}^d.$$

Proposition 3.1.8. *The canonical tropicalization map $\mathrm{trop}_E : E^{\mathrm{an}} \to N_{\mathbb{R}}$ is a harmonic tropicalization map.*

Proof. [GJR21, Proposition 16.2] □

Remark 3.1.9. For any non-archimedean field extension l/k, there is a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{T}^{\mathrm{an}} \arrow[r] & E^{\mathrm{an}} \arrow[r, "\mathfrak{q}"] & B^{\mathrm{an}} \arrow[r] & 0 \\
1 \arrow[r] & \mathbb{T}^{\mathrm{an}}\hat{\otimes}_k l \arrow[r] \arrow[u] & E^{\mathrm{an}}\hat{\otimes}_k l \arrow[r] \arrow[u] & B^{\mathrm{an}}\hat{\otimes}_k l \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

with exact rows, where the vertical maps are the structure morphisms. Then the fact that the diagram

$$\begin{tikzcd}
\mathbb{T}^{\mathrm{an}}\hat{\otimes}_k l \arrow[rr] \arrow[dr, "\mathrm{trop}"'] & & \mathbb{T}^{\mathrm{an}} \arrow[dl, "\mathrm{trop}"] \\
& N_{\mathbb{R}} &
\end{tikzcd}$$

commutes together with the construction of the Raynaud extension and the map $\mathrm{trop}_E : E^{\mathrm{an}} \to N_{\mathbb{R}}$ gives that the formation of the Raynaud extension and its canonical tropicalization is compatible with base change.

Lemma 3.1.10. *There is an affine Zariski-open subset $\mathcal{W} \subseteq \mathcal{B}$ with generic fiber W in B and associated affinoid domain $\mathcal{W}^{\beth} = \{x \in W^{\mathrm{an}} \mid |f(x)| \leq 1 \, \forall f \in \mathcal{O}_{\mathcal{W}}(\mathcal{W})\} \subseteq B^{\mathrm{an}}$ with $\xi_B \in \mathcal{W}^{\beth} \subseteq W^{\mathrm{an}}$ and an isomorphism $\phi : \mathfrak{q}^{-1}(W^{\mathrm{an}}) \simeq \mathbb{T}^{\mathrm{an}} \times_k W^{\mathrm{an}}$ such that the restriction of the isomorphism ϕ to $\mathfrak{q}^{-1}(\mathcal{W}^{\beth})$ yields a commutative diagram*

$$\begin{tikzcd} \mathfrak{q}^{-1}(\mathcal{W}^{\beth}) \arrow[r, "\phi|_{\mathfrak{q}^{-1}(\mathcal{W}^{\beth})}"] \arrow[d, "trop_E"] & \mathbb{T}^{\mathrm{an}} \times_k \mathcal{W}^{\beth} \arrow[dl, "trop \times \bullet"] \\ \mathbb{R}^d & \end{tikzcd}$$

where $trop \times \bullet : \mathbb{T}^{\mathrm{an}} \times \mathcal{W}^{\beth} \subseteq \mathbb{T}^{\mathrm{an}} \times_k W^{\mathrm{an}} \to \mathbb{R}^d$ denotes the composition $trop \circ pr_{\mathbb{T}^{\mathrm{an}}}$ of the projection $pr_{\mathbb{T}^{\mathrm{an}}} : \mathbb{T}^{\mathrm{an}} \times_k \mathcal{W}^{\beth} \to \mathbb{T}^{\mathrm{an}}$ from the fibre product with the canonical tropicalization map trop on $\mathbb{T}^{\mathrm{an}}$.

Proof. We use the description of the canonical tropicalization map trop_E from Remark 3.1.7. Let $\{\mathcal{U}_j\}_{j \in j}$ be an affine open cover of $\mathcal{B}$ as in Remark 3.1.7, leading to an affine open cover of B by the corresponding generic fibers $\{U_j\}_{j \in J}$. Using that the algebraic morphism $\mathfrak{q} : E \to B$ is locally trivial, by possibly refining the cover $\{\mathcal{U}_j\}_{j \in j}$, we may assume that there is an index $j \in J$ such that $\xi_B \in \mathcal{U}_j^{\beth} \subseteq U_j^{\mathrm{an}}$ and such that $\mathfrak{q}^{-1}(U_j^{\mathrm{an}}) \simeq U_j^{\mathrm{an}} \times_k \mathbb{T}^{\mathrm{an}}$ and $\mathfrak{q}_1^{-1}(\mathcal{U}_j^{\beth}) \simeq \mathcal{U}_j^{\beth} \times_k \mathbb{T}_1^{\mathrm{an}}$. We fix sections $s_j^{\beth} : \mathcal{U}_j^{\beth} \to G$ and $s_j : U_j^{\mathrm{an}} \to E^{\mathrm{an}} = (G \times_k \mathbb{T}^{\mathrm{an}}) / \mathbb{T}_1^{\mathrm{an}}$ for those trivializations. Then the commutativity of the diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{T}^{\mathrm{an}} \arrow[r] & E^{\mathrm{an}} \arrow[r, "\mathfrak{q}"] & B^{\mathrm{an}} \arrow[r] & 0 \\ 1 \arrow[r] & \mathbb{T}_1^{\mathrm{an}} \arrow[r] \arrow[u, hook] & G \arrow[r, "\mathfrak{q}_1"] \arrow[u, hook, "\iota_G"] & B^{\mathrm{an}} \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

shows that there exists some isomorphism $f : \mathbb{T}^{\mathrm{an}} \to \mathbb{T}^{\mathrm{an}}$ of analytic tori such that $\iota_G \circ s_j^{\beth} = (\mathrm{id}_G \times f) \circ s_j|_{\mathcal{U}_j^{\beth}}$, where $(\mathrm{id}_G \times f) : (G \times_k \mathbb{T}^{\mathrm{an}}) / \mathbb{T}_1^{\mathrm{an}} \to (G \times_k \mathbb{T}^{\mathrm{an}}) / \mathbb{T}_1^{\mathrm{an}} = E^{\mathrm{an}}$. If we now replace the trivialization s_j by the trivialization $(\mathrm{id}_G \times f) \circ s_j$, we get the claim for $\mathcal{W} := \mathcal{U}_j$. $\square$

Remark 3.1.11. Following [Gub10, 4.2], the canonical tropicalization map trop_E on E^{an} induces a canonical map $\mathrm{trop}_A : A^{\mathrm{an}} \to N_{\mathbb{R}}/\Lambda$ given by $\mathrm{trop}_A(x) := \mathrm{trop}_E(\tilde{x}) + \Lambda$ for all $\tilde{x} \in E^{\mathrm{an}}$ with corresponding $x := \tilde{x} + \Lambda := \mathfrak{p}(\tilde{x}) \in A^{\mathrm{an}} = E^{\mathrm{an}}/\Lambda$:

$$\begin{tikzcd} \Lambda \arrow[r, hook] & E^{\mathrm{an}} \arrow[r, "\mathrm{trop}_E"] \arrow[d, two heads, "\mathfrak{p}"] & N_{\mathbb{R}} \arrow[d, two heads] \\ & A^{\mathrm{an}} \arrow[r, "\mathrm{trop}_A"] & N_{\mathbb{R}}/\Lambda. \end{tikzcd}$$

We note that trop_A is nevertheless only locally a harmonic tropicalization map, which means that we can not define weakly smooth forms on A^{an} using trop_A globally, but we can make use of it by local considerations.

Definition 3.1.12. The subgroup $\Lambda \subseteq E(K)$ (resp. the complete lattice $\Lambda \subseteq N_{\mathbb{R}}$) acts on E^{an} (resp. $N_{\mathbb{R}}$) by translation. For $\lambda \in \Lambda \subseteq E^{\mathrm{an}}$ (resp. $\lambda' \in \Lambda \subseteq N_{\mathbb{R}}$) we denote the corresponding translation map by $\tau_\lambda : E^{\mathrm{an}} \to E^{\mathrm{an}}$ (resp. $\tau_{\lambda'} : N_{\mathbb{R}} \to N_{\mathbb{R}}$).

Definition 3.1.13. Let $\Omega \subseteq N_{\mathbb{R}}$ be an arbitrary subset. Then Ω is called Λ-*invariant* if for all $\lambda \in \Lambda$ and $v \in \Omega$, also $v + \lambda \in \Omega$. Furthermore, Ω is called Λ-*small* if for all non-zero elements $\lambda \in \Lambda \setminus \{0\}$ the equality $\Omega \cap \tau_\lambda^{-1}(\Omega) = \emptyset$ holds. An arbitrary subset $\tilde{U} \subseteq E^{\mathrm{an}}$ is defined to be Λ-*small* resp. Λ-*invariant* in exactly the same way.

Definition 3.1.14. Let $\Omega \subseteq N_{\mathbb{R}}$ be a Λ-invariant open subset and $\alpha' \in \mathcal{A}^{p,q}(\Omega)$ a (p,q)-Lagerberg form on Ω. Then we say that α' is Λ-*invariant* if

$$\tau_{\lambda'}^* \alpha' = \alpha' \text{ for all } \lambda' \in \Lambda.$$

Using a description for α' as

$$\alpha' = \sum_{|I|=p,|J|=q} \alpha'_{IJ} d'x_I \wedge d''x_J, \quad \alpha'_{IJ} \in C^\infty(\Omega)$$

this means that

$$\tau_{\lambda'}^* \alpha' = \sum_{|I|=p,|J|=q} \alpha'_{IJ} \circ \tau_{\lambda'} d'x_I \wedge d''x_J = \sum_{|I|=p,|J|=q} \alpha'_{IJ} d'x_I \wedge d''x_J = \alpha'$$

for all $\lambda' \in \Lambda$, i.e. the functions $\alpha'_{IJ} : \Omega \to \mathbb{R}$ are Λ-invariant. We denote by $\mathcal{A}^{p,q}(\Omega)^\Lambda$ the *space of* Λ-*invariant Lagerberg forms of bidegree* (p,q) *on* Ω. For a Λ-invariant open subset $\tilde{U}$ of E^{an}, the *space of* Λ-*invariant weakly smooth forms of bidegree* (p,q) *on* $\tilde{U}$ is defined in exactly the same way.

Lemma 3.1.15. *Every* Λ*-invariant Lagerberg form* $\alpha' \in \mathcal{A}^{p,q}(N_{\mathbb{R}})^\Lambda$ *on* $N_{\mathbb{R}}$ *induces in a canonical way a* Λ*-invariant weakly smooth form* $\tilde{\alpha} \in \mathcal{A}^{p,q}(E^{\mathrm{an}})^\Lambda$ *on* E^{an} *and a weakly smooth form* $\alpha \in \mathcal{A}^{p,q}(A^{\mathrm{an}})$ *on* A^{an}.

Proof. Let $\alpha' \in \mathcal{A}^{p,q}(N_{\mathbb{R}})^\Lambda$ be a Λ-invariant Lagerberg form on the real vector space $N_{\mathbb{R}}$. Let $\{U_i\}_{i\in I}$ be a cover of E^{an} by Λ-invariant open subsets such that the closure $\overline{U_i}$ is a compact and Λ-small analytic domain in E^{an}. Such a cover always exists; one explicit construction can be found in the proof of Theorem 5.2.8: The space E^{an} can be covered by the interiors of the preimages $\mathrm{trop}_E^{-1}(\Delta)$ of the polyhedra Δ from Lemma 5.2.5 and all its Λ-translates. For all $i \in I$ let now

$$h_i := \mathrm{trop}_E|_{\overline{U_i}} : \overline{U_i} \to N_{\mathbb{R}} \text{ and } \alpha'_i := \alpha'|_{\mathrm{trop}_E(\overline{U_i})}.$$

Then h_i is a harmonic tropicalization map on $\overline{U_i}$ for all $i \in I$, and the family $(h_i, \alpha'_i)_{i\in I}$ defines a weakly smooth form in $\mathcal{A}^{p,q}(E^{\mathrm{an}})$. We write

$$\alpha' = \sum_{\substack{I,J\subseteq\{1,\dots,d\} \\ |I|=p,|J|=q}} \alpha'_{IJ} d'x_I \wedge d''x_J \text{ for suitable } \alpha'_{IJ} \in C^\infty(N_{\mathbb{R}}).$$

For $I, J \subseteq \{1, \ldots, d\}$ with $|I| = p, |J| = q$, the Λ-invariance of α'_{IJ} together with the fact that trop_E maps $\Lambda \subseteq E(k)$ isomorphically onto Λ and $E^{\mathrm{an}}/\Lambda \cong A^{\mathrm{an}}$ shows that $(h_i, \alpha'_i)_{i \in I}$ is Λ-invariant, and that there are induced well-defined maps $\alpha_{IJ} : A^{\mathrm{an}} \to \mathbb{R}, x \mapsto \alpha'_{IJ} \circ \operatorname{trop}_E(\tilde{x})$ for $\tilde{x} \in E^{\mathrm{an}}$ with $x = \tilde{x} + \Lambda$ (i.e. $\mathfrak{p}(\tilde{x}) = x$) and $\overline{\alpha'_{IJ}} : N_{\mathbb{R}}/\Lambda \to \mathbb{R}$ such that the diagram

$$\begin{tikzcd}
E^{\mathrm{an}} \arrow[r, "\operatorname{trop}_E"] \arrow[dd, two heads, "p"'] & N_{\mathbb{R}} \arrow[r, "\alpha'_{IJ}"] \arrow[d, two heads] & \mathbb{R} \\
& N_{\mathbb{R}}/\Lambda \arrow[ur, dashed, "\overline{\alpha'_{IJ}}"] & \\
A^{\mathrm{an}} \arrow[ur, "\operatorname{trop}_A"] \arrow[uurr, dashed, bend right, "\alpha_{IJ}"'] & &
\end{tikzcd}$$

commutes. For $i \in I$, the compact analytic domain $\overline{U_i}$ is Λ-small, hence $\mathfrak{p}|_{\overline{U_i}}$ is a homeomorphism and

$$\left(\operatorname{trop}_E|_{\overline{U_i}} \circ \left(\mathfrak{p}|_{\overline{U_i}}\right)^{-1}\right) \left(\mathfrak{p}\left(\overline{U_i}\right)\right)$$

is a Λ-small subset of $N_{\mathbb{R}}$. The commutativity of the diagram above yields that

$$h_i^A := \operatorname{trop}_E|_{\overline{U_i}} \circ \left(\mathfrak{p}|_{\overline{U_i}}\right)^{-1} : \mathfrak{p}(\overline{U_i}) \to N_{\mathbb{R}}$$

is a harmonic tropicalization map for all $i \in I$ and the family $\left(h_i^A, \alpha'_i\right)_{i \in I}$ defines a weakly smooth form $\alpha \in \mathcal{A}^{p,q}(A^{\mathrm{an}})$. $\square$

3.2 The torus part of the Dolbeault cohomology of abelian varieties

Using the theory from the last section, we can now formulate one of the main results with regards to the Dolbeault cohomology of abelian varieties. In order to do this, let A be an abelian variety which is split over k.

Remark 3.2.1. The space of *Lagerberg forms with constant coefficients on* $N_{\mathbb{R}}$ is defined as $\bigwedge^{p,q} M_{\mathbb{R}} := \bigwedge^p M_{\mathbb{R}} \otimes_{\mathbb{R}} \bigwedge^q M_{\mathbb{R}} \subseteq \mathcal{A}^{p,q}(N_{\mathbb{R}})$. They are d''-closed Λ-invariant Lagerberg forms on $N_{\mathbb{R}}$, so in particular there is a canonical map

$$\Psi : \bigwedge^{p,q} M_{\mathbb{R}} \to H^q(\mathcal{A}^{p,\bullet}(N_{\mathbb{R}})^{\Lambda}, d''), \alpha' \mapsto \left[\alpha'\right] \tag{3.5}$$

given by mapping a (p,q)-Lagerberg form with constant coefficients on $N_{\mathbb{R}}$ onto its class in cohomology. By [Sto21, Proposition 3.4.26], this canonical map is an isomorphism of real vector spaces.

Theorem 3.2.2. *For all $p, q \in \mathbb{N}$ there exists a canonical injective morphism*

$$\bigwedge^{p,q} M_{\mathbb{R}} \hookrightarrow H^{p,q}(A^{\mathrm{an}})$$

which is induced by applying Lemma 3.1.15 to Lagerberg forms with constant coefficients on $N_{\mathbb{R}}$.

Proof. Later, in 5.3.6. □

Remark 3.2.3. We observe that the construction of the morphism $\bigwedge^{p,q} M_{\mathbb{R}} \hookrightarrow H^{p,q}(A^{\mathrm{an}})$ is based on the torus part of the Raynaud extension. The abelian variety B of good reduction from Remark/ Definition 3.1.4 does not appear in this construction, but it will be needed to prove the result. More precisely, we have to define a so-called δ-form β_0 on B^{an} which induces a suitable strong current T_β on A^{an} such that weakly smooth forms α induced by Lagerberg forms with constant coefficients on the torus vector space $N_{\mathbb{R}}$ do not vanish, i.e. such that $T_\beta(\alpha) \neq 0$. In order to do so, we first have to introduce the theory of δ-forms. We note here that the δ-forms introduced by Gubler and Künnemann in [GK17] contain the δ-form β_0 on B^{an} which is the one that we need, and moreover, in their paper [GK17], they study this particular δ-form in great detail. Furthermore, they explain how to pair their δ-forms with so-called piecewise smooth forms, and thus in particular with weakly smooth forms. We can furthermore pull-back the δ-form β_0 to the Raynaud extension E^{an} via the algebraic morphism $\mathfrak{q} : E \to B$ as in [GK17]. However, in order to pass to the analytification A^{an} of our original abelian variety A, we have to use the covering map $\mathfrak{p} : E^{\mathrm{an}} \to A^{\mathrm{an}}$ which is an analytic morphism. For this reason, the more general theory of δ-forms by Mihatsch from [Mih21] will be the one that we use here. Furthermore, the sheaf of weakly smooth forms embeds into the sheaf of δ-forms in the sense of [Mih21] which allows us to do all our computations in this space of δ-forms.

Chapter 4

δ-forms on Berkovich spaces

In this chapter, we follow the two papers [Mih23] and [Mih21] by Mihatsch. We introduce δ-forms in the sense of Mihatsch - which generalize the δ-forms introduced by Gubler and Künnemann in [GK17] in the algebraic setting. First we consider δ-forms on $\mathbb{R}$-vector spaces $\mathbb{R}^n$, then on so-called tropical spaces (X, μ, L) and then on non-archimedean spaces (without boundary).

Throughout this chapter, let k be a field that is complete with respect to a non-trivial non-archimedean absolute value $|.| : k \to \mathbb{R}$.

4.1 The sheaf of δ-forms on $\mathbb{R}^n$

In his paper [Mih23], Mihatsch introduced δ-forms on real vector spaces $\mathbb{R}^n$. They generalise classical tropical intersection theory. Roughly speaking, δ-forms are sums of products of smooth differential forms on polyhedra with integration currents, which fullfill some balancing condition. They come with differentials and a wedge product extending the tropical intersection product on cycles.

Definition 4.1.1. i) A *polyhedral current* on $\mathbb{R}^n$ is a current which is a locally finite sum of currents of the form $\alpha_\sigma \wedge [\sigma, \mu]$ for a weighted polyhedron $[\sigma, \mu]$ in $\mathbb{R}^n$ and a Lagerberg form α_σ of any bidegree on σ. We write $P(\mathbb{R}^n)$ for the space of all polyhedral currents on $\mathbb{R}^n$.

ii) We write $P^{p,q,r}(\mathbb{R}^n) \subseteq \mathcal{D}^{p+r,q+r}(\mathbb{R}^n)$ for the space of all polyhedral currents which may be written as locally finite sum of currents $\alpha_\sigma \wedge [\sigma, \mu]$, where $\sigma \subseteq \mathbb{R}^n$ is of codimension r, and $\alpha_\sigma \in \mathcal{A}^{p,q}(\sigma)$. We call elements of $P^{p,q,r}(\mathbb{R}^n)$ *trihomogeneous of tridegree (p,q,r)*.

Remark 4.1.2. i) There is a direct sum decomposition $P(\mathbb{R}^n) = \bigoplus_{p,q,r \in \mathbb{N}} P^{p,q,r}(\mathbb{R}^n)$.

ii) Let $T = \sum_{i \in I} \alpha_i \wedge [\sigma_i, \mu_i]$ be a representation for a polyhedral current $T \in P(\mathbb{R}^n)$. Then the datum of all $(\alpha_i, \sigma_i, \mu_i)_{i \in I}$ representing T is unique up to locally finitely many operations of the following kinds: Subdividing the polyhedra $\sigma_i, i \in I$, adding or removing terms with $\alpha = 0$, replacing a triple (α, σ, μ) by $(\lambda\alpha, \sigma, \lambda^{-1}\mu)$ for some $\lambda \in \mathbb{R}_{>0}$, and writing $(\alpha_{i_1}, \sigma, \mu) + (\alpha_{i_2}, \sigma, \mu)$ instead of $(\alpha_{i_1} + \alpha_{i_2}, \sigma, \mu)$ or the other way round.

Definition 4.1.3. Let $T = \sum_{i\in I} \alpha_i \wedge [\sigma_i, \mu_i]$ be a polyhedral current in $P(\mathbb{R}^n)$. We define the *polyhedral derivatives of* T as the polyhedral currents

$$d'_P T := \sum_{i\in I} \left(d'\alpha_i\right) \wedge [\sigma_i, \mu_i] \text{ and } d''_P T := \sum_{i\in I} \left(d''\alpha_i\right) \wedge [\sigma_i, \mu_i] .$$

Remark 4.1.4. By [GK17, Remark 2.4 (iii)], the polyhedral derivatives d'_p and d''_P on $P(\mathbb{R}^n)$ do in general not coincide with the derivatives d' and d'' in the sense of currents on $P(\mathbb{R}^n) \subseteq \mathcal{D}(\mathbb{R}^n)$. The derivatives $d'T$ and $d''T$ of a polyhedral current $T \in P(\mathbb{R}^n)$ might even be non-polyhedral. An example is given in [Mih23, Example 2.10]. The δ-forms introduced by Mihatsch are polyhedral currents that have the property that their derivatives as currents are again polyhedral.

Definition 4.1.5. Let $T \in P(\mathbb{R}^n)$. Then a polyhedral complex $\mathcal{T}$ in $\mathbb{R}^n$ is called *subordinate to* T if T can be represented as $T = \sum_{\sigma\in\mathcal{T}} \alpha_\sigma \wedge [\sigma, \mu_\sigma]$.

Remark 4.1.6. In the situation of Definition 4.1.5, the datum $(\alpha_\sigma, \mu_\sigma)_{\sigma\in\mathcal{T}}$ is uniquely determined up to replacing a tuple $(\alpha_\sigma, \mu_\sigma)$ by $(\lambda\alpha_\sigma, \lambda^{-1}\mu_\sigma)$ for some $\lambda \in \mathbb{R}_{>0}$.

Definition 4.1.7. i) Let $\sigma \subseteq \mathbb{R}^n$ be a polyhedron. Then a *piecewise smooth form on* σ is the datum - up to subdivision - of a polyhedral complex $\mathcal{T}$ in $\mathbb{R}^n$ with support σ together with smooth forms $\alpha_\rho \in \mathcal{A}(\rho)$ for all $\rho \in \mathcal{T}$ such that $\alpha_\rho|_\tau = \alpha_\tau$ for all $\tau, \rho \in \mathcal{T}$ with $\tau \subseteq \rho$. We denote the space of piecewise smooth forms on σ by $PS(\sigma)$, and the space of all piecewise smooth forms on σ of the form $(\alpha_\rho)_{\rho\in\mathcal{T}}$ with $\alpha_\rho \in \mathcal{A}^{p,q}(\rho)$ for all $\rho \in \mathcal{T}$ by $PS^{p,q}(\sigma)$.

ii) Let $[\sigma, \mu]$ be a weighted d-dimensional polyhedron in $\mathbb{R}^n$, and let $\alpha = (\alpha_\rho)_{\rho\in\mathcal{T}}$ be a piecewise smooth form on σ. We observe that μ defines a weight on all $\rho \in \mathcal{T}$ with $\dim \rho = \dim \sigma$ since $N_\rho = N_\sigma$ for dimension reasons. We define the polyhedral current

$$\alpha \wedge [\sigma, \mu] : \mathcal{A}_c(\sigma) \to \mathbb{R}, \eta \mapsto \left(\sum_{\substack{\rho\in\mathcal{T} \\ \dim \rho = \dim \sigma}} \alpha_\rho \wedge [\rho, \mu] \right) (\eta) .$$

iii) Let $[\sigma, \mu]$ be a weighted d-dimensional polyhedron in $\mathbb{R}^n$. Using (ii), there is an integral

$$\int_{[\sigma,\mu]} : PS_c^{d,d}(\sigma) \to \mathbb{R}.$$

Remark 4.1.8. From Definition 4.1.7 ii) we obtain for every polyhedron $\sigma \subseteq \mathbb{R}^n$ and a fixed weight μ on σ an embedding $PS(\sigma) \subseteq P(\mathbb{R}^n) \subseteq \mathcal{D}(\mathbb{R}^n)$.

Remark 4.1.9. i) For an exact sequence of finite dimensional $\mathbb{R}$-vector spaces

$$0 \to N_1 \to N_2 \to N_3 \to 0,$$

there is a canonical isomorphism $\det N_2 = \det N_1 \otimes_{\mathbb{R}} \det N_3$. So given weights μ_i for N_i for two out of $\{N_1, N_2, N_3\}$, they uniquely determine a weight for the third space through the relation

$$\mu_2 = \mu_1 \wedge \mu_3 := \mu_1 \wedge \tilde{\mu_3}, \tag{4.1}$$

where $\tilde{\mu_3} \in \bigwedge^{\dim N_3} N_2$ is any lift of μ_3, see [Mih23, (2.12)].

ii) Let $\sigma \subseteq \mathbb{R}^n$ be a polyhedron and $f : \mathbb{R}^n \to \mathbb{R}^m$ an affine map. Then f maps σ onto a polyhedron $f(\sigma) \subseteq \mathbb{R}^m$. Let μ be a weight on σ, and ν a weight on $f(\sigma)$. Let $K := \ker\big((f - f(0))|_{N_\sigma} : N_\sigma \to N_{f(\sigma)}\big)$. Then the short exact sequence

$$0 \to K \to N_\sigma \to N_{f(\sigma)} \to 0$$

together with i) acquires a weight δ for $K \subseteq N_\sigma \subseteq \mathbb{R}^n$, called *fibre weight on K*, and there is a natural *fibre integration map*

$$f_{\delta,*} : \mathcal{A}_c^{p,q}(\sigma) \to PS^{p-\dim K, q-\dim K}(f(\sigma)).$$

iii) For $\alpha \in PS(\sigma)$ with corresponding polyhedral current $\alpha \wedge [\sigma, \mu]$, we obtain a projection formula

$$\begin{aligned}(\alpha \wedge [\sigma, \mu])\,(f^*\eta) &= \int_{[\sigma,\mu]} \alpha \wedge f^*\eta \\ &= \int_{[f(\sigma),\nu]} (f_{\delta,*}\alpha) \wedge \eta \\ &= (f_{\delta,*}\alpha \wedge [f(\sigma), \nu])\,(\eta)\end{aligned}$$

for all $\eta \in \mathcal{A}_c(f(\sigma))$.

iv) In other words, fibre integration provides a representative for the push-forward of currents in the sense that for all $\alpha \in PS(\sigma)$, the identity

$$f_*\,(\alpha \wedge [\sigma, \mu]) = (f_{\delta,*}\alpha) \wedge [f(\sigma), \nu]$$

holds, see [Mih23, (2.14)]. This shows in particular that the push-forward of a polyhedral current with relatively compact support is polyhedral again.

v) Given a surjective affine map $f : \mathbb{R}^n \to \mathbb{R}^m$ and a current T on $\mathbb{R}^m$, we define the *pull-back current* f^*T on $\mathbb{R}^n$ by

$$(f^*T)(\eta) := T(f_*\eta)$$

for a Lagerberg form $\eta \in \mathcal{A}_c(\mathbb{R}^n)$, where $f_*\eta$ denotes the fibre integral of η with respect to the standard weights on $\mathbb{R}^n$ and $\mathbb{R}^m$. We observe that $f_*\eta$ is again a Lagerberg form on $\mathbb{R}^m$.

vi) We note that the differentials d' and d'' commute with taking pull-backs.

Definition 4.1.10. A *δ-form on* $\mathbb{R}^n$ is a polyhedral current $T \in P(\mathbb{R}^n)$ such that the differentials $d'T$ and $d''T$ are again polyhedral.

Remark/Definition 4.1.11. Let $[\sigma, \mu]$ be a weighted polyhedron in $\mathbb{R}^n$, and let $\tau \preceq \sigma$ be a facet together with a weight ν on τ. Then there is a unique vector $\overline{n_{\sigma,\tau}} \in N_\sigma / N_\tau$ that points in direction of σ and is such that $\mu = \nu \wedge \overline{n_{\sigma,\tau}}$ in the sense of (4.1). A *normal vector for* $\tau \preceq \sigma$ is any choice of a lift $n_{\sigma,\tau} \in N_\sigma$ for $\overline{n_{\sigma,\tau}}$.

Definition 4.1.12. Let $\mathcal{T}$ be a polyhedral complex in $\mathbb{R}^n$ together with weights μ_σ and smooth forms $\alpha_\sigma \in \mathcal{A}(\sigma)$ for all its polyhedra $\sigma \in \mathcal{T}$. This datum $(\mu_\sigma, \alpha_\sigma)_{\sigma \in \mathcal{T}}$ is called *balanced* if the following holds: For all $\tau \in \mathcal{T}$, the sum

$$\sum_{\substack{\sigma \in \mathcal{T} \\ \tau \preceq \sigma \text{ facet}}} \alpha_\sigma|_\tau \otimes_{\mathbb{R}} n_{\sigma,\tau} \in \mathcal{A}(\tau) \otimes_{\mathbb{R}} \mathbb{R}^n$$

lies in the subspace $\mathcal{A}(\tau) \otimes_{\mathbb{R}} M_\tau$, where the normal vectors $n_{\sigma,\tau}$ are taken for the weights μ_σ for σ and μ_τ for τ.

Remark 4.1.13. i) We consider a datum $(\mu_\sigma, \alpha_\sigma)_{\sigma \in \mathcal{T}}$ as in Definition 4.1.12, defining a current

$$T = \sum_{\sigma \in \mathcal{T}} \alpha_\sigma \wedge [\sigma, \mu_\sigma] \,.$$

Then, for the datum $(\mu_\sigma, \alpha_\sigma)_{\sigma \in \mathcal{T}}$, being balanced is stable under the operations in Remark 4.1.2 ii), which yields that it only depends on the current T, not on its representation.

ii) Writing a polyhedral current $T = \sum_{\sigma \in \mathcal{T}} \alpha_\sigma \wedge [\sigma, \mu_\sigma]$ as a sum $T = \sum_{p,q,r \in \mathbb{N}} T^{p,q,r}$ with $T^{p,q,r} \in P^{p,q,r}(\mathbb{R}^n)$ the current T is balanced if and only if $T^{p,q,r}$ is balanced for all $p, q, r \in \mathbb{N}$, see [Mih23, Theorem 3.3].

Definition 4.1.14. Let $T \in P(\mathbb{R}^n)$ be a polyhedral current. Then T is called *balanced* if there is a datum representing T which is balanced. We observe that this is well-defined by Remark 4.1.13 ii).

Theorem 4.1.15. *A polyhedral current $T \in P(\mathbb{R}^n)$ is a δ-form if and only if it is balanced. In particular - writing T as a sum $T = \sum_{p,q,r \in \mathbb{N}} T^{p,q,r}$ with $T^{p,q,r} \in P^{p,q,r}(\mathbb{R}^n)$ - T is a δ-form if and only if $T^{p,q,r}$ is a δ-form for all $p, q, r \in \mathbb{N}$. Furthermore, T is already a δ-form if one out of $d'T$ and $d''T$ is polyhedral.*

Proof. [Mih23, Theorem 3.3] □

Remark/Definition 4.1.16. i) For $p, q, r \in \mathbb{N}$, we denote by $B^{p,q,r}(\mathbb{R}^n)$ the space of *δ-forms on* $\mathbb{R}^n$ *of tridegree* (p, q, r), and by $B(\mathbb{R}^n) = \bigoplus_{p,q,r} B^{p,q,r}(\mathbb{R}^n)$ the space of all δ-forms. Furthermore, we denote by $B^{p,q}(\mathbb{R}^n) = \bigoplus_{r \in \mathbb{N}} B^{p-r,q-r,r}(\mathbb{R}^n)$ the space of δ-forms of bidegree (p, q) in the sense of currents.

ii) There are derivatives

$$d' : B^{p,q}(\mathbb{R}^n) \to B^{p+1,q}(\mathbb{R}^n) \text{ and } d'' : B^{p,q}(\mathbb{R}^n) \to B^{p,q+1}(\mathbb{R}^n)$$

for all $p, q \in \mathbb{N}$ obtained by restricting the corresponding maps on currents.

iii) The polyhedral derivatives restrict to operators

$$d'_p : B^{p,q,r}(\mathbb{R}^n) \to B^{p+1,q,r}(\mathbb{R}^n) \text{ and } d''_p : B^{p,q,r}(\mathbb{R}^n) \to B^{p,q+1,r}(\mathbb{R}^n)$$

for all $p, q, r \in \mathbb{N}$.

iv) We define boundary operators $\partial' := d'_P - d'$ and $\partial'' := d''_P - d''$, which can be shown to be trihomogeneous in the sense that they are maps

$$\partial' : B^{p,q,r}(\mathbb{R}^n) \to B^{p,q-1,r+1} \text{ and } \partial'' : B^{p,q,r}(\mathbb{R}^n) \to B^{p-1,q,r+1}.$$

Lemma 4.1.17. *i) Let $T \in B(\mathbb{R}^n)$ be a δ-form on $\mathbb{R}^n$ with compact support, and $f : \mathbb{R}^n \to \mathbb{R}^m$ an affine map such that T has relatively compact support with respect to f. Then the push-forward of currents $f_*T \in B(\mathbb{R}^m)$ is also a δ-form.*

ii) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a surjective affine map and $S \in B(\mathbb{R}^m)$ a δ-form. Then the pull-back $f^*S \in B(\mathbb{R}^n)$ is again a δ-form.

Proof. [Mih23, Lemma 3.5] □

Lemma 4.1.18. *For, $p, q \in \mathbb{N}$, the δ-forms $B^{p,q,0}(\mathbb{R}^n)$ are precisely the currents of the form $\alpha \wedge [\mathbb{R}^n, \mu_{\mathbb{R}^n}]$ for $\alpha \in PS^{p,q}(\mathbb{R}^n)$.*

Proof. [Mih23, Lemma 3.7] □

Example/Definition 4.1.19. Let $\alpha \in PS(\mathbb{R}^n)$ be a piecewise smooth form and $T \in P(\mathbb{R}^n)$ a polyhedral current on $\mathbb{R}^n$. Let $\mathcal{T}$ be a polyhedral complex in $\mathbb{R}^n$ which is subordinate to both T and α. We write

$$\alpha = (\alpha_\rho)_{\rho \in \mathcal{T}} \text{ and } T = \sum_{\rho \in \mathcal{T}} \beta_\rho \wedge [\rho, \mu_\rho]$$

for suitable $\alpha_\rho, \beta_\rho \in \mathcal{A}(\rho)$ and weights μ_ρ for all $\rho \in \mathcal{T}$. Then the *product of α and T* is defined as

$$\alpha T := \sum_{\rho \in \mathcal{T}} \alpha_\rho \wedge \beta_\rho \wedge [\rho, \mu_\rho] \, .$$

We note that if T is assumed to be a δ-form, then αT is also a δ-form, see [Mih23, Example 3.8].

Remark 4.1.20. i) Following [Mih23, §4], there is an *exterior product of currents*, given as follows: For homogeneous currents $T_1 \in \mathcal{D}(\mathbb{R}^n)$ and $T_2 \in \mathcal{D}(\mathbb{R}^m)$ the exterior product $T_1 \boxtimes T_2$ of T_1 and T_2 is the unique current $T_1 \boxtimes T_2 \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}^m)$ such that

$$(T_1 \boxtimes T_2)(p_1^*\eta_1 \wedge p_2^*\eta_2) = (-1)^{\deg(T_1)\deg(T_2)} T_1(\eta_1) T_2(\eta_2)$$

for $\eta_1 \in \mathcal{A}_c(\mathbb{R}^n)$ and $\eta_2 \in \mathcal{A}_c(\mathbb{R}^n)$. The exterior product of currents preserves polyhedral currents since for weighted polyhedra (σ_n, μ_n) in $\mathbb{R}^n$, (σ_m, μ_m) in $\mathbb{R}^m$ and smooth Lagerberg forms α_n on σ_n and α_m on σ_m the equality

(4.2)
$$\alpha_n \wedge [\sigma_n, \mu_n] \boxtimes \alpha_m \wedge [\sigma_m, \mu_m] = p_1^*\alpha_n \wedge p_2^*\alpha_m \wedge [\sigma_n \times \sigma_m, \mu_n \wedge \mu_m]$$

holds (see [Mih23, (4.3)]), where $\sigma_n \times \sigma_m := i_1(\sigma_n) \times i_2(\sigma_m) \subseteq \mathbb{R}^n \times \mathbb{R}^m$ for the inclusions $i_1 : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ and $i_2 : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$, and where $\mu_n \wedge \mu_m$ is defined as in Remark 4.1.9 with respect to the canonical short exact sequence

$$0 \to N_{\sigma_n} \to N_{\sigma_n} \oplus N_{\sigma_m} \to N_{\sigma_m} \to 0.$$

By [Mih23, 4.1], the exterior product of δ-forms is again a δ-form.

ii) Now, in [Mih23, §4], Mihatsch defines a $\wedge$-product on δ-forms, which coincides with the tropical intersection product on tropical cycles for δ-forms of degree $(0,0,r)$, $r \in \mathbb{N}$ (those δ-forms are tropical cycles, see [Mih23, Example 3.8]). By [Mih23, Lemma 4.14], the identity of δ-forms

$$S \boxtimes T = p_1^*S \wedge p_2^*T$$

holds, where the left-hand side denotes the exterior product of S and T as currents, and the right-hand side denotes the $\wedge$-product of δ-forms. We observe that the two terms also coincide with the intersection product $p_1^*S \cdot p_2^*T$ of S and T as tropical cycles, see [Mih23, 4.1].

Notation 4.1.21. In the following, we denote by $\Delta = (\mathrm{id}, \mathrm{id})_* [\mathbb{R}^n, \mu_{\mathrm{std}}] \in B^{0,0,n}(\mathbb{R}^n \times \mathbb{R}^n)$ the *diagonal* viewed as a δ-form.

Theorem 4.1.22. *There is a unique way to define an associative product*

$$\wedge : B(\mathbb{R}^n) \times B(\mathbb{R}^n) \to B(\mathbb{R}^n)$$

that satisfies the Leibniz rules with respect to d' and d'', extends the definition

$$(\alpha \wedge [\mathbb{R}^n, \mu_{\mathbb{R}^n}]) \wedge T := \alpha T$$

for all $\alpha \in PS(\mathbb{R}^n), T \in B(\mathbb{R}^n)$, and can be computed by restriction to the diagonal, meaning that

$$S \wedge T = p_{1,*}\left(\Delta \wedge (S \boxtimes T)\right)$$

for all $S, T \in B(\mathbb{R}^n)$. This product $\wedge$ has the following additional properties:

i) It is graded commutative and trihomogeneous in the sense that it restricts to maps

$$\wedge : B^{p,q,r}(\mathbb{R}^n) \times B^{s,t,u}(\mathbb{R}^n) \to B^{p+s,q+t,r+u}(\mathbb{R}^n)$$

for all $p, q, r, s, t, u \in \mathbb{N}$. In particular, it satisfies the Leibniz rule with respect to the operators $\partial', \partial'', d'_P$ and d''_P.

ii) *It commutes with pull-backs, i.e. for all surjective affine maps* $f : \mathbb{R}^n \to \mathbb{R}^m$ *and* δ*-forms* $S, T \in B(\mathbb{R}^m)$*, the equality*

$$f^*(T \wedge S) = f^*T \wedge f^*S$$

holds.

iii) *It satisfies the projection formula, i.e. for all surjective affine maps* $f : \mathbb{R}^n \to \mathbb{R}^m$*,* δ*-forms* $S \in B(\mathbb{R}^m)$ *and* $T \in B(\mathbb{R}^n)$ *with support compact over* $\mathbb{R}^m$*, the equality*

$$f_*(T \wedge f^*S) = f_*T \wedge S$$

holds.

Proof. [Mih23, Theorem 4.1] □

4.2 The sheaf of δ-forms on Berkovich spaces

In this section we introduce δ-forms on non-archimedean analytic spaces following [Mih21]. Roughly speaking, they are obtained by locally pulling back δ-forms on real vector spaces via smooth tropicalization maps. In order to make this precise, at first we have to define δ-forms on tropical spaces - hence locally on skeleta of tropical coordinates. Then, there is an integration theory of δ-forms leading to a product formula for fibre products which will be needed to prove the main theorem 5.2.8.

For the rest of this chapter, we fix a field k that is complete with respect to a non-trivial non-archimedean absolute value $|.| : k \to \mathbb{R}$. In this section, an analytic space is always meant to be a good strictly k-analytic space in the sense of [Ber93] which is Hausdorff and paracompact.

4.2.1 The sheaf of δ-forms on tropical spaces

In the setting of non-archimedean analytic spaces together with tropical coordinates, tropical spaces arise in a natural way. Locally, they are roughly speaking just tropical cycles in the classical sense, together with a so-called sheaf of linear functions which is given by the coordinate functions of the tropical coordinates. Then we can define δ-forms on such spaces using the δ-forms on real vector spaces defined before, and pulling them back under tropical coordinates.

Definition 4.2.1. Let $C \subseteq \mathbb{R}^n$ be a polyhedral set.

i) A map $f : C \to \mathbb{R}$ is called *piecewise linear* if it is continuous and if C can be written as a locally finite union of polyhedra $C = \bigcup_{i\in I} \sigma_i$ in $\mathbb{R}^n$ such that the restriction $f|_{\sigma_i}$ is affine for all $i \in I$.

ii) A map $f : U \to \mathbb{R}$ on an open subset $U \subseteq C$ is called *piecewise linear* if for every polyhedron $\sigma \subseteq U \subseteq \mathbb{R}^n$, the restriction $f|_\sigma$ is piecewise linear. We denote the so-defined *sheaf of piecewise linear functions on C* by Λ_C.

iii) A *piecewise linear space* is a pair (X, Λ_X) of the following kind: X is a paracompact Hausdorff topological space, and $\Lambda_X \subseteq \mathcal{C}(-,\mathbb{R})$ is a subsheaf of the sheaf of real-valued continuous functions on X such that for every $x \in X$ there is a neighbourhood V of x in X together with a polyhedral set D and a homeomorphism $\varphi : V \to D$ such that $\Lambda_D \circ \varphi = \Lambda_X|_V$. Here $\Lambda_X|_V := \operatorname{Im}\left(i_V^{-1} : \Lambda_X \to \mathcal{C}(-,\mathbb{R})\right)$ denotes the sheaf on V of those continuous $\mathbb{R}$-valued functions that locally extend to a section of Λ_X, where $i_V : V \subseteq X$. In the above situation, we often only write X instead of (X, Λ_X).

iv) A *map of piecewise linear spaces* $\varphi : (X, \Lambda_X) \to (Y, \Lambda_Y)$ is a continuous map $\varphi : X \to Y$ of topological spaces such that $\Lambda_Y \circ \varphi = \Lambda_X$.

v) Let $\varphi : (C, \Lambda_C) \to (D, \Lambda_D)$ be a piecewise linear map between polyhedral sets. Then polyhedral complex structures $\mathcal{C}$ for C and $\mathcal{D}$ for D are called *subordinate to* φ if for all $\sigma \in \mathcal{C}$ there exists some $\rho \in \mathcal{D}$ such that $\varphi(\sigma) \subseteq \rho$ and such that $\varphi|_\sigma : \sigma \to f(\sigma)$ is a linear map for all $\sigma \in \mathcal{C}$.

vi) A piecewise linear space (X, Λ_X) is called *polyhedral* if it is itself isomorphic to some polyhedral set (C, Λ_C).

Remark/Definition 4.2.2. Let $C \subseteq \mathbb{R}^n$ be a polyhedral set.

i) For $p, q \in \mathbb{N}$, a *piecewise smooth form of bidegree* (p, q) on C is the datum of - up to refinement - a locally finite decomposition $C = \bigcup_{i\in I} \sigma_i$ of C as a union of polyhedra together with Lagerberg forms $\omega_i \in \mathcal{A}^{p,q}(\sigma_i)$ for all $i \in I$ such that $\omega_i|_{\sigma_i \cap \sigma_j} = \omega_j|_{\sigma_i\cap\sigma_j}$ for all $i, j \in I$.

ii) For all $p, q \in \mathbb{N}$, the piecewise smooth forms of bidegree (p, q) on C define a sheaf on C, denoted by $PS_C^{p,q}$ or $PS^{p,q}$ if the polyhedral set is clear from the context. We define the *sheaf of piecewise smooth forms on* C by $PS := PS_C := \bigoplus_{p,q\in\mathbb{N}} PS_C^{p,q}$.

iii) The piecewise smooth forms on C are equipped with a $\wedge$-product and differentials: The $\wedge$-product and the differentials d' and d'' on Lagerberg forms are computed on piecewise smooth forms polyhedron-by-polyhedron. We denote the resulting operators by $\wedge, d'_P$ and d''_P.

iv) For a piecewise linear map $\varphi : (C, \Lambda_C) \to (D, \Lambda_D)$ of polyhedral sets, there is a *pull-back morphism*

$$f^* : f^{-1}PS_D \to PS_C$$

which is computed by using the pull-back of Lagerberg forms polyhedron-by-polyhedron.

Remark 4.2.3. The definitions from 4.2.2 on polyhedral sets $C \subseteq \mathbb{R}^n$ extend by locality to all piecewise linear spaces (X, Λ_X). For a piecewise linear space (X, Λ_X), we denote the *sheaf of piecewise linear forms on* X by $PS_X = \bigoplus_{p,q\in\mathbb{N}} PS_X^{p,q}$, we write $\wedge$ for the defined wedge product, d'_P resp. d''_P for the differentials, and for a piecewise linear map $f : (X, \Lambda_X) \to (Y, \Lambda_Y)$ to some piecewise linear space (Y, Λ_Y), write f^* for the induced pull-back.

Definition 4.2.4. Let C be a polyhedral set and $\eta \in PS_C(C)$ a piecewise smooth form on C. Then a polyhedral complex structure $\mathcal{C}$ for C is called *subordinate to* η if the restrictions $\eta|_\sigma$ are Lagerberg forms for all $\sigma \in \mathcal{C}$.

Definition 4.2.5. Let $C \subseteq \mathbb{R}^n$ be a polyhedral set. The *local dimension* of C in a point $x \in C$ is defined as

$$\dim_x C := \max_{\substack{\sigma \subseteq C \text{ polyhedron} \\ x \in \sigma}} \dim \sigma.$$

The *dimension of* C is defined as $\dim C := \max_{x \in C} \dim_x C$. We call C *pure of dimension* d if $\dim_x C = d$ for all $x \in C$.

Remark 4.2.6. The above definition can be extended from polyhedral sets $C \subseteq \mathbb{R}^n$ to more general piecewise linear spaces X.

Definition 4.2.7. Let $C \subseteq \mathbb{R}^n$ be a purely d-dimensional polyhedral set. Then a *familiy of weights for* C is the datum - up to refinement - of a locally finite decomposition $C = \bigcup_{i \in I} \sigma_i$ of C into d-dimensional polyhedra $\sigma_i \subseteq \mathbb{R}^n$ and weights $\mu = (\mu_i)_{i \in I}$ for $(\sigma_i)_{i \in I}$ such that

$$\mu_i|_{\sigma_i \cap \sigma_j} = \mu_j|_{\sigma_i \cap \sigma_j} \text{ whenever } \dim(\sigma_i \cap \sigma_j) = d.$$

The pair (C, μ) is then called a *weighted polyhedral set.*

Definition 4.2.8. Enhancing Definition 4.1.7, for all weighted polyhedral sets (C, μ), there is a natural integral

$$\int_{[C,\mu]} : PS_c^{d,d}(C) \to \mathbb{R}.$$

Remark 4.2.9. i) Note that by convention, all weighted polyhedral sets are already pure dimensional.

ii) Let (C, μ) be a d-dimensional weighted polyhedral set and let D be another pure dimensional polyhedral set, together with an isomorphism of piecewise linear spaces $f : (C, \Lambda_C) \xrightarrow{\sim} (D, \Lambda_D)$. Then there is a unique way to define a weight $f_*\mu$ for D such that for all $\eta \in PS_c^{d,d}(D)$, the equality

$$\int_{[D,f_*\mu]} \eta = \int_{[C,\mu]} f^*\eta$$

holds. To construct this weight $f_*\mu$ for D, let $C = \bigcup_{i \in I} \sigma_i$ be a decomposition of C into d-dimensional polyhedra, fine enough such that $f|_{\sigma_i}$ is linear for all $i \in I$. Then, the push-forward weights are given by $f_*\mu = (f(\mu_i))_{i \in I}$, where for all $i \in I$, $f(\mu_i) \in \det N_{f(\sigma_i)}$ denotes the image of μ_i under the corresponding map of determinant spaces.

iii) Let (X, Λ_X) be a piecewise linear space of some pure dimension. Using ii), the definition of being weighted extends from polyhedral sets to piecewise linear spaces, such that for every d-dimensional weighted polyhedral set $(C, \mu) \subseteq X$ and every $\alpha \in PS_c^{p,q}(C)$, there is a linear functional

$$\alpha \wedge [C, \mu] : PS_c^{d-p,d-q}(X) \to \mathbb{R}, \eta \mapsto \int_{[C,\mu]} \alpha \wedge \eta|_C.$$

Definition 4.2.10. Let X be a weighted piecewise linear space, and $U \subseteq X$ an open subset. Then a *polyhedral current on* U is defined to be an element of $\mathrm{Hom}(PS_c(U), \mathbb{R})$ which is a locally finite sum of currents of the form $\alpha \wedge [C, \mu]$ for a weighted polyhedral set (C, μ) in X and a piecewise smooth form $\alpha \in PS_c(C)$.

Remark 4.2.11. Let X be a weighted piecewise linear space. Since by [Mih21, Proposition 2.2], piecewise linear spaces admit piecewise smooth partitions of unity, polyhedral currents form a sheaf on X, which is denoted by P or P_X. Furthermore, there is a grading $P_X = \bigoplus_{p,q,d \in \mathbb{N}} P_d^{p,q}$, where the sheaf $P_d^{p,q}$ is generated by the currents of the form $\alpha \wedge [C, \mu]$ for a d-dimensional weighted polyhedral complex (C, μ) in X and $\alpha \in PS_c^{p,q}(C)$.

Remark 4.2.12. As an extension of Remark 4.1.9, we consider a linear map $f : \sigma \to \mathbb{R}^r$ on a d-dimensional polyhedron $\sigma \subseteq \mathbb{R}^n$. Let μ be a weight for σ, ν a weight for $f(\sigma)$, and let δ be the corresponding fibre weight with the property that $\mu = \nu \wedge \delta$ in the sense of Remark 4.1.9. Then fibre integration is the unique map

$$f_* : PS_c(\sigma) \to PS(f(\sigma))$$

with the following properties (see [Mih21, (2.14), (2.15)]):

i) For $\alpha \in PS_c^{d,d}(\sigma)$, the image $f_*\alpha \in PS^{0,0}(f(\sigma))$ is the piecewise smooth function on $f(\sigma)$ which is pointwise given by

$$(f_*\alpha)(y) := \int_{[f^{-1}(y),\delta]} \alpha|_{f^{-1}(y)} \text{ for all } y \in f(\sigma), \tag{4.3}$$

where (4.3) is defined to be zero if $f^{-1}(y)$ has dimension less than d.

ii) The projection formula

$$f_*(f^*\alpha \wedge \beta) = \alpha \wedge f_*\beta$$

holds for all $\alpha \in PS^{d-p,d-q}(f(\sigma))$ and $\beta \in PS_c^{p,q}(\sigma)$.

Proposition 4.2.13. *Let $f : X \to Y$ be a map of weighted piecewise linear spaces and let $T \in P_d^{p,q}(X)$ be a polyhedral current on X such that* $\mathrm{supp}(T) \to Y$ *is proper. Then there exists a polyhedral current $f_*T \in P(Y)$ on Y such that for all $\eta \in PS^{d-p,d-q}(Y)$, the equality*

$$(f_*T)(\eta) = T(f^*\eta)$$

holds.

Proof. [Mih21, Proposition 2.8] □

Definition 4.2.14. Let $f : (X, \mu) \to (Y, \nu)$ be a map of weighted piecewise linear spaces. The map f is said to be *flat* if for every $\eta \in PS_c(X)$ there exists a piecewise smooth form $f_*\eta \in PS(Y)$ such that

$$f_* (\eta \wedge [X, \mu]) = (f_*\eta) \wedge [Y, \nu] .$$

In that case, the element $f_*\eta$ is called the *push-forward of η along f*.

Proposition 4.2.15. *Let $f : (X, \mu) \to (Y, \nu)$ be a flat map of weighted piecewise linear spaces, and $T \in P(Y)$ a polyhedral current on Y. Then there is a polyhedral current $f^*T \in P(X)$ such that for all $\alpha \in PS_c(X)$, the formula*

$$(f^*T)(\alpha) = T(f_*\alpha)$$

holds.

Proof. [Mih21, Proposition 2.14] □

Definition 4.2.16. Let (X, Λ_X) be a piecewise linear space and $L \subseteq \Lambda_X$ a subsheaf. Then, for any polyhedral set $i : K \subseteq X$, we denote by

$$L|_K : \operatorname{Im}\left(i^{-1} : L \to \mathcal{C}(-, \mathbb{R})\right)$$

the *restriction of L to K* and by $L(K) = L|_K(K)$ the space of piecewise linear functions on K that locally extend to a section of L.

Definition 4.2.17. Let (X, Λ_X) be a piecewise linear space. A *sheaf of linear functions on X* is a subsheaf of $\mathbb{R}$-vector spaces $L \subseteq \Lambda_X$ such that the following holds:

i) L contains all constants, i.e. $\mathbb{R} \subseteq L$.

ii) For every $x \in X$ there exists a polyhedral neighbourhood K, some $r \in \mathbb{N}$ and $f = (f_1, \dots, f_r) \in L(K)^r$ such that $f : K \to \mathbb{R}$ has finite fibres.

iii) For every polyhedral set K in X, the restriction $L|_K$ is finitely generated in the following sense: There is a polyhedral structure $\mathcal{K}$ for K, called *subordinate to L*, such that for all open subsets $U \subseteq X$, every $\phi \in L(U)$ and every $\sigma \in \mathcal{K}$, the restriction $\phi|_{\sigma \cap U}$ is linear.

Definition 4.2.18. i) A *piecewise linear space with linear functions* is a tuple (X, L_X) where $X = (X, \Lambda_X)$ is a piecewise linear space and L_X is a sheaf of linear functions on X.

ii) Let (X, L_X) and (Y, L_Y) be piecewise linear spaces with linear functions. Then a *linear map* $f : X \to Y$ is defined to be a map of piecewise linear spaces such that $L_Y \circ f \subseteq L_X$.

Example 4.2.19. Considering $\mathbb{R}^r$ as a piecewise linear space with the usual sheaf of linear functions, a linear map $x : X \to \mathbb{R}^r$ from some piecewise linear space with linear functions X is the same as an r-tuple $(x_1, \dots, x_r) \in L(X)^r$ of linear functions.

Definition 4.2.20. Let (X, L_X) be a piecewise linear space with linear functions. A *smooth* (p,q)*-form on* X is a piecewise smooth form on X that is locally on $U \subseteq X$ open of the form $x^*\alpha'$ for a linear map $x : U \to \mathbb{R}^r$ and a Lagerberg form $\alpha' \in \mathcal{A}^{p,q}(\mathbb{R}^r)$. The resulting *sheaf of smooth* (p,q)*-forms on* X is denoted by $\mathcal{A}_X^{p,q}$ or $\mathcal{A}^{p,q}$. Furthermore, we write $\mathcal{A}_X := \mathcal{A} := \bigoplus_{p,q\in\mathbb{N}} \mathcal{A}_X^{p,q}$ for the sheaf of smooth forms on X.

Remark 4.2.21. Let (X, L_X) and (Y, L_Y) be piecewise linear spaces with linear functions. Then by [Mih21, §2], the following holds:

i) Smooth forms on X are stable under the $\wedge$-product and the polyhedral derivatives d'_P and d''_P of piecewise smooth forms. For a smooth form α on X, we write $d'\alpha$ resp. $d''\alpha$ for the polyhedral derivatives of $\alpha \in \mathcal{A}_X(X) \subseteq PS_X(X)$.

ii) For a linear map $f : (X, L_X) \to (Y, L_Y)$ and a smooth form $\alpha \in \mathcal{A}_Y(Y)$, the *pull-back* $f^*\alpha$ *of* α *via* f is defined as the pull-back of α viewed as a piecewise smooth form on Y. We note here that the pull-back of a smooth form on Y is again a smooth form on X since for any tuple $x \in L(Y)^r$ of linear functions and any Lagerberg form $\alpha' \in \mathcal{A}(\mathbb{R}^r)$, the equality $f^*(x^*\alpha') = (x \circ f)^* \alpha'$ holds.

Remark 4.2.22. Let (X, L_X) be a piecewise linear space with linear functions and $U \subseteq X$ an open subset. We equip the smooth forms with compact support on U, hence $\mathcal{A}_c(U) \subseteq \mathcal{A}_X(U)$, with the following topology: A sequence $(\alpha_i)_{i\in I}$ of smooth forms on U is said to be convergent to a smooth form α on U if and only if the following holds: There exist finitely many compact polyhedral sets $K_1, \ldots, K_n \subseteq X$ and presentations $K_l \simeq \bigcup_{j\in J_l} \sigma_{lj} \subseteq \mathbb{R}^{r_l}$ as unions of polyhedra such that

i) $\operatorname{supp}(\alpha), \operatorname{supp}(\alpha_i) \subseteq K_1 \cup \ldots \cup K_n$ for all $i \in I$;

ii) for all $i \in I, l \in \{1, \ldots, n\}, j \in J_l$, the restriction $\alpha_i|_{\sigma_{lj}}$ is smooth, i.e. a Lagerberg form in $\mathcal{A}(\sigma_{lj})$;

iii) for all $l \in \{1, \ldots, n\}, j \in J_l$, the sequences $\left(\alpha_i|_{\sigma_{lj}}\right)_{i\in I}$ converge to $\alpha|_{\sigma_{lj}}$ in the Schwartz sense, i.e. all higher partial derivatives of all coefficients of $\alpha_i|_{\sigma_{lj}}$ converge uniformly.

Definition 4.2.23. Let (X, L_X) be a purely n-dimensional piecewise linear space with linear functions.

i) We define the *sheaf of* (p,q)*-currents* $\mathcal{D}^{p,q} = \mathcal{D}_X^{p,q}$ *on* X by mapping open subsets U of X to the space of continuous linear functionals on the space of smooth $(n-p, n-q)$-forms $\mathcal{A}_c^{n-p,n-q}(U)$ on U with compact support.

ii) We endow the space of currents on X with partial derivatives d' and d'' which are defined by duality as

$$\left(d'T\right)(\eta) = (-1)^{\deg T+1} T(d'\eta) \text{ and } \left(d''T\right)(\eta) = (-1)^{\deg T+1} T(d''\eta)$$

for a current T and a smooth form η on X.

iii) We define an associative pairing

$$\mathcal{A}_X^{p_1,q_1} \times \mathcal{D}_X^{p_2,q_2} \to \mathcal{D}_X^{p_1+p_2,q_1+q_2}$$

by $(\alpha, T) \mapsto (-1)^{\deg\alpha\cdot\deg T} T(\alpha\wedge -)$.

iv) Let $f : (X, L_X) \to (Y, L_Y)$ be a linear map of piecewise linear spaces with linear functions. We define the *push-forward* $f_* : \mathcal{D}_c(X) \to \mathcal{D}_c(Y)$ of currents with compact support as the dual of pull-back of smooth forms on piecewise linear spaces.

Remark 4.2.24. In the situation of Definition 4.2.23, the restriction maps $\mathcal{D}_X^{p,q}(U) \to \mathcal{D}_X^{p,q}(V)$ are given as the dual maps to the inclusions $\mathcal{A}_c(V) \to \mathcal{A}_c(U)$ of smooth forms, for all open subsets $V \subseteq U \subseteq X$, see [Mih21, Proposition 2.19].

Remark 4.2.25. Let (X, L_X) be a piecewise linear space with linear functions. Then every polyhedral current on X defines a current on X, and one can show that this realizes $P_X(U)$ as a subset of $\mathcal{D}_X(U)$ for all open subsets U of X. In particular, we obtain derivatives d' and d'' on polyhedral currents on X (which do not coincide with d'_P and d''_P in general).

Remark 4.2.26. Let (X, L_X) be a piecewise linear space with linear functions. Then there is a balancing condition for polyhedral currents $T \in P_X(X)$. It demands that T is balanced on the interior of all polyhedral sets contained in X. For a polyhedral set C in X, being balanced means the following: Let $\mathcal{C}$ be a polyhedral complex structure that is subordinate to T and L_X, and write $T = \sum_{\sigma\in\mathcal{C}} \alpha_\sigma \wedge [\sigma, \mu_\sigma]$. Then the condition is that for all $U \subseteq C$ open, $\tau \in \mathcal{C}$ and $\phi \in L_X(U)$ with $\phi|_{\tau\cap U}$ constant the sum

$$\sum_{\substack{\sigma\in\mathcal{C}\\ \tau\preceq\sigma\text{facet}}} \frac{\partial\phi|_\sigma}{\partial n_{\sigma,\tau}} \alpha_\sigma|_\tau$$

vanishes, where the normal vectors $n_{\sigma,\tau}$ are taken with respect to the weights μ_τ and μ_σ.

Proposition 4.2.27. *Let (X, L_X) be a piecewise linear space with linear functions, and let $T \in P_X(X)$. Then the following are equivalent:*

i) T is balanced

ii) Both derivatives $d'T$ and $d''T$ are again polyhedral.

iii) One out of $d'T$ and $d''T$ is polyhedral.

iv) For every compact polyhedral set $K \subseteq X$ and every tuple of linear functions $f \in L|_K(K)^r$, the push-forward $f_*\left(T|_{K\setminus f^{-1}(f(\partial K))}\right)$ is balanced (i.e. a δ-form) away from $f(\partial K)$.

Proof. [Mih21, Proposition 2.23] $\square$

Definition 4.2.28. A *tropical space* is a triple (X, μ, L) that consists of a weighted piecewise linear space (X, μ) with linear functions L such that the fundamental cycle $[X, \mu]$ - viewed as a current for smooth forms on X with respect to L - is closed with respect to d' and d''.

Remark 4.2.29. Let (X, μ) be a weighted piecewise linear space with linear functions L. Then by [Mih21, Corollary 2.26], the following are equivalent:

i) The triple (X, μ, L) is a tropical space.

ii) $[X, \mu]$ is balanced with respect to L.

iii) For every compact polyhedral set $K \subseteq X$ and every linear map $f : K \to \mathbb{R}^r$, the push-forward $f_* \left(K, \mu|_{K \setminus f^{-1}(f(\partial K))}\right)$ is balanced away from $f(\partial K)$.

Definition 4.2.30. Let (X, L_X) be a piecewise linear space with linear functions. Then a *refinement* of a linear map $f : X \to \mathbb{R}^r$ is a pair (g, p), where $g : X \to \mathbb{R}^s$ is a linear map and $p : \mathbb{R}^s \to \mathbb{R}^r$ is an affine map such that $f = p \circ g$.

Lemma 4.2.31. *Let X be a piecewise linear space, $T \in P(X)$ a polyhedral current on X and $f : X \to \mathbb{R}^r$ a piecewise linear map with finite fibres. Then T is uniquely determined by all push-forwards $f_* \left(T|_{K \setminus f^{-1}(f(\partial K))}\right)$ for $K \subseteq X$ a compact polyhedral set.*

Proof. [Mih21, Lemma 2.28] □

Proposition 4.2.32. *Let (X, μ, L) be an n-dimensional tropical space, $f : X \to \mathbb{R}^r$ a linear map and $\gamma \in B(\mathbb{R}^r)$ a δ-form. Then there is a unique polyhedral current $f^\star\gamma \in P(X)$ such that for all compact polyhedral sets K and all refinements (g, p) of f such that $g : K \to \mathbb{R}^s$ has finite fibres, the identity*

$$g_* \left((f^\star\gamma) |_{K \setminus g^{-1}(g(\partial K))} \right) = g_* K \wedge p^* T \tag{4.4}$$

holds away from $g(\partial K)$. On the right-hand-side, the $\wedge$-product is meant in the sense of δ-forms, where $g_* K$ is viewed as tropical cycle via Remark 4.2.29 (away from $g(\partial K)$).

Proof. [Mih21, Proposition 2.29] □

Definition 4.2.33. Let (X, μ, L) be a tropical space.

i) A *δ-form* on (X, μ, L) is a polyhedral current on X which is locally of the form $f^\star\gamma$ for a linear map f mapping to $\mathbb{R}^r$ for some $r \in \mathbb{N}$, and a δ-form $\gamma \in B(\mathbb{R}^r)$. The resulting *sheaf of δ-forms* on X is denoted by B_X or B.

ii) A δ-form ω on X is said to have *degree* (p, q, r) if this is the case for T viewed as a polyhedral current. We denote the *sheaf of δ-forms of degree* (p, q, r) on X by $B_X^{p,q,r}$ or $B^{p,q,r}$.

Remark 4.2.34. Let (X, μ, L) be a tropical space. Then a δ-form ω on X is of degree (p, q, r) if and only if ω is locally of the form $f^\star\gamma$ for a linear map f to some $\mathbb{R}^r$ and a δ-form γ on $\mathbb{R}^r$ of degree (p, q, r). Furthermore, there is a direct sum decomposition $B_X = \bigoplus_{p,q,r \in \mathbb{N}} B_X^{p,q,r}$.

Remark/Definition 4.2.35. Let (X, μ, L) be a tropical space. There is a $\wedge$-product and differential operators $d', d'', d'_P, d''_P, \partial'$ and ∂'' on δ-forms on X, obtained from the corresponding operators on δ-forms on spaces $\mathbb{R}^r$: They are computed locally in charts as

$$(f^\star\gamma_1) \wedge (f^\star\gamma_2) := f^\star (\gamma_1 \wedge \gamma_2) \text{ and } d\,(f^\star\gamma) := f^\star\,(d\gamma)$$

for $d \in \{d', d'', d'_P, d''_P, \partial', \partial''\}$, linear maps f to some $\mathbb{R}^r$ and δ-forms $\gamma, \gamma_1, \gamma_2 \in B(\mathbb{R}^r)$.

Proposition 4.2.36. *Let (X, μ, L) be a tropical space of dimension n. Then for any δ-form $\omega \in B_X(X)$ on X, its derivatives $d'\omega$ and $d''\omega$ as current and as δ-form agree. In particular, Stokes' formula holds, i.e.*

$$\int_X d'\alpha = 0 = \int_X d''\beta$$

for all $\alpha \in B_c^{n-1,n}(X)$ and $\beta \in B_c^{n,n-1}(X)$.

Proof. [Mih21, Proposition 2.31] □

Proposition 4.2.37. *Let $\phi : X \to Y$ be a flat linear map of tropical spaces. Then the pull-back $\phi^*\omega$ of any δ-form $\omega \in B_Y(Y)$ is again a δ-form on X. More precisely, if $\omega \in B_Y(Y)$ is locally given by a linear map f to some $\mathbb{R}^r$ and a δ-form $\gamma \in B(\mathbb{R}^r)$, then*

$$\phi^*\,(f^\star\gamma) = (f \circ \phi)^\star\,(\gamma).$$

Proof. [Mih21, Proposition 2.43] □

Remark/Definition 4.2.38. Let X be a compact purely n-dimensional analytic space, and let $f : X \to \mathbb{G}_m^{r,\mathrm{an}}$ be tropical coordinates on X. We denote by $T(X,f) \subseteq T'(X,f)$ the n-dimensional locus of the tropical variety $T'(X,f) = t_f(X)$ from Remark 2.2.9, i.e. the union of all n-dimensional polyhedra contained in $T'(X,f)$. Following [Mih21, Definition 3.3, Theorem 3.4] resp. [CD12, §2], there exists a weighted polyhedral complex structure $(\mathcal{T}, \mu)$ for $T(X,f)$ such that $(T(X,f), \mu)$ is a tropical cycle away from $t_f(\partial X)$, obtained as follows: We choose a polyhedral complex structure $\mathcal{T}$ for $T(X,f)$ such that $t_f(\partial X)$ is contained in its $(n-1)$-skeleton. For $\sigma \in \mathcal{T}_n$ we choose a surjective morphism $q : \mathbb{G}_m^{r,\mathrm{an}} \to \mathbb{G}_m^{n,\mathrm{an}}$ such that $q|_\sigma$ is injective, where q also denotes the induced map $q : \mathbb{R}^r \to \mathbb{R}^n$ on tropicalizations. Then it can be shown that the map

$$q \circ f|_{t_f^{-1}(\sigma)} : t_f^{-1}(\sigma) \to \mathbb{G}_m^{n,\mathrm{an}}(q(\sigma))$$

is finite flat of some degree $d_{\sigma,q}$ over a neighbourhood of the relative interior $q(\sigma^\circ)$, viewed as a subset of the canonical skeleton $\Sigma(\mathbb{G}_m^n)$ via $\iota_{\mathbb{G}_m^n}$. The polyhedron $\sigma \in \mathcal{T}_n$ is then endowed with the weight $\mu_\sigma := d_{\sigma,q} \cdot q^{-1}(\mu_{\mathbb{R}^n})$, where $\mu_{\mathbb{R}^n}$ denotes the standard weight on $\mathbb{R}^n$ induced by the lattice $\mathbb{Z}^n \subseteq \mathbb{R}^n$, see [CD12, (2.4.4)]. This construction is independent of the choice of q, see [GJR21, §4].

Remark/Definition 4.2.39. Let X be a compact analytic space of pure dimension n, and $f : X \to \mathbb{G}_m^{r,\mathrm{an}}$ tropical coordinates on X. By [Duc12, Théorème 5.1], the closed subspace

$$\Sigma'(X,f) = \bigcup_{q:\mathbb{G}_m^{r,\mathrm{an}} \to \mathbb{G}_m^{n,\mathrm{an}}} (q \circ f)^{-1} (\Sigma(\mathbb{G}_m^n))$$

of X is naturally a piecewise linear space, where q runs through all morphisms $q : \mathbb{G}_m^{r,\mathrm{an}} \to \mathbb{G}_m^{n,\mathrm{an}}$ of analytic tori. By [Mih21, Theorem 3.1], $\Sigma'(X,f)$ is of dimension at most n, and $\Sigma'(X,f) \cap \partial X$ is contained in a piecewise linear subspace of dimension at most $n-1$. Let $\Sigma(X,f)$ denote the n-dimensional locus of $\Sigma'(X,f)$, called *skeleton of f in X*.

Remark 4.2.40. In the situation of Remark/Definition 4.2.39, [GJR21, Proposition 3.12] states that $\Sigma'(X,f) \setminus \partial X$ agrees with all $x \in X \setminus \partial X$ such that $\dim t_f(U) = n$ for all neighbourhoods U of x in X. In particular $\Sigma'(X,f) \setminus \partial X$ only depends on the tropicalization map t_f (and not on the moment map itself). Furthermore, the equality

$$\Sigma'(X,f) \setminus \partial X = \Sigma(X,f) \setminus \partial X$$

holds.

Remark/Definition 4.2.41. Let X be a compact analytic space of pure dimension n, and $f : X \to \mathbb{G}_m^{r,\mathrm{an}}$ tropical coordinates on X. Then there exists a weighted piecewise linear space structure $(\mathcal{T}, \mu)$ for $\Sigma(X,f)$ such that for the sheaf L on $\Sigma(X,f)$ generated by $t_{f_1}, \ldots, t_{f_r}$, the triple $(\Sigma(X,f) \setminus \partial X, \mu, L)$ is a tropical space. Here t_{f_i} is the composition $\pi_i \circ t_f$ of the tropicalization map t_f with the projection $\pi_i : \mathbb{R}^r \to \mathbb{R}$ onto the i-th coordinate for all $i \in \{1, \ldots, r\}$. The construction of $(\mathcal{T}, \mu)$ is as follows: Working locally, we may assume that $\Sigma(X,f)$ is a polyhedral set. Let $\mathcal{T}$ be a polyhedral complex structure for $\Sigma(X,f)$ such that $\Sigma(X,f) \cap \partial X$ is contained in its $(n-1)$-skeleton. For each $\sigma \in \mathcal{T}_n$ we choose a morphism $q : \mathbb{G}_m^{r,\mathrm{an}} \to \mathbb{G}_m^{n,\mathrm{an}}$ such that $(q \circ f)\,|_\sigma$ is injective. Then the weight μ_σ on σ is defined as $\mu_\sigma := d_{\sigma,q} \cdot t_{q\circ f}^{-1}(\mu_{\mathbb{R}^n})$, where $d_{\sigma,q}$ is the degree of X over $(q \circ f)(\sigma) \subseteq \Sigma(\mathbb{G}_m^{n,\mathrm{an}})$ near the interior of σ. The independence of the choice of q is shown in [GJR21, §4], and also a more detailed construction can be found in [GJR21, §3, §4].

Remark 4.2.42. Let X be a compact analytic space of pure dimension n, and $f : X \to \mathbb{G}_m^{r,\mathrm{an}}$ tropical coordinates on X. Then there is an equality of tropical cycles

$$T(X,f) = (t_f)_* \Sigma(X,f),$$

away from $t_f(\Sigma(X,f) \cap \partial X)$, see [Mih21, (3.3), Corollary 3.7].

Remark 4.2.43. Proposition 2.2.19 gives the following: Let $h : X \to \mathbb{R}^n$ be a piecewise linear tropicalization map on a compact purely d-dimensional analytic space. Then there exists a $(\mathbb{Z}, \Gamma)$-polytopal complex Π of dimension at most

d with support $h(X)$ such that $h(\partial X)$ is contained in a subcomplex of dimension at most $d-1$, and such that using G-locally tropical multiplicities, there are weights μ_σ for each d-dimensional polyhedron in the polyhedral complex structure, such that $h(X)_d$ satisfies the balancing condition away from $h(\partial X)$. The construction of the polyhedral complex structure and the weights is done in [GJR21, §8], and the proof of the balancing property is [GJR21, Theorem 9.7]. We consider $h(X)_d$ in the subsequent always with those weights.

Remark 4.2.44. We assume that $\partial X = \emptyset$. Then each harmonic tropicalization map $h : X \to \mathbb{R}^n$ defines a tropical space, where the sheaf of linear functions is the one generated by the harmonic functions $h_1, \dots h_n$.

4.2.2 The sheaf of δ-forms on Berkovich spaces

The aim of this section is to finally introduce δ-forms on non-archimedean analytic spaces following [Mih21, §4]. This theory relies on the theory of δ-forms on tropical spaces from the previous section. The existence of differential operators and wedge products can be deduced from the theory of δ-forms on tropical spaces introduced before. Furthermore, we will give some examples of already known classes of δ-forms in this section. In particular, the in Chapter 2.2 considered sheaf of weakly smooth forms by [GJR21] can naturally be embedded into the sheaf of δ-forms on Berkovich analytic spaces, as well as the sheaf of piecewise smooth and the sheaf of smooth forms, which we will define here. The fact that weakly smooth forms can be considered as δ-forms allows to pair weakly smooth forms with δ-forms later, which is essential for the proof of the main theorem 5.2.8.

Definition 4.2.45. Let $H \subseteq K \subseteq \mathbb{R}^m$ be polyhedral sets and $T \in P_K(K)$ a polyhedral current on K. Let $\mathcal{K}$ be a polyhedral complex structure for K which is also subordinate to T and H, and let $\alpha_\sigma \in \mathcal{A}(\sigma)$ be Lagerberg forms for all $\sigma \in \mathcal{K}$ such that $T = \sum_{\sigma \in \mathcal{K}} \alpha_\sigma \wedge [\sigma, \mu_\sigma]$ for weights μ_σ on each polyhedron $\sigma \in \mathcal{K}$. Then the *(polyhedral) restriction of T to H* is defined as

$$T|_H^P := \sum_{\substack{\sigma \in \mathcal{K} \\ \sigma \subseteq H}} \alpha_\sigma \wedge [\sigma, \mu_\sigma] \in P_H(H).$$

Remark 4.2.46. In the situation of Definition 4.2.45, the term $T|_H^P$ is independent of the choice of the polyhedral complex $\mathcal{K}$. Furthermore, given an isomorphism $K \simeq K' \subseteq \mathbb{R}^r$ of polyhedral sets, the image of H under this map is also a polyhedral set since $\Lambda_K|_H = \Lambda_H$. Then the definition of $T|_H^P$ is independent of the representation of K as a polyhedral set. This gives that for inclusions of piecewise linear spaces $(Y, \Lambda_Y) \subseteq (X, \Lambda_X)$ with the property that $(Y, \Lambda_X|_Y)$ is also a piecewise linear space, the above construction defines a *(polyhedral) restriction*

$$P_X(X) \to P_Y(Y), T \mapsto T|_Y^P,$$

which is locally given as in Definition 4.2.45.

Lemma 4.2.47. *Let (Σ, μ, L) be an n-dimensional tropical space and let $f = (f_1, \ldots, f_r) \in L(\Sigma)^r$ be a tuple of linear functions on Σ. We denote by Σ' the union of all n-dimensional polyhedra σ of Σ such that $f|_\sigma$ is injective. Let $\gamma \in B(\mathbb{R}^r)$ be a δ-form. Then the following holds:*

i) The triple (Σ', μ', L') for $\mu' := \mu|_{\Sigma'}$ and $L' := \mathbb{R} + \langle f_1, \ldots, f_r\rangle$ is a tropical space.

ii) There is an equality of polyhedral currents

$$f^\star(\gamma)|^P_{\Sigma'} = (f|_{\Sigma'})^\star(\gamma).$$

Remark 4.2.48. Let X be an analytic space. For an open subset U of X and tropical coordinates f and g on U such that g refines f, the inclusion of skeletons

$$\Sigma(U, f) \subseteq \Sigma(U, g)$$

is of the type considered in Lemma 4.2.47 with respect to the tuple of linear functions given by t_f. This follows from Remark 4.2.40.

Remark/Definition 4.2.49. Let U be an open subset of an analytic space X, and let f and g be tropical coordinates on U. We assume that g maps to $\mathbb{R}^r$ and let $\gamma \in B(\mathbb{R}^r)$ be a δ-form. We define a polyhedral current, the restriction $t_g^\star(\gamma)|_{\Sigma(U,f)}$ of $t_g^\star(\gamma)$ from $\Sigma(U, g)$ to $\Sigma(U, f)$ as follows:

i) First we consider the case that g refines f. Then $\Sigma(U, f) \subseteq \Sigma(U, g)$ is an inclusion of tropical spaces as in Lemma 4.2.47. In this case, we set

$$t_g^\star(\gamma)|_{\Sigma(U,f)} := t_g^\star(\gamma)|^P_{\Sigma(U,f)}.$$

ii) Next we consider the case that g does not refine f. We choose any common refinement (h, p) of both f and g and set

$$t_g^\star(\gamma)|_{\Sigma(U,f)} := t_h^\star(p^*\gamma)|^P_{\Sigma(U,f)}.$$

This is well-defined by Lemma 4.2.47.

Definition 4.2.50. Let X be an analytic space. A *skeleton in X* is a locally closed subset $\Sigma \subseteq X$ which is locally a piecewise linear subspace of some $\Sigma(U, f)$ for an open subset U of X and tropical coordinates f on U.

Remark 4.2.51. We observe that there is no natural tropical space structure on skeletons in analytic spaces. They might not even be pure dimensional.

Remark 4.2.52. Let X be an analytic space and let U be an open subset of X together with a tropical coordinates $f : U \to \mathbb{G}_m^{r,\mathrm{an}}$ and a δ-form $\gamma \in B(\mathbb{R}^r)$. Let Σ be a skeleton in X with $\Sigma \subseteq U$. We obtain by locality a well-defined restriction $t_f^\star(\gamma)|_\Sigma$.

Definition 4.2.53. Let X be an analytic space of pure dimension n with $\partial X = \emptyset$.

i) A *δ-form on X* is the datum of a polyhedral current ω_Σ for every skeleton Σ in X such that every $x \in X$ has an open neighbourhood U together with tropical coordinates $f : U \to \mathbb{G}_m^{r,\mathrm{an}}$ and a δ-form $\gamma \in B(\mathbb{R}^r)$ such that for all skeletons Σ in X with $\Sigma \subseteq U$:

$$\omega_\Sigma = t_f^\star(\gamma)|_\Sigma.$$

ii) For a δ-form $\omega = (\omega_\Sigma)_{\Sigma \subseteq X \text{ skeleton}}$ we write $\omega|_\Sigma := \omega_\Sigma$ for every skeleton Σ in X.

iii) For an open subset U of X together with tropical coordinates $f : U \to \mathbb{G}_m^{r,\mathrm{an}}$ and a δ-form $\gamma \in B(\mathbb{R}^r)$ we denote the δ-form which is given by the assignment

$$\Sigma \mapsto t_f^\star(\gamma)|_\Sigma \tag{4.5}$$

for any skeleton Σ in X by $t_f^\star(\gamma)$.

Remark/Definition 4.2.54. Let X be an analytic space of pure dimension n with $\partial X = \emptyset$.

i) Definition 4.2.53 gives rise to a sheaf, the *sheaf of δ-forms on X* which is denoted by B_X or B. There is a natural trigrading $B_X = \bigoplus_{p,q,r \in \mathbb{N}} B_X^{p,q,r}$ stemming from the trigrading of δ-forms on real vector spaces $\mathbb{R}^r$.

ii) For a tropical space (X, μ, L), the operators $\wedge, d', d'', d'_P, d''_P, \partial'$ and ∂'' on δ-forms on (X, μ, L) were defined through charts. This fact together with Lemma 4.2.47 yields that they all commute with formation of the polyhedral restriction of polyhedral currents. We endow the sheaf B_X of δ-forms on the analytic space X with those seven operators by setting

$$\begin{aligned}
(t_f^\star\alpha) \wedge (t_f^\star\beta) &:= t_f^\star(\alpha \wedge \beta), & d'(t_f^\star\alpha) &:= t_f^\star(d'\alpha)\\
d''(t_f^\star\alpha) &:= t_f^\star(d''\alpha), & d'_P(t_f^\star\alpha) &:= t_f^\star(d'_P\alpha)\\
d''_P(t_f^\star\alpha) &:= t_f^\star(d''_P\alpha), & \partial'(t_f^\star\alpha) &:= t_f^\star(\partial'\alpha) \text{ and}\\
\partial''(t_f^\star\alpha) &:= t_f^\star(\partial''\alpha)
\end{aligned}$$

for all open subsets $U \subseteq X$, tropical coordinates $f : U \to \mathbb{G}_m^{r,\mathrm{an}}$ and δ-forms $\alpha, \beta \in B(\mathbb{R}^r)$.

Remark 4.2.55. Let X be a pure dimensional analytic space with $\partial X = \emptyset$ and $\omega \in B_X(X)$ a δ-form on X. Then he restrictions $(\omega|_\Sigma)_{\Sigma \subseteq X \text{skeleton}}$ form a compatible family in the sense that for all inclusions of skeletons $\Sigma' \subseteq \Sigma \subseteq X$, there is an equality

$$(\omega|_\Sigma)\,|_{\Sigma'} = \omega_{\Sigma'}.$$

Theorem 4.2.56. *Let $\pi : X \to Y$ be a morphism of pure dimensional analytic spaces with $\partial X = \emptyset = \partial Y$. Then there is a well-defined pull-back map $\pi^* : \pi^{-1}B_Y \to B_X$ which satisfies*

$$\pi^*(t_f^\star(\gamma)) = t_{f\circ\pi}^\star(\gamma)$$

for all open subsets U of X, tropical coordinates $f : U \to \mathbb{G}_m^{r,\mathrm{an}}$ on U and $\gamma \in B(\mathbb{R}^r)$.

Proof. [Mih21, Theorem 4.5]. □

Remark 4.2.57. In the situation of Theorem 4.2.56, the pull-back map π^* commutes with the operators $\wedge, d', d'', d'_P, d''_P, \partial'$ and ∂'' on δ-forms on analytic spaces since they are computed in charts. In the case of immersions $\pi := i : X \hookrightarrow Y$, we write $\omega|_X$ instead of $i^*\omega$ for δ-forms $\omega \in B_Y(Y)$.

Remark/Definition 4.2.58. Let A be an abelian variety that is split over k, and let

$$1 \to \mathbb{T}^{\mathrm{an}} \xrightarrow{\iota} E^{\mathrm{an}} \xrightarrow{\mathfrak{q}} B^{\mathrm{an}} \to 0$$

denote the Raynaud extension of A from Remark/Definition 3.1.4. We recall that there is a morphism $\mathfrak{p} : E^{\mathrm{an}} \twoheadrightarrow A^{\mathrm{an}}$ which is a covering map in the topological sense. Its kernel $\Lambda := \ker(\mathfrak{p})$ is a discrete subgroup of $E(k)$, and $\mathfrak{p}$ induces an isomorphism $E^{\mathrm{an}}/\Lambda \simeq A^{\mathrm{an}}$. For any $a \in \Lambda$, there is a natural translation map $\tau_a : E^{\mathrm{an}} \to E^{\mathrm{an}}$. We call a δ-form $\tilde{\omega} \in B_{E^{\mathrm{an}}}(E^{\mathrm{an}})$ *Λ-invariant* if for all $a \in \Lambda$, the equality $\tau_a^*(\tilde{\omega}) = \tilde{\omega}$ holds. We denote the space of Λ-invariant δ-forms on E^{an} by $B_{E^{\mathrm{an}}}(E^{\mathrm{an}})^\Lambda$.

Lemma 4.2.59. *In the situation of Remark/Definition 4.2.58, every Λ-invariant δ-form $\tilde{\omega} \in B_{E^{\mathrm{an}}}(E^{\mathrm{an}})^\Lambda$ descends to a δ-form $\omega \in B_{A^{\mathrm{an}}}(A^{\mathrm{an}})$ via $\mathfrak{p}$ in the sense that there is a δ-form $\omega \in B_{A^{\mathrm{an}}}(A^{\mathrm{an}})$ such that $\mathfrak{p}^*(\omega) = \tilde{\omega}$.*

Proof. Let $\tilde{\omega} = (\tilde{\omega}_\Sigma)_{\Sigma \subseteq E^{\mathrm{an}} \text{ skeleton}} \in B_{E^{\mathrm{an}}}(E^{\mathrm{an}})^\Lambda$ be a Λ-invariant δ-form on E^{an}. We choose a Λ-periodic open cover $\{\tilde{U}_i\}_{i\in I}$ of E^{an} by Λ-small open subsets together with tropical coordinates $f_i : \tilde{U}_i \to \mathbb{G}_m^{r_i,\mathrm{an}}$ and δ-forms $\gamma_i \in B(\mathbb{R}^{r_i})$ for all $i \in I$ such that for all skeletons Σ in E^{an} with $\Sigma \subseteq \tilde{U}_i$ for some $i \in I$, the equality $\tilde{\omega}_\Sigma = t_{f_i}^\star(\gamma_i)|_\Sigma$ holds. The property Λ-periodic is meant in the sense that for all $i \in I$ and all $a \in \Lambda$ there is some $j \in I$ such that $\tau_a(\tilde{U}_i) = \tilde{U}_j$. Furthermore, for simplicity we assume that for all $i \neq j \in I$, $\tilde{U}_i$ is not properly contained in $\tilde{U}_j$. The existence of such triples $(\tilde{U}_i, f_i, \gamma_i)_{i\in I}$ follows from the definition of δ-forms once we observe that a suitable cover of E^{an} can always be constructed by taking the preimages of the interiors of suitable (refinements of) polyhedra Δ in $N_\mathbb{R}$ as in Lemma 5.2.5, together with all their Λ-translates. For each $i \in I$ we denote by $U_i := \mathfrak{p}(\tilde{U}_i) \in A^{\mathrm{an}}$ the image of $\tilde{U}_i$ under $\mathfrak{p}$. We note here that for all $i \in I$, the sets $\tilde{U}_i$ and U_i are isomorphic via $\mathfrak{p}$ since $\tilde{U}_i$ was chosen to be Λ-small. In this way, we obtain an open cover $\{U_i\}_{i\in I}$ of A^{an}. Now we define a δ-form $\omega \in B_{A^{\mathrm{an}}}(A^{\mathrm{an}})$ by setting

$$\omega_\Sigma := t^\star_{f_i \circ \left(\mathfrak{p}|_{\tilde{U}_i}\right)^{-1}}(\gamma_i)|_\Sigma$$

for all skeletons Σ in A^{an} with $\Sigma \subseteq U_i$ for some $i \in I$. We note here that for all $i \in I$, the restriction $\mathfrak{p}|_{\tilde{U}_i}$ is an isomorphism since $\tilde{U}_i$ is Λ-small. We have to show that this is well-defined. Using the assumptions on our cover $\{\tilde{U}_i\}_{i\in I}$, we only have to show that for $i \neq j \in I$ such that $U_i = \mathfrak{p}(\tilde{U}_i) = \mathfrak{p}(\tilde{U}_j) = U_j$, the equality

$$t^\star_{f_j \circ \left(\mathfrak{p}|_{\tilde{U}_j}\right)^{-1}}(\gamma_j)|_\Sigma = t^\star_{f_i \circ \left(\mathfrak{p}|_{\tilde{U}_i}\right)^{-1}}(\gamma_i)|_\Sigma$$

holds for all skeletons Σ in A^{an} with $\Sigma \subseteq U_i = U_j$. So let $i \neq j \in I$ with $U_i = U_j$. Then, using the assumptions on our covering $\{\tilde{U}_i\}_{i\in I}$, there is some $a \in \Lambda$ such that $\tilde{U}_i = \tau_a(\tilde{U}_j)$, i.e. such that the diagram

$$\begin{tikzcd}
\tilde{U}_i \arrow[rr, "\tau_a"] \arrow[dr, "\mathfrak{p}|_{\tilde{U}_i}"'] & & \tilde{U}_j \arrow[dl, "\mathfrak{p}|_{\tilde{U}_j}"] \\
& U_i = U_j &
\end{tikzcd} \tag{4.6}$$

commutes. Since $\tilde{\omega}$ is a Λ-invariant δ-form on E^{an}, we have that $\tau_a^*\tilde{\omega} = \tilde{\omega}$ and hence in particular

$$\tau_a^*\left(t^\star_{f_j}(\gamma_j)\right) = t^\star_{f_i}(\gamma_i) \tag{4.7}$$

holds on every skeleton Σ in E^{an} with $\Sigma \subseteq \tilde{U}_i$. Together we obtain that for every skeleton Σ in A^{an} with $\Sigma \subseteq U_i = U_j$ we have

$$\begin{aligned}
t^\star_{f_j \circ \left(\mathfrak{p}|_{\tilde{U}_j}\right)^{-1}}(\gamma_j)|_\Sigma &\overset{(4.6)}{=} t^\star_{f_j \circ \tau_a \circ \left(\mathfrak{p}|_{\tilde{U}_i}\right)^{-1}}(\gamma_j)|_\Sigma \\
&\overset{4.2.56}{=} \left(\left(\mathfrak{p}|_{\tilde{U}_i}\right)^{-1}\right)^* \left(t^\star_{f_j\circ\tau_a}(\gamma_j)\right)|_\Sigma \\
&\overset{4.2.56}{=} \left(\left(\mathfrak{p}|_{\tilde{U}_i}\right)^{-1}\right)^* \left(\tau_a^*\left(t^\star_{f_j}(\gamma_j)\right)\right)|_\Sigma \\
&\overset{(4.7)}{=} \left(\left(\mathfrak{p}|_{\tilde{U}_i}\right)^{-1}\right)^* \left(t^\star_{f_i}(\gamma_i)\right)|_\Sigma \overset{4.2.56}{=} t^\star_{f_i \circ \left(\mathfrak{p}|_{\tilde{U}_i}\right)^{-1}}(\gamma_i)|_\Sigma.
\end{aligned}$$

This shows that $\omega \in B_{A^{\mathrm{an}}}(A^{\mathrm{an}})$ is well-defined. Furthermore, for every $i \in I$, on every skeleton Σ in E^{an} with $\Sigma \subseteq \tilde{U}_i$ we have that

$$\begin{aligned}
\tilde{\omega}|_\Sigma = t^\star_{f_i}(\gamma_i)|_\Sigma &= t^\star_{f_i \circ \left(\mathfrak{p}|_{\tilde{U}_i}\right)^{-1} \circ \mathfrak{p}|_{\tilde{U}_i}}(\gamma_i)|_\Sigma \\
&\overset{4.2.56}{=} \left(\mathfrak{p}|_{\tilde{U}_i}\right)^* \left(t^\star_{f_i \circ \left(\mathfrak{p}|_{\tilde{U}_i}\right)^{-1}}(\gamma_i)\right)|_\Sigma = \left(\mathfrak{p}|_{\tilde{U}_i}\right)^* (\omega|_{U_i})|_\Sigma.
\end{aligned}$$

Since $E^{\mathrm{an}} = \bigcup_{i\in I} \tilde{U}_i$ this shows that $\mathfrak{p}^*\omega = \tilde{\omega}$, and the claim follows. $\square$

Definition 4.2.60. Let X be an analytic space. Then Chambert-Loir and Ducros define in [CD12] the sheaf $\mathcal{A}_{sm}$ of smooth forms on X as follows: We consider the presheaf Q on X, where

$$Q(U) = \{(f, \eta) \mid f : U \to \mathbb{G}_m^{s,\mathrm{an}} \text{ moment map}, \eta \in \mathcal{A}(\mathbb{R}^s)\}$$

for every $U \subseteq X$ open. For $U \subseteq X$ open, two elements $(f_1, \eta_1), (f_2, \eta_2) \in Q(U)$ are defined to be equivalent if for every affinoid domain $K \subseteq U$, the equality

$$p_1^*(\eta_1)|_{T'(K,f_1 \times f_2)} = p_2^*(\eta_2)|_{T'(K,f_1 \times f_2)}$$

holds, where $f_1 : U \to \mathbb{G}_m^{s_1,\mathrm{an}}$, $f_2 : U \to \mathbb{G}_m^{s_2,\mathrm{an}}$ and p_1 and p_2 denote the canonical projection maps on $\mathbb{R}^{s_1} \times \mathbb{R}^{s_2}$. The above defines an equivalence relation on Q, and the *sheaf* $\mathcal{A}_{sm}$ *of smooth forms* on X is defined as the sheafification of $Q/\sim$. For $(f, \eta) \in Q(U)$ for some $U \subseteq X$ open, we denote the associated *smooth form* on U by $f^*\eta$.

Proposition 4.2.61. *Let X be an affinoid space, and let $\alpha \in \mathcal{A}_{sm}(X)$ be a smooth form on X. Then there exists a finite family $(g_1, \ldots, g_m)$ of holomorphic functions on X with the following property:*
Let $x \in X$, and let $I := \{i \in \{1, \ldots, m\} \mid g_i(x) \neq 0\}$. Let $U := X \setminus \bigcup_{i \in I} Z(g_i)$ denote the Zariski-open subset of X which is defined as the complement of the union of the zero loci of all g_i for $i \in I$. Then $g := (g_i)_{i \in I} : U \to \mathbb{G}_m^{\#I,\mathrm{an}}$ defines a moment map on U such that $\alpha|_{t_g^{-1}(\Omega)} = g^*(\eta)$ for some open neighbourhood Ω of $t_g(x)$ in $\mathbb{R}^{\#I}$ and some Lagerberg form $\eta \in \mathcal{A}(\mathbb{R}^{\#I})$.

Proof. [CD12, Proposition (3.4.1)] □

Definition 4.2.62. Let X be an analytic space. Then there is a sheaf $PS_X = PS$ of piecewise smooth forms on X defined as follows: We consider the presheaf Q on X, where

$$Q(U) = \{(f, \eta) \mid f : U \to \mathbb{G}_m^{s,\mathrm{an}} \text{ moment map}, \eta \in PS(\mathbb{R}^s)\}$$

for every $U \subseteq X$ open. For $U \subseteq X$ open, two elements $(f_1, \eta_1), (f_2, \eta_2) \in Q(U)$ are defined to be equivalent if for every k-affinoid domain $K \subseteq U$, the equality

$$p_1^*(\eta_1)|_{T'(K,f_1 \times f_2)} = p_2^*(\eta_2)|_{T'(K,f_1 \times f_2)}$$

holds, where $f_1 : U \to \mathbb{G}_m^{s_1,\mathrm{an}}$, $f_2 : U \to \mathbb{G}_m^{s_2,\mathrm{an}}$ and p_1 and p_2 denote the canonical projection maps on $\mathbb{R}^{s_1} \times \mathbb{R}^{s_2}$. The above defines an equivalence relation on Q, and the *sheaf* PS_X *of piecewise smooth forms* on X is defined as the sheafification of $Q/\sim$. For $(f, \eta) \in Q(U)$ for some $U \subseteq X$ open, we denote the associated *piecewise smooth form* on U by $f^*\eta$.

Remark 4.2.63. Let X be an analytic space. Then by [GJR21, Proposition 8.3, Remark 10.15], a piecewise smooth form ω on X is the datum of a G-covering $X = \bigcup_{i \in I} X_i$ of X together with smooth forms $\omega_i \in \mathcal{A}_{sm}(X_i)$ for each $i \in I$ such that $\omega_i|_{X_i \cap X_j} = \omega_j|_{X_i \cap X_j}$ for all $i, j \in I$; up to refinement.

Remark 4.2.64. Let X be an analytic space.

i) The topology on weakly smooth forms $\mathcal{A}_c^{p,q}(X)$ is compatible with the inclusion of sheaves $\mathcal{A}_{sm} \subseteq \mathcal{A}$ in the sense that the topology on the smooth forms $\mathcal{A}_{sm,c}^{p,q}(X)$ on X with compact support defined in [CD12, (4.1.1)] coincides with the subspace topology induced by the inclusion $\mathcal{A}_{sm,c,}^{p,q}(X) \subseteq \mathcal{A}_c^{p,q}(X)$.

ii) Following [GJR21, 12.2], the restriction of a strong current $S \in \mathcal{D}_{p,q}(X)$ to the compactly supported smooth forms $\mathcal{A}^{p,q}_{sm,c}(X)$ is a current in the sense of [CD12, 4.2]. This yields a canonical linear map $\mathcal{D}_{p,q}(X) \to \mathcal{D}^{sm}_{p,q}(X)$, where $\mathcal{D}^{sm}_{p,q}(X)$ denotes the space of currents on X, i.e. the topological dual of $\mathcal{A}^{p,q}_{sm,c}(X)$.

Remark 4.2.65. Let X be an analytic space. Then by [GJR21, Proposition 10.14], there are natural inclusions of sheaves of bigraded differential $\mathbb{R}$-algebras

$$\mathcal{A}_{sm} \hookrightarrow \mathcal{A} \hookrightarrow PS.$$

Furthermore, assuming that $\partial X = \emptyset$, [Mih21, 4.2] shows that there is a natural inclusion

$$\mathcal{A} \hookrightarrow B_X \tag{4.8}$$

which is compatible with derivatives, the $\wedge$-product and pull-backs. Also by [Mih21, 4.2], the δ-forms B_X^{GK} of Gubler and Künnemann [GK17] embed into the spaces of δ-forms B_X in the case where X is the Berkovich analytification of an algebraic variety over k.

Proposition 4.2.66. *Let X be a pure dimensional analytic space with $\partial X = \emptyset$. Then the piecewise smooth forms of bidegree (p, q) on X are precisely the δ-forms of tridegree $(p, q, 0)$ on X.*

Proof. [Mih21, Proposition 4.6] □

Remark 4.2.67. Let X be the Berkovich analytification of an algebraic variety over k. Let $\alpha \in \mathcal{A}(X)$ be a weakly smooth form on X and let $\beta \in B_X^{GK}(X)$ be a δ-form on X in the sense of [GK17]. Then the wedge product of α and β considered as elements of $B_X(X)$ via the natural inclusion maps coincides with the wedge product of α and β in the space $PSP(X)$ as in [GK17], where α is considered as piecewise smooth form on X via the natural inclusion $\mathcal{A} \hookrightarrow PS$.

4.2.3 Integration of δ-forms on Berkovich spaces

There is an integration theory for δ-forms of top degree on non-archimedean analytic spaces. Integration can be defined in terms of integration along tropical spaces, or equivalently through partitions of unity. There is also an inclusion-exclusion type formula. All this is due to [Mih21, 4.3].

Lemma 4.2.68. *Let (Σ, μ, L) be an n-dimensional tropical space and $\omega = t_f^\star(\gamma) \in B_\Sigma^{p,q}(\Sigma)$ a presented δ-form.*

i) If $\dim f(K) < \max\{p, q\}$ for every compact polyhedral set $K \subseteq \Sigma$, then $\omega = 0$.

ii) We assume $\max\{p, q\} = n$ and let $i : \Sigma' \subseteq \Sigma$ be the tropical space defined by f as in Lemma 4.2.47. Then $\mathrm{supp}(\omega) \subseteq \Sigma'$, or more precisely

$$\omega = i_* \left(t^\star_{f|_{\Sigma'}}(\gamma) \right).$$

Proof. [Mih21, Lemma 4.7] □

Corollary 4.2.69. *Let X be a pure dimensional analytic space with $\partial X = \emptyset$, and let $\omega \in B_X^{p,q}(X)$ be a δ-form.*

i) The equality $\operatorname{supp}(\omega) \cap \{x \in X \mid \mathrm{d}(x) < \max\{p,q\}\} = \emptyset$ *holds.*

ii) We assume $\max\{p,q\} = n$ *and that* $\omega = t_f^\star(\gamma)$ *is a presented δ-form for tropical coordinates* $f : X \to \mathbb{G}_m^{r,\mathrm{an}}$ *and a δ-form* $\gamma \in B(\mathbb{R}^r)$*. Then* $\operatorname{supp}(\omega) \subseteq \Sigma(X,f)$ *and for every refinement* (g,p) *of* f*, the equality*

$$\omega|_{\Sigma(X,g)} = i_* \left(\omega|_{\Sigma(X,f)}\right)$$

holds, where $i : \Sigma(X,f) \to \Sigma(X,g)$ *denotes the inclusion map.*

Proof. [Mih21, Corollary 4.8] □

Remark 4.2.70. Let X be a purely n-dimensional analytic space with $\partial X = \emptyset$, and let $\omega \in B_c^{p,q}(X)$ be a δ-form on X with compact support and with $p = n$ or $q = n$.

i) There exists a skeleton $\Sigma \subseteq X$ with $\operatorname{supp}(\omega) \subseteq \Sigma$: We pick a covering $\operatorname{supp}(\omega) \subseteq \bigcup_{i \in I} K_i^\circ$ by a finite union of the interiors of affinoid domains K_i in X such that for all $i \in I$ there exist tropical coordinates $f_i : K_i^\circ \to \mathbb{G}_m^{r_i,an}$ extending to K_i and δ-forms $\gamma_i \in B(\mathbb{R}^{r_i})$ such that $\omega|_{K_i^\circ} = t_{f_i}^\star(\gamma_i)$. Then $\operatorname{supp}(\omega) \subseteq \Sigma := \bigcup_{i \in I} \Sigma(K_i, f_i)$.

ii) Let Σ be a skeleton in X with $\operatorname{supp}(\omega) \subseteq \Sigma$. Then by Corollary 4.2.69 the equality $i_* \left(\omega|_\Sigma\right) = \omega|_{\Sigma'}$ holds for every skeleton Σ' in X with $i : \Sigma \subseteq \Sigma'$.

Definition 4.2.71. Let X be a purely n-dimensional analytic space with $\partial X = \emptyset$ and $\omega \in B_c^{n,n}(X)$ a compactly supported δ-form of top bidegree. Then the *integral of* ω *along* X is defined as

$$\int_X \omega := \int_\Sigma \omega|_\Sigma,$$

where $\Sigma \subseteq X$ is any skeleton in X with $\operatorname{supp}(\omega) \subseteq \Sigma$.

Remark 4.2.72. The definition of integration of δ-forms on analytic spaces is well-defined by Remark 4.2.70.

Remark 4.2.73. There is an equivalent definition of the integral of δ-forms through partitions of unity of piecewise linear spaces. Furthermore, there is an inclusion-exclusion type formula, see [Mih21, 4.9 (3)].

Remark 4.2.74. Let K be an n-dimensional affinoid space, $f : K \to \mathbb{G}_m^{r,\mathrm{an}}$ tropical coordinates, and $\gamma \in B_c^{n,n}(\mathbb{R}^r)$ a δ-form with $\operatorname{supp}(\gamma) \subseteq \mathbb{R}^r \setminus f\left(\partial K \cap \Sigma(K,f)\right)$. Then for the presented δ-form $\omega := t_f^\star(\gamma) \in B_c^{n,n}(K \setminus \partial K)$, the integral of ω along K is given by

$$\int_K \omega = \int_{\mathbb{R}^r} T(K,f) \wedge \gamma, \tag{4.9}$$

see [Mih21, 4.9 (4)].

Remark 4.2.75. We assume that k is algebraically closed and let X be an algebraic variety over k of dimension n with associated Berkovich analytification X^{an}. Then the injective map from the space of δ-forms on X^{an} in the sense of [GK17] to the space $B_{X^{\mathrm{an}}}(X^{\mathrm{an}})$ of δ-forms on X^{an} in the sense of [Mih21] is compatible with integration: Let ω be a compactly supported δ-form of top bidegree (n,n) on X^{an} in the sense of [GK17]. Using a partition of unity argument, it suffices to consider the case that ω is supported on an open subset U of X^{an} where ω can be described in terms of tropical coordinates $f : U \to \mathbb{G}_m^{r,\mathrm{an}}$ and a δ-form $\gamma \in B(\mathbb{R}^n)$. More precisely, U, f and γ are chosen with the property that $t_f^*(\gamma)$ represents $\omega|_U$ as a δ-form in the sense of [GK17, Proposition 4.18]. Following [Mih21, §4.2], the corresponding δ-form $\omega|_U \in B_{X^{\mathrm{an}}}(U)$ in the sense of [Mih21] is then given by $t_f^\star(\gamma)$. The integral of $t_f^\star(\gamma)$ in [Mih21, Definition 4.9 (4)] as well as the integral of $t_f^*(\gamma)$ in [GK17, 5.1] is now computed by passing to the associated tropical variety $T(U,f)$. This shows that the integral of ω along X^{an} in the sense of [GK17] agrees with its integral in the sense of [Mih21].

Lemma 4.2.76. *Let X be a purely n-dimensional analytic space and $\omega_1, \omega_2 \in B_c^{n,n}(X)$ compactly supported δ-forms of top bidegree. Furthermore, let $\Sigma \subseteq X$ be a skeleton such that $\mathrm{supp}(\omega_1), \mathrm{supp}(\omega_2) \subseteq \Sigma$ and such that $\omega_1|_\Sigma = \omega_2|_\Sigma$. Then the integrals of ω_1 and ω_2 along X agree, i.e.*

$$\int_X \omega_1 = \int_X \omega_2.$$

Proof. This follows directly from the definition of the integral of δ-forms on analytic spaces, see Definition 4.2.71. □

Definition 4.2.77. Let X be a purely n-dimensional analytic space with $\partial X = \emptyset$, and let $\omega \in B_X^{p,q}(X)$. We define a linear functional associated to ω by

$$[\omega] : \mathcal{A}_{sm,c}^{n-p,n-q}(X) \to \mathbb{R}, \eta \mapsto \int_X \omega \wedge \eta,$$

where the $\wedge$-product is meant in the sense of δ-forms, by viewing the smooth form η as a δ-form.

Proposition 4.2.78. *Let X be a purely n-dimensional analytic space with $\partial X = \emptyset$, and let $\omega \in B_X^{p,q}(X)$.*

i) Then $[\omega]$ is a current, i.e. the map is continuous with respect to the Schwartz topology on the source space.

ii) The natural map $B_X(X) \to \mathcal{D}(X), \omega \mapsto [\omega]$ is injective and commutes with the differentials d' and d''.

iii) For any $\alpha \in B_c^{n-1,n}(X)$ and $\beta \in B_c^{n,n-1}(X)$, the equality

$$\int_X d'\alpha = \int_X d''\beta = 0$$

holds, i.e. there is a Stokes' Theorem for δ-forms on analytic spaces.

Proof. [Mih21, Proposition 4.10] □

Lemma 4.2.79. *Let X be an analytic space of pure dimension n with $\partial X = \emptyset$. Let $\omega \in B_X^{p,q}(X)$ be a δ-form on X. We associate to the δ-form $\omega \in B_X^{p,q}(X)$ the linear functional*

$$T_\omega : \mathcal{A}_c^{n-p,n-q}(X) \to \mathbb{R}, \eta \mapsto \int_X \omega \wedge \eta,$$

where the $\wedge$-product is the $\wedge$-product of δ-forms. Then T_ω is a strong current on X. Furthermore, the resulting map

$$B_X(X) \to \mathcal{D}(X), \omega \mapsto T_\omega$$

from the space of δ-forms on X to the space of strong currents on X is injective and commutes with the differentials d' and d''.

Proof. The proof works analogously to the proof of [Mih21, Proposition 4.10] once we observe the following: The topology on the space of weakly smooth forms on X is the one adapted from the space of smooth forms on X in [CD12, (4.1.1)], only replacing smooth by harmonic tropicalization maps. And furthermore, G-locally, every harmonic tropicalization map is given by a smooth tropicalization map. Hence, more precisely, the proof is literally the same as the proof of [Mih21, Proposition 4.10] if we replace all smooth tropicalization maps by harmonic tropicalization maps, and then - using [GJR21, 8.6] - pass to a finite affinoid refinement of the affinoid covering such that the restriction of the harmonic tropicalization maps to any covering element is smooth. Then the claim follows exactly by the arguments named in the proof of [Mih21, Proposition 4.10]. □

Remark 4.2.80. Lemma 4.2.79 can be considered as an extension of [Mih21, Proposition 10.4] from smooth forms to weakly smooth forms.

Lemma 4.2.81. *Let X be an analytic space of some pure dimension n with $\partial X = \emptyset$ and let $\omega \in B_X^{p,q}(X)$ be a d''-closed δ-form on X. Then the map*

$$T_\omega : \mathcal{A}_c^{n-p,n-q}(X) \to \mathbb{R}, \eta \mapsto \int_X \omega \wedge \eta$$

from Lemma 4.2.79 induces a linear form on the Dolbeault cohomology.

Proof. Let $\eta \in \mathcal{A}_c^{n-p,n-q}(X)$ such that $[\eta] = 0 \in H^{n-p,n-q}(X)$. Then there is some weakly smooth form $\tilde{\eta} \in \mathcal{A}^{n-p,n-q-1}(X)$ such that $\eta = d''\tilde{\eta}$. Then using the Leibniz formula and Stokes' Theorem we obtain that

$$\begin{aligned}
T_\omega(\eta) &= \int_X \omega \wedge \eta \\
&= \int_X \omega \wedge d''\tilde{\eta} \\
&\overset{4.1.22}{=} \pm \int_X d''(\tilde{\eta} \wedge \omega) \pm (-1)^{2n-p-q-1} \tilde{\eta} \wedge d''\omega \\
&= \pm \int_X d''(\tilde{\eta} \wedge \omega) \pm \int_X \tilde{\eta} \wedge d''\omega \\
&\overset{4.2.79}{=} 0 \pm \int_X \tilde{\eta} \wedge d''\omega = 0
\end{aligned}$$

since ω is d''-closed. This gives the claim. □

4.2.4 The product formula for fibre products

This section serves as one main ingredient for the proof of the main result. Namely, there is a product formula for the integration theory of δ-forms in the sense of Mihatsch. This allows to split integration of forms on the Berkovich analytification of abelian varieties into two parts: The torus part, which is very explicit, and the part of good reduction, where Gubler and Künnemann explain useful results in their paper [GK17]. Here in this section, we prove the product formula in a general setting. Everything relies purely on the theory of δ-forms in [Mih23] and [Mih21].

For analytic spaces X and Y, we denote by $X \times_k Y$ its fibre product in the category of k-analytic spaces in the sense of [Ber93]. We note here that the fibre product exists for k-analytic spaces, but in general not for analytic spaces over k, i.e. X and Y need to be defined over the same ground field. For more details, see [Ber93, 1.4]. Furthermore, for a k-affinoid algebra $\mathcal{G}$ we denote by $\mathcal{M}(\mathcal{G})$ its Berkovich spectrum. For any point x in an analytic space X we denote by $\mathcal{H}(x)$ the corresponding completed residue field. For more details on this theory we refer to [Ber90] and [Ber93].

Remark 4.2.82. Let $[\sigma, \mu_\sigma]$ resp. $[\rho, \mu_\rho]$ be weighted polyhedra in $\mathbb{R}^n$ resp. $\mathbb{R}^m$ and let

$$\sigma \times \rho := i_1(\sigma) \times i_2(\rho)$$

for the canonical inclusions $i_1 : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ and $i_2 : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$. Let furthermore $p_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $p_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ denote the canonical projection maps, and let $\varphi \in C^\infty(\sigma)$ and $\phi \in C^\infty(\rho)$ be smooth functions with compact support. Then Fubini's Theorem shows that the equality

$$\int_{[\sigma\times\rho, \mu_\sigma \wedge \mu_\rho]} p_1^*\varphi \wedge p_2^*\phi = \int_\sigma \varphi \, d\lambda_{\mu_\sigma} \cdot \int_\rho \phi \, d\lambda_{\mu_\rho}$$

holds, where $\mu_\sigma \wedge \mu_\rho$ is meant in the sense of Remark 4.1.9.

Remark 4.2.83. For analytic spaces X and Y, there is an inclusion

$$\operatorname{Int}(X) \times_k \operatorname{Int}(Y) \subseteq \operatorname{Int}(X \times_k Y), \tag{4.10}$$

where $\operatorname{Int}(Z) = Z \setminus \partial Z$ denotes the *interior* of the morphism $Z \to \mathcal{M}(k)$ in the sense of [Ber93, Definition 1.5.4] for any analytic space Z. Indeed, denoting by $\mathrm{pr}_X : X \times_k Y \to X$ and $\mathrm{pr}_Y : X \times_k Y \to Y$ the canonical projection maps from the fibre product, [Ber93, Proposition 1.5.5 (iii)] applied to $X \to \mathcal{M}(k)$ resp. $Y \to \mathcal{M}(k)$ yields

$$\mathrm{pr}_X^{-1}(\operatorname{Int}(X)) \subseteq \operatorname{Int}(X \times_k Y/X) \text{ resp. } \mathrm{pr}_Y^{-1}(\operatorname{Int}(Y)) \subseteq \operatorname{Int}(X \times_k Y/Y). \tag{4.11}$$

Here $\operatorname{Int}(Z/Z') = Z \setminus \partial(Z/Z')$ denotes the *relative interior* of the morphism $Z \to Z'$ in the sense of [Ber93, Definition 1.5.4], where Z and Z' are analytic spaces. Applying [Ber93, Proposition 1.5.5 (ii)] to $X \times_k Y \xrightarrow{\mathrm{pr}_X} X \longrightarrow \mathcal{M}(k)$ resp. $X \times_k Y \xrightarrow{\mathrm{pr}_Y} Y \longrightarrow \mathcal{M}(k)$ furthermore gives

$$\begin{aligned} &\mathrm{pr}_X^{-1}(\operatorname{Int}(X)) \cap \operatorname{Int}(X \times_k Y/X) \subseteq \operatorname{Int}(X \times_k Y) \text{ resp.} \\ &\mathrm{pr}_Y^{-1}(\operatorname{Int}(Y)) \cap \operatorname{Int}(X \times_k Y/Y) \subseteq \operatorname{Int}(X \times_k Y). \end{aligned} \tag{4.12}$$

Hence (4.11) and (4.12) together imply that

$$\mathrm{pr}_X^{-1}(\operatorname{Int}(X)), \mathrm{pr}_Y^{-1}(\operatorname{Int}(Y)) \subseteq \operatorname{Int}(X \times_k Y)$$

which finally gives (4.10).

Lemma 4.2.84. *Let X resp. Y be affinoid spaces – i.e. analytic spaces that are already affinoid – of pure dimension n resp. m. Let $f : X \to \mathbb{G}_m^{r,\mathrm{an}}$ resp. $g : Y \to \mathbb{G}_m^{s,\mathrm{an}}$ be tropical coordinates on X resp. Y. Then the equality*

$$T(X \times_k Y, f \times g) = p_1^* T(X, f) \wedge p_2^* T(Y, g)$$

of δ-forms holds away from $(t_f(\partial X) \times t_g(Y)) \cup (t_f(X) \times t_g(\partial Y))$.

Proof. Let $t_f \times t_g := t_{f \times g}$. We denote by $pr_X : X \times_k Y \to X$ and $pr_Y : X \times_k Y \to Y$ the projection maps coming from the definition of the fibre product and by pr_1 and pr_2 the canonical projection maps on the $\mathbb{R}$-vector space $\mathbb{R}^r \times \mathbb{R}^s$. Then the morphism $f \times g$ is by definition the unique morphism $f \times g : X \times_k Y \to \mathbb{G}_m^{r,\mathrm{an}} \times_k \mathbb{G}_m^{s,\mathrm{an}}$ making the diagram

$$\begin{tikzcd}
X \times_k Y \arrow[dr, dashed, "f \times g"] \arrow[drr, bend left, "f \circ pr_X"] \arrow[ddr, bend right, "g \circ pr_Y"'] & & \\
& \mathbb{G}_m^{r,\mathrm{an}} \times_k \mathbb{G}_m^{s,\mathrm{an}} \arrow[r, "pr_{\mathbb{G}_m^{r,\mathrm{an}}}"] \arrow[d, "pr_{\mathbb{G}_m^{s,\mathrm{an}}}"] & \mathbb{G}_m^{r,\mathrm{an}} \arrow[d] \\
& \mathbb{G}_m^{s,\mathrm{an}} \arrow[r] & \mathcal{M}(k)
\end{tikzcd}$$

commute. Hence the resulting bigger diagram obtained by applying the canonical tropicalization maps to the tori

$$\begin{tikzcd}
X \times_k Y \arrow[drr, "pr_X"] \arrow[dd, "pr_Y"'] \arrow[ddr, "f\times g"] \arrow[dddrrrr, bend left, "t_f \circ pr_X"] \arrow[dddddrr, bend right, "t_g \circ pr_Y"'] & & & & \\
& & X \arrow[dr, "f"] & & \\
Y \arrow[ddr, "g"] & \mathbb{G}_m^{r,\mathrm{an}} \times_k \mathbb{G}_m^{s,\mathrm{an}} \arrow[rr, "pr_{\mathbb{G}_m^{r,\mathrm{an}}}"] \arrow[dd, "pr_{\mathbb{G}_m^{s,\mathrm{an}}}"] \arrow[dr, "\mathrm{trop}"] & & \mathbb{G}_m^{r,\mathrm{an}} \arrow[dr, two heads, "\mathrm{trop}"] \arrow[dd] & \\
& & \mathbb{R}^r \times \mathbb{R}^s \arrow[rr, two heads] \arrow[dd, two heads] & & \mathbb{R}^r \\
& \mathbb{G}_m^{s,\mathrm{an}} \arrow[rr] \arrow[dr, "\mathrm{trop}"'] & & \mathcal{M}(k) & \\
& & \mathbb{R}^s & &
\end{tikzcd}$$

also commutes, which shows that the inclusion $t_{f\times g}(X\times_k Y) \subseteq t_f(X)\times t_g(Y) \subseteq \mathbb{R}^r \times \mathbb{R}^s$ holds. The diagram above also shows that for the converse inclusion, it suffices to show that for each $x \in X$ and each $y \in Y$, the intersection of fibres $pr_X^{-1}(\{x\}) \cap pr_Y^{-1}(\{y\}) \subseteq X \times_k Y$ is not empty. So let $x \in X$ and $y \in Y$. Then there exist affinoid neighbourhoods $\mathcal{M}(\mathcal{A}_x)$ of x in X and $\mathcal{M}(\mathcal{A}_y)$ of y in Y for strictly k-affinoid algebras $\mathcal{A}_x$ and $\mathcal{A}_y$. Using the well-known natural homeomorphisms

$$\begin{aligned}
pr_X|^{-1}_{\mathcal{M}\left(\mathcal{A}_x\hat{\otimes}_k\mathcal{A}_y\right)}(\{x\}) &\simeq \mathcal{M}\left((\mathcal{A}_x\hat{\otimes}_k\mathcal{A}_y)\hat{\otimes}_{\mathcal{A}_x}\mathcal{H}(x)\right) \text{ and}\\
pr_Y|^{-1}_{\mathcal{M}\left(\mathcal{A}_x\hat{\otimes}_k\mathcal{A}_y\right)}(\{y\}) &\simeq \mathcal{M}\left((\mathcal{A}_x\hat{\otimes}_k\mathcal{A}_y)\hat{\otimes}_{\mathcal{A}_y}\mathcal{H}(y)\right)
\end{aligned}$$

giving rise to a natural homeomorphism

$$\begin{aligned}
pr_X|^{-1}_{\mathcal{M}\left(\mathcal{A}_x\hat{\otimes}_k\mathcal{A}_y\right)}(\{x\}) \cap pr_Y|^{-1}_{\mathcal{M}\left(\mathcal{A}_x\hat{\otimes}_k\mathcal{A}_y\right)}(\{y\}) &\simeq \mathcal{M}(\mathcal{H}(x)\hat{\otimes}_k\mathcal{H}(y))\\
&\subseteq X \times_k Y,
\end{aligned}$$

the Berkovich spectrum $\mathcal{M}(\mathcal{H}(x)\hat{\otimes}_k\mathcal{H}(y))$ of the completed tensor product of the fields $\mathcal{H}(x)$ and $\mathcal{H}(y)$ is non-empty by [Gru66, Sect. 2, Theo.1]. Alltogether we obtain an equality $t_{f\times g}(X \times_k Y) = t_f(X) \times t_g(Y) \subseteq \mathbb{R}^r \times \mathbb{R}^s$ of sets. By [CD12, (2.3.3)] $t_f(X)$ is a polyhedral set of dimension $\leq n$, $t_g(Y)$ is a polyhedral set of dimension $\leq m$, and $(t_f \times t_g)(X \times_k Y)$ is a polyhedral set of dimension $\leq n+m$. Hence it follows that, as a set, the $n+m$-dimensional locus of $(t_f \times t_g)(X \times_k Y)$ is given by the union of all products $i_1(\sigma) \times i_2(\rho) \subseteq \mathbb{R}^r \times \mathbb{R}^s$, where $\sigma \subseteq \mathbb{R}^r$ runs through all n-dimensional polyhedra in $t_f(X)$ and $\rho \subseteq \mathbb{R}^s$ runs through all m-dimensional polyhedra in $t_g(Y)$, and $i_1 : \mathbb{R}^r \to \mathbb{R}^r \times \mathbb{R}^s$ resp. $i_2 : \mathbb{R}^s \to \mathbb{R}^r \times \mathbb{R}^s$ denote the natural inclusions. This gives the equality $T(X \times_k Y, f \times g) = i_1(T(X,f)) \times i_2(T(Y,g))$ of sets. Now we consider the weights on $T(X,f), T(Y,g)$ and $T(X\times_k Y, f\times g)$ that Chambert-Loir and Ducros assign to those polyhedral sets. For this, we follow [Mih21, Definition 3.3]. We choose polyhedral complex structures $\mathcal{T}_X$ for $T(X,f)$ such that $t_f(\partial X)$ is contained in its $(n-1)$-skeleton, and $\mathcal{T}_Y$ for $T(Y,g)$ such that $t_g(\partial Y)$ is contained in its $(m-1)$-skeleton. Furthermore, let $\mathcal{T} := \mathcal{T}_X \times \mathcal{T}_Y := \{i_1(\sigma)\times i_2(\rho) \mid \sigma\in \mathcal{T}_X, \rho\in\mathcal{T}_Y\}$. Then $\mathcal{T}$ is a polyhedral complex structure for $T(X\times_k Y, f\times g)$ by the above set-theoretical observations. Furthermore, using that by (4.10) the inclusion

$$\mathrm{Int}(X)\times_k \mathrm{Int}(Y) \subseteq \mathrm{Int}\,(X\times_k Y)$$

holds and that the boundary of a Berkovich analytic space is by definition the complement of its interior, we obtain that

$$t_{f\times g}\left(\partial\left(X\times_k Y\right)\right) \subseteq \left(i_1(t_f(\partial X))\times i_2(t_g(Y))\right)\cup\left(i_1(t_f(X))\times i_2(t_g(\partial Y))\right),$$

hence $t_{f\times g}\left(\partial\left(X\times_k Y\right)\right)$ is contained in the $(m+n-1)$-skeleton of $\mathcal{T}$. Now let $\tau\in\mathcal{T}_X\times\mathcal{T}_Y$ be an $(n+m)$-dimensional polyhedron, and let $\sigma\in\mathcal{T}_{X,n}, \rho\in\mathcal{T}_{Y,m}$ such that $\tau = i_1(\sigma)\times i_2(\rho)$. Let $q_X : \mathbb{G}_m^{r,\mathrm{an}} \to \mathbb{G}_m^{n,\mathrm{an}}$ be a surjective morphism of analytic tori, inducing a map $\mathbb{R}^r\to\mathbb{R}^n$ which is also denoted by q_X, such that $q_X|_\sigma$ is injective. Following [Mih21, Definition 3.3], the composite map $q_x\circ f : t_f^{-1}(\sigma)\to \mathbb{G}_m^{n,\mathrm{an}}(q_X(\sigma))$ is then finite flat of some degree d_{σ,q_X} over a neighbourhood of the relative interior $q_X(\sigma^\circ)$ viewed as a subset of the skeleton $\Sigma(\mathbb{G}_m^n)$. Denoting by $\mu_{\mathbb{R}^n}$ the standard weight on $\mathbb{R}^n$ induced by the standard basis of $\mathbb{Z}^n$, σ is then endowed with the weight $\mu_\sigma := d_{\sigma,q_X}\cdot q_X^{-1}(\mu_{\mathbb{R}^n})$. By [GJR21, §4] this weight is independent of the choice of the morphism q_X. Now we do the same to endow ρ with a weight μ_ρ. So let $q_Y : \mathbb{G}_m^{s,\mathrm{an}}\to\mathbb{G}_m^{m,\mathrm{an}}$ be a surjective morphism inducing a map q_Y on the corresponding real vector spaces with the property that $q_Y|_\rho$ is injective. Let d_{ρ,q_Y} be the degree of the composite map $q_Y\circ g$ over a neighbourhood of the relative interior $q_Y(\rho^\circ)\subseteq\Sigma(\mathbb{G}_m^m)$ where $q_Y\circ g$ is finite flat. Let $\mu_{\mathbb{R}^m}$ denote the standard weight on $\mathbb{R}^m$ induced by the standard basis of $\mathbb{Z}^m$ and we equip ρ with the weight $\mu_\rho := d_{\rho,q_Y}\cdot q_Y^{-1}(\mu_{\mathbb{R}^m})$. We consider the short exact sequence of real vector spaces

$$0\to N_\sigma\to N_\tau\to N_\rho\to 0$$

obtained by the observation

$$\begin{aligned}
N_\tau &= \mathrm{span}\{z-z' \mid z,z'\in\tau\}\\
&= \mathrm{span}\{(x_1,y_1)-(x_2,y_2)\mid x_1,x_2\in\sigma, y_1,y_2\in\rho\}\\
&= \mathrm{span}\{(x_1-x_2,y_1-y_2)\mid x_1,x_2\in\sigma,y_1,y_2\in\rho)\}\\
&= \mathrm{span}\{x_1-x_2\mid x_1,x_2\in\sigma\}\oplus\mathrm{span}\{y_1-y_2\mid y_1,y_2\in\rho\} = N_\sigma\oplus N_\rho.
\end{aligned}$$

As in Remark 4.1.9, there is a canonical isomorphism $\det N_\tau = \det N_\sigma\otimes_{\mathbb{R}}\det N_\rho$, and the weights μ_σ and μ_ρ uniquely determine a weight μ_τ for $\tau\subseteq\mathbb{R}^r\times\mathbb{R}^s$ by $\mu_\tau := \mu_\sigma\wedge\mu_\rho := \mu_\sigma\wedge\tilde{\mu_\rho}$, where $\tilde{\mu_\rho}\in\bigwedge^m N_\tau$ is any lift of μ_ρ. Now we consider the surjective morphism

$$q := q_X\times q_Y : \mathbb{G}_m^{r,\mathrm{an}}\times_k\mathbb{G}_m^{s,\mathrm{an}}\to\mathbb{G}_m^{n,\mathrm{an}}\times_k\mathbb{G}_m^{m,\mathrm{an}}$$

with corresponding map $q : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n \times \mathbb{R}^m$ on real vector spaces. Since q_X is injective on σ, q_Y is injective on ρ and $\tau = i_1(\sigma) \times i_2(\rho)$, $q|_\tau$ is injective as well. The composite map

$$q \circ (f \times g) = (q_X \circ f) \times (q_Y \circ g) : t_{f\times g}^{-1}(\tau) \to \mathbb{G}_m^{n+m,\mathrm{an}}(q(\tau))$$

is then finite flat of degree $d_{\tau,q} = d_{\sigma,q_X} \cdot d_{\rho,q_Y}$ over a neighbourhood of the relative interior $q(\tau^\circ) \subseteq \Sigma(\mathbb{G}_m^{n+m})$. Following definition [Mih21, Definition 3.3], τ considered as an element of the polyhedral complex structure $\mathcal{T}_X \times \mathcal{T}_Y$ for the polyhedral set $T(X \times_k Y, f \times g)$ is then endowed with the weight $d_{\tau,q} \cdot q^{-1}(\mu_{\mathbb{R}^n\times\mathbb{R}^m})$, where $\mu_{\mathbb{R}^n\times\mathbb{R}^m}$ denotes the standard weight on $\mathbb{R}^n \times \mathbb{R}^m$. Note here again that $d_{\tau,q} \cdot q^{-1}(\mu_{\mathbb{R}^n\times\mathbb{R}^m})$ is independent of the choice of q. The canonical weight on $\tau \subseteq T(X \times_k Y, f \times g)$ is hence given by

$$\begin{aligned} &d_{\tau,q} \cdot q^{-1}(\mu_{\mathbb{R}^n\times\mathbb{R}^m}) = d_{\tau,q} \cdot (q_X \times q_Y)^{-1}(\mu_{\mathbb{R}^n} \wedge \mu_{\mathbb{R}^m}) \\ =&d_{\sigma,q_X} \cdot d_{\rho,q_Y} \cdot q_X^{-1}(\mu_{\mathbb{R}^n}) \wedge q_Y^{-1}(\mu_{\mathbb{R}^m}) = \mu_\sigma \wedge \mu_\rho = \mu_\tau, \end{aligned}$$

where the wedge products of the weights $\mu_{\mathbb{R}^n}$ and $\mu_{\mathbb{R}^m}$ and their preimages under q_X and q_Y (resp. μ_σ and μ_ρ) are meant as in Remark 4.1.9 with respect to the canonical short exact sequences $0 \to \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m \to 0$ and the induced short exact sequence by q (resp. $0 \to N_\sigma \to N_\sigma \oplus N_\rho \to N_\rho \to 0$). This identity together with the construction of the wedge product, or more precisely with (4.2), yields that the weights of the tropical cycles $p_1^*T(X,f) \wedge p_2^*T(Y,g) = T(X,f) \boxtimes T(Y,g)$ and $T(X \times_k Y, f \times g)$ (away from the boundaries) agree. Alltogether this proves the claim. □

Lemma 4.2.85. *Let $r, s \in \mathbb{N}$ and let $\alpha \in B_c^{r,r}(\mathbb{R}^r)$ and $\beta \in B_c^{s,s}(\mathbb{R}^s)$ be δ-forms with compact support. Let $p_1 : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^r$ resp. $p_2 : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^s$ denote the projection maps. Then*

$$\int_{\mathbb{R}^r\times\mathbb{R}^s} p_1^*\alpha \wedge p_2^*\beta = \int_{\mathbb{R}^r} \alpha \cdot \int_{\mathbb{R}^s} \beta.$$

Proof. We choose weighted polyhedral complexes of definition $(\mathcal{C}, \mu)$ in $\mathbb{R}^r$ resp. $(\mathcal{D}, \nu)$ in $\mathbb{R}^s$ for α resp. β. Let furthermore $(\alpha_\sigma)_{\sigma\in\mathcal{C}}$ and $(\beta_\rho)_{\rho\in\mathcal{D}}$ with $\alpha_\sigma \in \mathcal{A}(\sigma), \beta_\rho \in \mathcal{A}(\rho)$ for $\sigma \in \mathcal{C}, \rho \in \mathcal{D}$ such that

$$\alpha = \sum_{\sigma\in\mathcal{C}} \alpha_\sigma \wedge [\sigma, \mu_\sigma] \text{ and } \beta = \sum_{\rho\in\mathcal{D}} \beta_\rho \wedge [\rho, \nu_\rho].$$

For every $\sigma \in \mathcal{C}$ and every $\rho \in \mathcal{D}$ we fix $d_\sigma := \dim\sigma, d_\rho := \dim\rho$ and pick bases $e_1, \ldots, e_{d_\sigma} \in N_\sigma, a_1, \ldots, a_{d_\rho} \in N_\rho$ with dual bases $x_1, \ldots, x_{d_\sigma} \in M_\sigma, y_1, \ldots, y_{d_\rho} \in M_\rho$ such that $\mu_\sigma = \pm e_1 \wedge \ldots \wedge e_{d_\sigma}$ and $\nu_\rho = \pm a_1 \wedge \ldots \wedge a_{d_\rho}$. Let $\varphi_\sigma \in C^\infty(\sigma), \phi_\rho \in C^\infty(\rho)$ such that

$$\begin{aligned} &\alpha_\sigma = \varphi_\sigma d'x_1 \wedge d''x_1 \wedge \ldots \wedge d'x_{d_\sigma} \wedge d''x_{d_\sigma} \text{ and} \\ &\beta_\rho = \phi_\rho d'y_1 \wedge d''y_1 \wedge \ldots \wedge d'y_{d_\rho} \wedge d''y_{d_\rho}. \end{aligned}$$

Then

$$p_1^*\alpha_\sigma \wedge p_2^*\beta_\rho = (p_1^*\varphi_\sigma \wedge p_2^*\phi_\rho)\, \mu_\sigma \wedge \nu_\rho \tag{4.13}$$

for all $\sigma \in \mathcal{C}, \rho \in \mathcal{D}$, where the product $\mu_\sigma \wedge \nu_\rho$ is meant in the sense of Remark 4.1.9. Furthermore

$$p_1^*\alpha \wedge p_2^*\beta \overset{4.1.20\text{ii})}{=} \alpha \boxtimes \beta \overset{(4.2)}{=} \sum_{\substack{\sigma \in \mathcal{C} \\ \rho \in \mathcal{D}}} p_1^*\alpha_\sigma \wedge p_2^*\beta_\rho \wedge [\sigma \times \rho, \mu_\sigma \wedge \nu_\rho], \tag{4.14}$$

where $\sigma \times \rho := i_1(\sigma) \times i_2(\rho)$ for the canonical inclusions $i_1 : \mathbb{R}^r \to \mathbb{R}^r \times \mathbb{R}^s$ and $i_2 : \mathbb{R}^s \to \mathbb{R}^r \times \mathbb{R}^s$. Integration of the δ-forms α and β in terms of integration of Lagerberg forms on polyhedra is by definition given as

$$\int_{\mathbb{R}^r} \alpha = \sum_{\sigma \in \mathcal{C}} \int_\sigma \varphi_\sigma \, d\lambda_{\mu_\sigma} \text{ and } \int_{\mathbb{R}^s} \beta = \sum_{\rho \in \mathcal{D}} \int_\rho \phi_\rho \, d\lambda_{\mu_\rho}.$$

This yields

$$\begin{aligned}
\int_{\mathbb{R}^r} \alpha \cdot \int_{\mathbb{R}^s} \beta &= \sum_{\sigma \in \mathcal{C}} \int_\sigma \varphi_\sigma \, d\lambda_{\mu_\sigma} \cdot \sum_{\rho \in \mathcal{D}} \int_\rho \phi_\rho \, d\lambda_{\mu_\rho} \\
&= \sum_{\substack{\sigma \in \mathcal{C} \\ \rho \in \mathcal{D}}} \int_\sigma \varphi_\sigma \, d\lambda_{\mu_\sigma} \cdot \int_\rho \phi_\rho \, d\lambda_{\mu_\rho} \\
&\overset{4.2.82}{=} \sum_{\sigma \times \rho \in \mathcal{C} \times \mathcal{D}} \int_{\sigma \times \rho} p_1^*\varphi_\sigma \wedge p_2^*\phi_\rho \, d\lambda_{\mu_\sigma} \, d\lambda_{\mu_\rho} \\
&\overset{(4.13)}{=} \sum_{\sigma \times \rho \in \mathcal{C} \times \mathcal{D}} \int_{[\sigma \times \rho, \mu_\sigma \wedge \nu_\rho]} p_1^*\alpha_\sigma \wedge p_2^*\beta_\rho \overset{(4.14)}{=} \int_{\mathbb{R}^r \times \mathbb{R}^s} p_1^*\alpha \wedge p_2^*\beta
\end{aligned}$$

and hence the claim. □

Theorem 4.2.86. *Let X and Y be analytic spaces with $\partial X = \emptyset = \partial Y$. We assume that X is of pure dimension r and Y of pure dimension s for $r, s \in \mathbb{N}$. Let $\alpha \in B_c^{r,r}(X)$ and $\beta \in B_c^{s,s}(Y)$ be δ-forms of top bidegree with compact support, and we denote by $p_X : X \times_k Y \to X$ resp. $p_Y : X \times_k Y \to Y$ the canonical projection morphisms. Then the equality*

$$\int_{X \times_k Y} p_X^*\alpha \wedge p_Y^*\beta = \int_X \alpha \cdot \int_Y \beta$$

holds.

Proof. First we note that $p_X^*\alpha \wedge p_Y^*\beta \in B_c^{r+s,r+s}(X \times_k Y)$ is a δ-form on the purely $(r+s)$-dimensional analytic space $X \times_k Y$ of top bidegree, hence the left hand side in the equation is well-defined. Now let $(X_i)_{i \in I}$ be a finite collection of affinoid domains in X such that the finite union $\bigcup_{i \in I} X_i^\circ$ contains $\mathrm{supp}(\alpha)$ and such that we may write

$$\alpha|_{X_i^\circ} = t_{f_{X_i}}^\star(\alpha_i')$$

for suitable tropical coordinates $f_{X_i} : X_i^\circ \to \mathbb{G}_m^{r_i,\mathrm{an}}$ extending to a moment map on X_i, and δ-forms $\alpha_i' \in B(\mathbb{R}^{r_i})$. Such a covering exists since X is a good analytic space in the sense of [Ber93]. Now let $(\lambda_i)_{i\in I} \subseteq \mathcal{A}^{0,0}(X)$ be a smooth partition of unity subordinate to $(X_i^\circ)_{i\in I}$, which exists by [CD12, Proposition (3.3.6)]. For all $i \in I$ we proceed now as follows: Using Proposition 4.2.61, there is an $m_i \in \mathbb{N}$ and a finite family of holomorphic functions $g_i = \big(g_{i_1}, \dots, g_{i_{m_i}}\big) : X_i \to \mathbb{A}^{m_i,\mathrm{an}}$ such that λ_i is given by $t^\star_{g_i}(\lambda'_i)$ on the open subset $U_i := X_i \setminus \bigcup_{j=1}^{m_i} Z(g_{i_j})$ for a suitable $\lambda'_i \in \mathcal{A}^{0,0}(\mathbb{R}^{m_i})$. Let A_i be a strictly k-affinoid algebra with Berkovich spectrum $\mathcal{M}(A_i) = X_i$, and let $\mathfrak{p}_{i_1}, \dots, \mathfrak{p}_{i_{l_i}}$ denote the finitely many minimal prime ideals of the noetherian ring A_i. Then, passing to the Zariski-irreducible components $\mathcal{M}(A/\mathfrak{p}_{i_1}), \dots \mathcal{M}(A/\mathfrak{p}_{i_l})$ of $X_i = \mathcal{M}(A_i)$, we may assume that none of the chosen holomorphic functions $g_{i_1}, \dots, g_{i_{m_i}}$ is zero or a zero divisor, once we observe that the image of $\mathrm{trop}\circ g = t_g : X_i \backslash Z(g) \to \mathbb{R}^{r_g}$ is just a single point for a nilpotent holomorphic function $g : X_i \to \mathbb{A}^{r_g,\mathrm{an}}$. Hence we may assume that $\mathrm{codim}(Z(g_{i_j})) = 1$ for all $j \in \{1, \dots, m_i\}$ using Krull's principal ideal theorem. Then the inclusion $\Sigma(X_i, f_{X_i}) \subseteq U_i$ holds since elements x in a skeleton always have maximal local dimension $\mathrm{d}(x)$ (they are Abhyankar points, see [GJR21, Proposition 3.12]). Since $\mathrm{supp}(\lambda_i) \subseteq X_i^\circ$, in particular $\mathrm{supp}(\lambda_i|_{U_i}) \cap \partial X_i = \emptyset$ holds, and the equality $\mathrm{supp}\,(\lambda_i|_{U_i}) = t_{g_i}^{-1}\,(\mathrm{supp}\,\lambda'_i)$ yields that $t_{g_i}(\partial X_i) \cap \mathrm{supp}(\lambda'_i) = \emptyset$. This gives

$$(t_{f_{X_i}} \times t_{g_i})(\partial X_i \cap U_i) \cap \mathrm{supp}\,\big(p_1^*\alpha'_i \wedge p_2^*\lambda'_i\big) = \emptyset,$$

where $p_1 : \mathbb{R}^{r_i} \times \mathbb{R}^{m_i} \to \mathbb{R}^{r_i}$ and $p_2 : \mathbb{R}^{r_i} \times \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$ are the canonical projection maps, and $p_1^*\alpha'_i \wedge p_2^*\lambda'_i$ denotes the wedge product of δ-forms, whence the smooth form $p_2^*\lambda'_i$ is considered as δ-form. Alltogether, this gives

$$\begin{aligned}
(4.15)\qquad \int_X \alpha &= \sum_{i\in I} \int_X \lambda_i \cdot \alpha|_{X_i} \\
&= \sum_{i\in I} \int_{\Sigma(X_i, g_i \times f_{X_i})} t^\star_{g_i}(\lambda'_i) \wedge t^\star_{f_{X_i}}(\alpha'_i) \\
&= \sum_{i\in I} \int_{\Sigma(X_i, g_i \times f_{X_i})} (t_{g_i} \times t_{f_{X_i}})^\star(p_1^*(\lambda'_i)) \wedge (t_{g_i} \times t_{f_{X_i}})^\star(p_2^*(\alpha'_i)) \\
&= \sum_{i\in I} \int_{\Sigma(X_i, g_i \times f_{X_i})} (t_{g_i} \times t_{f_{X_i}})^\star(p_1^*(\lambda'_i) \wedge p_2^*(\alpha'_i)) \\
&= \sum_{i\in I} \int_{\mathbb{R}^{m_i} \times \mathbb{R}^{r_i}} T(X_i, g_i \times f_{X_i}) \wedge (p_1^*(\lambda'_i) \wedge p_2^*(\alpha'_i))
\end{aligned}$$

using integration theory via partitions of unity from [Mih21, (4.8)], the definition of the wedge product of δ-forms on non-archimedean spaces and the calculation of the integral in a special case as in Remark 4.2.74. Doing exactly the same procedure for β we obtain

$$(4.16)\qquad \int_Y \beta = \sum_{j\in J} \int_{\mathbb{R}^{l_j} \times \mathbb{R}^{s_j}} T(Y_j, h_j \times f_{Y_j}) \wedge (p_1^*(\rho'_j) \wedge p_2^*(\beta'_j)).$$

Here, $(Y_j)_{j\in J}$ is a finite collection of affinoid domains in Y such that $\mathrm{supp}(\beta) \subseteq \bigcup_{j\in J} Y_j^\circ$ and such that $\beta|_{Y_j^\circ} = t^\star_{f_{Y_j}}(\beta'_j)$ for suitable tropical coordinates f_{Y_j} :

$Y_j \to \mathbb{G}_m^{s_i,\mathrm{an}}$ and δ-forms $\beta'_j \in B(\mathbb{R}^{s_i})$. The sequence $(\rho_j)_{j\in J} \subseteq \mathcal{A}^{0,0}(Y)$ denotes a smooth partition of unity subordinate to $(Y_j^\circ)_{j\in J}$, and $h_j : V_j \to \mathbb{G}_m^{l_i,\mathrm{an}}$ are suitable tropical coordinates such that $\rho_j|_{V_j} = t_{h_j}^\star(\rho'_j)$ on some open subset V_j of Y_j with $\Sigma(Y_j, f_{Y_j}) \subseteq V_j$, where $\rho'_j \in \mathcal{A}^{0,0}(\mathbb{R}^{l_i})$. Now we use Lemma 4.2.84 and 4.2.85 to compute the product of the two integrals. For $i \in I$ let $\alpha_i := p_1^*(\lambda'_i) \wedge p_2^*(\alpha'_i)$ and $t_{\alpha_i} := t_{g_i} \times t_{f_{X_i}} : X_i \to \mathbb{R}^{m_i} \times \mathbb{R}^{r_i}$, where p_1 and p_2 denote the canonical projection maps on $\mathbb{R}^{m_i} \times \mathbb{R}^{r_i}$. For $j \in J$ let $\beta_j := p_1^*(\rho'_j) \wedge p_2^*(\beta'_j)$ and $t_{\beta_i} := t_{h_i} \times t_{f_{Y_j}} : Y_j \to \mathbb{R}^{l_i} \times \mathbb{R}^{s_i}$, where p_1 and p_2 denote the canonical projection maps on $\mathbb{R}^{l_i} \times \mathbb{R}^{s_i}$ here. Now let $p_1 : \mathbb{R}^{m_i} \times \mathbb{R}^{r_i} \times \mathbb{R}^{l_i} \times \mathbb{R}^{s_i} \to \mathbb{R}^{m_i} \times \mathbb{R}^{r_i}$ and $p_2 : \mathbb{R}^{m_i} \times \mathbb{R}^{r_i} \times \mathbb{R}^{l_i} \times \mathbb{R}^{s_i} \to \mathbb{R}^{l_i} \times \mathbb{R}^{s_i}$ be the natural projection maps. We note that $(p_X^*\lambda_i \wedge p_Y^*\rho_j)_{(i,j)\in I\times J}$ is a partition of unity subordinate to the open cover $(X_i^\circ \times_k Y_j^\circ)_{(i,j)\in I\times J}$ of $X \times_k Y$. Then we obtain

$$
\begin{aligned}
&\int_{X\times_k Y} p_X^*\alpha \wedge p_Y^*\beta = \\
&= \sum_{i\in I}\sum_{j\in J}\int_{X\times_k Y} p_X^*\lambda_i \wedge p_Y^*\rho_j \wedge p_X^*\alpha \wedge p_Y^*\beta \\
&= \sum_{i\in I}\sum_{j\in J}\int_{X\times_k Y} p_X^*(\lambda_i\cdot\alpha) \wedge p_Y^*(\rho_j\cdot\beta) \\
&= \sum_{i\in I}\sum_{j\in J}\int_{X\times_k Y} p_X^*(t_{\alpha_i}^\star(\alpha_i)) \wedge p_Y^*(t_{\beta_i}^\star(\beta_j)) \\
&= \sum_{i\in I}\sum_{j\in J}\int_{X\times_k Y} (t_{\alpha_i}\times t_{\beta_j})^\star(p_1^*\alpha_i \wedge p_2^*\beta_j) \\
&= \sum_{i\in I}\sum_{j\in J}\int_{\mathbb{R}^{m_i}\times\mathbb{R}^{r_i}\times\mathbb{R}^{l_i}\times\mathbb{R}^{s_i}} T(X_i\times Y_j, (g_i\times f_{X_i})\times(h_j\times f_{Y_j})) \wedge p_1^*\alpha_i \wedge p_2^*\beta_j \\
&\overset{4.2.84}{=} \sum_{i\in I}\sum_{j\in J}\int_{\mathbb{R}^{m_i}\times\mathbb{R}^{r_i}\times\mathbb{R}^{l_i}\times\mathbb{R}^{s_i}} p_1^*T(X_i, g_i\times f_{X_i}) \wedge p_2^*T(Y_j, h_j\times f_{Y_j}) \wedge p_1^*\alpha_i \wedge p_2^*\beta_j \\
&= \sum_{i\in I}\sum_{j\in J}\int_{\mathbb{R}^{m_i}\times\mathbb{R}^{r_i}\times\mathbb{R}^{l_i}\times\mathbb{R}^{s_i}} p_1^*\left(T(X_i, g_i\times f_{X_i})\wedge\alpha_i\right) \wedge p_2^*\left(T(Y_j, h_j\times f_{Y_j})\wedge\beta_j\right) \\
&\overset{4.2.85}{=} \sum_{i\in I}\sum_{j\in J}\int_{\mathbb{R}^{m_i}\times\mathbb{R}^{r_i}} T(X_i, g_i\times f_{X_i})\wedge\alpha_i \cdot \int_{\mathbb{R}^{l_i}\times\mathbb{R}^{s_i}} T(Y_j, h_j\times f_{Y_j})\wedge\beta_j \\
&= \sum_{i\in I}\int_{\mathbb{R}^{m_i}\times\mathbb{R}^{r_i}} T(X_i, g_i\times f_{X_i})\wedge\alpha_i \cdot \sum_{j\in J}\int_{\mathbb{R}^{l_i}\times\mathbb{R}^{s_i}} T(Y_j, h_j\times f_{Y_j})\wedge\beta_j \\
&\overset{(4.15),(4.16)}{=} \int_X \alpha \cdot \int_Y \beta.
\end{aligned}
$$

This gives the claim. $\square$

Chapter 5

Tropical Dolbeault cohomology of abelian varieties

Having introduced δ-forms on non-archimedean analytic spaces, we can now introduce the key δ-form of this thesis: It is subject of the main theorem 5.2.8 which will then allow us to investigate the Dolbeault cohomology of abelian varieties. More precisely, we will see that there are two non-trivial parts that contribute to the Dolbeault cohomology of abelian varieties: One part coming from the algebraic torus associated to the abelian variety, and one part coming from the good reduction part of the abelian variety. The first part comes from the canonical tropicalization trop_E of the Raynaud extension and was already introduced in Section 3.2, whereas the latter is given by the announced key δ-form which we will introduce in Section 5.1.

Throughout this chapter, let k be a field that is complete with respect to a non-trivial non-archimedean absolute value $|.| : k \to \mathbb{R}$. Let k° denote the valuation ring of k.

5.1 The good reduction part of the Dolbeault cohomology of abelian varieties

Here in this section, we consider the part of good reduction B of the Raynaud extension of an abelian variety A over an algebraically closed field. There exists some δ-form β_0 on the analytification B^{an}, the so-called Chern-δ-form associated to some metrized line bundle on B which defines a δ-form in the sense of [GK17], and hence in particular in the sense of [Mih21] as introduced in the last chapter. In [Gub10] and [GK17], those metrized line bundles and their corresponding δ-forms are studied in more detail, and the definitions and results in this section are due to them. One crucial aspect is that a suitable choice of a metrized line bundle gives rise to a Chern δ-form on B^{an} which also descends to the original analytified abelian variety A^{an}. Furthermore, integration of this δ-form can be done in terms of the degree of the line bundle, which is very explicit, too. All together, the constructed Chern δ-form will on the one hand allow to manifest the non-trivial part of the Dolbeault cohomology of A^{an} coming from the algebraic torus associated to A, and on the other hand it will induce a non-zero element in the cohomology of strong currents on A^{an}. We will see this later in Section 5.3 as a corollary to the main theorem 5.2.8.

In this first section, we assume that k is algebraically closed. Let A be an abelian variety over k with Raynaud extension

$$1 \to \mathbb{T} \to E \to B \to 0$$

from Remark/ Definition 3.1.4. We note here that in the case of an algebraically closed field k, A is always split over k.

Convention 5.1.1. We use the notations and definitions from Remark/ Definition 3.1.4. In particular, let b be the dimension of B and let $\mathcal{B}$ be the abelian scheme over $\mathrm{Spec}(k^\circ)$ with generic fibre B. We note here again that this exists since B is of good reduction, and we denote the unique Shilov point of B^{an} by ξ_B. Furthermore, let $\mathfrak{q}$ denote the (algebraic) morphism $E \to B$, and we denote the corresponding analytic morphism $E^{\mathrm{an}} \to B^{\mathrm{an}}$ also by $\mathfrak{q}$. Furthermore, let $\Lambda \subseteq E(k)$ be the normal subgroup such that $A^{\mathrm{an}} \simeq E^{\mathrm{an}}/\Lambda$ via the topological covering map $\mathfrak{p} : E^{\mathrm{an}} \twoheadrightarrow A^{\mathrm{an}}$. For every $a \in \Lambda$, we denote the associated translation map $E^{\mathrm{an}} \to E^{\mathrm{an}}$ by τ_a.

Remark/Definition 5.1.2. Let L be an ample line bundle on the abelian variety B, and let $\mathcal{L}$ be a cubic model for L on $\mathcal{B}$ over $\mathrm{Spec}(k^\circ)$. The corresponding metric $\| \,.\, \|_{\mathcal{L}}$ is a canonical metric, see [GK17, Example 8.15]. Following [GK17, Remark 9.16], [GK17, Proposition 8.11] yields that $\| \,.\, \|_{\mathcal{L}}$ is a δ-metric in the sense of [GK17, Definition 9.9]. Then the so-called *first Chern δ-form* of $(L, \| \,.\, \|_{\mathcal{L}})$ from [GK17, Definition 9.12], denoted by $c_1\,(L, \| \,.\, \|_{\mathcal{L}})$, defines a δ-form in the sense of Gubler and Künnemann, i.e. an element $c_1\,(L, \| \,.\, \|_{\mathcal{L}}) \in B^{1,1}_{GK}(B^{\mathrm{an}}) \subseteq B^{1,1}_{B^{\mathrm{an}}}(B^{\mathrm{an}})$ by [GK17, Proposition 9.11]. Following [GK17, 10.1], we denote the associated δ-form of top bidegree on B^{an} by $\beta_0 := c_1\,(L, \| \,.\, \|_{\mathcal{L}})^{\wedge b} \in B^{b,b}_{B^{\mathrm{an}}}(B^{\mathrm{an}})$. Locally on a trivialization U of L with nowhere vanishing section $s \in \Gamma(U, L)$, $c_1\,(L, \| \,.\, \|_{\mathcal{L}})$ is given by $d'd''\,[-\log \| \, s|_{U^{\mathrm{an}}} \, \|_{\mathcal{L}}]$, see [GK17, 7.7].

Remark 5.1.3. Let L' be a line bundle on a proper smooth algebraic variety over k which is algebraically equivalent to zero. Then by [GK19, Remark 8.11], we have the following results: Using [GK17, Example 8.16], the line bundle L' carries a canonical metric $\| \,.\, \|_{\mathrm{can}}$ which is a δ-metric since a positive tensor power $\| \,.\, \|_{\mathrm{can}}^{\otimes j}$ is piecewise linear. Furthermore, the canonical metric is obtained from a canonical metric on an odd line bundle on an abelian variety via pull-back, and using [GK19, Example 8.10] it then follows that $c_1\,(L', \| \,.\, \|_{\mathrm{can}})$ vanishes.

Lemma 5.1.4. *In the situation of Remark/Definition 5.1.2, the pull-back*

$$\omega := \mathfrak{q}^* c_1(L, \| \,.\, \|_{\mathcal{L}}) \in B^{1,1}(E^{\mathrm{an}})$$

is translation invariant under rational points $E(k)$. In particular, ω is translation invariant under Λ.

Proof. Let $a \in E(k)$ be an arbitrary element with corresponding point $\bar{a} = \mathfrak{q}(a) \in B(k)$, and let $\tau_a : E^{\mathrm{an}} \to E^{\mathrm{an}}$ resp. $\tau_{\bar{a}} : B^{\mathrm{an}} \to B^{\mathrm{an}}$ denote the corresponding translation map. Using that the formation of the first Chern δ-form is compatible with tensor products of metrized line bundles and with pull-backs we obtain

$$\tau_a^*\omega - \omega = \mathfrak{q}^* c_1(\tau_{\bar{a}}^* L \otimes L^{-1}, \| \,.\, \|_{\tau_{\bar{a}}^*\mathcal{L}\otimes\mathcal{L}^{-1}}) = 0,$$

where the second equality follows from Remark 5.1.3. □

Remark/Definition 5.1.5. We consider the situation of Remark/ Definition 5.1.2. Let

$$\beta_0 := c_1(L, \| \,.\, \|_{\mathcal{L}})^{\wedge b} \in B^{b,b}_{B^{\mathrm{an}}}(B^{\mathrm{an}}).$$

Then Lemma 5.1.4 together with Lemma 4.2.59 shows there is some δ-form $\beta \in B^{b,b}_{A^{\mathrm{an}}}(A^{\mathrm{an}})$ such that $\mathfrak{q}^*\beta_0 = \mathfrak{p}^*\beta$.

Proposition 5.1.6. *In the situation of Remark/ Definition 5.1.5 we have that*

$$\int_{B^{\mathrm{an}}} \beta_0 = \deg_{L,\ldots,L}(B) > 0.$$

Proof. The first equality follows directly from [GK17, Proposition 10.4] once we observe that β_0 defines a δ-form on B^{an} in the sense of [GK17] and using that the injective map $B_{GK}(B^{\mathrm{an}}) \to B_{B^{\mathrm{an}}}(B^{\mathrm{an}})$ from the space of δ-forms in the sense of [GK17] to the space of δ-forms in the sense of [Mih21] is compatible with integration, see Remark 4.2.75. The fact that $\deg_{L,\ldots,L}(B) > 0$ holds by Kleiman's Criterion since L is ample. □

Remark 5.1.7. In the situation of Remark/ Definition 5.1.5, following [GK17, 10.1], more generally the following holds: The wedge product $\beta_0 = c_1(L, \| \,.\, \|_{\mathcal{L}})^{\wedge b}$ defines a Radon measure on B^{an}. The measure associated to β_0 is called *Monge-Ampère measure.*

5.2 The product formula for abelian varieties

The goal of this section is to prove the main theorem 5.2.8. It gives a product formula for the integration of wedge products on the analytification of an abelian variety A, where one factor is given by any weakly smooth form on A^{an} coming from the torus part of A, and the other factor is the δ-form from the last section which is based on the good reduction part of A.

In this section, we assume that k is algebraically closed. Furthermore, we fix an abelian variety A over k with associated Raynaud extension

$$1 \to \mathbb{T}^{\mathrm{an}} \xrightarrow{\iota} E^{\mathrm{an}} \xrightarrow{\mathfrak{q}} B^{\mathrm{an}} \to 0$$

as in Remark/ Definition 3.1.4. We recall some of the definitions and results explained there: We denote by $\mathfrak{p} : E^{\mathrm{an}} \to A^{\mathrm{an}}$ the topological covering map whose kernel Λ is a discrete subgroup of $E(k)$ giving rise to an isomorphism

$A^{\mathrm{an}} \simeq E^{\mathrm{an}}/\Lambda$. Furthermore, we denote the character lattice of the algebraic torus $\mathbb{T}$ associated to A by M, and we let N be the cocharacter lattice. We let $d := \dim \mathbb{T}$, $b := \dim B$ and $n := b + d$. We denote the canonical tropicalization map $E^{\mathrm{an}} \to N_{\mathbb{R}}$ by trop_E, and we recall that it is a harmonic tropicalization map in the sense of [GJR21]. Furthermore, it maps $\Lambda \subseteq E^{\mathrm{an}}$ isomorphically onto a complete lattice in $N_{\mathbb{R}}$ which we also denote by Λ. We fix a $\mathbb{Z}$-basis $T_1, \ldots, T_d$ of the character lattice M of $\mathbb{T}$ giving rise to coordinates $x_1, \ldots, x_d$ on $N_{\mathbb{R}}$ and a natural isomorphism $N_{\mathbb{R}} \cong \mathbb{R}^d$. We will use this isomorphism to identify $N_{\mathbb{R}}$ with $\mathbb{R}^d$, and we also identify the lattice $\Lambda \subseteq N_{\mathbb{R}}$ with a lattice in $\mathbb{R}^d$ via $N_{\mathbb{R}} \cong \mathbb{R}^d$. We denote this lattice in $\mathbb{R}^d$ also by Λ. Furthermore, we denote by trop resp. trop_E not only the smooth resp. harmonic tropicalization map $\mathrm{trop} : \mathbb{T}^{\mathrm{an}} \to N_{\mathbb{R}}$ resp. $\mathrm{trop}_E : E^{\mathrm{an}} \to N_{\mathbb{R}}$, but also its composition with the identification $N_{\mathbb{R}} \cong \mathbb{R}^d$.

Definition 5.2.1. Let $\lambda_{N_{\mathbb{R}}}$ be the Haar measure on $N_{\mathbb{R}}$ with the property that $\lambda_{N_{\mathbb{R}}}(F_N) = 1$, where F_N denotes a fundamental domain for the lattice N in $N_{\mathbb{R}}$. We consider the lattice $\Lambda \subseteq N_{\mathbb{R}}$ and define the *covolume* of Λ by $\mathrm{covol}(\Lambda) := \lambda_{N_{\mathbb{R}}}(F_\Lambda)$, where $F_\Lambda \subseteq N_{\mathbb{R}}$ is a fundamental domain for $\Lambda \subseteq N_{\mathbb{R}}$.

Remark 5.2.2. The covolume of Λ in Definition 5.2.1 does not depend on the choice of the fundamental domain F_Λ for Λ since the Haar measure is translation invariant. Furthermore, it is strictly positive since Λ is a complete lattice in $N_{\mathbb{R}}$.

Remark 5.2.3. Considering Λ as a lattice in $\mathbb{R}^d$ via the fixed isomorphism $N_{\mathbb{R}} \cong \mathbb{R}^d$, the covolume of Λ is equivalently given by $\lambda_{\mathbb{R}^d}(F_\Lambda)$ where $F_\Lambda \subseteq \mathbb{R}^d$ is any fundamental domain for Λ in $\mathbb{R}^d$ and $\lambda_{\mathbb{R}^d}$ denotes the Lebesgue measure on $\mathbb{R}^d$.

Convention 5.2.4. Let $(b_1, \ldots, b_d)$ be a $\mathbb{Z}$-basis for the lattice Λ in $\mathbb{R}^d$, i.e. $b_1, \ldots, b_d \in \mathbb{R}^d$ are linearly independent vectors such that $\Lambda = \{\sum_{l=1}^d a_l b_l \mid a_l \in \mathbb{Z}\, \forall l \in \{1, \ldots, d\}\}$. Furthermore, let $F_\Lambda \subseteq \mathbb{R}^d$ be the fundamental domain for the lattice Λ in $\mathbb{R}^d$ given by $F'_\Lambda + v'$ where $F'_\Lambda = \{\sum_{l=1}^d \lambda_l b_l \mid \lambda_l \in [0,1)\, \forall l \in \{1, \ldots, d\}\}$ and $v' = \left(-\frac{1}{4}\right) \sum_{l=1}^d b_l \in \mathbb{R}^d$. We keep these notations for the whole section.

Lemma 5.2.5. *There exists a finite family $\{\Delta_i\}_{i \in I}$ with $\# I = 2^d$ of Λ-small $(\mathbb{Q}, \mathbb{R})$-polyhedra in $\mathbb{R}^d$ such that the following holds:*

i) $\overline{F_\Lambda} \subseteq \bigcup_{i \in I} \Delta_i^\circ$.

ii) $\bigcup_{i \in I} \left(\bigcup_{\lambda \in \Lambda} \tau_\lambda(\Delta_i^\circ) \right) = \mathbb{R}^d$, *i.e. the set* $\{\bigcup_{\lambda \in \Lambda} \tau_\lambda(\Delta_i^\circ)\}_{i \in I}$ *is an open cover of* $\mathbb{R}^d$.

iii) There exists a Λ-invariant smooth partition of unity $\{\rho'_i\}_{i \in I}$ on $\mathbb{R}^d$ subordinate to the finite open cover $\{\bigcup_{\lambda \in \Lambda} \tau_\lambda(\Delta_i^\circ)\}_{i \in I}$ of $\mathbb{R}^d$ such that for every $i \in I$ there is some vector $v_i \in \mathbb{R}^d$ with the property that

$$\Delta_i^\circ \subseteq F_\Lambda^\circ + v_i := \{x + v_i \mid x \in F_\Lambda^\circ\} \text{ and } \mathrm{supp}(\rho'_i) \cap (F_\Lambda + v_i) \subseteq \Delta_i^\circ.$$

iv) The finite set $\{\mathfrak{p}\left(trop_E^{-1}(\Delta_i^\circ)\right)\}_{i\in I}$ *defines an open cover of* A^{an} *which is induced by the* Λ*-small open subsets* $trop_E^{-1}(\Delta_i^\circ) \subseteq E^{\mathrm{an}}$ *via the covering map* $\mathfrak{p}$.

Proof. For $v \in \mathbb{R}^d$ and $\epsilon \in \mathbb{R}_{>0}$ let

$$\Delta(v,\epsilon) := \left\{ v + \epsilon \cdot \sum_{l=1}^{d} \lambda_l b_l \mid \lambda_l \in [-1,1]\ \forall l \in \{1,\dots,d\} \right\}$$

denote the d-dimensional cube around v with edge length 2ϵ. We consider the set

$$Z := \left\{ \sum_{l=1}^{d} \lambda_l b_l \mid \lambda_l \in \left\{0, \frac{1}{2}\right\}\ \forall l \in \{1,\dots,d\} \right\}$$

of vectors in $\mathbb{R}^d$ with $\#Z = 2^d$. For $v \in Z$ let

$$\Delta_v := \Delta\left(v, \frac{1}{3}\right).$$

We claim that the family $\{\Delta_v\}_{v\in Z}$ - that is obviously a family of Λ-small $(\mathbb{Q},\mathbb{R})$-polyhedra in $\mathbb{R}^d$ - fullfills the desired properties. First of all, we show that $\overline{F_\Lambda} \subseteq \bigcup_{v\in Z} \Delta_v^\circ$. So let $x \in \overline{F_\Lambda}$. Then there exist $\lambda_1,\dots,\lambda_d \in [0,1]$ such that $x = v' + \sum_{l=1}^{d} \lambda_l b_l$. Now let

$$\varphi : [0,1] \to \left\{0, \frac{1}{2}\right\}, a \mapsto \begin{cases} 0, & \text{if } a \leq \frac{1}{2} \\ \frac{1}{2}, & \text{if } a > \frac{1}{2}. \end{cases}$$

Then $a - \varphi(a) \in \left[0, \frac{1}{2}\right]$ for all $a \in [0,1]$, and for all $l \in \{1,\dots,d\}$ we set $\mu_l := -\frac{3}{4} + 3(\lambda_l - \varphi(\lambda_l)) \in \left[-\frac{3}{4}, \frac{3}{4}\right]$. For $v := \sum_{l=1}^{d} \varphi(\lambda_l) b_l \in Z$ we obtain that $x \in \Delta_v^\circ$ since

$$\begin{aligned} x = v' + \sum_{l=1}^{d} \lambda_l b_l = \sum_{l=1}^{d} \left(\lambda_l - \frac{1}{4}\right) b_l &= \sum_{l=1}^{d} \left(\varphi(\lambda_l) - \frac{1}{4} + \lambda_l - \varphi(\lambda_l)\right) b_l \\ &= \sum_{l=1}^{d} \varphi(\lambda_l) b_l + \frac{1}{3} \sum_{l=1}^{d} \mu_l b_l \\ &= v + \frac{1}{3} \sum_{l=1}^{d} \mu_l b_l \in \Delta_v^\circ. \end{aligned}$$

Furthermore, by definition of a fundamental domain, for an arbitrary $x \in \mathbb{R}^d$ there is always some $\lambda \in \Lambda$ with $\tau_\lambda(x) \in F_\Lambda$. Hence the fact that $\overline{F_\Lambda} \subseteq \bigcup_{v\in Z} \Delta_v^\circ$ also shows that the set $\{\bigcup_{\lambda\in\Lambda} \tau_\lambda(\Delta_v^\circ)\}_{v\in Z}$ is an open cover of $\mathbb{R}^d$. Now let us consider the subsets $\mathrm{trop}_E^{-1}(\Delta_v^\circ) \subseteq E^{\mathrm{an}}$ for arbitrary $v \in Z$. Since Δ_v is Λ-small and trop_E maps the subgroup $\Lambda \subseteq E(k)$ homeomorphically onto the complete lattice $\Lambda \subseteq \mathbb{R}^d \cong N_{\mathbb{R}}$, the preimage $\mathrm{trop}_E^{-1}(\Delta_v^\circ)$ of Δ_v° under trop_E is Λ-small in E^{an}. Since $A^{\mathrm{an}} \simeq E^{\mathrm{an}}/\Lambda$ via the topological covering map $\mathfrak{p} : E^{\mathrm{an}} \to A^{\mathrm{an}}$ this gives that $\mathfrak{p}\left(\mathrm{trop}_E^{-1}(\Delta_v^\circ)\right) \simeq \mathrm{trop}_E^{-1}(\Delta_v^\circ)$ via $\mathfrak{p}$, and since $\overline{F_\Lambda} \subseteq \bigcup_{w\in Z} \Delta_w^\circ$ the above shows that the sets $\mathfrak{p}\left(\mathrm{trop}_E^{-1}(\Delta_w^\circ)\right) \subseteq A^{\mathrm{an}}$ with w varying over Z cover A^{an}. So now it is left to show that there exists a Λ-invariant smooth partition of unity $\{\rho'_w\}_{w\in Z}$ on $\mathbb{R}^d$ subordinate to the finite open cover $\{\bigcup_{\lambda\in\Lambda} \tau_\lambda(\Delta_w^\circ)\}_{w\in Z}$ of $\mathbb{R}^d$ such that for every $w \in Z$ there is some vector $v_w \in \mathbb{R}^d$ with the property that

$$\Delta_w^\circ \subseteq F_\Lambda^\circ + v_w := \{x + v_w \mid x \in F_\Lambda^\circ\} \text{ and } \mathrm{supp}(\rho'_w) \cap (F_\Lambda + v_w) \subseteq \Delta_w^\circ.$$

For every $w \in Z$ we denote by $\overline{\Delta}_w \subseteq \mathbb{R}^d/\Lambda$ the image of the polyhedron Δ_w under the canonical projection map $\pi : \mathbb{R}^d \to \mathbb{R}^d/\Lambda$. Since $\mathbb{R}^d/\Lambda$ is a manifold, there is a smooth partition of unity $\{\overline{\rho_w}\}_{w\in Z}$ subordinate to the open cover $\left\{\overline{\Delta}_w^\circ\right\}_{w\in Z}$ of $\mathbb{R}^d/\Lambda$ which is given by the topological interiors of $\overline{\Delta}_w$ in $\mathbb{R}^d/\Lambda$ for $w \in Z$, where $\mathbb{R}^d/\Lambda$ is equipped with the quotient topology. Now we let $\rho'_w := \overline{\rho_w} \circ \pi \in \mathcal{A}^{0,0}(\mathbb{R}^d)^\Lambda$ for all $w \in Z$. Then $\{\rho'_w\}_{w\in Z}$ is a Λ-invariant smooth partition of unity on $\mathbb{R}^d$ subordinate to the finite open cover $\{\bigcup_{\lambda\in\Lambda} \tau_\lambda(\Delta_w^\circ)\}_{w\in Z}$ of $\mathbb{R}^d$. If we now fix for every $w \in Z$ the vector $v_w := \sum_{l=1}^d \psi(\lambda_l) b_l \in \mathbb{R}^d$, where $\lambda_1, \dots, \lambda_l \in \left\{0, \frac{1}{2}\right\}$ such that $w = \sum_{l=1}^d \lambda_l b_l$ and

$$\psi : \left\{0, \frac{1}{2}\right\} \to \left\{-\frac{1}{6}, \frac{1}{6}\right\}, 0 \mapsto -\frac{1}{6}, \frac{1}{2} \mapsto \frac{1}{6},$$

we get the desired properties that $\Delta_w^\circ \subseteq F_\Lambda^\circ + v_w$ and $\mathrm{supp}(\rho'_w) \cap (F_\Lambda + v_w) \subseteq \Delta_w^\circ$ for all $w \in Z$. This shows the claim. □

Convention 5.2.6. Let b be the dimension of B, and let $\mathcal{B}$ be the abelian scheme over $\mathrm{Spec}(k^\circ)$ with generic fibre B. Let L be an ample line bundle on B, and $\mathcal{L}$ a cubic model for L on $\mathcal{B}$ over $\mathrm{Spec}(k^\circ)$. Let $\beta_0 := c_1(L, \|\ .\ \|_{\mathcal{L}})^{\wedge b} \in B^{b,b}_{B^{\mathrm{an}}}(B^{\mathrm{an}})$, and let $\beta \in B^{b,b}_{A^{\mathrm{an}}}(A^{\mathrm{an}})$ such that $\tilde{\beta} := \mathfrak{q}^*\beta_0 = \mathfrak{p}^*\beta \in B^{b,b}_{E^{\mathrm{an}}}(E^{\mathrm{an}})^\Lambda$ as in Remark/ Definition 5.1.5.

Convention 5.2.7. In the following, we consider all weakly smooth forms also as δ-forms via the natural inclusion of the sheaf of weakly smooth forms into the sheaf of δ-forms on Berkovich analytic spaces, see Remark 4.2.65.

Theorem 5.2.8. *For arbitrary $\alpha' \in \mathcal{A}^{d,d}(N_\mathbb{R})^\Lambda$ with corresponding weakly smooth form $\alpha \in \mathcal{A}^{d,d}(A^{\mathrm{an}})$ induced by α' and trop_E (see Lemma 3.1.15) and β as before we have*

$$T_\beta(\alpha) = \int_{A^{\mathrm{an}}} \alpha \wedge \beta = \int_{F_\Lambda^\circ} \alpha' \cdot \deg_{L,\dots,L}(B),$$

where the wedge product is meant in the sense of δ-forms, F_Λ° denotes the interior of F_Λ in $\mathbb{R}^d$, and T_β denotes the strong current associated to the δ-form $\beta \in B^{b,b}_{A^{\mathrm{an}}}(A^{\mathrm{an}})$ as in Lemma 4.2.79.

Proof. We are going to use a partition of unity argument to reduce to the case that the support of α' is contained in the interior Δ° of some Λ-small $(\mathbb{Q},\mathbb{R})$-polyhedron Δ in $N_\mathbb{R}$. First we note that $\alpha' \in \mathcal{A}^{d,d}(N_\mathbb{R})^\Lambda$ may always be considered as a Λ-invariant Lagerberg form on $\mathbb{R}^d$ using the fixed isomorphism $N_{\mathbb{R}} \cong \mathbb{R}^d$. Let $I := \{1, \ldots, 2^d\}$ and let $\{\Delta_i\}_{i \in I}$ be a family of Λ-small $(\mathbb{Q}, \mathbb{R})$-polyhedra in $\mathbb{R}^d$ as in Lemma 5.2.5. For $i \in I$ let $\rho'_i \in \mathcal{A}^{0,0}(\mathbb{R}^d)^\Lambda$ such that $\{\rho'_i\}_{i \in I}$ defines a smooth partition of unity on $\mathbb{R}^d$ subordinate to the finite open cover $\{\bigcup_{\lambda \in \Lambda} \tau_\lambda(\Delta_i^\circ)\}_{i \in I}$ of $\mathbb{R}^d$ as in Lemma 5.2.5. Hence for every $i \in I$ there is some vector $v_i \in \mathbb{R}^d$ with the property that

$$\Delta_i^\circ \subseteq F_\Lambda^\circ + v_i := \{x + v_i \mid x \in F'_\Lambda\} \text{ and } \operatorname{supp}(\rho'_i) \cap (F_\Lambda + v_i) \subseteq \Delta_i^\circ.$$

For every $i \in I$, we fix such a $v_i \in \mathbb{R}^d$, and furthermore we define

$$\alpha'_i := \rho'_i \cdot \alpha' \in \mathcal{A}^{d,d}(\mathbb{R}^d)^\Lambda.$$

Using Lemma 3.1.15, the harmonic tropicalization map trop_E on E^{an} then gives for all $i \in I$ corresponding weakly smooth forms

$$\tilde{\alpha}_i \in \mathcal{A}^{d,d}(E^{\mathrm{an}}) \text{ and } \alpha_i \in \mathcal{A}^{d,d}(A^{\mathrm{an}}) \text{ with } \operatorname{supp}(\alpha_i) \subseteq \mathfrak{p}(\operatorname{trop}_E^{-1}(\Delta_i^\circ)).$$

We claim now that it suffices to show that for $i \in I$, the equality

$$\int_{\mathfrak{p}(\operatorname{trop}_E^{-1}(\Delta_i^\circ))} \alpha_i \wedge \beta = \int_{\Delta_i^\circ} \alpha'_i \cdot \deg_{L,\ldots,L}(B) \tag{5.1}$$

holds. So let us assume that (5.1) holds for all $i \in I$. By Lemma 5.2.5, A^{an} is covered by the sets $\mathfrak{p}(\operatorname{trop}_E^{-1}(\Delta_i^\circ))$ for $i \in I$. Furthermore, using that $\{\rho'_i\}_{i \in I}$ was chosen as a Λ-invariant smooth partition of unity on $\mathbb{R}^d$ we obtain that $\sum_{i \in I} \alpha'_i = \alpha' \in \mathcal{A}^{d,d}(\mathbb{R}^d)^\Lambda$ and $\sum_{i \in I} \alpha_i = \alpha \in \mathcal{A}^{d,d}(A^{\mathrm{an}}) \subseteq B^{d,d}_{A^{\mathrm{an}}}(A^{\mathrm{an}})$. Hence summing up over $i \in I$ gives

$$\begin{aligned}
\int_{A^{\mathrm{an}}} \alpha \wedge \beta &= \int_{A^{\mathrm{an}}} \sum_{i \in I} \alpha_i \wedge \beta \\
&= \sum_{i \in I} \int_{A^{\mathrm{an}}} \alpha_i \wedge \beta \\
&= \sum_{i \in I} \int_{\mathfrak{p}(\operatorname{trop}_E^{-1}(\Delta_i^\circ))} \alpha_i \wedge \beta \\
&\overset{(5.1)}{=} \sum_{i \in I} \int_{\Delta_i^\circ} \alpha'_i \cdot \deg_{L,\ldots,L}(B) \\
&= \sum_{i \in I} \int_{F_\Lambda^\circ + v_i} \alpha'_i \cdot \deg_{L,\ldots,L}(B) \\
&= \sum_{i \in I} \int_{F_\Lambda^\circ} \alpha'_i \cdot \deg_{L,\ldots,L}(B) \\
&= \int_{F_\Lambda^\circ} \sum_{i \in I} \alpha'_i \cdot \deg_{L,\ldots,L}(B) \\
&= \int_{F_\Lambda^\circ} \alpha' \cdot \deg_{L,\ldots,L}(B),
\end{aligned}$$

where in the third-to-last step we use the Λ-invariance of α_i'.
The above now allows us to assume in the following that there exists some Λ-small $(\mathbb{Q},\mathbb{R})$-polyhedron Δ in $\mathbb{R}^d$ such that $\operatorname{supp}(\alpha') \subseteq \Delta^\circ$. We fix such a polyhedron $\Delta \subseteq \mathbb{R}^d$. Now we turn to the element β in the considered $\wedge$-product $\alpha \wedge \beta \in B^{n,n}_{A^{\mathrm{an}}}(A^{\mathrm{an}})$. By [GK17, Proposition 10.4], the support of β_0 is just the Shilov point ξ_B of B^{an}. Furthermore, by [GK17, §5], there is an open neighbourhood of ξ_B in B^{an} of the form W^{an} for $W \subseteq B$ affine open, a smooth tropicalization map $t_\psi : W^{\mathrm{an}} \to \mathbb{R}^r$ induced by a moment map $\psi : W^{\mathrm{an}} \to \mathbb{G}_m^{r,\mathrm{an}}$ for some $r \in \mathbb{N}$ and a δ-form $\beta_{0,\mathrm{trop}} \in B(\mathbb{R}^r)$ such that on W^{an}, β_0 is given by $t_\psi^\star(\beta_{0,\mathrm{trop}})$. Using Lemma 3.1.10, by possibly shrinking W (i.e. intersecting W with the generic fiber of a suitable Zariski-dense open subset $\mathcal{W}$ of the model $\mathcal{B}$ of B over k°), we may assume that there is an isomorphism

$$\mathfrak{q}^{-1}(W^{\mathrm{an}}) \overset{\phi}{\simeq} \mathbb{T}^{\mathrm{an}} \times_k W^{\mathrm{an}}$$

together with an affinoid domain $\mathcal{W}^{\sqsupset} \subseteq B^{\mathrm{an}}$ with $\xi_B \in \mathcal{W}^{\sqsupset} \subseteq W^{\mathrm{an}}$ such that $\phi|_{\mathcal{W}^{\sqsupset}}$ is compatible with the tropicalization maps trop on $\mathbb{T}^{\mathrm{an}}$ and trop_E on $\mathfrak{q}^{-1}(\mathcal{W}^{\sqsupset})$, i.e. such that the diagram

$$\begin{tikzcd}
\mathfrak{q}^{-1}(\mathcal{W}^{\sqsupset}) \arrow[r, "\phi"] \arrow[d, "\mathrm{trop}_E"] & \mathbb{T}^{\mathrm{an}} \times_k \mathcal{W}^{\sqsupset} \arrow[dl, "\mathrm{trop}\times\bullet"] \\
\mathbb{R}^d &
\end{tikzcd} \tag{5.2}$$

commutes. Then $\operatorname{supp}(\beta_0) \subseteq \mathcal{W}^{\sqsupset} \subseteq W^{\mathrm{an}}, \operatorname{supp}(\tilde{\beta}) = \operatorname{supp}(q^*\beta_0) \subseteq \mathfrak{q}^{-1}(\mathcal{W}^{\sqsupset}) \subseteq \mathfrak{q}^{-1}(W^{\mathrm{an}})$ and $\operatorname{supp}(\beta) \subseteq \mathfrak{p}(\mathfrak{q}^{-1}(\mathcal{W}^{\sqsupset})) \subseteq \mathfrak{p}(\mathfrak{q}^{-1}(W^{\mathrm{an}}))$. We note here that in the resulting bigger diagram

$$\begin{tikzcd}
\mathfrak{q}^{-1}(W^{\mathrm{an}}) \arrow[r, phantom, "\supseteq"] \arrow[ddrr, "\mathrm{trop}_E"'] & \mathfrak{q}^{-1}(\mathcal{W}^{\sqsupset}) \arrow[rr, "\phi"] \arrow[dr, "\mathrm{trop}_E"'] & & \mathbb{T}^{\mathrm{an}} \times_k \mathcal{W}^{\sqsupset} \arrow[r, phantom, "\subseteq"] \arrow[dl, "\mathrm{trop}\times\bullet"] & \mathbb{T}^{\mathrm{an}} \times_k W^{\mathrm{an}} \arrow[ddll, "\mathrm{trop}\times\bullet"] \\
& & \mathbb{R}^d & & \\
& & \mathbb{R}^d & &
\end{tikzcd} \tag{5.3}$$

only the inner triangle commutes, and the outer triangle does not in general. Now the idea is to pass to small enough subsets of E^{an} resp. A^{an} where we can concretely describe $\tilde{\alpha} \wedge \tilde{\beta}$ resp. $\alpha \wedge \beta$, but which are also big enough such that they contain the support of $\tilde{\alpha} \wedge \tilde{\beta}$ resp. $\alpha \wedge \beta$. The subsets we consider are given by the Λ-small subsets (Λ-smallness follows from the Λ-smallness of the polyhedron Δ)

$$\mathfrak{q}^{-1}(W^{\mathrm{an}}) \cap \mathrm{trop}_E^{-1}(\Delta^\circ) \subseteq E^{\mathrm{an}} \text{ and } \mathfrak{q}^{-1}(\mathcal{W}^{\sqsupset}) \cap \mathrm{trop}_E^{-1}(\Delta^\circ) \subseteq E^{\mathrm{an}}$$

resp. their images under the covering map $\mathfrak{p} : E^{\mathrm{an}} \to A^{\mathrm{an}}$. We note here that since every Λ-small subset of E^{an} is isomorphic to its image under the map $\mathfrak{p}$, the restriction to Λ-small subsets allows us to work with $\tilde{\alpha} \wedge \tilde{\beta}$ instead of $\alpha \wedge \beta$.

Now we consider the restriction of the isomorphism ϕ to $\mathfrak{q}^{-1}(\mathcal{W}^{\beth}) \cap \mathrm{trop}_E^{-1}(\Delta^\circ) \subseteq E^{\mathrm{an}}$ and to $\mathfrak{q}^{-1}(W^{\mathrm{an}}) \cap \mathrm{trop}_E^{-1}(\Delta^\circ) \subseteq E^{\mathrm{an}}$. Since the inner triangle of (5.3) commutes, ϕ restricts to an isomorphism

$$\mathfrak{q}^{-1}(\mathcal{W}^{\beth}) \cap \mathrm{trop}_E^{-1}(\Delta^\circ) \overset{\phi}{\simeq} \mathrm{trop}^{-1}(\Delta^\circ) \times \mathcal{W}^{\beth} \subseteq \mathbb{T}^{\mathrm{an}} \times_k W^{\mathrm{an}}.$$

Since $\mathfrak{q}^{-1}(W^{\mathrm{an}}) \cap \mathrm{trop}_E^{-1}(\Delta^\circ) \subseteq E^{\mathrm{an}}$ is open and so is its image under ϕ, there exist open subsets $V_{\mathbb{T}} \subseteq \mathbb{T}^{\mathrm{an}}$ and $W_B \subseteq W^{\mathrm{an}}$ such that

$$\begin{aligned} \phi(\mathfrak{q}^{-1}(\mathcal{W}^{\beth}) \cap \mathrm{trop}_E^{-1}(\Delta^\circ)) &= \mathrm{trop}^{-1}(\Delta^\circ) \times \mathcal{W}^{\beth} \\ &\subseteq V_{\mathbb{T}} \times_k W_B \\ &\subseteq \phi(\mathfrak{q}^{-1}(W^{\mathrm{an}}) \cap \mathrm{trop}_E^{-1}(\Delta^\circ)) \\ &\subseteq \mathbb{T}^{\mathrm{an}} \times_k W^{\mathrm{an}}. \end{aligned}$$

The tuple $(\mathrm{trop}|_{V_{\mathbb{T}}}, \alpha')$ defines an element in $\mathcal{A}^{d,d}(V_{\mathbb{T}})$ with corresponding δ-form

$$\alpha^{\mathbb{T}} := \left(\mathrm{trop}^\star \left(\alpha' \wedge \left[\mathbb{R}^d, \mu_{\mathbb{R}^d}\right]\right)\right)|_{V_{\mathbb{T}}} \in B^{d,d}_{\mathbb{T}^{\mathrm{an}}}(V_{\mathbb{T}})$$

and well-defined restriction $\alpha^{\mathbb{T}}|_{\mathrm{trop}^{-1}(\Delta^\circ)} \in B^{d,d}_{\mathbb{T}^{\mathrm{an}}}(\mathrm{trop}^{-1}(\Delta^\circ))$. By commutativity of the triangle

$$\begin{tikzcd} \mathbb{T}^{\mathrm{an}} \arrow[r, "\iota"] \arrow[d, "\mathrm{trop}"] & E^{\mathrm{an}} \arrow[dl, "\mathrm{trop}_E"] \\ \mathbb{R}^d & \end{tikzcd},$$

the equality of δ-forms $\iota^*(\tilde{\alpha})|_{V_{\mathbb{T}}} = \alpha^{\mathbb{T}}$ holds. We consider now furthermore the δ-form

$$\beta_0|_{W_B} = t_\psi^\star(\beta_{0,\mathrm{trop}})|_{W_B} \in B^{b,b}_{B^{\mathrm{an}}}(W_B)$$

on $W_B \subseteq W^{\mathrm{an}} \subseteq B^{\mathrm{an}}$. Let $p_{V_{\mathbb{T}}}$ and p_{W_B} denote the canonical projection maps on $V_{\mathbb{T}} \times_k W_B$. Then we consider the δ-forms

$$\omega := \left(\phi|_{\phi^{-1}(V_{\mathbb{T}} \times_k W_B)}\right)^* \left(p^*_{V_{\mathbb{T}}}\left(\alpha^{\mathbb{T}}\right) \wedge p^*_{W_B}\left(\beta_0|_{W_B}\right)\right) \in B^{b+d,b+d}_{E^{\mathrm{an}}}(\phi^{-1}(V_{\mathbb{T}} \times_k W_B))$$

and

$$\begin{aligned} \left(\tilde{\alpha} \wedge \tilde{\beta}\right)|_{\phi^{-1}(V_{\mathbb{T}} \times_k W_B)} &= \tilde{\alpha}|_{\phi^{-1}(V_{\mathbb{T}} \times_k W_B)} \wedge \left(\mathfrak{q}^*\beta_0\right)|_{\phi^{-1}(V_{\mathbb{T}} \times_k W_B)} \\ &\in B^{b+d,b+d}_{E^{\mathrm{an}}}(\phi^{-1}(V_{\mathbb{T}} \times_k W_B)) \end{aligned}$$

on the open subset $\phi^{-1}(V_{\mathbb{T}} \times_k W_B) \subseteq E^{\mathrm{an}}$. Furthermore, we consider the skeleton

$$\begin{aligned} \Sigma_{\mathcal{W}^{\beth},\Delta} &:= \Sigma\left(\mathfrak{q}^{-1}(\mathcal{W}^{\beth}) \cap \mathrm{trop}^{-1}(\Delta), (\mathrm{trop} \times t_\psi) \circ \phi\right) \\ &\subseteq \mathfrak{q}^{-1}(\mathcal{W}^{\beth}) \cap \mathrm{trop}^{-1}(\Delta) \subseteq E^{\mathrm{an}} \end{aligned}$$

obtained from the - by [Gub10, 4.7] - affinoid domain $\mathfrak{q}^{-1}(\mathcal{W}^{\sqsupset}) \cap \mathrm{trop}^{-1}(\Delta)$ in E^{an} and the tropicalization map

$$(\mathrm{trop} \times t_\psi) \circ \phi : \mathfrak{q}^{-1}(\mathcal{W}^{\sqsupset}) \cap \mathrm{trop}^{-1}(\Delta) \overset{\phi}{\simeq} \mathrm{trop}^{-1}(\Delta) \times \mathcal{W}^{\sqsupset} \overset{\mathrm{trop}\times t_\psi}{\longrightarrow} \mathbb{R}^d \times \mathbb{R}^r.$$

Now the facts that $\mathfrak{q}^{-1}(\mathcal{W}^{\sqsupset}) \cap \mathrm{trop}_E^{-1}(\Delta^\circ) \subseteq \phi^{-1}(V_{\mathbb{T}} \times_k W_B)$ and $\iota^*(\tilde{\alpha})|_{V_{\mathbb{T}}} = \alpha^{\mathbb{T}} = \left(\mathrm{trop}^\star(\alpha' \wedge \left[\mathbb{R}^d, \mu_{\mathbb{R}^d}\right])\right)|_{V_{\mathbb{T}}}$ together with the commutativity of the inner triangle of (5.3) and the facts that $\mathrm{supp}(\alpha') \subseteq \Delta^\circ$ and $\mathrm{supp}(\beta_0) = \{\xi_B\} \subseteq \mathcal{W}^{\sqsupset} \subseteq W_B$ yield that

$$\omega|_{\Sigma_{\mathcal{W}^{\sqsupset},\Delta}} = \left(\tilde{\alpha} \wedge \tilde{\beta}\right)|_{\Sigma_{\mathcal{W}^{\sqsupset},\Delta}} \text{ and } \mathrm{supp}(\omega), \mathrm{supp}(\tilde{\alpha} \wedge \tilde{\beta}) \subseteq \Sigma_{\mathcal{W}^{\sqsupset},\Delta}.$$

Applying Lemma 4.2.76, using the projection formula and the earlier proved product formula 4.2.86 we obtain

(5.4)
$$\begin{aligned}
&\int_{\phi^{-1}(V_{\mathbb{T}} \times_k W_B)} \left(\tilde{\alpha} \wedge \tilde{\beta}\right)|_{\phi^{-1}(V_{\mathbb{T}} \times_k W_B)} \\
&\qquad = \int_{\phi^{-1}(V_{\mathbb{T}} \times_k W_B)} \omega \\
&\qquad = \int_{\phi^{-1}(V_{\mathbb{T}} \times_k W_B)} \left(\phi|_{\phi^{-1}(V_{\mathbb{T}} \times_k W_B)}\right)^* \left(p_{V_{\mathbb{T}}}^* \left(\alpha^{\mathbb{T}}\right) \wedge p_{W_B}^* \left(\beta_0|_{W_B}\right)\right) \\
&\qquad = \int_{V_{\mathbb{T}} \times_k W_B} p_{V_{\mathbb{T}}}^* \left(\alpha^{\mathbb{T}}\right) \wedge p_{W_B}^* \left(t_\psi^\star(\beta_{0,\mathrm{trop}})|_{W_B}\right) \\
&\qquad \overset{4.2.86}{=} \int_{V_{\mathbb{T}}} \alpha^{\mathbb{T}} \cdot \int_{W_B} \beta_0|_{W_B}.
\end{aligned}$$

The fact that $\mathrm{supp}(\alpha \wedge \beta) \subseteq \mathfrak{p}\left(\phi^{-1}(V_{\mathbb{T}} \times_k W_B)\right) \cong \phi^{-1}(V_{\mathbb{T}} \times_k W_B)$ then yields

(5.5)
$$\begin{aligned}
\int_{\mathfrak{p}(\mathrm{trop}_E^{-1}(\Delta^\circ))} \alpha \wedge \beta &= \int_{\mathfrak{p}(\phi^{-1}(V_{\mathbb{T}} \times_k W_B))} \alpha|_{\mathfrak{p}(\phi^{-1}(V_{\mathbb{T}} \times_k W_B))} \wedge \beta|_{\mathfrak{p}(\phi^{-1}(V_{\mathbb{T}} \times_k W_B))} \\
&= \int_{\phi^{-1}(V_{\mathbb{T}} \times_k W_B)} \tilde{\alpha}|_{\phi^{-1}(V_{\mathbb{T}} \times_k W_B)} \wedge \tilde{\beta}|_{\phi^{-1}(V_{\mathbb{T}} \times_k W_B)} \\
&= \int_{\phi^{-1}(V_{\mathbb{T}} \times_k W_B)} \left(\tilde{\alpha} \wedge \tilde{\beta}\right)|_{\phi^{-1}(V_{\mathbb{T}} \times_k W_B)} \\
&\overset{(5.4)}{=} \int_{V_{\mathbb{T}}} \alpha^{\mathbb{T}} \cdot \int_{W_B} \beta_0|_{W_B} \\
&= \int_{\mathrm{trop}^{-1}(\Delta^\circ)} \alpha^{\mathbb{T}}|_{\mathrm{trop}^{-1}(\Delta^\circ)} \cdot \int_{B^{\mathrm{an}}} \beta_0 \\
&= \int_{\Delta^\circ} \alpha' \cdot \int_{B^{\mathrm{an}}} \beta_0,
\end{aligned}$$

where in the second-to-last step we use that the support of β_0 is just the Shilov point $\xi_B \in B^{\mathrm{an}}$ which already lies in W_B, and that $\mathrm{supp}(\alpha^{\mathbb{T}}) \subseteq \mathrm{trop}^{-1}(\Delta^\circ)$. In the last step we just use the definition of the δ-form $\alpha^{\mathbb{T}}$. Since $\int_{B^{\mathrm{an}}} \beta_0 = \deg_{L,\ldots,L}(B)$ by Proposition 5.1.6, the claim follows. $\square$

Corollary 5.2.9. *For $\omega' := d'x_1 \wedge d''x_1 \wedge \ldots \wedge d'x_d \wedge d''x_d \in \bigwedge^{d,d} M_{\mathbb{R}} \subseteq \mathcal{A}^{d,d}(N_{\mathbb{R}})^{\Lambda}$ with corresponding weakly smooth form $\omega \in \mathcal{A}^{d,d}(A^{\mathrm{an}})$ induced by ω' and trop_E (see Lemma 3.1.15) we have the formula*

$$T_{\beta}(\omega) = \mathrm{covol}(\Lambda) \cdot \deg_{L,\ldots,L}(B), \tag{5.6}$$

where T_{β} denotes the strong current associated to the δ-form β as in Lemma 4.2.79.

Proof. This follows directly from Theorem 5.2.8. □

5.3 Conclusions: The Dolbeault cohomology of abelian varieties

In this last section, we will use the main theorem 5.2.8 to investigate the Dolbeault cohomology of the analytification of an abelian variety A. We will show that the two forms occurring in Theorem 5.2.8 contribute to the cohomology of A^{an} as non-trivial elements: For the weakly smooth form α, this contribution is directly onto the Dolbeault cohomology of A^{an}, whereas the δ-form β may be considered as a non-trivial element in the cohomology of strong currents on A^{an}.

In this section, we do not assume that k is algebraically closed. We fix an abelian variety A over k, and assume that A is split over k. We note here again that by [Ber90, 6.5], this can always be achieved by passing to a suitable finite separable extension of k. Furthermore, we use the notations from the last section with regards to the Raynaud extension of A.

Proposition 5.3.1. *There is a canonical morphism of real vector spaces*

$$\phi_A^{p,q} : \mathcal{A}^{p,q}(N_{\mathbb{R}})^{\Lambda} \to \mathcal{A}^{p,q}(A^{\mathrm{an}})$$

which is compatible with the differential d'' and the $\wedge$-product.

Proof. The existence of the canonical morphism $\phi_A^{p,q}$ follows directly from Lemma 3.1.15. Let $\alpha' \in \mathcal{A}^{p,q}(N_{\mathbb{R}})^{\Lambda}$ and let $\{U_i\}_{i\in I}$ be an open cover of E^{an} such that the closure $\overline{U_i}$ is a compact and Λ-small analytic domain in E^{an}. For all $i \in I$ let

$$h_i := \mathrm{trop}_E|_{\overline{U_i}} : \overline{U_i} \to N_{\mathbb{R}} \text{ and } \alpha'_i := \alpha'|_{\mathrm{trop}_E(\overline{U_i})}.$$

Then $\phi_A^{p,q}(\alpha')$ is given by the family $(h_i, \alpha'_i)_{i\in I}$, and $d''\left(\phi_A^{p,q}(\alpha')\right)$ is given by $(h_i, d''\alpha'_i)_{i\in I}$ by definition of the differential d'' on weakly smooth forms. Hence $\phi_A^{p,q}$ is d''-compatible. Now we consider another Λ-invariant Lagerberg form $\beta' \in \mathcal{A}^{p',q'}(N_{\mathbb{R}})^{\Lambda}$ and let $\beta'_i := \beta'|_{\mathrm{trop}_E(\overline{U_i})}$ for all $i \in I$. Then $\phi_A^{p+p',q+q'}(\alpha' \wedge \beta')$ is by definition given by the weakly smooth preforms $(h_i, \alpha'_i \wedge \beta'_i)_{i\in I}$. It agrees with $\phi_A^{p,q}(\alpha') \wedge \phi_A^{p',q'}(\beta')$ which is by definition given by $((h_i, h_i), p_1^*\alpha'_i \wedge p_2^*\beta'_i)_{i\in I}$ since the pull-back of Lagerberg forms commutes with the $\wedge$-product. Here, p_1 and p_2 denote the canonical projection maps on $N_{\mathbb{R}} \times N_{\mathbb{R}}$. □

Remark 5.3.2. We observe that the morphism $\phi_A^{p,q}$ from Proposition 5.3.1 induces a morphism

$$\mathcal{A}_{\mathrm{cl}}^{p,q}(N_{\mathbb{R}})^{\Lambda} \to H^{p,q}(A^{\mathrm{an}}), \alpha \mapsto \left[\phi_A^{p,q}(\alpha)\right] \tag{5.7}$$

using d''-compatibility.

Remark 5.3.3. We consider the commutative diagram

$$\begin{tikzcd} \mathcal{A}_{\mathrm{cl}}^{p,q}(N_{\mathbb{R}})^{\Lambda} \arrow[r, "(5.7)"] & H^{p,q}(A^{\mathrm{an}}) \\ \bigwedge^{p,q} M_{\mathbb{R}} \arrow[u, phantom, "\cup|"] \arrow[ur, bend right] & \end{tikzcd} .$$

This gives the desired canonical morphism of real vector spaces

$$\Phi_A^{p,q} : \bigwedge^{p,q} M_{\mathbb{R}} \to H^{p,q}(A^{\mathrm{an}}).$$

It is now left to show that the map is injective. We keep this notation for the rest of the chapter.

Remark 5.3.4. We assume that k is algebraically closed and let $\beta \in B_{A^{\mathrm{an}}}^{b,b}(A^{\mathrm{an}})$ be the element from Remark/Definition 5.1.5. The δ-form β is d''-closed by [GK17, 9.14] together with the Leibniz formula and the fact that the differential d'' commutes with pull-back morphisms of analytic spaces. Then the map

$$T_\beta : \mathcal{A}_c^{d,d}(A^{\mathrm{an}}) \to \mathbb{R}, \eta \mapsto \int_{A^{\mathrm{an}}} \beta \wedge \eta$$

from Lemma 4.2.79 induces a linear form on the cohomology $H^{d,d}(A^{\mathrm{an}})$ by Lemma 4.2.81.

Lemma 5.3.5. *Let $\alpha' \in \bigwedge^{p,q} M_{\mathbb{R}}$ be a non-zero element. Then there exists some $\alpha'' \in \bigwedge^{d-p,d-q} M_{\mathbb{R}}$ such that*

$$\alpha' \wedge \alpha'' = d'x_1 \wedge d''x_1 \wedge \ldots \wedge d'x_d \wedge d''x_d.$$

Proof. For all $I, J \subseteq \{1, \ldots, d\}$ with $|I| = p$, $|J| = q$ let $\tilde{I} := \{1, \ldots, d\} \setminus I$, $\tilde{J} := \{1, \ldots, d\} \setminus J$ and let $\alpha'_{IJ} \in \mathbb{R}$ such that

$$\alpha' = \sum_{|I|=p, |J|=q} \alpha'_{IJ} d'x_I \wedge d''x_J.$$

We consider the set

$$Z := \{(I, J) \subseteq \{1, \ldots, d\}^2 \mid |I| = p \wedge |J| = q \wedge \alpha'_{IJ} \neq 0\}.$$

Note that $Z \neq \emptyset$ since α' is non-zero. For $(I, J) \in Z$ we define

$$s(I, J) := \begin{cases} 1, \text{ if } d'x_I \wedge d''x_J \wedge d'x_{\tilde{I}} \wedge d''x_{\tilde{J}} = d'x_1 \wedge d''x_1 \wedge \ldots \wedge d'x_d \wedge d''x_d \\ -1, \text{ else} \end{cases}$$ and

$$\alpha''_{\tilde{I}\tilde{J}} := s(I, J) \cdot \frac{1}{\#Z} \cdot (\alpha'_{IJ})^{-1}$$

and for all $I, J \subseteq \{1, \ldots, d\}$ with $|I| = p$, $|J| = q$ and $(I, J) \notin Z$ let $\alpha''_{\tilde{I}\tilde{J}} := 0$. Now let

$$\alpha'' := \sum_{|I|=p,|J|=q} \alpha''_{\tilde{I}\tilde{J}} d'x_{\tilde{I}} \wedge d''x_{\tilde{J}} \in \bigwedge^{d-p,d-q} M_{\mathbb{R}},$$

Then

$$\begin{aligned}
&\alpha' \wedge \alpha'' \\
=&\left(\sum_{|I|=p,|J|=q} \alpha'_{IJ} d'x_I \wedge d''x_J\right) \wedge \left(\sum_{|I|=p,|J|=q} \alpha''_{\tilde{I}\tilde{J}} d'x_{\tilde{I}} \wedge d''x_{\tilde{J}}\right) \\
=&\sum_{|I|=p,|J|=q} \sum_{|I|=p,|J|=q} \alpha'_{IJ} \alpha''_{\tilde{I}\tilde{J}} d'x_I \wedge d''x_J \wedge d'x_{\tilde{I}} \wedge d''x_{\tilde{J}} \\
=&\sum_{|I|=p,|J|=q} \alpha'_{IJ} \alpha''_{\tilde{I}\tilde{J}} d'x_I \wedge d''x_J \wedge d'x_{\tilde{I}} \wedge d''x_{\tilde{J}} \\
=&\sum_{(I,J)\in Z} \alpha'_{IJ} \alpha''_{\tilde{I}\tilde{J}} d'x_I \wedge d''x_J \wedge d'x_{\tilde{I}} \wedge d''x_{\tilde{J}} \\
=&\sum_{(I,J)\in Z} \alpha'_{IJ} \cdot s(I,J) \cdot \frac{1}{\#Z} \cdot (\alpha'_{IJ})^{-1} d'x_I \wedge d''x_J \wedge d'x_{\tilde{I}} \wedge d''x_{\tilde{J}} \\
=&\sum_{(I,J)\in Z} \frac{1}{\#Z} d'x_1 \wedge d''x_1 \wedge \ldots \wedge d'x_d \wedge d''x_d \\
=&d'x_1 \wedge d''x_1 \wedge \ldots \wedge d'x_d \wedge d''x_d.
\end{aligned}$$

□

Corollary 5.3.6. *For all $p, q \in \mathbb{N}$, the canonical morphism*

$$\Phi_A^{p,q} : \bigwedge^{p,q} M_{\mathbb{R}} \to H^{p,q}(A^{\mathrm{an}}), \alpha' \mapsto \left[\phi_A^{p,q}(\alpha')\right]$$

is injective. In particular, if the dimension $\dim \mathbb{T}$ *of the algebraic torus associated to A is non-zero, then the Dolbeault cohomology $H^{p,q}(A^{\mathrm{an}})$ of A^{an} is non-trivial for all $p, q \in \mathbb{N}$ with* $\max\{p, q\} \leq d = \dim \mathbb{T}$.

Proof. Let $\alpha' \in \bigwedge^{p,q} M_{\mathbb{R}}$ be a non-zero element. Then by Lemma 5.3.5 there exists some $\alpha'' \in \bigwedge^{d-p,d-q} M_{\mathbb{R}}$ such that

$$\omega' := \alpha' \wedge \alpha'' = d'x_1 \wedge d''x_1 \wedge \ldots \wedge d'x_d \wedge d''x_d \in \bigwedge^{d,d} M_{\mathbb{R}}.$$

By Remark 2.2.45, base change induces a morphism on the Dolbeault cohomology groups of weakly smooth forms, and by Remark 3.1.9, base change is compatible with the formation of the Raynaud extension and its canonical (harmonic) tropicalization map. Hence we may assume that k is algebraically closed by considering the commutative diagram

$$\begin{tikzcd}
\bigwedge^{p,q} M_{\mathbb{R}} \arrow[rr, "\phi^{p,q}_{A\hat{\otimes}_k\widehat{\overline{k}}}"] \arrow[dr, "\phi^{p,q}_A"'] & & H^{p,q}(A^{\mathrm{an}}\hat{\otimes}_k\widehat{\overline{k}}) \\
& H^{p,q}(A^{\mathrm{an}}) \arrow[ur] &
\end{tikzcd}$$

and observing that the completion $\widehat{\overline{k}}$ of an algebraic closure $\overline{k}$ of k is again algebraically closed by Krasner's Lemma. Let L be an ample line bundle on B, $\mathcal{B}$ the abelian scheme over $\mathrm{Spec}(k^\circ)$ with generic fibre B and $\mathcal{L}$ a cubic model for L on $\mathcal{B}$ over $\mathrm{Spec}(k^\circ)$. Let

$$\beta_0 := c_1(L, \| \,.\, \|_{\mathcal{L}})^{\wedge b}$$

as in Remark/ Definition 5.1.5. Then there is some δ-form $\beta \in B^{b,b}_{A^{\mathrm{an}}}(A^{\mathrm{an}})$ such that $\mathfrak{q}^*\beta_0 = \mathfrak{p}^*\beta$. We obtain

$$T_\beta(\phi^{d,d}_A(\omega')) \overset{(5.6)}{=} \mathrm{covol}(\Lambda) \cdot \deg_{L,\dots,L}(B) \neq 0.$$

By Remark 5.3.4, T_β induces a linear form on the cohomology $H^{d,d}(A^{\mathrm{an}})$. This gives that the linear functional T_β applied to the class $\left[\phi^{d,d}_A(\omega')\right] \in H^{d,d}(A^{\mathrm{an}})$ is non-zero, and since $\left[\phi^{d,d}_A(\omega')\right] = \left[\phi^{p,q}_A(\alpha')\right] \wedge \left[\phi^{d-p,d-q}_A(\alpha'')\right]$ this gives the claim. □

Corollary 5.3.7. *We assume that k is algebraically closed. Then there is an injective morphism*

$$\bigwedge^{p,q} M_{\mathbb{R}} \hookrightarrow H^{b+p,b+q}_{\mathcal{D}}(A^{\mathrm{an}})$$

to the cohomology of strong currents on A^{an}.

Proof. We consider an ample line bundle L on B together with a cubic model $\mathcal{L}$ for L on $\mathcal{B}$ over $\mathrm{Spec}(k^\circ)$. We denote the associated δ-form, the first Chern δ-form of $(L, \| \,.\, \|_{\mathcal{L}})$ from Remark/Definition 5.1.2 by $\hat{\beta}_0 := c_1(L, \| \,.\, \|_{\mathcal{L}}) \in B^{1,1}_{B^{\mathrm{an}}}(B^{\mathrm{an}})$, and define $\beta_0 := \hat{\beta}_0^{\wedge b} \in B^{b,b}_{B^{\mathrm{an}}}(B^{\mathrm{an}})$. By Lemma 5.1.4, the δ-form $\mathfrak{q}^*(\beta_0) \in B^{b,b}_{E^{\mathrm{an}}}(E^{\mathrm{an}})$ is Λ-invariant, and using Lemma 4.2.59 we get an induced δ-form $\beta \in B^{b,b}_{A^{\mathrm{an}}}(A^{\mathrm{an}})$ on A^{an} with $\mathfrak{p}^*(\beta) = \mathfrak{q}^*(\beta_0)$ as in Remark/ Definition 5.1.5. We consider the strong current

$$T_\beta : \mathcal{A}^{d,d}(A^{\mathrm{an}}) \to \mathbb{R}, \eta \mapsto \int_{A^{\mathrm{an}}} \eta \wedge \beta$$

on A^{an}, see Lemma 4.2.79. We recall here that $\dim(A) = n = b + d$ where $b = \dim(B)$ and $d = \dim(N_{\mathbb{R}}) = \dim(\mathbb{T})$. By [GK17, 9.14], $\hat{\beta}_0$ is d''-closed, and hence – using the Leibniz formula and the fact that the differential d'' commutes with pull-back maps – β is d''-closed as well. Since the map $B(A^{\mathrm{an}}) \to$

$\mathcal{D}(A^{\mathrm{an}})$ from Lemma 4.2.79 which maps the δ-form β to the strong current T_β is compatible with the differential d'', it follows that $T_\beta \in \mathcal{D}^{b,b}(A^{\mathrm{an}})$ is a d''-closed strong current on A^{an}. Furthermore, for every (p,q)-Lagerberg form $\alpha' \in \bigwedge^{p,q} M_{\mathbb{R}}$ with constant coefficients on $N_{\mathbb{R}}$, the associated weakly smooth form $\phi_A^{p,q}(\alpha') \in \mathcal{A}^{p,q}(A^{\mathrm{an}})$ induced by trop_E is d''-closed since α' is d''-closed and $\phi_A^{p,q}$ is compatible with d'' by Proposition 5.3.1. Hence by Lemma 2.2.42, we obtain a d''-closed strong current

$$\phi_A^{p,q}(\alpha') \wedge T_\beta : \mathcal{A}^{d-p,d-q}(A^{\mathrm{an}}) \to \mathbb{R},$$
$$\eta \mapsto T_\beta(\phi_A^{p,q}(\alpha') \wedge \eta) = \int_{A^{\mathrm{an}}} \phi_A^{p,q}(\alpha') \wedge \eta \wedge \beta$$

and hence an element $\left[\phi_A^{p,q}(\alpha') \wedge T_\beta\right] \in H_{\mathcal{D}}^{b+p,b+q}(A^{\mathrm{an}})$ in the cohomology of strong currents for every $\alpha' \in \bigwedge^{p,q} M_{\mathbb{R}}$. Using this observation, we can now define a morphism

$$\psi : \bigwedge^{p,q} M_{\mathbb{R}} \to H_{\mathcal{D}}^{b+p,b+q}(A^{\mathrm{an}}), \alpha' \mapsto \left[\phi_A^{p,q}(\alpha') \wedge T_\beta\right]$$

and we claim that this map is injective. So let $\alpha' \in \bigwedge^{p,q} M_{\mathbb{R}}$ be a non-zero Lagerberg form with constant coefficients on N_R. By Lemma 5.3.5 there is some $\alpha'' \in \bigwedge^{d-p,d-q} M_{\mathbb{R}}$ such that

$$\alpha' \wedge \alpha'' = d'x_1 \wedge d''x_1 \wedge \ldots \wedge d'x_d \wedge d''x_d =: \omega'$$

and we obtain

$$\begin{aligned}
\left(\phi_A^{p,q}(\alpha') \wedge T_\beta\right)\left(\phi_A^{d-p,d-q}(\alpha'')\right) &= T_\beta\left(\phi_A^{p,q}(\alpha') \wedge \phi_A^{d-p,d-q}(\alpha'')\right) \\
&\overset{5.3.1}{=} T_\beta\left(\phi_A^{d,d}(\omega')\right) \\
&\overset{(5.6)}{=} \mathrm{covol}(\Lambda) \cdot \deg_{L,\ldots,L}(B) \neq 0.
\end{aligned}$$

We can now conclude the injectivity of ψ using the following observation: For every strong current $S \in \mathcal{D}^{b+q,b+q-1}(A^{\mathrm{an}})$ and every $\eta'' \in \bigwedge^{d-p,d-q} M_{\mathbb{R}}$ we have that

$$\begin{aligned}
d''S\left(\phi_A^{d-p,d-q}(\eta'')\right) &= S\left(d''\phi_A^{d-p,d-q}(\eta'')\right) \\
&\overset{5.3.1}{=} S\left(\phi_A^{d-p,d-q+1}(d''\eta'')\right) \\
&= S(0) = 0,
\end{aligned}$$

since every Lagerberg form with constant coefficients on $N_{\mathbb{R}}$ is d''-closed. Since $\left(\phi_A^{p,q}(\alpha') \wedge T_\beta\right)\left(\phi_A^{d-p,d-q}(\alpha'')\right) \neq 0$ we get that $\psi(\alpha') = \left[\phi_A^{p,q}(\alpha') \wedge T_\beta\right] \in H_{\mathcal{D}}^{b+p,b+q}(A^{\mathrm{an}})$ is non-zero in the cohomology of strong currents on A^{an}. $\square$

Remark 5.3.8. In the situation of Corollary 5.3.7, roughly speaking, the injective map

$$\bigwedge^{p,q} M_{\mathbb{R}} \hookrightarrow H_{\mathcal{D}}^{b+p,b+q}(A^{\mathrm{an}})$$

represents the contribution of the good reduction part B^{an} of the Raynaud extension to the Dolbeault cohomology of A^{an}. Also roughly speaking, this contribution is is given by the first Chern δ-form $c_1\,(L, \|\,.\,\|_{\mathcal{L}})$ of an ample line bundle L on B together with a cubic model $\mathcal{L}$ for L on $\mathcal{B}$.

Bibliography

[Ber90] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

[Ber93] Vladimir G. Berkovich. Étale cohomology for non-Archimedean analytic spaces. *Inst. Hautes Études Sci. Publ. Math.*, (78):5–161 (1994), 1993.

[BL93] Siegfried Bosch and Werner Lütkebohmert. Formal and rigid geometry. I. Rigid spaces. *Math. Ann.*, 295(2):291–317, 1993.

[CD12] Antoine Chambert-Loir and Antoine Ducros. Formes différentielles réelles et courants sur les espaces de Berkovich. `arXiv:1204.6277`, 2012.

[Duc12] Antoine Ducros. Espaces de Berkovich, polytopes, squelettes et théorie des modèles. *Confluentes Math.*, 4(4):1250007, 57, 2012.

[GGJ$^+$20] José Ignacio Burgos Gil, Walter Gubler, Philipp Jell, Klaus Künnemann, and Florent Martin. Differentiability of non-archimedean volumes and non-archimedean Monge-Ampère equations. *Algebr. Geom.*, 7(2):113–152, 2020. With an appendix by Robert Lazarsfeld.

[GJR21] Walter Gubler, Philipp Jell, and Joseph Rabinoff. Forms on Berkovich spaces based on harmonic tropicalizations. `arXiv:2111.05741`, 2021.

[GK17] Walter Gubler and Klaus Künnemann. A tropical approach to nonarchimedean Arakelov geometry. *Algebra Number Theory*, 11(1):77–180, 2017.

[GK19] Walter Gubler and Klaus Künnemann. Positivity properties of metrics and delta-forms. *J. Reine Angew. Math.*, 752:141–177, 2019.

[Gru66] Laurent Gruson. Catégories d'espaces de Banach ultramétriques. *Bull. Soc. Math. France*, 94:287–299, 1966.

[Gub10] Walter Gubler. Non-Archimedean canonical measures on abelian varieties. *Compos. Math.*, 146(3):683–730, 2010.

[Gub16] Walter Gubler. Forms and current on the analytification of an algebraic variety (after Chambert-Loir and Ducros). In *Nonarchimedean and tropical geometry*, Simons Symp., pages 1–30. Springer, [Cham], 2016.

[Jel16] Philipp Jell. *Differential forms on Berkovich analytic spaces and their cohomology.* PhD thesis, Universität Regensburg, 2016. `urn:nbn:de:bvb:355-epub-347884`.

[JSS19] Philipp Jell, Kristin Shaw, and Jascha Smacka. Superforms, tropical cohomology, and Poincaré duality. *Adv. Geom.*, 19(1):101–130, 2019.

[Lag12] Aron Lagerberg. Super currents and tropical geometry. *Math. Z.*, 270(3-4):1011–1050, 2012.

[Mih21] Andreas Mihatsch. δ-forms on Lubin–Tate Space. `arXiv:2112.10018`, 2021.

[Mih23] Andreas Mihatsch. On tropical intersection theory. *Selecta Math. (N.S.)*, 29(2):Paper No. 17, 34, 2023.

[Sto21] Martino Stoffel. *Real-valued differential forms on non-archimedean abelian varieties with totally degenerate reduction.* PhD thesis, Universität Regensburg, 2021. `urn:nbn:de:bvb:355-epub-442639`.