Phase-Field Methods for Spectral Shape and Topology Optimization

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Abstract

We consider a variety of shape and topology optimization problems in which the cost functional always contains a finite selection of eigenvalues of the shape that is to be optimized. The governing state equations in our framework are either the Laplace equation with Dirichlet or Neumann boundary data or the equations of linear elasticity with mixed boundary data. The control variable is given in the framework of a diffuse interface approach as a possibly vector-valued phase-field variable, allowing for structures to be composed of multiple materials.

In a first step, in each case, we analyze the continuity and differentiablity of the involved eigenvalues and eigenfunctions in order to arrive at well-posedness of the associated optimization problem and first order necessary optimality conditions. Subsequently, we perform the sharp-interface limit in order to link the diffuse interface approach with optimization problems which only involve phases in their pure form, i.e., we analyze the limit passage when the thickness of the diffuse interface gets infinitesimally small. In the case of the Dirichlet–Laplacian we will rigorously perform this limit in the sense of Γ-convergence.

The theory developed here allows us to state a novel phase-field version of the Faber–Krahn theorem which then in the limit provides a version of the Faber–Krahn theorem in the class of functions of bounded variation.

In the setting of linear elasticity we will apply the ansatz of formally matched asymptotic expansions in order to derive the sharp-interface problem including first-order conditions. Eventually, we present and discuss several numerical simulations for concrete spectral optimization problems.

Zusammenfassung

Wir betrachten diverse Probleme der Form- und Topologieoptimierung in denen das Zielfunktional immer eine endliche Auswahl an Eigenwerten der Form beinhaltet, welche optimiert werden soll. Die zugrundeliegenden Zustandsgleichungen unseres Modells sind entweder die Laplace Gleichung mit Dirichlet oder Neumann Randwerten oder die Gleichungen der linearen Elastizität mit gemischten Randdaten. Die Kontrollvariable ist im Rahmen des Ansatzes diffuser Grenzschichten als gegebenenfalls vektorwertiges Phasenfeld gegeben. Dies ermöglicht die Konstruktion von Strukturen aus mehreren Materialien. Zunächst analysieren wir jeweils die Stetigkeit und Differenzierbarkeit der involvierten Eigenwerte und Eigenfunktionen, um zur Wohlgestelltheit des zugehörigen Optimierungsproblems und notwendigen Optimalitätsbedingungen erster Ordnung zu gelangen. Daraufhin studieren wir den sogenannten "scharfen Grenzschicht Limes" um die diffusen Grenzschichtmodelle mit Optimierungsproblemen in Verbindung zu bringen, welche die Phasen ausschließlich in ihrer Reinform beinhalten, d.h. wir analysieren denjenigen Grenzübergang für den die Dicke der diffusen Grenzschicht beliebig klein wird. Im Falle des Dirichlet–Laplace werden wir diesen Grenzübergang rigoros im Sinne der Γ-Konvergenz durchführen.
Die hier entwickelte Theorie erlaubt es uns eine neuartige Phasenfeld Version des Faber–Krahn Theorems herzuleiten, welche uns dann im Grenzwert eine Version des Faber–Krahn Theorems in der Klasse der Funktionen mit beschränkter Variation liefert.
Im Kontext der linearen Elastizität werden wir den Ansatz der formalen asymptotischen Entwicklungen nutzen, um das zugehörige scharfe Grenzschicht Problem inklusive ersten Ordnungbedingungen herzuleiten.
Abschließend diskutieren wir diverse numerische Simulationen für konkrete Eigenwertoptimierungsprobleme.

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There's a crack in everything
that's how the light gets in
Leonard Cohen

Contents

Chapter 1

Introduction

1.1. General overview and relation to the literature

Optimization encounters everyone of us in everyday life and is deeply encoded into our human nature. “What is the shortest way to the next gas station?”, “How to arrange my weekend plans in order to have the most fun, while respecting my needs for sleep and the limited time of the weekend in general?”, “What is the least amount of exercises that I have to solve on my exercise sheets but still pass the course?”.
All of this boils down to the question: “How can we get the most out of something with the least amount of effort?”. Science is aiming to give specific answers to this question across the whole spectrum of its disciplines, from mathematics and engineering over economics to philosophy. For a nice introduction to the topic of optimization from a mathematical engineering point of view including historical background, we refer to [158].
An ancient problem we want to mention at this stage, as it is also closely related to the contents of this thesis, is the foundation of the so called class of *isoperimetric problems* (*iso* meaning *same* and *perimeter* denoting the length of the boundary of a shape). It is commonly known as Dido’s problem dating back to the 9th century B.C., see [24, 110]. According to roman mythology, queen Dido founded the city of Carthage by taking the hide of a bull and enclosing the largest possible area within it. Doing so, she composed a *half circle* out of narrow strips of the hide, as she was allowed to use the coast line as a natural boundary. In other words, Dido intuitively solved the problem of finding among all curves of *given length* a curve which encloses *maximal area*. It took mankind over two thousand years to give a rigorous proof to the fact that the circle in the plane is indeed a minimizer of the Euclidean isoperimetric problem. This was basically proven by the ideas of J. Steiner (1796-1863) and his famous symmetrization technique, see [121, Section 14] for a complete mathematical treatment of this problem. We note that symmetrization techniques are also the main tools in Chapter 4 for proving a phase-field version of the Faber–Krahn theorem, which belongs also to the class of isoperimetric problems.
By the way Dido’s problem also motivated Euler in the 18th century to found a whole new mathematical discipline, namely the *calculus of variations*, which happens to be the mathematical field this thesis is located in.

Let us come to the more specific branch of optimization this thesis aims at. The field of shape and topology optimization is a huge area of research as it tries to answer the question in which way an object consisting of possibly many materials has to be constructed in order

to have optimal properties, while simultaneously satisfying initially imposed constraints. In pure shape optimization indeed only the shape of the object is considered as control in the optimization process, whereas in topology optimization also topological changes such as the creation or nucleation of holes *within* the object are admissible. Note that in the literature the notions are not consistent and sometimes topology optimization is also included in the category of shape optimization.
The object varied along the optimization process, which is commonly referred to as *shape*, is surrounded by an ambient domain which does not vary along the optimization process. It is called the *design domain* or *reference domain*. Thus, the main objective in shape and topology optimization is to distribute material and void within this design domain in order to arrive at an optimal structure. Mathematically this is done by minimizing a so called *cost functional*, also often called *objective functional*. The quantities that are to be optimized are then often characterized as solutions of one or multiple partial differential equations, which are referred to as the *state equations*.
Of course this gives rise to the question which features of the structure should be optimal and how this optimality is measured. A fundamental problem in this context is the so called *minimum compliance problem*. Here the final structure should minimally comply when a load is applied to its boundary and/or forces act within the structure. Formulated as in [4, Section 1.2], the less the work of the structure under applied forces is, the more rigid it is.
Taking now the *eigenvalues* of the structure into account, which is the key problem in this thesis, goes beyond the compliance problem: It is known that structures with a high fundamental eigenfrequency tend to be reasonably stiff for *all* conceivable loads, see [29, 158].
Apart from this observation, problems in dynamics are also relevant on their own. The eigenvalues model the so called *resonant frequencies* of a structure, i.e., they correspond to those frequencies that the structure freely vibrates with. Thus, in many applications it is desirable to control these frequencies with the goal of either tuning the structure towards some desired frequency response or preventing unwanted resonance.
We want to illustrate this situation briefly with the following example of [29] which is a motivating example for considering eigenvalues in linear elasticity in Chapter 5. Consider an airplane driven by an engine. Of course the working engine vibrates, but at frequencies that we can assume to be rather low. These vibrations are forced to some extent also on the other parts of the airplane, such as the wings. In order to avoid a resonance disaster, the resonant frequencies of the wings must be kept as far away as possible from the frequencies of the engine. Mathematically speaking, this goal can be realized by maximizing the smallest non-trivial eigenvalue of the wings, because then also all larger eigenvalues of the wings are separated from the low frequencies of the engine.
Of course in all these problems additional constraints, such as volume restrictions and prescribed solid or void regions have to be taken into account. These constraints will also be incorporated into our mathematical framework. For a comprehensive overview over shape and topology optimization, we refer to [4, 29, 88, 158].

The mathematical treatment of shape and topology optimization problems is a large field of research and many techniques have been analyzed in order to tackle these problems. The first intuition when optimizing the shape of a domain is to deform its boundary in order to arrive at an optimal domain. Thus, the traditional approach is to consider boundary variations by computing so called shape derivatives. These shape derivatives

give then suitable descent directions in order to decrease the value of the cost functional. In this context shape derivatives are obtained by perturbing the shape via a family of diffeomorphisms and then considering the first variation of this perturbation in the sense of Fréchet-derivatives, see e.g., [49,110,128,138,145,146] and the references cited therein. We will compare our formally derived first-order conditions on the sharp-interface level in Chapter 6 with those obtained in the framework of shape derivatives.
From a computational point of view the drawbacks of the technique of boundary variation are its high costs, especially due to frequent re-meshing, and no opportunity to include topological changes, see [6, 134]. These drawbacks, in some situations, can be overcome by homogenization methods (see, e.g., [3]) or variants of this approach such as the SIMP (Solid Isotropic Material with Penalization) method (see, e.g., [29]). Having the application to spectral shape and topology optimization problems in mind, the SIMP method has a further drawback. In the framework of this method it is rather delicate to deal with spurious eigenmodes localizing in void areas of the structure. These modes are problematic when their associated eigenvalues fall within the part of the spectrum we are trying to optimize, see [29, 136]. In the framework of formally matched asymptotics we will see in Chapter 6 that our approach is able to deal with localized modes.

A further method enjoying a huge popularity in the recent literature is the level-set method. Originally developed in [132], it was frequently used also in the context of spectral shape and topology optimization (see, e.g., [5, 6, 17, 60, 74, 115, 131, 134, 157]). In this approach the evolution of the shape in the optimization process is driven by a Hamilton–Jacobi equation. The involved normal velocities evolving the boundary are again computed with the above mentioned classical shape calculus via shape derivatives. The main advantages of the level-set method are its moderate numerical costs and that it allows for topological changes. Nevertheless, the creation of new holes can be a challenge. This drawback can be overcome by incorporating so called topological derivatives, see [59, 129].

In this thesis we pursue the so called *phase-field approach*, also known as *diffuse interface approach*. The key idea and at the same time a huge benefit compared to a technical geometrical evolution of the shape interface is the following. Instead of directly varying the shape, we formulate an approximate eigenvalue problem involving a phase-field variable on the fixed design domain. Any shape within this domain is now implicitly represented by this phase-field.
More precisely, instead of interpreting the shape as the unknown quantity in the optimization process, we describe it via a phase-field φ_ε, which we assume to be a scalar valued and suitably regular function for the sake of this introduction. It attains the value $+1$ in most parts of the shape and the value -1 in most parts of its complement (with respect to the design domain). These values will also be referred to as the *pure phases*, because they represent material or void in its pure unmixed form. The most important feature of the phase-field approach is that it approximates the sharp shape boundary by a so called *diffuse interface* in the following way. We consider a tubular neighborhood around the shape boundary of thickness proportional to a small parameter $\varepsilon > 0$, which is commonly called the *interface parameter*. In this small region the phase-field now undergoes a transition between the values $+1$ and -1, meaning that in this region the phases mix, creating a diffuse interface. From an analytical point of view it is important to mention that with respect to the space variable the phase-field φ does not change its values abruptly along the diffuse interface. In other words the phase-field does not simply jump between pure

phases but exhibits a "continuous" transition, that will be specified later. This regularity is a main feature of the phase-field approach, as it makes the models we intend to study very well suited to be treated with methods from functional analysis.
The phase-field method was applied to topology optimization for the first time in [43] and afterwards reached a wide popularity in the shape and topology optimization community, e.g., in [18, 20, 31, 32, 34, 60, 68, 82, 84, 95, 96, 122, 137, 154–156]. The main advantage of this approach compared to the ones discussed above is that the possibility of topological changes is directly incorporated within the diffuse interface framework. Especially, the creation of holes within the shape does not pose any problems, as there is no need for a priori information on the shape boundary.

The goal of this thesis is to study spectral shape and topology optimization problems formulated in the phase-field framework. Regardless of the specific governing state equations we will always proceed as follows. First of all, we will assure the existence of solutions to the eigenvalue problem serving as state equation and analyze some useful properties of this eigenvalue equation. Then we will rigorously analyze the continuity and differentiability of eigenvalues and eigenfunctions with respect to the phase-field variable in the diffuse interface setting, i.e., for fixed $\varepsilon > 0$. Subsequently, we will derive from this analysis first-order necessary optimality conditions providing us with an optimality system for the underlying optimization problem.
Large parts of this thesis are then devoted to either rigorously or formally pass the optimization problems formulated in the diffuse interface framework to the sharp-interface limit. Identifying such a limit optimization problem is of major importance, because this will guarantee that the phase-field optimization problems are indeed suitable approximations of the physically reasonable optimization problems formulated on the sharp-interface level.

This thesis is divided into the study of two different partial differential equations and associated optimization problems, which we want to briefly explain at this point.
In Part I the governing problem is to optimize the eigenvalues of the Laplacian with either homogeneous Dirichlet or homogeneous Neumann boundary conditions. More explicitly, the aim is to find a shape D such that a finite selection of eigenvalues of

$$\begin{cases} -\Delta w = \lambda w & \text{in } D, \\ w = 0 & \text{on } \partial D, \end{cases} \quad \text{or} \quad \begin{cases} -\Delta w = \mu w & \text{in } D, \\ \partial_{\boldsymbol{n}} w = 0 & \text{on } \partial D, \end{cases} \tag{CL}$$

is optimal. Here the notion of optimality of course depends on the specific application and will be further specified in the according chapters. As explained above, our analysis of these problems will be based on a phase-field approximation of these classical problems. At this point it is worth mentioning that the Neumann problem is in general harder to deal with, because the eigenvalues in this case carry some deep stability issues. We will comment on that at the end of this introduction.
In the Dirichlet case the situation is still delicate but easier to deal with and for the sake of this introduction let us focus on this case. The approximate eigenvalue problem, which will serve as the state equation in the forthcoming optimization problems, is given as

$$\begin{cases} -\nabla \cdot [a_\varepsilon(\varphi)\nabla w^{\varepsilon,\varphi}] + b_\varepsilon(\varphi) w^{\varepsilon,\varphi} = \lambda^{\varepsilon,\varphi} c_\varepsilon(\varphi) w^{\varepsilon,\varphi} & \text{in } \Omega, \\ w^{\varepsilon,\varphi} = 0 & \text{on } \partial\Omega, \end{cases} \tag{AL}$$

where $\varepsilon > 0$ is the interface parameter and $a_\varepsilon, b_\varepsilon, c_\varepsilon : \mathbb{R} \to \mathbb{R}$ are functions depending on this parameter and the phase-field variable φ. The key idea is to choose these functions

such that in the sharp-interface limit $\varepsilon \to 0$, at least formally, we recover the classical eigenvalue problem (CL). This kind of relaxation was also applied in [41, 96, 97] in the modeling and analysis of fluid dynamics.

In Part II of the thesis we will focus on the the system of equations of linear elasticity. Here we allow now for structures which are composed of multiple materials, i.e., we pass from a scalar-valued phase-field to a vector-valued one. The governing system of equations is given as

$$\begin{cases} -\nabla \cdot [\mathbb{C}_\varepsilon(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{w}^{\varepsilon,\boldsymbol{\varphi}})] = \lambda^{\varepsilon,\boldsymbol{\varphi}} \rho_\varepsilon(\boldsymbol{\varphi}) \boldsymbol{w}^{\varepsilon,\boldsymbol{\varphi}} & \text{in } \Omega, \\ \boldsymbol{w}^{\varepsilon,\boldsymbol{\varphi}} = \mathbf{0} & \text{on } \Gamma_D, \\ [\mathbb{C}_\varepsilon(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{w}^{\varepsilon,\boldsymbol{\varphi}})]\, \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma_0, \end{cases} \tag{AE}$$

with the disjoint splitting $\partial\Omega = \Gamma_D \cup \Gamma_0$. Here the elasticity tensor $\mathbb{C}_\varepsilon$ and the density ρ_ε are from a modeling point of view closely linked to the coefficient functions $a_\varepsilon, c_\varepsilon$ appearing in the Laplace problem (AL). The corresponding sharp-interface problem is of Neumann type and thus a rigorous analysis of the sharp-interface limit is quite delicate and to the best of the author's knowledge not performed yet. Nevertheless, in Chapter 6 we will derive the following sharp-interface limit system together with first-order optimality conditions in the framework of formally matched asymptotic expansions

$$\begin{cases} -\nabla \cdot [\mathbb{C}\mathcal{E}(\boldsymbol{w})] &= \lambda \rho \boldsymbol{w} \qquad \text{in } D, \\ [\mathbb{C}\mathcal{E}(\boldsymbol{w})]\ \boldsymbol{n} &= \mathbf{0} \qquad \text{on } \partial D, \end{cases} \tag{CE}$$

which we simplified for the sake of this introduction.

Now let us come to the formulation of the optimization problems which are given analogously in both parts of the thesis. This is why we will for the moment consider the scalar-valued case. A key ingredient in the optimization problems studied in this thesis will be the so called *Ginzburg–Landau* energy

$$E^\varepsilon_{\mathrm{GL}}(\varphi) = \int_\Omega \frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon} \psi(\varphi) \, \mathrm{d}x,$$

where $\psi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is a bulk potential. This energy was first introduced in [67] as it describes the demixing of two materials or fluids as $\varepsilon \to 0$. In optimization problems it serves as a penalizing term accounting for the desired phase-field structure and providing us with well-posed problems, see e.g., [32,39,43,96]. In the works [23,126,127,135,147,148] this energy is studied in the framework of Γ-convergence as an approximation of the perimeter of the limit shape both in the scalar-valued and in the vectorial case. Of course the analysis of this energy depends on the specific choice of the potential ψ. In this thesis we mostly work with the non-smooth double obstacle potential, rigorously studied in this context in [37], but we will also allow for the standard continuous double-well potential studied in [126, 127, 147]. The double obstacle potential is obtained via a so called deep quench limit from the originally proposed logarithmic energy potential in [67]. From a physical viewpoint this means that using such kind of a non-smooth potential models the behavior of phase separation when the temperature is small compared to the critical temperature, see [37, 130].

It is the goal of this thesis to understand the cost functional

$$J^\varepsilon_l(\varphi) \coloneqq \Psi(\lambda^{\varepsilon,\varphi}_{i_1}, \dots, \lambda^{\varepsilon,\varphi}_{i_l}) + \gamma E^\varepsilon(\varphi),$$

where $\lambda_{i_1}^{\varepsilon,\varphi}, \dots, \lambda_{i_l}^{\varepsilon,\varphi}$ is a finite selection of eigenvalues either to (AL) or (AE) and $\gamma > 0$ is a parameter weighting the perimeter penalization. The function $\Psi : (\mathbb{R}_{>0})^l \to \mathbb{R}$ can be chosen quite general and thus, allows us to formulate a large variety of optimization problems involving the above selection of eigenvalues, such as linear combinations of these eigenvalues. Hence, this general formulation includes problems ranging from classical ones such as minimizing the first Dirichlet eigenvalue, see [90, 108, 117, 139] to problems which especially numerically have not been considered yet, such as the simultaneous minimization and maximization of different eigenvalues. Thus, studying the phase-field approach has two main benefits. On the one hand it is a framework very well suited for mathematical analysis and numerical computations and on the other hand by considering the sharp-interface limit we can justify that the phase-field formulation is indeed a suitable approximation of the aforementioned classical spectral shape and topology optimization problems.

To conclude this introduction let us give an overview over the existing literature in order to put our theory into further context both from an analytical and a numerical viewpoint. In the pioneering papers [61, 63, 73] a relaxed formulation is introduced for shape optimization problems where the state equation is given as a Dirichlet problem. Their theory is based on the relaxation of Dirichlet problems derived in [77]. Relaxation in this context means that the intuitive optimization problem formulated in the class of open sets is replaced by a much more general problem where not the shapes directly but non-negative Borel measures serve as controls in the optimization problem. This relaxation is indeed necessary, because general shape optimization problems on the sharp-interface level are not well-posed in the class of open sets, see e.g., [49, Section 4]. Based on this relaxed formulation the authors of [64] show well-posedness of above optimization problems in the class of so called *quasi-open sets*. In particular, spectral optimization problems associated to the Dirichlet-Laplacian can be treated in this framework, see also [62]. Quasi-openness is a generalization of the concept of openness and is very naturally associated to the Sobolev space $H^1(\Omega)$, see [110, Section 3.3.4] and Section 2.2.6.
This theory will also be of crucial importance when considering the Γ-limit of the cost functionals studied in this thesis in Chapter 3, because there we will perform a smoothing argument on the sharp-interface level which requires the continuity of eigenvalues with respect to this approximation. Note that in our analysis and in above references the monotonicity of the function under which the eigenvalues enter the cost function, in our notation the function Ψ, is crucial, because only then the monotonicity of Dirichlet eigenvalues with respect to set inclusion is inherited by the whole cost functional. In some cases there are results in the literature when this monotonicity assumption does not need to be satisfied. In this spirit in [28, 50] an existence result is shown when the cost functional depends only on the two lowest eigenvalues.
Furthermore it is crucial in all of the above results and also in our analysis that all shapes belonging to the admissible set are contained in a *bounded* design domain. By a deep geometric analysis the existence result of [64] is generalized in [123] to the unbounded case. Independently this generalization was also proven in [48] which relies on the concept of shape subsolutions allowing the author to deduce that minimizers of the k-th eigenvalue are sets of finite perimeter. A further interesting approach is performed in [106] for the minimization of the principal Dirichlet eigenvalue with a volume constraint. There the author shows the existence of a Lipschitz regular minimizer, particularly proving that the minimization in the case of the first Dirichlet Laplace eigenvalue is indeed well-posed in

the class of open sets, not only in the larger class of quasi-open sets. This is of course clear when a ball satisfying the volume constraint fits into the surrounding design domain, due to the Faber-Krahn theorem. The approach was already applied by the same author to general Dirichlet problems with a given force on the right hand-side of the equation instead of the eigenvalue, see [105]. It uses a beautiful approximate problem where a relaxation function g_ε enters which is reminiscent of our penalization function b_ε. Here no phase-field enters but the eigenfunction u_ε of the approximate problem is plugged into g_ε allowing the author to show via a clever Lagrange multiplier technique the convergence of principal eigenvalues of the approximate problem to the initial non-relaxed principal eigenvalue.
Let us also relate our analysis to the work in [12]. The authors there investigate the continuity of eigenvalues of penalized Dirichlet eigenvalues. Similar to the works [38,40,42,133] which will be discussed below, the penalizing coefficient function is given as λa where λ is some constant (not the eigenvalue) sent to infinity in the limit and a is a suitably nice Borel function which is fixed along the limit process and only depends on the spatial variable. Our case of $b_\varepsilon(\varphi_\varepsilon)$ is much more general than just scaling by a constant, as we do not have a priori knowledge on the phase-fields as $\varepsilon \to 0$ other than L^1-convergence and a rate condition. Let us also mention the work [65] which tackles shape optimization problems where the controls are given as Schrödinger operators. As mentioned there, certain types of Schrödinger operators can be used to approximate classical shape optimization problems similarly to the b_ε term in our phase-field approach.

A very vivid field of research is also the analysis of the regularity of minimizing shapes for spectral shape optimization problems, see [56, 66, 80, 81, 118, 124, 125]. Note that our analysis does not provide regularity results but also does not rely on further knowledge on the regularity of minimizers as the sharp-interface problem in our case is always formulated in the class of sets of finite perimeter.

Now let us comment on the numerical aspects. The numerical simulations discussed in this thesis are implemented by Dr. Christian Kahle in a finite element setting via the variable metric projected gradient (`VMPT`) method in order to solve the optimization problem. This method is very well suited for problems formulated in the diffuse interface framework as it extends the projected gradient method to the setting of non-reflexive Banach spaces, which is the class of spaces the optimization problems we deal with are formulated in. The method was developed in [36] and applied to shape and topology optimization, e.g., in [33,98]. We will see that one of the strengths of the phase-field approach in combination with this method is that it allows for a large variety of optimization problems that have not been simulated in the existing literature.
At this point let us review the existing literature concerning numerical approaches for spectral shape and topology optimization problems. In [38, 40, 42, 133] the authors use a phase-field like penalization (also there referred to as *fictious domain method*) in order to obtain a relaxation of an optimal partition problem for the first Dirichlet-Laplace eigenvalue which is suited for numerical computations. Note that their justification of the convergence of the relaxed problem to the desired optimization problem is limited to the *first* Dirichlet eigenvalue as a concavity arguments is exploited. As also stated by the authors this argumentation does not extend to the larger Dirichlet eigenvalues. More precisely, it is only shown that the *values* of the relaxed cost functionals converge to the values of the initial cost functional, see [42, Theorem 2.4]. But for eigenvalues other than the first one it is not clear whether the minimizers of the relaxed problem

converge to a characteristic function representing the desired partitioned shape. In this thesis however, the rigorous study of the sharp-interface limit goes beyond these results as the Γ-convergence of the cost functionals in Chapter 3 is shown no matter which Dirichlet eigenvalues are involved.
Also in the context of optimal partition problems for the first Dirichlet-Laplace eigenvalue, [153] proposes a relaxation approach via a diffusion generated method. Only the continuity of the first eigenvalue is discussed with respect to the relaxation and no Γ-convergence result is provided.
A beautiful numerical comparison between the classical method of boundary variation and a relaxed approach involving the level-set method for the optimization of Dirichlet eigenvalues is contained in [134]. Here the relaxation is directly based on the initial measure theoretic relaxation for shape optimization problems of [63] described above, combined with a finite element method. The results there were improved and also extended to the Neumann case in [15,17] by solving the eigenvalue problem via the method of fundamental solutions, see [13], and the optimization problem via a gradient method evolving the shape with the shape derivative of eigenvalues. Results using the same method for the associated perimeter constrained problem were simulated in [16]. In [15, 16] the shape is parameterized in polar coordinates which restricts their method to the class of star-shaped domains. In [17] the shapes are parameterized by a level-set like method using Fourier series, which also allows for multiple connected components. Let us also mention the related work [30] which uses a finite element discretization combined with the previously mentioned Fourier series parameterization to compute the eigenvalues and boundary variations to solve the optimization problem.

Finally [39] uses a phase-field approach similar to the one proposed in this thesis combined with a finite difference discretization in order to tackle the minimization of Dirichlet eigenvalues with perimeter constraint. Nevertheless, the theory in the thesis at hand allows for more general penalizing terms in the approximate eigenvalue problem and is able to deal with additional volume constraints and point-wise constraints. Furthermore our proof of Γ-convergence is more flexible regarding the construction of the recovery sequence as it only relies on the continuity properties rigorously derived in this thesis when passing from diffuse to sharp-interfaces. More precisely opposed to [39], we do not require the monotonicity of eigenvalues with respect to the recovery sequence in order to obtain the lim sup inequality. In particular, we cover not only the case of smooth potentials in the Ginzburg–Landau energy but account for the non-smooth double-obstacle potential.

In the numerical section of Chapter 3 we will reproduce the results of [17] but simulate several other non-standard examples such as simultaneous minimization and maximization of eigenvalues or the presence of obstacles within the design domain. In the multi-phase case in Chapter 5 we will reproduce results of [5] obtained via the level-set method but also simulate other non-standard examples such as the maximization of the sum of the first two eigenvalues and joint compliance and spectral optimization.

So far we have mostly focused on eigenvalues to the Dirichlet problem. To conclude this introduction let us discuss the issues occurring when considering Neumann boundary conditions. On the sharp-interface level the spectrum of the Neumann-Laplacian can be extremely unstable under general domain perturbations as stated in [49, Section 7]. As shown in a classical example by Courant and Hilbert, see [108, Section 2.3.4], the first non-trivial Neumann eigenvalue can degenerate along a sequence of shapes γ-converging

to a rectangle. Obviously a rectangle has non-trivial second Neumann eigenvalue. So there is in general no hope to obtain continuity of the Neumann spectrum. For the concept of γ-convergence see Section 2.2.6. Note that as opposed to the Neumann eigenvalues, the *Dirichlet* eigenvalues are continuous with respect to γ-convergence.
The main problem, in particular in the light of a Γ-convergence approach, is that the Neumann problem as opposed to the Dirichlet or the Robin problem does not involve an energy which controls the behavior of eigenfunctions at the boundary, see also [54] where the additional boundary energy in the Robin problem is crucial for the analysis in the SBV setting.
Nevertheless, there is very recent progress concerning the Neumann problem also from the viewpoint of a suitable numerical approximation, see [55]. There the authors use degenerate densities which also naturally arise in our phase-field approach as the coefficient functions $a_\varepsilon(\varphi)$ and $c_\varepsilon(\varphi)$. In [55, Lemma 14] they prove an approximation result for the Neumann eigenvalues which we believe can also be used to prove the continuity when passing from the diffuse interface level to the sharp-interface level if the final shape is assumed to be smooth enough. But a delicate subsequent step would be the approximation of an arbitrary finite perimeter set by smooth sets on the sharp-interface level and to obtain the continuity of the Neumann eigenvalue with respect to this approximation in analogy to the Γ-convergence proof in the Dirichlet case in Chapter 3. Note that using our approach we can guarantee the convergence of shapes not only in symmetric difference but the associated characteristic functions converge strictly in BV which is a slightly better property, as this guarantees that in the limit no perimeter is lost, which is a key issue in the counter example by Courant and Hilbert mentioned above.
This discussion motivates why in Part II of this thesis in Chapter 6 we pursue a formal approach in order to derive a limit optimality system. As noted above, we will see in Chapter 6 that (AE) corresponds to the Neumann problem (CE) on the sharp-interface level, so a rigorous limit passage at the moment seems out of reach. Nevertheless, the spaces of generalized functions of bounded variation $GSBV$ applied to the Ambrosio-Tortorelli functional to provide a phase-field relaxation of the Mumford-Shah problem and generalized functions of bounded deformation $GSBD$ applied to linear elasticity are verified to be the adequate function spaces for Γ-convergence results when no boundary energy is present, see [45, 69–71, 76, 92]. Thus, it would be a promising further research topic to use theses spaces in order to derive Γ-convergence results for optimization problems involving the Neumann spectrum.

1.2. Structure and main contributions of this thesis

In Chapter 2 we will formulate the problems, make the necessary assumptions and give the most important theory applied in this thesis.
In Chapter 3 we will focus on general optimization problems involving the eigenvalues of the Laplacian with either Dirichlet or Neumann boundary data.
Section 3.2.1 comprises the analysis of the optimization problems associated to the Dirichlet and Neumann-Laplace eigenvalues, i.e., it studies the existence of the spectrum and the continuity and differentiability of spectral quantities with respect to the phase-field variable. In Section 3.2.2 we will state the optimality system resulting from these previous properties. In particular the Fréchet-differentiability of simple eigenvalues is used to arrive at a variational inequality serving as first-order optimality condition. Here,

the assumption that the considered eigenvalues are simple will be essential, as otherwise Fréchet-differentiability can not be guaranteed. Note that in these two sections we keep the proofs short as they directly follow from their counterparts in linear elasticity proven in full detail in Chapter 5.
In Section 3.3 we perform an in depth and rigorous study of the sharp-interface limit of the Dirichlet problem in the framework of Γ-convergence. As a first step we will analyze the sharp-interface limit problem and especially the continuity of spectral quantities when passing from diffuse to sharp-interfaces, see Section 3.3.1. On the basis of these results we will then show in Section 3.3.2 the Γ-convergence of cost functionals in the Dirichlet case. In a first step in Theorem 3.3.11 we will show Γ-convergence of cost functionals which do not involve the volume constraint and then in the final proof of Theorem 3.3.9 we will see how we can modify the recovery sequence constructed in Theorem 3.3.11 in order to account for the desired mean value. Finally, in Section 3.4 we provide numerical simulations, which were carried out by Dr. Christian Kahle from the University of Koblenz. Through many numerical examples we show the strength of the phase-field approach applied to shape and topology optimization problems. In particular our approach allows us to reproduce former results obtained by [17] but is also capable of dealing with a broad variety of spectral shape optimization problems exceeding the classical examples given in the literature.

In Chapter 4 we then apply the results of the previous chapter in the context of the most fundamental problem when it comes to spectral optimization, namely the Faber–Krahn theorem. The intention of this chapter is to derive a novel phase-field version of this celebrated theorem. Our main results show that all minimizers φ_ε of this optimization problem are radially symmetric-decreasing functions which indeed exhibit a phase-field structure (see Theorem 4.3.7 and Theorem 4.3.8). This radial symmetry of the phase-fields is the natural analogue to the radial symmetry of the balls in the Faber–Krahn inequality. Furthermore, by means of Γ-convergence we link this diffuse interface version to a general Faber–Krahn theorem in the framework of functions of bounded variations, see Theorem 4.3.15 and Corollary 4.3.16. Note that in this chapter a homogeneous Dirichlet boundary condition is crucially imposed on the *phase-field* variable in order to apply the theory of symmetric-decreasing rearrangements. This gives rise to a Γ-convergence proof in the case of boundary data extending the theory of the previous chapter. At this stage we will give two versions of the proof, one inspired by [43] and the other inspired by the more general framework of [135].

In Chapter 5 we consider eigenvalues in the framework of linear elasticity. As explained above we now analyze a system of equations in the multi-phase case. As this system is more involved we present here all the proofs in detail which then are easily carried over to the proofs of Section 3.2. First of all in Section 5.2 we prove the existence of spectral quantities solving the state equation. Afterwards in Section 5.3 we prove continuity of eigenvalues and eigenfunctions with respect to the phase-field variable. Here the notion of continuity is specifically tailored to the one induced by the optimization problem. More precisely, we show continuity of spectral quantities when the phase-fields only weakly converge in $H^1(\Omega;\mathbb{R}^d)$, see Theorem 5.3.4. However, under stronger assumptions on the convergence of the phase-fields we will even show Lipschitz continuity of eigenvalues, see Lemma 5.3.5.
In Section 5.4 we show the Fréchet-differentiability of simple eigenvalues and eigenfunctions in Theorem 5.4.3 and provide an ansatz for still obtaining directional derivatives in

the case of a multiple principal eigenvalue. Afterwards, in Section 5.5 the well-posedness of the associated optimization problem is shown. Finally, in Theorem 5.5.3, we arrive at first-order necessary optimality conditions which involve a variational inequality crucially depending on the derivative of the involved eigenvalues. Finally in Section 5.6 we combine the eigenvalue problem with compliance minimization problems as studied, e.g., in [32].

In Chapter 6 we use the method of formally matched asymptotic expansions in order derive the sharp-interface limit of the problem studied in Chapter 5. This method will not only provide us with an eigenvalue equation in the limit but also with a gradient equality, see Section 6.7. This gradient equality is obtained in two steps. First, in Section 6.3, we carefully analyze the variational *inequality*, obtained in the previous chapter, via the regularization procedure from [35] in order to arrive at an *equality* on the diffuse interface level which is suited for the asymptotic process. In order to arrive at this equality a regularity assumption on the eigenfunctions involved in the gradient inequality will be crucial.
The asymptotic process itself will be carried out in several steps. In Section 6.4.1 we derive the state equations of the limit problem via outer expansions which capture the behavior of the phase-field in regions away from an interface. Furthermore in Subsection 6.4.2 we will include a discussion on how an adequate choice of model parameters is able to deal with localized eigenmodes. This will in particular motivate the model used for the final numerical computations.
The foundations for inner expansions are layed in Section 6.4.3 in full detail where a rescaled coordinate system is introduced in order to capture rapid changes of the phase-field variable in interfacial layers. In intermediate layers both expansions are matched in Section 6.5. Finally in Section 6.6 both the state equation and the gradient equality are considered to leading orders. This procedure allows us to derive boundary conditions on the free boundary prescribed by the materials in the sharp-interface setting. Furthermore it will provide us with a limit of the gradient equality. Section 6.7 then comprehensively states the limit optimality system. In Section 6.8 we will relate this equality to the first-order condition of the corresponding optimization problem formulated in the framework of classical shape derivatives from [5]. In the numerical simulations, carried out by Dr. Christian Kahle, in Section 6.9 we will justify our model by successfully comparing it to the cantilever beam examples in [5]. Furthermore we will provide simulations for the simultaneous maximization of the first two eigenvalues and a joint optimization of compliance and principal eigenvalue.

The main results of this thesis appear in the following publications:

- H. Garcke, P. Hüttl, C. Kahle, P. Knopf, and T. Laux, Phase-field methods for spectral shape and topology optimization, *ESAIM Control Optim. Calc. Var.*, 29 (2023), Paper No. 10.
- P. Hüttl, P. Knopf, and T. Laux, A phase-field version of the Faber–Krahn theorem. Preprint: arXiv:2207.10946, 2022.
- H. Garcke, P. Hüttl, and P. Knopf, Shape and topology optimization involving the eigenvalues of an elastic structure: A multi-phase-field approach, *Adv. Nonlinear Anal.*, 11 (2022), pp. 159–197.
- H. Garcke, P. Hüttl, P. Knopf, and C. Kahle, Sharp-interface limit of a multi-phase spectral shape optimization problem for elastic structures.
  Preprint: arXiv:2304.02477, 2023.

Chapter 2

Preliminaries

2.1. Formulation of the problem

2.1.1. The design domain and the phase-field variable

Throughout this thesis we fix a bounded design domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ with Lipschitz boundary. Whenever we are concerned with Γ-convergence, i.e. in Section 3.3.2 and Theorem 4.3.17 we need to strengthen this assumption to $d \geq 2$ because there we are in need of the theory of quasi-open sets, see Section 2.2.6, but we will recall this fact later on. In Chapter 4, we will always restrict the design domain to the case $\Omega = B_R(0)$, i.e., Ω is the ball centered at the origin with finite radius $R > 0$.
Within the design domain now lives a set D which from a mathematical point of view in our model will be quite general namely a set of finite perimeter, see Section 2.2.4. This set is the quantity that is to be optimized and thus we denote it as *shape*. More precisely, the goal of our shape and topology optimization problems is to minimize a cost functional involving a selection of eigenvalues by adjusting the shape D in an optimal way.

The key concept in this thesis is to replace this sharp interface problem by a diffuse interface relaxation via a *phase-field approach*. Here we need to differentiate two different models according to the two different parts of this thesis.
In Chapter 3, to approximate the shape D, we use a scalar-valued phase-field function $\varphi_\varepsilon : \Omega \to [-1, 1]$ which attains the value $+1$ in most parts of D and the value -1 in most parts of the relative complement $\Omega \backslash D$. These extremal values of the phase-field will be referred to as the pure phases as they capture either the shape we are interested in or its complement. In physical terms the set $\{x \in \Omega \mid \varphi_\varepsilon(x) = 1\}$ is the set where the design domain is entirely filled by one material, whereas the set $\{x \in \Omega \mid \varphi_\varepsilon(x) = -1\}$ corresponds to the void part. In Chapter 4 we are going to replace the void value -1 by the value 0 in order to account for an elegant application of the theory of symmetric-decreasing rearrangements, see Section 2.2.3.
In Part II of the thesis we pass to the so called *multi-phase approach*, by considering a vector-valued phase-field variable. Here the distribution of $N \in \mathbb{N}$ materials is represented by the vector-valued phase-field $\boldsymbol{\varphi}_\varepsilon : \Omega \to \mathbb{R}^N$. This means, for any $i \in \{1, ..., N\}$, the component $\varphi_{i,\varepsilon}$ ranging from 0 to 1 can be interpreted as the concentration of the i-th material. In this regard, $\varphi_{i,\varepsilon} = 0$ describes the absence of the i-th material, whereas $\varphi_{i,\varepsilon} = 1$ means that only the i-th material is present. We use the convention that void is

also interpreted as a sort of material, whose distribution is given by the N-th component of the vector $\boldsymbol{\varphi}_\varepsilon$. Thus, in the vector-valued case the pure phases are represented by the family of standard basis vectors $\boldsymbol{e}_i \in \mathbb{R}^N$ for $i = 1, \dots, N$.

In any of these cases, the free boundary ∂D of the sharp shape is approximated by a thin interfacial layer in which the phase-field variable performs a suitably regular transition between the pure phases. This regularity will be understood in the weak sense of Sobolev spaces in the following section. The thickness of this transition region, also referred to as *diffuse interface*, is proportional to a small parameter $\varepsilon > 0$ also indicated in the phase-field variable. In our notation we will often drop this ε dependence when $\varepsilon > 0$ is fixed.

In the sharp interface limit, i.e., sending the interface thickness $\varepsilon \to 0$, we expect a corresponding sequence of phase-fields to converge to a function with much less regularity. Loosely speaking, this limit function will exhibit jumps across the sharp interfaces separating the pure phases. Here the space of functions of bounded variation, see Section 2.2.4, will provide an adequate formulation. The rigorous analysis of the sharp interface limit for the Dirichlet-Laplace spectral optimization problem in Chapter 3 and the formal analysis of the sharp interface limit for the spectral optimization problem in the framework of linear elasticity in Chapter 6 are at the heart of this thesis.

As the notation and the specific assumptions vary between the two-phase case in Part I and the multi-phase case in Part II, we will at first focus on the preliminaries of the first part and then explain the necessary adaptions in upcoming sections introducing the second part.

2.1.2. The constraints on the phase-field in Part I

Having optimization problems in mind we will now formulate the constraints that we demand the phase-fields to satisfy. For the moment let us focus on the setting of Chapter 3. First of all, we prescribe regions within the design domain Ω where the pure phases are prescribed. Mathematically speaking, we fix two disjoint measurable sets $S_0, S_1 \subset \Omega$ such that $\tilde{\Omega} \coloneqq \Omega \backslash (S_0 \cup S_1)$ is a domain with Lipschitz boundary. This is, for example, the case if we choose S_0 and S_1 as closed balls which keep a fixed, positive distance between themselves and towards the boundary $\partial\Omega$ of the design domain. Thus, the set representing these constraints on φ is given as

$$\mathcal{U} \coloneqq \left\{ \varphi \in L^1(\Omega) \,\middle|\, \varphi = -1 \text{ a.e. on } S_0 \text{ and } \varphi = 1 \text{ a.e. on } S_1 \right\}. \tag{2.1.1}$$

As the values of $\varphi \in \mathcal{U}$ are fixed on $S_0 \cup S_1$, the relevant set in our optimization process is given as $\tilde{\Omega}$ which will play also an important role in the definition of the Ginzburg–Landau energy in Section 2.1.4.

We additionally prescribe general bounds on the mean value of φ in order to take volume constraints into account. To this end, we impose the general constraint

$$\beta_1 |\tilde{\Omega}| \leq \int_{\tilde{\Omega}} \varphi \, \mathrm{d}x \leq \beta_2 |\tilde{\Omega}|,$$

with $\beta_1, \beta_2 \in \mathbb{R}$, $\beta_1 \leq \beta_2$ and $\beta_1, \beta_2 \in (-1, 1)$. This condition ensures that in the sharp interface case, where $\varphi \in BV(\tilde{\Omega}, \{\pm 1\})$, the sets $\{\varphi = 1\} \cap \tilde{\Omega}$ and $\{\varphi = -1\} \cap \tilde{\Omega}$ both have

strictly positive measure. Hence, the trivial cases are excluded. Note that for the choice $\beta_1 = \beta_2$ we obtain an equality constraint for the mean value.

Furthermore, we require sufficient regularity of the phase-field, namely $H^1(\tilde{\Omega})$, in order to account for suitably smooth transitions between two pure phases and for the Ginzburg–Landau energy (that will be introduced in the next subsection) to be well-defined. All these constraints are summarized in the set

$$\mathcal{G}^\beta = \left\{ \varphi \in H^1(\tilde{\Omega}) \,\middle|\, |\varphi| \leq 1, \beta_1 |\tilde{\Omega}| \leq \int_{\tilde{\Omega}} \varphi \,\mathrm{d}x \leq \beta_2 |\tilde{\Omega}| \right\}. \tag{2.1.2}$$

Finally, the admissible set, i.e., the set over which we will formulate our optimization problem in Chapter 3, see also Section 2.1.7, is given as $\Phi_{\mathrm{ad}} \coloneqq \mathcal{G}^\beta \cap \mathcal{U}$. Note that we only demand $\varphi \in H^1(\tilde{\Omega})$, i.e., we do only prescribe regularity in the region of the design domain where the phase-field is not fixed by the prescribed values in $\mathcal{U}$. We will comment on that in further detail in Section 2.1.4.

We point out that for the upcoming analysis in Section 3.2, we could also include a constraint preventing the shape to touch the boundary, i.e., the Dirichlet boundary condition on the phase-field $\varphi = -1$ on $\partial\Omega$, which is used also in some of the numerical simulations presented in Section 3.4. In order for this constraint to be well-defined in the trace sense, and not to interfere with the one formulated via $\mathcal{U}$ in (2.1.1), we would also need to demand $S_1 \subset\subset \Omega$ to be compactly contained. Nevertheless, we do not include this Dirichlet condition in the discussion of the sharp interface limit in Section 3.3 as this would produce an additional term in the Γ-limit of the Ginzburg-Landau energy as explained in the following section.

In Chapter 4 we will rigorously incorporate this boundary condition on the phase-field and especially prove the Γ-limit result in this more complex setting requiring to carefully adapt the Γ-limit proof of Theorem 3.3.11 to obtain Theorem 4.3.17. Note that we are also forced to do so, because without the inclusion of this Dirichlet condition the application of the technique of symmetric-decreasing rearrangements is bound to fail, see Remark 2.2.13. Let us now introduce the remaining constraint formulated in Chapter 4. There we are concerned with the Faber–Krahn theorem where only a volume constraint is present. Thus, we do not prescribe the point-wise constraint (2.1.1) there. Furthermore above volume constraint is simplified to

$$⨍_\Omega \varphi \,\mathrm{d}x = m,$$

with prescribed $m \in (0,1)$. Recall that here the pure phase -1 used in Chapter 3 in order to model void is replaced by the value 0, thus the volume constraint exactly corresponds to the case when $\beta_1 = \beta_2$ previously mentioned. Combining this volume constraint with the appropriate regularity of the phase-field and the above mentioned Dirichlet boundary condition we arrive at the admissible set

$$\Phi_{\mathrm{ad}} := \left\{ \varphi \in H^1_0(\Omega) \,\middle|\, \begin{array}{l} 0 \leq \varphi(x) \leq 1 \ \text{ for a.e. } x \in \Omega, \\ ⨍_\Omega \varphi \,\mathrm{d}x = m \end{array} \right\} \subset H^1_0(\Omega) \cap L^\infty(\Omega).$$

2.1.3. The constraints on the phase-field in Part II

Let us now consider the constraints formulated for the vector-valued phase-field variable $\boldsymbol{\varphi} : \Omega \to \mathbb{R}^N$ used in Chapter 5 and 6. We want to prescribe the total spatial amount of

each phase. To this end, we impose the mean value constraint

$$\fint_\Omega \boldsymbol{\varphi}\,\mathrm{d}x = \boldsymbol{m} = (m_i)_{i=1}^N,$$

where $m^i \in (0,1)$ is a fixed given number for any $i \in \{1,\dots,N\}$. In addition, we want $\boldsymbol{\varphi}(x)$ for a.e. $x \in \Omega$ and thus, also the vector $\boldsymbol{m}$ to be an element of the set

$$\Sigma^N = \left\{ \boldsymbol{\xi} \in \mathbb{R}^N \,\middle|\, \sum_{i=1}^N \xi_i = 1 \right\}.$$

This constraint is a plausible consequence of the physical fact that at each point in space the volume fractions of the materials should sum up to 1. Furthermore, being a volume fraction, each component of $\boldsymbol{\varphi}(x)$ clearly has to be non-negative. For the upcoming analysis, in analogy to the two-phase case we want to prescribe a suitable regularity for the phase-field, namely $H^1(\Omega;\mathbb{R}^N)$. All these constraints are expressed in the set

$$\boldsymbol{\mathcal{G}}^{\boldsymbol{m}} = \left\{ \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}} \,\middle|\, \fint_\Omega \boldsymbol{\varphi}\,\mathrm{d}x = \boldsymbol{m} \right\}.$$

Here, $\boldsymbol{\mathcal{G}}$ is given by

$$\boldsymbol{\mathcal{G}} = \left\{ \boldsymbol{\varphi} \in H^1(\Omega;\mathbb{R}^N) \,\middle|\, \boldsymbol{\varphi}(x) \in \boldsymbol{G} \text{ for almost all } x \in \Omega \right\},$$

where $\boldsymbol{G} = \mathbb{R}^N_+ \cap \Sigma^N$ with

$$\mathbb{R}^N_+ = \left\{ \boldsymbol{\xi} \in \mathbb{R}^N \,\middle|\, \xi_i \geq 0 \quad \forall i \in \{1,\dots,N\} \right\}.$$

The set $\boldsymbol{G}$ is referred to as the *Gibbs-Simplex*.

In Chapter 5 we additionally impose a point-wise constraint in analogy to the previous section. Thus, we fix two disjoint measurable sets $S_i \subset \Omega$ with $i \in \{0,1\}$ and define the set

$$\boldsymbol{U}_c \coloneqq \left\{ \boldsymbol{\varphi} \in H^1(\Omega;\mathbb{R}^N) \,|\, \varphi_N = 0 \text{ a.e. on } S_0 \text{ and } \varphi_N = 1 \text{ a.e. on } S_1 \right\},$$

to fix material on S_0 and complete void on S_1. Note that as we are not concerned with the sharp interface limit in Chapter 5 we can simply impose the H^1-regularity on the whole of Ω without worrying about the regularity of the sets S_0, S_1. Nevertheless, in order to obtain a well-posed optimization problem in Section 2.1.11, these sets have to be chosen such that $\boldsymbol{U}_c \neq \emptyset$, which is for example guaranteed if the sets keep a positive fixed distance.

In Chapter 6 however, we omit above constraint as our analysis in Section 6.3 is not able deal with it. More precisely, in the regularization process applied to the gradient inequality, we would get additional terms resulting from the constraint $\boldsymbol{U}_c$ in which we are not able to pass to the limit. Therefore we omit this constraint in order to recover a strong formulation of the original non-regularized gradient inequality by passing the regularization parameter to the limit, see also Theorem 6.3.12.

2.1.4. The Ginzburg–Landau energy

Let us give a general introduction to the Ginzburg–Landau energy which in phase-field models is a key ingredient in the objective functional. Let us assume for the moment that we can formulate this energy on the whole design domain Ω. Then the Ginzburg–Landau energy is given as

$$E^{\varepsilon}_{\mathrm{GL}}(\varphi) = \int_{\Omega} \frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon} \psi(\varphi) \, \mathrm{d}x, \quad \varepsilon > 0. \tag{2.1.3}$$

Here, the potential $\psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is assumed to have exactly the two global minimum points at the pure phases, which we denote for the sake of this general introduction with -1 and $+1$, with

$$\min_{\mathbb{R}} \psi = \psi(\pm 1) = 0.$$

We have already indicated in the introduction that the minimization of the Ginzburg–Landau energy enforces the demixing of phases into their pure form. Heuristically, this is seen as follows. The gradient term in (2.1.3) wishes the phase-field to be as constant as possible or in other words to have as less transitions as possible. Our volume constraints on φ of the previous section are now chosen in such a way that the design domain is neither completely occupied by the shape neither completely empty. Thus, transitions need to take place. Naively, the gradient term in the energy can be kept small by choosing this transition region to be very thin. But the thinner this region gets the steeper the phase-field transition gets and we pay with a large gradient.
These two effects now obviously compete and this is where the interface parameter ε comes into play. Let us assume that the thickness of the transition region corresponding to φ_ε is of order $\mathcal{O}(\varepsilon)$. Then formally the gradient is of qualitative order

$$\|\nabla \varphi_\varepsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\varepsilon^{-1}).$$

In combination, we deduce that as long as the transition region is of order $\mathcal{O}(\varepsilon)$ the gradient term in the energy (2.1.3) remains finite as $\varepsilon \to 0$.
Now let us come to the potential term. So far the gradient term only tells us that transitions should happen at a length-scale of order $\mathcal{O}(\varepsilon)$ but not in between which specific values this takes place. So it is not surprising that the potential is constructed in such a way that it favours the pure phases. More precisely by considering the above transition φ_ε enforced by the gradient term, we arrive at

$$\int_{\Omega} \psi(\varphi_\varepsilon) \, \mathrm{d}x = \mathcal{O}(\varepsilon),$$

as ψ vanishes outside the transition layer which is of order $\mathcal{O}(\varepsilon)$. Thus, this discussion justifies why we expect minimizers of the Ginzburg–Landau energy to perform transitions between the pure phases on a legthscale proportional to ε. At this point we want to mention that from a numerical point of view the constants appearing in such qualitative estimates can in general be quite large and thus "proportional" especially in the quantitative sense has to be understood with care.

As already indicated in the introduction and as we will also see explicitly in Chapter 4 there are two qualitatively different cases depending on the explicit choice of the bulk

potential ψ. We will refer to these cases as the smooth and the non-smooth case and give an in depth mathematical treatment of the following discussion in Chapter 4 and especially the proofs of Theorem 4.3.17.
If ψ is chosen to be a function belonging to $C^{1,1}_{loc}(\mathbb{R},\mathbb{R})$ we speak of the smooth case. A classical choice here is the quartic double-well potential $\psi(\varphi) = \frac{1}{4}(1-\varphi^2)^2$, see [126,147]. In this case, although the transition layer is qualitatively of order $\mathcal{O}(\varepsilon)$, the pure phases will be reached only asymptotically. In particular we will see in the construction of the optimal profile in Theorem 4.3.17 that a stretching of the transition layer combined with a suitable interpolation will be necessary in order for the phase-field to reach the pure phase values on a *finite* length-scale, see also the construction in [135,147] our analysis in this case is based on. A beautiful side note, also mentioned in [135], is that this stretching introduces an *intermediate region* which is crucial in the theory of formally matched asymptotic expansions because in this region both inner and outer expansions meet and via matching conditions one is able to deduce boundary conditions on the sharp-interface boundary, see Chapter 6.
Now let us come to the non-smooth case, which will be the one mostly used in this thesis. Let ψ be decomposed as $\psi(\varphi) = \psi_0(\varphi) + I_{[-1,1]}(\varphi)$ with $I_{[-1,1]} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ being the indicator functional

$$I_{[-1,1]}(\xi) = \begin{cases} 0 & \text{if } \xi \in [-1,1], \\ +\infty & \text{otherwise.} \end{cases}$$

and $\psi_0 \in C^{1,1}_{loc}(\mathbb{R};\mathbb{R})$, denoted as the regular part of ψ. A classical choice here is the quadratic potential $\psi_0(\varphi) = \frac{1}{2}(1-\varphi^2)$, see [37]. Now ψ is a non-smooth potential taking the value infinity outside the pure-phase interval. This choice is referred to as *double-obstacle potential* as it enforces φ to stay within the pure phase interval $[-1,1]$ as this interval is not penalized by the indicator functional. We refer to [87] who first introduced this obstacle formulation of the energy $E^\varepsilon_{\mathrm{GL}}$. Compared to the smooth case this choice has the benefit that it is possible to construct an optimal profile directly connecting the pure phase values on a length-scale of $\mathcal{O}(\varepsilon)$, i.e., the pure phases are reached in finite time, see the proofs of Theorem 4.3.17. Note that in the framework of [37] the energy is scaled with $\sqrt{\varepsilon}$ and not with ε as in our setting.
If we do not give detailed assumptions on the choice of the potential in the upcoming analysis of this thesis we will allow both for the smooth case and for the non-smooth case, with general $\psi \in C^{1,1}_{loc}$ in the smooth case and $\psi_0 \in C^{1,1}_{loc}$ in the non-smooth case. In particular if we are in the smooth case we will identify $\psi = \psi_0$ in our notation.

In the light of first-order optimality conditions, see Chapter 3 and Chapter 5, we also wish to derive the Ginzburg–Landau energy with respect to φ. In our optimization problems, we impose the phase-field constraint $\varphi \in \Phi_{\mathrm{ad}}$, which implies that the phase-field variable does not leave the pure phase interval. Thus, it suffices to include the regular part

$$E^\varepsilon(\varphi) := \int_\Omega \left(\frac{\varepsilon}{2} |\nabla\varphi|^2 + \frac{1}{\varepsilon}\psi_0(\varphi) \right) \mathrm{d}x, \tag{2.1.4}$$

of the Ginzburg–Landau energy in the cost functional, since $E^\varepsilon(\varphi) = E^\varepsilon_{\mathrm{GL}}(\varphi)$ for all $\varphi \in \Phi_{\mathrm{ad}}$. This is important for the analysis as it allows us to compute directional derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t} E^\varepsilon(\varphi + t(\tilde\varphi - \varphi))\big|_{t=0} = \int_\Omega \varepsilon \nabla\varphi \cdot \nabla(\tilde\varphi - \varphi)\,\mathrm{d}x + \int_\Omega \frac{1}{\varepsilon}\psi_0'(\varphi)(\tilde\varphi - \varphi)\,\mathrm{d}x,$$

in *every* direction $\tilde{\varphi} - \varphi$ with $\tilde{\varphi} \in \Phi_{\mathrm{ad}}$.

Let us mention a further detail which will be important for Chapter 3 in view of the Γ-limit performed in Section 3.3.2. In the light of the point-wise constraint (2.1.1) the relevant set of our optimization process is $\tilde{\Omega}$ thus, in Chapter 3 we consider the Ginzburg–Landau energy restricted to this set $\tilde{\Omega}$, i.e.,

$$E^{\varepsilon}_{\mathrm{GL}}(\varphi) = \int_{\tilde{\Omega}} \frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon} \psi(\varphi) \, \mathrm{d}x.$$

Thus, it is also natural to include only the regularity $H^1(\tilde{\Omega})$ in the definition of $\mathcal{G}^{\beta}$ in (2.1.2). If we demanded $\varphi \in H^1(\Omega) \cap \mathcal{U}$, we would obtain Dirichlet conditions

$$\begin{aligned} \varphi &= -1 \quad \text{on } \partial S_0, \\ \varphi &= 1 \quad \text{on } \partial S_1. \end{aligned}$$

This would produce an additional contact energy term in the Γ-limit of E^{ε} as $\varepsilon \to 0$, see [135]. In order to avoid this phenomenon and to obtain the classical Γ-limit as studied in [37,126,147], the energy and the H^1-regularity are restricted to the subset $\tilde{\Omega} \subset \Omega$, which also reveals the regularity assumption on the sets S_0, S_1 with respect to Ω. Nevertheless, as mentioned above, we will rigorously include a homogeneous Dirichlet boundary condition on the phase-field in Chapter 4.

In Part II of this thesis the Ginzburg–Landau energy is defined as in (2.1.3) but now of course depending on the vector-valued phase-field $\boldsymbol{\varphi} : \Omega \to \mathbb{R}^N$. Thus, we suitably modify the bulk potential ψ. The function $\psi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ should attain exactly N global minima of value zero attained at the unit vectors $\boldsymbol{e}_i \in \mathbb{R}^N$, i.e., for all $i \in \{1, ..., N\}$,

$$\min \psi = \psi(\boldsymbol{e}_i) = 0.$$

Recall from Section 2.1.3 that the unit vectors exactly correspond to the pure phases in our model. Furthermore, in the whole of Part II, ψ is assumed to exhibit the decomposition $\psi(\boldsymbol{\varphi}) = \psi_0(\boldsymbol{\varphi}) + I_{\boldsymbol{G}}(\boldsymbol{\varphi})$ with $\psi_0 \in C^{1,1}(\mathbb{R}^N, \mathbb{R})$ and $I_{\boldsymbol{G}}$ being the indicator functional

$$I_{\boldsymbol{G}}(\boldsymbol{\varphi}) = \begin{cases} 0 & \text{if } \boldsymbol{\varphi} \in \boldsymbol{G}, \\ +\infty & \text{otherwise.} \end{cases}$$

Analogously as for the scalar case, this type of obstacle functional is used to enforce that $\boldsymbol{\varphi}$ attains its values only in $\boldsymbol{G}$. In our optimization problems, we will impose the phase-field constraint $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$ which ensures that $\boldsymbol{\varphi}(x) \in \boldsymbol{G}$ for almost all $x \in \Omega$. Thus, as in the previous discussion for the scalar case it suffices to include the regular part (2.1.4) in the cost functional.

2.1.5. The approximate eigenvalue problems in Part I

For any $\varepsilon > 0$, we now introduce approximate eigenvalue problems with Dirichlet boundary condition and Neumann boundary condition, respectively. These problems will be governing the analysis in Chapter 3 and 4. They will act as the state equation in the forthcoming optimization problems. We either consider

$$\begin{cases} -\nabla \cdot [a_\varepsilon(\varphi) \nabla w] + b_\varepsilon(\varphi) w = \lambda^{\varepsilon,\varphi} c_\varepsilon(\varphi) w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1.5}$$

or

$$\begin{cases} -\nabla \cdot [a_\varepsilon(\varphi)\nabla w] = \mu^{\varepsilon,\varphi} c_\varepsilon(\varphi) w & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1.6}$$

Here $\boldsymbol{\nu}$ is the outer unit normal vector on $\partial\Omega$, and $a_\varepsilon, b_\varepsilon, c_\varepsilon : \mathbb{R} \to \mathbb{R}$ are coefficient functions which depend on the phase-field φ and the interface parameter $\varepsilon > 0$. Note that these equations are imposed on the whole design domain and not just on the restriction $\tilde{\Omega}$. Of course this is necessary as the set S_1 is fixed to be part of the final shape on which the eigenvalue equation in the sharp interface limit shall hold.

For fixed $\varepsilon > 0$, we demand that $a_\varepsilon, c_\varepsilon > C_\varepsilon > 0$ in order to avoid degeneration for fixed ε and $b_\varepsilon \geq 0$ in $\mathbb{R}$. Note that if we would just choose $a_\varepsilon = c_\varepsilon \equiv 1$ and $b_\varepsilon \equiv 0$ we would obtain the classical Dirichlet and Neumann eigenvalue problems formulated on the *whole* of Ω. The beauty of these coefficient functions is that they will enforce the Dirichlet and Neumann problems on a shape *within* the design domain. We further assume $a_\varepsilon, b_\varepsilon, c_\varepsilon \in C^{1,1}_{loc}(\mathbb{R})$. These properties allow us to define the following scalar products on $L^2(\Omega)$ depending on the phase-field $\varphi \in L^\infty(\Omega)$:

$$(u,\eta)_{a_\varepsilon(\varphi)} \coloneqq \int_\Omega a_\varepsilon(\varphi) u\eta \,\mathrm{d}x, \quad (u,\eta)_{c_\varepsilon(\varphi)} \coloneqq \int_\Omega c_\varepsilon(\varphi) u\eta \,\mathrm{d}x, \quad u,\eta \in L^2(\Omega).$$

The induced norms on $L^2(\Omega)$ are

$$\|u\|_{a_\varepsilon(\varphi)} = (u,u)^{\frac{1}{2}}_{a_\varepsilon(\varphi)}, \quad \|u\|_{c_\varepsilon(\varphi)} = (u,u)^{\frac{1}{2}}_{c_\varepsilon(\varphi)}. \tag{2.1.7}$$

In the following, we use the notation $L^2_\varphi(\Omega)$ to indicate that $L^2(\Omega)$ is equipped with the φ-dependent scalar product $(\cdot,\cdot)_{c_\varepsilon(\varphi)}$. Similarly, we equip the spaces $H^1_0(\Omega)$ and

$$H^1_{(0),\varphi}(\Omega) = \left\{ w \in H^1(\Omega) \,\middle|\, \int_\Omega c_\varepsilon(\varphi) w \,\mathrm{d}x = 0 \right\}$$

with the scalar product $(\nabla\cdot, \nabla\cdot)_{a_\varepsilon(\varphi)}$. For the purpose of a clearer presentation we further define a positive semi-definite bi-linear form $(\cdot,\cdot)_{b_\varepsilon(\varphi)}$ in the same fashion as for the coefficient functions $a_\varepsilon, c_\varepsilon$. However, this bi-linear form does not define a scalar product as it possibly degenerates.

In the subsequent analysis, we will work with the weak formulations of the approximate problems (2.1.5) and (2.1.6) which are given as

$$(\nabla w, \nabla\eta)_{a_\varepsilon(\varphi)} + (w,\eta)_{b_\varepsilon(\varphi)} = \lambda^{\varepsilon,\varphi} (w,\eta)_{c_\varepsilon(\varphi)} \quad \text{for all } \eta \in H^1_0(\Omega), \tag{2.1.8}$$

and

$$(\nabla w, \nabla\eta)_{a_\varepsilon(\varphi)} = \mu^{\varepsilon,\varphi} (w,\eta)_{c_\varepsilon(\varphi)} \quad \text{for all } \eta \in H^1(\Omega), \tag{2.1.9}$$

respectively. In Theorem 3.2.2 we will see that for any $\varphi \in L^\infty(\Omega)$, all eigenvalues in either the Dirichlet or the Neumann case can be written as a sequence

$$0 < \lambda^{\varepsilon,\varphi}_1 \leq \lambda^{\varepsilon,\varphi}_2 \leq \lambda^{\varepsilon,\varphi}_3 \leq \cdots \to \infty,$$

or

$$0 = \mu_0^{\varepsilon,\varphi} < \mu_1^{\varepsilon,\varphi} \leq \mu_2^{\varepsilon,\varphi} \leq \mu_3^{\varepsilon,\varphi} \leq \cdots \to \infty,$$

respectively.

In the sharp interface limit studied in Section 3.3.2 and in the modeling of the Faber–Krahn theorem studied in Chapter 4 we will choose $a_\varepsilon \equiv c_\varepsilon \equiv 1$ in the approximate problem (2.1.5), because the appropriate choice of the functions $(b_\varepsilon)_{\varepsilon>0}$ will provide the desired relaxation of the Dirichlet-Laplace problem on the sharp shape D as we will see formally in the next section. Nevertheless, we include the coefficients $a_\varepsilon, c_\varepsilon$ in the Dirichlet problem in order to keep the model as general as possible. As we will see also in the next section, for the Neumann problem the coefficients $a_\varepsilon, c_\varepsilon$ are indeed essential.

2.1.6. The sharp interface limit: A formal discussion

Before we formulate the optimization problems in which (2.1.8) and (2.1.9) serve as the state equations, we formally discuss their behavior when taking the limit $\varepsilon \to 0$.

In both cases (2.1.5) and (2.1.6) we want to ensure that the boundary condition is not only fulfilled on the fixed boundary $\partial\Omega$ but also on the free boundary obtained in the sharp interface limit $\varepsilon \to 0$. By our diffuse interface approach we want to approximate this behavior.

Although the analytical results for $\varepsilon > 0$ are independent of the following considerations as they can be carried out under the general assumptions on the coefficient functions made in Section 2.1.5, we want to formally discuss how the coefficient functions need to be chosen explicitly in order to obtain the desired properties in the sharp-interface limit.

For a sequence of phase-field functions $(\varphi_\varepsilon)_{\varepsilon>0}$ that is expected to converge to φ_0 in the sharp-interface limit $\varepsilon \to 0$ (with φ_0 attaining only the values -1 and $+1$), we define

$$\begin{aligned}
\Omega_+^\varepsilon &:= \{\boldsymbol{x} \in \Omega |\, \varphi_\varepsilon(\boldsymbol{x}) \geq 0\}, \\
\Omega_-^\varepsilon &:= \{\boldsymbol{x} \in \Omega |\, \varphi_\varepsilon(\boldsymbol{x}) < 0\}, \\
\Omega_\pm &:= \{\boldsymbol{x} \in \Omega |\, \varphi_0(\boldsymbol{x}) = \pm 1\}, \\
\Gamma &:= \partial\Omega_+ \cap \Omega,
\end{aligned}$$

where $\boldsymbol{n}_\varepsilon$ is the outer unit normal vector field on $\partial\Omega_+^\varepsilon \cap \Omega$ and $\boldsymbol{n}$ is the outer unit normal vector field on Γ. An illustration of the diffuse interface and the sharp interface limit can be found in Figure 2.1.

Now, the coefficient functions are to be chosen in such a way that they enforce the boundary condition

$$w = 0 \quad \text{on } \Gamma, \tag{2.1.10}$$

in the Dirichlet case and the boundary condition

$$\frac{\partial w}{\partial \boldsymbol{n}} = 0 \quad \text{on } \Gamma, \tag{2.1.11}$$

in the Neumann case. We now present suitable choices for the coefficients a_ε, b_ε and c_ε and we formally discuss how the boundary conditions (2.1.10) and (2.1.11) are obtained in the sharp interface limit.

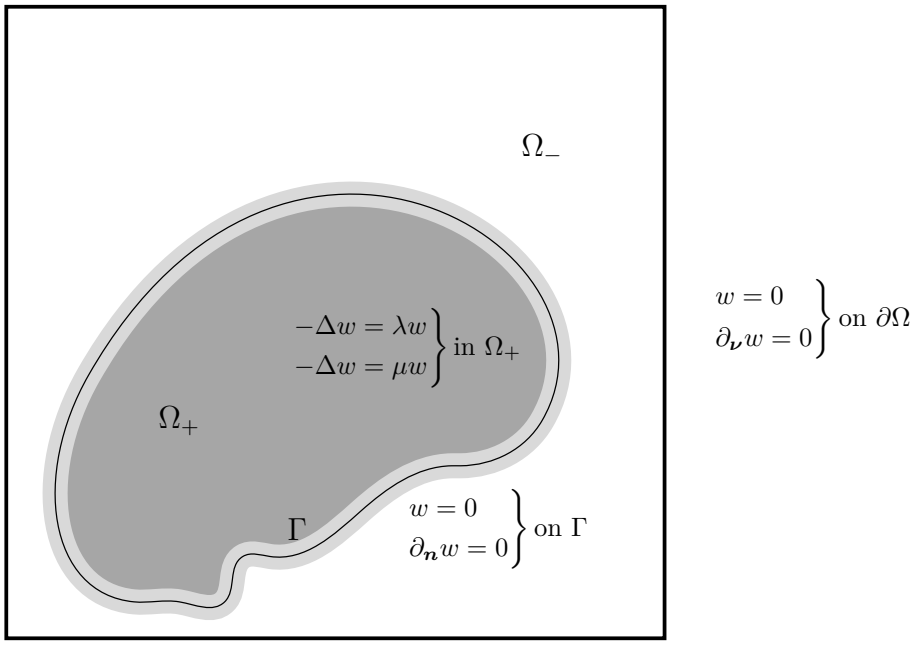

Figure 2.1: The classical eigenvalue problems on $D = \Omega_+$ approximated by the diffuse interface approach. The diffuse interface is represented by the light gray surrounding of Γ.

In the Neumann case, we choose

$$a_\varepsilon(1) = c_\varepsilon(1) = 1, \qquad a_\varepsilon(-1) = a\varepsilon, \qquad c_\varepsilon(-1) = c\varepsilon, \tag{2.1.12}$$

with constants $a, c > 0$. Then, condition (2.1.11) will be implicitly enforced in the following sense. The weak formulation of (2.1.6) is given by

$$\int_\Omega a_\varepsilon(\varphi_\varepsilon)\nabla w^{\varepsilon,\varphi_\varepsilon} \cdot \nabla\eta \,\mathrm{d}x = \mu^{\varepsilon,\varphi_\varepsilon} \int_\Omega c_\varepsilon(\varphi_\varepsilon) w^{\varepsilon,\varphi_\varepsilon}\eta \,\mathrm{d}x \quad \text{for all } \eta \in H^1(\Omega). \tag{2.1.13}$$

Assuming that the convergence $\varphi_\varepsilon \to \varphi_0$ implies the convergence of all appearing ε-dependent quantities, we use (2.1.12) to recover

$$\int_{\Omega_+} \nabla w^{\varphi_0} \cdot \nabla\eta \,\mathrm{d}x = \mu^{\varphi_0} \int_{\Omega_+} w^{\varphi_0}\eta \,\mathrm{d}x \quad \text{for all } \eta \in H^1(\Omega),$$

that is the weak formulation of the classical eigenvalue problem with Neumann boundary data

$$\begin{aligned} -\Delta w^{\varphi_0} &= \mu^{\varphi_0} w^{\varphi_0} \quad \text{in} \quad D = \Omega_+, \\ \frac{\partial w^{\varphi_0}}{\partial \boldsymbol{n}} &= 0 \qquad\qquad \text{on } \partial D = \Gamma, \end{aligned}$$

by formally sending $\varepsilon \to 0$. Note that in order to arrive at this limit problem the degeneracy of $a_\varepsilon(-1)$ and $c_\varepsilon(-1)$ as $\varepsilon \to 0$ is essential.

For the Dirichlet case let us fix $a_\varepsilon = c_\varepsilon \equiv 1$. Then condition (2.1.10) will be ensured by the coefficient function b_ε appearing in the state equation (2.1.5) by prescribing the following properties

$$b_\varepsilon(1) = 0, \qquad \lim_{\varepsilon \searrow 0} b_\varepsilon(-1) = \infty. \tag{2.1.14}$$

The idea of adding such a coefficient function comes from the porous medium approach that is used to model fluid dynamics phenomena, see e.g., [41, 96, 97].

For $\varepsilon > 0$, let us consider the weak formulation of the Dirichlet problem (2.1.8) which now reads as

$$\int_\Omega \nabla w^{\varepsilon,\varphi_\varepsilon} \nabla \eta \, \mathrm{d}x + \int_\Omega b_\varepsilon(\varphi_\varepsilon) w^{\varepsilon,\varphi_\varepsilon} \eta \, \mathrm{d}x = \lambda^{\varepsilon,\varphi_\varepsilon} \int_\Omega w^{\varepsilon,\varphi_\varepsilon} \eta \, \mathrm{d}x \quad \text{for all } \eta \in H_0^1(\Omega).$$

Assuming again that the convergence $\varphi_\varepsilon \to \varphi_0$ implies the convergence of all appearing ε-dependent quantities, we infer that

$$\sup \left\{ \int_{\Omega_-^\varepsilon} b_\varepsilon(\varphi_\varepsilon) \, |w^{\varepsilon,\varphi_\varepsilon}|^2 \, \mathrm{d}x \;\middle|\; \varepsilon > 0 \right\} < \infty. \tag{2.1.15}$$

In the light of the choice (2.1.14) this can only be the case if $w^{\varphi_0} = 0$ almost everywhere on the set $\Omega \backslash \Omega_+ = \{\varphi_0 = -1\}$. Proceeding as in the Neumann case and formally passing to the limit $\varepsilon \to 0$ in the weak formulation, we conclude that w^{φ_0} is a solution to the classical eigenvalue problem on Ω_+ with Dirichlet boundary data, that is

$$\begin{cases} -\Delta w^{\varphi_0} = \lambda^{\varphi_0} w^{\varphi_0} & \text{in } \quad D \quad = \Omega_+, \\ \quad w^{\varphi_0} = 0 & \text{on } \partial D \quad = \Gamma. \end{cases}$$

For a detailed rigorous analysis of the sharp interface limit in the Dirichlet case we refer to Section 3.3.

However, a rigorous analysis of the Neumann problem in our framework at the moment is not possible as the coefficient function a_ε chosen in (2.1.12) degenerates outside the prescribed shape. More explicitly, testing the weak formulation of the Neumann problem (2.1.13) with the eigenfunction $w^{\varepsilon,\varphi_\varepsilon}$ yields

$$\int_\Omega a_\varepsilon(\varphi_\varepsilon) \, |\nabla w^{\varepsilon,\varphi_\varepsilon}|^2 \, \mathrm{d}x = \mu^{\varepsilon,\varphi_\varepsilon},$$

as we can assume the eigenfunction to be normalized with respect to $\|\cdot\|_{c_\varepsilon(\varphi)}$. To apply classical compactness results, we need to control the Dirichlet energy of the eigenfunctions, but a_ε as chosen in (2.1.12) degenerates in the phase $\{\varphi_\varepsilon = -1\}$ as $\varepsilon \to 0$, i.e., on the left-hand side we obtain the term

$$a\varepsilon \int_{\{\varphi_\varepsilon = -1\}} |\nabla w^{\varepsilon,\varphi_\varepsilon}|^2 \, \mathrm{d}x.$$

In other words, knowing that the sequence of eigenvalues $(\mu^{\varepsilon,\varphi_\varepsilon})_{\varepsilon>0}$ is bounded, does not imply that, on the whole of Ω, also the Dirichlet energy is bounded.
Nevertheless, we believe that by choosing the convergence orders of $a_\varepsilon(-1)$ and $c_\varepsilon(-1)$ in an appropriate way similarly to [55, Lemma 14] one can still obtain the desired convergence

if the shape D is smooth enough. The key idea there in the notation of our setting is to use the model parameters

$$a_\varepsilon(-1) = a\varepsilon, \quad c_\varepsilon(-1) = c\varepsilon^2,$$

and then apply this particular scaling to the Rayleigh quotient

$$\frac{\int_\Omega a_\varepsilon(\varphi_\varepsilon) \left|\nabla w^{\varepsilon,\varphi^\varepsilon}\right|^2 \mathrm{d}x}{\int_\Omega c_\varepsilon(\varphi_\varepsilon) \left|w^{\varepsilon,\varphi_\varepsilon}\right|^2 \mathrm{d}x} = \mu^{\varepsilon,\varphi_\varepsilon},$$

in order to obtain the convergence

$$\lim_{\varepsilon \searrow 0} \mu^{\varepsilon,\varphi_\varepsilon} = \mu^{\varphi_0},$$

at least if the sequence $(\varphi_\varepsilon)_{\varepsilon>0}$ is a suitably nice recovery sequence for φ_0. Note that this choice of model parameters is also reflected in the discussion about the avoidance of spurious eigenmodes in Section 6.4.2. In the elasticity problem there it is also essential to scale the void part of the stiffness with a lower ε order compared to the mass in order to exclude the case of physically unreasonable localized eigenmodes. A rigorous analysis of the Γ-limit for the phase-field approximation of the Neumann problem is definitely a fascinating future research project.

Having motivated the constraints and the state equations we are now in a position to introduce the optimization problems studied in Chapter 3 and Chapter 4 for $\varepsilon > 0$.

2.1.7. The optimization problems in Part I

For any fixed $l \in \mathbb{N}$ and indices $i_1, \dots, i_l \in \mathbb{N}$ with $1 \leq i_1 < i_2 < \dots < i_l$, we include a finite selection of eigenvalues $\lambda_{i_1}^{\varepsilon,\varphi}, \dots, \lambda_{i_l}^{\varepsilon,\varphi}$ of (2.1.5) or $\mu_{i_1}^{\varepsilon,\varphi}, \dots, \mu_{i_l}^{\varepsilon,\varphi}$ of (2.1.6), respectively, in the cost functional via the function

$$\Psi : (\mathbb{R}_{>0})^l \to \mathbb{R},$$

which is assumed to be of class C^1. As mentioned above, the Ginzburg–Landau energy also needs to be included in the cost functional in order to guarantee for the desired phase-field structure and well-posed optimization problems. Hence, for $\varepsilon > 0$, we define the objective functional as

$$J_l^{D,\varepsilon}(\varphi) \coloneqq \Psi(\lambda_{i_1}^{\varepsilon,\varphi}, \dots, \lambda_{i_l}^{\varepsilon,\varphi}) + \gamma E_{\mathrm{GL}}^\varepsilon(\varphi), \tag{2.1.16}$$

in the Dirichlet case, and

$$J_l^{N,\varepsilon}(\varphi) \coloneqq \Psi(\mu_{i_1}^{\varepsilon,\varphi}, \dots, \mu_{i_l}^{\varepsilon,\varphi}) + \gamma E_{\mathrm{GL}}^\varepsilon(\varphi), \tag{2.1.17}$$

in the Neumann case, where $\gamma > 0$ is a weighting parameter. In the light of (2.1.1) and (2.1.2), we recall the set of admissible phase-fields as $\Phi_{\mathrm{ad}} = \mathcal{G}^\beta \cap \mathcal{U}$. Now, the optimization problem reads as

$$\left\{ \begin{array}{ll} \min & J_l^{D,\varepsilon}(\varphi), \\ \text{s.t.} & \varphi \in \Phi_{\mathrm{ad}}, \\ & \lambda_{i_1}^{\varepsilon,\varphi}, \dots, \lambda_{i_l}^{\varepsilon,\varphi} \text{ are eigenvalues of (2.1.8)} \end{array} \right. \tag{$\mathcal{P}_l^{D,\varepsilon}$}$$

in the Dirichlet case, and

$$\begin{cases} \min & J_l^{N,\varepsilon}(\varphi), \\ \text{s.t.} & \varphi \in \Phi_{\text{ad}}, \\ & \mu_{i_1}^{\varepsilon,\varphi}, \dots, \mu_{i_l}^{\varepsilon,\varphi} \text{ are eigenvalues of (2.1.9)} \end{cases} \tag{$\mathcal{P}_l^{N,\varepsilon}$}$$

in the Neumann case.

Note that we do not need an additional assumption on the function Ψ to be bounded from below in order for the minimization problem to possess a minimizer, as we can show that any eigenvalue of our approximate problem is bounded by the corresponding eigenvalue of the classical eigenvalue problem where the shape is chosen to be the whole design domain Ω, see Lemma 3.2.7.
This allows us to cover a large variety of optimization problems. For example, the Faber–Krahn problem and the Szegő–Weinberger problem, see (3.1.2) and (3.1.3) in the introduction of Chapter 3, can be formulated within our framework by choosing

$$\Psi(\lambda_1^{\varepsilon,\varphi}) = \lambda_1^{\varepsilon,\varphi}, \quad \Psi(\mu_1^{\varepsilon,\varphi}) = -\mu_1^{\varepsilon,\varphi},$$

in $(\mathcal{P}_l^{D,\varepsilon})$ and $(\mathcal{P}_l^{N,\varepsilon})$, respectively. In Section 3.4, we will further demonstrate that the optimization of linear combinations of eigenvalues

$$\Psi(\lambda_{i_1}^{\varepsilon,\varphi}, \dots, \lambda_{i_l}^{\varepsilon,\varphi}) = \sum_{j=1}^{l} \alpha_j \lambda_{i_j}, \quad \Psi(\mu_{i_1}^{\varepsilon,\varphi}, \dots, \mu_{i_l}^{\varepsilon,\varphi}) = \sum_{j=1}^{l} \alpha_j \mu_{i_j},$$

(with coefficients $\alpha_j \in \mathbb{R}$) can also be handled at least numerically. In the Dirichlet case, if we additionally assume the coefficients $\alpha_j \geq 0$ for $j = 1, \dots, l$, linear combinations are even included in the setting of our rigorous sharp-interface analysis in Section 3.3.2.

Now that we have explained all the necessary preliminaries for Part I, we will focus on Part II of this thesis which covers spectral optimization problems in the framework of linear elasticity.

2.1.8. The density function

In analogy to the previous state equations, for the equations of linear elasticity we will both allow for a phase-field dependent density distribution ρ_ε and a phase-field dependent elasticity tensor $\mathbb{C}_\varepsilon$. Compared to the approximate Neumann problem introduced in (2.1.6) the density plays now the role of the coefficient function c_ε and the elasticity tensor plays the role of the coefficient function a_ε. The density distribution ρ_ε depends directly on the phase-field $\boldsymbol{\varphi}$ and this way, ρ_ε is not just a given function but represents the density of the actual structure we want to optimize.

To this end, we assume that the density function ρ_ε belongs to $C^{1,1}_{\text{loc}}(\mathbb{R}^N;\mathbb{R})$ and is uniformly positive, i.e., for $\varepsilon > 0$ there exists a constant $\rho_{\varepsilon,0} > 0$ such that $\rho(\boldsymbol{\varphi}) \geq \rho_{\varepsilon,0}$ for all $\boldsymbol{\varphi} \in \mathbb{R}^N$. This directly yields

$$\rho_{\varepsilon,0} \left|\boldsymbol{u}\right|^2 \leq \rho(\boldsymbol{\varphi}) \left|\boldsymbol{u}\right|^2, \tag{2.1.18}$$

for all $\boldsymbol{\varphi}, \boldsymbol{u} \in \mathbb{R}^N$. For any fixed $\boldsymbol{\varphi} \in \mathbb{R}^N$ we infer from the above properties that there exist constants $C_{\varepsilon,\boldsymbol{\varphi}}, C'_{\varepsilon,\boldsymbol{\varphi}} > 0$ (that may locally depend on $\boldsymbol{\varphi}$, i.e., $C_{\varepsilon,\boldsymbol{\varphi}}$ and $C'_{\varepsilon,\boldsymbol{\varphi}}$ can be

chosen uniformly on bounded sets), such that

$$\begin{aligned} |\rho_\varepsilon(\boldsymbol{\varphi})\boldsymbol{u}\cdot\boldsymbol{v}| &\le C_{\varepsilon,\boldsymbol{\varphi}}\,|\boldsymbol{u}|\,|\boldsymbol{v}|\,, \\ |\rho'_\varepsilon(\boldsymbol{\varphi})\boldsymbol{h}\boldsymbol{u}\cdot\boldsymbol{v}| &\le C'_{\varepsilon,\boldsymbol{\varphi}}\,|\boldsymbol{h}|\,|\boldsymbol{u}|\,|\boldsymbol{v}|\,, \end{aligned} \tag{2.1.19}$$

for all $\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^d$ and $\boldsymbol{h}\in\mathbb{R}^N$. Next, for any function $\varrho\in L^\infty(\Omega)$, we define

$$(\boldsymbol{f},\boldsymbol{g})_\varrho := \int_\Omega \varrho\,\boldsymbol{f}\cdot\boldsymbol{g}\,\mathrm{d}x \quad \text{for all } \boldsymbol{f},\boldsymbol{g}\in L^2(\Omega;\mathbb{R}^d).$$

Due to the above assumptions, we can use this notation to define a family of scalar products on $L^2(\Omega;\mathbb{R}^d)$ depending on $\boldsymbol{\varphi}\in L^\infty(\Omega;\mathbb{R}^N)$ by

$$(\boldsymbol{f},\boldsymbol{g})_{\rho_\varepsilon(\boldsymbol{\varphi})} := \int_\Omega \rho_\varepsilon(\boldsymbol{\varphi})\boldsymbol{f}\cdot\boldsymbol{g}\,\mathrm{d}x \quad \text{for all } \boldsymbol{f},\boldsymbol{g}\in L^2(\Omega;\mathbb{R}^d). \tag{2.1.20}$$

These scalar products canonically induce norms that are all equivalent to the standard norm on $L^2(\Omega;\mathbb{R}^d)$. To indicate the norm we consider $L^2(\Omega;\mathbb{R}^d)$ to be equipped with, we will use the notation $L^2_{\rho_\varepsilon(\boldsymbol{\varphi})}(\Omega;\mathbb{R}^d)$ or simply $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$ when $\varepsilon>0$ is fixed. On the other hand $(\cdot,\cdot)$ denotes the classical scalar product on $L^2(\Omega;\mathbb{R}^d)$.

A reasonable choice of ρ would be

$$\rho_\varepsilon(\boldsymbol{\varphi}) = \overline{\rho}(\boldsymbol{\varphi}) + \rho_{\varepsilon,N}\varphi_N = \sum_{i=1}^{N-1}\rho_i\,\varphi_i + \varepsilon\tilde{\rho}_N\varphi_N, \quad \boldsymbol{\varphi}\in\boldsymbol{G}. \tag{2.1.21}$$

Here, for any $i\in\{1,...,N-1\}$, the coefficient $\rho_i>0$ stands for the density of the i-th material which is assumed to be constant. In our model we interpret the void as a material of very low density. Hence, we chose $\rho_{\varepsilon,N}=\varepsilon\tilde{\rho}_N$ as corresponding density, where $\tilde{\rho}_N>0$ is a fixed constant.
Note that from the view of formally matched asymptotic expansions in Chapter 6, any scaling of $\tilde{\rho}^N$ with ε^p and $p>0$ would be admissible here. In Section 6.4.2 we will give a quadratic decomposition of (2.1.21) and an explicit ε scaling which we will see, in combination with a related decomposition of the elasticity tensor $\mathbb{C}$, is able to deal with the phenomenon of spurious eigenmodes. As mentioned also in the introduction, from a numerical viewpoint the occurrence of such modes localizing in void areas are problematic if the associated eigenvalues fall into the lower part of the spectrum our optimization problem deals with, see [5, 29, 55, 136]. Thus, we will verify that choosing above model parameters adequately, we can guarantee that eigenmodes localizing in void areas will only produce eigenvalues λ^ε which will become arbitrarily large as $\varepsilon\to 0$ and therefore leave the relevant part of the spectrum for ε small enough. This model is then also successfully implemented in the numerical simulations in Section 6.9.
Note that whenever we are not concerned with the limit process $\varepsilon\to 0$ we will drop this index in order to account for a more elegant depiction.

For the sake of mathematical analysis in Chapter 6, we have to extend the choice of ρ_ε in (2.1.21) onto the whole of $\mathbb{R}^N$. To this end, proceeding as in [32, Sect. 2.2], we define the cut-off function

$$\sigma_\delta:\mathbb{R}\to\mathbb{R} \quad s\mapsto \begin{cases} -\delta & \text{if } s\le -\delta, \\ a_\delta & \text{if } -\delta < s < 0, \\ s & \text{if } 0\le s\le 1, \\ b_\delta & \text{if } 1 < s < 1+\delta, \\ 1+\delta & \text{if } s\ge 1+\delta, \end{cases} \tag{2.1.22}$$

for any $\delta > 0$ which will be specified later. Here, a_δ and b_δ are monotonically increasing $C^{1,1}$-functions such that $\sigma_\delta \in C^{1,1}(\mathbb{R};\mathbb{R})$. We now define the function ρ by

$$\rho : \mathbb{R}^N \to \mathbb{R}, \quad \boldsymbol{\varphi} \mapsto \sum_{i=1}^{N} \rho_i \, \sigma_\delta(P_\Sigma(\boldsymbol{\varphi})_i), \tag{2.1.23}$$

where P_Σ denotes the ℓ^2-orthogonal projection of $\mathbb{R}^N$ onto Σ^N defined as

$$P_\Sigma(\boldsymbol{\varphi}) = \operatorname*{arg\,min}_{\boldsymbol{v} \in \Sigma^N} \frac{1}{2} \left\| \boldsymbol{\varphi} - \boldsymbol{v} \right\|_{\ell^2}^2,$$

or equivalently

$$P_\Sigma(\boldsymbol{\varphi}) = P_{T\Sigma}(\boldsymbol{\varphi}) + \frac{1}{N}\mathbf{1},$$

where $\mathbf{1} = (1, ..., 1)^T \in \mathbb{R}^N$ and $P_{T\Sigma}$ denotes the (linear) ℓ^2-orthogonal projection onto the tangent space

$$T\Sigma^N = \left\{ \boldsymbol{\xi} \in \mathbb{R}^N \,\middle|\, \sum_{i=1}^{N} \xi_i = 0 \right\}.$$

Obviously, it holds that $\rho \in C^{1,1}(\mathbb{R}^N;\mathbb{R})$ and since $\sigma_\delta(P_\Sigma(\boldsymbol{\varphi})_i) = \varphi_i$ for all $i \in \{1, ..., N\}$ as long as $\boldsymbol{\varphi} \in \boldsymbol{G}$, the relation (2.1.21) holds true for this definition.

It remains to show that ρ is uniformly positive, at least if δ is chosen sufficiently small. To this end, we fix an arbitrary vector $\boldsymbol{\varphi} \in \Sigma^N$ and define the index sets

$$I := \{1, ..., N\}, \quad I_{<0} := \{i \in I \,|\, \varphi_i < 0\}, \quad I_{\geq 0} := I \setminus I_{<0}.$$

Recalling the definition of Σ^N, we infer that

$$\sum_{I_{\geq 0}} \varphi_i \geq 1, \quad \text{and thus also} \quad \sum_{I_{\geq 0}} \sigma_\delta(\varphi_i) \geq 1.$$

Choosing

$$M := \max_{i \in I} \rho_i, \quad m := \min_{i \in I} \rho_i, \quad \delta := \frac{m}{2MN} > 0, \quad \text{and} \quad \rho_0 := \frac{m}{2} > 0$$

we conclude the estimate

$$\rho(\boldsymbol{\varphi}) = \sum_{I_{\geq 0}} \rho_i \, \sigma_\delta(\boldsymbol{\varphi}_i) \; + \; \sum_{I_{<0}} \rho_i \, \sigma_\delta(\boldsymbol{\varphi}_i) \geq m - \delta MN = \rho_0 > 0.$$

Since $\boldsymbol{\varphi} \in \Sigma^N$ was arbitrary, this estimate holds for all $\boldsymbol{\varphi} \in \Sigma^N$ and by application of the projection in (2.1.23) also for all $\boldsymbol{\varphi} \in \mathbb{R}^N$. We point out that ρ_0 does not depend on $\boldsymbol{\varphi}$ and thus, this estimate is uniform. This means that the function ρ defined in (2.1.23) is admissible as it exhibits all properties that we demand for general density distributions.

2.1.9. The elasticity tensor

The underlying quantities appearing in linear elasticity are the tensors appearing in Hooke's Law (see, e.g., [86, 102]), namely the strain and the elasticity tensor which describe the stress tensor. To introduce the strain tensor we consider the displacement vector $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ that describes the deformation of the structure under applied forces or vibrations. Now, the strain tensor of $\boldsymbol{u}$ can be defined as

$$\mathcal{E}(\boldsymbol{u}) \coloneqq (\nabla \boldsymbol{u})^{\mathrm{sym}},$$

where $\mathcal{A}^{\mathrm{sym}} \coloneqq \frac{1}{2}(\mathcal{A} + \mathcal{A}^T)$ is the symmetrized gradient for any matrix $\mathcal{A} \in \mathbb{R}^{d\times d}$. For $\varepsilon > 0$ the elasticity tensor $\mathbb{C}_\varepsilon$ is a fourth order tensor whose components are demanded to fulfill $\mathbb{C}_{ijkl} \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R})$ as well as the symmetry properties

$$\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{ijlk} = \mathbb{C}_{klij}. \tag{2.1.24}$$

for all $i, j, k, l \in \{1, \dots, d\}$. From the regularity property we conclude that for any $\boldsymbol{\varphi} \in \mathbb{R}^N$, there exist constants $\Lambda_{\varepsilon,\boldsymbol{\varphi}}, \Lambda'_{\varepsilon,\boldsymbol{\varphi}} > 0$ locally depending on $\boldsymbol{\varphi}$ such that

$$\begin{aligned} |\mathbb{C}_\varepsilon(\boldsymbol{\varphi})\mathcal{A} : \mathcal{B}| &\le \Lambda_{\varepsilon,\boldsymbol{\varphi}} |\mathcal{A}| \, |\mathcal{B}|, \\ |\mathbb{C}'_\varepsilon(\boldsymbol{\varphi})\boldsymbol{h}\mathcal{A} : \mathcal{B}| &\le \Lambda'_{\varepsilon,\boldsymbol{\varphi}} |\boldsymbol{h}| \, |\mathcal{A}| \, |\mathcal{B}|, \end{aligned} \tag{2.1.25}$$

for all symmetric matrices $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{d\times d}\setminus\{\mathbf{0}\}$ and $\boldsymbol{h} \in \mathbb{R}^N$, where

$$\mathcal{A} : \mathcal{B} \coloneqq \sum_{i,j=1}^{d} \mathcal{A}_{ij}\mathcal{B}_{ij},$$

and

$$\mathbb{C}'_\varepsilon(\boldsymbol{\varphi})\boldsymbol{h} = \left(\sum_{m=1}^{N} \partial_m \mathbb{C}_{ijkl}(\boldsymbol{\varphi}) h_m \right)_{i,j,k,l=1}^{d},$$

denotes the derivative of $\mathbb{C}_\varepsilon(\boldsymbol{\varphi})$ in the direction $\boldsymbol{h}$. Furthermore, we demand that there exists a constant $\theta_\varepsilon > 0$ such that for all symmetric matrices $\mathcal{A} \in \mathbb{R}^{d\times d}\setminus\{\mathbf{0}\}$ and for all $\boldsymbol{\varphi} \in \mathbb{R}^N$ it holds

$$\theta_\varepsilon |\mathcal{A}|^2 \le \mathbb{C}_\varepsilon(\boldsymbol{\varphi})\mathcal{A} : \mathcal{A}. \tag{2.1.26}$$

Recall that the application of a fourth order tensor onto a quadratic matrix is given by

$$(\mathbb{C}\mathcal{A})_{ij} = \sum_{k,l=1}^{d} \mathbb{C}_{ijkl}\mathcal{A}_{kl}.$$

A concrete choice of the elasticity tensor in analogy to the construction of ρ is

$$\mathbb{C}_\varepsilon(\boldsymbol{\varphi}) = \overline{\mathbb{C}}(\boldsymbol{\varphi}) + \mathbb{C}_{\varepsilon,N}\varphi_N = \sum_{i=1}^{N-1} \mathbb{C}_i\varphi_i + \varepsilon\tilde{\mathbb{C}}_N\varphi_N, \quad \boldsymbol{\varphi} \in \boldsymbol{G},$$

where for $i = 1, \dots, N-1$, $\mathbb{C}_i$ denote constant material specific elasticity tensors. To guarantee (2.1.26) we need to assume the existence of positive constants $\tilde{\Theta}_i, \Theta_i$ such that for all $\mathcal{A} \in \mathbb{R}^{d\times d}\setminus\{\mathbf{0}\}$, it holds that

$$\tilde{\Theta}_i |\mathcal{A}|^2 \le \mathbb{C}_i\mathcal{A} : \mathcal{A} \le \Theta_i |\mathcal{A}|^2,$$

for all $i = 1, \dots, N-1$ and the existence of positive constants $\tilde{\Theta}_N, \Theta_N$ such that

$$\tilde{\Theta}_N \left|\mathcal{A}\right|^2 \leq \tilde{\mathbb{C}}_N \mathcal{A} : \mathcal{A} \leq \Theta_N \left|\mathcal{A}\right|^2,$$

for all $\mathcal{A} \in \mathbb{R}^{d\times d} \setminus \{\mathbf{0}\}$. Now, proceeding similarly as for the density ρ, we can construct an extension to $\mathbb{R}^N$ satisfying the above properties.

2.1.10. The state equations in Part II

We now introduce the system of equations approximating the vibration of an elastic structure which will serve as state equation in the optimization problems studied in Part II. Let $\varepsilon > 0$, then consider the eigenvalue problem

$$\begin{cases} -\nabla \cdot [\mathbb{C}_\varepsilon(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{w}^{\varepsilon,\boldsymbol{\varphi}})] = \lambda^{\varepsilon,\boldsymbol{\varphi}} \rho_\varepsilon(\boldsymbol{\varphi}) \boldsymbol{w}^{\varepsilon,\boldsymbol{\varphi}} & \text{in } \Omega, \\ \boldsymbol{w}^{\varepsilon,\boldsymbol{\varphi}} = \mathbf{0} & \text{on } \Gamma_D, \\ [\mathbb{C}_\varepsilon(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{w}^{\varepsilon,\boldsymbol{\varphi}})]\, \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma_0. \end{cases} \tag{2.1.27}$$

Here, $\boldsymbol{n}$ is the outer unit normal vector to the boundary of the design domain $\partial\Omega = \overline{\Gamma_D \cup \Gamma_0}$. The subsets $\Gamma_D, \Gamma_0 \subset \partial\Omega$ are relatively open and satisfy $\Gamma_D \cap \Gamma_0 = \emptyset$ and $\mathcal{H}^{d-1}(\Gamma_D) > 0$, where $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. These assumptions are crucial for Korn's inequality, see Theorem 2.2.2 and [160, Section 62.15]. To consider above system in the weak sense, we define the closed subspace

$$H^1_D(\Omega;\mathbb{R}^d) := \left\{ \boldsymbol{\eta} \in H^1(\Omega;\mathbb{R}^d) \;\middle|\; \boldsymbol{\eta} = \mathbf{0} \;\text{ on } \Gamma_D \right\} \subset H^1(\Omega;\mathbb{R}^d).$$

Endowed with the standard inner product and norm given by

$$(\cdot,\cdot)_{H^1_D(\Omega;\mathbb{R}^d)} := (\cdot,\cdot)_{H^1(\Omega;\mathbb{R}^d)}, \quad \|\cdot\|_{H^1_D(\Omega;\mathbb{R}^d)} := \|\cdot\|_{H^1(\Omega;\mathbb{R}^d)},$$

$H^1_D(\Omega;\mathbb{R}^d)$ is a Hilbert space. For any matrices $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{d\times d}$ and any fourth-order tensor $\mathcal{C} \in \mathbb{R}^{d\times d\times d\times d}$, we introduce the notation

$$\langle \mathcal{A}, \mathcal{B} \rangle_{\mathcal{C}} := \int_\Omega \mathcal{C}\mathcal{A} : \mathcal{B} \,\mathrm{d}x.$$

Let $\boldsymbol{\varphi} \in L^\infty(\Omega;\mathbb{R}^N)$, then the mapping

$$\langle \mathcal{E}(\cdot), \mathcal{E}(\cdot) \rangle_{\mathbb{C}_\varepsilon(\boldsymbol{\varphi})} : H^1_D(\Omega;\mathbb{R}^d) \times H^1_D(\Omega;\mathbb{R}^d) \to \mathbb{R}, \quad (\boldsymbol{w}, \boldsymbol{\eta}) \mapsto \langle \mathcal{E}(\boldsymbol{w}), \mathcal{E}(\boldsymbol{\eta}) \rangle_{\mathbb{C}_\varepsilon(\boldsymbol{\varphi})} \tag{2.1.28}$$

defines a scalar product on $H^1_D(\Omega;\mathbb{R}^d)$. As for ρ we only write the ε dependence in $\mathbb{C}$ explicitly when we consider the sharp interface limit. By Korn's inequality, the norm induced by this inner product is equivalent to the standard norm on $H^1_D(\Omega;\mathbb{R}^d)$. In what follows, we will always choose for a given $\boldsymbol{\varphi} \in L^\infty(\Omega;\mathbb{R}^N)$ this inner product and induced norm on $H^1_D(\Omega;\mathbb{R}^d)$.

Using this notation and invoking the symmetry property (2.1.24), the weak formulation of (2.1.27) can be expressed as

$$\langle \mathcal{E}(\boldsymbol{w}^{\varepsilon,\boldsymbol{\varphi}}), \mathcal{E}(\boldsymbol{\eta}) \rangle_{\mathbb{C}_\varepsilon(\boldsymbol{\varphi})} = \lambda^{\varepsilon,\boldsymbol{\varphi}} \left(\boldsymbol{w}^{\varepsilon,\boldsymbol{\varphi}}, \boldsymbol{\eta}\right)_{\rho_\varepsilon(\boldsymbol{\varphi})} \quad \text{for all } \boldsymbol{\eta} \in H^1_D(\Omega;\mathbb{R}^d). \tag{2.1.29}$$

In Section 5.2, we will see that for any $\boldsymbol{\varphi} \in L^\infty(\Omega, \mathbb{R}^N)$, there exists a sequence of eigenvalues

$$0 < \lambda_1^{\varepsilon,\boldsymbol{\varphi}} \leq \lambda_2^{\varepsilon,\boldsymbol{\varphi}} \leq \lambda_3^{\varepsilon,\boldsymbol{\varphi}} \leq \cdots \to \infty$$

and corresponding eigenfunctions $\{\boldsymbol{w}_1^{\varepsilon,\boldsymbol{\varphi}}, \boldsymbol{w}_2^{\varepsilon,\boldsymbol{\varphi}}, ...\} \subset H_D^1(\Omega; \mathbb{R}^d)$ which form an orthonormal basis of $L^2_{\rho_\varepsilon(\boldsymbol{\varphi})}(\Omega; \mathbb{R}^d)$.

Next, we introduce the structural optimization problem in which the system (2.1.27) can be regarded as the state equation.

2.1.11. The optimization problem in Part II

For $l \in \mathbb{N}$ and $i_1, \ldots, i_l \in \mathbb{N}$, the finite selection of eigenvalues $\lambda_{i_1}^{\varepsilon,\boldsymbol{\varphi}}, ..., \lambda_{i_l}^{\varepsilon,\boldsymbol{\varphi}}$ of (2.1.27) is to be penalized via a function

$$\Psi : (\mathbb{R}_{>0})^l \to \mathbb{R},$$

which is assumed to be of class C^1. Hence, in analogy to the two-phase case we define the objective functional as

$$J_l^\varepsilon(\boldsymbol{\varphi}) \coloneqq \Psi(\lambda_{i_1}^{\varepsilon,\boldsymbol{\varphi}}, \ldots, \lambda_{i_l}^{\varepsilon,\boldsymbol{\varphi}}) + \gamma E_{\mathrm{GL}}^\varepsilon(\boldsymbol{\varphi}), \tag{2.1.30}$$

with $\gamma > 0$. Consequently, the overall optimization problem reads as

$$\left\{ \begin{array}{ll} \min & J_l^\varepsilon(\boldsymbol{\varphi}), \\ \text{s.t.} & \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c, \\ & \lambda_{i_1}^{\varepsilon,\boldsymbol{\varphi}}, \ldots, \lambda_{i_l}^{\varepsilon,\boldsymbol{\varphi}} \text{ are eigenvalues of (2.1.29).} \end{array} \right. \tag{$\mathcal{P}_l^\varepsilon$}$$

Recall from Section 2.1.4 that it suffices to include the regular part of the Ginzburg–Landau energy here, because $E^\varepsilon(\boldsymbol{\varphi}) = E_{\mathrm{GL}}^\varepsilon(\boldsymbol{\varphi})$ for all $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c \subset \boldsymbol{\mathcal{G}}$. Note that here we of course need to assume that the sets S_0, S_1 introduced in Section 2.1.3 are chosen such that the admissible set $\boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c$ is non-empty, in order to allow for a well-posed optimization problem. In the light of the discussion in Section 2.1.3 this is guaranteed if the sets S_0, S_1 keep a positive fixed distance and $|S_0| < |\Omega|\,(1 - m_N)$ and $|S_1| < |\Omega|\,m_N$, because then a phase-field $\boldsymbol{\varphi} \in H^1(\Omega; \mathbb{R}^N)$ has enough freedom to meet the mean value constraint imposed by $\boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$.

2.1.12. A combination of compliance and eigenvalue optimization

As mentioned also in the introduction of this thesis, minimizing the compliance of a structure is a fundamental task in shape and topology optimization. Thus, we want to analytically and numerically consider also a problem of combining both spectral and compliance optimization, see Section 5.6 and Section 6.9.3.

In [32], the problem of minimizing the mean compliance

$$F(\boldsymbol{u}, \boldsymbol{\varphi}) = \int_\Omega (1 - \varphi^N)\, \boldsymbol{f} \cdot \boldsymbol{u} \,\mathrm{d}x + \int_{\Gamma_g} \boldsymbol{g} \cdot \boldsymbol{u} \,\mathrm{d}\Gamma,$$

with $\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^d)$ and $\boldsymbol{g} \in L^2(\Gamma_g,\mathbb{R}^d)$, and the deviation with respect to a target displacement $\boldsymbol{u}_\Omega \in L^2(\Omega;\mathbb{R}^d)$ given by

$$J_0(\boldsymbol{u},\boldsymbol{\varphi}) = \left(\int_\Omega c(1-\varphi^N)\,|\boldsymbol{u}-\boldsymbol{u}_\Omega|^2\,\mathrm{d}x\right)^\nu, \quad \nu \in (0,1],$$

is considered. Here, $c \in L^\infty(\Omega)$ denotes a function with $|\operatorname{supp} c| > 0$, where $|\operatorname{supp} c|$ stands for the Lebesgue measure of the support. The boundary $\partial\Omega$ is split into two relatively open, disjoint subsets $\Gamma_C, \Gamma_g \subset \partial\Omega$ such that $\partial\Omega = \overline{\Gamma_C \cup \Gamma_g}$ and $\mathcal{H}^{d-1}(\Gamma_C) > 0$. Moreover, the state equation is determined by the mean compliance in order to obtain

$$\boldsymbol{u} \in H^1_C(\Omega;\mathbb{R}^d) \coloneqq \left\{\boldsymbol{\eta} \in H^1(\Omega;\mathbb{R}^d) \,\middle|\, \boldsymbol{\eta} = \boldsymbol{0} \text{ on } \Gamma_C\right\}$$

as the displacement vector under the given forces. It reads as

$$\begin{cases} -\nabla\cdot[\mathbb{C}(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{u})] &= (1-\varphi^N)\boldsymbol{f} & \text{in } \Omega,\\ \boldsymbol{u} &= \boldsymbol{0} & \text{on } \Gamma_C,\\ [\mathbb{C}(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{u})]\,\boldsymbol{n} &= \boldsymbol{g} & \text{on } \Gamma_g. \end{cases} \tag{2.1.31}$$

Combining this problem with the one discussed in the previous section, we obtain a structure that is on the one hand as stiff as possible (i.e., it has small compliance) and on the other hand realizes the desired vibration properties (e.g., a large first eigenvalue). Note that in order to account for a general model the Dirichlet boundary of the compliance problem Γ_C needs not to be identical to the Dirichlet boundary of the spectral problem Γ_D. The combination of $(\mathcal{P}^\varepsilon)$ in [32] and $(\mathcal{P}_l^\varepsilon)$ reads as

$$\begin{cases} \min & I_l^\varepsilon(\boldsymbol{u},\boldsymbol{\varphi}) = \alpha F(\boldsymbol{u},\boldsymbol{\varphi}) + \beta J_0(\boldsymbol{u},\boldsymbol{\varphi}) + \gamma E^\varepsilon(\boldsymbol{\varphi}) + \Psi(\lambda_{i_1}^{\boldsymbol{\varphi}},\dots,\lambda_{i_l}^{\boldsymbol{\varphi}})\\ \text{s.t.} & (\boldsymbol{u},\boldsymbol{\varphi}) \in H^1_C(\Omega,\mathbb{R}^d)\times H^1(\Omega,\mathbb{R}^N),\\ & (2.1.31) \text{ is fulfilled}, \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c,\\ & \text{and } \lambda_{i_1}^{\boldsymbol{\varphi}},\dots,\lambda_{i_l}^{\boldsymbol{\varphi}} \text{ are eigenvalues of (2.1.29)}, \end{cases} \tag{$\mathcal{K}_l^\varepsilon$}$$

where $\alpha,\beta \geq 0$, $\gamma,\varepsilon > 0$, $\boldsymbol{m} \in (0,1)^N \cap \Sigma^N$. In Section 5.6, we will present an existence result as well as the variational inequality for this combined problem. Numerically we will revisit this problem at the end of the thesis in Section 6.9.3.

2.2. Mathematical tools

Let us now summarize some of the most important mathematical tools that are used in this thesis. We will tacitly assume that we are in the setting of the previous section.

2.2.1. General notation

Let us give some general notation that we will use throughout this thesis. $\mathcal{L}^d$ stands for the d-dimensional Lebesgue measure and $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure.
We write $\mathbb{R}_0^+ = [0, \infty)$ to denote the interval of non-negative real numbers and $\mathbb{R}_{>0}$ to denote the interval of positive real numbers $(0, \infty)$.

As we have also seen in the introduction of the double obstacle potential it is useful to consider cost functions taking also the value $+\infty$. Therefore we introduce the extended real numbers, see [142, 1.22] and [10, Example 2.8].

Definition 2.2.1. *We define the extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, with the standard ordering on $\mathbb{R}$ and $-\infty < a < +\infty$ for $a \in \mathbb{R}$. We use the standard arithmetic operations and especially the convention $\pm\infty \cdot 0 = 0$. Furthermore*

$$\begin{aligned} d : \overline{\mathbb{R}} \times \overline{\mathbb{R}} &\to [-1, 1] \\ d(x, y) &\coloneqq |g(x) - g(y)| \end{aligned}$$

with

$$g(x) \coloneqq \begin{cases} -1 & \text{for } x = -\infty \\ \frac{x}{1+|x|} & \text{for } x \in \mathbb{R} \\ +1 & \text{for } x = +\infty, \end{cases}$$

defines a metric on $\overline{\mathbb{R}}$.

In particular for a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ we understand the limit

$$\lim_{n\to\infty} a_n = +\infty \quad \text{in } \overline{\mathbb{R}},$$

with respect to this metric.

With $\triangle$ we denote the symmetric difference of sets, which is defined as

$$A \triangle B = (A \backslash B) \cup (B \backslash A) = (A \cup B) \backslash (A \cap B).$$

For a function $u : \Omega \to \mathbb{R}$ and $t \in \mathbb{R}$ we use the standard abbreviations

$$\begin{aligned} \{u > t\} &\coloneqq \{x \in \Omega \mid u(x) > t\}, \\ u^+ &\coloneqq \max\{0, u\}. \end{aligned}$$

We will always indicate weak convergence with the symbol $\rightharpoonup$ and strong convergence with the symbol $\to$.

2.2.2. Tools from functional and geometric analysis

First of all we will give Korn's inequality which will allow us to give a suitable norm on

$$H^1_D(\Omega;\mathbb{R}^d) := \left\{ \boldsymbol{\eta} \in H^1(\Omega;\mathbb{R}^d) \;\middle|\; \boldsymbol{\eta} = \mathbf{0} \;\text{ on } \Gamma_D \right\} \subset H^1(\Omega;\mathbb{R}^d),$$

in terms of the symmetrized gradient. As indicated in Section 2.1.10 this is crucial for the theory in Part II, because this will tell us that the bi-linear form (2.1.28) induced by the state equation is coercive.

Recall that $\Omega \subset \mathbb{R}^d$ was chosen to be a bounded Lipschitz domain.

Theorem 2.2.2 (Korn's inequality)**.** *Let $\Gamma_D \subset \partial\Omega$ be a relatively open subset with*

$$\mathcal{H}^{d-1}(\Gamma_D) > 0.$$

Then there is a $C > 0$ such that

$$\int_\Omega \mathcal{E}(\boldsymbol{u}) : \mathcal{E}(\boldsymbol{u}) \,\mathrm{d}x \geq C \left\|\boldsymbol{u}\right\|^2_{H^1(\Omega,\mathbb{R}^d)},$$

for all $\boldsymbol{u} \in H^1_D(\Omega;\mathbb{R}^d)$.

Remark 2.2.3.

1. For a proof we refer to [75, 6.15-4] or [160, Section 62.15].

2. It is essential in above assertion that the Hausdorff measure of Γ_D is positive. If $\Gamma_D = \emptyset$ then it holds

$$\begin{aligned} &\left\{ \boldsymbol{u} \in H^1(\Omega;\mathbb{R}^d) \;\middle|\; \mathcal{E}(\boldsymbol{u}) = \mathbf{0} \quad \text{a.e. in } \Omega \right\} \\ =& \left\{ x \mapsto \mathcal{B}x + \boldsymbol{a} \;\middle|\; \mathcal{B} \in \mathbb{R}^{d\times d}, \mathcal{B}^T = -\mathcal{B}, \boldsymbol{a} \in \mathbb{R}^d \right\}. \end{aligned}$$

   The space on the right-hand side is called the space of *infinitesimally rigid motions*, see [160, Lemma 62.15] or [86, Section 6.1.9]. In physical terms this means that rotating and translating a shape which can freely move within the design domain does not affect its elastic energy.

3. Note that Korn's inequality is a very deep result which is delicate to prove due to the mixed terms $\partial_i u_j \partial_j u_i$ appearing in $\mathcal{E}(\boldsymbol{u}) : \mathcal{E}(\boldsymbol{u})$. Nevertheless, under the additional assumption that $\boldsymbol{u} \in H^2(\Omega;\mathbb{R}^d)$ and $\Gamma_D = \partial\Omega$ the mixed terms can be transformed into quadratic terms via integration by parts, see [86, Theorem 6.14].

The following theorem is useful when the classical Lebesgue's theorem is not applicable, but the integrand can be majorized by a suitable sequence of functions.

Theorem 2.2.4 (Lebesgue's general convergence theorem)**.** *Let $1 \leq p < \infty$, $f, f_k : \Omega \to \mathbb{R}^N$ and $g_k \to g$ in $L^1(\Omega,\mathbb{R})$ for $k \to \infty$.*
Furthermore let

$$\begin{aligned} f_k &\to f \quad \text{a.e. in } \Omega \text{ for } k \to \infty, \\ |f_k|^p &\leq g_k \quad \text{a.e. in } \Omega \text{ for all } k \in \mathbb{N}. \end{aligned}$$

Then it holds

$$f_k \to f \quad \text{in } L^p(\Omega,\mathbb{R}^N).$$

For a proof see [10, Theorem 3.25].

In order to compute first-order optimality conditions in Chapter 3 and Chapter 5 we will derive the cost functional with respect to the phase-field variable. Here the notion of Fréchet- and Gâteaux-derivative is important. For a comprehensive overview see also [159].

Definition 2.2.5. Let X, Y be Banach spaces with $x \in X$. Denote $U(x) \subset X$ an open neighborhood around x. Consider the function $f : U(x) \to Y$.

1. The function f is said to be *Fréchet-differentiable* at x, if there is a continuous linear operator $T \in L(X, Y)$, such that

$$f(x+h) - f(x) = Th + o(\|h\|), \tag{2.2.1}$$

for all h in a sufficiently small neighborhood around 0 denoted by $U(0) \subset X$. This property uniquely defines $T \in L(X, Y)$. We call T the Fréchet-derivative of f at x and write $f'(x) = T$. Furthermore for given $h \in X$ we denote the Fréchet-derivative of f at x in direction h with $f'(x)h$.

2. The function f is said to be *Gâteaux-differentiable* at x, if there is a continuous linear operator $T \in L(X, Y)$, such that

$$f(x+tk) - f(x) = tTk + o(t), \tag{2.2.2}$$

for all $t \in \mathbb{R}$ with $t \to 0$ and all $k \in X$ with $\|k\| = 1$ and $x + tk \in U(x)$. This property uniquely defines $T \in L(X, Y)$. We call T the Gâteaux-derivative of f at x an write $f'(x) = T$. For given $h \in X$ we denote the Gâteaux-derivative of f at x in direction h with $f'(x)h$.

3. Let $A \subset X$ be an open subset. If the Fréchet- (respectively Gâteaux-) derivative exists for all $x \in A$ then we call the mapping

$$\begin{aligned} f' : A \subset X &\to L(X, Y) \\ x &\mapsto f'(x), \end{aligned}$$

the *Fréchet- (respectively Gâteaux-) derivative* on A.

4. Let Z be a further Banach space and consider $f : D(f) \subset X \times Y \to Z$, where $D(f)$ denotes the domain of definition of f. Then the *partial Fréchet-(respectively Gâteaux-) derivative* with respect to the x coordinate, denoted by $f_{/x}$, is defined as the Fréchet- (respectively Gâteaux-) derivative of

$$\begin{aligned} g : U(x) \subset X &\to Z \\ x &\mapsto f(x, y), \end{aligned}$$

where $y \in Y$ is fixed. The derivative with respect to the y coordinate is defined analogously.

Of course one can easily prove that all the classical properties such as the linearity, product rule and chain rule hold also for the Fréchet- and Gâteaux-derivative. Furthermore, we see that Fréchet-differentiability always implies Gâteaux-differentiability. We want to give one further essential property that will be used in the main proof concerning Fréchet-differentiability of the control-to-state operator of our optimization problem, see Theorem 5.4.3.

Proposition 2.2.6. *Let $f : D(f) \subset X \times Y \to Z$ be partially Fréchet-differentiable with respect to both the x and y coordinate in a neighborhood $U(x,y) \subset D(f)$ around the point $(x,y) \in D(f)$. Furthermore let both $f_{\prime x} : U(x,y) \to L(X,Z)$ and $f_{\prime y} : U(x,y) \to L(Y,Z)$ be continuous at (x,y). Then f is Fréchet-differentiable at (x,y) and the Fréchet-derivative satisfies*

$$f'(x,y)(h_1,h_2) = f_{\prime x}(x,y)h_1 + f_{\prime y}(x,y)h_2,$$

for all $h_1 \in X$ and $h_2 \in Y$.

For a proof we refer to [159, Proposition 4.8].

In the theory of Γ-limits one has to show a lim sup and a lim inf inequality, see Section 2.2.5. For the lim inf inequality Fatou's lemma is a helpful tool. Note that Fatou's lemma holds true on the extended positive real line and thus, is applicable in the context of the coefficient functions $(b_\varepsilon)_{\varepsilon>0}$ that will take the value ∞ outside the shape in the limit sharp-interface case, see Lemma 3.3.5.

Lemma 2.2.7 (Fatou's lemma)**.** *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of $\mathcal{L}^d$ measurable functions with*

$$f_k : \Omega \to [0,\infty]$$

for each $k \in \mathbb{N}$. Then defining $f(x) \coloneqq \liminf_{k\to\infty} f_k(x) \in [0,\infty]$ for almost every $x \in \Omega$, it holds

$$\int_\Omega f \, dx \le \liminf_{k\to\infty} \int_\Omega f_k \, dx.$$

For a proof see [89, Theorem 1.17].

As we are concerned with the eigenvalues of elliptic operators, we of course heavily rely on the following spectral theorem for compact self-adjoint operators.

Theorem 2.2.8. *Let H be a Hilbert space over $\mathbb{R}$ with scalar product $(\cdot,\cdot)_H$ and $\mathcal{T} : H \to H$ a compact, self-adjoint, positive-semi definite continuous linear operator. Then the following holds true:*

- *There is an orthonormal system $(e_k)_{k\in N} \subset H$, with $N \subset \mathbb{N}$, and a "sequence" $(\lambda_k)_{k\in N} \subset (0,\infty)$ with*

  $$\mathcal{T}e_k = \lambda_k e_k \quad \text{for all } k \in N.$$

- *If N is infinite then $\lambda_k \to 0$ for $k \to \infty$.*

- *The sequence $(\lambda_k)_{k\in N}$ comprises all non-trivial eigenvalues, i.e., if there is an element $e \in H$ and a $\lambda \in \mathbb{R}$ satisfying*

  $$\mathcal{T}e = \lambda e,$$

  then there is either a $k \in N$ such that $\lambda_k = \lambda$ or $\lambda = 0$.

- *The space H can be orthogonally decomposed as*

$$H = N(\mathcal{T}) \perp \overline{\langle e_1, e_2, \dots \rangle}_{\text{span}}.$$

Here $N(\mathcal{T}) \subset H$ denotes the kernel of the operator $\mathcal{T}$ and $\overline{\langle e_1, e_2, \dots \rangle}_{\text{span}} \subset H$ is the closure of the subspace spanned by the orthonormal system $(e_k)_{k\in N} \subset H$, more precisely $u \in \overline{\langle e_1, e_2, \dots \rangle}_{\text{span}}$ if and only if for $k \in N$ there are $\alpha_k \in \mathbb{R}$ such that

$$u = \sum_{k\in N} \alpha_k e_k,$$

where the convergence of the series has to take place in H.

- *The operator $\mathcal{T}$ is characterized by the eigenpairs $(\lambda_k, e_k)_{k\in N}$ via the decomposition*

$$\mathcal{T}x = \sum_{k\in N} \lambda_k (x, e_k)_H e_k,$$

for any $x \in H$.

Proof. The proof for the case where $\mathbb{R}$ is replaced by $\mathbb{C}$ and the condition $\mathcal{T}$ self-adjoint is replaced by $\mathcal{T}$ normal is found in [10, Theorem 12.12] combined with [10, Remark 12.13]. The real case is easily obtained from the complex case, by using the complexification argument in [10, 11.14]. □

Let us conclude this section with giving one of the most important tools in geometric analysis, which is the coarea formula. This formula is at the heart of the proofs of the lim sup inequality for the Ginzburg–Landau energy in [37, 126, 135, 147] and will also play a crucial role in the explicit construction of a recovery sequence in the proof of Theorem 4.3.17 in order to study the Γ-limit of our spectral optimization problem.

Theorem 2.2.9. *Let $m \le d$, $f : \mathbb{R}^d \to \mathbb{R}^m$ be Lipschitz and $g \in L^1(\mathbb{R}^d; \mathbb{R})$. Then for $\mathcal{L}^m$-almost every $y \in \mathbb{R}^m$ the integral*

$$\int_{\{f=y\}} |g(x)| \, \mathrm{d}\mathcal{H}^{d-m}(x),$$

is finite and it holds

$$\int_{\mathbb{R}^d} g(x) Jf(x) \, \mathrm{d}\mathcal{L}^d(x) = \int_{\mathbb{R}^m} \left[\int_{\{f=y\}} g(x) \, \mathrm{d}\mathcal{H}^{d-m}(x) \right] \mathrm{d}\mathcal{L}^m(y).$$

Here the coarea factor is defined as

$$Jf(x) \coloneqq \sqrt{\det(Df(x) Df(x)^T)}.$$

As we will see in Theorem 4.3.17, when $f = d$ is the signed distance function with respect to a smooth surface this will allow for a beautiful transformation of the integrals coming from the Ginzburg–Landau energy when the recovery sequence is inserted.

2.2.3. Symmetric-decreasing rearrangements

As explained also in the introduction, symmetrization techniques are a traditional and crucial tool in order to compare arbitrary shapes to more symmetric shapes or even the ball. We will see that for our purposes in Chapter 4, especially in Theorem 4.3.7, the symmetric-decreasing rearrangement will be a powerful tool.

For functions $f : \mathbb{R}^d \to \mathbb{R}_0^+$ vanishing at infinity (i.e., the level sets $\{x \in \mathbb{R}^d \,|\, f(x) > t\}$ have finite Lebesgue-measure for all $t > 0$), a definition of their radially symmetric-decreasing rearrangement can be found in [119, Section 3.3]. We can easily adapt this definition to functions $f : \Omega \to \mathbb{R}_0^+$ where $\Omega = B_R(0)$ is an open ball in $\mathbb{R}^d$ with radius $R > 0$ centered at the origin, which will be the setting in Chapter 4. Note that if Ω would be an arbitrary non-symmetric bounded design domain then the symmetric-decreasing rearrangement will also change the design domain, which is not reasonable, as the design domain shall remain fixed in our optimization problems.

Definition 2.2.10. *Let $\Omega = B_R(0)$ be an open ball in $\mathbb{R}^d$ centered at the origin with a given radius $R > 0$.*

(a) *A measurable function $f : \Omega \to \mathbb{R}$ is called **(radially) symmetric-decreasing**, if any fixed representative of the equivalence class of f satisfies the properties*

$$\begin{cases} f(x) = f(y) & \text{if} \quad |x| = |y|\,, \\ f(x) \geq f(y) & \text{if} \quad |x| \leq |y| \end{cases} \tag{2.2.3}$$

for almost all $x, y \in \Omega$. If additionally

$$f(x) > f(y) \quad \text{if} \quad |x| < |y|$$

*for almost all $x, y \in \Omega$, then f is called **strictly (radially) symmetric-decreasing**.*

(b) *For any measurable set $A \subseteq \Omega$ with $\mathcal{L}^d(A) < \infty$, its **(radially) symmetric rearrangement** A^* is defined to be the open ball centered at the origin whose volume is equal to that of A. This means that*

$$A^* = \{x \in \Omega \,:\, |x| < r\} \quad \text{where } r \geq 0 \text{ satisfies} \quad \mathcal{L}^d(B^d)\, r^d = \mathcal{L}^d(A).$$

Here, B^d denotes the d-dimensional ball with radius 1.

(c) *Let $f : \Omega \to \mathbb{R}_0^+$ be any measurable function. Then its **(radially) symmetric-decreasing rearrangement** f^* is defined as*

$$f^*(x) = \int_0^\infty \mathbb{1}_{\{f>t\}^*}(x)\, \mathrm{d}t$$

for all $x \in \Omega$.

Remark 2.2.11. Let $\Omega = B_R(0)$ be an open ball in $\mathbb{R}^d$ centered at the origin with a given radius $R > 0$.

(a) It obviously holds $\Omega^* = \Omega$, and for any measurable function $f : \Omega \to \mathbb{R}_0^+$, we have $f^{**} = f^*$.

(b) For any measurable function $f : \Omega \to \mathbb{R}_0^+$, its trivial extension $f_0 : \mathbb{R}^n \to \mathbb{R}_0^+$ with $f_0|_\Omega = f$ and $f_0|_{\mathbb{R}^d \setminus \Omega} = 0$ is measurable and naturally vanishes at infinity. In particular, we have $f_0^*|_\Omega = f^*$, where the symmetric-decreasing rearrangement f_0^* of the extension f_0 is defined as in [119, Section 3.3].

Some important properties of the symmetric-decreasing rearrangement are collected in the following lemma.

Lemma 2.2.12. *Let $\Omega = B_R(0)$ be an open ball in $\mathbb{R}^d$ centered at the origin with a given radius $R > 0$, and let $f, g : \Omega \to \mathbb{R}_0^+$ be arbitrary measurable functions. Then the following statements hold:*

(a) *f^* is measurable and symmetric-decreasing. Moreover, f^* is defined everywhere in Ω. In particular, the condition (2.2.3) is satisfied everywhere in Ω.*

(b) *The level sets of f^* are the rearrangements of the level sets of f, meaning that*

$$\{x \in \Omega : f^*(x) > t\} = \{x \in \Omega : f(x) > t\}^*$$

up to a Lebesgue null set in $\mathbb{R}^d$. In particular, if $f \in L^p(\Omega)$ for some $p \in [1, \infty]$, it holds that $f^* \in L^p(\Omega)$ with

$$\|f^*\|_{L^p(\Omega)} = \|f\|_{L^p(\Omega)}.$$

(c) *Let $\Phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a non-decreasing, lower semi-continuous function. Then it holds that*

$$(\Phi \circ f)^* = \Phi \circ f^* \quad \text{a.e. in } \Omega.$$

(d) *Let $\Psi \in C^1([0,1])$ with $\Psi(0) = 0$. If $0 \le f \le 1$ a.e. in Ω, it holds that*

$$\int_\Omega \Psi \circ f^* \, \mathrm{d}x = \int_\Omega \Psi \circ f \, \mathrm{d}x. \tag{2.2.4}$$

(e) ***Hardy–Littlewood inequality:*** *It holds that*

$$\int_\Omega f\, g \, \mathrm{d}x \le \int_\Omega f^*\, g^* \, \mathrm{d}x \tag{2.2.5}$$

with the convention that when the left-hand side is infinite, then also the right-hand side is infinite.

(f) ***Nonexpansivity of the rearrangement:*** *Let $F : \mathbb{R} \to \mathbb{R}_0^+$ be a convex function such that $F(0) = 0$. Then*

$$\int_\Omega F \circ (f^* - g^*) \, \mathrm{d}x \le \int_\Omega F \circ (f - g) \, \mathrm{d}x.$$

(g) ***Pólya–Szegő inequality:*** *Suppose that* $f \in H_0^1(\Omega;\mathbb{R}_0^+)$. *Then,* $f^* \in H_0^1(\Omega;\mathbb{R}_0^+)$ *with*

$$\int_\Omega |\nabla f^*|^2 \,\mathrm{d}x \le \int_\Omega |\nabla f|^2 \,\mathrm{d}x. \tag{2.2.6}$$

Moreover, if $f > 0$ *almost everywhere in* Ω *and*

$$\mathcal{L}^n\big(\{x \in \Omega \,|\, \nabla f^*(x) = 0\}\big) = 0, \tag{2.2.7}$$

then equality in (2.2.6) *holds if and only if* $f = f^*$ *almost everywhere in* Ω.

We first assure that the basic properties of radially symmetric-decreasing rearrangements in $\mathbb{R}^d$ carry over to our local case.

Proof. In view of Remark 2.2.11(b), the statements (a)–(c), (e) and (f) are direct consequences of the results in [119, Sections 3.3–3.5].

To prove (d), we use the decomposition $\Psi' = \Psi'_+ - \Psi'_-$, where $\Psi'_+ := \max(\Psi', 0)$ and $\Psi'_- := -\min(\Psi', 0)$ denote the positive part and the negative part of Ψ', respectively. Now, we define

$$\Psi_1(t) := \int_0^t \Psi'_+(s)\,\mathrm{d}s \quad \text{and} \quad \Psi_2(t) := \int_0^t \Psi'_-(s)\,\mathrm{d}s \quad \text{for all } t \in [0,1].$$

Recalling $\Psi(0) = 0$, we apply the fundamental theorem of calculus to derive the decomposition $\Psi = \Psi_1 - \Psi_2$. As the functions Ψ_1 and Ψ_2 are non-decreasing, (2.2.4) follows directly from [119, Section 3.3(iv)].

To prove (g), let $f \in H_0^1(\Omega;\mathbb{R}_0^+)$ be any function, and let $f_0 : \mathbb{R}^n \to \mathbb{R}_0^+$ denote its trivial extension as in Remark 2.2.11(b). This means that $f_0 \in H^1(\mathbb{R}^n)$ is a non-negative function with compact support. We further define

$$A : [0,\infty) \to [0,\infty),\ x \mapsto x^2.$$

Hence $A \in C^2([0,\infty)$ is strictly increasing, $A(0) = 0$ and $A^{\frac{1}{2}}$ is convex. Thus, as all conditions are fulfilled, we can apply the first part of [47, Theorem 1.1] and obtain

$$\int_{\mathbb{R}^n} \left|\nabla f_0^*\right|^2 \mathrm{d}x^n \le \int_{\mathbb{R}^n} \left|\nabla f_0\right|^2 \mathrm{d}x^n, \tag{2.2.8}$$

which directly implies (2.2.6) since $\nabla f_0 = 0$ and $\nabla f_0^* = 0$ almost everywhere on $\mathbb{R}^n\backslash\Omega$.

In addition, let us now assume that condition (2.2.7) holds true and that $f > 0$ almost everywhere in Ω. Since $f_0^* = 0$ on $\mathbb{R}^n\backslash\Omega$, we have $\{f_0^* > 0\} \subset \Omega$ and thus,

$$\mathcal{L}^n\Big(\{x \in \mathbb{R}^n \,|\, \nabla f_0^*(x) = 0\} \cap \big(f_0^*\big)^{-1}\big((0,\infty)\big)\Big) \le \mathcal{L}^n\big(\{x \in \Omega \,|\, \nabla f^*(x) = 0\}\big) = 0.$$

Therefore, [47, Theorem 1.1] states that equality in (2.2.8) holds if and only if f_0 is a translate of f_0^*. This directly entails that equality in (2.2.6) holds if and only if f is a translate of f^*. However, since $f \in H_0^1(\Omega)$ with $f > 0$ almost everywhere in $\Omega = B_R(0)$, this is possible if and only if $f = f^*$ almost everywhere in Ω, which proves the claim. □

Remark 2.2.13. We point out that the condition that f has a vanishing trace on $\partial\Omega$ is actually a necessary assumption for the Pólya–Szegő inequality (Lemma 2.2.12(g)). In general, as the following example shows, there exist functions $f \in H^1(\Omega;\mathbb{R}_0^+)$ such that f^* does not even belong to $H^1(\Omega;\mathbb{R}_0^+)$.

Counterexample to the Pólya–Szegő inequality for functions in $H^1(\Omega;\mathbb{R}_0^+)$.
Let $\Omega = B_1(0)$ be the open unit ball in $\mathbb{R}^2$. We consider the function

$$f : \Omega \to \mathbb{R}_0^+, \quad x \mapsto |x| \ ,$$

which obviously belongs to $H^1(\Omega;\mathbb{R}_0^+)$ but not to $H_0^1(\Omega;\mathbb{R}_0^+)$. Its symmetric-decreasing rearrangement f^* is given by

$$f^* : \Omega \to \mathbb{R}_0^+, \quad x \mapsto \sqrt{1-|x|^2}\,.$$

Hence, f^* is weakly differentiable with

$$\nabla f^*(x) = \frac{-x}{\sqrt{1-|x|^2}} \quad \text{for all } x \in \Omega\backslash\{0\}.$$

However, it is easy to see that the blow-up at $|x| = 1$ causes

$$\int_\Omega |\nabla f^*|^2 \ \mathrm{d}x^2 = +\infty.$$

This means that $f^* \notin H^1(\Omega;\mathbb{R}_0^+)$ and in particular, the Pólya–Szegő inequality (2.2.6) does not hold.

2.2.4. Functions of bounded variation and sets of finite perimeter

As the sequence of phase-fields will naturally develop jumps on the sharp-interface level we need to relax the space $H^1(\Omega)$ in order to account for this behavior while still having some control on the jumps and good compactness results.
As characteristic functions of finite perimeter sets are an important sub-class of functions of bounded variation, the theory of sets of finite perimeter and functions of bounded variation is closely related. We refer to [14, 89, 121] for more details. Let me note that the book [14] has become one of my favourite mathematical books during my doctoral project.

Definition 2.2.14. *The space of functions of bounded variation in Ω with values in $\mathbb{R}$, also referred to as BV functions, is defined as*

$$BV(\Omega) \coloneqq \left\{ u \in L^1(\Omega) \;\middle|\; V(u,\Omega) < \infty \right\}.$$

Here $V(u,\Omega)$ denotes the variation of a function $u \in L^1_{\mathrm{loc}}(\Omega)$ defined as

$$V(u,\Omega) \coloneqq \sup\left\{ \int_\Omega u \ \mathrm{div}\,\boldsymbol{\xi} \ \mathrm{d}x \;\middle|\; \boldsymbol{\xi} \in C_0^1(\Omega,\mathbb{R}^d),\ \|\boldsymbol{\xi}\|_{L^\infty(\Omega;\mathbb{R}^d)} \le 1 \right\}.$$

Remark 2.2.15. • The more abstract but equivalent definition of the space $BV(\Omega)$ is the following: $u \in L^1(\Omega)$ belongs to $BV(\Omega)$ if and only if its distributional derivative is representable by a finite Radon-measure, see also [14, Definition 1.40], i.e., there is a $\mathbb{R}^d$-valued Radon-measure $Du = (D_1u, \dots, D_du)$ such that

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} \,\mathrm{d}x = -\int_\Omega \phi \,\mathrm{d}D_iu, \tag{2.2.9}$$

for $i = 1, \dots, d$ and $|Du|(\Omega) < \infty$. Here $|\mu|$ denotes the total variation of a measure, see [14, Definition 1.4, Proposition 1.47]. It is important to note that $|Du|(\Omega) = V(u, \Omega)$ for any $u \in BV(\Omega)$, see [14, Proposition 3.6]

- Endowed with the norm

$$\|u\|_{BV(\Omega)} \coloneqq \|u\|_{L^1(\Omega)} + V(u, \Omega),$$

$BV(\Omega)$ is a Banach space, see [14, Proposition 3.6]

- Due to above definition, any $u \in W^{1,1}(\Omega)$ belongs to $BV(\Omega)$ because by definition of the weak derivative (2.2.9) is fulfilled for the finite Radon-measure $Du \coloneqq \nabla u \mathcal{L}^d$.

As indicated in the beginning of this section, the space $BV(\Omega)$ is equipped with the following compactness. This compactness is the key to prove that the Ginzburg–Landau energy provides us with compactness in our optimization problems, see Proposition 2.2.25.

Theorem 2.2.16. *Let $(u_k)_{k\in\mathbb{N}} \subset BV(\Omega)$ be a bounded sequence with respect to the BV-norm. Then there exists a (non-relabeled) subsequence and a $u \in BV(\Omega)$, such that*

$$u_k \to u \quad \text{in } L^1(\Omega),$$

for $k \to \infty$.

However, for some practical purposes, the topology induced by the above BV-norm is too strong. For this reason, the concept of strict convergence is commonly used.

Definition 2.2.17. *We say that a sequence $u_k \in BV(\Omega)$ strictly converges to $u \in BV(\Omega)$ if*

$$u_k \to u \quad \text{in } L^1(\Omega) \quad \text{and} \quad V(u_k, \Omega) \to V(u, \Omega),$$

as $k \to \infty$.

Note that in this definition we do not require for the sequence u_k to converge in variation but that only the variations of u_k converge.

One of the fine properties of the space $BV(\Omega)$, recalling that $\Omega \subset \mathbb{R}^d$ is Lipschitz in our setting, is that it allows for a well-defined trace.

Theorem 2.2.18. *Let $u \in BV(\Omega)$, then there exists a function*

$$u_{|\partial\Omega} \in L^1((\partial\Omega, \mathcal{H}^{d-1} \llcorner \partial\Omega); \mathbb{R}),$$

which satisfies

$$\lim_{\rho \searrow 0} \rho^{-d} \int_{\Omega \cap B_\rho(x)} \left| u(y) - u_{|\partial\Omega}(x) \right| \, dy = 0, \tag{2.2.10}$$

for $\mathcal{H}^{d-1}$ a.e. $x \in \partial\Omega$. Here, $\mathcal{H}^{d-1} \llcorner \partial\Omega$ denotes the restriction of the Hausdorff measure $\mathcal{H}^{d-1}$ to the boundary $\partial\Omega$ and $L^1\big((\partial\Omega, \mathcal{H}^{d-1} \llcorner \partial\Omega); \mathbb{R}\big)$ is the space of L^1-functions on $\partial\Omega$ with respect to the measure $\mathcal{H}^{d-1} \llcorner \partial\Omega$. $u_{|\partial\Omega}$ is uniquely defined as an element in $L^1\big((\partial\Omega, \mathcal{H}^{d-1} \llcorner \partial\Omega); \mathbb{R}\big)$. We call $u_{|\partial\Omega}$ the trace of u on $\partial\Omega$.

For a proof we refer to [14, Theorem 3.87].
In the following, we will simply write $L^1(\partial\Omega)$ instead of $L^1\big((\partial\Omega, \mathcal{H}^{d-1} \llcorner \partial\Omega); \mathbb{R}\big)$. The corresponding norm on $L^1(\partial\Omega)$ is given by

$$\|\cdot\|_{L^1(\partial\Omega)} = \int_{\partial\Omega} |\cdot| \, \mathrm{d}\mathcal{H}^{d-1}. \tag{2.2.11}$$

Finally an important property is the continuity of the trace operator in $BV(\Omega)$ not only with respect to the classical norm but with respect to strict convergence, which will be essential in the proof of Theorem 4.3.17.

Theorem 2.2.19. *The operator*

$$\begin{aligned} BV(\Omega) &\to L^1(\partial\Omega), \\ u &\mapsto u_{|\partial\Omega}, \end{aligned}$$

is continuous with respect to strict convergence in $BV(\Omega)$.

For a proof see [14, Theorem 3.88].

Having understood the theory of the space $BV(\Omega)$, it is only a small step to understand sets of finite perimeter, as they are defined as follows.

Definition 2.2.20. *The relative perimeter in Ω of a measurable set $E \subset \mathbb{R}^d$ is defined as*

$$P_\Omega(E) \coloneqq V(\chi_E, \Omega) = |D\chi_E|(\Omega).$$

We say that the set E is of relative finite perimeter in Ω if it holds

$$P_\Omega(E) < \infty.$$

Remark 2.2.21.

- A deep relation which is due to [14, Theorem 3.61] is

  $$P_\Omega(E) = \mathcal{H}^{d-1}(\Omega \cap \partial^* E),$$

  for any set of finite perimeter $E \subset \mathbb{R}^d$.
  Here the so called essential boundary $\partial^* E$ is a suitable measure theoretic concept for the boundary of finite perimeter sets, see [14, Definition 3.60]. If E is a bounded Lipschitz set, then ∂E coincides $\mathcal{H}^{d-1}$ almost everywhere with $\partial^* E$ by [121, Example

12.6]. This gives sense to the notion "perimeter", as $P_\Omega(E)$ in this case simply measures the boundary of E that lies within Ω.

In general, for $d \geq 2$, there are pretty wild sets of finite perimeter. It is really crucial to have these examples in mind, because they are intuitively hard to grasp. For any $\varepsilon > 0$, [121, Example 12.25] constructs open finite perimeter sets E_ε contained in the unit ball B with $\mathcal{L}^d(E_\varepsilon) \leq \varepsilon$ but $\mathcal{L}^d(\partial E_\varepsilon) > 0$. Hence, in particular $\mathcal{H}^{d-1}(\partial E_\varepsilon) = \infty$. The idea of this construction is to take a dense sequence $(x_k)_{k\in\mathbb{N}} \subset B$ and define for $\varepsilon > 0$

$$E_\varepsilon := \cup_{k\in\mathbb{N}} B_{r_k}(x_k) \subset B,$$

with sufficiently small radii $r_k(\varepsilon) \in (0, \varepsilon)$. It is then seen in [121, Example 12.25] that $\mathcal{L}^d(E_\varepsilon) \leq \varepsilon$ and $P_{\mathbb{R}^d}(E_\varepsilon) \leq 1$.
Taking the complement of this set we arrive at **a bounded set of finite perimeter with *empty* interior but *positive* Lebesgue measure**! More precisely just take one of the sets E_ε from above with $\varepsilon > 0$ small enough and consider the set $E_\varepsilon^c := B \backslash E_\varepsilon$ which then satisfies

$$\begin{aligned} \mathcal{L}^d(E_\varepsilon^c) &\geq \mathcal{L}^d(B) - \varepsilon \\ P_{\mathbb{R}^d}(E_\varepsilon^c) &\leq P_{\mathbb{R}^d}(B) + P_{\mathbb{R}^d}(E_\varepsilon) < \infty, \end{aligned}$$

using [121, Lemma 12.22].

- It is crucial not to confuse the relative perimeter with the global perimeter, i.e., P_Ω vs. $P_{\mathbb{R}^d}$. The relative perimeter arises as the Γ-limit of the Ginzburg–Landau energy when no boundary condition is imposed on the phase-fields on $\partial\Omega$. When a Dirichlet boundary condition is imposed, this will yield the global perimeter as we have already discussed at the end of Section 2.1.4 and which will also be seen rigorously in Theorem 4.3.17.

A final property is the lower semi-continuity of the perimeter with respect to local convergence in measure.

Proposition 2.2.22. *Let $(E_n)_{n\in\mathbb{N}}, E \subset \mathbb{R}^d$ be sets of relative finite perimeter in Ω. Furthermore for any open set $A \subset\subset \Omega$ let $\mathcal{L}^d(A \cap (E_n \triangle E)) \to 0$ for $n \to \infty$. Then it holds*

$$P_\Omega(E) \leq \liminf_{n\to\infty} P_\Omega(E_n).$$

For a proof see [14, Proposition 3.38]. Note that the convergence in measure with respect to Ω, i.e., $\mathcal{L}^d(\Omega \cap E_n \triangle E) \to 0$, implies local convergence in measure. Furthermore, the convergence in measure is equivalent to the convergence $\chi_{E_n} \to \chi_E$ in $L^1(\Omega)$, see [14, Remark 3.37].

2.2.5. Γ-convergence

The concept of Γ-convergence going back to E. De Giorgi is a tool that is very well suited in order to analyze minimization problems. In particular Γ-convergence of cost functionals implies the convergence of minimizers if additionally a certain compactness holds. For a comprehensive and accessible overview over Γ-convergence we refer to [44]. In this section let X be a metric space. In our applications this will mostly be the space $L^1(\Omega)$.

Definition 2.2.23. *A sequence of functionals $J_k : X \to \overline{\mathbb{R}}$ is said to Γ-converge in X to $J : X \to \overline{\mathbb{R}}$ if for all $x \in X$ the following two conditions are satisfied:*

- *For every sequence $(x_k)_{k\in\mathbb{N}}$ converging to x it holds*

$$J(x) \leq \liminf_{k\to\infty} J_k(x_k).$$

  This is called the lim inf *inequality.*

- *There exists a sequence $(x_k)_{k\in\mathbb{N}}$ converging to x such that*

$$\limsup_{k\to\infty} J_k(x_k) \leq J(x).$$

  This is called the lim sup *inequality.*

If the above properties are satisfied we write $J_k \xrightarrow{\Gamma} J$.

As a rule of thumb the lim sup inequality is always harder to prove, because there one has to explicitly construct a so called recovery sequence which is specifically tailored to the sequence of functionals in order to satisfy this inequality.

Notation. In this thesis we we will often consider sequences depending on the interface parameter ε which will be sent to 0. In order to study sequences $(\zeta_\varepsilon)_{\varepsilon>0}$ depending on the continuous parameter ε we will use the following notation. $(\zeta_\varepsilon)_{\varepsilon>0}$ stands for $(\zeta_{\varepsilon_k})_{k\in\mathbb{N}}$ where $(\varepsilon_k)_{k\in\mathbb{N}}$ denotes an arbitrary sequence with $\varepsilon_k \to 0$ as $k \to \infty$. In this sense, a subsequence extraction from $(\zeta_\varepsilon)_{\varepsilon>0}$ is to be understood as a subsequence extraction from the associated sequence $(\varepsilon_k)_{k\in\mathbb{N}}$. For convenience, our subsequences will not be relabeled, meaning that for any subsequence, we will use the same notation as for the whole sequence it was extracted from.

As mentioned above let us state the main reason why Γ-convergence is such a useful tool.

Proposition 2.2.24. *Let $(\overline{x}_k)_{k\in\mathbb{N}} \subset X$ be a sequence of global minimizers of J_k over X, i.e.,*

$$J_k(\overline{x}_k) = \min_{x\in X} J_k(x),$$

and let furthermore $J_k \xrightarrow{\Gamma} J$.
If there is a subsequence of $(\overline{x}_k)_{k\in\mathbb{N}}$ (which we do not relabel here) converging to some $x \in X$, then it holds

$$\lim_{k\to\infty} J_k(\overline{x}_k) = J(x),$$

along this subsequence and x is a global minimizer of J over X.

For a proof we refer to [44, Remark 1.22]. So this proposition tells us that if we have an accumulation point of a sequence of minimizers $(\overline{x}_k)_{k\in\mathbb{N}}$ then Γ-convergence implies that the accumulation point is a minimizer of the limit functional. In particular, although proving Γ-convergence itself is always a difficult business, one has to make sure that there *exists* an accumulation point of a sequence of minimizers. In our case this will always be gained by exploiting the following compactness of the Ginzburg–Landau energy. For full generality we will state this proposition under minimal assumptions on the potential ψ.

Proposition 2.2.25. *Let $\psi_0 \in C^0([-1,1])$ with $\psi_0 > 0$ on $(-1,1)$. Furthermore let $(\varphi_\varepsilon)_{\varepsilon>0} \subset H^1(\Omega;[-1,1])$ be a sequence such that there is a $C \geq 0$ with*

$$E^\varepsilon(\varphi_\varepsilon) = \int_\Omega \frac{\varepsilon}{2} |\nabla \varphi_\varepsilon|^2 + \frac{1}{\varepsilon}\psi_0(\varphi_\varepsilon)\, dx \leq C,$$

for all $\varepsilon > 0$. Then there is a $\varphi_0 \in L^1(\Omega;[-1;1])$ and a non-relabeled subsequence such that

$$\varphi_\varepsilon \to \varphi_0 \quad \text{in } L^1(\Omega),$$

for $\varepsilon \to 0$.

Proof. This beautiful proof is given in [126, Proposition 3(a)], [147, Remark (1.35)] and [37, Theorem 3.7]. As it reveals some nice properties of the Ginzburg–Landau energy and its interplay with the theory of the space $BV(\Omega)$ let us give this proof here.

Let us define the anti-derivative of the continuous function $\sqrt{2\psi_0}$ on $[-1,1]$ as

$$\begin{aligned} \Phi : [-1,1] &\to [0,c_0] \\ s &\mapsto \int_{-1}^{s} \sqrt{2\psi_0(t)}\, \mathrm{d}t. \end{aligned} \tag{2.2.12}$$

Here $c_0 := \int_{-1}^{1} \sqrt{2\psi_0(t)}\, \mathrm{d}t$ is an important constant appearing in the Γ-limit of the Ginzburg–Landau energy later on, see e.g., Theorem 3.3.11.

The function Φ has now some nice properties which we will exploit in the following. First of all clearly $\Phi \in C^1([-1,1])$ with $\Phi'(s) = \sqrt{2\psi_0(s)}$ for all $s \in [-1,1]$. Additionally as $\psi_0 > 0$ on $(-1,1)$, we know that Φ is strictly increasing and thus invertible with continuous inverse $\Phi^{-1} : [0,c_0] \to [-1,1]$. Due to [99, Lemma 7.5], the composition satisfies $\Phi \circ \varphi_\varepsilon \in H^1(\Omega)$ with $\nabla(\Phi \circ \varphi_\varepsilon) = \Phi'(\varphi_\varepsilon)\nabla\varphi_\varepsilon = \sqrt{2\psi_0(\varphi_\varepsilon)}\nabla\varphi_\varepsilon$. The beauty of this formula is that this is now closely related to the Ginzburg–Landau energy as follows

$$\int_\Omega |\nabla(\Phi \circ \varphi_\varepsilon)|\, \mathrm{d}x = \int_\Omega \left|\sqrt{2\psi_0(\varphi_\varepsilon)}\nabla\varphi_\varepsilon\right| \mathrm{d}x \leq \int_\Omega \frac{\varepsilon}{2} |\nabla\varphi_\varepsilon|^2 + \frac{1}{\varepsilon}\psi_0(\varphi_\varepsilon)\, \mathrm{d}x$$

This last ingenious step, which is an application of Young's inequality, is often referred to as the *Modica–Mortola trick*. Now the space $W^{1,1}(\Omega)$ has the severe deficit that there is no similar compactness as for example in $H^1(\Omega)$. This is now what exactly motivates the space $BV(\Omega)$. Namely, due to the compactness of this space, see Theorem 2.2.16,we can extract a subsequence such that

$$\Phi \circ \varphi_\varepsilon \to \phi_0 \quad \text{in } L^1(\Omega),$$

for $\varepsilon \to 0$ and a suitable $\phi_0 \in BV(\Omega;[0,c_0])$. Exploiting the continuity and boundedness of Φ^{-1} this provides us with $\varphi_0 = \Phi^{-1} \circ \phi_0 \in L^1(\Omega;[-1,1])$ and thus the proof is complete. □

2.2.6. Quasi-open sets and γ-convergence

As mentioned already in the introduction, on the sharp-interface level one needs a certain notion of convergence under which eigenvalues behave continuously. This will be the

so called γ-convergence. As a first step we will introduce the notion of quasi-openness which is a generalization of openness naturally appearing in the classical relaxation of the Dirichlet-Laplace problem studied in the seminal work [64]. For a comprehensive overview we refer to [19, 49, 110].

The concept of quasi-open sets heavily relies on the notion of capacity. From an intuitive point of view the benefit of the capacity is that it measures sets in a finer way than the Lebesgue measure does. In other words there are Lebesgue null sets with positive capacity, see [19, Section 5.8.2].

Definition 2.2.26. *The capacity of a measurable set $E \subset \mathbb{R}^d$ is defined as*

$$\operatorname{cap}(E) = \inf \left\{ \int_{\mathbb{R}^d} |\nabla u|^2 + u^2 \, dx \,\middle|\, u \in \mathcal{U}_E \right\},$$

where $\mathcal{U}_E$ is the set of functions $u \in H^1(\mathbb{R}^d)$ such that $u \geq 1$ a.e. in a neighborhood of E. Here, the expression "a.e. in a neighborhood" means that there exists an open set U containing E such that $u \geq 1$ a.e. in U.
We say that a relation holds quasi-everywhere (short q.e.) if it holds up to a set of zero capacity.

This allows us to introduce the notion of quasi-open sets. Intuitively a set is quasi open if it can be approximated by open sets in capacity. The analogous idea is also used to introduce the notion of quasi-continuity.

Definition 2.2.27. *A set $\omega \subset \mathbb{R}^d$ is called quasi-open if for every $\delta > 0$ there is an open set $\omega_\delta \subset \mathbb{R}^d$ such that $\operatorname{cap}(\omega \triangle \omega_\delta) < \delta$.*
A function $u : \Omega \to \mathbb{R}$ is called quasi-continuous if for any $\delta > 0$ there exists a continuous function $u_\delta : \Omega \to \mathbb{R}$ such that $\operatorname{cap}(\{x \in \Omega \mid u(x) \neq u_\delta(x)\}) < \delta$.

The concept of capacity is beautifully related to the Sobolev space $H^1(\Omega)$ as the following proposition shows. In particular quasi-open sets naturally arise as super-level sets of H^1 functions. This is a beautiful relaxation of the fact that super-level sets of continuous functions are open.

Proposition 2.2.28.

1. *Let $u \in H^1(\Omega)$ then there is a quasi-continuous representative of u which coincides with u almost everywhere. This representative is uniquely defined up to a set of zero capacity. We will always identify u with this representative. Whenever we want to distinguish between u and its quasi continuous representative we denote this representative with $\hat{u}$.*

2. *Let $A \subset \Omega$ be an open subset, then it holds*

$$H_0^1(A) = \left\{ u \in H_0^1(\Omega) \,\middle|\, u = 0 \text{ q.e. in } \Omega \backslash A \right\}. \tag{2.2.13}$$

3. *Let $u \in H^1(\Omega)$ and $t \in \mathbb{R}$ then the set $\{u > t\} \coloneqq \{x \in \Omega \mid u(x) > t\}$ (where we recall that u is the quasi-continuous representative) is quasi-open.*

Note that here in (2.2.13) on the left-hand side, as A is open, the space $H^1_0(A)$ is defined in the classical way as closure of $C^\infty_0(A)$ with respect to the associated H^1-norm. For a proof we refer to [51, Theorem 4.1.1, Theorem 4.1.3] and [110, Proposition 3.3.41].

In the light of the characterization (2.2.13), we want to give sense to the space $H^1_0(E)$ when E is merely a measurable set.

Definition 2.2.29. *Let $E \subset \Omega$ be measurable. Then we define*

$$H^1_0(E) = \left\{ u \in H^1_0(\Omega) \,\middle|\, u = 0 \text{ q.e. in } \Omega\backslash E \right\}.$$

In the sharp-interface limit problems studied in Chapter 3 and Chapter 4 we want to formulate the Dirichlet-Laplace eigenvalue problem on a shape which is quite irregular, namely a set of finite perimeter which could in general even have empty interior, see Remark 2.2.21.
In the light of the previous discussion the natural candidate for the Sobolev space on which we formulate the Dirichlet eigenvalue problem in the sharp-interface case would be $H^1_0(E)$ as defined above. But let us recall the formal discussion from Section 2.1.6 and especially (2.1.15) which reads as

$$\sup \left\{ \int_{\Omega^\varepsilon_-} b_\varepsilon(\varphi_\varepsilon) \, |w^{\varepsilon,\varphi_\varepsilon}|^2 \, \mathrm{d}x \,\middle|\, \varepsilon > 0 \right\} < \infty. \tag{2.2.14}$$

Based on this boundedness we will show in Theorem 3.3.7 that if the sequence of phase-fields $(\varphi_\varepsilon)_{\varepsilon>0} \subset H^1(\Omega)$ converge in $L^1(\Omega)$ to some $\varphi \in BV(\Omega;\{\pm 1\})$ then there is a non-relabeled subsequence such that

$$\lim_{\varepsilon \searrow 0} \|w^{\varepsilon,\varphi_\varepsilon} - u\|_{H^1(\Omega)} = 0$$

and

$$\lim_{\varepsilon \searrow 0} \int_\Omega b_\varepsilon(\varphi_\varepsilon) \, |w^{\varepsilon,\varphi_\varepsilon}|^2 \, \mathrm{d}x = \int_\Omega b_0(\varphi) \, |u|^2 \, \mathrm{d}x = 0.$$

Clearly u plays the role of an eigenfunction for the sharp-interface problem. Recalling the construction of the coefficient function b_ε in (2.1.14) we have $b_0(1) = 0$ and $b_0(-1) = \infty$. Thus, the condition

$$\int_\Omega b_0(\varphi) \, |u|^2 \, \mathrm{d}x = 0.$$

is equivalent to $u = 0$ *almost everywhere* in $\{\varphi = -1\}$. So we see that here the underlying measure is indeed the Lebesgue measure and not the capacity, thus motivating the introduction of the following relaxed "Sobolev-like" space.

Definition 2.2.30. *For any measurable set $E \subset \Omega$, we define the space*

$$\tilde{H}^1_0(E) = \left\{ u \in H^1_0(\Omega) \,\middle|\, u = 0 \text{ a.e. in } \Omega\backslash E \right\}. \tag{2.2.15}$$

It is clear that $\tilde{H}^1_0(E) \subset H^1_0(\Omega)$ is a closed subspace equipped with the norm induced by $H^1_0(\Omega)$. In general, even for open sets $E \subset \Omega$, the inclusion $H^1_0(E) \subset \tilde{H}^1_0(E)$ might be strict, see [80, Remark 2.1]. However, if the open set E additionally has a Lipschitz boundary, then equality holds, see [51].

Now we note the following crucial relation between these definitions, which will be also used in our Γ-convergence proof, see Theorem 3.3.11.

Proposition 2.2.31. *Let $E \subset \Omega$ be a measurable set. Then there exists a unique quasi-open set $\omega \subset \Omega$ such that*

$$\tilde{H}_0^1(E) = H_0^1(\omega) = \left\{ u \in H_0^1(\Omega) \,\middle|\, u = 0 \text{ q.e. in } \Omega \backslash \omega \right\}. \tag{2.2.16}$$

Here uniqueness is to be understood up to a set of zero capacity. Furthermore this set satisfies $\omega \subset E$ a.e.

Note that [110, Prop. 3.3.44] only asserts the existence of a unique quasi-open set $\omega \subset E$ such that $H_0^1(E) = H_0^1(\omega)$, so there the left hand side is the classical space from Definition 2.2.29. Although Proposition 2.2.31 has been frequently used in the literature, see [39, 51, 81], we give the proof here for the sake of completeness.

Proof. As $\tilde{H}_0^1(E) \subset H_0^1(\Omega)$ is a closed subspace and $H_0^1(\Omega)$ is separable, we infer the existence of a dense sequence $(u_n)_{n\in\mathbb{N}} \subset \tilde{H}_0^1(E)$. Now we define

$$\omega := \cup_{n\in\mathbb{N}} \{\hat{u}_n \neq 0\} = \cup_{n\in\mathbb{N}} (\{\hat{u}_n > 0\} \cup \{\hat{u}_n < 0\}),$$

which due to Proposition 2.2.28 is a quasi-open subset of Ω as the countable union of quasi-open sets is quasi-open, see [110, Proposition 3.3.40]. By definition of the space $\tilde{H}_0^1(E)$ we know for any $n \in \mathbb{N}$

$$\{\hat{u}_n \neq 0\} \subset E \quad \text{a.e.},$$

and thus, we directly infer $\omega \subset E$ a.e. and

$$H_0^1(\omega) \subset \tilde{H}_0^1(\omega) \subset \tilde{H}_0^1(E),$$

by noting that vanishing quasi everywhere is a stronger condition than vanishing almost everywhere.

Now for the other inclusion let $u \in \tilde{H}_0^1(E)$ be arbitrary. Then by construction we can extract a non-relabeled subsequence from the initially fixed dense sequence such that

$$u_n \to u \quad \text{in } \tilde{H}_0^1(E).$$

Due to [49, Theorem 4.1.2] it holds by extracting possibly a further subsequence

$$\hat{u}_n \to \hat{u} \quad \text{q.e. in } \Omega$$

and thus we infer

$$\{\hat{u} \neq 0\} \subset \cup_{n\in\mathbb{N}} \{\hat{u}_n \neq 0\} = \omega \quad \text{q.e.}$$

and thus by Definition 2.2.29 we infer $\tilde{H}_0^1(E) \subset H_0^1(\omega)$, as $u \in \tilde{H}_0^1(E)$ was arbitrary. The uniqueness follows directly from the fact

$$\omega_1, \omega_2 \subset \Omega \quad \text{quasi-open with } H_0^1(\omega_1) \subset H_0^1(\omega_2) \Rightarrow \omega_1 \subset \omega_2 \quad \text{q.e.},$$

see the proof of [110, Prop. 3.3.44]. □

Remark 2.2.32.

- From this constructive proof we see that $\omega \subset \Omega$ is chosen to be the largest quasi-open set contained in E with the property $H_0^1(\omega) \subset \tilde{H}_0^1(E)$. It is important to notice that, in general, the inclusion $\omega \subset E$ is strict, even in the almost everywhere sense. Indeed, the case $\mathcal{L}^d(E\backslash\omega) > 0$ may occur, see [51].
  In this line of thought it would be interesting to know whether there is an example of a finite perimeter set with positive Lebesgue measure for which $\tilde{H}_0^1(E) = \{0\}$? Of course in that case E has to have empty interior. This question was tackled in [2, Section 11.3]. Due to [2, Theorem 11.3.2] for a Borel set $E \subset \mathbb{R}^d$ with empty interior it holds $H_0^1(E) = \{0\}$ if and only if

$$\operatorname{cap}(\omega\backslash E) = \operatorname{cap}(\omega), \quad \text{for all open sets } \omega \subset \mathbb{R}^d. \tag{2.2.17}$$

  As discussed in [2], with the analogous construction as for the wild set of finite perimeter in Remark 2.2.21, one obtains an example for which condition (2.2.17) is not fulfilled. In other words there are measurable sets E with empty interior for which the space $H_0^1(E)$ and thus also the space $\tilde{H}_0^1(E)$ is non-trivial. On the other hand it is in general not clear whether (2.2.17) already implies that E is a Lebesgue null set.
  This discussion motivates the assumption for any admissible shape E^φ in the sharp-interface setting to contain an open ball, see Assumption **(A4)** in Chapter 3, as this obviously guarantees $\tilde{H}_0^1(E^\varphi)$ to be a non-trivial infinite dimensional vector space.

The relation (2.2.16) can now further be refined via the following Laplace equation. Thus, let $E \subset \Omega$ be a measurable set and $\omega \subset E$ be the quasi-open set provided by Proposition 2.2.31.
Let $u_E \in \tilde{H}_0^1(E)$ be the unique solution of

$$\int_\Omega \nabla u_E \cdot \nabla v \,\mathrm{d}x = \int_\Omega 1v \,\mathrm{d}x \quad \forall v \in \tilde{H}_0^1(E) \tag{2.2.18}$$

and $u_\omega \in H_0^1(\omega)$ be the unique solution of

$$\int_\Omega \nabla u_\omega \cdot \nabla v \,\mathrm{d}x = \int_\Omega 1v \,\mathrm{d}x \quad \forall v \in H_0^1(\omega). \tag{2.2.19}$$

We thus infer $u_E = u_\omega$ in $H_0^1(\omega)$. We now associate the quasi-open set ω with the Borel measure

$$\infty_{\Omega\backslash\omega}(B) = \begin{cases} 0 & \text{if } \operatorname{cap}(B \cap (\Omega\backslash\omega)) = 0, \\ +\infty & \text{else,} \end{cases}$$

for any Borel-set B. Using the notation $\mu_\omega := \infty_{\Omega\backslash\omega}$, this allows us to define the so called *relaxed Dirichlet problem* associated to (2.2.19):
Find $u_{\mu_\omega} \in X_{\mu_\omega}(\Omega) := H_0^1(\Omega) \cap L^2_{\mu_\omega}(\Omega)$

$$\int_\Omega \nabla u_{\mu_\omega} \cdot \nabla v \,\mathrm{d}x + \int_\Omega u_{\mu_\omega} v \,\mathrm{d}\mu_\omega = \int_\Omega 1v \,\mathrm{d}x \quad \forall v \in X_{\mu_\omega}(\Omega). \tag{2.2.20}$$

Here $H_0^1(\Omega) \cap L^2_{\mu_\omega}(\Omega)$ denotes the space of functions $v \in H_0^1(\Omega)$ fulfilling

$$\int_\Omega v^2 \,\mathrm{d}\mu_\omega < \infty.$$

This problem is studied in depth in [19,49,63,78,110] as this is the canonical limit problem when passing to the limit with a sequence of solutions of the classical Dirichlet–Laplace problem formulated on open sets $(\Omega_n)_{n\in\mathbb{N}}$ in the sharp-interface case. Due to [49,63], X_{μ_ω} is a Hilbert space with the scalar product

$$(u,v)_{X_{\mu_\omega}} := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega uv \,\mathrm{d}\mu_\omega.$$

Using [49, Ex. 4.3.2] and the fact that there exists a finely open set $\mathcal{A} \subset \Omega$ and a set of zero capacity $\mathcal{N}$ such that $\omega = \mathcal{A} \cup \mathcal{N}$ (see [49, Thm. 4.1.5]), one deduces that

$$H_0^1(\omega) = X_{\mu_\omega},$$

and therefore $u_{\mu_\omega} = u_\omega$ in $H_0^1(\omega)$.
Note that the comparison principle known for classical elliptic PDEs still holds for the relaxed Dirichlet problem, see [78, Prop. 2.6], which allows us to deduce that $u_{\mu_\omega} \geq 0$ q.e. in Ω and $u_{\mu_\omega} \in L^\infty(\Omega)$. Furthermore, using this comparison principle along with the relations shown above, we eventually obtain

$$\tilde{H}_0^1(E) = H_0^1(\omega) = H_0^1(\{u_\omega > 0\}) = H_0^1(\{u_{\mu_\omega} > 0\}) = H_0^1(\{u_E > 0\}).$$

Using these identities, it is easy to see that

$$H_0^1(\{u_\omega > 0\}) = \tilde{H}_0^1(\{u_\omega > 0\}),$$

because $u_\omega \in H_0^1(\omega) = \tilde{H}_0^1(E)$. The equality between these spaces will be crucially exploited in the proof of Theorem 3.3.11

In particular, for any quasi-open set $\omega \subset \Omega$, this discussion shows us that solving the relaxed Dirichlet problem (2.2.20) is equivalent to solve the classical Dirichlet problem (2.2.19) over $H_0^1(\omega)$. Thus, as also done in [39, 51], we can adapt the density result [78, Proposition 5.5] to our setting.

Proposition 2.2.33. *Let $\omega \subset \Omega$ be a quasi-open set and $u_\omega \in H_0^1(\omega)$ be the unique solution of* (2.2.19)*. Then*

$$\{u_\omega \phi \mid \phi \in C_0^\infty(\Omega)\} \subset H_0^1(\omega) = H_0^1(\{u_\omega > 0\}). \tag{2.2.21}$$

is dense.

This result is particularly helpful to show γ-convergence and we will apply this result also in the proof of Theorem 3.3.11.

Now let us finally come to the notion of γ-convergence. The γ-convergence can be abstractly defined in terms of Γ-convergence, see [49, Definition 3.3.1]. For our purposes the following working definition will be enough, see also [49, Proposition 4.5.3, Remark 4.5.5] for the equivalence of both definitions.

Definition 2.2.34. *Let $(\omega_n)_{n\in\mathbb{N}} \subset \Omega$ and $\omega \subset \Omega$ be quasi-open sets. We say that ω_n γ-converges to ω and write $\omega_n \xrightarrow{\gamma} \omega$ if*

$$u_{\omega_n} \to u_\omega \quad \text{in } H_0^1(\Omega),$$

where $u_{\omega_n} \in H_0^1(\omega_n)$, $u_\omega \in H_0^1(\omega)$ denote the unique solutions associated to equation (2.2.19)*.*

This means that the γ-convergence of quasi-open shapes is defined via the convergence of solutions to a generic state equation on each shape, namely Poisson's equation with right-hand side 1.

As mentioned above let us give a characterization of γ-convergence in terms of Mosco-convergence and therefore define convergence in the sense of Mosco at first.

Definition 2.2.35. *Let $(\omega_n)_{n\in\mathbb{N}} \subset \Omega$ and $\omega \subset \Omega$ be quasi-open sets. We say that $H_0^1(\omega_n)$ converges in the sense of Mosco to $H_0^1(\omega)$ if the following two conditions are satisfied.*

(M1) For all $\phi \in H_0^1(\omega)$ there exists a sequence $\phi_n \in H_0^1(\omega_n)$ such that

$$\phi_n \to \phi \quad \text{in } H_0^1(\Omega).$$

(M2) Let $\phi_n \in H_0^1(\omega_n)$ be an arbitrary sequence which weakly converges in $H_0^1(\Omega)$ to some $\phi \in H_0^1(\Omega)$.
Then it holds $\phi \in H_0^1(\omega)$.

This convergence reveals the close link to Γ-convergence from Definition 2.2.23, because in proving (M1) one has to *construct* a recovery sequence, which is reminiscent of the lim sup inequality and in proving (M2) one has to show a property for the limit of *any* sequence, which is reminiscent of the lim inf inequality.

Now the following characterization holds due to [51, Proposition 4.5.3].

Theorem 2.2.36. *Let $(\omega_n)_{n\in\mathbb{N}} \subset \Omega$ and $\omega \subset \Omega$ be quasi-open sets. Then $\omega_n \xrightarrow{\gamma} \omega$ if and only if $H_0^1(\omega_n)$ converges to $H_0^1(\omega)$ in the sense of Mosco.*

Finally, let us mention the stability of γ-convergence with respect to intersection.

Proposition 2.2.37. *Let $(A_n)_{n\in\mathbb{N}}, A, (B_n)_{n\in\mathbb{N}}, B \subset \Omega$ be quasi-open sets with $A_n \xrightarrow{\gamma} A$ and $B_n \xrightarrow{\gamma} B$ then it holds*

$$A_n \cap B_n \xrightarrow{\gamma} A \cap B.$$

For a proof we refer to [49, Proposition 4.5.6].

Part I

Spectral optimization problems for the Laplace operator

Chapter 3

The Laplace problem

3.1. Introduction

Poisson's equation is one of the most fundamental equations in physics and mathematics. From the perspective of vibration phenomena the eigenvalue equation associated to the Laplace operator for example models the vibration of a membrane. Furthermore it serves also as a model problem to understand the behavior of eigenvalues for more general elliptic partial differential equations.
The key motivation for this chapter is to find a model that is well suited for the mathematical analysis of the problems

$$\begin{cases} -\Delta w = \lambda w & \text{in } D, \\ w|_{\partial D} = 0 & \text{on } \partial D, \end{cases} \quad \text{or} \quad \begin{cases} -\Delta w = \mu w & \text{in } D, \\ \partial_{\boldsymbol{n}} w = 0 & \text{on } \partial D, \end{cases} \tag{CL}$$

where D is a varying shape. As also discussed in the general introduction, a rich part of the literature is devoted to classical shape optimization. Classical here means that the shape itself is varied along the optimization process, i.e., the shapes here serve as control in the optimization problems, see [5, 49, 51, 64, 108, 109].

In order to approximate the classical problems (CL) we introduce a phase-field formulation of these equations. Even though our approach is similar to the one used in [39] we propose a model with a more general relaxation term in the approximate eigenvalue problem. Furthermore, our framework allows us to deal with additional volume constraints and gives us the possibility to place obstacles within the design domain and to initially prescribe regions containing either material or void throughout the optimization process. Finally, our proof for the sharp interface limit in the Dirichlet case works for the non-smooth double obstacle potential in the Ginzburg–Landau energy and not only for smooth potentials. This is due to the fact that we prove the continuity of eigenvalues when passing from diffuse to sharp interfaces.
Let us now shortly explain the strategy we will pursue.

As mentioned in the overall introduction we approximate (CL) by introducing a phase-field variable allowing us to formulate an elliptic eigenvalue problem on the whole design domain. This problem now involves coefficient functions depending on the phase-field which are chosen such that the associated classical eigenvalue problem is approximately satisfied on the shape given by the super-level set $\Omega_+^\varepsilon = \{\varphi_\varepsilon \geq 0\}$, see also Section 2.1.6.

Thus, Ω_+^ε in turn can be understood as approximate shape which is implicitly recovered from the phase-field variable. In Section 2.1.6 we have already discussed choices of these coefficient functions which will formally lead to the classical eigenvalue problem (CL) on the shape $D = \{\varphi_0 = 1\}$ when passing the interface parameter $\varepsilon \to 0$.

We refer to the books [19, 49, 108, 110] for in depth discussions of analytical results about classical shape variation, especially the continuity of eigenvalues under perturbation of the shape and shape differentiability. A crucial tool introduced there is the notion of γ-convergence, see Section 2.2.6. This concept for the sharp-interface case is constructed in such a way that the γ-convergence of the shapes implies the convergence of the associated eigenvalues. In our framework we also need to rely on this theory as the Γ-limit of the associated cost functionals involves an approximation of finite perimeter sets by smooth sets and thus, we particularly need the continuity of eigenvalues with respect to this approximation on the sharp-interface level. Here on the sharp-interface level, we will follow the lines of reasoning of [39], but need to be careful in distinguishing the relative perimeter P_Ω from the global perimeter $P_{\mathbb{R}^d}$, see Section 3.3.2.

In order to illustrate the behavior of solutions (λ, w) to (CL) let us consider this problem in the class of *rectangular* shapes. Therefore let the shape be $D = D_{L,l} = (0, L) \times (0, l)$ with $L > l > 0$. The eigenvalues in both problems of (CL) in this two dimensional example can be directly computed as

$$\pi^2 \left(\frac{m^2}{L^2} + \frac{k^2}{l^2} \right). \tag{3.1.1}$$

The crucial difference is that in the Neumann case $m, k \in \mathbb{N}_0$ can be zero whereas in the Dirichlet case both $m \in \mathbb{N}$ and $k \in \mathbb{N}$ have to be strictly positive. The corresponding $L^2(D)$-normalized eigenfunctions are

$$w^D_{m,k}(x, y) = \frac{2}{\sqrt{Ll}} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{k\pi y}{l}\right) \quad m, k \geq 1,$$

in the Dirichlet case, and

$$w^N_{m,k}(x, y) = \frac{2}{\sqrt{Ll}} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{k\pi y}{l}\right) \quad m, k \geq 0,$$

in the Neumann case, see e.g., [108, Proposition 1.2.13].
Having optimization problems in mind we observe the following. Let us consider the class of above rectangles which additionally satisfies the area constraint $Ll = 1$. Then for fixed L, l the smallest non-trivial Neumann eigenvalue of (CL) is given by $\frac{\pi^2}{L^2}$. This eigenvalue converges to zero for $L \to \infty$ and $l = \frac{1}{L} \to 0$. In other words taking long and thin rectangles, the first non-trivial Neumann eigenvalue degenerates and thus, the minimization of this eigenvalue in the class of rectangles of prescribed area is not well-posed. On the other hand the smallest Dirichlet eigenvalue of (CL) is given as $\pi^2(\frac{1}{L^2} + \frac{1}{l^2})$. Hence, choosing long and thin rectangles will produce arbitrarily large principal eigenvalues. This indicates from a mathematical viewpoint that the maximization of the smallest non-trivial Neumann eigenvalue and the minimization of the smallest Dirichlet eigenvalue are well-posed problems.

Let us now mention two classical results concerning the optimization of eigenvalues on the sharp-interface level. Definitely the most fundamental problem in this context is given

by the theorem of Faber–Krahn (see, e.g., [108, Theorem 3.2.1]), which states the ball $B \subset \mathbb{R}^n$ of prescribed volume $|B| = c > 0$ satisfies

$$\lambda_1(B) = \min\{\lambda_1(D) \,|\, D \text{ open subset of } \mathbb{R}^n \text{ with } |D| = c\}. \tag{3.1.2}$$

From a physical point of view, this theorem, which was first conjectured by Lord Rayleigh in 1877 (see [139]), states in particular that the lowest possible frequency under all clamped membrane of fixed area is achieved by a circular membrane. Loosely speaking: "*All drums are round.*" Chapter 4 of this thesis is entirely devoted to the study of this celebrated result in the phase-field framework.
In the Neumann case the analogon to the Faber–Krahn theorem is the theorem of Szegő and Weinberger (see, e.g., [108, Theorem 7.1.1]) which states that the ball of fixed volume $|B| = c$ maximizes the first non-trivial eigenvalue as follows

$$\mu_1(B) = \max\left\{\mu_1(D) \,\middle|\, \begin{array}{l} D \text{ open subset of } \mathbb{R}^n \text{ with} \\ \text{Lipschitz boundary and } |D| = c \end{array}\right\}. \tag{3.1.3}$$

Note that this Lipschitz condition here is indeed necessary in order to obtain a sequence of increasingly Neumann eigenvalues by classical spectral theory, see Theorem 2.2.8 and [108, Theorem 1.2.8] using the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$.

For the sake of completeness let us briefly recall the formulation of the optimization problems studied in this chapter, see also Section 2.1.7. Let the interface parameter $\varepsilon > 0$ be fixed. For some phase-field $\varphi = \varphi_\varepsilon$, let $(\lambda_k^{\varepsilon,\varphi})_{k\in\mathbb{N}}$ and $(\mu_k^{\varepsilon,\varphi})_{k\in\mathbb{N}}$ comprise all the Dirichlet and Neumann eigenvalues of the equations approximating (CL), respectively. In the Dirichlet case, we minimize the functional

$$J_\varepsilon^D(\varphi) = \Psi(\lambda_{i_1}^{\varepsilon,\varphi}, \dots, \lambda_{i_l}^{\varepsilon,\varphi}) + \gamma E_{\mathrm{GL}}^\varepsilon(\varphi), \tag{3.1.4}$$

and in the Neumann case, we minimize the functional

$$J_\varepsilon^N(\varphi) = \Psi(\mu_{i_1}^{\varepsilon,\varphi}, \dots, \mu_{i_l}^{\varepsilon,\varphi}) + \gamma E_{\mathrm{GL}}^\varepsilon(\varphi), \tag{3.1.5}$$

where the indices $i_1, \dots, i_l \in \mathbb{N}$ select eigenvalues from the above sequences and $E_{\mathrm{GL}}^\varepsilon$ is the Ginzburg–Landau energy.
One of the strengths of our model lies in the feasible set Φ_{ad}, these problems are formulated on, see Section 2.1.2. On the one hand $\varphi \in \Phi_{\mathrm{ad}}$ satisfies a volume constraint, in the sense that its integral lies in between some initially imposed barriers and on the other hand we initially prescribe regions within the design domain which are either completely filled with material or have to remain empty along the optimization process. We will see in Section 3.3.2 that these additional constraints combined with the perimeter penalization coming from the Ginzburg–Landau energy require a quite delicate analysis which goes beyond the existing Γ-convergence proof for the Dirichlet Laplace problem in [39].

In summary, this chapter comprises the analysis of the optimization problems associated to (3.1.4) and (3.1.5), an in depth and rigorous study of the sharp-interface limit of the Dirichlet problem in the framework of Γ-convergence and finally numerical simulations which through many examples show the strength of the phase-field approach applied to shape and topology optimization problems.

3.2. Analysis of the diffuse interface problem

3.2.1. The state equations and their properties

In this section we fix $\varepsilon > 0$ and therefore, we will just write φ instead of φ_ε. For a cleaner presentation, we also omit the superscript ε when denoting eigenvalues and eigenfunctions as the ε-dependency is indicated by the coefficient functions.

Definition 3.2.1 (Definition of eigenvalues and eigenfunctions)**.** *Let $\varphi \in L^\infty(\Omega)$ be arbitrary.*

(a) *λ^φ is called a Dirichlet eigenvalue of the state equation* (2.1.8) *if there exists a non-trivial weak solution $w^{D,\varphi}$ to* (2.1.8)*, i.e., $0 \neq w^{D,\varphi} \in H_0^1(\Omega)$, and it holds that*

$$\left(\nabla w^{D,\varphi}, \nabla \eta\right)_{a_\varepsilon(\varphi)} + \left(w^{D,\varphi}, \eta\right)_{b_\varepsilon(\varphi)} = \lambda^\varphi \left(w^{D,\varphi}, \eta\right)_{c_\varepsilon(\varphi)} \quad \text{for all } \eta \in H_0^1(\Omega). \tag{3.2.1}$$

In this case, the function $w^{D,\varphi}$ is called an eigenfunction to the eigenvalue λ^φ.

(b) *μ^φ is called a Neumann eigenvalue of the state equation* (2.1.9) *if there exists a non-trivial weak solution $w^{N,\varphi}$ to* (2.1.9)*, i.e., $0 \neq w^{N,\varphi} \in H^1(\Omega)$, and it holds that*

$$\left(\nabla w^{N,\varphi}, \nabla \eta\right)_{a_\varepsilon(\varphi)} = \mu^\varphi \left(w^{N,\varphi}, \eta\right)_{c_\varepsilon(\varphi)} \quad \text{for all } \eta \in H^1(\Omega). \tag{3.2.2}$$

In this case, the function $w^{N,\varphi}$ is called an eigenfunction to the eigenvalue μ^φ.

The properties and assumptions of the preliminary section allow us to prove two classical functional analytic results concerning the eigenvalues and eigenfunctions.

Theorem 3.2.2 (Existence and properties of eigenvalues and eigenfunctions)**.**
Let $\varphi \in L^\infty(\Omega)$ be arbitrary.

(a) *There exist sequences*

$$\left(w_k^{D,\varphi}, \lambda_k^\varphi\right)_{k\in\mathbb{N}} \subset H_0^1(\Omega) \times \mathbb{R}, \quad \left(w_k^{N,\varphi}, \mu_k^\varphi\right)_{k\in\mathbb{N}_0} \subset H^1(\Omega) \times \mathbb{R},$$

possessing the following properties:

- *For all $k \in \mathbb{N}$, $w_k^{D,\varphi}$ is an eigenfunction to the eigenvalue λ_k^φ and $w_k^{N,\varphi}$ is an eigenfunction to the eigenvalue μ_k^φ in the sense of Definition 3.2.1.*
- *The eigenvalues $\lambda_k^\varphi, \mu_k^\varphi$ (which are repeated according to their multiplicity) can be ordered in the following way:*

$$\begin{aligned} 0 < \lambda_1^\varphi \leq \lambda_2^\varphi \leq \lambda_3^\varphi \leq \cdots, \\ 0 = \mu_0^\varphi < \mu_1^\varphi \leq \mu_2^\varphi \leq \mu_3^\varphi \leq \cdots. \end{aligned}$$

Moreover, it holds that $\lambda_k^\varphi, \mu_k^\varphi \to \infty$ as $k \to \infty$, and there exist no further eigenvalues of the state equation (3.2.1) *and* (3.2.2)*.*

- *Both the Dirichlet eigenfunctions* $\{w_1^{D,\varphi}, w_2^{D,\varphi}, \dots\} \subset H_0^1(\Omega)$ *and the Neumann eigenfunctions* $\{w_0^{N,\varphi}, w_1^{N,\varphi}, \dots\} \subset H^1(\Omega)$ *form an orthonormal basis of the space* $L^2_\varphi(\Omega)$. *Furthermore, the eigenfunctions* $\{w_1^{N,\varphi}, w_2^{N,\varphi}, \dots\}$ *belong to the space* $H^1_{(0),\varphi}(\Omega)$ *and form an* $L^2_\varphi(\Omega)$*-orthonormal basis of the space*

$$L^2_{(0),\varphi}(\Omega) = \left\{ w \in L^2_\varphi(\Omega) \,\middle|\, \int_\Omega c_\varepsilon(\varphi) w \,\mathrm{d}x = 0 \right\}.$$

In particular, this implies that any eigenfunction to a non-trivial eigenvalue belongs to the space $H^1_{(0),\varphi}(\Omega)$.

(b) *For* $k \in \mathbb{N}$, *we have the Courant–Fischer characterizations*

$$\lambda_k^\varphi = \max_{V \in \mathcal{S}_{k-1}} \min \left\{ \frac{\int_\Omega a_\varepsilon(\varphi) |\nabla v|^2 \, dx + \int_\Omega b_\varepsilon(\varphi) |v|^2 \,\mathrm{d}x}{\int_\Omega c_\varepsilon(\varphi) |v|^2 \,\mathrm{d}x} \,\middle|\, \begin{matrix} v \in V^{\perp, L^2_\varphi(\Omega)} \cap H_0^1(\Omega), \\ v \neq 0 \end{matrix} \right\}, \tag{3.2.3}$$

and

$$\mu_k^\varphi = \max_{V \in \mathcal{S}_{k-1}} \min \left\{ \frac{\int_\Omega a_\varepsilon(\varphi) |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega c_\varepsilon(\varphi) |v|^2 \,\mathrm{d}x} \,\middle|\, \begin{matrix} v \in V^{\perp, L^2_\varphi(\Omega)} \cap H^1_{(0),\varphi}(\Omega), \\ v \neq 0 \end{matrix} \right\}. \tag{3.2.4}$$

Here, $\mathcal{S}_{k-1}$ *denotes the collection of all* $(k-1)$*-dimensional subspaces of* $L^2_\varphi(\Omega)$.

The set $V^{\perp, L^2_\varphi(\Omega)}$ *denotes the orthogonal complement of* $V \subset L^2(\Omega)$ *with respect to the scalar product* $(\cdot,\cdot)_{c_\varepsilon(\varphi)}$ *on* $L^2_\varphi(\Omega)$.

Moreover, the maximum is attained at the subspace

$$V = \langle w_1^{D,\varphi}, \dots, w_{k-1}^{D,\varphi} \rangle_{\mathrm{span}} \quad \textit{and} \quad V = \langle w_1^{N,\varphi}, \dots, w_{k-1}^{N,\varphi} \rangle_{\mathrm{span}},$$

respectively.

(c) *We can choose the eigenfunction* $w_1^{D,\varphi}$ *such that it is positive almost everywhere in* Ω. *Furthermore, every solution* $w \in H_0^1(\Omega)$ *of*

$$(\nabla w, \nabla \eta)_{a_\varepsilon(\varphi)} + (w, \eta)_{b_\varepsilon(\varphi)} = \lambda_1^\varphi (w, \eta)_{c_\varepsilon(\varphi)} \quad \textit{for all } \eta \in H_0^1(\Omega),$$

is a multiple of $w_1^{D,\varphi}$, *i.e., there is a constant* $\xi \in \mathbb{R}$ *such that* $w = \xi w_1^{D,\varphi}$ *almost everywhere in* Ω. *This means that the eigenspace to* λ_1^φ *is simple.*

Proof of Theorem 3.2.2. (a) The assertion is a direct consequence of the spectral theorem for compact self-adjoint operators, see Theorem 2.2.8 and also the proof of Theorem 5.2.2 where we go through the main steps which then directly apply to our situation.

(b) The claim is established in the same fashion as we will do later on for the problem in linear elasticity, see Theorem 5.2.2.

(c) The assertion follows directly from [99, Theorem 8.38]. □

Remark 3.2.3. In the following, we impose weaker assumptions on our phase field φ compared to the analysis in Chapter 5, namely we only consider $\varphi \in H^1(\tilde{\Omega}) \cap L^\infty(\Omega)$ instead of $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, where $\tilde{\Omega} = \Omega \backslash (S_0 \cap S_1)$ with S_0 and S_1 being the sets appearing in the point-wise constraint (2.1.1). As explained in Section 2.1.4, we consider this reduction of H^1-regularity in order to avoid additional Dirichlet boundary conditions for the phase-field on the boundaries of S_0 and S_1 which would complicate the sharp-interface limit $\varepsilon \to 0$ that is discussed in Section 3.3.

Nevertheless, one can still formulate and prove all the continuity and differentiability results established in Chapter 5 as the results in Section 5.3.1 merely rely on the point-wise almost everywhere convergence of phase-field sequences $\varphi_k \to \varphi$ in Ω and the boundedness in $L^\infty(\Omega)$ in order to apply Lebesgue's dominated convergence theorem and the results from Section 5.3.2 and Section 5.4 only rely on the strong convergence in $L^\infty(\Omega; \mathbb{R}^d)$ as this is sufficient to exploit the local Lipschitz continuity of $\mathbb{C}$ and ρ there.

Following this remark, we only display the most important results of Chapter 5 adapted to our setting, namely the continuity of eigenvalues and eigenfunctions as well as the Fréchet-differentiability of simple eigenvalues.

Theorem 3.2.4 (Continuity properties for the eigenvalues and their eigenfunctions)**.** *Let $j \in \mathbb{N}$ be arbitrary and let $(\varphi_k)_{k\in\mathbb{N}} \subset L^\infty(\Omega)$ be a bounded sequence with*

$$\varphi_k \to \varphi \quad \text{a.e. in } \Omega \text{ as } k \to \infty.$$

Moreover, let $(u_j^{D,\varphi_k})_{k\in\mathbb{N}} \subset H_0^1(\Omega)$ and $(u_j^{N,\varphi_k})_{k\in\mathbb{N}} \subset H^1_{(0),\varphi}(\Omega)$ be sequences of $L^2_{\varphi_k}(\Omega)$-normalized eigenfunctions to the eigenvalues $(\lambda_j^{\varphi_k})_{k\in\mathbb{N}}$ and $(\mu_j^{\varphi_k})_{k\in\mathbb{N}}$ respectively.
Then it holds that

$$\lambda_j^{\varphi_k} \to \lambda_j^{\varphi} \quad \text{and} \quad \mu_j^{\varphi_k} \to \mu_j^{\varphi} \quad \text{as } k \to \infty.$$

Furthermore, there exist $L^2_\varphi(\Omega)$-normalized eigenfunctions $\overline{u}_j^D \in H_0^1(\Omega)$, $\overline{u}_j^N \in H^1_{(0),\varphi}(\Omega)$ to the eigenvalue $\lambda_j^\varphi, \mu_j^\varphi$ respectively, such that for $\zeta \in \{D, N\}$,

$$u_j^{\zeta,\varphi_k} \rightharpoonup \overline{u}_j^\zeta \quad \text{in } H^1(\Omega), \quad \text{and} \quad u_j^{\zeta,\varphi_k} \to \overline{u}_j^\zeta \quad \text{in } L^2_\varphi(\Omega),$$

as $k \to \infty$ along a non-relabeled subsequence.

Proof. The assertion can be established inductively by proceeding as in Theorem 5.3.4, using the Courant–Fischer characterization from Theorem 3.2.2, and the Banach–Alaoglu theorem applied to the sequence of eigenfunctions. □

In the remaining analysis of this section, namely the Fréchet-differentiability of eigenvalues, the assumption that the considered eigenvalues are simple is crucial. For instance, it already becomes clear in finite dimension (see [108, Sec. 2.5] or Section 5.4.2) that multiple eigenvalues are in general not differentiable as two different eigenvalues can cross in a non-differentiable way. To overcome this issue, one can switch to a weaker notion of differentiability such as directional differentiability or semi-differentiability, cf. [108] or Section 5.4.2. Nevertheless, our numerical method needs the first-order conditions to be

formulated in the framework of classical Fréchet-differentiability. However multiple eigenvalues can to some extent still be handled numerically by adaptively reformulating the cost functional as we will see in Section 3.4.

Let us first recall the *sign condition* for simple eigenvalues that is introduced in Section 5.3.3.

Lemma 3.2.5. *Let $i \in \mathbb{N}$ and $\varphi \in L^\infty(\Omega)$ be arbitrary. We suppose that the eigenvalue λ_i^φ (or μ_i^φ) is simple. Let $w_i^{D,\varphi} \in H_0^1(\Omega)$ (or $w_i^{N,\varphi} \in H^1_{(0),\varphi}(\Omega)$) be an $L^2_\varphi(\Omega)$-normalized eigenfunction to the eigenvalue λ_i^φ (or μ_i^φ).*

Then, for all $0 < \sigma < 1$, there exists $\delta > 0$ such that for all

$$h \in L^\infty(\Omega) \quad \text{with} \quad \|h\|_{L^\infty(\Omega)} < \delta,$$

there exist a unique $L^2_{\varphi+h}(\Omega)$-normalized eigenfunction $w_i^{D,\varphi+h} \in H_0^1(\Omega)$ (or $w_i^{N,\varphi+h} \in H^1_{(0),\varphi}(\Omega)$) to the eigenvalue $\lambda_i^{\varphi+h}$ (or $\mu_i^{\varphi+h}$) satisfying the condition

$$0 < \sigma < \left(w_i^{D,\varphi+h}, w_i^{D,\varphi}\right)_{c_\varepsilon(\varphi)} \qquad \left(\text{ or } \ 0 < \sigma < \left(w_i^{N,\varphi+h}, w_i^{N,\varphi}\right)_{c_\varepsilon(\varphi)} \right). \tag{3.2.5}$$

In particular, the eigenvalue $\lambda_i^{\varphi+h}$ (or $\mu_i^{\varphi+h}$) is simple.

In the following, for the derivatives of the coefficient functions, we will use also the notation

$$(u,v)_{a'_\varepsilon(\varphi)h} \coloneqq \int_\Omega a'_\varepsilon(\varphi)h\, uv \,\mathrm{d}x$$

to provide a clearer presentation.

Theorem 3.2.6 (Fréchet-differentiability of simple eigenvalues and their eigenfunctions)**.** *Let $\varphi \in L^\infty(\Omega)$ be arbitrary and suppose that for $i \in \mathbb{N}$, the eigenvalue λ_i^φ (or μ_i^φ) is simple. We further fix a corresponding $L^2_\varphi(\Omega)$-normalized eigenfunction $w_i^{D,\varphi}$ (or $w_i^{N,\varphi}$).*

Then there exist constants δ_i^φ, $r_i^\varphi > 0$ such that the operator

$$\begin{cases} S_i^{D,\varphi} : B_{\delta_i^\varphi}(\varphi) \subset L^\infty(\Omega) \to B_{r_i^\varphi}\big((w_i^{D,\varphi}, \lambda_i^\varphi)\big) \subset H_0^1(\Omega) \times \mathbb{R} \\ \qquad\qquad\qquad\qquad \vartheta \mapsto (w_i^{D,\vartheta}, \lambda_i^\vartheta) \end{cases}$$

$$\left(\text{or} \quad \begin{cases} S_i^{N,\varphi} : B_{\delta_i^\varphi}(\varphi) \subset L^\infty(\Omega) \to B_{r_i^\varphi}\big((w_i^{N,\varphi}, \mu_i^\varphi)\big) \subset H^1_{(0),\varphi}(\Omega) \times \mathbb{R}, \\ \qquad\qquad\qquad\qquad \vartheta \mapsto (w_i^{N,\vartheta}, \mu_i^\vartheta) \end{cases} \right),$$

is well-defined and continuously Fréchet-differentiable.

Here, for any $\vartheta \in B_{\delta_i^\varphi}(\varphi)$, $w_i^{D,\vartheta}$ (or $w_i^{N,\vartheta}$) denotes the unique $L^2_\vartheta(\Omega)$-normalized eigenfunction to the eigenvalue λ_i^ϑ (or μ_i^ϑ) satisfying the sign condition (3.2.5) written for $h = \vartheta - \varphi$.

Moreover, for any $h \in L^\infty(\Omega)$, the Fréchet-derivative $(\lambda_i^\vartheta)'h$ of the Dirichlet eigenvalue λ_i^ϑ at $\vartheta \in B_{\delta_i^\varphi}(\varphi)$ in the direction h reads as

$$\begin{aligned} (\lambda_i^\vartheta)'h &= \left(S_{i,2}^{D,\varphi}(\vartheta)\right)' h \\ &= \left(\nabla w_i^{D,\vartheta}, \nabla w_i^{D,\vartheta}\right)_{a'_\varepsilon(\vartheta)h} + \left(w_i^{D,\vartheta}, w_i^{D,\vartheta}\right)_{b'_\varepsilon(\vartheta)h} - \lambda_i^\vartheta \left(w_i^{D,\vartheta}, w_i^{D,\vartheta}\right)_{c'_\varepsilon(\vartheta)h}. \end{aligned} \tag{3.2.6}$$

On the other hand, for any $h \in L^\infty(\Omega)$, the Fréchet-derivative $(\mu_i^\vartheta)'h$ of the Neumann eigenvalue μ_i^ϑ at $\vartheta \in B_{\delta_i^\varphi}(\varphi)$ in the direction h reads as

$$(\mu_i^\vartheta)'h = \left(S_{i,2}^{N,\varphi}(\vartheta)\right)' h = \left(\nabla w_i^{N,\vartheta}, \nabla w_i^{N,\vartheta}\right)_{a_\varepsilon'(\vartheta)h} - \mu_i^\vartheta \left(w_i^{N,\vartheta}, w_i^{N,\vartheta}\right)_{c_\varepsilon'(\vartheta)h}. \tag{3.2.7}$$

Proof. This is proven as in Theorem 5.4.3 by combining the implicit function theorem with the Fredholm alternative from Lemma 5.4.4. □

3.2.2. First-order optimality conditions

We now intend to apply the theory developed in Section 3.2 to show that the optimization problems $(\mathcal{P}_l^{D,\varepsilon})$ and $(\mathcal{P}_l^{N,\varepsilon})$ (that were introduced in Section 2.1.7) possess a minimizer. However, in the Neumann case, we first need to establish an additional boundedness property.

Recall that one possible application of our model is to *maximize* the first non-trivial Neumann eigenvalue. Since $(\mathcal{P}_l^{N,\varepsilon})$ is formulated as a minimization problem, we thus allow for functions Ψ that are not bounded from below. A possible choice to realize a maximization of the first Neumann eigenvalue would be $\Psi(\mu_1^\varphi) = -\mu_1^\varphi$ (meaning that $\Psi(x) = -x$ for all $x \in \mathbb{R}$). To apply the direct method in the calculus of variations, we need to show that the cost functional $J_1^\varepsilon(\varphi) = \Psi(\mu_1^\varphi) + \gamma E^\varepsilon(\varphi)$ remains bounded from below on the admissible set Φ_{ad}, even if Ψ is not bounded from below.

Our goal is to show that any Dirichlet eigenvalue of (3.2.1) and any Neumann eigenvalue of (3.2.2), is uniformly bounded by expressions involving the corresponding eigenvalue of the classical eigenvalue problem on the *whole* design domain Ω. This allows us to deduce that each Dirichlet and Neumann eigenvalue belongs to a compact subset of $\mathbb{R}_{>0}$. Hence, as the function Ψ is assumed to be continuous on $(\mathbb{R}_{>0})^l$, it is bounded on such compact sets.

Lemma 3.2.7. *Let $k \in \mathbb{N}$ and let λ_k^{LD} denote the k-th eigenvalue of the classical Dirichlet eigenvalue problem, i.e.,*

$$(\nabla w, \nabla \eta) = \lambda_k^{LD}(w, \eta) \quad \forall \eta \in H_0^1(\Omega),$$

and let μ_k^{LD} denote the k-th eigenvalue of the classical Neumann eigenvalue problem, i.e.,

$$(\nabla w, \nabla \eta) = \mu_k^{LD}(w, \eta) \quad \forall \eta \in H^1(\Omega),$$

where $(\cdot,\cdot)$ denotes the standard scalar product on $L^2(\Omega)$. Then there exist constants $C_{1,\varepsilon}$, $C_{2,\varepsilon} > 0$ depending only on the choice of coefficient functions a_ε, b_ε, and c_ε such that

$$\begin{aligned} C_{1,\varepsilon}\, \lambda_k^{LD} &\le \lambda_k^\varphi \le C_{2,\varepsilon}\, (\lambda_k^{LD} + 1), \\ C_{1,\varepsilon}\, \mu_k^{LD} &\le \mu_k^\varphi \le C_{2,\varepsilon}\, \mu_k^{LD}, \end{aligned}$$

for all $\varphi \in \Phi_{\mathrm{ad}}$.

Proof. Let us start with the Neumann case. We will work with a Courant–Fischer characterization which is equivalent to (3.2.4) namely

$$\mu_k^\varphi = \min_{V \in \tilde{\mathcal{S}}_{k+1}} \max \left\{ \frac{\int_\Omega a_\varepsilon(\varphi)\, |\nabla v|^2 \, \mathrm{d}x}{\int_\Omega c_\varepsilon(\varphi)\, |v|^2 \, \mathrm{d}x} \,\middle|\, \begin{matrix} v \in V, \\ v \neq 0 \end{matrix} \right\}, \tag{3.2.8}$$

see [108, Sec. 1.3.1]. Here, $\tilde{S}_{k+1}$ denotes the collection of all $(k+1)$-dimensional subspaces of $H^1(\Omega)$. Note that compared to [108] we have to consider dimension $(k+1)$ instead of k, as we start our labeling of Neumann eigenvalues with the index 0 and not with 1. Obviously, we obtain the characterization of the classical Neumann eigenvalue by setting $a_\varepsilon \equiv c_\varepsilon \equiv 1$, i.e.,

$$\mu_k^{LD} = \min_{V \in \tilde{S}_{k+1}} \max \left\{ \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega |v|^2 \,\mathrm{d}x} \,\middle|\, v \in V, \atop v \neq 0 \right\}. \tag{3.2.9}$$

We now recall the assumptions on the coefficient functions from Section 2.1.5. In particular, we know that there is a constant $\tilde{C}_\varepsilon > 0$ such that $a_\varepsilon(\varphi), c_\varepsilon(\varphi) < \tilde{C}_\varepsilon$ for all $\varphi \in \Phi_{\mathrm{ad}}$, as $a_\varepsilon, c_\varepsilon \in C^0(\mathbb{R})$ and $|\varphi| \leq 1$. Furthermore, we assumed $a_\varepsilon, c_\varepsilon$ to be uniformly bounded from below, i.e., $a_\varepsilon, c_\varepsilon \geq C_\varepsilon$ for some constant $C_\varepsilon > 0$. Thus, we deduce that there are constants $C_{1,\varepsilon}, C_{2,\varepsilon} > 0$ that only depend on the choice of the real functions $a_\varepsilon, c_\varepsilon$, such that

$$C_{1,\varepsilon} \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega |v|^2 \,\mathrm{d}x} \leq \frac{\int_\Omega a_\varepsilon(\varphi) |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega c_\varepsilon(\varphi) |v|^2 \,\mathrm{d}x} \leq C_{2,\varepsilon} \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega |v|^2 \,\mathrm{d}x},$$

for all $0 \neq v \in H^1(\Omega)$ and $\varphi \in \Phi_{\mathrm{ad}}$. Comparing (3.2.8) and (3.2.9), this directly allows us to deduce the claim in the Neumann case.

The Dirichlet case works completely analogously but one has to mind the additional term coming from the coefficient function $b_\varepsilon \in C^{1,1}_{loc}(\mathbb{R})$ which is assumed to be non-negative. Thus we obtain here

$$C_{1,\varepsilon} \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega |v|^2 \,\mathrm{d}x} \leq \frac{\int_\Omega a_\varepsilon(\varphi) |\nabla v|^2 + b_\varepsilon(\varphi) |v|^2 \,\mathrm{d}x}{\int_\Omega c_\varepsilon(\varphi) |v|^2 \,\mathrm{d}x} \leq C_{2,\varepsilon} \left(\frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega |v|^2 \,\mathrm{d}x} + 1 \right),$$

for all $0 \neq v \in H^1_0(\Omega)$ and $\varphi \in \Phi_{\mathrm{ad}}$. □

Theorem 3.2.8 (Existence of a minimizer to $(\mathcal{P}_l^{D,\varepsilon})$ and $(\mathcal{P}_l^{N,\varepsilon})$)**.** *The problems $(\mathcal{P}_l^{D,\varepsilon})$ and $(\mathcal{P}_l^{N,\varepsilon})$ possess a minimizer $\varphi^D \in \mathcal{G}^\beta \cap \mathcal{U}$ and $\varphi^N \in \mathcal{G}^\beta \cap \mathcal{U}$, respectively.*

Proof. We proceed by applying the direct method in the calculus of variations. First of all, the feasible set $\mathcal{G}^\beta \cap \mathcal{U}$ is non-empty since it contains the function which is identical to β_1 in $\tilde{\Omega}$, -1 in S_0 and 1 in S_1. By the previous discussion we already know that the objective functionals $J_l^{D,\varepsilon}$ and $J_l^{N,\varepsilon}$ are bounded from below. Since $\varphi \in \mathcal{G}^\beta \subset L^\infty(\Omega)$, the term

$$\gamma E^\varepsilon(\varphi) = \gamma \int_{\tilde{\Omega}} \left(\frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon} \psi_0(\varphi) \right) \mathrm{d}x,$$

in the cost functional can be used to control φ in the $H^1(\tilde{\Omega})$-norm, but not in the whole $H^1(\Omega)$-norm. This means that for any minimizing sequence, we can only apply compactness on $\tilde{\Omega}$ which implies strong convergence in $L^2(\tilde{\Omega})$. However, as the elements of this minimizing sequence are additionally contained in the feasible set $\mathcal{G}^\beta \cap \mathcal{U}$, their values on $S_0 \cup S_1$ are a priori fixed, which gives us the desired point-wise almost everywhere convergence on the whole domain Ω. This allows us to apply Theorem 3.2.4 which provides the continuity of the eigenvalues with respect to the phase-field variable. □

Now, invoking the differentiability properties established in Section 3.2, we can derive a first-order necessary condition for local optimality. The following variational inequalities follow directly from the fact that for $\zeta \in \{D, N\}$ and $\vartheta \in \mathcal{G}^\beta \cap \mathcal{U}$

$$0 \leq \frac{\mathrm{d}}{\mathrm{d}t}\left[J_l^{\zeta,\varepsilon}(\varphi^\zeta + t(\vartheta - \varphi^\zeta))\right]_{|_{t=0}} = \left(J_l^{\zeta,\varepsilon}\right)'(\varphi^\zeta)(\vartheta - \varphi^\zeta),$$

as the convexity of $\mathcal{G}^\beta \cap \mathcal{U}$ implies that $\varphi^\zeta + t(\vartheta - \varphi^\zeta) \in \mathcal{G}^\beta \cap \mathcal{U}$ for $t \in [0,1]$.

Theorem 3.2.9 (The optimality system to $(\mathcal{P}_l^{D,\varepsilon})$)**.** *Let $\varphi \in \mathcal{G}^\beta \cap \mathcal{U}$ be a local minimizer of the optimization problem $(\mathcal{P}_l^{D,\varepsilon})$, i.e., there exists $\delta > 0$ such that*

$$J_l^{D,\varepsilon}(\vartheta) \geq J_l^{D,\varepsilon}(\varphi) \quad \text{for all } \vartheta \in \mathcal{G}^\beta \cap \mathcal{U} \text{ with } \|\vartheta - \varphi\|_{L^\infty(\Omega)} < \delta.$$

Suppose that the eigenvalues $\lambda_{i_1}^\varphi, \dots, \lambda_{i_l}^\varphi$ are simple and let us fix associated $L^2_\varphi(\Omega)$-normalized eigenfunctions $w_{i_1}^{D,\varphi}, \dots, w_{i_l}^{D,\varphi} \in H_0^1(\Omega)$.
Then the following optimality system is satisfied:

- *The state equations*

$$\begin{cases} -\nabla \cdot \left[a_\varepsilon(\varphi)\nabla w_{i_j}^{D,\varphi}\right] + b_\varepsilon(\varphi) w_{i_j}^{D,\varphi} = \lambda_{i_j}^\varphi c_\varepsilon(\varphi) w_{i_j}^{D,\varphi} & \text{in } \Omega, \\ w_{i_j}^{D,\varphi} = 0 & \text{on } \partial\Omega \end{cases} \tag{SD_j}$$

  are fulfilled in the weak sense for all $j \in \{1, \dots, l\}$.

- *The variational inequality*

$$\begin{aligned} 0 \leq{}& \gamma\varepsilon \int_{\tilde{\Omega}} \nabla\varphi \cdot \nabla(\vartheta - \varphi)\,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_{\tilde{\Omega}} \psi_0'(\varphi)(\vartheta - \varphi)\,\mathrm{d}x \\ &+ \sum_{j=1}^{l} \left\{ [\partial_{\lambda_{i_j}}\Psi](\lambda_{i_1}^\varphi, \dots, \lambda_{i_l}^\varphi) \left(\left(\nabla w_{i_j}^{D,\varphi}, \nabla w_{i_j}^{D,\varphi}\right)_{a_\varepsilon'(\varphi)(\vartheta-\varphi)} + \left(w_{i_j}^{D,\varphi}, w_{i_j}^{D,\varphi}\right)_{b_\varepsilon'(\varphi)(\vartheta-\varphi)} \right.\right. \\ &\qquad \left.\left. - \lambda_{i_j}^\varphi \left(w_{i_j}^{D,\varphi}, w_{i_j}^{D,\varphi}\right)_{c_\varepsilon'(\varphi)(\vartheta-\varphi)} \right) \right\} \end{aligned} \tag{VD}$$

  is satisfied for all $\vartheta \in \mathcal{G}^\beta \cap \mathcal{U}$.

Theorem 3.2.10 (The optimality system to $(\mathcal{P}_l^{N,\varepsilon})$)**.** *Let $\varphi \in \mathcal{G}^\beta \cap \mathcal{U}$ be a local minimizer of the optimization problem $(\mathcal{P}_l^{N,\varepsilon})$, i.e., there exists $\delta > 0$ such that*

$$J_l^{N,\varepsilon}(\vartheta) \geq J_l^{N,\varepsilon}(\varphi) \quad \text{for all } \vartheta \in \mathcal{G}^\beta \cap \mathcal{U} \text{ with } \|\vartheta - \varphi\|_{L^\infty(\Omega)} < \delta.$$

Suppose that the eigenvalues $\mu_{i_1}^\varphi, \dots, \mu_{i_l}^\varphi$ are simple and let us fix associated $L^2_\varphi(\Omega)$-normalized eigenfunctions $w_{i_1}^{N,\varphi}, \dots, w_{i_l}^{N,\varphi} \in H^1_{(0),\varphi}(\Omega)$.
Then the following optimality system is satisfied:

- *The state equations*

$$\begin{cases} -\nabla \cdot \left[a_\varepsilon(\varphi) \nabla w_{i_j}^{N,\varphi} \right] = \mu_{i_j}^\varphi c_\varepsilon(\varphi) w_{i_j}^{N,\varphi} & \text{in } \Omega, \\ \dfrac{\partial w_{i_j}^{N,\varphi}}{\partial \boldsymbol{\nu}} = 0 & \text{on } \partial\Omega \end{cases} \tag{SN_j}$$

  are fulfilled in the weak sense for all $j \in \{1, \dots, l\}$.

- *The variational inequality*

$$\begin{aligned} 0 \le \gamma\varepsilon \int_{\tilde{\Omega}} \nabla\varphi \cdot \nabla(\vartheta - \varphi) \,\mathrm{d}x + \frac{\gamma}{\varepsilon} \int_{\tilde{\Omega}} \psi_0'(\varphi)(\vartheta - \varphi) \,\mathrm{d}x \\ + \sum_{j=1}^{l} \left\{ [\partial_{\lambda_{i_j}} \Psi](\mu_{i_1}^\varphi, \dots, \mu_{i_l}^\varphi) \left(\left(w_{i_j}^{N,\varphi}, w_{i_j}^{N,\varphi} \right)_{a_\varepsilon'(\varphi)(\vartheta-\varphi)} \right.\right. \\ \left.\left. - \mu_{i_j}^\varphi \left(w_{i_j}^{N,\varphi}, w_{i_j}^{N,\varphi} \right)_{c_\varepsilon'(\varphi)(\vartheta-\varphi)} \right) \right\} \end{aligned} \tag{VN}$$

  is satisfied for all $\vartheta \in \mathcal{G}^\beta \cap \mathcal{U}$.

3.3. Sharp-interface asymptotics for the Dirichlet case

In this section, we want to discuss the sharp-interface asymptotics for the Dirichlet eigenvalue optimization problem $(\mathcal{P}_l^{D,\varepsilon})$, i.e., its behavior when $\varepsilon \to 0$. For the sake of a rigorous discussion we need to make additional assumptions that are supposed to hold throughout the remainder of this section.

(A1) We assume that the design domain Ω is a bounded Lipschitz domain in $\mathbb{R}^d$ with $d \ge 2$.

(A2) We fix $a_\varepsilon = c_\varepsilon = 1$ on $[-1, 1]$.

(A3) Let

$$b_\varepsilon : [-1, 1] \to [0, \bar{b}_\varepsilon], \quad \varepsilon > 0, \quad \text{and} \quad b_0 : [-1, 1] \to [0, +\infty], \tag{3.3.1}$$

be functions with

- b_ε is decreasing, continuous and surjective
- b_0 is continuous at the point 1
- $b_0(0) < +\infty$,
- $b_\varepsilon \to b_0$ point-wise on $[-1, 1]$ as $\varepsilon \to 0$,
- and $b_\delta \ge b_\varepsilon$ on $[-1, 1]$ for all $0 \le \delta \le \varepsilon$.

Here, the interval $[0, +\infty]$ is to be understood as a subset of the extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, on which we use the common conventions $\pm\infty \cdot 0 = 0$ and $0^{-1} = +\infty$, see also Section 2.2.1.

Moreover, the numbers $\overline{b}_\varepsilon$ in (3.3.1) are chosen such that

$$\lim_{\varepsilon\to 0} \overline{b}_\varepsilon = +\infty \quad \text{with} \quad \overline{b}_\varepsilon = o(\varepsilon^{-\kappa}) \quad \text{as } \varepsilon \to 0,$$

where, depending on the dimension d,

$$\begin{cases} \kappa \in (0,1) & \text{if } d = 2, \\ \kappa = \frac{2}{d} & \text{if } d \geq 3. \end{cases}$$

(A4) In the following, we only consider elements $\varphi \in BV(\tilde{\Omega}, \{\pm 1\}) \cap \mathcal{U}$ such that the set

$$E^\varphi := \{x \in \Omega \,|\, \varphi(x) = 1\}$$

contains an open ball B. From this assumption we infer $C_0^\infty(B) \subset V^\varphi$ hence this excludes the pathological case that the space

$$V^\varphi := \left\{\eta \in H_0^1(\Omega) \,|\, \eta = 0 \text{ a.e. in } \Omega\backslash E^\varphi\right\} = \tilde{H}_0^1(E^\varphi)$$

is trivial or finite dimensional. See also the discussion in Remark 2.2.32. Recalling the definition of the set $\mathcal{U}$ in Section 2.1.7, this condition on E^φ can be implemented in the constraint $\varphi \in \mathcal{U}$ by simply demanding $B \subset S_1$ for any prescribed open ball $B \subset \Omega$. Later, in Subsection 3.3.1, we will discuss how V^φ is related to "Sobolev-like" spaces in the context of quasi-open sets. We further define the space

$$H^\varphi := \left\{\eta \in L^2(\Omega) \,|\, \eta = 0 \text{ a.e. in } \Omega\backslash E^\varphi\right\}.$$

(A5) In addition to the assumptions of Section 2.1.2, we demand that S_0 and S_1 are sets of finite perimeter in Ω. Then [14, Theorem 3.87] yields that any $\varphi \in BV(\tilde{\Omega}, \{\pm 1\}) \cap \mathcal{U}$ is indeed an element of $BV(\Omega, \{\pm 1\})$. Hence, in particular, E^φ is a set of finite perimeter in Ω. Therefore, we consider $\varphi \in BV(\Omega, \{\pm 1\}) \cap \mathcal{U}$ in the following.

(A6) For the potential ψ appearing in the Ginzburg–Landau energy, we choose the double-obstacle potential whose regular part is given as $\psi_0(\varphi) = \frac{1}{2}(1 - \varphi^2)$, see also Section 2.1.4.

Remark 3.3.1.

(a) The case of dimension $d = 1$ needs to be excluded as here some of the fundamental theorems about quasi-open sets and capacity theory are not true, see, e.g., [49, Chap. 4].

(b) The growth condition $\overline{b}_\varepsilon = o(\varepsilon^{-\kappa})$ is chosen in order to obtain the desired convergence of the term involving b_ε in Lemma 3.3.5 and Theorem 3.3.7 via a Hölder estimate in dependence of the dimension $d \geq 2$, see [96, Proof of Lemma 3, 2nd step] for the case $d = 3$. As explained in [96, Rem. 2], for $d = 2$, this growth condition can be weakened to $\overline{b}_\varepsilon = o(\varepsilon^{-\kappa})$ for any $\kappa \in (0,1)$. We will go through all the details in *Step 2* of the proof of Lemma 3.3.5.

(c) According to [107, Lemma 3.2], we find a crack free representative E_c^φ of E^φ that is a set of finite perimeter with $\mathrm{int}(E_c^\varphi) = \mathrm{int}(\overline{E_c^\varphi})$ (where int denotes the interior) and

$$|E^\varphi \triangle E_c^\varphi| = 0, \quad \text{where} \quad E^\varphi \triangle E_c^\varphi = (E^\varphi \cup E_c^\varphi)\backslash(E^\varphi \cap E_c^\varphi).$$

We further point out that assumption **(A4)** guarantees that $\mathrm{int}(E_c^\varphi) \neq \emptyset$ and we can thus apply [83, Theorem 6.3] from which we infer

$$V^\varphi \subset \left\{ v \in H_0^1(\Omega) \,|\, v = 0 \text{ a.e. in } \Omega\backslash E_c^\varphi \right\} = H_0^1(\mathrm{int}(E_c^\varphi)).$$

This means any function in the abstract space V^φ can be seen as an element of the restricted Sobolev space $H_0^1(\mathrm{int}(E_c^\varphi))$. This will help us to understand the limit problem in the remainder of this section.

(d) Instead of employing the potential as declared in **(A6)**, it would also be possible to use different choices. For instance, the quartic regular part $\psi_0(\varphi) = \frac{1}{4}(1-\varphi^2)^2$ could also be chosen. This choice would only affect the choice of profiles used to construct a recovery sequence in *Step 1* of the proof of Theorem 3.3.11 but our theory would still remain valid.

Under the above assumptions, we can prove that eigenfunctions of (3.2.1) for $\varepsilon > 0$ converge to eigenfunctions of a limit problem as $\varepsilon \searrow 0$, and these limit functions have suitable properties. To this end, we first want to develop a better understanding of the limit problem.

3.3.1. The limit problem and its properties

In the following, we discuss the limit eigenvalue problem and its most important properties. It is well known that due to well-posedness of the minimization problem on the sharp-interface level, we need to consider the Dirichlet eigenvalue problem in its relaxed form using Borel measures as introduced, e.g., in [19, 49, 63, 78, 110]. This theory we have discussed in Section 2.2.6, so we can directly start with the analysis of the limit problem.

First of all, we now analyze the limit eigenvalue problem. After that, we will establish its connection to the diffuse eigenvalue problem (3.2.1).

Theorem 3.3.2. *In addition to the assumptions made in Section 2.1, we suppose that the assumptions* **(A1)**–**(A6)** *are fulfilled. For any given $\varphi \in BV(\Omega, \{\pm 1\}) \cap \mathcal{U}$, we consider the following eigenvalue problem: Find $(w, \lambda) \in (V^\varphi\backslash\{0\}) \times \mathbb{R}$ such that*

$$\int_\Omega \nabla w \cdot \nabla \eta \,\mathrm{d}x = \lambda \int_\Omega w\eta \,\mathrm{d}x \quad \forall \eta \in V^\varphi. \tag{3.3.2}$$

Then the following holds true:

(a) *There exists a sequence*

$$\left(w_k^{0,\varphi}, \lambda_k^{0,\varphi}\right)_{k\in\mathbb{N}} \subset (V^\varphi\backslash\{0\}) \times \mathbb{R},$$

having the following properties:

- *For all indices $k \in \mathbb{N}$, $w_k^{0,\varphi}$ is an $L^2(\Omega)$-normalized eigenfunction of (3.3.2) to the eigenvalue $\lambda_k^{0,\varphi}$ and these eigenfunctions are pairwise orthogonal with respect to the canonical scalar product on $L^2(\Omega)$ denoted by $(\cdot,\cdot)$.*
- *The eigenvalues $\lambda_k^{0,\varphi}$ (which are repeated according to their multiplicity) can be ordered in the following way:*

$$0 < \lambda_1^{0,\varphi} \leq \lambda_2^{0,\varphi} \leq \lambda_3^{0,\varphi} \leq \cdots .$$

Moreover, it holds that $\lambda_k^{0,\varphi} \to \infty$ as $k \to \infty$, and there exist no further eigenvalues of the limit problem (3.3.2).

- *The normalized eigenfunctions*

$$\left\{ \frac{w_1^{0,\varphi}}{\sqrt{\lambda_1^{0,\varphi}}}, \frac{w_2^{0,\varphi}}{\sqrt{\lambda_2^{0,\varphi}}}, \dots \right\} \subset V^\varphi,$$

form an orthonormal basis of the space V^φ and any $u \in V^\varphi$ can be expressed as

$$u = \sum_{i=1}^{\infty} (u, w_i^{0,\varphi}) w_i^{0,\varphi},$$

where the series converges in V^φ.

(b) *For any $k \in \mathbb{N}$, we have the Courant–Fischer characterization*

$$\lambda_k^{0,\varphi} = \max_{U \in \mathcal{S}_{k-1}} \min \left\{ \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega |v|^2 \,\mathrm{d}x} \,\middle|\, \begin{array}{l} v \in U^\perp \cap V^\varphi, \\ v \neq 0 \end{array} \right\}. \tag{3.3.3}$$

Here, $\mathcal{S}_{k-1}$ denotes the collection of all $(k-1)$-dimensional subspaces of $L^2(\Omega)$. The set $U^\perp$ denotes the orthogonal complement of $U \subset L^2(\Omega)$ with respect to the canonical scalar product.

Moreover, the maximum is attained at the subspace

$$U = \langle w_1^{0,\varphi}, \dots, w_{k-1}^{0,\varphi} \rangle_{\mathrm{span}}.$$

Remark 3.3.3. Let us relate this definition to the one of [51, Section 6] in the framework of the relaxed Dirichlet problem, also introduced in Section 2.2.6.
For $\omega \subset \Omega$ quasi-open the eigenvalue problem

$$\int_\Omega \nabla w \cdot \nabla \eta \,\mathrm{d}x + \int_\Omega w\eta \,\mathrm{d}\mu_\omega = \lambda \int_\Omega w\eta \,\mathrm{d}x \quad \text{for all } \eta \in X_{\mu_\omega}(\Omega),$$

is considered in [51, Section 6], where $\mu_\omega = \infty_{\Omega\setminus\omega}$ as in Section 2.2.6. Due to the discussion there, see also [51, Remark 6.1.4], this problem is equivalent to

$$\int_\Omega \nabla w \cdot \nabla \eta \,\mathrm{d}x = \lambda(\omega) \int_\Omega w\eta \,\mathrm{d}x \quad \text{for all } \eta \in H_0^1(\omega). \tag{3.3.4}$$

Thus, for $E^\varphi = \{\varphi = 1\}$ in above theorem, recalling Proposition 2.2.31, there is a unique quasi-open set $\omega \subset E$ such that $V^\varphi = \tilde{H}_0^1(E^\varphi) = H_0^1(\omega)$. Thus, we obtain that $\lambda_k^{0,\varphi}$ is identical to the k-th eigenvalue $\lambda_k(\omega)$ of (3.3.4). We will stick to this notation in the

following. Note that for convenience we also define $\lambda_k^0(E) \coloneqq \lambda_k^{0,\chi_E}$ for any finite perimeter set E.
In other words $\lambda_k(\omega)$ plays the role of a "classical" eigenvalue as it is formulated over the classical Sobolev space $H_0^1(\omega)$, whereas the limit eigenvalue $\lambda_k^0(E)$ is formulated over the Sobolev-like space $\tilde{H}_0^1(E)$.

Now a key argument in the Γ-convergence proof in Theorem 3.3.11, as performed also in [51, Theorem 3.6] and [39, Theorem 3.1], will be to exploit this relation and the continuity of eigenvalues with respect to γ-convergence established in [49, Corollary 6.1.8, Remark 6.1.10], which reads as follows:
Let $k \in \mathbb{N}$. If $(\omega_n)_{n\in\mathbb{N}}, \tilde{\omega} \subset \Omega$ are quasi-open sets

$$\omega_n \xrightarrow{\gamma} \tilde{\omega} \quad \Rightarrow \quad \lambda_k(\omega_n) \to \lambda_k(\tilde{\omega}), \tag{3.3.5}$$

for $n \to \infty$. A further crucial property that we will use is the monotonicity of these eigenvalues with respect to set inclusion, i.e., for quasi-open set $\omega_2 \subset \omega_1 \subset \Omega$ it holds

$$\lambda_k(\omega_1) \leq \lambda_k(\omega_2). \tag{3.3.6}$$

This is simply due to the fact that $H_0^1(\omega_2) \subset H_0^1(\omega_1)$.

Proof of Theorem 3.3.2. *Proof of* (a). Equipping $V^\varphi \subset H_0^1(\Omega)$ with the canonical $H_0^1(\Omega)$ scalar product

$$(\cdot,\cdot)_{V^\varphi} : V^\varphi \times V^\varphi \to \mathbb{R}, \quad (v,w)_{V^\varphi} := \int_\Omega \nabla v \cdot \nabla w \,\mathrm{d}x,$$

it is a closed subspace of $H_0^1(\Omega)$ and hence, it is a Hilbert space. Using standard arguments, we conclude the existence of a self-adjoint and compact solution operator

$$\mathcal{T} : H^\varphi \to V^\varphi \hookrightarrow H^\varphi, \quad \mathcal{T}(f) := v_f,$$

where v_f denotes the unique solution of

$$\int_\Omega \nabla v_f \cdot \nabla \eta \,\mathrm{d}x = \int_\Omega f\eta \,\mathrm{d}x \quad \forall \eta \in V^\varphi. \tag{3.3.7}$$

We point out that the operator $\mathcal{T}$ is not necessarily injective, because we can in general not take $\eta = f$ as test function, as f possesses only L^2 regularity. Additionally in the spirit of the fundamental lemma in the calculus of variations we can only test with functions $C_0^\infty(\Omega\backslash\overline{E^\varphi})$ because the set $\Omega\backslash E^\varphi$ is in general not open. Recall that the finite perimeter set can be *wild* in the sense of Remark 2.2.21 and thus, in particular $\overline{E^\varphi} = \overline{\Omega}$ is possible. However, by assumption **(A4)**, we have $C_0^\infty(B) \subset V^\varphi \subset H^\varphi$ and the operator $\mathcal{T}$ restricted to $C_0^\infty(B)$ is obviously injective due to the fundamental lemma in the calculus of variations. We thus conclude that the image $\mathcal{T}(H^\varphi)$ of the non-restricted operator is an infinite dimensional space.

Thus, the spectral theorem for compact self-adjoint linear operators, see Theorem 2.2.8, yields an infinite sequence $(\hat{\lambda}_k)_{k\in\mathbb{N}}$ comprising all non-trivial eigenvalues of $\mathcal{T}$ that converges to 0. Furthermore it provides us with the decomposition

$$H^\varphi = N(\mathcal{T}) \perp \langle w_1^{0,\varphi}, w_2^{0,\varphi}, \dots \rangle_{\mathrm{span}}, \tag{3.3.8}$$

where $N(\mathcal{T})$ denotes the kernel of $\mathcal{T}$ and $(w_k)_{k\in\mathbb{N}} \subset V^\varphi$ are the eigenfunctions corresponding to $(\hat{\lambda}_k)_{k\in\mathbb{N}}$. The sequence $(\lambda_k)_{k\in\mathbb{N}}$ is now obtained by defining $\lambda_k = \hat{\lambda}_k^{-1}$ and a suitable reordering due to the construction of $\mathcal{T}$.
Now we have to be very precise with the question if $(\lambda_k)_{k\in\mathbb{N}}$ indeed comprises *all* eigenvalues of problem (3.3.2) in view of the in general non-trivial kernel $N(\mathcal{T})$. As this sequence comprises of course all *non-trivial* eigenvalues, the question is answered if we show that $\lambda = 0$ is not an eigenvalue in (3.3.2). Thus, assume that there is a non-trivial $w \in V^\varphi \subset H_0^1(\Omega)$ such that

$$\int_\Omega \nabla w \cdot \nabla \eta \,\mathrm{d}x = 0 \int_\Omega w\eta \,\mathrm{d}x \quad \text{for all } \eta \in V^\varphi.$$

By Poincaré's inequality this directly implies $w \equiv 0$, and thus all eigenvalues of (3.3.2) are positive.

The remaining assertions are obtained in complete analogy to Theorem 3.2.2, see also Theorem 5.2.2, but in order to be precise let us comment on the fact that

$$\left\{ \frac{w_1^{0,\varphi}}{\sqrt{\lambda_1^{0,\varphi}}}, \frac{w_2^{0,\varphi}}{\sqrt{\lambda_2^{0,\varphi}}}, \dots \right\} \subset V^\varphi,$$

forms an orthonormal *basis* of V^φ even though

$$\left\{ w_1^{0,\varphi}, w_2^{0,\varphi}, \dots \right\} \subset H^\varphi,$$

is in general not an orthonormal basis of H^φ in the light of the non-trivial kernel $N(\mathcal{T})$. Therefore let $u \in V^\varphi$ be arbitrary such that

$$0 = \int_\Omega \nabla u \cdot \nabla w_k^{0,\varphi} \,\mathrm{d}x = \lambda_k^{0,\varphi} \int_\Omega u w_k^{0,\varphi} \,\mathrm{d}x,$$

for all $k \in \mathbb{N}$. Due to the decomposition (3.3.8) and the crucial fact that $\lambda_k^{0,\varphi} > 0$ this implies that $u \in N(\mathcal{T})$. In the light of (3.3.7) this means

$$0 = \int_\Omega \nabla \mathcal{T}(u) \cdot \nabla \eta \,\mathrm{d}x = \int_\Omega u\eta \,\mathrm{d}x \quad \forall \eta \in V^\varphi.$$

Now the crucial fact that $u \in V^\varphi$ and not only in H^φ allows us to test with $\eta = u$ and infer $u \equiv 0$. In other words this shows that $V^\varphi \cap N(\mathcal{T}) = \{0\}$.

□

In general, due to possible cracks within the set E^φ, we cannot guarantee that an eigenfunction w of the limit problem vanishes on the whole of ∂E^φ. Nevertheless, we know from Remark 3.3.1.(c) that $w = 0$ on $\partial(\mathrm{int}(E_c^\varphi))$ in the trace sense (provided that the boundary is sufficiently smooth), meaning that w has trace zero at least on the *outer* boundary of E^φ. For more details we refer to [107, Sec. 3]. If E^φ is actually an open set with Lipschitz boundary, then indeed $w \in H_0^1(E^\varphi) = \tilde{H}_0^1(E^\varphi)$ where $H_0^1(E^\varphi)$ can be understood in the standard sense, meaning that the trace of w vanishes on ∂E^φ. We thus interpret (3.3.2) as the weak formulation of the classical eigenvalue problem

$$\begin{cases} -\Delta w = \lambda w & \text{in } E^\varphi, \\ w = 0 & \text{on } \partial E^\varphi, \end{cases}$$

where the boundary condition is included in the space V^φ of test functions in a relaxed way.

Remark 3.3.4. Recall from Theorem 3.2.2 that for given $\varepsilon > 0$ and $\varphi \in L^\infty(\Omega)$, we can fix the sequence

$$(w_k^{\varepsilon,\varphi}, \lambda_k^{\varepsilon,\varphi})_{k\in\mathbb{N}} \subset H_0^1(\Omega) \times \mathbb{R}^+$$

of eigenfunctions and eigenvalues to (3.2.1), where $\{w_1^{\varepsilon,\varphi}, w_2^{\varepsilon,\varphi}, \dots, \} \subset L^2(\Omega)$ forms an orthonormal basis. This notation will be used throughout this chapter, and we will drop the additional index ε if the context is clear.

The following lemma will now link the diffuse interface problem (3.2.1) to the sharp-interface problem (3.3.2) for the first eigenvalue and it will serve as initial case for all higher eigenvalues. This rigorous continuity analysis is new to the best of the authors' knowledge. Recall from the introduction that there are further results in optimal partitioning for the principal eigenvalue using a phase-field approach which show the convergence of minimizing eigenvalues of the optimization problem, see [40, 42].

Lemma 3.3.5. *In addition to the assumptions made in Section 2.1, we suppose that the assumptions* **(A1)**–**(A6)** *are fulfilled. Let* $(\varphi_\varepsilon)_{\varepsilon>0} \subset L^1(\Omega)$ *with* $|\varphi_\varepsilon| \leq 1$ *almost everywhere in* Ω *and let* $\varphi \in BV(\Omega, \{\pm 1\}) \cap \mathcal{U}$ *such that*

$$\lim_{\varepsilon \searrow 0} \|\varphi_\varepsilon - \varphi\|_{L^1(\Omega)} = 0, \tag{3.3.9}$$

and the convergence exhibits the rate

$$\|\varphi_\varepsilon - \varphi\|_{L^1(E^\varphi \cap \{\varphi_\varepsilon < 0\})} = \mathcal{O}(\varepsilon).$$

Then there exists an eigenfunction $u \in V^\varphi$ *of the limit problem* (3.3.2) *to the eigenvalue* $\lambda_1^{0,\varphi}$ *such that*

$$\lim_{\varepsilon \searrow 0} \int_\Omega b_\varepsilon(\varphi_\varepsilon)\, |w_1^{\varphi_\varepsilon}|^2 \, \mathrm{d}x = \int_\Omega b_0(\varphi)\, |u|^2 \, \mathrm{d}x = 0,$$

as well as

$$\lim_{\varepsilon \searrow 0} \|w_1^{\varphi_\varepsilon} - u\|_{H^1(\Omega)} = 0 \quad \textit{and} \quad \lim_{\varepsilon \searrow 0} \lambda_1^{\varepsilon,\varphi_\varepsilon} = \lambda_1^{0,\varphi},$$

up to subsequence extraction.

Remark 3.3.6.

(a) We point out that we will always use the letter u (or $u_1, u_2, \dots$) to denote the limit of eigenfunctions. This is done in order to avoid confusion with the orthogonal system

$$\{w_1^{0,\varphi}, w_2^{0,\varphi}, \dots\} \subset V^\varphi$$

of eigenfunctions to the limit problem we obtained in Theorem 3.3.2. From the above lemma, we merely know that u belongs to the first eigenspace which is spanned by the first eigenfunctions $w_i^{0,\varphi}$ in accordance with the multiplicity of the space. However, we cannot relate u and $w_i^{0,\varphi}$ any further.

(b) Note that, in contrast to [96], it would suffice to demand the above convergence rate condition only on the set $\tilde{E}^\varphi \cap \{\varphi_\varepsilon < 0\}$ with $\tilde{E}^\varphi := \{x \in \tilde{\Omega} \,|\, \varphi(x) = 1\} \subset \tilde{\Omega}$ instead of $E^\varphi = \{x \in \Omega \,|\, \varphi(x) = 1\} \subset \Omega$. As we have $\varphi_\varepsilon, \varphi \in \mathcal{U}$, the difference $\varphi_\varepsilon - \varphi$ vanishes on $S_0 \cup S_1$ anyway.

Proof of Lemma 3.3.5. Some ideas of the proof are the same as in [96, Lemma 3], especially *Step 1* and *Step 2* carry over in large parts. However, for the sake of readability we will include all the details here. We will also divide the proof into several steps.
In the following, due to (3.3.9), we may consider a non-relabeled subsequence of $(\varphi_\varepsilon)_{\varepsilon>0}$ such that $\varphi_\varepsilon \to \varphi$ a.e. in Ω.

Step 1: For almost every $x \in \Omega$, it holds that

$$\lim_{\varepsilon \searrow 0} b_\varepsilon(\varphi_\varepsilon(x)) = b_0(\varphi(x)) \quad \textit{in } [0, +\infty]$$

We will show this by proving

$$\liminf_{\varepsilon \searrow 0} b_\varepsilon(\varphi_\varepsilon) = \limsup_{\varepsilon \searrow 0} b_\varepsilon(\varphi_\varepsilon) = b_0(\varphi_0) \quad \text{in } [0, +\infty]. \tag{3.3.10}$$

For the lim sup inequality

$$\limsup_{\varepsilon \searrow 0} b_\varepsilon(\varphi_\varepsilon(x)) \leq b_0(\varphi(x)), \tag{3.3.11}$$

in the proof of Step 1 of [96, Lemma 1], the additionally assumed continuity of b_0 in the point 1 given in **(A3)** is needed. More precisely, as $\varphi \in BV(\Omega, \{\pm 1\})$ the inequality is clear if $\varphi(x) = -1$ because then $b_0(\varphi(x)) = +\infty$. If $\varphi(x) = 1$ then due to the point-wise convergence $\varphi_\varepsilon(x) \to \varphi(x)$, the continuity of b_0 in 1 and the fact $b_\varepsilon \leq b_0$ point-wise, we obtain

$$\limsup_{\varepsilon \searrow 0} b_\varepsilon(\varphi_\varepsilon(x)) \leq b_0(\varphi(x)).$$

To prove the lim inf inequality

$$b_0(\varphi(x)) \leq \liminf_{\varepsilon \searrow 0} b_\varepsilon(\varphi_\varepsilon(x)) \tag{3.3.12}$$

one can proceed exactly as in [96]. So for fixed $\varepsilon > 0$ let us choose a $\delta > 0$ with $\varepsilon < \delta$. Now exploiting the monotonicity assumption $b_\delta \leq b_\varepsilon$ from **(A3)** we obtain that for almost every $x \in \Omega$

$$\lim_{\varepsilon \searrow 0} b_\delta(\varphi_\varepsilon(x)) \leq \liminf_{\varepsilon \searrow 0} b_\varepsilon(\varphi_\varepsilon(x))$$

and thus, due to the point-wise convergence $b_\delta \to b_0$ on $[-1, 1]$ from assumption **(A3)** and the continuity of b_δ we have

$$b_0(\varphi(x)) = \lim_{\delta \searrow 0} b_\delta(\varphi(x)) = \lim_{\delta \searrow 0} (\lim_{\varepsilon \searrow 0} b_\delta(\varphi_\varepsilon(x))) \leq \lim_{\delta \searrow 0} (\liminf_{\varepsilon \searrow 0} b_\varepsilon(\varphi_\varepsilon(x))) = \liminf_{\varepsilon \searrow 0} b_\varepsilon(\varphi_\varepsilon(x)).$$

Hence, we arrive at (3.3.12). Thus combining (3.3.11) and (3.3.12) we proved at (3.3.10). We point out that for this step, the convergence rate imposed on $(\varphi_\varepsilon)_{\varepsilon>0}$ was not needed.

Step 2: For any $v \in H_0^1(\Omega)$ with

$$v = 0 \quad \text{a.e. in } \Omega \backslash E^\varphi \tag{3.3.13}$$

i.e., $v \in V^\varphi$, it holds that

$$\lim_{\varepsilon \searrow 0} \int_\Omega b_\varepsilon(\varphi_\varepsilon) \, |v|^2 \, \mathrm{d}x = \int_\Omega b_0(\varphi) \, |v|^2 \, \mathrm{d}x = 0.$$

This step can be established as in [96] since by assumption **(A3)**, the coefficient function b_ε possesses all the properties of [96]. Nevertheless, as this is a key part of the proof we provide it here. Note also that this step heavily relies on the convergence rate imposed on $(\varphi_\varepsilon)_{\varepsilon>0}$.

So let us fix a $v \in H_0^1(\Omega)$ satisfying (3.3.13), then we know from *Step 1*

$$\lim_{\varepsilon \searrow 0} b_\varepsilon(\varphi_\varepsilon(x)) \, |v(x)|^2 = b_0(\varphi(x)) \, |v(x)|^2 = 0, \tag{3.3.14}$$

for almost every $x \in \Omega$, as $b_0(\varphi) = 0$ a.e on E^φ by assumption **(A3)**.

Now the strategy is to split the integral over Ω into two integrals over $\Omega_\varepsilon^+ := \{\varphi_\varepsilon \geq 0\}$ and $\Omega_\varepsilon^- := \{\varphi_\varepsilon < 0\}$ respectively, recall also the formal motivation in Section 2.1.6.
On Ω_ε^+ the goal is to apply Lebesgue's theorem. Noticing that for fixed $\varepsilon > 0$, b_ε is decreasing on $[-1, 1]$ and $b_\varepsilon \leq b_0$ point-wise on $[-1, 1]$ we obtain

$$b_\varepsilon(\varphi_\varepsilon(x)) \, |v(x)|^2 \leq b_\varepsilon(0) \, |v(x)|^2 \leq b_0(0) \, |v(x)|^2 \, ,$$

for almost every $x \in \Omega_\varepsilon^+$. As $b_0(0) < +\infty$ by assumption **(A3)**, we deduce

$$\chi_{\Omega_\varepsilon^+} b_\varepsilon(\varphi_\varepsilon) \, |v|^2 \leq C \, |v|^2 \, .$$

As $v \in L^2(\Omega)$, Lebesgue's theorem and using the point-wise convergence (3.3.14) yield

$$\lim_{\varepsilon \searrow 0} \int_{\Omega_\varepsilon^+} b_\varepsilon(\varphi_\varepsilon) \, |v|^2 \, \mathrm{d}x = \lim_{\varepsilon \searrow 0} \int_\Omega \chi_{\Omega_\varepsilon^+} b_\varepsilon(\varphi_\varepsilon) \, |v|^2 \, \mathrm{d}x = 0. \tag{3.3.15}$$

Now we need to deal with the set Ω_ε^-, which is more involved as here b_0 becomes unbounded.
By definition of E^φ we obtain

$$\{\, x \in \Omega |\, v(x) \neq 0\} \subset \{\, x \in \Omega |\, \varphi(x) = 1\} \quad \text{a.e.},$$

as we have chosen v with the property (3.3.13). Hence, for almost every $x \in \Omega_\varepsilon^-$ we infer

$$b_\varepsilon(\varphi_\varepsilon(x)) \, |v(x)|^2 \leq \bar{b}_\varepsilon \, |\varphi_\varepsilon(x) - \varphi(x)| \, |v(x)|^2 \, \chi_{E^\varphi}(x), \tag{3.3.16}$$

as per assumption **(A3)** $b_\varepsilon : [-1, 1] \to [0, \bar{b}_\varepsilon]$ and noting that $|\varphi_\varepsilon(x) - \varphi(x)| \geq 1$ on $\Omega_\varepsilon^- \cap E^\varphi$. Recalling that $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain we have the embedding

$$H^1(\Omega) \hookrightarrow \begin{cases} L^q(\Omega) & \text{for any } q \in [1, \infty) \text{ if } d = 2, \\ L^{\frac{2d}{d-2}}(\Omega) & \text{if } d > 2, \end{cases} \tag{3.3.17}$$

see e.g. [10, 10.9].
As indicated in Remark 3.3.1, if $d > 2$, we now apply Hölder's inequality with $p := \frac{d}{2}, p' := \frac{d}{d-2}$ in the following way

$$\int_\Omega \chi_{\Omega_\varepsilon^- \cap E^\varphi} \left|\varphi_\varepsilon - \varphi\right| |v|^2 \, \mathrm{d}x \leq \left(\int_\Omega \left| \chi_{\Omega_\varepsilon^- \cap E^\varphi} \left(\varphi_\varepsilon - \varphi\right) \right|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_\Omega \left(|v|^2 \right)^{p'} \mathrm{d}x \right)^{\frac{1}{p'}}. \tag{3.3.18}$$

As we know $|\varphi_\varepsilon| \leq 1$ and $|\varphi_0| \leq 1$, we find a constant $C_p > 0$ such that

$$|\varphi_\varepsilon - \varphi|^p \leq C_p \left|\varphi_\varepsilon - \varphi\right|,$$

almost everywhere in Ω. Thus, we deduce

$$\int_\Omega \chi_{\Omega_\varepsilon^- \cap E^\varphi} \left|\varphi_\varepsilon - \varphi\right| |v|^2 \, \mathrm{d}x \leq C \left\|\varphi_\varepsilon - \varphi\right\|_{L^1(\Omega_\varepsilon^- \cap E^\varphi)}^{\frac{2}{d}} \left\|v\right\|_{L^{\frac{2d}{d-2}}(\Omega)}^2.$$

Now we can use the rate condition from the assumption, i.e.,

$$\left\|\varphi_\varepsilon - \varphi\right\|_{L^1(E^\varphi \cap \{\varphi_\varepsilon < 0\})} = \mathcal{O}(\varepsilon),$$

and combine it with the rate of $\overline{b}_\varepsilon$ from assumption **(A3)**, namely

$$\overline{b}_\varepsilon = o\big(\varepsilon^{-\kappa}\big),$$

with $\kappa = \frac{2}{d}$ (as we are in the case $d > 2$ for the moment) to infer

$$\lim_{\varepsilon \searrow 0} \left(\overline{b}_\varepsilon \int_\Omega \chi_{\Omega_\varepsilon^- \cap E^\varphi} \left|\varphi_\varepsilon - \varphi\right| |v|^2 \, \mathrm{d}x \right) = 0. \tag{3.3.19}$$

Due to the embedding (3.3.17) for the case $d = 2$ we can use any $p' \in [1, \infty)$ in the estimate (3.3.18) and thus see that in this case the assumption $\overline{b}_\varepsilon = o\big(\varepsilon^{-\kappa}\big)$ with arbitrary $\kappa \in (0, 1)$ in **(A3)** is sufficient to deduce the convergence (3.3.19).

Using the estimate (3.3.16) this yields

$$\lim_{\varepsilon \searrow 0} \int_{\Omega_\varepsilon^-} b_\varepsilon(\varphi_\varepsilon(x)) \, |v|^2 \, \mathrm{d}x = 0,$$

and together with (3.3.15) we obtain the desired convergence, as by construction

$$\int_\Omega b_0(\varphi) \, |v|^2 \, \mathrm{d}x = 0,$$

because $b_0(\varphi)$ vanishes a.e. on E^φ.

In the remainder of this proof, we establish the convergence properties for the eigenvalues and eigenfunctions using the Courant–Fisher characterization. In the following, we write w_ε and λ^ε instead of $w_1^{\varphi_\varepsilon}$ and $\lambda_1^{\varepsilon, \varphi_\varepsilon}$, respectively, for convenience.

Step 3: We find a subsequence of $(w_\varepsilon)_{\varepsilon > 0}$ and an $L^2(\Omega)$-normalized function $u \in V^\varphi$ such that

$$w_\varepsilon \rightharpoonup u \quad \text{in } H^1(\Omega) \quad \text{and} \quad w_\varepsilon \to u \quad \text{in } L^2(\Omega). \tag{3.3.20}$$

For any eigenfunction solving (3.2.1) to the smallest eigenvalue λ^ε we recall the Courant–Fischer characterization (3.2.3) which simplifies to

$$\begin{aligned}\lambda^\varepsilon &= \min\left\{ \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_\Omega b_\varepsilon(\varphi_\varepsilon)\,|v|^2 \,\mathrm{d}x}{\int_\Omega |v|^2 \,\mathrm{d}x} \middle| \begin{array}{l} v \in H^1_0(\Omega), \\ v \neq 0 \end{array} \right\} \\ &= \min\left\{ \int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_\Omega b_\varepsilon(\varphi_\varepsilon)\,|v|^2 \,\mathrm{d}x \middle| \begin{array}{l} v \in H^1_0(\Omega), \\ \|v\|_{L^2(\Omega)} = 1 \end{array} \right\},\end{aligned}$$

as we have fixed the coefficient functions a_ε and c_ε in assumption **(A2)**. We define

$$F_\varepsilon : Q \to \mathbb{R}^+_0, \quad v \mapsto \int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_\Omega b_\varepsilon(\varphi_\varepsilon)\,|v|^2 \,\mathrm{d}x,$$

with $Q \coloneqq \{v \in H^1_0(\Omega) \mid \|v\|_{L^2(\Omega)} = 1\}$. In this way, $w_\varepsilon \in Q$ fulfills

$$F_\varepsilon(w_\varepsilon) = \min_{v \in Q} F_\varepsilon(v). \tag{3.3.21}$$

To describe the limit situation, we similarly define

$$F_0 : Q \to [0, +\infty], \quad v \mapsto \int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_\Omega b_0(\varphi)\,|v|^2 \,\mathrm{d}x.$$

By assumption **(A4)**, V^φ is non-trivial and hence, there exists a function $\overline{v} \in Q$ satisfying property (3.3.13). Then, (3.3.21) obviously entails that

$$F_\varepsilon(w_\varepsilon) \leq F_\varepsilon(\overline{v}),$$

and from *Step 2*, we already know that there is a constant $C > 0$ such that for all $\varepsilon > 0$,

$$\int_\Omega b_\varepsilon(\varphi_\varepsilon)\,|\overline{v}|^2 \,\mathrm{d}x \leq C.$$

We thus infer that

$$\|\nabla w_\varepsilon\|^2_{L^2(\Omega)} \leq F_\varepsilon(w_\varepsilon) \leq F_\varepsilon(\overline{v}) \leq C,$$

and combining these two bounds, the Banach–Alaoglu theorem implies the desired convergences (3.3.20) up to subsequence extraction.

Step 4: The function $u \in Q$ is a minimizer of F_0 and we have

$$\lim_{\varepsilon \searrow 0} F_\varepsilon(w_\varepsilon) = F_0(u), \tag{3.3.22}$$

along a non-relabeled subsequence.

If we can show that F_ε Γ-converges to F_0 on Q with respect to the weak topology on $H^1(\Omega)$, we can apply Proposition 2.2.24 which give exactly the claimed properties. To prove Γ-convergence we have to verify the corresponding lim sup and lim inf inequalities.

To verify the lim sup inequality, for any $v \in Q$ we need to find a so called recovery sequence $(v_\varepsilon)_{\varepsilon>0} \subset Q$ that converges to $v \in Q$ weakly in $H^1(\Omega)$ and satisfies

$$\limsup_{\varepsilon \searrow 0} F_\varepsilon(v_\varepsilon) \leq F_0(v). \tag{3.3.23}$$

Here, we can simply choose the constant sequence $v_\varepsilon := v \in Q$. Without loss of generality, we can assume that $F_0(v) < +\infty$, as otherwise (3.3.23) is trivially fulfilled. This assumption implies that

$$\int_\Omega b_0(\varphi)\,|v|^2\,\mathrm{d}x < \infty,$$

and since $b_0(-1) = +\infty$ and $\varphi \in BV(\Omega, \{\pm 1\})$, we conclude that $v \in V^\varphi$. Hence, we infer from *Step 2* that

$$\int_\Omega b_0(\varphi)\,|v|^2\,\mathrm{d}x = \lim_{\varepsilon \searrow 0} \int_\Omega b_\varepsilon(\varphi_\varepsilon)\,|v|^2\,\mathrm{d}x = 0. \tag{3.3.24}$$

By construction of F_ε and F_0, this already implies (3.3.23).

For the lim inf inequality we need to show that for any sequence $(v_\varepsilon)_{\varepsilon>0} \subset Q$ converging to a $v \in Q$ weakly in $H^1(\Omega)$ topology, it holds

$$F_0(v) \le \liminf_{\varepsilon \searrow 0} F_\varepsilon(v_\varepsilon). \tag{3.3.25}$$

By the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ we know that $v_\varepsilon \to v$ almost everywhere in Ω up to subsequence extraction. Furthermore, we have already seen in *Step 1* that for almost every $x \in \Omega$,

$$\lim_{\varepsilon \searrow 0} b_\varepsilon(\varphi_\varepsilon(x)) = b_0(\varphi(x)).$$

Therefore, we deduce

$$\begin{aligned}
\int_\Omega b_0(\varphi_0(x))\,|v(x)|^2\,\mathrm{d}x &= \int_\Omega \lim_{\varepsilon \searrow 0} b_\varepsilon(\varphi_\varepsilon(x)) \lim_{\varepsilon \searrow 0} |v_\varepsilon(x)|^2\,\mathrm{d}x \\
&= \int_\Omega \liminf_{\varepsilon \searrow 0} \left[b_\varepsilon(\varphi_\varepsilon(x))\,|v_\varepsilon(x)|^2 \right] \mathrm{d}x \\
&\le \liminf_{\varepsilon \searrow 0} \int_\Omega b_\varepsilon(\varphi_\varepsilon(x))\,|v_\varepsilon(x)|^2\,\mathrm{d}x,
\end{aligned}$$

by means of Fatou's lemma, see Lemma 2.2.7. Noting that the $H^1_0(\Omega)$ norm $\|\nabla \cdot\|_{L^2(\Omega)}$ is weakly lower semi- continuous this yields the lim inf inequality (3.3.25).

In summary, this means that

$$F_\varepsilon \xrightarrow{\Gamma} F_0,$$

with respect to the weak $H^1(\Omega)$-topology. In the light of *Step 3* we can apply Proposition 2.2.24 to deduce the claim of *Step 4*.

We now want to complete the proof by applying the results established in the previous steps. Proceeding as in *Step 4*, and using the convergence properties in (3.3.20), we deduce

$$\begin{aligned}
\int_\Omega b_0(\varphi)\,|u|^2\,\mathrm{d}x &\le \liminf_{\varepsilon \searrow 0} \int_\Omega b_\varepsilon(\varphi_\varepsilon)\,|w_\varepsilon|^2\,\mathrm{d}x, \\
\int_\Omega |\nabla u|^2\,\mathrm{d}x &\le \liminf_{\varepsilon \searrow 0} \int_\Omega |\nabla w_\varepsilon|^2\,\mathrm{d}x.
\end{aligned}$$

As both sequences are bounded from below by zero, we can use (3.3.22) along with [96, Lemma 4] to infer

$$\int_\Omega b_0(\varphi)\,|u|^2\,\mathrm{d}x = \lim_{\varepsilon\searrow 0}\int_\Omega b_\varepsilon(\varphi_\varepsilon)\,|w_\varepsilon|^2\,\mathrm{d}x,$$
$$\int_\Omega |\nabla u|^2\,\mathrm{d}x = \lim_{\varepsilon\searrow 0}\int_\Omega |\nabla w_\varepsilon|^2\,\mathrm{d}x.$$

From the second convergence and the weak convergence (3.3.20) from *Step 3*, we conclude that

$$\lim_{\varepsilon\searrow 0}\|w_\varepsilon - u\|_{H^1(\Omega)} = 0.$$

Furthermore, we have seen in the previous step that $u \in Q$ minimizes F_0. Hence,

$$\int_\Omega b_0(\varphi)\,|u|^2\,\mathrm{d}x < +\infty,$$

and arguing as in (3.3.24), we find that

$$\int_\Omega b_0(\varphi)\,|u|^2\,\mathrm{d}x = 0. \tag{3.3.26}$$

To complete the proof, we still have to prove the following assertion:

Step 5: The function $u \in V^\varphi$ solves

$$\int_\Omega \nabla u\cdot\nabla\eta\,\mathrm{d}x = \lambda_1^{0,\varphi}\int_\Omega u\eta\,\mathrm{d}x \quad \textit{for all } \eta \in V^\varphi, \tag{3.3.27}$$

and $\lambda_1^{0,\varphi} = \lim_{\varepsilon\searrow 0}\lambda^\varepsilon$.

If we can show that $u \in V^\varphi \cap Q$ is even a minimizer of

$$v \mapsto \frac{\int_\Omega |\nabla v|^2\,\mathrm{d}x}{\int_\Omega |v|^2\,\mathrm{d}x} \quad \text{subject to} \quad v \in V^\varphi, v \neq 0, \tag{3.3.28}$$

we directly infer from the first-order condition associated with this minimization problem and the Courant–Fischer characterization (3.3.3) that u solves (3.3.2) to the eigenvalue $\lambda_1^{0,\varphi}$.

As $u \in V^\varphi$ is a minimizer of F_0 over Q we use (3.3.26) to deduce

$$\begin{aligned}
\min\left\{\frac{\int_\Omega |\nabla v|^2\,\mathrm{d}x}{\int_\Omega |v|^2\,\mathrm{d}x}\middle|\begin{matrix} v\in V^\varphi,\\ v\neq 0\end{matrix}\right\} &\leq \frac{\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\int_\Omega |u|^2\,\mathrm{d}x} = F_0(u)\\
&= \min\left\{\frac{\int_\Omega |\nabla v|^2\,\mathrm{d}x + \int_\Omega b_0(\varphi)\,|v|^2\,\mathrm{d}x}{\int_\Omega |v|^2\,\mathrm{d}x}\middle|\begin{matrix} v\in H_0^1(\Omega),\\ v\neq 0\end{matrix}\right\}\\
&\leq \min\left\{\frac{\int_\Omega |\nabla v|^2\,\mathrm{d}x + \int_\Omega b_0(\varphi)\,|v|^2\,\mathrm{d}x}{\int_\Omega |v|^2\,\mathrm{d}x}\middle|\begin{matrix} v\in V^\varphi,\\ v\neq 0\end{matrix}\right\}.
\end{aligned}$$

Here, the last inequality holds because $V^\varphi \subset H_0^1(\Omega)$. However, we already know from *Step 2* that $\int_\Omega b_0(\varphi)\,|v|^2\,\mathrm{d}x = 0$ for all $v \in V^\varphi$. Hence, we conclude from the above estimate that $u \in V^\varphi$ minimizes (3.3.28), and in particular, $F_0(u) = \lambda_1^{0,\varphi}$. Now, the second claim of *Step 5* follows from (3.3.22) since $F_\varepsilon(w_\varepsilon) = \lambda^\varepsilon$ holds by construction.

The proof of Lemma 3.3.5 is now complete. □

This lemma now serves as initial step to show the analogous properties also for all higher eigenvalues and eigenfunctions via induction.

Theorem 3.3.7. *In addition to the assumptions made in Section 2.1, we suppose that the assumptions* **(A1)**–**(A6)** *are fulfilled. Let $k \in \mathbb{N}$, and suppose that $(\varphi_\varepsilon)_{\varepsilon>0} \subset L^1(\Omega)$ and $\varphi \in BV(\Omega, \{\pm 1\}) \cap \mathcal{U}$ fulfill the same assumptions as in Lemma 3.3.5.*

Then, there exists an eigenfunction $u_k \in V^\varphi$ of the limit problem (3.3.2) *to the eigenvalue $\lambda_k^{0,\varphi}$ such that the convergences*

$$\lim_{\varepsilon \searrow 0} \int_\Omega b_\varepsilon(\varphi_\varepsilon) \left| w_k^{\varphi_\varepsilon} \right|^2 \, \mathrm{d}x = \int_\Omega b_0(\varphi) \left| u_k \right|^2 \, \mathrm{d}x = 0,$$
$$\lim_{\varepsilon \searrow 0} \left\| w_k^{\varphi_\varepsilon} - u_k \right\|_{H^1(\Omega)} = 0, \quad \lim_{\varepsilon \searrow 0} \lambda_k^{\varphi_\varepsilon} = \lambda_k^{0,\varphi}$$

hold up to subsequence extraction.

Proof. We prove the assertion via induction. The initial step was carried out in Lemma 3.3.5.

Let us now assume that the assertion is already established for the first $k-1$ eigenfunctions $w_1^\varepsilon := w_1^{\varphi_\varepsilon}, \dots, w_{k-1}^\varepsilon := w_{k-1}^{\varphi_\varepsilon}$ of (3.2.1).

To prove the result for $w_k^\varepsilon := w_k^{\varphi_\varepsilon}$, let $W_\varepsilon := \langle w_1^\varepsilon, \dots, w_{k-1}^\varepsilon \rangle_{\mathrm{span}} \subset L^2(\Omega)$ denote the space spanned by the first $k-1$ eigenfunctions and $W_\varepsilon^\perp$ its orthogonal complement with respect to the canonical scalar product on $L^2(\Omega)$ that is denoted by $(\cdot, \cdot)$. We further define the space

$$Q_\varepsilon := \left\{ v \in H_0^1(\Omega) \,\middle|\, \|v\|_{L^2(\Omega)} = 1 \text{ and } v \in W_\varepsilon^\perp \right\},$$

as well as the operator

$$F_\varepsilon : Q_\varepsilon \to [0, +\infty], \quad v \mapsto \int_\Omega |\nabla v|^2 \, \mathrm{d}x + \int_\Omega b_\varepsilon(\varphi_\varepsilon) \, |v|^2 \, \mathrm{d}x.$$

Then, from the Courant-Fischer characterization (3.2.3) we infer that

$$\lambda_k^\varepsilon := \lambda_k^{\varphi_\varepsilon} = \min_{v \in Q_\varepsilon} F_\varepsilon(v).$$

In the limit situation, we define the space $W_0 := \langle u_1, \dots, u_{k-1} \rangle_{\mathrm{span}}$ where the functions $u_i \in V^\varphi$, $i = 1, \dots, k-1$ are determined by the induction hypothesis as the limits

$$w_i^\varepsilon \to u_i \quad \text{in } H^1(\Omega) \text{ as } \varepsilon \to 0,$$

up to subsequence extractions, as discussed in Lemma 3.3.5. In particular, for $i, j = 1, \dots, k-1$ with $i \neq j$, we have $(u_i, u_j) = 0$ since $(w_i^\varepsilon, w_j^\varepsilon) = 0$. Moreover, we know that $\|w_i^\varepsilon\|_{L^2(\Omega)} = 1$ and hence, we also have $\|u_i\|_{L^2(\Omega)} = 1$ for $i = 1, \dots, k-1$. This means that $\{u_1, \dots, u_{k-1}\} \subset L^2(\Omega)$ is an orthonormal basis of the $(k-1)$-dimensional space $W_0 \subset V^\varphi$. We further set

$$Q_0 := \left\{ v \in H_0^1(\Omega) \,\middle|\, \|v\|_{L^2(\Omega)} = 1 \text{ and } v \in W_0^\perp \right\},$$

and we define the operator

$$F_0 : Q_0 \to [0, +\infty], \quad v \mapsto \int_\Omega |\nabla v|^2 \, \mathrm{d}x + \int_\Omega b_0(\varphi) \, |v|^2 \, \mathrm{d}x.$$

Now we introduce the orthogonal projections

$$P_\varepsilon : L^2(\Omega) \to W_\varepsilon, \quad v \mapsto \sum_{i=1}^{k-1} (v, w_i^\varepsilon) w_i^\varepsilon,$$
$$P_0 : L^2(\Omega) \to W_0, \quad v \mapsto \sum_{i=1}^{k-1} (v, u_i) u_i.$$

In the following, these projections will be a useful tool to construct recovery sequences.

Per construction, $w_k^\varepsilon \in H_0^1(\Omega)$ is a minimizer of F_ε. Now, we need to show that there exists a constant $C > 0$ that does not depend on $\varepsilon > 0$ such that

$$F_\varepsilon(w_k^\varepsilon) \leq C, \tag{3.3.29}$$

as this allows us to bound $(w_k^\varepsilon)_{\varepsilon>0}$ in the $H^1(\Omega)$ norm.

As in *Step 3* of the proof of Lemma 3.3.5, we want to choose suitable elements v^ε in the feasible sets Q_ε for which we can bound the sequence $(F_\varepsilon(v^\varepsilon))_{\varepsilon>0}$. Here, the situation is more complicated compared to Lemma 3.3.5 as the feasible set Q_ε depends on ε.

Due to assumption **(A4)**, we find $v^0 \in V^\varphi$ such that

$$v^0 \in W_0^\perp \setminus \{0\}.$$

Otherwise, V^φ would be a subset of the $(k-1)$-dimensional space W_0, which is a contradiction to the fact that V^φ is infinite dimensional. Let us define the sequence

$$v^\varepsilon := v^0 - \sum_{i=1}^{k-1} \left(v^0, w_i^\varepsilon\right) w_i^\varepsilon = v^0 - P_\varepsilon(v^0) \in H_0^1(\Omega) \cap W_\varepsilon^\perp.$$

Now, by the induction hypothesis, for every $i = 1, \dots, k-1$, we know that

$$w_i^\varepsilon \to u_i \quad \text{in } H^1(\Omega), \tag{3.3.30}$$

along a suitable non-relabeled subsequence. Hence, from the construction of v^ε, we infer

$$v^\varepsilon \to v^0 \quad \text{in } H^1(\Omega). \tag{3.3.31}$$

In particular, for $\varepsilon > 0$ sufficiently small, we thus have $v^\varepsilon \neq 0$. Altogether this allows us to define the sequence

$$\overline{v}^\varepsilon = \frac{v^\varepsilon}{\|v^\varepsilon\|_{L^2(\Omega)}} \in Q_\varepsilon,$$

which fulfills the convergence

$$\overline{v}^\varepsilon \to \overline{v}^0 := \frac{v^0}{\|v^0\|_{L^2(\Omega)}} \in Q_0 \quad \text{in } H^1(\Omega).$$

If we can now verify that

$$F_\varepsilon(\overline{v}^\varepsilon) \leq C, \tag{3.3.32}$$

uniformly in ε, (3.3.29) directly follows as our minimizer $w_k^\varepsilon \in Q_\varepsilon$ obviously fulfills

$$F_\varepsilon(w_k^\varepsilon) \leq F_\varepsilon(\overline{v}^\varepsilon).$$

Therefore, we recall that

$$F_\varepsilon(\overline{v}^\varepsilon) = \int_\Omega |\nabla \overline{v}^\varepsilon|^2 \, \mathrm{d}x + \int_\Omega b_\varepsilon(\varphi_\varepsilon) \, |\overline{v}^\varepsilon|^2 \, \mathrm{d}x.$$

For the first summand on the right-hand side, we obtain

$$\int_\Omega |\nabla \overline{v}^\varepsilon|^2 \, \mathrm{d}x = \frac{1}{\|v^\varepsilon\|^2_{L^2(\Omega)}} \left(\int_\Omega |\nabla v^\varepsilon|^2 \, \mathrm{d}x \right).$$

Hence, this term is bounded because of (3.3.31) which further entails the convergence $\|v^\varepsilon\|_{L^2(\Omega)} \to \|v^0\|_{L^2(\Omega)} > 0$. For the second summand, we use Young's inequality to obtain

$$\int_\Omega b_\varepsilon(\varphi_\varepsilon) \, |\overline{v}^\varepsilon|^2 \, \mathrm{d}x \leq \frac{2}{\|v^\varepsilon\|^2_{L^2(\Omega)}} \left(\int_\Omega b_\varepsilon(\varphi_\varepsilon) |v^0|^2 \, \mathrm{d}x + \int_\Omega b_\varepsilon(\varphi_\varepsilon) |P_\varepsilon(v^0)|^2 \, \mathrm{d}x \right).$$

As, per construction, $v^0 \in V^\varphi$ fulfills property (3.3.13) we can apply *Step 2* of the proof of Lemma 3.3.5 which yields

$$\lim_{\varepsilon \searrow 0} \int_\Omega b_\varepsilon(\varphi_\varepsilon) |v^0|^2 \, \mathrm{d}x = 0.$$

Furthermore, Lemma 3.3.5 implies that for $i = 1, \ldots, k-1$,

$$\lim_{\varepsilon \searrow 0} \int_\Omega b_\varepsilon(\varphi_\varepsilon) |w_i^\varepsilon|^2 \, \mathrm{d}x = 0.$$

Now, as

$$P_\varepsilon(v^0) = \sum_{i=1}^{k-1} \left(v^0, w_i^\varepsilon \right) w_i^\varepsilon,$$

we obtain

$$\lim_{\varepsilon \searrow 0} \int_\Omega b_\varepsilon(\varphi_\varepsilon) |P_\varepsilon(v^0)|^2 \, \mathrm{d}x = 0,$$

by applying Young's inequality again. Altogether, we deduce

$$\lim_{\varepsilon \searrow 0} \int_\Omega b_\varepsilon(\varphi_\varepsilon) \, |\overline{v}^\varepsilon|^2 \, \mathrm{d}x = 0.$$

This proves the estimate (3.3.32) which directly entails the uniform bound (3.3.29). In particular, we have

$$\int_\Omega |\nabla w_k^\varepsilon|^2 \, \mathrm{d}x \leq F_\varepsilon(w_k^\varepsilon) \leq C.$$

Applying the Banach–Alaoglu theorem and the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ we infer the existence of a limit $u_k \in H_0^1(\Omega)$ such that

$$w_k^\varepsilon \rightharpoonup u_k \quad \text{in } H^1(\Omega), \quad w_k^\varepsilon \to u_k \quad \text{in } L^2(\Omega), \quad w_k^\varepsilon \to u_k \quad \text{a.e. on } \Omega \tag{3.3.33}$$

as $\varepsilon \to 0$ up to subsequence extraction.

Our next task is to show that u_k belongs to Q_0 and fulfills

$$F_0(u_k) = \min_{v \in Q_0} F_0(v). \tag{3.3.34}$$

First of all, we can use the convergence (3.3.30) of the first $k-1$ eigenfunctions along with (3.3.33) to obtain the convergence

$$\lim_{\varepsilon \searrow 0} (w_k^\varepsilon, w_i^\varepsilon) = (u_k, u_i),$$

for $i = 1, \dots, k-1$. However, by the orthogonality of the eigenfunctions for $\varepsilon > 0$, we know $0 = (w_k^\varepsilon, w_i^\varepsilon)$, and thus $(u_k, u_i) = 0$, $i = 1, \dots, k-1$. This already proves that $u_k \in W_0^\perp$. Notice that $\|u_k\|_{L^2(\Omega)} = 1$, as the w_k^ε are assumed to be $L^2(\Omega)$-normalized. All in all, we get $u_k \in Q_0$.

To verify (3.3.34) we cannot directly apply the theory of Γ-convergence as in Lemma 3.3.5 but we can establish similar estimates that will help us to obtain the desired properties. For the sake of a clearer presentation we divide this part of the proof into several steps.

Step 1: The following $\liminf$ *inequality holds:*

$$F_0(u_k) \le \liminf_{\varepsilon \searrow 0} F_\varepsilon(w_k^\varepsilon). \tag{3.3.35}$$

To prove the assertion, we recall that

$$F_0(u_k) = \int_\Omega |\nabla u_k|^2 \, \mathrm{d}x + \int_\Omega b_0(\varphi_0) \, |u_k|^2 \, \mathrm{d}x.$$

For the gradient term, we obtain the inequality

$$\int_\Omega |\nabla u_k|^2 \, \mathrm{d}x \le \liminf_{\varepsilon \searrow 0} \int_\Omega |\nabla w_k^\varepsilon|^2 \, \mathrm{d}x,$$

by using the weak lower semi-continuity of this expression. Now, due to the convergence properties (3.3.33), we are exactly in the same situation as in *Step 4* of the proof of Lemma 3.3.5. Hence, Fatou's lemma yields

$$\int_\Omega b_0(\varphi) \, |u_k|^2 \, \mathrm{d}x \le \liminf_{\varepsilon \searrow 0} \int_\Omega b_\varepsilon(\varphi_\varepsilon) \, |w_k^\varepsilon|^2 \, \mathrm{d}x.$$

In summary, we infer (3.3.35).

Step 2: For any $v \in Q_0$ *there exists a sequence* $(\overline{v}^\varepsilon)_{\varepsilon > 0} \subset Q_\varepsilon$ *which satisfies*

$$\limsup_{\varepsilon \searrow 0} F_\varepsilon(\overline{v}^\varepsilon) \le F_0(v). \tag{3.3.36}$$

Here, finding such a recovery sequence is more complicated than in *Step 4* of Lemma 3.3.5, as we cannot just take the constant sequence v. Without loss of generality, we assume that $F_0(v) < +\infty$ as otherwise (3.3.36) is trivially fulfilled. This guarantees that

$$\int_\Omega b_0(\varphi_0)\,|v|^2\,\mathrm{d}x < +\infty.$$

Hence, v fulfills property (3.3.13) (i.e., $v \in V^\varphi$) which will be needed later. Analogously to the beginning of this proof, we now define the sequence

$$v^\varepsilon := v - \sum_{i=1}^{k-1}(v, w_i^\varepsilon)w_i^\varepsilon = v - P_\varepsilon(v) \in W_\varepsilon^\perp.$$

Exactly as in (3.3.31), we obtain the convergence

$$v^\varepsilon \to v - \sum_{i=1}^{k-1}(v, u_i)u_i = v - P_0(v) \in W_0^\perp \quad \text{in } H^1(\Omega).$$

However, since $v \in Q_0$, we also have $v \in W_0^\perp$ meaning that $P_0(v) = 0$. We thus get $v^\varepsilon \to v$ in $H^1(\Omega)$. Furthermore, $v \in Q_0$ ensures that $\|v\|_{L^2(\Omega)} = 1 > 0$ and hence, we infer that for $\varepsilon > 0$ sufficiently small, it holds that $\|v^\varepsilon\|_{L^2(\Omega)} > 0$. We can thus consider the normalized sequence

$$\overline{v}^\varepsilon := \frac{v^\varepsilon}{\|v^\varepsilon\|_{L^2(\Omega)}} \in Q_\varepsilon, \quad \text{which fulfills} \quad \overline{v}^\varepsilon \to v \quad \text{in } H^1(\Omega). \tag{3.3.37}$$

We now prove (3.3.36) by again considering the gradient term and the term involving b_ε appearing in F_ε separately. Using (3.3.37), we infer that

$$\lim_{\varepsilon \searrow 0}\int_\Omega |\nabla \overline{v}^\varepsilon|^2\,\mathrm{d}x = \int_\Omega |\nabla v|^2\,\mathrm{d}x.$$

For the second term, considering the representation

$$\overline{v}^\varepsilon = \frac{1}{\|v^\varepsilon\|}\left[v - \sum_{i=1}^{k-1}(v, w_i^\varepsilon)w_i^\varepsilon\right],$$

we see that this sequence has exactly the same properties as the same-named sequence in the beginning of this proof. Hence, proceeding as above, we use the convergence properties known for w_i^ε, the convergence $\|v^\varepsilon\|_{L^2(\Omega)} \to \|v\|_{L^2(\Omega)} > 0$ and the crucial fact that $v \in V^\varphi$ to deduce

$$\lim_{\varepsilon \searrow 0}\int_\Omega b_\varepsilon(\varphi^\varepsilon)\,|\overline{v}^\varepsilon|\,\mathrm{d}x = 0 = \int_\Omega b_0(\varphi)\,|v|^2\,\mathrm{d}x.$$

Hence, in particular, this verifies (3.3.36).

Now, combining these two steps, we obtain for any arbitrary $v \in Q_0$,

$$F_0(u_k) \le \liminf_{\varepsilon \searrow 0} F_\varepsilon(w_k^\varepsilon) \le \limsup_{\varepsilon \searrow 0} F_\varepsilon(w_k^\varepsilon) \le \limsup_{\varepsilon \searrow 0} F_\varepsilon(\overline{v}^\varepsilon) \le F_0(v),$$

since w_k^ε is a minimizer of F_ε over Q_ε. As $v \in Q_0$ was arbitrary, this finally shows that

$$F_0(u_k) = \min_{v \in Q_0} F_0(v).$$

Furthermore, plugging $v = u_k$ into the above chain of estimates we get

$$F_0(u_k) = \liminf_{\varepsilon \searrow 0} F^\varepsilon(w_k^\varepsilon) = \limsup_{\varepsilon \searrow 0} F^\varepsilon(w_k^\varepsilon),$$

which directly yields

$$\lim_{\varepsilon \searrow 0} F_\varepsilon(w_k^\varepsilon) = F_0(u_k). \tag{3.3.38}$$

As in the proof of Lemma 3.3.5, this allows us to deduce

$$\lim_{\varepsilon \searrow 0} \|w_k^\varepsilon - u_k\|_{H^1(\Omega)} = 0, \quad \text{and} \quad u_k \in V^\varphi.$$

Therefore, it only remains to prove the following statement.

Step 3: The function $u_k \in V^\varphi$ solves

$$\int_\Omega \nabla u_k \cdot \nabla \eta \,\mathrm{d}x = \lambda_k^{0,\varphi} \int_\Omega u_k \eta \,\mathrm{d}x \quad \textit{for all } \eta \in V^\varphi, \tag{3.3.39}$$

and it holds that $\lambda_k^{0,\varphi} = \lim_{\varepsilon \searrow 0} \lambda_k^{\varphi_\varepsilon}$.

With an analogous reasoning as in *Step 5* of the proof of Lemma 3.3.5, we see that

$$\tilde{\lambda} \coloneqq \min \left\{ \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega |v|^2 \,\mathrm{d}x} \middle| \begin{matrix} v \in V^\varphi \cap W_0^\perp, \\ v \neq 0 \end{matrix} \right\} = \frac{\int_\Omega |\nabla u_k|^2 \,\mathrm{d}x}{\int_\Omega |u_k|^2 \,\mathrm{d}x} = F_0(u_k). \tag{3.3.40}$$

As $W_0 = \langle u_1, \dots, u_{k-1} \rangle_{\mathrm{span}}$ is a $(k-1)$-dimensional subspace of V^φ, the Courant–Fischer characterization (3.3.3) entails that $\lambda_k^{0,\varphi} \geq \tilde{\lambda}$.

Furthermore, for $\varepsilon > 0$, we have $\lambda_k^{\varphi_\varepsilon} \geq \lambda_{k-1}^{\varphi_\varepsilon}$, and by (3.3.38) and the induction hypothesis we infer

$$\lim_{\varepsilon \searrow 0} \lambda_k^{\varphi_\varepsilon} = \lim_{\varepsilon \searrow 0} F_\varepsilon(w_k^\varepsilon) = F_0(u_k) = \tilde{\lambda}, \quad \text{and} \quad \lim_{\varepsilon \searrow 0} \lambda_{k-1}^{\varphi_\varepsilon} = \lambda_{k-1}^{0,\varphi}.$$

This proves that $\tilde{\lambda} \geq \lambda_{k-1}^{0,\varphi}$.

Now, to show that $\tilde{\lambda} = \lambda_k^{0,\varphi}$, we need to consider two cases.

Case 1: It holds that $\lambda_k^{0,\varphi} = \lambda_{k-1}^{0,\varphi}$. Then, from the above considerations we already infer that $\lambda_k^{0,\varphi} = \tilde{\lambda} = \lambda_{k-1}^{0,\varphi}$.

Case 2: It holds that $\lambda_k^{0,\varphi} > \lambda_{k-1}^{0,\varphi}$. Then, the span of the eigenfunctions $\{w_1^{0,\varphi}, \dots, w_{k-1}^{0,\varphi}\} \subset V^\varphi$ (given as in Theorem 3.3.2) contains the union of all eigenspaces belonging to the eigenvalues $\lambda_1^{0,\varphi}, \dots, \lambda_{k-1}^{0,\varphi}$. On the other hand, we know from the induction hypothesis that $\{u_1, \dots, u_{k-1}\}$ is a linearly independent family of eigenfunctions belonging to the aforementioned eigenvalues. Hence, we conclude that

$$W_0 = \langle u_1, \dots, u_{k-1} \rangle_{\mathrm{span}} = \langle w_1^{0,\varphi}, \dots, w_{k-1}^{0,\varphi} \rangle_{\mathrm{span}}. \tag{3.3.41}$$

This means that W_0 is exactly the $(k-1)$-dimensional vector space on which the maximum in the Courant–Fischer characterization (3.3.3) is attained. Thus, by (3.3.40), we infer $\tilde{\lambda} = \lambda_k^{\varphi}$.

Now, computing the first-order optimality condition (Euler–Lagrange equation) of the minimization problem associated with (3.3.40), we conclude that $u_k \in V^{\varphi}$ and λ_k^{φ} fulfill the weak formulation (3.3.2) of the limit problem for all test functions in $V^{\varphi} \cap W_0^{\perp}$. However, we know from (3.3.41) that W_0 is spanned by eigenfunctions to the limit problem (3.3.2). This means that u_k and λ_k^{φ} trivially satisfy the weak formulation (3.3.2) for all test functions in W_0. In summary, this proves that u_k is an eigenfunction of the limit problem (3.3.2) to the eigenvalue $\lambda_k^{0,\varphi}$.

This completes the proof by induction. □

3.3.2. Sharp-interface limit of the optimization problem

We now show that a sequence of minimizers of the cost functionals for $\varepsilon > 0$ converges, as $\varepsilon \to 0$, to a minimizer of the cost functional associated with the sharp-interface setting which will be defined in the following.

First of all, we demand again that the coefficient functions satisfy the assumptions **(A2)** and **(A3)**. For $\varepsilon > 0$, we extend the cost functional of the problem $(\mathcal{P}_l^{D,\varepsilon})$ to the space $L^1(\Omega)$ by defining

$$J^{\varepsilon}(\varphi) \coloneqq \begin{cases} \Psi(\lambda_{i_1}^{\varepsilon,\varphi}, \dots, \lambda_{i_l}^{\varepsilon,\varphi}) + \gamma E_{\mathrm{GL}}^{\varepsilon}(\varphi) & \text{if } \varphi \in \Phi_{\mathrm{ad}}, \\ +\infty & \text{if } \varphi \in L^1(\Omega) \backslash \Phi_{\mathrm{ad}}. \end{cases} \tag{3.3.42}$$

Here, for any $k \in \mathbb{N}$, $\lambda_k^{\varepsilon,\varphi}$ denotes the k-th eigenvalue of the Dirichlet problem (3.2.1) with $\varepsilon > 0$ and $\varphi \in \Phi_{\mathrm{ad}} \subset L^{\infty}(\Omega)$. In the sharp-interface situation, we consider

$$\tilde{\mathcal{G}}^{\beta} = \left\{ \varphi \in L^1(\tilde{\Omega}) \,\middle|\, |\varphi| \leq 1,\ \beta_1 |\tilde{\Omega}| \leq \int_{\tilde{\Omega}} \varphi \,\mathrm{d}x \leq \beta_2 |\tilde{\Omega}| \right\},$$

and define the cost functional as

$$J^0(\varphi) \coloneqq \begin{cases} \Psi(\lambda_{i_1}^{0,\varphi}, \dots, \lambda_{i_l}^{0,\varphi}) + \gamma c_0 P_{\tilde{\Omega}}(\tilde{E}^{\varphi}) & \text{if } \varphi \in \Phi_{\mathrm{ad}}^0, \\ +\infty & \text{if } \varphi \in L^1(\Omega) \backslash \Phi_{\mathrm{ad}}^0, \end{cases} \tag{3.3.43}$$

where $\Phi_{\mathrm{ad}}^0 \coloneqq BV(\Omega, \{\pm 1\}) \cap \mathcal{U} \cap \tilde{\mathcal{G}}^{\beta}$.

Here, for any $k \in \mathbb{N}$, $\lambda_k^{0,\varphi}$ denotes the k-th eigenvalue of the limit problem (3.3.2) that was introduced in Theorem 3.3.2 for $\varphi \in BV(\Omega, \{\pm 1\}) \cap \mathcal{U}$.

Let $P_{\tilde{\Omega}}(\tilde{E}^{\varphi})$ denote the relative perimeter in $\tilde{\Omega}$ of the set $\tilde{E}^{\varphi} \coloneqq \{x \in \tilde{\Omega} \,|\, \varphi(x) = 1\}$, i.e.,

$$P_{\tilde{\Omega}}(\tilde{E}^{\varphi}) \coloneqq \sup \left\{ \int_{\tilde{E}^{\varphi}} \mathrm{div} \boldsymbol{\zeta} \,\mathrm{d}x \,\middle|\, \boldsymbol{\zeta} \in C_0^1(\tilde{\Omega}, \mathbb{R}^d), \|\zeta\|_{L^{\infty}(\tilde{\Omega})} \leq 1 \right\}.$$

We further set

$$c_0 \coloneqq \int_{-1}^{1} \sqrt{2\psi_0(x)} \,\mathrm{d}x,$$

where ψ_0 is the potential appearing in the regularized Ginzburg–Landau energy

$$E^\varepsilon(\varphi) = \int_{\tilde{\Omega}} \left(\frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon} \psi_0(\varphi) \right) \mathrm{d}x, \quad \varepsilon > 0.$$

Additionally to the assumptions in Section 2.1.7, we make the following assumption that is supposed to hold throughout the remainder of this section.

(A7) Ψ is bounded from below, i.e., we find a constant $C_\Psi > 0$ such that $\Psi(\boldsymbol{x}) \geq -C_\Psi$ for all $\boldsymbol{x} \in (\mathbb{R}_{>0})^l$. Without loss of generality, we assume $C_\Psi = 0$.

Now, the goal is to show Γ-convergence of the cost functionals as this yields that a subsequence of minimizers φ_ε of J^ε converges in $L^1(\Omega)$ to a minimizer of J^0. In this sense, the diffuse interface optimization problem can be regarded as an approximation of the sharp-interface optimization problem.

In our previous considerations, we needed to impose the rate condition

$$\|\varphi_\varepsilon - \varphi_0\|_{L^1(E^{\varphi_0} \cap \{\varphi_\varepsilon < 0\})} = \mathcal{O}(\varepsilon), \tag{3.3.44}$$

in order to show the desired properties such as the convergence of the eigenvalues as $\varepsilon \to 0$. However, to obtain a true unconditional Γ-convergence result, we do not want to impose such an additional assumption on our sequence of minimizers. The lim inf inequality can be shown for general cost functionals, i.e., for Ψ fulfilling only the current assumptions. Furthermore, the proof does not rely on the continuity of eigenvalues when passing from diffuse to sharp-interfaces. Therefore, no rate condition needs to be assumed.

For the lim sup inequality, the classical recovery sequence for the Ginzburg–Landau energy constructed in [37] fulfills the rate condition. However, it is a delicate aspect that this recovery sequence can only be constructed explicitly for sets fulfilling suitable regularity assumptions, but not for general finite perimeter sets. As also seen in [37, 126], one therefore needs to approximate finite perimeter sets on the sharp-interface level in a suitable way such that the perimeter converges and, in our framework, also the eigenvalues. This convergence of eigenvalues on the sharp-interface level was studied in [39, 51] and can be applied here also in a slightly modified way in order to take care of the constraint formulated in $\mathcal{U}$. As done there we also need to assume that the cost functional satisfies a component-wise monotonicity. Note that in [39] the Γ-convergence was studied without any additional volume constraint, which allows the usage of the recovery sequence of [127]. After the authors had shown the convergence of eigenvalues on the sharp-interface level, their lim sup inequality on the diffuse interface level was a direct consequence of the monotonicity of the cost functional. Hence, no continuity property for the eigenvalues was required. In our situation with an additional volume constraint, even though we also need to assume the monotonicity of the cost functional, we can rely on the continuity of eigenvalues on the diffuse interface level in the sense of Theorem 3.3.7. This allows us to use the recovery sequence from [96] for the double obstacle potential which is based on the construction of [37, 126].

To motivate the additional monotonicity assumption on Ψ, we first establish the following lemma.

Lemma 3.3.8. *Let $X \subset \mathbb{R}^l$. We consider a continuous function*

$$f : X \to \mathbb{R}.$$

Then the following assertions are equivalent.

(a) *For any sequence* $(\boldsymbol{x}_k)_{k\in\mathbb{N}} \subset X$ *and* $\boldsymbol{x} \in X$ *fulfilling*

$$\boldsymbol{x} \leq \liminf_{k\to\infty} \boldsymbol{x}_k \in X \text{ component-wise},$$

it holds

$$f(\boldsymbol{x}) \leq \liminf_{k\to\infty} f(\boldsymbol{x}_k).$$

(b) f *is monotonically increasing in the sense that for* $\boldsymbol{x}, \boldsymbol{y} \in X$,

$$\boldsymbol{x} \leq \boldsymbol{y} \text{ component-wise } \Rightarrow f(\boldsymbol{x}) \leq f(\boldsymbol{y}). \tag{3.3.45}$$

Proof. The implication (a)⇒(b) follows by choosing the constant sequence $x_k = y$ for all $k \in \mathbb{N}$. In order to show (b)⇒(a) we recall the definition of the limes inferior, and we use the monotonicity and the continuity of f to obtain

$$f(\boldsymbol{x}) \leq f\left(\liminf_{k\to\infty} \boldsymbol{x}_k\right) = \lim_{n\to\infty} f\left(\inf\left\{\boldsymbol{x}_k \middle| k \geq n\right\}\right).$$

Exploiting again the monotonicity of f, we deduce that for all $n \in \mathbb{N}$,

$$f\left(\inf\left\{\boldsymbol{x}_k \middle| k \geq n\right\}\right) \leq \inf\left\{f(\boldsymbol{x}_k) \middle| k \geq n\right\}.$$

This implies that

$$f(\boldsymbol{x}) \leq \lim_{n\to\infty} \inf\left\{f(\boldsymbol{x}_k) \middle| k \geq n\right\} = \liminf_{k\to\infty} f(\boldsymbol{x}_k),$$

an thus, the claim is established. □

In order to be able to apply Lemma 3.3.8, we make the following additional assumption on the function Ψ, which is supposed to hold throughout the remainder of this section.

(A8) The function $\Psi : (\mathbb{R}_{>0})^l \to \mathbb{R}_{\geq 0}$ is assumed to be monotonically increasing in the sense of Lemma 3.3.8 and exhibit the coercivity property

$$(\Psi(\boldsymbol{x}_k))_{k\in\mathbb{N}} \text{ is bounded} \Rightarrow (\boldsymbol{x}_k)_{k\in\mathbb{N}} \text{ is bounded}, \tag{3.3.46}$$

for any sequence $(\boldsymbol{x}_k)_{k\in\mathbb{N}} \subset (\mathbb{R}_{>0})^l$.

These properties are for example fulfilled if Ψ is given as a positive linear combination of the components, i.e.,

$$\Psi(\boldsymbol{x}) = \sum_{j=1}^{l} \alpha_j x_j,$$

where $\alpha_j > 0$ for $j = 1, \ldots, l$. In the context of our cost functional this would mean that linear combinations of eigenvalues $\lambda_{i_j}^{\varepsilon,\varphi}$ and $\lambda_{i_j}^{0,\varphi}$ respectively are involved in our

optimization process. In particular, by choosing $l = 1$ and $\psi(x) = x$ for all $x \in \mathbb{R}_{\geq 0}$, the minimization of just one single eigenvalue of course also fulfills the assumption on Ψ.

Assumption **(A8)** might look a bit technical at first sight, but it is exactly what we need in order to establish the lim inf inequality. The monotonicity of Ψ combined with Lemma 3.3.8 allows us to infer the lim inf inequality for the cost functional from the lim inf inequality for the eigenvalues. On the other hand the coercivity property (3.3.46) entails that the sequence of eigenvalues is bounded uniformly in ε if the cost functionals stay bounded.

Under Assumption **(A8)** it is now possible to establish an unconditional Γ-convergence result.

Theorem 3.3.9. *In addition to the assumptions made in Section 2.1, we suppose that the assumptions* **(A1)**–**(A8)** *are fulfilled. Then, it holds that*

$$J^\varepsilon \xrightarrow{\Gamma} J^0 \quad \text{as } \varepsilon \to 0.$$

Due to the compactness of the Ginzburg–Landau energy from Proposition 2.2.25 and the abstract convergence of minimizers from Proposition 2.2.24 we directly draw the following corollary.

Corollary 3.3.10. *In addition to the assumptions made in Section 2.1, we suppose that the assumptions* **(A1)**–**(A8)** *are fulfilled. Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be a sequence of minimizers of the functionals $(J^\varepsilon)_{\varepsilon>0}$. Then there exists a function $\varphi_0 \in L^1(\Omega)$, such that*

$$\lim_{\varepsilon \searrow 0} \|\varphi_\varepsilon - \varphi_0\|_{L^1(\Omega)} = 0, \qquad \lim_{\varepsilon \searrow 0} J^\varepsilon(\varphi_\varepsilon) = J^0(\varphi_0),$$

and φ_0 is a minimizer of J^0. In particular, this means that $\varphi_0 \in \Phi^0_{\mathrm{ad}} \subseteq BV(\Omega, \{\pm 1\})$.

We now conclude this section by presenting the proof of the above theorem. In order to tackle the volume constraint

$$\beta_1 |\tilde{\Omega}| \leq \int_{\tilde{\Omega}} \varphi \, \mathrm{d}x \leq \beta_2 |\tilde{\Omega}|,$$

we first show a Γ-convergence result similar to [39, Theorem 3.1], where the volume constraint is omitted and then, in a further step, we suitably modify the recovery sequence such that it actually fulfills the volume constraint.

Theorem 3.3.11. *In addition to the assumptions made in Section 2.1, we suppose that the assumptions* **(A1)**–**(A8)** *are fulfilled. Let*

$$I^\varepsilon(\varphi) \coloneqq \begin{cases} \Psi(\lambda^{\varepsilon,\varphi}_{i_1}, \dots, \lambda^{\varepsilon,\varphi}_{i_l}) + \gamma E^\varepsilon_{\mathrm{GL}}(\varphi) & \text{if } \varphi \in \Lambda_{\mathrm{ad}}, \\ +\infty & \text{if } \varphi \in L^1(\Omega) \backslash \Lambda_{\mathrm{ad}} \end{cases} \tag{3.3.47}$$

and

$$I^0(\varphi) \coloneqq \begin{cases} \Psi(\lambda^{0,\varphi}_{i_1}, \dots, \lambda^{0,\varphi}_{i_l}) + \gamma c_0 P_{\tilde{\Omega}}(\tilde{E}^\varphi) & \text{if } \varphi \in \Lambda^0_{\mathrm{ad}}, \\ +\infty & \text{if } \varphi \in L^1(\Omega) \backslash \Lambda^0_{\mathrm{ad}}, \end{cases} \tag{3.3.48}$$

with

$$\Lambda_{\mathrm{ad}} \coloneqq \left\{ \varphi \in H^1(\tilde{\Omega}) \,\middle|\, |\varphi| \leq 1 \right\} \cap \mathcal{U},$$

$$\Lambda^0_{\mathrm{ad}} \coloneqq BV(\Omega, \{\pm 1\}) \cap \mathcal{U}.$$

Then, it holds that $I^\varepsilon \xrightarrow{\Gamma} I$ *as* $\varepsilon \to 0$.

To proof the assertion we will follow partly the reasoning in [39]. Although, the arguments in [39] contain highly valuable ideas, we have the impression that at some points the authors do not distinguish carefully enough between the global perimeter $P_{\mathbb{R}^d}$ on $\mathbb{R}^d$ and the relative perimeter $P_{\tilde{\Omega}}$ on $\tilde{\Omega}$ which does not see the boundary $\partial\tilde{\Omega}$, see also Remark 2.2.21. This plays a crucial role when the Γ-convergence results of [37, 126, 127] are applied. In the following, we thus present a very detailed proof where we take care that all steps are applicable for the relative perimeter $P_{\tilde{\Omega}}$.

We further point out that in contrast to [39], our proof does *not* rely on the property that the recovery sequence $(\varphi_\varepsilon)_{\varepsilon>0}$ constructed in [127] fulfills the inclusion

$$\{\varphi = 1\} \subset \{\varphi_\varepsilon = 1\} \quad \text{for all } \varepsilon > 0,$$

where $\varphi_\varepsilon \to \varphi$ in $L^1(\Omega)$. In [39], this inclusion is crucial to obtain the lim sup inequality for the eigenvalues. As we do not require this condition, we can use the construction of [37, 96] for the double obstacle potential. Our strategy is based on the continuity properties of eigenvalues shown in the previous section. In this way, we achieve that our Γ-convergence result holds for any general coefficient function b_ε fulfilling Assumption **(A3)**. In particular, this means that the coefficient function can be chosen in a more general way compared to the explicit affine linear construction in [39].

Proof of Theorem 3.3.11. As previously explained, we need to approximate any general finite perimeter set by a sequence of (sufficiently) smooth sets in order to construct a recovery sequence for the lim sup inequality. The construction of such an approximate sequence of smooth sets is presented now.

Step 1: For $\tilde{E} \subset \tilde{\Omega}$ *with* $P_{\tilde{\Omega}}(\tilde{E}) < \infty$ *there exists a sequence of bounded smooth open sets* $E_k \subset \mathbb{R}^d$ *fulfilling*

$$\begin{cases} \mathcal{H}^{d-1}(\partial E_k \cap \partial\tilde{\Omega}) = 0, \\ \lim\limits_{k\to\infty} P_{\tilde{\Omega}}(E_k) = P_{\tilde{\Omega}}(\tilde{E}), \\ \lim\limits_{k\to\infty} \varphi_k = \varphi \text{ in } L^1(\Omega), \\ \limsup\limits_{k\to\infty} \lambda^{0,\varphi_k} \le \lambda^{0,\varphi}, \end{cases} \tag{3.3.49}$$

where $\varphi_k \coloneqq 2\chi_{E_k^{\tilde{\Omega}} \cup S_1} - 1$ *and* $\varphi \coloneqq 2\chi_{\tilde{E}\cup S_1} - 1$*, and for* $m \in \mathbb{N}$*,* $\lambda^{0,\varphi} = \lambda^{0,\varphi}_m$*, stands for an arbitrary eigenvalue. Here, for any set* $A \subset \mathbb{R}^d$*, we use the notation* $A^{\tilde{\Omega}} := A \cap \tilde{\Omega}$.

To construct an approximate sequence of bounded smooth open sets, we follow the proof of [126, Lem. 1] which can also be found in [140, Lemma 13.9]. For the sake of readability, we explain the key steps of this construction and will stick to the notation of [140]. Note that we cannot assume that the finite perimeter set $\tilde{E}$ contains an open ball. For pure perimeter minimization this assumption would be justified by [100, Theorem 1] if only the convergence of minimizers is to be shown. For that reason, we cannot easily adjust the volume of the approximating sets E_k by including or excluding balls as it was done

in [126,140]. To overcome this, we will adjust the volume of the recovery sequence only at the diffuse interface level which will eventually be done in the proof of Theorem 3.3.9. An alternative way of tackling the volume constraint on the sharp-interface level is performed in [147] which does not need the finite perimeter set $\tilde{E}$ to contain any open ball. There, the approximating sequence E_k is modified by adding or subtracting suitable hyper-cubes, but due to the rather technical construction this would require a delicate discussion in order to analyze the lim sup inequality of eigenvalues in (3.3.49).

The key idea of constructing a sequence $(E_k)_{k\in\mathbb{N}}$ of bounded smooth open sets is to extend $\chi_E \in BV(\tilde{\Omega}) \cap L^\infty(\tilde{\Omega})$ to a function $v \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with $|Dv|(\partial\tilde{\Omega}) = 0$ (which is possible as $\tilde{\Omega}$ is assumed to be a bounded Lipschitz domain, see [14, Prop. 3.21]). Note that the set $\{v \neq 0\}$ is still bounded. Here, Dv denotes the Radon measure associated with $v \in BV(\mathbb{R}^d)$ and $|\cdot|$ denotes the total variation. It is crucial that $|Dv|(\partial\tilde{\Omega}) = 0$ as we want to approximate the *relative* perimeter which does not see the boundary of the design domain $\partial\tilde{\Omega}$ but only the parts of the boundary of $\tilde{E}$ lying within $\tilde{\Omega}$.

Now, in order to construct a sequence of smooth approximating sets E_k fulfilling (3.3.49), we choose a standard sequence of mollifieres $(\rho_n)_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ and consider the super-level sets

$$\{v_n > t\}, \quad \text{where } v_n \coloneqq v * \rho_n,$$

for $t \in (0,1)$. In contrast to [126,140], where for each $n \in \mathbb{N}$, a specific $t_n \in (\frac{1}{n}, 1-\frac{1}{n})$ is selected in order to show the convergence of the corresponding super level sets with respect to perimeter and measure, we use the ideas of [39, Proof of Theorem 3.1] to obtain these convergences even for almost all $t \in (0,1)$.

Due to our extension, we have $|Dv|(\partial\tilde{\Omega}) = 0$. Proceeding as in [140], we thus get

$$\lim_{n\to\infty} \int_{\tilde{\Omega}} |\nabla v_n| \,\mathrm{d}x = P_{\tilde{\Omega}}(\tilde{E}).$$

In combination with the coarea formula for the relative perimeter (see [14, Theorem 3.40]) and Fatou's lemma, we deduce as in [39] that

$$P_{\tilde{\Omega}}(\tilde{E}) = \lim_{n\to\infty} \int_0^1 P_{\tilde{\Omega}}(\{v_n > t\}) \,\mathrm{d}t \geq \int_0^1 \liminf_{n\to\infty} P_{\tilde{\Omega}}(\{v_n > t\}) \,\mathrm{d}t.$$

On the other hand, as in [140], we infer that for almost every $t \in (0,1)$,

$$\left|\left(\{v_n > t\} \cap \tilde{\Omega}\right) \triangle \tilde{E}\right| \to 0$$

as $n \to \infty$ and thus,

$$P_{\tilde{\Omega}}(\tilde{E}) \leq \liminf_{n\to\infty} P_{\tilde{\Omega}}(\{v_n > t\}),$$

due to the lower semi-continuity of the perimeter, see Proposition 2.2.22. Combining the previous inequalities, we conclude

$$P_{\tilde{\Omega}}(\tilde{E}) = \liminf_{n\to\infty} P_{\tilde{\Omega}}(\{v_n > t\}) \tag{3.3.50}$$

for almost every $t \in (0,1)$. Now, according to [140], the properties

$$\nabla v_n(x) \neq 0 \text{ for all } x \in \mathbb{R}^d \text{ with } v_n(x) = t \quad \text{and} \tag{3.3.51}$$

$$\mathcal{H}^{d-1}\left(\left\{ x \in \partial\tilde{\Omega} \middle| \, v_n(x) = t \right\}\right) = 0, \tag{3.3.52}$$

hold for all $n \in \mathbb{N}$ and almost all $t \in (0,T)$. In summary, this means that we can choose a Lebesgue null set $\mathcal{N} \subset (0,1)$ such that for every $t \in (0,1)\backslash\mathcal{N}$ the sets $E_{n,t} \coloneqq \{v_n > t\}$ are bounded, smooth and fulfill the transversality condition

$$\mathcal{H}^{d-1}(\partial E_{n,t} \cap \partial\tilde{\Omega}) = 0.$$

After extracting a suitable (non-relabeled) subsequence (possibly depending on the choice of t), we further infer the convergence properties

$$\begin{cases} \lim\limits_{n\to\infty} P_{\tilde{\Omega}}(E_{n,t}) = P_{\tilde{\Omega}}(\tilde{E}), \\ \qquad \lim\limits_{n\to\infty} \varphi_{n,t} = \varphi \quad \text{in } L^1(\Omega), \end{cases} \tag{3.3.53}$$

where $\varphi_{n,t} \coloneqq 2\chi_{E_{n,t}^{\tilde{\Omega}} \cup S_1} - 1$.

It thus remains to establish the lim sup inequality for the eigenvalues. As the eigenvalue equation is formulated on the whole of Ω (not only $\tilde{\Omega}$), we now consider $E \coloneqq \tilde{E} \cup S_1$. Here we can exactly apply the strategy employed in [39, Theorem 3.1] which can also be found in [51, Theorem 3.5]. For the sake of readability, we explain the key steps.

By the theory of Section 2.2.6, there is a quasi-open set $\omega \subset \Omega$ such that $V^{\varphi} = \tilde{H}_0^1(E) = H_0^1(\omega)$. Now, we choose $u_\omega \in H_0^1(\omega)$ as the solution of the Laplace equation (2.2.19). It then holds $H_0^1(\omega) = H_0^1(\{u_\omega > 0\}) = \tilde{H}_0^1(\{u_\omega > 0\})$, see Section 2.2.6. In the light of Remark 3.3.3 we know $\lambda_k^0(E) = \lambda_k(\{u_\omega > 0\})$, where we recall that the eigenvalue $\lambda_k(\{u_\omega > 0\})$ denotes the one formulated over the classical Sobolev space $H_0^1(\{u_\omega > 0\})$. Furthermore, we know from Section 2.2.6 that $u_\omega \in L^\infty(\Omega)$ and hence, without loss of generality, we may assume that $u_\omega \leq 1$ a.e. on Ω. Due to the inclusion $\{u_\omega > 0\} \subset \omega \subset E$, we have $u_\omega \leq \chi_E = v$ a.e. on Ω and hence up to a Lebesgue null set,

$$\{u_\omega * \rho_n > t\} \cap \Omega \subset \{v * \rho_n > t\} \cap \Omega \subset E_{n,t}^{\Omega} \subset E_{n,t}^{\tilde{\Omega}} \cup S_1,$$

for all $t \in (0,1)$ and $n \in \mathbb{N}$. In particular, we have

$$\{u_\omega * \rho_n > t\} \cap \omega \subset E_{n,t}^{\tilde{\Omega}} \cup S_1, \tag{3.3.54}$$

up to a Lebesgue null set, for all $t \in (0,1)$ and $n \in \mathbb{N}$. Hence, due to the monotonicity of eigenvalues with respect to set inclusion (3.3.6), it holds

$$\lambda^{0,\varphi_{n,t}} \leq \lambda(\{u_\omega * \rho_n > t\} \cap \omega) \leq \lambda(\{u_\omega * \rho_n > t\} \cap \{u_\omega > t\} \cap \omega). \tag{3.3.55}$$

Note that here we used Proposition 2.2.31 for the first inequality in order to find the quasi-open set $\omega_{n,t} \subset E_{n,t}^{\tilde{\Omega}} \cup S_1$ that satisfies

$$\tilde{H}_0^1(E_{n,t}^{\tilde{\Omega}} \cup S_1) = H_0^1(\omega_{n,t}),$$

and due to inclusion (3.3.54) this quasi-open set in turn obviously satisfies

$$H_0^1(\{u_\omega * \rho_n > t\} \cap \omega) \subset H_0^1(\omega_{n,t}),$$

because $\{u_\omega * \rho_n > t\} \cap \omega$ is quasi-open, see also the remark below Proposition 2.2.31. Now, using the density result Proposition 2.2.33, it was shown in [39, 51] that for all $t \in (0,1)\backslash\mathcal{N}$,

$$\{u_\omega * \rho_n > t\} \cap \{u_\omega > t\} \xrightarrow{\gamma} \{u_\omega > t\},$$

as $n \to \infty$. For the sake of clarity let us comment on this crucial point a little further. Due to the fact

$$\int_\Omega \nabla(u_\omega - t)^+ \cdot \nabla\eta \,\mathrm{d}x = \int_\Omega \nabla u_\omega \cdot \nabla\eta \,\mathrm{d}x \quad \text{for all } \eta \in H_0^1(\{u_\omega > t\}),$$

the function $(u_\omega - t)^+ \in H_0^1(\{(u_\omega - t)^+ > 0\}) = H_0^1(\{u_\omega > t\})$ is the unique solution of (2.2.19) in $H_0^1(\{u_\omega > t\})$. An amusing way to reformulate this statement is

$$(u_\omega - t)^+ = u_{\{u_\omega > t\}} \quad \text{in } H_0^1(\{u_\omega > t\}).$$

Thus, the density result Proposition 2.2.33 yields that

$$\left\{(u_\omega - t)^+\phi \;\middle|\; \phi \in C_0^\infty(\Omega)\right\} \subset H_0^1(\{u_\omega > t\}).$$

is dense. In the light of this density result and due to the characterization of γ-convergence via Mosco convergence in Theorem 2.2.36 it suffices now to find for fixed $\phi \in C_0^\infty(\Omega)$ a sequence $\phi_n \in H_0^1(\{u_\omega * \rho_n > t\} \cap \{u_\omega > t\})$ such that

$$\phi_n \to \phi(u_\omega - t)^+ \quad \text{in } H_0^1(\Omega), \tag{3.3.56}$$

for $n \to \infty$ in order to show the first Mosco condition (M1). The second Mosco condition (M2) is obviously fulfilled due to the inclusion

$$H_0^1(\{u_\omega * \rho_n > t\} \cap \{u_\omega > t\}) \subset H_0^1(\{u_\omega > t\}).$$

So in order to satisfy (M1), as proposed in [39], one chooses

$$\phi_n \coloneqq \phi \cdot \min\left\{(u_\omega * \rho_n - t)^+, (u_\omega - t)^+\right\} \in H_0^1(\{u_\omega * \rho_n > t\} \cap \{u_\omega > t\}).$$

The convergence (3.3.56) now easily follows from the convergence

$$u_\omega * \rho_n \to u_\omega \quad \text{in } H_0^1(\Omega),$$

for $n \to \infty$.

As the γ-convergence is stable under intersection with quasi-open sets, see Proposition 2.2.37, we conclude

$$\{u_\omega * \rho_n > t\} \cap \{u_\omega > t\} \cap \omega \xrightarrow{\gamma} \{u_\omega > t\} \cap \omega.$$

Now due to the continuity of eigenvalues with respect to γ-convergence, see Remark 3.3.3 we have

$$\lim_{n\to\infty} \lambda(\{u_\omega * \rho_n > t\} \cap \{u_\omega > t\} \cap \omega) = \lambda(\{u_\omega > t\} \cap \omega),$$

and thus in the light of (3.3.55),

$$\limsup_{n\to\infty} \lambda^{0,\varphi_{n,t}} \leq \lambda(\{u_\omega > t\} \cap \omega).$$

For the limit $t \to 0$ we again want to apply the density result Proposition 2.2.33 in order to show that for any zero sequence $(t_n)_{n\in\mathbb{N}} \subset (0,1)$ it holds

$$\{u_\omega > t_n\} \overset{\gamma}{\to} \{u_\omega > 0\}, \tag{3.3.57}$$

as $n \to \infty$. In the same line of reasoning as above, noting the fact that $\{u_\omega > t_n\} \subset \{u_\omega > 0\}$, it suffices to construct for an arbitrary $\phi \in C_0^\infty(\Omega)$ a sequence $\phi_n \in H_0^1(\{u_\omega > t_n\})$ such that

$$\phi_n \to \phi(u_\omega)^+ = \phi u_\omega \quad \text{in } H_0^1(\Omega),$$

for $n \to \infty$. Here the natural choice is

$$\phi_n \coloneqq \phi \cdot (u_\omega - t_n)^+ \in H_0^1(\{u_\omega > t_n\}).$$

This sequence converges to $\phi(u_\omega)^+$ due to the point-wise a.e. convergence of $(u_\omega - t_n)^+ \to (u_\omega)^+$ and the fact that

$$\nabla(u_\omega - t_n)^+ = \chi_{\{u_\omega > t_n\}} \nabla u_\omega \quad \text{a.e. in } \Omega$$

So finally from (3.3.57) we deduce

$$\lim_{n\to\infty} \lambda(\{u_\omega > t_n\} \cap \omega) = \lambda(\{u_\omega > 0\} \cap \omega) = \lambda(\{u_\omega > 0\}) = \lambda^{0,\varphi},$$

where the second equality is valid due to the inclusion $\{u_\omega > 0\} \subset \omega$.

Hence, by a diagonal sequence argument, we can now choose a zero sequence $(t_k)_{k\in\mathbb{N}} \subset (0,1)\backslash\mathcal{N}$ and a sequence of indices $(n_k)_{k\in\mathbb{N}} \in \mathbb{N}$ such that $E_k \coloneqq E_{n_k,t_k}$ fulfills the desired properties (3.3.49).

Step 2: Let $\varphi \in L^1(\Omega)$ be arbitrary. There exists a recovery sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset L^1(\Omega)$ with

$$\lim_{\varepsilon \searrow 0} \|\varphi_\varepsilon - \varphi\|_{L^1(\Omega)} = 0,$$

such that the lim sup *inequality*

$$\limsup_{\varepsilon \searrow 0} I^\varepsilon(\varphi_\varepsilon) \leq I^0(\varphi),$$

holds.

Without loss of generality, we assume $I^0(\varphi) < \infty$. We thus have $\varphi \in \Lambda_{\text{ad}}^0 \subseteq BV(\Omega, \{\pm 1\})$. Due to the previous step there exists a sequence of bounded smooth open sets $(E_k)_{k\in\mathbb{N}} \subset \mathbb{R}^d$ approximating $\tilde{E}^\varphi$ satisfying all the properties in (3.3.49). Now, the idea is to construct for each k a recovery sequence $(\varphi_{k,\varepsilon})_{\varepsilon>0} \subset \Lambda_{\text{ad}}$ for $\varphi_k \coloneqq 2\chi_{E_k^{\tilde{\Omega}} \cup S_1} - 1 \in \Lambda_{\text{ad}}^0$. Due to the properties of the set E_k we can proceed as in [96, Theorem 2] (which relies on the ideas of [37, 126, 147]). Note that we operate on the open subset $\tilde{\Omega} \subset \Omega$ where the point-wise

constraints incorporated in $\mathcal{U}$ do not play any role. In this way, for every $k \in \mathbb{N}$, we obtain a recovery sequence

$$(\varphi_{k,\varepsilon})_{\varepsilon>0} \subset \left\{ \varphi \in H^1(\tilde{\Omega}) \middle| \, |\varphi| \leq 1 \right\},$$

which satisfies

$$\limsup_{\varepsilon \searrow 0} \int_{\tilde{\Omega}} \frac{\gamma\varepsilon}{2} |\nabla \varphi_{k,\varepsilon}|^2 + \frac{\gamma}{\varepsilon} \psi(\varphi_{k,\varepsilon}) \, \mathrm{d}x \leq \gamma c_0 P_{\tilde{\Omega}}(\tilde{E}^{\varphi_k}), \tag{3.3.58}$$

and

$$\|\varphi_{k,\varepsilon} - \varphi_k\|_{L^1(\tilde{\Omega})} = \mathcal{O}(\varepsilon).$$

For any $\varepsilon > 0$, the function $\varphi_{k,\varepsilon}$ can be extended onto the whole design domain Ω by choosing $\varphi_{k,\varepsilon} := -1$ on S_0 and $\varphi_{k,\varepsilon} := 1$ on S_1. In particular, $\varphi_\varepsilon \in L^1(\Omega)$ for all $\varepsilon > 0$, and it holds that

$$\|\varphi_{k,\varepsilon} - \varphi_k\|_{L^1(\Omega)} = \|\varphi_{k,\varepsilon} - \varphi_k\|_{L^1(\tilde{\Omega})} = \mathcal{O}(\varepsilon). \tag{3.3.59}$$

It is worth mentioning that the constant hiding in $\mathcal{O}(\varepsilon)$ might strongly depend on k, see also Remark 3.3.12. Now, Theorem 3.3.7 implies that for $k \in \mathbb{N}$ and for each $m = 1, \ldots, l$ we have

$$\lambda_m^{\varphi_{k,\varepsilon}} \to \lambda_m^{0,\varphi_k} \quad \text{for } \varepsilon \to 0, \tag{3.3.60}$$

along a non-relabeled subsequence, where $\lambda_m^{\varphi_{k,\varepsilon}}$ and λ_m^{0,φ_k} denote the m-th eigenvalues of the diffuse interface problem (3.2.1) and the limit problem (3.3.2), respectively. Recalling that Ψ is continuous, we use (3.3.58) and (3.3.60) to conclude that

$$\limsup_{\varepsilon \searrow 0} I^\varepsilon(\varphi_{k,\varepsilon}) \leq I^0(\varphi_k).$$

By *Step 1*, we also know from the properties (3.3.49) and Assumption **(A8)** that

$$\limsup_{k \to \infty} I^0(\varphi_k) \leq I^0(\varphi).$$

Therefore, by a diagonal sequence argument, we find a zero sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that

$$\limsup_{k \to \infty} I^{\varepsilon_k}(\varphi_{k,\varepsilon_k}) \leq I^0(\varphi).$$

This proves *Step 2*.

Step 3: Let $\varphi \in L^1(\Omega)$ be arbitrary. For any sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset L^1(\Omega)$ with

$$\lim_{\varepsilon \searrow 0} \|\varphi_\varepsilon - \varphi\|_{L^1(\Omega)} = 0,$$

it holds that

$$I^0(\varphi) \leq \liminf_{\varepsilon \searrow 0} I^\varepsilon(\varphi_\varepsilon).$$

This is also shown in the setting of [39] using the compactness of the γ-convergence (see e.g., [49, Prop. 4.3.7]). Here, we provide an alternative proof which does not rely on γ-convergence but directly uses the Courant–Fischer characterization of eigenvalues.

Without loss of generality, we may assume

$$\liminf_{\varepsilon \searrow 0} I^\varepsilon(\varphi_\varepsilon) < +\infty. \tag{3.3.61}$$

Moreover, after extracting a suitable subsequence, we have

$$\lim_{\varepsilon \searrow 0} I^\varepsilon(\varphi_\varepsilon) = \liminf_{\varepsilon \searrow 0} I^\varepsilon(\varphi_\varepsilon) < +\infty. \tag{3.3.62}$$

Applying [37, Prop. 3.8], we conclude again that $\varphi \in \Lambda^0_{\mathrm{ad}}$ and that

$$\gamma c_0 P_{\tilde{\Omega}}(\tilde{E}^\varphi) \leq \liminf_{\varepsilon \searrow 0} \int_{\tilde{\Omega}} \left(\frac{\gamma\varepsilon}{2} |\nabla \varphi_\varepsilon|^2 + \frac{\gamma}{\varepsilon} \psi(\varphi_\varepsilon) \right) \mathrm{d}x. \tag{3.3.63}$$

Therefore, for $n = i_j \in \mathbb{N}$ with $j = 1, \dots, l$ recall the Courant–Fischer characterization of the diffuse interface problem (3.2.3) for $\varepsilon > 0$, that is

$$\lambda_n^{\varepsilon,\varphi} = \max_{W \in \mathcal{S}_{n-1}} \min \left\{ \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_\Omega b_\varepsilon(\varphi)\, |v|^2 \,\mathrm{d}x}{\int_\Omega |v|^2 \,\mathrm{d}x} \middle| \begin{array}{l} v \in H^1_0(\Omega) \cap W^\perp, \\ v \neq 0 \end{array} \right\}, \tag{3.3.64}$$

and the Courant–Fischer characterization for the sharp-interface problem (3.3.3), that is

$$\lambda_n^{0,\varphi} = \max_{W \in \mathcal{S}_{n-1}} \min \left\{ \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega |v|^2 \,\mathrm{d}x} \middle| \begin{array}{l} v \in V^\varphi \cap W^\perp, \\ v \neq 0 \end{array} \right\}, \tag{3.3.65}$$

where in both cases the maximum is taken over all $(n-1)$-dimensional subspaces of $L^2(\Omega)$.

Now, our goal is to show that

$$\lambda_n^{0,\varphi} \leq \liminf_{\varepsilon \searrow 0} \lambda_n^{\varepsilon,\varphi_\varepsilon} \in \mathbb{R}_{>0}, \tag{3.3.66}$$

since then Lemma 3.3.8 implies that

$$\Psi(\lambda_{i_1}^{0,\varphi}, \dots, \lambda_{i_l}^{0,\varphi}) \leq \liminf_{\varepsilon \searrow 0} \Psi(\lambda_{i_1}^{\varepsilon_k,\varphi_\varepsilon}, \dots, \lambda_{i_l}^{\varepsilon_k,\varphi_\varepsilon}),$$

and along with (3.3.62) and (3.3.63), this proves the assertion of *Step 3*.

First of all, Theorem 3.3.2 yields that the maximum in (3.3.65) is attained in the space $W \coloneqq \langle w_1^{0,\varphi}, \dots, w_{n-1}^{0,\varphi} \rangle_{\mathrm{span}} \subset L^2(\Omega)$, where $w_1^{0,\varphi}, \dots, w_{n-1}^{0,\varphi} \in V^\varphi$ are the first $n-1$ eigenfunctions of the limit problem (3.3.2). Hence, we can reformulate the Courant–Fischer characterization as

$$\lambda_n^{0,\varphi} = \min \left\{ \int_\Omega |\nabla v|^2 \,\mathrm{d}x \middle| \begin{array}{l} v \in V^\varphi \cap W^\perp, \\ \|v\|_{L^2(\Omega)} = 1 \end{array} \right\}. \tag{3.3.67}$$

Since $W \subset L^2(\Omega)$ is a $(n-1)$-dimensional subspace, we infer from (3.3.64) that

$$\lambda_n^{\varepsilon,\varphi_\varepsilon} \geq \min \left\{ \int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_\Omega b_\varepsilon(\varphi_\varepsilon)\, |v|^2 \,\mathrm{d}x \middle| \begin{array}{l} v \in H^1_0(\Omega) \cap W^\perp, \\ \|v\|_{L^2(\Omega)} = 1 \end{array} \right\}. \tag{3.3.68}$$

By means of the direct method in the calculus of variations it is straightforward to show that for any fixed $\varepsilon > 0$, there exists a $L^2(\Omega)$-normalized function $v_\varepsilon \in H^1_0(\Omega) \cap W^\perp$ at which the minimum in (3.3.68) is attained.

Next, from (3.3.62) we deduce that the sequence $\left(\Psi(\lambda_{i_1}^{\varepsilon,\varphi_\varepsilon}, \dots, \lambda_{i_l}^{\varepsilon,\varphi_\varepsilon})\right)_{\varepsilon>0}$ is bounded. Hence, using condition (3.3.46) from Assumption **(A8)**, we conclude that the sequence $(\lambda_n^{\varepsilon,\varphi_\varepsilon})_{\varepsilon>0}$ is also bounded and in particular the limes inferior of this sequence exists.

Now, using (3.3.68), we infer that $(\|\nabla v_\varepsilon\|_{L^2(\Omega)})_{\varepsilon>0}$ is bounded and hence, we find a function $\overline{v} \in H^1_0(\Omega)$ such that the convergences

$$v_\varepsilon \rightharpoonup \overline{v} \quad \text{in } H^1_0(\Omega), \qquad v_\varepsilon \to \overline{v} \quad \text{in } L^2(\Omega), \qquad v_\varepsilon \to \overline{v} \quad \text{a.e. in } \Omega \tag{3.3.69}$$

hold along a non-relabeled subsequence. Moreover, we have

$$b_\varepsilon(\varphi_\varepsilon(x)) \to b_0(\varphi(x)) \quad \text{as } \varepsilon \to 0 \text{ for almost all } x \in \Omega,$$

up to subsequence extraction. This convergence was shown in *Step 1* of the proof of Lemma 3.3.5 and its proof did not require any rate assumption on $(\varphi_\varepsilon)_{\varepsilon>0}$. We thus obtain

$$\begin{aligned}\liminf_{\varepsilon \searrow 0} \lambda_n^{\varepsilon,\varphi_\varepsilon} &\geq \liminf_{\varepsilon \searrow 0} \left[\int_\Omega |\nabla v_\varepsilon|^2 \,\mathrm{d}x + \int_\Omega b_\varepsilon(\varphi_\varepsilon)\, |v_\varepsilon|^2 \,\mathrm{d}x\right] \\ &\geq \int_\Omega |\nabla \overline{v}|^2 \,\mathrm{d}x + \int_\Omega b_0(\varphi)\, |\overline{v}|^2 \,\mathrm{d}x,\end{aligned}$$

by applying Fatou's lemma on the b_ε term, and employing the weak lower semi-continuity of $\|\nabla \cdot\|_{L^2(\Omega)}$. In particular, this implies that

$$\int_\Omega b_0(\varphi)\, |\overline{v}|^2 \,\mathrm{d}x < +\infty.$$

Hence, recalling that $\varphi \in \Phi^0_{\mathrm{ad}}$, we conclude that $\overline{v} \in V^\varphi$ which in turn implies

$$\int_\Omega b_0(\varphi)\, |\overline{v}|^2 \,\mathrm{d}x = 0.$$

Now, by construction, we have $\|v_\varepsilon\|_{L^2(\Omega)} = 1$ and $v_\varepsilon \in H^1_0(\Omega) \cap W^\perp$. Due to the convergences in (3.3.69) the same holds for the limit $\overline{v} \in V^\varphi$. This means that $\overline{v}$ belongs to the set appearing in (3.3.67), and we finally deduce

$$\liminf_{\varepsilon \searrow 0} \lambda_n^{\varepsilon,\varphi_\varepsilon} \geq \int_\Omega |\nabla \overline{v}|^2 \,\mathrm{d}x \geq \min\left\{ \int_\Omega |\nabla v|^2 \,\mathrm{d}x \,\middle|\, \begin{matrix} v \in V^\varphi \cap W^\perp, \\ \|v\|_{L^2(\Omega)} = 1 \end{matrix} \right\} = \lambda_n^{0,\varphi} > 0,$$

which proves (3.3.66). This means that *Step 3* is established and thus, the proof of Theorem 3.3.11 is complete. □

Remark 3.3.12. Let us comment more on the rate condition (3.3.59) given as

$$\|\varphi_{k,\varepsilon} - \varphi_k\|_{L^1(\Omega)} = \|\varphi_{k,\varepsilon} - \varphi_k\|_{L^1(\tilde{\Omega})} = \mathcal{O}(\varepsilon), \tag{3.3.70}$$

which is crucially used in order to pass the eigenvalues to the sharp-interface limit. Written more explicitly, we know that for any $k \in \mathbb{N}$ there is a constant $C_k > 0$ and a $\delta_k > 0$ such that

$$\|\varphi_{k,\varepsilon} - \varphi_k\|_{L^1(\Omega)} \leq C_k \varepsilon, \tag{3.3.71}$$

for all $\varepsilon \in (0, \delta_k)$, which is enough for the previous proof. We can now sharpen this estimate by showing that the constant C_k is independent of $k \in \mathbb{N}$ as follows. The essential contribution to this constant is a term of the form

$$\sup_{|s| \leq \sqrt{\varepsilon}} \Big(\mathcal{H}^{d-1}(\{d_{\Gamma_k} = s\} \cap \Omega) \Big),$$

where d_{Γ_k} denotes the signed distance to the boundary of the smooth set E_k, see e.g. [37, Proof of Proposition 3.11]. We will also perform these explicit computations in the alternative proof of Theorem 4.3.17. As we will also see there, *for fixed* k one can show that

$$\lim_{\varepsilon \searrow 0} \sup_{|s| \leq \sqrt{\varepsilon}} \Big(\mathcal{H}^{d-1}(\{d_{\Gamma_k} = s\} \cap \Omega) \Big) = \mathcal{H}^{d-1}(\{d_{\Gamma_k} = 0\} \cap \Omega) = P_\Omega(E_k).$$

Thus, choosing for any $k \in \mathbb{N}$ an $\varepsilon_k > 0$ small enough, we deduce by a diagonal sequence argument, exploiting $P_\Omega(E_k) \to P_\Omega(E)$ for $k \to \infty$ (see (3.3.49)), that

$$\sup_{|s| \leq \sqrt{\varepsilon}} \mathcal{H}^{d-1}(\{d_{\Gamma_k} = s\} \cap \Omega) \leq P_\Omega(E) + 1 =: C,$$

Thus we sharpen (3.3.71) to

$$\|\varphi_{k,\varepsilon_k} - \varphi_k\|_{L^1(\Omega)} \leq C \varepsilon_k,$$

where now the constant C is *independent* of k. Nevertheless, this is not enough to deduce

$$\|\varphi_{k,\varepsilon_k} - \varphi\|_{L^1(\Omega)} \leq C \varepsilon_k, \tag{3.3.72}$$

because we only know

$$\lim_{k \to \infty} \|\varphi_k - \varphi\|_{L^1(\Omega)} = 0,$$

from *Step 1* of the previous proof. Of course for any given $\varepsilon > 0$ we can choose $k_\varepsilon \in \mathbb{N}$ so small that

$$\|\varphi_{k_\varepsilon} - \varphi\|_{L^1(\Omega)} \leq \varepsilon,$$

but we have already chosen ε_k in dependence of k, so this results in a vicious circle! On the other hand if we would not choose ε_k in dependence of k we loose the uniform constant C above.

The problem in the Γ-convergence proof [96, Step 1 of proof of Theorem 2] is that the authors also rely on the rate (3.3.72), and not as stated there on the rate $\mathcal{O}(k^{-1})$, in order to apply [96, Lemma 3]. In order to see if we can correct this mistake we recall that in our proof of Theorem 3.3.11 we were obliged in *Step 1* to perform an analysis of the *sharp-interface problem*, especially of the continuity properties of the quantities appearing in

our optimization problem before entering the proof of the diffuse to sharp-interface limit, for which then the rate for fixed $k \in \mathbb{N}$ in (3.3.70) is indeed enough.

Unfortunately we are not able to correct the proof of [96, Theorem 2]. This is due to the fact that the sharp-interface analysis in *Step 1* of the proof of Theorem 3.3.11 heavily relies on the monotonicity of eigenvalues with respect to set inclusion, in particular the estimate (3.3.55). Even if we include some additional monotonicity assumption we can not make the proof work with the tools we have at hand, due to the additional impressibility condition hidden in the underlying space. Divergence free spaces are known to be difficult to handle with γ-convergence, see [49, Remark 4.10.5]. Let us see why the impressibility does not allow us to perform the γ-convergence approach.

Let us first introduce the setting of [97]. The sharp-interface cost functional in [96] reads as

$$\begin{aligned} J(\varphi, \boldsymbol{u}) &= \int_\Omega h(x, \boldsymbol{u}, D\boldsymbol{u}) \,\mathrm{d}x + \gamma c_0 P_\Omega(E^\varphi), \\ \varphi \in \Phi^0_{\mathrm{ad}} &:= \left\{ \varphi \in BV(\Omega; \{\pm 1\}) \;\middle|\; \int_\Omega \varphi \,\mathrm{d}x \leq \beta \,|\Omega| \right\} \end{aligned}$$

with $\beta \in (0,1)$, where the function

$$\begin{aligned} H^1(\Omega; \mathbb{R}^d) &\to \mathbb{R}, \\ \boldsymbol{u} &\mapsto \int_\Omega h(x, \boldsymbol{u}, D\boldsymbol{u}) \,\mathrm{d}x, \end{aligned} \tag{3.3.73}$$

is continuous. Furthermore in the minimization of J, the state $\boldsymbol{u} \in \boldsymbol{V}^\varphi$ is always chosen to be the unique solution of the state equation

$$\mu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{\eta} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\eta} \,\mathrm{d}x, \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{V}^\varphi, \tag{3.3.74}$$

where

$$\begin{aligned} \boldsymbol{V}^\varphi &:= \{ \boldsymbol{\eta} \in \boldsymbol{V} \mid \boldsymbol{\eta} = 0 \quad \text{a.e. in } \Omega \backslash E^\varphi \}, \\ \boldsymbol{V} &:= \left\{ \boldsymbol{\eta} \in H^1_0(\Omega; \mathbb{R}^d) \;\middle|\; \mathrm{div} \boldsymbol{\eta} = 0 \quad \text{a.e. in } \Omega \right\}. \end{aligned}$$

This solution $\boldsymbol{u}$ is proven to exist for any $\varphi \in L^1(\Omega)$ in [96, Lemma 2]. In that case, using the same notation as there, we denote $S_0(\varphi) := \boldsymbol{u}$. This allows us to introduce the reduced cost functional

$$j_0(\varphi) := \begin{cases} J_0(\varphi, S_0(\varphi)) & \text{if } \varphi \in \Phi^0_{\mathrm{ad}}, \\ +\infty & \text{otherwise,} \end{cases}$$

and thus to consider the minimization problem

$$\min_{\varphi \in L^1(\Omega)} j_0(\varphi) = \min_{\varphi \in L^1(\Omega)} \int_\Omega h(x, S_0(\varphi), D(S_0(\varphi))) \,\mathrm{d}x + \gamma c_0 P_\Omega(E^\varphi).$$

Note that in order to adapt the theory of [96] to our setting we only allow for homogeneous boundary data on $\partial\Omega$, or in the notation of [96] $\boldsymbol{g} = \boldsymbol{0}$, which in particular implies that we do not need to distinguish, as done there, between the spaces $\boldsymbol{V}^\varphi$ and $\boldsymbol{U}^\varphi$. Furthermore let us note that the impressibility property $\mathrm{div}\boldsymbol{\eta} = 0$ will play a crucial role in the following.

As indicated above, additionally to the assumptions made in [96] let us make the assumption:

(Mon) *Let $\omega_2 \subset \omega_1 \subset \Omega$ be quasi-open sets. Then it shall hold*

$$\int_\Omega h(x, S(\omega_1), D(S(\omega_1)))\,\mathrm{d}x \leq \int_\Omega h(x, S(\omega_2), D(S(\omega_2)))\,\mathrm{d}x.$$

Note that in the same line of reasoning as in Remark 3.3.3 we define here for any quasi-open set $\omega \subset \Omega$ the function $S(\omega) \in H^1(\omega;\mathbb{R}^d) \cap \boldsymbol{V} =: \boldsymbol{V}^\omega$ to be the unique solution of

$$\mu \int_\Omega \nabla S(\omega) : \nabla \boldsymbol{\eta}\,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\eta}\,\mathrm{d}x, \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{V}^\omega.$$

Let us remark, that here the term $\int_\Omega h(x, S(\varphi), D(S(\varphi)))\,\mathrm{d}x$ plays the same role as a limit eigenvalue $\lambda^{0,\varphi}$ and $\int_\Omega h(x, S(\omega), D(S(\omega)))\,\mathrm{d}x$ plays the role of a "classical" eigenvalue $\lambda(\omega)$.

As an example of h which satisfies (Mon) we could choose h to be the total potential power

$$h(x, \boldsymbol{u}, D\boldsymbol{u}) = \frac{\mu}{2}\,|D\boldsymbol{u}|^2 - \boldsymbol{f}(x) \cdot \boldsymbol{u},$$

as this definition allows for the variational characterization

$$\int_\Omega h(x, S(\omega), D(S(\omega)))\,\mathrm{d}x = \min\left\{ \frac{\mu}{2} \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v}\,\mathrm{d}x - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}x \;\middle|\; \boldsymbol{v} \in H^1_0(\omega;\mathbb{R}^d) \cap \boldsymbol{V} \right\},$$

and $H^1_0(\omega_2;\mathbb{R}^d) \subset H^1_0(\omega_1;\mathbb{R}^d)$ for $\omega_2 \subset \omega_1$.

Now let us try to apply the strategy of *Step 1* of the proof of Theorem 3.3.11 to the above setting. Denoting $\mathcal{A}(\Omega)$ the class of quasi-open sets in Ω, let us introduce

$$\begin{aligned} &\mathcal{A}(\Omega) \to \mathbb{R}, \\ &\qquad \omega \mapsto \int_\Omega h(x, S(\omega), D(S(\omega)))\,\mathrm{d}x. \end{aligned} \tag{3.3.75}$$

Note that now also in the vector valued case, by the theory introduced in Section 2.2.6, we find for any measurable set $E \subset \Omega$ a unique quasi-open set $\omega \subset E$ such that

$$\tilde{H}^1_0(E;\mathbb{R}^d) = H^1_0(\omega;\mathbb{R}^d) = H^1_0(\{u_\omega > 0\};\mathbb{R}^d) = \tilde{H}^1_0(\{u_\omega > 0\};\mathbb{R}^d), \tag{3.3.76}$$

as a function belongs to a vector valued Sobolev space if and only if each component belongs the according scalar valued Sobolev space. In particular it holds

$$S(\omega) = S(\chi_E) \in H^1_0(\omega),$$

as $\boldsymbol{V}^\omega = H^1_0(\omega;\mathbb{R}^d) \cap \boldsymbol{V} = \tilde{H}^1_0(E;\mathbb{R}^d) \cap \boldsymbol{V} = \boldsymbol{V}^{\chi_E}$.

Thus, if we could show that the function (3.3.75) is continuous with respect to γ-convergence, we can apply *Step 1* of the proof of Theorem 3.3.11 in complete analogy to the cost functional j_0 which would fix then the mistake in [96, Step 1 of proof of Theorem 2], as then the rate (3.3.70) is sufficient to perform a diagonal sequence argument as in our proof of Theorem 3.3.11 to conclude the lim sup inequality.

Now let us see why this continuity is not obvious. In order to discuss the desired continuity, the characterization of γ-convergence via Mosco convergence from Theorem 2.2.36 will be helpful. The following attempted proof follows a classical line of reasoning, see [49, Proposition 4.5.3], but we will see where the impressibility property makes things complicated. So let $(\omega_n)_{n\in\mathbb{N}}, \omega \subset \Omega$ be quasi-open sets with $\omega_n \xrightarrow{\gamma} \omega$. First of all we know that for $n \in \mathbb{N}$ the function $\boldsymbol{v}_{\omega_n} \coloneqq S(\omega_n) \in \boldsymbol{V}^{\omega_n}$ satisfies

$$\mu \int_\Omega \nabla \boldsymbol{v}_{\omega_n} : \nabla \boldsymbol{\eta} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\eta} \,\mathrm{d}x, \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{V}^{\omega_n}, \tag{3.3.77}$$

and thus in particular $(\boldsymbol{v}_{\omega_n})_{n\in\mathbb{N}} \subset \boldsymbol{V}$ is bounded. Therefore we can extract a non-relabeled subsequence such that

$$\boldsymbol{v}_{\omega_n} \rightharpoonup \tilde{\boldsymbol{v}} \quad \text{in } H^1_0(\Omega;\mathbb{R}^d), \tag{3.3.78}$$

for $n \to \infty$ and a suitable $\tilde{\boldsymbol{v}} \in \boldsymbol{V}$, because $\boldsymbol{V} \subset H^1_0(\Omega;\mathbb{R}^d)$ is a closed subspace. The second Mosco-condition (M2) exactly tells us that $\tilde{\boldsymbol{v}} \in H^1_0(\omega;\mathbb{R}^d)$, hence $\tilde{\boldsymbol{v}} \in \boldsymbol{V}^\omega$.
Now fix an arbitrary test function of the limit problem $\boldsymbol{\eta} \in V^\omega$. We know due to the first Mosco-condition (M1) that there is a recovery sequence $\boldsymbol{\eta}_n \in H^1_0(\omega_n;\mathbb{R}^d)$ such that

$$\boldsymbol{\eta}_n \to \boldsymbol{\eta} \quad \text{in } H^1_0(\Omega;\mathbb{R}^d), \tag{3.3.79}$$

for $n \to \infty$. Now we have to be cautious because $\boldsymbol{\eta}_n$ is not yet an admissible test function in (3.3.77), as it is *not divergence free*. For this purpose, for $n \in \mathbb{N}$ let us denote the orthogonal projection with

$$P_n : H^1_0(\omega_n;\mathbb{R}^d) \to \boldsymbol{V}^{\omega_n}.$$

Note that the underlying scalar product is $(\nabla\cdot, \nabla\cdot)_{L^2(\Omega)}$ as $\boldsymbol{V}^{\omega_n} \subset H^1_0(\omega_n;\mathbb{R}^d) \subset H^1_0(\Omega)$ are closed subspaces of one another. Now consider the left hand side of equation (3.3.77) tested with $P_n(\boldsymbol{\eta}_n) \in \boldsymbol{V}^{\omega_n}$

$$\mu \int_\Omega \nabla \boldsymbol{v}_{\omega_n} : \nabla P_n(\boldsymbol{\eta}_n) \,\mathrm{d}x = \mu \int_\Omega \nabla \boldsymbol{v}_{\omega_n} : \nabla \boldsymbol{\eta}_n \,\mathrm{d}x \to \mu \int_\Omega \nabla \tilde{\boldsymbol{v}} : \nabla \boldsymbol{\eta} \,\mathrm{d}x,$$

for $n \to \infty$. In the equality we used the fact that $\boldsymbol{v}_{\omega_n} \in \boldsymbol{V}^{\omega_n}$ and thus the projection P_n can be dropped. Furthermore we used the convergence (3.3.78) combined with (3.3.79).
Now the problem for the right hand side of (3.3.77) is that we can not just drop the projection, as $\boldsymbol{f}$ only belongs to $L^2(\Omega;\mathbb{R}^d)$ and is not even divergence free in the weak sense. Even if it would be, the L^2 scalar product is not the one associated to the projection P_n.
On the other hand one only knows that there is *some* $\tilde{\boldsymbol{\eta}} \in \boldsymbol{V}^\omega$ such that

$$P_n(\boldsymbol{\eta}_n) \rightharpoonup \tilde{\boldsymbol{\eta}} \quad \text{in } H^1_0(\Omega),$$

by exploiting (M2). Thus, we can only deduce the limit of (3.3.77) to be

$$\mu \int_\Omega \nabla \tilde{\boldsymbol{v}} : \nabla \boldsymbol{\eta} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \tilde{\boldsymbol{\eta}} \,\mathrm{d}x.$$

So as long as we do not know that $\tilde{\boldsymbol{\eta}} = \boldsymbol{\eta}$ the proof can not be completed. If we knew $\tilde{\boldsymbol{\eta}} = \boldsymbol{\eta}$ we would be done, because then $\tilde{\boldsymbol{v}}$ satisfies the desired equation and thus, due to uniqueness we arrive at $\tilde{\boldsymbol{v}} = \boldsymbol{v}_\omega$. Then by standard arguments one could sharpen the weak convergence in (3.3.78) to strong convergence. Then by the H^1-continuity of (3.3.73) this would give the desired continuity in (3.3.75).

Let us return to proving the sharp-interface limit of our spectral optimization problem. Using Theorem 3.3.11, we can finally prove the desired Γ-convergence result $J^\varepsilon \overset{\Gamma}{\to} J^0$ where the volume constraint is now incorporated.

Proof of Theorem 3.3.9. Due to the inclusion $\Phi_{\mathrm{ad}} \subset \Lambda_{\mathrm{ad}}$ we have $I^\varepsilon(\varphi) \leq J^\varepsilon(\varphi)$ for all $\varphi \in L^1(\Omega)$. Furthermore for a sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset \Phi_{\mathrm{ad}}$ with $\varphi_\varepsilon \to \varphi$ in $L^1(\Omega)$ we deduce $\varphi \in \tilde{\mathcal{G}}^\beta$ and therefore $I^0(\varphi) = J^0(\varphi)$. Hence, the lim inf inequality for J^ε directly follows from Theorem 3.3.11.

It remains to prove that for any $\varphi \in \Phi^0_{\mathrm{ad}}$, there exists a recovery sequence $(\tilde{\varphi}_\varepsilon)_{\varepsilon>0} \subset \Phi_{\mathrm{ad}}$ such that

$$\lim_{\varepsilon \searrow 0} \|\tilde{\varphi}_\varepsilon - \varphi\|_{L^1(\Omega)} = 0, \tag{3.3.80}$$

$$\limsup_{\varepsilon \searrow 0} J^\varepsilon(\tilde{\varphi}_\varepsilon) \leq J^0(\varphi). \tag{3.3.81}$$

Here our strategy is now to use the recovery sequence from Theorem 3.3.11. In the following, it will be denoted by $(\varphi_\varepsilon)_{\varepsilon>0} \subset \Lambda_{\mathrm{ad}}$. For any $\varepsilon > 0$, we now carefully modify the function φ_ε via a diffeomorphism in order to ensure that it additionally fulfills the volume constraint comprised in $\tilde{\mathcal{G}}^\beta$. In the following, we will always understand the functions $\varphi_\varepsilon, \varphi \in L^1(\mathbb{R}^d)$ as being trivially extended onto $\mathbb{R}^d$, i.e., these functions are constant zero on $\mathbb{R}^d \backslash \Omega$.

The key idea is to construct for any $\varepsilon > 0$ a suitable transformation $T_{s(\varepsilon)} : \mathbb{R}^d \to \mathbb{R}^d$ with $T_{s(\varepsilon)}(\tilde{\Omega}) = \tilde{\Omega}$ such that the modified functions $\tilde{\varphi}_\varepsilon := \varphi_\varepsilon(T^{-1}_{s(\varepsilon)})$ belong to Φ_{ad} and satisfy the convergence properties (3.3.80) and (3.3.81). This is a common method in geometric analysis and a similar procedure in the sharp-interface case can be found for example in the proof of [121, Theorem 19.8].

We now fix an arbitrary function $\varphi \in \Phi^0_{\mathrm{ad}}$. First of all, we find a vector field $\boldsymbol{\xi} \in C^1_0(\tilde{\Omega}; \mathbb{R}^d)$ such that

$$\int_{\tilde{\Omega}} \varphi \nabla \cdot \boldsymbol{\xi} \, \mathrm{d}x > 0,$$

as otherwise the total variation of the associated Radon measure would vanish, i.e., $|D\varphi|(\tilde{\Omega}) = 0$, which would imply that φ is constant almost everywhere in $\tilde{\Omega}$. However, this is not possible as neither $\varphi \equiv 1$ nor $\varphi \equiv -1$ in $\tilde{\Omega}$ would fulfill the mean value constraint in $\tilde{\mathcal{G}}^\beta$ due to the choice of β_1, β_2 in (2.1.2).

Using the vector field $\boldsymbol{\xi}$, we now define a family of transformations

$$T_s : \mathbb{R}^d \to \mathbb{R}^d, \quad x \mapsto x + s\boldsymbol{\xi}(x),$$

for $s \in \mathbb{R}$. As this map is a perturbation of the identity via a C^1-map with compact support in $\tilde{\Omega}$, it is clear that, for $|s|$ sufficiently small, T_s is a C^1-diffeomorphism with $T_s(\tilde{\Omega}) = \tilde{\Omega}$. Hence,

$$T_s|_{\tilde{\Omega}} : \tilde{\Omega} \to \tilde{\Omega}$$

is also a C^1-diffeomorphism. Moreover, the chain rule for Sobolev functions (see e.g., [10, 4.26]) implies

$$\varphi_\varepsilon \circ T_s^{-1} \in H^1(\tilde{\Omega}).$$

Since $\varphi_\varepsilon \in \mathcal{U}$ and $T_s|_{\mathbb{R}^d\setminus\tilde{\Omega}} = \mathrm{id}_{\mathbb{R}^d\setminus\tilde{\Omega}}$, we infer

$$\int_{\mathbb{R}^d} \varphi_\varepsilon \circ T_s^{-1}\,\mathrm{d}x = \int_{\tilde{\Omega}} \varphi_\varepsilon \circ T_s^{-1}\,\mathrm{d}x + |S_1| - |S_0|\,,$$

due to the trivial extension of φ_ε on $\mathbb{R}^d\setminus\Omega$. Moreover, we use the representation

$$\int_{\mathbb{R}^d} \varphi_\varepsilon \circ T_s^{-1}\,\mathrm{d}x = \int_{\mathbb{R}^d} \varphi_\varepsilon\,|\det DT_s|\;\mathrm{d}x. \tag{3.3.82}$$

Recalling that the determinant is a multi-linear form, a straightforward computation reveals that there is a $\delta > 0$ and a constant $C > 0$ depending only on δ and $\boldsymbol{\xi}$ such that for all $x \in \mathbb{R}^d$ and $s \in (-\delta, \delta)$,

$$\frac{1}{2} \leq 1 - C\,|s| \leq \det DT_s(x) \leq 1 + C\,|s|\,. \tag{3.3.83}$$

In the following, we use the convenient notation $\varphi_0 := \varphi$. We now define the function

$$f : B_\delta(0) \subset \mathbb{R}^2 \to \mathbb{R}, \quad (\varepsilon, s) \mapsto \int_{\mathbb{R}^d} \varphi_{|\varepsilon|} \circ T_s^{-1}\,\mathrm{d}x - \int_{\tilde{\Omega}} \varphi\,\mathrm{d}x - |S_1| + |S_0|\,,$$

where $0 < \delta < 1$ is chosen sufficiently small in order to ensure that $T_s|_{\tilde{\Omega}} : \tilde{\Omega} \to \tilde{\Omega}$ is a C^1-diffeomorphism and (3.3.83) holds for all $s \in (-\delta, \delta)$.

Now our next goal is to apply the implicit function theorem formulated in [159, Theorem 4.B] to the equation $f(\varepsilon, s) = 0$. First of all, $f(0,0) = 0$ is clear since $\varphi \in \Phi^0_{\mathrm{ad}}$. We next prove that f is continuous at $(0,0)$. To this end, let us choose zero sequences $(\varepsilon_k)_{k\in\mathbb{N}}, (s_k)_{k\in\mathbb{N}}$. Due to the symmetry of f with respect to its first argument, we may assume without loss of generality that $\varepsilon_k \geq 0$. As per construction, we find a non-relabeled subsequence $(\varphi_{\varepsilon_k})_{k\in\mathbb{N}}$ which converges to φ almost everywhere as $k \to \infty$, and since $\|\varphi_{\varepsilon_k}\|_{L^\infty(\mathbb{R}^n)} \leq 1$, we apply Lebesgue's dominated convergence theorem to the right-hand side of (3.3.82). Along with (3.3.83), we deduce

$$f(\varepsilon_k, s_k) \to f(0,0) \quad \text{for } k \to \infty.$$

As this limit does not depend on the choice of the subsequence, this argument can be repeated for any subsequence, thus, continuity of f at $(0,0)$ is shown.

In order to apply the implicit function theorem it remains to show that $\frac{\partial}{\partial s} f$ exists on $B_\delta(0)$, is continuous at $(0,0)$ and does not vanish at $(0,0)$. First of all, due to (3.3.83), the modulus in (3.3.82) can be omitted. Furthermore, the proof of [143, Lemma 1] implies that for fixed $x \in \mathbb{R}^d$ and $t \in (-\delta, \delta)$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\left[\det DT_s\right]_{|s=t} &= \mathrm{tr}\left(\frac{\mathrm{d}}{\mathrm{d}s}\left[DT_s\right]_{|s=t}(DT_t)^{-1}\right)\det DT_t \\
&= \mathrm{tr}\left(\nabla\boldsymbol{\xi}(\mathrm{Id} + t\nabla\boldsymbol{\xi})^{-1}\right)\det(\mathrm{Id} + t\nabla\boldsymbol{\xi}) \\
&= \mathrm{tr}\left(\nabla\boldsymbol{\xi}\sum_{k=0}^{\infty}(-t\nabla\boldsymbol{\xi})^k\right)\det(\mathrm{Id} + t\nabla\boldsymbol{\xi}).
\end{aligned} \tag{3.3.84}$$

As $\boldsymbol{\xi} \in C^1_0(\tilde{\Omega}, \mathbb{R}^d)$, we directly see that

$$\sup_{t\in(-\delta,\delta)} \left\| \frac{\mathrm{d}}{\mathrm{d}s}\left[\det DT_s\right]_{|s=t} \right\|_{C^0(\tilde{\Omega})} < \infty.$$

Noticing again that $\|\varphi_\varepsilon\|_{L^\infty(\Omega)} \leq 1$ for all $\varepsilon \geq 0$, we deduce via Lebesgue's dominated convergence theorem that for any $(\varepsilon, t) \in B_\delta(0)$, we have

$$\frac{\partial f}{\partial s}(\varepsilon, t) = \int_{\mathbb{R}^n} \varphi_\varepsilon \operatorname{tr}\left(\nabla \boldsymbol{\xi} \sum_{k=0}^{\infty} (-t\nabla \boldsymbol{\xi})^k\right) \det(\mathrm{Id} + t\nabla \boldsymbol{\xi}) \, \mathrm{d}x.$$

Therefore we directly infer

$$\frac{\partial f}{\partial s}(0,0) = \int_{\mathbb{R}^n} \varphi_0 \nabla \cdot \boldsymbol{\xi} \, \mathrm{d}x = \int_{\tilde{\Omega}} \varphi \nabla \cdot \boldsymbol{\xi} \, \mathrm{d}x > 0,$$

by our choice of $\boldsymbol{\xi} \in C_0^1(\tilde{\Omega}, \mathbb{R}^n)$ at the beginning of this proof. Now, the continuity of $\frac{\partial f}{\partial s}$ at $(0,0)$ follows again via Lebesgue's dominated convergence theorem using that for any $x \in \mathbb{R}^d$,

$$\operatorname{tr}\left(\nabla \boldsymbol{\xi} \sum_{k=0}^{\infty} (-t\nabla \boldsymbol{\xi})^k\right) \det(\mathrm{Id} + t\nabla \boldsymbol{\xi}).$$

is continuous at $t = 0$.

Now that we have checked all the assumptions of the implicit function theorem [159, Theorem 4.B], we deduce the existence of a $\tilde{\delta} > 0$ and a function

$$s : (-\tilde{\delta}, \tilde{\delta}) \to (-\delta, \delta),$$

which is continuous at 0, such that

$$\begin{aligned} f(\varepsilon, s(\varepsilon)) = 0 \quad &\text{for all } \varepsilon \in (-\tilde{\delta}, \tilde{\delta}) \text{ and} \\ s(0) = 0. \end{aligned}$$

In our framework, this means that having started with the recovery sequence $(\varphi_\varepsilon)_{\varepsilon>0}$ of Theorem 3.3.11, we now know

$$\tilde{\varphi}_\varepsilon \coloneqq \varphi_\varepsilon \circ T_{s(\varepsilon)}^{-1} \in \Phi_{\mathrm{ad}},$$

i.e., we have constructed an admissible sequence. Note that the point-wise constraints $\tilde{\varphi}_\varepsilon = 1$ a.e. in S_1 and $\tilde{\varphi}_\varepsilon = -1$ a.e. in S_0 are fulfilled since $T_s|_{\mathbb{R}^d \setminus \tilde{\Omega}} = \mathrm{id}_{\mathbb{R}^d \setminus \tilde{\Omega}}$. Hence, it remains to show

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \|\tilde{\varphi}_\varepsilon - \varphi\|_{L^1(\Omega)} &= 0, \\ \limsup_{\varepsilon \searrow 0} J^\varepsilon(\tilde{\varphi}_\varepsilon) &\leq J^0(\varphi). \end{aligned}$$

The L^1 convergence follows from the triangle inequality

$$\left\|\varphi_\varepsilon \circ T_{s(\varepsilon)}^{-1} - \varphi\right\|_{L^1(\Omega)} \leq \left\|(\varphi_\varepsilon - \varphi) \circ T_{s(\varepsilon)}^{-1}\right\|_{L^1(\Omega)} + \left\|\varphi \circ T_{s(\varepsilon)}^{-1} - \varphi\right\|_{L^1(\Omega)}.$$

Here, the convergence of the first summand can be shown by the same argumentation as for the continuity of f at 0, whereas the convergence of the second summand can be established via Lebesgue's dominated convergence theorem after approximating φ_0 by a sequence of $C_0^0(\Omega)$ functions.

To verify the lim sup inequality, let us first consider the Ginzburg–Landau energy separately. For the potential term we compute with the help of (3.3.83)

$$\int_{\tilde{\Omega}} \psi(\tilde{\varphi}_\varepsilon)\,\mathrm{d}x = \int_{\tilde{\Omega}} \psi(\varphi_\varepsilon)\left|\det DT_{s(\varepsilon)}\right|\,\mathrm{d}x \leq (1 + C\,|s(\varepsilon)|)\int_{\tilde{\Omega}} \psi(\varphi_\varepsilon)\,\mathrm{d}x.$$

Using the fact that for every $x \in \mathbb{R}^d$,

$$\Big(D(T_s^{-1})\Big)(x) = \Big(DT_s(T_s^{-1}(x))\Big)^{-1},$$

we infer that the gradient term can be expressed as

$$\begin{aligned}
\int_{\tilde{\Omega}} |\nabla\tilde{\varphi}_\varepsilon(x)|^2\,\mathrm{d}x &= \int_{\tilde{\Omega}} \left|\Big(DT_{s(\varepsilon)}(T_{s(\varepsilon)}^{-1}(x))\Big)^{-T}\nabla\varphi_\varepsilon\Big(T_{s(\varepsilon)}^{-1}(x)\Big)\right|^2\,\mathrm{d}x\\
&= \int_{\tilde{\Omega}} \left|\big(\mathrm{Id} + s(\varepsilon)\nabla\boldsymbol{\xi}(x)\big)^{-T}\nabla\varphi_\varepsilon(x)\right|^2\left|\det DT_{s(\varepsilon)}(x)\right|\,\mathrm{d}x\\
&= \int_{\tilde{\Omega}} \left|\left(\sum_{k=0}^{\infty}(-s(\varepsilon)\nabla\boldsymbol{\xi})^k\right)^T\nabla\varphi_\varepsilon(x)\right|^2\left|\det DT_{s(\varepsilon)}(x)\right|\,\mathrm{d}x\\
&\leq (1 + C\,|s(\varepsilon)|\,)\int_{\tilde{\Omega}} |\nabla\varphi_\varepsilon(x)|^2\,\mathrm{d}x,
\end{aligned}$$

where we use (3.3.83) and a straightforward computation involving the geometrical series. Therefore, the constant C only depends on $\tilde{\delta}$ and $\boldsymbol{\xi}$. Altogether, we deduce

$$\begin{aligned}
\limsup_{\varepsilon\searrow 0}\int_{\tilde{\Omega}} \frac{\varepsilon}{2}\,|\nabla\tilde{\varphi}_\varepsilon|^2 + \frac{1}{\varepsilon}\psi(\tilde{\varphi}_\varepsilon)\,\mathrm{d}x &\leq \lim_{\varepsilon\searrow 0}(1 + C\,|s(\varepsilon)|)\limsup_{\varepsilon\searrow 0}\int_{\tilde{\Omega}} \frac{\varepsilon}{2}\,|\nabla\varphi_\varepsilon|^2 + \frac{1}{\varepsilon}\psi(\varphi_\varepsilon)\,\mathrm{d}x\\
&\leq c_0\,P_{\tilde{\Omega}}(\tilde{E}^\varphi),
\end{aligned} \tag{3.3.85}$$

as $(\varphi_\varepsilon)_{\varepsilon>0}$ was the recovery sequence for φ of Theorem 3.3.11.

Now, we consider the eigenvalue term of the cost functional J^ε. Due to Theorem 3.3.11, we already know

$$\limsup_{\varepsilon\searrow 0}\lambda_{i_j}^{\varepsilon,\varphi_\varepsilon} \leq \lambda_{i_j}^{0,\varphi}, \tag{3.3.86}$$

for $j = 1,\dots,l$. In the following, to provide a cleaner presentation, we will write $k \coloneqq i_j$. We intend to show that there exists a sequence $(\alpha_{s(\varepsilon)})_{\varepsilon>0}$ (that may depend on k) with

$$\lim_{\varepsilon\searrow 0}\alpha_{s(\varepsilon)} = 1,$$

such that for all $\varepsilon > 0$ small enough

$$\alpha_{s(\varepsilon)}\lambda_k^{\tilde{\varphi}_\varepsilon} \leq \lambda_k^{\varphi_\varepsilon}. \tag{3.3.87}$$

Using (3.3.86) this then directly gives

$$\limsup_{\varepsilon\searrow 0}\lambda_k^{\varepsilon,\tilde{\varphi}_\varepsilon} \leq \lambda_k^{0,\varphi}. \tag{3.3.88}$$

So for $\varepsilon > 0$ let us consider the Courant–Fischer characterization of $\lambda_k^{\varphi_\varepsilon}$ which due to Theorem 3.2.2 reads as

$$\lambda_k^{\varphi_\varepsilon} = \max_{V \in \mathcal{S}_{k-1}} \min \left\{ \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_\Omega b_\varepsilon(\varphi_\varepsilon)\,|v|^2 \,\mathrm{d}x}{\int_\Omega |v|^2 \,\mathrm{d}x} \middle| \begin{array}{l} v \in V^{\perp, L^2(\Omega)} \cap H_0^1(\Omega), \\ v \neq 0 \end{array} \right\}. \tag{3.3.89}$$

Now let us choose the subspace

$$V_{T_\varepsilon} \coloneqq \left\{ \left(w_1^{\tilde{\varphi}_\varepsilon} \circ T_{s(\varepsilon)} \right) \left| \det T_{s(\varepsilon)} \right|, \ldots, \left(w_{k-1}^{\tilde{\varphi}_\varepsilon} \circ T_{s(\varepsilon)} \right) \left| \det T_{s(\varepsilon)} \right| \right\} \subset L^2(\Omega).$$

As the family of eigenfunctions

$$W_\varepsilon \coloneqq \left\{ w_1^{\tilde{\varphi}_\varepsilon}, \ldots, w_{k-1}^{\tilde{\varphi}_\varepsilon} \right\} \subset L^2(\Omega)$$

is linearly independent, $V_{T_\varepsilon} \subset L^2(\Omega)$ is indeed a $(k-1)$-dimensional subspace. Hence, we infer from (3.3.89) that

$$\lambda_k^{\varphi_\varepsilon} \geq \min \left\{ \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_\Omega b_\varepsilon(\varphi_\varepsilon)\,|v|^2 \,\mathrm{d}x}{\int_\Omega |v|^2 \,\mathrm{d}x} \middle| \begin{array}{l} v \in V_{T_\varepsilon}^{\perp, L^2(\Omega)} \cap H_0^1(\Omega), \\ v \neq 0 \end{array} \right\}.$$

Now we want to show that

$$\begin{aligned} &\min \left\{ \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_\Omega b_\varepsilon(\varphi_\varepsilon)\,|v|^2 \,\mathrm{d}x}{\int_\Omega |v|^2 \,\mathrm{d}x} \middle| \begin{array}{l} v \in V_{T_\varepsilon}^{\perp, L^2(\Omega)} \cap H_0^1(\Omega), \\ v \neq 0 \end{array} \right\} \\ \geq &\min \left\{ \frac{\int_\Omega \left| \nabla \left(v \circ T_{s(\varepsilon)} \right) \right|^2 \,\mathrm{d}x + \int_\Omega b_\varepsilon(\varphi_\varepsilon) \left| v \circ T_{s(\varepsilon)} \right|^2 \,\mathrm{d}x}{\int_\Omega \left| v \circ T_{s(\varepsilon)} \right|^2 \,\mathrm{d}x} \middle| \begin{array}{l} v \in W_\varepsilon^{\perp, L^2(\Omega)} \cap H_0^1(\Omega), \\ v \neq 0 \end{array} \right\}. \end{aligned} \tag{3.3.90}$$

To verify this, denote with $0 \neq \overline{v} \in V_{T_\varepsilon}^{\perp, L^2(\Omega)} \cap H_0^1(\Omega)$ a function at which the minimum in the first line is attained. Then, by the transformation formula it holds for $m = 1, \ldots, k-1$

$$0 = \int_\Omega \overline{v} \left(w_m^{\tilde{\varphi}_\varepsilon} \circ T_{s(\varepsilon)} \right) \left| \det T_{s(\varepsilon)} \right| \,\mathrm{d}x = \int_\Omega \left(\overline{v} \circ T_{s(\varepsilon)}^{-1} \right) w_m^{\tilde{\varphi}_\varepsilon} \,\mathrm{d}x.$$

As we additionally know $T_{s(\varepsilon)}(\partial\Omega) = \partial\Omega$, the function

$$0 \neq \left(\overline{v} \circ T_{s(\varepsilon)}^{-1} \right) \in W_\varepsilon^{\perp, L^2(\Omega)} \cap H_0^1(\Omega)$$

is admissible and, per construction, (3.3.90) holds. For any arbitrary $v \in H_0^1(\Omega)$, we find that

$$\begin{aligned} \int_\Omega \left| \nabla \left(v \circ T_{s(\varepsilon)} \right)(x) \right|^2 \,\mathrm{d}x &= \int_\Omega \left| (\mathrm{Id} + s(\varepsilon) \nabla \xi(x))^T \, \nabla v \left(T_{s(\varepsilon)}(x) \right) \right|^2 \,\mathrm{d}x \\ &\geq (1 - C\,|s(\varepsilon)|) \int_\Omega \left| \nabla v \left(T_{s(\varepsilon)}(x) \right) \right|^2 \,\mathrm{d}x, \end{aligned}$$

with a constant $C > 0$ depending only on $\tilde{\delta}$ and $\boldsymbol{\xi}$. Hence, invoking the transformation theorem and using (3.3.83), we conclude for the remaining terms in (3.3.90) that

$$\min \left\{ \frac{\int_\Omega \left| \nabla \left(v \circ T_{s(\varepsilon)} \right) \right|^2 \,\mathrm{d}x + \int_\Omega b_\varepsilon(\varphi_\varepsilon) \left| v \circ T_{s(\varepsilon)} \right|^2 \,\mathrm{d}x}{\int_\Omega \left| v \circ T_{s(\varepsilon)} \right|^2 \,\mathrm{d}x} \middle| \begin{array}{l} v \in W_\varepsilon^{\perp, L^2(\Omega)} \cap H_0^1(\Omega), \\ v \neq 0 \end{array} \right\}$$

$$\geq (1 - C\,|s(\varepsilon)|) \min \left\{ \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_\Omega b_\varepsilon(\tilde{\varphi}_\varepsilon)\,|v|^2 \,\mathrm{d}x}{\int_\Omega |v|^2 \,\mathrm{d}x} \,\middle|\, \begin{array}{l} v \in W_\varepsilon^{\perp, L^2(\Omega)} \cap H_0^1(\Omega), \\ v \neq 0 \end{array} \right\}.$$

Combining all the previous computations and recalling that W_ε is the space spanned by the first $k-1$ eigenfunctions corresponding to $\tilde{\varphi}_\varepsilon = \varphi_\varepsilon \circ T_{s(\varepsilon)}^{-1}$, we use Theorem 3.2.2 to conclude (3.3.87) with $\alpha(\varepsilon) := (1 - C\,|s(\varepsilon)|)$, and we eventually arrive at (3.3.88).

Finally, we use the monotonicity of Ψ from Assumption **(A8)** along with (3.3.85) to deduce

$$\limsup_{\varepsilon \searrow 0} J^\varepsilon(\tilde{\varphi}_\varepsilon) \leq J^0(\varphi).$$

This completes the proof. □

Remark 3.3.13. Assume that the sets S_0 and S_1 are compactly contained in the design domain Ω (i.e., $\overline{S_0}, \overline{S_1} \subset \Omega$). Then, the minimization of J^ε implicitly enforces the Neumann boundary condition $\frac{\partial \varphi_\varepsilon}{\partial \boldsymbol{\nu}} = 0$ on $\partial\Omega$. As discussed in Section 2.1.2, we could alternatively impose the Dirichlet condition $\varphi_\varepsilon = -1$ on $\partial\Omega$ which produces a different limiting cost functional for $\varepsilon \to 0$ in which a term penalizing the energy of transitions from $\varphi_0 = 1$ to $\varphi_0 = -1$ when approaching $\partial\Omega$ needs to be added; see Chapter 4 for a detailed discussion. In Figure 3.7, we show a numerical example where the Dirichlet condition $\varphi_\varepsilon = -1$ on $\partial\Omega$ is explicitly imposed in order for the boundary of Ω to act as an obstacle.

3.4. Numerical computations

In this section, we validate our approach by presenting several numerical examples. In Section 3.4.1, we describe the methods we use for the numerical approximation of a solution to $(\mathcal{P}_l^{D,\varepsilon})$ or $(\mathcal{P}_l^{N,\varepsilon})$, respectively. In Section 3.4.2, we study the solutions to some standard examples with known analytical solution to fix the parameters b_ε and c_ε in our approach. Thereafter, in Section 3.4.3, we show further capabilities of our proposed method by solving problems whose solution is analytically unknown.

3.4.1. The numerical realization

To discretize $(\mathcal{P}_l^{D,\varepsilon})$ and $(\mathcal{P}_l^{N,\varepsilon})$, we use standard piece-wise linear and globally continuous finite elements, provided by the finite element toolbox `FEniCS` [9, 120], for all appearing functions to obtain finite dimensional approximations φ^h of φ and w_i^h of w_i, where i corresponds to the index of the eigenvalue. The finite dimensional variants of the state equations (2.1.8) and (2.1.9) are solved by the eigenvalue solver `SLEPc` [112] provided by `PETSc` [21, 22] to obtain approximate eigenvalues λ_i^h and μ_i^h. The optimization problems $(\mathcal{P}_l^{D,\varepsilon})$ and $(\mathcal{P}_l^{N,\varepsilon})$ are treated by the `VMPT` method, see [36]. As mentioned in the introduction, this method can be understood as an extension of the projected gradient method to non-reflexive Banach spaces. In our setting, we consider $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$. As part of this method we need to solve projection-type sub-problems, that have the form of linear-quadratic optimization problems. These are solved using the package `IPOPT`, see [151].

Since the phase field φ^h changes its value between -1 and 1 over a length-scale of size ε, a very high resolution of certain parts of the computational domain is required. Here, we use

locally refined meshes. For given $\varepsilon > 0$, we fix the mesh parameter $h_{\min}$ as $h_{\min} = s\varepsilon$, with $s \approx 0.08$. This leads to about 12 cells over a length of ε. We start the optimization with a very coarse mesh and use the `VMPT` method until convergence to a numerical solution φ^h occurs. Thereafter, we adapt the mesh and refine all cells K with diameter larger than $h_{\min}$ which satisfy $|\varphi^h(K_m)| \leq 0.99$ (where K_m denotes the midpoint of K), whereas cells that satisfy $|\varphi^h(K_m)| \geq 1.0$ are coarsened. We then optimize again on the the new mesh. This loop is executed until no refinement is performed during the adaptation step. Alternative concepts for local error estimation might be used, e.g., residual based error estimation or dual weighted residuals, but we stress that, in any case, a high resolution of the interface $|\varphi^h| \leq 0.99$ is required for successful numerical calculations.

We point out that for small values of ε and γ, the interfaces tend to become very thin and thus, starting with a coarse mesh is numerically not feasible. In this situation a homotopy starting from larger values for γ is used. We choose γ as homotopy parameter because γ can be varied over larger scales than ε.

3.4.2. Fixing the model parameters

The considered optimization problems $(\mathcal{P}_l^{D,\varepsilon})$ and $(\mathcal{P}_l^{N,\varepsilon})$ involve several parameters that need to be chosen. Here, we fix some of them, and we will stick to this choice unless stated differently. We fix $\Omega = (0,1)^d$, $d \in \{2,3\}$, $\int_\Omega \frac{\varphi+1}{2} = (1/2)^d|\Omega|$, i.e., $\beta_1 = \beta_2 = -0.5$ if $d = 2$ and $\beta_1 = \beta_2 = -0.75$ if $d = 3$, and start from a constant initial value φ^0. Moreover, we use $\varepsilon = 0.02, \gamma = 1$ and the classical double-obstacle potential with regular part $\psi_0(\varphi) = \frac{1}{2}(1-\varphi^2)$.

The phase field approximations (2.1.8) and (2.1.9) involve three model functions, namely $a_\varepsilon(\varphi)$, $b_\varepsilon(\varphi)$, and $c_\varepsilon(\varphi)$. Here we make the following settings

- We fix $a_\varepsilon(\varphi) = \frac{1-\varepsilon}{2}\varphi + \frac{1+\varepsilon}{2}$, meaning that $a_\varepsilon(1) = 1$ and $a_\varepsilon(-1) = \varepsilon$.
- We fix $c_\varepsilon(\varphi) = \frac{1-c\varepsilon}{2}\varphi + \frac{1+c\varepsilon}{2}$ with some $c > 0$, meaning that $c_\varepsilon(1) = 1$ and $c_\varepsilon(-1) = c\varepsilon$. In case we consider Dirichlet boundary data we fix $c = 1$.
- We fix $b_\varepsilon(\varphi) = b\frac{1-\varphi}{2\varepsilon^{4/3}}$ with some $b > 0$, meaning that $b_\varepsilon(1) = 0$ and $b_\varepsilon(-1) = \frac{b}{\varepsilon^{4/3}}$.

We note that, in the following, the rate $\varepsilon^{4/3}$ appearing in b_ε leads to a common choice for b independent of ε. In summary, in case that we apply Dirichlet boundary data, we have one unknown parameter, namely b, and in case that we use Neumann boundary data, we have one unknown parameter, namely c.

Remark 3.4.1. In case of Dirichlet boundary data, we could potentially set $a_\varepsilon(\varphi) \equiv c_\varepsilon(\varphi) \equiv 1$ as stated in Assumption **(A2)**. However, in this setting, we experienced when solving the minimization problem that the shape tends to attain the form of one large ball, even in cases where, for instance, two balls are the optimal solution. Nevertheless, using $a_\varepsilon(-1) = \varepsilon$ and $c_\varepsilon(-1) = c\varepsilon$ as introduced above does not conflict with the assumptions made in Section 2.1.5. Hence, the analytical results obtained in Section 3.2 are valid in the current setting.

Functions like a_ε are often chosen as polynomials to mimic the SIMP approach (see, e.g., [136]). However, during the minimization process such an approach would lead to

	2D, $\lvert D\rvert = \frac{1}{4}$		3D, $\lvert D\rvert = \frac{1}{8}$	
	one ball	two balls	one ball	two balls
r	$\sqrt{\frac{1}{4\pi}}$	$\sqrt{\frac{1}{8\pi}}$	$\sqrt[3]{\frac{3}{32\pi}}$	$\sqrt[3]{\frac{3}{64\pi}}$
λ_1	72.67	145.34	102.59	162.84
λ_2	184.50	145.34	209.86	162.84
λ_3	184.50	369.00	209.86	333.14
λ_4	331.43	369.00	209.86	333.14

Table 3.1: The first eigenvalues of the Laplace operator with Dirichlet boundary condition in two dimensions on a ball of volume $\frac{1}{4}$ and on two balls of volume $\frac{1}{8}$ each, and in three dimensions on a ball of volume $\frac{1}{8}$ and two balls of volume $\frac{1}{16}$ each. The value r denotes the radius of one ball. We refer to [108, Proposition 1.2.14] on how to compute these values.

very thin interfaces that can barely be resolved by a finite element mesh. As we solve a minimization problem, the final shape of the interface is adjusted in an optimal way in terms of the chosen parameter functions.

Fixing b for Dirichlet boundary data. To fix b we solve the minimization problem related to minimizing the first eigenvalue λ_1 for the Laplace operator with Dirichlet boundary data for sequences of b and ε. The analytical result is given by Theorem 3.4.2 which is known as the Faber–Krahn theorem, see also Chapter 4.

Theorem 3.4.2 (Faber–Krahn, cf. [108, Theorem 3.2.1])**.** *The minimum of $\lambda_1(D)$ among all bounded open sets $D \subset \mathbb{R}^d$, $d \in \mathbb{N}$, with given volume is achieved by one ball.*

In Table 3.1, we present the first four analytical eigenvalues on one ball of the given volume and, for later reference, on two balls with the given volume in sum.

We solve the minimization problem related to Theorem 3.4.2 for $b \in \{300, 400, 500, 550, 600, 700, 800\}$ and $\varepsilon \in \{0.04, 0.02, 0.01, 0.005\}$ and compare the numerically found eigenvalue λ_1^h to the analytical known values λ_1 provided in Table 3.1. The relative errors $\eta_1^\lambda := |\lambda_1 - \lambda_1^h|/\lambda_1$ are presented in Figure 3.1. From Figure 3.1, we obtain that for the scaling $b(-1) = b\varepsilon^{-4/3}$ the choice of $b = 550$ is optimal in this situation and thus, in the following, we fix $b = 550$.

For $\varepsilon = 0.02$ and $b = 550$, the eigenfunction $w_1^{D,h}$ related to the eigenvalue λ_1^h approximates the analytical w_1^D with a relative error $\|w_1^D - w_1^{D,h}\|_{L^2(\Omega)}/\|w_1^D\|_{L^2(\Omega)} = 12 \cdot 10^{-4}$. Here, w_1^D is a scaled Bessel function, see [108, Proposition 1.2.14]. It is extended to the whole computational domain Ω with the constant value zero.

We validate our choice by solving the optimization problem related to λ_2 and λ_3 for $\varepsilon = 0.02$. The global optimal solutions are stated in Theorem 3.4.3 and Theorem 3.4.4.

Theorem 3.4.3 ([108, Theorem 4.1.1])**.** *The minimum of $\lambda_2(D)$ among all bounded open sets $D \subset \mathbb{R}^d$, $d \in \mathbb{N}$, with given volume is achieved by the union of two identical disjoint balls.*

Theorem 3.4.4 ([108, Corollary 5.2.2])**.** *The minimum of $\lambda_3(D)$ among all bounded open sets $D \subset \mathbb{R}^d$, $d \in \{2, 3\}$, with given volume is achieved by one ball.*

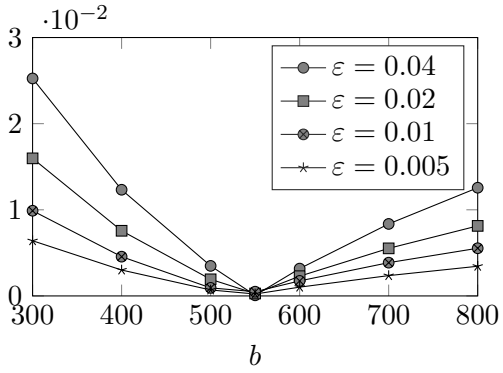

Figure 3.1: The relative error $\eta_1^\lambda = |(\lambda_1 - \lambda_1^h)|/\lambda_1$ when minimizing λ_1 in two dimensions for several values of b and ε. Here $b_\varepsilon(-1) = b\varepsilon^{-4/3}$.

Remark 3.4.5 (Eigenvalues with multiplicity larger one)**.** In the situation of Theorem 3.4.2, λ_2 has multiplicity two, while in the situation of Theorem 3.4.4, λ_3 has multiplicity equal to the spatial dimension. If eigenvalues have multiplicity larger one, the corresponding gradient is no longer unique and depends on the random ordering that the eigenvalue solver provides for these identical eigenvalues. This problem can be detected by considering the relative difference of subsequent eigenvalues during the optimization run. If this problem is detected for an eigenvalue of multiplicity two, we modify the objective functional to minimize the arithmetic mean of these equal eigenvalues. As both eigenvalues are equal, this does not change the value attained by the objective at the current local optimum. We stress, that changing the objective functional in advance does lead to a different optimization problem and thus, we typically obtain different local minimizers. We also refer to [134, Section 4.5] for more details.

In practice, we notice that this modification does not work for eigenvalues with a multiplicity larger than two. This is because it is rather unlikely, that the pairwise relative difference of more than two eigenvalues becomes small at the same time and thus jointly trigger the modification of the objective. In this situation, the above modification actually changes the objective and we would thus solve a different problem. This situation appears, for instance, when minimizing λ_3 in three spatial dimensions, where $\lambda_2 = \lambda_3 = \lambda_4$ holds for the optimal shape, namely a ball. Luckily, in this situation, by solving the modified optimization problem we still detect the correct minimizer predicted by Theorem 3.4.4 if γ is initially chosen sufficiently large and decreased subsequently.

The global optimal solutions are successfully found in two and three spatial dimensions. In Figure 3.2 we show optimal shapes for minimizing λ_i, $i = 1, 2, 3$ in two spatial dimensions. We note, that in case of minimizing λ_3 there also is an attracting local minimum, containing three small balls. Here, we need to start with a large value of $\gamma = 10^3$ to guide the optimization process to the correct global optimum in combination with a homotopy of decreasing values for γ towards $\gamma = 1$. Moreover, in three dimensions we need to substitute the minimization problem for λ_3 by $\frac{1}{3}(\lambda_2 + \lambda_3 + \lambda_4)$ to deal with the multiple eigenvalue. In two dimensions this problem is handled as described in Remark 3.4.5. The correct topologies are found and in Table 3.2, we present our numerical results in terms of the eigenvalues that we obtained.

	2D, $\lvert\{\varphi > 0\}\rvert = \frac{1}{4}$			3D, $\lvert\{\varphi > 0\}\rvert = \frac{1}{8}$		
k	λ_k	λ_k^h	η_k^λ	λ_k	λ_k^h	η_k^λ
1	72.67	72.68	$1 \cdot 10^{-4}$	102.59	101.43	$113 \cdot 10^{-4}$
2	145.34	143.16	$150 \cdot 10^{-4}$	162.84	162.40	$27 \cdot 10^{-4}$
3	184.50	183.53	$54 \cdot 10^{-4}$	209.86	209.21	$31 \cdot 10^{-4}$

Table 3.2: Analytical and numerically found eigenvalues related to minimizing λ_k, $k \in \{1, 2, 3\}$ in two and three spatial dimensions. Here λ_k denotes the analytical value of the k-th eigenvalue, λ_k^h denotes the numerical found approximation of this value and $\eta_k^\lambda := |\lambda_k - \lambda_k^h|/\lambda_k$ denotes the related relative numerical error.

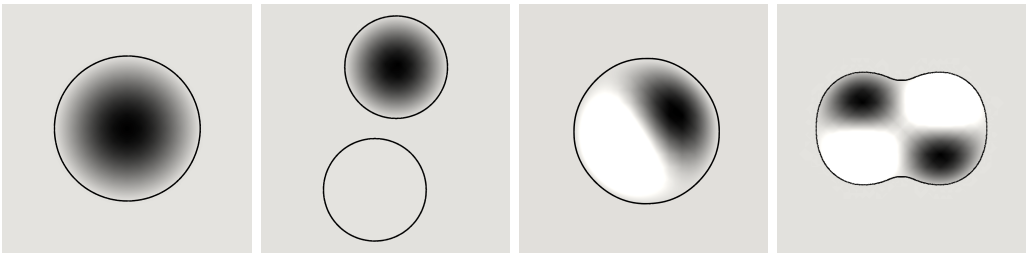

Figure 3.2: The optimal shape for minimization of λ_i, $i = 1, 2, 3, 5$ (left to right) is presented by the zero level line of φ^h black in each case, while the value of the corresponding eigenfunction $w_i^{D,h}$ is shown in gray scale. The gray outer domain corresponds to $w_i^{D,h} \approx 0$. Note that in the case of minimizing λ_2, there is a second eigenfunction (to same eigenvalue) that is supported on the bottom circle, while in the case of minimizing λ_3, there is a second eigenfunction (to same eigenvalue) that is rotated by 90°.

One additional example with known solution. As another example for which a reference solution exists, we consider the minimization of λ_5. From [17, Figure 11.1] we expect a butterfly-like shape and the proposed eigenvalue is $\lambda_5 = 312.60$. In contrast to the simulation above, we use $\gamma = 0.1$ to get closer to the features of the butterfly. In Figure 3.2 (right), we show our numerically obtained shape together with the corresponding eigenfunction $w_5^{D,h}$. We obtain the eigenvalue $\lambda_5^h = 311.59$. The normalized amplitude of $w_5^{D,h}$ is $\|w_5^{D,h}\|_{L^\infty(\Omega)} = 3.94$, while the amplitude of $w_5^{D,h}$ on the zero-level line of φ^h is of order $\|w_5^{D,h}\|_{L^\infty(\{\varphi^h=0\})} = 0.50$. We point out that the result proposed in [17, Figure 11.1] is more pronounced in the middle part and also on the left and right there are additional small deflections.

Fixing c for Neumann boundary data. Here we proceed as in the case of fixing b. We solve the corresponding maximization problem for μ_1 for

$$c \in \{0.2, 0.15, 0.1, 0.05\} \quad \text{and } \varepsilon \in \{0.04, 0.02, 0.01, 0.005\}$$

on $\Omega = (0, 1)^2$. In this situation, the optimal shape is a disc of radius $r = \sqrt{1/(4\pi)}$ with first eigenvalue $\mu_1 = 42.6002$. This can be obtained from [108, Proposition 1.2.14], see also the theorem of Szegő and Weinberger from the introduction of this chapter.

For the sake of brevity, we omit the presentation of relative errors as in Figure 3.1 and just state that we observe that $c = 0.1$ is a good choice independent of ε. In the following, we fix $c = 0.1$.

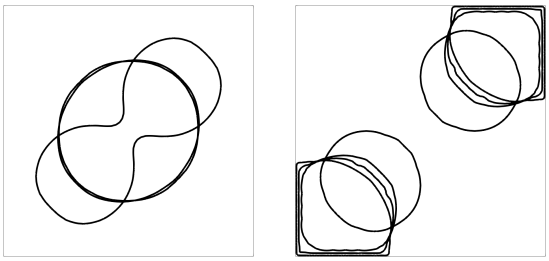

Figure 3.3: The zero level lines of φ^h for minimizing μ_1 with $\gamma \in \{10^3, 10^2, 10^1\}$ (left) and with $\gamma \in \{10^0, 10^{-1}, 10^{-2}, 10^{-3}\}$ (right). We observe that a large value of γ leads to a circle and that decreasing γ allows the optimizer to find topologies with longer boundaries.

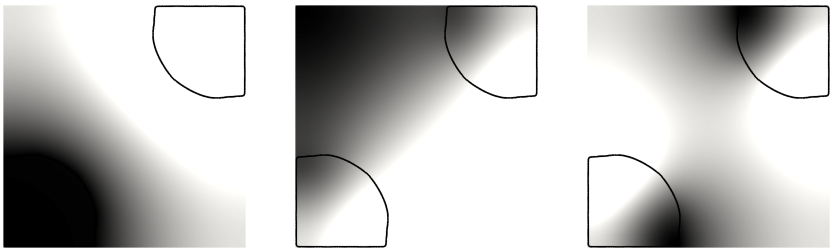

Figure 3.4: When minimizing μ_1 we obtain the shape that is indicated by black lines, which is the zero-level line of φ^h. From left to right, we show the first three (non-trivial) eigenfunctions $w_1^{N,h}$, $w_2^{N,h}$, and $w_3^{N,h}$ on this shape. Here, gray corresponds to $w_i^{N,h} \approx 0$. By our approach, the eigenfunctions are defined on the complete domain Ω, and as we are considering the Neumann case, they do not degenerate on the complement of the shape. The corresponding eigenvalues are $\mu_1^h = 0.40$, $\mu_2^h = 67.55$, and $\mu_3^h = 68.23$. In this example we chose $\gamma = 10^{-3}$.

3.4.3. Numerical examples without known solution

In the following, we investigate some numerical examples where the analytical solution is unknown in order to show the strength of our approach in finding unknown shapes with a priori unknown topologies. We also remark that the boundary of Ω acts as an obstacle which can be seen in several computations below. We refer to [103, 111] for more information on obstacle type problems for eigenvalues.

Minimization of μ_1. As discussed above, the *maximization* of μ_1 leads to a disc. Now we ask for the optimal shape and topology when *minimizing* μ_1 on $\Omega = (0,1)^2$. In Figure 3.3, we present the found optimal topology and the influence of γ on the result by showing the iterates for a homotopy reducing γ from 10^3 to 10^{-3}. In Figure 3.4, we present the first three corresponding non-trivial eigenfunctions on the optimal topology for $\gamma = 10^{-3}$. We observe that the boundary of Ω might act as an obstacle for the shape optimization problem.

Mixing minimization and maximization. In this example, we consider the minimization of a weighted sum of eigenvalues. Especially, we consider weights with different signs which leads to simultaneous minimization and maximization of certain eigenvalues.

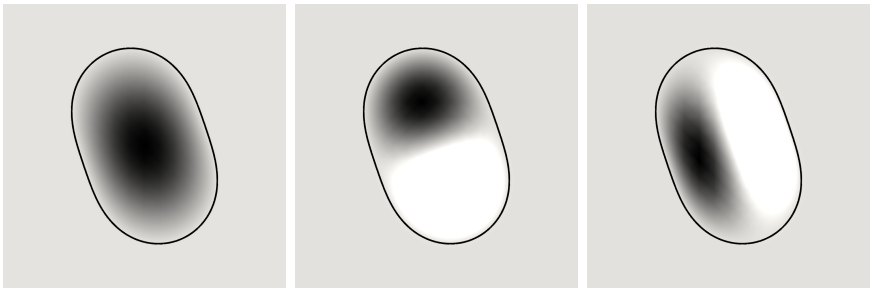

Figure 3.5: Optimal shapes for an example of mixed minimization and maximization, namely $J^\lambda = \frac{6\lambda_1 - \lambda_3}{7} + E^{0.02}(\varphi)$. The optimal shape is indicated by the zero-level line of φ^h in black, and we show $w_1^{D,h}$, $w_2^{D,h}$, and $w_3^{D,h}$ in gray scale (left to right). Gray indicates the zero level of the eigenfunctions. The corresponding eigenvalues are $\lambda_1^h = 81.32$, $\lambda_2^h = 161.01$, and $\lambda_3^h = 255.42$.

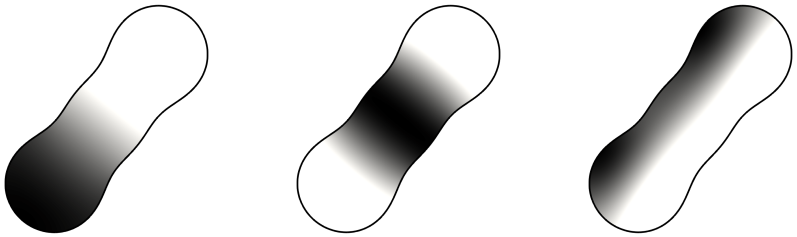

Figure 3.6: Optimal shapes for an example of mixed minimization and maximization, namely $J^\mu = \frac{12\mu_1 - \mu_3 - \mu_4}{14} + 10E^{0.02}(\varphi)$. The optimal shape is indicated by the zero-level line of φ^h in black, and we show $w_1^{N,h}$, $w_2^{N,h}$, and $w_3^{N,h}$ in gray scale (left to right). Gray indicates the zero level of the eigenfunctions. The corresponding eigenvalues are $\mu_1^h = 12.94$, $\mu_2^h = 56.94$, and $\mu_3^h = 115.27$. Here, the eigenfunctions $w_i^{N,h}$, $i = 1, 2, 3$, are plotted only on the actual shape.

In Figure 3.5, we present numerical results for the objective $J^\lambda = \frac{6\lambda_1 - \lambda_3}{7} + E^{0.02}(\varphi)$ and in Figure 3.6, we present numerical results for the objective $J^\mu = \frac{12\mu_1 - \mu_3 - \mu_4}{14} + 10E^{0.02}(\varphi)$.

Influences from Ω. As stated in Theorem 3.4.2, the minimizer of the first eigenvalue is one single disc of diameter $d = 2\sqrt{1/(4\pi)} \approx 0.56$. Here, we show the optimal shape in the case that this ball does not fit into the computational domain. This leads to an obstacle like problem where the boundary of Ω acts as an obstacle, see also [108, Section 3.4].

We consider two cases, namely $\Omega_1 = (0.0, 2.5) \times (0.0, 0.4)$ which is a rectangular domain of height $0.4 \leq d$, and $\Omega_2 = (0, 1.45)^2 \backslash (0.4, 1.45)^2$ which is an L-shaped domain. In both cases, a disc of diameter $d \approx 0.56$ does not fit into the domain. In this example we fix $\varphi^h = -1$ on $\partial\Omega$ to prevent the shape from touching the boundary. In Figure 3.7 we present numerical results for the minimization of λ_1 in this situation.

Prescribing parts of the optimal topology. Finally, we show another aspect of the flexibility of the proposed approach. We present two examples, in which we a-priori fix certain parts of the design domain. In Figure 3.8, we present numerical results obtained by either fixing some part of the domain as shape (left) or as void (right). In both cases,

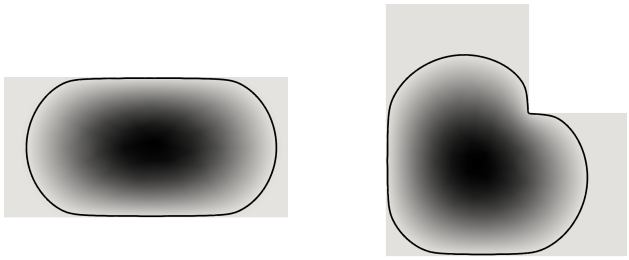

Figure 3.7: The optimal shapes for minimizing λ_1 on the rectangular domain $\Omega_1 = (0.0, 2.5) \times (0.0, 0.4)$ (left) and the L-shaped domain $\Omega_2 = (0, 1.45)^2 \backslash (0.4, 1.45)^2$ (right). The shapes are indicated by the zero level line of φ_ε^h in black and we show the corresponding first eigenfunction $w_1^{D,h}$ in gray-scale. Gray indicates the zero level of $w_1^{D,h}$. We show only the relevant part of the computational domain. The corresponding eigenvalues are $\lambda_1^h(\Omega_1) = 85.54$ and $\lambda_1^h(\Omega_2) = 77.13$. A disc of the same size would lead to $\lambda_1 = 72.68$ as stated in Table 3.2.

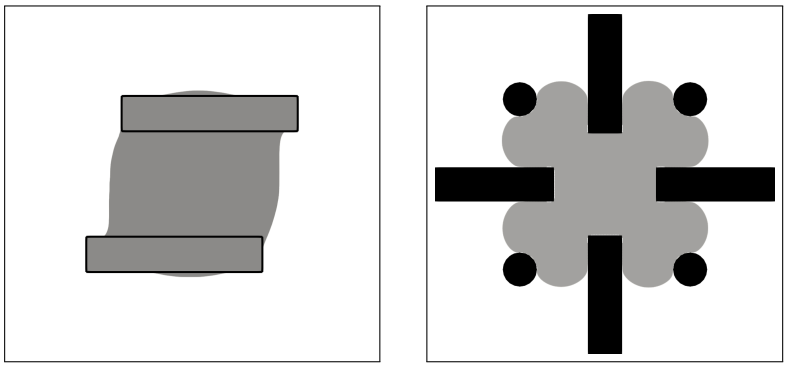

Figure 3.8: Numerically obtained optimal shapes in gray for minimizing λ_1. On the left we fix the domains inside the gray boxes as part of the shape, i.e., $\varphi^h = 1$, while on the right we fix the black domains to be void, i.e., $\varphi^h = -1$.

we minimize λ_1 and we fix $\gamma = 0.01$.

Chapter 4

A phase-field version of the Faber–Krahn theorem

4.1. Introduction

As already mentioned in the introduction of the preceding chapter, the *Faber–Krahn theorem* is one of the most fundamental theorems when it comes to spectral shape optimization. This theorem asserts that in the class of open sets $E \subset \mathbb{R}^d$ with fixed volume $|E| = 1$, the smallest eigenvalue $\lambda(E) > 0$ of the eigenvalue problem

$$-\Delta w = \lambda w \quad \text{in } E, \tag{4.1.1a}$$

$$w = 0 \qquad \text{on } \partial E, \tag{4.1.1b}$$

attains a global minimum if E is a ball B of volume $|B| = 1$. This result was first proven in dimension $d = 2$ independently by Faber [90] and Krahn [117], not in this whole generality but with suitable assumptions on the regularity of the shape boundary. By now it is well known that the result holds in full generality as stated above for any dimension $d \geq 2$, see, e.g., [108, Section 3.2]. A direct consequence of this theorem is the so called *Faber–Krahn inequality* which states that for any open ball $B \subset \mathbb{R}^d$ and any open set $E \subset \mathbb{R}^d$ it holds

$$|B|^{\frac{2}{d}} \lambda(B) \leq |E|^{\frac{2}{d}} \lambda(E) \tag{4.1.2}$$

see, e.g., [94].

The goal of this chapter is to prove a version of this result in our phase-field framework. Let us recall from Section 2.1.5 that our phase-field approach approximates (4.1.1) via the equation

$$-\Delta w^{\varepsilon,\varphi} + b^\varepsilon(\varphi) w^{\varepsilon,\varphi} = \lambda^{\varepsilon,\varphi} w^{\varepsilon,\varphi} \quad \text{in } \Omega, \tag{4.1.3a}$$

$$w^{\varepsilon,\varphi} = 0 \qquad \text{on } \partial\Omega, \tag{4.1.3b}$$

where the coefficient function $b^\varepsilon : [0,1] \to \mathbb{R}$ is chosen analogously as in the previous chapter in order to enforce homogeneous Dirichlet boundary conditions in the sharp-interface limit. Considering the Faber–Krahn theorem, the cost functional studied in the previous chapter, see also Section 2.1.7, simplifies to

$$J_\gamma^\varepsilon(\varphi) = \lambda_1^{\varepsilon,\varphi} + \gamma E^\varepsilon(\varphi), \tag{4.1.4}$$

where $\lambda_1^{\varepsilon,\varphi}$ denotes the principal eigenvalue of (4.1.3).

The proof of our phase-field version of the Faber–Krahn theorem given in Theorem 4.3.7 is based on a symmetrization technique. The key idea which is also a fundamental tool in the original proofs of Faber and Krahn [90, 117] is to decrease the cost functional by, plainly speaking, making the phase-field and the eigenfunction associated to the principal eigenvalue more symmetric. This is accomplished by using the theory of symmetric-decreasing rearrangements, also frequently refereed to as *Schwarz rearrangements*, see Section 2.2.3. Breakthrough results such as the quantitative isoperimetric inequality [93] underline the strength of such symmetrization techniques.
In the light of the cost functional (4.1.4), the main task in proving a phase-field version of the Faber–Krahn theorem will be to show that the principal eigenvalue $\lambda_1^{\varepsilon,\varphi}$ and the Ginzburg–Landau energy $E^\varepsilon(\varphi)$ are non-increasing under radially symmetric-decreasing rearrangements. The latter can be seen as a phase-field version of the *Euclidean isoperimetric problem* which, recalling it as Dido's problem from the introduction of this thesis, states that in the class of measurable sets of fixed volume the ball has minimal perimeter, see [121, Section 14].

As opposed to the theory developed in Chapter 3 it will be crucial to choose the design domain to be an open ball in $\mathbb{R}^d$ centered at the origin. This is due to the fact that the ball is obviously invariant under symmetric rearrangement, see also Section 2.2.3.
Let us emphasize a further consequence of exploiting symmetrization techniques. Opposed to the previous chapter the admissible set Φ_{ad} in the minimization of (4.1.4) must impose a homogeneous Dirichlet boundary condition on the phase-field variable in order for the proof of our main theorem to work. This means that not only the eigenfunction w^φ but also the phase-field φ has to satisfy a Dirichlet boundary condition on the boundary of the design domain. This condition is crucial in order to apply the famous *Pólya–Szegő* inequality asserting the non-expansivity of the Dirichlet energy under rearrangement, see especially Remark 2.2.13.
In turn this additional Dirichlet condition requires a delicate analysis when performing the Γ-limit for the cost functional, as a further contact energy enters the cost functional in the sharp-interface setting. Building up on the theory developed in the previous chapter we will perform the Γ-limit rigorously in this more complex setting and even give two versions of the proof. The first one uses the approach of [43] where the additional term in the limit energy is handled by a cut-off procedure on the *sharp-interface level*. This ansatz has the drawback that it only works for constant boundary data. The second version of the proof is inspired by the more general framework of [135]. Their ansatz allows even for non-constant boundary data. We will apply this ansatz in our proof just for homogeneous Dirichlet data but we believe that our proof is rather illustrative also for general Dirichlet boundary data. Furthermore this version of the proof involves the technical construction of a recovery sequence explicitly and in full detail, which we believe to be quite informative for the reader. Here the key idea is to perform a smooth cut-off by multiplying the classical recovery sequence with an optimal profile on the *diffuse interface level*.
Performing the sharp-interface limit we will finally arrive at a generalized version of the Faber–Krahn theorem formulated in the framework of functions of bounded variation, see Theorem 4.3.15 and Corollary 4.3.16.

To conclude this introduction let us review some further approaches in the literature deal-

ing with the Faber–Krahn inequality. In order to give an alternative proof of the classical Faber–Krahn inequality for the Dirichlet-Laplacian, Bucur and Freitas [53] use methods developed by Alt and Caffarelli [11] (see also [58] for a comprehensive overview) by giving sense to this inequality in a free boundary value problem framework. More precisely, the key idea is to pass from finding an optimal domain to finding the optimal support of the first eigenfunction, i.e., the shapes under consideration are associated to the super-level sets $\{w > 0\}$, see also [152]. Furthermore, in order to prove radial symmetry of an optimal shape, the authors do not exploit the classical Schwarz rearrangement but the Steiner symmetrization which reflects a given shape with respect to a hyperplane.
An interesting alteration of the Faber–Krahn problem is studied in [57] where no longer the Laplace operator but a general elliptic operator in divergence form with space depending coefficients is considered. These non-constant coefficients destroy the symmetry of the Laplacian which makes the application of symmetrization techniques impossible. Nevertheless, the authors can still show existence of minimizers via a concentration-compactness argument. A natural further development of the Faber–Krahn theorem is to replace the Dirichlet boundary condition by a boundary condition of Robin type interpolating in between Dirichlet and Neumann boundary conditions. In [79] the corresponding Faber–Krahn inequality for the Robin Laplacian is established via a level-set characterization of the principal eigenvalue. This characterization proves to be useful in order to give an estimate on the principal eigenvalue for which then a symmetrization argument is applicable in order to show the desired inequality in the class of Lipschitz domains. [52] generalizes this result for the p-Laplacian subject to a Robin boundary condition. In order to avoid the constraint of competing shapes to possess Lipschitz regularity, Bucur and Giacomini [54] characterize the Robin-Laplace eigenvalues via a Courant–Fischer characterization in the framework of special functions of bounded variation. The Faber–Krahn inequality is then obtained with the level-set characterization introduced in [79].

4.2. Additional assumptions

Additionally to the general preliminary assumptions from Chapter 2 the following assumptions on the potential $\psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ and the coefficients b^ε are supposed to hold throughout this chapter:

(A1) $\psi \in C^2([0,1])$, $\psi(0) = \psi(1) = 0$ and $\psi > 0$ in $(0,1)$.

(A2) The minima of ψ at 0 and 1 are non-degenerate in the sense that for $x \in \{0,1\}$, we either have $\psi'(x) \neq 0$ or $\psi'(x) = 0 < \psi''(x)$.

(A3) For any $\varepsilon > 0$, the coefficient b^ε is a function

$$b^\varepsilon : [0,1] \to [0, \beta^\varepsilon] \tag{4.2.1}$$

for some real number $\beta^\varepsilon > 0$. We demand that b^ε is continuous, strictly decreasing and surjective onto $[0, \beta^\varepsilon]$.

(A4) In complete analogy to the setting of Section 3.3 we make the following assumptions on the coefficient functions b_ε. The numbers $\beta^\varepsilon = b^\varepsilon(0)$ satisfy

$$\lim_{\varepsilon \searrow 0} \beta^\varepsilon = +\infty \quad \text{and} \quad \beta^\varepsilon = o\big(\varepsilon^{-\kappa}\big) \quad \text{with} \quad \begin{cases} \kappa \in (0,1) & \text{if } d = 2, \\ \kappa = \frac{2}{d} & \text{if } d \geq 3. \end{cases} \tag{4.2.2}$$

Moreover, there exists a limit function

$$b^0 : [0,1] \to [0,+\infty] \tag{4.2.3}$$

satisfying

- $b^0(\frac{1}{2}) < +\infty$,
- $b^\varepsilon \to b^0$ pointwise in $[0,1]$ as $\varepsilon \to 0$,
- $b^\delta \geq b^\varepsilon$ on $[0,1]$ for all $0 \leq \delta \leq \varepsilon$.

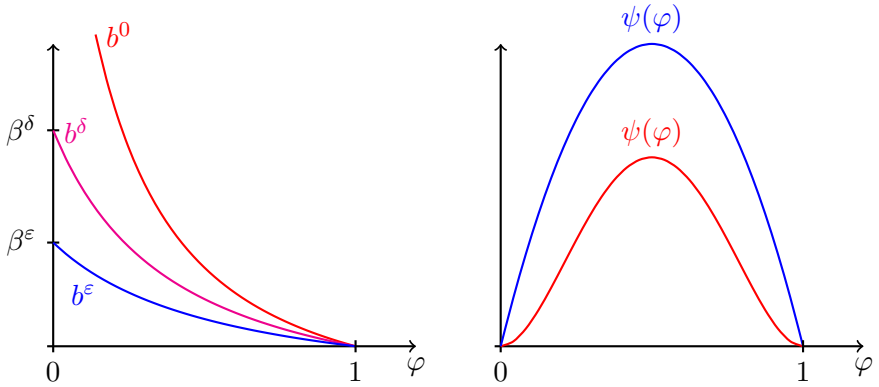

Figure 4.1: Sketch of a possible choice for the coefficient functions b^ε and b^0 (left) and the potential ψ (right).

Remark 4.2.1. The two classical choices we have in mind for the potential ψ are either the smooth quartic double-well potential $\psi(x) = \frac{1}{4}(1-x)^2x^2$ (which satisfies $\psi'(x) = 0 < \psi''(x)$ for $x \in \{0,1\}$) or the non-smooth double-obstacle potential

$$\psi(x) = \begin{cases} \frac{1}{2}(1-x)x & \text{if } x \in [0,1], \\ +\infty & \text{else.} \end{cases}$$

(which satisfies $\psi'(x) \neq 0$ for $x \in \{0,1\}$). However, the assumptions (A1) and (A2) allow for very general potentials. In particular, asymmetric potentials satisfying $\psi'(0) \neq 0$ and $\psi'(1) = 0 < \psi''(1)$ (or vice versa) can also be included.

Note that in the case of a smooth potential as studied in [126,147], opposed to their theory, we do not need a growth condition as in [126, Proposition 3(b)] or [147, Proposition 3] since we additionally incorporate the box constraint $\varphi \in [0,1]$ in our set of admissible phase-fields. Therefore, depending on the choice of ψ, one of the results [126, Proposition 3(a)], [147, Remark (1.35)] and [37, Theorem 3.7] can be applied, which are summarized in our general setting in Proposition 2.2.25, providing us with compactness of the Ginzburg–Landau energy.

The assumptions on ψ in [147, Theorem 1] differ from (A1) only in the fact that global continuity is assumed. However, due to the box constraint $\varphi \in [0,1]$, our phase-fields may not leave the interval $[0,1]$ and thus, such an assumption is not necessary.

The crucial difference between [147] and [37] is that in [147], the potentials need to satisfy (A2) with $\psi'(x) = 0 < \psi''(x)$ for $x \in \{0,1\}$, whereas [37] only covers the case $\psi'(x) \neq 0$ for

$x \in \{0,1\}$. However, we will see that also the mixed case $\psi'(0) \neq 0$ and $\psi'(1) = 0 < \psi''(1)$ (or vice versa) can be handled by combining the proofs of [147, Theorem 1] and [37, Proposition 3.11]. This is possible since their construction of a recovery sequence remains practicable as the ODE

$$\eta'(t) = \sqrt{2\psi(\eta(t))}, \tag{4.2.4}$$

which is used to define the profile at the diffuse interface, possesses a global solution that is strictly increasing as long as $\eta(t) \in (0,1)$. In [147], any solution of (4.2.4) satisfies $\eta(t) \in (0,1)$ for all $t \in \mathbb{R}$, whereas in [37], there exist $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$ such that

$$\eta(t) \begin{cases} = 0, & \text{if } t \in (-\infty, t_0], \\ \in (0,1), & \text{if } t \in (t_0, t_1), \\ = 1, & \text{if } t \in [t_1, \infty). \end{cases} \tag{4.2.5}$$

In our Γ-convergence proof of Theorem 4.3.17, we will take care of both cases simultaneously. Therefore, proceeding as in [147], we interpolate the solution η of (4.2.4) in such a way that the interpolated solution exhibits the behavior described in (4.2.5). The solvability of (4.2.4) and further properties of solutions to this ODE will be analyzed in depth in the proof of Theorem 4.3.17.

4.3. The phase-field Faber–Krahn theorem and its sharp-interface limit

4.3.1. Recalling the state equation

In the following, we consider the design domain $\Omega = B_R(0) \subset \mathbb{R}^d$ with $d \geq 2$ and some finite radius $R > 0$.

For functions $\varphi \in L^\infty(\Omega; [0,1])$, the eigenvalue problem introduced in Section 2.1.5 reads as

$$-\Delta w + b^\varepsilon(\varphi) w = \lambda w \qquad \text{in } \Omega, \tag{4.3.1a}$$

$$w|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega. \tag{4.3.1b}$$

In view of the term $b^\varepsilon(\varphi)w$, this problem can be understood as a phase-field approximation of the classical Dirichlet–Laplace eigenvalue problem on the shape represented by the set $\{\varphi = 1\}$. For a detailed motivation and introduction of this eigenvalue problem we refer to Section 2.1. To specify the notion of weak solutions, eigenvalues and eigenfunctions, we recall the following definition from Definition 3.2.1 adapted to our setting.

Definition 4.3.1. *Let $\varepsilon > 0$ and $\varphi \in L^\infty(\Omega; [0,1])$ be arbitrary.*

(a) *For any given $\lambda \in \mathbb{R}$, a function $w \in H^1_0(\Omega)$ is called a **weak solution** of the system (4.3.1) if the weak formulation*

$$\int_\Omega \nabla w \cdot \nabla \eta + b^\varepsilon(\varphi)\, w\eta \,\mathrm{d}x = \lambda \int_\Omega w\eta \,\mathrm{d}x \tag{4.3.2}$$

is satisfied for all test functions $\eta \in H^1_0(\Omega)$.

(b) *A real number $\lambda^{\varepsilon,\varphi}$ is called an **eigenvalue** associated with φ, if there exists at least one non-trivial weak solution $w^{\varepsilon,\varphi} \in H_0^1(\Omega)$ of the eigenvalue problem* (4.3.1) *written for $\lambda = \lambda^{\varepsilon,\varphi}$.*

*In this case, $w^{\varepsilon,\varphi}$ is called an **eigenfunction** to the eigenvalue $\lambda^{\varepsilon,\varphi}$.*

We further recall some important properties of the eigenvalue problem (4.3.1), which were already obtain in Theorem 3.2.2 but we restate it here for the purposes of this chapter.

Proposition 4.3.2. *Let $\varepsilon > 0$ and $\varphi \in L^\infty(\Omega)$ be arbitrary.*

(a) *The eigenvalue problem* (4.3.1) *has countably many eigenvalues and each of them has a finite dimensional eigenspace. Repeating each eigenvalue according to its multiplicity, we can write them as a sequence $\left(\lambda_k^{\varepsilon,\varphi}\right)_{k\in\mathbb{N}}$ with*

$$0 < \lambda_1^{\varepsilon,\varphi} \le \lambda_2^{\varepsilon,\varphi} \le \lambda_3^{\varepsilon,\varphi} \le \dots \quad \text{and} \quad \lambda_k^{\varepsilon,\varphi} \to \infty \ \text{ as } \ k \to \infty.$$

(b) *There exists an orthonormal basis $\left(w_k^{\varepsilon,\varphi}\right)_{k\in\mathbb{N}}$ of $L^2(\Omega)$ where for every $k \in \mathbb{N}$, $w_k^{\varepsilon,\varphi}$ is an eigenfunction to the eigenvalue $\lambda_k^{\varepsilon,\varphi}$.*

(c) *The eigenvalue $\lambda_1^{\varepsilon,\varphi}$ is called the **principal eigenvalue**. It can be represented via the Courant–Fischer characterization*

$$\lambda_1^{\varepsilon,\varphi} = \min_{w\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla w|^2 + b^\varepsilon(\varphi) w^2 \,\mathrm{d}x}{\|w\|_{L^2(\Omega)}^2}\,. \tag{4.3.3}$$

Any function $w \in H_0^1(\Omega)\setminus\{0\}$ at which this minimum is attained is an eigenfunction to the eigenvalue $\lambda_1^{\varepsilon,\varphi}$.

Moreover, the eigenspace of $\lambda_1^{\varepsilon,\varphi}$ is one-dimensional, and there exists a unique eigenfunction $\overline{w} \in H_0^1(\Omega)\setminus\{0\}$ corresponding to this eigenvalue which fulfills

$$\overline{w} > 0 \quad \text{a.e. in } \Omega, \quad \text{and} \quad \|\overline{w}\|_{L^2(\Omega)} = 1. \tag{4.3.4}$$

*We call $\overline{w}$ the **positive-normalized eigenfunction**. Without loss of generality, as the choice the sign does not matter, we will always choose $w_1^{\varepsilon,\varphi} = \overline{w}$ in the orthonormal basis given by part* (b).

Applying [99, Theorem 8.12] we directly infer the following regularity statement.

Corollary 4.3.3. *Let $\varepsilon > 0$ and $\varphi \in \Phi_m$ be arbitrary, and let w be an eigenfunction to the eigenvalue $\lambda_1^{\varepsilon,\varphi}$ in the sense of Definition 4.3.1. Then it holds that $w \in H_0^1(\Omega) \cap H^2(\Omega)$ and w is a strong solution of the system* (4.3.1)*, meaning that*

$$\begin{aligned} -\Delta w + b^\varepsilon(\varphi) w &= \lambda_1^{\varepsilon,\varphi} w && \text{a.e. in } \Omega, \\ w|_{\partial\Omega} &= 0 && \text{a.e. on } \partial\Omega. \end{aligned}$$

Now we formulate the shape optimization problem for the principal eigenvalue. This can be regarded as a special case of the framework introduced in Section 2.1.7 by choosing $\Psi(\lambda_1) = \lambda_1$ there. Hence, we only briefly summarize the main aspects concerning this optimization problem at this point.

For any prescribed $m \in (0,1)$, we define the set of admissible controls

$$\Phi_m := \left\{ \varphi \in H_0^1(\Omega) \;\middle|\; \begin{array}{l} 0 \le \varphi(x) \le 1 \ \text{ for a.e. } x \in \Omega, \\ \fint_\Omega \varphi \,\mathrm{d}x = m \end{array} \right\} \subset H_0^1(\Omega) \cap L^\infty(\Omega)\,.$$

4.3.2. The phase-field Faber–Krahn theorem

For $\varepsilon > 0$ and $\varphi \in \Phi_m$, we now introduce the Ginzburg–Landau energy

$$E^\varepsilon(\varphi) := \int_\Omega \frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon} \psi(\varphi) \, \mathrm{d}x. \tag{4.3.6}$$

This term regularizes the optimization problem in order for it to be well-posed. We observe that the Ginzburg–Landau energy is decreasing with respect to symmetric-decreasing rearrangement of its argument. This can be interpreted as a phase-field version of the isoperimetric inequality.

Lemma 4.3.4 (Phase-field isoperimetric inequality)**.** *Let $\varepsilon > 0$ be arbitrary. Then, for all $\varphi \in H_0^1(\Omega; [0,1])$, we have*

$$E^\varepsilon(\varphi^*) \leq E^\varepsilon(\varphi). \tag{4.3.7}$$

Furthermore we will prove the following phase-field version of the Faber–Krahn inequality on the diffuse interface level.

Theorem 4.3.5 (Phase-field Faber–Krahn inequality)**.** *Let $\varepsilon > 0$ be arbitrary. Then, for all $\varphi \in L^\infty(\Omega; [0,1])$, we have*

$$\lambda_1^{\varepsilon,\varphi^*} \leq \lambda_1^{\varepsilon,\varphi}.$$

In order to recover the classical Faber–Krahn inequality in the sharp-interface limit $\varepsilon \to 0$, we consider the following optimization problem:

$$\begin{cases} \text{Minimize} & J_\gamma^\varepsilon(\varphi) = \lambda_1^{\varepsilon,\varphi} + \gamma E^\varepsilon(\varphi) \\ \text{subject to} & \varphi \in \Phi_m \, . \end{cases} \tag{$\mathrm{OP}_\gamma^\varepsilon$}$$

Here, $\lambda_1^{\varepsilon,\varphi}$ denotes the principal eigenvalue corresponding to the function φ as introduced in Proposition 4.3.2, and $\gamma > 0$ is the surface tension. Here, the additional summand $\gamma E^\varepsilon(\varphi)$ acts as a regularization term which ensures well-posedness of the optimization problem and is further used to gain additional information about its minimizers. After passing to the sharp-interface limit $\varepsilon \to 0$, we recover the classical Faber–Krahn inequality in the framework of BV functions by sending $\gamma \to 0$, which is possible as we will see later that the minimizer on the sharp-interface level does not depend on γ.

The existence of a minimizer $\varphi \in \Phi_m$ of the optimization problem ($\mathrm{OP}_\gamma^\varepsilon$) was established in Theorem 3.2.8. This means that the following lemma holds.

Lemma 4.3.6. *Let $\varepsilon, \gamma > 0$ be arbitrary. Then, the optimization problem ($\mathrm{OP}_\gamma^\varepsilon$) possesses a minimizer $\varphi \in \Phi_m$.*

Now, based on Lemma 4.3.4 and Theorem 4.3.5, the following theorem shows that minimizers of ($\mathrm{OP}_\gamma^\varepsilon$) are necessarily symmetric-decreasing. The same holds for the positive-normalized eigenfunction of the corresponding principal eigenvalue.

Theorem 4.3.7 (Phase-field Faber–Krahn theorem)**.** *Let $\varepsilon, \gamma > 0$ be arbitrary, $m \in (0,1)$, and let $\varphi \in \Phi_m$ be any minimizer of the optimization problem ($\mathrm{OP}_\gamma^\varepsilon$). Then, $\varphi = \varphi^*$ almost everywhere in Ω, meaning that φ is symmetric-decreasing, and the positive-normalized eigenfunction $w_1^{\varepsilon,\varphi}$ to the principal eigenvalue $\lambda_1^{\varepsilon,\varphi}$ also fulfills $w_1^{\varepsilon,\varphi} = (w_1^{\varepsilon,\varphi})^*$ almost everywhere in Ω.*

4.3.3. The sharp-interface limit

Before entering the sharp-interface limit let us mention a nice fact concerning the thickness of the diffuse interface. The following theorem, which is a direct consequence of the boundedness of the Ginzburg–Landau energy along a sequence of minimizers for $\varepsilon \to 0$, shows that the thickness of the interface up to an infinitesimally small error is $\mathcal{O}(\varepsilon)$, which is a rigorous justification of the formal discussion contained in Section 2.1.4.

Theorem 4.3.8. *Let $\gamma > 0$ and $m \in (0,1)$ be arbitrary. For any $\varepsilon > 0$, let φ_ε be a minimizer of the optimization problem $(\mathrm{OP}_\gamma^\varepsilon)$. Then, there is a constant $C > 0$ such that for all $0 < \delta < \frac{1}{2}$ it holds that*

$$\mathcal{L}^d\left(\{\delta \le \varphi_\varepsilon \le 1-\delta\}\right) \le \frac{C\varepsilon}{\alpha_\delta \gamma} \quad \text{with} \quad \alpha_\delta := \min_{[\delta,1-\delta]} \psi > 0. \tag{4.3.8}$$

The proofs of Theorem 4.3.7 and Theorem 4.3.8 are presented in Section 4.4.

Combining the preceding two results, we deduce that every minimizer φ_ε of $(\mathrm{OP}_\gamma^\varepsilon)$ is symmetric-decreasing and exhibits the expected phase-field structure, i.e., for any $0 < \delta < \frac{1}{2}$, the width of the annulus on which φ_ε attains values between δ and $1-\delta$ is of order ε. This behavior is illustrated in Figure 4.2.

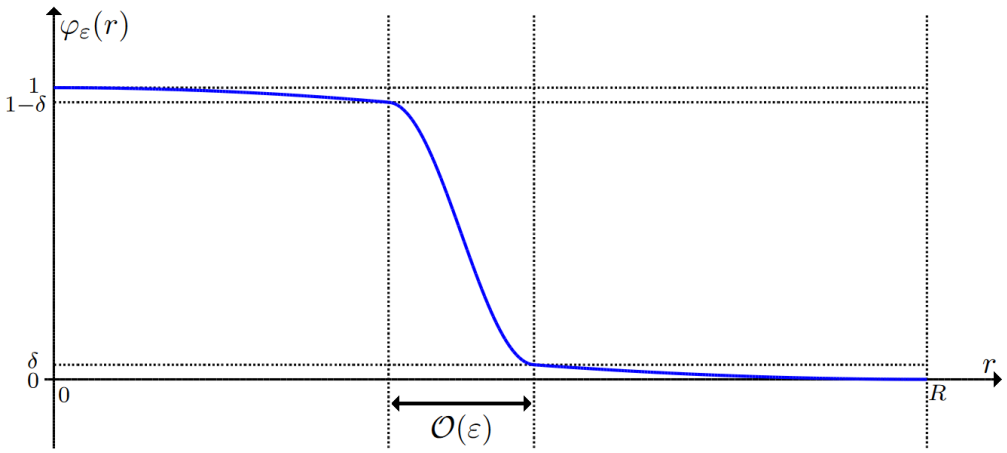

Figure 4.2: Schematic sketch of a minimizer φ_ε in radial direction $r = |x|$.

Now, we investigate the limit $\varepsilon \to 0$. Therefore, let us fix a sequence of minimizers $(\varphi_\varepsilon)_{\varepsilon>0}$ of $(\mathrm{OP}_\gamma^\varepsilon)$. We intend to show that this sequence converges to the characteristic function of the ball centered at the origin with volume $m\,|\Omega|$ and that this ball is a minimizer of a suitable limit cost functional (see Theorem 4.3.15). To this end, we recall the most important aspects from Chapter 3 in our setting.

First of all, we recall the limit eigenvalue problem, i.e., the eigenvalue problem corresponding to (4.3.2) on the sharp-interface level. For any given $\varphi \in BV(\Omega;\{\pm 1\})$, we want to solve

$$-\Delta w = \lambda w \qquad \text{in } E^\varphi, \tag{4.3.9a}$$

$$w|_{\partial E^\varphi} = 0 \qquad \text{on } \partial E^\varphi, \tag{4.3.9b}$$

where

$$E^\varphi = \{x \in \Omega \mid \varphi(x) = 1\}.$$

Note that, in general, E^φ is only a set of finite perimeter and therefore, it merely enjoys a very weak regularity. However as in Theorem 3.3.2, the following definition turns out to be the suitable notion of weak solution as it is compatible with the sharp-interface limit $\varepsilon \to 0$ (see Proposition 4.3.12).

Definition 4.3.9. *Let*

$$\varphi \in \Phi_m^0 := \{\varphi \in BV(\Omega; \{0,1\}) \mid \fint_\Omega \varphi \,\mathrm{d}x = m\}$$

be arbitrary.

(a) *For any given $\lambda \in \mathbb{R}$, a function $w \in V^\varphi$ is called a **weak solution** of the system* (4.3.9) *if the weak formulation*

$$\int_\Omega \nabla w \cdot \nabla \eta \,\mathrm{d}x = \lambda \int_\Omega w\eta \,\mathrm{d}x \tag{4.3.10}$$

is satisfied for all test functions $\eta \in V^\varphi$, where

$$V^\varphi = \left\{\eta \in H_0^1(\Omega) \;\middle|\; \eta = 0 \text{ a.e. in } \Omega\backslash E^\varphi\right\} = \tilde{H}_0^1(E^\varphi),$$

see Section 2.2.6.

(b) *A real number $\lambda^{0,\varphi}$ is called an **eigenvalue** associated with φ, if there exists at least one non-trivial weak solution $w^{0,\varphi} \in H_0^1(\Omega)$ of the eigenvalue problem* (4.3.9) *written for $\lambda = \lambda^{0,\varphi}$.*

*In this case, $w^{0,\varphi}$ is called an **eigenfunction** to the eigenvalue $\lambda^{0,\varphi}$.*

We recall the following proposition which is a direct consequence of Theorem 3.3.2.

Proposition 4.3.10. *Suppose that $\varphi \in \Phi_m^0$ with $V^\varphi \neq \{0\}$.*

(a) *The minimum in*

$$\min\left\{\frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega |v|^2 \,\mathrm{d}x}\,\middle|\, v \in V^\varphi\backslash\{0\}\right\} =: \lambda_1^{0,\varphi}. \tag{4.3.11}$$

is attained and any minimizer $w \in V^\varphi\backslash\{0\}$ is an eigenfunction of the limit problem (4.3.9) *to the eigenvalue $\lambda_1^{0,\varphi}$ in the sense of Definition 4.3.9*(b).

(b) *$\lambda_1^{0,\varphi} > 0$ is the smallest eigenvalue of the limit problem* (4.3.9) *in the sense of Definition 4.3.9*(b).

Remark 4.3.11. Note that in Theorem 3.3.2, we are in the situation that V^φ is an infinite dimensional vector space. In this chapter, we only assume that V^φ is non-trivial, but the above proposition can be established analogously using classical spectral theory, see Theorem 2.2.8. If $V^\varphi = \{0\}$, we set $\lambda_1^{0,\varphi} = +\infty$, which is consistent with the above proposition.

Furthermore recall the following continuity result for the principal eigenvalues in the limit $\varepsilon \to 0$ established in Theorem 3.3.5, which will again be crucial for our Γ-limit proof.

Proposition 4.3.12. *Let $(\varphi_\varepsilon)_{\varepsilon>0} \subset L^1(\Omega;[0,1])$ and suppose that $\varphi \in BV(\Omega,\{0,1\})$ with $V^\varphi \neq \{0\}$ such that*

$$\lim_{\varepsilon \searrow 0} \|\varphi_\varepsilon - \varphi\|_{L^1(\Omega)} = 0,$$

Moreover, we demand the additional convergence rate

$$\|\varphi_\varepsilon - \varphi\|_{L^1\left(E^\varphi \cap \{\varphi_\varepsilon < \frac{1}{2}\}\right)} = \mathcal{O}(\varepsilon).$$

Then, there exists an eigenfunction $u \in V^\varphi$ of the limit problem (4.3.9) to the eigenvalue $\lambda_1^{0,\varphi}$ such that

$$\lim_{\varepsilon \searrow 0} \int_\Omega b^\varepsilon(\varphi_\varepsilon) \left| w_1^{\varepsilon,\varphi_\varepsilon} \right|^2 \,\mathrm{d}x = \int_\Omega b^0(\varphi)\, |u|^2 \,\mathrm{d}x = 0,$$

as well as

$$\lim_{\varepsilon \searrow 0} \|w_1^{\varepsilon,\varphi_\varepsilon} - u\|_{H^1(\Omega)} = 0 \quad \textit{and} \quad \lim_{\varepsilon \searrow 0} \lambda_1^{\varepsilon,\varphi_\varepsilon} = \lambda_1^{0,\varphi}.$$

We point out that in Theorem 3.3.5, the above result was established for any eigenvalue in the case where V^φ is an infinite dimensional vector space. However, as we only consider the principal eigenvalue, the Rayleigh quotient merely needs to be minimized over the set $V^\varphi \backslash \{0\}$ (cf. (4.3.11)). It is thus clear that the proof of Theorem 3.3.5 also works under the weaker assumption $V^\varphi \neq \{0\}$.

Since $\lambda_1^{0,\varphi} = +\infty$ if $V^\varphi = \{0\}$, the following corollary is a direct consequence.

Corollary 4.3.13. *Let the assumptions of the previous theorem be fulfilled, but allow for the case $V^\varphi = \{0\}$. Then, it still holds*

$$\limsup_{\varepsilon \searrow 0} \lambda_1^{\varepsilon,\varphi_\varepsilon} \leq \lambda_1^{0,\varphi}.$$

Finally, we consider the limit cost functional

$$J_\gamma^0(\varphi) \coloneqq \begin{cases} \lambda_1^{0,\varphi} + \gamma c_0 \left(P_\Omega(E^\varphi) + \displaystyle\int_{\partial\Omega} \varphi_{|\partial\Omega} \,\mathrm{d}\mathcal{H}^{d-1} \right) & \text{if } \varphi \in \Phi_m^0, \\ +\infty & \text{if } \varphi \in L^1(\Omega) \backslash \Phi_m^0, \end{cases} \tag{4.3.12}$$

where $c_0 = \int_0^1 \sqrt{2\psi(t)}\,\mathrm{d}t$ and $\varphi_{|\partial\Omega} \in L^1(\partial\Omega)$ denotes the trace of the BV function φ (see Section 2.2.4).

Remark 4.3.14. We note that in [135], where the phase-fields are subject to a more complex inhomogeneous space dependent Dirichlet boundary condition, the corresponding term in the limit cost functional resulting from the Ginzburg–Laundau energy is written (transferred to our notation) as

$$\int_\Omega |\nabla \Phi(\varphi)| \,\mathrm{d}x + \int_{\partial\Omega} \left| \Phi(\varphi_{|\partial\Omega}) \right| \,\mathrm{d}\mathcal{H}^{d-1}.$$

Here, the function Φ is defined by

$$\Phi(s) \coloneqq \int_0^s \sqrt{2\psi(t)}\,\mathrm{d}t, \tag{4.3.13}$$

see also (2.2.12) and $V(\Phi(\varphi),\Omega) = \int_\Omega |\nabla\Phi(\varphi)|\ \mathrm{d}x$ denotes the variation of $\Phi(\varphi) \in L^1(\Omega)$ as given in Section 2.2.4. As obviously $\Phi(\varphi) = \Phi(1)\chi_{E^\varphi}$ in $BV(\Omega,\{0,1\})$ we obtain

$$\int_\Omega |\nabla\Phi(\varphi)|\ \mathrm{d}x = c_0 P_\Omega(E^\varphi).$$

Furthermore, due to the definition of the trace in (2.2.10) and $\varphi \in BV(\Omega,\{0,1\})$, we see that also $\varphi_{|\partial\Omega}$ only attains the values 0 and 1. Hence, we have $\Phi(\varphi_{|\partial\Omega}) = \Phi(1)\varphi_{|\partial\Omega}$ in $L^1(\partial\Omega)$, which yields

$$\int_{\partial\Omega} \left|\Phi(\varphi_{|\partial\Omega})\right|\ \mathrm{d}\mathcal{H}^{d-1} = c_0 \int_{\partial\Omega} \varphi_{|\partial\Omega}\ \mathrm{d}\mathcal{H}^{d-1}.$$

Note that, as we are imposing a homogeneous Dirichlet boundary condition, we do not need to rely on the very technical construction of a recovery sequence presented in [135] as we can simply perform a cut-off procedure as in [43]. The idea in [43] is to approximate any finite perimeter set by truncated sets that are compactly contained within Ω. For these truncated sets, we then perform a diffuse interface approximation in the spirit of [126,147]. Using this approach, the need of the additional boundary integral in the limit cost functional can be clearly seen: In the course of this approximation, the boundaries of the truncated sets are getting closer and closer to the boundary of Ω. Therefore, the whole boundary of the limit set has to be perceived by the limit energy. For more details, we refer to the proof of Theorem 4.3.17 given in Section 4.4.
Nevertheless, we think that the technical construction [135] adapted to our setting of homogeneous boundary data may be rather illustrative and thus, for the sake of completeness we additionally include an alternative constructive proof of Theorem 4.3.17 in Section 4.4

The previous discussion allows us to state the desired theorem which states the convergence of minimizers as ε tends to zero.

Theorem 4.3.15. *Let $\varphi_0 \in \Phi_m^0$ be the characteristic function of the ball centered at the origin with volume $m\,|\Omega|$ and let $(\varphi_\varepsilon)_{\varepsilon>0}$ be a sequence of minimizers of $(\mathrm{OP}_\gamma^\varepsilon)$.*
Then

$$\lim_{\varepsilon\searrow 0} \|\varphi_\varepsilon - \varphi_0\|_{L^1(\Omega)} = 0, \quad \lim_{\varepsilon\searrow 0} J_\gamma^\varepsilon(\varphi_\varepsilon) = J_\gamma^0(\varphi_0),$$

and φ_0 is a minimizer of J_γ^0.

As a direct consequence we finally obtain the classical Faber–Krahn theorem in our framework by sending the surface tension parameter γ to zero.

Corollary 4.3.16 (Faber–Krahn theorem for BV functions)**.** *Let $\varphi_0 \in \Phi_m^0$ be the characteristic function of the ball centered at the origin with volume $m\,|\Omega|$. Then, it holds that*

$$\lambda_1^{0,\varphi_0} = \min\left\{\lambda_1^{0,\varphi} \;\middle|\; \varphi \in \Phi_m^0\right\}.$$

We point out that this result at first sight extends the classical Faber–Krahn theorem which merely states that an (open) ball is a minimizer among all *open* sets of the same volume. Of course, this result can also be obtained if one takes the classical Faber–Krahn theorem for granted and then performs the above described regularization of finite perimeter sets. Furthermore the classical Faber–Krahn theorem is stated without the constraint of a surrounding design domain, which requires for a more delicate analysis, as also indicated in the introduction of this thesis. Nevertheless, the purpose of this chapter is not to generalize the Faber–Krahn theorem but to understand the classical Faber–Krahn theorem within our approach as the sharp-interface limit of the Faber–Krahn theorem on the diffuse interface level obtained in Theorem 4.3.7.
In another direction, in [53], it was shown that the Faber–Krahn theorem remains correct if the minimization problem is formulated for the class of *quasi-open* sets.

The proof of Theorem 4.3.15 can be found in Section 4.4. In this proof, the key step is to show that $J_\gamma^\varepsilon \xrightarrow{\Gamma} J_\gamma^0$ as $\varepsilon \to 0$, i.e., J_γ^ε converges to J_γ^0 in the sense of Γ-convergence. Our strategy will be similar as in Section 3.3.2 of the previous Chapter.

The first step is to prove the Γ-convergence for slightly modified functionals F_ε^γ where the corresponding set of admissible phase-fields does not contain a volume constraint. In the proof, we need to revisit the construction of the recovery sequence in [147] as we allow for more general potentials ψ. In order to tackle the Dirichlet boundary constraint hidden in $H_0^1(\Omega)$, we apply in the first proof the idea of [43, Theorem 3.1] and in the second the idea of [135, Lemma 2]. In the first proof, in order to construct a recovery sequence for any given $\varphi \in BV(\Omega;\{0,1\})$, we approximate the corresponding set $\{\varphi = 1\}$ by truncated sets on the *sharp-interface level* which are compactly contained in Ω. In the second proof we introduce an additional profile introducing a "smooth" cut-off on the *diffuse interface level* in order to guarantee the Dirichlet boundary condition.
The Γ-convergence result is stated by the following theorem.

Theorem 4.3.17. *For any $\varepsilon, \gamma > 0$, let the functions $F_\varepsilon^\gamma, F_0^\gamma : L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ be defined as*

$$F_\varepsilon^\gamma(\varphi) = \begin{cases} \lambda_1^{\varepsilon,\varphi_\varepsilon} + \gamma \int_\Omega \frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\,\mathrm{d}x & \text{if } \varphi \in H_0^1(\Omega;[0,1]), \\ +\infty & \text{else,} \end{cases}$$

and

$$F_0^\gamma(\varphi) = \begin{cases} \lambda_1^{0,\varphi} + c_0\gamma \left(P_\Omega(E^\varphi) + \int_{\partial\Omega} \varphi_{|\partial\Omega}\, \mathrm{d}\mathcal{H}^{d-1} \right) & \text{if } \varphi \in BV(\Omega;\{0,1\}), \\ +\infty & \text{else.} \end{cases}$$

Then $F_\varepsilon^\gamma \xrightarrow{\Gamma} F_0^\gamma$.

The second step is to modify the recovery sequence obtained by Theorem 4.3.17, as in Theorem 3.3.9, via suitable C^1-diffeomorphisms such that the modified sequence is actually a recovery sequence for J_γ^ε satisfying the volume constraint included in Φ_m. This is done in the following theorem.

Theorem 4.3.18. *For any $\varepsilon, \gamma > 0$, let the functions $J_\gamma^\varepsilon, J_\gamma^0 : L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ be defined as*

$$J_\gamma^\varepsilon(\varphi) = \begin{cases} \lambda_1^{\varepsilon,\varphi} + \gamma \int_\Omega \frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon}\psi(\varphi)\,\mathrm{d}x & \text{if } \varphi \in \Phi_m, \\ +\infty & \text{else,} \end{cases}$$

and

$$J_\gamma^0(\varphi) = \begin{cases} \lambda_1^{0,\varphi} + c_0\gamma\left(P_\Omega(E^\varphi) + \int_{\partial\Omega} \varphi_{|\partial\Omega}\, \mathrm{d}\mathcal{H}^{d-1}\right) & \text{if } \varphi \in \Phi_m^0, \\ +\infty & \text{else.} \end{cases}$$

Then $J_\gamma^\varepsilon \xrightarrow{\Gamma} J_\gamma^0$.

The proofs of Theorem 4.3.17 and Theorem 4.3.18 are presented in Section 4.4.

Remark 4.3.19. Although for our purposes we have fixed $\Omega = B_R(0)$ the results presented in Theorem 4.3.17 and Theorem 4.3.18 hold true for any bounded, open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, as these proofs do not rely on symmetrization techniques.

4.4. Proofs

We will now present the proofs of above results.

Proof of Theorem 4.3.4. In view of the definition of E^ε given in (4.3.6), the assertion follows directly by using the Pólya–Szegő inequality (Lemma 2.2.12(g)) to estimate the gradient term, and by applying Lemma 2.2.12(d) to the potential term. □

Proof of Theorem 4.3.5. Let $\varphi \in H_0^1(\Omega;[0,1])$ be arbitrary.

First of all, we derive some general inequalities. Therefore, let $w \in H_0^1(\Omega)\backslash\{0\}$ with $w \geq 0$ a.e. in Ω be arbitrary. In view of (A3), the coefficient b^ε is continuous, decreasing and $b^\varepsilon(0) = \beta^\varepsilon$, we infer that the function

$$B^\varepsilon : [0,1] \to [0,\beta^\varepsilon], \quad s \mapsto b^\varepsilon(0) - b^\varepsilon(s) \tag{4.4.1}$$

is continuous and increasing. Hence, according to Lemma 2.2.12(c), we have

$$\left(B^\varepsilon(\varphi)\right)^* = B^\varepsilon(\varphi^*) \quad \text{and} \quad \left(w^2\right)^* = \left(w^*\right)^2$$

almost everywhere in Ω. Applying Lemma 2.2.12(b) and (e), we thus obtain

$$\begin{aligned} \int_\Omega b^\varepsilon(\varphi) w^2 \,\mathrm{d}x &= b^\varepsilon(0)\int_\Omega w^2 \,\mathrm{d}x - \int_\Omega B^\varepsilon(\varphi) w^2 \,\mathrm{d}x \\ &\geq b^\varepsilon(0)\int_\Omega \left(w^*\right)^2 \mathrm{d}x - \int_\Omega \left(B^\varepsilon(\varphi)\right)^* \left(w^2\right)^* \mathrm{d}x \\ &= b^\varepsilon(0)\int_\Omega \left(w^*\right)^2 \mathrm{d}x - \int_\Omega B^\varepsilon(\varphi^*) \left(w^*\right)^2 \mathrm{d}x \\ &= \int_\Omega b^\varepsilon(\varphi^*) \left(w^*\right)^2 \mathrm{d}x. \end{aligned} \tag{4.4.2}$$

In particular, since $\varphi^{**} = \varphi^*$, this already entails that

$$\int_\Omega b^\varepsilon(\varphi^*) w^2 \,\mathrm{d}x \geq \int_\Omega b^\varepsilon(\varphi^*) \left(w^*\right)^2 \mathrm{d}x. \tag{4.4.3}$$

These general estimates can now be used to prove the assertion $\lambda_1^{\varepsilon,\varphi^*} \leq \lambda_1^{\varepsilon,\varphi}$. Therefore, we define the functional

$$\mathcal{R} : H_0^1(\Omega)\backslash\{0\} \to \mathbb{R}, \quad w \mapsto \frac{\int_\Omega |\nabla w|^2 + b^\varepsilon(\varphi^*)\, w^2 \,\mathrm{d}x}{\|w\|^2_{L^2(\Omega)}}\,. \tag{4.4.4}$$

We now consider the positive-normalized eigenfunction $w_1^{\varepsilon,\varphi^*}$ associated with φ, which is obviously a minimizer of $\mathcal{R}$. Using (4.4.3) along with the Pólya–Szegő inequality (Lemma 2.2.12(g)) and Lemma 2.2.12(b), we find that

$$\mathcal{R}\Big((w_1^{\varepsilon,\varphi^*})^*\Big) \leq \mathcal{R}\Big(w_1^{\varepsilon,\varphi^*}\Big).$$

Thus, $(w_1^{\varepsilon,\varphi^*})^*$ is also a minimizer of $\mathcal{R}$ and thus, due to Proposition 4.3.2(c), it is an eigenfunction to the eigenvalue $\lambda_1^{\varepsilon,\varphi^*}$. As $(w_1^{\varepsilon,\varphi^*})^*$ is non-negative and L^2-normalized, this is enough to deduce

$$(w_1^{\varepsilon,\varphi^*}) = (w_1^{\varepsilon,\varphi^*})^* \quad \text{a.e. in } \Omega,$$

as the eigenspace to $\lambda_1^{\varepsilon,\varphi^*}$ is one-dimensional. On the other hand, the Courant–Fischer characterization (4.3.3) yields that for any $w \in H_0^1(\Omega)\backslash\{0\}$ with $w \geq 0$ a.e. in Ω, we have

$$\lambda_1^{\varepsilon,\varphi^*} \leq \frac{\int_\Omega |\nabla w^*|^2 + b^\varepsilon(\varphi^*)\,(w^*)^2 \,\mathrm{d}x}{\|w^*\|^2_{L^2(\Omega)}} \leq \frac{\int_\Omega |\nabla w|^2 + b^\varepsilon(\varphi)\,(w)^2 \,\mathrm{d}x}{\|w\|^2_{L^2(\Omega)}}. \tag{4.4.5}$$

Here, we applied the Pólya–Szegő inequality (Lemma 2.2.12(g)), estimate (4.4.2) and Lemma 2.2.12(b). Hence, choosing $w = w_1^{\varepsilon,\varphi}$ we use Proposition 4.3.2(c) to conclude

$$\lambda_1^{\varepsilon,\varphi^*} \leq \lambda_1^{\varepsilon,\varphi} \tag{4.4.6}$$

and thus, the proof is complete. □

Proof of Theorem 4.3.7. Let $\varphi \in \Phi_m$ be any minimizer of the optimization problem $(\mathrm{OP}_\gamma^\varepsilon)$. Combining the estimates (4.3.7) and (4.4.6), we deduce that $J_\gamma^\varepsilon(\varphi^*) \leq J_\gamma^\varepsilon(\varphi)$.Since φ is a minimizer, this implies that

$$J_\gamma^\varepsilon(\varphi^*) = J_\gamma^\varepsilon(\varphi). \tag{4.4.7}$$

This proves that the symmetric-decreasing rearrangement φ^* is also a minimizer of the optimization problem $(\mathrm{OP}_\gamma^\varepsilon)$.

Therefore, it remains to prove that the eigenfunction $w_1^{\varepsilon,\varphi}$ and the minimizer φ are symmetric-decreasing, meaning that $\varphi = \varphi^*$ and $w_1^{\varepsilon,\varphi} = (w_1^{\varepsilon,\varphi})^*$ a.e. in Ω.

First of all, using (4.3.7) and (4.4.7), we obtain the estimate

$$\lambda_1^{\varepsilon,\varphi} = J_\gamma^\varepsilon(\varphi) - \gamma E^\varepsilon(\varphi) \leq J_\gamma^\varepsilon(\varphi^*) - \gamma E^\varepsilon(\varphi^*) = \lambda_1^{\varepsilon,\varphi^*}.$$

Hence, in combination with (4.4.6), we conclude that

$$\lambda := \lambda_1^{\varepsilon,\varphi} = \lambda_1^{\varepsilon,\varphi^*}. \tag{4.4.8}$$

As above let $w = w_1^{\varepsilon,\varphi}$ be the positive-normalized eigenfunction corresponding to the principal eigenvalue λ associated with the minimizer φ. Combining (4.4.5) with (4.4.8) we arrive at

$$\lambda_1^{\varepsilon,\varphi^*} = \frac{\int_\Omega |\nabla w^*|^2 + b^\varepsilon(\varphi^*)\,(w^*)^2 \,\mathrm{d}x}{\|w^*\|^2_{L^2(\Omega)}} = \frac{\int_\Omega |\nabla w|^2 + b^\varepsilon(\varphi)\,(w)^2 \,\mathrm{d}x}{\|w\|^2_{L^2(\Omega)}} = \lambda_1^{\varepsilon,\varphi}. \tag{4.4.9}$$

Since, according to Proposition 4.3.2(c), the eigenspace to the eigenvalue λ is one-dimensional, we conclude

$$w^* = \big(w_1^{\varepsilon,\varphi}\big)^* = w_1^{\varepsilon,\varphi^*} \quad \text{a.e. in } \Omega. \tag{4.4.10}$$

Moreover, Proposition 4.3.2(c) further yields $w^* > 0$ a.e. in Ω. As w^* is a symmetric-decreasing rearrangement, it follows from Lemma 2.2.12(a) that $w^* > 0$ actually holds *everywhere* in Ω, which will be crucial in the following. Since $\|w^*\|_{L^2(\Omega)} = \|w\|_{L^2(\Omega)} = 1$, (4.4.9) entails that

$$\int_\Omega |\nabla w|^2 + b^\varepsilon(\varphi) w^2 \,\mathrm{d}x = \int_\Omega |\nabla w^*|^2 + b^\varepsilon(\varphi^*)(w^*)^2 \,\mathrm{d}x. \tag{4.4.11}$$

Invoking (4.4.2), we thus obtain

$$\int_\Omega |\nabla w|^2 \,\mathrm{d}x - \int_\Omega |\nabla w^*|^2 \,\mathrm{d}x = \int_\Omega b^\varepsilon(\varphi^*)(w^*)^2 \,\mathrm{d}x - \int_\Omega b^\varepsilon(\varphi) w^2 \,\mathrm{d}x \le 0. \tag{4.4.12}$$

Hence, we have equality in the Pólya–Szegő inequality (Lemma 2.2.12(g)):

$$\int_\Omega |\nabla w|^2 \,\mathrm{d}x = \int_\Omega |\nabla w^*|^2 \,\mathrm{d}x. \tag{4.4.13}$$

In order to prove $w = w^*$ a.e. in Ω, we now intend to show that w^* is even *strictly* symmetric-decreasing. Therefore, we argue by contradiction and assume that this is not the case. This means that there exists a direction $x \in \mathbb{R}^d$ with $|x| = 1$ as well as $0 \le s < t \le R$ such that $w^*(sx) = w^*(tx) =: c$. As the function w^* is positive in Ω, we deduce that $c > 0$. Moreover, since w^* is non-increasing in radial direction, we infer that $w^*(\tau x) = c$ for all $\tau \in [s,t]$. Because of spherical symmetry, this already implies

$$w^* = c \quad \text{in } \big\{x \in \Omega \,\big|\, s \le |x| \le t\big\}. \tag{4.4.14}$$

We further know from Corollary 4.3.3 that $w^* \in H^2(\Omega)$ is a strong solution of the eigenvalue problem (4.3.1). Hence, we have

$$0 = \Delta w^* = \big(b^\varepsilon(\varphi^*) - \lambda\big) w^* \quad \text{a.e. in } \big\{x \in \Omega \,\big|\, s < |x| < t\big\}.$$

As $w^* > 0$ in Ω and since $b^\varepsilon(\varphi^*) - \lambda$ is non-decreasing and defined *everywhere* in Ω, we infer that

$$\begin{aligned} &b^\varepsilon(\varphi^*) - \lambda = 0 \quad \text{in } \big\{x \in \Omega \,\big|\, s < |x| < t\big\}, \\ &b^\varepsilon(\varphi^*) - \lambda \ge 0 \quad \text{in } \ A \coloneqq \big\{x \in \Omega \,\big|\, s < |x| < R\big\}, \end{aligned}$$

which in turn implies

$$\Delta w^* = \big(b^\varepsilon(\varphi^*) - \lambda\big) w^* \ge 0 \quad \text{a.e. in } A. \tag{4.4.15}$$

Due to (4.4.14), we have

$$\sup_A w^* = c,$$

since w^* is symmetric-decreasing. Applying the strong maximum principle for the Laplace operator (see, e.g., [99, Theorem 8.19] with $L = \Delta$), we infer that

$$w^* = c \quad \text{in } A.$$

However, since $c > 0$, this is a contradiction to the zero-trace condition hidden in $w^* \in H_0^1(\Omega)$. We have thus proven that w^* is *strictly* symmetric-decreasing.

As a consequence, we have $\nabla w^* \neq 0$ almost everywhere in Ω, meaning that

$$\mathcal{L}^d(\{x \in \Omega \,|\, \nabla w^* = 0\}) = 0.$$

Recalling that $w^* > 0$ in Ω, we use Lemma 2.2.12(g) along with (4.4.13) to conclude

$$w = w^* \quad \text{a.e. in } \Omega,$$

meaning that w is symmetric-decreasing. Plugging this into (4.4.11) we arrive at

$$\int_\Omega (b^\varepsilon(\varphi^*) - b^\varepsilon(\varphi))(w^*)^2 \,\mathrm{d}x = 0.$$

Since $w^* > 0$ in Ω, we infer

$$b^\varepsilon(\varphi^*) = b^\varepsilon(\varphi) \quad \text{a.e. in } \Omega.$$

This directly implies $\varphi = \varphi^*$ a.e. in Ω as b^ε is strictly decreasing (and thus injective). This proves that φ is symmetric-decreasing and hence, the proof is complete. □

Proof of Theorem 4.3.17. The lim inf inequality of the Ginzburg–Landau part directly carries over from [135, Lemma 1] as in our case, we only need to consider trivial extensions of $\varphi_\varepsilon, \varphi$ instead of the more complicated boundary value function h^ε discussed there. We point out that (A1) is sufficient for the proof to work, as only the continuity in $[0,1]$ and the non-negativity of the potential ψ is needed to ensure that the function Φ defined in (4.3.13) is well-defined and differentiable. Note that we actually need to include the factor $\sqrt{2}$ in the definition of Φ in order for the Modica–Mortola trick

$$\int_\Omega |\nabla \Phi(\varphi_\varepsilon)| \,\mathrm{d}x = \int_\Omega \sqrt{2\psi(\varphi_\varepsilon)} |\nabla \varphi_\varepsilon| \,\mathrm{d}x \leq \int_\Omega \frac{\varepsilon}{2} |\nabla \varphi_\varepsilon|^2 + \frac{1}{\varepsilon}\psi(\varphi_\varepsilon) \,\mathrm{d}x = E^\varepsilon(\varphi_\varepsilon)$$

in the proof of the lim inf inequality to work, see e.g., also [37, Formula (3.61)]. In [135], the factor 2 is used which is due to the fact that there the gradient term in the energy is not scaled with $\frac{1}{2}$.

To verify the lim inf inequality for the eigenvalue term, we proceed as in Theorem 3.3.11. However, we need to be careful with the constraints for the limit cost functional. In Chapter 3, the additional constraint $\varphi \in \mathcal{U}$ was imposed which fixes a non-trivial open set S_1 such that $S_1 \subset \{\varphi = 1\}$, see also Section 2.1.2. This guaranteed that all the eigenvalues are finite.

In our framework, we now additionally need to consider the case $\lambda_1^\varphi = +\infty$. Therefore we consider $\varphi_\varepsilon \to \varphi$ in $L^1(\Omega)$ such that

$$\liminf_{\varepsilon \searrow 0} F_\varepsilon^\gamma(\varphi_\varepsilon) < +\infty.$$

Applying Fatou's Lemma to the potential term as in [126, Proposition 1] (which only requires the continuity of ψ demanded in (A1)), we obtain that $\varphi \in BV(\Omega; \{0,1\})$ and, up to subsequence extraction, that the sequence of eigenvalues $(\lambda^{\varepsilon,\varphi_\varepsilon})_{\varepsilon>0}$ is bounded. Hence, as in *Step 3* of the proof of Theorem 3.3.11, the sequence of minimizers $(v_\varepsilon)_{\varepsilon>0}$ of the problem

$$\min\left\{ \int_\Omega |\nabla v|^2 \, \mathrm{d}x + \int_\Omega b_\varepsilon(\varphi_\varepsilon)\, |v|^2 \, \mathrm{d}x \,\middle|\, \begin{array}{l} v \in H_0^1(\Omega), \\ \|v\|_{L^2(\Omega)} = 1 \end{array} \right\}$$

fulfills

$$v_\varepsilon \rightharpoonup \overline{v} \quad \text{in } H_0^1(\Omega), \qquad v_\varepsilon \to \overline{v} \quad \text{in } L^2(\Omega), \qquad v_\varepsilon \to \overline{v} \quad \text{a.e. in } \Omega,$$

after another subsequence extraction. Now due to the boundedness of the sequence $\int_\Omega b_\varepsilon(\varphi_\varepsilon)\, |v_\varepsilon|^2 \, \mathrm{d}x$ we proceed as in *Step 4* of the proof of Theorem 3.3.5 and use Fatou's lemma to infer $\overline{v} \in V^\varphi$. Consequently, V^φ is non-trivial and $\lambda_1^{0,\varphi} < \infty$. This means that the case $\lambda_1^{0,\varphi} = +\infty$ can only occur if also

$$\liminf_{\varepsilon \searrow 0} F_\varepsilon^\gamma(\varphi_\varepsilon) = +\infty.$$

Therefore, the lim inf inequality is established as in Theorem 3.3.11.

It remains to prove the lim sup inequality. First of all, as already mentioned in Remark 4.2.1, we want to show that the proof given in [147, Theorem 1] for the smooth double well potential carries over to general potentials satisfying assumptions (A1) and (A2). The key step is to consider the ordinary differential equation

$$\begin{cases} \eta'(t) = \sqrt{2\psi(\eta(t))}, \\ \eta(0) = \frac{1}{2}. \end{cases} \tag{4.4.16}$$

As the right hand side is locally Lipschitz away from $\eta(t) = 0$ and $\eta(t) = 1$, the Picard–Lindelöf theorem provides the existence of a unique maximal solution η. Moreover, since the right-hand side is non-negative, we know that η is non-decreasing. Now, depending on the choice of the potential ψ, the values 0 and/or 1 can either be reached for finite t or the solution tends to those values asymptotically, meaning that

$$\lim_{t\to\infty} \eta(t) = 1 \quad \text{and/or} \quad \lim_{t\to-\infty} \eta(t) = 0. \tag{4.4.17}$$

If the solution η satisfies $\eta(t_0) = 0$ for some $t_0 < 0$ (or $\eta(t_1) = 1$ for some $t_1 > 0$), it holds that $\eta(t) = 0$ for all $t \le t_0$ (or $\eta(t) = 1$ for all $t \ge t_1$). In particular, in any case, the solution exists for all $t \in \mathbb{R}$. These properties follow from classical ODE theory, exploiting that, due to (A1), the right-hand side of (4.4.16) is strictly positive whenever $\eta(t) \in (0,1)$.

As in [147, (1.22)], we construct the profile function $\rho_\varepsilon^P : \mathbb{R} \to [0,1]$ by defining

$$\rho_\varepsilon^P(t) \coloneqq \begin{cases} 1 & \text{for } t > 2\sqrt{\varepsilon}, \\ 1 + \left(1 - \eta\left(\varepsilon^{-\frac{1}{2}}\right)\right)\left(\frac{t-2\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\right), & \text{for } \sqrt{\varepsilon} \le t \le 2\sqrt{\varepsilon}, \\ \eta\left(\frac{t}{\varepsilon}\right), & \text{for } |t| \le \sqrt{\varepsilon}, \\ \eta\left(-\varepsilon^{-\frac{1}{2}}\right)\left(\frac{t+2\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\right), & \text{for } -2\sqrt{\varepsilon} \le t \le -\sqrt{\varepsilon}, \\ 0, & \text{for } t < -2\sqrt{\varepsilon}. \end{cases} \tag{4.4.18}$$

The idea behind these profiles is to use the solution of (4.4.16) and possibly linearly interpolate the values where η is close to 0 or 1. This interpolation is necessary to obtain a transition from 0 to 1 on a *finite* interval scaling suitably with ε, even though the solution of (4.4.16) does possibly not reach these values in finite time.

In case η actually reaches the values 0 and/or 1, the interpolations in the second and fourth line of this definition become trivial and therefore negligible, provided that $\varepsilon > 0$ is sufficiently small.

In case the values 0 and/or 1 are only reached asymptotically, we still need the following exponential convergence rates in order for the proof in the spirit of [147] to work out:

- If the value 1 is only reached asymptotically, there exists $T_1 > 0$ as well as constants $C_1, a_1 > 0$ such that

$$|1 - \eta(t)| \le C_1 \exp(-a_1 t) \quad \text{for all } t \ge T_1. \tag{4.4.19}$$

- If the value 0 is only reached asymptotically, there exists $T_0 > 0$ as well as constants $C_0, a_0 > 0$ such that

$$|\eta(t)| \le C_0 \exp(-a_0 t) \quad \text{for all } t \le -T_0. \tag{4.4.20}$$

We will only prove estimate (4.4.19) since estimate (4.4.20) can be established completely analogously. Therefore, we assume that η reaches the value 1 only asymptotically, as otherwise above estimates are trivial. Due to the monotonicity of η, we have $\eta(t) \in \left[\frac{1}{2}, 1\right)$ and thus $\psi(\eta(t)) > 0$ for all $t \in [0,\infty)$. Hence, η is twice continuously differentiable with

$$\eta''(t) = \frac{1}{2\sqrt{2\psi(\eta(t))}} 2\psi'(\eta(t))\eta'(t) = \psi'(\eta(t)) \quad \text{for all } t \in [0,\infty).$$

Combining (4.4.17) and (4.4.16), we further deduce

$$\lim_{t\to\infty} \eta'(t) = 0.$$

Hence, for any $\xi \in [0,\infty)$, we have

$$\int_\xi^\infty \psi'(\eta(t))\,\mathrm{d}t = -\eta'(\xi). \tag{4.4.21}$$

Let us now assume that $\psi'(1) \neq 0$. Since $\psi\colon [0,1] \to \mathbb{R}$ possesses a local minimum at 1, this already entails $\psi'(1) < 0$. Hence, due to the continuity of ψ', we have

$$\psi'(s) < \frac{1}{2}\psi'(1) < 0$$

for all $s \in (0,1)$ in a suitably small neighborhood around 1. However, this is an obvious contradiction to the finiteness of the integral in (4.4.21). We thus conclude

$$\psi'(1) = 0. \tag{4.4.22}$$

This equality will now be the crucial ingredient in applying the comparison principle for ODEs. Recalling that $\psi \in C^2([0,1])$, for any $s \in (0,1]$, we consider the Taylor expansion

$$\psi(s) = \psi(1) + \psi'(1)(s-1) + \frac{1}{2}\psi''(\xi_s)(s-1)^2 = \frac{1}{2}\psi''(\xi_s)(s-1)^2, \tag{4.4.23}$$

for a ξ_s between 1 and s. In the light of (4.4.22) and (A2), there exist $c, \delta > 0$ such that

$$|\psi''(x)| \geq c > 0, \quad \text{for all } x \in B_\delta(1) \cap [0,1].$$

Hence, defining $a_1 := \sqrt{c}$, we obtain

$$\sqrt{2\psi(s)} \geq a_1(1-s) \tag{4.4.24}$$

for all for any $s \in B_\delta(1) \cap [0,1]$. Due to (4.4.17), there exists $t_1 \in \mathbb{R}$ such that $s_1 := \eta(t_1) \in B_\delta(1) \cap (0,1)$. We now consider the initial value problem

$$\begin{aligned} \mu'(t) &= a_1(1-\mu(t)), \\ \mu(t_1) &= s_1, \end{aligned} \tag{4.4.25}$$

which possesses the unique global solution

$$\mu : \mathbb{R} \to \mathbb{R}, \quad \mu(t) = (s_1 - 1)\exp(-a_1(t-t_1)) + 1.$$

Due to (4.4.24) and the fact that η solves (4.4.25) with the right-hand side being replaced by $\sqrt{2\psi(\eta(t))}$, the comparison principle for ODEs implies

$$\eta(t) \geq \mu(t) \quad \text{for all } t \geq t_1.$$

Hence, we conclude

$$|\eta(t) - 1| = 1 - \eta(t) \leq (1 - s_1)\exp(-a_1(t-t_1)) \quad \text{for all } t \geq t_1, \tag{4.4.26}$$

which proves (4.4.19) with $\bar{C}_1 := (1-s_1)\exp(a_1 t_1) > 0$ and $T_1 := t_1$.

Now, the estimates (4.4.19) and (4.4.20) allow us to continue as in [147]. Let us explain the general strategy first. First of all we need to approximate a general finite perimeter set by suitable smooth sets, because as in the proof of [147, Theorem 1], one knows that for any smooth, bounded, open set $E \subset \mathbb{R}^d$ having finite perimeter and satisfying the transversality condition

$$\mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0,$$

there exists a recovery sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset H^1(\Omega;[0,1])$ which satisfies

$$\begin{cases} \limsup\limits_{\varepsilon \searrow 0} \displaystyle\int_\Omega \frac{\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{1}{\varepsilon}\psi(\varphi_\varepsilon)\,\mathrm{d}x \leq c_0 \mathrm{Per}_\Omega(E), \\ \|\varphi_\varepsilon - \chi_E\|_{L^1(\Omega)} = \mathcal{O}(\varepsilon). \end{cases} \tag{4.4.27}$$

We point out that the above L^1-convergence rate can be obtained by following the line of argument in Step 1 of [147, Theorem 1]. The key fact is that, due to (4.4.19) and (4.4.20), the profile ρ_ε^P converges in the interpolation parts exponentially to 0 and 1, respectively. In the middle part, ρ_ε^P is scaled with ε such that, using the coarea formula and a change of variables, we obtain the desired rate.

Obviously, this recovery sequence is not yet admissible as (in general) it does not fulfill the homogeneous Dirichlet boundary condition hidden in $H_0^1(\Omega)$. Following the idea in [43, Theorem 3.1], we make the following observation: If the finite perimeter set E is *compactly* contained in Ω, then $(\varphi_\varepsilon)_{\varepsilon>0} \subset H_0^1(\Omega)$ is guaranteed provided that $\varepsilon > 0$ is sufficiently small. This directly follows from the construction of φ_ε via the optimal profile ρ_ε^P which vanishes in all points $t < -2\sqrt{\varepsilon}$. This means that outside of a small tubular neighborhood around the boundary of E we indeed have $\varphi_\varepsilon = 0$.
Therefore, we now approximate any finite perimeter set $E \subset \Omega$ by smooth, open finite perimeter sets which are compactly contained in Ω. Although the line of argument is outlined in the proof of [43, Theorem 3.1], we highlight the key steps in order to present a comprehensive proof.

Let now $\varphi \in BV(\Omega; \{0,1\})$ be arbitrary and $E := \{\varphi = 1\}$. In the following, we use the notation

$$E^\Omega := E \cap \Omega, \quad \lambda_1^0(E) := \lambda_1^{0,\chi_{E^\Omega}} \quad \text{and} \quad F_0^\gamma(E) := F_0^\gamma(\chi_{E^\Omega}).$$

In order to construct a recovery sequence in $H_0^1(\Omega; [0,1])$, we first approximate the finite perimeter set E by a sequence $(E_k)_{k\in\mathbb{N}}$ of bounded, smooth, open sets $E_k \subset \mathbb{R}^d$ fulfilling

$$\begin{cases} \mathcal{H}^{n-1}(\partial E_k \cap \partial\Omega) = 0, \\ \qquad \mathrm{Per}_\Omega(E_k) \to \mathrm{Per}_\Omega(E) \quad \text{for } k \to \infty, \\ \qquad\qquad \chi_{E_k^\Omega} \to \chi_E \quad \text{in } L^1(\Omega) \text{ for } k \to \infty, \\ \limsup\limits_{k\to\infty} \lambda_1^0(E_k) \le \lambda_1^0(E). \end{cases} \tag{4.4.28}$$

Such a sequence is constructed in the proof of Theorem 3.3.11 relying on ideas of [39, 126, 140]. Note that the second and the third property of (4.4.28) mean that

$$\chi_{E_k^\Omega} \to \chi_E \text{ } \textit{strictly} \text{ in } BV(\Omega) \tag{4.4.29}$$

as $k \to \infty$, see Subsection 2.2.4. Due to the continuity of the trace operator with respect to strict BV-convergence, see Theorem 2.2.19, we further know

$$\|\chi_{E_k^\Omega}\|_{L^1(\partial\Omega)} \to \|\chi_E\|_{L^1(\partial\Omega)} = \int_{\partial\Omega} \varphi_{|\partial\Omega} \, \mathrm{d}\mathcal{H}^{d-1}, \tag{4.4.30}$$

for $k \to \infty$.

Now, the crucial idea of [43, Theorem 3.1] is to perform a further approximation by cutting off E_k in a tubular neighborhood around the boundary of Ω such that the truncated set is compactly contained in Ω. As we fix $k \in \mathbb{N}$ in the following, we omit this index for a cleaner presentation.
For any $\delta > 0$, the truncated set is defined as $E^\delta := E \cap B_\delta$ with

$$B_\delta := \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \delta\}.$$

Obviously, E^δ is compactly contained in Ω and it also is a set of finite perimeter. We now intend to show that

$$\begin{cases} \limsup_{\delta\to 0} F_0^\gamma(E^\delta) \le F_0^\gamma(E), \\ \lim_{\delta\to 0} \left\|\chi_{E^\delta} - \chi_E\right\|_{L^1(\Omega)} = 0. \end{cases} \tag{4.4.31}$$

Then, applying a diagonal sequence argument will yield the desired lim sup inequality, see below.

The L^1 convergence in (4.4.31) is clear by construction. To establish the first line of (4.4.31), we consider the eigenvalue term and the perimeter term separately.
For the eigenvalue term, we again need to rely on the concept of γ-convergence. First of all by using the characterization via Mosco convergence, see Theorem 2.2.36, we can show

$$B_\delta \xrightarrow{\gamma} \Omega,$$

$\delta \to 0$. For the condition (M1) we note that for any $\phi \in H_0^1(\Omega)$ we find a sequence $(\phi_\delta)_{\delta>0} \subset C_0^\infty(\Omega)$ with

$$\phi_\delta \to \phi \quad \text{in } H_0^1(\Omega).$$

Due to the construction of B_δ and the fact that each ϕ_δ has compact support in Ω we know $\phi_\delta \in H_0^1(B_\delta)$ (after possibly relabeling the index δ). The condition (M2) is clear as $B_\delta \subset \Omega$.
Hence, we infer with Proposition 2.2.37

$$E^\delta = E \cap B_\delta \xrightarrow{\gamma} E \cap \Omega = E^\Omega.$$

Note that at this stage we have to be careful not to confuse the notion of eigenvalues, see also Remark 3.3.3, as continuity with respect to γ-convergence is only formulated for the notion of eigenvalue defined on the classical Sobolev space, i.e.,

$$\lambda_1(\omega) := \min\left\{ \frac{\int_\Omega |\nabla v|^2 \, \mathrm{d}x}{\int_\Omega |v|^2 \, \mathrm{d}x} \middle| v \in H_0^1(\omega)\backslash\{0\} \right\},$$

with $\omega \subset \Omega$ quasi-open. Recalling Definition 4.3.9, our notion of the limit eigenvalue is defined as

$$\lambda_1^0(\tilde{E}) := \min\left\{ \frac{\int_\Omega |\nabla v|^2 \, \mathrm{d}x}{\int_\Omega |v|^2 \, \mathrm{d}x} \middle| v \in \tilde{H}_0^1(\tilde{E})\backslash\{0\}, \right\},$$

for any finite perimeter set $\tilde{E} \subset \Omega$. Nevertheless as $E \subset \mathbb{R}^d$ and $B^\delta \subset \Omega$ are smooth and open sets we infer from the theorey in Section 2.2.6 that it holds

$$\tilde{H}_0^1(E \cap B_\delta) = H_0^1(E \cap B_\delta),$$

and thus in our case $\lambda_1^0(E^\delta) = \lambda_1(E^\delta)$. Therefore we can apply the γ-continuity of eigenvalues (3.3.5) and arrive at

$$\lim_{\delta \searrow 0} \lambda_1^0(E^\delta) = \lambda_1^0(E). \tag{4.4.32}$$

For the perimeter term, we obtain

$$P_\Omega(E^\delta) = P_\Omega(E \cap B_\delta) \leq \mathcal{H}^{d-1}(\partial(E \cap B_\delta)) \tag{4.4.33}$$

due to [14, Proposition 3.62], see also Remark 2.2.21. Applying [14, Proposition 2.95], we deduce that

$$\mathcal{H}^{d-1}(\partial E \cap \partial B_\delta) = \mathcal{H}^{d-1}(\partial E \cap \{\operatorname{dist}(\cdot, \partial\Omega) = \delta\}) = 0.$$

for almost every $\delta > 0$. Using the simple fact

$$\partial(E \cap B_\delta) \subset (\partial E \cap \overline{B_\delta}) \cup (\partial B_\delta \cap \overline{E}),$$

we arrive at

$$\begin{aligned}\mathcal{H}^{d-1}(\partial(E \cap B_\delta)) &\leq \mathcal{H}^{d-1}(\partial E \cap B_\delta) + \mathcal{H}^{d-1}(E \cap \partial B_\delta) \\ &= P_{B_\delta}(E) + \mathcal{H}^{d-1}(E \cap \{\operatorname{dist}(\cdot, \partial\Omega) = \delta\}).\end{aligned} \tag{4.4.34}$$

Here, for the equality in the second line, the smoothness of E is crucial. The first summand in the second line of (4.4.34) side can be expressed as $P_{B_\delta}(E) = |D\chi_E|(B_\delta)$, where $|D\chi_E|$ denotes the total variation of the Radon-measure $D\chi_E$ associated with $\chi_E \in BV(\Omega)$ (see, e.g., [121, Chapter 12] or Section 2.2.4). From to the σ-additivity of $|D\chi_E|$, we directly infer

$$\lim_{\delta\to 0} P_{B_\delta}(E) = \lim_{\delta\to 0} |D\chi_E|(B_\delta) = |D\chi_E|(\Omega) = P_\Omega(E). \tag{4.4.35}$$

For the second summand in the second line of (4.4.34), we use the transversality condition $\mathcal{H}^{d-1}(\partial E \cap \partial\Omega) = 0$ in order to apply [140, Lemma 13.9]. This yields

$$\lim_{\delta\to 0} \mathcal{H}^{d-1}(E \cap \{\operatorname{dist}(\cdot, \partial\Omega) = \delta\}) = \mathcal{H}^{d-1}(E \cap \partial\Omega) = \int_{\partial\Omega} \varphi_{|\partial\Omega}\, \mathrm{d}\mathcal{H}^{d-1}. \tag{4.4.36}$$

Noticing that the trace of φ_δ vanishes on $\partial\Omega$ in the sense of Section 2.2.4, we combine (4.4.33)–(4.4.36) to obtain

$$\limsup_{\delta\to 0} P_\Omega(E^\delta) \leq P_\Omega(E). \tag{4.4.37}$$

Combining (4.4.32) and (4.4.37), we eventually conclude (4.4.31).

Now, we close the proof by means of a final diagonal sequence argument. Therefore, we reinstate the index k. Note that, without loss of generality, we can assume E_k^δ, which is compactly contained in Ω, is smooth by performing again the approximation of (4.4.28). We point out that by performing this approximation of E_k^δ, the approximation sets are still compactly contained in Ω. This is because the corresponding proof of Theorem 3.3.11 is based on classical convolution with mollifiers and thus, the set E_k^δ is only modified up to a small tubular neighborhood.

As we now take for granted that the sets E_k^δ are smooth, recalling Remark 4.2.1, it is shown as in [37, 147] that, depending on the potential ψ, for any $k \in \mathbb{N}$ there exists a recovery sequence $\varphi_\varepsilon^{k,\delta} \subset H_0^1(\Omega; [0,1])$ fulfilling (4.4.27) with E replaced by E_k^δ. The interested reader can also look at the next proof where we explicitly construct optimal profiles and perform the necessary computations.

Using the convergence rate and the upper semi-continuity of eigenvalues provided by Corollary 4.3.13, we further know for any for fixed $k \in \mathbb{N}$ and $\delta > 0$,

$$\limsup_{\varepsilon \searrow 0} \lambda_1^{\varepsilon, \varphi_\varepsilon^{k,\delta}} \leq \lambda_1(E_k^\delta)$$

and consequently,

$$\limsup_{\varepsilon \searrow 0} F_\varepsilon^\gamma(\varphi_\varepsilon^{k,\delta}) \leq F_0^\gamma(E_k^\delta).$$

Now, according to (4.4.28),(4.4.30) and (4.4.31), for every $k \in \mathbb{N}$ we can find a sufficiently small $\delta_k > 0$ such that

$$\begin{aligned} \limsup_{k\to\infty} F_0^\gamma(E_k^{\delta_k}) &\leq F_0^\gamma(E) \quad \text{and} \\ \lim_{k\to\infty} \left\| \chi_{E_k^{\delta_k}} - \chi_E \right\|_{L^1(\Omega)} &= 0. \end{aligned}$$

This in turn allows us now to choose also $\varepsilon_k > 0$ small enough such that finally

$$\begin{aligned} \limsup_{k\to\infty} F_{\varepsilon_k}^\gamma(\varphi_{\varepsilon_k}^{k,\delta_k}) &\leq F_0^\gamma(E) \quad \text{and} \\ \lim_{k\to\infty} \left\| \varphi_{\varepsilon_k}^{k,\delta_k} - \chi_E \right\|_{L^1(\Omega)} &= 0. \end{aligned}$$

Thus, the proof is complete. □

Now as already mentioned we want to give an alternative proof of Theorem 4.3.17 which uses the approach of [135] and does not rely on the cut-off procedure of the previous proof, but uses a further profile in order to provide a "smooth" cut-off. Together with the proof of Theorem 3.3.11 this is my personal favorite proof in this thesis.

Alternative proof of Theorem 4.3.17. The first steps of this proof are exactly the same as in the previous proof and we re-enter the previous proof right before the construction of the optimal profile in (4.4.18). We will construct the profile here in the same spirit as in the previous proof, taking into account whether the values 0 and 1 are reached in finite time or not. Thus, for a fixed $p \in (\frac{1}{2}, 1)$ we define the profile

$$\rho_\varepsilon^P(t) \coloneqq \begin{cases} 0 & \text{for } t \geq 0, \\ \eta(-\varepsilon^{p-1})(\frac{-t}{\varepsilon^p}), & \text{for } -\varepsilon^p \leq t \leq 0, \\ \eta\left(\frac{-t-2\varepsilon^p}{\varepsilon}\right), & \text{for } -3\varepsilon^p \leq t \leq -\varepsilon^p, \\ 1 + (1 - \eta\left(\varepsilon^{p-1}\right))\left(\frac{-t-4\varepsilon^p}{\varepsilon^p}\right), & \text{for } -4\varepsilon^p \leq t \leq -3\varepsilon^p, \\ 1, & \text{for } t \leq -4\varepsilon^p. \end{cases} \tag{4.4.38}$$

Here it will be necessary that in contrast to the construction in (4.4.18) we make the transition along a region of thickness $\mathcal{O}(\varepsilon^p)$ with $p > \frac{1}{2}$. We will see the necessity of this fudge parameter in *Part 3* of the proof, as we need that the intersection of two transversal interfaces scales with order $o(\varepsilon)$. Here we see again, that in the case that η reaches 0 or 1 in finite time the interpolations in the second and fourth line of this definition are trivial for $\varepsilon > 0$ small enough and therefore negligible. In these interpolation areas, for $\varepsilon > 0$ small enough, the exponential rate (4.4.19) and (4.4.20) of course holds true as in the

previous proof.

Our goal is to show that for $\varphi \in BV(\Omega;\{0,1\})$ we find a recovery sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset H^1_0(\Omega;[0,1])$ such that

$$\limsup_{\varepsilon \searrow 0} F^\gamma_\varepsilon(\varphi_\varepsilon) \leq F^\gamma_0(\varphi). \tag{4.4.39}$$

First of all as in the previous proof we need to approximate the finite perimeter set $E \coloneqq E^\varphi = \{\varphi = 1\} \subset \Omega$ with a sequence of bounded, open, smooth sets $E_k \subset \mathbb{R}^d$ fulfilling

$$\begin{aligned} \mathcal{H}^{d-1}(\partial E_k \cap \partial\Omega) &= 0, \\ \mathrm{Per}_\Omega(E_k) &\to \mathrm{Per}_\Omega(E) \quad \text{for } k \to \infty, \\ \chi_{E_k^\Omega} &\to \chi_E \quad \text{in } L^1(\Omega) \text{ for } k \to \infty, \\ \limsup_{k\to\infty} \lambda^{0,\varphi_k} &\leq \lambda^{0,\varphi}, \end{aligned} \tag{4.4.40}$$

where $\varphi_k \coloneqq \chi_{E_k^\Omega}$. From the previous proof we additionally recall the trace continuity

$$\|\chi_{E_k^\Omega}\|_{L^1(\partial\Omega)} \to \|\chi_E\|_{L^1(\partial\Omega)} = \int_{\partial\Omega} \varphi \,\mathrm{d}\mathcal{H}^{d-1}, \tag{4.4.41}$$

for $k \to \infty$.

As the sets E_k enjoy the nice regularity properties mentioned above and recalling Remark 4.2.1, it is shown as in [37, 147] that, depending on the potential ψ, for any $k \in \mathbb{N}$ there is a recovery sequence $(\tilde\varphi^k_\varepsilon)_{\varepsilon>0} \subset H^1(\Omega;[0,1])$, constructed via a profile around Γ_k analogous to (4.4.38), fulfilling

$$\limsup_{\varepsilon\searrow 0} \int_\Omega \frac{\varepsilon}{2}\left|\nabla\tilde\varphi^k_\varepsilon\right|^2 + \frac{1}{\varepsilon}\psi(\tilde\varphi^k_\varepsilon)\,\mathrm{d}x \leq c_0 \mathrm{Per}_\Omega(E_k). \tag{4.4.42}$$

Note crucially, that these functions are in general not in our admissible set $H^1_0(\Omega;[0,1])$ but merely in $H^1(\Omega;[0,1])$. These profiles are constructed in the analogous fashion as the upcoming smooth cut-off ρ_ε for which we will perform all the details.
Now as opposed to the previous proof our goal is to correct the possibly wrong boundary conditions of $\tilde\varphi^k_\varepsilon$ on the diffuse interface level by multiplication with a "smooth" cut-off induced directly by the profile ρ^P_ε. Recall that in the previous proof we have considered a non-smooth cut-off on the sharp-interface level.
We consider

$$\rho_\varepsilon\tilde\varphi^k_\varepsilon, \quad \text{where} \quad \rho_\varepsilon = \rho^P_\varepsilon \circ d_{\partial\Omega},$$

and where $d_{\partial\Omega}$ denotes the signed distance with respect to Ω, which is smooth in any tubular neighborhood of $\partial\Omega$ that does not contain the origin, as $\Omega = B_R(0)$, for more details concerning the signed distance function we refer to [99, Section 14.6].
On the other hand the signed distance function d_{Γ_k} with respect to E_k, where we denote $\Gamma_k = E_k$, plays a crucial role in the construction of $\tilde\varphi^k_\varepsilon$ in the classical proofs of [37,126,147]. We use the convention that signed distance functions are negative within the set they are relative to and positive outside the set. Our aim is to derive the sharp upper bound

including the boundary contact energy $\|\chi_{E_k}\|_{L^1(\partial\Omega)}$ for the modified sequence $(\rho_\varepsilon\tilde{\varphi}_\varepsilon^k)_{\varepsilon>0} \subset H_0^1(\Omega;[0,1])$:

$$\limsup_{\varepsilon\searrow 0} \int_\Omega \frac{\varepsilon}{2}\left|\nabla\left(\rho_\varepsilon\tilde{\varphi}_\varepsilon^k\right)\right|^2 + \frac{1}{\varepsilon}\psi(\rho_\varepsilon\tilde{\varphi}_\varepsilon^k)\,\mathrm{d}x \leq c_0\mathrm{Per}_\Omega(E_k) + c_0\,\|\chi_{E_k}\|_{L^1(\partial\Omega)}\,. \tag{4.4.43}$$

Due to the construction in [147, Theorem 1] or [37, Proposition 3.11] we know that $\tilde{\varphi}_\varepsilon^k$ takes the values 0 and 1 at least up to a tubular neighborhood $\{|d_{\Gamma_k}| \leq 2\sqrt{\varepsilon}\}$. But in analogy to (4.4.38) we can here also take a slightly smaller interface, namely $\{|d_{\Gamma_k}| \leq 2\varepsilon^p\}$ by adapting the profile, which does not change the steps of the proof in [147, Theorem 1]. Furthermore, for $\varepsilon > 0$ small enough, in this neighborhood d_{Γ_k} is known to be smooth as ∂E_k is smooth.
Both functions fulfill $|\nabla d| = 1$ almost everywhere in $\mathbb{R}^d$, see [89, Theorem 3.14]. Note that as explained there the weak gradient exists almost everywhere due to Rademacher's theorem, as d is globally Lipschitz.
Note that for the $H^1(\Omega)$ regularity of $\rho_\varepsilon\tilde{\varphi}_\varepsilon^k$ the fact $\rho_\varepsilon, \tilde{\varphi}_\varepsilon^k \in H^1(\Omega)\cap L^\infty(\Omega)$ is crucial and for the zero trace condition the construction of the optimal profile in (4.4.38) provides us with the homogeneous Dirichlet boundary condition. Therefore indeed $\rho_\varepsilon\tilde{\varphi}_\varepsilon^k \in H_0^1(\Omega;[0,1])$ is admissible.

Next, for our purposes we need to introduce a further regularization concerning the sets E_k. Therefore, for $k \in \mathbb{N}$ let us define a sequence of sets E_k^δ which is given as

$$E_k^\delta \coloneqq \{d_{\Gamma_k} < \delta\}\,,$$

where we choose $\delta_0^k > 0$ so small that for all $\delta \in (0,\delta_0^k)$ the signed distance function d_{Γ_k} is smooth. In other words we simply shift the boundary of E_k in normal direction with the factor $\delta > 0$. The key advantage of this approximation is, that it guarantees us that the interfaces $\partial\Omega$ and $\Gamma_k^\delta \coloneqq \partial E_k^\delta$ intersect at an *angle* which for fixed $\delta > 0$ does *not degenerate*. This property which is crucial to *Part 3* of our proof will be discussed there.

As we now have passed from E_k to E_k^δ we of course need to justify why the properties (4.4.40) still hold true if we replace E_k by E_k^δ, and E by E_k there and pass $\delta \to 0$. The boundedness, openness and smoothness are of course not affected as we only shift the boundary in normal direction. The transversality condition

$$\mathcal{H}^{n-1}(\partial E_k^\delta \cap \partial\Omega) = 0, \tag{4.4.44}$$

is true for almost every $\delta \in (0,\delta_0^k)$, as a direct consequence of a "Sard-like" lemma [14, Lemma 2.95] using the transversality of E_k and $\partial E_k^\delta = \{d_{\Gamma_k} = \delta\}$.
The convergence in relative perimeter

$$\mathrm{Per}_\Omega(E_k^\delta) \to \mathrm{Per}_\Omega(E_k) \quad \text{for } \delta \to 0,$$

is true for almost every $\delta \in (0,\delta_0^k)$, as a direct consequence of [140, Lemma 13.9] exploiting the above transversality, due to the relation

$$\mathrm{Per}_\Omega(E_k^\delta) = \mathcal{H}^{d-1}(\{d_{\Gamma_k} = \delta\}\cap\Omega),$$

as the E_k^δ are smooth, see also Remark 2.2.21. The convergence

$$\chi_{E_k^\delta} \to \chi_{E_k} \quad \text{in } L^1(\Omega),$$

is a direct consequence of the fact $E_k \subset E_k^\delta$ and [89, Theorem 3.14] which tells us that

$$\mathcal{L}^d((E_k^\delta \setminus E_k)) = \mathcal{L}^d(\{0 \leq d_{\Gamma_k} < \delta\}) = \int_0^\delta \mathcal{H}^{d-1}(\{d_{\Gamma_k} = t\})\,\mathrm{d}t \to 0,$$

as $\delta \to 0$. Note that [89, Theorem 3.14] is stated for the classical distance function, but this theorem also holds true for the signed distance function as a corollary of the coarea formula Theorem 2.2.9 due to its Lipschitz continuity of d_{Γ_k} and the fact $|\nabla d_{\Gamma_k}| = 1$. Finally, the lim sup inequality of eigenvalues

$$\limsup_{\delta \searrow 0} \lambda^0(E_k^\delta \cap \Omega) \leq \lambda^0(E_k \cap \Omega),$$

is trivial, as by the inclusion $E_k \subset E_k^\delta$ we have

$$\lambda^0(E_k^\delta \cap \Omega) \leq \lambda^0(E_k \cap \Omega).$$

It is now our goal to show that for each $\delta > 0$ and $k \in \mathbb{N}$

$$\limsup_{\varepsilon \searrow 0} \int_\Omega \frac{\varepsilon}{2} \left|\nabla\left(\rho_\varepsilon \tilde{\varphi}_\varepsilon^{k,\delta}\right)\right|^2 + \frac{1}{\varepsilon}\psi(\rho_\varepsilon \tilde{\varphi}_\varepsilon^{k,\delta})\,\mathrm{d}x \leq c_0 \mathrm{Per}_\Omega(E_k^\delta) + c_0 \|\chi_{E_k^\delta}\|_{L^1(\partial\Omega)} \quad \text{and} \tag{4.4.45}$$

$$\|\varphi_\varepsilon^{k,\delta} - \chi_{E_k^\delta}\|_{L^1(\Omega)} = \mathcal{O}(\varepsilon), \tag{4.4.46}$$

where $\varphi_\varepsilon^{k,\delta} := \rho_\varepsilon \tilde{\varphi}_\varepsilon^{k,\delta} \in H_0^1(\Omega;[0,1])$ and the $(\tilde{\varphi}_\varepsilon^{k,\delta})_{\varepsilon>0} \in H^1(\Omega;[0,1])$ is the recovery sequence constructed as in [37, 147] as explained above but now with respect to E_k^δ. Then we can use the above properties (4.4.40), (4.4.41) and the upper semi-continuity of eigenvalues given in Corollary 4.3.13 to choose a suitable diagonal sequence such that

$$\limsup_{k\to\infty} \int_\Omega \frac{\varepsilon_k}{2} \left|\nabla\left(\rho_{\varepsilon_k} \tilde{\varphi}_{\varepsilon_k}^{k,\delta_k}\right)\right|^2 + \frac{1}{\varepsilon_k}\psi(\rho_{\varepsilon_k} \tilde{\varphi}_{\varepsilon_k}^{k,\delta_k})\,\mathrm{d}x \leq c_0 \mathrm{Per}_\Omega(E) + c_0 \|\chi_E\|_{L^1(\partial\Omega)},$$

$$\limsup_{k\to\infty} \lambda_1^{\varepsilon_k, \varphi_{\varepsilon_k}^{k,\delta_k}} \leq \lambda_1^{0,\varphi} \quad \text{and}$$

$$\lim_{k\to\infty} \left\|\varphi_{\varepsilon_k}^{k,\delta_k} - \varphi\right\|_{L^1(\Omega)} = 0,$$

and the proof is done.

In the following, unless mentioned explicitly, we fix a $\delta \in (0, \delta_0^k)\setminus\mathcal{N}$ where $\mathcal{N}$ is a Lebesgue null set chosen such that all the above properties hold true for δ, recalling the restrictions on δ induced by (4.4.44). Therefore by abuse of notation we write E_k for E_k^δ and Γ_k for Γ_k^δ.

The distances d_{Γ_k} and $d_{\partial\Omega}$ will help us in the following to separate the relevant transition parts of Ω into three regions, Region I, Region II, and Region III, see Figure 4.3.
Region I denotes the transition in the bulk, i.e., here we pass from E_k into Ω or vice versa. This region will provide us with the perimeter term in (4.4.45). Region II is the transition from E_k to the boundary $\partial\Omega$ which enforces a contact energy due to the homogeneous Dirichlet boundary condition on the phase-field. Thus, this region will provide us with boundary integral in (4.4.45). Finally in Region III the previous two regions intersect. Intuitively in this region two interface scaling both with $\mathcal{O}(\varepsilon^p)$ meet transversally, recalling

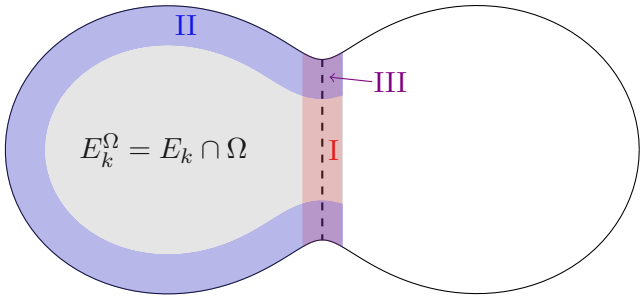

Figure 4.3: The three relevant regions for energy contributions of the recovery sequence. The bell-shaped region is the domain Ω. The set E_k is on the left; its boundary Γ_k within Ω is indicated by the dashed vertical line.

(4.4.44), thus intuitively this junction region scales with $o(\varepsilon)$ as $p > \frac{1}{2}$ and thus we will show that this region does not contribute to the energy.
With these preliminary remarks we can now enter the proof.

Region I: Transition in the bulk. Let us consider those points in Ω which have a sufficiently large distance from $\partial\Omega$ namely those which fulfill $d_{\partial\Omega}(x) \leq -4\varepsilon^p$. This choice is due to the construction of ρ_ε^P in (4.4.38) which guarantees that in this part $\rho_\varepsilon = 1$. Thus, we obtain

$$\begin{aligned}\int_{\{d_{\partial\Omega}\leq -4\varepsilon^p\}} \frac{\varepsilon}{2}\left|\nabla\left(\rho_\varepsilon\tilde{\varphi}_\varepsilon^k\right)\right|^2 + \frac{1}{\varepsilon}\psi(\rho_\varepsilon\tilde{\varphi}_\varepsilon^k)\,\mathrm{d}x &= \int_{\{d_{\partial\Omega}\leq -4\varepsilon^p\}} \frac{\varepsilon}{2}\left|\nabla\tilde{\varphi}_\varepsilon^k\right|^2 + \frac{1}{\varepsilon}\psi(\tilde{\varphi}_\varepsilon^k)\,\mathrm{d}x \\ &\leq \int_\Omega \frac{\varepsilon}{2}\left|\nabla\tilde{\varphi}_\varepsilon^k\right|^2 + \frac{1}{\varepsilon}\psi(\tilde{\varphi}_\varepsilon^k)\,\mathrm{d}x.\end{aligned}$$

Hence (4.4.42) implies

$$\limsup_{\varepsilon\searrow 0}\int_{\{d_{\partial\Omega}\leq -4\varepsilon^p\}} \frac{\varepsilon}{2}\left|\nabla\left(\rho_\varepsilon\tilde{\varphi}_\varepsilon^k\right)\right|^2 + \frac{1}{\varepsilon}\psi(\rho_\varepsilon\tilde{\varphi}_\varepsilon^k)\,\mathrm{d}x \leq c_0\mathrm{Per}_\Omega(E_k).$$

Region II: Boundary contact energy. Now we consider the part where a transition to the boundary $\partial\Omega$ via ρ_ε takes place but does not intersect with a transition coming from $\tilde{\varphi}_\varepsilon^k$. Speaking in the language of distances we consider the set

$$\{d_{\partial\Omega} \geq -4\varepsilon^p\} \cap (\Omega \setminus \{|d_{\Gamma_k}| \leq 2\varepsilon^p\}) \subset \Omega.$$

Here now $\tilde{\varphi}_\varepsilon^k(x) = 1$ or $\tilde{\varphi}_\varepsilon^k(x) = 0$ and we compute with the help of [99, Lemma 7.7]

$$\begin{aligned}&\int_{\{d_{\partial\Omega}\geq -4\varepsilon^p\}\cap\left(\Omega\setminus\left\{\left|d_{\Gamma_k}\right|\leq 2\varepsilon^p\right\}\right)} \frac{\varepsilon}{2}\left|\nabla\left(\rho_\varepsilon\tilde{\varphi}_\varepsilon^k\right)\right|^2 + \frac{1}{\varepsilon}\psi(\rho_\varepsilon\tilde{\varphi}_\varepsilon^k)\,\mathrm{d}x \\ \leq &\int_{\{d_{\partial\Omega}\geq -4\varepsilon^p\}\cap E_k^\Omega} \frac{\varepsilon}{2}\left|\nabla\rho_\varepsilon\right|^2 + \frac{1}{\varepsilon}\psi(\rho_\varepsilon)\,\mathrm{d}x,\end{aligned}$$

because

$$\left\{\tilde{\varphi}_\varepsilon^k = 1\right\} \subset E_k^\Omega$$

due to the fact that the recovery sequence $\tilde{\varphi}_\varepsilon^k$ satisfies $\tilde{\varphi}_\varepsilon^k < 1$ in $\Omega \setminus E_k$, see [147, (1.22)] or [37, (3.66)]. Recalling that $\rho_\varepsilon = \rho_\varepsilon^P \circ d_{\partial\Omega}$, it is now our task to prove

$$\limsup_{\varepsilon \searrow 0} \int_{\{d_{\partial\Omega} \geq -4\varepsilon^p\} \cap E_k^\Omega} \frac{\varepsilon}{2} \left|\nabla(\rho_\varepsilon^P \circ d_{\partial\Omega})\right|^2 + \frac{1}{\varepsilon}\psi(\rho_\varepsilon^P \circ d_{\partial\Omega}) \, \mathrm{d}x \leq c_0 \left\|\chi_{E_k}\right\|_{L^1(\partial\Omega)}. \quad (4.4.47)$$

We will split the integral in (4.4.47) according to the different cases in (4.4.38) and follow the lines of reasoning of [135] although many steps simplify in our setting as our Dirichlet boundary on Ω is fixed to be 0. In order to provide a self-contained proof we go through the steps here. For a cleaner presentation we set $d := d_{\partial\Omega}$.
So for the first non-trivial case in (4.4.38) we have the integral

$$\int_{\{-\varepsilon^p < d < 0\}} \chi_{E_k^\Omega} \frac{\varepsilon}{2} \left|\nabla\left(\eta\left(-\varepsilon^{p-1}\right)\left(\frac{-d}{\varepsilon^p}\right)\right)\right|^2 \, \mathrm{d}x + \int_{\{-\varepsilon^p < d < 0\}} \chi_{E_k^\Omega} \frac{1}{\varepsilon}\psi\left(\eta\left(-\varepsilon^{p-1}\right)\left(\frac{-d}{\varepsilon^p}\right)\right) \, \mathrm{d}x. \quad (4.4.48)$$

Using the coarea formula from Theorem 2.2.9 and the fact that $|\nabla d| = 1$ almost everywhere, we can estimate the first integral by

$$\begin{aligned}\int_{\{-\varepsilon^p < d < 0\}} \chi_{E_k^\Omega} \frac{\varepsilon}{2} \left|\nabla\left(\eta\left(-\varepsilon^{p-1}\right)\left(\frac{-d}{\varepsilon^p}\right)\right)\right|^2 \, \mathrm{d}x &\leq \frac{1}{2}\Big(\frac{\eta\left(-\varepsilon^{p-1}\right)}{\varepsilon^p}\Big)^2 \int_{-\varepsilon^p}^0 \mathcal{H}^{d-1}(\{d = s\} \cap E_k) \, \mathrm{d}s \\ &\leq \frac{1}{2}\Big(\frac{\eta\left(-\varepsilon^{p-1}\right)}{\varepsilon^p}\Big)^2 \sup_{s \in [-\varepsilon^p, 0]} \Big(\mathcal{H}^{d-1}(\{d = s\} \cap E_k)\Big) \\ &\to 0\end{aligned} \quad (4.4.49)$$

as $\varepsilon \to 0$, due to (4.4.20) (recalling $p \in (\frac{1}{2}, 1)$) and the boundedness of the sup, which we see as follows. We can use [140, Lemma 13.9] whose assumptions are fulfilled as per construction in (4.4.40)

$$\mathcal{H}^{d-1}(\partial\Omega \cap \partial E_k) = 0. \quad (4.4.50)$$

Note that compared to our situation the roles of Ω and E are switched in [140]. Anyways one obtains

$$\mathcal{H}^{d-1}\left(\{d = s\} \cap E_k\right) \to \mathcal{H}^{d-1}(\{d = 0\} \cap E_k) = \mathcal{H}^{d-1}\left(E_k \cap \partial\Omega\right) = \left\|\chi_{E_k^\Omega}\right\|_{L^1(\partial\Omega)} \quad (4.4.51)$$

as $s \to 0$. The last equality holds, because using (2.2.10) the trace of $u := \chi_{E_k^\Omega} \in BV(\Omega)$ is given as

$$u_{|\partial\Omega}(x) = \begin{cases} 1 & \text{if } x \in E_k \cap \partial\Omega, \\ 0 & \text{if } x \notin E_k \cap \partial\Omega. \end{cases}$$

Note that this exploits the transversality (4.4.50) in the sense that it suffices to know

$$u_{|\partial\Omega}(x) = 0 \quad \text{if } x \in \Omega \backslash \overline{E}_k, \quad (4.4.52)$$

in order to to deduce

$$u_{|\partial\Omega}(x) = 0 \quad \text{if } x \in \Omega \backslash E_k.$$

(4.4.52) is clear from the definition (2.2.10).
The convergence (4.4.51) now tells us that for a suitable neighborhood $U(0) \subset \mathbb{R}$ around 0 the function

$$\begin{aligned} U(0) &\to \mathbb{R} \\ s &\mapsto \mathcal{H}^{d-1}(\{d = s\} \cap E_k), \end{aligned}$$

is continuous at $s = 0$. Now considering the sup term in (4.4.49) for any $\varepsilon > 0$ there is an $s_\varepsilon \in [-\varepsilon^p, 0]$ such that

$$\mathcal{H}^{d-1}(\{d = 0\} \cap E_k) \leq \sup_{s\in[-\varepsilon^p,0]} \mathcal{H}^{d-1}(\{d = s\} \cap E_k) \leq \mathcal{H}^{d-1}(\{d = s_\varepsilon\} \cap E_k) + \varepsilon,$$

by definition of the supremum. Combined with above continuity we thus infer that

$$\lim_{\varepsilon \searrow 0} \sup_{s\in[-\varepsilon^p,0]} \mathcal{H}^{n-1}(\{d = s\} \cap E_k) = \left\|\chi_{E_k^\Omega}\right\|_{L^1(\partial\Omega)}. \tag{4.4.53}$$

In particular we now know that the sup term in (4.4.49) has to be bounded for $\varepsilon > 0$ small enough.

So let us move on to the potential term in (4.4.48). We see with the coarea formula and a change of variables

$$\int_{\{-\varepsilon^p<d<0\}} \chi_{E_k^\Omega} \frac{1}{\varepsilon} \psi\left(\eta\left(-\varepsilon^{p-1}\right)\left(\frac{-d}{\varepsilon^p}\right)\right) \,\mathrm{d}x = \varepsilon^{p-1} \int_{-1}^{0} \int_{\{d=s\varepsilon^p\}} \chi_{E_k^\Omega} \psi(-\eta(-\varepsilon^{p-1})s) \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}s.$$

Taylor and using (A1) yields

$$\psi(-\eta(-\varepsilon^{p-1})s) = \psi(0) - \psi'(\xi_{\varepsilon,s})\eta(-\varepsilon^{p-1})s = -\psi'(\xi_{\varepsilon,s})\eta(-\varepsilon^{p-1})s$$

for some $\xi_{\varepsilon,s} \in [0, -\eta(-\varepsilon^{p-1})s] \subset [0, 1]$ with $s \in [-1, 0]$. Again by the exponential decay of η for $t \to -\infty$ in (4.4.20) we infer

$$\begin{aligned} &\varepsilon^{p-1} \int_{-1}^{0} \int_{\{d=s\varepsilon^p\}} \chi_{E_k^\Omega} \psi(-\eta(-\varepsilon^{p-1})s) \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}s \\ &\leq C_0\Big(\sup_{[0,1]}|\psi'|\Big)\varepsilon^{p-1} \exp(-a_0\varepsilon^{p-1})\mathcal{H}^{d-1}(\{d = s\varepsilon^p\} \cap E_k) \to 0, \end{aligned}$$

for $\varepsilon \to 0$.
We note that due to the construction via the linear interpolation the fourth case in (4.4.18) works completely analogously and hence also vanishes for $\varepsilon \to 0$.

So we pass on to the third case in (4.4.38), i.e.,

$$\begin{aligned} &\int_{\{-3\varepsilon^p<d<-\varepsilon^p\}} \chi_{E_k^\Omega} \frac{\varepsilon}{2} \left|\nabla\left(\eta\left(\frac{-d-2\varepsilon^p}{\varepsilon}\right)\right)\right|^2 \,\mathrm{d}x \\ &+ \int_{\{-3\varepsilon^p<d<-\varepsilon^p\}} \chi_{E_k^\Omega} \frac{1}{\varepsilon} \psi\left(\eta\left(\frac{-d-2\varepsilon^p}{\varepsilon}\right)\right) \,\mathrm{d}x. \end{aligned}$$

For the first term we compute using the ODE (4.4.16)

$$\int_{\{-3\varepsilon^p<d<-\varepsilon^p\}} \chi_{E_k^\Omega} \frac{\varepsilon}{2} \left|\nabla\left(\eta\left(\frac{-d-2\varepsilon^p}{\varepsilon}\right)\right)\right|^2 \,\mathrm{d}x = \frac{1}{\varepsilon} \int_{\{-3\varepsilon^p<d<-\varepsilon^p\}} \chi_{E_k^\Omega} \psi\left(\eta\left(\frac{-d-2\varepsilon^p}{\varepsilon}\right)\right) \,\mathrm{d}x.$$

Hence, adding this with the above potential term, in the light of (4.4.45) it remains to show

$$\limsup_{\varepsilon\searrow 0}\frac{2}{\varepsilon}\int_{\{-3\varepsilon^p<d<-\varepsilon^p\}}\chi_{E_k^\Omega}\psi\left(\eta\left(\frac{-d-2\varepsilon^p}{\varepsilon}\right)\right)\,\mathrm{d}x\le c_0\|\chi_{E_k^\Omega}\|_{L^1(\partial\Omega)},\tag{4.4.54}$$

for $\varepsilon\to 0$. Applying the coarea formula and a change of variables we obtain the the left-hand side in (4.4.54) transforms to

$$\begin{aligned}&2\varepsilon^{p-1}\int_{-3}^{-1}\int_{\{d=s\varepsilon^p\}}\chi_{E_k^\Omega}\psi\left(\eta\left((-s-2)\varepsilon^{p-1}\right)\right)\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}s\\ \le&\Big(\sup_{-3\le s\le -1}\mathcal{H}^{d-1}\left(\{d=s\varepsilon^p\}\cap E_k\right)\Big)\int_{-3}^{-1}2\varepsilon^{p-1}\psi\left(\eta\left((-s-2)\varepsilon^{p-1}\right)\right)\,\mathrm{d}s.\end{aligned}$$

Due to (4.4.53) we obtain

$$\sup_{-3\le s\le -1}\mathcal{H}^{d-1}\left(\{d=s\varepsilon^p\}\cap E_k\right)\to\|\chi_{E_k^\Omega}\|_{L^1(\partial\Omega)},$$

as $\varepsilon\to 0$.

Using the ODE (4.4.16) and recalling that Φ defined in (4.3.13) is the primitive of $\sqrt{2\psi}$ the remaining factor can be written as

$$\begin{aligned}\int_{-3}^{-1}2\varepsilon^{p-1}\psi\left(\eta\left((-s-2)\varepsilon^{p-1}\right)\right)\,\mathrm{d}s&=-\int_{-3}^{-1}\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left(\eta\left((-s-2)\varepsilon^{p-1}\right)\right)\,\mathrm{d}s\\ &=\Phi\left(\eta\left(\varepsilon^{p-1}\right)\right)-\Phi\left(\eta\left(-\varepsilon^{p-1}\right)\right)\to\Phi(1)-\Phi(0)=\Phi(1)\end{aligned}$$

as $\varepsilon\to 0$, recalling (4.4.17). Now $c_0=\Phi(1)$ per definition and therefore we get (4.4.54). Here we see beautifully how the potential dependent coefficient c_0 explicitly shows up in the limit energy.
Putting everything together, this shows (4.4.47).

Region III: Junction. In the following now the additional approximation in E_k^δ, i.e., shifting the boundary Γ_k in normal direction with the factor $\delta>0$ (see the beginning of the proof), will play a key role. That is why we will write the δ dependence now again explicitly and fix a $\delta>0$ and a $k\in\mathbb{N}$ in the following.
It remains to consider the part where the transitions ρ_ε and $\tilde{\varphi}_\varepsilon^{k,\delta}$ *both* are non-constant, i.e., we now are in the set

$$\Omega_{k,\varepsilon}^\delta:=\{d_{\partial\Omega}\ge -4\varepsilon^p\}\cap\left\{|d_{\Gamma_k^\delta}|\le 2\varepsilon^p\right\}\cap\Omega.$$

As mentioned earlier our goal is to use the "size" of this junction region in order to show that this part has no contribution to the energy. First of all we notice that there is a constant $C_k^\delta>0$ such that for all $x\in\Omega_{k,\varepsilon}^\delta$ and $\varepsilon>0$ in dependence of k small enough

$$\frac{\varepsilon}{2}\left|\nabla\left(\rho_\varepsilon\tilde{\varphi}_\varepsilon^{k,\delta}\right)\right|^2\le\varepsilon\left(\left|\rho_\varepsilon\nabla\tilde{\varphi}_\varepsilon^{k,\delta}\right|^2+\left|\tilde{\varphi}_\varepsilon^{k,\delta}\nabla\rho_\varepsilon\right|^2\right)\le\varepsilon\left(\left|\nabla\tilde{\varphi}_\varepsilon^{k,\delta}\right|^2+|\nabla\rho_\varepsilon|^2\right)\le\frac{C_k^\delta}{\varepsilon}.\tag{4.4.55}$$

This is due to the construction in [37,147] and the computations in *Region II* which both crucially depend on the fact that the profiles are constructed via the solution of the ODE (4.4.16) and that the profiles are scaled at most with order ε^{-1}, as in (4.4.38). What we mean by that is that as in the calculation of the gradient term in (4.4.49) due to the chain rule, at most a factor in $\mathcal{O}(\varepsilon^{-1})$ shows up in the gradient of an optimal transition. This exactly corresponds to our formal considerations of the Ginzburg–Landau energy in Section 2.1.4.
Using (4.4.55) we obtain from the fact that ψ is bounded on $[0,1]$ by (A1)

$$\int_{\Omega^\delta_{k,\varepsilon}} \frac{\varepsilon}{2} \left|\nabla \left(\rho_\varepsilon \tilde{\varphi}^{k,\delta}_\varepsilon\right)\right|^2 + \frac{1}{\varepsilon}\psi(\rho_\varepsilon \tilde{\varphi}^{k,\delta}_\varepsilon)\,\mathrm{d}x \leq \frac{C^\delta_k}{\varepsilon}\int_{\Omega^\delta_{k,\varepsilon}} 1\,\mathrm{d}x. \tag{4.4.56}$$

Now our goal is to apply the general coarea formula Theorem 2.2.9 in *codimension* 2 to the Lipschitz function

$$\begin{aligned} f^\delta_k : \mathbb{R}^d &\to \mathbb{R}^2, \\ x &\mapsto \begin{pmatrix} d_{\Gamma^\delta_k}(x) \\ d_{\partial\Omega}(x) \end{pmatrix}. \end{aligned}$$

This approach is natural as, loosely speaking, the junction of two interface gives rise to a manifold of codimension 2. Using the notation of Theorem 2.2.9 we denote the corresponding coarea factor by

$$Jf^\delta_k(x) \coloneqq \sqrt{\det Df^\delta_k(x)(Df^\delta_k)^T(x)},$$

which is well defined for almost every $x \in \mathbb{R}^d$, as the distances are Lipschitz. We notice that

$$Df^\delta_k = \begin{pmatrix} (\nabla d_\Gamma)^T \\ (\nabla d_{\partial\Omega})^T \end{pmatrix} \in \mathbb{R}^{2\times d}$$

and thus using

$$\left|\nabla d_{\Gamma^\delta_k}\right| = |\nabla d_{\partial\Omega}| = 1 \quad \text{a.e.} \in \mathbb{R}^d, \tag{4.4.57}$$

implies

$$Df^\delta_k(Df^\delta_k)^T = \begin{pmatrix} 1 & \nabla d_{\partial\Omega}\cdot\nabla d_{\Gamma^\delta_k} \\ \nabla d_{\partial\Omega}\cdot\nabla d_{\Gamma^\delta_k} & 1 \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

This gives rise to

$$Jf^\delta_k = \sqrt{1-(\nabla d_{\partial\Omega}\cdot\nabla d_{\Gamma^\delta_k})^2}. \tag{4.4.58}$$

In the light of (4.4.56), our aim is to estimate the volume $\mathcal{L}^d(\Omega^\delta_{k,\varepsilon})$. In order to proceed this way, we show that the δ shift of the boundary of E_k defining the set E^δ_k gives, for each fixed $\delta > 0$, a uniform *lower* bound on Jf^δ_k in $\Omega_{k,\varepsilon}$ (with respect to $\varepsilon \to 0$).
In other words we will now see that this additional approximation will guarantee that for any $\delta > 0$ the interfaces Γ^δ_k and $\partial\Omega$ intersect at least at an angle α_δ that is bounded away

from 0, which then will justify the uniform positivity of the corresponding coarea factor Jf_k^δ in $\Omega_{k,\varepsilon}^\delta$. This non-degeneracy of the angle both interfaces meet with is then crucial to obtain the appropriate scaling of the junction region.
First of all Sard's theorem [121, Theorem 13.15] applied to the smooth function $d_{\Gamma_k} - d_{\partial\Omega}$ (in a suitably small neighborhood around these boundaries in order to guarantee smoothness) tells us that for almost every $\delta \in (0, \delta_0^k)$ it holds

$$\nabla(d_{\Gamma_k} - d_{\partial\Omega}) \neq 0 \quad \text{for all } x \in \Gamma_k^\delta \cap \partial\Omega, \tag{4.4.59}$$

where $\Gamma_k^\delta := \{d_{\Gamma_k} = \delta\} = \partial E_k^\delta$. Here we used the simple relation

$$d_{\Gamma_k} - d_{\partial\Omega} = d_{\Gamma_k} = \delta \quad \text{for all } x \in \Gamma_k^\delta \cap \partial\Omega.$$

We denote $\mathcal{N}$ a Lebesgue null set such that the relation (4.4.59) holds for any $\delta \in (0, \delta_0^k)\backslash\mathcal{N}$. We can repeat this argument to deduce that also for almost every $\delta \in (0, \delta_0^k)\backslash\mathcal{N}$ it holds (by possibly adapting the null set $\mathcal{N}$)

$$\nabla(d_{\Gamma_k} + d_{\partial\Omega}) \neq 0 \quad \text{for all } x \in \Gamma_k^\delta \cap \partial\Omega. \tag{4.4.60}$$

Thus, for $\delta \in (0, \delta_0^k)\backslash\mathcal{N}$, due to the compactness of $\Gamma_k^\delta \cap \partial\Omega \subset \mathbb{R}^d$ and noticing that by construction,

$$d_{\Gamma_k} = d_{\Gamma_k^\delta} + \delta, \tag{4.4.61}$$

we deduce, the existence of two numbers $\alpha_k^{\delta,1}, \alpha_k^{\delta,2} > 0$ such that

$$\alpha_k^{\delta,1} < \left|\nabla d_{\Gamma_k^\delta} - \nabla d_{\partial\Omega}\right|^2 = \left|\nabla d_{\Gamma_k^\delta}\right|^2 - 2\nabla d_{\Gamma_k^\delta} \cdot \nabla d_{\partial\Omega} + \left|\nabla d_{\partial\Omega}\right|^2 = 2(1 - \nabla d_{\Gamma_k^\delta} \cdot \nabla d_{\partial\Omega}) \quad \text{and}$$
$$\alpha_k^{\delta,2} < \left|\nabla d_{\Gamma_k^\delta} + \nabla d_{\partial\Omega}\right|^2 = \left|\nabla d_{\Gamma_k^\delta}\right|^2 + 2\nabla d_{\Gamma_k^\delta} \cdot \nabla d_{\partial\Omega} + \left|\nabla d_{\partial\Omega}\right|^2 = 2(1 + \nabla d_{\Gamma_k^\delta} \cdot \nabla d_{\partial\Omega}),$$

for all $x \in \Gamma_k^\delta \cap \partial\Omega$. Thus, we finally infer that for any $\delta \in (0, \delta_0^k)\backslash\mathcal{N}$ there is an $\alpha_k^\delta > 0$ such that

$$\sqrt{1 - (\nabla d_{\Gamma_k^\delta} \cdot \nabla d_{\partial\Omega})^2} > 2\alpha_k^\delta \quad \text{for all } x \in \Gamma_k^\delta \cap \partial\Omega. \tag{4.4.62}$$

Using the smoothness of $d_{\Gamma_k^\delta}$ and $d_{\partial\Omega}$ we see that for $\varepsilon > 0$ small enough it holds in (4.4.58)

$$Jf_k^\delta(x) \geq \alpha_k^\delta \quad \text{for all } x \in \Omega_{k,\varepsilon}^\delta,$$

as $\Omega_{k,\varepsilon}^\delta$ is the junction region where both distances are small.
In the light of (4.4.56) it is now our goal to show

$$\mathcal{L}^d(\Omega_{k,\varepsilon}^\delta) \leq c_{k,\varepsilon}^\delta\, \varepsilon,$$

as $\varepsilon \to 0$, where for each $k \in \mathbb{N}$ and $\delta \in (0, \delta_0^k)\backslash\mathcal{N}$, $(c_{k,\varepsilon}^\delta)_{\varepsilon>0} \subset \mathbb{R}$ is a zero sequence. We will see that we can show an even better result giving us an explicit rate in ε induced by the thickness of the two interfaces forming the junction.

As mentioned above we apply the coarea formula Theorem 2.2.9 in codimension 2 which gives us

$$\alpha_k^\delta \mathcal{L}(\Omega_{k,\varepsilon}^\delta) \leq \int_{\mathbb{R}^d} \chi_{\Omega_{k,\varepsilon}^\delta} Jf_k^\delta \,\mathrm{d}x = \int_{(-2\varepsilon^p, 2\varepsilon^p)\times(0,4\varepsilon^p)} \mathcal{H}^{d-2}\left(\left\{f_k^\delta = \boldsymbol{s}\right\}\right) \mathrm{d}\mathcal{L}^2(\boldsymbol{s}). \tag{4.4.63}$$

Now we intend to show that the function

$$\begin{aligned} H : (-2\varepsilon^p, 2\varepsilon^p) \times (0, 4\varepsilon^p) &\to \mathbb{R} \\ \boldsymbol{s} &\to \mathcal{H}^{d-2}\left(\left\{ f_k^\delta = \boldsymbol{s} \right\}\right) \end{aligned} \tag{4.4.64}$$

is bounded for $\varepsilon > 0$ small enough. This then allows us to infer in (4.4.63)

$$\mathcal{L}^d(\Omega_{k,\varepsilon}^\delta) \leq C_k^\delta \frac{1}{\alpha_k^\delta} \varepsilon^{2p}, \tag{4.4.65}$$

with a constant $C_k^\delta > 0$ and $2p > 1$, because in the beginning of the proof we have chosen $p \in (\frac{1}{2}, 1)$.
So lets prove that H is bounded. For $\boldsymbol{s} \in \mathbb{R}^2$ small we rewrite the level set of f_k^δ as

$$\left\{ f_k^\delta = \boldsymbol{s} \right\} = \left\{ x \in \mathbb{R}^d \;\middle|\; d_{\Gamma_k^\delta}(x) - s_1 = 0,\ d_{\partial\Omega} - s_2 = 0 \right\} =: \mathcal{M}_{\boldsymbol{s}}.$$

Let us now consider the situation locally. Therefore we fix $\bar{x} \in \mathcal{M}_0 = \Gamma_k^\delta \cap \partial\Omega$. Due to the angle condition (4.4.62) and the Eikonal equality(4.4.57) we know that $\nabla d_{\Gamma_k^\delta}(\bar{x}), \nabla d_{\partial\Omega}(\bar{x}) \in \mathbb{R}^d$ are linearly independent and therefore we can add $n-2$ standard basis vectors in $\mathbb{R}^d$ to obtain a basis in $\mathbb{R}^d$. Thus, without loss of generality we assume that

$$\left\{ e_1, \dots, e_{d-2}, \nabla d_{\Gamma_k^\delta}(\bar{x}), \nabla d_{\partial\Omega}(\bar{x}) \right\} \subset \mathbb{R}^d$$

forms a basis, where $e_1, \dots, e_{d-2} \in \mathbb{R}^d$ denote the first $(d-2)$ standard basis vectors.
Now we define the function

$$\begin{aligned} \Theta : \mathbb{R}^d &\to \mathbb{R}^d, \\ x &\mapsto (x_1, \dots, x_{d-2}, d_{\Gamma_k^\delta}(x), d_{\partial\Omega}(x))^T. \end{aligned} \tag{4.4.66}$$

This function is smooth around a tubular neighborhood of $\mathcal{M}_0$ and per construction $D\Theta(\bar{x})$ is invertible, because the rows of $D\Theta(\bar{x})$ are $e_1^T, \dots, e_{d-2}^T, \nabla d_{\Gamma_k^\delta}(\bar{x})^T, \nabla d_{\partial\Omega}(\bar{x})^T$. Hence, the inverse function theorem, see e.g., [149, Theorem 17.7.2], provides neighborhoods $U \subset \mathbb{R}^d$ around $\bar{x}$ and $V \subset \mathbb{R}^d$ around $\Theta(\bar{x})$ such that

$$\Theta : U \to V$$

is a bijection and its inverse is a C^1-mapping. As $\mathcal{M}_0 \subset \mathbb{R}^d$ is compact and $\bar{x} \in \mathcal{M}_0$ was arbitrary, we find a finite covering of such neighborhoods, i.e., there is an $N \in \mathbb{N}$ and $U_i, V_i \subset \mathbb{R}^d$ bounded and open such that

$$\mathcal{M}_0 \subset \cup_{i=1}^N U_i,$$

and

$$\Theta_i : U_i \to V_i,$$

are C^1-diffeomorphisms. Now for $|\boldsymbol{s}|$ small enough $\mathcal{M}_{\boldsymbol{s}} \subset \cup_{i=1}^N U_i$, as any sequence of points $x_{\boldsymbol{s}} \in \mathcal{M}_{\boldsymbol{s}}$ possesses a subsequence such that its limit is an element of $\mathcal{M}_0$.
Therefore after a classical partition of unity argument, it suffices to consider the situation on $\mathcal{M}_{\boldsymbol{s}}$ again locally. So denote by U one of the U_i and by V the corresponding image

under Θ_i, denoted by Θ for brevity. As up to switching the entries where the distances appear in (4.4.66), we have

$$\Theta(\mathcal{M}_{\boldsymbol{s}} \cap U) \subset \mathcal{E}_{\boldsymbol{s}} \cap V, \tag{4.4.67}$$

where

$$\mathcal{E}_{\boldsymbol{s}} \coloneqq \left\{ (x', s_1, s_2) \;\middle|\; x' \in \mathbb{R}^{n-2} \right\}.$$

Hence denoting by $V' \coloneqq P_{d-2}(V) \subset \mathbb{R}^{d-2}$ the projection of V to its first $(d-2)$ components we see that defining

$$\begin{aligned} Q_{\boldsymbol{s},d-2} : V' &\to \mathcal{E}_{\boldsymbol{s}} \cap V, \\ x' &\mapsto (x', s_1, s_2), \end{aligned}$$

we have

$$\Theta_{\boldsymbol{s}} \coloneqq \Theta^{-1} \circ Q_{\boldsymbol{s},d-2} : V' \to \Theta^{-1}(\mathcal{E}_{\boldsymbol{s}} \cap V)$$

is a C^1-parametrization of $\Theta^{-1}(\mathcal{E}_{\boldsymbol{s}} \cap V)$. In the light of (4.4.67) obviously $\mathcal{M}_{\boldsymbol{s}} \cap U \subset \Theta_{\boldsymbol{s}}(V')$. Thus, the area formula for manifolds [89, 3.3.4 D] yields

$$\mathcal{H}^{d-2}(\mathcal{M}_{\boldsymbol{s}} \cap U) \leq \mathcal{H}^{d-2}(\Theta_{\boldsymbol{s}}(V')) = \int_{V'} \sqrt{\det G_{\Theta_{\boldsymbol{s}}}}\, \mathrm{d}\mathcal{L}^{d-2},$$

where $G_{\Theta_{\boldsymbol{s}}} = (D\Theta_{\boldsymbol{s}})^T D\Theta_{\boldsymbol{s}}$ is the metric tensor coming from the parametrization $\Theta_{\boldsymbol{s}}$. As $D\Theta_{\boldsymbol{s}}$ depends continuously on $\boldsymbol{s}$ via $Q_{\boldsymbol{s},d-2}$ and $V' \subset \mathbb{R}^{n-2}$ is a bounded open set we have now proven that H, as defined in (4.4.64), is bounded in $\boldsymbol{s}$ for $|\boldsymbol{s}|$ small enough.

Recalling (4.4.63) we obtain

$$\begin{aligned} \alpha_k^\delta \mathcal{L}^d(\Omega_{k,\varepsilon}^\delta) \leq \int_{\mathbb{R}^d} \chi_{\Omega_{k,\varepsilon}^\delta} J f_k^\delta \,\mathrm{d}x &= \int_{(-2\varepsilon^p, 2\varepsilon^p)\times(0,4\varepsilon^p)} \mathcal{H}^{d-2}\left(\left\{ f_k^\delta = \boldsymbol{s} \right\}\right) \mathrm{d}\mathcal{L}^2(\boldsymbol{s}) \\ &\leq 16\varepsilon^{2p} \sup_{\boldsymbol{s} \in (-2\varepsilon^p, 2\varepsilon^p)\times(0,4\varepsilon^p)} \mathcal{H}^{d-2}\left(\left\{ f_k^\delta = \boldsymbol{s} \right\}\right) \\ &\leq C_k^\delta \varepsilon^{2p}. \end{aligned}$$

Thus, we have proven (4.4.65).

Hence, in the light of (4.4.56) we have now that for fixed $\delta > 0$ and $k \in \mathbb{N}$

$$\lim_{\varepsilon \searrow 0} \int_{\left\{ d_{\partial\Omega} \geq -4\sqrt{\varepsilon} \right\} \cap \left\{ \left| d_{\Gamma_k^\delta} \right| \leq 2\varepsilon^p \right\}} \frac{\varepsilon}{2} \left| \nabla \left(\rho_\varepsilon \tilde{\varphi}_\varepsilon^{k,\delta} \right) \right|^2 + \frac{1}{\varepsilon} \psi(\rho_\varepsilon \tilde{\varphi}_\varepsilon^{k,\delta}) = 0,$$

as $2p > 1$. Combining all the three big parts of the proof we performed so far, we arrive at (4.4.45), because in the parts of Ω not contained in one of the above transition regions we simply have that $\rho_\varepsilon \tilde{\varphi}_\varepsilon^{k,\delta}$ is constantly 0 or 1 and thus these do not contribute to the energy in (4.4.45).

Therefore it only remains to show (4.4.46), because then the proof is finished by a diagonal sequence argument as explained in the beginning of the proof. We compute

$$\int_\Omega \left| \varphi_\varepsilon^{k,\delta} - \chi_{E_k^\delta} \right| \,\mathrm{d}x \leq \int_\Omega \left| \rho_\varepsilon \left(\tilde{\varphi}_\varepsilon^{k,\delta} - \chi_{E_k^\delta} \right) \right| + \int_\Omega \left| \rho_\varepsilon \chi_{E_k^\delta} - \chi_{E_k^\delta} \right| \,\mathrm{d}x$$

$$\leq \int_\Omega \left|\tilde{\varphi}_\varepsilon^{k,\delta} - \chi_{E_k^\delta}\right| + \int_\Omega |\rho_\varepsilon - \chi_\Omega| \, \mathrm{d}x.$$

Now it can be seen completely analogous to the proof of Step 1 of [147, Theorem 1] that both summands converge with order $\mathcal{O}(\varepsilon)$ as both are constructed via a profile as in (4.4.38) which converges exponentially in the interpolating parts and scales with order ε in between. Here they key steps are analogous to those performed in *Region II* of our proof, i.e., using a change of variables combined with the coarea formula results in a term of order $\mathcal{O}(\varepsilon)$. This finishes the proof.

□

Proof of Theorem 4.3.18. We have $\Phi_m \subset H_0^1(\Omega;[0,1])$ and $\Phi_m^0 \subset BV(\Omega;0,1)$ and we know that the volume constraint is preserved under L^1 convergence. Hence, the lim inf inequality is a direct consequence of Theorem 4.3.17 as now there are less admissible sequences.

It remains to prove the lim sup inequality, namely that for every $\varphi \in \Phi_m^0$ there exists a sequence $(\tilde{\varphi}_\varepsilon)_{\varepsilon>0} \subset \Phi_m$ fulfilling

$$\lim_{\varepsilon \searrow 0} \|\tilde{\varphi}_\varepsilon - \varphi\|_{L^1(\Omega)} = 0, \tag{4.4.68}$$

$$\limsup_{\varepsilon \searrow 0} J_\gamma^\varepsilon(\tilde{\varphi}_\varepsilon) \leq J_\gamma^0(\varphi). \tag{4.4.69}$$

For any $\varphi \in \Phi_m^0$, a recovery sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset H_0^1(\Omega;[0,1])$ for the functional F_ε^γ was constructed in Theorem 4.3.17. It was shown that this recovery sequence converges in $L^1(\Omega)$ and fulfills the lim sup inequality for F_ε^γ. Now, our goal is to carefully modify this recovery sequence such that it preserves these properties but additionally fulfills the mean value constraint. For this modification, we proceed completely analogously as in the proof of Theorem 3.3.9.
Let us recall the most important steps. Since $\varphi \in \Phi_m^0$, it is non-constant. Hence, we can find a function $\boldsymbol{\xi} \in C_0^1(\Omega,\mathbb{R}^d)$ such that

$$\int_\Omega \varphi \nabla \cdot \boldsymbol{\xi} \, \mathrm{d}x > 0.$$

For any $s \in \mathbb{R}$, we define the function

$$\begin{aligned} T_s : \mathbb{R}^d &\to \mathbb{R}^d, \\ x &\mapsto x + s\boldsymbol{\xi}(x), \end{aligned}$$

which is a C^1-diffeomorphism if s is sufficiently small. By means of the implicit function theorem, we deduce that for any sufficiently small $\varepsilon > 0$, there exists $s(\varepsilon) \in \mathbb{R}$ such that

$$\tilde{\varphi}_\varepsilon := \varphi_\varepsilon \circ T_{s(\varepsilon)}^{-1} \in \Phi_m,$$

and $s(\varepsilon) \to 0$ as $\varepsilon \to 0$. In particular, the property $\tilde{\varphi}_\varepsilon \in H_0^1(\Omega)$ holds since for x close to $\partial\Omega$, we have $T_{s(\varepsilon)}(x) = x$ due to the fact that $\boldsymbol{\xi}$ has compact support in Ω. Eventually, we show that the sequence $(\tilde{\varphi}_\varepsilon)_{\varepsilon>0}$ satisfies the properties (4.4.68) and (4.4.69) and thus, it is a suitable recovery sequence. □

Proof of Theorem 4.3.8. In the proof of Theorem 4.3.18, for any admissible $\varphi \in \Phi_m^0$, we have constructed a recovery sequence $(\tilde{\varphi}_\varepsilon)_{\varepsilon>0}$. In particular, this implies that the cost functional J_γ^ε is bounded uniformly in ε along any sequence $(\varphi_\varepsilon)_{\varepsilon>0}$ of minimizers to the optimization problem $(\mathrm{OP}_\gamma^\varepsilon)$. Consequently, there exists a constant $C > 0$ independent of ε and γ such that

$$\int_\Omega \psi(\varphi_\varepsilon)\,\mathrm{d}x \le \varepsilon E^\varepsilon(\varphi_\varepsilon) \le \frac{\varepsilon}{\gamma} J_\gamma^\varepsilon(\varphi_\varepsilon) \le \frac{C\varepsilon}{\gamma}, \tag{4.4.70}$$

for all $\varepsilon > 0$. Recalling that $\psi \in C^2([0,1])$ is non-negative, we thus have

$$\begin{aligned}\Big(\min_{[\delta,1-\delta]} \psi\Big)\, \mathcal{L}^d\left(\{\delta \le \varphi_\varepsilon \le 1-\delta\}\right) &\le \int_{\{\delta\le\varphi_\varepsilon\le 1-\delta\}} \psi(\varphi_\varepsilon)\,\mathrm{d}x \\ &\le \int_\Omega \psi(\varphi_\varepsilon)\,\mathrm{d}x \le \frac{C\varepsilon}{\gamma},. \end{aligned} \tag{4.4.71}$$

Since $\psi > 0$ on $[\delta, 1-\delta]$ due to (A1), the assertion directly follows. □

Proof of Theorem 4.3.15. We apply the compactness of the Ginzburg Landau energy from Proposition 2.2.25 combined with the convergence of minimizers under Γ-convergence from Proposition 2.2.24. Consequently, in the light of Theorem 4.3.18 which states that $J_\gamma^\varepsilon \xrightarrow{\Gamma} J_\gamma^0$, there exists a function $\varphi_0 \in BV(\Omega, \{0,1\})$ such that

$$\lim_{\varepsilon \searrow 0} \|\varphi_\varepsilon - \varphi_0\|_{L^1(\Omega)} = 0 \tag{4.4.72}$$

along a non-relabeled subsequence of $(\varphi_\varepsilon)_{\varepsilon>0}$ and φ_0 is a minimizer of J_γ^0. We further recall that $\varphi_\varepsilon = \varphi_\varepsilon^*$ according to Theorem 4.3.7. Hence, the non-expansivity of the rearrangement (see Lemma 2.2.12(f)) yields

$$\begin{aligned}\int_\Omega |\varphi_0 - \varphi_0^*|\ \mathrm{d}x &\le \int_\Omega |\varphi_0 - \varphi_\varepsilon|\ \mathrm{d}x + \int_\Omega |\varphi_\varepsilon^* - \varphi_0^*|\ \mathrm{d}x \\ &\le 2\int_\Omega |\varphi_\varepsilon - \varphi_0|\ \mathrm{d}x.\end{aligned}$$

Hence, we infer $\varphi_0 = \varphi_0^*$ almost everywhere in Ω. As the mean value is preserved under L^1 convergence this is already enough to deduce that φ_0 is the characteristic function of the ball centered at the origin with volume $m\,|\Omega|$. Obviously the limit φ_0 does not depend on the choice of the subsequence of $(\varphi_\varepsilon)_{\varepsilon>0}$. Hence, the convergence (4.4.72) even holds for the *whole* sequence. □

Part II

Spectral optimization problems in linear elasticity

Chapter 5

Analysis of the diffuse interface problem

5.1. Introduction

After having mainly focused on the Dirichlet-Laplace problem in the previous two chapters we will incorporate now more involved state equations into our optimization problem, namely the equations of linear elasticity, see Section 2.1.11. Therefore we consider the elliptic system

$$\begin{cases} -\nabla \cdot [\mathbb{C}(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{w})] = \lambda^{\boldsymbol{\varphi}} \rho(\boldsymbol{\varphi}) \boldsymbol{w} & \text{in } \Omega, \\ \boldsymbol{w} = \mathbf{0} & \text{on } \Gamma_D, \\ [\mathbb{C}(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{w})]\, \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma_0, \end{cases} \tag{5.1.1}$$

with the disjoint splitting $\partial\Omega = \overline{\Gamma_D \cup \Gamma_0}$. In order to exclude the case of trivial eigenvalues we demand Γ_D to have strictly positive Hausdorff measure, see Section 2.1.10 for details. Here, $\mathbb{C}$ denotes the elasticity tensor, $\mathcal{E}(\boldsymbol{w})$ is the symmetrized gradient of $\boldsymbol{w}$, where $\boldsymbol{w} = \boldsymbol{w}^{\boldsymbol{\varphi}}$ is an eigenfunction to the eigenvalue $\lambda^{\boldsymbol{\varphi}}$, both depending on the phase-field variable $\boldsymbol{\varphi} : \Omega \to \mathbb{R}^N$. ρ is the phase-field dependent density function introduced in Section 2.1.8. Opposed to the previous chapters the eigenfunctions $\boldsymbol{w} : \Omega \to \mathbb{R}^d$ are now given as vector-valued displacement fields. Furthermore, also the phase-field is vector-valued meaning that the structure that is to be optimized can now be constructed out of multiple materials, see Section 2.1.1. Problem (5.1.1) now actually models the physical example of the airplane wings given in the introduction of this thesis, as the wings can be understood as an elastic structure freely vibrating except for the part of the boundary which is attached to the body of the airplane.

In [5, 32, 33, 150] models similar to the one we intend to study are investigated. In these papers either no density distribution ρ appears or it is assumed to depend only on the spatial variable $x \in \Omega$. This is a simplification meaning that the specific structure (represented by the phase-field $\boldsymbol{\varphi}$) has no influence on the density. However, as the objective is indeed to optimize the material distribution within the design domain, the optimal density distribution is a priori not known.
At this point we mention also the recent works [7, 8] considering an optimization problem in the context of elastoplasticity. The authors there provide an analytical treatment of

existence, first-order conditions and a sharp-interface limit result. As in our model they allow also for a phase-field dependent density on the right-hand side. However we consider an eigenvalue problem and thus, we need to carefully compute and analyze the Fréchet-derivative of eigenvalues and eigenfunctions which is a non-trivial task as the phase-field enters these quantities in a highly implicit and non-linear way.
Note that in this chapter we are only concerned with the analysis of the diffuse interface problem and therefore will mostly drop the dependence on the parameter $\varepsilon > 0$.

The goal of this chapter is to fully analyze the state equation (5.1.1) and the optimization problem associated to minimizing the cost functional

$$J_l^\varepsilon(\boldsymbol{\varphi}) = \Psi(\lambda_{i_1}^{\boldsymbol{\varphi}}, \dots, \lambda_{i_l}^{\boldsymbol{\varphi}}) + \gamma E_{\mathrm{GL}}^\varepsilon(\boldsymbol{\varphi}), \tag{5.1.2}$$

where we recall that $\Psi : (\mathbb{R}_{>0})^l \to \mathbb{R}$ is a function penalizing the finite selection of eigenvalues $\lambda_{i_1}^{\boldsymbol{\varphi}}, \dots, \lambda_{i_l}^{\boldsymbol{\varphi}}$, see Section 2.1.11. Furthermore we recall the *Ginzburg–Landau energy* from Section 2.1.4

$$E_{\mathrm{GL}}^\varepsilon(\boldsymbol{\varphi}) = \int_\Omega \frac{\varepsilon}{2} |\nabla \boldsymbol{\varphi}|^2 + \frac{1}{\varepsilon} \psi(\boldsymbol{\varphi}) \, \mathrm{d}x, \tag{5.1.3}$$

where $\varepsilon > 0$ is proportional to the thickness of the diffuse interface and ψ is a bulk potential being of double obstacle type in our analysis. The phase-field $\boldsymbol{\varphi}$ in our optimization will be subject to certain physical constraints, such as a volume constraint and a pointwise constraint prescribing void and material in certain parts of the design domain Ω, see Section 2.1.3.
After having proven the existence of eigenvalues and eigenfunctions for problem (5.1.1) we will focus on continuity properties of these spectral quantities that will be particularly useful in order to show well-posedness of above optimization problem. We show continuity of spectral quantities when the phase-fields weakly converge in $H^1(\Omega; \mathbb{R}^d)$, see Theorem 5.3.4. Under stronger assumptions on the convergence of the phase-fields we will even show Lipschitz continuity of eigenvalues, see Lemma 5.3.5.
The largest part of this chapter is then devoted to rigorously prove first-order necessary optimality conditions. This requires knowledge about the derivative of eigenvalues with respect to the control variable $\boldsymbol{\varphi}$. In order to show the Fréchet-differentiability of *simple* eigenvalues and eigenfunctions we rely on the Fredholm alternative and the implicit function theorem, which is also the classical strategy in order to prove shape differentiability, see [110, Section 5.7]. A positive side benefit is that the implicit function theorem not only provides us with the Fréchet-derivative of a simple eigenvalue but also with the differentiability of corresponding eigenfunctions and equations that these derivatives satisfy, see Theorem 5.4.3.
Here the keyword *simple* eigenvalue is crucial. As we will also discuss in Section 5.4.2 even in a finite dimensional setting the differentiability of multiple eigenvalues in the classical sense of Fréchet-derivatives is not possible. Although from a phenomenological point of view it can be argued that by imperfect symmetries in nature and numerical computations multiple eigenvalues are unlikely to occur, see also the discussion in Section 5.4.2, it is observed that in shape optimization problems often multiple eigenvalues occur at an optimal shape, see [15, 39, 134]. In order to circumvent the strong requirement of classical Fréchet differentiability it is natural to consider only directional derivatives and thus obtain also multiple sensitivities of multiple eigenvalues, see [85, 104] and [108, Section 2.5.3]. Numerically we already have seen in Section 3.4 the application of the strategy of [15, 134] in

order to deal with multiple eigenvalues. Here, if two eigenvalues meet each other along the optimization process, the cost functional is modified in order to contain not only one of these eigenvalues but the arithmetic mean of both. This has the advantage that now also two eigenfunctions are contained in the derivative of the cost functional thus preventing oscillatory behavior due to a random selection of associated eigenfunctions. In particular, using this strategy the first-order conditions derived via the classical Fréchet-derivative in this chapter and in Chapter 3 are applicable in our simulations.
Nevertheless, in Section 5.4.2 we will discuss the concept of *semi-differentiability* of [141] applied to our setting in more detail, which will allow us to tackle the issue of non-differentiability if the principal eigenvalue is multiple.

In this context we also want to further discuss the work [141]. The authors there analyze a general eigenvalue problem of the form

$$a(\boldsymbol{\varphi}; w(\boldsymbol{\varphi}), \boldsymbol{\eta}) = \lambda^{\boldsymbol{\varphi}} b(\boldsymbol{\varphi}; w(\boldsymbol{\varphi}), \boldsymbol{\eta}), \quad \boldsymbol{\eta} \in H. \tag{5.1.4}$$

Here H is a Hilbert space, $\boldsymbol{\eta} \in H$ a test function and $w(\boldsymbol{\varphi})$ an eigenfunction with corresponding eigenvalue $\lambda^{\boldsymbol{\varphi}}$. In their analysis the bi-linear forms a and b possess the representation

$$a(\boldsymbol{\varphi}; w, \boldsymbol{\eta}) = \left(A(\boldsymbol{\varphi}) w(\boldsymbol{\varphi}), \boldsymbol{\eta}\right)_H, \quad b(\boldsymbol{\varphi}; w, \boldsymbol{\eta}) = \left(B(\boldsymbol{\varphi}) w(\boldsymbol{\varphi}), \boldsymbol{\eta}\right)_H, \tag{5.1.5}$$

where A and B are linear, continuous operators, B is compact, and $(\cdot,\cdot)_H$ denotes the inner product on H.
In [141], for this abstract setting, the authors prove continuity and (semi-)differentiability of $\lambda^{\boldsymbol{\varphi}}$ and $\boldsymbol{w}(\boldsymbol{\varphi})$ with respect to $\boldsymbol{\varphi}$ using an approach involving inverse operators. However, optimization problems are not addressed in [141]. Our approach for continuity and differentiability of spectral quantities is specifically tailored to the optimization problems we intend to study, thus we prove continuity results not only for strongly but also weakly converging sequences of phase-fields that appear naturally in the application of the direct method in the calculus of variation.

So summing up this introduction the current chapter contains the existence analysis for the state equation (5.1.1), a rigorous treatment of continuity and differentiability of spectral quantities, well-posedness of the optimization problem associated to (5.1.2), first-order necessary optimality conditions and eventually also a further optimization problem where we combine the classical compliance problem with eigenvalue optimization.

5.2. Analysis of the state equation

Definition 5.2.1 (Definition of eigenvalues and eigenfunctions)**.** *Let $\boldsymbol{\varphi} \in L^\infty(\Omega; \mathbb{R}^N)$ be arbitrary. Then $\lambda^{\boldsymbol{\varphi}}$ is called an* eigenvalue *of the state equation* (2.1.27) *if there exists a non-trivial weak solution $\boldsymbol{w}^{\boldsymbol{\varphi}}$ to the system* (2.1.27)*, i.e., $\mathbf{0} \neq \boldsymbol{w}^{\boldsymbol{\varphi}} \in H^1_D(\Omega; \mathbb{R}^d)$ and it holds that*

$$\left\langle \mathcal{E}\left(\boldsymbol{w}^{\boldsymbol{\varphi}}\right), \mathcal{E}\left(\boldsymbol{\eta}\right)\right\rangle_{\mathbb{C}(\boldsymbol{\varphi})} = \lambda^{\boldsymbol{\varphi}} \left(\boldsymbol{w}^{\boldsymbol{\varphi}}, \boldsymbol{\eta}\right)_{\rho(\boldsymbol{\varphi})} \quad \text{for all } \boldsymbol{\eta} \in H^1_D(\Omega, \mathbb{R}^d). \tag{5.2.1}$$

In this case, the function $\boldsymbol{w}^{\boldsymbol{\varphi}}$ is called an eigenfunction *to the eigenvalue $\lambda^{\boldsymbol{\varphi}}$.*

The assumptions of the previous section allow us to prove two classical functional analytic results in our setting.

Theorem 5.2.2 (Existence and properties of eigenvalues and eigenfunctions)**.**
Let $\boldsymbol{\varphi} \in L^\infty(\Omega;\mathbb{R}^N)$ be arbitrary.

(a) *There exists a sequence*

$$(\boldsymbol{w}_k^{\boldsymbol{\varphi}}, \lambda_k^{\boldsymbol{\varphi}})_{k\in\mathbb{N}} \subset H_D^1(\Omega;\mathbb{R}^d) \times \mathbb{R}$$

possessing the following properties:

- *For all $k \in \mathbb{N}$, $\boldsymbol{w}_k^{\boldsymbol{\varphi}}$ is an eigenfunction to the eigenvalue $\lambda_k^{\boldsymbol{\varphi}}$ in the sense of Definition 5.2.1.*
- *The eigenvalues $\lambda_k^{\boldsymbol{\varphi}}$ (which are repeated according to their multiplicity) can be ordered in the following way:*

$$0 < \lambda_1^{\boldsymbol{\varphi}} \le \lambda_2^{\boldsymbol{\varphi}} \le \lambda_3^{\boldsymbol{\varphi}} \le \cdots .$$

  Moreover, it holds that $\lambda_k^{\boldsymbol{\varphi}} \to \infty$ as $k \to \infty$, and there exist no further eigenvalues of the state equation (5.2.1)*.*
- *The eigenfunctions $\{\boldsymbol{w}_1^{\boldsymbol{\varphi}}, \boldsymbol{w}_2^{\boldsymbol{\varphi}}, \dots\} \subset H_D^1(\Omega;\mathbb{R}^d)$ form an $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-orthonormal basis of the space $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$.*

(b) *For $k \in \mathbb{N}$ we have the Courant–Fischer characterization*

$$\lambda_k^{\boldsymbol{\varphi}} = \max_{V\in\mathcal{S}_{k-1}} \min \left\{ \frac{\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u})\rangle_{\mathbb{C}(\boldsymbol{\varphi})}}{\|\boldsymbol{u}\|^2_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)}} \,\middle|\, \begin{array}{l} \boldsymbol{u} \in V^{\perp, L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} \cap H_D^1(\Omega;\mathbb{R}^d), \\ \boldsymbol{u} \neq \boldsymbol{0} \end{array} \right\}.$$

Here, $\mathcal{S}_{k-1}$ denotes the collection of all $(k-1)$-dimensional subspaces of $H_D^1(\Omega;\mathbb{R}^d)$. The set $V^{\perp, L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)}$ denotes the orthogonal complement of $V \subset L^2(\Omega;\mathbb{R}^d)$ with respect to the scalar product on $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$.

Moreover, the maximum is attained at the subspace

$$V = \langle \boldsymbol{w}_1^{\boldsymbol{\varphi}}, \dots, \boldsymbol{w}_{k-1}^{\boldsymbol{\varphi}} \rangle_{\text{span}}.$$

Proof. Using the Lax–Milgram theorem and the fact that $\mathcal{H}^{d-1}(\Gamma_D) > 0$, we conclude that for any $\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^d)$, there exists a unique function $\boldsymbol{v}_{\boldsymbol{f}} \in H_D^1(\Omega;\mathbb{R}^d)$ solving the equation

$$\langle \mathcal{E}(\boldsymbol{v}_{\boldsymbol{f}}), \mathcal{E}(\boldsymbol{\eta})\rangle_{\mathbb{C}(\boldsymbol{\varphi})} = \int_\Omega \rho(\boldsymbol{\varphi}) \boldsymbol{f} \cdot \boldsymbol{\eta} \,\mathrm{d}x \quad \text{for all } \boldsymbol{\eta} \in H_D^1(\Omega, \mathbb{R}^d).$$

This allows us to define a solution operator

$$\mathcal{T} : L^2(\Omega;\mathbb{R}^d) \to H_D^1(\Omega;\mathbb{R}^d) \subset L^2(\Omega;\mathbb{R}^d), \quad \boldsymbol{f} \mapsto \boldsymbol{v}_{\boldsymbol{f}}.$$

Since $H_D^1(\Omega;\mathbb{R}^d)$ is compactly embedded in $L^2(\Omega;\mathbb{R}^d)$, we can easily show that $\mathcal{T}$ is a compact, self-adjoint, and bounded linear operator. Thus, the assertions in (a) directly follow from the spectral theorem for compact self-adjoint operators, see Theorem 2.2.8.

To prove (b), we first infer from (5.2.1) that the sequence

$$\left(\frac{\boldsymbol{w}_k^{\boldsymbol{\varphi}}}{\sqrt{\lambda_k}} \right)_{k\in\mathbb{N}} \subset H_D^1(\Omega;\mathbb{R}^d),$$

forms an orthonormal basis of $H^1_D(\Omega;\mathbb{R}^d)$ when taking the inner product (2.1.28). For any $\boldsymbol{v} \in H^1_D(\Omega;\mathbb{R}^d)$, this yields the representation

$$\boldsymbol{v} = \sum_{i=1}^{\infty} \left(\boldsymbol{v}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}\right)_{\rho(\boldsymbol{\varphi})} \boldsymbol{w}_i^{\boldsymbol{\varphi}},$$

where the series on the right-hand side converges in $H^1_D(\Omega;\mathbb{R}^d)$. In the following, we will sometimes omit the exponent $\boldsymbol{\varphi}$ for a more convenient depiction.

To establish the Courant–Fischer characterization we now fix an arbitrary subspace $V \in \mathcal{S}_{k-1}$. Let us denote the orthogonal projection from $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$ to V with respect to the scalar product on $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$ by

$$P_{\boldsymbol{\varphi}} : L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d) \to V \subset L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d).$$

Since V is a $(k-1)$-dimensional subspace, the family

$$\{P_{\boldsymbol{\varphi}}(\boldsymbol{w}_1), \dots, P_{\boldsymbol{\varphi}}(\boldsymbol{w}_k)\} \subset V,$$

must be linearly dependent. Hence, for every $i \in \{1,\dots,k\}$, we find coefficients $\alpha_i \in \mathbb{R}$ that are not all equal to zero such that

$$P_{\boldsymbol{\varphi}}\left(\sum_{i=1}^{k} \alpha_i \boldsymbol{w}_i\right) = \sum_{i=1}^{k} \alpha_i P_{\boldsymbol{\varphi}}(\boldsymbol{w}_i) = \boldsymbol{0}.$$

Per construction of the orthogonal projection this is equivalent to

$$\boldsymbol{v} \coloneqq \sum_{i=1}^{k} \alpha_i \boldsymbol{w}_i \in V^{\perp, L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} \cap H^1_D(\Omega;\mathbb{R}^d).$$

As not all of the coefficients vanish, and since the eigenfunctions $\{\boldsymbol{w}_1,\dots,\boldsymbol{w}_k\}$ are linearly independent, we know that $\boldsymbol{v} \neq \boldsymbol{0}$. Using the orthogonality of eigenfunctions and the fact, that the sequence of eigenvalues increases, we conclude that

$$\begin{aligned} &\inf\left\{ \frac{\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u})\rangle_{\mathbb{C}(\boldsymbol{\varphi})}}{\|\boldsymbol{u}\|^2_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)}} \,\middle|\, \begin{array}{l} \boldsymbol{u} \in V^{\perp, L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} \cap H^1_D(\Omega;\mathbb{R}^d), \\ \boldsymbol{u} \neq \boldsymbol{0} \end{array} \right\} \\ &\quad \leq \frac{\langle \mathcal{E}(\boldsymbol{v}), \mathcal{E}(\boldsymbol{v})\rangle_{\mathbb{C}(\boldsymbol{\varphi})}}{\|\boldsymbol{v}\|^2_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)}} = \frac{\sum_{i=1}^{k} \alpha_i^2 \lambda_i}{\sum_{i=1}^{k} \alpha_i^2} \leq \lambda_k. \end{aligned} \tag{5.2.2}$$

As the infimum in (5.2.2) obviously exists, we can find a minimizing sequence

$$(\boldsymbol{u}_l)_{l\in\mathbb{N}} \subset \left\{V^{\perp, L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} \cap H^1_D(\Omega;\mathbb{R}^d)\right\} \setminus \{\boldsymbol{0}\},$$

such that

$$\lim_{l\to\infty} \frac{\langle \mathcal{E}(\boldsymbol{u}_l), \mathcal{E}(\boldsymbol{u}_l)\rangle_{\mathbb{C}(\boldsymbol{\varphi})}}{\|\boldsymbol{u}_l\|^2_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)}} = \inf\left\{ \frac{\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u})\rangle_{\mathbb{C}(\boldsymbol{\varphi})}}{\|\boldsymbol{u}\|^2_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)}} \,\middle|\, \begin{array}{l} \boldsymbol{u} \in V^{\perp, L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} \cap H^1_D(\Omega;\mathbb{R}^d), \\ \boldsymbol{u} \neq \boldsymbol{0} \end{array} \right\}.$$

Now, recalling that the inner product $\langle \mathcal{E}(\cdot), \mathcal{E}(\cdot) \rangle_{\mathbb{C}(\boldsymbol{\varphi})}$ induces a norm on $H^1_D(\Omega;\mathbb{R}^d)$, this implies that the sequence

$$(\tilde{\boldsymbol{u}}_l)_{l\in\mathbb{N}} \coloneqq \left(\frac{\boldsymbol{u}_l}{\|\boldsymbol{u}_l\|_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)}} \right)_{l\in\mathbb{N}},$$

is bounded in $H^1_D(\Omega;\mathbb{R}^d)$. Hence, due to the Banach–Alaoglu theorem and the compact embedding $H^1_D(\Omega;\mathbb{R}^d) \subset L^2(\Omega;\mathbb{R}^d)$, there exists a function $\tilde{\boldsymbol{u}} \in H^1_D(\Omega;\mathbb{R}^d)$ such that

$$\tilde{\boldsymbol{u}}_l \rightharpoonup \tilde{\boldsymbol{u}} \quad \text{in } H^1_D(\Omega;\mathbb{R}^d), \quad \text{and} \quad \tilde{\boldsymbol{u}}_l \to \tilde{\boldsymbol{u}} \quad \text{in } L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d),$$

along a non-relabeled subsequence. In particular, since all members of the sequence $(\tilde{\boldsymbol{u}}_l)_{l\in\mathbb{N}}$ are normalized with respect to the $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-norm, it follows that $\tilde{\boldsymbol{u}} \neq \boldsymbol{0}$.

Furthermore, $V^{\perp,L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} \cap H^1_D(\Omega;\mathbb{R}^d)$ is a convex and closed subset of $H^1_D(\Omega;\mathbb{R}^d)$. Hence, it is also weakly (sequentially) closed and we thus know that

$$\tilde{\boldsymbol{u}} \in \left\{ V^{\perp,L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} \cap H^1_D(\Omega;\mathbb{R}^d) \right\} \setminus \{\boldsymbol{0}\}.$$

Using the fact that norms are always weakly lower semi-continuous we infer that $\tilde{\boldsymbol{u}}$ is a minimizer of the expression in (5.2.2) via the direct method in the calculus of variations.

Since this holds for any arbitrary $(k-1)$-dimensional subspace $V \subset H^1_D(\Omega;\mathbb{R}^d)$, we conclude that

$$\sup_{V\in\mathcal{S}_{k-1}} \min \left\{ \frac{\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u}) \rangle_{\mathbb{C}(\boldsymbol{\varphi})}}{\|\boldsymbol{u}\|^2_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)}} \middle| \begin{array}{l} \boldsymbol{u} \in V^{\perp,L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} \cap H^1_D(\Omega;\mathbb{R}^d), \\ \boldsymbol{u} \neq \boldsymbol{0} \end{array} \right\} \leq \lambda_k.$$

We now select a special $(k-1)$-dimensional subspace defined by

$$V \coloneqq \langle \boldsymbol{w}_1, \dots \boldsymbol{w}_{k-1} \rangle_{\text{span}} \subset H^1_D(\Omega;\mathbb{R}^d).$$

Then the definition of the orthogonal complement yields

$$V^{\perp,L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} = \langle \boldsymbol{w}_k, \boldsymbol{w}_{k+1}, \dots \rangle_{\text{span}} \subset L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d).$$

Hence, any $\boldsymbol{v} \in V^{\perp,L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} \cap H^1_D(\Omega;\mathbb{R}^d)$ can be represented as

$$\boldsymbol{v} = \sum_{i=k}^{\infty} (\boldsymbol{v}, \boldsymbol{w}_i)_{\rho(\boldsymbol{\varphi})} \boldsymbol{w}_i,$$

where the series converges in $H^1_D(\Omega;\mathbb{R}^d)$. Consequently, we obtain

$$\langle \mathcal{E}(\boldsymbol{v}), \mathcal{E}(\boldsymbol{v}) \rangle_{\mathbb{C}(\boldsymbol{\varphi})} = \sum_{i=k}^{\infty} (\boldsymbol{v}, \boldsymbol{w}_i)^2_{\rho(\boldsymbol{\varphi})} \lambda_i \geq \lambda_k \|\boldsymbol{v}\|^2_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)},$$

because of the identity

$$\sum_{i=k}^{\infty} (\boldsymbol{v}, \boldsymbol{w}_i)^2_{\rho(\boldsymbol{\varphi})} = \|\boldsymbol{v}\|^2_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)}.$$

Altogether, we conclude that

$$\sup_{\substack{V\subset\mathcal{S}_{k-1} \\ \dim(V)=k-1}} \min \left\{ \frac{\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u}) \rangle_{\mathbb{C}(\boldsymbol{\varphi})}}{\|\boldsymbol{u}\|^2_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)}} \middle| \begin{array}{l} \boldsymbol{u} \in V^{\perp,L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} \cap H^1_D(\Omega;\mathbb{R}^d), \\ \boldsymbol{u} \neq \boldsymbol{0} \end{array} \right\} = \lambda_k.$$

This means that the maximum is attained at the subspace $V = \langle \boldsymbol{w}_1, \dots \boldsymbol{w}_{k-1} \rangle_{\text{span}}$ at $\boldsymbol{0} \neq \boldsymbol{w}_k \in V^{\perp,L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} \cap H^1_D(\Omega;\mathbb{R}^d)$, which proves the claim. □

5.3. Continuity of eigenvalues and eigenfunctions

5.3.1. Weak sequential continuity of eigenvalues

First of all we only consider the first eigenvalue λ_1 to establish continuity results with respect to the phase-field $\boldsymbol{\varphi}$. Afterwards, we proceed inductively to obtain these results also for all the other eigenvalues.

We consider the mapping

$$\lambda_1 : H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N) \to \mathbb{R}_{>0}, \quad \boldsymbol{\varphi} \mapsto \lambda_1^{\boldsymbol{\varphi}}$$

associated with the first eigenvalue.

The first continuity result for the eigenvalue λ_1 is obtained by proving lower and upper semi-continuity. Lower semi-continuity is established by the following lemma.

Lemma 5.3.1. *Let $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)$ be a bounded sequence with respect to the $L^\infty(\Omega;\mathbb{R}^N)$-norm satisfying*

$$\boldsymbol{\varphi}_k \rightharpoonup \boldsymbol{\varphi} \quad \text{in } H^1(\Omega;\mathbb{R}^N) \text{ as } k \to \infty.$$

Then it holds that

$$\lambda_1^{\boldsymbol{\varphi}} \leq \liminf_{k\to\infty} \lambda_1^{\boldsymbol{\varphi}_k},$$

along a non-relabeled subsequence.

Proof. We first notice that the assumptions of Lemma 5.3.1 imply that $\boldsymbol{\varphi} \in L^\infty(\Omega;\mathbb{R}^N)$. Let $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots\} \subset L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$ denote an orthonormal basis of eigenfunctions corresponding to the sequence of eigenvalues $(\lambda_i^{\boldsymbol{\varphi}})_{i\in\mathbb{N}}$ from Theorem 5.2.2.

Now, for any $k \in \mathbb{N}$, we choose an arbitrary $L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)$-normalized eigenfunction $\boldsymbol{u}^k$ that fulfills (5.2.1) for $\lambda_1^{\boldsymbol{\varphi}_k}$. This choice is not necessarily unique up to multiplication with ± 1, as we do not assume simplicity of $\lambda_1^{\boldsymbol{\varphi}_k}$ or $\lambda_1^{\boldsymbol{\varphi}}$ yet.

Using the Courant–Fischer representation from Theorem 5.2.2(b) for the first eigenvalue and the continuity of $\mathbb{C}$ and ρ, we see that the sequence $(\boldsymbol{u}^k)_{k\in\mathbb{N}} \subset H^1_D(\Omega;\mathbb{R}^d)$ is bounded. By the Banach–Alaoglu theorem and the compact embedding $H^1_D(\Omega;\mathbb{R}^d) \subset L^2(\Omega;\mathbb{R}^d)$, we infer the existence of a function $\overline{\boldsymbol{u}} \in H^1_D(\Omega;\mathbb{R}^d)$ with

$$\boldsymbol{u}^k \rightharpoonup \overline{\boldsymbol{u}} \quad \text{in } H^1_D(\Omega;\mathbb{R}^d), \quad \text{and} \quad \boldsymbol{u}^k \to \overline{\boldsymbol{u}} \quad \text{in } L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d), \tag{5.3.1}$$

as $k \to \infty$, up to a subsequence. With the help of Lebesgue's theorem and the assumptions on the sequence $\boldsymbol{\varphi}_k$, this yields

$$\left(\boldsymbol{u}^k, \boldsymbol{u}^k\right)_{\rho(\boldsymbol{\varphi}_k)} \to \left(\overline{\boldsymbol{u}}, \overline{\boldsymbol{u}}\right)_{\rho(\boldsymbol{\varphi})},$$

as $k \to \infty$ after another subsequence extraction. This implies $\|\overline{\boldsymbol{u}}\|_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} = 1$ since the members $\boldsymbol{u}^k$ were chosen as $L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)$-normalized eigenfunctions. In particular, this implies that

$$1 = \sum_{i=1}^{\infty} \left(\overline{\boldsymbol{u}}, \boldsymbol{w}_i\right)^2_{\rho(\boldsymbol{\varphi})}. \tag{5.3.2}$$

Plugging $\overline{\boldsymbol{u}}$ into the continuous bi-linear form $\langle \mathcal{E}(\cdot), \mathcal{E}(\cdot)\rangle_{\mathbb{C}(\boldsymbol{\varphi})}$ on $H^1_D(\Omega;\mathbb{R}^d)$, and invoking the increasing order of the sequence $(\lambda_i^{\boldsymbol{\varphi}})_{i\in\mathbb{N}}$, we conclude that

$$\langle \mathcal{E}(\overline{\boldsymbol{u}}), \mathcal{E}(\overline{\boldsymbol{u}})\rangle_{\mathbb{C}(\boldsymbol{\varphi})} = \sum_{i=1}^{\infty} (\overline{\boldsymbol{u}}, \boldsymbol{w}_i)^2_{\rho(\varphi)}\, \lambda_i^{\boldsymbol{\varphi}} \geq \lambda_1^{\boldsymbol{\varphi}}.$$

If we can now show that

$$\liminf_{k\to\infty} \lambda_1^{\boldsymbol{\varphi}_k} \geq \langle \mathcal{E}(\overline{\boldsymbol{u}}), \mathcal{E}(\overline{\boldsymbol{u}})\rangle_{\mathbb{C}(\boldsymbol{\varphi})}, \tag{5.3.3}$$

the proof would be complete. Using the convergence results we have just established, the Cauchy–Schwarz inequality and the weak formulation (5.2.1), we infer that

$$\begin{aligned}
\liminf_{k\to\infty} \lambda_1^{\boldsymbol{\varphi}_k} &= \left(\liminf_{k\to\infty} \lambda_1^{\boldsymbol{\varphi}_k}\right)\left(\lim_{k\to\infty} \left\|\boldsymbol{u}^k\right\|_{L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)}\right)\left(\lim_{k\to\infty} \|\overline{\boldsymbol{u}}\|_{L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)}\right)\\
&= \liminf_{k\to\infty} \left[\lambda_1^{\boldsymbol{\varphi}_k} \|\boldsymbol{u}_k\|_{L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)} \|\overline{\boldsymbol{u}}\|_{L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)}\right]\\
&\geq \liminf_{k\to\infty} \left[\lambda_1^{\boldsymbol{\varphi}_k} (\boldsymbol{u}^k, \overline{\boldsymbol{u}})_{\rho(\boldsymbol{\varphi_k})}\right]\\
&= \liminf_{k\to\infty} \langle \mathcal{E}(\boldsymbol{u}^k), \mathcal{E}(\overline{\boldsymbol{u}})\rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)}.
\end{aligned}$$

We further know that

$$\begin{aligned}
&\langle \mathcal{E}(\boldsymbol{u}^k), \mathcal{E}(\overline{\boldsymbol{u}})\rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)} - \langle \mathcal{E}(\overline{\boldsymbol{u}}), \mathcal{E}(\overline{\boldsymbol{u}})\rangle_{\mathbb{C}(\boldsymbol{\varphi})}\\
&\quad = \left[\langle \mathcal{E}(\boldsymbol{u}^k), \mathcal{E}(\overline{\boldsymbol{u}})\rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)} - \langle \mathcal{E}(\boldsymbol{u}^k), \mathcal{E}(\overline{\boldsymbol{u}})\rangle_{\mathbb{C}(\boldsymbol{\varphi})}\right]\\
&\qquad + \left[\langle \mathcal{E}(\boldsymbol{u}^k), \mathcal{E}(\overline{\boldsymbol{u}})\rangle_{\mathbb{C}(\boldsymbol{\varphi})} - \langle \mathcal{E}(\overline{\boldsymbol{u}}), \mathcal{E}(\overline{\boldsymbol{u}})\rangle_{\mathbb{C}(\boldsymbol{\varphi})}\right].
\end{aligned}$$

Using Lebesgue's convergence theorem, the boundedness of $(\boldsymbol{u}^k)_{k\in\mathbb{N}}$, the local Lipschitz continuity of $\mathbb{C}$ and the assumptions on $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}}$, we conclude that the first summand converges to zero along a non-relabeled subsequence. The second summand converges to zero as a direct consequence of (5.3.1).

In summary, we obtain that

$$\liminf_{k\to\infty} \lambda_1^{\boldsymbol{\varphi}_k} \geq \liminf_{k\to\infty} \langle \mathcal{E}(\boldsymbol{u}^k), \mathcal{E}(\overline{\boldsymbol{u}})\rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)} = \langle \mathcal{E}(\overline{\boldsymbol{u}}), \mathcal{E}(\overline{\boldsymbol{u}})\rangle_{\mathbb{C}(\boldsymbol{\varphi})} \geq \lambda_1^{\boldsymbol{\varphi}}.$$

which completes the proof. □

Now, we establish the corresponding result for weak upper semi-continuity.

Lemma 5.3.2. *Let $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)$ be a bounded sequence with respect to the $L^\infty(\Omega;\mathbb{R}^N)$-norm satisfying*

$$\boldsymbol{\varphi}_k \rightharpoonup \boldsymbol{\varphi} \quad \text{in } H^1(\Omega;\mathbb{R}^N) \text{ as } k\to\infty.$$

Then it holds that

$$\lambda_1^{\boldsymbol{\varphi}} \geq \limsup_{k\to\infty} \lambda_1^{\boldsymbol{\varphi}_k},$$

along a non-relabeled subsequence.

Proof. As $\mathbb{C}$ and ρ satisfy suitable continuity properties we can proceed as in [108, Theorem 8.1.3] and use once more the Courant–Fischer representation for the first eigenvalue to prove the claim. □

Combining both lemmata we can conclude that λ_1 is weakly sequentially continuous.

Corollary 5.3.3. *Let $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ be a bounded sequence with respect to the $L^\infty(\Omega;\mathbb{R}^N)$-norm satisfying*

$$\boldsymbol{\varphi}_k \rightharpoonup \boldsymbol{\varphi} \quad \text{in } H^1(\Omega;\mathbb{R}^N) \text{ as } k \to \infty,$$

and let $(\boldsymbol{u}^k)_{k\in\mathbb{N}} \subset H^1_D(\Omega;\mathbb{R}^d)$ be a sequence of $L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)$-normalized eigenfunctions to the eigenvalues $(\lambda_1^{\boldsymbol{\varphi}_k})_{k\in\mathbb{N}}$, i.e., $\boldsymbol{u}_k$ satisfies (5.2.1) *written for $\lambda_1^{\boldsymbol{\varphi}_k}$ for every $k \in \mathbb{N}$.*

Then it holds that

$$\lambda_1^{\boldsymbol{\varphi}_k} \to \lambda_1^{\boldsymbol{\varphi}}, \quad \text{as } k \to \infty, \tag{5.3.4}$$

i.e., the whole *sequence of eigenvalues converges and not just a subsequence.*

Furthermore, there exists a $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-normalized eigenfunction $\overline{\boldsymbol{u}} \in H^1_D(\Omega;\mathbb{R}^d)$ to the eigenvalue $\lambda_1^{\boldsymbol{\varphi}}$ such that

$$\boldsymbol{u}^k \rightharpoonup \overline{\boldsymbol{u}} \quad \text{in } H^1_D(\Omega;\mathbb{R}^d), \quad \text{and} \quad \boldsymbol{u}^k \to \overline{\boldsymbol{u}} \quad \text{in } L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d),$$

as $k \to \infty$, along a non-relabeled subsequence.

Proof. The convergence $\lambda_1^{\boldsymbol{\varphi}_k} \to \lambda_1^{\boldsymbol{\varphi}}$ as $k \to \infty$ follows from Lemma 5.3.1 and Lemma 5.3.2 after extraction of a subsequence. Moreover, as the limit $\lambda_1^{\boldsymbol{\varphi}}$ does not depend on the choice of the subsequence, we conclude by a standard contradiction argument that the convergence remains true for the whole sequence.

The convergence properties of $(\boldsymbol{u}^k)_{k\in\mathbb{N}} \subset H^1_D(\Omega;\mathbb{R}^d)$ and the fact that the weak limit $\overline{\boldsymbol{u}} \in H^1_D(\Omega;\mathbb{R}^d)$ is $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-normalized have already been established in (5.3.1). Hence, it remains to show that $\overline{\boldsymbol{u}} \in H^1_D(\Omega;\mathbb{R}^d)$ is an eigenfunction corresponding to the eigenvalue $\lambda_1^{\boldsymbol{\varphi}}$. By construction, we know that for any $k \in \mathbb{N}$,

$$\left\langle \mathcal{E}(\boldsymbol{u}^k), \mathcal{E}(\boldsymbol{\eta}) \right\rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)} = \lambda_1^{\boldsymbol{\varphi}_k} \left(\boldsymbol{u}^k, \boldsymbol{\eta}\right)_{\rho(\boldsymbol{\varphi}_k)}, \tag{5.3.5}$$

for all test functions $\boldsymbol{\eta} \in H^1_D(\Omega;\mathbb{R}^d)$. Using the convergence of eigenvalues (5.3.4) and proceeding as in the proof of Lemma 5.3.1, we infer that for any any $\boldsymbol{\eta} \in H^1_D(\Omega;\mathbb{R}^d)$,

$$\begin{aligned} \left\langle \mathcal{E}(\boldsymbol{u}^k), \mathcal{E}(\boldsymbol{\eta}) \right\rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)} &\to \left\langle \mathcal{E}\left(\overline{\boldsymbol{u}}\right), \mathcal{E}\left(\boldsymbol{\eta}\right) \right\rangle_{\mathbb{C}(\boldsymbol{\varphi})}, \\ \left(\boldsymbol{u}^k, \boldsymbol{\eta}\right)_{\rho(\boldsymbol{\varphi}_k)} &\to \left(\overline{\boldsymbol{u}}, \boldsymbol{\eta}\right)_{\rho(\boldsymbol{\varphi})}, \end{aligned}$$

as $k \to \infty$, after extraction of a subsequence. Hence, we can pass to the limit in equation (5.3.5) to obtain

$$\left\langle \mathcal{E}\left(\overline{\boldsymbol{u}}\right), \mathcal{E}\left(\boldsymbol{\eta}\right) \right\rangle_{\mathbb{C}(\boldsymbol{\varphi})} = \lambda_1^{\boldsymbol{\varphi}} \left(\overline{\boldsymbol{u}}, \boldsymbol{\eta}\right)_{\rho(\boldsymbol{\varphi})},$$

which proves that $\overline{\boldsymbol{u}} \in H^1_D(\Omega;\mathbb{R}^d)$ is indeed an eigenfunction corresponding to $\lambda_1^{\boldsymbol{\varphi}}$. □

Corollary 5.3.3 now serves as initial case for the following inductive proof which yields convergence of all eigenvalues.

Theorem 5.3.4 (Continuity properties for the eigenvalues and their eigenfunctions)**.** *Let $j \in \mathbb{N}$ be arbitrary and let $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ be a bounded sequence with respect to the $L^\infty(\Omega;\mathbb{R}^N)$-norm satisfying*

$$\boldsymbol{\varphi}_k \rightharpoonup \boldsymbol{\varphi} \quad \text{in } H^1(\Omega;\mathbb{R}^N) \text{ as } k \to \infty.$$

Moreover, let $(\boldsymbol{u}_j^k)_{k\in\mathbb{N}} \subset H^1_D(\Omega;\mathbb{R}^d)$ be a sequence of $L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)$-normalized eigenfunctions to the eigenvalues $(\lambda_j^{\boldsymbol{\varphi}_k})_{k\in\mathbb{N}}$, i.e., $\boldsymbol{u}_j^k$ satisfies (5.2.1) *written for $\lambda_j^{\boldsymbol{\varphi}_k}$ for every $k \in \mathbb{N}$. Then it holds that*

$$\lambda_j^{\boldsymbol{\varphi}_k} \to \lambda_j^{\boldsymbol{\varphi}}, \quad \text{as } k \to \infty,$$

for the whole *sequence of eigenvalues and not just a subsequence.*

Furthermore, there exists a $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-normalized eigenfunction $\overline{\boldsymbol{u}}_j \in H^1_D(\Omega;\mathbb{R}^d)$ to the eigenvalue $\lambda_j^{\boldsymbol{\varphi}}$ such that

$$\boldsymbol{u}_j^k \rightharpoonup \overline{\boldsymbol{u}}_j \quad \text{in } H^1_D(\Omega;\mathbb{R}^d), \quad \text{and} \quad \boldsymbol{u}_j^k \to \overline{\boldsymbol{u}}_j \quad \text{in } L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$$

as $k \to \infty$ along a non-relabeled subsequence.

Proof. As mentioned before we proceed by induction. The initial step has already been established in Corollary 5.3.3.

Now, we assume that the statement is already verified for the index $(j-1) \in \mathbb{N}$. Our task is to prove that the assertion is true for the j-th eigenvectors and the associated eigenfunctions. In this regard, the Courant–Fischer representation of Theorem 5.2.2(b) will be a helpful tool.

For $k \in \mathbb{N}$, we fix the $(j-1)$-dimensional subspace of $H^1_D(\Omega;\mathbb{R}^d)$ that realizes the maximum in the Courant–Fischer representation discussed in Theorem 5.2.2(b), namely

$$V_k \coloneqq \langle \boldsymbol{w}_1^{\boldsymbol{\varphi}_k}, \dots, \boldsymbol{w}_{j-1}^{\boldsymbol{\varphi}_k} \rangle_{\text{span}}.$$

Analogously, we define

$$V \coloneqq \langle \boldsymbol{w}_1^{\boldsymbol{\varphi}}, \dots, \boldsymbol{w}_{j-1}^{\boldsymbol{\varphi}} \rangle_{\text{span}}.$$

Then by the induction hypothesis we know that for every $i = 1, \dots, j-1$, there exists a $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-normalized eigenfunction $\overline{\boldsymbol{u}}_i \in H^1_D(\Omega;\mathbb{R}^d)$ to the eigenvalue $\lambda_i^{\boldsymbol{\varphi}}$ such that

$$\boldsymbol{w}_i^{\boldsymbol{\varphi}_k} \rightharpoonup \overline{\boldsymbol{u}}_i \quad \text{in } H^1_D(\Omega;\mathbb{R}^d), \quad \text{and} \quad \boldsymbol{w}_i^{\boldsymbol{\varphi}_k} \to \overline{\boldsymbol{u}}_i \quad \text{in } L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d), \tag{5.3.6}$$

as $k \to \infty$ along a non-relabeled subsequence. As $\{\boldsymbol{w}_1^{\boldsymbol{\varphi}_k}, \boldsymbol{w}_2^{\boldsymbol{\varphi}_k}, \dots\} \subset L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)$ form an orthonormal basis we infer that

$$(\overline{\boldsymbol{u}}_m, \overline{\boldsymbol{u}}_l)_{\rho(\boldsymbol{\varphi})} = 0, \tag{5.3.7}$$

for $m \neq l$, using the convergence properties of the sequence $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}}$ along with Lebesgue's convergence theorem. In particular, the family $\{\overline{\boldsymbol{u}}_1, \dots, \overline{\boldsymbol{u}}_{j-1}\} \subset L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$ is linearly

independent, which yields that all eigenfunctions to eigenvalues strictly smaller than $\lambda_j^{\boldsymbol{\varphi}}$ are contained in its span $W \coloneqq \langle \overline{\boldsymbol{u}}_1, \dots, \overline{\boldsymbol{u}}_{j-1} \rangle_{\mathrm{span}}$. Hence, we conclude that

$$\min \left\{ \frac{\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u}) \rangle_{\mathbb{C}(\boldsymbol{\varphi})}}{\|\boldsymbol{u}\|^2_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)}} \middle| \boldsymbol{u} \in W^{\perp, L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} \cap H^1_D(\Omega;\mathbb{R}^d), \boldsymbol{u} \neq \boldsymbol{0} \right\} \geq \lambda_j^{\boldsymbol{\varphi}}. \tag{5.3.8}$$

As the minimum is attained, we infer that we find a non-trivial function $\boldsymbol{v} \in H^1_D(\Omega;\mathbb{R}^d)$ with

$$(\boldsymbol{v}, \overline{\boldsymbol{u}}_i)_{\rho(\boldsymbol{\varphi})} = 0, \tag{5.3.9}$$

for all $i = 1, \dots, j-1$ such that

$$\frac{\langle \mathcal{E}(\boldsymbol{v}), \mathcal{E}(\boldsymbol{v}) \rangle_{\mathbb{C}(\boldsymbol{\varphi})}}{\|\boldsymbol{v}\|^2_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)}} = \lambda_j^{\boldsymbol{\varphi}}. \tag{5.3.10}$$

Otherwise the inequality in (5.3.8) would be strict, which would be a contradiction to Theorem 5.2.2. This means we have shown the existence of a function

$$\boldsymbol{v} \in W^{\perp, L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} \cap H^1_D(\Omega;\mathbb{R}^d) \quad \text{with} \quad \boldsymbol{v} \neq \boldsymbol{0}, \tag{5.3.11}$$

fulfilling (5.3.10).

Let now the sequence $(\boldsymbol{v}_k)_{k\in\mathbb{N}}$ be defined by

$$\boldsymbol{v}_k \coloneqq \boldsymbol{v} - \sum_{i=1}^{j-1} (\boldsymbol{v}, \boldsymbol{w}_i^{\boldsymbol{\varphi}_k})_{\rho(\boldsymbol{\varphi}_k)} \boldsymbol{w}_i^{\boldsymbol{\varphi}_k}. \tag{5.3.12}$$

for all $k \in \mathbb{N}$. By this construction, we immediately observe that

$$\boldsymbol{v}_k \in V_k^{\perp, L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)} \cap H^1_D(\Omega;\mathbb{R}^d).$$

We now intend to show that the convergences

$$\langle \mathcal{E}(\boldsymbol{v}_k), \mathcal{E}(\boldsymbol{v}_k) \rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)} \to \langle \mathcal{E}(\boldsymbol{v}), \mathcal{E}(\boldsymbol{v}) \rangle_{\mathbb{C}(\boldsymbol{\varphi})}, \tag{5.3.13}$$

$$\|\boldsymbol{v}_k\|_{L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)} \to \|\boldsymbol{v}\|_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)}, \tag{5.3.14}$$

as $k \to \infty$, hold along a non-relabeled subsequence.

To verify (5.3.13), we consider the decomposition

$$\begin{aligned} &\langle \mathcal{E}(\boldsymbol{v}_k), \mathcal{E}(\boldsymbol{v}_k) \rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)} \\ &\quad = \langle \mathcal{E}(\boldsymbol{v}), \mathcal{E}(\boldsymbol{v}) \rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)} - 2 \left\langle \mathcal{E}(\boldsymbol{v}), \mathcal{E}\left(\sum_{i=1}^{j-1} (\boldsymbol{v}, \boldsymbol{w}_i^{\boldsymbol{\varphi}_k})_{\rho(\boldsymbol{\varphi}_k)} \boldsymbol{w}_i^{\boldsymbol{\varphi}_k} \right) \right\rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)} \\ &\qquad + \left\langle \mathcal{E}\left(\sum_{i=1}^{j-1} (\boldsymbol{v}, \boldsymbol{w}_i^{\boldsymbol{\varphi}_k})_{\rho(\boldsymbol{\varphi}_k)} \boldsymbol{w}_i^{\boldsymbol{\varphi}_k} \right), \mathcal{E}\left(\sum_{m=1}^{j-1} (\boldsymbol{v}, \boldsymbol{w}_m^{\boldsymbol{\varphi}_k})_{\rho(\boldsymbol{\varphi}_k)} \boldsymbol{w}_m^{\boldsymbol{\varphi}_k} \right) \right\rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)}. \end{aligned} \tag{5.3.15}$$

For the first product on the right-hand side, we directly obtain the convergence

$$\langle \mathcal{E}(\boldsymbol{v}), \mathcal{E}(\boldsymbol{v}) \rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)} \to \langle \mathcal{E}(\boldsymbol{v}), \mathcal{E}(\boldsymbol{v}) \rangle_{\mathbb{C}(\boldsymbol{\varphi})}, \quad \text{as } k \to \infty,$$

along a suitable subsequence. As the functions $\boldsymbol{w}_i^{\boldsymbol{\varphi}_k}$ are $L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)$-normalized eigenfunctions, we obtain from (5.2.1) the following representation of the third product on the right-hand side of (5.3.15):

$$\begin{aligned}&\left\langle \mathcal{E}\left(\sum_{i=1}^{j-1}\left(\boldsymbol{v},\boldsymbol{w}_i^{\boldsymbol{\varphi}_k}\right)_{\rho(\boldsymbol{\varphi}_k)}\boldsymbol{w}_i^{\boldsymbol{\varphi}_k}\right),\mathcal{E}\left(\sum_{m=1}^{j-1}\left(\boldsymbol{v},\boldsymbol{w}_m^{\boldsymbol{\varphi}_k}\right)_{\rho(\boldsymbol{\varphi}_k)}\boldsymbol{w}_m^{\boldsymbol{\varphi}_k}\right)\right\rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)}\\ &\quad=\sum_{i=1}^{j-1}\left(\boldsymbol{v},\boldsymbol{w}_i^{\boldsymbol{\varphi}_k}\right)^2_{\rho(\boldsymbol{\varphi}_k)}\lambda_i^{\boldsymbol{\varphi}_k}.\end{aligned}$$

As the sum takes only the indices $i=1,\dots,j-1$ into account, we can again use the induction hypothesis to obtain

$$\sum_{i=1}^{j-1}\left(\boldsymbol{v},\boldsymbol{w}_i^{\boldsymbol{\varphi}_k}\right)^2_{\rho(\boldsymbol{\varphi}_k)}\lambda_i^{\boldsymbol{\varphi}_k}\to\sum_{i=1}^{j-1}\left(\boldsymbol{v},\overline{\boldsymbol{u}}_i\right)^2_{\rho(\boldsymbol{\varphi})}\lambda_i^{\boldsymbol{\varphi}},\quad\text{as } k\to\infty,$$

along a suitable subsequence. Hence, (5.3.9) directly yields that the third product on the right-hand side of (5.3.15) converges to zero. The second product can be handled similarly, and we can also show that it tends to zero as $k\to\infty$. In summary, we get

$$\left\langle\mathcal{E}\left(\boldsymbol{v}_k\right),\mathcal{E}\left(\boldsymbol{v}_k\right)\right\rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)}\to\left\langle\mathcal{E}\left(\boldsymbol{v}\right),\mathcal{E}\left(\boldsymbol{v}\right)\right\rangle_{\mathbb{C}(\boldsymbol{\varphi})},\quad\text{as } k\to\infty.$$

This proves (5.3.13). The claim (5.3.14) can easily be verified using the induction hypothesis.

In particular, since $\boldsymbol{v}\neq\boldsymbol{0}$, we obtain that $\boldsymbol{v}_k\neq\boldsymbol{0}$ for all $k\in\mathbb{N}$ sufficiently large. For such $k\in\mathbb{N}$, we obtain the estimate

$$\begin{aligned}\lambda_j^{\boldsymbol{\varphi}_k}&=\min\left\{\frac{\left\langle\mathcal{E}\left(\boldsymbol{u}\right),\mathcal{E}\left(\boldsymbol{u}\right)\right\rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)}}{\|\boldsymbol{u}\|^2_{L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)}}\,\middle|\,\boldsymbol{u}\in V_k^{\perp,L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)}\cap H^1_D(\Omega;\mathbb{R}^d),\boldsymbol{u}\neq\boldsymbol{0}\right\}\\ &\leq\frac{\left\langle\mathcal{E}\left(\boldsymbol{v}_k\right),\mathcal{E}\left(\boldsymbol{v}_k\right)\right\rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)}}{\|\boldsymbol{v}_k\|^2_{L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)}}.\end{aligned}$$

Using (5.3.13) and (5.3.14), we conclude from (5.3.10) that, along a non-relabeled subsequence,

$$\limsup_{k\to\infty}\lambda_j^{\boldsymbol{\varphi}_k}\leq\limsup_{k\to\infty}\frac{\left\langle\mathcal{E}\left(\boldsymbol{v}_k\right),\mathcal{E}\left(\boldsymbol{v}_k\right)\right\rangle_{\mathbb{C}(\boldsymbol{\varphi}_k)}}{\|\boldsymbol{v}_k\|^2_{L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)}}=\frac{\left\langle\mathcal{E}\left(\boldsymbol{v}\right),\mathcal{E}\left(\boldsymbol{v}\right)\right\rangle_{\mathbb{C}(\boldsymbol{\varphi})}}{\|\boldsymbol{v}\|^2_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)}}=\lambda_j^{\boldsymbol{\varphi}}.\tag{5.3.16}$$

In particular, this implies that the subsequence $(\lambda_j^{\boldsymbol{\varphi}_k})_{k\in\mathbb{N}}$ is bounded.

As in Theorem 5.2.2, for $k\in\mathbb{N}$, we consider the j-th basis function $\boldsymbol{w}_j^{\boldsymbol{\varphi}_k}$ which is $L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)$-orthogonal to V_k. As a consequence of (5.3.16), due to the eigenvalue equation (5.2.1), the sequence $(\boldsymbol{w}_j^{\boldsymbol{\varphi}_k})_{k\in\mathbb{N}}\subset H^1_D(\Omega;\mathbb{R}^d)$ is bounded. Applying the Banach–Alaoglu theorem, we can thus extract a subsequence such that

$$\boldsymbol{w}_j^{\boldsymbol{\varphi}_k}\rightharpoonup\overline{\boldsymbol{w}}\quad\text{in } H^1_D(\Omega;\mathbb{R}^d),\quad\text{and}\quad\boldsymbol{w}_j^{\boldsymbol{\varphi}_k}\to\overline{\boldsymbol{w}}\quad\text{in } L^2(\Omega;\mathbb{R}^d),\tag{5.3.17}$$

as $k \to \infty$, where $\overline{\boldsymbol{w}} \in H^1_D(\Omega;\mathbb{R}^d)$ is $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-normalized. However, it is a priori not necessarily an eigenfunction to the eigenvalue $\lambda_j^{\boldsymbol{\varphi}}$, as the convergence of the corresponding eigenvalues is still unknown.

Proceeding as in Subsection 5.3.1, we want to show that

$$\lambda_j^{\boldsymbol{\varphi}} \leq \liminf_{k\to\infty} \lambda_j^{\boldsymbol{\varphi}_k}. \tag{5.3.18}$$

As in the proof of Corollary 5.3.3 in combination with (5.3.16), we can then conclude the desired convergence $\lambda_j^{\boldsymbol{\varphi}_k} \to \lambda_j^{\boldsymbol{\varphi}}$ for the whole sequence as $k \to \infty$.

To verify (5.3.18), we first observe that due to the orthogonality of the basis functions, we have

$$\left(\boldsymbol{w}_j^{\boldsymbol{\varphi}_k}, \boldsymbol{w}_m^{\boldsymbol{\varphi}_k}\right)_{\rho(\boldsymbol{\varphi}_k)} = 0,$$

for all $m \in \{1,\dots,j-1\}$ and all $k \in \mathbb{N}$. Recalling the assumptions on $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}}$, we can use (5.3.17) and (5.3.6) to infer that

$$\left(\overline{\boldsymbol{w}}, \overline{\boldsymbol{u}}_m\right)_{\rho(\boldsymbol{\varphi})} = 0,$$

for all $m \in \{1,\dots,j-1\}$. Now let us choose $j^* \leq j-1$ as the maximal index such that $\lambda_{j^*}^{\boldsymbol{\varphi}} < \lambda_j^{\boldsymbol{\varphi}}$. By this choice, we know from the orthogonality (5.3.7) that

$$\langle \overline{\boldsymbol{u}}_1,\dots,\overline{\boldsymbol{u}}_{j^*}\rangle_{\mathrm{span}} = \langle \boldsymbol{w}_1^{\boldsymbol{\varphi}},\dots,\boldsymbol{w}_{j^*}^{\boldsymbol{\varphi}}\rangle_{\mathrm{span}} \subset L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d).$$

This leads to the representation

$$\overline{\boldsymbol{w}} = \sum_{m=1}^{\infty} \left(\overline{\boldsymbol{w}}, \boldsymbol{w}_m^{\boldsymbol{\varphi}}\right)_{\rho(\boldsymbol{\varphi})} \boldsymbol{w}_m^{\boldsymbol{\varphi}} = \sum_{m=j^*+1}^{\infty} \left(\overline{\boldsymbol{w}}, \boldsymbol{w}_m^{\boldsymbol{\varphi}}\right)_{\rho(\boldsymbol{\varphi})} \boldsymbol{w}_m^{\boldsymbol{\varphi}}.$$

As the series converges in $H^1_D(\Omega;\mathbb{R}^d)$, we can use (5.2.1) to obtain

$$\left\langle \mathcal{E}\left(\overline{\boldsymbol{w}}\right), \mathcal{E}\left(\overline{\boldsymbol{w}}\right)\right\rangle_{\mathbb{C}(\boldsymbol{\varphi})} = \sum_{m=j^*+1}^{\infty} \lambda_m^{\boldsymbol{\varphi}} \left(\overline{\boldsymbol{w}}, \boldsymbol{w}_m^{\boldsymbol{\varphi}}\right)^2_{\rho(\boldsymbol{\varphi})} \geq \lambda_j^{\boldsymbol{\varphi}},$$

as $\overline{\boldsymbol{w}} \in H^1_D(\Omega;\mathbb{R}^d)$ is $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-normalized.

Hence, it only remains to show that

$$\liminf_{k\to\infty} \lambda_j^{\boldsymbol{\varphi}_k} \geq \left\langle \mathcal{E}\left(\overline{\boldsymbol{w}}\right), \mathcal{E}\left(\overline{\boldsymbol{w}}\right)\right\rangle_{\mathbb{C}(\boldsymbol{\varphi})}.$$

This, however, can be proven completely analogously as in the proof of Lemma 5.3.1.

In summary, we obtain the convergence

$$\lambda_j^{\boldsymbol{\varphi}_k} \to \lambda_j^{\boldsymbol{\varphi}}, \quad \text{as } k \to \infty, \tag{5.3.19}$$

along a non-relabeled subsequence. Since the limit does not depend on any subsequence extraction, this convergence holds true for the whole sequence. As in Corollary 5.3.3, we conclude that $\overline{\boldsymbol{w}} \in H^1_D(\Omega;\mathbb{R}^d)$ is an eigenfunction to the eigenvalue $\lambda_j^{\boldsymbol{\varphi}}$.

Now if we replace the special choice $\boldsymbol{w}_j^{\boldsymbol{\varphi}_k}$ by $\boldsymbol{u}_j^k \in H^1_D(\Omega;\mathbb{R}^d)$ being an arbitrary $L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)$-normalized eigenfunction to the eigenvalue $\lambda_j^{\boldsymbol{\varphi}_k}$, we again know that $(\boldsymbol{u}_j^k)_{k\in\mathbb{N}} \subset H^1_D(\Omega;\mathbb{R}^d)$ is bounded. Hence, we can extract a subsequence converging weakly in $H^1_D(\Omega;\mathbb{R}^d)$ and by proceeding as in Corollary 5.3.3, we use (5.3.19) to directly conclude that the weak limit is an eigenfunction to the eigenvalue $\lambda_j^{\boldsymbol{\varphi}}$. □

5.3.2. Local Lipschitz continuity of eigenvalues

The following lemma shows that all eigenvalues are locally Lipschitz continuous with respect to $\boldsymbol{\varphi}$.

Lemma 5.3.5 (Local Lipschitz continuity of the eigenvalues)**.** *Let $i \in \mathbb{N}$ be any index and let $\boldsymbol{\varphi} \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ be arbitrary. Then there exist $\delta_i^{\boldsymbol{\varphi}}$, $C_{\boldsymbol{\varphi}}^i > 0$ such that*

$$\left|\lambda_i^{\boldsymbol{\varphi}} - \lambda_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right| \leq C_{\boldsymbol{\varphi}}^i \left\|\boldsymbol{h}\right\|_{H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)},$$

for all $\boldsymbol{h} \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ with $\left\|\boldsymbol{h}\right\|_{H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)} < \delta_i^{\boldsymbol{\varphi}}$. This means that the mapping

$$\lambda_i : H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N) \to \mathbb{R}_{>0}, \quad \boldsymbol{\varphi} \mapsto \lambda_i^{\boldsymbol{\varphi}},$$

is locally Lipschitz continuous.

Proof. Let $\boldsymbol{w}_i \in H_D^1(\Omega;\mathbb{R}^d)$ denote a $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-normalized eigenfunction to the eigenvalue $\lambda_i^{\boldsymbol{\varphi}}$. In the same fashion, let $\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}} \in H_D^1(\Omega;\mathbb{R}^d)$ denote a $L^2_{\boldsymbol{\varphi}+\boldsymbol{h}}(\Omega;\mathbb{R}^d)$-normalized eigenfunction to the eigenvalue $\lambda_i^{\boldsymbol{\varphi}+\boldsymbol{h}}$. Then, if $\delta_i^{\boldsymbol{\varphi}}$ is sufficiently small, we obtain the estimate

$$\begin{aligned}
\left|(\lambda_i^{\boldsymbol{\varphi}} - \lambda_i^{\boldsymbol{\varphi}+\boldsymbol{h}})(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}, \boldsymbol{w}_i)_{\rho(\boldsymbol{\varphi})}\right| &\leq \left|\lambda_i^{\boldsymbol{\varphi}}(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}, \boldsymbol{w}_i)_{\rho(\boldsymbol{\varphi})} - \lambda_i^{\boldsymbol{\varphi}+\boldsymbol{h}}(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}, \boldsymbol{w}_i)_{\rho(\boldsymbol{\varphi}+\boldsymbol{h})}\right| \\
&\quad + \left|\lambda_i^{\boldsymbol{\varphi}+\boldsymbol{h}}(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}, \boldsymbol{w}_i)_{\rho(\boldsymbol{\varphi}+\boldsymbol{h})} - \lambda_i^{\boldsymbol{\varphi}+\boldsymbol{h}}(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}, \boldsymbol{w}_i)_{\rho(\boldsymbol{\varphi})}\right| \\
&= \left|\langle \mathcal{E}(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}), \mathcal{E}(\boldsymbol{w}_i)\rangle_{\mathbb{C}(\boldsymbol{\varphi})-\mathbb{C}(\boldsymbol{\varphi}+\boldsymbol{h})}\right| \\
&\quad + \left|\lambda_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\Big((\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}, \boldsymbol{w}_i)_{\rho(\boldsymbol{\varphi}+\boldsymbol{h})} - (\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}, \boldsymbol{w}_i)_{\rho(\boldsymbol{\varphi})}\Big)\right| \\
&\leq C_{\boldsymbol{\varphi}}^i \left\|\boldsymbol{h}\right\|_{L^\infty(\Omega;\mathbb{R}^N)},
\end{aligned}$$

where the last inequality holds due to the local Lipschitz continuity of $\mathbb{C}$ and ρ, and the boundedness of $\lambda_i^{\boldsymbol{\varphi}+\boldsymbol{h}}$ which follows from Theorem 5.3.4. Note that the constant $C_i^{\boldsymbol{\varphi}}$ may depend on $\lambda_i^{\boldsymbol{\varphi}}$ but not on the eigenfunctions we have chosen, as they were assumed to be normalized.

Suppose now that there exists a zero sequence $(\boldsymbol{h}_k)_{k\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ such that

$$\left|\lambda_i^{\boldsymbol{\varphi}} - \lambda_i^{\boldsymbol{\varphi}+\boldsymbol{h}_k}\right| > k \left\|\boldsymbol{h}_k\right\|_{H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)},$$

as $k \to \infty$. For the corresponding sequence of eigenfunctions $(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}_k})_{k\in\mathbb{N}} \subset H_D^1(\Omega;\mathbb{R}^d)$ for the eigenvalues $(\lambda_i^{\boldsymbol{\varphi}+\boldsymbol{h}_k})_{k\in\mathbb{N}}$, we know from Theorem 5.3.4 that we find a $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-normalized eigenfunction $\overline{\boldsymbol{w}}$ to the eigenvalue $\lambda_i^{\boldsymbol{\varphi}}$ such that

$$\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}_k} \to \overline{\boldsymbol{w}} \quad \text{ in } L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d),$$

as $k \to \infty$, up to subsequence extraction. In particular, for k sufficiently large, we know that the members of this subsequence satisfy

$$(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}_k}, \overline{\boldsymbol{w}})_{\rho(\boldsymbol{\varphi})} > \frac{1}{2}$$

and thus,

$$k \left\| \boldsymbol{h}_k \right\|_{H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)} < 2C^i_{\boldsymbol{\varphi}} \left\| \boldsymbol{h}_k \right\|_{H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)},$$

which is an obvious contradiction. This proves the claim. □

5.3.3. A sign convention for eigenfunctions

In the previous analysis there was no need to assume that the eigenspaces are one-dimensional. However, in Section 5.4, we want to show that the eigenvalues are Fréchet-differentiable with respect to the phase-field. Therefore, it will be necessary to assume that for fixed $\boldsymbol{\varphi} \in H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)$ the eigenspace corresponding to the considered eigenvalue $\lambda_i^{\boldsymbol{\varphi}}$ is one-dimensional. In this case the eigenvalue is called *simple*.

Simplicity of $\lambda_i^{\boldsymbol{\varphi}}$ allows us to choose a corresponding eigenfunction $\boldsymbol{w}_i^{\boldsymbol{\varphi}} \in H^1_D(\Omega;\mathbb{R}^d)$ that is normalized with respect to the scalar product on $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$ and *unique* up to multiplication by ± 1. We call such an eigenfunction a *representative* corresponding to $\lambda_i^{\boldsymbol{\varphi}}$.

In general, any eigenspace could be higher dimensional. For numerically motivated examples showing that even the simplicity of the first eigenvalue of a scalar elliptic regular PDE is no longer fulfilled in the vector-valued case, see [72]. However, in concrete applications, there are physical and numerical justifications for assuming simple eigenvalues. This is due to the fact that nature as well as numerical simulations on computers lead to perturbations of the case of equal eigenvalues.

As a classical two dimensional example to illustrate this behavior, an eigenvalue problem associated with the Laplacian subject to Dirichlet boundary conditions can be considered, as in Chapter 3. If the domain is a perfect circle, eigenvalues with higher multiplicity will occur. However, as soon as the perfect circular shape of the domain is perturbed by small imperfections, these eigenvalues will become different and simple. For more details see [150]. Apart from this theoretical discussion we have also seen in the numerical section of Chapter 3 a practical way to deal with multiple eigenvalues, see Remark 3.4.5.

In the following lemma, we will introduce a condition to fix a sequence of representatives whose elements $\boldsymbol{w}_i^{\boldsymbol{\varphi}_k}$ are uniquely determined if $\boldsymbol{\varphi}_k \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ is sufficiently close to $\boldsymbol{\varphi}$. In particular, we see that it is possible to deduce simplicity of the eigenvalues $\lambda_i^{\boldsymbol{\varphi}_k}$ in a suitable neighborhood of $\lambda_i^{\boldsymbol{\varphi}}$.

Lemma 5.3.6. *Let $i \in \mathbb{N}$ and $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ be a sequence such that*

$$\boldsymbol{\varphi}_k \to \boldsymbol{\varphi} \quad \text{in } H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N),$$

for $k \to \infty$. Moreover, we assume that $\lambda_i^{\boldsymbol{\varphi}}$ is a simple eigenvalue of (5.2.1) and let $\boldsymbol{w}_i^{\boldsymbol{\varphi}}$ be a corresponding $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-normalized eigenfunction.

Then for any $\varepsilon \in (0,1)$, we can find a $K_i^\varepsilon > 0$ such that for any $k > K_i^\varepsilon$, there exists a unique $L^2_{\boldsymbol{\varphi}_k}(\Omega;\mathbb{R}^d)$-normalized eigenfunction $\boldsymbol{w}_i^{\boldsymbol{\varphi}_k} \in H^1_D(\Omega;\mathbb{R}^d)$ to the eigenvalue $\lambda_i^{\boldsymbol{\varphi}_k}$ satisfying

$$\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}_k}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}\right)_{\rho(\boldsymbol{\varphi})} > \varepsilon. \tag{5.3.20}$$

In particular, the eigenvalues $\lambda_i^{\boldsymbol{\varphi}_k}$ with $k > K_i^\varepsilon$ are simple.

Proof. Note that we did not make any assumptions on the simplicity of the eigenspaces corresponding to $\lambda_i^{\varphi_k}$ for $k \in \mathbb{N}$. However, this can be established if $\boldsymbol{\varphi}_k$ is close to $\boldsymbol{\varphi}$ by invoking the simplicity of the eigenspace corresponding to $\lambda_i^{\boldsymbol{\varphi}}$ and using the continuity properties known from Theorem 5.3.4.

In the following, we will assume, without loss of generality, that k is large enough to ensure that all eigenspaces to the eigenvalues $\lambda_i^{\varphi_k}$ are simple. If we are now able to find a sequence of representatives $\boldsymbol{w}_i^{\varphi_k}$ that fulfills (5.3.20) for a suitable $K_i^\varepsilon \in \mathbb{N}$, then the uniqueness assertion is clear since the eigenfunctions are normalized and their sign is fixed by (5.3.20).

To prove the existence of such a sequence, we argue once more by contradiction. Let $\varepsilon \in (0,1)$ be arbitrary and let us assume that there is no $K_i^\varepsilon \in \mathbb{N}$ such that (5.3.20) is fulfilled. Hence, after possibly swapping some of the signs, we can extract a subsequence such that

$$\left|\left(\boldsymbol{w}_i^{\varphi_k}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}\right)_{\rho(\boldsymbol{\varphi})}\right| \leq \varepsilon < 1, \quad \text{for all } k \in \mathbb{N}. \tag{5.3.21}$$

Using Theorem 5.3.4 we obtain a weak limit $\overline{\boldsymbol{w}}$ of a non-relabeled subsequence of $\left(\boldsymbol{w}_i^{\varphi_k}\right)_{k\in\mathbb{N}}$ and infer from the simplicity of $\lambda_i^{\boldsymbol{\varphi}}$ that $\overline{\boldsymbol{w}} = \pm \boldsymbol{w}_i^{\boldsymbol{\varphi}}$. Hence, using (5.3.21), we obtain

$$1 = \left(\boldsymbol{w}_i^{\boldsymbol{\varphi}}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}\right)_{\rho(\boldsymbol{\varphi})} = \pm \left(\overline{\boldsymbol{w}}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}\right)_{\rho(\boldsymbol{\varphi})} < 1,$$

which is obviously a contradiction.

Eventually, this means that condition (5.3.20) allows us to pick a *unique* representative $\boldsymbol{w}_i^{\varphi_k}$ for every $k \in \mathbb{N}$ sufficiently large such that the obtained sequence fulfills

$$\boldsymbol{w}_i^{\varphi_k} \rightharpoonup \boldsymbol{w}_i^{\boldsymbol{\varphi}} \quad \text{in } H_D^1(\Omega;\mathbb{R}^d),$$

as $k \to \infty$, up to subsequence extraction. □

The following corollary is a direct consequence of Lemma 5.3.6.

Corollary 5.3.7. *For $i \in \mathbb{N}$ and $\boldsymbol{\varphi} \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$, we suppose that the eigenvalue $\lambda_i^{\boldsymbol{\varphi}}$ is simple. Let $\boldsymbol{w}_i^{\boldsymbol{\varphi}}$ be a $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-normalized eigenfunction to the eigenvalue $\lambda_i^{\boldsymbol{\varphi}}$.*

Then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all

$$\boldsymbol{h} \in L^\infty(\Omega;\mathbb{R}^N) \cap H^1(\Omega;\mathbb{R}^N) \quad \textit{with} \quad \|\boldsymbol{h}\|_{H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)} < \delta$$

there exists a unique $L^2_{\boldsymbol{\varphi}+\boldsymbol{h}}(\Omega;\mathbb{R}^d)$-normalized eigenfunction $\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}$ to the eigenvalue $\lambda_i^{\boldsymbol{\varphi}+\boldsymbol{h}}$ satisfying the condition

$$\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}\right)_{\rho(\boldsymbol{\varphi})} > \varepsilon > 0. \tag{5.3.22}$$

In particular, the eigenvalues $\lambda_i^{\boldsymbol{\varphi}+\boldsymbol{h}}$ are simple.

This means that, if $\boldsymbol{h}$ is sufficiently small, the signs of the eigenfunctions $\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}$ can be uniquely fixed in accordance with the sign of $\boldsymbol{w}_i^{\boldsymbol{\varphi}}$ by the sign condition (5.3.22).

5.3.4. Continuity of eigenfunctions

In view of the sign convention from Corollary 5.3.7, we can now prove the following continuity result.

Lemma 5.3.8 (Continuity of eigenfunctions to simple eigenvalues)**.** *Let $\boldsymbol{\varphi} \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ be arbitrary and let $\boldsymbol{w}_i^{\boldsymbol{\varphi}}$ denote a $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-normalized eigenfunction to the eigenvalue $\lambda_i^{\boldsymbol{\varphi}}$ which is assumed to be simple. For any $\varepsilon > 0$, we assume that $\delta > 0$, $\boldsymbol{h}$ and the eigenfunctions $\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}$ to the eigenvalues $\lambda_i^{\boldsymbol{\varphi}+\boldsymbol{h}}$ are all chosen in such a way that the sign condition* (5.3.22) *is satisfied.*

Then the eigenfunctions $\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}$ are uniquely determined and it holds that

$$\left\|\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}} - \boldsymbol{w}_i^{\boldsymbol{\varphi}}\right\|_{H^1_D(\Omega;\mathbb{R}^d)} \to 0, \tag{5.3.23}$$

as $\boldsymbol{h} \to \boldsymbol{0}$ in $H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$. This means that the mapping

$$\boldsymbol{w}_i : H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N) \to H^1_D(\Omega;\mathbb{R}^d), \quad \boldsymbol{\varphi} \mapsto \boldsymbol{w}_i^{\boldsymbol{\varphi}}$$

is (strongly sequentially) continuous with respect to the norm on $H^1_D(\Omega;\mathbb{R}^d)$.

Proof. Let $(\boldsymbol{h}_k)_{k\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ be any arbitrary sequence satisfying

$$\|\boldsymbol{h}_k\|_{H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)} < \delta \quad \text{for all } k \in \mathbb{N}. \tag{5.3.24}$$

Defining the sequence $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ by $\boldsymbol{\varphi}_k := \boldsymbol{\varphi} + \boldsymbol{h}_k$ for all $k \in \mathbb{N}$, we can apply Theorem 5.3.4 to conclude that

$$\boldsymbol{w}_i^{\boldsymbol{\varphi}_k} \to \boldsymbol{w}_i^{\boldsymbol{\varphi}} \quad \text{in } L^2(\Omega;\mathbb{R}^d) \text{ as } k \to \infty,$$

along a non-relabeled subsequence. However, as the limit does not depend on the extracted subsequence, this convergence even holds true for the whole sequence. Note that for this reasoning it is essential that all members of the sequence are fixed by the sign convention (5.3.22). As the sequence $(\boldsymbol{h}_k)_{k\in\mathbb{N}}$ was arbitrary, we further infer that

$$\left\|\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}} - \boldsymbol{w}_i^{\boldsymbol{\varphi}}\right\|_{L^2(\Omega;\mathbb{R}^d)} \to 0, \quad \text{as } \boldsymbol{h} \to \boldsymbol{0} \text{ in } H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N). \tag{5.3.25}$$

If we can now show that

$$\left\langle \mathcal{E}\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}} - \boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right), \mathcal{E}\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}} - \boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right)\right\rangle_{\mathbb{C}(\boldsymbol{\varphi})} \to 0, \tag{5.3.26}$$

as $\boldsymbol{h} \to \boldsymbol{0}$ in $H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$, the prove is completed.

To this end, let $\boldsymbol{h} \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ with

$$\|\boldsymbol{h}\|_{H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)} < \delta,$$

be arbitrary. We derive the identity

$$\begin{aligned}
&\left\langle \mathcal{E}\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}} - \boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right), \mathcal{E}\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}} - \boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right)\right\rangle_{\mathbb{C}(\boldsymbol{\varphi})} \\
&\quad = \left[\left\langle \mathcal{E}\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}}\right), \mathcal{E}\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}} - \boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right)\right\rangle_{\mathbb{C}(\boldsymbol{\varphi})} - \left\langle \mathcal{E}(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}), \mathcal{E}(\boldsymbol{w}_i^{\boldsymbol{\varphi}} - \boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}})\right\rangle_{\mathbb{C}(\boldsymbol{\varphi}+\boldsymbol{h})}\right]
\end{aligned}$$

$$+\left[\left\langle \mathcal{E}(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}), \mathcal{E}(\boldsymbol{w}_i^{\boldsymbol{\varphi}}-\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}})\right\rangle_{\mathbb{C}(\boldsymbol{\varphi}+\boldsymbol{h})} - \left\langle \mathcal{E}\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right), \mathcal{E}\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}}-\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right)\right\rangle_{\mathbb{C}(\boldsymbol{\varphi})}\right].$$

The second summand on the right-hand side converges to 0 in $H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)$ as $\boldsymbol{h}\to\mathbf{0}$, since the norm $\|\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\|_{H^1_D(\Omega;\mathbb{R}^d)}$ is bounded by a constant that may depend on δ but not on $\boldsymbol{h}$, and $\mathbb{C}$ is locally Lipschitz continuous. As $\boldsymbol{w}_i^{\boldsymbol{\varphi}}$ and $\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}$ are eigenfunctions, they satisfy the state equation (5.2.1) and we thus get

$$\begin{aligned}
&\left\langle \mathcal{E}\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}}\right), \mathcal{E}\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}}-\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right)\right\rangle_{\mathbb{C}(\boldsymbol{\varphi})} - \left\langle \mathcal{E}\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right), \mathcal{E}\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}}-\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right)\right\rangle_{\mathbb{C}(\boldsymbol{\varphi}+\boldsymbol{h})}\\
&= \lambda_i^{\boldsymbol{\varphi}}\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}-\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right)_{\rho(\boldsymbol{\varphi})} - \lambda_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}-\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right)_{\rho(\boldsymbol{\varphi}+\boldsymbol{h})}\\
&= \lambda_i^{\boldsymbol{\varphi}}\left[\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}-\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right)_{\rho(\boldsymbol{\varphi})} - \left(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}-\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right)_{\rho(\boldsymbol{\varphi})}\right]\\
&\quad + \lambda_i^{\boldsymbol{\varphi}}\left[\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}-\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right)_{\rho(\boldsymbol{\varphi})} - \left(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}-\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right)_{\rho(\boldsymbol{\varphi}+\boldsymbol{h})}\right]\\
&\quad + \left[\lambda_i^{\boldsymbol{\varphi}}-\lambda_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right]\left(\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}-\boldsymbol{w}_i^{\boldsymbol{\varphi}+\boldsymbol{h}}\right)_{\rho(\boldsymbol{\varphi}+\boldsymbol{h})}.
\end{aligned}$$

Here, the first summand converges to zero because of (5.3.25). The second summand converges to zero due to the local Lipschitz continuity of ρ and the last summand converges to zero as a consequence of Theorem 5.3.4. This verifies (5.3.26) and thus, the proof is complete. □

5.4. Differentiability of eigenvalues and eigenfunctions

5.4.1. A formal consideration

First of all, we want to discuss the desired differentiability results formally. To obtain the Fréchet-derivative of the functional

$$\lambda_i : H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)\to\mathbb{R}_{>0},\quad \boldsymbol{\varphi}\mapsto\lambda_i^{\boldsymbol{\varphi}},$$

for $i\in\mathbb{N}$, we formally differentiate the state equation in the Gâteaux sense. If $\boldsymbol{w}$ is an eigenfunction to the eigenvalue $\lambda_i^{\boldsymbol{\varphi}}$, we have

$$\langle \mathcal{E}(\boldsymbol{w}), \mathcal{E}(\boldsymbol{\eta})\rangle_{\mathbb{C}(\boldsymbol{\varphi})} = \lambda_i^{\boldsymbol{\varphi}}\int_\Omega \rho(\boldsymbol{\varphi})\boldsymbol{w}\cdot\boldsymbol{\eta}\,\mathrm{d}x \quad \text{for all } \boldsymbol{\eta}\in H^1_D(\Omega;\mathbb{R}^d). \tag{5.4.1}$$

Computing the first variation of (5.4.1) with respect to $\boldsymbol{\varphi}$ in the direction $\boldsymbol{h}\in H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)$, and choosing $\boldsymbol{w}$ and $\boldsymbol{\eta}$ as the $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-normalized eigenfunction $\boldsymbol{w}_i^{\boldsymbol{\varphi}}$ afterwards, we get

$$\left(\lambda_i^{\boldsymbol{\varphi}}\right)'\boldsymbol{h} = \langle \mathcal{E}(\boldsymbol{w}_i^{\boldsymbol{\varphi}}), \mathcal{E}(\boldsymbol{w}_i^{\boldsymbol{\varphi}})\rangle_{\mathbb{C}'(\boldsymbol{\varphi})\boldsymbol{h}} - \lambda_i^{\boldsymbol{\varphi}}\int_\Omega \rho'(\boldsymbol{\varphi})\boldsymbol{h}\left|\boldsymbol{w}_i^{\boldsymbol{\varphi}}\right|^2\,\mathrm{d}x. \tag{5.4.2}$$

Moreover, firstly plugging $\boldsymbol{w}=\boldsymbol{w}_i^{\boldsymbol{\varphi}}$ into (5.4.1), and then computing the first variation with respect to $\boldsymbol{\varphi}$ in the direction $\boldsymbol{h}\in H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)$ reveals that the formal

derivative $(\boldsymbol{w}_i^{\boldsymbol{\varphi}})' \boldsymbol{h}$ of $\boldsymbol{w}_i^{\boldsymbol{\varphi}}$ has to fulfill the equation

$$\begin{aligned}
&\left\langle \mathcal{E}\left((\boldsymbol{w}_i^{\boldsymbol{\varphi}})' \boldsymbol{h} \right), \mathcal{E}(\boldsymbol{\eta}) \right\rangle_{\mathbb{C}(\boldsymbol{\varphi})} - \lambda_i^{\boldsymbol{\varphi}} \int_\Omega \rho(\boldsymbol{\varphi}) (\boldsymbol{w}_i^{\boldsymbol{\varphi}})' \boldsymbol{h} \cdot \boldsymbol{\eta} \, \mathrm{d}x \\
&\quad = - \left\langle \mathcal{E}(\boldsymbol{w}_i^{\boldsymbol{\varphi}}), \mathcal{E}(\boldsymbol{\eta}) \right\rangle_{\mathbb{C}'(\boldsymbol{\varphi})\boldsymbol{h}} + \lambda_i^{\boldsymbol{\varphi}} \int_\Omega \rho'(\boldsymbol{\varphi}) \boldsymbol{h} \boldsymbol{w}_i^{\boldsymbol{\varphi}} \cdot \boldsymbol{\eta} \, \mathrm{d}x \\
&\qquad + (\lambda_i^{\boldsymbol{\varphi}})' \boldsymbol{h} \int_\Omega \rho(\boldsymbol{\varphi}) \boldsymbol{w}_i^{\boldsymbol{\varphi}} \cdot \boldsymbol{\eta} \, \mathrm{d}x.
\end{aligned} \tag{5.4.3}$$

In the following, we intend to verify these results rigorously. We already see that formula (5.4.2) is a priori not well-defined if there are at least two orthogonal eigenfunctions to the eigenvalue $\lambda_i^{\boldsymbol{\varphi}}$, i.e., if $\lambda_i^{\boldsymbol{\varphi}}$ is not simple. In the following approach, we will see that the simplicity of eigenvalues will play a crucial role in our analysis.

5.4.2. Semi-differentiability of the first eigenvalue

In [141, Section 4.2], the concept of *semi-differentiability* is introduced and applied to the first eigenvalue of an abstract problem discussed there. Semi-differentiability is a concept similar to Gateâux-differentiability, see Section 2.2.2, but the limit does not need to fulfill any linearity or continuity assumptions, and the variation is only performed along a *fixed positive direction*. The advantage becomes clear by the following example presented in [108, Section 2.5]. We consider the matrix-valued function

$$A : \mathbb{R} \to \mathbb{R}^{2\times 2}, \quad A(t) = \begin{pmatrix} 1-t & 0 \\ 0 & 1+t \end{pmatrix},$$

whose first eigenvalue

$$\lambda_1 : \mathbb{R} \to \mathbb{R}, \quad \lambda_1^t = 1 - |t|$$

is not simple at $t = 0$. Of course, λ_1 is not classically differentiable in $t = 0$, but we still obtain a well defined limit

$$\lim_{\substack{t\to 0 \\ t>0}} \frac{1 - |t| - 1}{t} = -1.$$

This means we can still compute some sort of derivative in a fixed positive direction.

We now give a precise definition of semi-differentiability which can be found, e.g., in [141, Definition 4.6].

Definition 5.4.1 (Definition of semi-differentiability)**.** *Let X, Y be Banach spaces and let $D \subseteq X$ be an open subset. Then the map $T : D \to Y$ is called* semi-differentiable *at the point $x \in D$ if for all $h \in X$, there exists $y(x,h) \in Y$ such that*

$$\lim_{\substack{t\to 0 \\ t>0}} \frac{T(x+th) - T(x)}{t} = y(x,h).$$

In this case we write $T'(x)h = y(x,h)$ to denote the semi-derivative of T at the point x with respect to the direction h.

We want to show that in our problem the first eigenvalue also fulfills this weaker notion of differentiability, which will be enough to deduce first-order necessary optimality conditions for the first eigenvalue as we only want to derive convex combinations where $t > 0$.

The advantage of this approach is that we do not have to assume simplicity of the first eigenvalue in order to obtain semi-differentiability, whereas as illustrated in Section 5.4.1, we need such simplicity assumptions to obtain classical differentiability.

The semi-differentiability of the first eigenvalue is established by the following lemma.

Theorem 5.4.2 (Semi-differentiability of the first eigenvalue)**.** *Let $\boldsymbol{\varphi}, \boldsymbol{h} \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ be arbitrary and let us define*

$$(\lambda_1^{\boldsymbol{\varphi}})'\boldsymbol{h} \coloneqq \inf \left\{ \langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u}) \rangle_{\mathbb{C}'(\boldsymbol{\varphi})\boldsymbol{h}} - \lambda_1^{\boldsymbol{\varphi}} (\boldsymbol{u}, \boldsymbol{u})_{\rho'(\boldsymbol{\varphi})\boldsymbol{h}} \,\middle|\, \begin{array}{l} \boldsymbol{u} \in H^1_D(\Omega;\mathbb{R}^d) \text{ is an} \\ \text{eigenfunction to } \lambda_1^{\boldsymbol{\varphi}} \\ \text{with } \|\boldsymbol{u}\|_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} = 1 \end{array} \right\}. \tag{5.4.4}$$

Then we have

$$\lim_{\substack{t \to 0 \\ t>0}} \frac{\lambda_1^{\boldsymbol{\varphi}+t\boldsymbol{h}} - \lambda_1^{\boldsymbol{\varphi}}}{t} = (\lambda_1^{\boldsymbol{\varphi}})' \boldsymbol{h}$$

and thus, the eigenvalue $\lambda_1^{\boldsymbol{\varphi}}$ is semi-differentiable with respect to $\boldsymbol{\varphi}$.

Proof. In this proof, the letter C denotes generic positive constants that may depend on $\boldsymbol{\varphi}$ and may change their value from line to line. We first prove that the infimum in (5.4.4) is actually attained by a minimizer. To this end, let $\boldsymbol{u} \in F_{\text{ad}}$ be arbitrary, where the feasible set is given as

$$F_{\text{ad}} \coloneqq \left\{ \boldsymbol{u} \in H^1_D(\Omega;\mathbb{R}^d) \,\middle|\, \begin{array}{l} \boldsymbol{u} \text{ is an eigenfunction to } \lambda_1^{\boldsymbol{\varphi}} \\ \text{with } \|\boldsymbol{u}\|_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} = 1 \end{array} \right\}.$$

By the differentiability and the local Lipschitz continuity of $\mathbb{C}$, we infer that there exists $t_0 > 0$ such that for all $t < t_0$,

$$\begin{aligned} &- C \|\boldsymbol{h}\|_{H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)} \|\boldsymbol{u}\|^2_{H^1_D(\Omega;\mathbb{R}^d)} \\ &\qquad \leq - \left| \frac{\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u}) \rangle_{\mathbb{C}(\boldsymbol{\varphi}+t\boldsymbol{h})} - \langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u}) \rangle_{\mathbb{C}(\boldsymbol{\varphi})}}{t} \right| \\ &\qquad \leq 1 + \langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u}) \rangle_{\mathbb{C}'(\boldsymbol{\varphi})\boldsymbol{h}}. \end{aligned} \tag{5.4.5}$$

Using (5.2.1), we conclude that

$$\|\boldsymbol{u}\|^2_{H^1_D(\Omega;\mathbb{R}^d)} \leq C\lambda_1^{\boldsymbol{\varphi}} \quad \text{for all } \boldsymbol{u} \in F_{\text{ad}}. \tag{5.4.6}$$

Moreover, due to (2.1.19) and (2.1.20), we have

$$\left| (\boldsymbol{u}, \boldsymbol{u})_{\rho'(\boldsymbol{\varphi})\boldsymbol{h}} \right| \leq C \|\boldsymbol{h}\|_{H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)} \quad \text{for all } \boldsymbol{u} \in F_{\text{ad}}. \tag{5.4.7}$$

Eventually, combining the above estimates, we conclude that

$$\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u}) \rangle_{\mathbb{C}'(\boldsymbol{\varphi})\boldsymbol{h}} - \lambda_1^{\boldsymbol{\varphi}} (\boldsymbol{u}, \boldsymbol{u})_{\rho'(\boldsymbol{\varphi})\boldsymbol{h}}$$

$$\geq -C\lambda_1^{\boldsymbol{\varphi}} \|\boldsymbol{h}\|_{H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)} - 1 \ > -\infty$$

for all $u \in F_{\mathrm{ad}}$. This directly implies that the infimum $(\lambda_1^{\boldsymbol{\varphi}})'\boldsymbol{h}$ exists.

Hence, we can find a minimizing sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}} \subset F_{\mathrm{ad}}$ such that

$$\lim_{n\to\infty} \left[\langle \mathcal{E}(\boldsymbol{u}_n), \mathcal{E}(\boldsymbol{u}_n)\rangle_{\mathbb{C}'(\boldsymbol{\varphi})\boldsymbol{h}} - \lambda_1^{\boldsymbol{\varphi}} (\boldsymbol{u}_n, \boldsymbol{u}_n)_{\rho'(\boldsymbol{\varphi})\boldsymbol{h}}\right] = (\lambda_1^{\boldsymbol{\varphi}})' \boldsymbol{h}.$$

Due to (5.4.6), there exists $\boldsymbol{u}^* \in H_D^1(\Omega;\mathbb{R}^d)$ such that

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}^* \quad \text{in } H_D^1(\Omega;\mathbb{R}^d), \quad \text{and} \quad \boldsymbol{u}_n \to \boldsymbol{u}^* \quad \text{in } L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$$

as $n \to \infty$, up to a subsequence extraction. In particular, this implies that $\boldsymbol{u}^* \in F_{ad}$ which leads to

$$\langle \mathcal{E}(\boldsymbol{u}_n - \boldsymbol{u}^*), \mathcal{E}(\boldsymbol{u}_n - \boldsymbol{u}^*)\rangle_{\mathbb{C}(\boldsymbol{\varphi})} = \lambda_1^{\boldsymbol{\varphi}} (\boldsymbol{u}_n - \boldsymbol{u}^*, \boldsymbol{u}_n - \boldsymbol{u}^*)_{\rho(\boldsymbol{\varphi})}.$$

This implies that $\boldsymbol{u}_n \to \boldsymbol{u}^*$ even strongly in $H_D^1(\Omega;\mathbb{R}^d)$. In particular we obtain

$$\begin{aligned}(\lambda_1^{\boldsymbol{\varphi}})' \boldsymbol{h} &= \lim_{n\to\infty} \left[\langle \mathcal{E}(\boldsymbol{u}_n), \mathcal{E}(\boldsymbol{u}_n)\rangle_{\mathbb{C}'(\boldsymbol{\varphi})\boldsymbol{h}} - \lambda_1^{\boldsymbol{\varphi}} (\boldsymbol{u}_n, \boldsymbol{u}_n)_{\rho'(\boldsymbol{\varphi})\boldsymbol{h}}\right] \\ &= \langle \mathcal{E}(\boldsymbol{u}^*), \mathcal{E}(\boldsymbol{u}^*)\rangle_{\mathbb{C}'(\boldsymbol{\varphi})\boldsymbol{h}} - \lambda_1^{\boldsymbol{\varphi}} (\boldsymbol{u}^*, \boldsymbol{u}^*)_{\rho'(\boldsymbol{\varphi})\boldsymbol{h}}.\end{aligned}$$

Hence, the infimum is attained at $\boldsymbol{u}^* \in H_D^1(\Omega;\mathbb{R}^d)$.

To prove

$$\lim_{\substack{t\to 0\\ t>0}} \frac{1}{t} \left|\lambda_1^{\boldsymbol{\varphi}+t\boldsymbol{h}} - \lambda_1^{\boldsymbol{\varphi}} - (\lambda_1^{\boldsymbol{\varphi}})'[t\boldsymbol{h}]\right| = 0, \tag{5.4.8}$$

it suffices to show that there exist functions $f, g : \mathbb{R} \to \mathbb{R}$ with $f, g \in o(t)$ as $t \to 0$ such that for all $t > 0$,

$$\lambda_1^{\boldsymbol{\varphi}+t\boldsymbol{h}} - \lambda_1^{\boldsymbol{\varphi}} - (\lambda_1^{\boldsymbol{\varphi}})'[t\boldsymbol{h}] \leq f(t), \tag{5.4.9}$$

$$-\lambda_1^{\boldsymbol{\varphi}+t\boldsymbol{h}} + \lambda_1^{\boldsymbol{\varphi}} + (\lambda_1^{\boldsymbol{\varphi}})'[t\boldsymbol{h}] \leq g(t). \tag{5.4.10}$$

By the construction of $\boldsymbol{u}^*$, we first observe that

$$\begin{aligned}&\lambda_1^{\boldsymbol{\varphi}+t\boldsymbol{h}} - \lambda_1^{\boldsymbol{\varphi}} - (\lambda_1^{\boldsymbol{\varphi}})'[t\boldsymbol{h}] \\ &\quad = \lambda_1^{\boldsymbol{\varphi}+t\boldsymbol{h}} - \langle \mathcal{E}(\boldsymbol{u}^*), \mathcal{E}(\boldsymbol{u}^*)\rangle_{\mathbb{C}(\boldsymbol{\varphi})} - \langle \mathcal{E}(\boldsymbol{u}^*), \mathcal{E}(\boldsymbol{u}^*)\rangle_{\mathbb{C}'(\boldsymbol{\varphi})t\boldsymbol{h}} + \lambda_1^{\boldsymbol{\varphi}} (\boldsymbol{u}^*, \boldsymbol{u}^*)_{\rho'(\boldsymbol{\varphi})t\boldsymbol{h}},\end{aligned}$$

since $\boldsymbol{u}^*$ is an $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-normalized eigenfunction to the eigenvalue $\lambda_1^{\boldsymbol{\varphi}}$. We compute

$$\begin{aligned}&\lambda_1^{\boldsymbol{\varphi}+t\boldsymbol{h}} - \langle \mathcal{E}(\boldsymbol{u}^*), \mathcal{E}(\boldsymbol{u}^*)\rangle_{\mathbb{C}(\boldsymbol{\varphi})} - \langle \mathcal{E}(\boldsymbol{u}^*), \mathcal{E}(\boldsymbol{u}^*)\rangle_{\mathbb{C}'(\boldsymbol{\varphi})t\boldsymbol{h}} + \lambda_1^{\boldsymbol{\varphi}} (\boldsymbol{u}^*, \boldsymbol{u}^*)_{\rho'(\boldsymbol{\varphi})t\boldsymbol{h}} \\ &\quad = \left(\lambda_1^{\boldsymbol{\varphi}} - \lambda_1^{\boldsymbol{\varphi}+t\boldsymbol{h}}\right) (\boldsymbol{u}^*, \boldsymbol{u}^*)_{\rho(\boldsymbol{\varphi}+t\boldsymbol{h})-\rho(\boldsymbol{\varphi})} + \langle \mathcal{E}(\boldsymbol{u}^*), \mathcal{E}(\boldsymbol{u}^*)\rangle_{\mathbb{C}(\boldsymbol{\varphi}+t\boldsymbol{h})-\mathbb{C}(\boldsymbol{\varphi})-\mathbb{C}'(\boldsymbol{\varphi})t\boldsymbol{h}} \\ &\qquad - \lambda_1^{\boldsymbol{\varphi}} (\boldsymbol{u}^*, \boldsymbol{u}^*)_{\rho(\boldsymbol{\varphi}+t\boldsymbol{h})-\rho(\boldsymbol{\varphi})-\rho'(\boldsymbol{\varphi})t\boldsymbol{h}} + \lambda_1^{\boldsymbol{\varphi}+t\boldsymbol{h}} (\boldsymbol{u}^*, \boldsymbol{u}^*)_{\rho(\boldsymbol{\varphi}+t\boldsymbol{h})} - \langle \mathcal{E}(\boldsymbol{u}^*), \mathcal{E}(\boldsymbol{u}^*)\rangle_{\mathbb{C}(\boldsymbol{\varphi}+t\boldsymbol{h})}.\end{aligned}$$

Now the first three summands on the right-hand side are clearly in $o(t)$ as $t \to 0$ since the eigenvalues converge, and the functions ρ and $\mathbb{C}$ are differentiable and locally Lipschitz

continuous. For the remaining summands we can use the Courant–Fischer representation for the first eigenvalue which yields

$$\lambda_1^{\varphi+t\boldsymbol{h}} = \min\left\{ \frac{\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u})\rangle_{\mathbb{C}(\varphi+t\boldsymbol{h})}}{\|\boldsymbol{u}\|^2_{L^2_{\varphi+t\boldsymbol{h}}(\Omega;\mathbb{R}^d)}} \middle| \begin{array}{l} \boldsymbol{u} \in H^1_D(\Omega;\mathbb{R}^d), \\ \boldsymbol{u} \neq \boldsymbol{0} \end{array} \right\} \leq \frac{\langle \mathcal{E}(\boldsymbol{u}^*), \mathcal{E}(\boldsymbol{u}^*)\rangle_{\mathbb{C}(\varphi+t\boldsymbol{h})}}{\|\boldsymbol{u}^*\|^2_{L^2_{\varphi+t\boldsymbol{h}}(\Omega;\mathbb{R}^d)}}.$$

This implies

$$\lambda_1^{\varphi+t\boldsymbol{h}} (\boldsymbol{u}^*, \boldsymbol{u}^*)_{\rho(\varphi+t\boldsymbol{h})} - \langle \mathcal{E}(\boldsymbol{u}^*), \mathcal{E}(\boldsymbol{u}^*)\rangle_{\mathbb{C}(\varphi+t\boldsymbol{h})} \leq 0,$$

and thus, (5.4.9) is established.

To prove (5.4.10), we argue by contradiction and assume that (5.4.10) does not hold. Then, there exists $\varepsilon > 0$ and a sequence $(t_k)_{k\in\mathbb{N}} \subset (0,1]$ with $t_k \to 0$ as $k \to \infty$ such that for all $k \in \mathbb{N}$,

$$-\lambda_1^{\varphi+t_k\boldsymbol{h}} + \lambda_1^{\varphi} + (\lambda_1^{\varphi})'[t_k\boldsymbol{h}] \geq \varepsilon t_k.$$

Then, according to Theorem 5.3.4, there exists a $L^2_{\varphi}(\Omega;\mathbb{R}^d)$-normalized eigenfunction $\boldsymbol{u}$ to the eigenvalue λ_1^{φ}, as well as a sequence $(\boldsymbol{u}^{\varphi+t_k\boldsymbol{h}})_{k\in\mathbb{N}}$ consisting of $L^2_{\varphi+t_k\boldsymbol{h}}(\Omega;\mathbb{R}^d)$-normalized eigenfunctions to the eigenvalues $(\lambda_1^{\varphi+t_k\boldsymbol{h}})_{k\in\mathbb{N}}$ such that

$$\boldsymbol{u}^{\varphi+t_k\boldsymbol{h}} \rightharpoonup \boldsymbol{u} \quad \text{in } H^1_D(\Omega;\mathbb{R}^d) \quad \text{and} \quad \boldsymbol{u}^{\varphi+t_k\boldsymbol{h}} \to \boldsymbol{u} \quad \text{in } L^2(\Omega;\mathbb{R}^d) \tag{5.4.11}$$

as $k \to \infty$, along a non-relabeled subsequence. Recalling the definition of $(\lambda_1^{\varphi})'\boldsymbol{h}$, we infer that

$$\begin{aligned}
&-\lambda_1^{\varphi+t_k\boldsymbol{h}} + \lambda_1^{\varphi} + (\lambda_1^{\varphi})'[t_k\boldsymbol{h}] \\
&\quad = -\lambda_1^{\varphi+t_k\boldsymbol{h}}(\boldsymbol{u}^{\varphi+t_k\boldsymbol{h}}, \boldsymbol{u}^{\varphi+t_k\boldsymbol{h}})_{\rho(\varphi+t_k\boldsymbol{h})} + \lambda_1^{\varphi} + (\lambda_1^{\varphi})'[t_k\boldsymbol{h}] \\
&\quad \leq -\big\langle \mathcal{E}\big(\boldsymbol{u}^{\varphi+t_k\boldsymbol{h}}\big), \mathcal{E}\big(\boldsymbol{u}^{\varphi+t_k\boldsymbol{h}}\big)\big\rangle_{\mathbb{C}(\varphi+t_k\boldsymbol{h})} + \lambda_1^{\varphi} + \big\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u})\big\rangle_{\mathbb{C}'(\varphi)t_k\boldsymbol{h}} \\
&\qquad - \lambda_1^{\varphi}\,(\boldsymbol{u}, \boldsymbol{u})_{\rho'(\varphi)t_k\boldsymbol{h}}\,.
\end{aligned}$$

Recalling the identities

$$\begin{aligned}
\big\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{\eta})\big\rangle_{\mathbb{C}(\varphi)} &= \lambda_1^{\varphi}(\boldsymbol{u}, \boldsymbol{\eta})_{\rho(\varphi)}, \\
\big\langle \mathcal{E}\big(\boldsymbol{u}^{\varphi+t_k\boldsymbol{h}}\big), \mathcal{E}(\boldsymbol{\eta})\big\rangle_{\mathbb{C}(\varphi+t_k\boldsymbol{h})} &= \lambda_1^{\varphi+t_k\boldsymbol{h}}\big(\boldsymbol{u}^{\varphi+t_k\boldsymbol{h}}, \boldsymbol{\eta}\big)_{\rho(\varphi+t_k\boldsymbol{h})},
\end{aligned}$$

for all $\boldsymbol{\eta} \in H^1_D(\Omega;\mathbb{R}^d)$, a straightforward computation reveals that

$$\begin{aligned}
&-\lambda_1^{\varphi+t_k\boldsymbol{h}} + \lambda_1^{\varphi} + (\lambda_1^{\varphi})'[t_k\boldsymbol{h}] \\
&\quad \leq \big\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u})\big\rangle_{\mathbb{C}'(\varphi)t_k\boldsymbol{h}+\mathbb{C}(\varphi)-\mathbb{C}(\varphi+t_k\boldsymbol{h})} + \lambda_1^{\varphi}\big(\boldsymbol{u}, \boldsymbol{u}\big)_{\rho(\varphi+t_k\boldsymbol{h})-\rho(\varphi)-\rho'(\varphi)t_k\boldsymbol{h}} \\
&\qquad + \big\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u}-\boldsymbol{u}^{\varphi+t_k\boldsymbol{h}})\big\rangle_{\mathbb{C}(\varphi+t_k\boldsymbol{h})-\mathbb{C}(\varphi)-\mathbb{C}'(\varphi)t_k\boldsymbol{h}} \\
&\qquad + \big\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u}-\boldsymbol{u}^{\varphi+t_k\boldsymbol{h}})\big\rangle_{\mathbb{C}'(\varphi)t_k\boldsymbol{h}} + \lambda_1^{\varphi+t_k\boldsymbol{h}}\big(\boldsymbol{u}, \boldsymbol{u}-\boldsymbol{u}^{\varphi+t_k\boldsymbol{h}}\big)_{\rho(\varphi)-\rho(\varphi+t_k\boldsymbol{h})} \\
&\qquad + (\lambda_1^{\varphi} - \lambda_1^{\varphi+t_k\boldsymbol{h}})\big(\boldsymbol{u}, \boldsymbol{u}-\boldsymbol{u}^{\varphi+t_k\boldsymbol{h}}\big)_{\rho(\varphi)} + \big(\lambda_1^{\varphi+t_k\boldsymbol{h}} - \lambda_1^{\varphi}\big)\big(\boldsymbol{u}, \boldsymbol{u}\big)_{\rho(\varphi+t_k\boldsymbol{h})-\rho(\varphi)}.
\end{aligned}$$

Recalling the convergence property (5.4.11), that ρ and $\mathbb{C}$ are of class C^1_{loc}, that $\boldsymbol{\varphi} \mapsto \lambda_1^{\boldsymbol{\varphi}}$ is locally Lipschitz continuous according to Lemma 5.3.5, and that

$$\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\cdot) \rangle_{\mathbb{C}'(\boldsymbol{\varphi})\boldsymbol{h}} \in \left(H^1_D(\Omega; \mathbb{R}^d) \right)^*,$$

we conclude that the right-hand side belongs to $o(t_k)$ as $k \to \infty$.

On the other hand we assumed

$$\varepsilon t_k \leq -\lambda_1^{\boldsymbol{\varphi}+t_k\boldsymbol{h}} + \lambda_1^{\boldsymbol{\varphi}} + (\lambda_1^{\boldsymbol{\varphi}})'[t_k\boldsymbol{h}],$$

which is obviously a contradiction as the inequality cannot hold for k sufficiently large. This proves (5.4.10).

Now, (5.4.8) directly follows from (5.4.9) and (5.4.10) and thus, the proof is complete. □

5.4.3. Fréchet-differentiability of eigenvalues and eigenfunctions

If the considered eigenvalue is simple, we can even obtain stronger differentiability results in the Fréchet sense. To be precise, if for $i \in \mathbb{N}$ and $\boldsymbol{\varphi} \in H^1(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$, the eigenvalue $\lambda_i^{\boldsymbol{\varphi}}$ associated with $\boldsymbol{\varphi}$ is simple, then $\lambda_i^{\boldsymbol{\varphi}}$ and any fixed $L^2_{\boldsymbol{\varphi}}(\Omega; \mathbb{R}^d)$-normalized eigenfunction $\boldsymbol{w}_i^{\boldsymbol{\varphi}}$ are even Fréchet-differentiable with respect to $\boldsymbol{\varphi}$. This is established by the following theorem:

Theorem 5.4.3 (Fréchet-differentiability of simple eigenvalues and their eigenfunctions)**.** *Let $\boldsymbol{\varphi} \in H^1(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$ be arbitrary and suppose that for $i \in \mathbb{N}$, the eigenvalue $\lambda_i^{\boldsymbol{\varphi}}$ is simple. We further fix a $L^2_{\boldsymbol{\varphi}}(\Omega; \mathbb{R}^d)$-normalized eigenfunction $\boldsymbol{w}_i^{\boldsymbol{\varphi}}$ to the eigenvalue $\lambda_i^{\boldsymbol{\varphi}}$.*

Then there exist constants $\delta_i^{\boldsymbol{\varphi}}, r_i^{\boldsymbol{\varphi}} > 0$ such that the operator

$$S_i^{\boldsymbol{\varphi}} : B_{\delta_i^{\boldsymbol{\varphi}}}(\boldsymbol{\varphi}) \subset H^1(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N) \to B_{r_i^{\boldsymbol{\varphi}}}\big((\boldsymbol{w}_i^{\boldsymbol{\varphi}}, \lambda_i^{\boldsymbol{\varphi}})\big) \subset H^1_D(\Omega; \mathbb{R}^d) \times \mathbb{R},$$
$$\boldsymbol{\vartheta} \mapsto (\boldsymbol{w}_i^{\boldsymbol{\vartheta}}, \lambda_i^{\boldsymbol{\vartheta}}),$$

is well-defined and continuously Fréchet-differentiable. Here, $\boldsymbol{w}_i^{\boldsymbol{\vartheta}}$ denotes the unique $L^2_{\boldsymbol{\vartheta}}(\Omega; \mathbb{R}^d)$-normalized eigenfunction to the eigenvalue $\lambda_i^{\boldsymbol{\vartheta}}$ satisfying the sign condition (5.3.22) *written for $\boldsymbol{h} = \boldsymbol{\vartheta} - \boldsymbol{\varphi}$.*

Moreover, for any $\boldsymbol{h} \in H^1(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$, the Fréchet-derivative $(\lambda_i^{\boldsymbol{\vartheta}})'\boldsymbol{h}$ of the eigenvalue $\lambda_i^{\boldsymbol{\vartheta}}$ at $\boldsymbol{\vartheta} \in B_{\delta_i^{\boldsymbol{\varphi}}}(\boldsymbol{\varphi})$ in the direction $\boldsymbol{h}$ reads as

$$(\lambda_i^{\boldsymbol{\vartheta}})'\boldsymbol{h} \coloneqq \left(S^{\boldsymbol{\varphi}}_{i,2}(\boldsymbol{\vartheta}) \right)' \boldsymbol{h} = \left\langle \mathcal{E}(\boldsymbol{w}_i^{\boldsymbol{\vartheta}}), \mathcal{E}(\boldsymbol{w}_i^{\boldsymbol{\vartheta}}) \right\rangle_{\mathbb{C}'(\boldsymbol{\vartheta})\boldsymbol{h}} - \lambda_i^{\boldsymbol{\vartheta}} \int_\Omega \rho'(\boldsymbol{\vartheta})\boldsymbol{h} \left| \boldsymbol{w}_i^{\boldsymbol{\vartheta}} \right|^2 \mathrm{d}x, \tag{5.4.12}$$

and the Fréchet-derivative $\left(\boldsymbol{w}_i^{\boldsymbol{\vartheta}} \right)' \boldsymbol{h} \coloneqq (S^{\boldsymbol{\varphi}}_{i,1}(\boldsymbol{\vartheta}))'\boldsymbol{h} \in H^1_D(\Omega; \mathbb{R}^d)$ of the corresponding eigenfunction $\boldsymbol{w}_i^{\boldsymbol{\vartheta}}$ at $\boldsymbol{\vartheta}$ in the direction $\boldsymbol{h}$ is the unique solution of

$$\begin{aligned} &\left\langle \mathcal{E}((\boldsymbol{w}_i^{\boldsymbol{\vartheta}})'\boldsymbol{h}), \mathcal{E}(\boldsymbol{\eta}) \right\rangle_{\mathbb{C}(\boldsymbol{\vartheta})} - \lambda_i^{\boldsymbol{\vartheta}} \int_\Omega \rho(\boldsymbol{\vartheta})(\boldsymbol{w}_i^{\boldsymbol{\vartheta}})'\boldsymbol{h} \cdot \boldsymbol{\eta} \,\mathrm{d}x \\ &\quad = -\left\langle \mathcal{E}(\boldsymbol{w}_i^{\boldsymbol{\vartheta}}), \mathcal{E}(\boldsymbol{\eta}) \right\rangle_{\mathbb{C}'(\boldsymbol{\vartheta})\boldsymbol{h}} + \lambda_i^{\boldsymbol{\vartheta}} \int_\Omega \rho'(\boldsymbol{\vartheta})\boldsymbol{h}\boldsymbol{w}_i^{\boldsymbol{\vartheta}} \cdot \boldsymbol{\eta} \,\mathrm{d}x \\ &\qquad + (\lambda_i^{\boldsymbol{\vartheta}})'\boldsymbol{h} \int_\Omega \rho(\boldsymbol{\vartheta})\boldsymbol{w}_i^{\boldsymbol{\vartheta}} \cdot \boldsymbol{\eta} \,\mathrm{d}x, \end{aligned} \tag{5.4.13}$$

for all $\boldsymbol{\eta} \in H^1_D(\Omega;\mathbb{R}^d)$*, that fulfills*

$$\left((\boldsymbol{w}_i^{\vartheta})'\boldsymbol{h}, \boldsymbol{w}_i^{\vartheta}\right)_{\rho(\vartheta)} = -\frac{1}{2}\int_\Omega \rho'(\vartheta)\boldsymbol{h}\left|\boldsymbol{w}_i^{\vartheta}\right|^2 dx. \tag{5.4.14}$$

To prove this theorem, we intend to apply the implicit function theorem (see, e.g., [159, Theorem 4.B]). Therefore, it is essential to show a bijectivity condition. In our setting this condition will be fulfilled if a certain PDE resulting from the eigenvalue equations has a unique solution. To show this existence and uniqueness we need to apply the Fredholm alternative established by Lemma 5.4.4.

In the following, we use the space

$$H^{-1}(\Omega;\mathbb{R}^d) \coloneqq \left(H^1_D(\Omega;\mathbb{R}^d)\right)^*$$

along with the canonical embedding

$$L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d) \hookrightarrow H^{-1}(\Omega,\mathbb{R}^d), \quad \boldsymbol{v} \mapsto \left(\boldsymbol{\eta} \mapsto (\boldsymbol{v},\boldsymbol{\eta})_{\rho(\boldsymbol{\varphi})}\right). \tag{5.4.15}$$

In particular, the duality pairing is given by

$$\langle\cdot,\cdot\rangle_{H^{-1},H^1} = (\cdot,\cdot)_{\rho(\boldsymbol{\varphi})}.$$

Lemma 5.4.4 (Fredholm alternative for the eigenvalue problem)**.** *Let* $\boldsymbol{\varphi} \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ *be arbitrary and suppose that for* $i \in \mathbb{N}$*, the eigenvalue* $\lambda_i^{\boldsymbol{\varphi}}$ *is simple. We further fix a* $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$*-normalized eigenfunction* $\boldsymbol{w}_i^{\boldsymbol{\varphi}}$ *to the eigenvalue* $\lambda_i^{\boldsymbol{\varphi}}$*.*

Then there exists a solution $\boldsymbol{u} \in H^1_D(\Omega;\mathbb{R}^d)$ *of the equation*

$$\langle\mathcal{E}(\boldsymbol{u}),\mathcal{E}(\boldsymbol{\eta})\rangle_{\mathbb{C}(\boldsymbol{\varphi})} - \lambda_i^{\boldsymbol{\varphi}}(\boldsymbol{u},\boldsymbol{\eta})_{\rho(\boldsymbol{\varphi})} = \langle\boldsymbol{f},\boldsymbol{\eta}\rangle_{H^{-1},H^1}, \tag{5.4.16}$$

for all $\boldsymbol{\eta} \in H^1_D(\Omega;\mathbb{R}^d)$ *if and only if* $\boldsymbol{f} \in H^{-1}(\Omega;\mathbb{R}^d)$ *fulfills*

$$\langle\boldsymbol{f},\boldsymbol{w}_i^{\boldsymbol{\varphi}}\rangle_{H^{-1},H^1} = 0.$$

In this case, there is a unique solution $\boldsymbol{u}^\perp$ *in* $H^1_D(\Omega;\mathbb{R}^d) \cap \langle\boldsymbol{w}_i^{\boldsymbol{\varphi}}\rangle_{\mathrm{span}}^{\perp,L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)}$*, and any other solution can be expressed as* $\boldsymbol{u}^\perp + \alpha\boldsymbol{w}_i^{\boldsymbol{\varphi}}$ *for some* $\alpha \in \mathbb{R}$*.*

Proof. Suppose that $\boldsymbol{f} \in H^{-1}(\Omega;\mathbb{R}^d)$. Then, $\boldsymbol{u} \in H^1_D(\Omega;\mathbb{R}^d)$ is a solution of (5.4.16) if and only if

$$\langle\mathcal{E}(\boldsymbol{u}),\mathcal{E}(\boldsymbol{\eta})\rangle_{\mathbb{C}(\boldsymbol{\varphi})} = \langle\lambda_i^{\boldsymbol{\varphi}}\boldsymbol{u} + \boldsymbol{f},\boldsymbol{\eta}\rangle_{H^{-1},H^1},$$

for all $\boldsymbol{\eta} \in H^1_D(\Omega;\mathbb{R}^d)$. As

$$\langle\mathcal{E}(\cdot),\mathcal{E}(\cdot)\rangle_{\mathbb{C}(\boldsymbol{\varphi})} : H^1_D(\Omega;\mathbb{R}^d) \times H^1_D(\Omega;\mathbb{R}^d) \to \mathbb{R},$$

is a continuous, coercive bi-linear form, we are able to define a continuous and compact operator

$$L^{-1} : H^{-1}(\Omega;\mathbb{R}^d) \to H^1_D(\Omega;\mathbb{R}^d) \hookrightarrow H^{-1}(\Omega;\mathbb{R}^d), \quad \boldsymbol{g} \mapsto \boldsymbol{u}^{\boldsymbol{g}},$$

that maps any right-hand side $\boldsymbol{g} \in H^{-1}(\Omega;\mathbb{R}^d)$ onto its unique solution $\boldsymbol{u}^{\boldsymbol{g}} \in H^1_D(\Omega;\mathbb{R}^d)$ of

$$\langle \mathcal{E}\left(\boldsymbol{u}^{\boldsymbol{g}}\right), \mathcal{E}\left(\boldsymbol{\eta}\right)\rangle_{\mathbb{C}(\boldsymbol{\varphi})} = \langle \boldsymbol{g}, \boldsymbol{\eta}\rangle_{H^{-1},H^1}\,, \tag{5.4.17}$$

for all $\boldsymbol{\eta} \in H^1_D(\Omega;\mathbb{R}^d)$.

In the following we understand $H^{-1}(\Omega;\mathbb{R}^d)$ as a Hilbert space endowed with the scalar product

$$(\boldsymbol{g},\boldsymbol{h})_{L^{-1}} \coloneqq \langle \mathcal{E}(L^{-1}\boldsymbol{g}), \mathcal{E}(L^{-1}\boldsymbol{h})\rangle_{\mathbb{C}(\boldsymbol{\varphi})}.$$

Indeed, this defines a scalar product as L^{-1} is injective. Note that due to (5.4.17) and the the fact that $\langle \mathcal{E}\left(\cdot\right), \mathcal{E}\left(\cdot\right)\rangle_{\mathbb{C}(\boldsymbol{\varphi})}$ is a scalar product on $H^1_D(\Omega;\mathbb{R}^d)$, the norm induced by $(\cdot,\cdot)_{L^{-1}}$ is equivalent to the canonical operator norm on $H^{-1}(\Omega;\mathbb{R}^d)$, which guarantees completeness of this space with respect to this new scalar product.

In the following, we write $R(\cdot)$ and $N(\cdot)$ denote the range and the null space of a linear operator, respectively. It is easy to see that L^{-1} is self-adjoint with respect to this scalar product. Furthermore the following equivalences are follow by a straightforward computation:

$$\begin{aligned}
&\exists \boldsymbol{u} \in H^1_D(\Omega;\mathbb{R}^d) \text{ that solves (5.4.16)},\\
&\quad\Leftrightarrow\quad \exists \boldsymbol{u} \in H^1_D(\Omega;\mathbb{R}^d):\ \ \boldsymbol{u} - \lambda_i^{\boldsymbol{\varphi}} L^{-1}\boldsymbol{u} = L^{-1}\boldsymbol{f},\\
&\quad\Leftrightarrow\quad L^{-1}\boldsymbol{f} \in R(Id - \lambda_i^{\boldsymbol{\varphi}} L^{-1}).
\end{aligned} \tag{5.4.18}$$

Since L^{-1} is compact, we have that

$$Id - \lambda_i^{\boldsymbol{\varphi}} L^{-1} : H^{-1}(\Omega;\mathbb{R}^d) \to H^{-1}(\Omega;\mathbb{R}^d),$$

is a Fredholm operator. In particular, we thus know that

$$R(Id - \lambda_i^{\boldsymbol{\varphi}} L^{-1}) \subset H^{-1}(\Omega;\mathbb{R}^d),$$

is closed and

$$R(Id - \lambda_i^{\boldsymbol{\varphi}} L^{-1}) = N(Id - \lambda_i^{\boldsymbol{\varphi}} L^{-1})^{\perp, H^{-1}(\Omega;\mathbb{R}^d)}.$$

Since L^{-1} is self-adjoint, we infer that

$$\begin{aligned}
L^{-1}\boldsymbol{f} \in R(Id - \lambda_i^{\boldsymbol{\varphi}} L^{-1}) \quad &\Leftrightarrow\quad \forall \boldsymbol{v} \in N(Id - \lambda_i^{\boldsymbol{\varphi}} L^{-1}):\ \left(L^{-1}\boldsymbol{f}, \boldsymbol{v}\right)_{L^{-1}} = 0,\\
&\Leftrightarrow\quad \forall \boldsymbol{v} \in N(Id - \lambda_i^{\boldsymbol{\varphi}} L^{-1}):\ \left(\boldsymbol{f}, L^{-1}\boldsymbol{v}\right)_{L^{-1}} = 0.
\end{aligned} \tag{5.4.19}$$

It further holds that

$$\begin{aligned}
&\boldsymbol{v} \in N(Id - \lambda_i^{\boldsymbol{\varphi}} L^{-1})\\
&\quad\Leftrightarrow\quad \forall \boldsymbol{\eta} \in H^1_D(\Omega;\mathbb{R}^d):\ \ \lambda_i^{\boldsymbol{\varphi}} \langle L^{-1}\boldsymbol{v}, \boldsymbol{\eta}\rangle_{H^{-1},H^1} = \langle \boldsymbol{v}, \boldsymbol{\eta}\rangle_{H^{-1},H^1}\\
&\quad\Leftrightarrow\quad \forall \boldsymbol{\eta} \in H^1_D(\Omega;\mathbb{R}^d):\ \ \lambda_i^{\boldsymbol{\varphi}} \left(L^{-1}\boldsymbol{v}, \boldsymbol{\eta}\right)_{\rho(\boldsymbol{\varphi})} = \langle \mathcal{E}(L^{-1}\boldsymbol{v}), \mathcal{E}(\boldsymbol{\eta})\rangle_{\mathbb{C}(\boldsymbol{\varphi})}\\
&\quad\Leftrightarrow\quad L^{-1}\boldsymbol{v} \in H^1_D(\Omega;\mathbb{R}^d) \text{ is an eigenfunction to the eigenvalue } \lambda_i^{\boldsymbol{\varphi}}\\
&\quad\Leftrightarrow\quad L^{-1}\boldsymbol{v} \in \langle \boldsymbol{w}_i^{\boldsymbol{\varphi}}\rangle_{\text{span}} \subset L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d),
\end{aligned}$$

where the last equivalence holds since $\lambda_i^{\boldsymbol{\varphi}}$ was assumed to be simple and therefore, the corresponding eigenspace is one-dimensional. In view of (5.4.19), this means that

$$\begin{aligned} L^{-1}\boldsymbol{f} \in R(Id - \lambda_i^{\boldsymbol{\varphi}} L^{-1}) \quad &\Leftrightarrow \quad \forall \boldsymbol{v} \in N(Id - \lambda_i^{\boldsymbol{\varphi}} L^{-1}) : \ (\boldsymbol{f}, L^{-1}\boldsymbol{v})_{L^{-1}} = 0 \\ &\Leftrightarrow \quad (\boldsymbol{f}, \boldsymbol{w}_i^{\boldsymbol{\varphi}})_{L^{-1}} = 0. \end{aligned} \tag{5.4.20}$$

We further know that

$$L^{-1}\boldsymbol{w}_i^{\boldsymbol{\varphi}} = \frac{1}{\lambda_i^{\boldsymbol{\varphi}}} \boldsymbol{w}_i^{\boldsymbol{\varphi}} \in H_D^1(\Omega; \mathbb{R}^d).$$

Hence, since $L^{-1}\boldsymbol{f}$ is a solution of (5.4.17), we have

$$\begin{aligned} (\boldsymbol{f}, \boldsymbol{w}_i^{\boldsymbol{\varphi}})_{L^{-1}} &= \left\langle \mathcal{E}\left(L^{-1}\boldsymbol{f}\right), \mathcal{E}\left(L^{-1}\boldsymbol{w}_i^{\boldsymbol{\varphi}}\right) \right\rangle_{\mathbb{C}(\boldsymbol{\varphi})} \\ &= \left\langle \boldsymbol{f}, L^{-1}\boldsymbol{w}_i^{\boldsymbol{\varphi}} \right\rangle_{H^{-1},H^1} = \frac{1}{\lambda_i^{\boldsymbol{\varphi}}} \left\langle \boldsymbol{f}, \boldsymbol{w}_i^{\boldsymbol{\varphi}} \right\rangle_{H^{-1},H^1}. \end{aligned} \tag{5.4.21}$$

Combining (5.4.18), (5.4.20) and (5.4.21), we conclude that

$$\begin{aligned} &\exists\, \boldsymbol{u} \in H_D^1(\Omega; \mathbb{R}^d) \text{ that solves (5.4.16)} \\ &\quad \Leftrightarrow \quad L^{-1}\boldsymbol{f} \in R(Id - \lambda_i^{\boldsymbol{\varphi}} L^{-1}) \\ &\quad \Leftrightarrow \quad \left\langle \boldsymbol{f}, \boldsymbol{w}_i^{\boldsymbol{\varphi}} \right\rangle_{H^{-1},H^1} = 0. \end{aligned}$$

This proves the first assertion.

Let us now assume that $\left\langle \boldsymbol{f}, \boldsymbol{w}_i^{\boldsymbol{\varphi}} \right\rangle_{H^{-1},H^1} = 0$ and let

$$P_i^{\boldsymbol{\varphi}} : L^2_{\boldsymbol{\varphi}}(\Omega; \mathbb{R}^d) \to \langle \boldsymbol{w}_i^{\boldsymbol{\varphi}} \rangle_{\text{span}} \subset L^2_{\boldsymbol{\varphi}}(\Omega; \mathbb{R}^d),$$

denote the orthogonal projection onto the linear subspace $\langle \boldsymbol{w}_i^{\boldsymbol{\varphi}} \rangle_{\text{span}}$ with respect to the scalar product on $L^2_{\boldsymbol{\varphi}}(\Omega; \mathbb{R}^d)$. For any solution $\boldsymbol{u} \in H_D^1(\Omega; \mathbb{R}^d)$ of (5.4.16) we obtain from the decomposition $\boldsymbol{u} = (\boldsymbol{u} - P_i^{\boldsymbol{\varphi}}(\boldsymbol{u})) + P_i^{\boldsymbol{\varphi}}(\boldsymbol{u})$ that

$$\langle \boldsymbol{f}, \boldsymbol{\eta} \rangle_{H^{-1},H^1} = \left\langle \mathcal{E}(\boldsymbol{u} - P_i^{\boldsymbol{\varphi}}(\boldsymbol{u})), \mathcal{E}(\boldsymbol{\eta}) \right\rangle_{\mathbb{C}(\boldsymbol{\varphi})} - \lambda_i^{\boldsymbol{\varphi}} (\boldsymbol{u} - P_i^{\boldsymbol{\varphi}}(\boldsymbol{u}), \boldsymbol{\eta})_{\rho(\boldsymbol{\varphi})}.$$

Hence, it also holds that

$$\boldsymbol{u}^{\perp} := \boldsymbol{u} - P_i^{\boldsymbol{\varphi}}(\boldsymbol{u}) \in H_D^1(\Omega; \mathbb{R}^d) \cap \langle \boldsymbol{w}_i^{\boldsymbol{\varphi}} \rangle_{\text{span}}^{\perp, L^2_{\boldsymbol{\varphi}}(\Omega; \mathbb{R}^d)}$$

fulfills equality (5.4.16). Uniqueness of the solution $\boldsymbol{u}^{\perp}$ follows from the simplicity of $\lambda_i^{\boldsymbol{\varphi}}$ and the linearity of equation (5.4.16). In particular, any solution $\boldsymbol{u} \in H_D^1(\Omega; \mathbb{R}^d)$ can be expressed as

$$\boldsymbol{u} = \boldsymbol{u}^{\perp} + P_i^{\boldsymbol{\varphi}}(\boldsymbol{u}) = \boldsymbol{u}^{\perp} + \alpha \boldsymbol{w}_i^{\boldsymbol{\varphi}}$$

for some $\alpha \in \mathbb{R}$. This completes the proof. □

We can now use the Fredholm alternative to prove Theorem 5.4.3.

Proof of Theorem 5.4.3. As mentioned above, we intend to apply the implicit function theorem to prove the assertion. To this end, we define the operator

$$F : \left(H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)\right) \times H^1_D(\Omega;\mathbb{R}^d) \times \mathbb{R} \to H^{-1}(\Omega;\mathbb{R}^d) \times \mathbb{R},$$
$$(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda) \mapsto \begin{pmatrix} -\nabla \cdot \mathbb{C}(\boldsymbol{\vartheta})\mathcal{E}(\boldsymbol{w}) - \lambda\rho(\boldsymbol{\vartheta})\boldsymbol{w} \\ (\boldsymbol{w},\boldsymbol{w})_{\rho(\boldsymbol{\vartheta})} - 1 \end{pmatrix}.$$

Here we canonically understand the first component of the right-hand side as an element of $H^{-1}(\Omega;\mathbb{R}^d)$, i.e.,

$$\langle F_1(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda), \boldsymbol{\eta}\rangle_{H^{-1},H^1_D} = \langle \mathcal{E}(\boldsymbol{w}), \mathcal{E}(\boldsymbol{\eta})\rangle_{\mathbb{C}(\boldsymbol{\vartheta})} - \lambda (\boldsymbol{w},\boldsymbol{\eta})_{\rho(\boldsymbol{\vartheta})},$$

for all $\boldsymbol{\eta} \in H^1_D(\Omega;\mathbb{R}^d)$. First of all, it is clear that $\boldsymbol{\varphi} \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$, the eigenvalue $\lambda_i^{\boldsymbol{\varphi}} \in \mathbb{R}$ and the corresponding representative $\boldsymbol{w}_i^{\boldsymbol{\varphi}} \in H^1_D(\Omega;\mathbb{R}^d)$ satisfy

$$F(\boldsymbol{\varphi}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}, \lambda_i^{\boldsymbol{\varphi}}) = \mathbf{0}.$$

To apply the implicit function theorem, we need to show that F is of class C^1 on a suitable neighborhood of $(\boldsymbol{\varphi}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}, \lambda_i^{\boldsymbol{\varphi}})$. For this purpose, we show that all partial Fréchet-derivatives are continuous at any point in the domain of definition of F, because by Theorem 2.2.6 this then yields the desired Fréchet-differentiability.
Formally computing the partial derivatives at a point

$$(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda) \in \left(H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)\right) \times H^1_D(\Omega;\mathbb{R}^d) \times \mathbb{R},$$

in the direction

$$(\boldsymbol{h}, \boldsymbol{u}, \mu) \in \left(H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)\right) \times H^1_D(\Omega;\mathbb{R}^d) \times \mathbb{R},$$

gives

$$\partial_{\boldsymbol{\vartheta}} F_1(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda)\boldsymbol{h} = -\nabla \cdot \mathbb{C}'(\boldsymbol{\vartheta})\boldsymbol{h}\mathcal{E}(\boldsymbol{w}) - \lambda\rho'(\boldsymbol{\vartheta})\boldsymbol{h}\boldsymbol{w}, \tag{5.4.22a}$$
$$\partial_{\boldsymbol{w}} F_1(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda)\boldsymbol{u} = -\nabla \cdot \mathbb{C}(\boldsymbol{\vartheta})\mathcal{E}(\boldsymbol{u}) - \lambda\rho(\boldsymbol{\vartheta})\boldsymbol{u}, \tag{5.4.22b}$$
$$\partial_{\lambda} F_1(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda)\mu = -\mu\rho(\boldsymbol{\vartheta})\boldsymbol{w}, \tag{5.4.22c}$$
$$\partial_{\boldsymbol{\vartheta}} F_2(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda)\boldsymbol{h} = \int_\Omega \rho'(\boldsymbol{\vartheta})\boldsymbol{h}\,|\boldsymbol{w}|^2 \,\mathrm{d}x, \tag{5.4.22d}$$
$$\partial_{\boldsymbol{w}} F_2(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda)\boldsymbol{u} = 2\int_\Omega \rho(\boldsymbol{\vartheta})\boldsymbol{w}\cdot\boldsymbol{u} \,\mathrm{d}x, \tag{5.4.22e}$$
$$\partial_{\lambda} F_2(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda)\mu = 0, \tag{5.4.22f}$$

where the first two identities are to be understood in a weak sense. We can rigorously prove that the above expressions are actually the partial Fréchet-derivatives. Here, we will present a detailed proof only for (5.4.22a) as all other derivatives can be verified analogously. We first notice that for any fixed $(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda) \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N) \times H^1_D(\Omega;\mathbb{R}^d) \times \mathbb{R}$, it holds that

$$\left[\boldsymbol{h} \mapsto \partial_{\boldsymbol{\vartheta}} F_1(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda)\boldsymbol{h}\right] \in \mathcal{L}\left(H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N), H^{-1}(\Omega;\mathbb{R}^d)\right). \tag{5.4.23}$$

Indeed, the linearity of the above mapping is clear and the boundedness properties (2.1.19) and (2.1.25) of $\mathbb{C}$ and ρ, respectively, along with Hölder's inequality imply the existence of a constant $C > 0$ such that

$$\begin{aligned}
&\|\partial_{\boldsymbol{\vartheta}} F_1(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda)\|_{\mathcal{L}(H^1 \cap L^\infty, H^{-1})} \\
&\quad = \sup_{\|\boldsymbol{h}\|_{H^1 \cap L^\infty} = 1} \sup_{\|\boldsymbol{\eta}\|_{H^1_D} = 1} \left| \langle \mathcal{E}(\boldsymbol{w}), \mathcal{E}(\boldsymbol{\eta}) \rangle_{\mathbb{C}'(\boldsymbol{\vartheta})\boldsymbol{h}} + \lambda \int_\Omega \rho'(\boldsymbol{\vartheta}) \boldsymbol{h} \boldsymbol{w} \cdot \boldsymbol{\eta} \, \mathrm{d}x \right| \\
&\quad \leq C \, \|\boldsymbol{w}\|_{H^1_D(\Omega;\mathbb{R}^d)} .
\end{aligned}$$

It further holds that

$$\begin{aligned}
&\|F_1(\boldsymbol{\vartheta} + \boldsymbol{h}, \boldsymbol{w}, \lambda) - F_1(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda) - \partial_{\boldsymbol{\vartheta}} F_1(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda)\boldsymbol{h}\|_{H^{-1}(\Omega;\mathbb{R}^d)} \\
&\quad \leq \sup_{\|\boldsymbol{\eta}\|_{H^1_D(\Omega;\mathbb{R}^d)} = 1} \left| \langle \mathcal{E}(\boldsymbol{w}), \mathcal{E}(\boldsymbol{\eta}) \rangle_{\mathbb{C}(\boldsymbol{\vartheta}+\boldsymbol{h})} - \langle \mathcal{E}(\boldsymbol{w}), \mathcal{E}(\boldsymbol{\eta}) \rangle_{\mathbb{C}(\boldsymbol{\vartheta})} - \langle \mathcal{E}(\boldsymbol{w}), \mathcal{E}(\boldsymbol{\eta}) \rangle_{\mathbb{C}'(\boldsymbol{\vartheta})\boldsymbol{h}} \right| \\
&\qquad + \sup_{\|\boldsymbol{\eta}\|_{H^1_D(\Omega;\mathbb{R}^d)} = 1} |\lambda| \int_\Omega \left| \rho(\boldsymbol{\vartheta} + \boldsymbol{h}) - \rho(\boldsymbol{\vartheta}) - \rho'(\boldsymbol{\vartheta})\boldsymbol{h} \right| |\boldsymbol{w} \cdot \boldsymbol{\eta}| \, \mathrm{d}x.
\end{aligned}$$

Now, proceeding similarly as in [32, Proof of Theorem 3.3], we invoke the local Lipschitz continuity of the first derivatives of $\mathbb{C}$ and ρ to conclude that

$$\begin{aligned}
\left\| \mathbb{C}(\boldsymbol{\vartheta} + \boldsymbol{h}) - \mathbb{C}(\boldsymbol{\vartheta}) - \mathbb{C}'(\boldsymbol{\vartheta})\boldsymbol{h} \right\|_{L^\infty(\Omega;\mathbb{R}^N)} &\in o\left(\|\boldsymbol{h}\|_{L^\infty(\Omega;\mathbb{R}^N)} \right), \\
\left\| \rho(\boldsymbol{\vartheta} + \boldsymbol{h}) - \rho(\boldsymbol{\vartheta}) - \rho'(\boldsymbol{\vartheta})\boldsymbol{h} \right\|_{L^\infty(\Omega;\mathbb{R}^N)} &\in o\left(\|\boldsymbol{h}\|_{L^\infty(\Omega;\mathbb{R}^N)} \right).
\end{aligned}$$

Hence, Hölder's inequality yields

$$\|F_1(\boldsymbol{\vartheta} + \boldsymbol{h}, \boldsymbol{w}, \lambda) - F_1(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda) - \partial_{\boldsymbol{\vartheta}} F_1(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda)\boldsymbol{h}\|_{H^{-1}(\Omega;\mathbb{R}^d)} \in o\left(\|\boldsymbol{h}\|_{H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)} \right),$$

which proves that $\partial_{\boldsymbol{\vartheta}} F_1(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda)$ is indeed the partial derivative of F_1 with respect to $\boldsymbol{\vartheta}$ in the Fréchet sense.

In the light of Theorem 2.2.6 it remains to prove the continuity of the partial Fréchet-derivatives. Here we also present the proof only for (5.4.22a) as the continuity of the other partial derivatives can be established similarly. Let $(\boldsymbol{\vartheta}_n, \boldsymbol{w}_n, \lambda_n)_{n \in \mathbb{N}}$ denote any sequence in $\left(H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)\right) \times H^1_D(\Omega;\mathbb{R}^d) \times \mathbb{R}$ satisfying

$$(\boldsymbol{\vartheta}_n, \boldsymbol{w}_n, \lambda_n) \to (\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda) \quad \text{in } \left(H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)\right) \times H^1_D(\Omega;\mathbb{R}^d) \times \mathbb{R},$$

for $n \to \infty$. Then it holds that

$$\begin{aligned}
&\sup_{\|\boldsymbol{h}\|_{H^1 \cap L^\infty} = 1} \|\partial_{\boldsymbol{\vartheta}} F_1(\boldsymbol{\vartheta}_n, \boldsymbol{w}_n, \lambda_n)\boldsymbol{h} - \partial_{\boldsymbol{\vartheta}} F_1(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda)\boldsymbol{h}\|_{H^{-1}(\Omega;\mathbb{R}^d)} \\
&\quad \leq \sup_{\|\boldsymbol{h}\|_{H^1 \cap L^\infty} = 1} \sup_{\|\boldsymbol{\eta}\|_{H^1_D} = 1} \left| \langle \mathcal{E}(\boldsymbol{w}_n), \mathcal{E}(\boldsymbol{\eta}) \rangle_{\mathbb{C}'(\boldsymbol{\vartheta}_n)\boldsymbol{h}} - \langle \mathcal{E}(\boldsymbol{w}), \mathcal{E}(\boldsymbol{\eta}) \rangle_{\mathbb{C}'(\boldsymbol{\vartheta})\boldsymbol{h}} \right| \\
&\qquad + \sup_{\|\boldsymbol{h}\|_{H^1 \cap L^\infty} = 1} \sup_{\|\boldsymbol{\eta}\|_{H^1_D} = 1} \left| \int_\Omega \lambda_n \rho'(\boldsymbol{\vartheta}_n) \boldsymbol{h} \boldsymbol{w}_n \cdot \boldsymbol{\eta} \, \mathrm{d}x - \int_\Omega \lambda \rho'(\boldsymbol{\vartheta}) \boldsymbol{h} \boldsymbol{w} \cdot \boldsymbol{\eta} \, \mathrm{d}x \right| .
\end{aligned}$$

Applying Hölder's inequality, and recalling again the local Lipschitz continuity of the first derivative of $\mathbb{C}$ and property (2.1.25), we infer that

$$\begin{aligned}
&\left|\langle \mathcal{E}(\boldsymbol{w}_n), \mathcal{E}(\boldsymbol{\eta})\rangle_{\mathbb{C}'(\boldsymbol{\vartheta}_n)\boldsymbol{h}} - \langle \mathcal{E}(\boldsymbol{w}), \mathcal{E}(\boldsymbol{\eta})\rangle_{\mathbb{C}'(\boldsymbol{\vartheta})\boldsymbol{h}}\right| \\
&\quad \leq \left|\langle \mathcal{E}(\boldsymbol{w}_n), \mathcal{E}(\boldsymbol{\eta})\rangle_{\mathbb{C}'(\boldsymbol{\vartheta}_n)\boldsymbol{h}} - \langle \mathcal{E}(\boldsymbol{w}_n), \mathcal{E}(\boldsymbol{\eta})\rangle_{\mathbb{C}'(\boldsymbol{\vartheta})\boldsymbol{h}}\right| \\
&\qquad + \left|\langle \mathcal{E}(\boldsymbol{w}_n), \mathcal{E}(\boldsymbol{\eta})\rangle_{\mathbb{C}'(\boldsymbol{\vartheta})\boldsymbol{h}} - \langle \mathcal{E}(\boldsymbol{w}), \mathcal{E}(\boldsymbol{\eta})\rangle_{\mathbb{C}'(\boldsymbol{\vartheta})\boldsymbol{h}}\right| \\
&\quad \leq C\left(\|\boldsymbol{\vartheta}_n - \boldsymbol{\vartheta}\|_{H^1\cap L^\infty} \|\boldsymbol{w}_n\|_{H^1_D(\Omega;\mathbb{R}^d)} + \|\boldsymbol{w}_n - \boldsymbol{w}\|_{H^1_D(\Omega;\mathbb{R}^d)}\right) \\
&\qquad \cdot \|\boldsymbol{h}\|_{H^1\cap L^\infty} \|\boldsymbol{\eta}\|_{H^1_D(\Omega;\mathbb{R}^d)}\,.
\end{aligned}$$

Arguing similarly for the second summand, we obtain

$$\begin{aligned}
&\left|\int_\Omega \lambda_n \rho'(\boldsymbol{\vartheta}_n)\boldsymbol{h}\boldsymbol{w}_n \cdot \boldsymbol{\eta}\,\mathrm{d}x - \int_\Omega \lambda \rho'(\boldsymbol{\vartheta})\boldsymbol{h}\boldsymbol{w} \cdot \boldsymbol{\eta}\,\mathrm{d}x\right| \\
&\quad \leq \left|\lambda_n \int_\Omega \rho'(\boldsymbol{\vartheta}_n)\boldsymbol{h}\boldsymbol{w}_n \cdot \boldsymbol{\eta} - \rho'(\boldsymbol{\vartheta})\boldsymbol{h}\boldsymbol{w}_n \cdot \boldsymbol{\eta}\,\mathrm{d}x\right| + \left|(\lambda_n - \lambda)\int_\Omega \rho'(\boldsymbol{\vartheta})\boldsymbol{h}\boldsymbol{w}_n \cdot \boldsymbol{\eta}\,\mathrm{d}x\right| \\
&\qquad + \left|\lambda \int_\Omega \rho'(\boldsymbol{\vartheta})\boldsymbol{h}\boldsymbol{w}_n \cdot \boldsymbol{\eta} - \rho'(\boldsymbol{\vartheta})\boldsymbol{h}\boldsymbol{w} \cdot \boldsymbol{\eta}\,\mathrm{d}x\right| \\
&\quad \leq C\Big(\|\boldsymbol{\vartheta}_n - \boldsymbol{\vartheta}\|_{H^1\cap L^\infty} \|\boldsymbol{w}_n\|_{H^1_D(\Omega;\mathbb{R}^d)} + |\lambda_n - \lambda|\, \|\boldsymbol{w}_n\|_{H^1_D(\Omega;\mathbb{R}^d)} \\
&\qquad\quad + \|\boldsymbol{w}_n - \boldsymbol{w}\|_{H^1_D(\Omega;\mathbb{R}^d)}\Big) \|\boldsymbol{h}\|_{H^1\cap L^\infty} \|\boldsymbol{\eta}\|_{H^1_D(\Omega;\mathbb{R}^d)}\,.
\end{aligned}$$

Hence, after taking the suprema, we conclude that

$$\|\partial_{\boldsymbol{\vartheta}} F_1(\boldsymbol{\vartheta}_n, \boldsymbol{w}_n, \lambda_n) - \partial_{\boldsymbol{\vartheta}} F_1(\boldsymbol{\vartheta}, \boldsymbol{w}, \lambda)\|_{\mathcal{L}\left(H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N), H^{-1}(\Omega;\mathbb{R}^d)\right)} \to 0,$$

as $n \to \infty$. In summary, this implies that

$$F : H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N) \times H^1_D(\Omega;\mathbb{R}^d) \times \mathbb{R} \to H^{-1}(\Omega;\mathbb{R}^d) \times \mathbb{R},$$

is continuously Fréchet-differentiable.

We next need to show that the partial derivative

$$\partial_{(\boldsymbol{w},\lambda)} F(\boldsymbol{\varphi}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}, \lambda_i^{\boldsymbol{\varphi}}) : H^1_D(\Omega;\mathbb{R}^d) \times \mathbb{R} \to H^{-1}(\Omega;\mathbb{R}^d) \times \mathbb{R}$$

is an isomorphism. Regarding injectivity, we consider

$$\partial_{\boldsymbol{w}} F(\boldsymbol{\varphi}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}, \lambda_i^{\boldsymbol{\varphi}})\boldsymbol{u} + \partial_\lambda F(\boldsymbol{\varphi}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}, \lambda_i^{\boldsymbol{\varphi}})\mu = \partial_{(\boldsymbol{w},\lambda)} F(\boldsymbol{\varphi}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}, \lambda_i^{\boldsymbol{\varphi}})(\boldsymbol{u}, \mu) = \boldsymbol{0},$$

which is equivalent to the equations

$$\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{\eta})\rangle_{\mathbb{C}(\boldsymbol{\varphi})} - \lambda_i^{\boldsymbol{\varphi}} (\boldsymbol{u}, \boldsymbol{\eta})_{\rho(\boldsymbol{\varphi})} - \mu (\boldsymbol{w}_i^{\boldsymbol{\varphi}}, \boldsymbol{\eta})_{\rho(\boldsymbol{\varphi})} = 0, \tag{5.4.24}$$

for all $\boldsymbol{\eta} \in H^1_D(\Omega;\mathbb{R}^d)$ and

$$2 (\boldsymbol{u}, \boldsymbol{w}_i^{\boldsymbol{\varphi}})_{\rho(\boldsymbol{\varphi})} = 0. \tag{5.4.25}$$

Testing (5.4.24) with $\boldsymbol{\eta} = \boldsymbol{w}_i^{\boldsymbol{\varphi}} \in H_D^1(\Omega;\mathbb{R}^d)$ immediately yields $\mu = 0$, as $\boldsymbol{w}_i^{\boldsymbol{\varphi}}$ is an $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-normalized eigenfunction. As $\lambda_i^{\boldsymbol{\varphi}}$ is assumed to be simple, we obtain from (5.4.25) that $\boldsymbol{u} = \mathbf{0}$. This shows that the operator is injective.

To verify surjectivity we take an arbitrary tuple $(\boldsymbol{f}, \kappa) \in H^{-1}(\Omega;\mathbb{R}^d) \times \mathbb{R}$ and we need to show that there exists a solution $(\boldsymbol{u}, \mu) \in H_D^1(\Omega;\mathbb{R}^d) \times \mathbb{R}$ of the system

$$\begin{aligned} \langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{\eta}) \rangle_{\mathbb{C}(\boldsymbol{\varphi})} - \lambda_i^{\boldsymbol{\varphi}} (\boldsymbol{u}, \boldsymbol{\eta})_{\rho(\boldsymbol{\varphi})} - \mu (\boldsymbol{w}_i^{\boldsymbol{\varphi}}, \boldsymbol{\eta})_{\rho(\boldsymbol{\varphi})} &= \langle \boldsymbol{f}, \boldsymbol{\eta} \rangle_{H^{-1},H^1}, \\ 2 (\boldsymbol{u}, \boldsymbol{w}_i^{\boldsymbol{\varphi}})_{\rho(\boldsymbol{\varphi})} &= \kappa, \end{aligned} \tag{5.4.26}$$

for all $\boldsymbol{\eta} \in H_D^1(\Omega;\mathbb{R}^d)$. Choosing $\mu = -\langle \boldsymbol{f}, \boldsymbol{w}_i^{\boldsymbol{\varphi}} \rangle_{H^{-1},H^1}$, we infer that a solution $\boldsymbol{u} \in H_D^1(\Omega;\mathbb{R}^d)$ needs to fulfill

$$\langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{\eta}) \rangle_{\mathbb{C}(\boldsymbol{\varphi})} - \lambda_i^{\boldsymbol{\varphi}} (\boldsymbol{u}, \boldsymbol{\eta})_{\rho(\boldsymbol{\varphi})} = \langle \boldsymbol{f}, \boldsymbol{\eta} \rangle_{H^{-1},H^1} - \langle \boldsymbol{f}, \boldsymbol{w}_i^{\boldsymbol{\varphi}} \rangle_{H^{-1},H^1} (\boldsymbol{w}_i^{\boldsymbol{\varphi}}, \boldsymbol{\eta})_{\rho(\boldsymbol{\varphi})}, \tag{5.4.27}$$

for all $\boldsymbol{\eta} \in H_D^1(\Omega;\mathbb{R}^d)$. Testing again with the normalized eigenfunction $\boldsymbol{w}_i^{\boldsymbol{\varphi}} \in H_D^1(\Omega;\mathbb{R}^d)$, we deduce that

$$\langle \boldsymbol{f}, \boldsymbol{w}_i^{\boldsymbol{\varphi}} \rangle_{H^{-1},H^1} - \langle \boldsymbol{f}, \boldsymbol{w}_i^{\boldsymbol{\varphi}} \rangle_{H^{-1},H^1} (\boldsymbol{w}_i^{\boldsymbol{\varphi}}, \boldsymbol{w}_i^{\boldsymbol{\varphi}})_{\rho(\boldsymbol{\varphi})} = 0.$$

Hence, Lemma 5.4.4 implies the existence of a function $\boldsymbol{u}^\perp \in H_D^1(\Omega;\mathbb{R}^d) \cap \langle \boldsymbol{w}_i^{\boldsymbol{\varphi}} \rangle_{\mathrm{span}}^{\perp, L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)}$, such that any solution of (5.4.27) can be written as $\boldsymbol{u}^\perp + \alpha \boldsymbol{w}_i^{\boldsymbol{\varphi}} \in H_D^1(\Omega;\mathbb{R}^d)$ and vice versa. Using the second equation of (5.4.26), we finally conclude that

$$\left(\boldsymbol{u}^\perp + \frac{\kappa}{2} \boldsymbol{w}_i^{\boldsymbol{\varphi}}, -\langle \boldsymbol{f}, \boldsymbol{w}_i^{\boldsymbol{\varphi}} \rangle \right) \in H_D^1(\Omega;\mathbb{R}^d) \times \mathbb{R}, \tag{5.4.28}$$

is a solution of (5.4.26).

In summary, this proves that

$$\partial_{(\boldsymbol{w},\lambda)} F(\boldsymbol{\varphi}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}, \lambda_i^{\boldsymbol{\varphi}}) : H_D^1(\Omega;\mathbb{R}^d) \times \mathbb{R} \to H^{-1}(\Omega;\mathbb{R}^d) \times \mathbb{R}$$

is bijective, and thus an isomorphism.

As now all requirements are verified, the implicit function theorem can be applied to the equation $F(\boldsymbol{\varphi}, \boldsymbol{w}_i^{\boldsymbol{\varphi}}, \lambda_i^{\boldsymbol{\varphi}}) = \mathbf{0}$. It implies that there exist radii $r_0, r_i^{\boldsymbol{\varphi}} > 0$ such that the mapping

$$S_i^{\boldsymbol{\varphi}} : B_{r_0}(\boldsymbol{\varphi}) \subset H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N) \to B_{r_i^{\boldsymbol{\varphi}}}((\boldsymbol{w}_i^{\boldsymbol{\varphi}}, \lambda_i^{\boldsymbol{\varphi}})) \subset H_D^1(\Omega;\mathbb{R}^d) \times \mathbb{R},$$

is well-defined, continuously Fréchet-differentiable and satisfies

$$F(\boldsymbol{\vartheta}, S_i^{\boldsymbol{\varphi}}(\boldsymbol{\vartheta})) = \mathbf{0},$$

for all $\boldsymbol{\vartheta} \in B_{r_0}(\boldsymbol{\varphi})$, where $S_i^{\boldsymbol{\varphi}}(\boldsymbol{\vartheta})$ is the *unique* element fulfilling $S_i^{\boldsymbol{\varphi}}(\boldsymbol{\vartheta}) \in B_{r_i^{\boldsymbol{\varphi}}}((\boldsymbol{w}_i^{\boldsymbol{\varphi}}, \lambda_i^{\boldsymbol{\varphi}}))$ and satisfying this equation. In particular, this means that

$$\begin{aligned} \left\langle \mathcal{E}\left(S_{i,1}^{\boldsymbol{\varphi}}(\boldsymbol{\vartheta})\right), \mathcal{E}(\boldsymbol{\eta}) \right\rangle_{\mathbb{C}(\boldsymbol{\vartheta})} &= S_{i,2}^{\boldsymbol{\varphi}}(\boldsymbol{\vartheta}) \int_\Omega \rho(\boldsymbol{\vartheta}) S_{i,1}^{\boldsymbol{\varphi}}(\boldsymbol{\vartheta}) \cdot \boldsymbol{\eta} \, \mathrm{d}x, \\ \int_\Omega \rho(\boldsymbol{\vartheta}) |S_{i,1}^{\boldsymbol{\varphi}}(\boldsymbol{\vartheta})|^2 \, \mathrm{d}x &= 1, \end{aligned} \tag{5.4.29}$$

for all $\boldsymbol{\eta} \in H^1_D(\Omega;\mathbb{R}^d)$, and thus, $S^{\boldsymbol{\varphi}}_{i,1}(\boldsymbol{\vartheta}) \in H^1_D(\Omega;\mathbb{R}^d)$ is a $L^2_{\boldsymbol{\vartheta}}(\Omega,\mathbb{R}^d)$-normalized eigenfunction to the eigenvalue $S^{\boldsymbol{\varphi}}_{i,2}(\boldsymbol{\vartheta})$.

However, it is still not clear whether the eigenvalues $S^{\boldsymbol{\varphi}}_{i,1}(\boldsymbol{\vartheta})$ and $\lambda_i^{\boldsymbol{\vartheta}}$ are actually identical. By construction, it holds that $S_i^{\boldsymbol{\varphi}}(\boldsymbol{\varphi}) = (\boldsymbol{w}_i^{\boldsymbol{\varphi}}, \lambda_i^{\boldsymbol{\varphi}})$. We now recall that, according to Lemma 5.3.5 and Lemma 5.3.8, both $\boldsymbol{\vartheta} \mapsto \lambda_i^{\boldsymbol{\vartheta}}$ and $\boldsymbol{\vartheta} \mapsto \boldsymbol{w}_i^{\boldsymbol{\vartheta}}$ are continuous on a suitable neighborhood around $\boldsymbol{\varphi} \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$. In particular, we recall from Lemma 5.3.8 that this neighborhood is chosen in such a way that $\lambda_i^{\boldsymbol{\vartheta}}$ is simple, which will be needed later. In combination with (5.4.29) and the uniqueness of $S_i^{\boldsymbol{\varphi}}(\boldsymbol{\vartheta})$, we conclude that there exists a radius $\delta_i^{\boldsymbol{\varphi}} \in (0, r_0]$ such that

$$(\boldsymbol{w}_i^{\boldsymbol{\vartheta}}, \lambda_i^{\boldsymbol{\vartheta}}) = S_i^{\boldsymbol{\varphi}}(\boldsymbol{\vartheta}) \quad \text{for all } \boldsymbol{\vartheta} \in B_{\delta_i^{\boldsymbol{\varphi}}}(\boldsymbol{\varphi}). \tag{5.4.30}$$

Via restriction to the ball $B_{\delta_i^{\boldsymbol{\varphi}}}(\boldsymbol{\varphi})$, we can thus rewrite the operator $S_i^{\boldsymbol{\varphi}}$ as

$$\begin{aligned} S_i^{\boldsymbol{\varphi}} : B_{\delta_i^{\boldsymbol{\varphi}}}(\boldsymbol{\varphi}) \subset H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N) &\to H^1_D(\Omega;\mathbb{R}^d) \times \mathbb{R}, \\ \boldsymbol{\vartheta} &\mapsto (\boldsymbol{w}_i^{\boldsymbol{\vartheta}}, \lambda_i^{\boldsymbol{\vartheta}}). \end{aligned}$$

It remains to compute the Fréchet-derivative of $S_i^{\boldsymbol{\varphi}}$ at a point $\boldsymbol{\vartheta} \in B_{\delta_i^{\boldsymbol{\varphi}}}(\boldsymbol{\varphi})$, which means computing the desired Fréchet-derivatives of the i-th eigenvalue and the corresponding eigenfunction with respect to $\boldsymbol{\vartheta}$.

To this end, let $\boldsymbol{\vartheta} \in B_{\delta_i^{\boldsymbol{\varphi}}}(\boldsymbol{\varphi})$ be arbitrary. Using the chain rule, we conclude that the Fréchet-derivative

$$\left(S_i^{\boldsymbol{\varphi}}(\boldsymbol{\vartheta})\right)' \boldsymbol{h} = \Big((\boldsymbol{w}_i^{\boldsymbol{\vartheta}})'\boldsymbol{h}, (\lambda_i^{\boldsymbol{\vartheta}})'\boldsymbol{h}\Big),$$

satisfies the equation

$$\partial_{(\boldsymbol{w},\lambda)} F(\boldsymbol{\vartheta}, S_i(\boldsymbol{\vartheta}))\Big((\boldsymbol{w}_i^{\boldsymbol{\vartheta}})'\boldsymbol{h}, (\lambda_i^{\boldsymbol{\vartheta}})'\boldsymbol{h}\Big) = -\partial_{\boldsymbol{\vartheta}} F(\boldsymbol{\vartheta}, S_i(\boldsymbol{\vartheta}))\boldsymbol{h} \quad \text{in } H^{-1}(\Omega;\mathbb{R}^d) \times \mathbb{R} \tag{5.4.31}$$

for any direction $\boldsymbol{h} \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$. Note that we will omit the exponent $\boldsymbol{\varphi}$ in $S_i^{\boldsymbol{\varphi}}$ for a clearer presentation. With the partial derivatives computed in (5.4.22a)-(5.4.22f) we obtain

$$\partial_{\boldsymbol{\vartheta}} F(\boldsymbol{\vartheta}, S_i(\boldsymbol{\vartheta}))\boldsymbol{h} = \begin{pmatrix} -\nabla \cdot \left[\mathbb{C}'(\boldsymbol{\vartheta})\boldsymbol{h}\mathcal{E}(\boldsymbol{w}_i^{\boldsymbol{\vartheta}})\right] - \lambda_i^{\boldsymbol{\vartheta}} \rho'(\boldsymbol{\vartheta})\boldsymbol{h}\boldsymbol{w}_i^{\boldsymbol{\vartheta}} \\ \int_\Omega \rho'(\boldsymbol{\vartheta})\boldsymbol{h} \left|\boldsymbol{w}_i^{\boldsymbol{\vartheta}}\right|^2 \,\mathrm{d}x \end{pmatrix} \in H^{-1}(\Omega;\mathbb{R}^d) \times \mathbb{R},$$

for all $\boldsymbol{h} \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$, and

$$\begin{aligned} \partial_{(\boldsymbol{w},\lambda)} F(\boldsymbol{\vartheta}, S_i(\boldsymbol{\vartheta}))(\boldsymbol{u}, \mu) &= \partial_{\boldsymbol{w}} F(\boldsymbol{\vartheta}, S_i(\boldsymbol{\vartheta}))\boldsymbol{u} + \partial_\lambda F(\boldsymbol{\vartheta}, S_i(\boldsymbol{\vartheta}))\mu \\ &= \begin{pmatrix} -\nabla \cdot \mathbb{C}(\boldsymbol{\vartheta})\mathcal{E}(\boldsymbol{u}) - \lambda_i^{\boldsymbol{\vartheta}} \rho(\boldsymbol{\vartheta})\boldsymbol{u} - \mu\rho(\boldsymbol{\vartheta})\boldsymbol{w}_i^{\boldsymbol{\vartheta}} \\ 2\int_\Omega \rho(\boldsymbol{\vartheta})\boldsymbol{w}_i^{\boldsymbol{\vartheta}} \cdot \boldsymbol{u} \,\mathrm{d}x \end{pmatrix} \in H^{-1}(\Omega;\mathbb{R}^d) \times \mathbb{R}, \end{aligned}$$

for all $(\boldsymbol{u}, \mu) \in H^1_D(\Omega;\mathbb{R}^d) \times \mathbb{R}$. Consequently, (5.4.31) is equivalent to the system (5.4.26) written for

$$(\boldsymbol{u}, \mu) = \Big((\boldsymbol{w}_i^{\boldsymbol{\vartheta}})'\boldsymbol{h}, (\lambda_i^{\boldsymbol{\vartheta}})'\boldsymbol{h}\Big),$$

$$\boldsymbol{f} = \nabla \cdot \left[\mathbb{C}'(\boldsymbol{\vartheta})\boldsymbol{h}\,\mathcal{E}(\boldsymbol{w}_i^{\boldsymbol{\vartheta}})\right] + \lambda_i^{\boldsymbol{\vartheta}}\rho'(\boldsymbol{\vartheta})\boldsymbol{h}\boldsymbol{w}_i^{\boldsymbol{\vartheta}},$$
$$\kappa = -\int_\Omega \rho'(\boldsymbol{\vartheta})\boldsymbol{h}|\boldsymbol{w}_i^{\boldsymbol{\vartheta}}|^2\,\mathrm{d}x,$$

with $\boldsymbol{\varphi}$ replaced by $\boldsymbol{\vartheta} \in B_{\delta_i^{\boldsymbol{\varphi}}}(\boldsymbol{\varphi})$. Recalling the Fredholm alternative 5.4.4, and noting that $\lambda_i^{\boldsymbol{\vartheta}}$ is simple by its construction, analogously as in (5.4.28) we deduce that $\mu = -\langle \boldsymbol{f}, \boldsymbol{w}_i^{\boldsymbol{\vartheta}}\rangle_{H^{-1},H^1}$, which directly yields

$$(\lambda_i^{\boldsymbol{\vartheta}})'\boldsymbol{h} = \left\langle \mathcal{E}(\boldsymbol{w}_i^{\boldsymbol{\vartheta}}), \mathcal{E}(\boldsymbol{w}_i^{\boldsymbol{\vartheta}})\right\rangle_{\mathbb{C}'(\boldsymbol{\vartheta})\boldsymbol{h}} - \lambda_i^{\boldsymbol{\vartheta}}\int_\Omega \rho'(\boldsymbol{\vartheta})\boldsymbol{h}|\boldsymbol{w}_i^{\boldsymbol{\vartheta}}|^2\,\mathrm{d}x,$$

Plugging this into (5.4.26) with the above choices for $\boldsymbol{u}$, μ $\boldsymbol{f}$ and κ, we conclude that $(\boldsymbol{w}_i^{\boldsymbol{\varphi}})'\boldsymbol{h}$ satisfies

$$\begin{aligned}
&\left\langle \mathcal{E}((\boldsymbol{w}_i^{\boldsymbol{\vartheta}})'\boldsymbol{h}), \mathcal{E}(\boldsymbol{\eta})\right\rangle_{\mathbb{C}(\boldsymbol{\vartheta})} - \lambda_i^{\boldsymbol{\vartheta}}\int_\Omega \rho(\boldsymbol{\vartheta})(\boldsymbol{w}_i^{\boldsymbol{\vartheta}})'\boldsymbol{h}\cdot\boldsymbol{\eta}\,\mathrm{d}x \\
&\quad = -\left\langle \mathcal{E}(\boldsymbol{w}_i^{\boldsymbol{\vartheta}}), \mathcal{E}(\boldsymbol{\eta})\right\rangle_{\mathbb{C}'(\boldsymbol{\vartheta})\boldsymbol{h}} + \lambda_i^{\boldsymbol{\vartheta}}\int_\Omega \rho'(\boldsymbol{\vartheta})\boldsymbol{h}\boldsymbol{w}_i^{\boldsymbol{\vartheta}}\cdot\boldsymbol{\eta}\,\mathrm{d}x \\
&\qquad + (\lambda_i^{\boldsymbol{\vartheta}})'\boldsymbol{h}\int_\Omega \rho(\boldsymbol{\vartheta})\boldsymbol{w}_i^{\boldsymbol{\vartheta}}\cdot\boldsymbol{\eta}\,\mathrm{d}x,
\end{aligned} \tag{5.4.32}$$

for all $\boldsymbol{\eta} \in H^1_D(\Omega;\mathbb{R}^d)$, and

$$\left((\boldsymbol{w}_i^{\boldsymbol{\vartheta}})'\boldsymbol{h}, \boldsymbol{w}_i^{\boldsymbol{\vartheta}}\right)_{\rho(\boldsymbol{\vartheta})} = \frac{\kappa}{2} = -\frac{1}{2}\int_\Omega \rho'(\boldsymbol{\vartheta})\boldsymbol{h}|\boldsymbol{w}_i^{\boldsymbol{\vartheta}}|^2\,\mathrm{d}x.$$

This completes the proof. □

5.5. The optimality system

We can now apply the theory developed in Section 5.3 and Section 5.4 to show that the optimization problem $(\mathcal{P}_l^\varepsilon)$ (that was introduced in Subsection 2.1.11) possesses a minimizer if the set $\boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c$ is non-empty. Here, the assumption that the set of admissible phase-fields is non-empty is actually necessary as the sets S_0 and S_1 could be chosen in such a way that no $\boldsymbol{\varphi} \in \boldsymbol{U}_c$ can have the desired regularity $H^1(\Omega;\mathbb{R}^N)$ or the mean value constraint imposed by $\boldsymbol{\mathcal{G}}^m$ is never met, see also the discussion in Section 2.1.11.

Remark 5.5.1. It is worth mentioning that we do not need any boundedness assumption on the penalizing function Ψ in order to prove the existence of a minimizer to $(\mathcal{P}_l^\varepsilon)$. In analogy to Lemma 3.2.7, one can show that there are constants $C_{1,\varepsilon}$, $C_{2,\varepsilon} > 0$ depending only on the choice of $\mathbb{C}_\varepsilon$ and ρ_ε such that

$$C_{1,\varepsilon}\lambda_k^M \le \lambda_k^{\varepsilon,\boldsymbol{\varphi}} \le C_{2,\varepsilon}\lambda_k^M \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}.$$

Here, λ_k^M denotes the k-th eigenvalue of the problem (5.2.1) with $\mathbb{C} \equiv \mathrm{Id}$ and $\rho \equiv 1$. Qualitatively speaking, λ_k^M denotes an eigenvalue in the situation when the whole design domain is occupied by one material.

Theorem 5.5.2 (Existence of a minimizer to $(\mathcal{P}_l^\varepsilon)$)**.** *Suppose that the set $\boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c$ is non-empty. Then the problem $(\mathcal{P}_l^\varepsilon)$ possesses a minimizer $\overline{\boldsymbol{\varphi}} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c$.*

Proof. To proof the assertion, we apply the direct method in the calculus of variations. Recalling that Ψ is continuous we first observe using Remark 5.5.1 that the objective functional is bounded below, i.e. there is a constant $C > 0$ such that

$$-C \leq J^\varepsilon(\boldsymbol{\varphi}) < \infty \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{F}_{\text{ad}} = \boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c.$$

Since $\mathcal{F}_{\text{ad}}$ is non-empty, the infimum

$$\bar{J} := \inf_{\boldsymbol{\varphi} \in \mathcal{F}_{\text{ad}}} J^\varepsilon(\boldsymbol{\varphi})$$

exists in $\mathbb{R}$. Thus, there exists a minimizing sequence $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}} \subset \mathcal{F}_{\text{ad}}$ with $J(\boldsymbol{\varphi}_k) \to \bar{J}$ as $k \to \infty$. Using the fact that $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}}$ is bounded in $H^1(\Omega;\mathbb{R}^N)$, we infer that

$$\boldsymbol{\varphi}_k \rightharpoonup \overline{\boldsymbol{\varphi}} \quad \text{in } H^1(\Omega;\mathbb{R}^N), \quad \text{as } k \to \infty,$$

along a non-relabeled subsequence. As the sequence $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}}$ lies in $\boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$ it is also bounded in $L^\infty(\Omega;\mathbb{R}^N)$. Hence, Theorem 5.3.4 implies that

$$\lambda_{i_j}^{\boldsymbol{\varphi}_k} \to \lambda_{i_j}^{\boldsymbol{\varphi}}, \quad \text{as } k \to \infty,$$

for all $j = 1, \dots, l$. As Ψ is continuous and the Ginzburg–Landau energy is weakly lower semi-continuous, we conclude that

$$J_l^\varepsilon(\overline{\boldsymbol{\varphi}}) \leq \liminf_{k\to\infty} J_l^\varepsilon(\boldsymbol{\varphi}_k) = \bar{J}.$$

This directly implies that $J_l^\varepsilon(\overline{\boldsymbol{\varphi}}) = \bar{J}$ and thus, $\overline{\boldsymbol{\varphi}}$ is a minimizer of the functional J on the set $\mathcal{F}_{\text{ad}}$. This completes the proof.

□

Now, invoking the differentiability properties established in Section 5.4, we can derive a first-order necessary condition for local optimality.

Theorem 5.5.3 (The optimality system to $(\mathcal{P}_l^\varepsilon)$)**.** *Let $\boldsymbol{\varphi} \in (\boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c)$ be a local minimizer of the optimization problem $(\mathcal{P}_l^\varepsilon)$, i.e., there exists $\delta > 0$ such that*

$$J_l^\varepsilon(\boldsymbol{\vartheta}) \geq J_l^\varepsilon(\boldsymbol{\varphi}) \quad \textit{for all } \boldsymbol{\vartheta} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c \textit{ with } \|\boldsymbol{\vartheta} - \boldsymbol{\varphi}\|_{H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)} < \delta.$$

Suppose that the eigenvalues $\lambda_{i_1}^{\boldsymbol{\varphi}}, \dots, \lambda_{i_l}^{\boldsymbol{\varphi}}$ are simple and let us fix $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-normalized eigenfunctions $\boldsymbol{w}_{i_1}^{\boldsymbol{\varphi}}, \dots, \boldsymbol{w}_{i_l}^{\boldsymbol{\varphi}} \in H_D^1(\Omega;\mathbb{R}^d)$ to the eigenvalues $\lambda_{i_1}^{\boldsymbol{\varphi}}, \dots, \lambda_{i_l}^{\boldsymbol{\varphi}}$, respectively.
Then the following optimality system is satisfied:

- *The state equations*

$$\left\{ \begin{array}{rll} -\nabla \cdot \left[\mathbb{C}(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{w}_{i_j}^{\boldsymbol{\varphi}})\right] &= \lambda_{i_j}^{\boldsymbol{\varphi}} \rho(\boldsymbol{\varphi}) \boldsymbol{w}_{i_j}^{\boldsymbol{\varphi}} & \textit{in } \Omega, \\ \boldsymbol{w}_{i_j}^{\boldsymbol{\varphi}} &= \boldsymbol{0} & \textit{on } \Gamma_D, \\ \left[\mathbb{C}(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{w}_{i_j}^{\boldsymbol{\varphi}})\right]\boldsymbol{n} &= \boldsymbol{0} & \textit{on } \Gamma_0, \end{array} \right. \tag{SE_j}$$

  are satisfied for all $j \in \{1, \dots, l\}$.

- *The variational inequality*

$$
\begin{aligned}
0 \leq \gamma\varepsilon \int_\Omega \nabla\boldsymbol{\varphi} : \nabla(\boldsymbol{\vartheta}-\boldsymbol{\varphi})\,dx + \frac{\gamma}{\varepsilon}\int_\Omega \psi_0'(\boldsymbol{\varphi})(\boldsymbol{\vartheta}-\boldsymbol{\varphi})\,dx \\
+ \sum_{j=1}^{l} \left\{ [\partial_{\lambda_{i_j}}\Psi](\lambda_{i_1}^{\boldsymbol{\varphi}},\dots,\lambda_{i_l}^{\boldsymbol{\varphi}}) \left(\left\langle \mathcal{E}(\boldsymbol{w}_{i_j}^{\boldsymbol{\varphi}}), \mathcal{E}(\boldsymbol{w}_{i_j}^{\boldsymbol{\varphi}}) \right\rangle_{\mathbb{C}'(\boldsymbol{\varphi})(\boldsymbol{\vartheta}-\boldsymbol{\varphi})} \right.\right. \\
\left.\left. - \lambda_{i_j}^{\boldsymbol{\varphi}} \int_\Omega \rho'(\boldsymbol{\varphi})\,(\boldsymbol{\vartheta}-\boldsymbol{\varphi})\,|\boldsymbol{w}_{i_j}^{\boldsymbol{\varphi}}|^2\,dx \right) \right\}
\end{aligned}
\tag{VI}
$$

is satisfied for all $\boldsymbol{\vartheta} \in (\boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c)$ *and all* $j \in \{1,\dots,l\}$.

Proof. Since $\boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c$ is convex, it holds that $\boldsymbol{\varphi} + t(\boldsymbol{\vartheta}-\boldsymbol{\varphi}) \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c$ for all $\boldsymbol{\vartheta} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c$ and all $t \in [0,1]$. As the objective functional J_l^ε is differentiable in every admissible direction $\boldsymbol{\vartheta}-\boldsymbol{\varphi}$, we know that

$$
0 \leq \frac{\mathrm{d}}{\mathrm{d}t} J_l^\varepsilon(\boldsymbol{\varphi} + t(\boldsymbol{\vartheta}-\boldsymbol{\varphi}))\big|_{t=0} = (J_l^\varepsilon)'(\boldsymbol{\varphi})(\boldsymbol{\vartheta}-\boldsymbol{\varphi}).
$$

Using (5.4.12) in Theorem 5.4.3 it is now straightforward to check that $(J_l^\varepsilon)'(\boldsymbol{\varphi})(\boldsymbol{\vartheta}-\boldsymbol{\varphi})$ is identical with the right-hand side of the variational inequality. This completes the proof. □

Remark 5.5.4. If the first eigenvalue $\lambda_1^{\boldsymbol{\varphi}}$ is not simple but only $\lambda_1^{\boldsymbol{\varphi}}$ and further simple eigenvalues appear in the objective functional, we can still derive a variational inequality by means of the semi-differentiability established in Theorem 5.4.2. This is because in the above proof only variations $\boldsymbol{\varphi} + t(\boldsymbol{\vartheta}-\boldsymbol{\varphi})$ with positive t are considered.

To be precise, let us assume that the multiplicity of the eigenvalue $\lambda_1^{\boldsymbol{\varphi}}$ is $M \in \mathbb{N}$. This means that

$$
\lambda_1^{\boldsymbol{\varphi}} = \lambda_2^{\boldsymbol{\varphi}} = \dots = \lambda_M^{\boldsymbol{\varphi}}.
$$

If now $\lambda_1^{\boldsymbol{\varphi}}$ appears in $(\mathcal{P}_l^\varepsilon)$ but none of the eigenvalues $\lambda_2^{\boldsymbol{\varphi}} = \dots = \lambda_M^{\boldsymbol{\varphi}}$ does, the term

$$
\langle \mathcal{E}(\boldsymbol{w}_1^{\boldsymbol{\varphi}}), \mathcal{E}(\boldsymbol{w}_1^{\boldsymbol{\varphi}}) \rangle_{\mathbb{C}'(\boldsymbol{\varphi})(\boldsymbol{\vartheta}-\boldsymbol{\varphi})} - \lambda_1^{\boldsymbol{\varphi}} \int_\Omega \rho'(\boldsymbol{\varphi})\,(\boldsymbol{\vartheta}-\boldsymbol{\varphi})\,|\boldsymbol{w}_1^{\boldsymbol{\varphi}}|^2,
$$

in the variational inequality has to be replaced by

$$
\inf \left\{ \langle \mathcal{E}(\boldsymbol{u}), \mathcal{E}(\boldsymbol{u}) \rangle_{\mathbb{C}'(\boldsymbol{\varphi})(\boldsymbol{\vartheta}-\boldsymbol{\varphi})} - \lambda_1^{\boldsymbol{\varphi}}\,(\boldsymbol{u},\boldsymbol{u})_{\rho'(\boldsymbol{\varphi})(\boldsymbol{\vartheta}-\boldsymbol{\varphi})} \,\middle|\, \begin{array}{l} \boldsymbol{u} \in H_D^1(\Omega;\mathbb{R}^d) \text{ is an} \\ \text{eigenfunction to } \lambda_1^{\boldsymbol{\varphi}} \\ \text{with } \|\boldsymbol{u}\|_{L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)} = 1 \end{array} \right\}.
$$

Of course, if $\lambda_1^{\boldsymbol{\varphi}}$ is simple (i.e., $M=1$) both terms coincide.

In the following we will only discuss the case of simple eigenvalues, but keep the fact in mind that it is not necessary to require simplicity of the first eigenvalue.

5.6. Combination of compliance and eigenvalue optimization

We now want to analyze the optimization problem $(\mathcal{K}_l^\varepsilon)$ (that was introduced in Section 2.1.12) by establishing results similar to those in Section 5.5 To this end, we will use the control-to-state operator

$$S : H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N) \to H_C^1(\Omega;\mathbb{R}^d), \quad \boldsymbol{\varphi} \mapsto \boldsymbol{u}(\boldsymbol{\varphi})$$

that was introduced in [32] and maps any $\boldsymbol{\varphi} \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ onto its corresponding solution $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{\varphi})$ of the state equation (2.1.31). This allows us to consider the reduced optimization problem

$$\begin{cases} \min & I_l^\varepsilon(\boldsymbol{\varphi}) = \alpha F(S(\boldsymbol{\varphi}),\boldsymbol{\varphi}) + \beta J_0(S(\boldsymbol{\varphi}),\boldsymbol{\varphi}) + \gamma E^\varepsilon(\boldsymbol{\varphi}) + \Psi(\lambda_{i_1}^{\boldsymbol{\varphi}},\dots,\lambda_{i_l}^{\boldsymbol{\varphi}}) \\ \text{s.t.} & \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c, (2.1.31) \text{ is fulfilled,} \\ & \text{and } \lambda_{i_1}^{\boldsymbol{\varphi}},\dots,\lambda_{i_l}^{\boldsymbol{\varphi}} \text{ are eigenvalues of (5.2.1)} \end{cases} \tag{$\mathcal{K}_l^{\varepsilon *}$}$$

(with $\alpha,\beta \geq 0$, $\gamma,\varepsilon > 0$ and $\boldsymbol{m} \in (0,1)^N \cap \Sigma^N$), which is obviously equivalent to the original problem $(\mathcal{K}_l^\varepsilon)$.

The following theorem ensures the existence of a minimizer to $(\mathcal{K}_l^{\varepsilon *})$ or $(\mathcal{K}_l^\varepsilon)$, respectively.

Theorem 5.6.1 (Existence of a minimizer to $(\mathcal{K}_l^{\varepsilon *})$)**.** *Suppose that the set $\boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c$ is non-empty. Then the problem $(\mathcal{K}_l^{\varepsilon *})$ has a minimizer $\overline{\boldsymbol{\varphi}} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c$.*

Proof. The assertion can be verified by simply combining the proof in [32, Theorem 4.1] and the proof of Theorem 5.5.2. Therefore we omit the details. □

From the differentiability properties deduced in [32] and in this chapter we obtain a variational inequality for the problem $(\mathcal{K}_l^\varepsilon)$. Note that for $\nu \in (0,1)$, the functional J_0 is in general not differentiable where the integral raised to the power ν is equal to zero. Hence, as in [32], we only consider $(\boldsymbol{u},\boldsymbol{\varphi})$ such that

$$\int_\Omega c(1-\varphi^N)\,|\boldsymbol{u}-\boldsymbol{u}_\Omega|^2 \,\mathrm{d}x \neq 0,$$

if $\beta \neq 0$.

Eventually, we can state the optimality system for the combined problem where the first-order necessary condition for local optimality is incorporated.

Theorem 5.6.2 (The optimality system to $(\mathcal{K}_l^{\varepsilon *})$)**.** *Let $\boldsymbol{\varphi} \in (\boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c)$ be a local minimizer of the optimization problem $(\mathcal{K}_l^{\varepsilon *})$, i.e., there exists $\delta > 0$ such that*

$$I_l^\varepsilon(\boldsymbol{\vartheta}) \geq I_l^\varepsilon(\boldsymbol{\varphi}) \quad \text{for all } \boldsymbol{\vartheta} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c \text{ with } \|\boldsymbol{\vartheta}-\boldsymbol{\varphi}\|_{H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)} < \delta.$$

Suppose that the eigenvalues $\lambda_{i_1}^{\boldsymbol{\varphi}},\dots,\lambda_{i_l}^{\boldsymbol{\varphi}}$ are simple and let us fix $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^d)$-normalized eigenfunctions $\boldsymbol{w}_{i_1}^{\boldsymbol{\varphi}},\dots,\boldsymbol{w}_{i_l}^{\boldsymbol{\varphi}} \in H_D^1(\Omega;\mathbb{R}^d)$ to the eigenvalues $\lambda_{i_1}^{\boldsymbol{\varphi}},\dots,\lambda_{i_l}^{\boldsymbol{\varphi}}$, respectively.

Then, there exist a state $\boldsymbol{u} \in H_C^1(\Omega;\mathbb{R}^d)$ and an adjoint state $\boldsymbol{p} \in H_C^1(\Omega;\mathbb{R}^d)$ such that the tuple

$$\begin{aligned} &\left(\boldsymbol{u},\boldsymbol{\varphi},\boldsymbol{p},(\boldsymbol{w}_{i_j}^{\boldsymbol{\varphi}})_{j=1}^l,(\lambda_{i_j})_{j=1}^l\right) \\ &\quad \in H_C^1(\Omega;\mathbb{R}^d) \times (\boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U_c}) \times H_C^1(\Omega;\mathbb{R}^d) \times \left(H_D^1(\Omega;\mathbb{R}^d)\right)^l \times \mathbb{R}^l \end{aligned}$$

fulfills the following optimality system:

- *The state equations*

$$\begin{cases} -\nabla \cdot [\mathbb{C}(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{u})] &= (1-\varphi^N)\boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} & \text{on } \Gamma_C, \\ [\mathbb{C}(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{u})]\,\boldsymbol{n} &= \boldsymbol{g} & \text{on } \Gamma_g, \end{cases} \tag{SE^*}$$

and

$$\begin{cases} -\nabla \cdot \left[\mathbb{C}(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{w}_{i_j}^{\boldsymbol{\varphi}})\right] &= \lambda_{i_j}^{\boldsymbol{\varphi}}\rho(\boldsymbol{\varphi})\boldsymbol{w}_{i_j}^{\boldsymbol{\varphi}} & \text{in } \Omega, \\ \boldsymbol{w}_{i_j}^{\boldsymbol{\varphi}} &= \boldsymbol{0} & \text{on } \Gamma_D, \\ \left[\mathbb{C}(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{w}_{i_j}^{\boldsymbol{\varphi}})\right]\boldsymbol{n} &= \boldsymbol{0} & \text{on } \Gamma_0, \end{cases} \tag{SE_j^*}$$

for $j = 1, \dots, l$, are satisfied in the weak sense.

- *The adjoint equation*

$$\begin{cases} -\nabla \cdot [\mathbb{C}(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{p})] = \alpha(1-\varphi^N)\boldsymbol{f} \\ \qquad\qquad + 2\beta\nu J_0(\boldsymbol{u},\boldsymbol{\varphi})^{\frac{\nu-1}{\nu}} c(1-\varphi^N)\,(\boldsymbol{u}-\boldsymbol{u}_\Omega) & \text{in } \Omega, \\ \boldsymbol{p} = \boldsymbol{0} & \text{on } \Gamma_C, \\ [\mathbb{C}(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{p})]\,\boldsymbol{n} = \alpha\boldsymbol{g} & \text{on } \Gamma_g, \end{cases} \tag{AE^*}$$

is satisfied in the weak sense.

- *The variational inequality*

$$\begin{aligned} 0 \leq\, & \gamma\varepsilon \int_\Omega \nabla\boldsymbol{\varphi} : \nabla(\boldsymbol{\vartheta}-\boldsymbol{\varphi})\,\mathrm{d}x + \frac{\gamma}{\varepsilon}\int_\Omega \psi_0'(\boldsymbol{\varphi})\cdot(\boldsymbol{\vartheta}-\boldsymbol{\varphi})\,\mathrm{d}x \\ & - \beta\nu J_0(\boldsymbol{u},\boldsymbol{\varphi})^{\frac{\nu-1}{\nu}} \int_\Omega c(\vartheta^N-\varphi^N)\,|\boldsymbol{u}-\boldsymbol{u}_\Omega|^2\,\mathrm{d}x \\ & - \int_\Omega (\vartheta^N-\varphi^N)\boldsymbol{f}\cdot(\alpha\boldsymbol{u}+\boldsymbol{p})\,\mathrm{d}x \;-\langle\mathcal{E}(\boldsymbol{p}),\mathcal{E}(\boldsymbol{u})\rangle_{\mathbb{C}'(\boldsymbol{\varphi})(\boldsymbol{\vartheta}-\boldsymbol{\varphi})} \\ & + \sum_{j=1}^{l}\Big\{[\partial_{\lambda_{i_j}}\Psi](\lambda_{i_1}^{\boldsymbol{\varphi}},\dots,\lambda_{i_l}^{\boldsymbol{\varphi}})\left(\left\langle\mathcal{E}(\boldsymbol{w}_{i_j}^{\boldsymbol{\varphi}}),\mathcal{E}(\boldsymbol{w}_{i_j}^{\boldsymbol{\varphi}})\right\rangle_{\mathbb{C}'(\boldsymbol{\varphi})(\boldsymbol{\vartheta}-\boldsymbol{\varphi})}\right. \\ & \qquad\qquad \left.-\lambda_{i_j}^{\boldsymbol{\varphi}}\int_\Omega \rho'(\boldsymbol{\varphi})\,(\boldsymbol{\vartheta}-\boldsymbol{\varphi})\,|\boldsymbol{w}_{i_j}^{\boldsymbol{\varphi}}|^2\,\mathrm{d}x\right)\Big\}, \end{aligned} \tag{VI^*}$$

is satisfied for all $\boldsymbol{\vartheta} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \cap \boldsymbol{U}_c$.

Proof. Using the properties of the control-to-state operator S, the assertion can be proved proceeding similarly as in the proof of Theorem 5.5.3. □

This combination of compliance and eigenvalue optimization will also be considered numerically in Section 6.9.3.

Chapter 6

Formal sharp-interface asymptotics

6.1. Introduction

In this chapter we will study the formal sharp-interface limit of the optimization problem studied in the previous chapter in order to relate the diffuse interface approach to the physically reasonable setting of sharp interfaces. In this context the main goal will be to derive the sharp-interface limit of the state equations and furthermore give an explicit formulation of the first-order optimality conditions on the sharp-interface level. Let us recall the state equations from the previous chapter given as

$$\begin{cases} -\nabla \cdot [\mathbb{C}_\varepsilon(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{w}^{\varepsilon,\varphi})] = \lambda^{\varepsilon,\boldsymbol{\varphi}}\rho_\varepsilon(\boldsymbol{\varphi})\boldsymbol{w}^{\varepsilon,\varphi} & \text{in } \Omega, \\ \boldsymbol{w}^{\varepsilon,\varphi} = \boldsymbol{0} & \text{on } \Gamma_D, \\ [\mathbb{C}_\varepsilon(\boldsymbol{\varphi})\mathcal{E}(\boldsymbol{w}^{\varepsilon,\varphi})]\,\boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma_0. \end{cases} \tag{SE^ε}$$

Let us assume for the sake of this introduction that we are in the two-phase case, i.e., only one material can be distributed within the design domain. We will show in the framework of formally matched asymptotic expansions that in the sharp-interface limit the eigenvalues and eigenfunctions of above problem will converge to a function $\boldsymbol{w}_0$ and a limit eigenvalue λ_0 satisfying

$$\begin{cases} -\nabla \cdot \left(\mathbb{C}^M\mathcal{E}(\boldsymbol{w}_0^{n_r})\right) = \lambda_0\rho^M\boldsymbol{w}_0 & \text{in } \Omega^M, \\ \mathbb{C}^M\mathcal{E}_M(\boldsymbol{w}_0)\,\boldsymbol{n}_{\Gamma_{MV}} = \boldsymbol{0} & \text{on } \Gamma_{MV}, \\ \boldsymbol{w}_0 = \boldsymbol{0} & \text{on } \Gamma_D^M, \\ \mathbb{C}^M\mathcal{E}(\boldsymbol{w}_0)\,\boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma_0^M. \end{cases} \tag{SE^{MV}}$$

Here Ω^M is the final shape composed of the material, Γ_{MV} is the free boundary between shape and void, and Γ_D^M, Γ_0^M are the Dirichlet and Neumann parts of the boundary that the

shape shares with the design domain. Furthermore we will pass the variational inequality

$$\begin{aligned}\sum_{r=1}^{l}\Bigg\{&[\partial_{\lambda_{n_r}}\Psi](\lambda_{n_1}^{\varepsilon,\boldsymbol{\varphi}},\dots,\lambda_{n_l}^{\varepsilon,\boldsymbol{\varphi}})\\ &\cdot\Big(\langle\mathcal{E}(\boldsymbol{w}_{n_r}^{\varepsilon,\boldsymbol{\varphi}}),\mathcal{E}(\boldsymbol{w}_{n_r}^{\varepsilon,\boldsymbol{\varphi}})\rangle_{\mathbb{C}'(\boldsymbol{\varphi})(\tilde{\boldsymbol{\varphi}}-\boldsymbol{\varphi})}-\lambda_{n_r}^{\varepsilon,\boldsymbol{\varphi}}\int_\Omega\rho'(\boldsymbol{\varphi})(\tilde{\boldsymbol{\varphi}}-\boldsymbol{\varphi})|\boldsymbol{w}_{n_r}^{\varepsilon,\boldsymbol{\varphi}}|^2\,\mathrm{d}x\Big)\Bigg\}\\ &+\gamma\varepsilon\int_\Omega\nabla\boldsymbol{\varphi}:\nabla(\tilde{\boldsymbol{\varphi}}-\boldsymbol{\varphi})\,\mathrm{d}x+\frac{\gamma}{\varepsilon}\int_\Omega\psi_0'(\boldsymbol{\varphi})(\tilde{\boldsymbol{\varphi}}-\boldsymbol{\varphi})\,\mathrm{d}x\ \geq\ 0\end{aligned}\tag{GI^ε}$$

derived in Theorem 5.5.3 to the sharp-interface limit, which then reads as

$$\begin{aligned}0=\gamma\,\sigma_{MV}\,\kappa_{MV}&+\sum_{r=1}^{l}[\partial_{\lambda_{n_r}}\Psi]\,(\lambda_{0,n_1},\dots,\lambda_{0,n_l})\,\lambda_{0,n_r}\rho^M\,|(\boldsymbol{w}_{0,n_r})_M|^2\\ &-\sum_{r=1}^{l}[\partial_{\lambda_{n_r}}\Psi]\,(\lambda_{0,n_1},\dots,\lambda_{0,n_l})\,\mathbb{C}^M\mathcal{E}_M(\boldsymbol{w}_{0,n_r}):\mathcal{E}_M(\boldsymbol{w}_{0,n_r})+\gamma\left(\vartheta_1^1-\vartheta_1^2\right)\quad\text{on }\Gamma_{MV}.\end{aligned}\tag{GE^{MV}}$$

Here σ_{MV} is the energy of an optimal transition from the material phase into the void, which plays the same role as the prefactor c_0 in the Γ-limit of the Ginzburg–Landau energy in the scalar case, see also Section 3.3.2. κ_{MV} denotes the mean curvature of the interface Γ_{MV} separating void and material and $\boldsymbol{\vartheta}$ is the Lagrange multiplier associated to the volume constraint imposed in the optimization problem. Eventually, we will see that using classical shape calculus (GE^{MV}) is exactly the first-order condition associated to the spectral shape optimization problem on the sharp-interface level.

Note that in (SE^{MV}) we see that on the free boundary Γ_{MV} a homogeneous Neumann boundary condition holds. As already discussed in the introduction of this thesis, spectral problems of Neumann type are harder to analyze rigorously than Dirichlet problems from the viewpoint of calculus of variations, due to the occurrence of severe instabilities of the spectrum.
This motivates the ansatz of *formally matched asymptotic expansions* in order to gain information about the sharp-interface limit. Let us mention that this approach enjoys a huge popularity in the framework of phase-field approaches and has been applied successfully especially to problems of Allen–Cahn or Cahn–Hilliard type in the literature, see [1,26,32,46,97]. We also refer to [86,91,116] for a comprehensive overview of formally matched asymptotic expansions.

Let us explain the general procedure of this technique for the sake of this introduction. The key *assumption* we make is that all appearing ε-dependent quantities such as the phase-field, the eigenfunctions and eigenvalues, each can be expanded as a power series in integer orders of ε. Each quantity will be associated with two different kinds of expansions, an *inner asymptotic expansion* and an *outer asymptotic expansion*. This distinction is necessary in view of the expected behavior of the phase-field modeling the diffuse interface: In regions "close" to the diffuse interface the phase-field exhibits rapid changes on a length-scale of ε, because here the transition between the pure phases takes place. On the other hand in regions "far" away from the interface the phase-field is close to the pure phases. The key idea in order to account for this distinction will be the introduction of a rescaled coordinate system which captures the ε-scaling of the inner region.

In a first step we will plug in the outer expansions in the state equations and compare the leading order terms in order to obtain a sharp-interface version of the state equation in outer regions. At this point we will include a discussion about localized eigenmodes into our analysis. As also mentioned in the introduction of this thesis, in numerical simulations the formation of eigenmodes supported only in void producing eigenvalues polluting the low part of the spectrum is a major problem. We will see how our asymptotic approach is able to deal with localized modes, more precisely we will see that if such modes appear, then as $\varepsilon \to 0$ the corresponding eigenvalues λ^ε will become arbitrarily large. Thus for $\varepsilon > 0$ sufficiently small, localized modes do not affect the lower part of the spectrum appearing in our optimization problem.
In order to obtain boundary conditions for the sharp-interface state equation, the key step will be to match the inner with the outer expansion within a so-called intermediate region. The idea here is to introduce a further intermediate coordinate system in order to make the two coordinate systems the inner and outer expansion are formulated in comparable. This comparison, the so called matching, will link the inner with the outer expansions in the sharp-interface limit $\varepsilon \to 0$. Considering now the leading order terms of the inner expansions in the state equation, this matching allows us to derive the desired boundary conditions.
As already mentioned above, we are also interested in the sharp-interface limit of the first-order optimality condition given in form of a gradient inequality derived in the previous chapter. As this variational inequality is given in integrated form we will first formulate it in terms of a point-wise equality which we then can analyze in the framework of formally matched asymptotics. This reformulation will be obtained by regularizing the non-linear point-wise Gibbs-Simplex constraint in a similar fashion as in [35]. Note that in order to pass from the variational formulation to a strong formulation we need to assume further regularity of the involved eigenfunctions in order to then prove H^2-regularity of the solution of this variational inequality. The main benefit of this procedure is that it allows us to explicitly construct Lagrange multipliers for all the constraints involved in our optimization problem. This construction provides us with crucial information that will be exploited in the asymptotic analysis.
Finally, after arriving at the sharp-interface limit of the gradient equality we will also motivate this specific equation from the viewpoint of classical shape calculus, see [5]. More precisely we will show that the gradient equality we have obtained via our formal approach indeed coincides with the classical shape derivative of the associated sharp-interface cost functional.
We end this chapter by presenting the numerical application of the phase-field approach to several concrete optimization problems occurring in linear elasticity both included in and exceeding the literature.

Concluding this introduction the key goal of this chapter is to pass the full optimality system obtained in the previous chapter to the sharp-interface limit in order to obtain state equations and a first-order necessary optimality condition in the form of a gradient equality.

6.2. Recalling the problem

In order to have a sound fundament for the upcoming analysis let us recall the overall optimization problem and the corresponding first-order condition formulated as gradient inequality in the previous chapter.

First let us recall the weak formulation of the state equation (SE^ε) from Definition 5.2.1. For $\varepsilon > 0$ and $\boldsymbol{\varphi} \in L^\infty(\Omega;\mathbb{R}^N)$ we seek for a solution $(\lambda^{\varepsilon,\boldsymbol{\varphi}}, \boldsymbol{w}^{\varepsilon,\boldsymbol{\varphi}}) \in \mathbb{R} \times H^1_D(\Omega;\mathbb{R}^d)\setminus\{0\}$ of

$$\langle \mathcal{E}(\boldsymbol{w}^{\varepsilon,\boldsymbol{\varphi}}), \mathcal{E}(\boldsymbol{\eta})\rangle_{\mathbb{C}_\varepsilon(\boldsymbol{\varphi})} = \lambda^{\varepsilon,\boldsymbol{\varphi}} (\boldsymbol{w}^{\varepsilon,\boldsymbol{\varphi}}, \boldsymbol{\eta})_{\rho_\varepsilon(\boldsymbol{\varphi})} \quad \text{for all } \boldsymbol{\eta} \in H^1_D(\Omega,\mathbb{R}^d). \tag{6.2.1}$$

Note that here the elasticity tensor $\mathbb{C}_\varepsilon$ and the density ρ_ε depend on the interface parameter ε. In order for our model to be adapted to the ansatz of formally matched asymptotic expansions, which expresses all quantities in power series with respect to ε, we choose for this chapter the decomposition

$$\begin{aligned} \mathbb{C}_\varepsilon(\boldsymbol{\varphi}) &= \overline{\mathbb{C}}(\boldsymbol{\varphi}) + \tilde{\mathbb{C}}^N \varepsilon \varphi^N = \sum_{i=1}^{N-1} \mathbb{C}^i \varphi^i + \tilde{\mathbb{C}}^N \varepsilon \varphi^N, \\ \rho_\varepsilon(\boldsymbol{\varphi}) &= \overline{\rho}(\boldsymbol{\varphi}) + \tilde{\rho}^N \varepsilon \varphi^N = \sum_{i=1}^{N-1} \rho^i \varphi^i + \tilde{\rho}^N \varepsilon \varphi^N, \end{aligned} \tag{6.2.2}$$

for any $\boldsymbol{\varphi} \in \boldsymbol{G}$, see Section 2.1.8 and Section 2.1.9. We note that the upcoming analysis will work for ε replaced in (6.2.2) by any ε^p with $p > 0$, thus the concrete scaling is not relevant for the analysis. Nevertheless, as mentioned also in Section 2.1.8, in Section 6.4.2 we will give a specific quadratic composition of $\mathbb{C}_\varepsilon$ and ρ_ε and a suited ε scaling of the void components $\tilde{\mathbb{C}}^N$ and $\tilde{\rho}^N$ in order to deal with localized eigenmodes. Note that for a cleaner depiction we will often omit the index ε in $\mathbb{C}$ and ρ.

The associated optimization problem studied in the previous chapter reads as

$$\begin{cases} \min & J_l^\varepsilon(\boldsymbol{\varphi}), \\ \text{over} & \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}}, \\ \text{s.t.} & \lambda^{\varepsilon,\boldsymbol{\varphi}}_{n_1}, \dots, \lambda^{\varepsilon,\boldsymbol{\varphi}}_{n_l}, \\ & \text{are eigenvalues of (6.2.1)}, \end{cases} \tag{$\mathcal{P}_l^\varepsilon$}$$

with

$$J_l^\varepsilon(\boldsymbol{\varphi}) \coloneqq \Psi(\lambda^{\varepsilon,\boldsymbol{\varphi}}_{n_1}, \dots, \lambda^{\varepsilon,\boldsymbol{\varphi}}_{n_l}) + \gamma E^\varepsilon_{\mathrm{GL}}(\boldsymbol{\varphi}) \quad \text{for } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$$

for some $l \in \mathbb{N}$, where $n_1, \dots, n_l \in \mathbb{N}$ indicate a selection of eigenvalues. Here, $\gamma > 0$ is a fixed constant related to surface tension, Ψ is a continuously differentiable penalization function and E^ε is the regular part of the Ginzburg–Landau energy $E^\varepsilon_{\mathrm{GL}}$, introduced in Section 2.1.4. Note that according to the discussion in Section 2.1.3, opposed to the previous chapter, we do not consider here the additional point-wise constraint $\boldsymbol{\varphi} \in \boldsymbol{U}_c$ in the admissible set of our optimization problem.
In Theorem 5.5.3, the following first-order necessary optimality condition was derived.

Theorem 6.2.1. *Let $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$ be a local minimizer of $(\mathcal{P}_l^\varepsilon)$, i.e., there exists $\delta > 0$ such that $J_l^\varepsilon(\boldsymbol{\varphi}) \le J_l^\varepsilon(\boldsymbol{\zeta})$ for all $\boldsymbol{\zeta} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$ with $\|\boldsymbol{\zeta} - \boldsymbol{\varphi}\|_{H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)} < \delta$. We further*

assume that the eigenvalues $\lambda_{n_1}^{\varepsilon,\boldsymbol{\varphi}},\dots,\lambda_{n_l}^{\varepsilon,\boldsymbol{\varphi}}$ are simple. Then the gradient inequality

$$\begin{aligned}\sum_{r=1}^{l}\Big\{&[\partial_{\lambda_{n_r}}\Psi](\lambda_{n_1}^{\varepsilon,\boldsymbol{\varphi}},\dots,\lambda_{n_l}^{\varepsilon,\boldsymbol{\varphi}})\\&\cdot\Big(\langle\mathcal{E}(\boldsymbol{w}_{n_r}^{\varepsilon,\boldsymbol{\varphi}}),\mathcal{E}(\boldsymbol{w}_{n_r}^{\varepsilon,\boldsymbol{\varphi}})\rangle_{\mathbb{C}'(\boldsymbol{\varphi})(\tilde{\boldsymbol{\varphi}}-\boldsymbol{\varphi})}-\lambda_{n_r}^{\varepsilon,\boldsymbol{\varphi}}\int_\Omega\rho'(\boldsymbol{\varphi})(\tilde{\boldsymbol{\varphi}}-\boldsymbol{\varphi})|\boldsymbol{w}_{n_r}^{\varepsilon,\boldsymbol{\varphi}}|^2\,\mathrm{d}x\Big)\Big\}\\&+\gamma\varepsilon\int_\Omega\nabla\boldsymbol{\varphi}:\nabla(\tilde{\boldsymbol{\varphi}}-\boldsymbol{\varphi})\,\mathrm{d}x+\frac{\gamma}{\varepsilon}\int_\Omega\psi_0'(\boldsymbol{\varphi})(\tilde{\boldsymbol{\varphi}}-\boldsymbol{\varphi})\,\mathrm{d}x\ \geq\ 0\end{aligned}\tag{GI^ε}$$

holds for all $\tilde{\boldsymbol{\varphi}}\in\boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$.

Recall from Theorem 5.2.2 that here for $\boldsymbol{\varphi}\in L^\infty(\Omega;\mathbb{R}^N)$ we denote with

$$\{\boldsymbol{w}_1^{\varepsilon,\boldsymbol{\varphi}},\boldsymbol{w}_2^{\varepsilon,\boldsymbol{\varphi}},\dots\}\subset H_D^1(\Omega;\mathbb{R}^d),$$

an $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^N)$ orthonormal basis of the space $L^2_{\boldsymbol{\varphi}}(\Omega;\mathbb{R}^N)$. In particular we have the normalization condition

$$1=\int_\Omega\rho(\boldsymbol{\varphi})\,|\boldsymbol{w}_i^{\varepsilon,\boldsymbol{\varphi}}|\,\mathrm{d}x,\tag{6.2.3}$$

for $i\in\mathbb{N}$.

The upcoming sharp-interface analysis will be concerned with passing to the limit in the state equation (SE^ε) as well as in the gradient inequality (GI^ε).

6.3. Analysis of the gradient inequality

In this section, we will show under a suitable regularity assumption on the eigenfunctions involved in (GI^ε) that there exists a solution of the above gradient inequality possessing even the regularity $\boldsymbol{\varphi}\in H^2(\Omega;\mathbb{R}^N)$. This will be carried out by applying a regularization process to the non-smooth potential ψ, which was employed in a similar fashion in [25, 35, 37, 87]. Our approach mainly follows the ideas of [35].

We regularize the gradient inequality in order to deal with the indicator functional $I_{\boldsymbol{G}}$ contained in the definition of the potential ψ. This will yield a sequence of H^2-regular approximating phase-fields $(\boldsymbol{\varphi}^\delta)_{\delta>0}$ solving regularized equations and converging to the desired phase-field $\boldsymbol{\varphi}$. Another convenient aspect of this procedure is that it will generate Lagrange multipliers that will allow us to transform the gradient inequality into an equality. This strong formulation of (GI^ε) will be the starting point for our asymptotic analysis in Section 6.6.

6.3.1. Regularization of ψ and rewriting the constraints

We notice that $\boldsymbol{\varphi}\in\boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$ needs to satisfy the constraint

$$\varphi^i(x)\geq 0,$$

for almost every $x \in \Omega$ and $i = 1, \dots, N$. To deal with this constraint we regularize the potential functional appearing in the Ginzburg-Landau energy which was initially given as

$$\psi(\boldsymbol{\varphi}) = \psi_0(\boldsymbol{\varphi}) + I_{\boldsymbol{G}}(\boldsymbol{\varphi}).$$

Definition 6.3.1. *For $\delta > 0$ we define the regularized potential*

$$\psi_\delta : \mathbb{R}^N \to \mathbb{R}, \quad \psi_\delta(\boldsymbol{\varphi}) = \psi_0(\boldsymbol{\varphi}) + \frac{1}{\delta}\hat{\psi}(\boldsymbol{\varphi}), \tag{6.3.1}$$

where

$$\hat{\psi}(\boldsymbol{\varphi}) \coloneqq \sum_{i=1}^{N} (\min(\varphi^i, 0))^2. \tag{6.3.2}$$

Remark 6.3.2. We see that the regularization now approximates the indicator functional $I_{\mathbb{R}^N_+}$ by the function $\frac{1}{\delta}\hat{\psi}$. For $\delta \searrow 0$ this exactly penalizes the negative parts of the components of $\boldsymbol{\varphi}$.

To deal with the remaining constraints hidden in $\boldsymbol{\mathcal{G}^m}$ namely the integral constraint $⨍_\Omega \boldsymbol{\varphi}\,\mathrm{d}x = \boldsymbol{m}$ and the sum constraint $\sum_{i=1}^N \varphi^i = 1$ a.e. in Ω, we introduce linear orthogonal projections.

Definition 6.3.3. *Let us define the linear orthogonal projections*

$$\begin{aligned} P_⨍ : L^2(\Omega;\mathbb{R}^N) &\to L^2_0(\Omega;\mathbb{R}^N), \\ \boldsymbol{u} &\mapsto \boldsymbol{u} - ⨍_\Omega \boldsymbol{u}\,\mathrm{d}x, \end{aligned} \tag{6.3.3}$$

with $L^2_0(\Omega;\mathbb{R}^N) \coloneqq \{\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^N) \,|\, ⨍_\Omega \boldsymbol{u}\,\mathrm{d}x = \boldsymbol{0}\}$ and

$$\begin{aligned} P_{T\Sigma} : L^2(\Omega;\mathbb{R}^N) &\to L^2_{T\Sigma}(\Omega;\mathbb{R}^N), \\ \boldsymbol{u} &\mapsto \boldsymbol{u} - \hat{\sum}\boldsymbol{u}, \end{aligned} \tag{6.3.4}$$

where for $\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^N)$

$$\hat{\sum}\boldsymbol{u} \coloneqq \left(\frac{1}{N}\sum_{i=1}^N u^i\right)\boldsymbol{1},$$

with $\boldsymbol{1} = (1, \dots, 1)^T \in \mathbb{R}^N$

$$L^2_{T\Sigma}(\Omega;\mathbb{R}^N) \coloneqq \left\{ \boldsymbol{u} \in L^2(\Omega;\mathbb{R}^N) \;\middle|\; \sum_{i=1}^N u^i = 0 \text{ a.e. in } \Omega \right\}.$$

To simplify the notation, we further define the composition $P \coloneqq P_{T\Sigma} \circ P_⨍ = P_⨍ \circ P_{T\Sigma}$.

Remark 6.3.4. Note that for the constraint $\varphi(x) \in \mathbb{R}^N_+$, we can not introduce a linear orthogonal projection as there is no vector space corresponding to this constraint. Thus, the approximation of the indicator function in Definition 6.3.1 is actually necessary.

6.3.2. Smoothness assumption and rewriting the gradient inequality

In order to obtain a suitable regularization of the gradient inequality, we need to find a way to test (GI^ε) with arbitrary functions in $H^1(\Omega;\mathbb{R}^N)$ and not only in $H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)$. Therefore sufficient smoothness of eigenfunctions has to be assumed.

We now fix a parameter $\varepsilon>0$ as well as a solution $\boldsymbol{\varphi}_\varepsilon\in\boldsymbol{\mathcal{G}}^{\boldsymbol{m}}\subset H^1(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)$ of (GI^ε). For a cleaner presentation we omit the superscript ε in the eigenvalues and eigenfunctions. A priori, the term

$$\langle\mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right),\mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right)\rangle_{\mathbb{C}'(\boldsymbol{\varphi})\boldsymbol{\eta}}=\int_\Omega\left[\mathbb{C}'(\boldsymbol{\varphi})\boldsymbol{\eta}\right]\mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right):\mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right)\,\mathrm{d}x$$

is well defined only for $\boldsymbol{\eta}\in L^\infty(\Omega;\mathbb{R}^N)$ as the expression $\mathcal{E}(\boldsymbol{w}_{n_r}):\mathcal{E}(\boldsymbol{w}_{n_r})$ merely belongs to $L^1(\Omega)$. However, in order to consider a suitable regularized problem associated to (GI^ε), we need this term to be an element in $L^2(\Omega)$. This implies that we need to require $\boldsymbol{w}_{n_r}\in W^{1,4}(\Omega)$. Therefore we now make the following crucial smoothness assumption which shall hold for the rest of this chapter.

(R) For $r=1,\dots,l$ let the eigenfunctions $\boldsymbol{w}_{n_r}$ involved in (GI^ε) be elements in the space $W^{1,4}(\Omega;\mathbb{R}^d)$.

Remark 6.3.5. Note that there exists regularity theory for the equations of linear and nonlinear elasticity, see, e.g. [113,144]. However, due to the fact that the coefficient $\mathbb{C}(\boldsymbol{\varphi})$ is only essentially bounded, one can only prove the existence of an (in general arbitrarily small) parameter $\iota>0$ such that

$$\mathcal{E}(\boldsymbol{w}_{n_r})\in L^{2+\iota}(\Omega). \tag{6.3.5}$$

Note that there exist counterexamples going back to De Giorgi for linear systems of elliptic PDEs (see, e.g., [27, Section 4.1]) providing *unbounded* solutions $\boldsymbol{u}\in W^{1,2}(B;\mathbb{R}^d)$ for $d\geq 3$ to a system of the form

$$\operatorname{div}(\mathcal{A}(x)D\boldsymbol{u}(x))=\boldsymbol{0}\text{ in }B\subset\mathbb{R}^d,$$

where $\mathcal{A}$ is bounded and coercive and B denotes the unit ball. In particular, in the physically relevant case $d=3$ where $W^{1,4}(\Omega;\mathbb{R}^d)\hookrightarrow C^0(\overline{\Omega};\mathbb{R}^d)$, the condition $\boldsymbol{w}_{n_r}\in W^{1,4}(\Omega;\mathbb{R}^d)$ seems to be a real assumption as unbounded eigenfunctions might exist.

In the following, let $(\cdot,\cdot)$ denote the classical scalar product on $L^2(\Omega;\mathbb{R}^N)$. Recalling the definition

$$\mathbb{C}'(\boldsymbol{\varphi})\boldsymbol{\eta}=\left(\sum_{m=1}^N\partial_m\mathbb{C}_{ijkl}(\boldsymbol{\varphi})\eta^m\right)_{i,j,k,l=1}^d$$

for $\boldsymbol{\eta}\in L^2(\Omega;\mathbb{R}^N)$, we have

$$\begin{aligned}\langle\mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right),\mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right)\rangle_{\mathbb{C}'(\boldsymbol{\varphi})\boldsymbol{\eta}}&=\int_\Omega\Big(\sum_{m=1}^N[\partial_m\mathbb{C}(\boldsymbol{\varphi})]\eta^m\Big)\mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right):\mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right)\,\mathrm{d}x\\&=\int_\Omega\sum_{m=1}^N\Big([\partial_m\mathbb{C}(\boldsymbol{\varphi})]\mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right):\mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right)\Big)\eta^m\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \sum_{m=1}^{N} \left[\left(\mathbb{C}'(\boldsymbol{\varphi}) \mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right) : \mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right) \right) \right]_m \eta^m \,\mathrm{d}x \\
&= \left(\mathbb{C}'(\boldsymbol{\varphi}) \mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right) : \mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right), \boldsymbol{\eta} \right).
\end{aligned}
$$

Note that the term in the last line is to be understood as

$$
\mathbb{C}'(\boldsymbol{\varphi}) \mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right) : \mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right) = \left([\partial_m \mathbb{C}(\boldsymbol{\varphi})] \mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right) : \mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right) \right)_{m=1}^{N} \in L^2(\Omega; \mathbb{R}^N).
$$

Thus, the projection of this term is well defined and the L^2 regularity of this object is ensured by the fact that $\mathbb{C}_{ijkl} \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R})$ and assumption (R). For later purposes, we point out that a straightforward computation reveals

$$
P_{T\Sigma}\left[\mathbb{C}'(\boldsymbol{\varphi}) \mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right) : \mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right) \right] = \left[\left(P_{T\Sigma}\left[\mathbb{C}'_{ijkl}(\boldsymbol{\varphi}) \right] \right)_{i,j,k,l=1}^{d} \right] \mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right) : \mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right),
$$

where

$$
\mathbb{C}'_{ijkl}(\boldsymbol{\varphi}) = (\partial_m \mathbb{C}_{ijkl})_{m=1}^{N} \in L^2(\Omega; \mathbb{R}^N).
$$

To have a more concise notation, we will write

$$
\langle \mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right) : \mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right) \rangle_{P_{T\Sigma}[\mathbb{C}'(\boldsymbol{\varphi})]} := P_{T\Sigma}\left[\mathbb{C}'(\boldsymbol{\varphi}) \mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right) : \mathcal{E}\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right) \right].
$$

Analogously, we use the notation

$$
\left(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}, \boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}\right)_{\rho'(\boldsymbol{\varphi})\boldsymbol{\eta}} = \left(\rho'(\boldsymbol{\varphi}) \boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}} \cdot \boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}, \boldsymbol{\eta}\right)
$$

for the density term. To reformulate the gradient inequality (GI^ε), we further define the function

$$
\begin{aligned}
\boldsymbol{f}^{\boldsymbol{\varphi}} := -\sum_{r=1}^{l} \Big\{ [\partial_{\lambda_{i_j}} \Psi](\lambda_{n_1}^{\boldsymbol{\varphi}}, \ldots, \lambda_{n_l}^{\boldsymbol{\varphi}}) \Big(\mathbb{C}'(\boldsymbol{\varphi}) \mathcal{E}(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}) : \mathcal{E}(\boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}}) \\
- \lambda_{n_r}^{\boldsymbol{\varphi}} \rho'(\boldsymbol{\varphi}) \boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}} \cdot \boldsymbol{w}_{n_r}^{\boldsymbol{\varphi}} \Big) \Big\} - \frac{\gamma}{\varepsilon} \psi_0'(\boldsymbol{\varphi}).
\end{aligned} \tag{6.3.6}
$$

From the above considerations, we infer $\boldsymbol{f}^{\boldsymbol{\varphi}} \in L^2(\Omega; \mathbb{R}^N)$. As $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}^m}$ is fixed, we write $\boldsymbol{f} = \boldsymbol{f}^{\boldsymbol{\varphi}}$ in the following. Using this notation, we directly obtain:

Proposition 6.3.6. *The gradient inequality (GI^ε) is equivalent to*

$$
\gamma\varepsilon \left(\nabla\boldsymbol{\varphi}, \nabla(\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) \right)_{L^2} \geq \left(\boldsymbol{f}, \tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi} \right)_{L^2} \quad \textit{for all } \tilde{\boldsymbol{\varphi}} \in \boldsymbol{\mathcal{G}^m}. \tag{6.3.7}
$$

6.3.3. The regularized problem and its limit

Now that we have introduced the regularized potential and suitable orthogonal projections, and have made the necessary regularity assumption (R), we can formulate a regularized problem which will approximate our initially fixed solution $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}^m}$ of (GI^ε) in order to provide the desired H^2-regularity of $\boldsymbol{\varphi}$.

Using all the previously introduced notation, we are now in a position to state the so-called regularized problem.

Definition 6.3.7. *Let*

$$\tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}} \coloneqq \left\{ \tilde{\boldsymbol{\varphi}} \in \ H^1(\Omega;\mathbb{R}^N) \,\middle|\, \fint_\Omega \tilde{\boldsymbol{\varphi}}\, dx = \boldsymbol{m} \text{ and } \sum_{i=1}^N \tilde{\varphi}^i = 1 \text{ a.e. in } \Omega \right\}. \tag{6.3.8}$$

We say that $\boldsymbol{\varphi}^\delta \in \tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}$ is a solution to the regularized problem if it solves

$$\gamma\varepsilon \left(\nabla\boldsymbol{\varphi}^\delta, \nabla\boldsymbol{\eta}\right) + \frac{\gamma}{\delta\varepsilon}\left(P[\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)], \boldsymbol{\eta}\right) = (P\boldsymbol{f}, \boldsymbol{\eta}) \quad \text{for all } \boldsymbol{\eta} \in H^1(\Omega;\mathbb{R}^N). \tag{RE}$$

Before proving the existence of a solution to (RE), we recall some properties proven in [35] that will be important for the upcoming analysis.

Proposition 6.3.8. *Let $\hat{\psi}$ be as defined in* (6.3.2)*. Then the following properties hold true.*

(a) *The weak derivative fulfills*

$$\nabla\hat{\psi} = \hat{\boldsymbol{\phi}} \tag{6.3.9}$$

where, for $\boldsymbol{\xi} \in \mathbb{R}^N$, $\hat{\phi}^i(\boldsymbol{\xi}) \coloneqq \hat{\phi}^i(\xi^i) \coloneqq 2\left[\xi^i\right]_-$ with $[s]_- \coloneqq \min(s,0)$ for all $s \in \mathbb{R}$.

(b) ***Monotonicity:*** *$\hat{\boldsymbol{\phi}}$ is non-decreasing in each component, i.e.,*

$$0 \le \left(\hat{\phi}^i(r) - \hat{\phi}^i(s)\right)(r-s) \tag{6.3.10}$$

for all $r, s \in \mathbb{R}$ and $i = 1, \dots, N$.

(c) ***Convexity:*** *$\hat{\psi}$ is convex, i.e.,*

$$(\boldsymbol{\xi} - \boldsymbol{\eta}) \cdot \hat{\boldsymbol{\phi}}(\boldsymbol{\eta}) \le \hat{\psi}(\boldsymbol{\xi}) - \hat{\psi}(\boldsymbol{\eta}) \tag{6.3.11}$$

for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N$.

Now, we are in a position to prove the existence result for the regularized problem. In order to show H^2-regularity of the solution we need the following regularity assumption on the boundary of the design domain which shall hold for the rest of the chapter.

(D) In addition to the assumptions on the Lipschitz design domain Ω in Section 2.1.10, we assume that Ω has at least one of the following properties:

(i) The boundary $\partial\Omega$ is of class $C^{1,1}$.

(ii) Ω is convex.

The well-posedness result for the regularized problem (RE) reads as follows.

Lemma 6.3.9. *For $\delta > 0$ there exists a unique solution $\boldsymbol{\varphi}^\delta \in \tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}} \subset H^1(\Omega;\mathbb{R}^N)$ of (RE). The solution possesses the regularity $\boldsymbol{\varphi}^\delta \in H^2(\Omega;\mathbb{R}^N)$ and it holds*

$$\begin{aligned} -\Delta\boldsymbol{\varphi}^\delta &= -\frac{1}{\delta\varepsilon^2}P[\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)] + \frac{1}{\gamma\varepsilon}P\boldsymbol{f} && \text{a.e. in } \Omega \\ \nabla\boldsymbol{\varphi}^\delta\,\boldsymbol{n} &= \boldsymbol{0} && \text{a.e. on } \partial\Omega, \end{aligned} \tag{PRE}$$

Proof. First of all, we want to show that there exists at most one solution to (RE). To this end, we assume that there are two solutions $\boldsymbol{\varphi}^{\delta,1}, \boldsymbol{\varphi}^{\delta,2} \in \tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}$. Then, by subtracting the corresponding equations, we obtain

$$\gamma\varepsilon \left(\nabla[\boldsymbol{\varphi}^{\delta,1} - \boldsymbol{\varphi}^{\delta,2}], \nabla\boldsymbol{\eta}\right) + \frac{\gamma}{\varepsilon\delta}\left(P[\hat{\phi}(\boldsymbol{\varphi}^{\delta,1}) - \hat{\phi}(\boldsymbol{\varphi}^{\delta,2})], \boldsymbol{\eta}\right) = 0$$

for all $\boldsymbol{\eta} \in H^1(\Omega;\mathbb{R}^N)$. Testing with $\boldsymbol{\varphi}^{\delta,1} - \boldsymbol{\varphi}^{\delta,2} \in L^2_{T\Sigma}(\Omega;\mathbb{R}^N) \cap L^2_0(\Omega;\mathbb{R}^N)$, we can drop the projection P in the second term. Using the monotonicity property (6.3.10), we infer

$$\gamma\varepsilon \left(\nabla[\boldsymbol{\varphi}^{\delta,1} - \boldsymbol{\varphi}^{\delta,2}], \nabla[\boldsymbol{\varphi}^{\delta,1} - \boldsymbol{\varphi}^{\delta,2}]\right) \leq 0.$$

This yields $\boldsymbol{\varphi}^{\delta,1} = \boldsymbol{\varphi}^{\delta,2} \in \tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}$ as these functions have identical mean value.

In order to prove the existence of a solution, we consider a suitable minimization problem. Therefore, we define the functional

$$I_\delta(\boldsymbol{\xi}) \coloneqq \frac{\gamma\varepsilon}{2}\int_\Omega |\nabla\boldsymbol{\xi}|^2 \,\mathrm{d}x + \frac{\gamma}{\varepsilon\delta}\int_\Omega \hat{\psi}(\boldsymbol{\xi})\,\mathrm{d}x - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\xi}\,\mathrm{d}x \tag{6.3.12}$$

for all $\boldsymbol{\xi} \in H^1(\Omega;\mathbb{R}^N)$. If we can now show that there exists a $\boldsymbol{\varphi}^\delta \in \tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}$ that solves the minimization problem

$$\min_{\boldsymbol{\xi}\in\tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}} I_\delta(\boldsymbol{\xi}), \tag{6.3.13}$$

the existence result is proven since then the Gâteaux derivative of $I'_\delta(\boldsymbol{\varphi}^\delta)$, which is given by

$$I'_\delta(\boldsymbol{\varphi}^\delta)\boldsymbol{\eta} = \gamma\varepsilon\left(\nabla\boldsymbol{\varphi}^\delta, \nabla\boldsymbol{\eta}\right) + \frac{\gamma}{\varepsilon\delta}\left(\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta), \boldsymbol{\eta}\right) - (\boldsymbol{f}, \boldsymbol{\eta}), \tag{6.3.14}$$

for all directions $\boldsymbol{\eta} \in H^1(\Omega;\mathbb{R}^N) \cap L^2_{T\Sigma}(\Omega;\mathbb{R}^N) \cap L^2_0(\Omega;\mathbb{R}^N)$, vanishes. By applying the projections $P_{T\Sigma}$ and $P_\int$ to any $\boldsymbol{\eta} \in H^1(\Omega;\mathbb{R}^N)$ and then switching them to the other component in the L^2 scalar product, it follows that solving (6.3.13) is equivalent to solving (RE).

Note that there is no need to project the gradient term. This is justified as follows. By construction, we have

$$P_{T\Sigma}\boldsymbol{\eta} = \boldsymbol{\eta} - \left[\frac{1}{N}\sum_{k=1}^N \eta^k\right]\mathbf{1}.$$

On the other hand, we compute

$$\nabla\left[\sum_{k=1}^N \eta^k\mathbf{1}\right] = \left(\left[\sum_{k=1}^N \partial_1\eta^k\right]\mathbf{1}, \ldots, \left[\sum_{k=1}^N \partial_d\eta^k\right]\mathbf{1}\right),$$

and therefore, the entries in each column are identical. Now, we compute

$$\nabla\boldsymbol{\varphi}^\delta : \nabla\left[\sum_{k=1}^N \eta^k\mathbf{1}\right] = \sum_{i=1}^N\left\{\left[\sum_{k=1}^N \partial_i\eta^k\right]\sum_{j=1}^N \partial_i\varphi^j_\delta\right\}. \tag{6.3.15}$$

We see that this term vanishes as by construction, as $\sum_{i=1}^{N} \varphi^{\delta,i} = 1$ a.e. in Ω because $\boldsymbol{\varphi}^\delta \in \tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}$. In other words, $\partial_i \boldsymbol{\varphi}^\delta \in L^2_{T\Sigma}(\Omega;\mathbb{R}^N)$.

As the gradient term is invariant under addition of constants we can also drop the projection $P_\int$.

It remains to show that there exists a minimizer of (6.3.13). By construction, $\hat{\psi} \geq 0$. Furthermore, using Young's inequality we find a constant $C > 0$ such that

$$\int_\Omega |\nabla \boldsymbol{\xi}|^2 \,\mathrm{d}x + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\xi} \,\mathrm{d}x \geq -C \quad \text{for all } \boldsymbol{\xi} \in \tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}. \tag{6.3.16}$$

This is obtained by absorbing the quantity $\|\boldsymbol{\xi}\|_{L^2}^2$ with the term $\|\nabla \boldsymbol{\xi}\|_{L^2}^2$ which controls the whole $H^1(\Omega;\mathbb{R}^N)$-norm as all $\boldsymbol{\xi} \in \tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}$ have a fixed mean value. Hence, I_δ is bounded from below on $\tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}$ and the infimum exists. We thus find a minimizing sequence $(\boldsymbol{\varphi}_k^\delta)_{k\in\mathbb{N}} \subset \tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}$ such that

$$\lim_{k\to\infty} I_\delta(\boldsymbol{\varphi}_{\delta,k}) = \inf_{\boldsymbol{\varphi}\in\tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}} I_\delta(\boldsymbol{\varphi}).$$

In particular, $\|\boldsymbol{\varphi}_k^\delta\|_{H^1(\Omega;\mathbb{R}^N)}$ remains bounded and thus, there exists a $\boldsymbol{\varphi}^\delta \in H^1(\Omega;\mathbb{R}^N)$ and a non-relabeled subsequence with the properties

$$\begin{aligned}
\boldsymbol{\varphi}_k^\delta &\rightharpoonup \boldsymbol{\varphi}^\delta \quad \text{in } H^1(\Omega;\mathbb{R}^N),\\
\boldsymbol{\varphi}_k^\delta &\to \boldsymbol{\varphi}^\delta \quad \text{in } L^2(\Omega;\mathbb{R}^N),\\
\boldsymbol{\varphi}_k^\delta &\to \boldsymbol{\varphi}^\delta \quad \text{a.e. in } \Omega.
\end{aligned}$$

From this convergence we deduce $\boldsymbol{\varphi}^\delta \in \tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}$. Noticing that $\hat{\psi}(\boldsymbol{\varphi}_k^\delta) \leq |\boldsymbol{\varphi}_{\delta,k}|^2$ a.e. in Ω and that $\hat{\psi}$ is continuous, we apply the generalized majorized convergence theorem of Lebesgue, see Theorem 2.2.4, to deduce

$$\int_\Omega \hat{\psi}(\boldsymbol{\varphi}_k^\delta) \,\mathrm{d}x \to \int_\Omega \hat{\psi}(\boldsymbol{\varphi}^\delta) \,\mathrm{d}x,$$

for $k \to \infty$. The weak lower semi-continuity of norms yields

$$\int_\Omega \left|\nabla \boldsymbol{\varphi}^\delta\right|^2 \mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega \left|\nabla \boldsymbol{\varphi}_k^\delta\right|^2 \mathrm{d}x$$

and it holds

$$\lim_{k\to\infty} \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}_k^\delta \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}^\delta \,\mathrm{d}x.$$

Altogether, we thus have

$$I_\delta(\boldsymbol{\varphi}^\delta) \leq \liminf_{k\to\infty} I_\delta(\boldsymbol{\varphi}_k^\delta) = \inf_{\boldsymbol{\varphi}\in\tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}} I_\delta(\boldsymbol{\varphi}).$$

This implies that $\boldsymbol{\varphi}^\delta \in \tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}$ is a minimizer.

Now that we have shown the existence of a solution $\boldsymbol{\varphi}^\delta \in H^1(\Omega;\mathbb{R}^N)$ to (RE), it remains to prove that it possesses the desired regularity $H^2(\Omega;\mathbb{R}^N)$. Since $\boldsymbol{\varphi}^\delta$ is a weak solution of (RE), it can be interpreted as a weak solution of

$$\begin{aligned}
-\Delta \boldsymbol{\varphi}^\delta &= \boldsymbol{\mathcal{F}} \qquad \text{in } \Omega,\\
\nabla \boldsymbol{\varphi}^\delta \, \boldsymbol{n} &= \boldsymbol{0} \qquad \text{on } \partial\Omega,
\end{aligned} \tag{6.3.17}$$

with

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}(\boldsymbol{\varphi}^\delta) = -\frac{1}{\delta\varepsilon^2} P[\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)] + \frac{1}{\gamma\varepsilon} P\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^N). \tag{6.3.18}$$

Due to assumption (D), elliptic regularity theory (see e.g., [101, Theorem 2.4.2.7] in the case (D)(i) or [101, Theorem 3.2.3.1] in the case (D)(ii), respectively) yields $\boldsymbol{\varphi}^\delta \in H^2(\Omega;\mathbb{R}^N)$. In particular, using this regularity, we conclude from (RE) that $\boldsymbol{\varphi}^\delta$ satisfies (PRE). □

As we want to pass to the limit in the regularized equation, we need some uniform bounds to apply classical compactness results.

Lemma 6.3.10. *Let $\boldsymbol{\varphi}^\delta \in H^2(\Omega;\mathbb{R}^N)$ be the solution of (RE). Then there exist a constant $C > 0$ such that*

$$\|\boldsymbol{\varphi}^\delta\|_{H^2(\Omega;\mathbb{R}^N)} \leq C, \tag{6.3.19}$$

$$\|[\boldsymbol{\varphi}^\delta]_-\|_{L^2(\Omega;\mathbb{R}^N)} \leq C\delta^{\frac{1}{2}}, \tag{6.3.20}$$

$$\frac{1}{\delta}\|P[\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)]\|_{L^2(\Omega;\mathbb{R}^N)} \leq C, \tag{6.3.21}$$

for all $\delta > 0$.

Proof. By the previous lemma, we know that $\boldsymbol{\varphi}^\delta$ minimizes I_δ (see (6.3.12)) over $\tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}$ (see (6.3.8)). Thus, we have

$$I_\delta(\boldsymbol{\varphi}^\delta) \leq I_\delta(\boldsymbol{\xi}), \quad \text{for all } \boldsymbol{\xi} \in \tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}.$$

If we now choose any $\boldsymbol{\xi} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}} \subset \tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}$, we know that it is additionally component-wise non-negative and therefore $\hat{\psi}(\boldsymbol{\xi}) = 0$ a.e. in Ω. In view of definition (6.3.12), this yields

$$\begin{aligned} &\frac{\gamma\varepsilon}{2}\int_\Omega |\nabla\boldsymbol{\varphi}^\delta|^2 \,\mathrm{d}x + \frac{\gamma}{\varepsilon\delta}\int_\Omega \hat{\psi}(\boldsymbol{\varphi}^\delta) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}^\delta \,\mathrm{d}x \\ &\quad\leq \frac{\gamma\varepsilon}{2}\int_\Omega |\nabla\boldsymbol{\xi}|^2 \,\mathrm{d}x - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\xi}\,\mathrm{d}x \\ &\quad\leq C, \end{aligned} \tag{6.3.22}$$

where $C > 0$ is a constant independent of δ. Recalling the absorption trick (6.3.16), we obtain

$$\|\boldsymbol{\varphi}^\delta\|_{H^1(\Omega;\mathbb{R}^N)} \leq C, \tag{6.3.23}$$

which will be needed in the end of the proof. Furthermore, using the definition of $\hat{\psi}$ (see (6.3.2)), we deduce that

$$\sum_{i=1}^N \|[\varphi_\delta^i]_-\|_{L^2}^2 \leq C\delta,$$

which directly leads to (6.3.20).

We notice that $\frac{1}{\delta}\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)$ is weakly differentiable (cf. [99, Lemma 7.6]) and belongs to $H^1(\Omega;\mathbb{R}^N)$. In order to prove (6.3.21), we test (RE) with $\boldsymbol{\eta} = \frac{1}{\delta}\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)$. We obtain

$$\begin{aligned}\frac{\gamma}{\varepsilon\delta}\left(\nabla\boldsymbol{\varphi}^\delta, \nabla\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)\right) + \frac{\gamma}{\delta^2\varepsilon}\int_\Omega \left|P[\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)]\right|^2 \mathrm{d}x \\ = \frac{1}{\delta}\int_\Omega \boldsymbol{f}\cdot P[\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)]\,\mathrm{d}x\end{aligned} \tag{6.3.24}$$

Applying [99, Lemma 7.6] to $\hat{\boldsymbol{\phi}}$, we further deduce

$$\frac{\gamma}{\varepsilon\delta}\left(\nabla\boldsymbol{\varphi}^\delta, \nabla\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)\right) \geq 0$$

since for a.e. x in Ω either $\nabla\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)(x) = \mathbf{0}$ or $\nabla\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)(x) = \nabla\boldsymbol{\varphi}^\delta(x)$. Applying Hölder's inequality in (6.3.24), we thus infer

$$\frac{\gamma}{\delta^2\varepsilon}\|P[\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)]\|_{L^2}^2 \leq \frac{C}{\delta}\|P[\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)]\|_{L^2},$$

and thus,

$$\frac{1}{\delta}\|P[\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)]\|_{L^2} \leq C.$$

As we now have bounded both the right-hand side of (PRE) and $\boldsymbol{\varphi}^\delta$ itself in $L^2(\Omega;\mathbb{R}^N)$ uniformly in δ (see (6.3.23)), we can again apply elliptic regularity theory (see [101, Theorem 2.3.1.5] or [101, Theorem 3.2.3.1]) to deduce (6.3.19). □

In order to reformulate (PRE) by means of Lagrange multipliers that are expected to converge in the weak sense, we need to get rid of the projection in (6.3.21).

Lemma 6.3.11. *There exists a constant $C > 0$ such that*

$$\frac{1}{\delta}\|\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)\|_{L^2} \leq C, \tag{6.3.25}$$

for all $\delta > 0$.

Proof. Note that the following proof works in complete analogy to [35, Theorem 2.1], but for the sake of completeness we go through the steps here.
Testing (RE) with $\boldsymbol{\varphi}^\delta \in H^2(\Omega;\mathbb{R}^N)\cap\tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}$, we obtain

$$\gamma\varepsilon\left(\nabla\boldsymbol{\varphi}^\delta, \nabla\boldsymbol{\varphi}^\delta\right) + \frac{\gamma}{\delta\varepsilon}\left(\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta), P_{\int}\boldsymbol{\varphi}^\delta\right) = \left(P\boldsymbol{f}, \boldsymbol{\varphi}^\delta\right), \tag{6.3.26}$$

where we used

$$\left(P[\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)], \boldsymbol{\varphi}^\delta\right) = \left(\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta), P_{\int}P_{T\Sigma}\boldsymbol{\varphi}^\delta\right) = \left(\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta), P_{\int}\boldsymbol{\varphi}^\delta\right),$$

as $P_{T\Sigma}\boldsymbol{\varphi}^\delta = \boldsymbol{\varphi}^\delta - \frac{1}{N}\mathbf{1}$ and $P_{\int}\mathbf{1} = \mathbf{0}$. Now taking any $\boldsymbol{\xi}\in\mathbb{R}^N$ and using the defintion of $P_{\int}$, we infer from (6.3.26) that

$$\begin{aligned}\frac{\gamma}{\delta\varepsilon}\left(\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta), \boldsymbol{\xi} - \fint_\Omega \boldsymbol{\varphi}^\delta\,\mathrm{d}x\right) &= \frac{\gamma}{\delta\varepsilon}\left(\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta), \boldsymbol{\xi} - \boldsymbol{\varphi}^\delta\right) - \gamma\varepsilon\left(\nabla\boldsymbol{\varphi}^\delta, \nabla\boldsymbol{\varphi}^\delta\right) + \left(P\boldsymbol{f}, \boldsymbol{\varphi}^\delta\right) \\ &\leq \frac{\gamma}{\delta\varepsilon}\left(\hat{\psi}(\boldsymbol{\xi}) - \hat{\psi}(\boldsymbol{\varphi}^\delta), 1\right) + C.\end{aligned} \tag{6.3.27}$$

For the last inequality, we used the convexity of $\hat{\psi}$ (see (6.3.11)). The constant $C > 0$ can be chosen uniformly in $\delta > 0$ due to the a priori estimate (6.3.19).

Choosing $\beta > 0$ with $\beta\mathbf{1} < \boldsymbol{m} < \mathbf{1}$ component-wise (which is possible as $\boldsymbol{m} \in (0,1)$ component-wise), we define

$$\boldsymbol{\xi} \coloneqq \fint_\Omega \boldsymbol{\varphi}^\delta \,\mathrm{d}x \pm \beta \boldsymbol{e}_i = \boldsymbol{m} \pm \beta \boldsymbol{e}_i,$$

where $\boldsymbol{e}_i \in \mathbb{R}^N$ denotes the i-th standard basis vector. Thus, by construction, $\boldsymbol{\xi} \geq 0$ component-wise. This directly implies $\hat{\psi}(\boldsymbol{\xi}) = 0$. Invoking $\hat{\psi} \geq 0$, we infer from (6.3.27) that

$$\frac{\gamma}{\delta\varepsilon}\left(\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta), \pm\beta\boldsymbol{e}_i\right) \leq C.$$

Consequently, we have

$$\frac{1}{\delta}\left|\fint_\Omega \hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)\,\mathrm{d}x\right| \leq C$$

for all $\delta > 0$. Therefore, using the definition of $P = P_{T\Sigma}P_{\int}$ from Definition 6.3.3, we deduce from (6.3.21) that

$$\frac{1}{\delta}\|P_{T\Sigma}\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)\|_{L^2} \leq C. \tag{6.3.28}$$

If we can finally find a uniform bound for $\frac{1}{\delta}\hat{\sum}\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)$ in $L^2(\Omega)$, we infer

$$\frac{1}{\delta}\|\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)\|_{L^2} \leq \frac{1}{\delta}\|P_{T\Sigma}\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)\|_{L^2} + \frac{1}{\delta}\|\hat{\sum}\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)\|_{L^2} \leq C \tag{6.3.29}$$

for all $\delta > 0$. To derive such a bound, we proceed as follows. Since $\boldsymbol{\varphi}^\delta \in \tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}$, we know that $\sum_{i=1}^N \varphi_\delta^i(x) = 1$ for almost all $x \in \Omega$. Thus, for almost every $x \in \Omega$ there must be a $j(x) \in \{1,\dots,N\}$ such that $[\boldsymbol{\varphi}^\delta(x)]^{j(x)} \geq 0$ and hence $[\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta(x))]^{j(x)} = 0$. Altogether, we obtain

$$[P_{T\Sigma}\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta(x))]^{j(x)} = [\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta(x))]^{j(x)} - \frac{1}{N}\sum_{i=1}^N \hat{\phi}^i(\boldsymbol{\varphi}^\delta(x)) = -\frac{1}{N}\sum_{i=1}^N \hat{\phi}^i(\boldsymbol{\varphi}^\delta(x)).$$

Recalling $\hat{\sum}\boldsymbol{\xi} = \frac{1}{N}\left(\sum_{i=1}^N \xi^i\right)\mathbf{1}$, we obtain from (6.3.28) that

$$\frac{1}{\delta}\|\hat{\sum}\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)\|_{L^2} \leq C.$$

Thus, (6.3.29) holds true and therefore, the claim is proven. □

Now, we introduce suitable Lagrange multipliers and pass to the limit in the the regularized equation.

Theorem 6.3.12. *The initially chosen solution $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$ of (GI^ε) possesses the regularity $\boldsymbol{\varphi} \in H^2(\Omega;\mathbb{R}^N)$. Furthermore, there are Lagrange multipliers $\boldsymbol{\Lambda}, \boldsymbol{\mu} \in L^2(\Omega;\mathbb{R}^N)$ and $\boldsymbol{\vartheta} \in \mathbb{R}^N$ such that*

$$\begin{aligned} -\gamma\varepsilon\Delta\boldsymbol{\varphi} &= \frac{1}{\varepsilon}(\boldsymbol{\Lambda} + \boldsymbol{\vartheta} + \boldsymbol{\mu}) + P_{T\Sigma}\boldsymbol{f}^{\boldsymbol{\varphi}} && \text{a.e. in } \Omega, \\ \nabla\boldsymbol{\varphi}\boldsymbol{n} &= \mathbf{0} && \text{on } \partial\Omega. \end{aligned} \tag{GS^ε}$$

with

$$\boldsymbol{\Lambda} = \Lambda \mathbf{1} \quad \textit{for a suitable } \Lambda \in L^2(\Omega), \tag{6.3.30}$$

$$\mu^i \geq 0 \textit{ and } \mu^i \varphi^i = 0 \quad \textit{a.e. in } \Omega \textit{ for all } i \in \{1, \dots, N\}, \tag{6.3.31}$$

$$\sum_{i=1}^{N} \vartheta^i = 0. \tag{6.3.32}$$

Proof. From (6.3.19) we deduce the existence of a function $\overline{\boldsymbol{\varphi}} \in H^2(\Omega; \mathbb{R}^N)$ such that

$$\begin{aligned} \boldsymbol{\varphi}^\delta &\rightharpoonup \overline{\boldsymbol{\varphi}} && \text{in } H^2(\Omega; \mathbb{R}^N), \\ \boldsymbol{\varphi}^\delta &\to \overline{\boldsymbol{\varphi}} && \text{in } H^1(\Omega; \mathbb{R}^N), \\ \boldsymbol{\varphi}^\delta &\to \overline{\boldsymbol{\varphi}} && \text{a.e. in } \Omega, \\ \hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta) &\to 0 && \text{in } L^2(\Omega; \mathbb{R}^N), \end{aligned} \tag{6.3.33}$$

as $\delta \to 0$ along a non-relabeled subsequence. This directly implies that $\overline{\boldsymbol{\varphi}}(x) \in \mathbb{R}^N_+$ for almost all $x \in \Omega$. Hence, since $\boldsymbol{\varphi}^\delta \in \tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{m}}$, we know that $\overline{\boldsymbol{\varphi}} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$.

Recalling the definition of $\boldsymbol{f} = \boldsymbol{f}^{\boldsymbol{\varphi}}$ in (6.3.6), we now define the Lagrange multipliers of the regularized problem as

$$\begin{aligned} \boldsymbol{\Lambda}^\delta &\coloneqq \hat{\sum} \left(\frac{\gamma}{\delta} \hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta) \right), \\ \boldsymbol{\vartheta}^\delta &\coloneqq \fint_\Omega P_{T\Sigma} \left(\frac{\gamma}{\delta} \hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta) - \varepsilon \boldsymbol{f} \right) \mathrm{d}x, \\ \boldsymbol{\mu}^\delta &\coloneqq -\frac{\gamma}{\delta} \hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta). \end{aligned} \tag{6.3.34}$$

The reason why we do not reformulate the projection term $P_{T\Sigma}\boldsymbol{f}$ by means of a Lagrange multiplier is that this is a term depending on x, which will produce terms of order $\mathcal{O}(\frac{1}{\varepsilon^2})$ when we consider the inner expansions in Section 6.6 due to the involved derivative of eigenfunctions.
Recalling Definition 6.3.3, we have

$$\boldsymbol{\Lambda}^\delta + \boldsymbol{\vartheta}^\delta + \boldsymbol{\mu}^\delta = -\frac{\gamma}{\delta} P_{T\Sigma}[\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta)] - \varepsilon \fint_\Omega P_{T\Sigma} \boldsymbol{f} \, \mathrm{d}x.$$

Hence, we can write (RE) as

$$\gamma\varepsilon \left(\nabla \boldsymbol{\varphi}^\delta, \nabla \boldsymbol{\eta} \right) - \frac{1}{\varepsilon} \left(\boldsymbol{\Lambda}^\delta + \boldsymbol{\vartheta}^\delta + \boldsymbol{\mu}^\delta, \boldsymbol{\eta} \right) = (P_{T\Sigma}\boldsymbol{f}, \boldsymbol{\eta}) \quad \text{for all } \boldsymbol{\eta} \in H^1(\Omega; \mathbb{R}^N). \tag{6.3.35}$$

We point out that the Lagrange multipliers are constructed in such a way that the factor $\frac{1}{\varepsilon}$ corresponding to the scaling of the original potential ψ is still present. This will be important in the next sections for the sharp-interface asymptotics.

We know from Lemma 6.3.11 that $\boldsymbol{\Lambda}^\delta, \boldsymbol{\mu}^\delta \in L^2(\Omega; \mathbb{R}^N)$ and $\boldsymbol{\vartheta}^\delta \in \mathbb{R}^N$ are bounded uniformly in δ. Hence, we find a subsequence and $\boldsymbol{\Lambda}, \boldsymbol{\mu} \in L^2(\Omega; \mathbb{R}^N)$ and $\boldsymbol{\vartheta} \in \mathbb{R}^N$ such that

$$\begin{aligned} \boldsymbol{\Lambda}^\delta &\rightharpoonup \boldsymbol{\Lambda} && \text{in } L^2(\Omega; \mathbb{R}^N), \\ \boldsymbol{\vartheta}^\delta &\to \boldsymbol{\vartheta} && \text{in } \mathbb{R}^N, \\ \boldsymbol{\mu}^\delta &\rightharpoonup \boldsymbol{\mu} && \text{in } L^2(\Omega; \mathbb{R}^N), \end{aligned} \tag{6.3.36}$$

for $\delta \to 0$. We additionally know from the definition of $\hat{\boldsymbol{\phi}}$ in (6.3.9) that $\boldsymbol{\mu} \geq 0$ component-wise as weak convergence in $L^2(\Omega;\mathbb{R}^N)$ preserves non-negativity. Furthermore from the construction in (6.3.34) we directly deduce (6.3.30) and (6.3.32).
Passing to the limit in (6.3.35), we infer

$$\gamma\varepsilon\left(\nabla\overline{\boldsymbol{\varphi}},\nabla\boldsymbol{\eta}\right)-\frac{1}{\varepsilon}\left(\boldsymbol{\Lambda}+\boldsymbol{\vartheta}+\boldsymbol{\mu},\boldsymbol{\eta}\right)=\left(P_{T\Sigma}\boldsymbol{f}^{\boldsymbol{\varphi}},\boldsymbol{\eta}\right),\quad \text{for all } \boldsymbol{\eta}\in H^1(\Omega;\mathbb{R}^N). \tag{6.3.37}$$

Thus, the regularity $\overline{\boldsymbol{\varphi}} \in H^2(\Omega;\mathbb{R}^N)$ and partial integration yield the equation

$$-\gamma\varepsilon\Delta\overline{\boldsymbol{\varphi}}=\frac{1}{\varepsilon}(\boldsymbol{\Lambda}+\boldsymbol{\vartheta}+\boldsymbol{\mu})+P_{T\Sigma}\boldsymbol{f}^{\boldsymbol{\varphi}}\qquad\text{a.e. in }\Omega. \tag{6.3.38}$$

If we can now show that for our initially fixed solution $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$ of (GI^ε) it holds $\boldsymbol{\varphi} = \overline{\boldsymbol{\varphi}}$, the proof is complete.

Let us consider the test function $\boldsymbol{\eta} := \overline{\boldsymbol{\varphi}} - \boldsymbol{\varphi} \in H^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$. First of all, we look at the Lagrange multipliers. Due to (6.3.30), we have $(\boldsymbol{\Lambda}^\delta, \boldsymbol{\eta}) = 0$, as $\sum_{i=1}^N \eta^i = 0$ because of $\overline{\boldsymbol{\varphi}}, \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$. In view of (6.3.32) we know that $(\boldsymbol{\vartheta}, \boldsymbol{\eta}) = 0$, because by construction $\int_\Omega \boldsymbol{\eta}\,\mathrm{d}x = 0$.

As already mentioned, we have $\boldsymbol{\mu}^\delta \geq 0$. Hence, using the monotonicity (6.3.10), we infer

$$\left(\boldsymbol{\mu}^\delta, \boldsymbol{\varphi}^\delta\right) = -\frac{1}{\delta}\left(\hat{\boldsymbol{\phi}}(\boldsymbol{\varphi}^\delta), \boldsymbol{\varphi}^\delta\right) \leq 0.$$

Using the convergences (6.3.33) and (6.3.36), we deduce $(\boldsymbol{\mu}, \overline{\boldsymbol{\varphi}}) \leq 0$. Recalling $\boldsymbol{\mu} \geq 0$ and that $\overline{\boldsymbol{\varphi}} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$ is component-wise non-negative, we already infer $(\boldsymbol{\mu}, \overline{\boldsymbol{\varphi}}) = 0$.

As also $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$ and $\boldsymbol{\varphi}$ is component-wise non-negative, we have $(\boldsymbol{\mu}, \boldsymbol{\varphi}) \geq 0$. Combining these results and testing (6.3.37) with our particular choice $\boldsymbol{\eta} = \overline{\boldsymbol{\varphi}} - \boldsymbol{\varphi}$, we get

$$\gamma\varepsilon\left(\nabla\overline{\boldsymbol{\varphi}}, \nabla[\overline{\boldsymbol{\varphi}} - \boldsymbol{\varphi}]\right) = -\frac{1}{\varepsilon}(\boldsymbol{\mu}, \boldsymbol{\varphi}) + (P_{T\Sigma}\boldsymbol{f}, \overline{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) \leq (P_{T\Sigma}\boldsymbol{f}, \overline{\boldsymbol{\varphi}} - \boldsymbol{\varphi}).$$

Considering on the other hand the gradient inequality (6.3.7) tested with $\tilde{\boldsymbol{\varphi}} = \overline{\boldsymbol{\varphi}} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$, we have

$$\gamma\varepsilon\left(\nabla\boldsymbol{\varphi}, \nabla[\overline{\boldsymbol{\varphi}} - \boldsymbol{\varphi}]\right) \geq (\boldsymbol{f}, \overline{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) = (P_{T\Sigma}\boldsymbol{f}, \overline{\boldsymbol{\varphi}} - \boldsymbol{\varphi}).$$

Hence, by subtracting both inequalities, we infer

$$\gamma\varepsilon\left(\nabla[\overline{\boldsymbol{\varphi}} - \boldsymbol{\varphi}], \nabla[\overline{\boldsymbol{\varphi}} - \boldsymbol{\varphi}]\right) \leq 0.$$

By $\int_\Omega \overline{\boldsymbol{\varphi}} - \boldsymbol{\varphi}\,\mathrm{d}x = 0$ the Poincaré inequality gives us the desired identity $\boldsymbol{\varphi} = \overline{\boldsymbol{\varphi}} \in H^2(\Omega;\mathbb{R}^N)$.
From the previous reasoning we know

$$\sum_{i=1}^N \int_\Omega \mu^i\varphi^i\,\mathrm{d}x = (\boldsymbol{\mu}, \boldsymbol{\varphi})_{L^2} = 0,$$

Furthermore, we know that $\boldsymbol{\mu}, \boldsymbol{\varphi} \geq 0$ component-wise and thus, each summand in above equality has to be identical to 0, providing us with (6.3.31). □

In the following, we use the above knowledge to show that our asymptotic expansions will produce a state equation and a gradient equality in the sharp-interface limit.

6.4. Asymptotic expansions

As mentioned above, we will now perform the procedure of sharp-interface asymptotics. Therefore, we start by analyzing outer and inner expansions approximating the quantities involved in our problem. The outer expansions are used to approximate these quantities in regions far away from the interfacial layers. They will be used to derive the state equation in the sharp-interface limit. The inner expansions are used in regions close to the interfacial layers where the phase transition takes place. They will provide boundary conditions for the equations obtained in the sharp-interface limit. As these layers are expected to scale proportionally to ε, a rescaling is needed here. By comparing the leading order equations, we will obtain jump conditions at the phase interfaces within the design domain and a sharp-interface version of the gradient equality (GS^ε).

In the following, we choose $(\boldsymbol{\varphi}_\varepsilon)_{\varepsilon>0} \subset \boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$ as a sequence of minimizers of the optimization problem $(\mathcal{P}_l^\varepsilon)$. For $r = 1, \dots, l$,

$$(\boldsymbol{w}_{n_r}^\varepsilon, \lambda_{n_r}^\varepsilon)_{\varepsilon>0} = (\boldsymbol{w}_{n_r}^{\varepsilon,\boldsymbol{\varphi}_\varepsilon}, \lambda_{n_r}^{\varepsilon,\boldsymbol{\varphi}_\varepsilon})_{\varepsilon>0} \subset H_D^1(\Omega;\mathbb{R}^d) \times \mathbb{R}$$

denotes the corresponding sequence of $L^2_{\boldsymbol{\varphi}_\varepsilon}(\Omega;\mathbb{R}^d)$ normalized eigenfunctions and eigenvalues, i.e., the solutions of the state equation (6.2.1) involved in the optimization problem $(\mathcal{P}_l^\varepsilon)$. Note that in order to have a nice depiction of all indices in the following, we will indicate the ε dependence in the exponent of each quantity.

6.4.1. Outer expansions

As in [32], we first consider the asymptotic expansion for regions far away from the interface. Therefore, we assume expansions of the form

$$\begin{aligned} \boldsymbol{\varphi}^\varepsilon(x) &= \sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{\varphi}_k(x), \\ \lambda_{n_r}^\varepsilon &= \sum_{k=0}^{\infty} \varepsilon^k \lambda_{k,n_r}, \\ \boldsymbol{w}_{n_r}^\varepsilon(x) &= \sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{w}_{k,n_r}(x), \end{aligned} \tag{6.4.1}$$

for all $x \in \Omega$. Furthermore, we demand for all $x \in \Omega$ that $\boldsymbol{\varphi}_0(x) \in \boldsymbol{G}$, $\boldsymbol{\varphi}_k(x) \in T\Sigma$, $\fint_\Omega \boldsymbol{\varphi}_0 \,\mathrm{d}x = \boldsymbol{m}$ and $\fint_\Omega \boldsymbol{\varphi}_k \,\mathrm{d}x = \boldsymbol{0}$ for $k \geq 1$, in order to be compatible with the constraints on the phase-field formulated in Section 2.1.3. As we are concerned with a formal limit process, we assume all the appearing quantities to possess a suitable regularity such that we can write the state equation (6.2.1) in its strong formulation (SE^ε).

To ensure that $\boldsymbol{\varphi}^\varepsilon$ exhibits the desired phase-field structure, we need to make the following assumption:

(E) There is a $C > 0$ such that

$$\limsup_{\varepsilon \searrow 0} E_{\mathrm{GL}}^\varepsilon(\boldsymbol{\varphi}^\varepsilon) \leq C.$$

Without this assumption, there would be no hope that for small ε the minimizers $\boldsymbol{\varphi}^\varepsilon$ of $(\mathcal{P}_l^\varepsilon)$ make transitions between pure phases only on a length-scale proportional to ε, compare in particular Proposition 2.2.25 for the two-phase case. The Γ-limit of the Ginzburg–Landau energy was studied rigorously in [23] in the multi-phase case. Note that for $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}$, we have

$$E_{\mathrm{GL}}^\varepsilon(\boldsymbol{\varphi}) = E^\varepsilon(\boldsymbol{\varphi}) = \int_\Omega \frac{\varepsilon}{2} |\nabla \boldsymbol{\varphi}|^2 + \frac{1}{\varepsilon} \psi_0(\boldsymbol{\varphi}) \,\mathrm{d}x,$$

since $\psi = \psi_0$ on $\boldsymbol{G}$, see Section 2.1.4. We further recall that ψ_0 is continuous, non-negative on $\boldsymbol{G}$, and its zeros in $\boldsymbol{\mathcal{G}}$ are exactly the unit vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_N$ (see Section 2.1.4). Hence, the theory of [23] can be applied.

Using assumption (E) and applying Fatou's lemma as in the proof of [23, Formula (2.8)], we now infer

$$0 \leq \int_\Omega \psi_0(\boldsymbol{\varphi}_0) \,\mathrm{d}x \leq \liminf_{\varepsilon \searrow 0} \int_\Omega \psi_0(\boldsymbol{\varphi}^\varepsilon) \,\mathrm{d}x \leq \liminf_{\varepsilon \searrow 0} \varepsilon E^\varepsilon(\boldsymbol{\varphi}^\varepsilon) = 0.$$

Consequently, as $\boldsymbol{\varphi}_0 \in \boldsymbol{\mathcal{G}}$

$$\int_\Omega \psi(\boldsymbol{\varphi}_0) \,\mathrm{d}x = \int_\Omega \psi_0(\boldsymbol{\varphi}_0) \,\mathrm{d}x = 0.$$

Hence, we can partition the domain as

$$\Omega = \bigcup_{i=1}^N \Omega_i \cup \mathcal{N} \quad \text{with} \quad \Omega_i \coloneqq \{\boldsymbol{\varphi}_0 = \boldsymbol{e}_i\}, \tag{6.4.2}$$

where $\mathcal{N} \subset \Omega$ is a Lebesgue null set. In general, the sets Ω_i are only of finite perimeter which is derived from the boundedness of the Ginburg–Landau energy, the inequality [23, (3.1)] and [23, Proposition 2.2]. Nevertheless, for our asymptotic analysis we assume them to be smooth enough.

With this knowledge we are in the position to derive the limit system resulting from (SE^ε) in the framework of outer expansions.

Claim 6.4.1. Recall the scaling of $\mathbb{C}$ and ρ in (6.2.2), i.e.,

$$\begin{aligned} \mathbb{C}(\boldsymbol{\varphi}) = \overline{\mathbb{C}}(\boldsymbol{\varphi}) + \tilde{\mathbb{C}}^N \varepsilon \varphi^N = \sum_{i=1}^{N-1} \mathbb{C}^i \varphi^i + \tilde{\mathbb{C}}^N \varepsilon \varphi^N, \\ \rho(\boldsymbol{\varphi}) = \overline{\rho}(\boldsymbol{\varphi}) + \tilde{\rho}^N \varepsilon \varphi^N = \sum_{i=1}^{N-1} \rho^i \varphi^i + \tilde{\rho}^N \varepsilon \varphi^N, \end{aligned} \tag{6.4.3}$$

for $\boldsymbol{\varphi} \in \boldsymbol{G}$. Then, for $r \in \{1, \dots, l\}$, we obtain that the pair $(\lambda_{0,n_r}, \boldsymbol{w}_{0,n_r})$ fulfills the eigenvalue equations in the material regions

$$\left\{ \begin{aligned} -\nabla \cdot \left[\mathbb{C}^i \mathcal{E}(\boldsymbol{w}_{0,n_r})\right] &= \lambda_{0,n_r} \rho^i \boldsymbol{w}_{0,n_r} && \text{in } \Omega_i, \\ \boldsymbol{w}_{0,n_r} &= \boldsymbol{0} && \text{on } \Gamma_D \cap \partial\Omega_i, \\ \left[\mathbb{C}^i \mathcal{E}(\boldsymbol{w}_{0,n_r})\right] \boldsymbol{n} &= \boldsymbol{0} && \text{on } \Gamma_0 \cap \partial\Omega_i, \end{aligned} \right. \tag{SE_0^i}$$

for $i = 1, \dots, N-1$. Furthermore, the normalization condition (6.2.3) is transferred to the limit eigenfunction $\boldsymbol{w}_{0,n_r}$ meaning that

$$1 = \int_\Omega \overline{\rho}(\boldsymbol{\varphi}_0) \, |\boldsymbol{w}_{0,n_r}|^2 \, \mathrm{d}x = \sum_{i=1}^{N-1} \int_{\Omega_i} \overline{\rho}(\boldsymbol{\varphi}_0) \, |\boldsymbol{w}_{0,n_r}|^2 \, \mathrm{d}x. \tag{6.4.4}$$

In particular, the eigenfunction $\boldsymbol{w}_{0,n_r}$ is non-trivial in Ω_i for at least one index $i \in \{1, \dots, N-1\}$. Thus, $\boldsymbol{w}_{0,n_r}$ cannot be a localized eigenmode as it cannot be supported only in the void region Ω_N.

Remark 6.4.2.

(a) Of course, the eigenvalue $\lambda^\varepsilon_{n_r}$ could degenerate in the limit, i.e., $\lambda_{0,n_r} = 0$. This is no contradiction to the normalization (6.4.4) because $\boldsymbol{w}_{0,n_r}$ could potentially be a non-trivial constant in each material region Ω_i. If each material region Ω_i shares a sufficiently nice part of the boundary with Γ_D, one can use Korn's inequality (see Section 2.2.2) to deduce that $\boldsymbol{w}_{0,n_r} = 0$ in each Ω_i, which would then indeed contradict (6.4.5). The inner expansions will provide us with boundary conditions that allow us to refine this statement, see Remark 6.7.1.

(b) In the case $\lambda_{0,n_r} > 0$, even though the limit eigenvalue equations (SE^i_0) hold for any $i \in \{1, \dots, N-1\}$, the eigenfunction $\boldsymbol{w}_{0,n_r}$ could potentially be non-trivial only in one particular material region Ω_i but vanish in all other material regions Ω_j with $j \in \{1, \dots, N-1\} \setminus \{i\}$. This means that a non-trivial equation might hold only in one single material region.

Let us show the above claim assuming that outer expansions of the form (6.4.1) exist. For the sake of a cleaner presentation, we will now fix the index $n_r \in \mathbb{N}$ and in the following, we omit the subscript n_r. In the spirit of formal asymptotics, we consider the state equation (SE^ε), i.e.,

$$-\nabla \cdot [\mathbb{C}(\boldsymbol{\varphi}^\varepsilon)\mathcal{E}(\boldsymbol{w}^\varepsilon)] = \lambda^\varepsilon \rho(\boldsymbol{\varphi}^\varepsilon)\boldsymbol{w}^\varepsilon \quad \text{a.e. in } \Omega,$$

and the normalization condition

$$1 = \int_\Omega \rho(\boldsymbol{\varphi}^\varepsilon) \, |\boldsymbol{w}^\varepsilon|^2 \, \mathrm{d}x \tag{6.4.5}$$

resulting from (6.2.3). Then, we plug in the asymptotic expansions (6.4.1) and consider each resulting order in ε separately.

We deduce that (6.4.5) reads to order $\mathcal{O}(1)$

$$1 = \int_\Omega \overline{\rho}(\boldsymbol{\varphi}_0) \, |\boldsymbol{w}_0|^2 \, \mathrm{d}x = \sum_{i=1}^{N-1} \int_{\Omega_i} \rho^i \, |\boldsymbol{w}_0|^2 \, \mathrm{d}x,$$

which proves (6.4.4). As a consequence, $\boldsymbol{w}_0$ has to be non-trivial in in Ω_i for at least one index $i \in \{1, \dots, N-1\}$.

Eventually, we compare the contributions of order $\mathcal{O}(1)$ in the state equation. We obtain

$$-\nabla \cdot \left[\overline{\mathbb{C}}(\boldsymbol{\varphi}_0)\mathcal{E}(\boldsymbol{w}_0)\right] = \lambda_0 \overline{\rho}(\boldsymbol{\varphi}_0)\boldsymbol{w}_0 \quad \text{a.e. in } \Omega, \tag{6.4.6}$$

which reads for each phase

$$-\nabla \cdot \left[\mathbb{C}^i \mathcal{E}(\boldsymbol{w}_0)\right] = \lambda_0 \rho^i \boldsymbol{w}_0 \quad \text{a.e. in } \Omega_i$$

for $i = 1, \dots, N-1$.

The remaining boundary conditions on the *outer* boundary Γ follow directly by plugging in the asymptotic expansion into (SE^ε). This completes our argumentation.

6.4.2. Intermezzo on spurious eigenmodes

As also mentioned in the introduction we want now to analytically justify the model that will be chosen for the numerical computations in order to avoid spurious eigenmodes. As we have seen in the previous proof, assuming outer expansions of the form (6.4.1) and a decomposition of $\mathbb{C}$ and ρ as in (6.4.3), we recover the desired limit system. Furthermore, we see that in (6.4.3) it is only important to scale the void contributions $\tilde{\mathbb{C}}^N$ and $\tilde{\rho}^N$ with *some* ε^p where $p > 0$, but the specific choice of $p > 0$ does not affect the steps of the proof. This is also why we keep this model for the analysis in the subsequent sections noting that also for all subsequent steps any scaling of the void contributions will work.
However, in numerical simulations the phenomenon of spurious eigenmodes is a serious issue, see [5, 29, 55, 136]. The problem is that if the model parameters are not suitably chosen, eigenmodes with support only in void regions can form. The associated eigenvalues are then of course nonphysical as void should not contribute to the resonance behavior of a structure. Nevertheless, in $(\mathcal{P}_l^\varepsilon)$ only a finite selection of eigenvalues enters our optimization problem, i.e., even though spurious eigenmodes might not be avoided in simulations, they do not pose a problem if the associated eigenvalues are large, because then these modes do not affect the optimization problem. Thus, the key idea as also observed in above literature is to choose the scaling in (6.4.3) in such a way that spurious eigenmodes will only produce large eigenvalues in simulations or more precisely eigenvalues λ^ε with $\lambda^\varepsilon \to \infty$ for $\varepsilon \to 0$. In other words this means that with an adequate model, spurious eigenmodes will not enter the sharp-interface limit as their eigenvalues leave the spectrum.
In order to include the possibility of spurious eigenmodes in our asymptotic expansions we have to allow also for terms of *negative* order in ε.

Claim 6.4.3. Assume the following outer asymptotic expansions

$$\begin{aligned} \boldsymbol{\varphi}^\varepsilon(x) &= \sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{\varphi}_k(x), \\ \lambda_{n_r}^\varepsilon &= \sum_{k=-m}^{\infty} \varepsilon^k \lambda_{k,n_r}, \\ \boldsymbol{w}_{n_r}^\varepsilon(x) &= \sum_{k=-m}^{\infty} \varepsilon^k \boldsymbol{w}_{k,n_r}(x), \end{aligned} \tag{6.4.7}$$

with $m \in \mathbb{N}$ arbitrary. Let $\mathbb{C}$ and ρ be given as

$$\begin{aligned} \mathbb{C}(\boldsymbol{\varphi}) &= \overline{\mathbb{C}}(\boldsymbol{\varphi}) + \tilde{\mathbb{C}}^N \varepsilon (\varphi^N)^2 = \sum_{i=1}^{N-1} \mathbb{C}^i (\varphi^i)^2 + \tilde{\mathbb{C}}^N \varepsilon (\varphi^N)^2, \\ \rho(\boldsymbol{\varphi}) &= \overline{\rho}(\boldsymbol{\varphi}) + \tilde{\rho}^N \varepsilon^2 (\varphi^N)^2 = \sum_{i=1}^{N-1} \rho^i (\varphi^i)^2 + \tilde{\rho}^N \varepsilon^2 (\varphi^N)^2, \end{aligned} \tag{6.4.8}$$

for $\boldsymbol{\varphi} \in \boldsymbol{G}$. Then, for $r \in \{1, \dots, l\}$, we obtain $\boldsymbol{w}_{k,n_r} = \mathbf{0}$ and $\lambda_{k,n_r} = 0$ for $k < -1$ and the pair $(\lambda_{-1,n_r}, \boldsymbol{w}_{-1,n_r})$ fulfills

$$\left\{ \begin{aligned} -\nabla \cdot \left[\tilde{\mathbb{C}}^N \mathcal{E}(\boldsymbol{w}_{-1,n_r}) \right] &= \lambda_{-1,n_r} \left[\tilde{\rho}^N + \overline{\rho}(\boldsymbol{\varphi}_1) \right] \boldsymbol{w}_{-1,n_r} && \text{in } \Omega_N, \\ \boldsymbol{w}_{-1,n_r} &= \mathbf{0} && \text{on } \Gamma_D \cap \partial\Omega_N, \\ \left[\tilde{\mathbb{C}}^N \mathcal{E}(\boldsymbol{w}_{-1,n_r}) \right] \boldsymbol{n} &= \mathbf{0} && \text{on } \Gamma_0 \cap \partial\Omega_N, \end{aligned} \right. \tag{SE_0^N}$$

and $w_{-1,n_r} = 0$ in $\Omega \backslash \Omega_N$.

Remark 6.4.4. The asymptotic analysis in the following argumentation is crucially based on the beautiful interplay of the *quadratic* interpolation of $\mathbb{C}$ and ρ in (6.4.8) and the scaling of the void components in (6.4.3). Note that these two features of the model proposed in this subsection are also reflected in the numerical part, which crucially depends on the quadratic interpolation on the one hand and the relatively lower scaling of the density compared to the elasticity in the void on the other hand. It is also seen in applications in the literature that a relatively lower scaling of mass compared to stiffness is an appropriate choice to deal with localized eigenmodes, see [5, 55, 136].

We now argue why Claim 6.4.3 is true. Therefore, consider the state equation (SE^ε) and the normalization (6.4.5). First of all we note that plugging in the asymptotic expansion of $\boldsymbol{\varphi}^\varepsilon$ into (6.4.8) yields

$$\begin{aligned} \mathbb{C}(\boldsymbol{\varphi}^\varepsilon) &= \overline{\mathbb{C}}(\boldsymbol{\varphi}_0) + \varepsilon \tilde{\mathbb{C}}^N (\boldsymbol{\varphi}_0^N)^2 + \varepsilon^2 \overline{\mathbb{C}}(\boldsymbol{\varphi}_1) + \mathcal{O}(\varepsilon^3) \\ \rho(\boldsymbol{\varphi}^\varepsilon) &= \overline{\rho}(\boldsymbol{\varphi}_0) + \varepsilon^2 (\overline{\rho}(\boldsymbol{\varphi}_1) + \tilde{\rho}^N (\boldsymbol{\varphi}_0^N)^2) + \mathcal{O}(\varepsilon^3). \end{aligned} \tag{6.4.9}$$

As a first step let us show that $\boldsymbol{w}_k = \mathbf{0}$ in Ω for $k = -m, -m+1, \dots, -2$. Therefore let us start to lowest order $\mathcal{O}(\varepsilon^{-2m})$ in (6.4.5), which reads as

$$0 = \int_\Omega \overline{\rho}(\boldsymbol{\varphi}_0) \, |\boldsymbol{w}_{-m}|^2 \, \mathrm{d}x. \tag{6.4.10}$$

This implies that $\boldsymbol{w}_{-m} = 0$ in $\Omega \backslash \Omega_N$ or in other words $\boldsymbol{w}_{-m}$ is localized in the void region. Now let us look at (6.4.5) to the order $\mathcal{O}(\varepsilon^{-2m+2})$ reading as

$$0 = \int_\Omega \overline{\rho}(\boldsymbol{\varphi}_0) \, |\boldsymbol{w}_{-m+1}|^2 + 2\overline{\rho}(\boldsymbol{\varphi}_0) \boldsymbol{w}_{-m} \cdot \boldsymbol{w}_{-m+2} + (\overline{\rho}(\boldsymbol{\varphi}_1) + \tilde{\rho}^N (\boldsymbol{\varphi}_0^N)^2) \, |\boldsymbol{w}_{-m}|^2 \, \mathrm{d}x, \tag{6.4.11}$$

in the light of (6.4.9). Note that here we used $-2m + 2 < 0$. As $\boldsymbol{w}_{-m}$ is localized in the void we infer

$$0 = \int_\Omega 2\overline{\rho}(\boldsymbol{\varphi}_0) \boldsymbol{w}_{-m} \cdot \boldsymbol{w}_{-m+2} \, \mathrm{d}x.$$

Thus, due to the non-negativity of the first summand in (6.4.11) we deduce

$$0 = \int_\Omega (\overline{\rho}(\boldsymbol{\varphi}_1) + \tilde{\rho}^N(\boldsymbol{\varphi}_0^N)^2)\,|\boldsymbol{w}_{-m}|^2 \,\mathrm{d}x. \tag{6.4.12}$$

In the light of (6.4.8) we have $\overline{\rho}(\boldsymbol{\varphi}_1) \geq 0$. Furthermore $\varphi_0 = \boldsymbol{e}_N$ in Ω_N and thus we deduce

$$0 = \int_{\Omega_N} \tilde{\rho}^N\,|\boldsymbol{w}_{-m}|^2 \,\mathrm{d}x. \tag{6.4.13}$$

Hence, due to the positivity of $\tilde{\rho}^N$ we infer $\boldsymbol{w}_{-m} = \boldsymbol{0}$ in Ω. These steps can now be repeated until the critical order $\mathcal{O}(1)$ is reached, because up to this order the normalization equation (6.4.5) possesses a trivial left hand side. Thus, this shows $\boldsymbol{w}_k = 0$ for $k = -m, -m+1, \ldots, -2$. We additionally deduce as in (6.4.10) that $\boldsymbol{w}_{-1} = 0$ in $\Omega\backslash\Omega_N$.

With this knowledge we are in the position to show $\lambda_k = 0$ for $k = -m, -m+1, \ldots, -2$. Therefore let us consider the energy associated with (SE^ε), i.e.,

$$\lambda^\varepsilon = \int_\Omega \mathbb{C}(\boldsymbol{\varphi}^\varepsilon)\mathcal{E}(\boldsymbol{w}^\varepsilon) : \mathcal{E}(\boldsymbol{w}^\varepsilon)\,\mathrm{d}x. \tag{6.4.14}$$

Due to the fact that $\boldsymbol{w}_k = \boldsymbol{0}$ in Ω for $k = -m, -m+1, \ldots, -2$ and $\boldsymbol{w}_{-1} = 0$ in $\Omega\backslash\Omega_N$ we deduce that the right hand side is of leading order $\mathcal{O}(\varepsilon^{-1})$, more precisely

$$\lambda_{-1} = \int_{\Omega_N} \tilde{\mathbb{C}}^N\mathcal{E}(\boldsymbol{w}_{-1})\mathcal{E}(\boldsymbol{w}_{-1})\,\mathrm{d}x,$$

meaning that $\lambda_k = 0$ for $k = -m, -m+1, \ldots, -2$.

Now it is our task to show that $(\lambda_{-1}, \boldsymbol{w}_{-1})$ solves the desired limit problem. Therefore we consider the state equation (SE^ε) to order $\mathcal{O}(1)$

$$\begin{aligned}-\nabla\cdot\left[\tilde{\mathbb{C}}^N\mathcal{E}(\boldsymbol{w}_{-1}) + \overline{\mathbb{C}}(\boldsymbol{\varphi}_0)\mathcal{E}(\boldsymbol{w}_0)\right] = \;& \lambda_1\overline{\rho}(\boldsymbol{\varphi}_0)\boldsymbol{w}_{-1} + \lambda_0\overline{\rho}(\boldsymbol{\varphi}_0)\boldsymbol{w}_0 + \\ & \lambda_{-1}\overline{\rho}(\boldsymbol{\varphi}_0)\boldsymbol{w}_1 + \lambda_{-1}(\tilde{\rho}^N + \overline{\rho}(\boldsymbol{\varphi}_1))\boldsymbol{w}_{-1}.\end{aligned}$$

In Ω_N this simplifies to

$$-\nabla\cdot\left[\tilde{\mathbb{C}}^N\mathcal{E}(\boldsymbol{w}_{-1})\right] = \lambda_{-1}(\tilde{\rho}^N + \overline{\rho}(\boldsymbol{\varphi}_1))\boldsymbol{w}_{-1} \quad \text{in } \Omega_N.$$

So summing up this intermezzo, even if spurious eigenmodes are not excluded the appropriate choice of the model parameters will force the associated eigenvalues to leave the spectrum as $\varepsilon \to 0$ and thus the spurious modes do not affect our optimization problem.

6.4.3. Inner expansions

In the interfacial regions, i.e., in layers separating two outer regions, we need to rescale our coordinate system in order to take into account that $\boldsymbol{\varphi}^\varepsilon$ changes rapidly in directions perpendicular to the interface.

This fact can, for example, be motivated by the following one dimensional computation that is also one of the crucial ideas for understanding the Γ-limit of the Ginzburg–Landau

energy, see also the discussion in Section 2.1.4. Let us consider the one dimensional Ginzburg–Landau energy on $[-1,1]$ with continuous potential ψ_0

$$\mathcal{F}_\varepsilon(v) = \int_{-1}^{1} \frac{\varepsilon}{2} \left|v'\right|^2 + \frac{1}{\varepsilon}\psi_0(v)\,\mathrm{d}x.$$

Then using the rescaling $v(x) = w(\frac{x}{\varepsilon})$ we obtain

$$\mathcal{F}_\varepsilon(v) = \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{1}{2} \left|w'\right|^2 + \psi_0(w)\,\mathrm{d}y.$$

Hence for $\varepsilon \to 0$ we obtain the energy

$$\mathcal{F}_*(w) = \int_{-\infty}^{\infty} \frac{1}{2} \left|w'\right|^2 + \psi_0(w)\,\mathrm{d}y.$$

Now it can be proved that the solution of the initial value problem

$$\begin{aligned} w_*' &= \sqrt{2\psi_0(w_*)}, \\ w_*(0) &= 0, \end{aligned} \tag{6.4.15}$$

is the unique minimizer of the energy $\mathcal{F}_*$ over the set

$$X \coloneqq \left\{ v \in H^1_{loc}(\mathbb{R}) \;\middle|\; \lim_{t\to\pm\infty} v(t) = \pm 1, v(0) = 0 \right\}.$$

In other words (6.4.15), allows us to characterize the (smooth) profile for the transition from -1 to 1 with the least energy. For instance, if the polynomial double-well potential $\psi_0(\varphi) = \frac{1}{2}(1-\varphi^2)^2$ is chosen, the solution of (6.4.15) is $w_* = \tanh$. Hence, by rescaling in accordance to our initial substitution we see a rapid change of the phase-field for small ε in the transition phase between -1 and 1. For an in detail discussion concerning the general Γ-limit of the Ginzburg–Landau energy in the scalar case we refer to [37,126,147] or the alternative proof of Theorem 4.3.17 where the ODE (6.4.15) is used to construct optimal profiles for the Ginzburg–Landau energy.

Let us come back to our multi-phase problem. For all $i,j = 1,\dots,N$, we write $\Gamma = \Gamma_{ij}$ to denote the sharp-interface separating Ω_i and Ω_j. Moreover, let $\boldsymbol{n}_{\Gamma_{ij}}$ denote the unit normal vector field on Γ pointing from Ω_i to Ω_j. In the following, we omit these indices to provide a cleaner presentation. We now introduce a suitable coordinate system that fits the geometry of the interface. The following discussion can be found, e.g., in [1] but for the sake of readability we summarize the most important arguments. Let us choose a local parametrization

$$\boldsymbol{\gamma} : U \to \mathbb{R}^d, \quad \boldsymbol{\gamma}(U) \subseteq \Gamma \tag{6.4.16}$$

of Γ, where U is an open subset of $\mathbb{R}^d$. We further define $\boldsymbol{\nu} \coloneqq \boldsymbol{n}_\Gamma \circ \boldsymbol{\gamma}$.

As we want to describe a whole neighborhood surrounding the local part of the interface $\boldsymbol{\gamma}(U) \subset \Gamma$, we introduce the signed distance function relative to Ω_i which satisfies $d(x) > 0$ if $x \in \Omega_j$ and $d(x) < 0$ if $x \in \Omega_i$. For more details concerning the signed distance function we refer the reader to [99, Sec. 14.6].

Now, let $x \in \Omega$ be an arbitrary point belonging to a sufficiently thin tubular neighborhood around $\boldsymbol{\gamma}(U)$. We define

$$\boldsymbol{s}(x) := \boldsymbol{\gamma}^{-1}\left(\min_{y\in\boldsymbol{\gamma}(U)} |x-y|\right) \in U, \quad z(x) := \tfrac{1}{\varepsilon} d(x) \in \mathbb{R}.$$

We further introduce the function

$$G^\varepsilon : U \times \mathbb{R}, \quad G^\varepsilon(\boldsymbol{s}, z) \coloneqq \boldsymbol{\gamma}(\boldsymbol{s}) + \varepsilon z \boldsymbol{\nu}(s).$$

This means that x satisfies the relation

$$x = G^\varepsilon\big(\boldsymbol{s}(x), z(x)\big).$$

This holds since the summand $\boldsymbol{\gamma}\big(\boldsymbol{s}(x)\big)$ denotes the orthogonal projection of x onto $\boldsymbol{\gamma}(U)$ and the summand

$$\varepsilon z(x) \boldsymbol{\nu}\big(s(x)\big) = d(x) \boldsymbol{n}_\Gamma\big(\boldsymbol{\gamma}(\boldsymbol{s}(x))\big)$$

shifts the point $\boldsymbol{\gamma}\big(\boldsymbol{s}(x)\big)$ back onto x. This allows us to express the point x by means of the new coordinates $\boldsymbol{s} = \boldsymbol{s}(x) \in U$ and $z = z(x) \in \mathbb{R}$. Hence, a sufficiently thin tubular neighborhood around $\boldsymbol{\gamma}(U)$ can be expressed by the coordinate system $(\boldsymbol{s}, z)$.

For any fixed $z \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$, we define the $(d-1)$-dimensional sub-manifold

$$\Gamma_{\varepsilon z} \coloneqq \{\boldsymbol{\gamma}(\boldsymbol{s}) + \varepsilon z \boldsymbol{\nu}(\boldsymbol{s}) \,|\, \boldsymbol{s} \in U\},$$

which describes a translation of Γ in the direction $\boldsymbol{\nu}$. In the light of our above considerations, for $\varepsilon > 0$ fixed we deduce that for any $\boldsymbol{s} = (s_1, \dots, s_{d-1}) \in U$ and $z \in \mathbb{R}$,

$$\big\{\partial_{s_1}\boldsymbol{\gamma}(\boldsymbol{s}) + \varepsilon z \partial_{s_1}\boldsymbol{\nu}(\boldsymbol{s}),\ \dots\ ,\ \partial_{s_{d-1}}\boldsymbol{\gamma}(\boldsymbol{s}) + \varepsilon z \partial_{s_{d-1}}\boldsymbol{\nu}(\boldsymbol{s})\big\}$$

is a basis of $\Gamma_{\varepsilon z}$, and

$$\big\{\partial_{s_1}\boldsymbol{\gamma}(\boldsymbol{s}) + \varepsilon z \partial_{s_1}\boldsymbol{\nu}(\boldsymbol{s}),\ \dots\ ,\ \partial_{s_{d-1}}\boldsymbol{\gamma}(\boldsymbol{s}) + \varepsilon z \partial_{s_{d-1}}\boldsymbol{\nu}(\boldsymbol{s}),\ \varepsilon\boldsymbol{\nu}(\boldsymbol{s})\big\},$$

is a basis of $\mathbb{R}^d$.

As we want to analyze the state and gradient equations of the eigenvalue problem, we need to understand how the differential operators behave under the coordinate transformation $x \mapsto (\boldsymbol{s}(x), z(x))$. Therefore, let us consider an arbitrary scalar function

$$b(x) = b(G^\varepsilon(\boldsymbol{s}(x), z(x)) = \hat{b}(\boldsymbol{s}(x), z(x)).$$

Using the above bases, we obtain the representations

$$\nabla_{\Gamma_{\varepsilon z}} \hat{b} = \sum_{k,l=1}^{d-1} g^{kl}\left(\partial_{s_k}\hat{b}\right)\partial_{s_l} G^\varepsilon, \tag{6.4.17}$$

for the surface gradient and

$$\nabla_x b = \sum_{k,l=1}^{d} g^{kl}\left(\partial_{s_k}\hat{b}\right)\partial_{s_l} G^\varepsilon, \tag{6.4.18}$$

for the full gradient (see, e.g., [114, Section 3.1]). Here, $(g^{kl})_{k,l=1}^{d}$ denotes the inverse of the metric tensor $(g_{kl})_{k,l=1}^{d}$, where

$$g_{ij} = \begin{cases} (\partial_{s_k}\boldsymbol{\gamma} + \varepsilon z \partial_{s_k}\boldsymbol{\nu}) \cdot (\partial_{s_l}\boldsymbol{\gamma} + \varepsilon z \partial_{s_l}\boldsymbol{\nu}) & \text{if } k,l \in \{1,...,d-1\}, \\ \varepsilon^2 \boldsymbol{\nu}\cdot\boldsymbol{\nu} = \varepsilon^2 & \text{if } k = l = d, \\ 0 & \text{else.} \end{cases} \tag{6.4.19}$$

Using $g^{dd} = \frac{1}{\varepsilon^2}$, we infer from (6.4.18) that

$$\nabla_x b = \sum_{k,l=1}^{d-1} g^{kl} \left(\partial_{s_k}\hat{b}\right) \partial_{s_l} G^\varepsilon + g^{dd}\left(\partial_{s_d}\hat{b}\right)\partial_{s_d} G^\varepsilon = \nabla_{\Gamma_{\varepsilon z}}\hat{b} + \frac{1}{\varepsilon}\left(\partial_z \hat{b}\right)\boldsymbol{\nu}. \tag{6.4.20}$$

Proceeding analogously, we deduce that the divergence of a vector-valued function $\boldsymbol{j}(x) = \hat{\boldsymbol{j}}(\boldsymbol{s}(x), z(x))$ can be expressed as

$$\nabla_x \cdot \boldsymbol{j} = \sum_{k,l=1}^{d-1} g^{kl}\left(\partial_{s_k}\hat{\boldsymbol{j}}\right)\cdot \partial_{s_l} G^\varepsilon + g^{dd}\left(\partial_{s_d}\hat{\boldsymbol{j}}\right)\cdot\partial_{s_d} G^\varepsilon = \nabla_{\Gamma_{\varepsilon z}}\cdot\hat{\boldsymbol{j}} + \frac{1}{\varepsilon}\partial_z\hat{\boldsymbol{j}}\cdot\boldsymbol{\nu}, \tag{6.4.21}$$

where $\nabla_{\Gamma_{\varepsilon z}}\cdot\hat{\boldsymbol{j}}$ stands for the surface divergence on $\Gamma_{\varepsilon z}$.

Now, we want to derive a formula for the Laplacian $\Delta_x b$. First of all, we know from (6.4.17) that $\nabla_{\Gamma_{\varepsilon z}}\hat{b}\cdot\boldsymbol{\nu} = 0$. This directly entails that $\partial_z(\nabla_{\Gamma_{\varepsilon z}}\hat{b}\cdot\boldsymbol{\nu}) = 0$. Thus, the product rule yields

$$\partial_z(\nabla_{\Gamma_{\varepsilon z}}\hat{b})\cdot\boldsymbol{\nu} = -\nabla_{\Gamma_{\varepsilon z}}\hat{b}\cdot\partial_z\boldsymbol{\nu} = 0.$$

By the definition of the signed distance function, we further have $\boldsymbol{\nu} = \nabla_x d$ (see, e.g., [99, Lemma 14.16]). Thus, employing (6.4.20) and (6.4.21), we obtain

$$\begin{aligned} \Delta_x b &= \nabla_x\cdot(\nabla_x b) = \nabla_x\cdot\left(\nabla_{\Gamma_{\varepsilon z}}\hat{b} + \frac{1}{\varepsilon}\left(\partial_z\hat{b}\right)\boldsymbol{\nu}\right) \\ &= \nabla_{\Gamma_{\varepsilon z}}\cdot\left(\nabla_{\Gamma_{\varepsilon z}}\hat{b} + \frac{1}{\varepsilon}\left(\partial_z\hat{b}\right)\boldsymbol{\nu}\right) + \frac{1}{\varepsilon}\partial_z\left(\nabla_{\Gamma_{\varepsilon z}}\hat{b} + \frac{1}{\varepsilon}\left(\partial_z\hat{b}\right)\boldsymbol{\nu}\right)\cdot\boldsymbol{\nu} \\ &= \Delta_{\Gamma_{\varepsilon z}}\hat{b} + \frac{1}{\varepsilon}\partial_z\hat{b}\nabla_{\Gamma_{\varepsilon z}}\cdot\boldsymbol{\nu} + \frac{1}{\varepsilon^2}\partial_{zz}\hat{b} \\ &= \Delta_{\Gamma_{\varepsilon z}}\hat{b} + \frac{1}{\varepsilon}\partial_z\hat{b}\nabla_x\cdot\boldsymbol{\nu} + \frac{1}{\varepsilon^2}\partial_{zz}\hat{b} \\ &= \Delta_{\Gamma_{\varepsilon z}}\hat{b} + \frac{1}{\varepsilon}\left(\Delta_x d\right)\partial_z\hat{b} + \frac{1}{\varepsilon^2}\partial_{zz}\hat{b}. \end{aligned} \tag{6.4.22}$$

Moreover, the surface Laplacian can be expressed as

$$\Delta_{\Gamma_{\varepsilon z}}\hat{b} = \frac{1}{\sqrt{g}}\sum_{k,l=1}^{d-1}\partial_{s_l}\left(\sqrt{g}\,g^{kl}\partial_{s_k}\hat{b}\right), \tag{6.4.23}$$

where $g \coloneqq \det\left((g_{kl})_{k,l=1}^{d-1}\right)$ (see, e.g., [114, Section 3.1]).

We now recall (6.4.19), where we computed the ε dependent components of the metric tensor. Moreover, by means of a Taylor expansion, we realize that

$$\frac{1}{1+\mathcal{O}(\varepsilon)} = 1 + \mathcal{O}(\varepsilon).$$

Hence, interpreting the representations of $\nabla_{\Gamma_{\varepsilon z}}, \nabla_{\Gamma_{\varepsilon z}}\cdot, \Delta_{\Gamma_{\varepsilon z}}$ (see (6.4.17), (6.4.21) and (6.4.23), respectively) as power series with respect to ε, we deduce

$$\begin{aligned}\nabla_{\Gamma_{\varepsilon z}}\hat{b}(s,z) &= \nabla_\Gamma \hat{b}(s,z) + \mathcal{O}(\varepsilon),\\ \Delta_{\Gamma_{\varepsilon z}}\hat{b}(s,z) &= \Delta_\Gamma \hat{b}(s,z) + \mathcal{O}(\varepsilon),\\ \nabla_{\Gamma_{\varepsilon z}}\cdot\hat{\boldsymbol{j}}(s,z) &= \nabla_\Gamma\cdot\hat{\boldsymbol{j}}(s,z) + \mathcal{O}(\varepsilon).\end{aligned} \tag{6.4.24}$$

Recall here the relation that Γ is locally represented by $\Gamma_{\varepsilon z}$ for $\varepsilon = 0$.

In the following, let $\mathcal{W}$ denote the *Weingarten map* associated with Γ that is given by

$$\mathcal{W}(x) \coloneqq -\nabla_\Gamma \boldsymbol{n}_\Gamma(x) \in \mathbb{R}^{d\times d} \quad \text{for all } x\in\Gamma, \tag{6.4.25}$$

see, e.g., [86, Appendix B]. Its non-trivial eigenvalues $\kappa_1,\dots,\kappa_{d-1}$ are the principal curvatures of Γ and its spectral norm can be expressed as

$$|\mathcal{W}| = \sqrt{\kappa_1^2 + \dots \kappa_{d-1}^2}.$$

Then, by means of [99, Lemma 14.17], we obtain the description

$$\Delta_x d = \sum_{i=1}^{d-1}\frac{-\kappa_i}{1-\kappa_i d} = \sum_{i=1}^{d-1}\frac{-\kappa_i}{1-\varepsilon\kappa_i z} = -\sum_{i=1}^{d-1}\kappa_i - \sum_{i=1}^{d-1}\varepsilon\kappa_i^2 z + \mathcal{O}(\varepsilon^2),$$

where we used the Taylor expansion of $z\mapsto(1-\varepsilon z)^{-1}$ at the point $z_0=0$ in the last step. Finally, combining (6.4.22) and (6.4.24), we obtain the representation

$$\Delta_x b = \frac{1}{\varepsilon^2}\partial_{zz}\hat{b} - \frac{1}{\varepsilon}\left(\hat{\kappa} + \varepsilon z|\hat{\mathcal{W}}|^2\right)\partial_z\hat{b} + \Delta_\Gamma\hat{b} + \mathcal{O}(\varepsilon), \tag{6.4.26}$$

where κ denotes the mean curvature which is defined as the sum of the principal curvatures of Γ. Note that in view of (6.4.25), κ can be expressed as

$$\kappa(x) = -\nabla_\Gamma\cdot\boldsymbol{n}_\Gamma(x) \quad \text{for all } x\in\Gamma, \tag{6.4.27}$$

which will be important for later purposes.

Furthermore, applying (6.4.20) to each vector component, the full gradient of a vector-valued function $\boldsymbol{j}(x) = \hat{\boldsymbol{j}}(\boldsymbol{s}(x), z(x))$ is given by

$$\nabla_x \boldsymbol{j} = \frac{1}{\varepsilon}\partial_z\hat{\boldsymbol{j}}\otimes\boldsymbol{\nu} + \nabla_\Gamma\hat{\boldsymbol{j}} + \mathcal{O}(\varepsilon), \tag{6.4.28}$$

where $\otimes$ denotes the dyadic product that is defined as $\boldsymbol{a}\otimes\boldsymbol{b} = (a_i b_j)_{i,j=1}^d$ for all $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^d$.

Analogously, for a matrix-valued function

$$\mathcal{A}(x) = (a_{ij}(x))_{i,j=1}^d = \hat{\mathcal{A}}(\boldsymbol{s}(x), z(x)),$$

we apply formula (6.4.21) to each component of the row-wise defined divergence $\nabla_x\cdot\mathcal{A}$. We obtain

$$\nabla_x\cdot\mathcal{A} = \nabla_\Gamma\cdot\hat{\mathcal{A}} + \frac{1}{\varepsilon}\partial_z\hat{\mathcal{A}}\boldsymbol{\nu} + \mathcal{O}(\varepsilon) \tag{6.4.29}$$

To conclude this section, we introduce the inner expansions that we will work with in the next section. Therefore, we make the ansatz

$$\begin{aligned} \boldsymbol{w}_{n_r}^{\varepsilon}(x) &= \sum_{k=0}^{\infty} \varepsilon^k \, \mathbf{W}_{k,n_r}\big(\boldsymbol{s}(x), z(x)\big), \\ \boldsymbol{\varphi}^{\varepsilon}(x) &= \sum_{k=0}^{\infty} \varepsilon^k \, \boldsymbol{\Phi}_k\big(\boldsymbol{s}(x), z(x)\big), \end{aligned} \tag{6.4.30}$$

where we assume $\boldsymbol{\Phi}_0(\boldsymbol{s}(x), z(x)) \in \boldsymbol{G}$ and $\boldsymbol{\Phi}_k(\boldsymbol{s}(x), z(x)) \in T\Sigma^N$ for all $k \geq 1$. In the next section, we will relate these inner expansions to the outer expansions that were introduced before.

Remark 6.4.5. Note that the eigenvalues $\lambda_{n_r}^{\varepsilon}$ do not depend locally on $x \in \Omega$ and thus, their inner expansion simply equals their outer expansion.

6.5. Derivation of the matching conditions

So far, we have constructed outer expansions which are supposed to hold inside the material regions Ω_i for $i = 1, \dots, N$ as well as inner expansions which are supposed to hold in a tubular neighborhood around the sharp-interfaces Γ_{ij}. Note that due to the construction in the previous section, the thickness of this tubular neighborhood is proportional to ε. In order to be compatible, both expansions must match in a suitable intermediate region by suitable matching conditions. This region is approximately given by all points $x \in \Omega$ with the property $\operatorname{dist}(x, \Gamma) \leq \varepsilon^{\theta}$ for some fixed $\theta \in (0, 1)$. This means we stretch the tubular neighborhood the inner expansions were constructed on from a thickness proportional to ε to a thickness proportional to ε^{θ} and relate both expansions in this region. These matching conditions will be expressed as limit conditions for the inner expansions when $\varepsilon \to 0$ or equivalently $z \to \pm\infty$ depending on which side we approach the interface from.

The definition of such an intermediate region is also required for a rigorous investigation of the Γ-limit of the Ginzburg–Landau energy (see, e.g., [147, formula (1.22)] or the construction of optimal profiles in the proofs of Theorem 4.3.17). There, the intermediate region of thickness $2\varepsilon^{\frac{1}{2}}$ is chosen to truncate the optimal profile such that it reaches the values -1 and 1 (which represent the pure phases), respectively.

In order to make both expansions compatible, we need to fix suitable variables which work for both expansions. In our case, we introduce the so-called *intermediate* variable $r_\theta(x) \coloneqq \varepsilon^{1-\theta} z(x) = \varepsilon^{-\theta} d(x)$, where $z(x) = \varepsilon^{-1} d(x)$ is the inner variable introduced in the previous section, which represents the suitably scaled signed distance of x to Γ. First of all, we want to keep z fixed and analyze the behavior of the outer expansions in which we insert $\varepsilon^{\theta} r_\theta(x) = d(x)$, and of the inner expansion in which we insert $\varepsilon^{\theta-1} r_\theta(x) = z(x)$. To obtain suitable relations between the inner and outer expansions we will need to make some assumptions on the single summands of these expansions.

Let us start with the outer expansions. Similarly as in the previous section but now *without* rescaling the distance coordinate, we write $\boldsymbol{\varphi}^{\varepsilon}(x) = \tilde{\boldsymbol{\varphi}}^{\varepsilon}\big(\boldsymbol{s}(x), d(x)\big)$ for any x in this intermediate region, provided that $\varepsilon > 0$ is sufficiently small. First of all, we assume that each summand of the outer expansion of $\tilde{\boldsymbol{\varphi}}^{\varepsilon}(\boldsymbol{s}, d)$ (see (6.4.1)) can be smoothly extended onto Γ_{ij} by approaching Γ_{ij} from both sides in normal direction. In the following, we use

the notation $\tilde{\boldsymbol{\varphi}}(s,0-)$ to denote the extension corresponding to the outer expansion in Ω_i and we write $\tilde{\boldsymbol{\varphi}}(s,0+)$ to denote the extension corresponding to the outer expansion in Ω_j. For sufficiently small $\varepsilon > 0$, this allows us to express $\tilde{\boldsymbol{\varphi}}(s,d)$ by means of a Taylor expansion at the point $(s,0\pm)$ as

$$\begin{aligned}\tilde{\boldsymbol{\varphi}}^\varepsilon(\boldsymbol{s},d) &= \tilde{\boldsymbol{\varphi}}^\varepsilon(\boldsymbol{s},\varepsilon^\theta r_\theta)\\ &= \begin{cases}\tilde{\boldsymbol{\varphi}}_0(\boldsymbol{s},0+) + \varepsilon^\theta \partial_d \tilde{\boldsymbol{\varphi}}_0(\boldsymbol{s},0+) r_\theta + \mathcal{O}(\varepsilon^{2\theta}) + \varepsilon\tilde{\boldsymbol{\varphi}}_1(\boldsymbol{s},0+) + \mathcal{O}(\varepsilon^{1+\theta})... & \text{if } d>0,\\ \tilde{\boldsymbol{\varphi}}_0(\boldsymbol{s},0-) + \varepsilon^\theta \partial_d \tilde{\boldsymbol{\varphi}}_0(\boldsymbol{s},0-) r_\theta + \mathcal{O}(\varepsilon^{2\theta}) + \varepsilon\tilde{\boldsymbol{\varphi}}_1(\boldsymbol{s},0-) + \mathcal{O}(\varepsilon^{1+\theta})... & \text{if } d<0.\end{cases}\end{aligned} \tag{6.5.1}$$

Here, as in the previous section, $\partial_d = \partial_{s_d}$ denotes the derivative with respect to the last component (i.e., the distance variable).

Now, for the summands of the inner expansion (see (6.4.30)), we assume that for large $|z|$ we have a polynomial approximation. More precisely, we assume that for all $k \in \mathbb{N}_0$, there exist coefficient functions $\boldsymbol{\Phi}^\pm_{k,i}$ and $p_k \in \mathbb{N}$ such that

$$\boldsymbol{\Phi}_k(\boldsymbol{s},z) \approx \begin{cases}\boldsymbol{\Phi}^+_{k,0}(\boldsymbol{s}) + \boldsymbol{\Phi}^+_{k,1}(\boldsymbol{s}) z + \cdots + \boldsymbol{\Phi}^+_{k,p_k}(\boldsymbol{s}) z^{p_k} & \text{if } z>0,\\ \boldsymbol{\Phi}^-_{k,0}(\boldsymbol{s}) + \boldsymbol{\Phi}^-_{k,1}(\boldsymbol{s}) z + \cdots + \boldsymbol{\Phi}^-_{k,p_k}(\boldsymbol{s}) z^{p_k} & \text{if } z<0,\end{cases} \tag{6.5.2}$$

Here, the symbol $\approx$ means that the difference of the left-hand side and the right-hand side as well as all its derivatives with respect to z tend to zero as $z \to \pm\infty$. This assumption is formally justified because the optimal profile computed in the scalar case for the Ginzburg–Landau energy tends to the pure phases even exponentially as $z \to \pm\infty$ (see e.g., (4.4.19) and (4.4.20) in the proof of Theorem 4.3.17). Thus, by plugging $\varepsilon^{\theta-1} r_\theta$ into the inner expansion (6.4.30), we obtain

$$\begin{aligned}\hat{\boldsymbol{\varphi}}^\varepsilon(\boldsymbol{s},\varepsilon^{\theta-1}r_\theta) \approx{}& \boldsymbol{\Phi}^\pm_{0,0}(\boldsymbol{s}) + \varepsilon^{\theta-1}\boldsymbol{\Phi}^\pm_{0,1}(\boldsymbol{s}) r_\theta + \cdots + \varepsilon^{p_0(\theta-1)}\boldsymbol{\Phi}^\pm_{0,p_0}(\boldsymbol{s}) r_\theta^{p_0}\\ &+ \varepsilon\boldsymbol{\Phi}^\pm_{1,0}(\boldsymbol{s}) + \varepsilon^\theta \boldsymbol{\Phi}^\pm_{1,1}(\boldsymbol{s}) r_\theta + \cdots + \varepsilon^{p_1(\theta-1)+1}\boldsymbol{\Phi}^\pm_{1,p_1}(\boldsymbol{s}) r_\theta^{p_1} + \ldots\end{aligned} \tag{6.5.3}$$

for $\varepsilon \to 0$ as then $z = \varepsilon^{\theta-1} r_\theta \to \pm\infty$ becomes large.

Now, we say that the inner expansions and the outer expansions *match* if they coincide in all orders of ε in the limit $z \to \pm\infty$, i.e., when we approach the sharp-interface from within Ω_j and Ω_i respectively. This matching makes sense because the coordinate $\varepsilon^\theta r_\theta = d$ in the outer expansions is supposed to match $\varepsilon^{\theta-1} r_\theta = z$ in the inner expansions due to our rescaling of the signed distance by the factor ε^{-1} in the derivation of the inner expansion.

Matching the outer and inner expansion (6.5.1) and (6.5.3) with respect to their first-orders and recalling that $\theta \in (0,1)$, we deduce

$$\begin{aligned}&\boldsymbol{\Phi}^\pm_{0,0}(\boldsymbol{s}) = \tilde{\boldsymbol{\varphi}}_0(\boldsymbol{s},0\pm), \quad \boldsymbol{\Phi}^\pm_{0,i}(\boldsymbol{s}) = 0 \quad \text{for } i>0,\\ &\boldsymbol{\Phi}^\pm_{1,0}(\boldsymbol{s}) = \tilde{\boldsymbol{\varphi}}_1(\boldsymbol{s},0\pm), \quad \boldsymbol{\Phi}^\pm_{1,1}(\boldsymbol{s}) = \partial_d\tilde{\boldsymbol{\varphi}}_0(\boldsymbol{s},0\pm),\\ &\boldsymbol{\Phi}^\pm_{1,i}(\boldsymbol{s}) = 0 \quad \text{for } i>1.\end{aligned}$$

In particular, using this information to simplify (6.5.2), we obtain

$$\begin{aligned}&\boldsymbol{\Phi}_0(\boldsymbol{s},z) \approx \boldsymbol{\Phi}^\pm_{0,0}(\boldsymbol{s}) && \text{as } z \to \pm\infty,\\ &\boldsymbol{\Phi}_1(\boldsymbol{s},z) \approx \boldsymbol{\Phi}^\pm_{1,0}(\boldsymbol{s}) + \boldsymbol{\Phi}^\pm_{1,1}(\boldsymbol{s}) z && \text{as } z \to \pm\infty.\end{aligned} \tag{6.5.4}$$

Recalling that for the outer expansions we have $\boldsymbol{\varphi}_0 \equiv \boldsymbol{e}_i$ in Ω_i for $i = 1, \dots, N$, we infer

$$\begin{aligned} \boldsymbol{\Phi}_0(\boldsymbol{s}, z) &\to \begin{cases} \tilde{\boldsymbol{\varphi}}_0(\boldsymbol{s}, 0+) = \boldsymbol{e}_j & \text{as } z \to +\infty, \\ \tilde{\boldsymbol{\varphi}}_0(\boldsymbol{s}, 0-) = \boldsymbol{e}_i & \text{as } z \to -\infty, \end{cases} \\ \partial_z \boldsymbol{\Phi}_0(\boldsymbol{s}, z) &\to 0 \quad \text{as } z \to \pm\infty \end{aligned} \tag{6.5.5}$$

and

$$\boldsymbol{\Phi}_1(\boldsymbol{s}, z) \approx \begin{cases} \tilde{\boldsymbol{\varphi}}_1(\boldsymbol{s}, 0+) + \partial_d \tilde{\boldsymbol{\varphi}}_0(\boldsymbol{s}, 0+) z & \text{as } z \to +\infty, \\ \tilde{\boldsymbol{\varphi}}_1(\boldsymbol{s}, 0-) + \partial_d \tilde{\boldsymbol{\varphi}}_0(\boldsymbol{s}, 0-) z & \text{as } z \to -\infty. \end{cases} \tag{6.5.6}$$

Reusing the notation from the previous section, where we indicated a function expressed with respect to the inner coordinates $(\boldsymbol{s}, z)$ with a hat, we obtain the relation

$$\hat{\boldsymbol{\varphi}}^\varepsilon(\boldsymbol{s}, z) = \boldsymbol{\varphi}^\varepsilon(x) = \tilde{\boldsymbol{\varphi}}^\varepsilon(\boldsymbol{s}, d).$$

Recalling that $z = \varepsilon^{-1} d$, we further deduce

$$\partial_d \tilde{\boldsymbol{\varphi}}_0(\boldsymbol{s}, d) = \frac{1}{\varepsilon} \partial_z \hat{\boldsymbol{\varphi}}_0(\boldsymbol{s}, z).$$

Thus, employing (6.4.20) and the fact that the surface gradient is orthogonal to $\boldsymbol{\nu}$, we arrive at $\partial_d \tilde{\boldsymbol{\varphi}}_0 = \nabla_x \boldsymbol{\varphi}_0 \cdot \boldsymbol{\nu}$. In the following, we will drop the index x and just write $\nabla = \nabla_x$. We further recall that locally $\boldsymbol{\nu}$ is given by $\boldsymbol{\nu} = \boldsymbol{n}_\Gamma \circ \boldsymbol{\gamma}$, where $\boldsymbol{\gamma} : U \to \mathbb{R}^d$ is a local parametrization of $\Gamma = \Gamma_{ij}$ (see (6.4.16)). In particular, using the notation

$$(\boldsymbol{v})_j(x) \coloneqq \lim_{\delta \searrow 0} \boldsymbol{v}(x \pm \delta \boldsymbol{n}_\Gamma(x)) \tag{6.5.7}$$

for any $x = \boldsymbol{\gamma}(\boldsymbol{s}) \in \Gamma = \Gamma_{ij}$, we infer from (6.5.6) that

$$\partial_z \boldsymbol{\Phi}_1(\boldsymbol{s}, z) \to \begin{cases} (\nabla \boldsymbol{\varphi}_0)_j(x)\, \boldsymbol{n}_\Gamma(x) & \text{as } z \to +\infty, \\ (\nabla \boldsymbol{\varphi}_0)_i(x)\, \boldsymbol{n}_\Gamma(x) & \text{as } z \to -\infty, \end{cases} \tag{6.5.8}$$

In this way, we can rewrite (6.5.5) and (6.5.6) as

$$\begin{aligned} \boldsymbol{\Phi}_0(\boldsymbol{s}, z) &\to \begin{cases} (\boldsymbol{\varphi}_0)_j(x) = \boldsymbol{e}_j & \text{as } z \to +\infty, \\ (\boldsymbol{\varphi}_0)_i(x) = \boldsymbol{e}_i & \text{as } z \to -\infty, \end{cases} \\ \partial_z \boldsymbol{\Phi}_0(\boldsymbol{s}, z) &= 0 \quad \text{as } z \to \pm\infty \end{aligned} \tag{6.5.9}$$

and

$$\boldsymbol{\Phi}_1(\boldsymbol{s}, z) \approx \begin{cases} (\boldsymbol{\varphi}_1)_j(x) + (\nabla \boldsymbol{\varphi}_0)_j(x)\, \boldsymbol{n}_\Gamma(x)\, z & \text{as } z \to +\infty, \\ (\boldsymbol{\varphi}_1)_i(x) + (\nabla \boldsymbol{\varphi}_0)_i(x)\, \boldsymbol{n}_\Gamma(x)\, z & \text{as } z \to -\infty \end{cases} \tag{6.5.10}$$

for all $x = \boldsymbol{\gamma}(\boldsymbol{s}) \in \Gamma = \Gamma_{ij}$. Similarly, we obtain analogous relations for the expansions of $\boldsymbol{w}^\varepsilon_{n_r}$.

In the following, we will also see that the *jump* across the interfaces Γ_{ij} is an important quantity. It is defined by

$$[\boldsymbol{v}]^j_i(x) \coloneqq \lim_{\delta \searrow 0} \Big(\boldsymbol{v}(x + \delta \boldsymbol{n}_\Gamma(x)) - \boldsymbol{v}(x - \delta \boldsymbol{n}_\Gamma(x)) \Big), \tag{6.5.11}$$

for any $x = \boldsymbol{\gamma}(\boldsymbol{s}) \in \Gamma = \Gamma_{ij}$.

Now, we have made all the necessary computations to analyze the state equations and the gradient equality near the interfaces Γ_{ij}. In particular, we are able to investigate their behavior as $\varepsilon \to 0$.

6.6. Comparison of the leading order terms

Now, we want to apply our knowledge about the inner and outer expansions to the optimality system consisting of (SE^ε) and (GS^ε). This means we apply the formulas for the differential operators discussed in Section 6.4.3 to the optimality system, compare the terms with same orders in ε and apply the matching conditions. In this section, we will suppress the index n_r to provide a clearer notation.

6.6.1. The state equation

Although some equalities are derived as in [32], we perform the computations here for completeness. Note that for the sake of a nice depiction we only write the dependence of the inner variables on the rescaled coordinates $(\boldsymbol{s}, z)$ when necessary. First of all, applying (6.4.28) on the lowest order contribution $\mathbf{W}_0$ of the inner expansion of $\boldsymbol{w}^\varepsilon$ (see (6.4.30)), we obtain

$$\mathcal{E}(\boldsymbol{w}^\varepsilon) = (\nabla_x \boldsymbol{w}^\varepsilon)^{\mathrm{sym}} = \left(\nabla_\Gamma \mathbf{W}_0 + \frac{1}{\varepsilon}\partial_z \mathbf{W}_0 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}}. \tag{6.6.1}$$

Using (6.4.29) with

$$\mathcal{A} = \left[\mathbb{C}(\boldsymbol{\varphi})\mathcal{E}\left(\boldsymbol{w}\right)\right], \quad \hat{\mathcal{A}} = \left[\overline{\mathbb{C}}(\boldsymbol{\Phi}_0)\left(\nabla_\Gamma \mathbf{W}_0 + \frac{1}{\varepsilon}\partial_z \mathbf{W}_0 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}}\right],$$

and recalling $\partial_z \boldsymbol{\nu} = \mathbf{0}$, we further deduce

$$\begin{aligned}
&-\nabla_x \cdot \left[\mathbb{C}(\boldsymbol{\varphi})\mathcal{E}\left(\boldsymbol{w}\right)\right] \\
&= -\nabla_\Gamma \cdot \left[\overline{\mathbb{C}}(\boldsymbol{\Phi}_0)\left(\nabla_\Gamma \mathbf{W}_0 + \frac{1}{\varepsilon}\partial_z \mathbf{W}_0 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}}\right] \\
&\quad - \frac{1}{\varepsilon}\partial_z \left[\overline{\mathbb{C}}(\boldsymbol{\Phi}_0)\left(\nabla_\Gamma \mathbf{W}_0 + \frac{1}{\varepsilon}\partial_z \mathbf{W}_0 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}}\right]\boldsymbol{\nu} \\
&= -\frac{1}{\varepsilon^2}\partial_z \left[\overline{\mathbb{C}}(\boldsymbol{\Phi}_0)\left(\partial_z \mathbf{W}_0 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}}\boldsymbol{\nu}\right] - \frac{1}{\varepsilon}\partial_z \left[\overline{\mathbb{C}}(\boldsymbol{\Phi}_0)\left(\nabla_\Gamma \mathbf{W}_0\right)^{\mathrm{sym}}\boldsymbol{\nu}\right] \\
&\quad - \frac{1}{\varepsilon}\nabla_\Gamma \cdot \left[\overline{\mathbb{C}}(\boldsymbol{\Phi}_0)\left(\partial_z \mathbf{W}_0 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}}\right] - \nabla_\Gamma \cdot \left[\overline{\mathbb{C}}(\boldsymbol{\Phi}_0)\left(\nabla_\Gamma \mathbf{W}_0\right)^{\mathrm{sym}}\right].
\end{aligned} \tag{6.6.2}$$

As no derivatives are involved in the right-hand side of the state equation (SE^ε), its expression via inner expansions only possesses non-negative orders of ε. Thus, for the contribution of leading order $\mathcal{O}(\varepsilon^{-2})$, we obtain

$$\partial_z \left[\overline{\mathbb{C}}(\boldsymbol{\Phi}_0)\left(\partial_z \mathbf{W}_0 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}}\boldsymbol{\nu}\right] = \mathbf{0} \quad \text{around } \Gamma_{ij}. \tag{6.6.3}$$

Here, the expression "around Γ_{ij}" means that the statement is valid in a sufficiently thin tubular neighborhood around Γ_{ij} where our inner expansions hold.

From (6.5.9), which holds analogously for $\mathbf{W}_0$ instead of $\boldsymbol{\Phi}_0$, we infer

$$\lim_{z\to\pm\infty} \partial_z \mathbf{W}_0(\boldsymbol{s}, z) = \mathbf{0} \quad \text{and} \quad \overline{\mathbb{C}}(\boldsymbol{\Phi}_0(\boldsymbol{s}, z)) \to \begin{cases} \mathbb{C}^j & \text{for } z \to +\infty, \\ \mathbb{C}^i & \text{for } z \to -\infty, \end{cases} \tag{6.6.4}$$

for all $\boldsymbol{s}$ such that $\gamma(\boldsymbol{s}) \in \Gamma_{ij}$ with $i, j \neq N$. Furthermore we obtain

$$\begin{aligned} \overline{\mathbb{C}}(\boldsymbol{\Phi}_0(\boldsymbol{s}, z)) \to 0 \quad \text{as } z \to +\infty, \quad \text{if } j = N, \\ \overline{\mathbb{C}}(\boldsymbol{\Phi}_0(\boldsymbol{s}, z)) \to 0 \quad \text{as } z \to -\infty, \quad \text{if } i = N \end{aligned} \tag{6.6.5}$$

due to the additional factor ε in the void contribution (see (6.2.2)). Of course, the relations (6.6.4) and (6.6.5) can be obtained analogously for $\overline{\rho}$ instead of $\overline{\mathbb{C}}$.

Now multiplying (6.6.3) with $\mathbf{W}_0$ and integrating by parts yield

$$0 = \int_{-\infty}^{+\infty} \overline{\mathbb{C}}(\boldsymbol{\Phi}_0) \left(\partial_z \mathbf{W}_0 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}} : \left(\partial_z \mathbf{W}_0 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}} \, \mathrm{d}z, \tag{6.6.6}$$

per definition of the dyadic product and the symmetry properties of $\mathbb{C}$ from Section 2.1.9. Note that the fact $\boldsymbol{\Phi}_0(x) \in \boldsymbol{G}$ according to (6.2.2) gives us the decomposition

$$\overline{\mathbb{C}}(\boldsymbol{\Phi}_0) = \sum_{i=1}^{N-1} \mathbb{C}^i \Phi_0^i. \tag{6.6.7}$$

Now the assumptions in Section 2.1.9 yield the non-degeneracy of each $\mathbb{C}^i$. Thus, the integrand in (6.6.6) is non-negative and thus we infer that for almost every $z \in (-\infty, \infty)$ either $\boldsymbol{\Phi}_0 = \boldsymbol{e}_N$ or $\partial_z \mathbf{W}_0 \otimes \boldsymbol{\nu} = \mathbf{0}$. But notice that we consider *inner* expansions, i.e., we are in a tubular neighborhood around Γ_{ij} where $\boldsymbol{\Phi}_0$ realizes a *transition* between the pure phases, but the pure phase values $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N$ are not attained within this region. In particular this gives

$$\left(\partial_z \mathbf{W}_0 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}} = 0 \quad \text{around } \Gamma_{ij},$$

for $i, j = 1, \ldots, N$. With a simple computation using the definition of the dyadic product and the properties of the unit normal, we obtain

$$\partial_z \mathbf{W}_0 = \mathbf{0} \quad \text{around } \Gamma_{ij}, \tag{6.6.8}$$

for $i, j = 1, \ldots, N$. Hence, $\mathbf{W}_0$ has to be constant in z across the interface Γ_{ij}. From the matching condition (6.5.9) written for $\mathbf{W}_0$, we obtain

$$\mathbf{W}_0(\boldsymbol{s}, z) \to \begin{cases} \left(\boldsymbol{w}_0\right)_j (x) & \text{for } z \to +\infty \\ \left(\boldsymbol{w}_0\right)_i (x) & \text{for } z \to -\infty, \end{cases} \tag{6.6.9}$$

for all $x = \gamma(\boldsymbol{s}) \in \Gamma_{ij}$. In combination with (6.6.8) we arrive at the jump condition $[\boldsymbol{w}_0]_i^j = \mathbf{0}$, for $i, j = 1, \ldots, N$. But note that practically the jump across an interface shared between a material and a void region is negligible as we do not know the exact behavior of $\boldsymbol{w}_0$ at the void boundary. In other words, we will obtain a closed system of PDEs forming the state equations of the sharp-interface problem in Section 6.7 without needing this additional jump condition at the void boundary.

Now we pass on to the next order $\mathcal{O}(\frac{1}{\varepsilon})$. Due to (6.6.8) we can drop all terms involving $\partial_z \mathbf{W}_0$ and obtain in (6.6.2) (now computed analogously also to higher orders in $\mathbf{W}$)

$$\mathbf{0} = \partial_z \left[\overline{\mathbb{C}}(\boldsymbol{\Phi}_0) \left(\partial_z \mathbf{W}_1 \otimes \boldsymbol{\nu} + \nabla_\Gamma \mathbf{W}_0\right)^{\mathrm{sym}} \boldsymbol{\nu} \right] \quad \text{around } \Gamma_{ij}, \tag{6.6.10}$$

for $i,j=1,\dots,N$. This means $\left[\overline{\mathbb{C}}(\mathbf{\Phi}_0)\left(\partial_z \mathbf{W}_1 \otimes \boldsymbol{\nu} + \nabla_\Gamma \mathbf{W}_0\right)^{\mathrm{sym}} \boldsymbol{\nu}\right]$ is constant in z across Γ_{ij}. Now first of all we recall that analogously to (6.4.18) it holds in the non-rescaled coordinates $(\boldsymbol{s},d)$

$$\nabla_x \boldsymbol{w}_0(x) = \nabla_\Gamma \tilde{\boldsymbol{w}}_0(\boldsymbol{s},d) + (\partial_d \tilde{\boldsymbol{w}}_0(\boldsymbol{s},d)) \otimes \boldsymbol{\nu}(\boldsymbol{s}), \tag{6.6.11}$$

for all $x=\gamma(\boldsymbol{s})\in\Gamma_{ij}$. In the notation of the previous section we know by matching

$$\mathbf{W}_0(\boldsymbol{s},z) \approx \mathbf{W}_{0,0}^{\pm}(\boldsymbol{s}) = \tilde{\boldsymbol{w}}_0(\boldsymbol{s},0\pm) \quad \text{for } z\to\pm\infty,$$

In particular we get

$$\nabla_\Gamma \mathbf{W}_0(\boldsymbol{s},z) \to \begin{cases} (\nabla_\Gamma \tilde{\boldsymbol{w}}_0(\boldsymbol{s},0+))_j & \text{for } z\to+\infty, \\ (\nabla_\Gamma \tilde{\boldsymbol{w}}_0(\boldsymbol{s},0-))_i & \text{for } z\to-\infty, \end{cases}$$

Combining this with (6.5.6) and using the decomposition (6.6.11) we arrive at

$$\nabla_\Gamma \mathbf{W}_0(\boldsymbol{s},z) + \partial_z \mathbf{W}_1(\boldsymbol{s},z)\otimes\boldsymbol{\nu}(\boldsymbol{s}) \to \begin{cases} (\nabla_x \boldsymbol{w}_0(x))_j & \text{for } z\to+\infty, \\ (\nabla_x \boldsymbol{w}_0(x))_i & \text{for } z\to-\infty, \end{cases} \tag{6.6.12}$$

for all $x=\gamma(\boldsymbol{s})\in\Gamma_{ij}$. So using (6.6.4) combined with (6.6.10) we obtain for all $i\neq N$

$$\mathbb{C}^i \mathcal{E}_i(\boldsymbol{w}_0)\boldsymbol{n}_\Gamma = \begin{cases} \mathbf{0} & \text{if } j=N, \\ \mathbb{C}^j \mathcal{E}_j(\boldsymbol{w}_0)\boldsymbol{n}_\Gamma & \text{if } j\neq N, \end{cases} \tag{6.6.13}$$

holds on each Γ_{ij} with $i\neq N$, where for $x\in\Gamma_{ij}$

$$\mathcal{E}_i(\boldsymbol{w}_0) \coloneqq \lim_{\delta\searrow 0} \mathcal{E}(\boldsymbol{w}_0)(x-\delta\boldsymbol{n}_\Gamma) \quad \text{and} \quad \mathcal{E}_j(\boldsymbol{w}_0) \coloneqq \lim_{\delta\searrow 0} \mathcal{E}(\boldsymbol{w}_0)(x+\delta\boldsymbol{n}_\Gamma).$$

6.6.2. The gradient equality

Now, we want to analyze the gradient equality (GS^ε), which reads as

$$\begin{aligned} \sum_{r=1}^{l} \Big\{ [\partial_{\lambda_{n_r}}\Psi]\left(\lambda_{n_1}^\varepsilon,\dots,\lambda_{n_l}^\varepsilon\right) \Big[&\langle \mathcal{E}(\boldsymbol{w}_{n_r}^\varepsilon), \mathcal{E}(\boldsymbol{w}_{n_r}^\varepsilon)\rangle_{P_{T\Sigma}[\mathbb{C}'(\boldsymbol{\varphi}^\varepsilon)]} \\ &- \lambda_{n_r}^\varepsilon \left(\boldsymbol{w}_{n_r}^\varepsilon, \boldsymbol{w}_{n_r}^\varepsilon\right)_{P_{T\Sigma}[\rho'(\boldsymbol{\varphi}^\varepsilon)]} \Big] \Big\} \\ = \gamma\varepsilon\Delta\boldsymbol{\varphi}^\varepsilon + \frac{1}{\varepsilon}(\boldsymbol{\Lambda}^\varepsilon + \boldsymbol{\vartheta}^\varepsilon + \boldsymbol{\mu}^\varepsilon) - \frac{\gamma}{\varepsilon} P_{T\Sigma}\left[\psi_0'(\boldsymbol{\varphi}^\varepsilon)\right]. \end{aligned} \tag{6.6.14}$$

Here, we recall that the Lagrange multipliers were constructed in Theorem 6.3.12 in such a way that their sum appearing in the gradient equality (6.6.14) is scaled by the factor $\frac{1}{\varepsilon}$. We now assume the Lagrange multipliers to have the following inner asymptotic expansions:

$$\boldsymbol{\Lambda}^\varepsilon(x) = \sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{\Lambda}_k(\boldsymbol{s},z) \quad \boldsymbol{\vartheta} = \sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{\vartheta}_k, \quad \boldsymbol{\mu}^\varepsilon(x) = \sum_{k=0}^{\infty} \varepsilon^k \boldsymbol{\mu}_k(\boldsymbol{s},z), \tag{6.6.15}$$

Furthermore, in order to deal with the nonlinear terms in (6.6.14) involving $\mathbb{C}', \rho', \psi_0', \partial_{\lambda_{nr}}\Psi$, we perform a (component-wise) Taylor expansion around the leading order term $\mathbf{\Phi}_0$ to obtain the inner expansions

$$\begin{aligned}
\mathbb{C}'(\boldsymbol{\varphi}^\varepsilon) &= \mathbb{C}'(\mathbf{\Phi}_0) + \mathcal{O}(\varepsilon),\\
\rho'(\boldsymbol{\varphi}^\varepsilon) &= \rho'(\mathbf{\Phi}_0) + \mathcal{O}(\varepsilon),\\
\psi_0'(\boldsymbol{\varphi}^\varepsilon) &= \psi_0'(\mathbf{\Phi}_0) + \mathcal{O}(\varepsilon),\\
[\partial_{\lambda_{nr}}\Psi](\lambda^\varepsilon_{n_1}, \dots, \lambda^\varepsilon_{n_l}) &= [\partial_{\lambda_{nr}}\Psi](\lambda_{0,n_1}, \dots, \lambda_{0,n_l}) + \mathcal{O}(\varepsilon).
\end{aligned}$$

In order to get rid of the projections appearing in the left-hand side of (6.6.14), we now take a closer look at the quantities $\mathbb{C}'(\mathbf{\Phi}_0)$ and $\rho'(\mathbf{\Phi}_0)$. To this end, we recall the definition of ρ in (6.2.2) and (2.1.23), which reads as

$$\rho : \mathbb{R}^N \to \mathbb{R}, \quad \boldsymbol{\varphi} \mapsto \sum_{i=1}^{N-1} \rho^i \sigma_\omega(P_\Sigma(\boldsymbol{\varphi})_i) + \tilde{\rho}^N \varepsilon \sigma_\omega(P_\Sigma(\boldsymbol{\varphi})_N),$$

where $\omega > 0$ is a sufficiently small parameter. Note that we can write the projection P_Σ as

$$P_\Sigma(\boldsymbol{\varphi}) = \boldsymbol{\varphi} - \left(\frac{1}{N}\sum_{i=1}^{N}\varphi^i\right)\mathbf{1} + \frac{1}{N}\mathbf{1},$$

for $\boldsymbol{\varphi} \in \mathbb{R}^N$, where $\mathbf{1} = (1, ..., 1)^T \in \mathbb{R}^N$. For the partial derivatives with respect to φ^j with $j \in \{1, ..., N\}$, we thus obtain

$$(\partial_j P_\Sigma)(\boldsymbol{\varphi}) = \boldsymbol{e}_j - \frac{1}{N}\mathbf{1}$$

and therefore,

$$(\partial_j \rho)(\boldsymbol{\varphi}) = \sum_{i=1}^{N} \rho^i \sigma_\omega'(P_\Sigma(\boldsymbol{\varphi})_i)\left(\delta_{ij} - \frac{1}{N}\right),$$

where δ_{ij} denotes the Kronecker delta and $\rho^N \coloneqq \varepsilon\tilde{\rho}^N$ to simplify the notation. Inserting $\mathbf{\Phi}_0$ (which belongs point-wise to Σ^N and thus, no projection is necessary) and recalling that σ_ω is the identity on $[0, 1]$, see (2.1.22), we arrive at

$$\rho'(\mathbf{\Phi}_0) = ((\partial_j\rho)(\mathbf{\Phi}_0))_{j=1}^N = (\rho^j - \frac{1}{N}\sum_{i=1}^{N}\rho^i)_{j=1}^N. \tag{6.6.16}$$

Keeping in mind that $\rho^N = \varepsilon\tilde{\rho}^N$ still produces terms of order $\mathcal{O}(\varepsilon)$, considering (6.6.16) to the lowest order $\mathcal{O}(1)$ gives

$$\overline{\rho}'(\mathbf{\Phi}_0) = \left(\rho^1 - \frac{1}{N}\sum_{i=1}^{N-1}\rho^i,\ \dots\ ,\ \rho^{N-1} - \frac{1}{N}\sum_{i=1}^{N-1}\rho^i, -\frac{1}{N}\sum_{i=1}^{N-1}\rho^i\right)^T. \tag{6.6.17}$$

Thus, it obviously holds $\overline{\rho}'(\mathbf{\Phi}_0) \in T\Sigma^N$. The function $\mathbb{C}'(\mathbf{\Phi}_0)$ can be expressed analogously. Altogether, this allows us to drop the projection acting on the left-hand side in (6.6.14) when considering only the lowest order contributions.

Now that we have considered all the quantities appearing in (6.6.14), we begin with our formal asymptotics. First of all, let us recall (6.6.1) from the beginning of the previous Subsection

$$\mathcal{E}(\boldsymbol{w}^\varepsilon) = (\nabla_x \boldsymbol{w}^\varepsilon)^{\mathrm{sym}} = \left(\nabla_\Gamma \mathbf{W}_0 + \frac{1}{\varepsilon}\partial_z \mathbf{W}_0 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}} + \mathcal{O}(\varepsilon).$$

Comparing the contributions of order $\mathcal{O}(\varepsilon^{-2})$ in (6.6.14), we thus arrive at

$$\begin{aligned}\mathbf{0} = \sum_{r=1}^{l} &[\partial_{\lambda_{n_r}}\Psi]\,(\lambda_{0,n_1},\dots,\lambda_{0,n_l}) \\ &\cdot \left[\overline{\mathbb{C}}'(\boldsymbol{\Phi}_0)(\partial_z \mathbf{W}_{0,n_r}\otimes\boldsymbol{\nu})^{\mathrm{sym}} : (\partial_z \mathbf{W}_{0,n_r}\otimes\boldsymbol{\nu})^{\mathrm{sym}}\right] \quad \text{around } \Gamma_{ij},\end{aligned}$$

where we have used (6.6.1). This equation is obviously fulfilled since $\partial_z \mathbf{W}_0$ vanishes according to (6.6.8).

Let us now consider (6.6.14) to order $\mathcal{O}(\varepsilon^{-1})$. First of all, we infer from (6.6.8) and (6.6.1) that the left-hand side has no contribution of order $\mathcal{O}(\varepsilon^{-1})$. We thus have

$$\mathbf{0} = \gamma\partial_{zz}\boldsymbol{\Phi}_0 + (\boldsymbol{\Lambda}_0 + \boldsymbol{\vartheta}_0 + \boldsymbol{\mu}_0) - \gamma P_{T\Sigma}\left[\psi_0'(\boldsymbol{\Phi}_0)\right] \quad \text{around } \Gamma_{ij}, \tag{6.6.18}$$

where we used the formula (6.4.26) to compute the Laplacian. Multiplying (6.6.18) by $\partial_z\boldsymbol{\Phi}_0$ and integrating with respect to z from $-\infty$ to ∞, we deduce

$$\begin{aligned}&-\int_{-\infty}^{\infty}(\boldsymbol{\Lambda}_0 + \boldsymbol{\vartheta}_0 + \boldsymbol{\mu}_0)\cdot\partial_z\boldsymbol{\Phi}_0\,\mathrm{d}z \\ &\quad= \gamma\int_{-\infty}^{\infty}\partial_{zz}\boldsymbol{\Phi}_0\cdot\partial_z\boldsymbol{\Phi}_0\,\mathrm{d}z - \gamma\int_{-\infty}^{\infty}P_{T\Sigma}\left[\psi_0'(\boldsymbol{\Phi}_0)\right]\partial_z\boldsymbol{\Phi}_0\,\mathrm{d}z.\end{aligned} \tag{6.6.19}$$

Now, we consider each of the terms in (6.6.19) separately. First of all, we see

$$\begin{aligned}\int_{-\infty}^{\infty}\partial_{zz}\boldsymbol{\Phi}_0\cdot\partial_z\boldsymbol{\Phi}_0\,\mathrm{d}z &= \int_{-\infty}^{\infty}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\left|\partial_z\boldsymbol{\Phi}_0\right|^2\,\mathrm{d}z \\ &= \frac{1}{2}\left(\lim_{z\to+\infty}\partial_z\boldsymbol{\Phi}_0(z) - \lim_{z\to-\infty}\partial_z\boldsymbol{\Phi}_0(z)\right) = \mathbf{0},\end{aligned} \tag{6.6.20}$$

where the last equality follows from the matching condition (6.5.9).

As $\boldsymbol{\Phi}_0 \in \boldsymbol{G}$ point-wise, we know that $\partial_z\boldsymbol{\Phi}_0 \in T\Sigma^N$ point-wise. Hence, we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}P_{T\Sigma}\left[\psi_0'(\boldsymbol{\Phi}_0)\right]\partial_z\boldsymbol{\Phi}_0\,\mathrm{d}z = \int_{-\infty}^{\infty}\psi_0'(\boldsymbol{\Phi}_0)\partial_z\boldsymbol{\Phi}_0\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}z}\left[\psi_0(\boldsymbol{\Phi}_0)\right]\,\mathrm{d}z = \lim_{z\to+\infty}\psi_0(\boldsymbol{\Phi}_0(z)) - \lim_{z\to-\infty}\psi_0(\boldsymbol{\Phi}_0(z)) = \mathbf{0}.\end{aligned} \tag{6.6.21}$$

For the last equality, we used the fact that ψ_0 vanishes on $\boldsymbol{e}_i$ for $i = 1,\dots,N$ along with the matching condition (6.5.9). We have thus shown that the right-hand side of (6.6.19) vanishes.

Now let us consider the left-hand side in (6.6.19). Recall from (6.3.30) that $\boldsymbol{\Lambda}^\varepsilon$ is identical in each component. It is therefore natural to assume that every term in the inner expansion of $\boldsymbol{\Lambda}^\varepsilon$ also has this property. Thus, recalling that $\partial_z\boldsymbol{\Phi}_0 \in T\Sigma^N$ point-wise, we infer

$$\int_{-\infty}^{\infty}\boldsymbol{\Lambda}_0\cdot\partial_z\boldsymbol{\Phi}_0\,\mathrm{d}z = \int_{-\infty}^{\infty}\Lambda_0\sum_{i=1}^{N}[\partial_z\boldsymbol{\Phi}_0]^i\,\mathrm{d}z = 0, \tag{6.6.22}$$

where Λ_0 denotes an arbitrary component of $\mathbf{\Lambda}_0$.

Recall from Theorem 6.3.12 that $\boldsymbol{\vartheta}^\varepsilon \in \mathbb{R}^N$ is constant. Thus, assuming that this property is transferred to the inner expansion, $\boldsymbol{\vartheta}_0$ is independent of z, we infer by means of the matching condition (6.5.9) that

$$\int_{-\infty}^{\infty} \boldsymbol{\vartheta}_0 \cdot \partial_z \mathbf{\Phi}_0 \,\mathrm{d}z = \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}z}[\boldsymbol{\vartheta}_0 \cdot \mathbf{\Phi}_0] \,\mathrm{d}z = \boldsymbol{\vartheta}_0 \cdot (\boldsymbol{e}_j - \boldsymbol{e}_i)\,. \tag{6.6.23}$$

Eventually, we want to justify that the remaining Lagrange multiplier fulfills

$$\int_{-\infty}^{\infty} \boldsymbol{\mu}_0 \cdot \partial_z \mathbf{\Phi}_0 \,\mathrm{d}z = 0. \tag{6.6.24}$$

Therefore, we recall (6.3.31) which tells us for $i = 1, \dots, N$ that

$$\mu_i^\varepsilon = 0 \quad \text{a.e. in } \Omega_i^+ = \{\boldsymbol{x} \in \Omega \,|\, \varphi_i^\varepsilon(\boldsymbol{x}) > 0\} = \Omega \backslash \{\boldsymbol{x} \in \Omega \;|\, \varphi_i^\varepsilon(\boldsymbol{x}) = 0\}.$$

Using [99, Lemma 7.7], we infer that for all $i \in \{1, \dots, N\}$,

$$\mu_i^\varepsilon \,\nabla_x \varphi_i^\varepsilon = \mathbf{0} \quad \text{a.e. in } \Omega. \tag{6.6.25}$$

Using (6.4.20) and comparing the terms of order $\mathcal{O}(\varepsilon^{-1})$, we deduce

$$\mu_0^i \,\partial_z \Phi_0^i \,\boldsymbol{\nu} = \mathbf{0} \quad \text{a.e. in } \Omega, \tag{6.6.26}$$

for all $i \in \{1, \dots, N\}$. In particular, by multiplying with $\boldsymbol{\nu}$ and integrating with respect to z from $-\infty$ to ∞, we arrive at

$$\int_{-\infty}^{\infty} \mu_0^i(z) \partial_z \Phi_0^i(z) \,\mathrm{d}z = 0.$$

for all $i \in \{1, \dots, N\}$. This proves (6.6.24).

Combining (6.6.20)–(6.6.24), we conclude from (6.6.19) that

$$\boldsymbol{\vartheta}_0 \cdot (\boldsymbol{e}_j - \boldsymbol{e}_i) = 0,$$

for all $i, j = 1, \dots, N$, meaning that all components of $\boldsymbol{\vartheta}_0$ are equal. Since $\boldsymbol{\vartheta}^\varepsilon \in T\Sigma^N$ in (6.3.32), we also assume $\boldsymbol{\vartheta}_0 \in T\Sigma^N$. This implies that $\boldsymbol{\vartheta}_0 = \mathbf{0}$ and thus, (6.6.18) can be rewritten as

$$\mathbf{0} = -\gamma \partial_{zz} \mathbf{\Phi}_0 + \gamma P_{T\Sigma}\left[\psi_0'(\mathbf{\Phi}_0)\right] - \mathbf{\Lambda}_0 - \boldsymbol{\mu}_0. \quad \text{around } \Gamma_{ij} \tag{6.6.27}$$

Let now $\tilde{z} \in \mathbb{R}$ be arbitrary. Multiplying (6.6.27) by $\partial_z \mathbf{\Phi}_0$ and integrating with respect to $\tilde{z}$ from $-\infty$ to $\tilde{z}$, we obtain

$$\int_0^{\tilde{z}} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}z} \left|\partial_z \mathbf{\Phi}_0\right|^2 \,\mathrm{d}z = \int_0^{\tilde{z}} \frac{\mathrm{d}}{\mathrm{d}z} \left[\psi_0(\mathbf{\Phi}_0)\right] \,\mathrm{d}z - \frac{1}{\gamma} \int_0^{\tilde{z}} (\mathbf{\Lambda}_0 + \boldsymbol{\mu}_0) \cdot \partial_z \mathbf{\Phi}_0 \,\mathrm{d}z.$$

Here, the last equality holds because of (6.6.22) and (6.6.24). By the fundamental theorem of calculus, we thus have

$$|\partial_z \mathbf{\Phi}_0(\tilde{z})|^2 - 2\psi_0(\mathbf{\Phi}_0(\tilde{z})) = |\partial_z \mathbf{\Phi}_0(0)|^2 - 2\psi_0(\mathbf{\Phi}_0(0))$$

for all $\tilde{z} \in \mathbb{R}$. We further know from the matching condition (6.5.9) that the left-hand side vanishes as $\tilde{z} \to \pm\infty$. This entails

$$|\partial_z \boldsymbol{\Phi}_0(0)|^2 - 2\psi_0(\boldsymbol{\Phi}_0(0)) = 0, \tag{6.6.28}$$

and thus, we arrive at

$$|\partial_z \boldsymbol{\Phi}_0(z)|^2 = 2\psi_0(\boldsymbol{\Phi}_0(z)) \quad \text{for all } z \in \mathbb{R}. \tag{6.6.29}$$

In order to obtain further information, we next show that (6.6.27) can be interpreted as the first-order optimality condition of a particular optimization problem that is similar to the minimization of the one-dimensional Ginzburg–Landau energy. Therefore, we first assume that

$$\sigma_{ij} \coloneqq \inf \left\{ \int_{-1}^{1} \sqrt{2\psi_0(\boldsymbol{\theta}(t))}\, |\boldsymbol{\theta}'(t)| \, \mathrm{d}t \;\middle|\; \begin{array}{l} \boldsymbol{\theta} \in C^{0,1}([0,1];\mathbb{R}^N),\ \boldsymbol{\theta} \in \boldsymbol{G} \text{ point-wise}, \\ \boldsymbol{\theta}(1) = \boldsymbol{e}_j \ \text{ and } \ \boldsymbol{\theta}(-1) = \boldsymbol{e}_i \end{array} \right\} \tag{6.6.30}$$

possesses a minimizer, which we call $\boldsymbol{\theta}_{ij}$. This means that $\boldsymbol{\theta}_{ij}$ is a geodesic with respect to the degenerate metric induced by the potential ψ_0 that connects the values $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$. Now, proceeding as in [148, proof of formula (15)], this geodesic can be used to construct a minimizer $\boldsymbol{\Phi}$ of the problem

$$\inf \left\{ \int_{-\infty}^{+\infty} |\partial_z \boldsymbol{\Phi}|^2 + 2\psi_0(\boldsymbol{\Phi})\, \mathrm{d}z \;\middle|\; \begin{array}{l} \boldsymbol{\Phi} \in C^{0,1}([0,1];\mathbb{R}^N),\ \boldsymbol{\Phi} \in \boldsymbol{G} \text{ point-wise}, \\ \lim\limits_{z\to\infty} \boldsymbol{\Phi}(z) = \boldsymbol{e}_j \ \text{ and } \ \lim\limits_{z\to-\infty} \boldsymbol{\Phi}(z) = \boldsymbol{e}_i \end{array} \right\}. \tag{6.6.31}$$

This means that $\boldsymbol{\Phi}$ describes an optimal transition between the values $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$. As in [148, proof of formula (15)], we further see that $\boldsymbol{\Phi}$ solves (6.6.27) and (6.6.29), where $\boldsymbol{\Lambda}_0 + \boldsymbol{\mu}_0$ is the Lagrange multiplier for the Gibbs–Simplex constraint. Consequently, choosing $\boldsymbol{\Phi}_0 = \boldsymbol{\Phi}$ we have found a solution of (6.6.27) and (6.6.29). Moreover, [148, formula (15)] states that $2\sigma_{ij}$ is exactly the value of the minimum sought in (6.6.31).

As the minimizer $\boldsymbol{\Phi}_0 = \boldsymbol{\Phi}$ of (6.6.31) satisfies (6.6.29), we further conclude

$$\sigma_{ij} = \int_{-\infty}^{\infty} |\partial_z \boldsymbol{\Phi}_0|^2 \, \mathrm{d}z = 2\int_{-\infty}^{\infty} \psi_0(\boldsymbol{\Phi}_0)\, \mathrm{d}z < \infty, \tag{6.6.32}$$

which will be important for later purposes.

Finally, we now consider (6.6.14) to the order $\mathcal{O}(1)$. Using (6.4.26) to reformulate the Laplacian, the $\mathcal{O}(1)$-contribution of the term $\gamma\varepsilon\Delta\boldsymbol{\varphi}$ reads as

$$\gamma \left(\partial_{zz} \boldsymbol{\Phi}_1 - \hat{\kappa} \partial_z \boldsymbol{\Phi}_0 \right).$$

For the $\mathcal{O}(1)$-contribution of the term $\frac{\gamma}{\varepsilon} P_{T\Sigma}[\psi_0'(\boldsymbol{\varphi})]$, we obtain

$$\gamma P_{T\Sigma} \left[\psi_0''(\boldsymbol{\Phi}_0) \boldsymbol{\Phi}_1 \right].$$

Recalling that (6.6.17) holds analogously for $\overline{\mathbb{C}}'(\boldsymbol{\Phi}_0)$ and using (6.6.8), we infer that the term

$$\sum_{r=1}^{l} \left([\partial_{\lambda_{nr}} \Psi](\lambda^{\varepsilon}_{n_1}, \ldots, \lambda^{\varepsilon}_{n_l}) P_{T\Sigma} \left[\mathbb{C}'(\boldsymbol{\varphi}^{\varepsilon}) \right] \mathcal{E}\left(\boldsymbol{w}^{\varepsilon}_{n_r} \right) : \mathcal{E}\left(\boldsymbol{w}^{\varepsilon}_{n_r} \right) \right),$$

has the $\mathcal{O}(1)$-contribution

$$\sum_{r=1}^{l}\Big([\partial_{\lambda_{n_r}}\Psi](\lambda_{0,n_1},\dots,\lambda_{0,n_l}) \\ \cdot\overline{\mathbb{C}}'(\mathbf{\Phi}_0)(\nabla_\Gamma \mathbf{W}_0+\partial_z\mathbf{W}_1\otimes\boldsymbol{\nu})^{\mathrm{sym}}:(\nabla_\Gamma \mathbf{W}_0+\partial_z\mathbf{W}_1\otimes\boldsymbol{\nu})^{\mathrm{sym}}\Big).$$

Moreover, the $\mathcal{O}(1)$-contribution of

$$\sum_{r=1}^{l}\left([\partial_{\lambda_{n_r}}\Psi](\lambda^\varepsilon_{n_1},\dots,\lambda^\varepsilon_{n_l})\lambda^\varepsilon_{n_r}P_{T\Sigma}\left[\rho'(\boldsymbol{\varphi}^\varepsilon)\right]|\boldsymbol{w}^\varepsilon_{n_r}|^2\right),$$

reads as

$$\sum_{r=1}^{l}\left([\partial_{\lambda_{n_r}}\Psi](\lambda_{0,n_1},\dots,\lambda_{0,n_l})\lambda_{0,n_r}\overline{\rho}'(\mathbf{\Phi}_0)|\mathbf{W}_{0,n_r}|^2\right).$$

Hence, dividing both sides resulting from (6.6.14) through γ we obtain

$$\begin{aligned}&\frac{1}{\gamma}(\mathbf{\Lambda}_1+\boldsymbol{\vartheta}_1+\boldsymbol{\mu}_1)+\partial_{zz}\mathbf{\Phi}_1-P_{T\Sigma}\left[\psi_0''(\mathbf{\Phi}_0)\mathbf{\Phi}_1\right]\\ &\quad=\hat\kappa\partial_z\mathbf{\Phi}_0+\frac{1}{\gamma}\sum_{r=1}^{l}\Big\{[\partial_{\lambda_{n_r}}\Psi](\lambda_{0,n_1},\dots,\lambda_{0,n_l})\\ &\qquad\cdot\Big[\overline{\mathbb{C}}'(\mathbf{\Phi}_0)(\nabla_\Gamma\mathbf{W}_{0,n_r}+\partial_z\mathbf{W}_{1,n_r}\otimes\boldsymbol{\nu})^{\mathrm{sym}}:(\nabla_\Gamma\mathbf{W}_{0,n_r}+\partial_z\mathbf{W}_{1,n_r}\otimes\boldsymbol{\nu})^{\mathrm{sym}}\\ &\qquad\quad-\lambda_{0,n_r}\overline{\rho}'(\mathbf{\Phi}_0)|\mathbf{W}_{0,n_r}|^2\Big]\Big\}\quad\text{around }\Gamma_{ij},.\end{aligned}\tag{6.6.33}$$

We now multiply this equation by $\partial_z\mathbf{\Phi}_0$ and integrate with respect to z from $-\infty$ to ∞. Let us consider each term of the resulting equation separately. Analogously to (6.6.22) and (6.6.23), we obtain

$$\int_{-\infty}^{\infty}\mathbf{\Lambda}_1\cdot\partial_z\mathbf{\Phi}_0\,\mathrm{d}z=0\quad\text{and}\tag{6.6.34}$$

$$\int_{-\infty}^{\infty}\boldsymbol{\vartheta}_1\cdot\partial_z\mathbf{\Phi}_0\,\mathrm{d}z=\boldsymbol{\vartheta}_1\cdot(\boldsymbol{e}_j-\boldsymbol{e}_i).\tag{6.6.35}$$

Considering the Lagrange multiplier $\boldsymbol{\mu}$, we recall (6.6.25),

$$\mu_i^\varepsilon\,\nabla_x\varphi_i^\varepsilon=\mathbf{0}\quad\text{a.e. in }\Omega.\tag{6.6.36}$$

Due to (6.4.20), its contribution of leading order $\mathcal{O}(1)$ in inner coordinates is given by

$$\mu_1^i\partial_z\Phi_0^i\boldsymbol{\nu}+\mu_0^i\nabla_\Gamma\Phi_0^i+\mu_0^i\partial_z\Phi_1^i\boldsymbol{\nu}=\mathbf{0}\quad\text{around }\Gamma_{ij},$$

for all $i\in\{1,\dots,N\}$. Multiplying this identity by $\boldsymbol{\nu}$ and integrating the resulting equation with respect to z, we infer

$$\int_{-\infty}^{\infty}\boldsymbol{\mu}_1\cdot\partial_z\mathbf{\Phi}_0\,\mathrm{d}z=-\int_{-\infty}^{\infty}\boldsymbol{\mu}_0\cdot\partial_z\mathbf{\Phi}_1\,\mathrm{d}z.\tag{6.6.37}$$

Furthermore, applying integration by parts twice and using that due to the matching condition (6.5.4) all derivatives of $\mathbf{\Phi}_0$ with respect to z tend to 0 as $z \to \pm\infty$, we obtain

$$\int_{-\infty}^{\infty} \partial_{zz}\mathbf{\Phi}_1 \cdot \partial_z\mathbf{\Phi}_0 \,\mathrm{d}z = \int_{-\infty}^{\infty} \partial_{zz}\left(\partial_z\mathbf{\Phi}_0\right) \cdot \mathbf{\Phi}_1 \,\mathrm{d}z. \tag{6.6.38}$$

As $\partial_z\mathbf{\Phi}_0$ attains its values only in $T\Sigma^N$, we deduce

$$\int_{-\infty}^{\infty} P_{T\Sigma}\left[\psi_0''(\mathbf{\Phi}_0)\mathbf{\Phi}_1\right] \cdot \partial_z\mathbf{\Phi}_0 \,\mathrm{d}z = \int_{-\infty}^{\infty} \psi_0''(\mathbf{\Phi}_0)\,\partial_z\mathbf{\Phi}_0 \cdot \mathbf{\Phi}_1 \,\mathrm{d}z \tag{6.6.39}$$

due to the symmetry of the Hessian matrix. Moreover, recalling that $\mathbf{W}_0$ is independent of z due to (6.6.8), a simple computation yields

$$\begin{aligned}\int_{-\infty}^{\infty} \overline{\rho}'(\mathbf{\Phi}_0)\partial_z\mathbf{\Phi}_0 \left|\mathbf{W}_0\right|^2 \,\mathrm{d}z &= \int_{-\infty}^{\infty} \left[\frac{\mathrm{d}}{\mathrm{d}z}\overline{\rho}(\mathbf{\Phi}_0)\right] \left|\mathbf{W}_0\right|^2 \,\mathrm{d}z \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}z}\left[\overline{\rho}(\mathbf{\Phi}_0)\left|\mathbf{W}_0\right|^2\right] \,\mathrm{d}z.\end{aligned} \tag{6.6.40}$$

Furthermore, by the definition of the dyadic product, it holds

$$\begin{aligned}&\left(\nabla_\Gamma\mathbf{W}_0 + \partial_z\mathbf{W}_1 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}} \boldsymbol{\nu} \cdot \partial_{zz}\mathbf{W}_1 \\ &\quad = \left(\nabla_\Gamma\mathbf{W}_0 + \partial_z\mathbf{W}_1 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}} : \left(\partial_{zz}\mathbf{W}_1 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}}.\end{aligned}$$

Now we use (6.6.8) (which directly entails $\partial_z\nabla_\Gamma\mathbf{W}_0 = \mathbf{0}$), (6.6.10) and $\partial_z\boldsymbol{\nu} = \mathbf{0}$ to deduce

$$\begin{aligned}&\int_{-\infty}^{\infty} \overline{\mathbb{C}}'(\mathbf{\Phi}_0)\partial_z\mathbf{\Phi}_0 \left(\nabla_\Gamma\mathbf{W}_0 + \partial_z\mathbf{W}_1 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}} : \left(\nabla_\Gamma\mathbf{W}_0 + \partial_z\mathbf{W}_1 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}} \,\mathrm{d}z \\ &\quad = \int_{-\infty}^{\infty} \left[\frac{\mathrm{d}}{\mathrm{d}z}\overline{\mathbb{C}}(\mathbf{\Phi}_0)\right] \left(\nabla_\Gamma\mathbf{W}_0 + \partial_z\mathbf{W}_1 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}} : \left(\nabla_\Gamma\mathbf{W}_0 + \partial_z\mathbf{W}_1 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}} \,\mathrm{d}z \\ &\quad = \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}z}\left[\overline{\mathbb{C}}(\mathbf{\Phi}_0) \left(\nabla_\Gamma\mathbf{W}_0 + \partial_z\mathbf{W}_1 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}} : \left(\nabla_\Gamma\mathbf{W}_0 + \partial_z\mathbf{W}_1 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}}\right] \,\mathrm{d}z \\ &\qquad - 2\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}z}\left[\overline{\mathbb{C}}(\mathbf{\Phi}_0) \left(\nabla_\Gamma\mathbf{W}_0 + \partial_z\mathbf{W}_1 \otimes \boldsymbol{\nu}\right)^{\mathrm{sym}} \boldsymbol{\nu} \cdot \partial_z\mathbf{W}_1\right] \,\mathrm{d}z\end{aligned} \tag{6.6.41}$$

by means of the product rule and integration by parts.

Collecting (6.6.34)–(6.6.41) and recalling (6.6.32), we eventually obtain

$$\begin{aligned}&\boldsymbol{\vartheta}_1 \cdot (\boldsymbol{e}_j - \boldsymbol{e}_i) \\ &\quad + \int_{-\infty}^{\infty} \left(\partial_{zz}\left(\partial_z\mathbf{\Phi}_0\right) - \psi_0''\left(\mathbf{\Phi}_0\right)\partial_z\mathbf{\Phi}_0\right) \cdot \mathbf{\Phi}_1 \,\mathrm{d}z - \int_{-\infty}^{\infty} \boldsymbol{\mu}_0 \cdot \partial_z\mathbf{\Phi}_1 \,\mathrm{d}z \\ &= -\frac{1}{\gamma}\sum_{r=1}^{l} \left[\partial_{\lambda_{n_r}}\Psi\right]\left(\lambda_{0,n_1}, \ldots, \lambda_{0,n_l}\right) \lambda_{0,n_r} \left(\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}z}\left(\overline{\rho}(\mathbf{\Phi}_0)\left|\mathbf{W}_{0,n_r}\right|^2\right) \,\mathrm{d}z\right) \\ &\quad + \sigma_{ij}\hat{\kappa} + \frac{1}{\gamma}\sum_{r=1}^{l} \Bigg\{\left[\partial_{\lambda_{n_r}}\Psi\right]\left(\lambda_{0,n_1}, \ldots, \lambda_{0,n_l}\right) \\ &\qquad \cdot \left[\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}z}\Big(\overline{\mathbb{C}}(\mathbf{\Phi}_0)(...)^{\mathrm{sym}} : (...)^{\mathrm{sym}}\Big) \,\mathrm{d}z \right. \\ &\qquad\quad \left. - 2\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}z}\left(\overline{\mathbb{C}}(\mathbf{\Phi}_0)(...)^{\mathrm{sym}}\boldsymbol{\nu} \cdot \partial_z\mathbf{W}_{1,n_r}\right) \,\mathrm{d}z\right]\Bigg\}\end{aligned} \tag{6.6.42}$$

on Γ_{ij}, where $(...)^{\mathrm{sym}}$ abbreviates $\big(\nabla_\Gamma \mathbf{W}_{0,n_r} + \partial_z \mathbf{W}_{1,n_r} \otimes \boldsymbol{\nu}\big)^{\mathrm{sym}}$. Next, we want to show that

$$\int_{-\infty}^{\infty} (\partial_{zz}(\partial_z \boldsymbol{\Phi}_0) - \psi_0''(\boldsymbol{\Phi}_0)\partial_z \boldsymbol{\Phi}_0) \cdot \boldsymbol{\Phi}_1 \,\mathrm{d}z - \int_{-\infty}^{\infty} \boldsymbol{\mu}_0 \cdot \partial_z \boldsymbol{\Phi}_1 \,\mathrm{d}z = 0. \tag{6.6.43}$$

Differentiating (6.6.27) with respect to z, multiplying by $\boldsymbol{\Phi}_1$ and integrating the resulting equation with respect to z, we deduce

$$\int_{-\infty}^{\infty} (\partial_{zz}(\partial_z \boldsymbol{\Phi}_0) - \psi_0''(\boldsymbol{\Phi}_0)\partial_z \boldsymbol{\Phi}_0) \cdot \boldsymbol{\Phi}_1 \,\mathrm{d}z = -\int_{-\infty}^{\infty} [\partial_z(\boldsymbol{\Lambda}_0 + \boldsymbol{\mu}_0)] \cdot \boldsymbol{\Phi}_1 \,\mathrm{d}z.$$

Thus, in order to prove (6.6.43), it suffices to show

$$\int_{-\infty}^{\infty} [\partial_z(\boldsymbol{\Lambda}_0 + \boldsymbol{\mu}_0)] \cdot \boldsymbol{\Phi}_1 \,\mathrm{d}z + \int_{-\infty}^{\infty} \boldsymbol{\mu}_0 \cdot \partial_z \boldsymbol{\Phi}_1 \,\mathrm{d}z = 0. \tag{6.6.44}$$

By means of integration by parts and the product rule, the left-hand side can be reformulated as

$$\begin{aligned}\int_{-\infty}^{\infty} [\partial_z(\boldsymbol{\Lambda}_0 + \boldsymbol{\mu}_0)] \cdot \boldsymbol{\Phi}_1 \,\mathrm{d}z = &-\int_{-\infty}^{\infty} (\boldsymbol{\Lambda}_0 + \boldsymbol{\mu}_0) \cdot \partial_z \boldsymbol{\Phi}_1 \,\mathrm{d}z \\ &+ \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}z} \left[(\boldsymbol{\Lambda}_0 + \boldsymbol{\mu}_0) \cdot \boldsymbol{\Phi}_1\right] \mathrm{d}z.\end{aligned}$$

Now as $\boldsymbol{\Phi}_1, \partial_z \boldsymbol{\Phi}_1 \in T\Sigma^N$ point-wise, we know as in (6.6.22)

$$\boldsymbol{\Lambda}_0 \cdot \partial_z \boldsymbol{\Phi}_1 = \boldsymbol{\Lambda}_0 \cdot \boldsymbol{\Phi}_1 = 0.$$

Thus, as we want to prove (6.6.44), it remains to show

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}z} \left[\boldsymbol{\mu}_0 \cdot \boldsymbol{\Phi}_1\right] \mathrm{d}z = 0. \tag{6.6.45}$$

Recalling once more formula (6.3.31), we infer

$$[\mu^\varepsilon]^i [\varphi^\varepsilon]^i = 0 \quad \text{a.e. in } \Omega$$

for all $i \in \{1, \dots, N\}$. Hence, for the $\mathcal{O}(1)$-contribution and the $\mathcal{O}(\varepsilon)$-contribution of the inner expansions, we obtain the relations

$$\mu_0^i(z)\, \Phi_0^i(z) = 0 \quad \text{and} \quad \mu_1^i(z)\, \Phi_0^i(z) = -\mu_0^i(z)\, \Phi_1^i(z), \tag{6.6.46}$$

respectively, for all $i \in \{1, \dots, N\}$ and $z \in \mathbb{R}$. Now, the first equation in (6.6.46) implies that for any $z \in \mathbb{R}$ with $\Phi_0^i(z) \neq 0$, we have $\mu_0^i(z) = 0$ and thus also $\mu_0^i(z)\Phi_1^i(z) = 0$. On the other, for all $z \in \mathbb{R}$ with $\Phi_0^i(z) = 0$, we infer from the second equation in (6.6.46) that $\mu_0^i(z)\Phi_1^i(z) = 0$. Combining both statements, we conclude

$$\mu_0^i(z)\, \Phi_1^i(z) = 0 \quad \text{for all } i \in \{1, \dots, N\} \text{ and } z \in \mathbb{R}.$$

This proves (6.6.45). By the above considerations, this verifies (6.6.44) which in turn implies equation (6.6.43).

To conclude this section, we recall the definition of the *jump*, see (6.5.11). Moreover, we recall from (6.4.27) that the mean curvature of Γ_{ij} is given by $\kappa_{ij} = -\nabla_{\Gamma_{ij}} \cdot \boldsymbol{n}_{\Gamma_{ij}}$. Using the matching conditions (6.5.8), (6.6.9) and (6.6.12), we finally infer from (6.6.42) that

$$\begin{aligned}(\vartheta_1^j - \vartheta_1^i) = \sigma_{ij}\kappa_{ij} &- \frac{1}{\gamma}\sum_{j=1}^{l}[\partial_{\lambda_{n_r}}\Psi]\,(\lambda_{0,n_1},\dots,\lambda_{0,n_l})\,\lambda_{0,n_r}\left[\overline{\rho}\,|\boldsymbol{w}_{0,n_r}|^2\right]_i^j \\ &+ \frac{1}{\gamma}\sum_{r=1}^{l}\Big\{[\partial_{\lambda_{n_r}}\Psi]\,(\lambda_{0,n_1},\dots,\lambda_{0,n_l}) \\ &\cdot\Big([\overline{\mathbb{C}}\mathcal{E}(\boldsymbol{w}_{0,n_r}) : \mathcal{E}(\boldsymbol{w}_{0,n_r})]_i^j - 2[\overline{\mathbb{C}}\mathcal{E}(\boldsymbol{w}_{0,n_r})\boldsymbol{\nu}\cdot\nabla\boldsymbol{w}_{0,n_r}\boldsymbol{\nu}]_i^j\Big)\Big\}\end{aligned} \tag{6.6.47}$$

on Γ_{ij} for all $i,j \in \{1,\dots,N-1\}$. In the case $j = N$ and $i \neq N$, equation (6.6.42) simplifies to

$$\begin{aligned}(\vartheta_1^j - \vartheta_1^i) = \;\; \sigma_{ij}\kappa_{ij} &+ \frac{1}{\gamma}\sum_{r=1}^{l}[\partial_{\lambda_{n_r}}\Psi]\,(\lambda_{0,n_1},\dots,\lambda_{0,n_l})\,\lambda_{0,n_r}\,\rho^i\,|(\boldsymbol{w}_{0,n_r})_i|^2 \\ &- \frac{1}{\gamma}\sum_{r=1}^{l}[\partial_{\lambda_{n_r}}\Psi]\,(\lambda_{0,n_1},\dots,\lambda_{0,n_l})\,\mathbb{C}^i\mathcal{E}_i(\boldsymbol{w}_{0,n_r}) : \mathcal{E}_i(\boldsymbol{w}_{0,n_r})\end{aligned} \tag{6.6.48}$$

on Γ_{iN} by the matching in (6.6.13) and (6.6.5).

To conclude this section we note that according to [32, Section 5.3] or [46, Section 2.4] equation (6.6.27) induces a further solvability condition, namely an angle condition for triple junctions. So let us assume that the regions $\Omega_i, \Omega_j, \Omega_k$ meet at a triple point m_{ijk} where $k \in \{1,\dots N\}\setminus\{i,j\}$. Then the angle condition is expressed via the normals of the three meeting interfaces as follows

$$\sigma_{ij}\boldsymbol{n}_{\Gamma_{ij}} + \sigma_{jk}\boldsymbol{n}_{\Gamma_{jk}} + \sigma_{ki}\boldsymbol{n}_{\Gamma_{ki}} = 0 \quad \text{in } m_{ijk}. \tag{6.6.49}$$

This relation immediately implies the angle condition

$$\frac{\sin(\theta_{ij})}{\sigma_{ij}} = \frac{\sin(\theta_{jk})}{\sigma_{jk}} = \frac{\sin(\theta_{ki})}{\sigma_{ki}},$$

where θ_{ij} denotes the angle between $\boldsymbol{n}_{\Gamma_{ij}}$. For the choice $\psi_0(\boldsymbol{\varphi}) = \frac{1}{2}(1-\boldsymbol{\varphi}\cdot\boldsymbol{\varphi})$ we deduce that the transition energy encoded by $\sigma_{ij},\sigma_{jk},\sigma_{ki}$ is always identical by exploiting the symmetry of this potential in (6.6.30). Thus, in this case we obtain that triple junctions occur always at a 120° contact angle.

6.7. The sharp-interface problem

Now we are in a position to state the complete problem that is obtained from (SE^ε) and (GI^ε) in the sharp-interface situation.

6.7.1. The sharp-interface limit of the state equation

Therefore, we recall that the domain Ω is partitioned into N regions Ω_i for $i = 1,\dots,N$ representing the presence of the i-th material ($i < N$) or void ($i = N$) in its pure form.

Those regions are separated by interfaces Γ_{ij}. Furthermore we have chosen $\boldsymbol{\eta}_{\Gamma_{ij}}$ to be the unit normal vector field on Γ_{ij} pointing from Ω_i into the region Ω_j. This means that

$$x + \delta\boldsymbol{\eta}_{\Gamma_{ij}}(x) \in \Omega_j \quad \text{and} \quad x - \delta\boldsymbol{\eta}_{\Gamma_{ij}}(x) \in \Omega_i \quad x \in \Gamma_{ij} \text{ and } \delta > 0.$$

To capture the behavior of a function $\boldsymbol{v}$ across the interface Γ_{ij}, we defined its *jump* by

$$[\boldsymbol{v}]_i^j(x) \coloneqq \lim_{\delta \searrow 0} \Big(\boldsymbol{v}(x + \delta\boldsymbol{\eta}_{\Gamma_{ij}}(x)) - \boldsymbol{v}(x - \delta\boldsymbol{\eta}_{\Gamma_{ij}}(x)) \Big),$$

for all $x \in \Gamma_{ij}$, see (6.5.11).

Combining the equations (SE_0^i) derived in Claim 6.4.1 and the jump conditions obtained in (6.6.9) and (6.6.13), we obtain the system

$$\begin{cases} -\nabla \cdot (\mathbb{C}^i \mathcal{E}(\boldsymbol{w}_{0,n_r})) &= \lambda_{0,n_r} \rho^i \boldsymbol{w}_{0,n_r} & \text{in } \Omega_i, \\ \left[\mathbb{C}\mathcal{E}(\boldsymbol{w}_{0,n_r})\boldsymbol{n}_{\Gamma_{ij}}\right]_i^j &= \boldsymbol{0} & \text{on } \Gamma_{ij}, \\ [\boldsymbol{w}_{0,n_r}]_i^j &= \boldsymbol{0} & \text{on } \Gamma_{ij}, \\ \mathbb{C}^i \mathcal{E}_i(\boldsymbol{w}_{0,n_r})\boldsymbol{n}_{\Gamma_{iN}} &= \boldsymbol{0} & \text{on } \Gamma_{iN}, \\ \mathbb{C}^i \mathcal{E}(\boldsymbol{w}_{0,n_r})\boldsymbol{n} &= \boldsymbol{0} & \text{on } \Gamma_0 \cap \partial\Omega_i, \\ \boldsymbol{w}_{0,n_r} &= \boldsymbol{0} & \text{on } \Gamma_D \cap \partial\Omega_i, \end{cases} \tag{SE_r^{ij}}$$

for $i, j = 1, \dots, N-1$ and $r = 1, \dots, l$, as the *sharp-interface limit of the state equation* (SE^ε). Here, $\boldsymbol{w}_{0,n_r}$ is normalized in the material regions, i.e.,

$$1 = \sum_{i=1}^{N-1} \int_{\Omega_i} \rho^i \, |\boldsymbol{w}_{0,n_r}|^2 \, \mathrm{d}x. \tag{6.7.1}$$

Furthermore, we infer from (6.6.13) that

$$[\boldsymbol{w}_{0,n_r}]_i^N = \boldsymbol{0} \quad \text{on } \Gamma_{iN} \tag{6.7.2}$$

for all $i \in \{1, \dots, N-1\}$ and each $r \in \{1, \dots, l\}$. However, this condition does not provide any additional information as we do not know how $\boldsymbol{w}_{0,n_r}$ behaves in the void region. In particular, we see that by interpreting (SE_r^{ij}) as one system of PDEs in the material region $\bigcup_{i=1}^{N-1} \Omega_i$, the homogeneous Neumann boundary condition in the fourth line of (SE_r^{ij}) is enough to obtain a closed system.

Combining the Neumann type jump condition on Γ_{ij} stated in the second line of (SE_r^{ij}) with the normality condition (6.7.1), we are able to obtain the beautiful relation

$$\int_{\Omega^M} \mathbb{C}^M \, \mathcal{E}(\boldsymbol{w}_{0,n_r}) : \mathcal{E}(\boldsymbol{w}_{0,n_r}) \, \mathrm{d}x = \lambda_{0,n_r}, \tag{6.7.3}$$

with

$$\Omega^M := \bigcup_{i=1}^{N-1} \Omega_i \quad \text{and} \quad \mathbb{C}^M := \left(\sum_{i=1}^{N-1} \mathbb{C}^i \, \mathbb{1}_{\Omega_i} \right),$$

where $\mathbb{1}_{\Omega_i}$ denotes the characteristic function on Ω_i. This means that the eigenvalue λ_{0,n_r} in the sharp-interface setting is indeed solely determined by an eigenvalue equation on the material region Ω^M but does not have any contribution from the void region.

To verify (6.7.3), we test (SE_r^{ij}) with $\boldsymbol{w}_{0,n_r}$ and integrate by parts. This yields

$$\begin{aligned}\int_{\Omega_i} \mathbb{C}^i \mathcal{E}(\boldsymbol{w}_{0,n_r}) : \mathcal{E}(\boldsymbol{w}_{0,n_r}) \,\mathrm{d}x - \int_{\partial\Omega_i} \mathbb{C}^i \mathcal{E}(\boldsymbol{w}_{0,n_r}) \boldsymbol{n}_{\Gamma_i} \cdot \boldsymbol{w}_{0,n_r} \,\mathrm{d}\Gamma = \lambda_{0,n_r} \\ = \int_{\Omega_i} \rho^i \left|\boldsymbol{w}_{0,n_r}\right|^2 \,\mathrm{d}x,\end{aligned} \tag{6.7.4}$$

for all $i \in \{1, \dots, N-1\}$, where $\boldsymbol{n}_{\Gamma_i}$ stands for the outer unit normal vector field of $\partial\Omega_i$. Noticing that the outer unit normal vector simply switches its sign on neighboring boundaries, we now use the second and the fourth line of (SE_r^{ij}) to infer

$$\sum_{i=1}^{N-1} \int_{\partial\Omega_i} \mathbb{C}^i \mathcal{E}(\boldsymbol{w}_{0,n_r}) \boldsymbol{n}_{\Gamma_i} \cdot \boldsymbol{w}_{0,n_r} \,\mathrm{d}\Gamma = 0.$$

Thus, summing the equations (6.7.4) from $i = 1$ to $N-1$ and using property (6.7.1), we conclude

$$\sum_{i=1}^{N-1} \int_{\Omega_i} \mathbb{C}^i \mathcal{E}(\boldsymbol{w}_{0,n_r}) : \mathcal{E}(\boldsymbol{w}_{0,n_r}) \,\mathrm{d}x = \lambda_{0,n_r}.$$

By the linearity of the integral, this directly proves (6.7.3).

Remark 6.7.1. As a refinement of Remark 6.4.2 (a), we now see that as long as at least one of the material regions $\Omega_1, \dots, \Omega_{N-1}$ shares a sufficiently nice part of its boundary with Γ_D, we can apply Korn's inequality in order to deduce that all λ_{0,n_r} are strictly positive. From a physical point of view, this is reasonable since if the material region Ω^M of the structure is not attached to some fixed boundary the shape can freely move within the design domain just by translation without exhibiting any vibrations, see also Remark 2.2.3.

6.7.2. The sharp-interface limit of the first-order optimality condition

Now let us turn to the limit of the gradient inequality (GI^ε). For the sake of completeness, let us restate our final results from the previous section, i.e., (6.6.47) and (6.6.48). We have

$$\begin{aligned}0 = \gamma\sigma_{ij}\kappa_{ij} - \sum_{j=1}^{l} [\partial_{\lambda_{n_r}}\Psi]\left(\lambda_{0,n_1}, \dots, \lambda_{0,n_l}\right) \lambda_{0,n_r} \left[\overline{\rho}\left|\boldsymbol{w}_{0,n_r}\right|^2\right]_i^j + \gamma(\vartheta_1^i - \vartheta_1^j) \\ + \sum_{r=1}^{l} \Big\{ [\partial_{\lambda_{n_r}}\Psi]\left(\lambda_{0,n_1}, \dots, \lambda_{0,n_l}\right) \\ \cdot \Big([\overline{\mathbb{C}}\mathcal{E}(\boldsymbol{w}_{0,n_r}) : \mathcal{E}(\boldsymbol{w}_{0,n_r})]_i^j - 2[\overline{\mathbb{C}}\mathcal{E}(\boldsymbol{w}_{0,n_r})\boldsymbol{\nu} \cdot \nabla \boldsymbol{w}_{0,n_r}\boldsymbol{\nu}]_i^j \Big) \Big\}\end{aligned} \tag{6.7.5}$$

on Γ_{ij} for all $i,j=1\dots,N-1$, and

$$\begin{aligned}0 &= \gamma\sigma_{iN}\kappa_{iN} + \sum_{r=1}^{l}[\partial_{\lambda_{n_r}}\Psi]\left(\lambda_{0,n_1},\dots,\lambda_{0,n_l}\right)\lambda_{0,n_r}\,\rho^i\,|(\boldsymbol{w}_{0,n_r})_i|^2 + \gamma\big(\vartheta_1^i-\vartheta_1^N\big)\\ &\quad - \sum_{r=1}^{l}[\partial_{\lambda_{n_r}}\Psi]\left(\lambda_{0,n_1},\dots,\lambda_{0,n_l}\right)\mathbb{C}^i\mathcal{E}_i(\boldsymbol{w}_{0,n_r}):\mathcal{E}_i(\boldsymbol{w}_{0,n_r})\end{aligned} \tag{6.7.6}$$

on Γ_{iN} for all $i=1\dots,N-1$ if $j=N$.

Here σ_{ij} is defined as in (6.6.30) and stands for the total energy of a transition across the interface Γ_{ij}. The vector $\boldsymbol{\vartheta}_1\in\mathbb{R}^N$ denotes the $\mathcal{O}(\varepsilon)$-contribution of the Lagrange-multiplier resulting from the integral constraint $\fint_\Omega\boldsymbol{\varphi}^\varepsilon\,\mathrm{d}x=\boldsymbol{m}$ that is hidden in the condition $\boldsymbol{\varphi}^\varepsilon\in\boldsymbol{\mathcal{G}}^{\boldsymbol{m}}$ (cf. Theorem 6.3.12). Recalling (6.6.49), we additionally have the triple junction condition at any junction point m_{ijk} with $i,j,k=1,\dots,N$

$$\sigma_{ij}\boldsymbol{n}_{\Gamma_{ij}}+\sigma_{jk}\boldsymbol{n}_{\Gamma_{jk}}+\sigma_{ki}\boldsymbol{n}_{\Gamma_{ki}}=0\quad\text{in } m_{ijk}.$$

6.7.3. The sharp-interface optimality system in the case of one material

We now want to state above equations for the simplest case of only one single material (i.e., $N=2$) as this is the scenario we further study in the subsequent sections.

In this case, we have $\Omega=\Omega^M\cup\Omega^V$, where Ω^M and Ω^V denote the material and the void parts of the domain, respectively. We now denote the interface separating the two phases by Γ_{MV}, its outer unit normal vector field by $\boldsymbol{n}_{\Gamma_{MV}}$ and its mean curvature by $\kappa_{MV}=-\nabla_{\Gamma_{MV}}\cdot\boldsymbol{n}_{\Gamma_{MV}}$. Using the notation $\Gamma_D^M\coloneqq\Gamma_D\cap\partial\Omega^M$ and $\Gamma_0^M\coloneqq\Gamma_0\cap\partial\Omega^M$, we obtain from (SE_r^{ij}), (6.7.1) and (6.7.2) the state equation

$$\begin{cases} -\nabla\cdot\Big(\mathbb{C}^M\mathcal{E}(\boldsymbol{w}_{0,n_r})\Big) &= \lambda_{0,n_r}\rho^M\boldsymbol{w}_{0,n_r} & \text{in } \Omega^M,\\ \mathbb{C}^M\mathcal{E}_M(\boldsymbol{w}_{0,n_r})\,\boldsymbol{n}_{\Gamma_{MV}} &= \boldsymbol{0} & \text{on } \Gamma_{MV},\\ \boldsymbol{w}_{0,n_r} &= \boldsymbol{0} & \text{on } \Gamma_D^M,\\ \mathbb{C}^M\mathcal{E}(\boldsymbol{w}_{0,n_r})\,\boldsymbol{n} &= \boldsymbol{0} & \text{on } \Gamma_0^M,\end{cases} \tag{SE_r^{MV}}$$

for $r=1,\dots,l$, along with the first-order necessary optimality condition

$$\begin{aligned}0 &= \gamma\,\sigma_{MV}\,\kappa_{MV} + \sum_{r=1}^{l}[\partial_{\lambda_{n_r}}\Psi]\left(\lambda_{0,n_1},\dots,\lambda_{0,n_l}\right)\lambda_{0,n_r}\rho^M\,|(\boldsymbol{w}_{0,n_r})_M|^2\\ &\quad - \sum_{r=1}^{l}[\partial_{\lambda_{n_r}}\Psi]\left(\lambda_{0,n_1},\dots,\lambda_{0,n_l}\right)\mathbb{C}^M\mathcal{E}_M(\boldsymbol{w}_{0,n_r}):\mathcal{E}_M(\boldsymbol{w}_{0,n_r})+\gamma\left(\vartheta_1^1-\vartheta_1^2\right)\end{aligned} \tag{6.7.7}$$

on Γ_{MV}. This means that the functions $\boldsymbol{w}_{0,n_r}$ are eigenfunctions to the eigenvalues λ_{0,n_r} which essentially solve the eigenvalue problem for the elasticity equation subject to a homogeneous Neumann boundary condition on the shape Ω^M.

Remark 6.7.2. Note that, in general, one cannot predict the behavior of solutions to (SE_r^{MV}). If Ω^M is merely a set of finite perimeter that does not have a Lipschitz boundary or if $\Gamma_{MV}\cap\Gamma_D^M=\emptyset$, the classical spectral theory from Theorem 2.2.8 does not necessarily

provide us with an infinite sequence of positive eigenvalues. Nevertheless, as we want to consider a well-posed minimization problem and want to calculate shape derivatives associated to this problem, we assume that these issues do not occur. In particular, we always assume Ω^M to be sufficiently smooth and $\partial\Omega^M$ to have a suitably nice intersection with Γ_D^M such that an infinite sequence of positive eigenvalues actually exists (see also Remark 6.7.1).

6.8. Relating the first-order optimality condition to classical shape calculus

We now want to compare the above results, especially (6.7.7), to the results in [5], which were obtained using shape calculus. Our goal is to justify that the gradient equality (6.7.7) is indeed the first-order condition of a sharp-interface eigenvalue optimization problem, which is formally the limit of the diffuse-interface problem we started with. Therefore, we need to fit the notation of [5] to our setting.

As above consider the situation $N = 2$, i.e., $\Omega = \Omega^M \cup \Omega^V$. Denote with $P_\Omega(\Omega^M)$ the relative perimeter of the shape Ω^M, which is given by the Hausdorff measure $\mathcal{H}^{d-1}(\partial\Omega^M \cap \Omega)$ provided that Ω^M is non-empty and sufficiently smooth, see Section 2.2.4. Furthermore, we consider a prescribed mass $m = |\Omega^M| < |\Omega|$. In order to be consistent with the notation used in the previous chapters, we choose $\boldsymbol{m} = (m_1, m_2)^T \in \Sigma^2$ with $m_1 = m|\Omega|^{-1}$ and $m_2 = 1 - m_1$. Then the sharp-interface structural optimization problem that we intend to approximate via our diffuse-interface problem $(\mathcal{P}_l^\varepsilon)$ reads as

$$
\begin{cases}
\min \quad J(\Omega^M) \coloneqq \Psi(\lambda_{n_1}, \dots, \lambda_{n_l}) + \gamma\, \sigma_{MV}\, P_\Omega(\Omega^M), \\
\text{over} \quad \mathcal{U}^{\mathrm{ad}} = \left\{ \Omega^M \subset \Omega : |\Omega^M| = m \right\}, \\
\text{s.t.} \quad (SE_r^{MV}) \begin{cases}
-\nabla \cdot \left(\mathbb{C}^M \mathcal{E}(\boldsymbol{w}_{n_r}) \right) &= \lambda_{n_r} \rho^M \boldsymbol{w}_{n_r} \qquad \text{in } \Omega^M, \\
\mathbb{C}^M \mathcal{E}_M(\boldsymbol{w}_{n_r})\, \boldsymbol{n}_{\Gamma_{MV}} &= \boldsymbol{0} \qquad \text{on } \Gamma_{MV}, \\
\mathbb{C}^M \mathcal{E}(\boldsymbol{w}_{n_r})\, \boldsymbol{n} &= \boldsymbol{0} \qquad \text{on } \Gamma_0^M, \\
\boldsymbol{w}_{n_r} &= \boldsymbol{0} \qquad \text{on } \Gamma_D^M,
\end{cases} \\
\qquad \text{for all } r \in \{1, \dots, l\}.
\end{cases}
\tag{$\mathcal{P}_l^0$}
$$

This problem is the sharp-interface limit problem associated to the diffuse-interface problem $(\mathcal{P}_l^\varepsilon)$, where the side condition is exactly the sharp-interface state equation (SE_r^{MV}) and the perimeter $\sigma_{MV} P_\Omega(\Omega^M)$ is the rigorous Γ-limit of the Ginzburg–Landau energy, see [23]. We recall that the constant σ_{MV} we obtained in (6.6.30) is exactly the one obtained in [23] in terms of the rigorous Γ-limit, which is denoted by $d(\boldsymbol{e}_i, \boldsymbol{e}_j)$ there. In particular, σ_{MV} is independent of the shape Ω^M.

In cases an ambiguity might arise, we indicate the shape dependency explicitly in the eigenfunctions and eigenvalues, i.e., we write $(\lambda_{n_r}(\Omega^M), \boldsymbol{w}_{n_r}(\Omega^M))$ for $r = 1, \dots, l$.

Now, we want to apply the calculus of shape derivatives from [5, Theorem 2.5] to our situation. We obtain the following statement.

Theorem 6.8.1. *Let Ω^M be a smooth bounded open set and let $\boldsymbol{\theta} \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ with*

$$\boldsymbol{\theta} \cdot \boldsymbol{n}_{\Gamma_{\partial\Omega^M}} = 0 \quad \text{on } \partial\Omega^M \backslash \Gamma_{MV}.$$

We further assume that for $r = 1, \dots, l$, the eigenfunctions $\boldsymbol{w}_{n_r}(\Omega^M)$ in (SE_r^{MV}) are sufficiently smooth, say $\boldsymbol{w}_{n_r}(\Omega^M) \in H^2(\Omega^M; \mathbb{R}^d)$.

Then, if the involved eigenvalues λ_{n_r} for $r = 1, \dots, l$ are all simple, the shape derivative of J at the shape Ω^M in the direction $\boldsymbol{\theta}$ fulfills the equation

$$\begin{aligned} J'(\Omega^M)(\boldsymbol{\theta}) = \sum_{r=1}^{l} \Bigg\{ & [\partial_{\lambda_{n_r}} \Psi](\lambda_{n_1}(\Omega^M), \dots, \lambda_{n_l}(\Omega^M)) \\ & \cdot \Bigg[\int_{\Gamma_{MV}} \mathbb{C}^M \mathcal{E}(\boldsymbol{w}_{n_r}(\Omega^M)) : \mathcal{E}(\boldsymbol{w}_{n_r}(\Omega^M)) \boldsymbol{\theta} \cdot \boldsymbol{n}_{\Gamma_{MV}} \, \mathrm{d}\mathcal{H}^{d-1} \\ & \quad - \lambda_{n_r}(\Omega^M) \int_{\Gamma_{MV}} \rho^M |\boldsymbol{w}_{n_r}(\Omega^M)|^2 \boldsymbol{\theta} \cdot \boldsymbol{n}_{\Gamma_{MV}} \, \mathrm{d}\mathcal{H}^{d-1} \Bigg] \Bigg\} \\ & - \int_{\Gamma_{MV}} \gamma \sigma_{MV} \, \kappa_{MV} \, \boldsymbol{\theta} \cdot \boldsymbol{n}_{\Gamma_{MV}} \, \mathrm{d}\mathcal{H}^{d-1}. \end{aligned} \tag{6.8.1}$$

Here, the shape derivative of J at a shape Ω^M is defined as the Fréchet-derivative of the functional

$$W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d) \to \mathbb{R}, \quad \boldsymbol{\zeta} \mapsto J((\mathrm{Id} + \boldsymbol{\zeta})\Omega^M)$$

evaluated at $\boldsymbol{\zeta} = \mathbf{0}$.

Remark 6.8.2.

(a) Note that the simplicity of eigenvalues is crucial here. Only then it is guaranteed that the eigenvalues and eigenfunctions depend on the domain Ω^M in a differentiable way, see also the discussion in Section 5.4.2. For a comprehensive overview over the differentiability of spectral quantities with respect to the domain we refer to [110, Section 5.7].

(b) For $\boldsymbol{\zeta} \in W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$ the application

$$T_{\boldsymbol{\zeta}} : \mathbb{R}^d \to \mathbb{R}^d, \quad x \mapsto (\mathrm{Id} + \boldsymbol{\zeta})(x),$$

is invertible if $\|\boldsymbol{\zeta}\|_{W^{1,\infty}} < 1$, and it holds $(\mathrm{Id} + \boldsymbol{\zeta})^{-1} - \mathrm{Id} \in W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$ with

$$\left\| (\mathrm{Id} + \boldsymbol{\zeta})^{-1} - \mathrm{Id} \right\|_{W^{1,\infty}} \leq \|\boldsymbol{\zeta}\|_{W^{1,\infty}} \, (1 - \|\boldsymbol{\zeta}\|_{W^{1,\infty}})^{-1}.$$

This means the family $(T_{\boldsymbol{\zeta}})_{\boldsymbol{\zeta} \in W^{1,\infty}}$ describes diffeomorphic perturbations of Ω^M "close" to Ω^M if $\|\boldsymbol{\zeta}\|_{W^{1,\infty}}$ is small, motivating the definition of the shape derivative above. For a detailed discussion of this concept, we refer to [110, Section 5.2].

Proof. We proceed analogously to [5, Theorem 2.5]. In the following, $\Omega_{\boldsymbol{\zeta}} = (\mathrm{Id} + \boldsymbol{\zeta})(\Omega^M)$ denotes the perturbation of Ω^M associated with a sufficiently small $\boldsymbol{\zeta} \in W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$. First of all, for $\boldsymbol{v}_{n_r} \in H^1(\mathbb{R}^d; \mathbb{R}^d)$ with $r = 1, \dots, l$, we introduce the Lagrangian

$$\mathcal{L}(\Omega_{\boldsymbol{\zeta}}, \boldsymbol{v}_{n_1}, \dots, \boldsymbol{v}_{n_l})$$

$$= \Psi\left(\frac{\int_{\Omega_\zeta} \mathbb{C}^M \mathcal{E}(\boldsymbol{v}_{n_1}) : \mathcal{E}(\boldsymbol{v}_{n_1})\,\mathrm{d}x}{\int_{\Omega_\zeta} \rho^M \left|\boldsymbol{v}_{n_1}\right|^2 \,\mathrm{d}x}, \dots, \frac{\int_{\Omega_\zeta} \mathbb{C}^M \mathcal{E}(\boldsymbol{v}_{n_l}) : \mathcal{E}(\boldsymbol{v}_{n_l})\,\mathrm{d}x}{\int_{\Omega_\zeta} \rho^M \left|\boldsymbol{v}_{n_l}\right|^2 \,\mathrm{d}x}\right) + \gamma\sigma_{MV} P(\Omega_\zeta)\,\mathrm{d}s.$$

For the partial Fréchet-derivatives of the Lagrangian with respect to $\boldsymbol{v}_{n_r}$ for $r = 1, \dots, l$ at the point $(\Omega_\zeta, \boldsymbol{w}_{n_1}(\Omega_\zeta), \dots, \boldsymbol{w}_{n_l}(\Omega_\zeta))$, we obtain

$$\partial_{\boldsymbol{v}_{n_r}} \mathcal{L}(\Omega_\zeta, \boldsymbol{w}_{n_1}(\Omega_\zeta), \dots, \boldsymbol{w}_{n_l}(\Omega_\zeta)) = 0. \tag{6.8.2}$$

This is simply due to the fact, that the derivative of the Rayleigh quotient

$$\mathcal{R}_\zeta : H^1(\mathbb{R}^d; \mathbb{R}^d) \to \mathbb{R}, \quad \boldsymbol{v} \mapsto \frac{\int_{\Omega_\zeta} \mathbb{C}^M \mathcal{E}(\boldsymbol{v}) : \mathcal{E}(\boldsymbol{v})\,\mathrm{d}x}{\int_{\Omega_\zeta} \rho^M \left|\boldsymbol{v}\right|^2 \,\mathrm{d}x},$$

evaluated at an eigenfunction $\boldsymbol{w}_n = \boldsymbol{w}_n(\Omega_\zeta)$ reads as

$$\mathcal{R}'_\zeta(\boldsymbol{w}_n)\boldsymbol{v} = \frac{2\int_{\Omega_\zeta} \mathbb{C}^M \mathcal{E}(\boldsymbol{w}_n) : \mathcal{E}(\boldsymbol{v})\,\mathrm{d}x \int_{\Omega_\zeta} \rho^M \left|\boldsymbol{w}_n\right|^2 \,\mathrm{d}x}{(\int_{\Omega_\zeta} \rho^M \left|\boldsymbol{w}_n\right|^2 \,\mathrm{d}x)^2} - \frac{2\int_{\Omega_\zeta} \mathbb{C}^M \mathcal{E}(\boldsymbol{w}_n) : \mathcal{E}(\boldsymbol{w}_n)\,\mathrm{d}x \int_{\Omega_\zeta} \rho^M \boldsymbol{w}_n \cdot \boldsymbol{v}\,\mathrm{d}x}{(\int_{\Omega_\zeta} \rho^M \left|\boldsymbol{w}_n\right|^2 \,\mathrm{d}x)^2}$$

and this vanishes due to (SE_r^{MV}).

On the other hand, recalling the definition of J in $(\mathcal{P}_l^0)$, we obviously have

$$J(\Omega_\zeta) = \mathcal{L}(\Omega_\zeta, \boldsymbol{w}_{n_1}(\Omega_\zeta), \dots, \boldsymbol{w}_{n_l}(\Omega_\zeta))$$

as the eigenvalues can be expressed by the corresponding Rayleigh quotients. Note that due to the differentiability of eigenfunctions as discussed in Remark 6.8.2, we can now apply the chain rule. Thus, using (6.8.2) we infer that the shape derivative is given by

$$\begin{aligned} J'(\Omega^M) &= \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\zeta}}[J((\mathrm{Id} + \boldsymbol{\zeta})(\Omega^M))]_{\boldsymbol{\zeta}=\mathbf{0}} \\ &= \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\zeta}}[\mathcal{L}((\mathrm{Id} + \boldsymbol{\zeta})(\Omega^M), \boldsymbol{w}_{n_1}(\Omega^M), \dots, \boldsymbol{w}_{n_l}(\Omega^M))]_{\boldsymbol{\zeta}=\mathbf{0}} \end{aligned}$$

Applying the formulas for shape derivatives in [5, Lemma 2.3], we deduce

$$\begin{aligned} J'(\Omega^M)(\boldsymbol{\theta}) = \sum_{r=1}^{l} \Bigg\{ & [\partial_{\lambda_{n_r}} \Psi](\lambda_{n_1}(\Omega^M), \dots, \lambda_{n_l}(\Omega^M)) \\ & \cdot \Bigg[\int_{\partial\Omega^M} \mathbb{C}^M \mathcal{E}(\boldsymbol{w}_{n_r}(\Omega^M)) : \mathcal{E}(\boldsymbol{w}_{n_r}(\Omega^M))\boldsymbol{\theta} \cdot \boldsymbol{n}_{\partial\Omega^M}\,\mathrm{d}\mathcal{H}^{d-1} \\ & - \lambda_{n_r}(\Omega^M) \int_{\partial\Omega^M} \rho^M \left|\boldsymbol{w}_{n_r}(\Omega^M)\right|^2 \boldsymbol{\theta} \cdot \boldsymbol{n}_{\partial\Omega^M}\,\mathrm{d}\mathcal{H}^{d-1} \Bigg] \Bigg\} \\ & - \int_{\partial\Omega^M} \gamma\sigma_{MV}\,\kappa_M\,\boldsymbol{\theta} \cdot \boldsymbol{n}_{\partial\Omega^M}\,\mathrm{d}\mathcal{H}^{d-1}. \end{aligned}$$

where κ_M denotes the mean curvature of $\partial\Omega^M$. By the assumption $\boldsymbol{\theta} \cdot \boldsymbol{n}_{\partial\Omega^M} = 0$ on $\partial\Omega^M \backslash \Gamma_{MV}$, the boundary integrals vanish on $\partial\Omega^M \backslash \Gamma_{MV}$ and we thus arrive at (6.8.1). Note that in [5], the mean curvature is defined as $\kappa = \nabla_{\partial\Omega^M} \cdot \boldsymbol{n}_{\partial\Omega^M}$, whereas (in accordance with (6.4.27)) our mean curvature is given by $\kappa = -\nabla_{\partial\Omega^M} \cdot \boldsymbol{n}_{\partial\Omega^M}$. This explains the negative sign of our term involving κ_M. □

Remark 6.8.3. The preceding theorem shows that using the approach of classical shape calculus and additionally taking the volume constraint $|\Omega^M| = m$ into account, we recover the gradient equality (6.7.7) since the volume constraint produces a Lagrange multiplier as in our previous analysis. This justifies our formal approach from the viewpoint of classical shape calculus since (6.7.7) can be interpreted as the first-order necessary optimality condition of the shape optimization problem $(\mathcal{P}_l^0)$.

6.9. Numerical examples

In the following, we present numerical results that illustrate the applicability of our approach to find optimal topologies. After a brief introduction of the numerical method, we investigate the dependence of solutions on the parameter ε in Section 6.9.1. Therefore, we study a particular setting of an elastic beam that is known from literature (cf. [5]). In Section 6.9.2, we consider a joint optimization of λ_1 and λ_2 for this beam setup, and in Section 6.9.3, we investigate an extended optimization problem to not only optimize the shape and topology of this beam with respect to its first eigenvalue but also its compliance.

As in Subsection 6.7.3 and Section 6.8, we restrict ourselves to the case of only two phases, i.e., material and void. In this situation, the vector-valued phase-field $\boldsymbol{\varphi} = (\varphi^1, \varphi^2)$ can be represented by a scalar order parameter

$$\varphi := \varphi^1 - \varphi^2 \in H^1(\Omega) \cap L^\infty(\Omega).$$

This means that φ attains its values in $[-1, 1]$, where „1“ represents the material and „-1“ represents the void. The elastic tensor $\mathbb{C}(\varphi)$ now is defined as

$$\mathbb{C}(\varphi)\mathcal{E}(w) := \alpha(\varphi)\big(2\mu\,\mathcal{E}(w) + \ell\,\mathrm{tr}\,\big(\mathcal{E}(w)\big)\,\mathcal{I}\big) \tag{6.9.1}$$

for Lamé parameters $\mu, \ell > 0$ and the quadratic interpolation function $\alpha(\varphi)$ satisfying $\alpha(1) = 1$, $\alpha(-1) = \underline{\alpha}\varepsilon^2$, and $\alpha'(-1) = 0$ for some constant $\underline{\alpha}$. The eigenvalue equation is given by

$$-\nabla \cdot [\mathbb{C}(\varphi)\mathcal{E}(w)] = \lambda\,\beta(\varphi)\rho\,w, \tag{6.9.2}$$

with the quadratic interpolation function $\beta(\varphi)$ satisfying $\beta(+1) = 1$, $\beta(-1) = \underline{\beta}\varepsilon^2$ and $\beta'(-1) = 0$ as well as an additional density ρ that might depend on the spatial variable. If not stated differently, we use $\underline{\alpha} = 10^{-2}$ and $\underline{\beta} = 10^{-4}$.

Remark 6.9.1. Recall the discussion about spurious eigenmodes in Section 6.4.2 which motivates the choice of the model in this numerical section. A slight difference compared to the setting proposed in Claim 6.4.3 is the scaling of the void components $\tilde{\mathbb{C}}^N$ and $\tilde{\rho}^N$ here reflected by $\alpha(-1)$ and $\beta(-1)$ respectively. Here the relatively lower scaling of α versus β in void regions is guaranteed by the prefactors $\underline{\alpha}$ and $\underline{\beta}$. Noting that in the

ε	$80{\cdot}10^{-3}$	$40{\cdot}10^{-3}$	$20\cdot10^{-3}$	$10\cdot10^{-3}$	$5\cdot10^{-3}$	$2.5\cdot10^{-3}$	$1.25\cdot10^{-3}$
$\gamma E^\varepsilon(\varphi)$	0.00119	0.00120	0.00117	0.00115	0.00114	0.00114	0.00114
λ_1	0.01577	0.01626	0.01658	0.01678	0.01692	0.01699	0.01703

Table 6.1: Scaled Ginzburg–Landau energy $\gamma E^\varepsilon(\varphi)$ and principal eigenvalue λ_1 of the optimal beam shape for decreasing values of ε. This indicates that the values $E^\varepsilon(\varphi)$ and λ_1 converge as ε decreases.

computations below ε is of order 10^{-2} one could absorb one ε into $\underline{\alpha}$ and arrive at the setting from Section 6.4.2, namely

$$\underline{\alpha} = \underline{\beta} = 10^{-4} \quad \text{and} \quad \alpha(-1) = \underline{\alpha}\varepsilon, \quad \beta(-1) = \underline{\beta}\varepsilon^2.$$

Numerical Solution Method. The numerical implementation is based on linear finite elements for all functions provided by the finite element package FEniCs [9,120] together with the PETSc linear algebra backend [21,22]. For the eigenvalue problem, we use the package SLEPc [112]. The optimization problem is solved by the VMPT method that is proposed in [36]. In our case, it can be understood as an extension of the projected gradient method into the space $H^1(\Omega) \cap L^\infty(\Omega)$. We refer to [36,98] and Chapter 3 for more details.

6.9.1. Numerical investigation of the sharp-interface limit $\varepsilon \to 0$

In this section, to illustrate the sharp-interface limit, we present numerical results for a sequence of decreasing values of ε.

We use the setup from [5, Sec. 7.1] to find a cantilever beam with maximal first eigenvalue, i.e., we choose $\Psi(\lambda_1) = -\lambda_1$. Our computational domain is given by $\Omega = (0,2) \times (0,1)$. The Young's modulus is $E = 1$ and Poisson's ratio is $\nu = 0.3$ leading to $\mu \approx 0.38$ and $\ell \approx 0.58$. We define the subset $\Omega_\rho = (1.9, 2.0) \times (0.45, 0.55)$ and set $\rho(x) = 1$ if $x \notin \Omega_\rho$ and $\rho(x) = 100$ if $x \in \Omega_\rho$. We also fix $\varphi(x) = 1$ for all $x \in \Omega_\rho$. The beam is supposed to be attached to the wall at the left boundary of Ω, i.e., at $\Gamma_D = \{(0,\eta) \mid \eta \in (0,1)\} \subset \partial\Omega$. This leads to the boundary condition $w = 0$ on Γ_D. We further set $\Gamma_0 = \partial\Omega \setminus \Gamma_D$ and we fix $\gamma = 10^{-4}$ and $\int_\Omega \varphi = 0$.

Similar as in [5, Sec. 7.1], we start our optimization process with a checkerboard type initial function given by $\varphi_0(x) = \operatorname{sign}\big(v(x)\big)\,|v(x)|^{0.3}$ with $v(x) = \cos(3\pi x_1)\cos(4\pi x_2)$ for all $x \in \Omega$. We want to emphasize that this problem is expected to have many local minima and thus, the choice of initial function can significantly influence the shape and topology of the local minimizer found by our numerical method.

We now solve the optimization problem for a decreasing sequence of values of ε. In Table 6.1, we present the values of ε together with the corresponding value of the Ginzburg–Landau energy $E^\varepsilon(\varphi) = \int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{2\varepsilon}(1-\varphi^2)\,\mathrm{d}x$ (i.e., the regular part is given as $\psi_0(s) = \frac{1}{2}(1-s^2)$) and the eigenvalue λ_1. Recall here that the values of the Ginzburg–Landau energy converge to a weighted perimeter of the shape in the sharp-interface limit $\varepsilon \to 0$. In Figure 6.1, we present the zero level lines of the (locally optimal) shapes we obtain for different values of ε. Here we started with $\varepsilon = 0.08$ and used the local optimum as initial value for subsequent simulations.

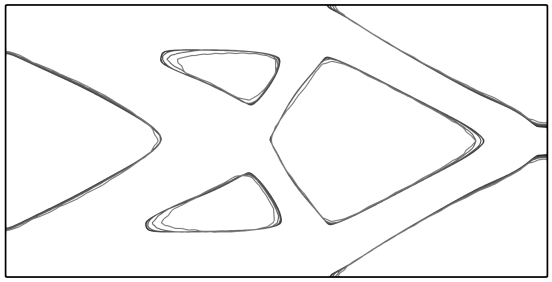

Figure 6.1: The zero level lines of the beam for the $\varepsilon \to 0$ test for all tested ε. The darker the line is, the smaller is ε. We observe that the interface seems to stabilize with decreasing values of ε and that it only mildly depends on ε.

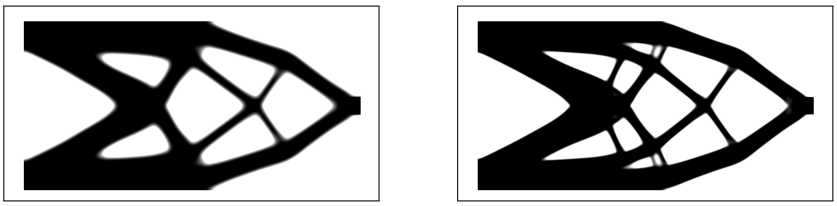

Figure 6.2: The optimal beam for $\Psi(\lambda_1) = -\lambda_1$, i.e., maximization of the principal eigenvalue for $\gamma = 10^{-4}$ (left) and $\gamma = 10^{-5}$ (right) with $\varepsilon = 0.02$. We clearly observe finer structures for smaller γ. We obtain $\lambda_1 = 1.68 \cdot 10^{-2}$ for $\gamma = 10^{-4}$ and $\lambda_1 = 1.72 \cdot 10^{-2}$ for $\gamma = 10^{-5}$. Thus, as expected, with less regularization we reach a larger value for λ_1.

6.9.2. Optimization of a beam

As a first test, we illustrate the influence of the regularization strength γ on the found structure. The parameter γ acts as a weight for the penalization of the length of the interface between void and material. Thus a smaller value of γ is expected to lead to thinner structures which contain more braces. Using the same setup as before, we solve again the optimization problem for the cantilever beam, but this time we fix $\varepsilon = 0.02$. We perform two simulations with $\gamma \in \{10^{-4}, 10^{-5}\}$. The smaller γ is chosen, the finer structures we expect. We also expect that we reach a larger value for λ_1, because less regularization is used.

In Figure 6.2, we present the found structures for these parameters. On the left we present the result for $\gamma = 10^{-4}$ and on the right for $\gamma = 10^{-5}$. As expected, it is clearly visible that the structure obtained for the smaller value of γ is finer and contains more braces. Additionally, decreasing γ also leads to sharper corners.

In a second test for the beam setup, we compare the numerical results for different choices of $\Psi(\lambda_1, \lambda_2)$ as a linear combination of λ_1 and λ_2. We set $\gamma = 10^{-4}$ and use the solution shown in Figure 6.2 as the initialization of the optimization method. In Figure 6.3, we present numerical results for this setting with the choice $\Psi(\lambda_1, \lambda_2) = -\lambda_1 - \alpha\lambda_2$ for $\alpha \in \{10^{-2}, 2 \cdot 10^{-2}, 6 \cdot 10^{-2}, 10^{-1}\}$. Moreover, in Table 6.2 we list the corresponding values of λ_1 and λ_2. Here, $\alpha = 0$ corresponds to the result shown in Figure 6.2 on the left.

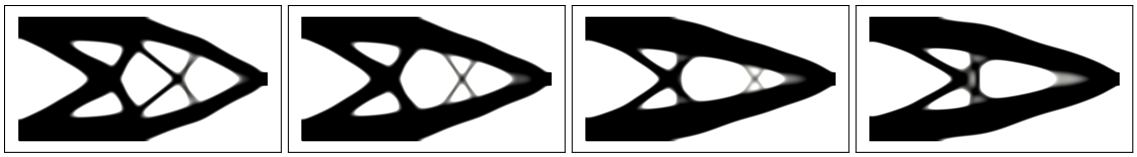

Figure 6.3: Optimization of the cantilever beam for $\gamma = 10^{-4}$ and $\Psi(\lambda_1, \lambda_2) = -\lambda_1 - \alpha\lambda_2$, where $\alpha \in \{10^{-2}, 2 \cdot 10^{-2}, 6 \cdot 10^{-2}, 10^{-1}\}$ (left to right). We observe that increasing the weight of λ_2 above a certain value reduces the amount of fine structures.

α	0	$1 \cdot 10^{-2}$	$2 \cdot 10^{-2}$	$6 \cdot 10^{-2}$	$1 \cdot 10^{-1}$
λ_1	$1.677 \cdot 10^{-2}$	$1.662 \cdot 10^{-2}$	$1.606 \cdot 10^{-2}$	$1.521 \cdot 10^{-2}$	$1.508 \cdot 10^{-2}$
λ_2	$9.181 \cdot 10^{-2}$	$11.874 \cdot 10^{-2}$	$15.178 \cdot 10^{-2}$	$17.663 \cdot 10^{-2}$	$18.047 \cdot 10^{-2}$

Table 6.2: The first and second eigenvalue (λ_1 and λ_2) for the optimal topologies for the beam example and $\Psi(\lambda_1, \lambda_2) = -\lambda_1 - \alpha\lambda_2$. As expected, for larger weights α we reach a lower value for λ_1 and a larger value for λ_2.

6.9.3. Joint optimization of compliance and principal eigenvalue

In this subsection, we extend the problem by using a linear combination of compliance and the first eigenvalue as objective, see Section 2.1.12. Recall that the state equation in the compliance problem asks for a displacement field $\boldsymbol{u} \in H^1(\Omega; \mathbb{R}^d)$ satisfying

$$\begin{aligned} -\nabla \cdot (\mathbb{C}(\varphi)\mathcal{E}(\boldsymbol{u})) &= \boldsymbol{0} \quad && \text{in } \Omega, \\ \boldsymbol{u_c} &= \boldsymbol{0} && \text{on } \Gamma_D \subset \partial\Omega, \\ [\mathbb{C}(\varphi)\mathcal{E}(\boldsymbol{u})] \cdot \boldsymbol{n} &= \boldsymbol{g} && \text{on } \Gamma_g \subset \partial\Omega, \\ [\mathbb{C}(\varphi)\mathcal{E}(\boldsymbol{u})] \cdot \boldsymbol{n} &= \boldsymbol{0} && \text{on } \Gamma_0 \subset \partial\Omega, \end{aligned} \tag{6.9.3}$$

which minimizes the objective $\int_{\Gamma_g} \boldsymbol{g} \cdot \boldsymbol{u} \, \mathrm{d}x$. Thus in the notation of Section 2.1.12 we choose the interior force $\boldsymbol{f} = \boldsymbol{0}$ in Ω and $\Gamma_C = \Gamma_D$.

Combining this with our eigenvalue optimization problem for $\Psi(\lambda_1) = -\alpha\lambda_1$ for some $\alpha > 0$, we arrive at

$$\begin{cases} \min J(\boldsymbol{u}, \lambda_1) = -\alpha\lambda_1 + \displaystyle\int_{\Gamma_g} \boldsymbol{g} \cdot \boldsymbol{u} \, \mathrm{d}x + \gamma E^\varepsilon(\varphi) \\ \qquad \text{s.t. } \boldsymbol{u} \text{ solves the compliance equation (6.9.3)}, \\ \qquad\quad \lambda_1 \text{ is the first eigenvalue of } (SE^\varepsilon). \end{cases} \tag{6.9.4}$$

This means that we are looking for a structure that simultaneously minimizes the compliance with respect to a given boundary force $\boldsymbol{g}$ and maximizes the first eigenvalue λ_1.

We use the same setup as in Section 6.9.2 for the beam example and fix $\gamma = 1 \cdot 10^{-3}$. Moreover, the exterior force is $\boldsymbol{g} = (0, -1)^T$ and acts on $\Gamma_g = \{(2.0, y) \mid y \in (0.45, 0.55)\}$. Note that Γ_g belongs to the boundary of the domain Ω_ρ on which we assume a higher value of the density ρ.

In Figure 6.4, we show numerical result for this setting for different values of α. We observe that the structures become finer when we increase the influence of the principal eigenvalue. In Table 6.3, we present the corresponding values for compliance and λ_1 for these shapes. As expected, we achieve a larger compliance when we increase the weight α

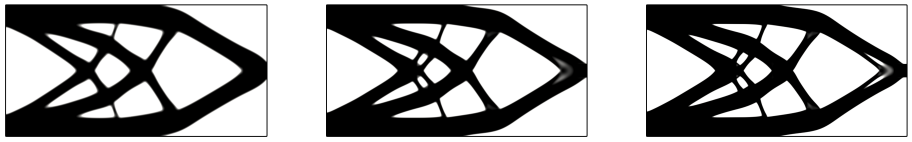

Figure 6.4: Numerical results for joint optimization of compliance and principle eigenvalue with weight $\alpha \in \{10, 100, 500\}$ (left to right). We observe that increasing the weight α of the first eigenvalue leads locally to a finer structure.

α	10	100	200	500
Compliance	0.5507	0.5629	0.5676	0.5769
λ_1	0.0164	0.0170	0.0172	0.0173

Table 6.3: Values of compliance and principal eigenvalue λ_1 for joint optimization of compliance and principle eigenvalue with weight $\alpha \in \{10, 100, 200, 500\}$. We observe that increasing α leads, as expected, to larger values of the principal eigenvalue and larger values for the compliance.

of the principal eigenvalue. Simultaneously, we also obtain larger values for the principal eigenvalue. It is worth mentioning that these results compare very well with the ones obtained in [5], where a level-set method was used to directly tackle the sharp-interface problem (see especially Fig. 2 and Fig. 5 in [5]).

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