Analysis of Fluid-Structure Interaction Problems with Growth

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Abstract

In this thesis, we study systems of nonlinear partial differential equations from applied science by mathematical analysis tools.

First, we focus on the local well-posedness of a series of fluid-structure interaction problems (FSI), which arise from plaque formation during the stage of the atherosclerotic lesion in human arteries. The blood is modeled by the incompressible Navier–Stokes equation, while the motion of the vessel is captured by nonlinear elasticity. The growth occurs when both cells in fluid and solid react, diffuse, and transport across the interface, resulting in the accumulation of foam cells, which are exactly seen as the plaques. We consider the following situations:

- A fluid-structure interaction problem with growth (FSIG) in a smooth domain with a kind of linearized Kelvin–Voigt viscoelasticity, including biochemical interactions and growth effects.
- A fluid-structure interaction problem with growth (FSIG) in a cylindrical domain, where fixed ninety-degree contact angles are concerned, leading to more difficulties for the analysis.
- A quasi-stationary fluid-structure interaction problem with growth (QFSIG) in a smooth domain, where the elasticity is assumed to be an equilibrium for each time, and the linearized system is a parabolic-elliptic mixed-type problem.

The proofs rely on a fixed-point argument, where the most crucial part is the analysis of the linearized systems, which causes remarkable differences and technical difficulties in different cases.

In the last part, we establish the existence of global weak solutions to a diffuse interface model of incompressible viscoelastic flows, which is in fact, a model proposed originally to handle the problem of fluid-structure interaction. More specifically, the fluids are assumed to be macroscopically immiscible, but with a small transition region, where the two components are partially mixed. Considering the elasticity of both components, one ends up with a coupled Oldroyd-B/Cahn−Hilliard type system, which describes the behavior of two-phase viscoelastic fluids. In particular cases by choosing suitable coefficients, one can recover the fluid-phase and elastic-phase respectively. We prove the existence of weak solutions to the system in two dimensions for general (unmatched) mass densities, variable viscosities, different shear moduli, and a class of physically relevant and singular free energy densities that guarantee that the order parameter stays in the physically reasonable interval. To this end, we propose a novel regularization of the original system and a new hybrid implicit time discretization for the regularized system, while new compactness arguments are used to pass to the final limit.

Zusammenfassung

In dieser Arbeit untersuchen wir Systeme nichtlinearer partieller Differentialgleichungen aus den angewandten Wissenschaften mit Methoden der mathematischen Analysis.

Zunächst konzentrieren wir uns auf die lokale Wohlgestelltheit einer Reihe von Fluid-Struktur-Interaktionsproblemen (FSI), die sich aus der Plaquebildung während des Stadiums der atherosklerotischen Läsion in menschlichen Arterien ergeben. Das Blut wird durch die inkompressible Navier–Stokes-Gleichung modelliert, während die Bewegung des Gefäßes durch eine nichtlineare Elastizitätsgleichung erfasst wird. Das Wachstum tritt auf, wenn sowohl Zellen in Flüssigkeit als auch in festem Material reagieren, diffundieren und über die Grenzfläche transportiert werden, was zu einer Ansammlung von Schaumzellen führt, die in Form von Plaque zu sehen sind. Wir betrachten die folgenden Situationen:

- Ein Fluid-Struktur-Interaktionsproblem mit Wachstum (FSIG) in einem glatten Gebiet mit einer Art linearisierter Kelvin–Voigt-Viskoelastizität, einschließlich biochemischer Wechselwirkungen und Wachstumseffekte.
- Ein Fluid-Struktur-Wechselwirkungsproblem mit Wachstum (FSIG) in einem zylindrischen Gebiet, bei dem feste Neunzig-Grad-Kontaktwinkel betrachtet werden, was zu größeren Schwierigkeiten bei der Analyse führt.
- Ein quasistationäres Fluid-Struktur-Interaktionsproblem mit Wachstum (QFSIG) in einem glatten Gebiet, bei dem die Elastizität für jede Zeit als Gleichgewicht angenommen wird und das linearisierte System ein parabolisch-elliptisches Problem gemischten Typs ist.

Die Beweise beruhen auf einem Fixpunktargument, wobei der wichtigste Teil die Analyse der linearisierten Systeme ist, die in verschiedenen Fällen bemerkenswerte Unterschiede aufweisen und zu technischen Schwierigkeiten führen.

Im letzten Teil wird die Existenz globaler schwacher Lösungen für ein diffuses Grenzflächenmodell für inkompressible viskoelastische Strömungen nachgewiesen. Genauer gesagt wird davon ausgegangen, dass die Flüssigkeiten makroskopisch nicht mischbar sind, jedoch mit einem kleinen Übergangsbereich, in dem die beiden Komponenten teilweise vermischt sind. Berücksichtigt man die Elastizität beider Komponenten, so erhält man ein gekoppeltes System vom Typ Oldroyd-B/Cahn–Hilliard, das das Verhalten zweiphasiger viskoelastischer Fluide beschreibt. In bestimmten Fällen kann man durch die Wahl geeigneter Koeffzienten die flüssige und die elastische Phase beschreiben. Wir beweisen die Existenz schwacher Lösungen des Systems in zwei Dimensionen für allgemeine (verschiedene) Massendichten, variable Viskositäten, verschiedene Schermodule und eine Klasse physikalisch relevanter, singulärer freier Energiedichten, die garantieren, dass der Ordnungsparameter im physikalisch sinnvollen Intervall bleibt. Zu diesem Zweck schlagen wir eine neuartige Regularisierung des ursprünglichen Systems und eine neue hybride implizite Zeitdiskretisierung für das regularisierte System vor, während neue Kompaktheitsargumente verwendet werden, um letztendlich zur Grenze überzugehen.

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To my parents and fiancée

Contents

Introduction

This thesis focuses on the study of nonlinear partial differential equations, which arise from physics, biology, and materials sciences, in particular, fluid mechanics, elasticity, and their interactions. Fluid-structure interactions involving the coupling of fluid mechanics and solid mechanics have been studied intensively by engineers, physicists, and also mathematicians, due to a broad range of applications in various areas, for example, hydro- and aero-elasticity, or biomechanics [BGN14, Dow15, GR10, Kal+18, Pai14, Ric17]. In this thesis, we investigate fluid-structure interaction problems that couple incompressible viscous flow, (visco-) morphoelasticity, biochemical processes, as well as the interactions on the interface.

Motivation from Biology

Atherosclerosis, a chronic disease of the arterial wall, has been a major cause of cardiovascular disease with high mortality rates worldwide for many years. Various compelling hypotheses regarding the pathophysiology of atherosclerotic lesion formation and complications such as myocardial infarction and stroke were proposed by constant research from medicine and biomedical engineering into the disease. The motivation for this thesis was triggered by the developed process of atherosclerotic lesions. In particular, we focus on a specific stage *plaque formation* during atherosclerosis, including the adhesion of blood leukocytes to an activated endothelial cell monolayer, the targeted migration of the bound leukocytes (white blood cells) into the intima, the maturation of monocytes (the most numerous of the leukocytes recruited) into macrophages, and their uptake of lipid, producing foam cells [LRH11]. This procedure is referred to as the concept of *atherogenesis*.

Mathematical Modeling: Free Boundary Problems

Before exploring detailed analysis, let us explain the processes briefly by means of partial differential equations in a domain as sketched in Figure 0.1, which is roughly a three-dimensional vertical section of half of the artery. The specific derivation from continuum mechanics is given in Chapter 1, while the original idea was developed in [Yan+16].

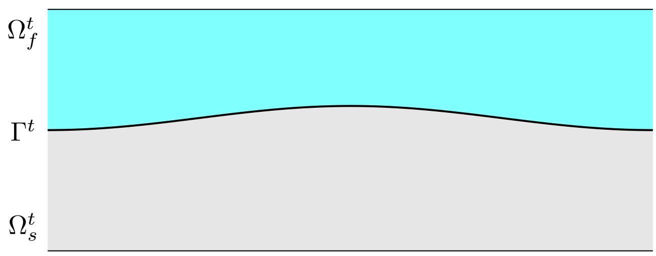

Figure 0.1: A two-dimensional sketch of a segment of artery

As shown in Figure 0.1, the blood flows in the upper part Ω_f^t and the vessel moves in the lower part Ω_s^t, while they are disjointly separated by a sharp interface Γ^t. Here the blood and the vessel are both assumed to be homogeneous, meaning that there is no stratification of the vessel (e.g. intima, medial, adventitia). This is rather a simplifying assumption to make the first-step analysis. The sharp interface Γ^t is referred to as the endothelial layer, where the targeted migration of monocytes (e.g. leukocytes) takes place through Γ^t.

Free interface – when fluid meets solid. Free boundary problems arise when the domain on which systems are solved changes with time t, such that the dynamics of the domain are determined by some unknowns. Physically, this occurs e.g. when a free interface between the fluid and its medium is present. Such situations are very common, for example, the ocean-atmosphere interface on earth, the elastic walls of blood vessels, or the surface of a star, where at least one side of them is modeled as a fluid. Thus, particularly in our case, it is reasonable and physical to consider the interface between blood and vessel to be *free*, indicated by t.

Blood – incompressible viscous fluid. The blood flow is assumed to be modeled by the classical incompressible Navier–Stokes system in Ω^t_f, $t > 0$, see also Section 1.3 and [BF13],

$$\begin{cases} \rho(\partial_t + v \cdot \nabla)v + \nabla p = \mu \Delta v, \\ \operatorname{div} v = 0. \end{cases}$$

where ρ is the material density, the unknowns $v(x,t) \in \mathbb{R}^d$ and $p(x,t) \in \mathbb{R}$ are the Eulerian velocity and pressure respective. Here the system describes the motion of an incompressible homogeneous viscous Newtonian fluid, by the momentum balance and mass conversation.

Vessel – incompressible morphoelastic solid. The vessel is regarded to as an incompressible hyperelastic solid involving volumetric growth in Ω^t_s, $t > 0$, in other word, *volumetric morphoelasticity*, see also Sections 1.4 and 1.6, and [Gor17, Part IV],

$$\begin{cases} F = F_e F_g, \\ \partial_t \rho + \operatorname{div}(\rho v) = \rho\gamma, \\ \rho(\partial_t + v \cdot \nabla)v + \nabla p = \operatorname{div} T, \\ T^\top = T, \\ T = \dfrac{\partial W}{\partial F_e} F_e^\top - pI, \end{cases}$$

where $F = \nabla\varphi$ is the induced deformation gradient by the motion φ, which is assumed further to satisfy a multiplicative decomposition (the first identity above). Similar to the fluid, ρ, v, p are the density, Eulerian velocity and pressure of the solid material, and the movement is captured by conservation of mass and momentum balance. Note that a growth rate function $\rho\gamma$ contributes to the continuity equation due to volumetric growth and the stress tensor T for the hyperelastic solid is endowed with a general form of the stored energy density function W in terms of F_e, which is exactly the evidence of the change of mechanical properties of the vessel due to the influence of foam cells.

In particular, we will further partially employ the incompressible Neo-Hookean material with stored energy density W satisfying

$$W(F_e) = \frac{\mu}{2} \operatorname{tr}(F_e F_e^\top - I).$$

Cells – advection-reaction-diffusion processes. The biochemical processes take place both in the blood and vessel, where the cells dynamics are depicted by the advection-reaction-diffusion equations, see Section 1.5, namely,

$$\begin{cases} \partial_t c_f + v \cdot \nabla c_f + \operatorname{div}(D_f \nabla c_f) = 0, \\ \partial_t c_s + v \cdot \nabla c_s + \operatorname{div}(D_s \nabla c_s) = -\beta c_s, \\ \partial_t c^* + v \cdot \nabla c^* = \beta c_s, \end{cases}$$

where D_f, D_s are the diffusion coefficients of cells in blood and vessel respectively, and β is a constant. In the blood, we only consider the concentration of monocytes denoted by c_f, such as leukocytes (white blood cells), while inside the vessel macrophages with concentration c_s uptake nutrients and lips, and produce foam cells whose concentration is indicated by c^*. The above equations can be interpreted as conservations of the concentrations of species, with change rate of cells which are determined by the transported amount of material, the diffusion in space, as well as the reactions between them. Here the accumulation of foam cells in the vessel turns to be the main source of plaque growth.

Coupling – interactions on interface. So far we only see separate partial differential equations on each domain. The coupling mechanism arise on the interface, where we have:

- Fluid-structure interaction:
  - Kinematic condition: The velocity of the fluid and the velocity of the solid are continuous on the interface.
  - Dynamic condition: The normal stresses of fluid and solid are continuous on the interface.
- Cells interaction (transmission conditions):
  - The normal concentration fluxes are continuous on the interface
  - The difference of cell concentrations across the interface is entailed by the concentration flux entering or leaving the blood domain and by the permeability of the vessel wall.

In addition to the coupling on the interface, there is a basic geometric condition for the *free interface*: we always assume that the interface does not develop singularities in the sense of self-intersection or boundary contact.

A diffuse interface approximation. An alternative method to model such multi-phase coupling free boundary problems is the so-called *phase-field approach* (also called *diffuse-interface model*). Alike classical free boundary models, it employs a continuum perspective to describe the evolution of each phase. Compared to sharp interface models, an advantage of diffuse interface models is that they allow for topology changes like break up and coalescence of interfaces, which is rather difficult and tricky for analysis encountering sharp interfaces. In addition, phase-field methods can be used numerically without an explicit tracking of the interface, saving much of the effort for numerical schemes.

The basic idea of phase-field models is to approximate the sharp interface by a small transition region with a certain small thickness, where the two components are partially mixed, see Figure 0.2. This can be realized by the *Ginzburg–Landau* free energy

$$\mathcal{E} = \int_\Omega \tilde{\sigma} \Big(\frac{\epsilon}{2} |\nabla \phi|^2 + \frac{1}{\epsilon} W(\phi) \Big) \mathrm{d}x,$$

where $\tilde{\sigma}$ is a coefficient related to the surface tensor and $\epsilon > 0$ is proportional to the thickness of the interface. The quantity ϕ is the order parameter indicating the different phases for the value

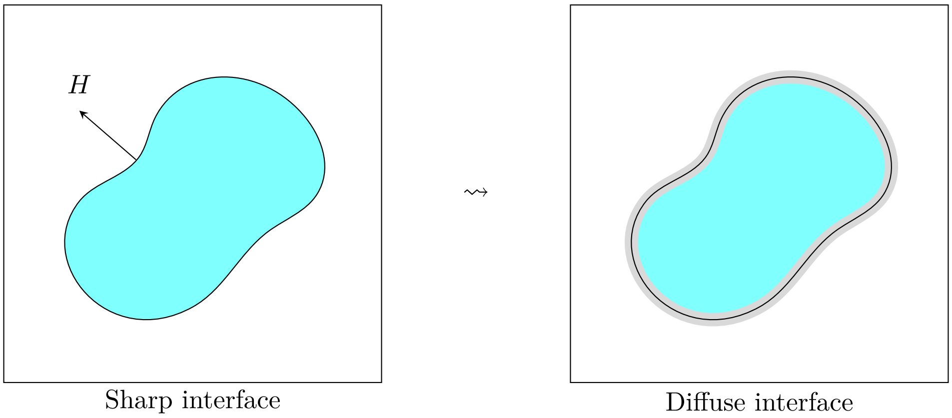

Figure 0.2: Approximation of the sharp interface

$\{\phi \gtrless \pm 1 \mp \delta\}$ for some small $\delta > 0$, while lies in the interval $(\delta - 1, 1 - \delta)$ in the small transition region, whose diagram of values looks like the function $\tanh(s)$, see Figure 0.3. The function W denotes the potential, which is typically endowed with the shape of "double-well", which is an approximation of the interfacial energy. Note that the first term in the *Ginzburg–Landau* energy penalizes the jumps of ϕ.

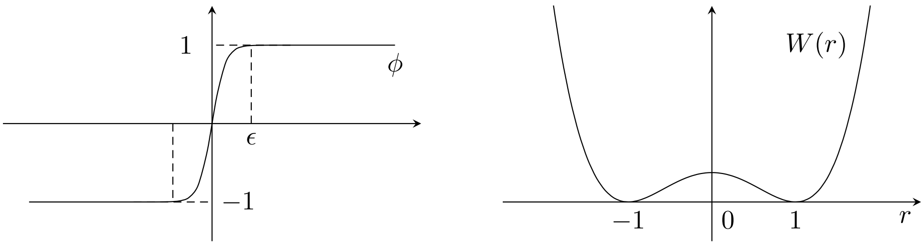

Figure 0.3: Typical shapes of ϕ (left) and W (right)

When including the kinetic energy and elastic energy of the solid, the total energy of the diffuse interface model becomes

$$\mathcal{E} := \underbrace{\int_\Omega \frac{\rho(\phi)}{2} |\mathbf{u}|^2 \, \mathrm{d}x}_{\text{Kinetic energy}} + \underbrace{\int_\Omega \frac{\mu(\phi)}{2} \operatorname{tr}(\mathbb{B} - \ln \mathbb{B} - \mathbb{I}) \, \mathrm{d}x}_{\text{Elastic energy}} + \underbrace{\int_\Omega \tilde{\sigma} \Big(\frac{\epsilon}{2} |\nabla \phi|^2 + \frac{1}{\epsilon} W(\phi) \Big) \, \mathrm{d}x}_{\text{Free energy}},$$

where $\rho(\phi)$ and $\mu(\phi)$ depending on ϕ are the density and elastic shear modulus of the material. The function $\mathbf{u}$ denotes the *volume-averaged velocity* and $\mathbb{B}$ is the Eulerian *left Cauchy–Green stress tensor*. For more detailed derivation of the model and explanations, see Section 1.8.

Goals and Rough Ideas

This thesis is devoted to studying the existence of solutions for such systems. Specifically, the local well-posedness of the sharp interface models and the global existence of weak solutions to the diffuse interface model will be explored.

For the sharp interface models, as the interface is *free*, it is usually convenient to perform the transformation to *Lagrangian coordinates* to get a *quasilinear* system with fixed interfaces. Then one is able to prove the short existence via the Banach fixed-point argument, for which we linearize the nonlinear problem around the initial state, and proceed with Lipschitz estimates for nonlinear parts in terms of the solution. The linear analysis relies on the *maximal L^p-regularity theory*, from which there are different difficulties depending on scenarios.

For the diffuse interface model, we will consider the case of unmatched densities, different viscosities and shear moduli, and a large type of logarithmic potential, which is rather physical and reasonable. The idea to show the existence of weak solutions is to use a regularization and weak convergence methods for nonlinear PDEs. More precisely, a *hybrid time-discretization scheme* is proposed to approximate the regularized system, and weak/weak* compactness arguments are employed for the time-averaged sequences. Then by good uniform estimates in terms of the regularization, one passes to the limit of the regularized system via weak convergence methods.

We shall state the main features and technical analysis with more details in each chapter.

Structure of the Thesis

This thesis is organized as follows. In Chapter 1, we recall the fundamental framework of continuum mechanics and derive the mathematical models we will deal with. The kinetics of continua is provided first, as well as several conservation laws. The specific cases of fluids and solids are discussed subsequently, and the morphoelasticity with respect to growth is pointed out since biochemical processes are considered. Then we couple the system at a free interface by certain interfacial conditions. Moreover, a thermodynamically consistent derivation of a diffuse interface model is presented for a two-phase incompressible viscoelastic flow. Finally, we summarize notations and results from basic tensor analysis.

In Chapter 2, we introduce some necessary function spaces with their properties that will be frequently used throughout the thesis. In addition, a short introduction of *maximal L^p-regularity theory* is enclosed.

From Chapter 3 to 5, we solve the fluid-structure interaction problems in different cases by a standard Banach fixed-point argument, while the linear analysis and nonlinear estimates distinguish a lot from each other. More specifically, a fluid-structure interaction problem in a smooth domain for plaque growth including viscoelasticity is solved in Chapter 3, where for the linearized system we prove the well-posedness of a two-phase Stokes problem with mixed boundary condition. Furthermore, in Chapter 4, we consider the situation when the fluid encounters a ninety-degree contact angle interacting with the solid, for which a localization procedure and reflection arguments are employed. Additionally in Chapter 5, a more realistic model consisting of a non-stationary system for an incompressible viscous fluid and a quasi-stationary system for a hyperelastic solid is carried out, which is motivated by the different time scales of the movement of fluid and solid. In this case, the system is of parabolic-elliptic type and one can make use of a lower-order anisotropic Bessel potential space for the solid displacement to ensure the regularities of solutions are compatible with the dynamic condition on the surface. In the end of each chapter, we include some necessary analysis of auxiliary problems.

Finally in Chapter 6, the global existence of weak solutions to a diffuse interface model for incompressible two-phase viscoelastic flows is shown via a regularization argument. More precisely, the well-posedness of an Oldroy-B type tensor-valued equation is proved by means of a standard Galerkin approximation and an entropy regularization. Combining a compactness argument, a hybrid time-discretization scheme is proposed to deal with the regularized system.

Disclaimer

The results of Chapters 3 to 6 are contained in [AL23a], [AL23b], [AL23c], and [LT22] respectively. In particular, all work contained in Chapters 3 to 5 is joint work with Helmut Abels, while the work in Chapter 6 is joint work with Dennis Trautwein.

Chapter 1

Mathematical Models

In this chapter, we briefly sketch the mathematical models of this thesis, including the necessary basis from continuum mechanics, cf. [BF13, Cia88, EGK17, GFA10, Gor17, KR19, Ric17].

Conventions of notations. Throughout this chapter, we, if applicable, employ the *Einstein summation convention* according to which summation over the range $1, \ldots, d$ is applied for any index that is repeated twice in any term. For example, in the expression $S_{ij}u_j$, the subscript i is free, because it is not summed over, while j is a dummy subscript, which can be replaced by any other symbol.

Overview of This Chapter

1.1. Kinematics of Continua

1.1.1. Motion of a body. In general, *a body* is modeled as an (open) subset of Euclidean space $\mathbb{R}^d$, which consisting of uncountably infinitely many other particles. It is identified with a regular domain $\Omega \subset \mathbb{R}^d$, occupied in some fixed configuration, called a *reference configuration*, which is arbitrarily chosen. We call a point in such body, described by its position $X \in \Omega$, a *material point*.

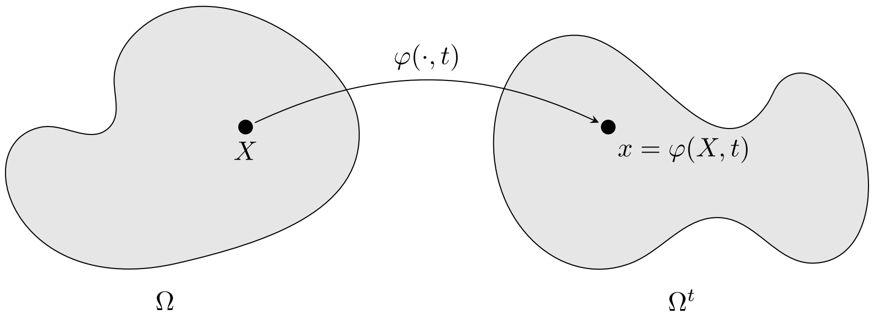

Figure 1.1: A depiction of deformation

Definition 1.1 (Motion of *a body*). A motion of a body Ω is a smooth function $\varphi : \Omega \times [0, \infty) \to \mathbb{R}^d$ that assigns to each *material point* X and the time t a point

$$x = \varphi(X, t).$$

Here x is referred to as the *spatial point* occupied by X at time t.

We call $\varphi^t := \varphi(\cdot, t)$, the function of X, the *deformation* at time $t \geq 0$. It is assumed as an *orientation-preserving* C^1-diffeomorphism onto its image in continuum mechanics, which descriptively means the body cannot penetrate itself. In some contexts, e.g. fluid mechanics, it is also identified with *flow map* with suitable regularity. Note that *orientation-preserving* property implies that the *volumetric Jacobian* (also called *Jacobian determinant*) $J(X, t)$ of the mapping φ^t at the material point X is strictly positive, i.e.,

$$J(X, t) := \det \nabla\varphi^t(X) > 0 \quad \text{for all } X \in \Omega,\ t \geq 0. \tag{1.1}$$

Then we write $\Omega^t := \varphi^t(\Omega)$ as the domain consisting of all the points x at time t, which is called *deformed configuration* (also called *current configuration* in some context). For a depiction of the motion, see Figure 1.1.

We now introduce the notion of Lagrangian and Eulerian coordinates. The motivation behind Lagrangian and Eulerian coordinates is that the former correspond to coordinates in the continuum's reference configuration Ω whilst the latter correspond to coordinates in an observer's frame of reference where the complete history of the trajectory of each point of the continuum is not kept track of.

Definition 1.2 (Lagrangian and Eulerian coordinates). Depending on the observing positions, we have:

(1) $(X, t) \in \Omega \times [0, \infty)$ is called *Lagrangian coordinates*: one considers a *material point* and follows its evolution.

(2) $(x, t) \in \Omega^t \times [0, \infty)$ is called *Eulerian coordinates*: one considers a fixed point in space and in general one will observe different material points at this point in space at different times.

In view of the setting above, we continue by introducing the velocity and acceleration in different configurations.

DEFINITION 1.3 (Velocity and acceleration). Let Ω, Ω^t and φ be as above. Define $\psi^t(x) := \psi(x, t) = \varphi^{-1}(x, t)$ for all $x \in \Omega^t$ and $t \geq 0$.

(1) $\partial_t \varphi : \Omega \times [0, \infty) \to \mathbb{R}^d$ and $\partial_t^2 \varphi : \Omega \times [0, \infty) \to \mathbb{R}^d$ are the *Lagrangian (material) velocity and acceleration.*

(2) $(\partial_t \varphi) \circ \psi^t : \bigcup_{t \geq 0} \Omega^t \times \{t\} \to \mathbb{R}^d$ and $(\partial_t^2 \varphi) \circ \psi^t : \bigcup_{t \geq 0} \Omega^t \times \{t\} \to \mathbb{R}^d$ are the *Eulerian (spatial) velocity and acceleration.*

By the definition of Eulerian velocity in terms of φ, we introduce the so-called *material time derivative* for any function defined in the Eulerian coordinates.

DEFINITION 1.4 (Material time derivative). Let Ω, Ω^t and φ be as above and v be the corresponding Eulerian velocity, i.e., $\partial_t \varphi = v \circ \varphi$. For any differentiable function $f : \bigcup_{t \geq 0} \Omega^t \times \{t\} \to \mathbb{R}^N$, $N \in \mathbb{N}$, we write

$$D_t f = (\partial_t + v \cdot \nabla) f$$

as the *material time derivative* of f, where $\nabla = \nabla_x$.

The material time derivative describes the change in time of the quantity f observed at a material point which at time t has the position x and the velocity $v(x, t)$. Now by virtue of the different descriptions, we are able to establish the derivatives of functions that defined in terms of the motion φ.

PROPOSITION 1.5. *Let Ω, Ω^t and φ be as above and v be the corresponding Eulerian velocity, i.e., $\partial_t \varphi = v \circ \varphi$. Let $f : \bigcup_{t \geq 0} \Omega^t \times \{t\} \to \mathbb{R}^N$, $N \in \mathbb{N}$ be a function, which is differentiable in both space and time. Then the derivatives of f induced by the motion (flow) φ is given as*

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi) = \big((\partial_t + v \cdot \nabla) f\big) \circ \varphi. \tag{1.2}$$

Moreover, the volumetric Jacobian $J = \det \nabla \varphi$ *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} J = \big((\operatorname{div} v) \circ \varphi\big) J. \tag{1.3}$$

Proof. A detailed computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi)(X, t) = \frac{\mathrm{d}}{\mathrm{d}t} f(\varphi(X, t), t) = \partial_t f \circ \varphi(X, t) + (\nabla f \circ \varphi)(X, t) \cdot \partial_t \varphi(X, t),$$

which finishes the proof of (1.2) combining $\partial_t \varphi = v \circ \varphi$.

To derive (1.3), on noting the derivative of determinants, i.e., (1.61) below, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} J = \frac{\mathrm{d}}{\mathrm{d}t} \det \nabla \varphi = \operatorname{tr} \Big((\nabla \varphi)^{-1} \partial_t \nabla \varphi \Big) J.$$

With the definition of v and chain rule, one calculates

$$\partial_t \nabla \varphi = \nabla(\partial_t \varphi) = \nabla(v \circ \varphi) = (\nabla v \circ \varphi) \nabla \varphi.$$

Then

$$\partial_t \nabla\varphi(\nabla\varphi)^{-1} = (\nabla v \circ \varphi)\nabla\varphi(\nabla\varphi)^{-1} = (\nabla v \circ \varphi)(\nabla\varphi(\nabla\varphi)^{-1}) = \nabla v \circ \varphi,$$

implying

$$\operatorname{tr}\left((\nabla\varphi)^{-1}\partial_t\nabla\varphi\right) = \operatorname{tr}\left(\partial_t\nabla\varphi(\nabla\varphi)^{-1}\right) = \operatorname{div} v \circ \varphi,$$

which completes the proof of (1.3). ☐

Corollary 1.6. *Under the assumptions of Proposition 1.5, the local change of volume is denoted by*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\Omega^t\right| = \int_{\Omega^t} \operatorname{div} v \,\mathrm{d}x \quad \text{for all } t \geq 0.$$

Proof. By definition and transformation formula, the volume is

$$\left|\Omega^t\right| = \int_{\Omega^t} 1 \,\mathrm{d}x = \int_{\Omega} J \,\mathrm{d}X \quad \text{for all } t \geq 0.$$

Then in light of (1.3),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\Omega^t\right| = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^t} 1 \,\mathrm{d}x = \int_{\Omega} \frac{\mathrm{d}}{\mathrm{d}t} J \,\mathrm{d}X = \int_{\Omega} \big((\operatorname{div} v) \circ \varphi\big) J \,\mathrm{d}X = \int_{\Omega^t} \operatorname{div} v \,\mathrm{d}x \quad \text{for all } t \geq 0.$$

☐

Theorem 1.7 (Reynolds' Transport Theorem). *Let $\mathcal{U} \subseteq \mathbb{R}^d$ be open, and $\varphi : \mathcal{U} \times [0,\infty) \to \mathbb{R}^d$ be the motion defined in $\mathcal{U}$, such that $\varphi^t := \varphi(\cdot, t)$ is an orientation-preserving C^1-diffeomorphism. Define $\mathcal{U}(t) := \varphi^t(\mathcal{U})$ and v as the corresponding Eulerian velocity, i.e., $\partial_t \varphi = v \circ \varphi$. For any sufficiently regular field $f : \bigcup_{t\geq 0} \mathcal{U}(t) \times \{t\} \to \mathbb{R}$, it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}(t)} f(x,t) \,\mathrm{d}x = \int_{\mathcal{U}(t)} \big(\partial_t f + \operatorname{div}(fv)\big) \,\mathrm{d}x \quad \text{for all } t \geq 0. \tag{1.4}$$

Proof. With the help of integration by substitution (Proposition 1.17) and Proposition 1.5, one obtains

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}(t)} f \,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}} (f\circ\varphi) \det\nabla\varphi \,\mathrm{d}X \\
&= \int_{\mathcal{U}} \big((\partial_t + v\cdot\nabla)f\big)\circ\varphi \det\nabla\varphi \,\mathrm{d}X + \int_{\mathcal{U}} (f\circ\varphi)\big((\operatorname{div} v)\circ\varphi\big) \det\nabla\varphi \,\mathrm{d}X \\
&= \int_{\mathcal{U}} \big(\partial_t f + \operatorname{div}(fv)\big)\circ\varphi \det\nabla\varphi \,\mathrm{d}X = \int_{\mathcal{U}(t)} \big(\partial_t f + \operatorname{div}(fv)\big)\mathrm{d}x \quad \text{for all } t \geq 0.
\end{aligned}$$

☐

Remark 1.8. By the Gauß Theorem, we know the second term on the right-hand side of (1.4) is equipped with the form

$$\int_{\mathcal{U}(t)} \operatorname{div}(fv) \,\mathrm{d}x = \int_{\partial\mathcal{U}(t)} f(v\cdot n) \,\mathrm{d}\mathcal{H}^{d-1} \quad \text{for all } t \geq 0.$$

which accounts for the change of volumes of $\mathcal{U}(t)$. Here $\partial\mathcal{U}(t)$ is the boundary of $\mathcal{U}(t)$ and n is the outer unit normal at $\partial\mathcal{U}(t)$ for all $t \geq 0$.

Remark 1.9. Let $u : \mathcal{U}(t) \times [0, \infty) \to \mathbb{R}^d$ be a sufficiently regular vector. Then the identity (1.4) is still valid with the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{U}(t)} u \,\mathrm{d}x = \int_{\mathcal{U}(t)} \big(\partial_t u + \operatorname{div}(u \otimes v)\big) \,\mathrm{d}x \quad \text{for all } t \geq 0. \tag{1.5}$$

In the following, we will discuss about the incompressibility of the material through the *deformation* $\varphi^t = \varphi(\cdot, t)$ with φ given in Definition 1.1. First recall the definition of *incompressibile* and *volume-preserving* motion.

Definition 1.10 (Incompressible flow). We say that a *motion (flow)* φ is incompressible if its Eulerian velocity v is divergence free, i.e., $\operatorname{div} v = 0$.

Definition 1.11 (Locally volume-preserving motion). Let $\Omega \subset \mathbb{R}^d$ be a regular domain, φ be a motion regarding Ω. For any measurable subset $\mathcal{U} \subseteq \Omega$, we define $\mathcal{U}(t) := \varphi^t(\mathcal{U})$. We say the motion φ is *locally volume-preserving*, if for all $t \geq 0$,

$$|\mathcal{U}| = |\mathcal{U}(t)| \quad \text{for all measurable } \mathcal{U} \subseteq \Omega.$$

Proposition 1.12 (Locally volume-preserving and incompressibility). *Let φ be a motion (flow). The following are equivalent:*

(1) φ is locally volume-preserving.

(2) $J = \det(\nabla\varphi) \equiv 1$.

(3) φ is incompressible.

Proof. *(1)* $\Leftrightarrow$ *(2)*: The proof is a direct computation for any measurable $\mathcal{U} \subset \Omega$:

$$\int_{\mathcal{U}} 1 \,\mathrm{d}X = \int_{\mathcal{U}(t)} 1 \,\mathrm{d}x = \int_{\mathcal{U}} J \,\mathrm{d}X \quad \text{for all } t \geq 0.$$

(2) $\Rightarrow$ *(3)*: On noting (1.3) and $J = 1$, one concludes

$$\operatorname{div} v \circ \varphi = 0,$$

which proves the assertion.

(2) $\Leftarrow$ *(3)*: In view of (1.3) and $\operatorname{div} v = 0$, one concludes

$$\frac{\mathrm{d}}{\mathrm{d}t} J = 0 \quad \text{for all } X \in \Omega, t \geq 0.$$

Then using the fact $\varphi^t|_{t=0} = \mathbf{id}$, we have $J = 1$. □

1.1.2. Deformation gradient. In continuum mechanics, we study the behavior of moving and deforming continua Ω^t over time.

Definition 1.13 (Deformation gradient). Let φ be the motion as above. Then the *deformation gradient* is defined as the matrix (tensor of order 2)

$$F = \nabla\varphi, \ \text{ i.e., } F_{ij} = \frac{\partial \varphi_i}{\partial X_j} \quad \text{for all } i, j = 1, ..., d. \tag{1.6}$$

By the orientation-preserving assumption, i.e., (1.1),

$$J = \det F > 0.$$

In the following we describe the relative change of positions.

Definition 1.14. Let φ be a motion and F be its deformation gradient as above. The *displacement* $u : \Omega \times [0, \infty) \to \mathbb{R}^d$ is defined for all $X \in \Omega$ and $t \geq 0$ as

$$u(X, t) = \varphi^t(X) - X,$$

whilst the *displacement gradient* reads

$$\nabla u = F - I,$$

where I is the identity matrix in $\mathbb{R}^{d\times d}$.

Very often, it will be necessary to rapidly switch between different viewpoints on the same physical problem. Sometimes, it is appropriate to consider the material in reference configuration Ω, while sometimes the Eulerian viewpoint of the current configuration Ω^t is better suited. In the following, we derive corresponding rules to map between both coordinate frames. Let φ be a motion with corresponding Eulerian velocity v, deformation gradient $F = \nabla\varphi$ and volumetric Jacobian $J = \det F$.

Proposition 1.15 (Material and spatial descriptions of functions). *Let $f : \bigcup_{t\geq 0} \Omega^t \times \{t\} \to \mathbb{R}$ and $u : \bigcup_{t\geq 0} \Omega^t \times \{t\} \to \mathbb{R}^d$ be a scalar function and a vector-valued function defined in spatial (Eulerian coordinates) that are continuously differentiable both in space and time, and the respecting material functions (Lagrangian coordinates) are defined by $\rho \coloneqq f \circ \varphi$, $w \coloneqq u \circ \varphi$ with a motion φ. Then it follows*

$$\begin{aligned} \partial_t f \circ \varphi &= (\partial_t - (F^{-1}(v \circ \varphi)) \cdot \nabla)\rho, \\ (\nabla f) \circ \varphi = F^{-\top}\nabla\rho, \quad (\nabla u) \circ \varphi &= \nabla w F^{-1}, \\ (\operatorname{div} u) \circ \varphi &= F^{-\top} : \nabla w, \end{aligned}$$

where $F = \nabla\varphi$ is the associated deformation gradient.

Proof. The last one follows from $(\operatorname{div} u) \circ \varphi = \operatorname{tr}(\nabla u) \circ \varphi = \operatorname{tr}(\nabla w F^{-1}) = F^{-\top} : \nabla w$, while others are the consequences of direct computations. □

Moreover, concerning the surface deformation, we have the following propositions.

Proposition 1.16 (Deformation of normals). *Consider a sufficiently smooth surface Γ^t with $\Gamma^t = \varphi^t(\Gamma)$, with a motion φ. Let n be the unit outward normal to Γ^t in the current configuration, and N be the unit outward normal to Γ in the reference configuration. Then it follows*

$$n = \frac{F^{-\top}N}{|F^{-\top}N|}.$$

Proof. See e.g. [KR19, Remark 1.1.11]. □

Proposition 1.17 (Deformation of volume and area). *Under the assumptions of Proposition 1.16, it follows*

$$\mathrm{d}\mathcal{H}^{d-1}\llcorner\Gamma^t = J\left|F^{-\top}N\right| \mathrm{d}\mathcal{H}^{d-1}\llcorner\Gamma,$$

where $\mathcal{H}^{d-1}$ is the $d-1$ dimensinal Hausdorff measure and

$$(\mathcal{H}^{d-1}\llcorner A)(B) = \mathcal{H}^{d-1}(A \cap B) \quad \textit{for all } A, B \subseteq \mathbb{R}^d \textit{ Borel measurable subsets.}$$

Proof. See e.g. [GFA10, Section 8.2 & 8.3]. □

Proposition 1.18 (Nanson's formula). *Under the assumptions of Proposition 1.16 it follows*

$$n\,\mathrm{d}\mathcal{H}^{d-1}\llcorner\Gamma^t = JF^{-\top}N\,\mathrm{d}\mathcal{H}^{d-1}\llcorner\Gamma.$$

Proof. This is a direct consequence of Propositions 1.16 and 1.17. □

1.1.3. Stretch and strain. In this section, we give the description of the strain in terms of the deformation gradient. First let us introduce the well-known *Polar Decomposition Theorem*:

Theorem 1.19 (Polar Decomposition Theorem [Cia88, GFA10, KR19]). *Let $F \in \mathbb{R}^{d\times d}$ be an invertible matrix with $\det F > 0$. Then there are unique positive-definite symmetric matrices $U, V \in \mathbb{R}^{d\times d}$ and a unique proper orthogonal matrix $R \in \mathbb{R}^{d\times d}$ such that*

$$F = RU = VR.$$

Here U and V are referred to as *right stretch tensor* and *left stretch tensor*, which are endowed with the explicit representations

$$U = \sqrt{F^\top F},$$
$$V = \sqrt{FF^\top}.$$

In particular, $V = RUR^\top$, $B = RCR^\top$, and

$$C = U^2 = F^\top F,$$
$$B = V^2 = FF^\top,$$

are called *right* and *left Cauchy–Green tensor.* Note that U, V, C, B are symmetric and positive-definite.

For F the deformation gradient in Definition 1.13, a particular application is the *Green–St. Venant strain tensor*

$$E = \frac{1}{2}(F^\top F - I) = \frac{1}{2}(C - I) = \frac{1}{2}(U^2 - I).$$

Being symmetric and positive-definite, U and V admit spectral representations of

$$U = \sum_{i=1}^d \lambda_i r_i \otimes r_i, \quad V = \sum_{i=1}^d \lambda_i l_i \otimes l_i,$$

where

- λ_i, $i \in \{1, ..., d\}$, the *principal stretches*, are the eigenvalues of U and V;
- r_i and l_i, $i \in \{1, ..., d\}$, the *right* and *left principal directions*, are the eigenvectors of U and V respectively satisfying
$$Ur_i = \lambda_i r_i, \quad Vl_i = \lambda_i l_i.$$

1.2. Conservation Laws

The fundamental equations of continuum mechanics are based on conservation laws for mass, (linear) momentum, angular momentum and energy. We will formulate these conservation laws in *Eulerian coordinates* in both *global and local form*. In the following we will consider

- $\mathcal{U}(t) = \{x : x = \varphi(X,t), X \in \mathcal{U}\} \subseteq \Omega^t$ for any $\mathcal{U} \subseteq \Omega$, where φ is the motion defined in Definition 1.1 with Ω the reference configuration.
- v, defined by $v \circ \varphi = \partial_t \varphi$, the corresponding Eulerian velocity.
- ρ, the Eulerian mass density.

1.2.1. Mass conservation. Without any source, for any spatial domain $\mathcal{U}$, the mass inside is conserved with the evolution of the body. Then the conservation of mass is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}(t)} \rho\,\mathrm{d}x = 0 \quad \text{for all } t \geq 0.$$

A direct application of Reynolds' Transport Theorem tells us the intergral balance of mass

$$\int_{\mathcal{U}(t)} \Big(\partial_t \rho + \operatorname{div}(\rho v)\Big)\,\mathrm{d}x = 0.$$

Since $\mathcal{U} \subseteq \Omega$ is arbitrary and $\varphi(\cdot,t)$ is diffeomorphism, it follows that the equation holds for arbitrary volume $\mathcal{U}(t) \subseteq \Omega^t$ as before. Then for continuously differentiable ρ and v, we obtain an equivalent local formulation for all (x,t)

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \tag{1.7}$$

which is called continuity equation.

Remark 1.20. By integration by substitution and a same localization procedure, one reaches

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(J(\rho \circ \varphi)\big) = 0.$$

If we define the density in Lagrangian coordinates $\rho_0(X,t)$, then the *mass conservation* in the reference configuration is simply

$$\rho_0 = J(\rho \circ \varphi), \qquad \partial_t \rho_0 = 0.$$

1.2.2. Momentum conservation. Within the framework of continuum mechanics, the basic balance laws for *linear and angular momentum* assert that, given any spatial region $\mathcal{U}(t)$ convecting with the body,

(1) the net force on $\mathcal{U}(t)$ is balanced by temporal changes in the *linear momentum* of $\mathcal{U}(t)$;

(2) the net moment on $\mathcal{U}(t)$ is balanced by temporal changes in the *angular momentum* of $\mathcal{U}(t)$.

First, let us recall the *linear and angular momentum* of $\mathcal{U}(t)$ respectively by

$$\ell(\mathcal{U}(t)) = \int_{\mathcal{U}(t)} \rho v \mathrm{d}x, \quad a(\mathcal{U}(t)) = \int_{\mathcal{U}(t)} (x - x_0) \times (\rho v)\mathrm{d}x,$$

where $x_0 \in \mathbb{R}^d$ is fixed. The momentum balances are hence written as

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\mathcal{U}(t)} \rho v \,\mathrm{d}x = \int_{\mathcal{U}(t)} \rho f \,\mathrm{d}x + \int_{\partial\mathcal{U}(t)} b \,\mathrm{d}\mathcal{H}^{d-1},$$

and

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\mathcal{U}(t)} (x - x_0) \times (\rho v) \,\mathrm{d}x = \int_{\mathcal{U}(t)} (x - x_0) \times (\rho f) \,\mathrm{d}x + \int_{\partial\mathcal{U}(t)} (x - x_0) \times b \,\mathrm{d}\mathcal{H}^{d-1},$$

for any given $\mathcal{U}(t) \subseteq \Omega^t$ sufficiently smooth as before, a force density $f(x,t) \in \mathbb{R}^d$ per unit mass defined in $\mathcal{U}(t)$, and a surface force density (traction) $b(n,x,t) \in \mathbb{R}^d$ defined for each outer unit normal n, $x \in \partial\mathcal{U}(t)$ and each t.

Theorem 1.21 (Cauchy stress tensor). *A consequence of the momentum balance is that there exists a* spatial tensor field $T = (T_{ij})_{i,j=1}^d$*, called* Cauchy stress tensor*, such that the traction* $b(n)$ *depending on* n *can be represented by*

$$b(n) = Tn = \left(\sum_{j=1}^d T_{ij} n_j \right)_{i=1}^d .$$

Proof. For the proof, we refer to e.g. [BF13, Theorem I.3.1] or [EGK17, Theorem 5.5]. □

Now the *linear and angular momentum balance* are

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\mathcal{U}(t)} \rho v \,\mathrm{d}x = \int_{\mathcal{U}(t)} \rho f \,\mathrm{d}x + \int_{\partial\mathcal{U}(t)} Tn \,\mathrm{d}\mathcal{H}^{d-1},$$

and

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\mathcal{U}(t)} (x - x_0) \times (\rho v) \,\mathrm{d}x = \int_{\mathcal{U}(t)} (x - x_0) \times (\rho f) \,\mathrm{d}x + \int_{\partial\mathcal{U}(t)} (x - x_0) \times (Tn) \,\mathrm{d}\mathcal{H}^{d-1}.$$

To derive the local momentum balance, employing the Reynolds' Transport Theorem (Remark 1.9) and the Gauß Theorem, one obtains

$$\int_{\mathcal{U}(t)} \big(\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v)\big) \,\mathrm{d}x = \int_{\mathcal{U}(t)} (\rho f + \operatorname{div} T) \,\mathrm{d}x,$$

which holds for arbitrary subsets $\mathcal{U}(t) \subseteq \Omega^t$ sufficiently smooth as before, since $\mathcal{U} \subseteq \Omega$ sufficiently smooth is arbitrary and $\varphi(\cdot,t)$ is a diffeomorphism. Then for sufficiently smooth ρ, v, T, f, it follows the pointwise PDE as the *local form of momentum balance*

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) = \operatorname{div} T + \rho f. \tag{1.8}$$

In particular, combining with the local form of continuity equation (1.7), one concludes

$$\rho \partial_t v + \rho \operatorname{div}(v \otimes v) = \operatorname{div} T + \rho f. \tag{1.9}$$

Now we give a fundamental property for T from the *angular momentum balance.*

Theorem 1.22 (Symmetry of Cauchy stress tensor). *Let us assume that the density field* ρ*, the velocity field* v*, and the body forces field* f *are smooth. Then the stress tensor* T *acting on the body in the motion (flow) is symmetric.*

Proof. For the proof we take the 3D case as an example. Let Λ be any *skew symmetric matrix*, for which by Proposition 1.55 below, there is a unique *axial vector* $\lambda \in \mathbb{R}^3$ such that

$$\Lambda = \lambda \times = (\epsilon_{ikj}\lambda_k)_{i,j=1}^3, \quad \Lambda w = \lambda \times w \;\text{ for all } w \in \mathbb{R}^3.$$

Applying the Reynolds' Transport Theorem (Remark 1.9) and multiplying the conservation of angular momentum with λ imply that

$$\begin{aligned}\lambda \cdot &\int_{\mathcal{U}(t)} (x - x_0) \times \big(\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v)\big)\,\mathrm{d}x \\ &= \lambda \cdot \int_{\mathcal{U}(t)} (x - x_0) \times (\rho f)\,\mathrm{d}x + \lambda \cdot \int_{\partial\mathcal{U}(t)} (x - x_0) \times (Tn)\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

With the help of the identify $a \cdot (b \times c) = (a \times b) \cdot c$ for $a, b, c \in \mathbb{R}^3$, as well as the Gauß Theorem, we have

$$\begin{aligned}\int_{\mathcal{U}(t)} &(\Lambda(x - x_0)) \cdot \big(\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) - \rho f\big)\,\mathrm{d}x \\ &= \int_{\partial\mathcal{U}(t)} \Lambda(x - x_0) \cdot (Tn)\,\mathrm{d}\mathcal{H}^{d-1} = \int_{\mathcal{U}(t)} \operatorname{div}\Big(T^\top\big(\Lambda(x - x_0)\big)\Big)\,\mathrm{d}x \qquad (1.10) \\ &= \int_{\mathcal{U}(t)} \big(\Lambda(x - x_0)\big) \operatorname{div} T\,\mathrm{d}x + \int_{\mathcal{U}(t)} \nabla\big(\Lambda(x - x_0)\big) : T\,\mathrm{d}x.\end{aligned}$$

Note that

$$\nabla\big(\Lambda(x - x_0)\big) = \Lambda^\top : \nabla(x - x_0) + (x - x_0) \cdot \operatorname{div}\Lambda^\top = \Lambda,$$

where the second term vanishes due to

$$\operatorname{div}\Lambda^\top = 0, \;\text{ as } \lambda_k \text{ is constant.}$$

Substituting the local balance of linear momentum conservation (1.8) into (1.10) yields

$$\int_{\mathcal{U}(t)} \Lambda : T = 0.$$

In view of the arbitrariness of $\mathcal{U}(t)$ as above, one has the pointwise identity

$$\Lambda : T = 0 \;\text{ for all } \Lambda \in \mathbb{R}^{3\times 3}_{\text{skew}},$$

where $\mathbb{R}^{3\times 3}_{\text{skew}} := \{A \in \mathbb{R}^{3\times 3} : A = -A^\top\}$, which means the Cauchy stress tensor T must be symmetric due to Proposition 1.60, i.e.,

$$T = T^\top, \quad T_{ij} = T_{ji},$$

since Λ is an arbitrarily chosen skew tensor. □

1.2.3. Energy conservation. In order to formulate the conservation of energy we introduce a specific internal energy density $e = e(x, t)$, a heat flux $q = q(x, t)$ and heat sources $g(x, t)$ which are defined per unit mass. The total energy denoted by $\mathcal{E}$ now reads

$$\mathcal{E} = \int_{\mathcal{U}(t)} \rho\left(\frac{1}{2}|v|^2 + e\right)\mathrm{d}x,$$

which consists of the *kinetic energy* $\int_{\mathcal{U}(t)} \frac{\rho}{2} |v|^2 \,\mathrm{d}x$ and the internal energy $\int_{\mathcal{U}(t)} \rho e \,\mathrm{d}x$. The *balance of energy* records as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{U}(t)} \rho \left(\frac{1}{2} |v|^2 + e\right) \mathrm{d}x \\ &\quad = \int_{\mathcal{U}(t)} \rho f \cdot v \,\mathrm{d}x + \int_{\partial\mathcal{U}(t)} Tn \cdot v \,\mathrm{d}\mathcal{H}^{d-1} + \int_{\mathcal{U}(t)} \rho g \,\mathrm{d}x - \int_{\partial\mathcal{U}(t)} q \cdot n \,\mathrm{d}\mathcal{H}^{d-1}. \end{aligned}$$

Here the first and second term on the right-hand side describe the work caused by volume and surface forces, while the third term denotes the changes of the energy due to outer heat sources and the last term accounts for the heat gain or loss from heat flux across outer boundary. Analogously, by the Reynolds' Transport Theorem and the Gauß Theorem, we derive

$$\begin{aligned} &\int_{\mathcal{U}(t)} \left(\partial_t\left(\frac{\rho}{2} |v|^2 + \rho e\right) + \operatorname{div}\left(\frac{\rho}{2} |v|^2 v + \rho e v\right)\right) \mathrm{d}x \\ &\qquad = \int_{\mathcal{U}(t)} \left(\rho f \cdot v + \operatorname{div}(T^\top v) + \rho g - \operatorname{div} q\right) \mathrm{d}x. \end{aligned}$$

Using the face that $\mathcal{U}$ is arbitrary, whence $\mathcal{U}(t)$ is arbitrary, one obtains the *local balance of energy* in a form of piecewise PDE

$$\partial_t\left(\frac{\rho}{2} |v|^2 + \rho e\right) + \operatorname{div}\left(\frac{\rho}{2} |v|^2 v + \rho e v\right) = \rho f \cdot v + \operatorname{div}(T^\top v) + \rho g - \operatorname{div} q,$$

which by hands of the continuity equation implies

$$\rho \partial_t\left(\frac{1}{2} |v|^2 + e\right) + \rho v \cdot \nabla\left(\frac{1}{2} |v|^2 + e\right) = \rho f \cdot v + \operatorname{div}(T^\top v) + \rho g - \operatorname{div} q,$$

Now recalling the conservation of linear momentum, we are able to eliminate the kinetic energy and get the *balance of internal energy*

$$\rho(\partial_t + v \cdot \nabla)e - T : \nabla v + \operatorname{div} q = \rho g, \tag{1.11}$$

where the identity

$$\operatorname{div}(T^\top v) = v \cdot \operatorname{div} T + T : \nabla v,$$

is employed

1.2.4. Mixture conservation laws. For mixtures consisting of several components we have in addition a conservation law for each component. The composition of the mixture can be described by a *concentration per unit mass* c_i of the component $i \in \{1, \ldots, M\}$, $M \in \mathbb{N}$. Moreover, we define a flux j_i of component i and a rate function r_i with which component i is produced or consumed. Then the *conservation law of species* i is

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{U}(t)} \rho c_i \,\mathrm{d}x = \int_{\mathcal{U}(t)} \rho r_i \,\mathrm{d}x - \int_{\partial\mathcal{U}(t)} j_i \cdot n \,\mathrm{d}\mathcal{H}^{d-1}.$$

Here $j_i \cdot n$ describes how much mass of component i flows through a unit surface with normal n per unit time. By virtue of the Reynolds' Transport Theorem and the Gauß Theorem, it follows

$$\int_{\mathcal{U}(t)} \left(\partial_t(\rho c_i) + \operatorname{div}(\rho c_i v) - \rho r_i + \operatorname{div} j_i\right) \mathrm{d}x = 0.$$

Hence the local version of *mixture conservation* now reads

$$\partial_t(\rho c_i) + \operatorname{div}(\rho c_i v) - \rho r_i + \operatorname{div} j_i = 0.$$

Finally, canceling with the continuity equation gives birth to the *conservation of the species*

$$\rho(\partial_t + v \cdot \nabla)c_i + \operatorname{div} j_i = \rho r_i \text{ for } i = 1, \dots, M. \tag{1.12}$$

Remark 1.23. By means of (1.9), (1.11), (1.12), we are able to describe a large class of continuum bodies independently of their specific material characteristics, including fluids and solids. To close the system of equations, *constitutive relationships* between stress, deformation gradient, rate of deformation, and density must be imposed to characterize the particular material under consideration.

1.3. Incompressible Viscous Fluids

In this section, we would like to derive the constitutive equation for the flow of an *incompressible viscous* fluid in the Eulerian coordinates. Let φ be the motion in Definition 1.1 and v be its Eulerian velocity. [BF13]

1.3.1. Newton's hypothesis. We start by recalling one of the fundamental properties of fluid. In a fluid at rest, the stress acting on a surface element of a fluid element acts in the direction opposite to the one of the outward normal of the surface. Moreover, the modulus of this stress is independent of direction. It is denoted as p and referred to as the *hydrostatic pressure* of the fluid. Then the stress at all points is $-pn$, resulting the form of stress tensor as

$$T = -pI.$$

When the fluid is in motion, the effects due to pressure and to motion are separated by expressing the stress tensor in the form

$$T = S - pI,$$

where the new symmetric tensor $S \in \mathbb{R}^{d\times d}$ is called the *viscous stress tensor*.

Now we want to specify the viscous stress tensor S. For the velocity v of the fluid, the strain rate tensor is defined by the following.

Definition 1.24 (strain rate tensor). The tensor $Dv = \frac{1}{2}(\nabla v + \nabla v^\top)$ is known as the *strain rate tensor* for the flow.

Note that by Definition 1.56, Dv is the *symmetric part* of ∇v. Then the *Newton's rheology hypothesis* is concluded as follows.

Definition 1.25 (Newtonian fluid). A fluid is said to be a *Newtonian fluid* if

(1) The viscous stress tensor S in a flow depends only on the strain rate tensor Dv.

(2) The dependence of S on Dv is linear.

(3) The relation linking S and Dv is isotropic.

Remark 1.26. Here we include *isotropic* assumption in Definition 1.25(3). In practice, the viscosity coefficients can be a constant tensor in front of the symmetric gradient.

Proposition 1.27. *For a* Newtonian fluid*, the law giving the viscous stress tensor as a function of the strain rate tensor is necessarily in the form*

$$S = 2\mu Dv + \lambda \operatorname{div} v I,$$

where μ *is called* shear viscosity *and the quantity* $\mu + \frac{2}{3}\lambda$ *is called* bulk viscosity*.*

Proof. For the proof, we refer to [BF13, Proposition I.4.4]. □

Remark 1.28. In elasticity theory μ and λ are called Lamé coefficients.

Consequently, we obtain the general form of the stress tensor for a *Newtonian fluid*

$$T = S - pI = 2\mu Dv + (\lambda \operatorname{div} v - p)I.$$

1.3.2. Incompressible and homogeneous fluid. With the constitutive relationship in the last subsection, we are able to establish a general system for the fluid. For further applications, we assume that the fluid described by the flow φ is incompressible, in the sense of Definition 1.10. Then its Eulerian velocity v is divergence free, i.e.,

$$\operatorname{div} v = 0.$$

Proposition 1.29. *Let* ρ *be the density of the fluid. Then the fluid is incompressible if and only if the density* ρ *is constant along the trajectories associated with the velocity field* v*.*

Proof. Recall the continuity equation,

$$\partial_t \rho + \operatorname{div}(\rho v) = 0,$$

that is,

$$D_t \rho + \rho \operatorname{div} v = 0,$$

where D_t is the material derivative portrayed by Definition 1.4. Then one observes that

$$\operatorname{div} v = 0 \Leftrightarrow D_t \rho = 0,$$

as long as ρ does not vanish. In fact, equivalence above exactly expresses that the incompressibility (by definition, divergence free condition on v) is equivalent to the fact that the density ρ is constant along the characteristic curves of v. □

Remark 1.30. By Proposition 1.29, a flow can be incompressible even if the density is not constant. It is only required that the density of a particle of fluid remain constant during the evolution.

Furthermore, we consider that the fluid is *homogeneous*, which roughly means the density does not depend on the material point, i.e.,

$$\rho \equiv \text{constant}.$$

With Proposition 1.29, one infers a corollary simultaneously.

Corollary 1.31. *For incompressible flows, if the density* ρ *is homogeneous at initial time, then the density remains constant*

$$\rho = \rho_0 \quad \textit{for all } t \geq 0.$$

In summary, for an *incompressible homogeneous viscous fluid*, the system consists of continuity equation (1.7) and momentum equation (1.9) reduces to the so-called *incompressible Navier–Stokes equation*

$$\begin{aligned} \rho(\partial_t + v \cdot \nabla)v + \nabla p &= \mu \Delta v + \rho f, \\ \operatorname{div} v &= 0. \end{aligned} \tag{1.13}$$

1.4. Elastic Solids

Note that the balance of energy provides restrictions on the form of constitutive relationships between stress, deformation gradient, rate of deformation, and density. In this section, we turn our attention to elastic solids. In particular, we are devoted to introducing the general mathematical model for *hyperelastic* materials.

1.4.1. Alternative stress measures. When working with solids, the use of a purely spatial description sometimes could be problematic. On the other hand, solids typically possess stress-free reference configurations with respect to which one may measure strain and develop constitutive equations. Therefore, it is often useful to measure contact forces with respect to areas measured initially in the reference configuration.

In this section, we consider

- $\mathcal{U}(t) = \{x : x = \varphi(X,t), X \in \mathcal{U}\} \subseteq \Omega^t$ for any $\mathcal{U} \subseteq \Omega$, where φ is the motion defined in Definition 1.1 with Ω the reference configuration.
- F, the deformation gradient of φ with its volumetric Jacobian $J := \det F$.
- T, the Cauchy stress tensor acting on $\partial\mathcal{U}(t)$.
- P, the stress measured per unit area in the reference body.
- n, the unit outer normal on $\partial\mathcal{U}(t)$.
- N, the unit outer normal on $\partial\mathcal{U}$.
- Tn, PN, the tractions on $\partial\mathcal{U}(t)$ and $\partial\mathcal{U}$ respectively.

In view of the invariant of the stress, see e.g. Figure 1.2, it follows from Propositions 1.16 and 1.18 that

$$\int_{\partial\mathcal{U}} PN \mathrm{d}\mathcal{H}^{d-1} = \int_{\partial\mathcal{U}(t)} Tn \mathrm{d}\mathcal{H}^{d-1} = \int_{\partial\mathcal{U}} JT \circ \varphi F^{-\top} N \mathrm{d}\mathcal{H}^{d-1}.$$

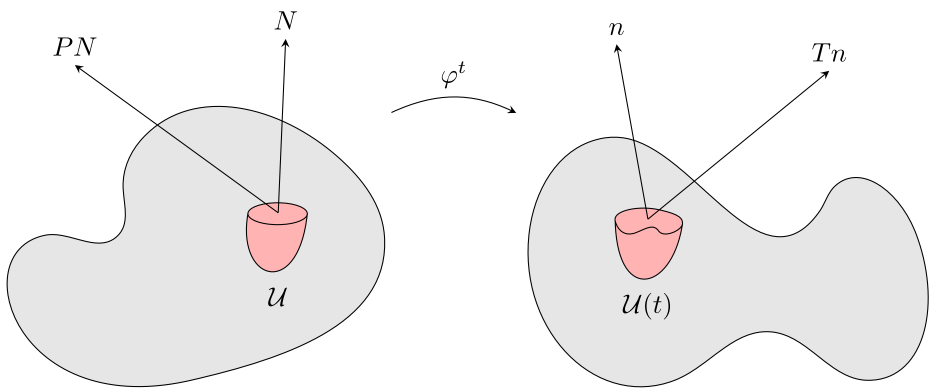

Figure 1.2: Quantities used in the definition of stress measures

By the arbitrariness of $\partial\mathcal{U}$, we give the first stress measure.

Definition 1.32 (First Piola–Kirchhoff stress tensor). We call P the *first Piola–Kirchhoff stress tensor* if it satisfies

$$P = JT \circ \varphi F^{-\top},$$

where T is the corresponding Cauchy stress tensor in deformed configuration, and J, F are defined above. Moreover, $P^\top = JF^{-1}T \circ \varphi$ is the *nominal stress tensor.*

Let us notice that P is not symmetric in general, whilst the tensor τ defined by

$$\tau = PF^\top = JT \circ \varphi,$$

is symmetric obviously, which is called the *Kirchhoff stress tensor.*

Definition 1.33 (Second Piola–Kirchhoff stress tensor). We call S the *second Piola–Kirchhoff stress tensor* if it satisfies

$$S = F^{-1}P = JF^{-1}(T \circ \varphi)F^{-\top}.$$

As a consequence of Definition 1.33, we know the second Piola–Kirchhoff stress tensor is symmetric.

Definition 1.34 (Piola transformation). Let φ be a motion, and F be its deformation gradient. Given any tensor T defined in the deformed domain Ω^t. Then we define P as the *Piola Transformation* of T in the reference configuration Ω by

$$P \coloneqq J(T \circ \varphi)F^{-\top}.$$

Now we record the property by the Piola transformation for smooth tensors.

Theorem 1.35 (Piola transformation [Cia88, KR19]). *Let φ be a motion. For a stress tensor T in the deformed configuration Ω^t, and the corresponding first Piola–Kirchhoff stress tensor P in the reference configuration Ω, we have:*

$$\operatorname{div} P = J(\operatorname{div} T) \circ \varphi,$$
$$\int_{\partial\mathcal{U}} PN \mathrm{d}\mathcal{H}^{d-1} = \int_{\partial\mathcal{U}(t)} Tn \mathrm{d}\mathcal{H}^{d-1},$$

for all $\mathcal{U} \subseteq \Omega$ and $\mathcal{U}(t) = \varphi^t(\mathcal{U}) \subseteq \Omega^t$, with N, n the corresponding unit outer normal to $\partial\mathcal{U}$ and $\mathcal{U}(t)$, respectively.

In the following, we present a useful identify, which will be frequently employed.

Lemma 1.36 (Piola's identity). *For a smooth motion φ with $F = \nabla\varphi$ and $J = \det \nabla\varphi$, it follows*

$$\operatorname{div}(JF^{-\top}) = 0. \tag{1.14}$$

Proof. See e.g. [Cia88, EGK17]. □

Remark 1.37. Piola transformation and Piola's identity provide us an efficient way to switch tensors defined by a certain smooth mapping between two different configurations, which is not necessarily a motion.

1.4.2. Conservation of elastic energy. The general principle for the balance of energy states that for any part of a body $\mathcal{U}(t) = \{x : x = \varphi(X,t), X \in \mathcal{U}\} \subseteq \Omega^t$ for any $\mathcal{U} \subseteq \Omega$, described by a motion φ with Ω the reference configuration, the rate of change of the total mechanical energy $\mathcal{E}$ is balanced by the power of the forces $\mathcal{P}$. If we ignore heat dissipation, the total energy for an elastic material is the sum of the kinetic energy and an internal elastic energy, that is,

$$\mathcal{E} = \int_{\mathcal{U}(t)} \frac{1}{2}\rho\,|v|^2 \,\mathrm{d}x + \int_{\mathcal{U}(t)} (J^{-1}W)\circ\varphi^{-1}\,\mathrm{d}x,$$

where W is the *internal elastic energy density* per unit *reference* volume.

The power of the forces acting on $\mathcal{U}$ is given by

$$\mathcal{P} = \int_{\mathcal{U}(t)} \rho f\cdot v\,\mathrm{d}x + \int_{\partial\mathcal{U}(t)} Tn\cdot v\,\mathrm{d}\mathcal{H}^{d-1},$$

In view of the *energy balance*, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \mathcal{P},$$

employing the same localization procedure as in Section 1.2, together with Reynolds' Transport Theorem, Gauß Theorem, continuity equation and linear momentum equation, gives birth to the equation

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \big(J(T\circ\varphi)F^{-\top}\big) : \dot{F} = P : \dot{F}, \tag{1.15}$$

pointwisely defined in the *reference configuration*, with $\dot{F} := \partial_t F = \nabla(v\circ\varphi)$, and P the first Piola–Kirchhoff stress tensor.

1.4.3. Hyperelasticity. In this section, we discuss about the *hyperelastic* material, namely,

Definition 1.38 (Hyperelasticity). A material is said to be *hyperelastic*, if the internal energy density W is a function of F alone.

Explicitly, with a little abuse of notation we posit that

$$W(X,t) = W(F(X,t)).$$

whence, W is referred to as the *strain energy function* (also called *stored energy function* in some contexts). Then the time derivative of W is

$$\frac{\mathrm{d}}{\mathrm{d}t}W(F) = \frac{\partial W}{\partial F} : \dot{F}.$$

Hence (1.15) is then endowed with the form of

$$\left(\frac{\partial W}{\partial F} - P\right) : \dot{F} = 0. \tag{1.16}$$

Since this identity holds for all motions pointwisely, one concludes from Lemma 1.59 that the contitutive relationship for the *strain energy density* W in terms of the deformation gradient F

$$\frac{\partial W}{\partial F} = P = J(T\circ\varphi)F^{-\top},$$

that is

$$T\circ\varphi = J^{-1}\frac{\partial W}{\partial F}F^{\top},$$

regarding to the Cauchy stress tensor T for compressible solids.

1.4.4. Incompressible material. In the case of an incompressible material, say, all the deformations are locally volume-preserving, it follows from Proposition 1.12 that

$$J = \det F = 1.$$

A simple way to ensure that a constraint holds is to introduce a Lagrangian multiplier $p = p(X,t)$ and modify accordingly the energy density $W \to W - p(J-1)$ so that Equation (1.16) reads now

$$\left(\frac{\partial}{\partial F}(W - p(J-1)) - P\right) : \dot{F} = 0,$$

leading to

$$\frac{\partial W}{\partial F} - p\frac{\partial J}{\partial F} = P = (T \circ \varphi)F^{-\top}.$$

This implies the *constitutive equation* for *incompressible hyperelastic materials*

$$T \circ \varphi = \frac{\partial W}{\partial F}F^{\top} - pI, \tag{1.17}$$

where by the Jacobi's formula,

$$\frac{\partial J}{\partial F} = \frac{\partial}{\partial F}\det F = JF^{-\top} = F^{-\top}.$$

Recalling that a *hydrostatic pressure* is a stress that is a multiple of the identity, we can identify the reaction stress in (1.17) with a hydrostatic pressure. Physically, we see that a pressure p is required to enforce locally the conservation of volume.

1.4.5. Choice of strain energy functions. In this section, we will present several typical choices of strain energy function densities. A basic principle underlying most of physics is that *physical laws should be independent of the frame of reference*, which is usually called *frame-indifference* or *objectivity* [GFA10]. For hyperelastic materials, the principle of frame-indifference implies the following.

Definition 1.39 (Frame-indifference). A *hyperelastic* material with strain energy density W is *frame-indifferent* or *objective*, if

$$W(QF) = W(F) \quad \text{for all } Q \in SO(d),$$

where $SO(d) := \{A \in \mathbb{R}^{d\times d} : A^{\top}A = I, \det A = 1\}$ denotes the set of all proper orthogonal matrices.

As we can always find a unique polar decomposition $F = RU$ with $U^2 = F^{\top}F$, a consequence of the frame-indifference is

$$W(F) = W(QF) = W(QRU) = W(QR\sqrt{C}) = \hat{W}(C), \tag{1.18}$$

if one takes $Q = R^{\top}$. One observation from (1.18) is that the first Piola–Kirchhoff stress tensor in fact can be expressed by the energy density regarding the right Cauchy–Green tensor C.

Proposition 1.40. *Let W be frame-indifferent as above. Then we have*

$$P = \frac{\partial W(F)}{\partial F} = 2F\frac{\partial \hat{W}(C)}{\partial C}. \tag{1.19}$$

Proof. By definition,

$$\frac{\partial W(F)}{\partial F_{ij}} = \frac{\partial \hat{W}(C)}{\partial C_{km}} \frac{\partial C_{km}}{\partial F_{ij}}.$$

On noting that $C = (C_{km})_{k,m=1}^d = (F_{lk}F_{lm})_{k,m=1}^d$, we find

$$\frac{\partial C_{km}}{\partial F_{ij}} = \frac{\partial F_{lk}}{\partial F_{ij}} F_{lm} + F_{lk} \frac{\partial F_{lm}}{F_{ij}} = \delta_{li}\delta_{kj}F_{lm} + F_{lk}\delta_{li}\delta_{mj} = \delta_{kj}F_{im} + F_{ik}\delta_{mj}.$$

Then

$$\begin{aligned} \frac{\partial \hat{W}(C)}{\partial C_{km}} \frac{\partial C_{km}}{\partial F_{ij}} &= \frac{\partial \hat{W}(C)}{\partial C_{km}} \delta_{kj} F_{im} + \frac{\partial \hat{W}(C)}{\partial C_{km}} F_{ik}\delta_{mj} \\ &= \frac{\partial \hat{W}(C)}{\partial C_{jm}} F_{im} + \frac{\partial \hat{W}(C)}{\partial C_{kj}} F_{ik} = 2F \frac{\partial \hat{W}(C)}{\partial C}, \end{aligned}$$

where the symmetry of C is employed, which finishes the proof. □

Note that Proposition 1.40 entails the formulation of second Piola–Kirchhoff stress tensor in terms of C by

$$S = F^{-1}P = 2\frac{\partial \hat{W}(C)}{\partial C}.$$

The second principle is the so-called *isotropy*, which means that at a given point of our material its response is the same in all directions, i.e., *isotropy* is directional uniformity, cf. [KR19]. Then for hyperelastic materials, we give the definition as follows.

DEFINITION 1.41 (Isotropy). A *hyperelastic* material with the strain energy function W is *isotropic* if

$$W(F) = W(FQ) \quad \text{for all } Q \in SO(d).$$

Then one sees that the stored energy density W of a hyperelastic isotropic frame-indifferent material satisfies

$$W(RFQ) = W(FQ) = W(F) \quad \text{for all } R, Q \in SO(d). \tag{1.20}$$

Thus by (1.18) and (1.20), we conclude that

$$\hat{W}(C) = W(F) = W(RFQ) = \hat{W}(Q^\top F^\top R^\top RFQ) = \hat{W}(Q^\top CQ) \quad \text{for all } R, Q \in SO(d).$$

In three dimensional case, the previous identity implies that $W(F) = \hat{W}(C)$ can be expressed in terms of *principal invariants*, i.e., for some function $\tilde{W}$ we have

$$W(F) = \tilde{W}(I_1, I_2, I_3),$$

where

$$\begin{aligned} I_1 &= \operatorname{tr} C = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= \frac{1}{2}((\operatorname{tr} C)^2 - \operatorname{tr}(C^2)) = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \\ I_3 &= \det C = \lambda_1^2\lambda_2^2\lambda_3^2, \end{aligned}$$

and λ_i^2 are eigenvalues of the right Cauchy–Green tensor $C = F^\top F$ or alternatively of left Cauchy–Green tensor $B = FF^\top$. Equivalently, it implies that W only depends on F through its *principal stretches* λ_1, λ_2, λ_3 (the square roots of the principal values of C or B).

St. Venant–Kirchhoff material. The simplest hyperelastic material model is the Saint Venant–Kirchhoff model which is just an extension of the geometrically linear elastic material model to the geometrically nonlinear regime.

$$W(F) = \frac{\lambda}{2}(\operatorname{tr} E)^2 + \mu\,|E|^2\,,$$

where λ, μ are Lamé constants (μ is also called shear modulus), and $E = \frac{1}{2}(C - I)$ is the *Green–St. Venant strain tensor*. Then we have

$$S = 2\frac{\hat{W}(C)}{\partial C} = \lambda \operatorname{tr} E + 2\mu E.$$

Mooney–Rivlin material. Mooney–Rivlin material is a hyperelastic material, where the strain energy density function W is a linear combination of two *principal invariants*:

$$W(F) = \frac{C_1}{2}(I_1 - 3) + \frac{C_2}{2}(I_2 - 3) = \frac{C_1}{2}\operatorname{tr}(C - I) + \frac{C_2}{2}\big(\frac{1}{2}((\operatorname{tr} C)^2 - \operatorname{tr}(C^2)) - 3\big),$$

where $C_1 + C_2 = \mu$ denotes the shear modulus. Then we have

$$P = \frac{W(F)}{\partial F} = 2F\frac{\hat{W}(C)}{\partial C} = C_1 F + C_2 F((\operatorname{tr} C)I - C).$$

Neo-Hookean material. As a special case of the Mooney–Rivlin material with $C_2 = 0$, we arrive at

$$W(F) = \frac{\mu}{2}(I_1 - 3) = \frac{\mu}{2}\operatorname{tr}(C - I),$$

and the first Piola–Kirchhoff stress tensor

$$P = \mu F.$$

It is generally known that Neo-Hookean material model does not predict accurate phenomena at large strains.

Ogden material. For modeling rubbery and biological materials at even higher strains, the more sophisticated Ogden material model has been developed. It is shown as a general expansion with N terms of the form

$$W(F) = W(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i}(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3).$$

Here μ_i, α_i are material constants, which are related to the shear modulus μ of small deformations by

$$2\mu = \sum_{i=1}^{N} \mu_i \alpha_i.$$

Remark 1.42 (Our choice). In particular, we will choose the Neo-Hookean material as our solid to perform the analysis later. We comment that in Chapter 5, an even more general nonlinear *hyperelastic* strain energy density function is considered.

1.4.6. A glance at viscoelasticity. In solid mechanics, the viscous nature of the material is a typical phenomenon. A basic example of rheological models that describe the viscoelastic response of materials is the so-called *Kelvin–Voigt material*, schematically depicted by an arrangement of a spring and a damper parallelly so that the stresses sum up [KR19], see Figure 1.3. More specifically, the solid is equipped with the total stress tensor

$$T = T_e + T_v,$$

where T_e is exactly the stress tensor due to elasticity as the T in last sections, and T_v denotes the stress tensor caused by the viscous damping.

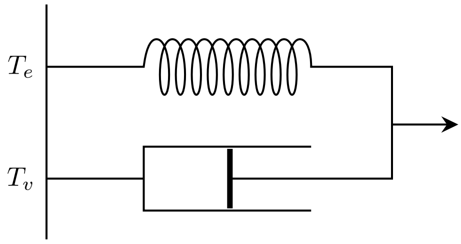

Figure 1.3: The simplest viscoelastic solid rheological model: *Kelvin–Voigt model*

For the *Kelvin–Voigt model*, the energy balance becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \mathcal{D} = \mathcal{P},$$

where $\mathcal{E}$ and $\mathcal{P}$ are given in Section 1.4.2, $\mathcal{D}$ is the dissipation

$$\mathcal{D} = \int_{\mathcal{U}(t)} (J^{-1}\Phi) \circ \varphi^{-1} \,\mathrm{d}x,$$

where Φ denotes the *specific dissipation rate* that can be identified with the form in terms of $\dot{F}$

$$\Phi(X,t) = \Phi(F(X,t), \dot{F}(X,t), X) = \frac{\partial \phi(F,\dot{F})}{\partial \dot{F}} : \dot{F},$$

with a slightly abuse of notation. Here ϕ is the *dissipative potential* to be determined. By the same argument as in Section 1.4.2, one obtains

$$\operatorname{tr}\left(\left(\frac{\partial W}{\partial F} - P^\top + \frac{\partial \phi(F,\dot{F})}{\partial \dot{F}}\right)\dot{F}\right) = 0.$$

That is

$$\frac{\partial W}{\partial F} - P^\top + \frac{\partial \phi(F,\dot{F})}{\partial \dot{F}} = 0. \tag{1.21}$$

Recall from hyperelasticity that $T_e \circ \varphi = J^{-1}\frac{\partial W}{\partial F}F^\top$. Then (1.21) reduces to

$$\frac{\partial \phi(F,\dot{F})}{\partial \dot{F}} = J(T_v \circ \varphi)F^{-\top},$$

leading to the viscous stress tensor

$$T_v \circ \varphi = J^{-1} \frac{\partial \phi(F, \dot{F})}{\partial \dot{F}} F^\top.$$

Now it remains to determine ϕ so that one can close the system. We will only present the linear case in the sense that $\dot{F} \mapsto \partial_{\dot{F}} \phi(F, \dot{F})$ is linear. A simple choice of dissipative potential for *linear Kelvin–Voigt models* is

$$\phi(F, \dot{F}) = \frac{\nu}{2} \left| \dot{F} + \dot{F}^\top \right|^2,$$

such that

$$\frac{\partial \phi(F, \dot{F})}{\partial \dot{F}} = \nu(\dot{F} + \dot{F}^\top).$$

Then the viscous stress tensor reads

$$T_v \circ \varphi = \nu J^{-1} (\dot{F} + \dot{F}^\top) F^\top. \tag{1.22}$$

Remark 1.43. The introduction of viscoelasticity not only initiated from physics, but also due to analytical reasons. In fact, when it comes to viscous dissipations, the system turns out to possess parabolic properties due to the higher-order regularity. For instance, in the case above, substituting T_v back into the momentum equation leads to the term of

$$\nu \operatorname{div}(\dot{F} + \dot{F}^\top) = \nu \operatorname{div} \left(\nabla(v \circ \varphi) + \nabla(v \circ \varphi)^\top \right) = \nu \Delta (v \circ \varphi),$$

for an incompressible solid in reference configurations. This term is of higher-order compared to elasticity (at least for a short time), and brings us hope to get the parabolic regularity of $v \circ \varphi$, in view of the structure

$$\partial_t (v \circ \varphi) - \nu \Delta (v \circ \varphi) = \text{lower order terms}$$

1.5. Biochemical Processes

One of the main problem in this thesis we want to discuss is the plaque formation, which is usually caused by the accumulation of foam cells resulting from biochemical processes in the blood flows and vessels, see the discussions in the Introduction. In particular, we will simply consider the dynamics of monocytes in the blood, and of macrophages and foam cells in the vessels. These are based on the assumptions in [Yan+16].

1.5.1. Advection-(reaction)-diffusion equations. Let φ, v, $\mathcal{U}(t) \subseteq \Omega^t$ be defined above. Let c_i be as in Section 1.2.4, the concentration of the i-th component, and r_i be a source function for the i-th component, which may depend on any c_k. Recall the local form of conservation of species (1.12) in Section 1.2.4

$$\rho(\partial_t + v \cdot \nabla) c_i + \operatorname{div} j_i = \rho r_i.$$

Concerning the *concentration flux*, we assume that j_i possesses the simplest constitutive relationship

$$j_i = -D_i \nabla c_i,$$

with a *diffusion coefficient* $D_i > 0$, which may depend on ρ, c_i. This is similar to the *Fourier's law* in thermodynamics, meaning diffusion leads to a flux from areas with a high concentration to areas with a low concentration, which can also be derived using stochastic analysis where a large quantity of "random walkers" are used to describe diffusion. Then we end up with the *advection-reaction-diffusion equation*

$$\rho(\partial_t + v \cdot \nabla)c_i - \operatorname{div}(D_i \nabla c_i) = \rho r_i.$$

Now we specify each component. Concerning the monocytes, say, identified with a superscript $i = f$, we assume there is no extra source in a volume of the domain $r_f = 0$. Then it follows an *advection-diffusion equation*

$$\rho(\partial_t + v \cdot \nabla)c_f - \operatorname{div}(D_f \nabla c_f) = 0.$$

When it comes to the macrophages and foam cells together in the vessels with associated quantities denoted by a superscript $i = s$ and a subscript $*$ respectively, we have an assumption that the concentration of foam cells produced only by the macrophages, and macrophages diffuse in the media and are consumed by foam cells in a volume of the domain. Thus, we have the equations

$$\rho(\partial_t + v \cdot \nabla)c_s - \operatorname{div}(D_s \nabla c_s) = -\rho r_s,$$

and

$$\rho(\partial_t + v \cdot \nabla)c^* = \rho r_s,$$

where r_s is a function that may depend on ρ, v, c_s.

In a particular situation, say, the density of material does not change, i.e., $\rho = \text{constant} > 0$, the diffusion coefficient D_i has the form of $\rho\tilde{D}_i$, and the reaction function r_s is imposed with a simple linear dependence on c_s, i.e., $r_s = \beta c_s$, one rewrites all the equations for the cells in a simple form with $\tilde{D}_i$ still denoted by D_i,

$$\partial_t c_f + v \cdot \nabla c_f - \operatorname{div}(D_f \nabla c_f) = 0, \tag{1.23}$$

$$\partial_t c_s + v \cdot \nabla c_s - \operatorname{div}(D_s \nabla c_s) = -\beta c_s, \tag{1.24}$$

$$\partial_t c^* + v \cdot \nabla c^* = \beta c_s, \tag{1.25}$$

where coefficient β may easily depend on some other quantities in chemical reactions.

Remark 1.44. Here we establish all the cells dynamics in a single body Ω with a velocity v. Note that this will be distinguished later when we couple the whole system together. Namely, the monocytes are involved in the blood, while the macrophages and foam cells are included in the vessel, transported by the fluid velocity and solid velocity respectively.

1.6. Growth in Continua

Aspects of growth and remodeling occur during the entire life of an organism. Therefore, growth fulfills many purposes and, accordingly, is associated with qualitatively different processes. Traditionally, a first classification is obtained by considering the way growth alters a body, either by changing its volume, its material properties, or by rearranging the relative position of material points. For example, the *growth*, which by itself refers to a change in mass, the *remodeling*, during which tissues may become stiffer or softer in the process of aging. We will not explore more about them here, and we refer to [Gor17] for more discussions. In this thesis, we will consider the so-called *Morphogenesis*, which is a biological process that causes a tissue or organ to develop its shape by controlling the spatial distribution of cells.

Concerning the plaque growth, it is not hard to see that during the processes the plaque (mainly of foam cells) becomes larger, ruptures and blocks the blood flow [LRH11]. So we assume that the growth is referred to as the *change of local volume*, say, *volumetric growth* (or *bulk growth*). *Volumetric growth* is typical of many developmental, physiological, and pathological processes and has been particularly well documented in arteries, muscles, solid tumors, and the heart [Cow04, Hum03, Tab95]. We mention that there are also other kinds of growth, through which the tissue grows on the boundary (surface), such as *tip growth* and *accretive growth* [Gor17]. Therefore, the next step is to describe the dynamics of this kind of *volumetric growth* in a hyperelastic material, by certain assumptions.

1.6.1. Kinematics of growth. Let Ω be a hyperelastic body with corresponding deformed configuration Ω^t, described by a motion φ with associated deformation gradient F, and Eulerian velocity v.

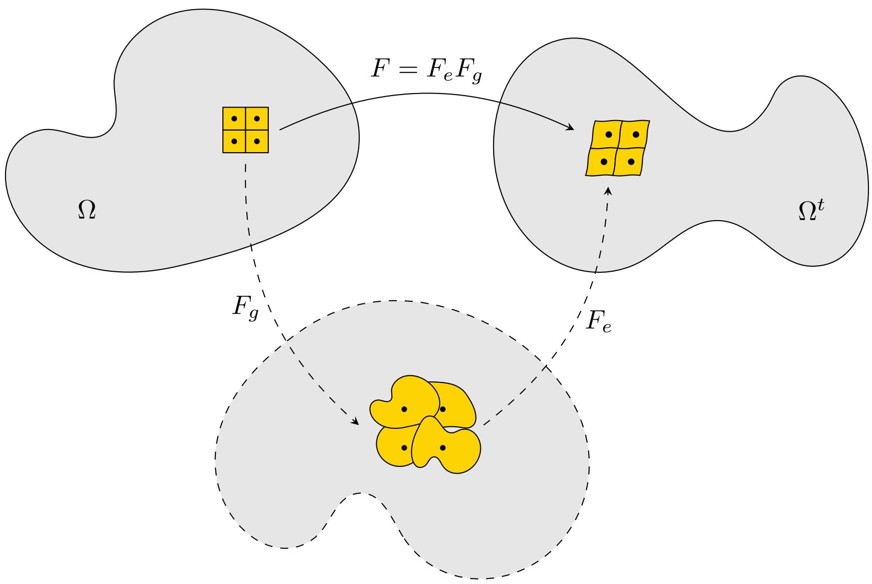

Figure 1.4: The *multiplicative decomposition*

In this thesis, we assume that the body Ω under consideration has a *residual stress field*, and does not rely on the existence of a global zero-stress state, which allows us to further define a *virtual configuration*, cf. [RHM94, Gor17]. Then we are able to assume a *multiplicative decomposition of the deformation gradient* as

$$F = F_e F_g, \tag{1.26}$$

where F_e is the *elastic deformation tensor* without growth, and F_g denotes the *local growth deformation tensor* (*growth tensor*) describing the change of shape and volume at all positions in the body due to growth. More precisely, for any point p in Ω with position $X \in \Omega$, the growth tensor F_g maps the tangential space $T_p\Omega$ at p to the linear space $T_p\Omega^g$ of a virtual state Ω^g for the same material point, that is,

$$F_g : T\Omega \to T\Omega^g.$$

See Figure 1.4. The virtual state Ω^g is called *natural configuration*, which consist of all the stress-released parts from deformed configuration Ω^t. Hence, it is locally stress free and describes the body after unconstrained growth. Such process is guaranteed by the *residual stress field* assumption above.

Remark 1.45. Note that the *natural configuration* in fact does not exist in Euclidean space. It is not a *domain* formulated after some transformation from Ω and is artificial for picturing the decomposition.

This idea has been employed and developed in the field of biomechanics, since the seminal work of Rodriguez–McCulloch–Hoger [RHM94]. Later on, the morphoelasticity systems with elastic equation and growth tensor were established in [AM02, JC12, Gor17, Yan+16] for different applications.

Remark 1.46 (Non-uniqueness of the decomposition [Gor17]). Given a deformation gradient F, the elastic and growth tensor F_e and F_g are not uniquely prescribed. Indeed, if F_e, F_g are such that $F = F_e F_g$, then the tensors

$$\tilde{F}_e = F_e R, \text{ and } \tilde{F}_g = R^\top F_g \quad \text{for all } R \in SO(d)$$

also provide a possible decomposition as $F = \tilde{F}_e \tilde{F}_g$.

1.6.2. Balance laws including growth. We follow the evolution of a subset $\mathcal{U}(t) \subseteq \Omega^t$ of a body Ω. In view of the conservation laws in Section 1.2, one further assumes that growth can occur through *volumetric growth* as above, which is identified by a growth rate function $\rho\gamma$, where γ may depends on the quantities during the biochemical processes. Namely, we have the *conservation of mass involving volumetric growth*:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{U}(t)} \rho \, \mathrm{d}x = \int_{\mathcal{U}(t)} \rho\gamma \, \mathrm{d}x.$$

In a same fashion as in Section 1.2.1, one ends up with the local version of the continuity equation for a growing continuum:

$$\partial_t \rho + \operatorname{div}(\rho v) = \rho\gamma. \tag{1.27}$$

Now we would like to describe the growth rate function in terms of the growth tensor F_g. To this end, one first rewrites the local continuity equation (1.27) in the *reference configuration* as

$$\frac{\mathrm{d}}{\mathrm{d}t}(J\rho) = J\rho\gamma, \tag{1.28}$$

where $J = \det F$ is the total volumetric Jacobian related to the motion φ. Moreover, by the multiplicative decomposition of the deformation gradient, one knows

$$J = \det F = \det(F_e F_g) = \det(F_e)\det(F_g) =: J_e J_g \quad \text{for all } X \in \Omega, t \geq 0,$$

which accounts for both elastic deformation and growth deformation. Furthermore, let ρ_g be the density defined in Ω with respect to the growing process without any stress. The invariance of mass during elastic process tells us that

$$\int_{\mathcal{U}} \rho_g \mathrm{d}X = \int_{\mathcal{U}(t)} \rho \mathrm{d}x = \int_{\mathcal{U}} \rho J_e \mathrm{d}X \quad \text{for all } t \geq 0,$$

implying

$$\rho_g = \rho J_e \quad \text{for all } X \in \Omega, t \geq 0.$$

Hence, the equation (1.28) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_g J_g)(X,t) = \rho_g J_g \gamma(X,t) \quad \text{for all } X \in \Omega, t \geq 0.$$

On noting (1.61), one records

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_g(X,t) + \rho_g \operatorname{tr}(F_g^{-1}\partial_t F_g)(X,t) = \rho_g \gamma(X,t) \quad \text{for all } X \in \Omega, t \geq 0. \tag{1.29}$$

By the formulation, it is obviously that the differential equation (1.29) is not solvable since neither the density ρ_g nor the growth tensor F_g is known. Thus, it is necessary to reduce (1.29) to some particular cases. There are two important limits when either the density or the volume does not change with growth.

- *Constant-density.* In this case, the density is unchanged by the growth process, which by the assumption of an incompressible tissue implies $\rho = \rho_g =$ constant. Then (1.29) is reduced to
$$\operatorname{tr}(F_g^{-1}\partial_t F_g) = \gamma \quad \text{for all } X \in \Omega, t \geq 0. \tag{1.30}$$
- *Constant-volume.* In this case, there is no change in volume, meaning $J_g = 1$. Then the tissue growth occurs by densification only, with
$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_g = \rho_g \gamma \quad \text{for all } X \in \Omega, t \geq 0. \tag{1.31}$$
where γ becomes a rate of densification, adding mass to the system without changing its volume.

Remark 1.47. Here the equations for either F_g or ρ_g are all defined in the *reference configurations*.

In addition to mass conservation, the balance of momentum is modified to take into account the possible contributions by growth. Here we do assume that there is no *non-compliant* sources acting to the system, which ensures that the added source possesses the same properties as the material itself [Gor17, JC12]. This is roughly a consequence of the so-called *slow-growth assumption* that the material is *hyperelastic* on short-time scales, due to the time scales of growth processes are much larger than any other time scales, see [Gor17, Section 13.1] for more discussions. Then *conservation of linear momentum involving volumetric growth* becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}(t)} \rho v \,\mathrm{d}x = \int_{\mathcal{U}(t)} \rho f \,\mathrm{d}x + \int_{\partial\mathcal{U}(t)} b \,\mathrm{d}\mathcal{H}^{d-1} + \int_{\mathcal{U}(t)} \rho\gamma v \,\mathrm{d}x,$$

as well as the *conservation of angular momentum*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}(t)} (x - x_0) \times (\rho v) \,\mathrm{d}x = \int_{\mathcal{U}(t)} (x - x_0) \times (\rho f) \,\mathrm{d}x + \int_{\partial\mathcal{U}(t)} (x - x_0) \times b \,\mathrm{d}\mathcal{H}^{d-1} \\ + \int_{\mathcal{U}(t)} (x - x_0) \times (\rho\gamma v) \,\mathrm{d}x.\end{aligned}$$

Employing the same argument as in Section 1.2.2, it holds the local version of momentum balance as

$$\rho\partial_t v + \rho \operatorname{div}(v \otimes v) = \operatorname{div} T + \rho f, \tag{1.32}$$

with

$$T = T^\top. \tag{1.33}$$

Analogously, the energy balance under the *non-compliant assumption* is endowed with the form of

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}(t)}\rho\left(\frac{1}{2}|v|^2+e\right)\mathrm{d}x=\int_{\mathcal{U}(t)}\rho f\cdot v\,\mathrm{d}x+\int_{\partial\mathcal{U}(t)}Tn\cdot v\,\mathrm{d}\mathcal{H}^{d-1}$$
$$+\int_{\mathcal{U}(t)}\rho g\,\mathrm{d}x-\int_{\partial\mathcal{U}(t)}q\cdot n\,\mathrm{d}\mathcal{H}^{d-1}$$
$$+\int_{\mathcal{U}(t)}\rho\gamma\left(\frac{1}{2}|v|^2+e\right)\mathrm{d}x,$$

where the last term on the right-hand side is the *compliant energy contribution*. Proceeding in a same manner as in Section 1.2.3 yields the local version of the energy balance

$$\rho(\partial_t+v\cdot\nabla)e-T:\nabla v+\operatorname{div}q=\rho g, \tag{1.34}$$

Remark 1.48. We observe that apart from possible *non-compliant* sources of energy, this balance (1.34) is the one that we would expect for a non-growing material. This is the same as for the momentum balances (1.32) and (1.33).

1.6.3. Elastic constitutive relationships. As discussed in the last section, the material is *hyperelastic* thanks to the *slow-growth assumption*. Moreover, we consider an isothermal situation. Thus, there is an internal energy density that depends only on the elastic tensor F_e, such that the total elastic energy has the form

$$\int_{\mathcal{U}(t)}J_e^{-1}W\mathrm{d}x,$$

for any sufficiently smooth $\mathcal{U}(t)\subseteq\Omega^t$, where $W=W(F_e)$ denotes the strain-energy density per unit volume (of the virtual *nature configuration*). Then arguing as in Sections 1.4.3 and 1.4.4 gives the Cauchy stress tensor

$$T\circ\varphi=J_e^{-1}\frac{\partial W(F_e)}{\partial F_e}F_e^{\top}-pI.$$

If the material is elastically incompressible, we know $J_e=1$ and p is the hydrostatic pressure, and for an elastically compressible material $p=0$.

Remark 1.49. Here we have a slight abuse of notation, since F_e, p on the right-hand side above are defined in the natural configuration, while the Cauchy stress tensor T is involved in deformed configuration and it is mapped to the reference configuration if composed with φ. However, we keep it consistent with the formulation in Section 1.4.3, when there is no danger of confusion.

1.6.4. Growth constitutive relationships. The growth tensor adds a kinematic descriptor to the theory of elasticity. Therefore, it requires a corresponding set of constitutive laws. In this section, we discuss how the growth tensor evolves, with some specific dynamics. Before that, we point out that in general, the evolution of the growth may depend on a large amount of phenomenon during the biochemical or physical processes. So one can write an equation for the growth as

$$\partial_t F_g F_g^{-1}=\mathcal{G}(T,F,F_g,f,t,X,x)\quad\text{for all }X\in\Omega,t\geq 0.$$

where $\mathcal{G}$ is a function of the stress tensor T, the deformation gradient F, the growth tensor itself F_g, the initial or current position, or any other chemical, biochemical, or physical field f. In actual applications, it is sometimes more convenient to possess the growth with more suitable assumptions.

Concerning the hyperelasticity, we have the following proposition.

Proposition 1.50. *The elastic energy density W of a growing isotropic hyperelasticity material only depends on the symmetric part of the growth tensor F_g.*

Proof. Observe that by the polar decomposition, there exist a proper orthogonal matrix $R_g \in SO(d)$ and a symmetric matrix U_g such that $F_g = R_g U_g$. Then in view of the isotropy of the material, one arrives at

$$W(F_e) = W(FF_g^{-1}) = W(FU_g^{-1}R_g^\top) = W(FU_g^{-1}R_g^\top Q).$$

Choosing $Q = R_g$ gives rise to

$$W(FF_g^{-1}) = W(FU_g^{-1}),$$

which means that the elastic energy of a growing isotropic material only depends on the symmetric part of the growth tensor. □

As discussed in Remark 1.46, the decomposition of F is not unique generally. Then it is motivated from non-uniqueness of decomposition and Proposition 1.50 that we may choose a non-relabeled growth tensor F_g to be *symmetric*. In the following, we record several typical types of symmetric growth law from particular applications. For more discussions, we refer to Goriely [Gor17].

- *Orthotropic growth.*

$$F_g = g_0 I + (g_1 - 1)\gamma_1 \otimes \gamma_1 + (g_2 - 1)\gamma_2 \otimes \gamma_2.$$

  Here g_0 represents the isotropic contribution to the growth process and g_i, $i \in \{1, 2\}$ are the anisotropic contributions, and γ_1, γ_2 are two unit vectors in the initial configuration.

- *Transversely isotropic growth.*

$$F_g = g_0 I + (g_1 - 1)\gamma \otimes \gamma.$$

  In this case, growth takes place isotropically in the directions normal to a unit vector γ.

- *Pure fiber growth.* If we further restrict growth along a single fiber we obtain

$$F_g = I + (g - 1)\gamma \otimes \gamma.$$

- *Area growth.* The particular case of in-plane growth obtained as a reduction of transversely isotropic growth is interesting as it provides a simple characterization for problems of growing plates and membranes

$$F_g = \sqrt{g} I + (1 - \sqrt{g})\gamma \otimes \gamma.$$

- *Isotropic growth.* The simplest nontrivial form for the growth tensor is to take it as a multiple of the identity, that is,

$$F_g = gI,$$

  with $g : \Omega \times [0, \infty) \to \mathbb{R}$ the *growth metric* function. Particularly, $\hat{g}$ indicates the resorption of materials for $0 < \hat{g} < 1$, while $\hat{g} > 1$ implies the growth. Moreover, we have

$$J_g = g^d,$$

representing the isotropic change of a volume element. Now one can link g to the growth rate function γ directly through (1.30)

$$\partial_t g = \frac{\gamma}{d} g \quad \text{for all } X \in \Omega, t \geq 0, \tag{1.35}$$

under the constant-density growth assumption.

1.7. Coupled Systems with Free Sharp Interfaces

In this section, we are devoted to presenting the final coupled systems we will concern, which includes the fluid dynamics, solid mechanics, biochemical processes, and growth.

First, let us record the main couplings of quantities between the fluid region and solid region, i.e., interfacial couplings. To begin with, we suppose that a spatial domain Ω^t — a bounded domain in $\mathbb{R}^d$ with smooth boundary — is filled with fluids and solids that occupy the regions Ω^t_f and Ω^t_s respectively. There is an interface between fluid and solid, which can be interpreted as the endothelial layer in a blood vessel that is supposed to be a sharp interface. The interface Γ^t separating these two phases will depend on time t. Here we assume that the fluid domain Ω^t_f is surrounded by the solid domain Ω^t_s in general. The outer unit normal at Γ^t with respect to Ω^t_f will be denoted by n_{Γ^t}, which depends on the points on Γ^t as well as on t.

For a quantity f, the double brackets $[\![f]\!]$ denotes the jump of values defined on Ω^t_f and Ω^t_s across Γ^t, namely,

$$[\![f]\!](x) := \lim_{\theta \to 0} f(x + \theta n_\Gamma(x)) - f(x - \theta n_\Gamma(x)), \quad \forall\, x \in \Gamma^t,$$

where n_Γ is the unit outer normal at Γ^t pointing from Ω^t_f to Ω^t_s.

Remark 1.51. We will not specify the outer boundaries and the boundary conditions here, since they vary a little bit in different contexts.

1.7.1. Interface conditions — couplings.

On the interface Γ^t, we have two principles concerning mechanics:

- Kinetic condition (continuity of velocities):

  $$[\![v]\!] = 0,$$

  meaning that there is no-slip between fluid velocity and solid velocity.

- Dynamical condition (continuity of normal stresses):

  $$[\![T]\!]\, n_{\Gamma^t} = 0,$$

  namely, the traction (normal stress tensor) on the interface is continuous.

Concerning the cells concentrations, the penetration of monocytes from the blood flow into the vessel wall is modeled by transmission conditions for the concentration of monocytes c_f and of macrophages c_s on the interface Γ^t

- Continuity of the normal fluxes:

  $$[\![D\nabla c]\!] \cdot n_{\Gamma^t} = 0.$$

- Concentration difference:

  $$\zeta\, [\![c]\!] = D_s \nabla c_s \cdot n_{\Gamma^t},$$

  which means the flux is related to the difference of concentrations across the interface. The coefficient ζ describes the permeability of the interface Γ^t with respect to the monocytes.

1.7.2. FSIG in a smooth domain. To distinguish the quantities in Ω_f^t and Ω_s^t, we specify them with a superscript f and s respectively. Particularly, the outer boundary is a smooth boundary denoted by $\Gamma_s^t = \partial\Omega^t$, which is a free boundary as well. Moreover, we assume that

- the blood is assumed to be an incompressible homogeneous viscous fluid.
- the vessel consists of an incompressible Neo-Hookean material with linear Kelvin–Voigt viscoelasticity.
- the reaction function r_s in Section 1.5.1 depends on macrophages c_s linearly, i.e., $r_s = \beta c_s$ [AM02, JC12, Yan+16].
- the plaque forms with constant-density growth.
- the growth evolves in an isotropic way, with the grow rate function endowed with the form of $\gamma\beta c_s$ for constant $\gamma > 0$, meaning that the mass of solid increases at a particular rate of macrophages reactions [AM02, JC12, Yan+16].
- the deformation gradient of the solid can be decomposed as in (1.26): $F_s = F_{s,e}F_{s,g} = gF_{s,e}$.

Now we summarize our first model in a smooth domain for $T > 0$:

$$\rho_f\left(\partial_t + v_f\cdot\nabla\right)v_f = \operatorname{div} T_f, \quad \text{in } \Omega_f^t,\ t\in(0,T), \tag{1.36a}$$
$$\operatorname{div} v_f = 0, \quad \text{in } \Omega_f^t,\ t\in(0,T), \tag{1.36b}$$
$$\rho_s\left(\partial_t + v_s\cdot\nabla\right)v_s = \operatorname{div} T_s, \quad \text{in } \Omega_s^t,\ t\in(0,T), \tag{1.36c}$$
$$\rho_s \operatorname{div} v_s = \gamma\beta c_s, \quad \text{in } \Omega_s^t,\ t\in(0,T), \tag{1.36d}$$
$$\partial_t c_f + v_f\cdot\nabla c_f - \operatorname{div}(D_f\nabla c_f) = 0, \quad \text{in } \Omega_f^t,\ t\in(0,T), \tag{1.36e}$$
$$\partial_t c_s + v_s\cdot\nabla c_s - \operatorname{div}(D_s\nabla c_s) = -\beta c_s, \quad \text{in } \Omega_s^t,\ t\in(0,T), \tag{1.36f}$$
$$\partial_t c_s^* + v_s\cdot\nabla c_s^* = \beta c_s, \quad \text{in } \Omega_s^t,\ t\in(0,T), \tag{1.36g}$$
$$\partial_t g + v_s\cdot\nabla g = \frac{\gamma\beta c_s}{d\rho_s}, \quad \text{in } \Omega_s^t,\ t\in(0,T), \tag{1.36h}$$
$$[\![v]\!] = 0, \quad [\![T]\!]\, n_{\Gamma^t} = 0, \quad \text{on } \Gamma^t,\ t\in(0,T), \tag{1.36i}$$
$$[\![D\nabla c]\!]\cdot n_{\Gamma^t} = 0, \quad \zeta\,[\![c]\!] - D_s\nabla c_s\cdot n_{\Gamma^t} = 0, \quad \text{on } \Gamma^t,\ t\in(0,T), \tag{1.36j}$$
$$T_s n_{\Gamma_s^t} = 0, \quad D_s\nabla c_s\cdot n_{\Gamma_s^t} = 0, \quad \text{on } \Gamma_s^t,\ t\in(0,T), \tag{1.36k}$$
$$v|_{t=0} = v^0,\ \ c|_{t=0} = c^0,\ \ c_s^*|_{t=0} = c_*^0,\ \ g|_{t=0} = g^0, \tag{1.36l}$$

where ρ_i are the densities and v_i are the velocities for $i\in\{f,s\}$. The tensor $T_f = -\pi_f\mathbb{I} + 2\nu_f Dv_f$ denotes the Cauchy stress tensor of the fluid, the function π_f is the unknown fluid pressure and the constant ν_f represents the fluid viscosity. The tensor T_s is the Cauchy stress tensor of the solid that includes incompressible Neo-Hookean elastic and viscoelastic effects. As discussed in Sections 1.4.4, 1.4.6 and 1.6.3, $T_s = T_s^e + T_s^v$ satisfying

$$T_s^e\circ\varphi = \mu_s(F_{s,e}F_{s,e}^\top - I) - \pi_s I, \quad T_s^v\circ\varphi = \nu_s J_s^{-1}(\dot F_s + \dot F_s^\top)F_s^\top.$$

In addition, c_f, c_s, c_s^* denote the concentrations of the monocytes, the macrophages and the foam cells, respectively. The constants $D_i > 0$, $i\in\{f,s\}$ are the diffusion coefficients in the blood and vessel, which are assumed to be constants.

1.7.3. FSIG in a cylindrical domain. Based on the assumptions in Section 1.7.2, our second model is defined in a cylindrical domain, where the boundary is no longer smooth anymore. Namely, the boundary has ninety-degree contact angles and moving contact lines on the cross sections. This is a rather realistic model to describe the blood flow surrounded by the vessels. More precisely, let $\Omega^t := \Omega^t_f \cup \Omega^t_s \cup \Sigma^t \subset \mathbb{R}^3$ (see Figure 4.1), with three disjoint parts, where Ω^t_f, Ω^t_s are piece-wise smooth domains for fluid and solid respectively, while Σ^t is a two dimensional sub-manifold of $\mathbb{R}^3$ with boundary $\partial\Sigma^t$. In particular, $\partial\Omega^t = \overline{G^t} \cup S$, $\partial\Omega^t_f = \overline{G^t_f} \cup \Sigma^t$ and $\partial\Omega^t_s = \overline{G^t_s} \cup \Sigma^t \cup S$, where $G^t := G^t_1 \cup G^t_2 \cup \partial\Sigma^t$ is a hypersurface with $G^t_\beta := G^t_{1,\beta} \cup G^t_{2,\beta}$, $\beta \in \{f, s\}$, $G^t_i = G^t_{i,f} \cup G^t_{i,s}$, $\overline{G^t_{i,f}} \subset G^t_i$, $i \in \{1, 2\}$, and S denotes the fixed surrounding surface, which is supposed to be perpendicular to G^t at ∂S. Moreover, Σ^t is assumed to be perpendicular to G^t at $\partial\Sigma^t$ as well. In such setting, the domain is endowed with the fixed contact line ∂S and moving contact line $\partial\Sigma^t$ with ninety-degree contact angles for a short time.

The system in the domain is same as (1.36), while the boundary and interface conditions read as

$$
\begin{aligned}
[\![v]\!] = 0, \quad [\![T]\!]\, n_{\Sigma^t} &= 0, && \text{on } \Sigma^t,\ t \in (0,T), && (1.37a)\\
[\![D\nabla c]\!] \cdot n_{\Sigma^t} = 0, \quad \zeta\, [\![c]\!] - D_s\nabla c_s \cdot n_{\Sigma^t} &= 0, && \text{on } \Sigma^t,\ t \in (0,T), && (1.37b)\\
v_s &= 0, && \text{on } S,\ t \in (0,T), && (1.37c)\\
D_s\nabla c_s \cdot n_S &= 0, && \text{on } S,\ t \in (0,T), && (1.37d)\\
\mathcal{P}_{G^t}(v) = 0, \quad (Tn_{G^t})n_{G^t} &= 0, && \text{on } G^t \setminus \Sigma^t,\ t \in (0,T), && (1.37e)\\
D\nabla c \cdot n_{G^t} &= 0, && \text{on } G^t \setminus \Sigma^t,\ t \in (0,T), && (1.37f)
\end{aligned}
$$

where $\mathcal{P}_{G^t} := I - n_{G^t} \otimes n_{G^t}$ denotes the tangential projection onto G^t.

1.7.4. QFSIG in a smooth domain. Note that the first two models all include artificial viscoelastic effects, which promote the regularity of solutions to the system. Now we introduce our third model in a smooth domain, which consists of an incompressible homogeneous viscous fluid and a general hyperelastic solid equation in equilibrium at each time, as well as the same biochemical processes as (1.36) and (1.37). Specifically, we neglect the kinetic energy of the solid reasonably, since the time-scale of the movement of the vessels is usually much larger than that of the blood. In this case, we do not require any viscous regularization any more and consider a general incompressible hyperelastic material with a nonlinear strain energy density function W in terms of F. The model is presented as

$$
\begin{aligned}
\rho_f\,(\partial_t + v_f \cdot \nabla)\, v_f &= \operatorname{div} T_f, && \text{in } \Omega^t_f,\ t \in (0,T), && (1.38a)\\
\operatorname{div} v_f &= 0, && \text{in } \Omega^t_f,\ t \in (0,T), && (1.38b)\\
\operatorname{div} T_s &= 0, && \text{in } \Omega^t_s,\ t \in (0,T), && (1.38c)\\
\rho_s \operatorname{div} v_s &= \gamma\beta c_s, && \text{in } \Omega^t_s,\ t \in (0,T), && (1.38d)\\
\partial_t c_f + v_f \cdot \nabla c_f - \operatorname{div}(D_f\nabla c_f) &= 0, && \text{in } \Omega^t_f,\ t \in (0,T), && (1.38e)\\
\partial_t c_s + v_s \cdot \nabla c_s - \operatorname{div}(D_s\nabla c_s) &= -\beta c_s, && \text{in } \Omega^t_s,\ t \in (0,T), && (1.38f)\\
\partial_t c^*_s + v_s \cdot \nabla c^*_s &= \beta c_s, && \text{in } \Omega^t_s,\ t \in (0,T), && (1.38g)\\
\partial_t g + v_s \cdot \nabla g &= \frac{\gamma\beta c_s}{d\rho_s}, && \text{in } \Omega^t_s,\ t \in (0,T), && (1.38h)\\
[\![v]\!] = 0, \quad [\![T]\!]\, n_{\Gamma^t} &= 0, && \text{on } \Gamma^t,\ t \in (0,T), && (1.38i)\\
[\![D\nabla c]\!] \cdot n_{\Gamma^t} = 0, \quad \zeta\, [\![c]\!] - D_s\nabla c_s \cdot n_{\Gamma^t} &= 0, && \text{on } \Gamma^t,\ t \in (0,T), && (1.38j)
\end{aligned}
$$

$$T_s n_{\Gamma_s^t} = 0, \quad D_s \nabla c_s \cdot n_{\Gamma_s^t} = 0, \qquad \text{on } \Gamma_s^t,\, t \in (0,T), \tag{1.38k}$$

$$v_f|_{t=0} = v_f^0,\;\; u_s|_{t=0} = u_s^0,\;\; c|_{t=0} = c^0,\;\; c_s^*|_{t=0} = c_*^0,\;\; g|_{t=0} = g^0. \tag{1.38l}$$

In particular, the Cauchy stress tensor T_s of the solid has the general form of

$$T_s \circ \varphi = J_{s,e}^{-1} \frac{\partial W(F_{s,e})}{\partial F_{s,e}} F_{s,e}^\top - \pi_s I,$$

with certain assumptions on the strain energy density function W, which we will specify for analysis in Section 5.1.1.

1.8. A Diffuse Interface Model for Two-Phase Flows

In this section, we provide the main arguments for a thermodynamically consistent derivation of a diffuse interface model for a two-phase incompressible viscoelastic flow, which particularly can describe a fluid-structure interaction problem. The general idea is to start from physical balance laws in a closed, isothermal system. After that, we state phenomenological assumptions such that the system fulfills the second law of thermodynamics. At that point, there are some general frameworks that can be used for the constitutive assumptions, such as the *Local Dissipation Inequality and the Lagrange Multiplier Approach* developed by Liu [Liu72], the *Onsager's variational principle* [Ons32] or the *General Equation for Non-Equilibrium Reversible-Irreversible Coupling* (GENERIC) framework of Gmerla–Öttinger [GO97, OG97]. We also refer to [GKL18, MP18]. In our case, we will follow [AGG12] and use the *Local Dissipation Inequality and the Lagrange Multiplier Approach*.

1.8.1. Local mass and momentum conservation laws. We suppose that every function is smooth enough for our arguments. We consider the time evolution of two fluids (indexed with $i = 1, 2$) in a smooth domain $\Omega \subset \mathbb{R}^d$ with $d \in \{2, 3\}$ on a time interval $[0, T]$ with $T > 0$. Note that here in fact, two fluids should be a fluid and a solid, while the solid is described as a fluid. Let $\rho_i : \Omega \times (0, T) \to \mathbb{R}$ be the mass density of the corresponding fluid i. The local mass conservation (1.7) read as

$$\partial_t \rho_i + \operatorname{div}(\rho_i \mathbf{u}_i) = 0, \tag{1.39}$$

with $\mathbf{u}_i$ be the velocity of fluid i. Let now $\phi_i : \Omega \times (0, T) \to \mathbb{R}$ be the volume fraction of two fluids. Concerning the specific constant mass densities $\tilde{\rho}_i$ of the pure phase of fluid i, the volume fraction is introduced by

$$\phi_i = \frac{\rho_i}{\tilde{\rho}_i}, \quad i = 1, 2.$$

The assumption that the excess volume is zero leads to

$$\phi_1 + \phi_2 = 1. \tag{1.40}$$

Define the order parameter ϕ by the difference of volume fractions

$$\phi = \phi_2 - \phi_1.$$

Moreover, we choose the volume averaged velocity $\mathbf{u}$ of the mixture as

$$\mathbf{u} = \phi_1 \mathbf{u}_1 + \phi_2 \mathbf{u}_2 = \frac{\rho_1}{\tilde{\rho}_1} \mathbf{u}_1 + \frac{\rho_2}{\tilde{\rho}_2} \mathbf{u}_2,$$

which satisfies the "natural" divergence-free condition

$$\operatorname{div}\mathbf{u} = \operatorname{div}\left(\frac{\rho_1}{\tilde{\rho}_1}\mathbf{u}_1\right) + \operatorname{div}\left(\frac{\rho_2}{\tilde{\rho}_2}\mathbf{u}_2\right) = -\partial_t\left(\frac{\rho_1}{\tilde{\rho}_1} + \frac{\rho_2}{\tilde{\rho}_2}\right) = -\partial_t 1 = 0, \tag{1.41}$$

in view of the continuity equation (1.39) and (1.40). Denote by $\mathbf{J}_i = \rho_i\mathbf{u}_i - \rho_i\mathbf{u}$ the mass flux of fluid i related to the velocity $\mathbf{u}$. Then the continuity equation (1.39) becomes

$$\partial_t\rho_i + \operatorname{div}(\rho_i\mathbf{u}) + \operatorname{div}\mathbf{J}_i = 0, \tag{1.42}$$

By definitions, it follows

$$\partial_t\phi + \operatorname{div}(\phi\mathbf{u}) + \operatorname{div}\mathbf{J}_\phi = 0, \tag{1.43}$$

where $\mathbf{J}_\phi = \frac{\mathbf{J}_2}{\tilde{\rho}_2} - \frac{\mathbf{J}_1}{\tilde{\rho}_1}$ is a diffusive flux.

The balance law of linear momentum (1.9) reads

$$\partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) = \operatorname{div}\tilde{\mathbf{T}}, \tag{1.44}$$

where $\rho = \rho_1 + \rho_2 = \frac{1-\phi}{2}\tilde{\rho}_1 + \frac{1+\phi}{2}\tilde{\rho}_2$ is the total mass and $\tilde{\mathbf{T}}$ is the full stress tensor of the system, which is symmetric due to the balance law of angular momentum. For simplicity, we assume that no external forces are present.

Remark 1.52. By the definition of ρ, one can rewrite the continuity equation (1.42) as

$$\partial_t\rho + \operatorname{div}(\rho\mathbf{u}) + \operatorname{div}\tilde{\mathbf{J}} = 0, \tag{1.45}$$

with $\tilde{\mathbf{J}} = \mathbf{J}_1 + \mathbf{J}_2$. Subsequently equation (1.44) can be transferred to a non-conservative formulation by eliminating (1.45).

1.8.2. Deformation gradient and Cauchy–Green stress tensor. We consider the Eulerian deformation gradient $\mathbb{F} : \Omega \times (0,T) \to \mathbb{R}^{d\times d}$ of the mixture, which by Definitions 1.3 and 1.13 entails

$$\partial_t\mathbb{F} + \mathbf{u}\cdot\nabla\mathbb{F} = \nabla\mathbf{u}\mathbb{F}. \tag{1.46}$$

For the derivation of the Oldroyd-B model with relaxation, we assume a virtual multiplicative decomposition of the deformation gradient into one part capturing the irreversible, dissipative processes and another part for the total elastic response of the material, i.e.,

$$\mathbb{F} = \mathbb{F}_e\mathbb{F}_d, \tag{1.47}$$

also see [MP18]. Then, the left Cauchy–Green tensor associated with the elastic part of the total mechanical response $\mathbb{B} \coloneqq \mathbb{F}_e\mathbb{F}_e^\top$ is the sought quantity for our model. Introducing the tensorial quantity $\mathbb{L}_d \coloneqq (\partial_t\mathbb{F}_d + \mathbf{u}\cdot\nabla\mathbb{F}_d)\mathbb{F}_d^{-1}$, one can obtain with a simple calculation, using (1.46) and (1.47), that

$$\partial_t\mathbb{F}_e + \mathbf{u}\cdot\nabla\mathbb{F}_e = \nabla\mathbf{u}\mathbb{F}_e - \mathbb{F}_e\mathbb{L}_d,$$

which gives for $\mathbb{B} = \mathbb{B}_e : \Omega\times(0,T) \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$

$$\partial_t\mathbb{B} + \mathbf{u}\cdot\nabla\mathbb{B} = \nabla\mathbf{u}\mathbb{B} + \mathbb{B}\nabla\mathbf{u}^\top - \mathbb{F}_e(\mathbb{L}_d + \mathbb{L}_d^\top)\mathbb{F}_e^\top. \tag{1.48}$$

Later, the dependence on $\mathbb{D}_d \coloneqq \frac{1}{2}(\mathbb{L}_d + \mathbb{L}_d^\top)$ will be removed with a constitutive relation.

1.8.3. Local energy dissipation laws. Assuming a general energy density of the form

$$e = \hat{e}(\phi, \nabla\phi, \mathbb{B}) + \frac{1}{2}\rho(\phi)\,|\mathbf{u}|^2\,,$$

composed of a (general) free energy density and the kinetic energy density of the system, the second law of thermodynamics for a closed physical system in the isothermal case gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}(t)} e(\phi, \nabla\phi, \mathbf{u}, \mathbb{B})\mathrm{d}x \le -\int_{\partial\mathcal{U}(t)} \mathbf{J}_e \cdot \mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1} + \int_{\partial\mathcal{U}(t)} (\tilde{\mathbf{T}}\mathbf{n}) \cdot \mathbf{u}\,\mathrm{d}\mathcal{H}^{d-1},$$

where $\mathcal{U}(t) \subseteq \Omega$ is an arbitrary open set, $\mathbf{n}$ is the outer unit normal to $\partial\mathcal{U}(t)$ and $\mathbf{J}_e$ is a dissipative energy flux yet to be determined. Roughly speaking, the change of the total energy in a test volume $\mathcal{U}(t)$ cannot exceed the change of energy due to diffusion and the working due to macroscopic stresses.

1.8.4. Lagrange multiplier and final model. With the help of Reynolds' Transport Theorem and Gauß Theorem, and as the test volume $\mathcal{U}(t) \subset \Omega$ is arbitrary, one can obtain a local inequality for the dissipation by

$$-\mathcal{D} := (\partial_t^\bullet e + e\,\mathrm{div}\,\mathbf{u} + \mathrm{div}\,\mathbf{J}_e) - \mathrm{div}(\tilde{\mathbf{T}}) \cdot \mathbf{u} - \tilde{\mathbf{T}} : \nabla\mathbf{u} - q(\partial_t^\bullet \phi + \phi\,\mathrm{div}\,\mathbf{u} + \mathrm{div}\,\mathbf{J}_\phi) \le 0,$$

where $\partial_t^\bullet := \partial_t + (\mathbf{u} \cdot \nabla)$ denotes the material derivative and q is a Lagrange multiplier for the balance law of mass (1.43) for the order parameter ϕ. Note that in general the unknowns $\mathbf{u}, \mathbf{J}_e, \tilde{\mathbf{T}}, \phi, \mathbf{J}_\phi, q$ (and their derivatives appearing in the local dissipation inequality) can attain arbitrary values for a given point in space and time. It can be checked with a straightforward computation, using (1.41), (1.44), (1.48) and various reformulations (see, e.g., [AGG12, Section 2.2] for a diffuse interface model for a two-phase flow of incompressible viscous fluids and, e.g., [MP18, Section 4.4] for the viscoelastic part) that this local dissipation inequality holds true if the following constitutive assumptions are applied:

$$\mathbf{J}_e = q\mathbf{J}_\phi + \partial_t^\bullet \phi \frac{\partial\hat{e}}{\partial\nabla e} - \frac{1}{2}\mathbf{J}\,|\mathbf{u}|^2\,, \quad \mathbf{J} = \rho'(\phi)\mathbf{J}_\phi, \quad \mathbf{J}_\phi = -m(\phi)\nabla q,$$
$$q = \frac{\partial\hat{e}}{\partial\phi} - \mathrm{div}\,\frac{\partial\hat{e}}{\partial\nabla\phi}, \quad \mathbb{D}_d = \frac{1}{\tilde{\lambda}(\phi)}\mathbb{F}_e^{-1}\frac{\partial\hat{e}}{\partial\mathbb{B}}\mathbb{F}_e,$$
$$\tilde{\mathbf{T}} = \mathbb{S} - p\mathbb{I} - (\mathbf{u} \otimes \mathbf{J}) - \frac{\partial\hat{e}}{\partial\nabla\phi} \otimes \nabla\phi, \quad \mathbb{S} = \nu(\phi)(\nabla\mathbf{u} + \nabla\mathbf{u}^\top) + 2\frac{\partial\hat{e}}{\partial\mathbb{B}}\mathbb{B},$$

where $m(\phi), \tilde{\lambda}(\phi), \nu(\phi)$ are positive functions corresponding to a mobility, a relaxation and a viscosity, respectively. Here, also a relative mass flux $\mathbf{J}$ and the viscoelastic stress tensor $\mathbb{S}$ were introduced, also see [AGG12, MAA18].

The constitutive system of equations (with a general energy density) reads

$$\begin{aligned}\partial_t(\rho(\phi)\mathbf{u}) + \mathrm{div}(\rho(\phi)\mathbf{u} \otimes \mathbf{u}) + \mathrm{div}(\mathbf{u} \otimes \mathbf{J}) + \nabla p&\\ - \mathrm{div}\,\big(\mathbb{S}(\nabla\mathbf{u}, \mathbb{B}, \phi)\big) = -\,\mathrm{div}\,\Big(\frac{\partial\hat{e}}{\partial\nabla\phi} \otimes \nabla\phi\Big)&,\end{aligned} \tag{1.49a}$$

$$\mathrm{div}\,\mathbf{u} = 0, \tag{1.49b}$$

$$\partial_t\mathbb{B} + \mathbf{u} \cdot \nabla\mathbb{B} + \frac{2}{\tilde{\lambda}(\phi)}\frac{\partial\hat{e}}{\partial\mathbb{B}}\mathbb{B} = \mathbb{B}\nabla\mathbf{u}^\top + \nabla\mathbf{u}\mathbb{B}, \tag{1.49c}$$

$$\partial_t\phi + \mathbf{u} \cdot \nabla\phi = \mathrm{div}(m(\phi)\nabla q), \tag{1.49d}$$

$$q = \frac{\partial\hat{e}}{\partial\phi} - \mathrm{div}\,\frac{\partial\hat{e}}{\partial\nabla\phi}\,. \tag{1.49e}$$

Here, the local dissipation is given by

$$\mathcal{D} = \frac{\nu(\phi)}{2} \left|\nabla\mathbf{u} + \nabla\mathbf{u}^\top\right|^2 + m(\phi)\left|\nabla q\right|^2 + \frac{2}{\tilde{\lambda}(\phi)} \left|\mathbb{F}_e^\top \frac{\partial \hat{e}}{\partial \mathbb{B}} \mathbb{F}_e\right|^2 \geq 0.$$

We note that the system (6.1) below can be recovered from (1.49) with the specific choice of the free energy density

$$\hat{e}(\phi, \nabla\phi, \mathbb{B}) = \frac{\tilde{\sigma}\epsilon}{2} \left|\nabla\phi\right|^2 + \frac{\tilde{\sigma}}{\epsilon} W(\phi) + \frac{\mu(\phi)}{2} \operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I}),$$

and a rescaling of the relaxation function $\tilde{\lambda}(\phi) = \lambda(\phi)\mu(\phi)/\alpha(\phi)$. In this case, cf. [MAA18], the (visco)elastic solid can be modeled by letting $\alpha = 0$, $\lambda > 0$, the viscous fluid is described with $\alpha > 0$, $\lambda = 0$, while a fluid-structure interaction problem can be recovered with $\alpha > 0$ and $\lambda > 0$.

1.9. Appendix: Scalars, Vectors, and Tensors

In this section, we recall and introduce some necessary algebra and analysis of functions, which can be a *scalar* $f(x,t) \in \mathbb{R}$ (density, temperature), a *vector* $v(x,t) \in \mathbb{R}^d$ (velocity, acceleration, force), or a *matrix* $F(x,t) \in \mathbb{R}^{d\times d}$ (deformation gradients or stress and strain tensors). In general, these different functions can all be understood as tensors of different orders. By definition, a scalar field is a tensor of order 0, a vector field is a first-order tensor, and a matrix is a tensor of order 2. Higher order tensors require the definition of the tensor product.

1.9.1. Algebra. Now we recall the fundamental algebraic relationships of different functions.

Definition 1.53 (Products of vectors). Let $u = (u_i)_{i=1}^d$, $v = (v_i)_{i=1}^d$ be vectors. We define the *cross product* for $d = 3$ by

$$u \times v = (\epsilon_{ijk} u_j v_k)_{i=1}^3 \in \mathbb{R}^3,$$

where ϵ_{ijk} is the permutation symbol (*Levi–Civita symbol*). Moreover, the *tensor product* is given by

$$u \otimes v = (u_i v_j)_{i,j=1}^d \in \mathbb{R}^{d\times d}.$$

Here the *Levi–Civita symbol* is defined as

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) \text{ is an even permutation,} \\ -1, & \text{if } (i,j,k) \text{ is an odd permutation,} \\ 0, & \text{if } (i,j,k) \text{ is not a permutation.} \end{cases} \tag{1.50}$$

For convenience, we give the so-called *epsilon-delta identities.*

Lemma 1.54 (Epsilon-delta identities). *The following identities hold:*

$$\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}, \quad \epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}.$$

Proposition 1.55 (Axial vector of a skew tensor [GFA10]). *Given any $\Omega \in \mathbb{R}^{3\times 3}_{\text{skew}}$, there is a unique vector ω, called the* axial vector of Ω*, such that*

$$\Omega = \omega\times, \ \textit{i.e.}\ \Omega_{ij} = \epsilon_{ikj} w_k \tag{1.51}$$

and, hence, such that

$$\Omega u = \omega \times u \quad \textit{for all vectors} \quad u.$$

Proof. The uniqueness follows from

$$a \times v = b \times v \text{ for all vectors } v \text{ if and only if } a = b.$$

To establish the existence of such a vector, if suffices to show that ω defined by

$$\omega_i = -\frac{1}{2}\epsilon_{ijk}\Omega_{jk} \tag{1.52}$$

satisfies (1.51). By (1.52) and the epsilon-delta identity,

$$\epsilon_{ipq}\omega_i = -\frac{1}{2}\epsilon_{ipq}\epsilon_{ijk}\Omega_{jk} = -\frac{1}{2}(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp})\Omega_{jk} = -\frac{1}{2}(\Omega_{pq} - \Omega_{qp}) = \Omega_{qp}.$$

Thus, $\Omega_{qp} = \epsilon_{ipq}\omega_i$, which implies (1.51). □

Definition 1.56 (Symmetric and skew tensors). Let $T = (T_{ij})_{i,j=1}^d$ be a matrix. We say it is *symmetric* or *skew* if

$$T = T^\top \quad \text{or} \quad T = -T^\top,$$

and we refer $\operatorname{sym} T$ and $\operatorname{skew} T$ respetively to the *symmetric part* and *skew part* of T, defined by

$$\operatorname{sym} T = \frac{1}{2}\left(T + T^\top\right), \quad \operatorname{skew} T = \frac{1}{2}\left(T - T^\top\right),$$

Definition 1.57 (Products of tensors). Let $T = (T_{ij})_{i,j=1}^d$, $S = (S_{ij})_{i,j=1}^d$ be matrices. We define the *inner product* of them by

$$T : S = \operatorname{tr}(T^\top S) = \operatorname{tr}(TS^\top) = T_{ij}S_{ij}.$$

The *inner product* is also called *Frobenius product* and its induced modulus is defined by

$$|T| = \sqrt{T : T}.$$

As a consequence, we have the following proposition.

Proposition 1.58. *Let $T = (T_{ij})_{i,j=1}^d$ be a matrix. Then*

$$\operatorname{skew}(\operatorname{sym} T) = \operatorname{sym}(\operatorname{skew} T) = 0,$$

and it is endowed with a unique *decomposition such that*

$$T = \operatorname{sym} T + \operatorname{skew} T. \tag{1.53}$$

Moreover, let $S = (S_{ij})_{i,j=1}^d$ be another matrix, then

$$T : S = (\operatorname{sym} T) : (\operatorname{sym} S) + (\operatorname{skew} T) : (\operatorname{skew} S). \tag{1.54}$$

In particular, $S = T$ implies

$$|T|^2 = |\operatorname{sym} T|^2 + |\operatorname{skew} T|^2.$$

If additionally S is symmetric and W is a skew matrix, then

$$S : T = S : T^\top = S : (\operatorname{sym} T), \tag{1.55}$$
$$W : T = -W : T^\top = W : (\operatorname{skew} T). \tag{1.56}$$

Proof. The first two assertions hold true directly by verifying the definition as above, while the third one follows from

$$4\,\mathrm{sym}\,T : \mathrm{skew}\,S = (T + T^\top) : (S - S^\top) = T : S - T^\top : S^\top + T^\top : S - T : S^\top = 0,$$
$$4\,\mathrm{skew}\,T : \mathrm{sym}\,S = (T - T^\top) : (S + S^\top) = T : S - T^\top : S^\top - T^\top : S + T : S^\top = 0.$$

Now we prove the last two identities. If S is symmetric, then $S = S^\top$, which implies

$$S : T = S^\top : T = S : T^\top.$$

Moreover, $S = \mathrm{sym}\,S$ and $\mathrm{skew}\,S = 0$. Then inserting it into (1.54) yields (1.55). (1.56) follows similarly. □

In view of the Frobenius product, it follows that

Lemma 1.59. *Let T be a matrix. If $T : S = 0$, for all $S \in \mathbb{R}^{d\times d}$, then*

$$T = 0.$$

Proof. We simply take $S = T$, then $T : T = |T|^2 = 0$, meaning $T = 0$. □

Proposition 1.60. *Let S be a matrix. If $T : S = 0$, for all skew (resp. symmetric) tensors T, then S is symmetric (resp. skew).*

Proof. By (1.55) (resp. (1.56)), we have $0 = T : S = T : \mathrm{sym}\,S$ (resp. $= T : \mathrm{skew}\,S$), which implies $\mathrm{sym}\,S = 0$ (resp. $\mathrm{skew}\,S = 0$) due to Lemma 1.59. Then $S = \mathrm{sym}\,S$ (resp. $S = \mathrm{sym}\,S$) is *skew* (resp. *symmetric*) in view of (1.53). □

Now we give an expansion of the determinant of matrices.

Proposition 1.61. *For a matrix $A = (A_{ij})_{i=1}^d$ and $\varepsilon > 0$,*

$$\det(I + \varepsilon A) = 1 + \varepsilon\,\mathrm{tr}\,A + \sum_{\ell=2}^{d} \varepsilon^\ell M_\ell(A), \tag{1.57}$$

where $M_\ell(A)$ is a homogeneous polynomial of degree ℓ in the entries of A.

Proof. The first part is obvious and we refer to [GFA10]. To prove (1.57), we first consider the Leibniz formula

$$\det(I + \varepsilon A) = \sum_{\pi \in S_d} \mathrm{sgn}(\pi) \prod_{i=1}^{d} (\delta_{i,\pi(i)} + \varepsilon A_{i,\pi(i)})$$

where sgn is the sign function of permutations in the permutation group S_d, which returns $+1$, -1 for even and odd permutations, respectively. Expanding the product and rearranging the terms by the exponent of the factor ε^ℓ and thus by the number of terms $\varepsilon^\ell A_{i,\pi(i)}$ directly yield the homogeneous polynomial $M_\ell(A)$. Namely,

$$\det(I + \varepsilon A) = \sum_{\ell=0}^{d} \varepsilon^\ell M_\ell(A).$$

For the order of ε^0, it is necessary that $\pi(i) = i$ since otherwise the product will be zero, while for the order ε^1, the same idea applies for εA_{ii}-term. Therefore, $M_0(A) = 1$ and $M_1(A) = \sum_{i=1}^d A_{ii} = \mathrm{tr}\,A$. The same argument can be found in [BKS23, Lemma A.1]. □

Then one reaches the following corollary.

Corollary 1.62. *Let* $T = (T_{ij})_{i,j=1}^d$, $S = (S_{ij})_{i,j=1}^d$ *be invertible tensors and* $\varepsilon > 0$. *Then*

$$\det(T + \varepsilon S) = \det T(1 + \varepsilon \operatorname{tr}(T^{-1}S) + \mathcal{O}(\varepsilon^2)). \tag{1.58}$$

Proof. By Proposition 1.61, we have

$$\begin{aligned}\det(T + \varepsilon S) = \det(T(I + \varepsilon(T^{-1}S)) &= \det T \det(I + \varepsilon \operatorname{tr}(T^{-1}S)) \\ &= \det T(1 + \varepsilon \operatorname{tr}(T^{-1}S) + \mathcal{O}(\varepsilon^2)),\end{aligned}$$

where the last identity follows from (1.57). □

1.9.2. Differentiation of scalar-, vector- and matrix-valued functions. With the notation $\partial_j := \partial_{x_j} := \frac{\partial}{\partial x_j}$ (sometimes mixed-used them depends on the context), and $\partial_t := \frac{\partial}{\partial t}$, now we turn to the derivatives of different fields.

Definition 1.63. Let $\mathcal{U} \subseteq \mathbb{R}^d$ be an open set. Let $f : \mathcal{U} \to \mathbb{R}$ be a scalar-valued function, $v : \mathcal{U} \to \mathbb{R}^d$ be a vector-valued function, and $T : \mathcal{U} \to \mathbb{R}^{d\times d}$ be a matrix-valued function, that are sufficiently differentiable. Then

(1) The gradient of f refers to a vector defined by its coordinates in any orthonormal basis of $\mathbb{R}^d$ by

$$\nabla f = \left(\frac{\partial f}{\partial x_j}\right)_{j=1}^d.$$

(2) The gradient of v refers to a matrix in $\mathbb{R}^{d\times d}$, and the divergence of v refers to the scalar field, respectively by

$$\nabla v = \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1}^d, \quad \operatorname{div} v = \sum_{j=1}^d \frac{\partial v_j}{\partial x_j}.$$

In the following, we will simply write $\nabla v^\top := (\nabla v)^\top$.

(3) In the case $d = 3$, the curl of v refers to a vector denoted as $\operatorname{curl} v$, defined by

$$\operatorname{curl} v = \nabla \times v = \left(\epsilon_{ijk}\frac{\partial v_k}{\partial x_j}\right)_{i=1}^3,$$

where ϵ_{ijk} is the permutation symbol defined as above.

(4) The gradient and divergence of T are defined by

$$\nabla T = \left(\frac{\partial T_{ij}}{\partial x_k}\right)_{i,j,k=1}^d, \quad \operatorname{div} T = \left(\sum_{j=1}^d \frac{\partial T_{ij}}{\partial x_j}\right)_{i=1}^d,$$

where the gradient of T is a tensor of oder 3, i.e., $\nabla T \in \mathbb{R}^{d\times d\times d}$.

(5) The *Laplacian* of f, v and T are given by

$$\Delta f = \operatorname{div}(\nabla f) = \sum_{j=1}^d \frac{\partial^2 f}{\partial x_j^2}, \ \Delta v = \operatorname{div}(\nabla v) = \left(\sum_{j=1}^d \frac{\partial^2 v_i}{\partial x_j^2}\right)_{i=1}^d, \ \Delta T = \left(\sum_{k=1}^d \frac{\partial^2 T_{ij}}{\partial x_k^2}\right)_{i,j=1}^d.$$

Moreover, we define the following product of vectors and matrices.

DEFINITION 1.64. Let $u = (u_i)_{i=1}^d$ be a vector and $T = (T_{ij})_{i,j=1}^d$, $S = (S_{ij})_{i,j=1}^d$ be matrices. Then we define

$$u \otimes T = (u_i T_{jk})_{i,j,k=1}^d, \quad u \cdot \nabla T = (u_k \partial_k T_{ij})_{i,j=1}^d,$$
$$(u \otimes T) : \nabla S = u_i T_{jk} \partial_i S_{jk}, \quad \nabla T : \nabla S = \partial_i T : \partial_i S = \partial_i T_{jk} \partial_i S_{jk}.$$

Based on the definition, one gives the following proposition providing counterparts, for functions, of the standard product rule for scalar functions of a scalar variable.

PROPOSITION 1.65 (Product rules). *Let $\mathcal{U} \subseteq \mathbb{R}^d$ be an open set. Let $f : \mathcal{U} \to \mathbb{R}$ be a scalar-valued function, $u, v : \mathcal{U} \to \mathbb{R}^d$ be vector-valued functions, and $T : \mathcal{U} \to \mathbb{R}^{d\times d}$ be a matrix-valued function, that are sufficiently differentiable. Then*

$$\begin{aligned}
\nabla(fv) &= f\nabla v + v \otimes \nabla f, \\
\nabla(u \cdot v) &= \nabla u^\top v + \nabla v^\top u, \\
\nabla(u \times v) &= (u\times)\nabla v - (v\times)\nabla u, \\
\operatorname{div}(fv) &= f \operatorname{div} v + v \cdot \nabla f, \\
\operatorname{div}(u \otimes v) &= \operatorname{div} vu - \nabla uv, \\
\operatorname{div}(Tv) &= T^\top : \nabla v + v \cdot \operatorname{div} T^\top, \\
\operatorname{div}(fT) &= f \operatorname{div} T + T\nabla f.
\end{aligned}$$

Proof. For the proof, we refer to [GFA10]. In fact, one can easily verify them by expressing them in terms of components and following the definition of the derivatives in Definition 1.63. □

1.9.3. Differentiation of a scalar function of a matrix-valued function. We start with the definition.

DEFINITION 1.66. Let $\mathcal{U} \subseteq \mathbb{R}^{d\times d}$ be an open set. Let $W : \mathcal{U} \to \mathbb{R}$ be a differentiable scalar function. The *derivative* of W in terms of F refers to a matrix defined by

$$DW(F) = \frac{\partial W(F)}{\partial F} = \left(\frac{\partial W(F)}{\partial F_{ij}} \right)_{i,j=1}^d \quad \text{for all } F \in \mathcal{U}.$$

In some places, we will alternatively use the notation $\partial_F W$ with the same definition. Moreover, let $G = (G_{ij})_{i,j=1}^d$, we define the "directional derivative" of $W(F)$ in the "direction" G by

$$\frac{\partial W(F)}{\partial F} : G = \frac{\partial}{\partial \varepsilon} W(F + \varepsilon G)\bigg|_{\varepsilon=0}.$$

Remark 1.67 ([GFA10]). In computing this derivative, care must be taken to respect the matrix space within which the domain of W lies. For example, if $\mathcal{U} \in \mathbb{R}^{d\times d}_{\text{sym}}$, then, $DW(F) \in \mathbb{R}^{d\times d}_{\text{sym}}$ for all $F \in \mathcal{U}$.

As a consequence of Definition 1.66, we have

LEMMA 1.68 (Jacobi's formula). *If $W(F) = \det F$ with $\det : \mathrm{GL}_d \subset \mathbb{R}^{d\times d} \to \mathbb{R}$, where $\mathrm{GL}_d := \{A \in \mathbb{R}^{d\times d} : A \text{ is invertible}\}$. Then*

$$\frac{\partial \det F}{\partial F} = (\det F)F^{-\top} \quad \text{for all } F \in \mathrm{GL}_d. \tag{1.59}$$

Proof. In view of (1.58), for each matrix $G \in \mathbb{R}^{d\times d}$

$$\frac{\partial \det F}{\partial F} : G = \frac{\partial}{\partial \varepsilon} \det(F + \varepsilon G)\bigg|_{\varepsilon=0} = \det F \operatorname{tr}(F^{-1}G) + \mathcal{O}(\varepsilon)\big|_{\varepsilon=0} = (\det F)F^{-\top} : G.$$

Then Lemma 1.59 implies the desired assertion. □

Based on Definition 1.66 and Lemma 1.68, one has following proposition.

Proposition 1.69 (Time derivative). *Let $W : \mathbb{R}^{d\times d} \to \mathbb{R}$ be differentiable and $F = (F_{ij})_{i,j=1}^{d} : M \subseteq \mathbb{R} \to \mathbb{R}^{d\times d}$ depending on $t \in M$ with M open, be differentiable. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} W(F) = \frac{\partial W(F)}{\partial F} : \dot{F} \quad \text{for all } t \in M, \tag{1.60}$$

where $\dot{F} = \partial_t F$. In particular, if $F(t)$ is invertible,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det F) = (\det F) \operatorname{tr}\left(F^{-1} \frac{\partial F}{\partial t}\right) \quad \text{for all } t \in M, \tag{1.61}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} F^{-1} = -F^{-1} \left(\frac{\mathrm{d}}{\mathrm{d}t} F\right) F^{-1} \quad \text{for all } t \in M. \tag{1.62}$$

Proof. By means of the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t} W(F) = \frac{\partial W(F)}{\partial F_{ij}} \frac{\partial F_{ij}}{\partial t} = \frac{\partial W(F)}{\partial F} : \dot{F} \quad \text{for all } t \in M.$$

Equation (1.61) follows from Jacobi's formula and (1.60). (1.62) can be derived easily by

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} I = \frac{\mathrm{d}}{\mathrm{d}t}(FF^{-1}) = \frac{\mathrm{d}}{\mathrm{d}t} F F^{-1} + F \frac{\mathrm{d}}{\mathrm{d}t} F^{-1} \quad \text{for all } t \in M.$$

□

Chapter 2

Function Spaces and Maximal Regularity Theory

2.1. Function Spaces

2.1.1. Notations. We denote by $\mathbb{N}_0 \coloneqq \mathbb{N} \cup \{0\}$. For metric spaces X, $B_X(x,r)$ denotes the open ball with radius $r > 0$ around $x \in X$. For normed spaces X, Y over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the set of bounded, linear operators $T : X \to Y$ is denoted by $\mathcal{L}(X,Y)$ and in particular, $\mathcal{L}(X) = \mathcal{L}(X,X)$. Moreover, a subset $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$ is called "domain", if Ω is open, nonempty and connected. We simply write $K(M)^N$ to be $K(M;\mathbb{R}^N)$ for $K \in \{L^p, W^k_p, H^k_p\}$, $N \in \mathbb{N}$. Sometimes we will mix the use of W^k_p and H^k_p, as there are equivalent for $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, to make the notations consistent.

Throughout the thesis, unless we give a special declaration, the letter C will denote a generic positive constant that may change its value from line to line, or even in the same line.

2.1.2. Continuous and continuously differentiable functions. Let $\Omega \subseteq \mathbb{R}^n$ be a domain and X be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The space of continuous functions $f : \Omega \to X$ is denoted by $C^0(\Omega;X)$ or $C(\Omega;X)$, and analogously we define $C(\overline{\Omega};X)$ as functions in $C(\Omega;X)$ that can be extended continuously to $\overline{\Omega}$. Moreover, $C_w(\overline{\Omega};X)$ is defined as the space of functions that are continuous on I with respect to the weak topology of X. For $k \in \mathbb{N}$ we define

$$
\begin{aligned}
&C^k(\Omega;X) \coloneqq \{f : \Omega \to X : \partial^\alpha f \in C(\Omega;X) \text{ for all } |\alpha| \leq k\},\\
&C^k_b(\Omega;X) \coloneqq \{f \in C^k(\Omega;X) : \partial^\alpha f \text{ is bounded on } \Omega \text{ for all } |\alpha| \leq k\},\\
&C^k(\overline{\Omega};X) \coloneqq \{f \in C^k(\Omega;X) : \partial^\alpha f \text{ has a unique continuous extension to } \overline{\Omega} \text{ for all } |\alpha| \leq k\},\\
&C^k_b(\overline{\Omega};X) \coloneqq \{f \in C^k_b(\Omega;X) : \partial^\alpha f \text{ has a unique continuous extension to } \overline{\Omega} \text{ for all } |\alpha| \leq k\},
\end{aligned}
$$

where

$$\partial^\alpha f \coloneqq \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n} \coloneqq \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n}},$$

for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n_0$ with $|\alpha| \coloneqq \sum_{i=1}^n \alpha_i$ and $\partial_{x_j} f : \Omega \to X$ is defined as

$$\partial_{x_j} f(x) \coloneqq \lim_{\sigma \to 0} \frac{f(x + \sigma e_j) - f(x)}{\sigma} \in X \quad \text{for all } x \in \Omega.$$

The infinitely differentiable function spaces $C^\infty(\Omega)$ are defined with intersections of all $C^k(\Omega)$, $k \in \mathbb{N}$. Moreover, $C^\infty_0(\Omega)$ is the set of $f \in C^\infty(\Omega;\mathbb{R})$ with compact support $\operatorname{supp} f \subseteq \Omega$. In addition, $C^\infty_0(\overline{\Omega}) \coloneqq \{f|_{\overline{\Omega}} : f \in C^\infty_0(\mathbb{R}^n)\}$.

The Hölder spaces $C^{0,\gamma}(\overline{\Omega};X)$, $\gamma \in (0,1]$ consists of all functions in $C^0_b(\overline{\Omega};X)$, which has a finite γ-th Hölder seminorm, i.e.,

$$[f]_{C^{0,\gamma}(\overline{\Omega};X)} \coloneqq \sup_{x,y \in \overline{\Omega}, x \neq y} \frac{\|f(x) - f(y)\|_X}{|x-y|^\gamma} < \infty.$$

In particular, the functions in $C^{0,1}(\Omega;X)$ are called *Lipschitz continuous*. For $k \in \mathbb{N}_0$, we define

$$C^{k,\gamma}(\overline{\Omega};X) = \{f \in C_b^k(\overline{\Omega};X) : [\partial^\alpha f]_{C^{0,\gamma}(\overline{\Omega};X)} < \infty \text{ for all } |\alpha| \le k\}.$$

Moreover, we introduce the concept concerning the regularity of domains, which is necessary for the definition of some function spaces in domains, as well as regularity theories in domains.

Definition 2.1 (Regularity of domains [Alt16, AF03, Leo17]). Let $k \in \mathbb{N}_0$ and $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be a domain. We say Ω is of class C^k, if for all $x_0 \in \partial\Omega$ there exist $r > 0$ and a C^k function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that – upon relabeling and reorienting the coordinates axes if necessary – we have

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) : x_n > \gamma(x_1, \dots, x_{n-1})\}.$$

In some contexts, we also say the boundary $\partial\Omega$ is C^k, if Ω is C^k. It is usually called *Lipschitz domain* if γ is a Lipschitz function. Moreover, the domain Ω is of class C^∞, if it is C^k for all $k \in \mathbb{N}_0$, and Ω is *smooth* if it is of C^∞.

2.1.3. Lebesgue and Sobolev spaces. Let $(M, \mathcal{A}, \mu)$ be a σ-finite, complete measure space and X be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with norm $\|\cdot\|_X$. Then one can define the notions of *(μ- or strongly-) measurable* and *(Bochner-) integrable* functions $f : M \to X$ and the *Bochner(-Lebesgue)-Integral* through *simple functions*, see Adams–Fournier [AF03, Chapter 1], Amann–Escher [AE09, Chapter X], Leoni [Leo17, Chapter 8] for the definitions and properties. In particular, the *Lebesgue spaces* $L^p(M;X)$ for $1 \le p \le \infty$ are defined by

Definition 2.2 (Lebesgue space). Let $(M, \mathcal{A}, \mu)$ be a measure space and X be a Banach space, and let $1 \le p \le \infty$. Then

$$L^p(M;X) := \{f : M \to X : f \text{ is strongly measurable }, \|f\|_{L^p(M;X)} < \infty\},$$

where for $1 \le p < \infty$

$$\|f\|_{L^p(M;X)} := \left(\int_M \|f\|_X^p \,\mathrm{d}\mu\right)^{\frac{1}{p}},$$

while if $p = \infty$,

$$\|f\|_{L^\infty(M;X)} = \operatorname*{esssup}_M \|f\|_X := \inf\{t \ge 0 : \|f\|_X \text{ for } \mu\text{-a.e. } x \in M\}.$$

If $M = (a, b)$, we write for simplicity $L^p(a, b; X)$. By simple computation, we have

$$\|f\|_{L^p(a,b;X)} \le |a - b|^{\frac{1}{p}} \|f\|_{L^\infty(a,b;X)}. \tag{2.1}$$

Additionally in the case of $X = \mathbb{R}$, we omit X in the notation, i.e., $L^p(M) = L^p(M;\mathbb{R})$. Moreover, a function $f : M \to [-\infty, \infty]$ is said to belong to $L^p_{loc}(M)$ if $f \in L^p(K)$ for every compact set $K \subseteq M$.

Now, it is necessary to introduce the definition of *weak derivative* of a function [AF03, AE09, Leo17], to define the Sobolev spaces.

Definition 2.3 (Weak derivative). Given an open set $\Omega \subseteq \mathbb{R}^n$, a multi-index $\alpha \in \mathbb{N}^n$, and $1 \le p \le \infty$. We say that a function $u \in L^1_{loc}(\Omega;\mathbb{R}^n)$ admits a *weak* or *distributional α-th derivative* in $L^p(\Omega;\mathbb{R}^n)$ if there exists a function $v_\alpha \in L^p(\Omega;\mathbb{R}^n)$ such that

$$\int_\Omega u\partial^\alpha \phi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v_\alpha \phi \,\mathrm{d}x \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

Then function v_α is denoted by $\partial^\alpha u$ or $\frac{\partial^{|\alpha|} u}{\partial x^\alpha}$.

DEFINITION 2.4 (Sobolev space). Given an open set $\Omega \subseteq \mathbb{R}^n$, $n, k \in \mathbb{N}$, and $1 \leq p \leq \infty$. The *Sobolev space* $W_p^k(\Omega)$ is defined by

$$W_p^k(\Omega) := \{f \in L^p(\Omega) : \partial^\alpha f \in L^p(\Omega) \text{ for all } |\alpha| \leq k\}.$$

Namely, the space of all functions $f \in L^p(\Omega)$, which admit weak derivatives $\partial^\alpha f \in L^p(\Omega)$ for every $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$. The space $W_p^k(\Omega)$ is endowed with the norm

$$\|f\|_{W_p^k(\Omega)} := \|f\|_{L^p(\Omega)} + \sum_{1 \leq |\alpha| \leq k} \|\partial^\alpha f\|_{L^p(\Omega)}.$$

In particular, $W_p^0(\Omega) = L^p(\Omega)$.

Let $\Omega \subseteq \mathbb{R}^n$ be a domain, we set

$$W_{p,0}^m(\Omega) = \overline{C_0^\infty(\Omega)}^{W_p^m(\Omega)}, \quad W_p^{-m}(\Omega) := [W_{p',0}^m(\Omega)]',$$
$$W_{p,(0)}^m(\Omega) = W_p^m(\Omega) \cap L_{(0)}^p(\Omega), \quad W_{p,(0)}^{-m}(\Omega) := [W_{p',(0)}^m(\Omega)]',$$

where p' is the conjugate exponent to p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$, and $[\cdot]'$ means the dual space. Here $L_{(0)}^p(M)$ is the mean value zero Lebesgue space $L_{(0)}^p(M) := \left\{f \in L^p(M) : \int_M f \mathrm{d}\mu = 0\right\}$ if $|M| < \infty$. Furthermore, we set the Sobolev space associated with $\Gamma \subseteq \partial\Omega$ as

$$W_{p,\Gamma}^m(\Omega) = \left\{\psi \in W_p^m(\Omega) : \psi|_\Gamma = 0\right\}, \quad W_{p,\Gamma}^{-m}(\Omega) := [W_{p',\Gamma}^m(\Omega)]'.$$

2.1.4. Vector-valued Sobolev, Slobodeckij, Besov and Bessel potential spaces. Now we record the vector-valued Sobolev, Slobodeckij and Bessel potential spaces, as well as their properties. For the properties of scalar-valued versions, we refer to [Tri78] for a complete theory.

By weak derivatives, one can establish the *vector-valued Sobolev space* as

DEFINITION 2.5 (Vector-valued Sobolev space). Let $\Omega \subseteq \mathbb{R}^n$ be either a bounded domain or $\mathbb{R}^n$ for $n \in \mathbb{N}$, $k \in \mathbb{N}$ be integer, X be a Banach space, and $1 \leq p \leq \infty$. The *Sobolev space* $W_p^k(\Omega; X)$ is defined by

$$W_p^k(\Omega; X) := \{f \in L^p(\Omega; X) : \partial^\alpha f \in L^p(\Omega; X) \text{ for all } |\alpha| \leq k\}.$$

Namely, the space of all functions $f \in L^p(\Omega; X)$, which admit weak derivatives $\partial^\alpha f \in L^p(\Omega; X)$ for every $\alpha \in \mathbb{N}^n$ with $1 \leq |\alpha| \leq k$. The space $W_p^k(\Omega; X)$ is endowed with the norm

$$\|f\|_{W_p^k(\Omega;X)} := \|f\|_{L^p(\Omega;X)} + \sum_{1 \leq |\alpha| \leq k} \|\partial^\alpha f\|_{L^p(\Omega;X)}.$$

In particular, we have $W_p^0(\Omega; X) = L^p(\Omega; X)$.

Remark 2.6. Another equivalent norm which we will sometimes use is given by

$$\|f\|_{W_p^k(\Omega;X)} := \left(\|f\|_{L^p(\Omega;X)}^p + \sum_{1 \leq |\alpha| \leq k} \|\partial^\alpha f\|_{L^p(\Omega;X)}^p\right)^{\frac{1}{p}},$$

for $1 \leq p < \infty$, and

$$\|f\|_{W_\infty^k(\Omega;X)} := \max\left\{\|\partial^\alpha f\|_{L^\infty(\Omega;X)} : |\alpha| \leq k\right\},$$

for $p = \infty$.

For a non-integer $s > 0$, we define the *Sobolev–Slobodeckij space* (for short *Slobodeckij space*, also called *fractional Sobolev space*) $W_p^s(\Omega; X)$ with Ω being either a bounded domain or $\mathbb{R}^n$.

Definition 2.7 (Sobolev–Slobodeckij space). Let $\Omega \subseteq \mathbb{R}^n$ be either a bounded domain or $\mathbb{R}^n$ for $n \in \mathbb{N}$, $s > 0$ be non-integer, X be a Banach space, and $1 \le p < \infty$. Then

$$W_p^s(\Omega; X) := \{f \in W_p^{\lfloor s \rfloor}(\Omega; X) : [f]_{W_p^s(\Omega;X)} < \infty\},$$

where $\lfloor s \rfloor = \max\{k \in \mathbb{Z} : k \le s\}$, and

$$[f]_{W_p^s(\Omega;X)} := \sum_{|\alpha| = \lfloor s \rfloor} \left(\int_\Omega \int_\Omega \left(\frac{\|\partial^\alpha f(x) - \partial^\alpha f(y)\|_X}{|x-y|^{s-\lfloor s \rfloor}} \right)^p \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n} \right)^{\frac{1}{p}}.$$

It is easy to see that the Sobolev–Slobodeckij space $W_p^s(\Omega; X)$ is a Banach space endowed with the norm

$$\|\cdot\|_{W_p^s(\Omega;X)} := \|\cdot\|_{W_p^{\lfloor s \rfloor}(\Omega;X)} + [\cdot]_{W_p^s(\Omega;X)}.$$

In particular, for $0 < s < 1$ and $\Omega = (0, T)$ with $T > 0$, we have

$$W_p^s(0, T; X) := \{f \in L^p(0, T; X) : [f]_{W_p^s(0,T;X)} < \infty\},$$

with

$$[f]_{W_p^s(0,T;X)} := \left(\int_0^T \int_0^T \left(\frac{\|f(t) - f(\tau)\|_X}{|t-\tau|^s} \right)^p \frac{\mathrm{d}t\mathrm{d}\tau}{|t-\tau|} \right)^{\frac{1}{p}}.$$

Now we introduce the vector-valued Besov space and Bessel potential space in $\mathbb{R}^n$ [Ama09, Hyt+16, Men21, MV12]. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi(\xi) = 1$ for $|\xi| < 1$ and $\psi(\xi) = 0$ for $|\xi| > 2$. Put

$$\psi_k(\xi) := \psi(2^{-k}\xi) - \psi(2^{-k+1}\xi) \quad \text{for } \xi \in \mathbb{R},\ k \in \mathbb{N},$$

and $\psi_k(D) := \mathscr{F}^{-1}\psi_k\mathscr{F}$, where $\mathscr{F}$ denotes the Fourier transformation on the space of all X-valued tempered distributions defined by $\mathscr{S}'(\mathbb{R}^n; X) = \mathcal{L}(\mathscr{S}(\mathbb{R}^n), X)$ with $\mathscr{S}(\mathbb{R}^n)$ the *Schwartz space* of *rapidly decreasing smooth functions* on $\mathbb{R}^n$. Then the *Besov space* is defined as follows.

Definition 2.8 (Besov space in $\mathbb{R}^n$). Let X be a Banach space, $s \in \mathbb{R}$, $1 \le p < \infty$. The *Besov space* $B_{pq}^s(\mathbb{R}^n; X)$ is defined as

$$B_{pq}^s(\mathbb{R}^n; X) := \{f \in \mathscr{S}'(\mathbb{R}^n; X) : \|f\|_{B_{pq}^s(\mathbb{R}^n;X)} < \infty\},$$

where if $1 \le q < \infty$

$$\|f\|_{B_{pq}^s(\mathbb{R}^n;X)} := \left(\sum_{k=0}^\infty 2^{skq} \|\psi_k(D) f\|_{L^p(\mathbb{R}^n;X)}^q \right)^{\frac{1}{q}},$$

while for $q = \infty$

$$\|f\|_{B_{pq}^s(\mathbb{R}^n;X)} := \sup_{k \in \mathbb{N}_0} 2^{sk} \|\psi_k(D) f\|_{L^p(\mathbb{R}^n;X)}.$$

In terms of the *Fourier transformation*, we give the *Bessel potential space*.

Definition 2.9 (Bessel potential space in $\mathbb{R}^n$). Let X be a Banach space, $s \in \mathbb{R}$, $1 \leq p < \infty$. The *Bessel potential space* $H_p^s(\mathbb{R}^n; X)$ is defined as

$$H_p^s(\mathbb{R}^n; X) \coloneqq \{f \in \mathscr{S}'(\mathbb{R}^n; X) : J_s f \in L^p(\mathbb{R}^n; X)\},$$

where J_s is the standard *Bessel potential* defined by

$$J_s f \coloneqq \mathscr{F}^{-1}[(1 + |\cdot|^2)^{\frac{s}{2}} \mathscr{F} f] \in \mathscr{S}'(\mathbb{R}^n; X).$$

The space is equipped with the norm

$$\|f\|_{H_p^s(\mathbb{R}^n;X)} \coloneqq \|J_s f\|_{L^p(\mathbb{R}^n;X)} .$$

For scalar-valued spaces, embeddings and interpolations hold well [Tri78], however, one can not generalize them to the vector-valued spaces directly. Instead, we need some more geometric assumptions on the spaces. Throughout we assume that the Banach space X is a so-called *UMD space* (*Unconditionality of Martingale Differences*), or equivalently, of class $\mathcal{HT}$. We refer to [Ama95, Sections III.4.3–5] and references therein for the definition and properties of such spaces. We note that Hilbert spaces are of class $\mathcal{HT}$, as well as the reflexive Lebesgue and Sobolev (–Slobodeckij) spaces. Now we give an equivalent characterization of Sobolev–Slobodeckij space by Bessel potential space and Besov space.

Theorem 2.10. *Let $s > 0$, $1 < p < \infty$, and X be a UMD space. Then*

$$W_p^s(\mathbb{R}^n; X) = \begin{cases} H_p^s(\mathbb{R}^n; X), & \text{if } s \in \mathbb{N}, \\ B_{pp}^s(\mathbb{R}^n; X), & \text{if } s \in \mathbb{R}_+ \setminus \mathbb{N}, \end{cases}$$

with equivalent norms.

Proof. See e.g. Zimmermann [Zim89] or [Hyt+16, Theorem 5.6.11] for the case of $s \in \mathbb{N}$, while for the case of $s \in \mathbb{R}_+ \setminus \mathbb{N}$ we refer to Amann [Ama97]. □

In the following, we present a theorem on the characterizations of Besov and Bessel potential spaces by real and complex interpolations of Sobolev spaces, respectively.

Theorem 2.11 (Equivalent norms in $\mathbb{R}^n$). *Let $n \in \mathbb{N}$, $1 < p < \infty$, $s > 0$ be non-integer and X be a UMD space. Then*

$$\begin{aligned} B_{pp}^s(\mathbb{R}^n; X) &= (W_p^{\lfloor s \rfloor}(\mathbb{R}^n; X), W_p^{\lfloor s \rfloor + 1}(\mathbb{R}^n; X))_{s - \lfloor s \rfloor, p}, \\ H_p^s(\mathbb{R}^n; X) &= [W_p^{\lfloor s \rfloor}(\mathbb{R}^n; X), W_p^{\lfloor s \rfloor + 1}(\mathbb{R}^n; X)]_{s - \lfloor s \rfloor}, \end{aligned}$$

with equivalent norms.

Proof. For the scalar-valued versions of them all, we refer to [Tri78, Chapter 2]. Concerning the vector-valued case of $B_{pp}^s(\mathbb{R}^n; X)$, we refer to [Ama97] for the discussions on the vector-valued Besov spaces and real interpolations of Sobolev spaces. For $H_p^s(\mathbb{R}^n; X)$, see e.g. [Hyt+16, Theorem 5.6.9] for the complex interpolation of Bessel potential spaces, combined with Theorem 2.10 of the case $s \in \mathbb{N}$. □

As we already established the spaces for functions in $\mathbb{R}^n$, now we turn to functions in domains, by means of *restrictions*. We refer to [Tri78, Chapter 4] for the discussion of scalar-valued spaces defined in domains.

Definition 2.12 (Besov space). Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be a domain. Given $s \in \mathbb{R}$, a Banach space X, and $1 \le p, q < \infty$. Then $B^s_{pq}(\Omega; X)$ is defined as the restriction of $B^s_{pq}(\mathbb{R}^n; X)$ to Ω, i.e.,

$$B^s_{pq}(\Omega; X) := \{ g|_\Omega : g \in B^s_{pq}(\mathbb{R}^n; X)\},$$

which is endowed with the norm

$$\|f\|_{B^s_{pq}(\Omega;X)} := \inf\left\{ \|g\|_{B^s_{pq}(\mathbb{R}^n;X)} : g|_\Omega = f,\ g \in B^s_{pq}(\mathbb{R}^n; X)\right\}.$$

Here $g|_\Omega \in [C^\infty_0(\Omega; X)]'$ denotes the restriction of $g \in C^\infty_0(\mathbb{R}^n; X)$ to Ω in the sense of distribution.

Similarly, one can define the Bessel potential space in domains.

Definition 2.13 (Bessel potential space). Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be a domain. Given $s \in \mathbb{R}$, a Banach space X of UMD, and $1 \le p < \infty$. Then $H^s_p(\Omega; X)$ is the restriction of $H^s_p(\mathbb{R}^n; X)$ to Ω,

$$H^s_p(\Omega; X) := \{ g|_\Omega : g \in H^s_p(\mathbb{R}^n; X)\},$$

which is endowed with the norm

$$\|f\|_{H^s_p(\Omega;X)} := \inf\left\{ \|g\|_{H^s_p(\mathbb{R}^n;X)} : g|_\Omega = f,\ g \in H^s_p(\mathbb{R}^n; X)\right\}.$$

In the following, we introduce the extensions for the vector-valued spaces defined in domains.

Theorem 2.14 (Extension theorem for smooth domains). *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and X be a Banach space. Then for all $1 \le p \le \infty$ there exists a continuous linear operator $\mathcal{E} : L^p(\Omega; X) \to L^p(\mathbb{R}^n; X)$ such that $\mathcal{E}|_{W^k_p} \in \mathcal{L}(W^k_p(\Omega; X), W^k_p(\mathbb{R}^n; X))$ for all $k \in \mathbb{N}$, $\mathcal{E}(f)|_\Omega = f$ for a.e. $x \in \Omega$, and for all $f \in W^k_p(\Omega; X)$,*

$$\|\mathcal{E}(f)\|_{L^p(\mathbb{R}^n;X)} \le C\, \|f\|_{L^p(\Omega;X)},$$
$$\|\partial^\alpha \mathcal{E}(f)\|_{L^p(\mathbb{R}^n;X)} \le C \sum_{i=0}^{\alpha} \left\|\partial^i f\right\|_{L^p(\Omega;X)},$$

for every multi-index $\alpha \in \mathbb{N}^n_0$ with $1 \le |\alpha| \le k$, where C depends on the Lipschitz constant of the boundary.

Proof. We refer to e.g., [Men21, Theorem B.17]. □

Theorem 2.15 (Equivalent norms). *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be a smooth domain, $1 < p < \infty$, $s > 0$ be non-integer and X be a Banach space of UMD. Then*

$$B^s_{pp}(\Omega; X) = (W^{\lfloor s \rfloor}_p(\Omega; X), W^{\lfloor s \rfloor+1}_p(\Omega; X))_{s-\lfloor s \rfloor, p},$$
$$H^s_p(\Omega; X) = [W^{\lfloor s \rfloor}_p(\Omega; X), W^{\lfloor s \rfloor+1}_p(\Omega; X)]_{s-\lfloor s \rfloor},$$

with equivalent norms.

Proof. Let $\mathcal{E}$ be the extension operator in Theorem 2.14 for Ω and $\mathcal{R} : W^k_p(\mathbb{R}^n; X) \to W^k_p(\Omega; X)$, $k \in \mathbb{N}$ be the restriction operator defined by $\mathcal{R}f = f|_\Omega$ for a.e. $x \in \Omega$, which is obviously a linear bounded operator. By Theorem 2.14, $\mathcal{E}$ is injective, and hence $\mathcal{R}$, which is left-inverse to $\mathcal{E}$ and does not depend on k, is surjective. Therefore, one derives

$$\begin{aligned} B^s_{pp}(\Omega; X) &= \mathcal{R}(B^s_{pp}(\mathbb{R}^n; X)) \\ &= \mathcal{R}\left((W_p^{\lfloor s\rfloor}(\mathbb{R}^n; X), W_p^{\lfloor s\rfloor+1}(\mathbb{R}^n; X))_{s-\lfloor s\rfloor,p}\right) \\ &= (W_p^{\lfloor s\rfloor}(\Omega; X), W_p^{\lfloor s\rfloor+1}(\Omega; X))_{s-\lfloor s\rfloor,p}, \end{aligned}$$

where the last identity follows from [Tri78, Theorem 1.2.4] such that $\mathcal{R}$ is an isomorphic mapping from $(W_p^{\lfloor s\rfloor}(\mathbb{R}^n; X), W_p^{\lfloor s\rfloor+1}(\mathbb{R}^n; X))_{s-\lfloor s\rfloor,p}$ to $(W_p^{\lfloor s\rfloor}(\Omega; X), W_p^{\lfloor s\rfloor+1}(\Omega; X))_{s-\lfloor s\rfloor,p}$. This proves the first assertion. For the Bessel potential space, it follows analogously by means of complex interpolation, on noting that [Tri78, Theorem 1.2.4] holds true for any kinds of interpolation functor. ☐

Theorem 2.16. *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be a smooth domain, $s > 0$, $1 < p < \infty$, and X be of UMD. Then*

$$W^s_p(\Omega; X) = \begin{cases} H^s_p(\Omega; X), & \text{if } s \in \mathbb{N}, \\ B^s_{pp}(\Omega; X), & \text{if } s \in \mathbb{R}_+ \setminus \mathbb{N}, \end{cases}$$

with equivalent norms.

Proof. In the case of $s \in \mathbb{R}_+ \setminus \mathbb{N}$, we refer to [Ama00, Corollary 4.3] for the details. If $s = k \in \mathbb{N}$, we first claim that each $f \in W^k_p(\Omega; X)$ is the restriction of some $g \in W^k_p(\mathbb{R}^n; X)$, i.e., $g|_\Omega = f$. To this end, we define the restriction map $\mathcal{R} : W^k_p(\mathbb{R}^n; X) \to W^k_p(\Omega; X)$. By Theorem 2.14, the restriction map $\mathcal{R}$ is surjective (onto), since $\mathcal{R} \circ \mathcal{E} = \mathbf{id}$ on $W^k_p(\Omega; X)$. Then one proceeds as

$$W^s_p(\Omega; X) = \mathcal{R}(W^s_p(\mathbb{R}^n; X)) = \mathcal{R}(H^s_p(\mathbb{R}^n; X)) = H^s_p(\Omega; X) \text{ for } s \in \mathbb{N},$$

which finishes the proof. ☐

In applications, one natural class of function spaces associated with parabolic systems is given by the so-called *anisotropic function spaces*, particularly, *anisotropic Sobolev–Slobodeckij and Bessel potential spaces*. To make it compatible with the thesis, we introduce them here.

Definition 2.17 (Anisotropic function spaces). Let $I \subseteq \mathbb{R}$ be an interval and $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{R}$ be a domain. For $r, s \geq 0$ and $1 \leq p < \infty$, the *anisotropic function spaces* $K^{r,s}_p$, $K \in \{W, H\}$ is defined as

$$K^{r,s}_p(\Omega \times I) := L^p(I; K^r_p(\Omega)) \cap K^s_p(I; L^p(\Omega)), \tag{2.2}$$

which is equipped with the norm

$$\|\cdot\|_{K^{r,s}_p(\Omega\times I)} := \|\cdot\|_{L^p(I;K^r_p(\Omega))} + \|\cdot\|_{K^s_p(I;L^p(\Omega))} .$$

2.1.5. Continuous embeddings. In this section, we record some embedding results for (anisotropic) (vector-valued) Sobolev–Slobodeckij spaces and Bessel potential spaces $K^s_p(\Omega; X)$, $K \in \{W, H\}$, in particular the case of Ω is a finite interval.

First we introduce two important properties of Sobolev–Slobedeckij and Bessel potential spaces, that is, multiplication and composition.

Proposition 2.18 (Multiplication). *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded Lipschitz domain. For $f, g \in K^s_p(\Omega)$ and $sp > d$ with $s > 0$, $1 < p < \infty$, we have*

$$\|fg\|_{K^s_p(\Omega)} \leq M_p \|f\|_{K^s_p(\Omega)} \|g\|_{K^s_p(\Omega)} ,$$

where M_p is a constant depending on p, but independent of f and g.

Proof. See [RS96, Theorem 4.6.1/1 (5)] for the case $K = H$ with $q = q_1 = q_2 = 2$ therein, [RS96, Theorem 4.6.1/2 (18)] for the case $K = W$ with $p = q = q_1 = q_2$ therein. □

Proposition 2.19 (Composition properties). *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded domain with boundary of C^1 class. Let $N \in \mathbb{N}$, $0 < s \leq 1$ and $1 \leq p < \infty$ with $s > d/p$. Then for all $f \in C^1(\mathbb{R}^N)$ and every $R > 0$ there exists a constant $C > 0$ depending on R such that for all $u \in K_p^s(\Omega)^N$ with $\|u\|_{K_p^s(\Omega)^N} \leq R$, it holds that $f(u) \in K_p^s(\Omega)$ and $\|f(u)\|_{K_p^s(\Omega)} \leq C(R)$. Moreover, if $f \in C^2(\mathbb{R}^N)$, then for all $R > 0$ there exists a constant $L > 0$ depending on R such that*

$$\|f(u) - f(v)\|_{K_p^s(\Omega)} \leq L(R)\, \|u - v\|_{K_p^s(\Omega)^N}$$

for all $u, v \in K_p^s(\Omega)^N$ with $\|u\|_{K_p^s(\Omega)^N}, \|v\|_{K_p^s(\Omega)^N} \leq R$.

Proof. The first part follows from Runst–Sickel [RS96, Theorem 5.5.1/1]. We note that in [RS96], the function spaces act on the full space $\mathbb{R}^n$. Here we just need to employ suitable extensions for Ω so that we can reduce to the case of a full space. For the second part, let u, v be arbitrary two functions in $K_p^s(\Omega)^N$ with $\|u\|_{K_p^s(\Omega)^N}, \|v\|_{K_p^s(\Omega)^N} \leq R$. By a simple calculation, one obtains

$$(f(u) - f(v))(x) = \int_0^1 Df(tu + (1-t)v)(x)\, \mathrm{d}t \cdot (u - v)(x), \tag{2.3}$$

where $[Df(u)]_j := \partial_{u_j} f(u)$, $j = 1, 2, ..., N$. Now let $g(u, v) := \int_0^1 Df(tu + (1-t)v)(x)\, \mathrm{d}t$, we have $g(u, v) \in C^1(\mathbb{R}^N \times \mathbb{R}^N; \mathbb{R}^N)$ since $f(u) \in C^2(\mathbb{R}^N)$. Then the first part implies that

$$\|g(u, v)\|_{K_p^s(\Omega)^N} \leq C(R),$$

which completes the proof with (2.3) and the multiplication property Lemma 2.18 with $s > d/p$.
For the case $s = 1$, we refer to [RS96]. □

Now we recall several embedding results and properties for vector-valued spaces that will be frequently used later, especially concerning the parabolic problems with time embeddings.

Lemma 2.20. *Suppose $0 < r < s \leq 1$ and $1 \leq p < \infty$. X is a Banach space and $I = (0, T) \subset \mathbb{R}$ is a finite interval for $0 < T < \infty$. Then $K_p^s(I; X) \hookrightarrow W_p^r(I; X)$ and for some $\delta > 0$*

$$[f]_{W_p^r(I;X)} \leq |I|^\delta\, [f]_{K_p^s(I;X)} \text{ for all } f \in K_p^s(I; X).$$

In particular, we have

$$W_p^1(I; X) \hookrightarrow W_p^\theta(I; X) \text{ for all } 0 < \theta < 1$$

and

$$[f]_{W_p^\theta(I;X)} \leq T^{1-\theta}\, \|\partial_t f\|_{L^p(I;X)} \text{ for all } f \in W_p^1(I; X).$$

Proof. The case $K = W$ was shown in Simon [Sim90, Corollary 17]. For the case $K = H$, taking t, satisfying $r < t < s$, one has

$$[f]_{W_p^r(I;X)} \leq C\, |I|^{t-r}\, [f]_{W_p^t(I;X)} \leq C\, |I|^{t-r}\, [f]_{K_p^s(I;X)},$$

where $C > 0$ is uniform in I, and the last inequality follows from the complex interpolations with Theorem 2.15, see e.g. [MS12]. The second assertion can be easily derived by means of the observation

$$f(t) - f(t-h) = h \int_0^1 \partial_t f(t + (\tau - 1)h) \mathrm{d}\tau$$

and the definition of Sobolev–Slobodeckij space. □

For convenience, with $T > 0$ we define the corresponding space with vanishing initial trace at $t = 0$ as

$$ {}_0K_p^s(0,T;X) = \left\{ f \in K_p^s(0,T;X) : f|_{t=0} = 0 \right\} \text{ for } s > \frac{1}{p}. $$

Then for $K_p^s(I;X)$, $K \in \{W, H\}$, it follows an embedding to continuous Banach space-valuned space

Proposition 2.21. *Let $0 < s < 1$, $1 < p < \infty$ satisfying $sp > 1$, X be a Banach space and $I = (0,T) \subset \mathbb{R}$ be a bounded interval for $0 < T < \infty$. Then*

$$ K_p^s(I;X) \hookrightarrow C(\bar{I};X). $$

Moreover, for some $\delta > 0$ and all $f \in {}_0K_p^s(I;X)$,

$$ \|f\|_{C(\bar{I};X)} \le CT^\delta \|f\|_{K_p^s(I;X)}, $$

where C is independent of I.

Proof. By Meyries–Schnaubelt [MS12, Proposition 2.10] with $\mu = 1$ there, one has the first assertion and for $K = W$, $1/p < r < s$,

$$ \|f\|_{C(\bar{I};X)} \le C \|f\|_{{}_0K_p^r(I;X)}, $$

where C is independent of I. Then it follows from Lemma 2.20 that

$$ \|f\|_{C(\bar{I};X)} \le C \|f\|_{{}_0K_p^r(I;X)} = C\big(\|f\|_{L^p(I;X)} + [f]_{{}_0K_p^r(I;X)} \big) \le C |I|^\delta [f]_{K_p^s(I;X)}, $$

for some $\delta > 0$. □

Proposition 2.22 (Trace spaces by real interpolation). *Let X_1, X_0 be two Banach spaces and $X_1 \hookrightarrow X_0$. Define $X_T = L^p(0,T;X_1) \cap W_p^1(0,T;X_0)$ for all $1 < p < \infty$ and $0 < T < \infty$. Then*

$$ X_T \hookrightarrow C\left([0,T];X_\gamma\right), $$

where

$$ X_\gamma = (X_0, X_1)_{1-\frac{1}{p},p} = \{ u|_{t=0} : u \in X_T \} $$

is the trace space. Moreover, if X_T is endowed with the norm

$$ \|u\|_{X_T} := \|u\|_{L^p(0,T;X_1)} + \|u\|_{W_p^1([0,T];X_0)} + \|u|_{t=0}\|_{X_\gamma}, $$

then there is some $C > 0$ independent of T such that for $T \in [0,\infty)$ and $u \in X_T$,

$$ \|u\|_{C(0,T;X_\gamma)} \le C \|u\|_{X_T}. $$

In particular, let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a bounded domain, $n < p < \infty$, and if $X_1 = W_p^2(\Omega)$, $X_0 = L^p(\Omega)$, then $X_\gamma = W_p^{2-\frac{2}{q}}(\Omega)$ and

$$ W_p^{2,1}(\Omega \times (0,T)) \hookrightarrow C([0,T]; W_p^{2-\frac{2}{p}}(\Omega)) \hookrightarrow C([0,T]; W_p^1(\Omega)), \tag{2.4} $$

together with

$$ \|u\|_{C([0,T];W_p^1(\Omega))} \le C(\|u\|_{W_p^{2,1}(\Omega\times(0,T))} + \|u_0\|_{W_p^{2-\frac{2}{p}}}), $$
$$ \|u-v\|_{C([0,T];W_p^1(\Omega))} \le C \|u-v\|_{W_p^{2,1}(\Omega\times(0,T))}, $$

for $u, v \in W_p^{2,1}(\Omega \times (0,T))$ with $u|_{t=0} = v|_{t=0} = u_0$.

Proof. We refer to [Ama95, Chapter III, Theorem 4.10.2] for the first assertion, while the particular case holds in view of the real interpolation $(L^p, W_p^2)_{1-\frac{1}{p},p} = W_p^{2-\frac{2}{p}}$. □

Lemma 2.23. *Let Σ be a compact sufficiently smooth hypersurface. For $1 < p < \infty$, $\frac{1}{p} < \alpha \leq 1$ and $0 < T < \infty$, define $X_T := L^p(0,T;W_p^{2\alpha}(\Sigma)) \cap W_p^{\alpha}(0,T;L^p(\Sigma))$, then*

$$X_T \hookrightarrow C\left([0,T];X_\gamma\right),$$

where

$$X_\gamma = \{u|_{t=0} : u \in X_T\} = W_p^{2\alpha-\frac{2}{p}}(\Sigma).$$

Moreover, if X_T is endowed with the norm

$$\|u\|_{X_T} := \|u\|_{L^p(0,T;X_1)} + \|u\|_{W_p^{\alpha}(0,T;X_0)} + \|u|_{t=0}\|_{X_\gamma},$$

then there is some $C > 0$ independent of T such that for all $u \in X_T$,

$$\|u\|_{C([0,T];X_\gamma)} \leq C\,\|u\|_{X_T}.$$

Proof. See e.g. [PS16, Section 3.4.6]. □

Adapting from Meyries–Schnaubelt [MS12, Proposition 3.2] and Prüss–Simonett [PS16, Section 4.5.5], we use the following time-space embedding results.

Proposition 2.24. *Let $1 < p < \infty$, $0 < \alpha, s < 2$ and $0 < r < s$, we have the embeddings*

$$H_p^s(0,T;L^p) \cap L^p(0,T;K_p^{\alpha}) \hookrightarrow H_p^r(0,T;K_p^{\alpha(1-\frac{r}{s})}).$$

In particular,

$$H_p^{\frac{1}{2}}(0,T;L^p) \cap L^p(0,T;W_p^1) \hookrightarrow H_p^{\frac{1}{4}}(0,T;H_p^{\frac{1}{2}}),$$
$$W_p^1(0,T;L^p) \cap L^p(0,T;W_p^2) \hookrightarrow H_p^{\frac{1}{2}}(0,T;W_p^1).$$

All these assertions remain true if one replaces W- and H- spaces by ${}_0W$- and ${}_0H$- spaces respectively, and the embedding constants in this case does not depend on $T > 0$.

In the following, an anisotropic trace lemma is introduced for a fractional order space.

Lemma 2.25 (Anisotropic trace on the boundary). *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded domain with $\Gamma := \partial\Omega$ of class C^1, $T > 0$, and*

$$X_T := H_p^{\frac{1}{2}}(0,T;L^p(\Omega)) \cap L^p(0,T;W_p^1(\Omega)).$$

Then there is a trace operator

$$\gamma : X_T \to X_{\gamma,T} := W_p^{\frac{1}{2}-\frac{1}{2p}}(0,T;L^p(\Gamma)) \cap L^p(0,T;W_p^{1-\frac{1}{p}}(\Gamma)),$$

such that $\gamma f = f|_\Gamma$ for $f \in X_T \cap C([0,T] \times \overline{\Omega})$ and

$$\|\gamma f\|_{X_{\gamma,T}} \leq C\,\|f\|_{X_T},$$

where $C > 0$ is independent of T and f. Moreover, it is surjective and has a continuous right-inverse.

Proof. By means of a coordinate transformation and a partition of unity of Ω, one can easily reduce it to case of a half-space $\mathbb{R}^{d-1} \times \mathbb{R}_+$. Then thanks to [MS12, Theorem 4.5] with $s = 1/2$, $m = 1$, $\mu = 1$ (see also [PS16, Proposition 6.2.4] with $m = 1$, $\mu = 1$ there), one completes the proof. □

2.1.6. Temporal extensions. In this section, we are intended to construct an extension operator from $W_p^s(0,T;X)$ to $W_p^s(0,\infty;X)$, where $s \in (\frac{1}{p}, 1]$ and X is a Banach space. The main feature is that the operator norms can be bounded independently of $T > 0$, compared to the extension theorem for general Sobolev-Slobodeckij spaces. The reason we made such modification here is that if the constant depends on T, then the extended norm may blow up for small T, which is the case we address in the following. For example, in the proof of Theorem 5.4 in [DNPV12], the extension from $W_p^s(\Omega)$ to $W_p^s(\mathbb{R}^n)$ with $0 < s < 1$, several smooth functions ψ_j satisfying $0 \leq \psi_j \leq 1$ and $\sum_{j=0}^k \psi_j = 1$ are chosen to construct the extension operator. In the case $|\Omega| \to 0$, we have $\nabla \psi_j \to \infty$, which means that the extension is not valid. To avoid such problem, we employ an *even* extension and make use of the embedding results in Simon [Sim90]. Now, we give the extension theorem.

Theorem 2.26. *Let $p \geq 1$, $s = 0$, or $s \in (\frac{1}{p}, 1]$, $T > 0$ and X be a Banach space. Then there exists an extension operator $E_T : {}_0W_p^s(0,T;X) \to W_p^s(0,\infty;X)$, where ${}_0W_p^s(0,T;X) = \{u \in W_p^s(0,T;X) : u|_{t=0} = 0, \text{ if } s > \frac{1}{p}\}$, such that $E_T(u)|_{[0,T]} = u$ and*

$$\|E_T(u)\|_{W_p^s(0,\infty;X)} \leq C \|u\|_{{}_0W_p^s(0,T;X)},$$

where $C > 0$ depends on s, p and does not depend on T.

Proof. The proof is divided into three cases, namely, $s = 0$, $\frac{1}{p} < s < 1$ and $s = 1$.
Case 1: $s = 0$. In this situation, $W_p^s(0,T;X)$ is just the Lebesgue space $L^p(0,T;X)$, which does not contain any time regularity. Hence for any function $u \in L^p(0,T;X)$, we can take the extension by zero.
Case 2: $s = 1$. With $u|_{t=0} = 0$, we apply an even extension to u in $[0,T]$ around T to $[0,2T]$ and zero extension for $T > 2T$. Then the extended function $\bar{u}$ is weakly differentiable with

$$\partial_t \bar{u}(t) = \begin{cases} \partial_t u(t), & \text{if } 0 \leq t \leq T, \\ -\partial_t u(2T - t), & \text{if } T < t \leq 2T, \\ 0, & \text{if } t > 2T. \end{cases}$$

Then we have

$$\|\bar{u}\|_{W_p^1(0,\infty;X)} = 2^{\frac{1}{p}} \|u\|_{W_p^1(0,T;X)}.$$

Case 3: $\frac{1}{p} < s < 1$. With the same extension as in Case 2, we define the same function $\tilde{u}$. Now we are in the position to show $\tilde{u} \in W_p^s(0,\infty;X)$, for which we only need to prove $[\tilde{u}]_{W_p^s(0,\infty;X)} \leq C [u]_{W_p^s(0,T;X)}$, where C is independent of T. From the definition of Sobolev–Slobodeckij space,

$$\begin{aligned} &[\tilde{u}]^p_{W_p^s(0,\infty;X)} \\ &= \int_0^T \int_0^T \frac{\|u(t) - u(\tau)\|_X^p}{|t-\tau|^{1+sp}} \mathrm{d}t\mathrm{d}\tau + \int_T^{2T} \int_T^{2T} \frac{\|u(2T-t) - u(2T-\tau)\|_X^p}{|t-\tau|^{1+sp}} \mathrm{d}t\mathrm{d}\tau \\ &\quad + 2 \int_0^T \int_T^{2T} \frac{\|u(t) - u(2T-\tau)\|_X^p}{|t-\tau|^{1+sp}} \mathrm{d}\tau\mathrm{d}t + 2 \int_0^{2T} \int_{2T}^{\infty} \frac{\|\tilde{u}(t)\|_X^p}{|t-\tau|^{1+sp}} \mathrm{d}\tau\mathrm{d}t =: \sum_{i=1}^4 Q_i. \end{aligned}$$

It is clear that

$$Q_1 + Q_2 = 2 [u]^p_{W_p^s(0,T;X)}.$$

Since $|t-\tau| \geq |t-(2T-\tau)|$ with $t \in [0,T]$ and $\tau \in [T,2T]$, we have

$$Q_3 \leq 2 \int_0^T \int_0^T \frac{\|u(t)-u(h)\|_X^p}{|t-h|^{1+sp}} \mathrm{d}h\mathrm{d}t = 2\,[u]^p_{W^s_p(0,T;X)}\,.$$

Noticing that $\tilde{u}|_{t=2T} = 0$ due to the even extension, we get

$$\begin{aligned} Q_4 &= \frac{2}{sp} \int_0^{2T} \frac{\|\tilde{u}(2T-h)-\tilde{u}(2T)\|_X^p}{h^{sp}} \mathrm{d}h \\ &\leq \frac{2}{sp} \int_0^{2T} \left(\frac{\|\tilde{u}(\cdot - h) - \tilde{u}(\cdot)\|_{L^\infty(h,2T;X)}}{h^{s-\frac{1}{p}}} \right)^p \frac{\mathrm{d}h}{h} = \frac{2}{sp}\,[\tilde{u}]^p_{B^{s-\frac{1}{p}}_{\infty,p}(0,2T;X)}\,, \end{aligned}$$

where the seminorm of $B^s_{p,}(0,T;X)$ is given by

$$[f]_{B^s_{p,q}(0,T;X)} = \left(\int_0^T \left(\frac{\|\Delta_h f(t)\|_{L^p(h,T;X)}}{h^s} \right)^q \frac{\mathrm{d}h}{h} \right)^{\frac{1}{q}}$$

for $0 < s < 1$ and $1 \leq p,q \leq \infty$. From Theorem 10 in Simon [Sim90], we know that for $\frac{1}{p} < s < 1$ and $p \geq 1$,

$$[f]_{B^{s-\frac{1}{p}}_{\infty,p}(0,T;X)} \leq \frac{3\theta}{s-\frac{1}{q}}\,[f]_{B^s_{p,p}(0,T;X)} = \frac{3\theta}{s-\frac{1}{p}}\,[f]_{W^s_p(0,T;X)}\,, \quad \forall\, f \in W^s_p(0,T;X),$$

where $\theta = 3^{1-(s-1/p)}$. Hence,

$$Q_4 \leq \frac{6\theta}{sp(sp-1)}\,[\tilde{u}]^p_{W^s_p(0,2T;X)} \leq \frac{24\theta}{sp(sp-1)}\,[u]^p_{W^s_p(0,T;X)}\,.$$

Combining the estimates of Q_i, $i = 1, \ldots, 4$, one obtains

$$[\tilde{u}]_{W^s_p(0,\infty;X)} \leq C\,[u]_{W^s_p(0,T;X)}\,,$$

where $C = \left(4 + \frac{24\theta}{sp(sp-1)}\right)^{1/p}$.

Now, let $E_T(u) = \tilde{u}$. Then $E_T(u)$ is well-defined from ${}_0W^s_p(0,T;X)$ to $W^s_p(0,T;X)$ as well as $E_T(u)|_{[0,T]} = u$ and

$$\|E_T(u)\|_{W^s_p(0,\infty;X)} \leq C\,\|u\|_{{}_0W^s_p(0,T;X)}\,,$$

where $C > 0$ depends on s, p and does not depend on T. □

Next, we give an extension theorem for general functions.

Theorem 2.27. *Let X_1, X_0 be two Banach spaces and $X_1 \hookrightarrow X_0$. For $1 < p < \infty$ and $0 < T < \infty$, define $X_T := L^p(0,T;X_1) \cap W^1_p(0,T;X_0)$ endowed with the norm*

$$\|u\|_{X_T} := \|u\|_{L^p(0,T;X_1)} + \|u\|_{W^1_p(0,T;X_0)} + \|\,u|_{t=0}\|_{X_\gamma}\,,$$

where $X_\gamma = (X_0, X_1)_{1-1/p,p}$. Then there exists an extension operator $\mathcal{E} \in \mathcal{L}(X_T, X_\infty)$ satisfying $\mathcal{E}(u)|_{[0,T]} = u$, for all $u \in X_T$. Moreover, there is a constant $C > 0$, independent of $0 < T < \infty$, such that

$$\|\mathcal{E}(u)\|_{X_\infty} \leq C\,\|u\|_{X_T}\,, \tag{2.5}$$

for all $u \in X_T$.

Proof. First of all, we consider the case $u|_{t=0} = 0$. Let E be the extension operator as in Theorem 2.26. Define $\tilde{u} = E(u)$. Then we have $\tilde{u}|_{[0,T]} = u$ and

$$\|\tilde{u}\|_{X_\infty} \le C \|u\|_{X_T},$$

where C does not depend on T.

Let $u_0 := u|_{t=0} \in X_\gamma$. Since $X_\gamma = (X_0, X_1)_{1-1/p,p}$, the trace method of interpolation implies that there exists a function $v \in X_\infty$ such that $v|_{t=0} = u_0$, see e.g. [Lun18, Proposition 1.13]. Moreover, it follows from the norm of X_T that there is a constant $C > 0$ such that

$$\|v\|_{X_\infty} \le C \|u|_{t=0}\|_{X_\gamma} \le C \|u\|_{X_T}.$$

Now for general $u \in X_T$, we define $w := u - v$. Then w is reduced to the case $w|_{t=0} = 0$ and can be extended to $E(w)$ in X_∞ like $\tilde{u}$. Now we define the extension operator as $\mathcal{E}(u) := w + v$. Then one obtains $\mathcal{E}(u)|_{[0,T]} = u$ and there is a constant, independent of T, such that

$$\|\mathcal{E}(u)\|_{X_\infty} \le C \|w\|_{X_\infty} + C \|v\|_{X_\infty} \le C \|u\|_{X_T},$$

for all $u \in X_T$, which completes the proof. □

With a similar argument, we have the following extension theorem for functions in $W_p^{2\alpha,\alpha}$.

Theorem 2.28. *Let Σ be a compact sufficiently smooth hypersurface. For $1 < p < \infty$, $1/p < \alpha \le 1$ and $0 < T < \infty$, let $W_p^{2\alpha,\alpha}(\Sigma \times (0,T)) := L^p(0,T; W_p^{2\alpha}(\Sigma)) \cap W_p^{\alpha}(0,T; L^p(\Sigma))$ be endowed with norm*

$$\|g\|_{W_p^{2\alpha,\alpha}(\Sigma\times(0,T))} := \|g\|_{L^p(0,T;W_p^{2\alpha}(\Sigma))} + \|g\|_{W_p^{\alpha}(0,T;L^p(\Sigma))} + \|g|_{t=0}\|_{W_p^{2\alpha-\frac{2}{q}}(\Sigma)}.$$

Then for $g \in W_p^{2\alpha,\alpha}(\Sigma \times (0,T))$, there exists an extension operator

$$\mathcal{E} \in \mathcal{L}(W_p^{2\alpha,\alpha}(\Sigma \times (0,T)), W_p^{2\alpha,\alpha}(\Sigma \times (0,\infty)))$$

satisfying $\mathcal{E}(g)|_{[0,T]} = g$. Moreover, there is a constant $C > 0$, independent of $0 < T < \infty$, such that

$$\|\mathcal{E}(g)\|_{W_p^{2\alpha,\alpha}(\Sigma\times(0,\infty))} \le C \|g\|_{W_p^{2\alpha,\alpha}(\Sigma\times(0,T))}. \tag{2.6}$$

Remark 2.29. The proof is similar to the part in Theorem 2.27, which relies on Theorem 2.26 for $1/q < \alpha < 1$ and the trace method interpolation, namely,

$$W_p^{2\alpha-\frac{2}{p}}(\Sigma) = \left\{ g(0) : g \in L^p(0,T; W_p^{2\alpha}(\Sigma)) \cap W_p^{\alpha}(0,T; L^p(\Sigma)) \right\},$$

see e.g., Lemma 2.23 or [PS16, Example 3.4.9(i)]. These results can also be extended to more general anisotropic Sobolev-Slobodeckij spaces with general trace theorem, see e.g., [PS16, Theorem 3.4.8].

2.2. Maximal L^p-regularity Theory

Throughout the thesis, we will proceed the analysis in the framework of the so-called *maximal regularity*. Thus, in this section, we follow [DHP03, PS16] to briefly introduce the *maximal L^p-regularity theory*. Note that here we specify with L^p, the Lebesgue space. There is also maximal regularity theory concerning the Hölder space $C^{0,\gamma}$, for which we refer to e.g. [Lun95].

Let X be a Banach space, and $A : D(A) \subseteq X \to X$ be a linear operator with domain $D(A)$. If A is closed, then $D(A)$ equipped with the graph norm of A, $\|x\|_A = \|x\| + \|Ax\|$, is a Banach space, for which the symbol X_A is employed. Let $J = \mathbb{R}_+$ or $(0, a)$ for some $a > 0$ and let $f : J \to X$. We consider the inhomogeneous initial value problem

$$\partial_t u(t) + Au(t) = f(t) \quad \text{for all } t \in J, \quad u(0) = u_0, \tag{2.7}$$

with $f \in L^p(J; X)$.

Definition 2.30 (Maximal L^p-regularity). Suppose $A : D(A) \subset X \to X$ is closed and densely defined. Then A is said to belong to the class $\mathcal{MR}_p(J; X)$ – and we say that there is *maximal L^p-regularity* for (2.7) – if for each $f \in L^p(J; X)$ there exists a unique $u \in H^1_p(J; X) \cap L^p(J; X_A)$ satisfying (2.7) a.e. in J, with $u_0 = 0$.

The closed graph theorem implies then that there exists a constant $C > 0$ such that

$$\|u\|_{L^p(J;X)} + \|\partial_t u\|_{L^p(J;X)} + \|Au\|_{L^p(J;X)} \le C \|f\|_{L^p(J;X)} .$$

In view of the trace method of interpolation, for $u_0 \in X_\gamma$ there is a $v \in \mathbb{E}_1(J)$ such that $v(0) = u_0$. Let $w = u - v$. We know w is the unique solution of

$$\partial_t w(t) + Aw(t) = \tilde{f}(t), \quad \text{for all } t \in J, \quad w(0) = 0,$$

where $\tilde{f} = f - \partial_t v - Av \in L^p(J; X)$. Then we have

$$\|u\|_{L^p(J;X)} + \|\partial_t u\|_{L^p(J;X)} + \|Au\|_{L^p(J;X)} \le C\big(\|u_0\|_{(X,X_A)_{1-1/p,p}} + \|f\|_{L^p(J;X)} \big).$$

Let $X_0 := X$, $X_1 := X_A$, and

$$\mathbb{E}_0(J) := L^p(J; X_0),\ \mathbb{E}_1(J) := H^1_p(J; X_0) \cap L^p(J; X_1),\ X_\gamma := (X_0, X_1)_{1-1/p,p}.$$

Then we have the following proposition concerning the *isomorphism* of an evolution problem operator depending on t, in view of the $\mathcal{MR}_p$-class operators, cf. [PS16, Proposition 3.5.6].

Proposition 2.31 (Isomorphism). *Suppose that $A \in C(J, \mathcal{L}(X_1, X_0))$ and $A(t) \in \mathcal{MR}_p(J, X_0)$ for each $t \in J = [0, a]$. Then*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + A(\cdot), \mathrm{tr} \right) \in \mathrm{Isom}(\mathbb{E}_1(J), \mathbb{E}_0(J) \times X_\gamma),$$

where $\mathrm{Isom}(X, Y)$ *denotes the set of all linear isomorphisms between X and Y. In particular, the non-autonomous problem*

$$\partial_t u + A(t)u = f(t), \quad t \in J, \quad u(0) = u_0,$$

admits for each $(f, u_0) \in \mathbb{E}_0(J) \times X_\gamma$ a unique solution $u \in \mathbb{E}_1(J)$.

Corollary 2.32. *Let $J = [0, a]$. If (2.7) admits a unique solution $u \in H^1_p(J; X) \cap L^p(J; X_A)$ for all $f \in L^p(J; X)$ and $u_0 \in X_\gamma$, then*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + A, \mathrm{tr} \right) \in \mathrm{Isom}(\mathbb{E}_1(J), \mathbb{E}_0(J) \times X_\gamma).$$

Proof. By Definition 2.30, $A \in \mathcal{MR}_p(J; X)$. Moreover, $A \in C(J; \mathcal{L}(X_A, X))$, since A does not depend on t. Then Proposition 2.31 yields the assertion. ☐

Chapter 3

Fluid-Structure Interaction Problem with Growth in Smooth Domain

We study a free-interface fluid-structure interaction problem for plaque growth with additional viscoelastic effects, which arises from the plaque formation in blood vessels. The fluid is described by the incompressible Navier–Stokes equations, while the structure is considered as a viscoelastic incompressible neo-Hookean material. Moreover, the growth due to the biochemical process is taken into account. Applying the maximal regularity theory to a linearization of the equations, along with a deformation mapping, we prove the well-posedness of the full nonlinear problem via the contraction mapping principle.

Overview of This Chapter

Notations. In this chapter, we consider specifically the following notations.

- $\Omega^t = \Omega_f^t \cup \Omega_s^t \cup \Gamma^t$, where $\Omega^t \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ is divided by the interface Γ^t into two disjoint parts, fluid domain Ω_f^t and solid domain Ω_s^t. Γ_s^t denotes the outer boundary of Ω^t. See Figure 3.1.
- $\mathbf{v}$, the Eulerian velocity
- $\hat{\mathbf{v}}$, the Lagrangian velocity
- c, $\hat{c}$, cell concentrations
- c^*, $\hat{c}^*$ foam cell concentration
- g, $\hat{g}$, growth metrics
- $\hat{\mathbf{F}}$, the deformation gradient in terms of $\hat{\mathbf{v}}$
- $\mathbf{F}$, the inverse deformation gradient

When there is no danger of confusion, we specify the quantities with a subscript "f" and "s" to identify those defined in fluid domain and solid domain respectively. In addition, without a special statement, the quantities or operators with a hat "$\hat{\cdot}$" will indicate those in Lagrangian coordinates.

3.1. Introduction

In this chapter, we consider a free-interface fluid-structure interaction problem with growth, which is used to describe the plaque formation in a human artery. The motion of the blood is assumed to be represented by the incompressible Navier–Stokes equations and the artery is modeled by an elastic equation with viscosity. The model was derived in Chapter 1 in the framework of continuum mechanics.

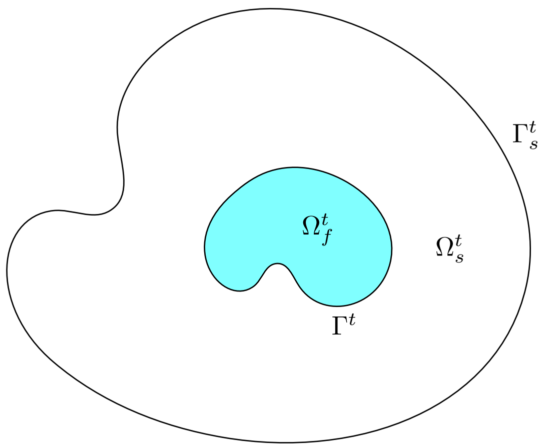

Figure 3.1: Domain Ω^t of the problem.

Here we consider the problem in a smooth bounded domain $\Omega^t \subset \mathbb{R}^n$, $n \geq 2$. See Figure 3.1. For convenience, we present the PDE system (1.36) again as follows.

$$\rho_f \left(\partial_t + \mathbf{v}_f \cdot \nabla\right) \mathbf{v}_f = \operatorname{div} \mathbb{T}_f, \quad \text{in } \Omega_f^t,\ t \in (0,T), \tag{3.1a}$$

$$\operatorname{div} \mathbf{v}_f = 0, \quad \text{in } \Omega_f^t,\ t \in (0,T), \tag{3.1b}$$

$$\rho_s \left(\partial_t + \mathbf{v}_s \cdot \nabla\right) \mathbf{v}_s = \operatorname{div} \mathbb{T}_s, \quad \text{in } \Omega_s^t,\ t \in (0,T), \tag{3.1c}$$

$$\rho_s \operatorname{div} \mathbf{v}_s = \gamma\beta c_s, \quad \text{in } \Omega_s^t,\ t \in (0,T), \tag{3.1d}$$

$$\partial_t c_f + \mathbf{v}_f \cdot \nabla c_f - \operatorname{div}(D_f \nabla c_f) = 0, \quad \text{in } \Omega_f^t,\ t \in (0,T), \tag{3.1e}$$

$$\partial_t c_s + \mathbf{v}_s \cdot \nabla c_s - \operatorname{div}(D_s \nabla c_s) = -\beta c_s, \quad \text{in } \Omega_s^t,\ t \in (0,T), \tag{3.1f}$$

$$\partial_t c_s^* + \mathbf{v}_s \cdot \nabla c_s^* = \beta c_s, \quad \text{in } \Omega_s^t,\ t \in (0,T), \tag{3.1g}$$

$$\partial_t g + \mathbf{v}_s \cdot \nabla g = \frac{\gamma\beta c_s}{n\rho_s}, \quad \text{in } \Omega_s^t,\ t \in (0,T), \tag{3.1h}$$

$$[\![\mathbf{v}]\!] = 0, \quad [\![\mathbb{T}]\!]\, \mathbf{n}_{\Gamma^t} = 0, \quad \text{on } \Gamma^t,\ t \in (0,T), \tag{3.1i}$$

$$[\![D\nabla c]\!] \cdot \mathbf{n}_{\Gamma^t} = 0, \quad \zeta\, [\![c]\!] - D_s \nabla c_s \cdot \mathbf{n}_{\Gamma^t} = 0, \quad \text{on } \Gamma^t,\ t \in (0,T), \tag{3.1j}$$

$$\mathbb{T}_s \mathbf{n}_{\Gamma_s^t} = 0, \quad D_s \nabla c_s \cdot \mathbf{n}_{\Gamma_s^t} = 0, \quad \text{on } \Gamma_s^t,\ t \in (0,T), \tag{3.1k}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}^0,\ c|_{t=0} = c^0,\ c_s^*|_{t=0} = 0,\ g|_{t=0} = 1, \tag{3.1l}$$

where $\rho_{f/s}$ are the densities and $\mathbf{v}_{f/s}$ are the velocities of the fluid and the solid respectively, the stress tensor $\mathbb{T}_f(\mathbf{v}_f, \pi_f) := -\pi_f \mathbb{I} + \nu_f(\nabla \mathbf{v}_f + \nabla \mathbf{v}_f^\top)$ denotes the Cauchy stress tensor of the fluid, π_f is the unknown fluid pressure and ν_f represents the fluid viscosity, while $\mathbb{T}_s$ is the

Cauchy stress tensor of the solid that includes viscoelastic effects. In addition, c_f, c_s, c_s^* denote the concentrations of the monocytes, the macrophages and the foam cells, respectively. The constant $D_f/s > 0$ are the diffusion coefficients in the blood and vessel, which are assumed to be constants. f_s^r is the reaction functions, modeling the rate of conversion from macrophages c_s into foam cells c_s^*.

Moreover, on the boundary $\mathbf{n}_{\Gamma^t}$ stands for the outer unit normal vector on Γ^t pointing from Ω_f^t to Ω_s^t and $\mathbf{n}_{\Gamma_s^t}$ is the unit outer normal vector on $\Gamma_s^t = \partial\Omega^t$. The constant ζ denotes the permeability of the interface Γ^t between blood and vessel regarding the cells.

Subsequently, let $\Omega = \Omega_f \cup \Gamma \cup \Omega_s$ be the initial configuration of Ω^t, φ be the motion in it, and $\mathbf{v}$ be its Eulerian velocity. We denote by $\hat{\mathbf{F}}$ the deformation gradient as in (1.6):

$$\hat{\mathbf{F}} = \frac{\partial}{\partial X}\varphi(X,t) = \hat{\nabla}\varphi(X,t) = \mathbb{I} + \int_0^t \hat{\nabla}\hat{\mathbf{v}}(X,\tau)\,\mathrm{d}\tau, \quad \forall\, X \in \Omega, \tag{3.2}$$

with initial deformation $\hat{\mathbf{F}}|_{t=0} = \mathbb{I}$ and by $\hat{J} = \det\hat{\mathbf{F}}$ its determinant. Conversely, the inverse deformation gradient is defined by $\mathbf{F} = \hat{\mathbf{F}}^{-1}$.

As discussed in Section 1.4, especially in Sections 1.4.5 and 1.4.6, the solid is assumed to be an incompressible viscous Neo-Hookean material, whose constitutive relationship of $\mathbb{T}_s$ is defined as $\mathbb{T}_s := \mathbb{T}_s^e + \mathbb{T}_s^v$ with

$$\begin{aligned}\mathbb{T}_s^e &= -\pi_s\mathbb{I} + \mu_s\left(\mathbf{F}_{s,e}^{-1}\mathbf{F}_{s,e}^{-\top} - \mathbb{I}\right),\\ \mathbb{T}_s^v &= \nu_s J_s^{-1}\left(\partial_t\mathbf{F}_s^{-1} + \partial_t\mathbf{F}_s^{-\top}\right)\mathbf{F}_s^{-\top},\end{aligned}$$

where the tensor $\mathbf{F}_{s,e}$ is the inverse elastic deformation gradient under the assumption of growth, which is discussed in Section 1.6. Here π_s is the unknown solid pressure, μ_s denotes the Lamé coefficient and ν_s represents the solid viscosity, which are all positive constants. We consider not only the elastic stress tensor $\mathbb{T}_s^e$, but also the viscoelastic stress tensor $\mathbb{T}_s^v$, which could be deduced by linearizing the Kelvin-Voigt stress tensor, see Section 1.4.6, and Mielke–Roubíček [MR20].

Remark 3.1. For short time existence, the Kelvin–Voigt viscous stress tensor $\mathbb{T}_s^v$ we introduced brings the parabolicity to the system for the solid, which dominates the regularity of solutions. Moreover, after linearization one obtains a two-phase Stokes type problem, which allows us to get the solvabilities and regularities of fluid and solid velocities by maximal regularity theory. In a recent work [BG21], a similar stress tensor of the solid part was also considered to investigate weak solutions of the interaction between an incompressible fluid and an incompressible immersed viscous-hyperelastic solid structure.

Remark 3.2. In [Yan+16, Tan+04], some numerical simulations are carried out by considering that μ_s depends on the concentration of some chemical species, and hence varies from healthy vessel to plaque area. In the case of viscoelasticity, ν_s may also vary over the solid domain. However, to simplify the model for the analysis, we assume that these coefficients are constant over the solid domain.

The interaction between the fluid and solid is modeled by transmission conditions (3.1i) on the interface Γ^t, which consists of the continuity of velocity and the balance of normal stresses. Moreover, to ensure the compatibility between growth and incompressibility, the boundary condition on Γ_s^t is assumed to be the so-called *"stress-free" boundary condition* (3.1k).

Remark 3.3. We choose the "stress-free" boundary condition for the velocity in (3.1k) to obtain physical compatibility. Since we consider the growth of the solid part and both the fluid and solid are incompressible, one can not impose some types of boundary conditions. For example, the no-slip condition $\mathbf{v}_s = 0$ on Γ_s^t (correspondingly, $\mathbf{v}_s = \partial_t \mathbf{u}_s = 0$ on Γ_s^t with $\mathbf{u}_s$ being the solid displacement) is incompatible with the incompressible growth assumption (see later in Section 3.1.1). Namely,

$$0 = \underbrace{\int_{\partial\Omega^t} \mathbf{v}_s \cdot \mathbf{n} \, \mathrm{d}\sigma = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^t} \mathrm{d}x}_{\text{by the Reynolds' Transport Theorem}} = \frac{\mathrm{d}}{\mathrm{d}t}\left|\Omega^t\right| = \int_{\Omega_s^t} \mathrm{div}\mathbf{v}_s \, \mathrm{d}x = \int_{\Omega_s^t} \gamma\beta c_s/\rho_s \, \mathrm{d}x \neq 0,$$

due to growth.

Remark 3.4. In this thesis, the fluid part is supposed to be surrounded by the solid part. In fact, if the solid is immersed in the fluid domain, there will be no essential difference in our framework of analysis. Specifically, the outer boundary will still be a Neumann-type boundary, which is a "do-nothing" outer boundary condition for fluid.

Remark 3.5. In general, the right-hand sides of (3.1d), (3.1e) and (3.1f) can be more general reaction functions that may depend on any quantities of the system. If we impose the Lipschitz condition for f in terms of c, the local well-posedness will not change too much. Thus, here for the sake of simplicity, we just assume a linear relation.

Remark 3.6. In addition to the process inside the fluid or solid domain, one needs to specify the interfacial laws for the cell interactions in (3.1j). The first one denotes the balance of the normal concentration flux at the interface, while due to the flux, cells move across the interface (penetration), which is the second equation in (3.1j). Here the permeability ζ of the interface Γ^t in general should depend on the hemodynamical stress $\mathbb{T}_f \cdot \mathbf{n}_{\Gamma^t}$, which, however, is supposed to be a constant for simplicity. The outer concentration flux is assumed to vanish on Γ_s^t as in (3.1k).

3.1.1. A recall of growth. In this section, we record the growth assumption, as introduced in Section 1.6. The first assumption is that we can always have the multiplicative decompostion of the deformation gradient $\hat{\mathbf{F}}_s$, namely,

$$\hat{\mathbf{F}}_s = \hat{\mathbf{F}}_{s,e}\hat{\mathbf{F}}_{s,g}.$$

Moreover, with constant-density growth hypothesis, one ends up with (see (1.30) for the derivation)

$$\mathrm{tr}(\hat{\mathbf{F}}_{s,g}^{-1}\partial_t \hat{\mathbf{F}}_{s,g}) = \gamma\beta c_s, \quad \text{in } \Omega_s,$$

and then the continuity equation of solid reduces to

$$\rho_s \, \mathrm{div}\, \mathbf{v}_s = \gamma\beta c_s \quad \text{in } \Omega_s^t.$$

In addition, the growth is assumed to be isotropic, i.e.,

$$\hat{\mathbf{F}}_{s,g} = \hat{g}\mathbb{I}, \quad \text{in } \Omega_s,$$

where $\hat{g} = \hat{g}(X,t)$ is the metric of growth, a scalar function depending on the concentration of macrophages. Hence,

$$\hat{\mathbf{F}}_{s,e} = \frac{1}{\hat{g}}\hat{\mathbf{F}}_s, \quad \hat{J}_{s,g} = \hat{g}^n,$$

where n is the dimension of space. As mentioned in [AM02], $\hat{g}$ describes the deformation state of the material, either growing if $\hat{g} > 1$ or resorbing if $0 < \hat{g} < 1$. Consequently, under the assumption of constant-density growth, one deduces the equation for growth in Lagrangian coordinates

$$\partial_t \hat{g} = \frac{\gamma\beta\hat{c}_s}{n\hat{\rho}_s}\hat{g}, \quad \text{in } \Omega_s. \tag{3.3}$$

This shows the specific dependence of $\hat{g}$ on $\hat{c}_s$.

3.1.2. Literature. During the last decades, fluid-structure interaction problems attracted much attention from mathematicians due to their strong applications in various areas, e.g., biomechanics, hemodynamics, aeroelasticity and hydroelasticity. Studies can be divided into two types depending on the dimensions of the fluid and the solid. They are for example 3d-3d coupled and 3d-2d coupled systems, where the solid is contained in the fluid and one part of the fluid's boundary, respectively.

In the case of 3d-3d model, the existence and uniqueness of strong solutions of such kind of models was firstly established by Coutand–Shkoller [CS05], where they investigated the interaction of the Navier–Stokes equation and a linear Kirchhoff elastic material. The results were extended to the quasilinear elastodynamics case by them in [CS06], where they regularized the hyperbolic elastic equation by a particular parabolic artificial viscosity and then obtained the existence of strong solutions by the a priori estimates. Subsequently, systems coupled the incompressible Navier–Stokes equation and the wave equation were continuously analyzed by Ignatova–Kukavica–Lasiecka–Tuffaha [Ign+14, Ign+17]. More specifically, In [Ign+14], a wave equation with several damping terms was considered and exponential decay of the energy was obtained. Later, they proved that the energy still decay without the boundary friction by introducing the tangential and time-tangential energy estimates. The coupling of the Navier–Stokes equation and the Lamé system was studied by Kukavica–Tuffaha [KT12a] with initial regularity $(v_0, w_1) \in H^3(\Omega_f) \times H^2(\Omega_s)$, while Raymond–Vanninathan [RV14] further obtained the same results by a weaker initial regularity $(v_0, w_1) \in H^{3/2+\varepsilon}(\Omega_f) \times H^{1+\varepsilon}(\Omega_s)$, $\varepsilon > 0$ arbitrarily small, with periodic boundary conditions. Recently, Boulakia–Guerrero–Takahashi [BGT19] showed a similar result for the Navier–Stokes–Lamé system in a smooth domain with reduced demand of the initial regularity.

Besides the standard models above, we refer to [BG10], for compressible fluid coupled with elastic bodies, where Boulakia and Guerrero addressed the short time existence and the uniqueness of regular solutions with the initial data $(\rho_0, u_0, w_0, w_1) \in H^3(\Omega_f) \times H^4(\Omega_f) \times H^3(\Omega_s) \times H^2(\Omega_s)$. Kukavica–Tuffaha [KT12b] improved the result by a weaker initial regularity $(\rho_0, u_0, w_1) \in H^3(\Omega_f) \times H^{3/2+r}(\Omega_f) \times H^{3/2+r}(\Omega_s)$, $r > 0$. More recently, Shen–Wang–Yang [SWY21] considered the magnetohydrodynamics (MHD)-structure interaction system, where the fluid is described by the incompressible viscous non-resistive MHD equation and the structure is modeled by the wave equation with superconductor material. They solved the existence of local strong solutions with penalization and regularization techniques.

As for 3d-2d/2d-1d systems, numerous models and results were established during the last twenty years. The widely investigated case is the fluid-beam/plate systems where the beam/plate equations were imposed with different mechanical mechanism (rigidity, stretching, friction, rotation, etc.), readers are refer to e.g. [Cha+05, Gra08, MČ13, TW20] for weak solution results. Considering strong solutions, one can find related results in e.g. [Vei04, DS20, GHL19, Leq11, Leq13, MT21, Mit20] and the references therein. Moreover, the fluid-structure interaction problems with linear/nonlinear shells were studied in e.g. [BS18, LR14, MČ15] for weak solutions and in e.g. [CCS07, CS10, MRR20] for strong solutions respectively. It is worth mentioning that in recent works [DS20, MT21], a maximal regularity framework, which requires lower initial regularity and less compatibility conditions compared to the energy method, was employed.

3.1.3. Mathematical strategy and features. The new difficulties arise from the plaque formation in the blood vessels, along with the interaction between the fluid and the solid separated by a free interface, the reaction and the diffusion of different cells and the growth of the vessel wall. Numerical computations were carried out in recent years [FRW16, Yan+16, Yan+17] to simulate the plaque formation and test the effects of different parameters. To our best knowledge, this is the first work concerning the existence of strong solutions to the fluid-structure interaction problems with growth. Unlike most of the literature above, where L^2-Sobolev spaces and energy methods are used, we establish our local strong solutions in the framework of maximal L^q-regularity for any space dimension ($n \geq 2$). The method is based on the Banach fixed-point theorem, for which we rewrite the free boundary problem established in Eulerian coordinates in Lagrangian coordinates, linearize the system at the initial configuration, construct a contraction mapping in a suitable ball and show the local existence and uniqueness of strong solutions. Throughout the proof, we point out the following features:

i) We adapt the maximal L^q-regularity theory for the Stokes system to solve our problem. Hence, there will be no "regularity loss" from the data to the solution spaces and only a few compatibility conditions are needed.

ii) The growth is considered to be of constant-density type. Then under the assumption of isotropy, the growth will be described by the metric function $\hat{g}$. An ordinary differential equation for $\hat{g}$ provides the regularity of $\hat{g}$ needed for the solid velocity and the concentration of macrophages.

iii) The Kelvin–Voigt viscous stress tensor $\mathbb{T}_s^v$, we introduced, brings parabolicity to the solid equation. For the linearization, we can use a two-phase Stokes type problem for the fluid-structure interaction problem. This ensures that we can get the solvabilities and regularities of fluid and solid velocities by maximal L^q-regularity theory.

iv) The transformed two-phase Stokes problem is endowed with a stress-free (Neumann-type) outer boundary condition, cf. Remark 3.3. One of our aims is to obtain the solvability of such system. To this end, reduction and truncation arguments are applied. More specifically, we first reduce the inhomogeneous linear system to a semi-homogeneous problem (with inhomogeneous boundary terms), in order to obtain the pressure regularities. Then by choosing a cutoff function (see (3.21)) which is supported in a subset $U \subseteq \Omega$ and imposing an artificial vanishing Dirichlet boundary on $\Gamma_s = \partial\Omega$, one obtains the solvability of the linear system since the two-phase Stokes problem with Dirichlet boundary is solved in Section 3.5.1.

3.1.4. Outline of the chapter. In Section 3.2 we transform the system in deformed configuration to the reference configuration by means of the Lagrangian coordinates, and present the main theorem for the transformed system. Section 3.3 is devoted to the analysis of the underlying linear problems, where three separate parts of the analysis are treated. The main results of this section are the maximal L^q-regularities for these linear problems. The first one is the two-phase Stokes problems with Neumann boundary conditions, to which reduction and truncation (localization) arguments are applied. The second problem consists of two reaction-diffusion systems with Neumann boundary conditions due to the decoupling of the transmission problem, while the last one is an ordinary differential equation for the growth of the foam cells. In Section 3.4, we first give some estimates related to the deformation gradient, which are of much importance when proving that the constructed nonlinear terms are well-defined and Lipschitz continuous. Then the full nonlinear system is shown to be well-posed locally in time via the Banach fixed-point theorem. Moreover, the cell concentrations are proved to be always nonnegative, provided that the initial data is nonnegative. Additionally, we introduce some necessary maximal L^q-regularity results of several linear systems in Section 3.5.

3.2. Linearization and Main Result

In this section, we transform the free-interface fluid-structure problem with growth from deformed configuration to a fixed reference configuration and state the main result. For quantities in different configurations, we define

$$\begin{aligned}&\hat{\mathbf{v}}(X,t)=\mathbf{v}(x,t),\ \hat{\pi}(X,t)=\pi(x,t),\ \hat{\mathbb{T}}(X,t)=\mathbb{T}(x,t),\\&\hat{\rho}(X,t)=\rho(x,t),\ \hat{\mu}(X,t)=\mu(x,t),\ \hat{\nu}(X,t)=\nu(x,t),\end{aligned}\tag{3.4}$$

for all $x=\varphi(X,t),\ X\in\Omega$ and $t\geq 0$. For the fluid part, it follows from Proposition 1.12 that

$$\hat{J}_f=1,\quad \text{in } \Omega_f.\tag{3.5}$$

For the solid part, since the deformation from the natural configuration Ω_s^g to the deformed configuration Ω_s^t conserves mass, incompressibility yields $\hat{J}_{s,e}=1$ and hence

$$\hat{J}_s=\hat{J}_{s,g}=\hat{g}^n,\quad \text{in } \Omega_s.$$

Now combining Propositions 1.12, 1.15 and 1.65, and Theorem 1.35, we rewrite the fluid-structure interaction problem (3.1) in the reference configuration Ω.

$$\left.\begin{aligned}\hat{\rho}_f\partial_t\hat{\mathbf{v}}_f-\widehat{\operatorname{div}}\left(\hat{\mathbb{T}}_f\hat{\mathbf{F}}_f^{-\top}\right)&=0\\ \hat{\mathbf{F}}_f^{-\top}:\hat{\nabla}\hat{\mathbf{v}}_f&=0\\ \partial_t\hat{c}_f-\hat{D}_f\,\widehat{\operatorname{div}}\left(\hat{\mathbf{F}}_f^{-1}\hat{\mathbf{F}}_f^{-\top}\hat{\nabla}\hat{c}_f\right)&=0\end{aligned}\right\}\quad \text{in } \Omega_f\times(0,T),\tag{3.6a}$$

$$\left.\begin{aligned}\hat{\rho}_s\partial_t\hat{\mathbf{v}}_s-\hat{J}_s^{-1}\,\widehat{\operatorname{div}}\left(\hat{J}_s\hat{\mathbb{T}}_s\hat{\mathbf{F}}_s^{-\top}\right)&=0\\ \hat{\mathbf{F}}_s^{-\top}:\hat{\nabla}\hat{\mathbf{v}}_s-\frac{\gamma\beta}{\hat{\rho}_s}\hat{c}_s&=0\\ \partial_t\hat{c}_s-\hat{D}_s\hat{J}_s^{-1}\,\widehat{\operatorname{div}}\left(\hat{J}_s\hat{\mathbf{F}}_s^{-1}\hat{\mathbf{F}}_s^{-\top}\hat{\nabla}\hat{c}_s\right)+\beta\hat{c}_s\left(1+\frac{\gamma}{\hat{\rho}_s}\hat{c}_s\right)&=0\\ \partial_t\hat{c}_s^*-\beta\hat{c}_s+\frac{\gamma\beta}{\hat{\rho}_s}\hat{c}_s\hat{c}_s^*=0,\quad \partial_t\hat{g}-\frac{\gamma\beta}{n\hat{\rho}_s}\hat{c}_s\hat{g}&=0\end{aligned}\right\}\quad \text{in } \Omega_s\times(0,T),\tag{3.6b}$$

$$\left.\begin{aligned}[\![\hat{\mathbf{v}}]\!]=0,\quad \left[\!\!\left[\hat{\mathbb{T}}\hat{\mathbf{F}}^{-\top}\right]\!\!\right]\hat{\mathbf{n}}_\Gamma=0,\quad \left[\!\!\left[\hat{D}\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}^{-\top}\hat{\nabla}\hat{c}\right]\!\!\right]\hat{\mathbf{n}}_\Gamma&=0\\ \zeta\,[\![\hat{c}]\!]-\hat{D}_s\hat{\mathbf{F}}_s^{-1}\hat{\mathbf{F}}_s^{-\top}\hat{\nabla}\hat{c}_s\cdot\hat{\mathbf{n}}_\Gamma&=0\end{aligned}\right\}\quad \text{on } \Gamma\times(0,T),\tag{3.6c}$$

$$\hat{\mathbb{T}}_s\hat{\mathbf{F}}_s^{-\top}\hat{\mathbf{n}}_{\Gamma_s}=0,\quad \hat{D}_s\hat{\mathbf{F}}_s^{-1}\hat{\mathbf{F}}_s^{-\top}\hat{\nabla}\hat{c}_s\cdot\hat{\mathbf{n}}_{\Gamma_s}=0\quad \text{on } \Gamma_s\times(0,T),\tag{3.6d}$$

$$\hat{\mathbf{v}}|_{t=0}=\hat{\mathbf{v}}^0,\quad \hat{c}|_{t=0}=\hat{c}^0\quad \text{in } \Omega\setminus\Gamma,\tag{3.6e}$$

$$\hat{c}_s^*|_{t=0}=0,\quad \hat{g}|_{t=0}=1\quad \text{in } \Omega_s,\tag{3.6f}$$

where the corresponding stress tensors are

$$\begin{aligned}\hat{\mathbb{T}}_f&=-\hat{\pi}_f\mathbb{I}+\hat{\nu}_f\left(\hat{\mathbf{F}}_f^{-1}\hat{\nabla}\hat{\mathbf{v}}_f+\hat{\nabla}\hat{\mathbf{v}}_f^\top\hat{\mathbf{F}}_f^{-\top}\right),\quad \hat{\mathbb{T}}_s=\hat{\mathbb{T}}_s^e+\hat{\mathbb{T}}_s^v,\\ \hat{\mathbb{T}}_s^e&=-\hat{\pi}_s\mathbb{I}+\hat{\mu}_s\left(\hat{\mathbf{F}}_{s,e}\hat{\mathbf{F}}_{s,e}^\top-\mathbb{I}\right)=-\hat{\pi}_s\mathbb{I}+\hat{\mu}_s\left(\frac{1}{(\hat{g})^2}\hat{\mathbf{F}}_s\hat{\mathbf{F}}_s^\top-\mathbb{I}\right),\\ \hat{\mathbb{T}}_s^v&=\hat{\nu}_sJ_s^{-1}\left(\hat{\nabla}\hat{\mathbf{v}}_s+\hat{\nabla}\hat{\mathbf{v}}_s^\top\right)\hat{\mathbf{F}}_s^\top.\end{aligned}$$

For the maximal L^q-regularity setting, we assume

$$\hat{\mathbf{v}}^0 \in B^{1-1/q}_{q,q}(\Omega)^n \cap B^{2(1-1/q)}_{q,q}(\widetilde{\Omega})^n, \quad \hat{c}^0 \in B^{2(1-1/q)}_{q,q}(\widetilde{\Omega}),$$

that is,

$$\hat{\mathbf{v}}^0 \in W^{1-1/q}_q(\Omega)^n \cap W^{2(1-1/q)}_q(\widetilde{\Omega})^n =: \mathcal{D}^1_q, \quad \hat{c}^0 \in W^{2(1-1/q)}_q(\widetilde{\Omega}) =: \mathcal{D}^2_q,$$

where we define $\widetilde{\Omega} = \Omega_f \cup \Omega_s$. The space $\mathcal{D}_q := \mathcal{D}^1_q \times \mathcal{D}^2_q$ will be the initial space for velocities and concentrations. Moreover, we introduce the compatibility conditions for $q > n+2$, which were also used in e.g. Abels [Abe05], Prüss–Simonett [PS16], Shibata–Shimizu [SS08], Shimizu [Shi08]:

$$\begin{aligned} \widehat{\operatorname{div}}\, \hat{\mathbf{v}}^0 = 0, \quad [\![\hat{\mathbf{v}}^0]\!]|_\Gamma = 0, \quad \left[\!\!\left[\mathcal{P}_{\hat{\mathbf{n}}_\Gamma}\left(\hat{\nu}\left(\hat{\nabla}\hat{\mathbf{v}}^0 + (\hat{\nabla}\hat{\mathbf{v}}^0)^\top\right)\hat{\mathbf{n}}_\Gamma\right)\right]\!\!\right]\Big|_\Gamma = 0, \\ \mathcal{P}_{\hat{\mathbf{n}}_{\Gamma_s}}\left(\hat{\nu}\left(\hat{\nabla}\hat{\mathbf{v}}^0 + (\hat{\nabla}\hat{\mathbf{v}}^0)^\top\right)\hat{\mathbf{n}}_{\Gamma_s}\right)\Big|_{\Gamma_s} = 0, \end{aligned} \tag{3.7}$$

and

$$\left(\zeta\, [\![\hat{c}^0]\!] - \hat{D}_s\hat{\nabla}\hat{c}^0_s\cdot\hat{\mathbf{n}}_\Gamma\right)\Big|_\Gamma = 0, \quad \left[\!\!\left[\hat{D}\hat{\nabla}\hat{c}^0\right]\!\!\right]\cdot\hat{\mathbf{n}}_\Gamma\Big|_\Gamma = 0, \quad \hat{D}_s\hat{\nabla}\hat{c}^0_s\cdot\hat{\mathbf{n}}_{\Gamma_s}\Big|_{\Gamma_s} = 0, \tag{3.8}$$

where $\mathcal{P}_{\hat{\mathbf{n}}}$ denotes the tangential part on the surface, namely, $\mathcal{P}_{\hat{\mathbf{n}}} = (\mathbb{I} - \hat{\mathbf{n}}\otimes\hat{\mathbf{n}})\cdot$. Besides this, we define the solution space for $(\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}^*_s, \hat{g})$ as $Y_T = Y^1_T \times Y^2_T \times Y^3_T \times Y^4_T \times Y^4_T$, where

$$\begin{aligned} Y^1_T &= L^q(0,T; W^2_q(\widetilde{\Omega})^n \cap W^1_q(\Omega)^n) \cap W^1_q(0,T; L^q(\Omega)^n), \\ Y^2_T &= \left\{ \begin{aligned} \hat{\pi} \in L^q(0,T; W^1_q(\Omega)) : [\![\hat{\pi}]\!] \in W^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}_q(\Gamma\times(0,T)) \\ \hat{\pi}|_{\Gamma_s} \in W^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}_q(\Gamma_s\times(0,T)) \end{aligned} \right\}, \\ Y^3_T &= L^q(0,T; W^2_q(\widetilde{\Omega})) \cap W^1_q(0,T; L^q(\Omega)), \\ Y^4_T &= W^1_q(0,T; W^1_q(\Omega_s)), \end{aligned}$$

equipped with norms

$$\begin{aligned} \|\hat{\mathbf{v}}\|_{Y^1_T} &= \|\hat{\mathbf{v}}\|_{L^q(0,T;W^2_q(\widetilde{\Omega})^n \cap W^1_{q,0}(\Omega)^n)} + \|\hat{\mathbf{v}}\|_{W^1_q(0,T;L^q(\Omega)^n)}, \\ \|\hat{\pi}\|_{Y^2_T} &= \|\hat{\pi}\|_{L^q(0,T;W^1_q(\Omega))} + \|[\![\hat{\pi}]\!]\|_{W^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}_q(\Gamma\times(0,T))} \\ &\quad + \left\| \hat{\pi}|_{\Gamma_s} \right\|_{W^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}_q(\Gamma_s\times(0,T))}, \\ \|\hat{c}\|_{Y^3_T} &= \|\hat{c}\|_{L^q(0,T;W^2_q(\widetilde{\Omega}))} + \|\hat{c}\|_{W^1_q(0,T;L^q(\Omega))}, \\ \|\hat{c}^*_s\|_{Y^4_T} &= \|\hat{c}^*_s\|_{W^1_q(0,T;W^1_q(\Omega_s))}, \quad \|\hat{g}\|_{Y^4_T} = \|\hat{g}\|_{W^1_q(0,T;W^1_q(\Omega_s))}. \end{aligned}$$

Moreover, we set $Y^v_T := Y^1_T \times Y^2_T$.

Remark 3.7. These spaces are constructed from the problem and the maximal regularity theory, endowed with the natural norms. In particular, $[\![\hat{\pi}]\!]$ and $\hat{\pi}|_{\Gamma_s}$ are determined by the regularities of the Neumann trace of $\hat{\mathbf{v}}$ on Γ and Γ_s respectively. Hence, we add the norm of $\|\, [\![\hat{\pi}]\!]\, \|_{W^{1-1/q,(1-1/q)/2}_q(\Gamma\times(0,T))}$ and $\|\hat{\pi}|_{\Gamma_s}\|_{W^{1-1/q,(1-1/q)/2}_q(\Gamma_s\times(0,T))}$ in Y^2_T-norm correspondingly. One can easily verify that all spaces are Banach spaces.

Now the main result in this chapter is given as follows.

Theorem 3.8 (Main theorem). *Let $q > n+2$. Assume that Γ, Γ_s are hypersurfaces of class C^3, $(\hat{\mathbf{v}}^0, \hat{c}^0) \in \mathcal{D}_q$, such that the compatibility conditions (3.7) and (3.8) hold, then there is a positive $T_0 = T_0(\hat{\mathbf{v}}^0, \hat{c}^0) < \infty$ such that there exists a unique strong solution $(\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}^*_s, \hat{g}) \in Y_{T_0}$ to system (3.6). Moreover, $\hat{c} \geq 0$ and $\hat{c}^*_s, \hat{g} > 0$, if $\hat{c}^0 \geq 0$.*

Remark 3.9. In this work, the boundary of the domain is supposed to be C^3. We remark here that if the domain is not smooth enough, for example, with boundary contact, it is still an open problem. We considered a similar model with ninety degree contact angles in Chapter 4.

The proof of Theorem 3.8 relies on the Banach fixed-point theorem. To this end, we need to linearize the nonlinear system (3.6). Since we consider a nonzero initial reference configuration, a standard perturbation method is applied to (3.6), for which we linearize the system at the initial deformation and move all remainder terms to the right-hand side, namely,

$$\left.\begin{aligned} \hat{\rho}_f \partial_t \hat{\mathbf{v}}_f - \widehat{\operatorname{div}}\, \mathbf{S}(\hat{\mathbf{v}}_f, \hat{\pi}_f) = \mathbf{K}_f \\ \widehat{\operatorname{div}}\, \hat{\mathbf{v}}_f = G_f \end{aligned}\right\} \quad \text{in } \Omega_f \times (0,T), \tag{3.9a}$$

$$\left.\begin{aligned} \hat{\rho}_s \partial_t \hat{\mathbf{v}}_s - \widehat{\operatorname{div}}\, \mathbf{S}(\hat{\mathbf{v}}_s, \hat{\pi}_s) = \bar{\mathbf{K}}_s + \mathbf{K}_s^g =: \mathbf{K}_s \\ \widehat{\operatorname{div}}\, \hat{\mathbf{v}}_s - \frac{\gamma\beta}{\hat{\rho}_s}\hat{c}_s = G_s \end{aligned}\right\} \quad \text{in } \Omega_s \times (0,T), \tag{3.9b}$$

$$[\![\hat{\mathbf{v}}]\!] = 0, \quad [\![\mathbf{S}(\hat{\mathbf{v}}, \hat{\pi})]\!]\, \hat{\mathbf{n}}_\Gamma = \mathbf{H}^1 \quad \text{on } \Gamma \times (0,T), \tag{3.9c}$$

$$\mathbf{S}(\hat{\mathbf{v}}_s, \hat{\pi}_s)\hat{\mathbf{n}}_{\Gamma_s} = \mathbf{H}^2 \quad \text{on } \Gamma_s \times (0,T), \tag{3.9d}$$

$$\hat{\mathbf{v}}|_{t=0} = \hat{\mathbf{v}}^0 \quad \text{in } \widetilde{\Omega}, \tag{3.9e}$$

$$\partial_t \hat{c}_f - \hat{D}_f \widehat{\Delta} \hat{c}_f = F_f^1 \quad \text{in } \Omega_f \times (0,T), \tag{3.9f}$$

$$\partial_t \hat{c}_s - \hat{D}_f \widehat{\Delta} \hat{c}_s = \bar{F}_s^1 + F_s^g =: F_s^1 \quad \text{in } \Omega_s \times (0,T), \tag{3.9g}$$

$$\left.\begin{aligned} \hat{D}_f \hat{\nabla} \hat{c}_f \cdot \hat{\mathbf{n}}_\Gamma = \hat{D}_s \nabla \hat{c}_s \cdot \hat{\mathbf{n}}_\Gamma + \bar{F}_f^2 =: F_f^2 \\ \hat{D}_s \hat{\nabla} \hat{c}_s \cdot \hat{\mathbf{n}}_\Gamma = \zeta\, [\![\hat{c}]\!] + \bar{F}_s^2 =: F_s^2 \end{aligned}\right\} \quad \text{on } \Gamma \times (0,T), \tag{3.9h}$$

$$\hat{D}_s \hat{\nabla} \hat{c}_s \cdot \hat{\mathbf{n}}_{\Gamma_s} = F^3 \quad \text{on } \Gamma_s \times (0,T), \tag{3.9i}$$

$$\hat{c}|_{t=0} = \hat{c}^0 \quad \text{in } \widetilde{\Omega}, \tag{3.9j}$$

$$\partial_t \hat{c}_s^* - \beta \hat{c}_s = F^4 \quad \text{in } \Omega_s \times (0,T), \tag{3.9k}$$

$$\hat{c}_s^*|_{t=0} = 0 \quad \text{in } \Omega_s, \tag{3.9l}$$

$$\partial_t \hat{g} - \frac{\gamma\beta}{n\hat{\rho}_s}\hat{c}_s = F^5 \quad \text{in } \Omega_s \times (0,T), \tag{3.9m}$$

$$\hat{g}|_{t=0} = 1 \quad \text{in } \Omega_s, \tag{3.9n}$$

where $\mathbf{S}(\hat{\mathbf{v}}, \hat{\pi}) = -\hat{\pi}\mathbb{I} + \hat{\nu}\left(\hat{\nabla}\hat{\mathbf{v}} + \hat{\nabla}\hat{\mathbf{v}}^\top\right)$ in $\widetilde{\Omega}$ and

$$\begin{aligned}
&\mathbf{K}_f = \widehat{\operatorname{div}}\, \tilde{\mathbf{K}}_f, \quad \bar{\mathbf{K}}_s = \widehat{\operatorname{div}}\, \tilde{\mathbf{K}}_s, \quad \mathbf{K}_s^g = -\left(\hat{\mathbb{T}}_s \hat{\mathbf{F}}_s^{-\top}\right) \frac{n\hat{\nabla}\hat{g}}{\hat{g}}, \\
&G = -\left(\hat{\mathbf{F}}^{-\top} - \mathbb{I}\right) : \hat{\nabla}\hat{\mathbf{v}}, \quad \mathbf{H}^1 = -\left[\!\!\left[\tilde{\mathbf{K}}\right]\!\!\right] \cdot \hat{\mathbf{n}}_\Gamma, \quad \mathbf{H}^2 = -\tilde{\mathbf{K}}_s \cdot \hat{\mathbf{n}}_{\Gamma_s}, \\
&F_f^1 = \widehat{\operatorname{div}}\, \tilde{F}_f, \quad \bar{F}_s^1 = \widehat{\operatorname{div}}\, \tilde{F}_s, \\
&F_s^g = -\beta \hat{c}_s \left(1 + \frac{\gamma}{\hat{\rho}_s}\hat{c}_s\right) - \frac{n\hat{\nabla}\hat{g}}{\hat{g}} \cdot \left(\hat{D}_s \hat{\mathbf{F}}_s^{-1} \hat{\mathbf{F}}_s^{-\top} \hat{\nabla}\hat{c}_s\right), \\
&\bar{F}_f^2 = -\left[\!\!\left[\tilde{F}\right]\!\!\right] \cdot \hat{\mathbf{n}}_\Gamma, \quad \bar{F}_s^2 = -\tilde{F}_s \cdot \hat{\mathbf{n}}_\Gamma, \quad F^3 = -\tilde{F}_s \cdot \hat{\mathbf{n}}_{\Gamma_s}, \\
&F^4 = -\frac{\gamma\beta}{\hat{\rho}_s}\hat{c}_s \hat{c}_s^*, \quad F^5 = -\frac{\gamma\beta}{n\hat{\rho}_s}\hat{c}_s (\hat{g} - 1),
\end{aligned} \tag{3.10}$$

with

$$
\begin{aligned}
\tilde{\mathbf{K}}_f &= -\hat{\pi}_f \left(\hat{\mathbf{F}}_f^{-\top} - \mathbb{I}\right) + \nu_f \left(\hat{\mathbf{F}}_f^{-1}\hat{\nabla}\hat{\mathbf{v}}_f + \hat{\nabla}\hat{\mathbf{v}}_f^{\top}\hat{\mathbf{F}}_f^{-\top}\right)\left(\hat{\mathbf{F}}_f^{-\top} - \mathbb{I}\right) \\
&\quad + \nu_f \left(\left(\hat{\mathbf{F}}_f^{-1} - \mathbb{I}\right)\hat{\nabla}\hat{\mathbf{v}}_f + \hat{\nabla}\hat{\mathbf{v}}_f^{\top}\left(\hat{\mathbf{F}}_f^{-\top} - \mathbb{I}\right)\right), \\
\tilde{\mathbf{K}}_s &= -\hat{\pi}_s \left(\hat{\mathbf{F}}_s^{-\top} - \mathbb{I}\right) + \mu_s \left(\frac{1}{\hat{g}^2}\left(\hat{\mathbf{F}}_s - \mathbb{I}\right) + \left(\frac{1}{\hat{g}^2} - 1\right)\mathbb{I} - \left(\hat{\mathbf{F}}_s^{-\top} - \mathbb{I}\right)\right), \\
\tilde{F} &= \hat{D}\left(\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}^{-\top} - \mathbb{I}\right)\hat{\nabla}\hat{c}.
\end{aligned}
$$

Then we analyze system (3.9), which is exactly (3.6).

Remark 3.10. It follows from the Piola identity (1.14) that

$$
\widehat{\operatorname{div}}\left(\hat{J}\hat{\mathbf{F}}^{-\top}\right) = 0.
$$

Then from Proposition 1.65,

$$
\hat{J}\hat{\mathbf{F}}^{-\top} : \hat{\nabla}\hat{\mathbf{v}} = \widehat{\operatorname{div}}\left(\hat{J}\hat{\mathbf{F}}^{-1}\hat{\mathbf{v}}\right).
$$

Hence, G possesses the form

$$
G_f = -\widehat{\operatorname{div}}\left(\left(\hat{\mathbf{F}}_f^{-1} - \mathbb{I}\right)\hat{\mathbf{v}}_f\right), \quad G_s = -\widehat{\operatorname{div}}\left(\left(\hat{\mathbf{F}}_s^{-1} - \mathbb{I}\right)\hat{\mathbf{v}}_s\right) + \hat{\mathbf{v}}_s \cdot \widehat{\operatorname{div}}\,\hat{\mathbf{F}}_s^{-\top}. \tag{3.11}
$$

Remark 3.11. The system (3.9f)–(3.9j) for the concentrations of monocytes and macrophages can be considered as a transmission problem in Ω_f and Ω_s with a common boundary Γ. However, if we use the concentration and stress jump condition as boundary condition on Γ, we will meet a regularity problem due to the high order term $D_s\hat{\nabla}\hat{c}_s \cdot \hat{\mathbf{n}}_\Gamma$ in $(3.9\text{h})_2$. More precisely, in our further perturbation argument, all perturbed or unrelated terms will be moved to the right-hand side of the equation and the regularities of both sides should coincide. The point is that in such argument, the right-hand side of $(3.9\text{h})_2$ contains $D_s\hat{\nabla}\hat{c}_s \cdot \hat{\mathbf{n}}_\Gamma$, which leads to a lower regularity, provided the same regularity of $\hat{c}$ on the both side.

Therefore, to avoid such awkward situation, we rewrite the transmission conditions as two Neumann type boundary conditions. Then the transmission problem can be decoupled into two separate parabolic system, which are both imposed with Neumann boundary and defined in Ω_f and Ω_s respectively. This is why we treat the boundary conditions on Γ as the form shown in (3.9h).

Consequently, given data $(\mathbf{K}, G, \mathbf{H}^1, \mathbf{H}^2, F^1, F^2, F^3, F^4, F^5)$ with suitable regularities, existence and uniqueness of $(\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s^*, \hat{g})$ in the associated spaces will be obtained by the well-poesdness of linear systems in the next section.

3.3. Analysis of the Linear Systems

As seen in (3.9), the linearized system can be seen as a two-phase Stokes type problem (3.9a)–(3.9e), two separate reaction-diffusion systems (3.9f)–(3.9j) and two ordinary differential equations (3.9k)–(3.9n) (equation for foam cells and growth, respectively). In this section, thanks to the maximal L^q-regularity theory, we establish the existence of strong solutions to these systems with prescribed initial data and source terms in appropriate spaces.

Let Ω be a bounded domain satisfying $\Omega = \Omega_f \cup \Gamma \cup \Omega_s$ with $\overline{\Omega_f} \subset \Omega$, $\Gamma = \partial\Omega_f$ a C^3 interface, and a boundary $\Gamma_s := \partial\Omega$ of class C^3.

3.3.1. Two-phase Stokes problems with Neumann boundary condition. Observing that $(\mathbf{K}, G, \mathbf{H}^1, \mathbf{H}^2)|_{t=0} = 0$, one replaces $(\mathbf{K}, G, \mathbf{H}^1, \mathbf{H}^2)$ in (3.9a)–(3.9e) by known functions $(\mathbf{k}, g, \mathbf{h}^1, \mathbf{h}^2)$ with $(\mathbf{k}, g, \mathbf{h}^1, \mathbf{h}^2)|_{t=0} = 0$ in (3.9b). Then we get the problem addressed in this subsection.

$$\begin{aligned} \rho\partial_t \mathbf{v} - \operatorname{div} \mathbf{S}(\mathbf{v}, \pi) &= \mathbf{k} && \text{in } \Omega \setminus \Gamma \times (0,T), \\ \operatorname{div} \mathbf{v} &= g && \text{in } \Omega \setminus \Gamma \times (0,T), \\ [\![\mathbf{v}]\!] &= 0 && \text{on } \Gamma \times (0,T), \\ [\![\mathbf{S}(\mathbf{v}, \pi)]\!]\, \mathbf{n}_\Gamma &= \mathbf{h}^1 && \text{on } \Gamma \times (0,T), \\ \mathbf{S}(\mathbf{v}_s, \pi_s)\mathbf{n}_{\Gamma_s} &= \mathbf{h}^2 && \text{on } \Gamma_s \times (0,T), \\ \mathbf{v}|_{t=0} &= \mathbf{v}^0 && \text{in } \Omega \setminus \Gamma, \end{aligned} \tag{3.12}$$

where $\mathbf{S}(\mathbf{v}, \pi) = -\pi\mathbb{I} + \nu(\nabla\mathbf{v} + \nabla\mathbf{v}^\top)$. $\rho, \nu > 0$ are the constant density and viscosity. $\mathbf{n}_\Gamma$, $\mathbf{n}_{\Gamma_s}$ denotes the unit outer normal vectors on Γ, Γ_s respectively. Now, we will prove the following theorem, namely, existence of unique solution to a two-phase Stokes problem with outer Neumann boundary condition.

Theorem 3.12. *Let $q > n + 2$, $T > 0$, Ω a bounded domain as before with $\Gamma_s \in C^3$, Γ a closed hypersurface of class C^3. Assume that $(\mathbf{k}, g, \mathbf{h}^1, \mathbf{h}^2)$ are known functions with regularity*

$$\mathbf{k} \in \mathbb{F}_{\mathbf{k}}(T) := L^q(0,T; L^q(\Omega \setminus \Gamma)^n),$$

$$g \in \mathbb{F}_g(T) := \left\{ \begin{aligned} & g \in L^q(0,T; W^1_q(\Omega \setminus \Gamma)) \cap W^1_q(0,T; W^{-1}_q(\Omega)) : \\ & \qquad g|_\Gamma \in W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\Gamma \times (0,T)), \\ & \qquad g|_{\Gamma_s} \in W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\Gamma_s \times (0,T)) \end{aligned} \right\}.$$

$$\mathbf{h}^1 \in \mathbb{F}_{\mathbf{h}^1}(T) := W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\Gamma \times (0,T))^n,$$

$$\mathbf{h}^2 \in \mathbb{F}_{\mathbf{h}^2}(T) := W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\Gamma_s \times (0,T))^n.$$

Then two-phase Stokes equation (3.12) admits a unique strong solution $(\mathbf{v}, \pi) \in \mathbb{E}(T) := \mathbb{E}_{\mathbf{v}}(T) \times \mathbb{E}_\pi(T)$ where

$$\mathbb{E}_{\mathbf{v}}(T) := L^q(0,T; W^2_q(\Omega \setminus \Gamma)^n) \cap W^1_q(0,T; L^q(\Omega)^n),$$

$$\mathbb{E}_\pi(T) := \left\{ \begin{aligned} & \pi \in L^q(0,T; W^1_q(\Omega \setminus \Gamma)) : [\![\pi]\!] \in W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\Gamma \times (0,T)) \\ & \qquad \pi|_{\Gamma_s} \in W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\Gamma_s \times (0,T)). \end{aligned} \right\}$$

In addition, the initial value $\mathbf{v}^0 \in X_\gamma := W_q^{1-1/q}(\Omega)^n \cap W_q^{2(1-1/q)}(\Omega \setminus \Gamma)^n$ with compatibility conditions

$$\operatorname{div} \mathbf{v}^0 = g|_{t=0}, \quad [\![\mathbf{v}^0]\!]\big|_\Gamma = 0, \quad [\![\mathcal{P}_{\mathbf{n}_\Gamma}\left(\nu\left(\nabla\mathbf{v}^0 + (\nabla\mathbf{v}^0)^\top\right)\mathbf{n}_\Gamma\right)]\!]\big|_\Gamma = 0,$$
$$\mathcal{P}_{\mathbf{n}_{\Gamma_s}}\left(\nu\left(\nabla\mathbf{v}^0 + (\nabla\mathbf{v}^0)^\top\right)\mathbf{n}_{\Gamma_s}\right)\big|_{\Gamma_s} = 0.$$

Then the Stokes problem (3.12) admits a unique strong solution $(\mathbf{v}, \pi)$ in $\mathbb{E}(T)$. Moreover, there exist a time $T_0 > 0$ and a constant $C = C(T_0) > 0$ such that for $0 < T \le T_0$,

$$\|(\mathbf{v}, \pi)\|_{\mathbb{E}(T)} \le C \left\|(\mathbf{k}, g, \mathbf{h}^1, \mathbf{h}^2, \mathbf{v}^0)\right\|_{{}_0\mathbb{F}(T) \times X_\gamma}, \tag{3.13}$$

where ${}_0\mathbb{F}(T)$ *has vanishing initial trace and* $\mathbb{F}(T) := \mathbb{F}_{\mathbf{k}}(T) \times \mathbb{F}_g(T) \times \mathbb{F}_{\mathbf{h}^1}(T) \times \mathbb{F}_{\mathbf{h}^2}(T)$*, endowed with the norms*

$$\begin{aligned}
\|\mathbf{k}\|_{\mathbb{F}_{\mathbf{k}}(T)} &= \|\mathbf{k}\|_{L^q(0,T;L^q(\Omega\setminus\Gamma)^n)}, \\
\|g\|_{\mathbb{F}_g(T)} &= \|g\|_{L^q(0,T;W^1_q(\Omega\setminus\Gamma))} + \|g\|_{W^1_q(0,T;W^{-1}_q(\Omega))} \\
&\quad + \|\mathrm{tr}_\Gamma(g)\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma\times(0,T))} + \|\mathrm{tr}_{\Gamma_s}(g)\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma_s\times(0,T))}, \\
\|\mathbf{h}^1\|_{\mathbb{F}_{\mathbf{h}^1}(T)} &= \|\mathbf{h}\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma\times(0,T))^n}, \quad \|\mathbf{h}^2\|_{\mathbb{F}_{\mathbf{h}^2}(T)} = \|\mathbf{h}\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma_s\times(0,T))^n}.
\end{aligned}$$

Reductions. To simplify the proof of Theorem 3.12, we reduce (3.12) to the case $(\mathbf{k}, g, \mathbf{v}^0) = 0$.

First of all, we define $\bar{\mathbf{v}}$ as the solution of the parabolic transmission problem

$$\begin{aligned}
\rho_f \partial_t \bar{\mathbf{v}} - \operatorname{div} \mathbf{S}(\bar{\mathbf{v}}, 0) &= \mathbf{k} && \text{in } \Omega\setminus\Gamma\times(0,T), \\
[\![\hat{\mathbf{v}}]\!] &= 0 && \text{on } \Gamma\times(0,T), \\
[\![\mathbf{S}(\bar{\mathbf{v}}, 0)]\!]\,\mathbf{n}_\Gamma &= 0 && \text{on } \Gamma\times(0,T), \\
\mathbf{S}(\bar{\mathbf{v}}_s, 0)\mathbf{n}_{\Gamma_s} &= 0 && \text{on } \Gamma_s\times(0,T), \\
\bar{\mathbf{v}}|_{t=0} &= \mathbf{v}^0 && \text{in } \Omega\setminus\Gamma,
\end{aligned} \tag{3.14}$$

with $\mathbf{k} \in L^q(\Omega\setminus\Gamma\times(0,T))$ and $\mathbf{v}^0 \in X_\gamma$. Since the Lopatinskii–Shapiro conditions are satisfied, (3.14) is uniquely solvable in $W^{2,1}_q(\Omega\times(0,T))$, thanks to [PS16, Theorem 6.5.1].

Now, we are in the position to reduce g to zero. To this end, we introduce an elliptic transmission problem with Dirichlet boundary

$$\begin{aligned}
\Delta\phi = g - \operatorname{div}\bar{\mathbf{v}} &=: \tilde{g} && \text{in } \Omega\setminus\Gamma, \\
[\![\rho\phi]\!] &= 0 && \text{on } \Gamma, \\
[\![\nabla\phi]\!]\cdot\mathbf{n}_\Gamma &= 0 && \text{on } \Gamma, \\
\rho_s\phi_s &= 0 && \text{on } \Gamma_s,
\end{aligned} \tag{3.15}$$

with $\tilde{g} \in L^q(\Omega\setminus\Gamma)$. Then (3.15) is uniquely solvable by Proposition 3.23. In addition, with the regularity of g and $\mathbf{v}$, the solution satisfies $\nabla\phi \in \mathbb{E}_{\mathbf{v}}(T)$. Employing the decomposition

$$(\mathbf{v}, \pi) = (\bar{\mathbf{v}} + \nabla\phi + \tilde{\mathbf{v}}, -\rho\partial_t\phi + \nu\Delta\phi + \tilde{\pi}), \tag{3.16}$$

we know that $(\tilde{\mathbf{v}}, \tilde{\pi})$ solves system (3.12) with $(\mathbf{k}, g, \mathbf{v}^0) = 0$ and modified nonvanishing data $(\mathbf{h}^1, \mathbf{h}^2)$ (not to be relabeled) in the right regularity classes having a vanishing trace at $t = 0$. Thus, we will focus on the reduced system in the case $(\mathbf{k}, g, \mathbf{v}^0) = 0$.

Remark 3.13. Because of the decomposition (3.16), the regularity of π given in $\mathbb{E}_\pi(T)$ indicates that $\partial_t\phi$ and $\Delta\phi$ must be contained in $\mathbb{E}_\pi(T)$. Since $\nabla\phi \in \mathbb{E}_{\mathbf{v}}(T) = L^q(0,T;W^2_q(\Omega\setminus\Gamma)^n \cap W^1_q(\Omega)^n) \cap W^1_q(0,T;L^q(\Omega)^n)$, it is clear that $\partial_t\phi, \Delta\phi \in L^q(0,T;W^1_q(\Omega\setminus\Gamma))$. Moreover:

i) The vanishing Dirichlet boundary conditions of ϕ on Γ and Γ_s lead to $[\![\partial_t\phi]\!]|_\Gamma = \partial_t\phi|_{\Gamma_s} = 0$, which naturally satisfy the boundary regularity

$$W_q^{1-1/q,(1-1/q)/2}(\Gamma\times(0,T)), \text{ and } W_q^{1-1/q,(1-1/q)/2}(\Gamma_s\times(0,T)).$$

Hence $\partial_t\phi \in \mathbb{E}_\pi(T)$.

ii) For $\Delta\phi = \tilde{g} = g - \operatorname{div}\bar{\mathbf{v}}$, the boundary regularity for $\operatorname{div}\bar{\mathbf{v}}$ is not a problem due to the zero Neumann boundary of $\bar{\mathbf{v}}$. Thus, to ensure the validation of the regularity for $\hat{\pi}$, we add trace regularities on Γ and Γ_s for g in $\mathbb{F}_g(T)$. Namely,

$$\operatorname{tr}_\Gamma(g) \in W_q^{1-1/q,(1-1/q)/2}(\Gamma \times (0,T)), \quad \operatorname{tr}_{\Gamma_s}(g) \in W_q^{1-1/q,(1-1/q)/2}(\Gamma_s \times (0,T)).$$

Consequently, $\Delta\phi \in \mathbb{E}_\pi(T)$.

Proof of Theorem 3.12. As stated in the last section, we analyze the reduced system of (3.12) with $(\mathbf{k}, g, \mathbf{v}^0) = 0$. Due to the outer Neumann boundary condition, the proof is proceeded by a truncation (localization) argument, based on the results given in Appendix 3.5. More precisely, with a suitable cutoff function, we decompose the system into a two-phase Stokes problem with Dirichlet boundary conditions and a one-phase nonstationary Stokes problem, which are uniquely solvable as in Section 3.5.1 and Abels [Abe10, Theorem 1.1] respectively.

Proof of Theorem 3.12. ***Step 1.*** The first step is finding $(\mathbf{v}^1, \pi^1)$ to solve

$$\begin{aligned}
\rho\partial_t\mathbf{v}^1 - \operatorname{div}\mathbf{S}(\mathbf{v}^1,\pi^1) &= 0 && \text{in } \Omega\setminus\Gamma\times(0,T),\\
\operatorname{div}\mathbf{v}^1 &= 0 && \text{in } \Omega\setminus\Gamma\times(0,T),\\
[\![\mathbf{v}^1]\!] &= 0 && \text{on } \Gamma\times(0,T),\\
[\![\mathbf{S}(\mathbf{v}^1,\pi^1)]\!]\,\mathbf{n}_\Gamma &= \mathbf{h}^1 && \text{on } \Gamma\times(0,T),\\
\mathbf{v}^1 &= 0 && \text{on } \Gamma_s\times(0,T),\\
\mathbf{v}^1\big|_{t=0} &= 0 && \text{in } \Omega\setminus\Gamma,
\end{aligned} \tag{3.17}$$

where $\mathbf{h}^1 \in \mathbb{F}_{\mathbf{h}^1}(T)$ with $\mathbf{h}^1\big|_{t=0} = 0$. Since $\mathbf{v}^1\big|_{t=0} = 0$, the compatibility conditions (3.40) hold true and then (3.17) admits a unique solution $(\mathbf{v}^1,\pi^1) \in \mathbb{E}(T)$, thanks to Proposition 3.39. In addition, we have the estimate

$$\left\|(\mathbf{v}^1,\pi^1)\right\|_{\mathbb{E}(T)} \le C\left\|\mathbf{h}^1\right\|_{\mathbb{F}_{\mathbf{h}^1}(T)}, \tag{3.18}$$

for some $C > 0$ independent of $\mathbf{v}^1, \pi^1, \mathbf{h}^1$.
Step 2. Now, we construct $(\mathbf{v}_s^2, \pi_s^2)$ to solve the Stokes problem with Neumann boundary condition, which reads

$$\begin{aligned}
\rho_s\partial_t\mathbf{v}_s^2 - \operatorname{div}\mathbf{S}(\mathbf{v}_s^2,\pi_s^2) &= 0 && \text{in } \Omega_s\times(0,T),\\
\operatorname{div}\mathbf{v}_s^2 &= 0 && \text{in } \Omega_s\times(0,T),\\
\mathbf{S}(\mathbf{v}_s^2,\pi_s^2)\mathbf{n}_\Gamma &= 0 && \text{on } \Gamma\times(0,T),\\
\mathbf{S}(\mathbf{v}_s^2,\pi_s^2)\mathbf{n}_{\Gamma_s} &= \mathbf{h}^2 && \text{on } \Gamma_s\times(0,T),\\
\mathbf{v}^2\big|_{t=0} &= 0 && \text{in } \Omega_s,
\end{aligned} \tag{3.19}$$

where $\mathbf{h}^2 \in \mathbb{F}_{\mathbf{h}^2}(T)$ with $\mathbf{h}^2\big|_{t=0} = 0$. Thanks to Theorem 1.1 in Abels [Abe10] with $\Gamma_1 = \emptyset$, (3.19) admits a unique solution $(\mathbf{v}^2,\pi^2)$ in $W_q^{2,1}(\Omega\setminus\Gamma)\times L^q(0,T;W_q^1(\Omega\setminus\Gamma))$. Due to $\mathbf{v}_s^2\big|_{t=0} = 0$, all the compatibility conditions are satisfied. Moreover,

$$\left\|(\mathbf{v}^2,\pi^2)\right\|_{W_q^{2,1}(\Omega\setminus\Gamma)\times L^q(0,T;W_q^1(\Omega\setminus\Gamma))} \le C\left\|\mathbf{h}^2\right\|_{\mathbb{F}_{\mathbf{h}^2}(T)}, \tag{3.20}$$

for some $C > 0$ independent of $\mathbf{v}^2, \pi^2, \mathbf{h}^2$.
Step 3. Finally, we combine the regularity results above by truncation. Specifically, let $\psi \in C_0^\infty(\Omega)$ be a cutoff function over Ω such that

$$\psi(x) = \begin{cases} 1, & \text{in a neighborhood of } \Omega_f, \\ 0, & \text{in a neighborhood of } \Gamma_s. \end{cases} \tag{3.21}$$

We define

$$\tilde{\mathbf{v}} := \psi \mathbf{v}^1 + (1-\psi)\mathbf{v}^2, \quad \tilde{\pi} := \psi \pi^1 + (1-\psi)\pi^2.$$

Then $(\tilde{\mathbf{v}}, \tilde{\pi}) \in \mathbb{E}(T)$ solves

$$\begin{aligned} \rho \partial_t \tilde{\mathbf{v}} - \operatorname{div} \mathbf{S}(\tilde{\mathbf{v}}, \tilde{\pi}) &= \mathbf{R}^1 && \text{in } \Omega \setminus \Gamma \times (0,T), \\ \operatorname{div} \tilde{\mathbf{v}} &= R^2 && \text{in } \Omega \setminus \Gamma \times (0,T), \\ [\![\tilde{\mathbf{v}}]\!] &= 0 && \text{on } \Gamma \times (0,T), \\ [\![\mathbf{S}(\tilde{\mathbf{v}}, \tilde{\pi})]\!] \mathbf{n}_\Gamma &= \mathbf{h}^1 && \text{on } \Gamma \times (0,T), \\ \mathbf{S}(\tilde{\mathbf{v}}_s, \tilde{\pi}_s)\mathbf{n}_{\Gamma_s} &= \mathbf{h}^2 && \text{on } \Gamma_s \times (0,T), \\ \tilde{\mathbf{v}}|_{t=0} &= 0 && \text{in } \Omega \setminus \Gamma, \end{aligned} \tag{3.22}$$

where $\mathbf{R}^1$ and R^2 vanish in Ω_f, while in Ω_s,

$$\begin{aligned} \mathbf{R}^1 = &- \mathbf{S}(\mathbf{v}_s^1 - \mathbf{v}_s^2, \pi_s^1 - \pi_s^2)\nabla\psi \\ &\qquad - 2\nu_s \left(\Delta\psi \left(\mathbf{v}_s^1 - \mathbf{v}_s^2\right) + \left(\nabla\mathbf{v}_s^1 - \nabla\mathbf{v}_s^2\right)\nabla\psi + \nabla^2\psi\left(\mathbf{v}_s^1 - \mathbf{v}_s^2\right)\right), \\ R^2 = &\nabla\psi \cdot \left(\mathbf{v}_s^1 - \mathbf{v}_s^2\right). \end{aligned}$$

Since the embedding

$${}_0W_q^{2,1}\left(\Omega_s \times (0,T)\right) \hookrightarrow {}_0W_q^{\frac{1}{2}}\left(0,T; W_q^1(\Omega_s)\right)$$

holds, we know $\hat{\mathbf{v}}^i \in {}_0W_q^{\frac{1}{2}}(0,T; W_q^1(\Omega_s))$, $i = 1, 2$. For the reduced system, Proposition 8.2.1 and 7.3.5 in Prüss–Simonett [PS16] imply that π^1 and π_s^2 enjoys extra time regularities $\pi^1 \in {}_0W_q^\alpha(0,T; L^q(\Omega))$ and $\pi_s^2 \in {}_0W_q^\alpha(0,T; L^q(\Omega_s))$ respectively for $0 < \alpha < \frac{1}{2}(1 - \frac{1}{q})$. Hence

$$\mathbf{R}^1 \in {}_0W_q^\alpha(0,T; L^q(\Omega_s)) \cap L^q(0,T; W_q^1(\Omega_s)),$$

for some fixed $0 < \alpha < \frac{1}{2}(1 - \frac{1}{q})$.

To complete the proof, we still need to prove that the right-hand side terms of (3.22) can be in fact substituted by the right-hand side terms of (3.12) in appropriate spaces. Since the regularity of $\hat{\mathbf{v}}_s^i$ and $\hat{\pi}_s^i$, $i = 1, 2$, are not enough to control $\mathbf{R}^1$ and R^2 for small times, we are going to remove the inhomogeneities $\mathbf{R}^1$ and R^2. For $\mathbf{R}^1$, we construct a $\bar{\phi}$ solving the problem

$$\begin{aligned} \bar{\phi}_f &= 0 && \text{in } \Omega_f, \\ \Delta\bar{\phi}_s &= \operatorname{div} \mathbf{R}^1 && \text{in } \Omega_s, \\ \bar{\phi}_s &= 0 && \text{on } \Gamma, \\ \bar{\phi}_s &= 0 && \text{on } \Gamma_s. \end{aligned} \tag{3.23}$$

Then we obtain $\nabla\bar{\phi}|_{t=0} = \mathbf{R}^1|_{t=0} = 0$. By elliptic theory and regularity of $\mathbf{R}^1$, (3.23) admits a unique solution $\bar{\phi}$ satisfying ${}_0W_q^\alpha(0,T; W_q^1(\Omega_s)) \cap L^q(0,T; W_q^2(\Omega_s))$. For R^2, we find a ϕ solving the elliptic transmission problem

$$\begin{aligned} \Delta\phi_f &= 0 && \text{in } \Omega_f, \\ \Delta\phi_s &= R^2 && \text{in } \Omega_s, \\ [\![\rho\phi]\!] &= 0 && \text{on } \Gamma, \\ [\![\nabla\phi]\!]\cdot \mathbf{n}_\Gamma &= 0 && \text{on } \Gamma, \\ \rho_s\phi_s &= 0 && \text{on } \Gamma_s. \end{aligned} \tag{3.24}$$

Then we have $\phi|_{t=0} = 0$. Since $\mathbf{v}_s^1 - \mathbf{v}_s^2 \in {}_0W_q^{2,1}(\Omega_s \times (0,T))^n$, $R^2 \in {}_0W_q^{2,1}(\Omega_s \times (0,T)) \hookrightarrow {}_0W_q^{1/2}(0,T;W_q^1(\Omega_s))$. Together with Proposition 3.23, one concludes that (3.24) admits a solution such that $\nabla\phi$ is unique, with regularity

$$\nabla\phi \in \mathbb{E}_0 \coloneqq {}_0W_q^1(0,T;W_q^1(\Omega\setminus\Gamma)^n) \cap {}_0W_q^{\frac14}(0,T;W_q^2(\Omega\setminus\Gamma)^n).$$

For its traces on Γ and Γ_s, we have

$$\begin{aligned} [\![\nabla\phi]\!] \in \mathbb{E}_1 &\coloneqq {}_0W_q^1(0,T;W_q^{1-\frac1q}(\Gamma)^n) \cap {}_0W_q^{\frac14}(0,T;W_q^{2-\frac1q}(\Gamma)^n), \\ \nabla\phi_s \in \mathbb{E}_1^s &\coloneqq {}_0W_q^1(0,T;W_q^{1-\frac1q}(\Gamma_s)^n) \cap {}_0W_q^{\frac14}(0,T;W_q^{2-\frac1q}(\Gamma_s)^n). \end{aligned}$$

Besides,

$$\begin{aligned} [\![\nu\nabla^2\phi]\!] \in \mathbb{E}_2 &\coloneqq {}_0W_q^{1-\frac{1}{2q}}(0,T;L^q(\Gamma)^{n\times n}) \cap {}_0W_q^{\frac14}(0,T;W_q^{1-\frac1q}(\Gamma)^{n\times n}), \\ \nu_s\nabla^2\phi_s \in \mathbb{E}_2^s &\coloneqq {}_0W_q^{1-\frac{1}{2q}}(0,T;L^q(\Gamma_s)^{n\times n}) \cap {}_0W_q^{\frac14}(0,T;W_q^{1-\frac1q}(\Gamma_s)^{n\times n}). \end{aligned}$$

Moreover, the following estimate holds for a constant C, independent of $0 < T < T_0$,

$$\begin{aligned} &\|\nabla\phi\|_{\mathbb{E}_0} + \|[\![\nabla\phi]\!]\|_{\mathbb{E}_1} + \|\nabla\phi_s\|_{\mathbb{E}_1^s} \\ &\qquad + \left\|[\![\nu\nabla^2\phi]\!]\right\|_{\mathbb{E}_2} + \left\|\nu_s\nabla^2\phi\right\|_{\mathbb{E}_2^s} \le C\left\|\mathbf{v}_s^1 - \mathbf{v}_s^2\right\|_{W_q^{2,1}(\Omega_s\times(0,T))^n}. \end{aligned}$$

Finally, define

$$\mathbf{v}^\sharp \coloneqq \tilde{\mathbf{v}} - \nabla\phi, \quad \pi^\sharp \coloneqq \tilde\pi + \rho\partial_t\phi - \bar\phi - 2\nu\Delta\phi.$$

Since $[\![\rho\phi]\!]\,|_\Gamma, \rho_s\phi|_{\Gamma_s} = 0$, we have $[\![\rho\partial_t\phi]\!]\,|_\Gamma, \rho_s\partial_t\phi|_{\Gamma_s} = 0$. Then $(\mathbf{v}^\sharp, \pi^\sharp)$ solves

$$\begin{aligned} \rho\partial_t\mathbf{v}^\sharp - \operatorname{div}\mathbf{S}(\mathbf{v}^\sharp,\pi^\sharp) &= \mathbf{R}^1 - \nabla\bar\phi =: \mathbf{R}^0 && \text{in } \Omega\setminus\Gamma\times(0,T), \\ \operatorname{div}\mathbf{v}^\sharp &= 0 && \text{in } \Omega\setminus\Gamma\times(0,T), \\ [\![\mathbf{v}^\sharp]\!] &= \mathbf{R}' && \text{on } \Gamma\times(0,T), \\ [\![\mathbf{S}(\mathbf{v}^\sharp,\pi^\sharp)]\!]\,\mathbf{n}_\Gamma &= \mathbf{h}^1 + \mathbf{R}^3 && \text{on } \Gamma\times(0,T), \\ \mathbf{S}(\mathbf{v}_s^\sharp,\pi_s^\sharp)\mathbf{n}_{\Gamma_s} &= \mathbf{h}^2 + \mathbf{R}^4 && \text{on } \Gamma_s\times(0,T), \\ \mathbf{v}^\sharp\big|_{t=0} &= 0 && \text{in } \Omega\setminus\Gamma, \end{aligned} \tag{3.25}$$

where

$$\begin{gathered} \operatorname{div}\mathbf{R}^0 = 0, \quad \mathbf{R}' = -[\![\nabla\phi]\!], \\ \mathbf{R}^3 = [\![2\nu\nabla^2\phi]\!]\,\mathbf{n}_\Gamma - [\![2\nu_s\Delta\phi]\!]\,\mathbf{n}_\Gamma, \quad \mathbf{R}^4 = 2\nu_s\nabla^2\phi_s\mathbf{n}_{\Gamma_s} - 2\nu_s\Delta\phi_s\mathbf{n}_{\Gamma_s}. \end{gathered}$$

$\mathbf{R}^0$ can be seen as a Helmholtz projection of $\mathbf{R}^1$ and

$$\mathbf{R}^0 \in {}_0W_q^{\alpha}(0,T;L^q(\Omega_s)^n) \cap L^q(0,T;W_q^{2\alpha}(\Omega_s)^n), \quad \text{for all } 0<\alpha<\frac{1}{2}-\frac{1}{2q}.$$

By Lemma 2.23,

$$\mathbf{R}^0 \in C([0,T];W_q^{2\alpha-\frac{2}{q}}(\Omega_s)^n) \hookrightarrow C([0,T];L^q(\Omega_s)^n)$$

holds for $\frac{1}{q}<\alpha<\frac{1}{2}-\frac{1}{2q}$. Hence, for $\mathbf{R}^0|_{t=0} = (\nabla\bar{\phi}-\mathbf{R}^1)|_{t=0}=0$,

$$\begin{aligned}\left\|\mathbf{R}^0\right\|_{\mathbb{F}_{\mathbf{k}}(T)} &\le CT^{\frac{1}{q}}\left\|\mathbf{R}^0\right\|_{C([0,T];L^q(\Omega_s)^n)} \le CT^{\frac{1}{q}}\left\|\mathbf{R}^0\right\|_{{}_0W_q^{\alpha}(0,T;L^q(\Omega_s)^n)\cap L^q(0,T;W_q^{2\alpha}(\Omega_s)^n)}\\ &\le CT^{\frac{1}{q}}\left(\max_{i=1,2}\left\|(\mathbf{v}^i,\pi^i)\right\|_{\mathbb{E}(T)}\right) \le CT^{\frac{1}{q}}\left\|(\mathbf{k},g,\mathbf{h}^1,\mathbf{h}^2,\mathbf{v}^0)\right\|_{\mathbb{F}(T)\times X_\gamma},\end{aligned}$$

for $0<T<T_0$. According to Appendix 3.5, the regularity space of $\mathbf{R}'$ is defined as $\mathbb{F}'(T) := W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\Gamma\times(0,T))$. Then with Lemma 2.20 and $W_q^s(0,T;X)\hookrightarrow C([0,T];X)$ for $sq>1$,

$$\begin{aligned}\|\mathbf{R}'\|_{\mathbb{F}'(T)} &\le C\Bigg(\|[\![\nabla\phi]\!]\|_{L^q(0,T;W_q^{2-\frac{1}{q}}(\Gamma))^n}\\ &\qquad + \|[\![\nabla\phi]\!]\|_{L^q(0,T;L^q(\Gamma))^n} + [[\![\nabla\phi]\!]]_{W_q^{1-\frac{1}{2q}}(0,T;L^q(\Gamma))^n}\Bigg)\\ &\le CT^{\frac{1}{q}}\|[\![\nabla\phi]\!]\|_{{}_0W_q^{\frac{1}{4}}(0,T;W_q^{2-\frac{1}{q}}(\Gamma))^n} + CT^{\frac{1}{2q}}[[\![\nabla\phi]\!]]_{{}_0W_q^{1}(0,T;W_q^{1-\frac{1}{q}}(\Gamma))^n}\\ &\le CT^{\frac{1}{2q}}\left(\max_{i=1,2}\left\|(\mathbf{v}^i,\pi^i)\right\|_{\mathbb{E}(T)}\right) \le CT^{\frac{1}{2q}}\left\|(\mathbf{k},g,\mathbf{h}^1,\mathbf{h}^2,\mathbf{v}^0)\right\|_{\mathbb{F}(T)\times X_\gamma}.\end{aligned}$$

Since Γ and Γ_s are of class C^3, $\mathbf{n}_\Gamma$ and $\mathbf{n}_{\Gamma_s}$ are contained in C^2. Then we obtain

$$\begin{aligned}\left\|\mathbf{R}^3\right\|_{\mathbb{F}_{\mathbf{h}^1}(T)} &\le C\Bigg(\left\|[\![\nabla^2\phi]\!]\right\|_{L^q(0,T;W_q^{1-\frac{1}{q}}(\Gamma)^{n\times n})}\\ &\qquad + \left\|[\![\nabla^2\phi]\!]\right\|_{L^q(0,T;L^q(\Gamma)^{n\times n})} + \left[[\![\nabla^2\phi]\!]\right]_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}(0,T;L^q(\Gamma)^{n\times n})}\Bigg)\\ &\le CT^{\frac{1}{q}}\left\|[\![\nabla^2\phi]\!]\right\|_{{}_0W_q^{\frac{1}{4}}(0,T;W_q^{1-\frac{1}{q}}(\Gamma)^{n\times n})} + CT^{\frac{1}{2}}\left[[\![\nabla^2\phi]\!]\right]_{{}_0W_q^{1-\frac{1}{2q}}(0,T;L^q(\Gamma)^{n\times n})}\\ &\le CT^{\frac{1}{q}}\left(\max_{i=1,2}\left\|(\mathbf{v}^i,\pi^i)\right\|_{\mathbb{E}(T)}\right) \le CT^{\frac{1}{q}}\left\|(\mathbf{k},g,\mathbf{h}^1,\mathbf{h}^2,\mathbf{v}^0)\right\|_{\mathbb{F}(T)\times X_\gamma},\end{aligned}$$

with the help of Lemma 2.20. Similarly,

$$\left\|\mathbf{R}^4\right\|_{\mathbb{F}_{\mathbf{h}^2}(T)} \le CT^{\frac{1}{q}}\left(\max_{i=1,2}\left\|(\mathbf{v}^i,\pi^i)\right\|_{\mathbb{E}(T)}\right) \le CT^{\frac{1}{q}}\left\|(\mathbf{k},g,\mathbf{h}^1,\mathbf{h}^2,\mathbf{v}^0)\right\|_{\mathbb{F}(T)\times X_\gamma}.$$

Taking T_0 sufficiently small such that $CT_0^{\frac{1}{2q}} \le \frac{1}{2}$, we have

$$\left\|\mathbf{R}^0(y)\right\|_{\mathbb{F}_{\mathbf{k}}(T)} + \|\mathbf{R}'(y)\|_{\mathbb{F}'(T)} + \left\|\mathbf{R}^3(y)\right\|_{\mathbb{F}_{\mathbf{h}^1}(T)} + \left\|\mathbf{R}^4(y)\right\|_{\mathbb{F}_{\mathbf{h}^2}(T)} \le \frac{1}{2}\|y\|_{\mathbb{F}(T)\times X_\gamma},$$

for $y=(\mathbf{k},g,\mathbf{h}^1,\mathbf{h}^2,\mathbf{v}^0)^\top$. By a Neumann series argument,

$$\Phi : \tilde{y} \mapsto \tilde{y} + (\mathbf{R}^0,0,\mathbf{R}',\mathbf{R}^3,\mathbf{R}^4,0)^\top(\tilde{y})$$

is invertible for $\tilde{y} = (\mathbf{k}, g, 0, \mathbf{h}^1, \mathbf{h}^2, \mathbf{v}^0)^\top$. Consequently, replacing $\tilde{y}$ by $\Phi^{-1}(\tilde{y})$ in (3.22) yields the solvability of (3.25) for $0 < T < T_0 \leq 1/(2C)^{2q}$. Solving (3.12) iteratively on $[0, T_0]$, $[T_0, 2T_0]$, $\dots$, with initial values $\mathbf{v}^0$, $\mathbf{v}^0|_{t=T_0}$, $\dots$, one obtains the solvability for any $T_0 > 0$. Additionally, estimate (3.13) is a result of (3.18) and (3.20). This completes the proof. $\square$

Remark 3.14. For $c \in L^q(0, T; W^2_q(\Omega\backslash\Gamma)) \cap W^1_q(0, T; L^q(\Omega))$, one can easily verify that $c \in \mathbb{F}_g(T)$. Hence, we replace g in (3.12) by $g + \frac{\gamma\beta}{\rho_s} c_s$ with the same existence and regularity results to the original linear system. To be more precise, we find $(\bar{\mathbf{v}}, \bar{\pi}) \in \mathbb{E}(T)$ to solve

$$\begin{aligned} \rho\partial_t\bar{\mathbf{v}} - \operatorname{div} \mathbf{S}(\bar{\mathbf{v}}, \bar{\pi}) &= 0 && \text{in } \Omega \setminus \Gamma \times (0, T), \\ \operatorname{div} \bar{\mathbf{v}} &= \frac{\gamma\beta}{\rho_s} c_s && \text{in } \Omega \setminus \Gamma \times (0, T), \\ [\![\bar{\mathbf{v}}]\!] &= 0 && \text{on } \Gamma \times (0, T), \\ [\![\mathbf{S}(\bar{\mathbf{v}}, \bar{\pi})]\!] \cdot \mathbf{n}_\Gamma &= 0 && \text{on } \Gamma \times (0, T), \\ \mathbf{S}(\bar{\mathbf{v}}_s, \bar{\pi}_s) \cdot \mathbf{n}_{\Gamma_s} &= 0 && \text{on } \Gamma_s \times (0, T), \\ \bar{\mathbf{v}}|_{t=0} &= 0 && \text{in } \Omega \setminus \Gamma, \end{aligned}$$

with $c \in \mathbb{F}_g(T)$, thanks to Theorem 3.12. Then $(\mathbf{v} + \bar{\mathbf{v}}, \pi + \bar{\pi})$ solves the original linear system of (3.9a)–(3.9e).

3.3.2. Parabolic equations with Neumann boundary conditions. Thanks to the general maximal regularity theory for parabolic problems, for example, Prüss–Simonett [PS16, Section 6.3], we obtain the solvability of parabolic systems with Neumann boundary conditions. Let $T > 0$, $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega$ of class C^{3-}. ν denotes the unit outer normal vectors on $\partial\Omega$. Consider the problem

$$\begin{aligned} \partial_t u - D\Delta u &= f, && \text{in } \Omega \times (0, T), \\ D\nabla u \cdot \nu &= g, && \text{on } \partial\Omega \times (0, T), \\ u|_{t=0} &= u_0, && \text{in } \Omega, \end{aligned} \tag{3.26}$$

where $D > 0$ is a constant. $u : \overline{\Omega} \times (0, T) \to \mathbb{R}$ stands for the system unknown, for example, the temperature or the concentration.

Theorem 3.15. *Let $3 < q < \infty$ and $T_0 > 0$. Assume that $u_0 \in W^{2-2/q}_q(\Omega)$ with the compatibility condition $D\nabla u_0|_{\partial\Omega} = g|_{t=0}$ holds. Given known functions (f, g) with regularity*

$$\begin{aligned} f &\in \mathbb{F}_f(T) := L^q(0, T; L^q(\Omega)), \\ g &\in \mathbb{F}_g(T) := W^{\frac{1}{2} - \frac{1}{2q}}_q(0, T; L^q(\partial\Omega)) \cap L^q(0, T; W^{1-\frac{1}{q}}_q(\partial\Omega)). \end{aligned}$$

Then the parabolic equation (3.26) admits a unique strong solution $u \in \mathbb{E}(T)$ where

$$\mathbb{E}(T) := L^q(0, T; W^2_q(\Omega)) \cap W^1_q(0, T; L^q(\Omega)).$$

Moreover, there is a constant $C > 0$ independent of f, g, u_0, T_0, such that for $0 < T < T_0$

$$\|u\|_{\mathbb{E}(T)} \leq C \left(\|f\|_{\mathbb{F}_f(T)} + \|g\|_{\mathbb{F}_g(T)} + \|u_0\|_{W^{2-2/q}_q(\Omega)} \right). \tag{3.27}$$

Proof. This theorem can be easily shown by means of Prüss–Simonett [PS16, Theorem 6.3.2], for which we need to extend the right-hand sides just as in the proof of Proposition 3.21 and construct a solution solving (6.45) in [PS16]. This can be done since we established general extension theorems in Section 2.1.6. $\square$

3.3.3. Ordinary differential equations for foam cells and growth. Let Ω be the domain defined in Section 3.3.2. Given a function $f \in \mathbb{F}(T) := L^q(0,T;W_q^1(\Omega))$, a constant $\gamma > 0$ and a function $u_0 \in W_q^1(\Omega)$, $u_0 \geq 0$, by ordinary differential equation theory,

$$\begin{aligned} \partial_t u - \gamma w &= f, \quad \text{in } \Omega \times (0,T), \\ u|_{t=0} &= u_0, \quad \text{in } \Omega. \end{aligned} \tag{3.28}$$

admits a unique solution

$$u \in \mathbb{E}(T) := W_q^1(0,T;W_q^1(\Omega)),$$

provided $w \in L^q(0,T;W_q^2(\Omega)) \cap W_q^1(0,T;L^q(\Omega))$. Moreover, for every $T_0 > 0$, there exists a constant $C > 0$ independent of f, u_0, T_0, such that for $0 < T < T_0$

$$\|u\|_{\mathbb{E}(T)} \leq C \left(\|f\|_{\mathbb{F}(T)} + \|w\|_{L^q(0,T;W_q^2(\Omega)) \cap W_q^1(0,T;L^q(\Omega))} + \|u_0\|_{W_q^1(\Omega)} \right). \tag{3.29}$$

3.4. Local in Time Existence

This section is intended to prove Theorem 3.8.

3.4.1. Some key estimates. Before showing Theorem 3.8, let us give some useful estimates with regard to the deformation gradient $\hat{\mathbf{F}}^{-1}$ and the Slobodeckij space $W_q^{1/2-\varepsilon}(0,T;L^q(\Omega))$.

LEMMA 3.16 (Estimates on deformation gradient). *Let $q > n$, $n \geq 2$ and $\hat{\mathbf{F}}(\hat{\mathbf{v}})$ be the deformation gradient defined in (3.2) corresponding to a function $\hat{\mathbf{v}} \in Y_T^1$. Then for every $R > 0$, there are a constant $C = C(R) > 0$ and a finite time $0 < T_R < 1$ depending on R such that for all $0 < T < T_R$, $\hat{\mathbf{F}}^{-1}$ exists and*

(1) $\left\|\hat{\mathbf{F}}^{-1}\right\|_{L^\infty(0,T;W_q^1(\widetilde{\Omega})^{n\times n})} \leq C, \quad \left\|\partial_t \hat{\mathbf{F}}^{-1}\right\|_{L^q(0,T;W_q^1(\widetilde{\Omega})^{n\times n})} \leq C \|\hat{\mathbf{v}}\|_{Y_T^1}$;

(2) $\left\|\hat{\mathbf{F}}^{-1} - \mathbb{I}\right\|_{L^\infty(0,T;W_q^1(\widetilde{\Omega})^{n\times n})} \leq CT^{\frac{1}{q'}} \|\hat{\mathbf{v}}\|_{Y_T^1}$;

(3) $\sup\limits_{0\leq t\leq T} \left(\int_0^t \frac{\left\|\Delta_h \left(\hat{\mathbf{F}}^{-1} - \mathbb{I}\right)(\cdot,t)\right\|^q_{W_q^1(\widetilde{\Omega})^{n\times n}}}{h^{1+\frac{q}{2q'}}} \mathrm{d}h \right)^{\frac{1}{q}} \leq CT^{\frac{1}{2q'}} \|\hat{\mathbf{v}}\|_{Y_T^1}$;

(4) $\left[\hat{\mathbf{F}}^{-1} - \mathbb{I}\right]_{W_q^{\frac{1}{2}(1-\frac{1}{q})}(0,T;W_q^1(\widetilde{\Omega})^{n\times n})} \leq CT^{\frac{1}{q}+\frac{1}{2q'}} \|\hat{\mathbf{v}}\|_{Y_T^1}$,

for all $\|\hat{\mathbf{v}}\|_{Y_T^1} \leq R$, where $\Delta_h f(t) := f(t) - f(t-h)$ is a difference of the time shift for a function f. Moreover, for another $\hat{\mathbf{u}} \in Y_T^1$ with $\|\hat{\mathbf{u}}\|_{Y_T^1} \leq R$ and $\hat{\mathbf{v}}|_{t=0} = \hat{\mathbf{u}}|_{t=0}$, we have

(5) $\left\|\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{L^\infty(0,T;W_q^1(\widetilde{\Omega})^{n\times n})} \leq CT^{\frac{1}{q'}} \|\hat{\mathbf{u}} - \hat{\mathbf{v}}\|_{Y_T^1}$;
$\left\|\partial_t \hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}}) - \partial_t \hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{L^q(0,T;L^\infty(\widetilde{\Omega})^{n\times n})} \leq CT^{\frac{1}{q}-\frac{1}{r}} \|\hat{\mathbf{u}} - \hat{\mathbf{v}}\|_{Y_T^1}$;

(6) $\sup\limits_{0\leq t\leq T} \left(\int_0^t \frac{\left\|\Delta_h \left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right)(\cdot,t)\right\|^q_{W_q^1(\widetilde{\Omega})^{n\times n}}}{h^{1+\frac{q}{2q'}}} \mathrm{d}h \right)^{\frac{1}{q}} \leq CT^{\frac{1}{2q'}} \|\hat{\mathbf{u}} - \hat{\mathbf{v}}\|_{Y_T^1}$;

(7) $\left[\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right]_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}(0,T;W_q^1(\widetilde{\Omega})^{n\times n})} \leq CT^{\frac{1}{q}+\frac{1}{2q'}}\left\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\right\|_{Y_T^1},$

where $r=\frac{q^2}{n}$.

Proof. Recall from (3.2) the definition of $\hat{\mathbf{F}}$ that

$$\hat{\mathbf{F}}(X,t)=\mathbb{I}+\int_0^t \hat{\nabla}\hat{\mathbf{v}}(X,\tau)\mathrm{d}\tau, \quad \forall X\in\Omega.$$

Then we have

$$\sup_{0\leq t\leq T}\left\|\hat{\mathbf{F}}-\mathbb{I}\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}=\sup_{0\leq t\leq T}\left\|\int_0^t \hat{\nabla}\hat{\mathbf{v}}(X,\tau)\mathrm{d}\tau\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}\leq CT^{\frac{1}{q'}}R,$$

for all $\|\hat{\mathbf{v}}\|_{Y_T^1}\leq R$. Choosing $T_R>T$ so small that $CT_R^{\frac{1}{q'}}R\leq\frac{1}{2M_q}$, we know

$$\sup_{0\leq t\leq T}\left\|\hat{\mathbf{F}}-\mathbb{I}\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}\leq\frac{1}{2M_q},$$

where M_q is the constant of multiplication in $W_q^1(\widetilde{\Omega})$, see Proposition 2.18. According to the Neumann series (see [Alt16, Section 5.7]), $\hat{\mathbf{F}}^{-1}$ does exist and

$$\hat{\mathbf{F}}^{-1}=\left(\hat{\mathbf{F}}-\mathbb{I}+\mathbb{I}\right)^{-1}=\left(\mathbb{I}-\left(\mathbb{I}-\hat{\mathbf{F}}\right)\right)^{-1}=\sum_{k=0}^{\infty}\left(\mathbb{I}-\hat{\mathbf{F}}\right)^k.$$

Then from Proposition 2.18, one obtains

$$\begin{aligned}\sup_{0\leq t\leq T}\left\|\hat{\mathbf{F}}^{-1}\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}&\leq\sup_{0\leq t\leq T}\sum_{k=0}^{\infty}\left\|\left(\mathbb{I}-\hat{\mathbf{F}}\right)^k\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}\\&\leq\frac{1}{M_q}\sum_{k=0}^{\infty}\left(M_q\sup_{0\leq t\leq T}\left\|\mathbb{I}-\hat{\mathbf{F}}\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}\right)^k\leq\frac{1}{M_q}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k=\frac{2}{M_q},\end{aligned}$$

Consequently, it follows from (1.62) and Proposition 2.18 that

$$\begin{aligned}&\left\|\partial_t\hat{\mathbf{F}}^{-1}\right\|_{L^q(0,T;W_q^1(\widetilde{\Omega})^{n\times n})}\\&\quad\leq M_q^2\left\|\hat{\mathbf{F}}^{-1}\right\|^2_{L^\infty(0,T;W_q^q(\widetilde{\Omega})^{n\times n})}\left\|\hat{\nabla}\hat{\mathbf{v}}\right\|_{L^q(0,T:W_q^1(\widetilde{\Omega})^{n\times n})}\leq C\left\|\hat{\mathbf{v}}\right\|_{Y_T^1},\end{aligned}$$

for all $0<T<T_R$ and

$$\left\|\hat{\mathbf{F}}^{-1}-\mathbb{I}\right\|_{L^\infty(0,T;W_q^1(\widetilde{\Omega})^{n\times n})}\leq\int_0^T\left\|\partial_t\hat{\mathbf{F}}^{-1}(\cdot,\tau)\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}\mathrm{d}\tau\leq CT^{\frac{1}{q'}}\left\|\hat{\mathbf{v}}\right\|_{Y_T^1},$$

where $C=C(R)$ depends on R. These estimates prove the first two statements.

For the third and fourth statements, we have

$$\left\|\Delta_h\left(\hat{\mathbf{F}}^{-1}-\mathbb{I}\right)(\cdot,t)\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}\leq\int_{t-h}^t\left\|\partial_t\hat{\mathbf{F}}^{-1}(\cdot,\tau)\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}\mathrm{d}\tau\leq Ch^{\frac{1}{q'}}\left\|\hat{\mathbf{v}}\right\|_{Y_T^1},$$

which can be used to deduce

$$\sup_{0\le t\le T}\left(\int_0^t \frac{\left\|\Delta_h\left(\hat{\mathbf{F}}^{-1}-\mathbb{I}\right)(\cdot,t)\right\|^q_{W^1_q(\widetilde{\Omega})^{n\times n}}}{h^{1+\frac{q}{2q'}}}\mathrm{d}h\right)^{\frac{1}{q}}$$

$$\le C\sup_{0\le t\le T}\left(\int_0^t h^{-1+\frac{q}{2q'}}\mathrm{d}h\right)^{\frac{1}{q}}\|\hat{\mathbf{v}}\|_{Y^1_T} = C2q'\sup_{0\le t\le T} t^{\frac{1}{2q'}}\|\hat{\mathbf{v}}\|_{Y^1_T}\le CT^{\frac{1}{2q'}}\|\hat{\mathbf{v}}\|_{Y^1_T},$$

and therefore from (2.1) and the definition of Sobolev–Slobodeckij space,

$$\left[\hat{\mathbf{F}}^{-1}-\mathbb{I}\right]_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}(0,T;W^1_q(\widetilde{\Omega})^{n\times n})}\le CT^{\frac{1}{q}+\frac{1}{2q'}}\|\hat{\mathbf{v}}\|_{Y^1_T}.$$

For the rest statements, we notice from (3.2) that

$$\hat{\mathbf{F}}(\hat{\mathbf{u}})-\hat{\mathbf{F}}(\hat{\mathbf{v}})=\int_0^t\left(\hat{\nabla}\hat{\mathbf{u}}-\hat{\nabla}\hat{\mathbf{v}}\right)(X,\tau)\mathrm{d}\tau.$$

Then for all $0<T<T_R$,

$$\sup_{0\le t\le T}\left\|\hat{\mathbf{F}}(\hat{\mathbf{u}})-\hat{\mathbf{F}}(\hat{\mathbf{v}})\right\|_{W^1_q(\widetilde{\Omega})^{n\times n}}\le CT^{\frac{1}{q'}}\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y^1_T}.$$

Since

$$\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})=-\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\left(\hat{\mathbf{F}}(\hat{\mathbf{u}})-\hat{\mathbf{F}}(\hat{\mathbf{v}})\right)\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}),$$

it follows from the multiplication property of $W^1_q(\widetilde{\Omega})$ again that for all $0<T<T_R$,

$$\begin{aligned}\sup_{0\le t\le T}&\left\|\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{W^1_q(\widetilde{\Omega})^{n\times n}}\\ &\le M_q^2\sup_{0\le t\le T}\left\|\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\right\|_{W^1_q(\widetilde{\Omega})^{n\times n}}\left\|\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{W^1_q(\widetilde{\Omega})^{n\times n}}\left\|\hat{\mathbf{F}}(\hat{\mathbf{u}})-\hat{\mathbf{F}}(\hat{\mathbf{v}})\right\|_{W^1_q(\widetilde{\Omega})^{n\times n}}\\ &\le CT^{\frac{1}{q'}}\|\hat{\mathbf{u}}-\hat{\mathbf{v}}\|_{Y^1_T}.\end{aligned}$$

Moreover, by (1.62)

$$\begin{aligned}\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})&=-\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\left(\hat{\mathbf{F}}(\hat{\mathbf{u}})-\hat{\mathbf{F}}(\hat{\mathbf{v}})\right)\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\\ &\quad-\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\partial_t\left(\hat{\mathbf{F}}(\hat{\mathbf{u}})-\hat{\mathbf{F}}(\hat{\mathbf{v}})\right)\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})-\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\left(\hat{\mathbf{F}}(\hat{\mathbf{u}})-\hat{\mathbf{F}}(\hat{\mathbf{v}})\right)\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}).\end{aligned}\tag{3.30}$$

Hence

$$\begin{aligned}&\left\|\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})-\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{L^q(0,T;L^\infty(\widetilde{\Omega})^{n\times n})}\\ &\quad\le\left\|\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\left(\hat{\mathbf{F}}(\hat{\mathbf{u}})-\hat{\mathbf{F}}(\hat{\mathbf{v}})\right)\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{L^q(0,T;L^\infty(\widetilde{\Omega})^{n\times n})}\\ &\qquad+\left\|\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\left(\hat{\nabla}\hat{\mathbf{u}}-\hat{\nabla}\hat{\mathbf{v}}\right)\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{L^q(0,T;L^\infty(\widetilde{\Omega})^{n\times n})}\\ &\qquad+\left\|\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\left(\hat{\mathbf{F}}(\hat{\mathbf{u}})-\hat{\mathbf{F}}(\hat{\mathbf{v}})\right)\partial_t\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right\|_{L^q(0,T;L^\infty(\widetilde{\Omega})^{n\times n})}=:F_1+F_2+F_3.\end{aligned}$$

From the embedding (2.4), we know that for $\hat{\mathbf{v}} \in Y_T^1$,

$$\sup_{0\le t\le T} \left\| \hat{\nabla}\hat{\mathbf{v}} \right\|_{L^q(\widetilde{\Omega})^{n\times n}} \le C \left(\|\hat{\mathbf{v}}\|_{Y_T^1} + \|\,\hat{\mathbf{v}}|_{t=0}\|_{W_q^1(\widetilde{\Omega})} \right).$$

The Gagliardo–Nirenberg inequality tells us

$$\left\| \hat{\nabla}\hat{\mathbf{v}} \right\|_{L^\infty(\widetilde{\Omega})^{n\times n}} \le C \left\| \hat{\nabla}\hat{\mathbf{v}} \right\|_{L^q(\widetilde{\Omega})^{n\times n}}^{1-\frac{n}{q}} \left\| \hat{\nabla}\hat{\mathbf{v}} \right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}^{\frac{n}{q}}.$$

For $r = \frac{q^2}{n} > q$, we obtain

$$\left\| \hat{\nabla}\hat{\mathbf{v}} \right\|_{L^r(0,T;L^\infty(\widetilde{\Omega})^{n\times n})} \le C \left\| \hat{\nabla}\hat{\mathbf{v}} \right\|_{L^\infty(0,T;L^q(\widetilde{\Omega})^{n\times n})}^{1-\frac{n}{q}} \left\| \hat{\nabla}\hat{\mathbf{v}} \right\|_{L^q(0,T;W_q^1(\widetilde{\Omega})^{n\times n})}^{\frac{n}{q}} \le C(R).$$

Then,

$$\left\| \hat{\nabla}\hat{\mathbf{v}} \right\|_{L^q(0,T;L^\infty(\widetilde{\Omega})^{n\times n})} \le T^{\frac{1}{q}-\frac{1}{r}} \left\| \hat{\nabla}\hat{\mathbf{v}} \right\|_{L^r(0,T;L^\infty(\widetilde{\Omega})^{n\times n})} \le C(R) T^{\frac{1}{q}-\frac{1}{r}},$$

and also, for $\hat{\mathbf{u}} \in Y_T^1$, $\|\hat{\mathbf{u}}\|_{Y_T^1} \le R$,

$$\left\| \hat{\nabla}\hat{\mathbf{v}} - \hat{\nabla}\hat{\mathbf{u}} \right\|_{L^q(0,T;L^\infty(\widetilde{\Omega})^{n\times n})} \le C(R) T^{\frac{1}{q}-\frac{1}{r}} \|\hat{\mathbf{v}} - \hat{\mathbf{u}}\|_{Y_T^1}. \tag{3.31}$$

Consequently, with $W_q^1(\widetilde{\Omega}) \hookrightarrow L^\infty(\widetilde{\Omega})$ for $q > n$,

$$\begin{aligned}
F_1 \le{}& \left\| \partial_t \hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}}) \right\|_{L^q(0,T;L^\infty(\widetilde{\Omega})^{n\times n})} \\
&\times \left\| \hat{\mathbf{F}}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}(\hat{\mathbf{v}}) \right\|_{L^\infty(0,T;L^\infty(\widetilde{\Omega})^{n\times n})} \left\| \hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}) \right\|_{L^\infty(0,T;L^\infty(\widetilde{\Omega})^{n\times n})} \\
\le{}& \left\| \hat{\nabla}\hat{\mathbf{u}} \right\|_{L^q(0,T;L^\infty(\widetilde{\Omega})^{n\times n})} \left\| \hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}}) \right\|_{L^\infty(0,T;L^\infty(\widetilde{\Omega})^{n\times n})}^2 \\
&\times \left\| \hat{\mathbf{F}}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}(\hat{\mathbf{v}}) \right\|_{L^\infty(0,T;L^\infty(\widetilde{\Omega})^{n\times n})} \left\| \hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}) \right\|_{L^\infty(0,T;L^\infty(\widetilde{\Omega})^{n\times n})} \\
\le{}& C T^{\frac{1}{q}-\frac{1}{r}} \|\hat{\mathbf{u}} - \hat{\mathbf{v}}\|_{Y_T^1}.
\end{aligned}$$

Similarly,

$$F_2 \le C T^{\frac{1}{q}-\frac{1}{r}} \|\hat{\mathbf{u}} - \hat{\mathbf{v}}\|_{Y_T^1}, \quad F_3 \le C T^{\frac{1}{q}-\frac{1}{r}} \|\hat{\mathbf{u}} - \hat{\mathbf{v}}\|_{Y_T^1}.$$

Thus,

$$\left\| \partial_t \hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}}) - \partial_t \hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}) \right\|_{L^q(0,T;L^\infty(\widetilde{\Omega})^{n\times n})} \le C T^{\frac{1}{q}-\frac{1}{r}} \|\hat{\mathbf{u}} - \hat{\mathbf{v}}\|_{Y_T^1}.$$

Moreover, we can also conclude from (3.30) that

$$\left\| \partial_t \hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}}) - \partial_t \hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}) \right\|_{L^q(0,T;W_q^1(\widetilde{\Omega})^{n\times n})} \le C \|\hat{\mathbf{u}} - \hat{\mathbf{v}}\|_{Y_T^1}.$$

Using that $\left(\hat{\mathbf{F}}_0^{-1}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}_0^{-1}(\hat{\mathbf{v}})\right) = 0$,

$$\left\|\Delta_h \left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right)(\cdot, t)\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}}$$
$$\leq \int_{t-h}^{t} \left\|\partial_t \left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right)(\cdot, \tau)\right\|_{W_q^1(\widetilde{\Omega})^{n\times n}} \mathrm{d}\tau \leq Ch^{\frac{1}{q'}} \|\hat{\mathbf{u}} - \hat{\mathbf{v}}\|_{Y_T^1}.$$

Therefore, for all $0 < T < T_R$,

$$\sup_{0\leq t\leq T} \left(\int_0^h \frac{\left\|\Delta_h \left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right)(\cdot, t)\right\|^q_{W_q^1(\widetilde{\Omega})^{n\times n}}}{h^{1+\frac{q}{2q'}}} \mathrm{d}h\right)^{\frac{1}{q}}$$
$$\leq C \sup_{0\leq t\leq T} t^{\frac{1}{2q'}} \|\hat{\mathbf{u}} - \hat{\mathbf{v}}\|_{Y_T^1} = CT^{\frac{1}{2q'}} \|\hat{\mathbf{u}} - \hat{\mathbf{v}}\|_{Y_T^1}.$$

Again with the help of (2.1) and the definition of Sobolev–Slobodeckij space, one obtains the last statement. This completes the proof. □

Lemma 3.17. *Under the assumption of Lemma 3.16, there exist a constant $C = C(R) > 0$ and a finite time $T_R > 0$ depending on R such that for all $0 < T < T_R$ and for two arbitrary functions $f \in L^q(0,T;W_q^1(\widetilde{\Omega}))$ and $\mathbf{f} \in L^q(0,T;W_q^2(\widetilde{\Omega})^n)$,*

(1) $\left\|\left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}) - \mathbb{I}\right) f\right\|_{L^q(0,T;W_q^1(\widetilde{\Omega})^n)} \leq CT^{\frac{1}{q'}} \|f\|_{L^q(0,T;W_q^1(\widetilde{\Omega}))} \|\hat{\mathbf{v}}\|_{Y_T^1}$;
$\left\|\left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}) - \mathbb{I}\right)\left(\hat{\nabla}\mathbf{f}\right)\right\|_{L^q(0,T;W_q^1(\widetilde{\Omega})^{n\times n})} \leq CT^{\frac{1}{q'}} \|\mathbf{f}\|_{L^q(0,T;W_q^2(\widetilde{\Omega})^n)} \|\hat{\mathbf{v}}\|_{Y_T^1}$;

(2) $\left\|\left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right) f\right\|_{L^q(0,T;W_q^1(\widetilde{\Omega})^n)} \leq CT^{\frac{1}{q'}} \|f\|_{L^q(0,T;W_q^1(\widetilde{\Omega}))} \|\hat{\mathbf{u}} - \hat{\mathbf{v}}\|_{Y_T^1}$;
$\left\|\left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right)\left(\hat{\nabla}\mathbf{f}\right)\right\|_{L^q(0,T;W_q^1(\widetilde{\Omega})^{n\times n})}$
$\leq CT^{\frac{1}{q'}} \|\mathbf{f}\|_{L^q(0,T;W_q^2(\widetilde{\Omega})^n)} \|\hat{\mathbf{u}} - \hat{\mathbf{v}}\|_{Y_T^1}$;

(3) $\left\|\left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}}) - \hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}})\right)\left(\hat{\nabla}\mathbf{f}\hat{\mathbf{F}}^{-1}(\hat{\mathbf{u}})\right)\right\|_{L^q(0,T;W_q^1(\widetilde{\Omega})^{n\times n})}$
$\leq CT^{\frac{1}{q'}} \|\mathbf{f}\|_{L^q(0,T;W_q^2(\widetilde{\Omega})^n)} \|\hat{\mathbf{u}} - \hat{\mathbf{v}}\|_{Y_T^1}$.

Proof. The key point to deduce these estimates is to use the multiplication property of $W_q^1(\widetilde{\Omega})$ with $q > n$, which was given in Proposition 2.18. Then Lemma 3.16 implies these results. □

Lemma 3.18. *Let $1 < q < \infty$, $T_0 > 0$ and $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded domain with $C^{1,1}$ boundary. Then*

$$\left[\hat{\nabla}\hat{\mathbf{v}}\right]_{W_q^{\frac{1}{2}-\varepsilon}(0,T;L^q(\Omega)^{n\times n})} \leq CT_0^{\varepsilon} [\hat{\mathbf{v}}]_{W_q^{2,1}(\Omega\times(0,T))^n},$$

for every $\hat{\mathbf{v}} \in W_q^{2,1}(\Omega \times (0,T))^n$, $\varepsilon \in (0, \frac{1}{2})$ and $0 < T < T_0$. Here C depends on ε.

Proof. The lemma can be easily proved by using the arguments in [Abe05, Lemma 4.2], where a layer-like domain with $C^{1,1}$ boundary is considered. Besides, it can be seen as a corollary of Lemma 2.20. □

3.4.2. Proof of Theorem 3.8. In this subsection, we prove Theorem 3.8 by applying the strategy of a fixed-point procedure, along with the Lipschitz estimates.

To this end, we define the function spaces for nonlinear terms $Z_T := \Pi_{j=1}^4 Z_T^j$, where

$$Z_T^1 := L^q(0,T;L^q(\widetilde{\Omega})^n),$$
$$Z_T^2 := \left\{ \begin{aligned} g \in L^q(0,T;W_q^1(\widetilde{\Omega})) \cap W_q^1(0,T;W_q^{-1}(\Omega)) : & \\ \mathrm{tr}_\Gamma(g) \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma \times (0,T)), & \\ \mathrm{tr}_{\Gamma_s}(g) \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma_s \times (0,T)) & \end{aligned} \right\},$$
$$Z_T^3 := W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma \times (0,T))^n, \quad Z_T^4 := W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma_s \times (0,T))^n.$$

We set $\mathbf{w} = (\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s^*, \hat{g})$, $\mathbf{w}_0 := (\hat{\mathbf{v}}, \hat{c}, \hat{c}_s^*, \hat{g})|_{t=0} = (\hat{\mathbf{v}}^0, \hat{c}^0, 0, 1)$ and reformulate the initial and boundary value problem (3.9) as an abstract equation:

$$\mathscr{L}(\mathbf{w}) = \mathscr{N}(\mathbf{w}, \mathbf{w}_0), \quad \text{for all } \mathbf{w} \in Y_T,\ (\hat{\mathbf{v}}^0, \hat{c}^0) \in \mathcal{D}_q, \tag{3.32}$$

where Y_T, $\mathcal{D}_q$ are defined in Section 3.2.

$$\mathscr{L}(\mathbf{w}) := \begin{pmatrix} \partial_t \hat{\mathbf{v}} - \widehat{\mathrm{div}}\, \mathbf{S}(\hat{\mathbf{v}}, \hat{\pi}) \\ \widehat{\mathrm{div}}\,(\hat{\mathbf{v}}) - \dfrac{\gamma\beta}{\hat{\rho}_s} \hat{c}_s \\ [\![\mathbf{S}(\hat{\mathbf{v}}, \hat{\pi})]\!] \cdot \hat{\mathbf{n}}_\Gamma \\ \mathbf{S}(\hat{\mathbf{v}}_s, \hat{\pi}_s) \cdot \hat{\mathbf{n}}_{\Gamma_s} \\ \partial_t \hat{c} - \hat{D} \hat{\Delta} \hat{c} \\ \hat{D} \hat{\nabla} \hat{c} \cdot \hat{\mathbf{n}}_\Gamma \\ \hat{D}_s \hat{\nabla} \hat{c}_s \cdot \hat{\mathbf{n}}_{\Gamma_s} \\ \partial_t \hat{c}_s^* - \beta \hat{c}_s \\ \partial_t \hat{g} - \dfrac{\gamma\beta}{n\hat{\rho}_s} \hat{c}_s \\ (\hat{\mathbf{v}}, \hat{c}, \hat{c}_s^*, \hat{g})|_{t=0} \end{pmatrix}, \quad \mathscr{N}(\mathbf{w}, \mathbf{w}_0) := \begin{pmatrix} \mathbf{K}(\mathbf{w}) \\ G(\mathbf{w}) \\ \mathbf{H}^1(\mathbf{w}) \\ \mathbf{H}^2(\mathbf{w}) \\ F^1(\mathbf{w}) \\ F^2(\mathbf{w}) \\ F^3(\mathbf{w}) \\ F^4(\mathbf{w}) \\ F^5(\mathbf{w}) \\ \mathbf{w}_0 \end{pmatrix}.$$

In the sequel, we focus on (3.32). For $\mathscr{L}$, we have the following proposition.

Proposition 3.19. *Let $\mathscr{L}$ be defined as in (3.32). Then $\mathscr{L}$ is an isomorphism from Y_T to $Z_T \times \mathcal{D}_q$.*

Proof. As $\mathscr{L} \in \mathcal{L}(Y_T, Z_T \times \mathcal{D}_q)$, it suffices to show that $\mathscr{L}$ is bijective, thanks to the bounded inverse theorem.

Injective. Take any $\mathbf{w}^1, \mathbf{w}^2 \in Y_T$. Then, from (3.13), (3.27) and (3.29), we have

$$\left\| \mathscr{L}(\mathbf{w}^1) - \mathscr{L}(\mathbf{w}^2) \right\|_{Z_T \times \mathcal{D}_q} \leq C \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T},$$

which implies the injectivity of $\mathscr{L}$.

Surjective. The existence of (3.12), (3.26) and (3.28) immediately yields the surjectivity of $\mathscr{L}$. $\square$

To employ the contraction mapping principle to (3.32), we investigate the dependence and contraction property of $(\mathbf{K}, G, \mathbf{H}^1, \mathbf{H}^2, F^1, F^2, F^3, F^4, F^5)$ on $(\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s^*, \hat{g})$. To this end, we define

$$\mathscr{M}(\mathbf{w}) := \left(\mathbf{K}(\mathbf{w}), G(\mathbf{w}), \mathbf{H}^1(\mathbf{w}), \mathbf{H}^2(\mathbf{w}), F^1(\mathbf{w}), F^2(\mathbf{w}), F^3(\mathbf{w}), F^4(\mathbf{w}), F^5(\mathbf{w})\right)^\top,$$

where the elements are given by (3.10). Then it is still needed to show that $\mathscr{M} : Y_T \to Z_T$ is well-defined for $\mathbf{w} = (\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s^*, \hat{g}) \in Y_T$ and to verify that $\mathscr{M}$ possesses the contraction property.

Proposition 3.20. *Let $q > n$ and $R > 0$. Assume $\mathbf{w} = (\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s^*, \hat{g}) \in Y_T$ with $\|\mathbf{w}\|_{Y_T} \leq R$, then there exist a constant $C = C(R) > 0$, a finite time $T_R > 0$ depending on R and $\delta > 0$ such that for $0 < T < T_R$, $\mathscr{M} : Y_T \to Z_T$ is well-defined and bounded along with the estimates:*

$$\|\mathscr{M}(\mathbf{w})\|_{Z_T} \leq C(R) T^\delta \left(\|\mathbf{w}\|_{Y_T} + 1\right). \tag{3.33}$$

Moreover, for $\mathbf{w}^1 = (\hat{\mathbf{v}}^1, \hat{\pi}^1, \hat{c}^1, \hat{c}_s^{*1}, \hat{g}^1), \mathbf{w}^2 = (\hat{\mathbf{v}}^2, \hat{\pi}^2, \hat{c}^2, \hat{c}_s^{*2}, \hat{g}^2) \in Y_T$ with $\mathbf{w}^1 \neq \mathbf{w}^2$, $\hat{c}^i|_{t=0} = \hat{c}^0$, $\hat{c}_s^*|_{t=0} = 0$, $\hat{g}^i|_{t=0} = 1$ and $\left\|\mathbf{w}^i\right\|_{Y_T} \leq R$ $(i = 1, 2)$, there exist a constant $C = C(R) > 0$, a finite time $T_R > 0$ depending on R and $\delta > 0$ such that for $0 < T < T_R$,

$$\left\|\mathscr{M}(\mathbf{w}^1) - \mathscr{M}(\mathbf{w}^2)\right\|_{Z_T} \leq C(R) T^\delta \left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T}. \tag{3.34}$$

Proof. First of all, we prove the second part. To this end, for $\left\|\mathbf{w}^i\right\|_{Y_T} \leq R$, $i = 1, 2$ we estimate the following terms respectively

$$\left\|\mathbf{K}(\mathbf{w}^1) - \mathbf{K}(\mathbf{w}^2)\right\|_{Z_T^1}, \quad \left\|G(\mathbf{w}^1) - G(\mathbf{w}^2)\right\|_{Z_T^2},$$
$$\left\|\mathbf{H}^j(\mathbf{w}^1) - \mathbf{H}^j(\mathbf{w}^2)\right\|_{Z_T^{j+2}}, \quad \left\|F^k(\mathbf{w}^1) - F^k(\mathbf{w}^2)\right\|_{Z_T^{k+4}}, \quad \left\|F^5(\mathbf{w}^1) - F^5(\mathbf{w}^2)\right\|_{Z_T^8},$$

where $j \in \{1, 2\}$, $k \in \{1, 2, 3, 4\}$. If $0 < T \leq 1$, we have $T^s < T^{s'}$ for $s > s' > 0$. In the sequel, we set a universal constant $\delta = \min\{\frac{1}{2q'}, \frac{1}{q} - \frac{1}{r}\}$, where $q' = \frac{q}{q-1}$, $r = \frac{q^2}{n}$.

Estimate of $\|\mathbf{K}(\mathbf{w}^1) - \mathbf{K}(\mathbf{w}^2)\|_{Z_T^1}$. For $\mathbf{K}_f = \widehat{\operatorname{div}}\, \tilde{\mathbf{K}}_f$ from (3.10), with the help of Proposition 2.18, Lemmas 3.16 and 3.17, we derive that

$$\begin{aligned}
&\left\|\tilde{\mathbf{K}}_f(\mathbf{w}^1) - \tilde{\mathbf{K}}_f(\mathbf{w}^2)\right\|_{L^q(0,T;W_q^1(\Omega_f)^{n\times n})} \\
&\leq \left\|\hat{\pi}_f^1 \left(\hat{\mathbf{F}}_f^{-\top}(\hat{\mathbf{v}}_f^1) - \hat{\mathbf{F}}_f^{-\top}(\hat{\mathbf{v}}_f^2)\right) + \left(\hat{\pi}_f^1 - \hat{\pi}_f^2\right)\left(\hat{\mathbf{F}}_f^{-\top}(\hat{\mathbf{v}}_f^2) - \mathbb{I}\right)\right\|_{L^q(0,T;W_q^1(\Omega_f)^{n\times n})} \\
&\quad + \nu_f \left\|\left(\hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}_f^1)\hat{\nabla}\hat{\mathbf{v}}_f^1 + (\hat{\nabla}\hat{\mathbf{v}}_f^1)^\top \hat{\mathbf{F}}_f^{-\top}(\hat{\mathbf{v}}_f^1)\right)\left(\hat{\mathbf{F}}_f^{-\top}(\hat{\mathbf{v}}_f^1) - \hat{\mathbf{F}}_f^{-\top}(\hat{\mathbf{v}}_f^2)\right)\right\|_{L^q(0,T;W_q^1(\Omega_f)^{n\times n})} \\
&\quad + 2\nu_f \left\|\left(\left(\hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}_f^1) - \hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}_f^2)\right)\hat{\nabla}\hat{\mathbf{v}}_f^1 + \hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}_f^2)\left(\hat{\nabla}\hat{\mathbf{v}}_f^1 - \hat{\nabla}\hat{\mathbf{v}}_f^2\right)\right)\right. \\
&\qquad\qquad \left.\times \left(\hat{\mathbf{F}}_f^{-\top}(\hat{\mathbf{v}}_f^2) - \mathbb{I}\right)\right\|_{L^q(0,T;W_q^1(\Omega_f)^{n\times n})} \\
&\quad + 2\nu_f \left\|\left(\hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}_f^1) - \hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}_f^2)\hat{\nabla}\hat{\mathbf{v}}_f^1\right)\left(\hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}_f^2) - \mathbb{I}\right)\left(\hat{\nabla}\hat{\mathbf{v}}_f^1 - \hat{\nabla}\hat{\mathbf{v}}_f^2\right)\right\|_{L^q(0,T;W_q^1(\Omega_f)^{n\times n})} \\
&\leq CT^{\frac{1}{q'}}\left(\|\hat{\pi}_f^1\|_{Y_T^2}\|\hat{\mathbf{v}}_f^1 - \hat{\mathbf{v}}_f^2\|_{Y_T^1} + \|\hat{\pi}_f^1 - \hat{\pi}_f^2\|_{Y_T^2}\|\hat{\mathbf{v}}_f^2\|_{Y_T^1}\right) + CT^{\frac{1}{q'}}\|\hat{\mathbf{v}}_f^1\|_{Y_T^1}\|\hat{\mathbf{v}}_f^1 - \hat{\mathbf{v}}_f^2\|_{Y_T^1} \\
&\quad + CT^{\frac{2}{q'}}\|\hat{\mathbf{v}}_f^1\|_{Y_T^1}\|\hat{\mathbf{v}}_f^1 - \hat{\mathbf{v}}_f^2\|_{Y_T^1}\|\hat{\mathbf{v}}_f^2\|_{Y_T^1} + CT^{\frac{1}{q'}}\|\hat{\mathbf{v}}_f^1 - \hat{\mathbf{v}}_f^2\|_{Y_T^1}\|\hat{\mathbf{v}}_f^2\|_{Y_T^1} \\
&\quad + CT^{\frac{1}{q'}}\left(\|\hat{\mathbf{v}}_f^1\|_{Y_T^1}\|\hat{\mathbf{v}}_f^1 - \hat{\mathbf{v}}_f^2\|_{Y_T^1} + \|\hat{\mathbf{v}}_f^1 - \hat{\mathbf{v}}_f^2\|_{Y_T^1}\|\hat{\mathbf{v}}_f^2\|_{Y_T^1}\right) \leq C(R)T^\delta\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T}.
\end{aligned}$$

Let $\hat{g} \in W^1_q(0,T;W^1_q(\Omega_s))$ with $\hat{g}|_{t=0} = 1$. Now we claim that there exists a time $T_R > 0$ such that for $0 < T < T_R$, $\hat{g} \geq \frac{1}{2} > 0$. Let $\hat{g}$ be such a function with $\|\hat{g}\|_{W^1_q(0,T;W^1_q(\Omega_s))} \leq R$ for some $R > 0$. Then for $0 < t < T$,

$$\|\hat{g}(t) - 1\|_{L^\infty(\Omega_s)} \leq C \left\| \int_0^t \partial_t \hat{g}(X,\tau) \mathrm{d}\tau \right\|_{W^1_q(\Omega_s)} \leq CT^{\frac{1}{q'}} R \leq \frac{1}{2},$$

where we choose $T_R > 0$ small enough such that $T_R^{1/q'} \leq \frac{1}{2CR}$. Hence,

$$\hat{g} \geq \frac{1}{2} > 0.$$

For $\mathbf{K}_s = \widehat{\operatorname{div}}\, \tilde{\mathbf{K}}_s + \bar{\mathbf{K}}^g_s$, the first part can be estimated similarly using

$$\left\| \tilde{\mathbf{K}}_s(\mathbf{w}^1) - \tilde{\mathbf{K}}_s(\mathbf{w}^2) \right\|_{L^q(0,T;W^1_q(\Omega_s)^{n\times n})} \leq C(R) T^\delta \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T}.$$

For the second part it follows from (2.1), Lemma 3.16 and 3.17 that

$$\begin{aligned}
&\left\| \bar{\mathbf{K}}^g_s(\mathbf{w}^1) - \bar{\mathbf{K}}^g_s(\mathbf{w}^2) \right\|_{L^q(0,T;L^q(\Omega_s)^{n\times n})} \\
&\leq \left\| \left(\hat{\mathbb{T}}_s(\hat{\mathbf{v}}^1_s, \hat{\pi}^1_s, \hat{g}^1) \hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}^1_s) - \hat{\mathbb{T}}_s(\hat{\mathbf{v}}^2_s, \hat{\pi}^2_s, \hat{g}^2) \hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}^2_s) \right) \frac{n \hat{\nabla} \hat{g}^1}{\hat{g}^1} \right\|_{L^q(0,T;L^q(\Omega_s)^{n\times n})} \\
&\quad + \left\| \hat{\mathbb{T}}_s(\hat{\mathbf{v}}^2_s, \hat{\pi}^2_s, \hat{g}^2) \hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}^2_s) \left(\frac{n \hat{\nabla} \hat{g}^1}{\hat{g}^1} - \frac{n \hat{\nabla} \hat{g}^2}{\hat{g}^2} \right) \right\|_{L^q(0,T;L^q(\Omega_s)^{n\times n})} =: N_1 + N_2.
\end{aligned}$$

From the definition of $\hat{\mathbb{T}}_s$ and $\hat{g} \geq 1/2$,

$$N_1 \leq C \left\| \hat{\nabla} \hat{g}^1 \right\|_{L^\infty(0,T;L^q(\Omega_s)^n)} N_1^1 \leq C(R) T^{\frac{1}{q'}} N_1^1,$$

where

$$\begin{aligned}
N_1^1 :=& \left\| \hat{\pi}^1_s \left(\hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}^1_s) - \hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}^2_s) \right) \right\|_{L^q(0,T;L^\infty(\Omega_s)^{n\times n})} \\
&+ \left\| \left(\hat{\pi}^1_s - \hat{\pi}^2_s \right) \hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}^2_s) \right\|_{L^q(0,T;L^\infty(\Omega_s)^{n\times n})} + \hat{\nu}_s \left\| \hat{\nabla} \hat{\mathbf{v}}^1_s - \hat{\nabla} \hat{\mathbf{v}}^2_s \right\|_{L^q(0,T;L^\infty(\Omega_s)^{n\times n})} \\
&+ \hat{\mu}_s \Bigg(\left\| \frac{1}{(\hat{g}^1)^2} \left(\hat{\mathbf{F}}_s(\hat{\mathbf{v}}^1_s) - \hat{\mathbf{F}}_s(\hat{\mathbf{v}}^2_s) \right) \right\|_{L^q(0,T;L^\infty(\Omega_s)^{n\times n})} \\
&\qquad + \left\| \left(\frac{1}{(\hat{g}^1)^2} - \frac{1}{(\hat{g}^2)^2} \right) \hat{\mathbf{F}}_s(\hat{\mathbf{v}}^2_s) \right\|_{L^q(0,T;L^\infty(\Omega_s)^{n\times n})} \\
&\qquad\qquad + \left\| \hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}^1_s) - \hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}^2_s) \right\|_{L^q(0,T;L^\infty(\Omega_s)^{n\times n})} \Bigg) \\
\leq& CT^{\frac{1}{q'}} \left\| \hat{\pi}^1_s \right\|_{Y^2_T} \left\| \hat{\mathbf{v}}^1_s - \hat{\mathbf{v}}^2_s \right\|_{Y^1_T} + C \left\| \hat{\pi}^1_s - \hat{\pi}^2_s \right\|_{Y^2_T} + C \left\| \hat{\mathbf{v}}^1_s - \hat{\mathbf{v}}^2_s \right\|_{Y^1_T} \\
&+ \hat{\mu}_s \Big(CT^{\frac{1}{q'}} \left\| \hat{\mathbf{v}}^1_s - \hat{\mathbf{v}}^2_s \right\|_{Y^1_T} + CT^{\frac{1}{q'}} \left\| \hat{g}^1 - \hat{g}^2 \right\|_{Y^4_T} \left\| \hat{\mathbf{v}}^2_s \right\|_{Y^1_T} \left(\left\| \hat{g}^1 \right\|_{Y^4_T} + \left\| \hat{g}^2 \right\|_{Y^4_T} \right) \\
&+ CT^{\frac{1}{q'}} \left\| \hat{\mathbf{v}}^1_s - \hat{\mathbf{v}}^2_s \right\|_{Y^1_T} \Big) \leq C(R) \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T}.
\end{aligned}$$

Then we get

$$N_1 + N_2 \le C(R)T^{\frac{1}{q'}} \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T}.$$

Consequently,

$$\left\| \mathbf{K}(\mathbf{w}^1) - \mathbf{K}(\mathbf{w}^2) \right\|_{Z_T^1} \le C(R)T^{\delta} \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T}. \tag{3.35}$$

Estimate of $\|G(\mathbf{w}^1) - G(\mathbf{w}^2)\|_{Z_T^2}$. From the definition of Z_T^2, we need to verify that $G(\mathbf{w}^1) - G(\mathbf{w}^2)$ is contained both in $L^q(0,T; W_q^1(\widetilde{\Omega}))$ and $W_q^1(0,T; W_q^{-1}(\Omega))$, as well as the trace regularity

$$\begin{aligned}
\operatorname{tr}_{\Gamma}(G(\mathbf{w}^1) - G(\mathbf{w}^2)) &\in W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\Gamma \times (0,T)), \\
\operatorname{tr}_{\Gamma_s}(G(\mathbf{w}^1) - G(\mathbf{w}^2)) &\in W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\Gamma_s \times (0,T)).
\end{aligned}$$

For the first regularity it follows easily from (3.10), Lemmas 3.16 and 3.17 that

$$\begin{aligned}
&\left\| G(\mathbf{w}^1) - G(\mathbf{w}^2) \right\|_{L^q(0,T;W_q^1(\widetilde{\Omega}))} \\
&\quad \le CT^{\frac{1}{q'}} \left\| \hat{\mathbf{v}}^1 \right\|_{Y_T^1} \left\| \hat{\mathbf{v}}^1 - \hat{\mathbf{v}}^2 \right\|_{Y_T^1} + CT^{\frac{1}{q'}} \left\| \hat{\mathbf{v}}^2 \right\|_{Y_T^1} \left\| \hat{\mathbf{v}}^1 - \hat{\mathbf{v}}^2 \right\|_{Y_T^1} \le CT^{\delta} R \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T}.
\end{aligned}$$

From the approximation argument in [AM18, Page 15], we know that a weak derivative with respect to time does exist for G. Hence, substituting G by the form (3.11) and using integration by parts, we have

$$\begin{aligned}
\langle \partial_t G(\cdot,t), \phi \rangle_{W_q^{-1} \times W_{q',0}^1} &= \frac{\mathrm{d}}{\mathrm{d}t} \langle G(\cdot,t), \phi \rangle_{W_q^{-1} \times W_{q',0}^1} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \left(\left\langle \left(\hat{\mathbf{F}}^{-1} - \mathbb{I} \right) \hat{\mathbf{v}}, \hat{\nabla} \phi \right\rangle_{L^q \times L^{q'}} - \left\langle \hat{\mathbf{v}}_s \cdot \widehat{\operatorname{div}}\, \hat{\mathbf{F}}_s^{-\top}, \phi \right\rangle_{L^q \times L^{q'}} \right) \\
&= \int_{\Omega} \left(\left(\partial_t \hat{\mathbf{F}}^{-1} \right) \hat{\mathbf{v}} + \left(\hat{\mathbf{F}}^{-1} - \mathbb{I} \right) \partial_t \hat{\mathbf{v}} \right) \cdot \hat{\nabla} \phi \mathrm{d}X \\
&\quad + \int_{\Omega_s} \left(\partial_t \hat{\mathbf{v}}_s \cdot \widehat{\operatorname{div}}\, \hat{\mathbf{F}}_s^{-\top} + \hat{\mathbf{v}}_s \cdot \widehat{\operatorname{div}}\, \partial_t \hat{\mathbf{F}}_s^{-\top} \right) \cdot \phi \mathrm{d}X \\
&= \int_{\Omega_f} \left(\partial_t \hat{\mathbf{F}}_f^{-1} \right) \hat{\mathbf{v}}_f \cdot \hat{\nabla} \phi \mathrm{d}X + \int_{\Omega} \left(\left(\hat{\mathbf{F}}^{-1} - \mathbb{I} \right) \partial_t \hat{\mathbf{v}} \right) \cdot \hat{\nabla} \phi \mathrm{d}X \\
&\quad + \int_{\Omega_s} \left(\partial_t \hat{\mathbf{v}}_s \cdot \widehat{\operatorname{div}}\, \hat{\mathbf{F}}_s^{-\top} + \partial_t \hat{\mathbf{F}}_s^{-\top} : \hat{\nabla} \hat{\mathbf{v}}_s \right) \cdot \phi \mathrm{d}X,
\end{aligned}$$

for every $\phi \in W_{q',0}^1(\Omega)$, where $\langle \cdot, \cdot \rangle_{X \times X'}$ denotes the duality product between a dual pair of spaces X and X'. Then according to (1.62), the Sobolev embedding $W_q^1(\Omega) \hookrightarrow C^{0,1-n/q}(\Omega) \hookrightarrow L^\infty(\Omega)$ and Lemma 3.16, one obtains

$$\begin{aligned}
&\left\| \partial_t G(\mathbf{w}^1) - \partial_t G(\mathbf{w}^2) \right\|_{L^q(0,T;W_q^{-1}(\Omega))} \\
&\le \left\| \left(\partial_t \hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}^1) - \partial_t \hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}_f^2) \right) \hat{\mathbf{v}}_f^1 + \partial_t \hat{\mathbf{F}}_f^{-1}(\hat{\mathbf{v}}_f^2) \left(\hat{\mathbf{v}}_f^1 - \hat{\mathbf{v}}_f^2 \right) \right\|_{L^q(0,T;L^q(\Omega)^n)} \\
&\quad + \left\| \left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}^2) \right) \partial_t \hat{\mathbf{v}}^1 + \left(\hat{\mathbf{F}}^{-1}(\hat{\mathbf{v}}^2) - \mathbb{I} \right) \left(\partial_t \hat{\mathbf{v}}^1 - \partial_t \hat{\mathbf{v}}^2 \right) \right\|_{L^q(0,T;L^q(\Omega)^n)} \\
&\quad + \left\| \partial_t \hat{\mathbf{v}}_s^1 \cdot \left(\widehat{\operatorname{div}}\, \hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}^1) - \widehat{\operatorname{div}}\, \hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}_s^2) \right) \right\|_{L^q(0,T;L^q(\Omega_s))} \\
&\qquad + \left\| \left(\partial_t \hat{\mathbf{v}}_s^1 - \partial_t \hat{\mathbf{v}}_s^2 \right) \cdot \widehat{\operatorname{div}}\, \hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}_s^2) \right\|_{L^q(0,T;L^q(\Omega_s))}
\end{aligned}$$

$$
\begin{aligned}
&+ \left\| \left(\partial_t \hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}_s^1) - \partial_t \hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}_s^2) \right) : \hat{\nabla} \hat{\mathbf{v}}_s^1 \right\|_{L^q(0,T;L^q(\Omega_s))} \\
&\qquad + \left\| \partial_t \hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}_s^2) : \left(\hat{\nabla} \hat{\mathbf{v}}_s^1 - \hat{\nabla} \hat{\mathbf{v}}_s^2 \right) \right\|_{L^q(0,T;L^q(\Omega_s))} \\
&\le CT^{\frac{1}{q}-\frac{1}{r}} \left\| \hat{\mathbf{v}}^1 - \hat{\mathbf{v}}^2 \right\|_{Y_T^1} \left(1 + T^{\frac{1}{q'}} \left\| \hat{\mathbf{v}}^1 \right\|_{Y_T^1} \right) + CT^{\frac{1}{q'}} \left\| \hat{\mathbf{v}}^2 \right\|_{Y_T^1} \left\| \hat{\mathbf{v}}^1 - \hat{\mathbf{v}}^2 \right\|_{Y_T^1} \\
&+ CT^{\frac{1}{q'}} \left\| \hat{\mathbf{v}}^1 - \hat{\mathbf{v}}^2 \right\|_{Y_T^1} \left\| \hat{\mathbf{v}}^1 \right\|_{Y_T^1} + CT^{\frac{1}{q'}} \left\| \hat{\mathbf{v}}^2 \right\|_{Y_T^1} \left\| \hat{\mathbf{v}}^1 - \hat{\mathbf{v}}^2 \right\|_{Y_T^1} \\
&+ CT^{\frac{1}{q'}} \left\| \hat{\mathbf{v}}^1 \right\|_{Y_T^1} \left\| \hat{\mathbf{v}}^1 - \hat{\mathbf{v}}^2 \right\|_{Y_T^1} + CT^{\frac{1}{q'}} \left\| \hat{\mathbf{v}}^1 - \hat{\mathbf{v}}^2 \right\|_{Y_T^1} \left\| \hat{\mathbf{v}}^2 \right\|_{Y_T^1} \\
&+ CT^{\frac{1}{q}-\frac{1}{r}} \left\| \hat{\mathbf{v}}^1 - \hat{\mathbf{v}}^2 \right\|_{Y_T^1} \left\| \hat{\mathbf{v}}^1 \right\|_{Y_T^1} + CT^{\frac{1}{q'}} \left\| \hat{\mathbf{v}}^2 \right\|_{Y_T^1} \left\| \hat{\mathbf{v}}^1 - \hat{\mathbf{v}}^2 \right\|_{Y_T^1} \\
&\le C(R) T^{\delta} \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T}.
\end{aligned}
$$

Then we are in the position to prove $\mathrm{tr}_\Gamma(G(\mathbf{w}^1) - G(\mathbf{w}^2)) \in W_q^{1-1/q,(1-1/q)/2}(\Gamma \times (0,T))$. We first write the norm explicitly:

$$
\begin{aligned}
&\left\| \mathrm{tr}_\Gamma \left(G(\mathbf{w}^1) - G(\mathbf{w}^2) \right) \right\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma \times (0,T))} \\
&= \left\| \left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2) \right) : \hat{\nabla} \hat{\mathbf{v}}^1 \right\|_{L^q\left(0,T;W_q^{1-\frac{1}{q}}(\Gamma)\right)} \\
&\qquad + \left\| \left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2) - \mathbb{I} \right) : \left(\hat{\nabla} \hat{\mathbf{v}}^1 - \hat{\nabla} \hat{\mathbf{v}}^2 \right) \right\|_{L^q\left(0,T;W_q^{1-\frac{1}{q}}(\Gamma)\right)} \\
&\quad + \left\| \left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2) \right) : \hat{\nabla} \hat{\mathbf{v}}^1 \right\|_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}(0,T;L^q(\Gamma))} \\
&\qquad + \left\| \left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2) - \mathbb{I} \right) : \left(\hat{\nabla} \hat{\mathbf{v}}^1 - \hat{\nabla} \hat{\mathbf{v}}^2 \right) \right\|_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}(0,T;L^q(\Gamma))} =: \sum_{i=1}^4 I_i.
\end{aligned}
$$

According to the trace theorem from $W_q^1(\widetilde{\Omega})$ into $W_q^{1-\frac{1}{q}}(\Gamma)$, Proposition 2.18, 3.16 and 3.17,

$$
\begin{aligned}
I_1 &\le C \left\| \left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2) \right) : \hat{\nabla} \hat{\mathbf{v}}^1 \right\|_{L^q(0,T;W_q^1(\widetilde{\Omega}))} \le C(R) T^{\delta} \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T}, \\
I_2 &\le C \left\| \left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2) - \mathbb{I} \right) : \left(\hat{\nabla} \hat{\mathbf{v}}^1 - \hat{\nabla} \hat{\mathbf{v}}^2 \right) \right\|_{L^q(0,T;W_q^1(\widetilde{\Omega}))} \le C(R) T^{\delta} \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T}.
\end{aligned}
$$

It follows from the definition of vector valued Sobolev–Slobodeckij spaces, Lemma 3.16 and 3.18 that

$$
\begin{aligned}
I_3 \le &\left(\int_0^T \int_0^t \frac{\left\| \Delta_h \left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2) \right)(t) : \hat{\nabla} \hat{\mathbf{v}}^1(t-h) \right\|_{L^q(\Gamma)}^q}{h^{1+\frac{q}{2}\left(1-\frac{1}{q}\right)}} \mathrm{d}h \mathrm{d}t \right)^{\frac{1}{q}} \\
&+ \left(\int_0^T \int_0^t \frac{\left\| \left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2) \right)(t) : \Delta_h \left(\hat{\nabla} \hat{\mathbf{v}}^1 \right)(t) \right\|_{L^q(\Gamma)}^q}{h^{1+\frac{q}{2}\left(1-\frac{1}{q}\right)}} \mathrm{d}h \mathrm{d}t \right)^{\frac{1}{q}} \\
\le &\sup_{0 \le t \le T} \left(\int_0^t \frac{\left\| \Delta_h \left(\hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2) \right) \right\|_{L^\infty(\Gamma)^{n \times n}}^q}{h^{1+\frac{q}{2}\left(1-\frac{1}{q}\right)}} \mathrm{d}h \right)^{\frac{1}{q}} \left\| \hat{\nabla} \hat{\mathbf{v}}^1 \right\|_{L^q(0,T;L^q(\Gamma)^{n \times n})}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sup_{0\le t\le T} \left\| \hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-\top}(\hat{\mathbf{v}}^2) \right\|_{W^1_q(\widetilde{\Omega})^{n\times n}} \left[\hat{\nabla}\hat{\mathbf{v}}^1 \right]_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}(0,T;L^q(\Gamma)^{n\times n})} \\
&\le C \left(T^{\frac{1}{2q'}} \left\|\hat{\mathbf{v}}^1\right\|_{Y_T^1} \left\|\hat{\mathbf{v}}^1 - \hat{\mathbf{v}}^2\right\|_{Y_T^1} + T^{\frac{1}{q'}} \left\|\hat{\mathbf{v}}^1 - \hat{\mathbf{v}}^2\right\|_{Y_T^1} \left\|\hat{\mathbf{v}}^1\right\|_{Y_T^1} \right) \\
&\le C(R) T^\delta \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T},
\end{aligned}
$$

where we used the property of Δ_h that $\Delta_h(fg)(t) = \Delta_h f(t) g(t-h) + f(t)\Delta_h g(t)$. Similarly,

$$
I_4 \le C(R) T^\delta \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T},
$$

Collecting I_i, $i = 1, \ldots, 4$, we get

$$
\left\| \mathrm{tr}_\Gamma \left(G(\mathbf{w}^1) - G(\mathbf{w}^2) \right) \right\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma\times(0,T))} \le C(R) T^\delta \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T}.
$$

Since the trace regularities for G on Γ and Γ_s are same, one also obtains

$$
\left\| \mathrm{tr}_{\Gamma_s} \left(G(\mathbf{w}^1) - G(\mathbf{w}^2) \right) \right\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Gamma_s\times(0,T))} \le C(R) T^\delta \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T}.
$$

Then

$$
\left\| G(\mathbf{w}^1) - G(\mathbf{w}^2) \right\|_{Z_T^2} \le C(R) T^\delta \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T},
$$

Estimate of $\|\mathbf{H}^1(\mathbf{w}^1) - \mathbf{H}^1(\mathbf{w}^2)\|_{Z_T^3}$, $\|\mathbf{H}^2(\mathbf{w}^1) - \mathbf{H}^2(\mathbf{w}^2)\|_{Z_T^4}$. Since Γ is of class C^3, $\hat{\mathbf{n}}_\Gamma \in C^2(\partial\Omega_f)$. Then by similar estimates as for $\mathrm{tr}_\Gamma(G(\mathbf{w}^1) - G(\mathbf{w}^2))$, the norm of $\mathbf{H}^1(\mathbf{w}^1) - \mathbf{H}^1(\mathbf{w}^2)$ in Z_T^3 can be estimated as

$$
\begin{aligned}
\left\| \mathbf{H}^1(\mathbf{w}^1) - \mathbf{H}^1(\mathbf{w}^2) \right\|_{Z_T^3} &= \left\| \left[\!\left[\tilde{\mathbf{K}}(\mathbf{w}^1) - \tilde{\mathbf{K}}(\mathbf{w}^2) \right]\!\right] \hat{\mathbf{n}}_\Gamma \right\|_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}(0,T;L^q(\Gamma)^n)} \\
&\quad + \left\| \left[\!\left[\tilde{\mathbf{K}}(\mathbf{w}^1) - \tilde{\mathbf{K}}(\mathbf{w}^2) \right]\!\right] \hat{\mathbf{n}}_\Gamma \right\|_{L^q(0,T;W_q^{1-\frac{1}{q}}(\Gamma)^n)} \\
&\le C \left\| \tilde{\mathbf{K}}_f(\mathbf{w}^1) - \tilde{\mathbf{K}}_f(\mathbf{w}^2) + \tilde{\mathbf{K}}_s(\mathbf{w}^1) - \tilde{\mathbf{K}}_s(\mathbf{w}^2) \right\|_{W_q^{\frac{1}{2}\left(1-\frac{1}{q}\right)}(0,T;L^q(\Gamma)^{n\times n})} \\
&\quad + C \left\| \left(\tilde{\mathbf{K}}(\mathbf{w}^1) - \tilde{\mathbf{K}}(\mathbf{w}^2) \right) \right\|_{L^q(0,T;W_q^1(\widetilde{\Omega})^{n\times n})} \le C(R) T^\delta \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T}.
\end{aligned}
$$

Similarly, we can easily derive

$$
\left\| \mathbf{H}^2(\mathbf{w}^1) - \mathbf{H}^2(\mathbf{w}^2) \right\|_{Z_T^4} \le C T^\delta (1+R)^2 \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T}.
$$

Estimate of $\|F^1(\mathbf{w}^1) - F^1(\mathbf{w}^2)\|_{Z_T^5}$. For $F_f^1 = \widehat{\mathrm{div}}\, \tilde{F}_f$, we have

$$
\begin{aligned}
&\left\| F_f^1(\mathbf{w}^1) - F_f^1(\mathbf{w}^2) \right\|_{Z_T^5} \\
&\quad \le \left\| \tilde{F}_f(\mathbf{w}^1) - \tilde{F}_f(\mathbf{w}^2) \right\|_{L^q(0,T;W_q^1(\Omega_f)^n)} \\
&\quad \le \hat{D}_f \left\| \left(\hat{\mathbf{F}}_f^{-1}(\mathbf{v}_f^1) \hat{\mathbf{F}}_f^{-\top}(\mathbf{v}_f^1) - \hat{\mathbf{F}}_f^{-1}(\mathbf{v}_f^2) \hat{\mathbf{F}}_f^{-\top}(\mathbf{v}_f^2) \right) \hat{\nabla}\hat{c}_f^1 \right\|_{L^q(0,T;W_q^1(\Omega_f)^n)} \\
&\qquad + \hat{D}_f \left\| \left(\hat{\mathbf{F}}_f^{-1}(\mathbf{v}_f^2) \hat{\mathbf{F}}_f^{-\top}(\mathbf{v}_f^2) - \mathbb{I} \right) \left(\hat{\nabla}\hat{c}_f^1 - \hat{\nabla}\hat{c}_f^2 \right) \right\|_{L^q(0,T;W_q^1(\Omega_f)^n)} =: \mathfrak{F}_1 + \mathfrak{F}_2.
\end{aligned}
$$

Lemma 3.16 and the multiplication property of $W_q^1(\Omega_f)$ in Proposition 2.18 imply that

$$\begin{aligned}\mathfrak{F}_1 &\leq C\left(\left\|\hat{\mathbf{F}}_f^{-1}(\mathbf{v}_f^1)\left(\hat{\mathbf{F}}_f^{-\top}(\mathbf{v}_f^1)-\hat{\mathbf{F}}_f^{-\top}(\mathbf{v}_f^2)\right)\right\|_{L^\infty(0,T;W_q^1(\Omega_f)^{n\times n})}\right.\\ &\qquad\left.+\left\|\left(\hat{\mathbf{F}}_f^{-1}(\mathbf{v}_f^1)-\hat{\mathbf{F}}_f^{-1}(\mathbf{v}_f^2)\right)\hat{\mathbf{F}}_f^{-\top}(\mathbf{v}_f^2)\right\|_{L^\infty(0,T;W_q^1(\Omega_f)^{n\times n})}\right)\left\|\hat{\nabla}\hat{c}_f^1\right\|_{L^q(0,T;W_q^1(\Omega_f)^n)}\\ &\leq C(R)T^{\frac{1}{q'}}\left\|\mathbf{w}^1-\mathbf{w}^2\right\|_{Y_T},\end{aligned}$$

and

$$\begin{aligned}\mathfrak{F}_2 &\leq C\left(\left\|\hat{\mathbf{F}}_f^{-1}(\mathbf{v}_f^2)\left(\hat{\mathbf{F}}_f^{-\top}(\mathbf{v}_f^2)-\mathbb{I}\right)\right\|_{L^\infty(0,T;W_q^1(\Omega_f)^{n\times n})}\right.\\ &\qquad\left.+\left\|\hat{\mathbf{F}}_f^{-1}(\mathbf{v}_f^2)-\mathbb{I}\right\|_{L^\infty(0,T;W_q^1(\Omega_f)^{n\times n})}\right)\left\|\hat{\nabla}\hat{c}_f^1-\hat{\nabla}\hat{c}_f^2\right\|_{L^q(0,T;W_q^1(\Omega_f)^n)}\\ &\leq C(R)T^{\frac{1}{q'}}\left\|\mathbf{w}^1-\mathbf{w}^2\right\|_{Y_T}.\end{aligned}$$

Then

$$\left\|F_f^1(\mathbf{w}^1)-F_f^1(\mathbf{w}^2)\right\|_{Z_T^5}\leq C(R)T^{\frac{1}{q'}}\left\|\mathbf{w}^1-\mathbf{w}^2\right\|_{Y_T}.$$

For $F_s^1=\bar{F}_s^1+F_s^g=\widehat{\operatorname{div}}\,\tilde{F}_s+F_s^g$, it can be deduced similarly as for F_f^1 that

$$\left\|\bar{F}_s^1(\mathbf{w}^1)-\bar{F}_s^1(\mathbf{w}^2)\right\|_{Z_T^5}\leq C(R)T^{\frac{1}{q'}}\left\|\mathbf{w}^1-\mathbf{w}^2\right\|_{Y_T}.$$

Moreover,

$$\begin{aligned}&\left\|F_s^g(\mathbf{w}^1)-F_s^g(\mathbf{w}^2)\right\|_{Z_T^5}\\ &\leq\beta\left\|(\hat{c}_s^1-\hat{c}_s^2)\left(1+\frac{\gamma}{\hat{\rho}_s}\hat{c}_s^1\right)\right\|_{L^q(\Omega_s\times(0,T))}+\beta\left\|\hat{c}_s^2\left(\hat{c}_s^1-\hat{c}_s^2\right)\right\|_{L^q(\Omega_s\times(0,T))}\\ &\quad+n\left\|\frac{\hat{\nabla}\hat{g}^1}{\hat{g}^1}\left(\tilde{F}_s(\mathbf{w}^1)-\tilde{F}_s(\mathbf{w}^2)+\left(\hat{\nabla}\hat{c}_s^1-\hat{\nabla}\hat{c}_s^2\right)\right)\right\|_{L^q(\Omega_s\times(0,T))}\\ &\quad+n\left\|\left(\frac{\hat{\nabla}\hat{g}^1}{\hat{g}^1}-\frac{\hat{\nabla}\hat{g}^2}{\hat{g}^2}\right)\hat{\mathbf{F}}_s^{-1}(\hat{\mathbf{v}}_s^2)\hat{\mathbf{F}}_s^{-\top}(\hat{\mathbf{v}}_s^2)\hat{\nabla}\hat{c}_s^2\right\|_{L^q(\Omega_s\times(0,T))}=:\sum_{i=1}^4\mathfrak{F}_i^g.\end{aligned}$$

Apparently, with $\hat{c}^i|_{t=0}=\hat{c}^0$, $i=1,2$,

$$\begin{aligned}\mathfrak{F}_1^g+\mathfrak{F}_2^g &\leq C\left\|\hat{c}_s^1-\hat{c}_s^2\right\|_{L^\infty(0,T;L^q(\Omega_s))}\left\|1+\frac{\gamma}{\hat{\rho}_s}\hat{c}_s^1\right\|_{L^q(0,T;L^\infty(\Omega_s))}\\ &\quad+C\left\|\hat{c}_s^1-\hat{c}_s^2\right\|_{L^\infty(0,T;L^q(\Omega_s))}\left\|\hat{c}_s^2\right\|_{L^q(0,T;L^\infty(\Omega_s))}\leq C(R)T^{\frac{1}{q'}}\left\|\mathbf{w}^1-\mathbf{w}^2\right\|_{Y_T}.\end{aligned}$$

Proceeding the same estimates as $\tilde{F}_f$ above, we have

$$\mathfrak{F}_3^g+\mathfrak{F}_4^g\leq C(R)T^{\frac{1}{q'}}\left\|\mathbf{w}^1-\mathbf{w}^2\right\|_{Y_T},$$

by $\hat{g}\geq\frac{1}{2}$ and Lemma 3.16. Collecting $\mathfrak{F}_i^g$, $i=1,...,4$ together, one concludes

$$\left\|F_s^1(\mathbf{w}^1)-F_s^1(\mathbf{w}^2)\right\|_{Z_T^5}\leq C(R)T^{\frac{1}{q'}}\left\|\mathbf{w}^1-\mathbf{w}^2\right\|_{Y_T}.$$

Estimate of $\|F^2(\mathbf{w}^1) - F^2(\mathbf{w}^2)\|_{Z_T^6}$, $\|F^3(\mathbf{w}^1) - F^3(\mathbf{w}^2)\|_{Z_T^7}$. Since the key ingredient here is to estimate $\tilde{F}(\mathbf{w}^1) - \tilde{F}(\mathbf{w}^2)$ in the space $W_q^{1-1/q,1/2-1/2q}(\Gamma \times (0,T))$, we only give the details to handle this term. By definition,

$$\begin{aligned} &\left\|\tilde{F}(\mathbf{w}^1) - \tilde{F}(\mathbf{w}^2)\right\|_{W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma\times(0,T))^n} \\ &= \left\|\tilde{F}(\mathbf{w}^1) - \tilde{F}(\mathbf{w}^2)\right\|_{L^q(0,T;W_q^{1-\frac{1}{q}}(\Gamma)^n)} + \left\|\tilde{F}(\mathbf{w}^1) - \tilde{F}(\mathbf{w}^2)\right\|_{W_q^{\frac{1}{2}(1-\frac{1}{q})}(0,T;L^q(\Gamma)^n)}. \end{aligned}$$

The first term can be controlled easily by the trace theorem for $q > n$ and the estimates of $\tilde{F}$ in $\widetilde{\Omega}$ above. Namely,

$$\begin{aligned} &\left\|\tilde{F}(\mathbf{w}^1) - \tilde{F}(\mathbf{w}^2)\right\|_{L^q(0,T;W_q^{1-\frac{1}{q}}(\Gamma)^n)} \\ &\qquad \le C\left\|\tilde{F}(\mathbf{w}^1) - \tilde{F}(\mathbf{w}^2)\right\|_{L^q(0,T;W_q^1(\widetilde{\Omega})^n)} \le C(R)T^{\frac{1}{q'}}\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T}. \end{aligned}$$

For the second term, again by the definition of vector-valued Sobolev–Slobodeckij space, we have

$$\left\|\tilde{F}(\mathbf{w}^1) - \tilde{F}(\mathbf{w}^2)\right\|_{W_q^{\frac{1}{2}(1-\frac{1}{q})}(0,T;L^q(\Gamma)^n)} \le C(R)T^{\frac{1}{2q'}}\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T},$$

following the argument of estimating $\mathrm{tr}_\Gamma(G(\mathbf{w}^1) - G(\mathbf{w}^2))$. Then,

$$\left\|F^2(\mathbf{w}^1) - F^2(\mathbf{w}^2)\right\|_{Z_T^6} + \left\|F^3(\mathbf{w}^1) - F^3(\mathbf{w}^2)\right\|_{Z_T^7} \le C(R)T^\delta\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T}.$$

Estimate of $\|F^4(\mathbf{w}^1) - F^4(\mathbf{w}^2)\|_{Z_T^8}$, $\|F^5(\mathbf{w}^1) - F^5(\mathbf{w}^2)\|_{Z_T^8}$. Observing that the nonlinearities in F^4 and F^5 are $\hat{c}_s\hat{c}_s^*$ and $\hat{c}_s\hat{g}$, which are all quadratic, we control them under the assumptions $\hat{c}^i\big|_{t=0} = \hat{c}^0$, $\hat{c}_s^*|_{t=0} = 0$, $\hat{g}^i\big|_{t=0} = 1$, $i = 1,2$, and by

$$\|uv\|_{L^q(0,T;W_q^1(\Omega_s))} \le M_q\|u\|_{L^\infty(0,T;W_q^1(\Omega_s))}\|v\|_{L^q(0,T;W_q^1(\Omega_s))},$$

for $u, v \in W_q^1(0,T;W_q^1(\Omega_s))$. Hence,

$$\left\|F^4(\mathbf{w}^1) - F^4(\mathbf{w}^2)\right\|_{Z_T^8} + \left\|F^5(\mathbf{w}^1) - F^5(\mathbf{w}^2)\right\|_{Z_T^8} \le C(R)T^{\frac{1}{q'}}\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T}.$$

Consequently, we derive (3.34). Now, choosing $\mathbf{w}^1 = \mathbf{w}$ and $\mathbf{w}^2 = (0,0,0,0,1)$ in (3.34), (3.33) follows immediately from the fact that $\mathscr{M}(0,0,0,0,1) = 0$. □

Proof of Theorem 3.8. Since $\mathscr{L} : Y_T \to Z_T \times \mathcal{D}_q$ is an isomorphism as shown in Proposition 3.19, and because of the estimates in Theorem 3.12, we can set a well-defined constant

$$C_{\mathscr{L}} := \sup_{0\le T\le 1}\left\|\mathscr{L}^{-1}\right\|_{\mathcal{L}(Z_T\times\mathcal{D}_q,Y_T)},$$

which is uniformly bounded as $T \to 0$ for trivial initial data. We choose $R > 0$ so large that $R \ge 2C_{\mathscr{L}}\left\|(\hat{\mathbf{v}}^0, \hat{c}^0)\right\|_{\mathcal{D}_q}$. Then

$$\left\|\mathscr{L}^{-1}\mathscr{N}(\bar{\mathbf{0}}, \mathbf{w}_0)\right\|_{Y_T} \le C_{\mathscr{L}}\left\|(\hat{\mathbf{v}}^0, \hat{c}^0)\right\|_{\mathcal{D}_q} \le \frac{R}{2}. \tag{3.36}$$

Here $\mathscr{N}(\bar{\mathbf{0}}, \mathbf{w}_0)$ is in the sense of trivial data $\bar{\mathbf{0}} = (0,0,0,0,1)$. For $\|\mathbf{w}^i\|_{Y_T} \le R$, $i = 1,2$, we take $T_R > 0$ small enough such that

$$C_{\mathscr{L}} C(R) T_R^\delta \le \frac{1}{2},$$

where $C(R)$ is the constant in (3.34). Then for $0 < T < T_R$, we infer from Theorem 3.20 that

$$\begin{aligned} &\left\|\mathscr{L}^{-1}\mathscr{N}(\mathbf{w}^1, \mathbf{w}_0) - \mathscr{L}^{-1}\mathscr{N}(\mathbf{w}^2, \mathbf{w}_0)\right\|_{Y_T} \\ &\qquad \le C_{\mathscr{L}} C(R) T^\delta \left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T} \le \frac{1}{2}\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T}, \end{aligned} \tag{3.37}$$

which implies the contraction property. From (3.36) and (3.37), we have

$$\begin{aligned} &\left\|\mathscr{L}^{-1}\mathscr{N}(\mathbf{w}, \mathbf{w}_0)\right\|_{Y_T} \\ &\qquad \le \left\|\mathscr{L}^{-1}\mathscr{N}(\bar{\mathbf{0}}, \mathbf{w}_0)\right\|_{Y_T} + \left\|\mathscr{L}^{-1}\mathscr{N}(\mathbf{w}, \mathbf{w}_0) - \mathscr{L}^{-1}\mathscr{N}(\bar{\mathbf{0}}, \mathbf{w}_0)\right\|_{Y_T} \le R. \end{aligned}$$

We define $\mathcal{M}_{R,T}$ by

$$\mathcal{M}_{R,T} := \left\{\mathbf{w} \in \overline{B_{Y_T}(\bar{\mathbf{0}}, R)} : \mathbf{w} = (\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s^*, \hat{g})\right\},$$

which is a closed subset of Y_T. Hence, $\mathscr{L}^{-1}\mathscr{N}(\cdot, \mathbf{w}_0) : \mathcal{M}_{R,T} \to \mathcal{M}_{R,T}$ is well-defined for all $0 < T < T_R$ and a strict contraction. Since Y_T is a Banach space, the Banach fixed-point Theorem implies the existence of a unique fixed-point of $\mathscr{L}^{-1}\mathscr{N}$ in $\mathcal{M}_{R,T}$, i.e., (3.9) admits a unique strong solution in $\mathcal{M}_{R,T}$ for small time $0 < T < T_R$.

In the following, we prove the uniqueness of solutions in Y_T by a continuity argument. Let $\mathbf{w}^1, \mathbf{w}^2 \in Y_T$ be two different solutions of (3.9) and $\widetilde{R} := \max\{\left\|\mathbf{w}^1\right\|_{Y_T}, \left\|\mathbf{w}^2\right\|_{Y_T}\}$, then there is a time $T_{\widetilde{R}} \le T$ such that $\mathscr{L}^{-1}\mathscr{N}(\cdot, \mathbf{w}_0) : \mathcal{M}_{\widetilde{R}, T_{\widetilde{R}}} \to \mathcal{M}_{\widetilde{R}, T_{\widetilde{R}}}$ is a contraction and therefore $\mathbf{w}^1|_{[0,T_{\widetilde{R}}]} = \mathbf{w}^2|_{[0,T_{\widetilde{R}}]}$. Now we argue by contradiction. We define $\widetilde{T}$ as

$$\widetilde{T} := \sup\left\{T' \in (0,T] : \mathbf{w}^1\big|_{[0,T']} = \mathbf{w}^2\big|_{[0,T']}\right\},$$

and assume $\widetilde{T} < T$. Since $\mathbf{w}^1|_{[0,\widetilde{T}]} = \mathbf{w}^2|_{[0,\widetilde{T}]}$, we consider $\mathbf{w}^1|_{t=\widetilde{T}} = \mathbf{w}^2|_{t=\widetilde{T}}$ as the initial value for (3.9). Repeating the argument above, we see that there is a time $\widehat{T} \in (\widetilde{T}, T)$ such that $\mathbf{w}^1|_{[\widetilde{T},\widehat{T}]} = \mathbf{w}^2|_{[\widetilde{T},\widehat{T}]}$, which contradicts the definition of $\widetilde{T}$.

In conclusion, (3.9) admits a unique solution in Y_T.

For the nonnegativity of $\hat{c}$, we show it in Eulerian coordinates. Let $U_T = (\Omega^t \backslash \Gamma^t) \times (0,T)$, $U_{f,T} = \Omega_f^t \times (0,T)$, $U_{s,T} = \Omega_s^t \times (0,T)$, and define the parabolic boundary $\partial_P U_{f,T} := (\overline{\Omega}_f^0 \times \{0\}) \cup (\Gamma^t \times [0,T])$, $\partial_P U_{s,T} := (\overline{\Omega}_s^0 \times \{0\}) \cup ((\Gamma^t \cup \Gamma_s^t) \times [0,T])$ and $\partial_P U_T := \partial_P U_{f,T} \cup \partial_P U_{s,T}$. First of all, we claim that $c \in C_{loc}^{2,1}(U_T) \cap C(\overline{U_T})$, where

$$C^{2s,s}(U_T) := \left\{c(\cdot,t) \in C^{2s}(\Omega^t \backslash \Gamma^t),\ c(x,\cdot) \in C^s(0,T),\ \forall x \in \Omega^t \backslash \Gamma^t, t \in (0,T)\right\},$$

for $s > 0$. As shown above, we assume that $c \in Y_T^3$ is the solution of

$$\partial_t c - D\Delta c = -(\mathbf{v} \cdot \nabla c + (\operatorname{div} \mathbf{v} + \beta) c) =: f. \tag{3.38}$$

With the regularity of $\mathbf{v}, c$ and embedding theorems, we know that $f \in C_{loc}^{\alpha,\alpha/2}(U_T)$ for some $0 < \alpha < 1$. By the local regularity theory for parabolic equations, one obtains

$$c \in C_{loc}^{2+\alpha, 1+\frac{\alpha}{2}}(U_T) \hookrightarrow C_{loc}^{2,1}(U_T).$$

The continuity of c can be derived directly from Proposition 2.22, especially (2.4) with

$$W_q^1 \hookrightarrow C^{1-\frac{n}{q}} \hookrightarrow C^0, \text{ for } q > n.$$

Now, given a nonnegative initial value $c^0(x) \geq 0$, $x \in \Omega^0$. Define $c_\lambda := e^{-\lambda t} c$ where $\lambda > 0$ is a constant, which will be assigned later. Adding cc_λ to the both sides of (3.38), we have the equation for c_λ

$$\partial_t c_\lambda - D\Delta c_\lambda + \mathbf{v}\cdot\nabla c_\lambda + (\operatorname{div}\mathbf{v} + c + \beta + \lambda)c_\lambda = c^2 e^{-\lambda t} \geq 0.$$

Taking λ sufficiently large such that

$$\beta + \lambda \geq \sup_{0\leq t\leq T, x\in\Omega^t\setminus\Sigma^t} |\operatorname{div}\mathbf{v}| + |c|,$$

one obtains

$$\operatorname{div}\mathbf{v} + c + \beta + \lambda \geq 0.$$

By the weak maximum principle for parabolic equations, we have

$$\min_{\overline{U}_{f,T}} c_f(x,t) \geq -\max_{\partial_P U_{f,T}} c_f^-(x,t), \quad \min_{\overline{U}_{s,T}} c_s(x,t) \geq -\max_{\partial_P U_{s,T}} c_s^-(x,\tau),$$

namely,

$$\min_{\overline{U}_T} c(x,t) \geq -\max_{\partial_P U_T} c^-(x,t),$$

where $c^-(x,t) := -\min\{c(x,t), 0\}$.

Since $c^0(x) \geq 0$, now we claim that $c(x,t) \geq 0$ for all $(x,t) \in (\Gamma^t \cup \Gamma_s^t) \times [0,T]$. To this end, we argue by contradiction. Assume that for some $t_0 \in (0,T]$, there exists a point $x_0 \in \Gamma^{t_0} \cup \Gamma_s^{t_0}$, such that

$$c(x_0,t_0) = -\max_{x\in\Gamma^{t_0}\cup\Gamma_s^{t_0}} c^-(x,t_0) < 0,$$

that is,

$$\min_{x\in\Gamma^{t_0}\cup\Gamma_s^{t_0}} \min\{c(x,t_0), 0\} < 0.$$

This implies that $x \mapsto \min\{c(x,t_0),0\}$ attains a negative minimum at x_0, i.e., $x \mapsto c(x,t_0)$ attains a negative minimum at x_0.

Case 1: $x_0 \in \Gamma^{t_0}$. For both $\Omega_f^{t_0}$ and $\Omega_s^{t_0}$, since Γ^{t_0} is assumed to be a C^{3-} interface, we infer from Hopf's Lemma that

$$D_f \nabla c_f \cdot \mathbf{n}_{\Gamma^{t_0}}(x_0) < 0, \quad D_s \nabla c_s \cdot \mathbf{n}_{\Gamma^{t_0}}(x_0) > 0, \text{ on } \Gamma^{t_0}.$$

Hence,

$$[\![D\nabla c]\!] \cdot \mathbf{n}_{\Gamma^{t_0}}(x_0) < 0,$$

which contradicts (3.1j).

Case 2: $x_0 \in \Gamma_s^{t_0}$. Again by Hopf's Lemma, one obtains

$$D\nabla c \cdot \mathbf{n}_{\Gamma_s^{t_0}}(x_0) < 0, \text{ on } \Gamma_s^{t_0},$$

which contradicts to (3.1k).

In summary, $c(x,t) \geq 0$ for all $(x,t) \in \overline{\Omega}^t \times [0,T]$.

For $\hat{c}_s^*$ and $\hat{g}$, we note that the equations for them in Lagrangian coordinates are ordinary differential equations with suitable $\hat{c}_s \geq 0$. Then

$$\hat{c}_s^* = \int_0^t e^{\int_t^\sigma \frac{\gamma\beta}{\hat{\rho}_s}\hat{c}_s(x,\tau)\mathrm{d}\tau} \beta\hat{c}_s(x,\sigma)\mathrm{d}\sigma > 0, \quad \hat{g} = e^{\int_0^t \frac{\gamma\beta}{n\hat{\rho}_s}\hat{c}_s(x,\tau)\mathrm{d}\tau} > 0,$$

which completes the proof. □

3.5. Appendix: Some Results on Linear Systems

In this section, we give several maximal L^q-regularity results of different problems, which are needed for the whole system.

3.5.1. Two-phase Stokes problems with Dirichlet boundary condition. In this section, we focus on the following nonstationary two-phase Stokes problem.

$$
\begin{aligned}
\varrho\partial_t v - \operatorname{div}(2\mu Dv) + \nabla p &= \varrho f_u, && \text{in } \Omega\backslash\Sigma \times (0,T),\\
\operatorname{div} v &= g_d, && \text{in } \Omega\backslash\Sigma \times (0,T),\\
v &= g_b, && \text{on } \partial\Omega \times (0,T),\\
[\![v]\!] &= g_u, && \text{on } \Sigma \times (0,T),\\
[\![-2\mu Dv + p\mathbb{I}]\!]\,\nu_\Sigma &= g, && \text{on } \Sigma \times (0,T),\\
v|_{t=0} &= v_0, && \text{in } \Omega\backslash\Sigma,
\end{aligned}
\tag{3.39}
$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with $\partial\Omega \in C^3$, $\Sigma \subset \Omega$ a closed hypersurface of class C^3. ϱ_j are positive constants, $j = 1, 2$. $v : \Omega \times (0,T) \to \mathbb{R}^n$ is the velocity of the fluid, $p : \Omega \times (0,T) \to \mathbb{R}$ denotes the pressure. $\mu > 0$ is the constant viscosity and $Dv = \frac{1}{2}\left(\nabla v + \nabla v^\top\right)$. ν_Σ represents the unit outer normal vector on Σ. f_u, g_d, g_b, g_u, g are given functions and v_0 is the prescribed initial value. System (3.39) has been investigated by many scholars in various aspects. We refer for the maximal L_q regularity results of such kind of two-phase Stokes problem to Prüss–Simonett [PS16]. Readers can also find similar results in Abels and Moser [AM18] for $(g_b, g_u) = 0$.

Proposition 3.21. *Let $q > n+2$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^3$, $\Sigma \subset \Omega$ a closed hypersurface of class C^3. Assume that $(f_u, g_d, g_b, g_u, g) \in Z_T$ where*

$$
Z_T := \left\{
\begin{aligned}
&f_u \in L^q(0,T; L^q(\Omega)^n), \quad g_d \in L^q(0,T; W^1_q(\Omega\backslash\Sigma)),\\
&g_b \in W_q^{2-\frac{1}{q}, 1-\frac{1}{2q}}(\partial\Omega \times (0,T))^n, \quad g_u \in W_q^{2-\frac{1}{q}, 1-\frac{1}{2q}}(\Sigma \times (0,T))^n,\\
&g \in W_q^{1-\frac{1}{q}, \frac{1}{2}\left(1-\frac{1}{q}\right)}(\Sigma \times (0,T))^n : (g_d, g_b \cdot \nu_{\partial\Omega}, g_u \cdot \nu_\Sigma) \in W^1_q(0,T; \widehat{W}_q^{-1}(\Omega))
\end{aligned}
\right\}
$$

and $v_0 \in W_q^{2-\frac{2}{q}}(\Omega\backslash\Sigma)^n$ satisfying the compatibility conditions

$$
\operatorname{div} v_0 = g_d|_{t=0}\,, \quad v_0|_{\partial\Omega} = g_b|_{t=0}\,, \quad [\![v_0]\!]|_\Sigma = g_u|_{t=0}\,, \quad [\![(2\mu Dv_0\nu_\Sigma)_\tau]\!]|_\Sigma = g_\tau|_{t=0}\,. \tag{3.40}
$$

Then two-phase Stokes problem (3.39) admits a unique solution (v, p) with regularity

$$
\begin{aligned}
v &\in L^q(0,T; W^2_q(\Omega\backslash\Sigma)^n) \cap W^1_q(0,T; L^q(\Omega)^n),\\
p &\in L^q(0,T; W^1_{q,(0)}(\Omega\backslash\Sigma)), \quad [\![p]\!] \in W_q^{1-\frac{1}{q}, \frac{1}{2}\left(1-\frac{1}{q}\right)}(\Sigma \times (0,T)).
\end{aligned}
$$

Moreover, for any fixed $0 < T_0 < \infty$, there is a constant C, independent of $T \in (0, T_0]$, such that

$$
\begin{aligned}
&\|v\|_{L^q(0,T;W^2_q(\Omega\backslash\Sigma)^n)} + \|v\|_{W^1_q(0,T;L^q(\Omega)^n)}\\
&\quad + \|p\|_{L^q(0,T;W^1_{q,(0)}(\Omega\backslash\Sigma))} + \|[\![p]\!]\|_{W_q^{1-\frac{1}{q}, \frac{1}{2}\left(1-\frac{1}{q}\right)}(\Sigma\times(0,T))}\\
&\leq C \Bigg(\|f_u\|_{L^q(0,T;L^q(\Omega)^n)} + \|g_d\|_{L^q(0,T;W^1_q(\Omega\backslash\Sigma))} + \|g_b\|_{W_q^{2-\frac{1}{q}, 1-\frac{1}{2q}}(\partial\Omega\times(0,T))^n}
\end{aligned}
\tag{3.41}
$$

$$+ \|g_u\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\Sigma\times(0,T))^n} + \|\partial_t(g_d, g_b \cdot \nu_{\partial\Omega}, g_u \cdot \nu_\Sigma)\|_{L^q(0,T;\widehat{W}_q^{-1}(\Omega))}$$
$$+ \|g\|_{W_q^{1-\frac{1}{q},\frac{1}{2}\left(1-\frac{1}{q}\right)}(\Sigma\times(0,T))^n} + \|v_0\|_{W_q^{2-\frac{2}{q}}(\Omega\backslash\Sigma)^n}\Bigg).$$

Here, $\widehat{W}_q^{-1}(\Omega)$ is the space of all triples $(\varphi, \psi, \chi) \in L^q(\Omega) \times W_q^{2-1/q}(\partial\Omega)^n \times W_q^{2-1/q}(\Sigma)^n$, which enjoy the regularity property $(\varphi, \psi \cdot \nu_{\partial\Omega}, \chi \cdot \nu_\Sigma) \in \dot{W}_q^{-1}(\Omega) = (\dot{W}_{q'}^1(\Omega))'$, where

$$\langle(\varphi, \psi \cdot \nu_{\partial\Omega}, \chi \cdot \nu_\Sigma), \phi\rangle := -\langle\varphi, \phi\rangle_\Omega + \langle\psi \cdot \nu_{\partial\Omega}, \phi\rangle_{\partial\Omega} + \langle\chi \cdot \nu_\Sigma, \phi\rangle_\Sigma, \tag{3.42}$$

for all $\phi \in \dot{W}_{q'}^1(\Omega)$.

Proof. We proceed to prove this theorem with Theorem 8.1.4 in [PS16], by which we need some special treatments for (3.39). The first one is to extend the quintuple (f_u, g_d, g_b, g_u, g) from Z_T to Z_∞. Since $f_u \in L^q(0,T; L^q(\Omega)^n)$ is without time derivatives, we simply extend it by zero to a new function $\bar{f}_u = \chi_{[0,T]} f_u \in L^q(0,\infty; L^q(\Omega)^n)$. Since $g_d \in L^q(0,T; W_q^1(\Omega\backslash\Sigma)) \cap W_q^1(0,T; W_q^{-1}(\Omega))$, by Theorem 2.27 with $X_1 = W_q^1(\Omega\backslash\Sigma)$, $X_0 = W_q^{-1}(\Omega)$, we obtain a new function $\bar{g}_d := \mathcal{E}(g_d) \in L^q(0,\infty; W_q^1(\Omega\backslash\Sigma)) \cap W_q^1(0,\infty; W_q^{-1}(\Omega))$, which is uniformly bounded for $T \le T_0$. For $(g_b, g_u, g) \in W_q^{2-1/q,1-1/2q}(\partial\Omega \times (0,T))^n \times W_q^{2-1/q,1-1/2q}(\Sigma \times (0,T))^n \times W_q^{1-1/q,(1-1/q)/2}(\Sigma \times (0,T))^n$, Theorem 2.28 with $\alpha = 1 - 1/2q > 1/q$ and $(1-1/q)/2 > 1/q$ respectively imply that they can be extended as $(\bar{g}_b, \bar{g}_u, \bar{g}) := \mathcal{E}(g_b, g_u, g) \in W_q^{2-1/q,1-1/2q}(\partial\Omega \times (0,\infty))^n \times W_q^{2-1/q,1-1/2q}(\Sigma \times (0,\infty))^n \times W_q^{1-1/q,(1-1/q)/2}(\Sigma \times (0,\infty))^n$, which are uniformly bounded for $T \le T_0$. In summary,

$$(\bar{f}_u, \bar{g}_d, \bar{g}_b, \bar{g}_u, \bar{g})\big|_{[0,T]} = (f_u, g_d, g_b, g_u, g)$$

and

$$(\bar{f}_u, \bar{g}_d, \bar{g}_b, \bar{g}_u, \bar{g}) \in Z_\infty.$$

Now, for a constant $\omega > \omega_0 \ge 0$, define

$$(\tilde{f}_u, \tilde{g}_d, \tilde{g}_b, \tilde{g}_u, \tilde{g})(t) = e^{-\omega t}(\bar{f}_u, \bar{g}_d, \bar{g}_b, \bar{g}_u, \bar{g})(t).$$

Then it is easy to verify that $(\tilde{f}_u, \tilde{g}_d, \tilde{g}_b, \tilde{g}_u, \tilde{g})$ is also contained in Z_∞, since $e^{-\omega t}$ is smooth and bounded with respect to time t.

Let (u, π) be the solution of (8.4) in [PS16] with $(f_u, g_d, g_b, g_u, g) = (\tilde{f}_u, \tilde{g}_d, \tilde{g}_b, \tilde{g}_u, \tilde{g})$ given above, as well as the constant viscosity $\mu > 0$ in (3.39). For all $t \in \mathbb{R}_+$, we define

$$v(t) = e^{\omega t} u(t), \quad p(t) = e^{\omega t}\pi(t),$$

then (v, p) solves (3.39) for $t \in [0,T]$. Consequently, existence and regularity of (u, π), which are given by Theorem 8.1.4 in [PS16], imply those of (v, p). Additionally, (3.41) holds under our construction of (v, p).

Finally, we need to show that our solution is unique. To this end, let $(v_1, p_1) \ne (v_2, p_2)$ be two solutions of (3.39) in $(0,T)$ with same source terms and initial value. Define $(v, p) = (v_1 - v_2, p_1 - p_2)$. Since (3.39) is linear, (v, p) satisfies

$$\begin{aligned}
\varrho\partial_t v - \operatorname{div}(2\mu D v) + \nabla p &= 0, && \text{in } \Omega\backslash\Sigma \times (0,T),\\
\operatorname{div} v &= 0, && \text{in } \Omega\backslash\Sigma \times (0,T),\\
v &= 0, && \text{on } \partial\Omega \times (0,T),\\
[\![v]\!] &= 0, && \text{on } \Sigma \times (0,T),\\
[\![-2\mu D v + p\mathbb{I}]\!]\,\nu_\Sigma &= 0, && \text{on } \Sigma \times (0,T),\\
v|_{t=0} &= 0, && \text{in } \Omega\backslash\Sigma.
\end{aligned} \tag{3.43}$$

Multiplying the first equation of (3.43) by v and integrating by parts over $\Omega\backslash\Sigma \times (0,t)$, one obtains

$$\int_{\Omega\backslash\Sigma} \frac{\varrho}{2} |v(t)|^2 \,\mathrm{d}x + \int_0^t \int_{\Omega\backslash\Sigma} 2\mu \,|Dv(x,t)|^2 \,\mathrm{d}x\mathrm{d}t = 0, \quad \text{for a.e. } t \in (0,T),$$

which implies the uniqueness and completes the proof. □

Remark 3.22. For $(g_d, g_b \cdot \nu_{\partial\Omega}, g_u \cdot \nu_\Sigma) \in W_q^1(0,T; \widehat{W}_q^{-1}(\Omega))$, we notice that

$$\int_\Omega g_d \mathrm{d}x = \int_{\partial\Omega} g_b \cdot \nu_{\partial\Omega} \mathrm{d}(\partial\Omega) - \int_\Sigma g_u \cdot \nu_\Sigma \mathrm{d}\Sigma,$$

when $\phi = 1$ in (3.42), the regularity property of $\widehat{W}_q^{-1}(\Omega)$. Thus, for the zero-Dirichlet problem, which means $g_b = g_u = 0$ in (3.39), one has an hidden compatibility condition

$$\int_\Omega g_d \mathrm{d}x = 0.$$

This is an important condition when we solve the Stokes type problems with homogeneous Dirichlet boundary conditions.

3.5.2. Laplacian transmission problems with Dirichlet boundary. In this section, we investigate a transmission problem for the Laplacian equation with Dirichlet boundary condition, which reads

$$\begin{aligned} -\Delta\psi &= f && \text{in } \Omega\backslash\Sigma, \\ [\![\partial_\nu \psi]\!] &= g && \text{on } \Sigma, \\ [\![\psi]\!] &= h && \text{on } \Sigma, \\ \psi &= g_b && \text{on } \partial\Omega. \end{aligned} \tag{3.44}$$

Here, we denote the inner domain by Ω^-, resp. outer domain by Ω^+ and the unit normal vector on $\Sigma = \partial\Omega^-$ by ν.

The second result concerns strong solutions.

Proposition 3.23. *Let $1 < q < \infty$, $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with boundary $\partial\Omega$ of class C^{3-}, and let $\Sigma \subset \Omega$ be a closed hypersurface of class C^{3-}, $s \in \{0,1\}$. For all $f \in W_q^s(\Omega\backslash\Sigma)$, $g \in W_q^{1+s-1/q}(\Sigma)$, $h \in W_q^{2+s-1/q}(\Sigma)$, $g_b \in W_q^{2+s-1/q}(\partial\Omega)$, the problem (3.44) admits a unique solution $\psi \in W_q^{2+s}(\Omega\backslash\Sigma)$. Moreover, there is a constant $C > 0$ such that*

$$\|\psi\|_{W_q^{2+s}} \leq C \left(\|f\|_{W_q^s} + \|g\|_{W_q^{1+s-\frac{1}{q}}} + \|h\|_{W_q^{2+s-\frac{1}{q}}} + \|g_b\|_{W_q^{2+s-\frac{1}{q}}} \right).$$

Proof. ***Step 1: Reduction.*** We first reduce to the case $(h, g_b) = 0$. To this end, we find a φ solving

$$\begin{aligned} -\Delta\varphi &= 0 && \text{in } \Omega^-, \\ \varphi &= h && \text{on } \Sigma, \end{aligned}$$

and

$$\begin{aligned} -\Delta\varphi &= 0 && \text{in } \Omega^+, \\ \varphi &= 0 && \text{on } \Sigma, \\ \varphi &= g_b && \text{on } \partial\Omega. \end{aligned}$$

The existence and uniqueness of these two systems are clear due to elliptic theory. Thanks to the trace theorem, the extra outer normal derivatives terms on Σ enjoys the same regularities as g. Substracting φ from ψ, we can investigate the reduced system (3.44) with $(h, g_b) = 0$.
Step 2: Weak solution with L^2-setting. Now, let $H^k = W_2^k$ and $H_0^k = W_{2,0}^k$ for $k \in \mathbb{N}$. Testing (3.44) by a function $\phi \in H_0^1(\Omega)$ and integrating by parts, one obtains

$$\int_{\Omega\backslash\Sigma} \nabla\psi \cdot \nabla\phi \mathrm{d}x = \int_{\Omega\backslash\Sigma} f\phi \mathrm{d}x - \int_{\Sigma} g\phi \mathrm{d}\Sigma =: \langle F, \phi \rangle_{H^{-1}\times H_0^1},$$

as a result of the regularities of f and g. The Lax-Milgram Lemma implies existence of a unique weak solution $\psi \in H_0^1(\Omega)$ to (3.44) with $(h, g_b) = 0$.
Step 3: Truncation. Since the problem (3.44) with Neumann boundary conditions on $\partial\Omega$ has been uniquely solved, see e.g. Prüss–Simonett [PS16, Proposition 8.6.1], we show the propostion by a truncation method. More specifically, we choose a cutoff function $\eta \in C_0^\infty(\Omega)$ such that

$$\eta(x) = \begin{cases} 1, & \text{in a neighborhood of } \Omega^-, \\ 0, & \text{in a neighborhood of } \Omega^+, \end{cases}$$

We decompose $\psi = \eta\psi + (1-\eta)\psi =: u_1 + u_2$, where u_1 solves

$$\begin{aligned} -\Delta u_1 &= \eta f - 2\nabla\eta \cdot \nabla\psi + \psi\Delta\eta =: f^1 && \text{in } \Omega\backslash\Sigma, \\ [\![\partial_\nu u_1]\!] &= [\![\partial_\nu \psi]\!] = g && \text{on } \Sigma, \\ [\![u]\!] &= [\![\psi]\!] = 0 && \text{on } \Sigma, \\ \partial_\nu u_1 &= 0 && \text{on } \partial\Omega, \end{aligned}$$

weakly and u_2 solves

$$\begin{aligned} -\Delta u_2 &= (1-\eta) f + 2\nabla\eta \cdot \nabla\psi - \psi\Delta\eta =: f^2 && \text{in } \Omega, \\ u_2 &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Step 4: Improving the regularity. From Step 2, we already know that (3.44) admits a unique weak solution ψ enjoying the regularity $\nabla\psi \in L^2(\Omega)$, which means $f^i \in L^2(\Omega)$ in Step 3. By classical elliptic theory and [PS16], one obtains $u_1 \in H^2(\Omega\backslash\Sigma)$, $u_2 \in H_0^1(\Omega) \cap H^2(\Omega)$. Then $\psi \in H_0^1(\Omega) \cap H^2(\Omega\backslash\Sigma)$. Moreover,

$$\nabla\psi \in H^1(\Omega\backslash\Sigma) \hookrightarrow \begin{cases} L^p(\Omega\backslash\Sigma), & \text{if } 1 \le p < \infty, & n = 2, \\ L^p(\Omega\backslash\Sigma), & \text{if } 1 \le p \le p^* := \dfrac{2n}{n-2}, & n > 2, \end{cases}$$

due to the Sobolev Embedding Theorem. For $n = 2$, the right-hand side terms f^1 and f^2 in Step 3 are contained in $L^p(\Omega\backslash\Sigma)$, $1 \le p < \infty$. Consequently with $p = q$, Proposition 8.6.1 and Corollary 7.4.5 in Prüss–Simonett [PS16] indicate that $u_1 \in W_q^2(\Omega\backslash\Sigma)$ and $u_2 \in W_q^2(\Omega)$, which implies $\psi \in W_q^2(\Omega\backslash\Sigma)$. For $n > 2$, we have $f^i \in L^{p^*}$, $i = 1, 2$. Again by regularity results in [PS16], we have $u_1 \in W_{p^*}^2(\Omega\backslash\Sigma)$ and $u_2 \in W_{p^*}^2(\Omega)$ and hence

$$\nabla\psi \in W_{p^*}^1(\Omega\backslash\Sigma) \hookrightarrow \begin{cases} L^p(\Omega\backslash\Sigma), & 1 \le p < \infty, & n = q^*, \\ L^p(\Omega\backslash\Sigma), & 1 \le p \le p^{**} := \dfrac{np^*}{n-p^*}, & n > p^*, \\ C^\alpha(\overline{\Omega\backslash\Sigma}), & 0 < \alpha \le 1 - \dfrac{n}{p^*} & 2 < n < p^*. \end{cases}$$

For the first and third cases, we find $f^i \in L^p(\Omega\backslash\Sigma)$, $i = 1, 2$, $1 \leq p < \infty$, and then get the regularity of ψ. For the second case, we know $p^{**} = \frac{np^*}{n-p^*} > p^*$. Therefore, by a bootstrapping argument, we can always increase the integration exponent until we obtain L^q. Thus, by Proposition 8.6.1 and Corollary 7.4.5 in Prüss–Simonett [PS16], one obtains $u_1 \in W_q^2(\Omega\backslash\Sigma)$ and $u_2 \in W_q^2(\Omega)$, i.e., $\psi \in W_q^2(\Omega\backslash\Sigma)$ with the estimate

$$\|\psi\|_{W_q^2(\Omega\backslash\Sigma)} \leq C \left(\|f\|_{L^q(\Omega\backslash\Sigma)} + \|g\|_{W_q^{1-\frac{1}{q}}(\Sigma)} + \|h\|_{W_q^{2-\frac{1}{q}}(\Sigma)} + \|g_b\|_{W_q^{2-\frac{1}{q}}(\partial\Omega)} \right),$$

for some constant $C > 0$. Then as above, one gets $f^i \in W_q^1(\Omega\backslash\Sigma)$, $i = 1, 2$. With the help of Proposition 8.6.1 and Corollary 7.4.5 in Prüss–Simonett [PS16], we have the desired regularity and estimate with $s = 1$. □

Chapter 4

Fluid-Structure Interaction Problem with Growth in Cylindrical Domain

This chapter concerns a free-interface fluid-structure interaction problem for plaque growth with additional viscoelastic effects, which was also investigated in Chapter 3 (see [AL23a] for the published version). Compared to it, the problem is posed in a cylindrical domain with ninety-degree contact angles, which brings additional difficulties when we deal with the linearization of the system. By a reflection argument, we obtain the existence and uniqueness of strong solutions to the model problems for the linear systems, which are then shown to be well-posed in a cylindrical (annular) domain via a localization procedure. Finally, we prove that the full nonlinear system admits a unique strong solution locally in time with the aid of the contraction mapping principle.

Overview of This Chapter

Notations. In this chapter, we consider specifically the following notations.

- $\mathbf{v}$, $\hat{\mathbf{v}}$, the Eulerian and Lagrangian velocity
- c, $\hat{c}$, cell concentrations
- c^*, $\hat{c}^*$ foam cell concentration
- g, $\hat{g}$, growth metrics
- $\hat{\mathbf{F}}$, the deformation gradient in terms of $\hat{\mathbf{v}}$
- $\mathbf{F}$, the inverse deformation gradient

When there is no danger of confusion, we specify the quantities with a subscript "f" and "s" to identify those defined in fluid domain and solid domain respectively. In addition, without a special statement, the quantities or operators with a hat "$\hat{\cdot}$" will indicate those in Lagrangian coordinates.

4.1. Introduction

In this chapter, we focus on a 3d free-boundary fluid-structure interaction problem for plaque growth, which was also addressed in Chapter 3. To be more precise, the blood is assumed to be described by the incompressible Navier–Stokes equation and the artery is modeled as an elastic material with viscoelasticity, while inside the blood flow and vessel wall the cells react, leading to the plaque formation, see e.g. Chapter 1 or Yang–Jäger–Neuss-Radu–Richter [Yan+16]. Define $\Omega^t := \Omega^t_f \cup \Omega^t_s \cup \Sigma^t \subset \mathbb{R}^3$ (see Figure 4.1), with three disjoint parts, where Ω^t_f, Ω^t_s are piece-wise smooth domains for the fluid and solid respectively, while Σ^t is a two dimensional sub-manifold of $\mathbb{R}^3$ with boundary $\partial\Sigma^t$. In particular, $\partial\Omega^t = \overline{G^t} \cup S$, $\partial\Omega^t_f = \overline{G^t_f} \cup \Sigma^t$ and $\partial\Omega^t_s = \overline{G^t_s} \cup \Sigma^t \cup S$, where

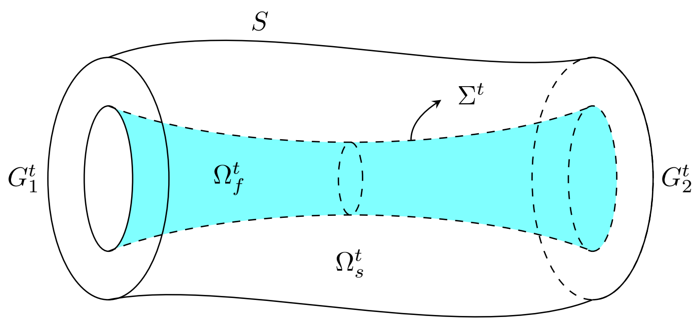

Figure 4.1: Deformed cylindrical domain.

$G^t := G^t_1 \cup G^t_2 \cup \partial\Sigma^t$ is a hypersurface with $G^t_\beta := G^t_{1,\beta} \cup G^t_{2,\beta}$, $\beta \in \{f,s\}$, $G^t_i = G^t_{i,f} \cup G^t_{i,s}$, $\overline{G^t_{i,f}} \subset G^t_i$, $i \in \{1,2\}$, and S denotes the fixed surrounding surface, which is supposed to be perpendicular to G^t at ∂S. Moreover, Σ^t is assumed to be perpendicular to G^t at $\partial\Sigma^t$ as well. In such setting, the domain is endowed with the fixed contact line ∂S and moving contact line $\partial\Sigma^t$ with ninety-degree contact angles for a short time, while in [AL23a], we considered a smooth domain without contact.

Recalling from (1.36) and (1.37), we have

$$\begin{aligned}
\rho_f\left(\partial_t + \mathbf{v}_f\cdot\nabla\right)\mathbf{v}_f &= \operatorname{div}\mathbb{T}_f, && \text{in } \Omega^t_f,\ t\in(0,T), && (4.1a)\\
\operatorname{div}\mathbf{v}_f &= 0, && \text{in } \Omega^t_f,\ t\in(0,T), && (4.1b)\\
\rho_s\left(\partial_t + \mathbf{v}_s\cdot\nabla\right)\mathbf{v}_s &= \operatorname{div}\mathbb{T}_s, && \text{in } \Omega^t_s,\ t\in(0,T), && (4.1c)\\
\rho_s\operatorname{div}\mathbf{v}_s &= \gamma\beta c_s, && \text{in } \Omega^t_s,\ t\in(0,T), && (4.1d)\\
\partial_t c_f + \mathbf{v}_f\cdot\nabla c_f - \operatorname{div}(D_f\nabla c_f) &= 0, && \text{in } \Omega^t_f,\ t\in(0,T), && (4.1e)\\
\partial_t c_s + \mathbf{v}_s\cdot\nabla c_s - \operatorname{div}(D_s\nabla c_s) &= -\beta c_s, && \text{in } \Omega^t_s,\ t\in(0,T), && (4.1f)\\
\partial_t c^*_s + \mathbf{v}_s\cdot\nabla c^*_s &= \beta c_s, && \text{in } \Omega^t_s,\ t\in(0,T), && (4.1g)\\
\partial_t g + \mathbf{v}_s\cdot\nabla g &= \frac{\gamma\beta c_s}{3\rho_s}, && \text{in } \Omega^t_s,\ t\in(0,T), && (4.1h)\\
[\![\mathbf{v}]\!] = 0, \quad [\![\mathbb{T}]\!]\,\mathbf{n}_{\Sigma^t} &= 0, && \text{on } \Sigma^t,\ t\in(0,T), && (4.1i)\\
[\![D\nabla c]\!]\cdot\mathbf{n}_{\Sigma^t} = 0, \quad \zeta\,[\![c]\!] - D_s\nabla c_s\cdot\mathbf{n}_{\Sigma^t} &= 0, && \text{on } \Sigma^t,\ t\in(0,T), && (4.1j)\\
\mathbf{v}_s &= 0, && \text{on } S,\ t\in(0,T), && (4.1k)\\
D_s\nabla c_s\cdot\mathbf{n}_S &= 0, && \text{on } S,\ t\in(0,T), && (4.1l)
\end{aligned}$$

$$\mathcal{P}_{G^t}(\mathbf{v}) = 0, \quad (\mathbb{T}\mathbf{n}_{G^t})\mathbf{n}_{G^t} = 0, \qquad \text{on } G^t \setminus \Sigma^t,\ t \in (0,T), \tag{4.1m}$$

$$D\nabla c \cdot \mathbf{n}_{G^t} = 0, \qquad \text{on } G^t \setminus \Sigma^t,\ t \in (0,T), \tag{4.1n}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}^0,\ c|_{t=0} = c^0,\ c_s^*|_{t=0} = 0,\ g|_{t=0} = 1, \tag{4.1o}$$

where $\rho_f > 0$ is the fluid density. The tensor $\mathbb{T}_f = -\pi_f\mathbb{I} + \nu_f(\nabla\mathbf{v}_f + \nabla\mathbf{v}_f^\top)$ denotes the Cauchy stress tensor, the unknown function π_f is the fluid pressure and $\nu_f > 0$ represents the fluid viscosity. For the solid, the constant $\rho_s > 0$ is the solid density. The tensor $\mathbb{T}_s = \mathbb{T}_s^e + \mathbb{T}_s^v$ stands for the stress tensor satisfying

$$\mathbb{T}_s^e = -\pi_s\mathbb{I} + \mu_s\left(\mathbf{F}_{s,e}^{-1}\mathbf{F}_{s,e}^{-\top} - \mathbb{I}\right), \quad \mathbb{T}_s^v = \nu_s\left(\partial_t\mathbf{F}_s^{-1} + \partial_t\mathbf{F}_s^{-\top}\right)\mathbf{F}_s^{-\top}.$$

Here, π_s denotes the solid pressure, μ_s, ν_s represents the Lamé coefficient and viscosity respectively, which are all constant. The elastic tensor $\mathbb{T}_s^e$ is given by the constitutive relation of an incompressible Neo-Hookean material, which is hyperelastic, isotropic and incompressible. Note that $\mathbf{F}_s = \hat{\mathbf{F}}_s^{-1}$ is the inverse deformation gradient and $\mathbf{F}_{s,e}$ denotes an inverse elastic tensor under the assumption of growth as in Section 3.1. Similarly, the Kelvin–Voigt stress tensor $\mathbb{T}_s^e$ is introduced. For more discussions about it, readers are referred to Chapter 1.

The conditions (4.1i) and (4.1j) on the interface are exactly the same as in Chapter 3. Moreover, S is supposed to be the rigid part of the boundary, i.e.,

$$\mathbf{v}_s = 0 \quad \text{on } S, \tag{4.2}$$

which means that the outside of the blood vessel is fixed.

Now for the boundary conditions on G^t, we need a more careful consideration to make sure that they are physically meaningful and compatible to the conditions on S and Σ^t respectively. In this work, the outflow conditions (4.1m) is employed, where $\mathcal{P}_{G^t} := I - \mathbf{n}_{G^t} \otimes \mathbf{n}_{G^t}$ denotes the tangential projection onto G^t.

Remark 4.1. We comment that G^t is a free surface. Since the vessel is cut in the Lagrangian coordinate, the cross sections then turn to be free when we recovered it in the Eulerian coordinate. In general, it can be static with suitable boundary conditions.

Remark 4.2. Now we discuss more about the choice of the outflow boundary conditions on G^t.

(1) The boundary condition is not possible to be of the Dirichlet type, i.e., *no-slip* condition. On one hand, a Dirichlet boundary yields a the rigid part. For example $\mathbf{v} = 0$ means that the displacement $\int_0^t \mathbf{v}\mathrm{d}t = 0$ vanishes on the boundary, which contradicts our consideration, as in Remark 4.1. On the other hand, a *no-slip* condition leads to an incompatibility at the moving contact lines $\partial\Sigma^t$. Specifically, on the interface Σ^t the normal velocity is $V_{\Sigma^t} := \mathbf{v} \cdot \mathbf{n}_{\Sigma^t}$, while at $\partial\Sigma^t$ one derives $V_{\Sigma^t} = 0$ according to the vanishing Dirichlet condition, contradicting a "moving contact line" setting.

(2) In some literature, e.g. [GT18, GT23], the well-known Navier-slip condition was employed for the slip of the fluid along the solid with dynamic contact lines, while in [Wil13], Wilke showed that an incompatibility with moving contact lines happens for a two-phase Navier–Stokes problem in a cylindrical domain. So they considered a so-called *pure-slip* condition. However, in our model, the surface G^t is not supposed to be fixed and it is physically meaningless if we take the *pure-slip* condition into account. Thus, an outflow boundary condition is proper and fits to the reality of the blood flow in a vessel if we cut the human vessels along the cross section to get a bounded domain.

(3) For the sake of analysis, the outflow condition allows for a reflection argument at the contact lines so that the reflected solutions of model problems are endowed with the symmetry, in view of the divergence-free condition of the solutions. See Section 4.3 for more details.

For further analysis, we define the initial domain as $\Omega = \Omega_f \cup \Omega_s \cup \Sigma$, where $\Omega_f := \Omega_f^0$, $\Omega_s := \Omega_s^0$ and $\Sigma := \Sigma^0$ that is supposed to be perpendicular to the assumed flat initial surface $G := G^0$ at $\partial\Sigma$ just like S. Then the deformation from initial configuration to current one is defined via a motion φ with

$$x = \varphi(X,t) = X + \int_0^t \hat{\mathbf{v}}(X,\tau)\mathrm{d}\tau, \quad \forall X \in \Omega, \tag{4.3}$$

as well as $x|_{t=0} = \varphi(X,0) = X$. Then the deformation gradient has the form of

$$\hat{\mathbf{F}}(\hat{\mathbf{v}})(X,t) = \frac{\partial}{\partial X}\varphi(X,t) = \hat{\nabla}\varphi(X,t) = \mathbb{I} + \int_0^t \hat{\nabla}\hat{\mathbf{v}}(X,\tau)\mathrm{d}\tau, \tag{4.4}$$

for all $X \in \Omega$ with initial deformation $\hat{\mathbf{F}}|_{t=0} = \mathbb{I}$ and by $\hat{J} := \det \hat{\mathbf{F}}$ its determinant. Conversely, we have the inverse deformation gradient by $\mathbf{F}(\mathbf{v})(x,t) = \hat{\mathbf{F}}^{-1}$. As in Section 1.6.1, we assume the decomposition of $\hat{\mathbf{F}}_s$ as

$$\hat{\mathbf{F}}_s = \hat{\mathbf{F}}_{s,e}\hat{\mathbf{F}}_{s,g}, \quad \text{in } \Omega_s,$$

where $\hat{\mathbf{F}}_{s,g}$ is the growth tensor and $\hat{\mathbf{F}}_{s,e}$ represents the elastic tensor. Then the corresponding determinants are

$$\hat{J}_{s,g} = \det \hat{\mathbf{F}}_{s,g}, \quad \hat{J}_{s,e} = \det \hat{\mathbf{F}}_{s,e}, \quad \text{in } \Omega_s,$$

with $\hat{J}_s = \hat{J}_{s,g}\hat{J}_{s,e}$.

Moreover, with constant-density growth hypothesis, one ends up with (see (1.30) for the derivation)

$$\mathrm{tr}(\hat{\mathbf{F}}_{s,g}^{-1}\partial_t\hat{\mathbf{F}}_{s,g}) = \gamma\beta c_s, \quad \text{in } \Omega_s.$$

In addition, the growth is assumed to be isotropic, i.e.,

$$\hat{\mathbf{F}}_{s,g} = \hat{g}\mathbb{I}, \quad \text{in } \Omega_s,$$

where $\hat{g} = \hat{g}(X,t)$ is the metric of growth, a scalar function depending on the concentration of macrophages. Hence,

$$\hat{\mathbf{F}}_{s,e} = \frac{1}{\hat{g}}\hat{\mathbf{F}}_s, \quad \hat{J}_{s,g} = \hat{g}^3,$$

where 3 is the dimension of space. Consequently, under the assumption of constant-density growth, one deduces the equation for growth in Lagrangian coordinates

$$\partial_t\hat{g} = \frac{\gamma\beta\hat{c}_s}{3\hat{\rho}_s}\hat{g}, \quad \text{in } \Omega_s. \tag{4.5}$$

This shows the specific dependence of $\hat{g}$ on $\hat{c}_s$.

4.1.1. A short review of contact angle problems. In Section 3.1.2, we already recalled the literature of analysis of fluid-structure interaction problems. Concerning contact angle problems in fluid dynamics, it is a challenging problem and is not yet well understood, especially with moving contact lines and dynamic contact angles.

For the mathematical analysis, there are only limited results. Schweizer [Sch01] studied a 2D Navier–Stokes problem with a fixed contact angle of $\frac{\pi}{2}$. Bodea [Bod06] analysis a similar problem with fixed $\frac{\pi}{2}$ contact angle in 3D channels with periodicity in one direction. The dynamics of a 2D drop with fixed contact angle when the fluid is assumed to be governed by Darcy' s law are investigated by Knüpfer–Masmoudi [KM13, KM15]. Related analysis of the fully stationary Navier−Stokes system with free, but unmoving boundary, was carried out in 2D by Solonnikov [Sol95] with contact angle fixed at π, by Jin [JJ05] in 3D with angle $\frac{\pi}{2}$. In recent work [GT18, GT23], Guo–Tice considered (Navier–)Stokes equations that integrate boundary conditions allowing for full motion of the contact points and angles. They prove that the solutions exists globally close to equilibrium with contact angles between 0 and π, and decay to equilibrium at an exponential rate. Tice–Wu [TW21] and [ZT17] proved corresponding results for the Stokes droplet problem and established local existence results.

As far as we know, with a ninety-degree contact angle, Wilke [Wil13, Wil20] gave a complete and outstanding analysis of two-phase Navier–Stokes equations with surface tension in cylindrical domains, using the framework of maximal L^p-regularity theory. Then Rauchecker [Rau20] and Rauchecker–Wilke [RW20] extended the results to a two-phase Navier–Stokes/Stefan problem and a two-phase Navier–Stokes/Mullins–Sekerka system with boundary contact, respectively. The general methods in these papers are the localization procedure and the key element is the reflection argument for the model problems with respect to the linearized systems, thanks to the assumption of *ninety-degree* contact angle. Considering various angles, Köhne–Saal–Westermann [KSW21] proved that the solution of the stationary and the instationary Stokes equations subject to perfect slip boundary conditions on a 2D wedge domain admits optimal regularity in the L^p-setting, in particular it is W_p^2 in space.

4.1.2. Main features. In Chapter 3, we firstly established the short time existence of strong solutions to the considered plaque growth model including viscoelastic effects in a smooth domain using the maximal regularity theory. Motivated by these preceding results above, especially Wilke [Wil20], we will investigate the local existence of strong solutions to (4.1) in a cylindrical domain. In other words, we attempt to extend the results in Chapter 3 to the system in a cylindrical domain with ninety-degree contact angles, which is close to human arteries.

First of all, let us comment that we include the viscoelastic effects into the model as in Chapter 3, which yields the parabolicity of the solid equation. To be precise, for a short time, the term describing the Kelvin–Voigt viscoelasticity leads to a principal regularity part of the equation, and yields a two-phase Stokes type problem for the linearized fluid-structure interaction problem. This provides the solvabilities and maximal regularities of the solutions. Throughout the proof based on the maximal regularity of type L^q that has been used in [PS10, PS16, Wil13], one obtains a semi-flow for the free boundary problem in a natural phase space. In particular, there is no loss of regularity and no more additional compatibility conditions needed. In this work, we only consider the three dimensional case for the sake of simplicity. In fact, it could be any dimension $d \geq 2$ as long as q has a modification restriction with respect to d. This is also an advantage of the maximal L^q-regularity theory.

In the present chapter, the boundary is not smooth any more since we have ninety-degree contact angles, so even for the linear system, there seem to be no well-posedness result that can be applied. To circumvent this problem, a localization procedure will be employed as in [PS16, Wil20]. More precisely, under suitable partition of unity for the cylindrical domain, we try to solve several kinds of model problems, for which we proceed using reflection arguments for the model problems in quarter (bent) spaces and half (bent) spaces, thanks to the ninety-degree contact angle. Then by a Neumann series argument, one can conclude the existence results of model problems and derive the well-posedness for the related linear system. However, we have no chance to consider all possible boundary conditions around the contact line, which play an essential role in both physics and mathematics. For example, as commented in [Wil20] no-slip condition may lead to a paradoxon for the moving contact line, see also e.g. [PS82, GT18, GT23] and references therein. For the purpose they imposed a pure-slip condition on the boundary with contact, which guarantees that a reflection argument can be used. As in Remark 4.2, an outflow boundary condition is chosen on the surface G^t in our work. Here on the one hand, the outflow condition coincides with the situation of blood flow in vessels and does not violate the physical reality of a moving contact line. On the other hand, the symmetry of the reflected solutions on the boundary can be ensured from the perspective of mathematical analysis. For instance in the three dimensional case, if one carries out the reflection argument for Stokes equations in a quarter space with respect to $x_2 = 0$, the divergence-free condition $\operatorname{div} u = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0$ should also hold on $x_2 = 0$. Then once u_1, u_3 are odd function with regard to $x_2 = 0$, $\partial_2 u_2$ must be an odd function, which is exactly associated with the outflow boundary condition.

Moreover, due to the presence of the outflow boundary condition on G in (4.10), when we applied the localization argument, several auxiliary problems, namely, elliptic/Laplace transmission problem in a cylindrical domain with a Dirichlet boundary condition on G and parabolic transmission problem with a Neumann boundary condition, are involved, see Section 4.7. Note that in [Wil20], similar auxiliary problems defined in a vertical cylindrical domain with a horizontal interface instead were analyzed. Our results in Section 4.7 are the extensions to [Wil20, Appendix 5.3], where a Neumann boundary condition was impose on the vertical surface.

Let us finally point out that one of our main results obtained is the well-posedness of the nonstationary two-phase Stokes problem in a general cylindrical domain with an outflow boundary condition and a ninety-degree contact angle, as well as the heat equation in both a cylindrical domain and a cylindrical annular domain. Since we considered the general situation of the domain, i.e., smooth hypersurfaces S and Σ, the results can be applied to solve broader nonlinear problems.

4.1.3. Outline. In Section 4.2 we reformulate the system by transferring the system from the deformed configuration to the reference one, and present the well-posedness theorems for the linearized systems in Subsection 4.2.2. Section 4.3 is devoted to the analysis of the corresponding model problems. The main results of this section are well-posedness of these new model problems with outflow boundary conditions. In particular, the reflection argument is employed to investigate the solvability of a two-phase Stokes equation with boundary contact, as well as two heat equations. In Section 4.4, we make an observation of additional regularity of the pressure for the first step. Then the reduction argument is carried out for the sake of analysis. Finally we prove the well-posedness of the two-phase Stokes equation in general cylindrical domain by a localization procedure. Based on the solvability results in the previous section, in Section 4.5 the full nonlinear system is shown to be well-posed locally via the Banach fixed-point theorem. In addition, the existence of a partition of our domain is given in Section 4.6 and the existence and uniqueness of auxiliary elliptic and parabolic transmission problems are analyzed in Section 4.7.

4.2. Preliminaries

4.2.1. Reformulation in the reference configuration. In this section, we transform the free-interface fluid-structure problem with growth from the deformed configuration to a fixed reference configuration and state the main result. For quantities in different configurations, we define

$$\begin{aligned} &\hat{\mathbf{v}}(X,t) = \mathbf{v}(x,t),\ \hat{\pi}(X,t) = \pi(x,t),\ \hat{\mathbb{T}}(X,t) = \mathbb{T}(x,t), \\ &\hat{\rho}(X,t) = \rho(x,t),\ \hat{\mu}(X,t) = \mu(x,t),\ \hat{\nu}(X,t) = \nu(x,t), \end{aligned} \tag{4.6}$$

for all $x = \varphi(X,t),\ X \in \Omega$ and $t \geq 0$. For the fluid part, it follows from Proposition 1.12 that

$$\hat{J}_f = 1, \quad \text{in } \Omega_f. \tag{4.7}$$

For the solid part, since the deformation from natural configuration Ω_s^g to the deformed configuration Ω_s^t conserves mass, incompressibility yields $\hat{J}_{s,e} = 1$ and hence

$$\hat{J}_s = \hat{J}_{s,g} = \hat{g}^n, \quad \text{in } \Omega_s.$$

Now combining Propositions 1.12, 1.15 and 1.65, and Theorem 1.35, we rewrite the fluid–structure interaction problem (4.1) in Lagrangian coordinate and rearrange the equation to obtain the system for $t \in J := (0,T)$, $T > 0$,

$$\begin{aligned}
\left.\begin{aligned}
\hat{\rho}_f \partial_t \hat{\mathbf{v}}_f - \widehat{\operatorname{div}}\, \mathbf{S}(\hat{\mathbf{v}}_f, \hat{\pi}_f) &= \mathbf{K}_f \\
\widehat{\operatorname{div}}\, \hat{\mathbf{v}}_f &= G_f
\end{aligned}\right\} &\quad \text{in } \Omega_f \times J, \\
\left.\begin{aligned}
\hat{\rho}_s \partial_t \hat{\mathbf{v}}_s - \widehat{\operatorname{div}}\, \mathbf{S}(\hat{\mathbf{v}}_s, \hat{\pi}_s) &= \bar{\mathbf{K}}_s + \mathbf{K}_s^g =: \mathbf{K}_s \\
\widehat{\operatorname{div}}\, \hat{\mathbf{v}}_s - \frac{\gamma\beta}{\hat{\rho}_s} \hat{c}_s &= G_s
\end{aligned}\right\} &\quad \text{in } \Omega_s \times J, \\
[\![\hat{\mathbf{v}}]\!] = 0, \quad [\![\mathbf{S}(\hat{\mathbf{v}}, \hat{\pi})]\!]\, \hat{\mathbf{n}}_\Sigma = \mathbf{H}^1 &\quad \text{on } \Sigma \times J, \\
\mathcal{P}_G(\hat{\mathbf{v}}) = \mathbf{H}^2, \quad \mathbf{S}(\hat{\mathbf{v}}, \hat{\pi}) \hat{\mathbf{n}}_G \cdot \hat{\mathbf{n}}_G = H^3 &\quad \text{on } G \backslash \Sigma \times J, \\
\hat{\mathbf{v}}_s = 0 &\quad \text{on } S \times J, \\
\hat{\mathbf{v}}|_{t=0} = \hat{\mathbf{v}}^0 &\quad \text{in } \Omega \backslash \Sigma, \\
\partial_t \hat{c}_f - \hat{D}_f \,\widehat{\ }\, \hat{c}_f = F_f^1 &\quad \text{in } \Omega_f \times J, \\
\partial_t \hat{c}_s - \hat{D}_f \,\widehat{\ }\, \hat{c}_s = \bar{F}_s^1 + F_s^g =: F_s^1 &\quad \text{in } \Omega_s \times J, \\
\left.\begin{aligned}
\hat{D}_f \hat{\nabla} \hat{c}_f \cdot \hat{\mathbf{n}}_\Sigma = \hat{D}_s \hat{\nabla} \hat{c}_s \cdot \hat{\mathbf{n}}_\Sigma + \bar{F}_f^2 &=: F_f^2 \\
\hat{D}_s \hat{\nabla} \hat{c}_s \cdot \hat{\mathbf{n}}_\Sigma = \zeta\, [\![\hat{c}]\!] + \bar{F}_s^2 &=: F_s^2
\end{aligned}\right\} &\quad \text{on } \Sigma \times J, \\
\hat{D} \hat{\nabla} \hat{c} \cdot \hat{\mathbf{n}}_G = F^3 &\quad \text{on } G \backslash \partial\Sigma \times J, \\
\hat{D}_s \hat{\nabla} \hat{c}_s \cdot \hat{\mathbf{n}}_S = F^4 &\quad \text{on } S \times J, \\
\hat{c}|_{t=0} = \hat{c}^0 &\quad \text{in } \Omega \backslash \Sigma, \\
\partial_t \hat{c}_s^* - \beta \hat{c}_s = F^5, \quad \partial_t \hat{g} - \frac{\gamma\beta}{n\hat{\rho}_s} \hat{c}_s = F^6 &\quad \text{in } \Omega_s \times J, \\
\hat{c}_s^*|_{t=0} = 0, \quad \hat{g}|_{t=0} = 1 &\quad \text{in } \Omega_s,
\end{aligned} \tag{4.8}$$

where $\mathbf{S}(\hat{\mathbf{v}}, \hat{\pi}) = -\hat{\pi}\mathbb{I} + \hat{\nu}\left(\hat{\nabla}\hat{\mathbf{v}} + \hat{\nabla}\hat{\mathbf{v}}^\top\right)$ and

$$\begin{aligned}
&\mathbf{K}_f = \widehat{\operatorname{div}}\, \tilde{\mathbf{K}}_f, \quad \bar{\mathbf{K}}_s = \widehat{\operatorname{div}}\, \tilde{\mathbf{K}}_s, \quad \mathbf{K}_s^g = -\left(\hat{\mathbb{T}}_s \hat{\mathbf{F}}_s^{-\top}\right) \frac{n \hat{\nabla} \hat{g}}{\hat{g}}, \\
&G = -\left(\hat{\mathbf{F}}^{-\top} - \mathbb{I}\right) : \hat{\nabla}\hat{\mathbf{v}}, \quad \mathbf{H}^1 = -\left[\!\!\left[\tilde{\mathbf{K}}\right]\!\!\right] \hat{\mathbf{n}}_\Sigma, \\
&\mathbf{H}^2 = -\left(\mathbb{I} - \left((\hat{\mathbf{F}}^{-\top}\hat{\mathbf{n}}_G) \otimes (\hat{\mathbf{F}}^{-\top}\hat{\mathbf{n}}_G) - \hat{\mathbf{n}}_G \otimes \hat{\mathbf{n}}_G\right)\right) \hat{\mathbf{v}}, \\
&H^3 = -\hat{\mathbb{T}}\hat{\mathbf{F}}^{-\top}\hat{\mathbf{n}}_G \cdot (\hat{\mathbf{F}}^{-\top}\hat{\mathbf{n}}_G) + \mathbf{S}(\hat{\mathbf{v}}, \hat{\pi}) \hat{\mathbf{n}}_G \cdot \hat{\mathbf{n}}_G, \quad F_f^1 = \widehat{\operatorname{div}}\, \tilde{F}_f, \\
&\bar{F}_s^1 = \widehat{\operatorname{div}}\, \tilde{F}_s, \quad F_s^g = -\beta \hat{c}_s \left(1 + \frac{\gamma}{\hat{\rho}_s} \hat{c}_s\right) - \frac{n \hat{\nabla} \hat{g}}{\hat{g}} \cdot \left(\hat{D}_s \hat{\mathbf{F}}_s^{-1} \hat{\mathbf{F}}_s^{-\top} \hat{\nabla} \hat{c}_s\right), \\
&\bar{F}_f^2 = -\left[\!\!\left[\tilde{F}\right]\!\!\right] \cdot \hat{\mathbf{n}}_\Sigma, \quad \bar{F}_s^2 = -\tilde{F}_s \cdot \hat{\mathbf{n}}_\Sigma, \quad F^3 = \tilde{F} \cdot \hat{\mathbf{n}}_G, \quad F^4 = -\tilde{F}_s \cdot \hat{\mathbf{n}}_S, \\
&F^5 = -\frac{\gamma\beta}{\hat{\rho}_s} \hat{c}_s \hat{c}_s^*, \quad F^6 = -\frac{\gamma\beta}{n\hat{\rho}_s} \hat{c}_s (\hat{g} - 1),
\end{aligned} \tag{4.9}$$

with

$$
\begin{aligned}
&\hat{\mathbb{T}}_f = -\hat{\pi}_f \mathbb{I} + \hat{\nu}_f \left(\hat{\mathbf{F}}_f^{-1} \hat{\nabla} \hat{\mathbf{v}}_f + \hat{\nabla} \hat{\mathbf{v}}_f^\top \hat{\mathbf{F}}_f^{-\top} \right), \quad \hat{\mathbb{T}}_s = \hat{\mathbb{T}}_s^e + \hat{\mathbb{T}}_s^v, \\
&\hat{\mathbb{T}}_s^e = -\hat{\pi}_s \mathbb{I} + \hat{\mu}_s \left(\frac{1}{(\hat{g})^2} \hat{\mathbf{F}}_s \hat{\mathbf{F}}_s^\top - \mathbb{I} \right), \quad \hat{\mathbb{T}}_s^v = \hat{\nu}_s \left(\hat{\nabla} \hat{\mathbf{v}}_s + \hat{\nabla} \hat{\mathbf{v}}_s^\top \right) \hat{\mathbf{F}}_s^\top, \\
&\tilde{\mathbf{K}}_f = -\hat{\pi}_f \left(\hat{\mathbf{F}}_f^{-\top} - \mathbb{I} \right) + \nu_f \left(\hat{\mathbf{F}}_f^{-1} \hat{\nabla} \hat{\mathbf{v}}_f + \hat{\nabla} \hat{\mathbf{v}}_f^\top \hat{\mathbf{F}}_f^{-\top} \right) \left(\hat{\mathbf{F}}_f^{-\top} - \mathbb{I} \right) \\
&\qquad\qquad + \nu_f \left(\left(\hat{\mathbf{F}}_f^{-1} - \mathbb{I} \right) \hat{\nabla} \hat{\mathbf{v}}_f + \hat{\nabla} \hat{\mathbf{v}}_f^\top \left(\hat{\mathbf{F}}_f^{-\top} - \mathbb{I} \right) \right), \\
&\tilde{\mathbf{K}}_s = -\hat{\pi}_s \left(\hat{\mathbf{F}}_s^{-\top} - \mathbb{I} \right) + \mu_s \left(\frac{1}{\hat{g}^2} \left(\hat{\mathbf{F}}_s - \mathbb{I} \right) + \left(\frac{1}{\hat{g}^2} - 1 \right) \mathbb{I} - \left(\hat{\mathbf{F}}_s^{-\top} - \mathbb{I} \right) \right), \\
&\tilde{F} = \hat{D} \left(\hat{\mathbf{F}}^{-1} \hat{\mathbf{F}}^{-\top} - \mathbb{I} \right) \hat{\nabla} \hat{c}.
\end{aligned}
$$

For (4.8), we introduce the corresponding function spaces for the solutions

$$
\begin{gathered}
\mathbb{E}_{\hat{\mathbf{v}}}(J) := \left\{ \mathbf{v} \in W_q^1(J; L^q(\Omega)^3) \cap L^q(J; W_q^2(\Omega \backslash \Sigma)^3) : [\![\mathbf{v}]\!] = 0, \ \mathbf{v}|_S = 0 \right\}, \\
\mathbb{E}_{\hat{\pi}}(J) := \left\{ \begin{aligned} &\pi \in L^q(J; \dot{W}_q^1(\Omega \backslash \Sigma)) : \\ &\qquad [\![\pi]\!] \in W_q^{\frac{1}{2} - \frac{1}{2q}}(J; L^q(\Sigma)) \cap L^q(J; W_q^{1-\frac{1}{q}}(\Sigma)), \\ &\qquad \pi|_G \in W_q^{\frac{1}{2} - \frac{1}{2q}}(J; L^q(G)) \cap L^q(J; W_q^{1-\frac{1}{q}}(G \backslash \partial\Sigma)) \end{aligned} \right\}, \\
\mathbb{E}_{\hat{c}}(J) := W_q^1(J; L^q(\Omega)) \cap L^q(J; W_q^2(\Omega \backslash \Sigma)), \\
\mathbb{E}_{\hat{c}_s^*}(J) := W_q^1(J; W_q^1(\Omega_s)), \quad \mathbb{E}_{\hat{g}}(J) := W_q^1(J; W_q^1(\Omega_s)), \\
\mathbb{E}(J) := \mathbb{E}_{\hat{\mathbf{v}}} \times \mathbb{E}_{\hat{\pi}} \times \mathbb{E}_{\hat{c}} \times \mathbb{E}_{\hat{c}_s^*} \times \mathbb{E}_{\hat{g}},
\end{gathered}
$$

as well as the spaces for initial data

$$
X_{\gamma, \hat{\mathbf{v}}} := W_q^{2-\frac{2}{q}}(\Omega \backslash \Sigma)^3, \quad X_{\gamma, c} := W_q^{2-\frac{2}{q}}(\Omega \backslash \Sigma), \quad X_\gamma := X_{\gamma, \hat{\mathbf{v}}} \times X_{\gamma, c}.
$$

Then the main result of (4.8) reads as follows.

Theorem 4.3. *Let $q > 5$, $\Omega \subset \mathbb{R}^3$ be a domain defined as above with Σ, G, S of class C^3 and ∂G, $\partial\Sigma$ of C^4 as well. Given $(\hat{\mathbf{v}}^0, \hat{c}^0) \in X_\gamma$ satisfying the compatibility conditions*

$$
\begin{gathered}
\operatorname{div} \hat{\mathbf{v}}^0 = 0, \ \mathcal{P}_G(\hat{\mathbf{v}}^0)\big|_G = 0, \ \hat{\mathbf{v}}^0\big|_S = 0, \ [\![\hat{\mathbf{v}}^0]\!] = 0, \ \mathcal{P}_\Sigma \left[\!\left[\mu(\nabla \hat{\mathbf{v}}^0 + (\nabla \hat{\mathbf{v}}^0)^\top) \nu_\Sigma \right]\!\right] = 0, \\
\zeta \left[\!\left[\hat{c}^0 \right]\!\right] - \hat{D}_s \hat{\nabla} \hat{c}_s^0 \cdot \hat{\mathbf{n}}_\Sigma = 0, \ \left[\!\left[\hat{D} \hat{\nabla} \hat{c}^0 \right]\!\right] \cdot \hat{\mathbf{n}}_\Sigma = 0, \ \hat{D} \hat{\nabla} \hat{c}^0 \cdot \hat{\mathbf{n}}_G\big|_{G \backslash \partial\Sigma} = 0.
\end{gathered}
$$

Then there exists a positive $T_0 = T_0(\|(\hat{\mathbf{v}}^0, \hat{c}^0)\|_{X_\gamma}) < \infty$ such that for $J := (0, T)$, $0 < T < T_0$, (4.8) admits a unique solution $(\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s^*, \hat{g}) \in \mathbb{E}(J)$.

The proof of Theorem 4.3 relies on the Banach fixed-point theorem. More specifically, we need to prove that the linearized operator defines an isomorphism and construct a contraction mapping from the solutions to the nonlinear data.

Remark 4.4. We comment here that in [AL23a], it was shown that the cell concentrations are nonnegative, which is not clear in the present case. The problem comes up with the contact line formulated in the problem, that is, the boundaries for both c_f and c_s are not smooth anymore.

Remark 4.5. Here for nonlinear well-posedness, we assume $q > 5$. In fact, $q > 3$ is enough to argue in the *Lagrangian coordinates*. If additionally, we have $q > 5 = 3 + 2$, one has the embedding

$$
W_q^1(J; L^q(\Omega)^3) \cap L^q(J; W_q^2(\Omega \backslash \Sigma)^3) \hookrightarrow C(\bar{J}; C^1(\overline{\Omega \backslash \Sigma})^3).
$$

Then the Lagrangian flow map φ defined in (4.3) above is endowed with regularity $C^1(\bar{J}; C^1(\overline{\Omega\backslash\Sigma})^3)$. Hence all the regularities can be transformed to Eulerian coordinates.

4.2.2. Linearization. Let $J := (0,T)$ with $T > 0$. Now we consider the linear systems separately, namely,

1). **Two-phase Stokes problem in a cylindrical domain**

$$\begin{aligned} \rho\partial_t u - \operatorname{div} S_\mu(u,\pi) &= f_u, && \text{in } \Omega\backslash\Sigma \times J, \\ \operatorname{div} u &= f_d, && \text{in } \Omega\backslash\Sigma \times J, \\ [\![u]\!] &= g_1, && \text{on } \Sigma \times J, \\ \left[\!\left[-\pi\mathbb{I} + \mu(\nabla u + \nabla u^\top)\right]\!\right]\nu_\Sigma &= g_2, && \text{on } \Sigma \times J, \\ (u_1,u_3)^\top &= g_3, && \text{on } G\backslash\partial\Sigma \times J, \\ -\pi + 2\mu\partial_2 u_2 &= g_4, && \text{on } G\backslash\partial\Sigma \times J, \\ u &= g_5, && \text{on } S \times J, \\ u(0) &= u_0, && \text{in } \Omega\backslash\Sigma. \end{aligned} \tag{4.10}$$

where $S_\mu(u,\pi) = -\pi\mathbb{I} + \mu(\nabla u + \nabla u^\top)$. $\rho,\mu > 0$ are constants representing the density and viscosity, respectively. ν_Σ denotes the unit normal vector on Σ, pointing from Ω_f to Ω_s.

For the maximal regularity setting, we define the suitable function spaces for solutions and data with $q > 3$, as

$$u \in \mathbb{E}_u(J) := W^1_q(J; L^q(\Omega)^3) \cap L^q(J; W^2_q(\Omega\backslash\Sigma)^3),$$

$$\pi \in \mathbb{E}_\pi(J) := \left\{ \begin{aligned} &\pi \in L^q(J; \dot{W}^1_q(\Omega\backslash\Sigma)) : \\ &\qquad [\![\pi]\!] \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\Sigma)) \cap L^q(J; W_q^{1-\frac{1}{q}}(\Sigma)), \\ &\qquad \pi|_G \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(G)) \cap L^q(J; W_q^{1-\frac{1}{q}}(G\backslash\partial\Sigma)) \end{aligned} \right\}.$$

The additional regularities for π on Σ and G come from the Neumann trace of u on both boundaries. If (u,π) is a solution of (4.10), the necessary regularity classes for the data are subsequently given by

$$\begin{aligned} u_0 \in X_{\gamma,u} := W_q^{2-\frac{2}{q}}(\Omega\backslash\Sigma)^3, \quad & f_u \in \mathbb{F}_1(J) := L^q(J; L^q(\Omega)^3), \\ f_d \in \mathbb{F}_2(J) &:= W^1_q(J; \dot{W}^{-1}_q(\Omega)) \cap L^q(J; W^1_q(\Omega\backslash\Sigma)), \\ g_1 \in \mathbb{F}_3(J) &:= W_q^{1-\frac{1}{2q}}(J; L^q(\Sigma)^3) \cap L^q(J; W_q^{2-\frac{1}{q}}(\Sigma)^3), \\ g_2 \in \mathbb{F}_4(J) &:= W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\Sigma)^3) \cap L^q(J; W_q^{1-\frac{1}{q}}(\Sigma)^3), \\ g_3 \in \mathbb{F}_5(J) &:= W_q^{1-\frac{1}{2q}}(J; L^q(G)^2) \cap L^q(J; W_q^{2-\frac{1}{q}}(G\backslash\partial\Sigma)^2), \\ g_4 \in \mathbb{F}_6(J) &:= W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(G)) \cap L^q(J; W_q^{1-\frac{1}{q}}(G\backslash\partial\Sigma)), \\ g_5 \in \mathbb{F}_7(J) &:= W_q^{1-\frac{1}{2q}}(J; L^q(S)) \cap L^q(J; W_q^{2-\frac{1}{q}}(S)^3). \end{aligned}$$

Then it is necessary to consider the compatibility conditions at $t = 0$, that is,

$$\begin{aligned} \operatorname{div} u_0 = f_d|_{t=0}, \quad ((u_0)_1, (u_0)_3)^\top|_G = g_3|_{t=0}, \quad u_0|_S = g_5|_{t=0}, \\ [\![u_0]\!] = g_1|_{t=0}, \quad \mathcal{P}_\Sigma \left[\!\left[\mu(\nabla u_0 + \nabla u_0^\top)\nu_\Sigma\right]\!\right] = \mathcal{P}_\Sigma g_2|_{t=0}, \end{aligned} \tag{4.11}$$

where $\mathcal{P}_\Sigma := I - \nu_\Sigma \otimes \nu_\Sigma$ denotes the tangential projection of ν_Σ.

Moreover, the following compatibility conditions at the contact lines ∂G and $\partial\Sigma$ must be satisfied as well.

$$\begin{aligned} g_3 &= ((g_5)_1, (g_5)_3)^\top, \text{ at } \partial G, \quad [\![g_3]\!] = ((g_1)_1, (g_1)_3)^\top, \text{ at } \partial\Sigma, \\ (g_2)_1 &= [\![2\mu\partial_1(g_3)_1\nu_\Sigma \cdot e_1 + \mu(\partial_1(g_3)_2 + \partial_3(g_3)_1)\nu_\Sigma \cdot e_3]\!] \\ &\quad + [\![g_4 - 2\mu f_d]\!]\, \nu_\Sigma \cdot e_1 + [\![2\mu(\partial_1(g_3)_1 + \partial_3(g_3)_2)]\!]\, \nu_\Sigma \cdot e_1, \text{ at } \partial\Sigma, \\ (g_2)_3 &= [\![2\mu\partial_3(g_3)_2\nu_\Sigma \cdot e_3 + \mu(\partial_1(g_3)_2 + \partial_3(g_3)_1)\nu_\Sigma \cdot e_1]\!] \\ &\quad + [\![g_4 - 2\mu f_d]\!]\, \nu_\Sigma \cdot e_3 + [\![2\mu(\partial_1(g_3)_1 + \partial_3(g_3)_2)]\!]\, \nu_\Sigma \cdot e_3, \text{ at } \partial\Sigma. \end{aligned} \tag{4.12}$$

Remark 4.6. For f_d, we assume the regularity $W^1_q(J; \dot{W}^{-1}_q(\Omega))$, which is due to the divergence equation in (4.10) and the regularity $u \in W^1_q(J; L^q(\Omega))$. A similar but different condition was employed in Wilke [Wil20, Section 1.2], where they considered the Dirichlet and pure-slip boundary conditions around the contact line. Such setting gives rise to a compatibility identity for f_d and data associated with the normal component of velocities on each boundary and the interface. Readers are also referred to e.g. Prüss–Simonett [PS16, Section 7.3 and (8.2)]. The difference comes up with the consideration of outflow boundary condition, which is not endowed with the normal component of velocity. In our previous work [AL23a], the stress-free boundary is considered and hence we also have an additional regularity for f_d without the hidden compatibility identity, see also Prüss–Simonett [PS16, Page 338].

Let $\mathbb{E}(J) := \mathbb{E}_u(J) \times \mathbb{E}_\pi(J)$ and $\mathbb{F}(J) := \Pi^7_{j=1}\mathbb{F}_j(J)$. Now we give the theorem for the two-phase Stokes problem in a cylindrical domain, whose proof is postponed in Section 4.4.3.

Theorem 4.7. *Let $\rho, \mu > 0$, $q > 3$. Assume that $\Omega \subset \mathbb{R}^3$ is the domain defined in Theorem 4.3 with Σ, G, S of class C^3 and ∂G, $\partial\Sigma$ of C^4 as well. Then there exists a unique solution*

$$(u, \pi) \in \mathbb{E}(J)$$

of (4.10) if and only if the data are subject to the following regularity and compatibility conditions:

(1) $(f_u, f_d, g_1, g_2, g_3, g_4, g_5) \in \mathbb{F}(J)$,

(2) $u_0 \in X_{\gamma,u}$,

(3) compatibility conditions (4.11) and (4.12) hold.

2). **Heat equation in a cylindrical domain**

$$\begin{aligned} \partial_t c_f - D_f \Delta c_f &= f_{c,f}, && \text{in } \Omega_f \times J, \\ D_f \nabla c_f \cdot \nu_{G_f \cup \Sigma} &= g_6, && \text{on } G_f \cup \Sigma \times J, \\ c_f(0) &= c_{0,f}, && \text{in } \Omega_f, \end{aligned} \tag{4.13}$$

where $D_f > 0$ is the diffusivity. As above, we need the regularity classes of solutions and data for the maximal regularity setting. More specifically,

$$c_f \in \mathbb{E}_{c,f}(J) := W^1_q(J; L^q(\Omega_f)) \cap L^q(J; W^2_q(\Omega_f)),$$

with the data

$$\begin{aligned} c_{0,f} \in X_{\gamma,c_f} &:= W^{2-\frac{2}{q}}_q(\Omega_f), \quad f_{c,f} \in \mathbb{F}_8(J) := L^q(J; L^q(\Omega_f)), \\ g_6 \in \mathbb{F}_9(J) &:= W^{\frac{1}{2}-\frac{1}{2q}}_q(J; L^q(G_f \cup \Sigma)) \cap L^q(J; W^{1-\frac{1}{q}}_q(G_f \cup \Sigma)). \end{aligned}$$

Moreover, concerning the compatibility conditions at $t = 0$ and the contact line $\partial\Sigma \cap G_f$, for $q > 3$ we have

$$\begin{aligned} D_f \nabla c_{0,f} \cdot \nu_{G_2 \cup \Sigma}|_{G_2 \cup \Sigma} &= g_6|_{t=0}\,, \text{ on } G_f \cup \Sigma, \\ \partial_{\nu_{G_f}}(g_6|_{\Sigma}) &= \partial_{\nu_\Sigma}(g_6|_{G_f}), \text{ at } \partial\Sigma. \end{aligned} \tag{4.14}$$

Theorem 4.8. *Let $D_f > 0$, $q > 3$. Assume that $\Omega_f \subset \mathbb{R}^3$ is the domain defined in Theorem 4.3 with Σ, G_f of class C^3 and $\partial\Sigma$ of C^4 as well. Then there exists a unique solution*

$$c_f \in \mathbb{E}_{c,f}(J)$$

of (4.13) if and only if the data are subject to the following regularity and compatibility conditions:

(1) $(f_{c,f}, g_6) \in \mathbb{F}_8(J) \times \mathbb{F}_9(J)$,

(2) $c_{0,f} \in X_{\gamma,c_f}$,

(3) compatibility condition (4.14) holds.

3). **Heat equation in a cylindrical ring**

$$\begin{aligned} \partial_t c_s - D_s \Delta c_s &= f_{c,s}, && \text{in } \Omega_s \times J, \\ D_s \nabla c_s \cdot \nu_{\partial\Omega_s} &= g_7, && \text{on } \partial\Omega_s \times J, \\ c_s(0) &= c_{0,s}, && \text{in } \Omega_s, \end{aligned} \tag{4.15}$$

where $\partial\Omega_s := \Sigma \cup S \cup G_s$, $D_s > 0$ is the diffusivity. As above, we need the regularity classes of solutions and data for the maximal regularity setting. More specifically,

$$c_s \in \mathbb{E}_{c,s}(J) := W^1_q(J; L^q(\Omega_s)) \cap L^q(J; W^2_q(\Omega_s)),$$

with the data

$$\begin{aligned} c_{0,s} \in X_{\gamma,c_f} := W_q^{2-\frac{2}{q}}(\Omega_s), \quad f_{c,s} \in \mathbb{F}_{10}(J) := L^q(J; L^q(\Omega_s)), \\ g_7 \in \mathbb{F}_{11}(J) := W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\partial\Omega_s)) \cap L^q(J; W_q^{1-\frac{1}{q}}(\partial\Omega_s)). \end{aligned}$$

Moreover, concerning the compatibility conditions at $t = 0$, for $q > 3$ we have

$$\begin{aligned} D_f \nabla c_{0,s} \cdot \nu_{\partial\Omega_s}|_{\partial\Omega_s} = g_7|_{t=0}\,, \\ \partial_{\nu_{G_s}}(g_7|_{\Sigma}) = \partial_{\nu_\Sigma}(g_7|_{G_s}), \text{ at } \partial\Sigma, \quad \partial_{\nu_{G_s}}(g_7|_S) = \partial_{\nu_S}(g_7|_{G_s}) \text{ at } \partial S. \end{aligned} \tag{4.16}$$

Theorem 4.9. *Let $D_s > 0$, $q > 3$. Assume that $\Omega_s \subset \mathbb{R}^3$ is the domain defined in Theorem 4.3 with Σ, G_s, S of class C^3 and $\partial\Sigma, \partial G$ of C^4 as well. Then there exists a unique solution*

$$c_s \in \mathbb{E}_{c,s}(J)$$

of (4.10) if and only if the data are subject to the following regularity and compatibility conditions:

(1) $(f_{c,s}, g_8) \in \mathbb{F}_{10}(J) \times \mathbb{F}_{11}(J)$,

(2) $c_{0,s} \in X_{\gamma,c_s}$,

(3) compatibility condition (4.16) holds.

4). **ODEs in a cylindrical ring**

$$\begin{aligned} \partial_t c_s^* - \beta_1 c_s &= f_{c^*}, && \text{in } \Omega_s \times J, \\ \partial_t g - \beta_2 c_s &= f_g, && \text{in } \Omega_s \times J, \\ c_s^*(0) = 0,\ g(0) &= 1, && \text{in } \Omega_s, \end{aligned} \tag{4.17}$$

where $\beta_1 = \beta$, $\beta_2 = \gamma\beta/3\rho_s$. Since both equations in (4.17) are ordinary differential equation, if c_s is the solution of (4.15), then one obtains

$$c_s^* \in W_q^1(J; W_q^1(\Omega_s)), \quad g \in W_q^1(J; W_q^1(\Omega_s)),$$

provided that

$$f_{c^*} \in L^q(J; W_q^1(\Omega_s)), \quad f_g \in L^q(J; W_q^1(\Omega_s)).$$

4.3. Model Problems

As in [PS16] and [Wil20], a localization argument will be employed to prove the existence and uniqueness of the solution to (4.10)–(4.17). Since the linearized decoupled systems are a two-phase Stokes problem, two heat equations and two ODEs, we apply the localization procedure to each system. To this end, we study nine different types model problems, which are:

- the two-phase Stokes equations with a flat interface and without any boundary condition
- the full space heat equations (without any boundary or interface conditions)
- the Stokes equations with outflow boundary conditions in a half-space and no interface
- the Stokes equations with no-slip boundary conditions in a half-space and no interface
- the heat equations with Dirichlet or Neumann boundary conditions in a half-space and no interface
- the Stokes equations in quarter-spaces with outflow conditions on one part of the boundary and no-slip boundary conditions on the other part
- the two-phase Stokes equations with outflow boundary conditions, a flat interface and a contact angle of 90 degrees in a half-space
- the heat equations in quarter-spaces with Neumann boundary conditions.

Remark 4.10. As the first five problems are well understood, see e.g. Prüss–Simonett [PS16], we focus on the remaining three ones. Note that in [Wil20], Wilke studied model problems for the Stokes equations with pure slip boundary conditions in a quarter-space, as well as for the two-phase Stokes equations with pure slip boundary conditions in a half-space, our consideration extends his results to such model problems with outflow boundary conditions, which is reasonable if we cut the human vessels virtually.

4.3.1. The Stokes equations in quarter-spaces. For convenience, we define $\Omega := \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$, $G := \mathbb{R} \times \{0\} \times \mathbb{R}_+$ and $S := \mathbb{R} \times \mathbb{R}_+ \times \{0\}$. Now let us firstly consider the system

$$\begin{aligned} \rho\partial_t u - \operatorname{div} S_\mu(u, \pi) &= f, && \text{in } \Omega \times J, \\ \operatorname{div} u &= f_d, && \text{in } \Omega \times J, \\ (u_1, u_3)^\top &= 0, && \text{on } G \times J, \\ -\pi + 2\mu\partial_2 u_2 &= g_2, && \text{on } G \times J, \\ u &= g_3, && \text{on } S \times J, \\ u(0) &= 0, && \text{in } \Omega \times J, \end{aligned} \tag{4.18}$$

where $S_\mu(u,\pi) = -\pi\mathbb{I} + \mu(\nabla u + \nabla u^\top)$. Then we have the following theorem related to (4.18).

Theorem 4.11. *Let $q > 3/2$, $q \neq 3$, $T > 0$, $\rho, \mu > 0$ and $J = (0,T)$. Assume that $G, S \in C^3$ and ∂G is of class C^4. Then there exists a unique solution*

$$u \in {}_0W^1_q(J; L^q(\Omega)^3) \cap L^q(J; W^2_q(\Omega)^3),$$
$$\pi \in L^q(J; \dot{W}^1_q(\Omega)), \quad \pi|_G \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(G)) \cap L^q(J; W_q^{1-\frac{1}{q}}(G))$$

of (4.18) if and only if the data satisfy the following regularity and compatibility conditions:

(1) $f \in L^q(J; L^q(\Omega)^3)$,

(2) $f_d \in {}_0W^1_q(J; \dot{W}^{-1}_q(\Omega)) \cap L^q(J; W^1_q(\Omega))$,

(3) $g_2 \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(G)) \cap L^q(J; W_q^{1-\frac{1}{q}}(G))$,

(4) $g_3 \in {}_0W_q^{1-\frac{1}{2q}}(J; L^q(S)^3) \cap L^q(J; W_q^{2-\frac{1}{q}}(S)^3)$,

(5) $((g_3)_1, (g_3)_3)^\top|_{\overline{G}\cap\overline{S}} = 0$.

Proof. To deal with the well-posedness of the Stokes equations in quarter-spaces, we extend the data suitably to the half-space and solve the Stokes equations in half space. For this purpose, we firstly extend the function

$$f_d \in {}_0W^1_q(J; \dot{W}^{-1}_q(\Omega)^3) \cap L^q(J; W^1_q(\Omega)^3)$$

with respect to x_3 by

$$\tilde{f}_d(t,x_1,x_2,x_3) = \begin{cases} f_d(t,x_1,x_2,x_3), & \text{if } x_3 > 0, \\ -f_d(t,x_1,x_2,-2x_3) + 2f_d(t,x_1,x_2,-x_3/2), & \text{if } x_3 < 0. \end{cases} \tag{4.19}$$

to

$$\tilde{f}_d \in {}_0W^1_q(J; \dot{W}^{-1}_q(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R})^3) \cap L^q(J; W^1_q(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R})^3).$$

By the same extension, we have the extended functions

$$\tilde{f} \in L^q(J; L^q(\mathbb{R}\times\mathbb{R}^+\times\mathbb{R})^3),$$
$$\tilde{g}_2 \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\mathbb{R}\times\{0\}\times\mathbb{R})) \cap L^q(J; W_q^{1-\frac{1}{q}}(\mathbb{R}\times\{0\}\times\mathbb{R})).$$

Now one solves the auxiliary half-space Stokes problem

$$\begin{aligned} \rho\partial_t u - \operatorname{div} S_\mu(u,\pi)\mu &= \tilde{f}, && \text{in } \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times J, \\ \operatorname{div} u &= \tilde{f}_d, && \text{in } \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times J, \\ (u_1,u_3)^\top &= 0, && \text{on } \mathbb{R}\times\{0\}\times\mathbb{R}\times J, \\ -\pi + 2\mu\partial_2 u_2 &= \tilde{g}_2, && \text{on } \mathbb{R}\times\{0\}\times\mathbb{R}\times J, \\ u(0) &= 0, && \text{in } \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}, \end{aligned} \tag{4.20}$$

and obtains a unique solution (see e.g. [BP07, Theorem 6.1] or [PS16, Theorem 7.2.1])

$$\tilde{u} \in {}_0W^1_q(J; L^q(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})^3) \cap L^q(J; W^2_q(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})^3),$$
$$\tilde{\pi} \in L^q(J; \dot{W}^1_q(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})), \quad \tilde{\pi}|_{x_2=0} \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\mathbb{R}^2)) \cap L^q(J; W_q^{1-\frac{1}{q}}(\mathbb{R}^2)).$$

If (u,π) is a solution of (4.18), then the function $(u-\tilde{u},\pi-\tilde{\pi})$ solves

$$\begin{aligned}
\rho\partial_t u - \Delta u + \nabla\pi &= 0, && \text{in } \Omega\times J,\\
\operatorname{div} u &= 0, && \text{in } \Omega\times J,\\
(u_1,u_3)^\top &= 0, && \text{on } G\times J,\\
-\pi + 2\mu\partial_2 u_2 &= 0, && \text{on } G\times J,\\
u &= g_3, && \text{on } S\times J,\\
u(0) &= 0, && \text{in } \Omega,
\end{aligned} \tag{4.21}$$

with a modified data g_3 (not to be relabeled) in the right regularity classes having a vanishing trace at $t=0$ and satisfying $((g_3)_1,(g_3)_3)^\top|_{\overline{G}\cap\overline{S}}=0$. Note that the odd reflection does work for functions in both $W^1_{q,0}(\mathbb{R}^n_+)$ and $W^2_q(\mathbb{R}^n_+)\cap W^1_{q,0}(\mathbb{R}^n_+)$, the real interpolation implies that the function in $W^s_q(\mathbb{R}^n_+)\cap W^1_{q,0}(\mathbb{R}^n_+)$ can be extended to $W^s_q(\mathbb{R}^n)$, $1<s<2$, as well. It follows from $\operatorname{div} u=0$ that $\partial_2(g_3)_2=\partial_2 u_2=-\partial_1 u_1-\partial_3 u_3=0$ on $\overline{G}\cap\overline{S}$. Thus, we extend $((g_3)_1,(g_3)_3)$ via odd reflection and $(g_3)_2$ via even reflection along x_2 to

$$\tilde{g}_3 \in {}_0W_q^{1-\frac{1}{2q}}(J;L^q(\mathbb{R}^2\times\{0\})^3)\cap L^q(J;W_q^{2-\frac{1}{q}}(\mathbb{R}^2\times\{0\})^3).$$

By Bothe–Prüss [BP07, Theorem 6.1] or Prüss–Simonett [PS16, Theorem 7.2.1], the half-space Stokes problem

$$\begin{aligned}
\rho\partial_t\bar{u} - \Delta\bar{u} + \nabla\bar{\pi} &= 0, && \text{in } \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+\times J,\\
\operatorname{div}\bar{u} &= 0, && \text{in } \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+\times J,\\
\bar{u} &= \tilde{g}_3, && \text{on } \mathbb{R}\times\mathbb{R}\times\{0\}\times J,\\
\bar{u}(0) &= 0, && \text{in } \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+,
\end{aligned} \tag{4.22}$$

admits a unique solution satisfying

$$\begin{aligned}
\bar{u} &\in {}_0W^1_q(J;L^q(\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+)^3)\cap L^q(J;W^2_q(\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+)^3),\\
\bar{\pi} &\in L^q(J;\dot{W}^1_q(\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+)).
\end{aligned}$$

With symmetry, we conclude that

$$(\breve{u},\breve{\pi})(t,x_1,x_2,x_3) := ((-\hat{u}_1,\hat{u}_2,-\hat{u}_3)^\top,-\hat{\pi})(t,x_1,-x_2,x_3)$$

is a solution pair to (4.22) as well. It follows from the uniqueness that

$$\breve{u}(t,x_1,x_2,x_3)=\hat{u}(t,x_1,x_2,x_3),\quad \breve{\pi}(t,x_1,x_2,x_3):=\hat{\pi}(t,x_1,x_2,x_3)$$

and this yields

$$(\hat{u}_1,\hat{u}_3)(t,x_1,0,x_3)=0,\quad \partial_2\hat{u}_2(t,x_1,0,x_3)=0,\quad \hat{\pi}(t,x_1,0,x_3)=0.$$

Therefore, the restricted pair $(u,\pi):=(\tilde{u}+\bar{u},\tilde{\pi}+\bar{\pi})$ is the desired solution to (4.18).

Now we give a brief proof of the uniqueness. To this end, for a solution u to (4.18) with data all reduced in a quarter-space, one reflects it by suitable extension operators with respect to x_2 to a function $\tilde{u}$ with $\tilde{u}|_{x_2>0}=u$ in a half-space. Note here reflecting along x_3 is also an option without a difference. Then by the uniqueness of Stokes equation in half-space, see e.g. [PS16, Theorem 7.2.1], one concludes that $\tilde{u}\equiv 0$, which implies the uniqueness of u. This completes the proof. □

4.3.2. The Stokes equations in bent quarter-spaces. Now we consider the case of bent quarter spaces. Let $\theta \in BC^2(\mathbb{R}^2)$ and $x = (x', x_3)$, $x' = (x_1, x_2)$ such that

$$\Omega_\theta := \left\{ x \in \mathbb{R}^3 : x_2 > 0, x_3 > \theta(x') \right\}.$$

Let $\nabla_{x'} = (\partial_1, \partial_2)^\top$. Assume that $\|\nabla_{x'}\theta\|_\infty \leq \eta$ and $\left\|\nabla^2_{x'}\theta\right\|_\infty \leq M$, $M > 0$, where $\eta > 0$ may be chosen as small as we need. Define

$$S_\theta := \left\{ x \in \mathbb{R}^3 : x_3 = \theta(x'), x_2 > 0 \right\}, \quad G_\theta := \left\{ x \in \mathbb{R}^3 : x_3 > \theta(x'), x_2 = 0 \right\}.$$

Moreover, denote the unit outer normal vector on S_θ by

$$\nu_{S_\theta} := \frac{1}{\sqrt{1 + |\nabla_{x'}\theta(x')|^2}} (\nabla_{x'}\theta(x'), -1)^\top.$$

Now we consider the problem

$$\begin{aligned} \rho\partial_t u - \operatorname{div} S_\mu(u, \pi) &= f, && \text{in } \Omega_\theta \times J, \\ \operatorname{div} u &= f_d, && \text{in } \Omega_\theta \times J, \\ (u_1, u_3)^\top &= g_1, && \text{on } G_\theta \times J, \\ -\pi + 2\mu\partial_2 u_2 &= g_2, && \text{on } G_\theta \times J, \\ u &= g_3, && \text{on } S_\theta \times J, \\ u(0) &= u_0, && \text{in } \Omega_\theta, \end{aligned} \tag{4.23}$$

where $S_\mu(u, \pi) = -\pi\mathbb{I} + \mu(\nabla u + \nabla u^\top)$ and $\rho, \mu > 0$ are given constants. For given data $(f, f_d, g_1, g_2, g_3, u_0)$, the following compatibility conditions hold:

$$\operatorname{div} u_0 = f_d|_{t=0}, \quad u_0|_{S_\theta} = g_3|_{t=0}, \quad ((u_0)_1, (u_0)_3)^\top = g_1|_{t=0}, \tag{4.24}$$

$$g_1 = ((g_3)_1, (g_3)_3)^\top, \text{ at the contact line } \overline{G_\theta} \cap \overline{S_\theta}. \tag{4.25}$$

Reduction. To solve (4.23), we shall reduce it to the case $(u_0, f, g_1, g_2) = 0$. For this purpose, it is possible to extend u_0 and f to some $\tilde{u}_0 \in W_q^{2-2/q}(\mathbb{R}^3)^3$ and $\tilde{f} \in L^q(J; L^q(\mathbb{R}^3)^3)$ respectively (for example, extend them to $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$ by the extension (4.41) below and then extend them to $\mathbb{R}^3$ by a standard extension operator). Then solving the full-space heat equation

$$\begin{aligned} \rho\partial_t u - \mu\Delta u &= f, && \text{in } \mathbb{R}^3 \times J, \\ u(0) &= \tilde{u}_0, && \text{in } \mathbb{R}^3, \end{aligned}$$

yields a solution

$$\tilde{u} \in W_q^1(J; L^q(\mathbb{R}^3)^3) \cap L^q(J; W_q^1(\mathbb{R}^3)^3).$$

For $\tilde{g}_1 := g_1 - ((\tilde{u})_1, (\tilde{u})_3)^\top|_{G_\theta}$, $\tilde{g}_2 := g_2 - 2\mu\partial_2\tilde{u}_2|_{G_\theta}$ and $\tilde{f}_d := f_d - \operatorname{div}\tilde{u}$, it holds that $\tilde{g}_1|_{t=0} = 0$ and $\tilde{f}_d|_{t=0} = 0$ from (4.24). Analogously, one can suitably extend them to some function

$$\begin{aligned} \hat{g}_1 &\in {}_0W_q^{1-\frac{1}{2q}}(J; L^q(\mathbb{R} \times \{0\} \times \mathbb{R})^2) \cap L^q(J; W_q^{2-\frac{1}{q}}(\mathbb{R} \times \{0\} \times \mathbb{R})^2), \\ \hat{g}_2 &\in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\mathbb{R} \times \{0\} \times \mathbb{R})^2) \cap L^q(J; W_q^{1-\frac{1}{q}}(\mathbb{R} \times \{0\} \times \mathbb{R})^2), \\ \hat{f}_d &\in {}_0W_q^1(J; \dot{W}_q^{-1}(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})) \cap L^q(J; W_q^1(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})). \end{aligned}$$

Then we solve the half-space Stokes equation with outflow boundary condition

$$\begin{aligned} \rho\partial_t u - S_\mu(u,\pi) &= 0, && \text{in } \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times J,\\ \operatorname{div} u &= \hat{f}_d, && \text{in } \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times J,\\ (u_1,u_3)^\top &= \hat{g}_1, && \text{on } \mathbb{R}\times\{0\}\times\mathbb{R}\times J,\\ -\pi + 2\mu\partial_2 u_2 &= \hat{g}_2, && \text{on } \mathbb{R}\times\{0\}\times\mathbb{R}\times J,\\ u(0) &= 0, && \text{in } \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}, \end{aligned}$$

and obtain a unique solution $(\hat{u},\hat{\pi})$ satisfying

$$\hat{u} \in {}_0W^1_q(J;L^q(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})^3)\cap L^q(J;W^1_q(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})^3),$$
$$\hat{\pi} \in L^q(J;\dot{W}^1_q(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})^3), \quad \hat{\pi}|_{x_2=0} \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J;L^q(\mathbb{R}^2))\cap L^q(J;W_q^{1-\frac{1}{q}}(\mathbb{R}^2)),$$

by Bothe–Prüss [BP07, Theorem 6.1] or Prüss–Simonett [PS16, Theorem 7.2.1]. If (u,π) is a solution of (4.23), then $(\check{u},\check{\pi}) := (u-\tilde{u}-\hat{u},\pi-\hat{\pi})$ solves (4.23) with reduced data $(u_0,f,f_d,g_1,g_2)=0$ and a modified function g_3 (not to be relabeled) in the right regularity classes satisfying the compatibility conditions $g_3|_{t=0}=0$ and $((g_3)_1,(g_3)_3)^\top = 0$ at the contact line $\overline{G_\theta}\cap\overline{S_\theta}$ by (4.25).

Transform to a quarter space. In this part, we transform the boundaries S_θ and G_θ to $S := \mathbb{R}\times\mathbb{R}_+\times\{0\}$ and $G := \mathbb{R}\times\{0\}\times\mathbb{R}_+$, respectively, and hence, Ω_θ to $\Omega := \mathbb{R}\times\mathbb{R}^2_+$. To this end, we introduce the new variables $\bar{x} := (\bar{x}_1,\bar{x}_2,\bar{x}_3)$ where $\bar{x}_1 = x_1$, $\bar{x}_2 = x_2$ and $\bar{x}_3 = x_3 - \theta(x')$ for $x\in\Omega_\theta$. For (u,π) a solution to (4.23), set

$$\bar{u}(t,\bar{x}) := u(t,\bar{x}_1,\bar{x}_2,\bar{x}_3+\theta(\bar{x}')), \quad \bar{\pi}(t,\bar{x}) := \pi(t,\bar{x}_1,\bar{x}_2,\bar{x}_3+\theta(\bar{x}')).$$

for $t\in J, \bar{x}\in\Omega$. In the same way we transform the data (f_d,g_2,g_3) to $(\bar{f}_d,\bar{g}_2,\bar{g}_3)$. Note that

$$\begin{aligned} \nabla\theta = (\nabla_{x'}\theta,0)^\top, &\quad \nabla u = \nabla\bar{u} - \partial_{\bar{x}_3}\bar{u}\otimes\nabla\theta,\\ \operatorname{div} u = \operatorname{div}\bar{u} - \nabla\theta\cdot\partial_{\bar{x}_3}\bar{u}, &\quad \operatorname{div} T = \operatorname{div}\overline{T} - \partial_{\bar{x}_3}\overline{T}\nabla\theta, \end{aligned} \tag{4.26}$$

where u, $\bar{u}$ are vector-valued and T, $\overline{T}$ are tensor-valued functions. Then

$$\begin{aligned} \Delta u = \operatorname{div}\nabla u &= \operatorname{div}(\nabla\bar{u} - \partial_{\bar{x}_3}\bar{u}\otimes\nabla\theta)\\ &= \Delta\bar{u} - \partial_{\bar{x}_3}\nabla\bar{u}\nabla\theta - \operatorname{div}(\partial_{\bar{x}_3}\bar{u}\otimes\nabla\theta) + \partial_{\bar{x}_3}(\partial_{\bar{x}_3}\bar{u}\otimes\nabla\theta)\nabla\theta\\ &= \Delta\bar{u} - \partial_{\bar{x}_3}\nabla\bar{u}\nabla\theta - (\nabla\partial_{\bar{x}_3}\bar{u})\nabla\theta + \Delta\theta\partial_{\bar{x}_3}\bar{u} + (\partial^2_{\bar{x}_3}\bar{u}\otimes\nabla\theta)\nabla\theta. \end{aligned} \tag{4.27}$$

In the following, we will write $\partial_{\bar{x}_3}$ as ∂_3 for the sake of readability. Consequently, for $(\bar{u},\bar{\pi})$, we have the problem

$$\begin{aligned} \rho\partial_t\bar{u} - \mu\Delta\bar{u} + \nabla\bar{\pi} &= M_1(\theta,\bar{u},\bar{\pi}), && \text{in } \Omega\times J,\\ \operatorname{div}\bar{u} &= M_2(\theta,\bar{u}), && \text{in } \Omega\times J,\\ (\bar{u}_1,\bar{u}_3)^\top &= 0, && \text{on } G\times J,\\ -\bar{\pi} + 2\mu\partial_2\bar{u}_2 &= M_3(\theta,\bar{u}), && \text{on } G\times J,\\ \bar{u} &= \bar{g}_3, && \text{on } S\times J,\\ \bar{u}(0) &= 0, && \text{in } \Omega, \end{aligned} \tag{4.28}$$

where M_1, M_2 and M_3 are given by

$$\begin{aligned} M_1(\theta,\bar{u},\bar{\pi}) &:= \mu\big(-2\partial_3\nabla\bar{u}\nabla\theta + \Delta\theta\partial_3\bar{u} + (\partial_3^2\bar{u}\otimes\nabla\theta)\nabla\theta\big) + \nabla\theta(\bar{x}')\partial_3\bar{\pi},\\ M_2(\theta,\bar{u}) &:= \nabla\theta(\bar{x}')\cdot\partial_3\bar{u}, \quad M_3(\theta,\bar{u}) := 2\mu\partial_2\theta(\bar{x}')\partial_3\bar{u}_2. \end{aligned}$$

Existence and uniqueness. Notice that (4.28) can be seen as a perturbation of (4.18) if data are in the right regularity classes and $\|\nabla\theta\|_\infty \le \eta$, where $\eta > 0$ is small enough. Since Theorem 4.11 holds, we are going to apply the Neumann series argument to (4.28). To this end, we employ the similar regularity classes for $(\bar{u}, \bar{\pi})$ and given data as in Wilke [Wil20, Section 1.3.2], namely, define

$$
{}_0\mathbb{E}(J) := {}_0\mathbb{E}_u(J) \times \mathbb{E}_\pi(J),
$$

where

$$
\begin{aligned}
{}_0\mathbb{E}_u(J) &:= \left\{ u \in {}_0W^1_q(J; L^q(\Omega)^3) \cap L^q(J; W^2_q(\Omega)^3) : (u_1, u_3)^\top\big|_G = 0 \right\}, \\
\mathbb{E}_\pi(J) &:= \left\{ \pi \in L^q(J; \dot{W}^1_q(\Omega)) : \pi|_G \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(G)) \cap L^q(J; W_q^{1-\frac{1}{q}}(G)) \right\},
\end{aligned}
$$

and

$$
\widetilde{\mathbb{F}}(J) := \mathbb{F}_1(J) \times {}_0\mathbb{F}_2(J) \times {}_0\mathbb{F}_3(J) \times \mathbb{F}_4(J),
$$

where

$$
\begin{aligned}
\mathbb{F}_1(J) &:= L^q(J; L^q(\Omega)^3), \quad {}_0\mathbb{F}_2(J) := {}_0W^1_q(J; \dot{W}^{-1}_q(\Omega)) \cap L^q(J; W^1_q(\Omega)), \\
{}_0\mathbb{F}_3(J) &:= {}_0W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(G)) \cap L^q(J; W_q^{1-\frac{1}{q}}(G)), \\
\mathbb{F}_4(J) &:= W_q^{1-\frac{1}{2q}}(J; L^q(S)^3) \cap L^q(J; W_q^{2-\frac{1}{q}}(S)^3).
\end{aligned}
$$

In addition, let

$$
{}_0\mathbb{F}(J) := \left\{ (f, f_d, g_2, g_3) \in \widetilde{\mathbb{F}}(J) : ((g_3)_1, (g_3)_3)^\top = 0 \text{ at the contact line } \overline{G} \cap \overline{S} \right\}.
$$

Now define an operator $L : {}_0\mathbb{E}(J) \to {}_0\mathbb{F}(J)$ by

$$
L(\bar{u}, \bar{\pi}) := \begin{bmatrix} \rho\partial_t \bar{u} - \mu\Delta\bar{u} + \nabla\bar{\pi} \\ \operatorname{div} \bar{u} \\ -\bar{\pi} + 2\mu\partial_2 \bar{u}_2|_G \\ \bar{u}|_S \end{bmatrix},
$$

which is an isomorphism by Theorem 4.11. For the right-hand side of (4.28), we set

$$
M(\theta, \bar{u}, \bar{\pi}) := (M_1(\theta, \bar{u}, \bar{\pi}), M_2(\theta, \bar{u}), M_3(\theta, \bar{u}), 0)^\top
$$

and

$$
F := (0, 0, 0, f_3)^\top, \quad f_3 := \bar{g}_3.
$$

It is clear that $F \in \widetilde{\mathbb{F}}(J)$, since $\theta \in C^3(\mathbb{R}^2)$. Noticing that $((\bar{g}_3)_1, (\bar{g}_3)_3)^\top = 0$, one obtains $F \in {}_0\mathbb{F}(J)$.

To proceed with the Neumann series argument, we focus on the perturbation term $M(\theta, \bar{u}, \bar{\pi})$. It can be verified easily that $M(\theta, \bar{u}, \bar{\pi}) \in {}_0\mathbb{F}(J)$ for each $(\bar{u}, \bar{\pi}) \in {}_0\mathbb{E}(J)$, combining the smoothness of θ. The only point that needs to be taken care of is $M_2(\theta, \bar{u}) \in {}_0W^1_q(J; \dot{W}^{-1}_q(\Omega))$. By integration by parts with respect to the $\bar{x}_3$, we have

$$
\int_\Omega M_2(\theta, \bar{u})\phi \mathrm{d}\bar{x} = -\int_\Omega \nabla\theta \cdot \bar{u}\partial_3\phi \mathrm{d}\bar{x}, \quad \text{for all } \phi \in W^1_{q',0}(\Omega), \tag{4.29}
$$

which yields the claim with $\bar{u} \in {}_0\mathbb{E}_u(J)$.

Then we rewrite (4.28) as

$$(\bar{u}, \bar{\pi}) = L^{-1}M(\theta, \bar{u}, \bar{\pi}) + L^{-1}F.$$

Our aim in the following is to show that for each $\varepsilon > 0$, there exist $T_0 > 0$ and $\eta_0 > 0$ such that

$$\|M(\theta, \bar{u}, \bar{\pi})\|_{{}_0\mathbb{F}(J)} \leq \varepsilon \|(\bar{u}, \bar{\pi})\|_{{}_0\mathbb{E}(J)}, \tag{4.30}$$

for $0 < T < T_0$ and $0 < \eta < \eta_0$.

Note that

$${}_0W^1_q(J; L^q(\Omega)) \cap L^q(J; W^2_q(\Omega)) \hookrightarrow {}_0W^{\frac{1}{2}}_q(J; W^1_q(\Omega)) \hookrightarrow L^{2q}(J; W^1_q(\Omega)),$$

is valid for every $q > 1$ and the embedding constant does not depend on $T > 0$, since $\bar{u}|_{t=0} = 0$. Then direct calculations yield

$$\begin{aligned}
&\|M_1(\theta, \bar{u}, \bar{\pi})\|_{\mathbb{F}_1(J)} \\
&\leq C \|\nabla\theta\|_\infty \|(\bar{u}, \bar{\pi})\|_{{}_0\mathbb{E}(J)} + \|\nabla\theta\|^2_\infty \|\bar{u}\|_{{}_0\mathbb{E}_u(J)} + \left\|\nabla^2\theta\right\|_\infty \|\nabla\bar{u}\|_{L^q(J;L^q(\Omega))} \\
&\leq C \|\nabla\theta\|_\infty \|(\bar{u}, \bar{\pi})\|_{{}_0\mathbb{E}(J)} + \|\nabla\theta\|^2_\infty \|\bar{u}\|_{{}_0\mathbb{E}_u(J)} + T^{\frac{1}{2q}} \left\|\nabla^2\theta\right\|_\infty \|\nabla\bar{u}\|_{L^{2q}(J;L^q(\Omega))} \\
&\leq C(\|\nabla\theta\|_\infty (1 + \|\nabla\theta\|_\infty) + T^{\frac{1}{2q}} \left\|\nabla^2\theta\right\|_\infty) \|(\bar{u}, \bar{\pi})\|_{{}_0\mathbb{E}(J)},
\end{aligned} \tag{4.31}$$

where $C > 0$ does not depend on $T > 0$. For $M_2(\theta, \bar{u}) = \nabla\theta(\bar{x}') \cdot \partial_3\bar{u}$, we have

$$\|M_2(\theta, \bar{u})\|_{{}_0\mathbb{F}_2(J)} = \|M_2(\theta, \bar{u})\|_{L^q(J;W^1_q(\Omega))} + \|M_2(\theta, \bar{u})\|_{{}_0W^1_q(J;\dot{W}^{-1}_q(\Omega))}.$$

The first term in the right-hand side above can be obtained by

$$\begin{aligned}
&\|M_2(\theta, \bar{u}, \bar{\pi})\|_{L^q(J;W^1_q(\Omega))} \\
&\quad\leq C \|\nabla\theta\|_\infty \|\bar{u}\|_{{}_0\mathbb{E}_u(J)} + \left\|\nabla^2\theta\right\|_\infty \|\bar{u}\|_{L^q(J;W^1_q(\Omega))} \\
&\quad\leq C \|\nabla\theta\|_\infty \|\bar{u}\|_{{}_0\mathbb{E}_u(J)} + T^{\frac{1}{2q}} \left\|\nabla^2\theta\right\|_\infty \|\bar{u}\|_{L^{2q}(J;W^1_q(\Omega))} \\
&\quad\leq C(\|\nabla\theta\|_\infty + T^{\frac{1}{2q}} \left\|\nabla^2\theta\right\|_\infty) \|\bar{u}\|_{{}_0\mathbb{E}_u(J)}
\end{aligned}$$

similarly to M_1, while the second term follows from (4.29) that

$$\|M_2(\theta, \bar{u}, \bar{\pi})\|_{{}_0W^1_q(J;\dot{W}^{-1}_q(\Omega))} \leq C \|\nabla\theta\|_\infty \|\bar{u}\|_{{}_0\mathbb{E}_u(J)}.$$

Then

$$\|M_2(\theta, \bar{u})\|_{{}_0\mathbb{F}_2(J)} \leq C(\|\nabla\theta\|_\infty + T^{\frac{1}{2q}} \left\|\nabla^2\theta\right\|_\infty) \|\bar{u}\|_{{}_0\mathbb{E}_u(J)}. \tag{4.32}$$

For $M_3(\theta, \bar{u}) = 2\mu\partial_2\theta(\bar{x}')\partial_3\bar{u}_2$, one may need to verify both the regularity of ${}_0W^{1/2-1/2q}_q(J; L^q(G))$ and $L^q(J; W^{1-1/q}_q(G))$. To this end, we proceed by the definition of Sobolev–Slobodeckij space that

$$\|M_3(\theta, \bar{u})\|_{{}_0W^{\frac{1}{2}-\frac{1}{2q}}_q(J;L^q(G))} \leq C \|\nabla\theta\|_\infty \|\nabla\bar{u}\|_{{}_0W^{\frac{1}{2}-\frac{1}{2q}}_q(J;L^q(G))} \leq C \|\nabla\theta\|_\infty \|\bar{u}\|_{{}_0\mathbb{E}_u(J)}.$$

With the help of Sobolev trace theory, one derives that

$$\|M_3(\theta, \bar{u})\|_{L^q(J;W^{1-\frac{1}{q}}_q(G))} \leq C \|\nabla\theta\|_\infty \|\nabla\bar{u}\|_{L^q(J;W^1_q(\Omega))} \leq C \|\nabla\theta\|_\infty \|\bar{u}\|_{{}_0\mathbb{E}_u(J)}.$$

Then we arrive at

$$\|M_3(\theta,\bar{u})\|_{_0\mathbb{F}_3(J)} \leq C\,\|\nabla\theta\|_\infty\,\|\bar{u}\|_{_0\mathbb{E}_u}\,. \tag{4.33}$$

Consequently, collecting (4.31) and (4.32) together yields

$$\|M(\theta,\bar{u},\bar{\pi})\|_{_0\mathbb{F}(J)} \leq C(\|\nabla\theta\|_\infty\,(1+\|\nabla\theta\|_\infty) + T^{\frac{1}{2q}}\,\|\nabla^2\theta\|_\infty)\,\|(\bar{u},\bar{\pi})\|_{_0\mathbb{E}(J)}\,.$$

Since $\|\nabla\theta\|_\infty \leq \eta$, $\|\nabla^2\theta\|_\infty \leq M$, by choosing $\eta > 0$ and $T > 0$ sufficiently small, one obtains (4.30). A Neumann series argument in ${}_0\mathbb{E}(J)$ finally implies that there exists a unique solution $(\bar{u},\bar{\pi}) \in {}_0\mathbb{E}(J)$ of $L(\bar{u},\bar{\pi}) = M(\theta,\bar{u},\bar{\pi}) + F$ or equivalently a solution (u,π) of (4.23), provided that the data satisfy all relevant compatibility conditions at the contact line $\overline{G}\cap\overline{S}$.

Conversely, (4.23) admits a solution operator $S_{QS} : \mathbb{F}_{QS} \to \mathbb{E}_{QS}$, where $\mathbb{F}_{QS}$ and $\mathbb{E}_{QS}$ are the solution space and data space, respectively, for the bent quarter-space and the data in $\mathbb{F}_{QS}$ satisfy the compatibility conditions at the contact line $\{(x_1,0,\theta(x_1)) : x_1 \in \mathbb{R}\}$.

4.3.3. The two-phase Stokes equations in half-spaces. Let $\Omega := \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}$, $G := \mathbb{R}\times\{0\}\times\mathbb{R}$, $\Sigma := \mathbb{R}\times\mathbb{R}_+\times\{0\}$ and $\partial\Sigma := \mathbb{R}\times\{0\}\times\{0\}$. Consider the problem

$$\begin{aligned}
\rho\partial_t u - \operatorname{div} S_\mu(u,\pi) &= f, && \text{in } \Omega\backslash\Sigma\times J,\\
\operatorname{div} u &= f_d, && \text{in } \Omega\backslash\Sigma\times J,\\
[\![u]\!] &= g_1, && \text{on } \Sigma\times J,\\
[\![-\pi e_3 + \mu\,(\partial_3 u + \nabla u_3)]\!] &= g_2, && \text{on } \Sigma\times J,\\
(u_1,u_3)^\top &= 0, && \text{on } G\backslash\partial\Sigma\times J,\\
-\pi + 2\mu\partial_2 u_2 &= g_4, && \text{on } G\backslash\partial\Sigma\times J,\\
u(0) &= 0, && \text{in } \Omega\backslash\Sigma.
\end{aligned} \tag{4.34}$$

Then we have the following well-posedness result.

Theorem 4.12. *Let $q > 3$, $T > 0$, $\rho_j,\mu_j > 0$, $j = 1,2$, and $J = (0,T)$. Assume that $\Sigma, G \in C^3$ and $\partial\Sigma$ is of class C^4. Then (4.34) admits a unique solution (u,π) with regularity*

$$\begin{aligned}
u &\in {}_0W^1_q(J;L^q(\Omega)^3)\cap L^q(J;W^2_q(\Omega\backslash\Sigma)^3),\\
\pi &\in L^q(J;\dot{W}^1_q(\Omega\backslash\Sigma)), \quad [\![\pi]\!] \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J;L^q(\Sigma))\cap L^q(J;W_q^{1-\frac{1}{q}}(\Sigma)),\\
\pi|_{x_2=0} &\in W_q^{\frac{1}{2}-\frac{1}{2q}}(J;L^q(G))\cap L^q(J;W_q^{1-\frac{1}{q}}(G)),
\end{aligned}$$

if and only if the data satisfy the following regularity and compatibility conditions:

(1) $f \in L^q(J;L^q(\Omega)^3)$,

(2) $f_d \in {}_0W^1_q(J;\dot{W}^{-1}_q(\Omega))\cap L^q(J;W^1_q(\Omega\backslash\Sigma))$,

(3) $g_1 \in {}_0W_q^{1-\frac{1}{2q}}(J;L^q(\Sigma)^3)\cap L^q(J;W_q^{2-\frac{1}{q}}(\Sigma)^3)$,

(4) $((g_2)_1,(g_2)_2)^\top \in {}_0W_q^{\frac{1}{2}-\frac{1}{2q}}(J;L^q(\Sigma)^2)\cap L^q(J;W_q^{1-\frac{1}{q}}(\Sigma)^2)$,

(5) $(g_2)_3 \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J;L^q(\Sigma))\cap L^q(J;W_q^{1-\frac{1}{q}}(\Sigma))$,

(6) $g_4 \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J;L^q(G))\cap L^q(J;W_q^{1-\frac{1}{q}}(G\backslash\partial\Sigma))$,

(7) $((g_1)_1, (g_1)_3)^\top = 0$, $(g_2)_1 = 0$, $(g_2)_3 = [\![g_4]\!] - [\![2\mu f_d]\!]$ *at* $\partial\Sigma$.

Proof. The idea of the proof is based on the procedure in Wilke [Wil20], while a different type of boundary conditions (*outflow* conditions) is considered in our case.

Reduction of f and f_d. Let us firstly reduce (f, f_d) to zero. To this end, we extend $f \in L^q(J; L^q(\Omega)^3)$ to some function $\tilde{f} \in L^q(J; L^q(\mathbb{R}^3)^3)$ and

$$f_d \in L^q(J; W_q^1(\Omega\backslash\Sigma)) \cap {}_0W_q^1(J; \dot{W}_q^{-1}(\Omega))$$

to some function

$$\tilde{f}_d \in L^q(J; W_q^1(\mathbb{R}^2 \times \dot{\mathbb{R}})) \cap {}_0W_q^1(J; \dot{W}_q^{-1}(\mathbb{R}^2 \times \dot{\mathbb{R}}))$$

by the extension (4.19) above with respect to $-x_2$ direction. Define

$$\tilde{f}^\pm := \tilde{f}\Big|_{x_3 \gtrless 0} \in L^q(J; L^q(\mathbb{R}^2 \times \mathbb{R}_\pm)),$$
$$\tilde{f}_d^\pm := \tilde{f}_d\Big|_{x_3 \gtrless 0} \in L^q(J; W_q^1(\mathbb{R}^2 \times \mathbb{R}_\pm)) \cap {}_0W_q^1(J; \dot{W}_q^{-1}(\mathbb{R}^2 \times \mathbb{R}_\pm)).$$

Now extend $\tilde{f}^+$ with respect to x_3 to some function $\hat{f}^+ \in L^q(J; L^q(\mathbb{R}^3)^3)$ and $\tilde{f}_d^+$ to some function $\hat{f}_d^+ \in L^q(J; W_q^1(\mathbb{R}^3)) \cap W_q^1(J; \dot{W}_q^{-1}(\mathbb{R}^3))$. Let $\mu^\pm := \mu|_{x_3 \gtrless 0}$, $\rho^\pm := \rho|_{x_3 \gtrless 0}$ and extend $\mu^+, \rho^+ \in \mathbb{R} \times \mathbb{R}_+^2$ to $\hat{\mu}^+ \equiv \mu^+, \hat{\rho}^+ \equiv \rho^+ \in \mathbb{R}^3$ by constants. Then solving the full space Stokes equation

$$\begin{aligned} \hat{\rho}^+ \partial_t \hat{u}^+ - \operatorname{div} S_{\hat{\mu}^+}(\hat{u}^+, \hat{\pi}^+) &= \hat{f}^+, && \text{in } \mathbb{R}^3 \times J, \\ \operatorname{div} \hat{u}^+ &= \hat{f}_d^+, && \text{in } \mathbb{R}^3 \times J, \\ \hat{u}^+(0) &= 0, && \text{in } \mathbb{R}^3, \end{aligned} \tag{4.35}$$

yields a unique solution $(\hat{u}^+, \hat{\pi}^+)$ satisfying

$$\hat{u}^+ \in {}_0W_q^1(J; L^q(\mathbb{R}^3)^3) \cap L^q(J; W_q^2(\mathbb{R}^3)^3), \quad \hat{\pi}^+ \in L^q(J; \dot{W}_q^1(\mathbb{R}^3)),$$

by Prüss–Simonett [PS16, Theorem 7.1.1]. Analogously, we extend $\tilde{f}^-$ and $\tilde{f}_d^-$ to some function $\hat{f}^- \in L^q(J; L^q(\mathbb{R}^3)^3)$ and $\hat{f}_d^- \in L^q(J; W_q^1(\mathbb{R}^3)) \cap W_q^1(J; \dot{W}_q^{-1}(\mathbb{R}^3))$ with respect to x_3 respectively. Let $\hat{\mu}^- \equiv \mu^-, \hat{\rho}^- \equiv \rho^-$. Then we solve (4.35) with all superscripts $^+$ replaced by $^-$ to obtain a unique solution

$$\hat{u}^- \in {}_0W_q^1(J; L^q(\mathbb{R}^3)^3) \cap L^q(J; W_q^2(\mathbb{R}^3)^3), \quad \hat{\pi}^- \in L^q(J; \dot{W}_q^1(\mathbb{R}^3)).$$

Based on these functions, we define

$$(\hat{u}, \hat{\pi}) := \begin{cases} (\hat{u}^+, \hat{\pi}^+)\big|_\Omega, & \text{if } x_3 > 0, \\ (\hat{u}^-, \hat{\pi}^-)\big|_\Omega, & \text{if } x_3 < 0. \end{cases}$$

Then

$$\hat{u} \in {}_0W_q^1(J; L^q(\Omega)^3) \cap L^q(J; W_q^2(\Omega\backslash\Sigma)^3), \quad \hat{\pi} \in L^q(J; \dot{W}_q^1(\Omega\backslash\Sigma)).$$

If (u, π) is a solution of (4.34), then $(u - \hat{u}, \pi - \hat{\pi})$ solves (4.34) with $(f, f_d) = 0$ and some modified data g_j, $j \in \{1, 2, 3, 4\}$ (not to be relabeled), in the right regularity classes, having vanishing traces at $t = 0$ whenever it exists and satisfying the compatibility conditions at $\partial\Sigma$ stated in Theorem 4.12.

Reduction of g_1 and g_4. Now we extend

$$g_4^+ := g_4|_{x_3>0} \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\mathbb{R}\times\{0\}\times\mathbb{R}_+)) \cap L^q(J; W_q^{1-\frac{1}{q}}(\mathbb{R}\times\{0\}\times\mathbb{R}_+))$$

by means of even reflection to functions

$$\tilde{g}_4^+ \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\mathbb{R}\times\{0\}\times\mathbb{R})) \cap L^q(J; W_q^{1-\frac{1}{q}}(\mathbb{R}\times\{0\}\times\mathbb{R})).$$

Let $\mu^+ := \mu|_{x_3>0}$, $\rho^+ := \rho|_{x_3>0}$. Then the half-space Stokes problem with the outflow boundary condition

$$\begin{aligned} \rho^+\partial_t\breve{u}^+ - \mu^+\Delta\breve{u}^+ + \nabla\breve{\pi}^+ &= 0, && \text{in } \Omega\times J,\\ \operatorname{div}\breve{u}^+ &= 0, && \text{in } \Omega\times J,\\ (\breve{u}_1^+, \breve{u}_3^+)^\top &= 0, && \text{on } G\times J,\\ -\breve{\pi}^+ + 2\mu^+\partial_2\breve{u}_2^+ &= \tilde{g}_4^+, && \text{on } G\times J,\\ \breve{u}^+(0) &= 0, && \text{in } \Omega, \end{aligned} \tag{4.36}$$

admits a unique solution $(\breve{u}^+, \breve{\pi}^+)$ with regularity

$$\breve{u}^+ \in {}_0W_q^1(J; L^q(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})^3) \cap L^q(J; W_q^2(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})^3),$$
$$\breve{\pi}^+ \in L^q(J; \dot{W}_q^1(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})), \quad \breve{\pi}^+\big|_{x_2=0} \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(G)) \cap L^q(J; W_q^{1-\frac{1}{q}}(G)),$$

by Bothe–Prüss [BP07, Theorem 6.1] or Prüss–Simonett [PS16, Theorem 7.2.1]. Repeating the same procedure with all superscripts $^+$ replaced by $^-$ in (4.36), where $\tilde{g}_4^-$ is the extension of $g_4^- := g_4|_{x_3<0}$ similar to $\tilde{g}_4^+$, we get the corresponding functions $(\breve{u}^-, \breve{\pi}^-)$ with regularity

$$\breve{u}^- \in {}_0W_q^1(J; L^q(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})^3) \cap L^q(J; W_q^2(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})^3),$$
$$\breve{\pi}^- \in L^q(J; \dot{W}_q^1(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})), \quad \breve{\pi}^-\big|_{x_2=0} \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(G)) \cap L^q(J; W_q^{1-\frac{1}{q}}(G)).$$

Define

$$(\breve{u}, \breve{\pi}) := \begin{cases} (\breve{u}^+, \breve{\pi}^+), & \text{if } x_3 > 0,\\ (\breve{u}^-, \breve{\pi}^-), & \text{if } x_3 < 0. \end{cases}$$

Then $(\bar{u}, \bar{\pi}) := (u - \breve{u}, \pi - \breve{\pi})$ (restricted) satisfies that $\bar{u}|_{t=0} = 0$, $[\![\bar{u}]\!] = g_1 - [\![\breve{u}]\!] =: k$ on Σ and

$$(\bar{u}_1, \bar{u}_3)^\top = 0, \quad -\bar{\pi} + 2\mu\partial_2\bar{u}_2 = 0,$$

on $G\backslash\partial\Sigma$, which also means $\partial_i\bar{u}_j = 0$ for $i, j = 1, 3$ on $G\backslash\partial\Sigma$. From the compatibility conditions, we know $k_1 = k_3 = 0$ on $\partial\Sigma$. Moreover, since $\operatorname{div}\bar{u} = 0$ up to the boundary, one obtains $\partial_2 k_2 = [\![\partial_2\bar{u}_2]\!] = \operatorname{div}\bar{u} - [\![\partial_1\bar{u}_1 + \partial_3\bar{u}_3]\!] = 0$ on $\partial\Sigma$. Therefore, one may extend

$$k \in {}_0W_q^{1-\frac{1}{2q}}(J; L^q(\Sigma)^3) \cap L^q(J; W_q^{2-\frac{1}{q}}(\Sigma)^3)$$

to some function

$$\tilde{k} \in {}_0W_q^{1-\frac{1}{2q}}(J; L^q(\mathbb{R}^2\times\{0\})^3) \cap L^q(J; W_q^{2-\frac{1}{q}}(\mathbb{R}^2\times\{0\})^3)$$

by odd reflection of k_1, k_3 as before and even reflection of k_2. Then the half-space Dirichlet Stokes equation

$$\begin{aligned} \rho\partial_t w - \mu\Delta w + \nabla p &= 0, && \text{in } \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+\times J,\\ \operatorname{div} w &= 0, && \text{in } \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+\times J,\\ w &= \tilde{k}, && \text{on } \mathbb{R}\times\mathbb{R}\times\{0\}\times J,\\ w(0) &= 0, && \text{in } \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+, \end{aligned} \tag{4.37}$$

admits a unique solution

$$w \in {}_0W^1_q(J; L^q(\mathbb{R}^2 \times \mathbb{R}_+)^3) \cap L^q(J; W^2_q(\mathbb{R}^2 \times \mathbb{R}_+)^3),\ p \in L^q(J; \dot{W}^1_q(\mathbb{R}^2 \times \mathbb{R}_+)).$$

By symmetry, it is easy to verify that the function

$$\bar{w}(t,x) = (-w_1, w_2, -w_3)^\top(t, x_1, -x_2, x_3),\ \bar{p}(t,x) = -p(t, x_1, -x_2, x_3)$$

is also a solution of (4.37). The uniqueness of (w,p) implies that $(w,p) = (\bar{w}, \bar{p})$ and therefore $w_1 = w_3 = 0$ as well as $\partial_2 w_2 = 0$, $p = 0$ on $G \backslash \partial\Sigma$. Let $(\bar{u}_\pm, \bar{\pi}_\pm) := (\bar{u}, \bar{\pi})|_{x_3 \gtrless 0}$ and define

$$(u^*, \pi^*) := \begin{cases} (\bar{u}_+, \bar{\pi}_+) - (w,p), & \text{if } x_3 > 0, \\ (\bar{u}_-, \bar{\pi}_-), & \text{if } x_3 < 0. \end{cases}$$

Then $[\![u^*]\!] = 0$ on Σ and

$$(u_1^*, u_3^*)^\top = 0, \quad -\pi^* + 2\mu\partial_2 u_2^* = 0,$$

on $G\backslash\partial\Sigma$. Hence, (u^*, π^*) (restricted) solves (4.34) with $(f, f_d, g_j) = 0$, $j \in \{1,4\}$ and the modified data g_2 (not to be relabeled) in proper regularity classes having vanishing trace at $t = 0$ whenever it exists and satisfying the compatibility conditions.

Proof of Theorem 4.12. From the compatibility conditions and boundary conditions on $G\backslash\partial\Sigma$, it is possible to extend $((g_2)_1, (g_2)_3)$ by odd reflection and $(g_2)_2$ by even reflection with respect to $-x_2$. In the following, we consider the reflected problem

$$\begin{aligned} \rho\partial_t\tilde{u} - \mu\Delta\tilde{u} + \nabla\tilde{\pi} &= 0, && \text{in } \mathbb{R}\times\mathbb{R}\times\dot{\mathbb{R}}\times J, \\ \operatorname{div}\tilde{u} &= 0, && \text{in } \mathbb{R}\times\mathbb{R}\times\dot{\mathbb{R}}\times J, \\ [\![u]\!] &= 0, && \text{on } \mathbb{R}\times\mathbb{R}\times\{0\}\times J, \\ [\![-\tilde{\pi}e_3 + \mu(\partial_3\tilde{u} + \nabla\tilde{u}_3)]\!] &= \tilde{g}_2, && \text{on } \mathbb{R}\times\mathbb{R}\times\{0\}\times J, \\ \tilde{u}(0) &= 0, && \text{in } \mathbb{R}\times\mathbb{R}\times\dot{\mathbb{R}}. \end{aligned} \tag{4.38}$$

with given reflected data

$$((\tilde{g}_2)_1, (\tilde{g}_2)_2) \in {}_0W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\mathbb{R}^2\times\{0\})^2) \cap L^q(J; W_q^{1-\frac{1}{q}}(\mathbb{R}^2\times\{0\})^2),$$
$$(\tilde{g}_2)_3 \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\mathbb{R}^2\times\{0\})) \cap L^q(J; \dot{W}_q^{1-\frac{1}{q}}(\mathbb{R}^2\times\{0\})).$$

By Prüss–Simonett [PS10, Theorem 3.1], (4.38) admits a unique solution $(\tilde{u}, \tilde{\pi})$ with

$$\tilde{u} \in W^1_q(J; L^q(\mathbb{R}^3)^3) \cap L^q(J; W^2_q(\mathbb{R}^2\times\dot{\mathbb{R}})^3),\ \tilde{\pi} \in L^q(J; \dot{W}^1_q(\mathbb{R}^2\times\dot{\mathbb{R}})),$$
$$[\![\tilde{\pi}]\!] \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\mathbb{R}^2\times\{0\})) \cap L^q(J; W_q^{1-\frac{1}{q}}(\mathbb{R}^2\times\{0\})).$$

Define $(\bar{u}, \bar{\pi})$ as

$$\bar{u}(t,x) := (-u_1, u_2, -u_3)^\top(t, x_1, -x_2, x_3),\ \bar{\pi}(t,x) := -\pi(t, x_1, -x_2, x_3).$$

It is clear that $(\bar{u}, \bar{\pi})$ is also a solution of (4.38). The uniqueness implies that $(\tilde{u}, \tilde{\pi}) = (\hat{u}, \hat{\pi})$ and therefore,

$$\tilde{u}_1 = \tilde{u}_3 = 0, \quad \partial_2\tilde{u}_2 = 0,$$

as well as $\tilde{\pi} = 0$ on $G\backslash\partial\Sigma$. Consequently, the restriction $(\tilde{u}, \tilde{\pi})|_\Omega$ is the strong solution of (4.34) with $(f, f_d, g_j) = 0$, $j \in \{1,4\}$. The uniqueness can be verified by a similar argument as in Section 4.3.1. More specifically, assume u is a solution of (4.34) with data all reduced in a half-space. Then one reflects it by suitable extension operators with respect to x_2 to a function $\tilde{u}$ with $\tilde{u}|_{x_2>0} = u$ in full space. By the uniqueness of two-phase Stokes equation in full space, see e.g. [PS10, Theorem 3.1], we know $\tilde{u} \equiv 0$ and hence obtain the uniqueness of u. This completes the proof. □

4.3.4. The two-phase Stokes equations in half-spaces with a bent interface. In this section, we consider the case of a bent interface for the two-phase Stokes equations in half-spaces. So let the interface Σ be given as a graph of a function $\theta : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ of class C^3. Moreover, let $x = (x', x_3)$, $x' = (x_1, x_2)$ and $\nabla_{x'} = (\partial_1, \partial_2)^\top$, we assume that $\|\nabla_{x'}\theta(x')\|_\infty \le \eta$ and $\left\|\nabla^i_{x'}\theta(x')\right\|_\infty \le M$, $i \in \{2,3\}$ for $M > 0$, where $\eta > 0$ will be chosen sufficiently small later. Then the interface is defined as

$$\Sigma_\theta := \left\{(x', x_3) \in \mathbb{R}^3 : x_3 = \theta(x'), x' \in \mathbb{R} \times \mathbb{R}_+\right\},$$

and the associated unit normal vector is given by

$$\nu_{\Sigma_\theta}(x') := \beta(x')(-\nabla_{x'}\theta(x'), 1)^\top, \quad \beta(x') := \frac{1}{\sqrt{1 + |\nabla_{x'}\theta(x')|^2}}.$$

Let $\Omega := \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$, $G := \mathbb{R} \times \{0\} \times \mathbb{R}$ and

$$\partial\Sigma_\theta := \{(x', x_3) : x_3 = \theta(x'), x' \in \mathbb{R} \times \{0\}\}.$$

We consider the problem

$$\begin{aligned}
\rho\partial_t u - \operatorname{div} S_\mu(u, \pi) &= f, && \text{in } \Omega\backslash\Sigma_\theta \times J,\\
\operatorname{div} u &= f_d, && \text{in } \Omega\backslash\Sigma_\theta \times J,\\
[\![u]\!] &= g_1, && \text{on } \Sigma_\theta \times J,\\
\left[\!\!\left[-\pi\mathbb{I} + \mu(\nabla u + \nabla u^\top)\right]\!\!\right] \nu_{\Sigma_\theta} &= g_2, && \text{on } \Sigma_\theta \times J,\\
(u_1, u_3)^\top &= g_3, && \text{on } G\backslash\partial\Sigma_\theta \times J,\\
-\pi + 2\mu\partial_2 u_2 &= g_4, && \text{on } G\backslash\partial\Sigma_\theta \times J,\\
u(0) &= u_0, && \text{in } \Omega\backslash\Sigma_\theta,
\end{aligned} \tag{4.39}$$

where $S_\mu(u, \pi) = -\pi\mathbb{I} + \mu(\nabla u + \nabla u^\top)$ and $\rho, \mu > 0$ are given constants. For given data $(f, f_d, g_1, g_2, g_3, g_4, u_0)$, we give the following compatibility conditions at time $t = 0$

$$\begin{gathered}
\operatorname{div} u_0 = f_d|_{t=0}, \quad [\![u_0]\!] = g_1|_{t=0}, \quad ((u_0)_1, (u_0)_3)^\top|_G = g_3|_{t=0},\\
\mathcal{P}_{\Sigma_\theta}\left[\!\!\left[\mu(\nabla u_0) + \nabla u_0^\top)\nu_{\Sigma_\theta}\right]\!\!\right] = \mathcal{P}_{\Sigma_\theta} g_2|_{t=0},
\end{gathered}$$

where $\mathcal{P}_{\Sigma_\theta} := I - \nu_{\Sigma_\theta} \otimes \nu_{\Sigma_\theta}$ denotes the tangential projection of ν_{Σ_θ}. Moreover, at the contact line $\partial\Sigma_\theta$, we have

$$\begin{aligned}
[\![g_3]\!] &= ((g_1)_1, (g_1)_3)^\top,\\
(g_2)_1 &= [\![2\mu\partial_1(g_3)_1\nu_{\Sigma_\theta} \cdot e_1 + \mu(\partial_1(g_3)_2 + \partial_3(g_3)_1)\nu_{\Sigma_\theta} \cdot e_3]\!],\\
&\quad + [\![g_4 - 2\mu f_d]\!]\, \nu_{\Sigma_\theta} \cdot e_1 + [\![2\mu(\partial_1(g_3)_1 + \partial_3(g_3)_2)]\!]\, \nu_{\Sigma_\theta} \cdot e_1,\\
(g_2)_3 &= [\![2\mu\partial_3(g_3)_2\nu_{\Sigma_\theta} \cdot e_3 + \mu(\partial_1(g_3)_2 + \partial_3(g_3)_1)\nu_{\Sigma_\theta} \cdot e_1]\!]\\
&\quad + [\![g_4 - 2\mu f_d]\!]\, \nu_{\Sigma_\theta} \cdot e_3 + [\![2\mu(\partial_1(g_3)_1 + \partial_3(g_3)_2)]\!]\, \nu_{\Sigma_\theta} \cdot e_3.
\end{aligned} \tag{4.40}$$

Reduction. As above, we want to reduce (4.39) to the case $(u_0, f, f_d, g_3, g_4) = 0$ for convenience. In a same way as in Section 4.3.2, one can reduce to the case $(u_0, f, f_d) = 0$, we do not give the details here. Then we have modified data g_j, $j \in \{1, 2, 3, 4\}$, in the right regularity classes and having vanishing traces at $t = 0$ whenever it exists.

Next, set $g_j^\pm := g_j|_{x_3 \gtrless \theta(x_1,0)}$, $j \in \{3,4\}$. By the extension

$$\tilde{g}_3^+(x_1,0,x_3) = \begin{cases} g_3^+(x_1, x_3 - \theta(x_1,0)), & \text{if } x_1 \in \mathbb{R}, x_3 > \theta(x_1,0), \\ -\,g_3^+(x_1, -2(x_3 - \theta(x_1,0))) & \\ \qquad +\,2g_3^+(x_1, -(x_3-\theta(x_1,0))/2), & \text{if } x_1 \in \mathbb{R}, x_3 < \theta(x_1,0), \end{cases} \tag{4.41}$$

we have

$$\tilde{g}_3^+ \in {}_0W_q^{1-\frac{1}{2q}}(J; L^q(\mathbb{R}\times\{0\}\times\mathbb{R})^2) \cap L^q(J; W_q^{2-\frac{1}{q}}(\mathbb{R}\times\{0\}\times\mathbb{R})^2).$$

Employing an even reflection to g_4^+, one obtains the extended function

$$\tilde{g}_4^+ \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\mathbb{R}\times\{0\}\times\mathbb{R})^2) \cap L^q(J; W_q^{1-\frac{1}{q}}(\mathbb{R}\times\{0\}\times\mathbb{R})^2).$$

Let $\mu^+ := \mu|_{x_3>0}$, $\rho^+ := \rho|_{x_3>0}$. Now consider the half-space Stokes equation with the outflow boundary condition

$$\begin{aligned} \rho^+\partial_t\check{u}^+ - \mu^+\Delta\check{u}^+ + \nabla\check{\pi}^+ &= 0, && \text{in } \Omega\times J, \\ \operatorname{div}\check{u}^+ &= 0, && \text{in } \Omega\times J, \\ (\check{u}_1^+, \check{u}_3^+)^\top &= \tilde{g}_3^+, && \text{on } G\times J, \\ -\check{\pi}^+ + 2\mu^+\partial_2\check{u}_2^+ &= \tilde{g}_4^+, && \text{on } G\times J, \\ \check{u}^+(0) &= 0, && \text{in } \Omega, \end{aligned} \tag{4.42}$$

which admits a unique solution $(\check{u}^+, \check{\pi}^+)$ with regularity

$$\begin{aligned} \check{u}^+ &\in {}_0W_q^1(J; L^q(\Omega)^3) \cap L^q(J; W_q^2(\Omega)^3), \\ \check{\pi}^+ &\in L^q(J; \dot{W}_q^1(\Omega)), \quad \check{\pi}^+|_{x_2=0} \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(G)) \cap L^q(J; W_q^{1-\frac{1}{q}}(G)), \end{aligned}$$

by Bothe–Prüss [BP07, Theorem 6.1] or Prüss–Simonett [PS16, Theorem 7.2.1]. Repeating the same procedure with all superscripts $^+$ replaced by $^-$ in (4.42), where $\tilde{g}_j^-$ are the extensions of $g_j^- := g_j|_{x_3<\theta(x_1,0)}$ similar to $\tilde{g}_j^+$, $j = 3,4$, we get the corresponding functions $(\check{u}^-, \check{\pi}^-)$ with regularity

$$\begin{aligned} \check{u}^- &\in {}_0W_q^1(J; L^q(\Omega)^3) \cap L^q(J; W_q^2(\Omega)^3), \\ \check{\pi}^- &\in L^q(J; \dot{W}_q^1(\Omega)), \quad \check{\pi}^-|_{x_2=0} \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(G)) \cap L^q(J; W_q^{1-\frac{1}{q}}(G)). \end{aligned}$$

We define

$$(\check{u}, \check{\pi}) := \begin{cases} (\check{u}^+, \check{\pi}^+), & \text{if } x_3 > \theta(x'), \\ (\check{u}^-, \check{\pi}^-), & \text{if } x_3 < \theta(x'). \end{cases}$$

Then $(\bar{u}, \bar{\pi}) := (u - \check{u}, \pi - \check{\pi})$ (restricted) solves (4.39) with $(u_0, f, f_d, g_3, g_4) = 0$ and modified data (g_1, g_2) (not to be relabeled) belonging to proper regularity classes with vanishing traces at $t = 0$ whenever they exist and satisfying

$$((g_1)_1, (g_1)_3)^\top = 0, \quad ((g_2)_1, (g_2)_3)^\top = 0, \qquad \text{on } G.$$

Transform to a half space. By the reduction argument explained above we may assume

$$(u_0, f, f_d, g_3, g_4) = 0.$$

Our aim now is to transform Σ_θ to $\Sigma := \mathbb{R} \times \mathbb{R}_+ \times \{0\}$ and $\partial\Sigma_\theta$ to $\partial\Sigma := \mathbb{R} \times \{0\} \times \{0\}$. To this end, we introduce the new variables $\bar{x}' = x'$ and $\bar{x}_3 = x_3 - \theta(x')$, where $\bar{x}' := (\bar{x}_1, \bar{x}_2)$, for $x \in \Omega$, as in Section 4.3.2. Then we define

$$\bar{u}(t, \bar{x}) := u(t, \bar{x}', \bar{x}_3 + \theta(\bar{x}')), \quad \bar{\pi}(t, \bar{x}) := \pi(t, \bar{x}', \bar{x}_3 + \theta(\bar{x}')),$$

for $t \in J$, $\bar{x}' \in \mathbb{R} \times \mathbb{R}_+$. In the same way, we transform the data (g_1, g_2) to $(\bar{g}_1, \bar{g}_2)$. In the following, we will write $\partial_{\bar{x}_3}$ as ∂_3 for the sake of readability. Then with the help of (4.26) and (4.27) in Section 4.3.2, one arrives at the transformed system

$$\begin{aligned}
\rho\partial_t\bar{u} - \mu\Delta\bar{u} + \nabla\bar{\pi} &= M_1(\theta, \bar{u}, \bar{\pi}), && \text{in } \Omega\backslash\Sigma \times J,\\
\operatorname{div}\bar{u} &= M_2(\theta, \bar{u}), && \text{in } \Omega\backslash\Sigma \times J,\\
[\![\bar{u}]\!] &= \bar{g}_1, && \text{on } \Sigma \times J,\\
[\![-\bar{\pi}e_3 + \mu(\partial_3\bar{u} + \nabla\bar{u}_3)]\!] &= M_4(\theta, \bar{u}) + (\bar{g}_2 + \nabla\theta(\bar{g}_2)_3)/\beta, && \text{on } \Sigma \times J,\\
(\bar{u}_1, \bar{u}_3)^\top &= 0, && \text{on } G\backslash\partial\Sigma \times J,\\
-\bar{\pi} + 2\mu\partial_2\bar{u}_2 &= M_5(\theta, \bar{u}), && \text{on } G\backslash\partial\Sigma \times J,\\
\bar{u}(0) &= 0, && \text{in } \Omega\backslash\Sigma,
\end{aligned} \tag{4.43}$$

where M_j, $j = \{1, 2, 4, 5\}$, are given by

$$\begin{aligned}
M_1(\theta, \bar{u}, \bar{\pi}) &:= \mu\big(-2\partial_3\nabla\bar{u}\nabla\theta + \Delta\theta\partial_3\bar{u} + (\partial_3^2\bar{u} \otimes \nabla\theta)\nabla\theta\big) + \nabla\theta(\bar{x}')\partial_3\bar{\pi},\\
M_2(\theta, \bar{u}) &:= \nabla\theta \cdot \partial_3\bar{u},\\
M_4(\theta, \bar{u}) &:= [\![\mu(\nabla\bar{u} + \nabla\bar{u}^\top)]\!]\,\nabla\theta - [\![\mu(\nabla\theta \otimes \partial_3\bar{u} + \partial_3\bar{u} \otimes \nabla\theta)]\!]\,\nabla\theta\\
&\qquad + ([\![\mu\partial_3\bar{u}_3]\!]\,/\beta^2 - [\![\mu(\partial_3\bar{u} + \nabla\bar{u}_3)]\!] \cdot \nabla\theta)\nabla\theta,\\
M_5(\theta, \bar{u}) &:= 2\mu\partial_2\theta(x')\partial_3\bar{u}_2.
\end{aligned}$$

Existence and uniqueness. As in Section 4.3.2 and [Wil20, Section 1.3.4], if the data are in the right regularity classes and $\|\nabla\theta\|_\infty \le \eta$, $\eta > 0$ is sufficiently small, one may consider (4.43) as a perturbation of (4.39) and apply the Neumann series argument to (4.43) via Theorem 4.12. To this end, we define the solution spaces as

$${}_0\mathbb{E}(J) := {}_0\mathbb{E}_u(J) \times \mathbb{E}_\pi(J),$$

where

$$\begin{aligned}
{}_0\mathbb{E}_u(J) &:= \left\{u \in {}_0W_q^1(J; L^q(\Omega)^3) \cap L^q(J; W_q^2(\Omega\backslash\Sigma)^3) : (u_1, u_3)^\top\big|_G = 0\right\},\\
\mathbb{E}_\pi(J) &:= \left\{\begin{aligned} &\pi \in L^q(J; \dot{W}_q^1(\Omega)) : [\![\pi]\!] \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\Sigma)) \cap L^q(J; W_q^{1-\frac{1}{q}}(\Sigma)),\\ &\qquad \pi|_G \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(G)) \cap L^q(J; W_q^{1-\frac{1}{q}}(G\backslash\partial\Sigma))\end{aligned}\right\},
\end{aligned}$$

as well as the data function spaces

$$\widetilde{\mathbb{F}}(J) := \mathbb{F}_1(J) \times {}_0\mathbb{F}_2(J) \times {}_0\mathbb{F}_3(J) \times {}_0\mathbb{F}_4(J) \times {}_0\mathbb{F}_5(J),$$

where

$$\begin{aligned}
\mathbb{F}_1(J) &:= L^q(J; L^q(\Omega)^3),\\
{}_0\mathbb{F}_2(J) &:= {}_0W^1_q(J; \dot{W}^{-1}_q(\Omega\backslash\Sigma)) \cap L^q(J; W^1_q(\Omega\backslash\Sigma)),\\
{}_0\mathbb{F}_3(J) &:= {}_0W^{1-\frac{1}{2q}}_q(J; L^q(\Sigma)^3) \cap L^q(J; W^{2-\frac{1}{q}}_q(\Sigma)^3),\\
{}_0\mathbb{F}_4(J) &:= {}_0W^{\frac{1}{2}-\frac{1}{2q}}_q(J; L^q(\Sigma)^3) \cap L^q(J; W^{1-\frac{1}{q}}_q(\Sigma)^3),\\
{}_0\mathbb{F}_5(J) &:= {}_0W^{\frac{1}{2}-\frac{1}{2q}}_q(J; L^q(G)) \cap L^q(J; W^{1-\frac{1}{q}}_q(G\backslash\partial\Sigma)).
\end{aligned}$$

In addition, let

$${}_0\mathbb{F}(J) := \left\{ (f_1, f_2, g_1, g_2, g_3) \in \widetilde{\mathbb{F}}(J) : \begin{array}{r} ((g_1)_1, (g_1)_3)^\top = 0,\ (g_2)_1 = 0,\ \text{at } \partial\Sigma \\ (g_2)_3 = [\![g_3 - 2\mu f_2]\!],\ \text{at } \partial\Sigma \end{array} \right\}.$$

Now we define an operator $L : {}_0\mathbb{E}(J) \to {}_0\mathbb{F}(J)$ by

$$L(\bar{u}, \bar{\pi}) := \begin{bmatrix} \rho\partial_t\bar{u} - \mu\Delta\bar{u} + \nabla\bar{\pi} \\ \operatorname{div}\bar{u} \\ [\![\bar{u}]\!] \\ [\![-\bar{\pi}e_3 + \mu(\partial_3\bar{u} + \nabla\bar{u}_3)]\!] \\ (-\bar{\pi} + 2\mu\partial_2\bar{u}_2)|_G \end{bmatrix}.$$

Then L is an isomorphism by Theorem 4.12. For the right-hand side of (4.28), we set

$$M(\theta, \bar{u}, \bar{\pi}) := (M_1(\theta, \bar{u}, \bar{\pi}), M_2(\theta, \bar{u}), 0, M_4(\theta, \bar{u}), M_5(\theta, \bar{u}))^\top$$

and

$$F := (0, 0, f_3, f_4, 0)^\top, \quad f_3 := \bar{g}_1, \quad f_4 := (\bar{g}_2 + \nabla\theta(\bar{g}_2)_3)/\beta.$$

It is clear that $F \in \widetilde{\mathbb{F}}(J)$, since $\theta \in C^3(\mathbb{R} \times \mathbb{R}_+)$. We note that $((\bar{g}_1)_1, (\bar{g}_1)_3)^\top = 0$, $((\bar{g}_2 + \nabla\theta(\bar{g}_2)_3)/\beta)_1 = 0$ as well as $(\bar{g}_2)_3 = 0$ at the contact line $\partial\Sigma$, one obtains $F \in {}_0\mathbb{F}(J)$.

To apply the Neumann series argument, we investigate the perturbation term $M(\theta, \bar{u}, \bar{\pi})$. Note that from the compatibility conditions, $(M_4(\theta, \bar{u}))_1 = 0$ as well as $(M_4(\theta, \bar{u}))_3 = 0$ must be true at the contact line $\partial\Sigma$, since $2\mu M_2(\theta, \bar{u}) = M_5(\theta, \bar{u})$ at the contact line $\partial\Sigma$. However, these do not hold in general, namely,

$$\begin{aligned}
(M_4(\theta, \bar{u}))_1 &= \partial_2\theta\mu(\partial_1\bar{u}_2 + \partial_2\bar{u}_1 + (1 + \partial_1\theta)\partial_3\bar{u}_2 + \partial_2\bar{u}_3),\ \text{at } \partial\Sigma,\\
(M_4(\theta, \bar{u}))_3 &= \partial_2\theta\mu(\partial_2\bar{u}_3 + \partial_3\bar{u}_2),\ \text{at } \partial\Sigma.
\end{aligned}$$

To overcome this trouble, we follow the argument in [Wil20] by introducing a modified $\widetilde{M_4}(\theta, \bar{u})$ as

$$\widetilde{M_4}(\theta, \bar{u}) = (M_4(\theta, \bar{u}))_1 - \partial_2\theta(K_1, 0, K_2)^\top,$$

where

$$\begin{aligned}
K_1(\theta, \bar{u}) &= \operatorname{ext}_\Sigma\left(\left[\!\!\left[\mu(\partial_1\bar{u}_2 + \partial_2\bar{u}_1 + (1 + \partial_1\theta)\partial_3\bar{u}_2 + \partial_2\bar{u}_3)|_{G\backslash\partial\Sigma}\right]\!\!\right]\right),\\
K_2(\theta, \bar{u}) &= \operatorname{ext}_\Sigma\left(\left[\!\!\left[\mu(\partial_2\bar{u}_3 + \partial_3\bar{u}_2)|_{G\backslash\partial\Sigma}\right]\!\!\right]\right).
\end{aligned}$$

Here, by Wilke [Wil20, Proposition 5.1], there exists a linear and bounded extension operator ext_Σ from

$$ {}_0W_q^{\frac{1}{2}-\frac{1}{q}}(J;L^q(\partial\Sigma))\cap L^q(J;W_q^{1-\frac{2}{q}}(\partial\Sigma)) $$

to

$$ {}_0W_q^{\frac{1}{2}-\frac{1}{2q}}(J;L^q(\Sigma))\cap L^q(J;W_q^{1-\frac{1}{q}}(\Sigma)), $$

such that $(\mathrm{ext}_\Sigma z)|_{\partial\Sigma} = z$ for all $z\in {}_0W_q^{1/2-1/q}(J;L^q(\partial\Sigma))\cap L^q(J;W_q^{1-2/q}(\partial\Sigma))$. Define

$$ \widetilde{M}(\theta,\bar u,\bar\pi) := (M_1(\theta,\bar u,\bar\pi), M_2(\theta,\bar u), 0, \widetilde{M}_4(\theta,\bar u), M_5(\theta,\bar u))^\top. $$

Then it can be verified easily that the modified perturbation $\widetilde{M}(\theta,\bar u,\bar\pi)\in {}_0\mathbb{F}(J)$ for each $(\bar u,\bar\pi)\in {}_0\mathbb{E}(J)$, combining the smoothness of θ. The only point that needs to be taken care of is $M_2(\theta,\bar u)\in {}_0W_q^1(J;\dot W_q^{-1}(\Omega\backslash\Sigma))$. By integration by parts with respect to the $\bar x_3$, we have

$$ \int_\Omega M_2(\theta,\bar u)\phi \mathrm{d}\bar x = \int_\Omega \nabla\theta\cdot\partial_3\bar u\phi\mathrm{d}\bar x = -\int_\Omega \nabla\theta\cdot\bar u\partial_3\phi\mathrm{d}\bar x, \tag{4.44} $$

for all $\phi\in W^1_{q',0}(\Omega\backslash\Sigma)$, which implies the claim with $\bar u\in {}_0\mathbb{E}_u(J)$.

Then we rewrite (4.43), with M_4 replaced by $\widetilde{M}_4$, as

$$ (\bar u,\bar\pi) = L^{-1}\widetilde{M}(\theta,\bar u,\bar\pi) + L^{-1}F. $$

Our goal in the following is to show that for each $\varepsilon>0$, there exist $T_0>0$ and $\eta_0>0$ such that

$$ \left\|\widetilde{M}(\theta,\bar u,\bar\pi)\right\|_{{}_0\mathbb{F}(J)} \le \varepsilon\left\|(\bar u,\bar\pi)\right\|_{{}_0\mathbb{E}(J)} \tag{4.45} $$

holds for $0<T<T_0$ and $0<\eta<\eta_0$.

We estimate $\widetilde{M}(\theta,\bar u,\bar\pi)$ term by term. Mimicking the estimates in Section 4.3.2, one obtains

$$ \|M_1(\theta,\bar u,\bar\pi)\|_{\mathbb{F}_1(J)} \le C(\|\nabla\theta\|_\infty + \|\nabla\theta\|_\infty^2 + T^{\frac{1}{2q}}\left\|\nabla^2\theta\right\|_\infty)\left\|(\bar u,\bar\pi)\right\|_{{}_0\mathbb{E}(J)}, \tag{4.46} $$

$$ \|M_2(\theta,\bar u)\|_{{}_0\mathbb{F}_2(J)} \le C(\|\nabla\theta\|_\infty + T^{\frac{1}{2q}}\left\|\nabla^2\theta\right\|_\infty)\left\|\bar u\right\|_{{}_0\mathbb{E}_u(J)}. \tag{4.47} $$

where $C>0$ does not depend on $T>0$. For $\widetilde{M}_4(\theta,\bar u)\in {}_0\mathbb{F}_4(J)$, we see that

$$ \left\|\widetilde{M}_4(\theta,\bar u)\right\|_{{}_0\mathbb{F}_4(J)} = \left\|\widetilde{M}_4(\theta,\bar u)\right\|_{{}_0W_q^{\frac{1}{2}-\frac{1}{2q}}(J;L^q(\Sigma)^3)} + \left\|\widetilde{M}_4(\theta,\bar u)\right\|_{L^q(J;W_q^{1-\frac{1}{q}}(\Sigma)^3)}. $$

The first term in the right-hand side above is estimated by

$$ \begin{aligned} \left\|\widetilde{M}_4(\theta,\bar u)\right\|_{{}_0W_q^{\frac{1}{2}-\frac{1}{2q}}(J;L^q(\Sigma)^3)} &\le C(\|\nabla\theta\|_\infty + \|\nabla\theta\|_\infty^2)\left\|\nabla\bar u\right\|_{{}_0W_q^{\frac{1}{2}-\frac{1}{2q}}(J;L^q(\Sigma)^{3\times3})} \\ &\le C(\|\nabla\theta\|_\infty + \|\nabla\theta\|_\infty^2)\left\|\bar u\right\|_{{}_0\mathbb{E}_u}, \end{aligned} $$

while the second term satisfies

$$ \begin{aligned} \left\|\widetilde{M}_4(\theta,\bar u)\right\|_{L^q(J;W_q^{1-\frac{1}{q}}(\Sigma)^3)} &\le C(\|\nabla\theta\|_\infty + \|\nabla\theta\|_\infty^2)\left\|\nabla\bar u\right\|_{L^q(J;W_q^{1-\frac{1}{q}}(\Sigma)^{3\times3})} \\ &\quad + C(1+\|\nabla\theta\|_\infty)\left\|\nabla^2\theta\right\|_\infty\left\|\nabla\bar u\right\|_{L^q(J;L^q(\Omega\backslash\Sigma)^{3\times3})} \\ &\le C(\|\nabla\theta\|_\infty + \|\nabla\theta\|_\infty^2)\left\|\bar u\right\|_{{}_0\mathbb{E}_u} + CT^{\frac{1}{2q}}(1+\|\nabla\theta\|_\infty)\left\|\nabla^2\theta\right\|_\infty\left\|\bar u\right\|_{{}_0\mathbb{E}_u}, \end{aligned} $$

by the trace theory of Sobolev spaces. Hence,

$$\begin{aligned}&\left\|\widetilde{M}_4(\theta,\bar{u})\right\|_{{}_0\mathbb{F}_4(J)}\\&\quad\leq C(\|\nabla\theta\|_\infty+\|\nabla\theta\|_\infty^2)\,\|\bar{u}\|_{{}_0\mathbb{E}_u}+CT^{\frac{1}{2q}}(1+\|\nabla\theta\|_\infty)\left\|\nabla^2\theta\right\|_\infty\|\bar{u}\|_{{}_0\mathbb{E}_u}.\end{aligned}\tag{4.48}$$

The same estimate as in (4.33) yields that

$$\|M_5(\theta,\bar{u})\|_{{}_0\mathbb{F}_5(J)}\leq C\,\|\nabla\theta\|_\infty\,\|\bar{u}\|_{{}_0\mathbb{E}_u}.\tag{4.49}$$

Collecting (4.46)–(4.49), we have

$$\begin{aligned}&\left\|\widetilde{M}(\theta,\bar{u},\bar{\pi})\right\|_{{}_0\mathbb{F}(J)}\\&\quad\leq C\left(\|\nabla\theta\|_\infty+\|\nabla\theta\|_\infty^2+CT^{\frac{1}{2q}}(1+\|\nabla\theta\|_\infty)\left\|\nabla^2\theta\right\|_\infty\right)\|(\bar{u},\bar{\pi})\|_{{}_0\mathbb{E}(J)}.\end{aligned}$$

Since $\|\nabla_{x'}\theta(x')\|_\infty\leq\eta$ and $\left\|\nabla_{x'}^2\theta(x')\right\|_\infty\leq M$, by choosing $\eta>0$ and $T>0$ sufficiently small, one obtains (4.45). A Neumann series argument in ${}_0\mathbb{E}(J)$ finally implies that there exists a unique solution $(\bar{u},\bar{\pi})\in{}_0\mathbb{E}(J)$ of $L(\bar{u},\bar{\pi})=\widetilde{M}(\theta,\bar{u},\bar{\pi})+F$ or equivalently a solution (u,π) of (4.39).

Consequently, (4.39) admits a solution operator $S_{HS}:\mathbb{F}_{HS}\to\mathbb{E}_{HS}$, where $\mathbb{F}_{HS}$ and $\mathbb{E}_{HS}$ are the solution space and data space respectively, for a half-space with a bent interface and the data in $\mathbb{F}_{HS}$ satisfy the compatibility conditions (4.40).

4.3.5. The heat equations in quarter-spaces. As in Section 4.3.1, define $\Omega:=\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+$, $G:=\mathbb{R}\times\{0\}\times\mathbb{R}_+$ and $S:=\mathbb{R}\times\mathbb{R}_+\times\{0\}$. Consider the problem

$$\begin{aligned}\partial_t u-D\Delta u&=f, &&\text{in }\Omega\times J,\\D\partial_2 u&=g_1, &&\text{on }G\times J,\\D\partial_3 u&=g_2, &&\text{on }S\times J,\\u(0)&=u_0, &&\text{in }\Omega,\end{aligned}\tag{4.50}$$

where $u:\mathbb{R}^3\times\mathbb{R}_+\to\mathbb{R}$ is the quantity of the system, e.g., the temperature, the concentration, etc. $D>0$ denotes the constant diffusivity.

Theorem 4.13. *Let $q>3$, $T>0$, $D>0$ and $J=(0,T)$. Assume that $G,S\in C^3$ and ∂G is of class C^4. Then there exists a unique solution*

$$u\in W_q^1(J;L^q(\Omega))\cap L^q(J;W_q^2(\Omega)),$$

of (4.50) if and only if the data satisfy the following regularity and compatibility conditions:

(1) $f\in L^q(J;L^q(\Omega))$,

(2) $g_1\in W_q^{\frac{1}{2}-\frac{1}{2q}}(J;L^q(G))\cap L^q(J;W_q^{1-\frac{1}{q}}(G))$,

(3) $g_2\in W_q^{\frac{1}{2}-\frac{1}{2q}}(J;L^q(S))\cap L^q(J;W_q^{1-\frac{1}{q}}(S))$,

(4) $u_0\in W_q^{2-\frac{2}{q}}(\Omega)$,

(5) $D\partial_2 u_0|_G=g_1|_{t=0}$, $D\partial_3 u_0|_S=g_2|_{t=0}$,

(6) $\partial_3 g_1|_{\overline{G}\cap\overline{S}} = \partial_2 g_2|_{\overline{G}\cap\overline{S}}$.

Proof. To prove Theorem 4.13, we extend the data suitably to the half-space and solve the heat equations in half space. To this end, one extends (f, g_1, u_0) with respect to x_3 by general extension to some functions

$$\tilde{f} \in L^q(J; L^q(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})), \quad \tilde{u}_0 \in W_q^{2-\frac{2}{q}}(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}),$$
$$\tilde{g}_1 \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q((\mathbb{R}\times\{0\}\times\mathbb{R}))) \cap L^q(J; W_q^{1-\frac{1}{q}}((\mathbb{R}\times\{0\}\times\mathbb{R}))).$$

Solving

$$\begin{aligned} \partial_t u - D\Delta u &= \tilde{f}, && \text{in } \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times J,\\ D\partial_2 u &= \tilde{g}_1, && \text{in } \mathbb{R}\times\{0\}\times\mathbb{R}\times J,\\ u(0) &= \tilde{u}_0, && \text{in } \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}, \end{aligned}$$

yields a unique solution

$$\tilde{u} \in W_q^1(J; L^q(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})) \cap L^q(J; W_q^2(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})),$$

by [PS16, Theorem 6.2.5]. Then $(u-\tilde{u})$ solves (4.50) with $(f, g_1, u_0) = 0$ and modified g_2 (not to be relabeled) satisfying $\partial_2 g_2|_S = 0$ with vanishing time trace at $t = 0$. Now extend g_2 by even reflection with respect to x_2 to functions

$$\tilde{g}_2 \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\mathbb{R}^2\times\{0\})) \cap L^q(J; W_q^{1-\frac{1}{q}}(\mathbb{R}^2\times\{0\})).$$

We solve

$$\begin{aligned} \partial_t u - D\Delta u &= 0, && \text{in } \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+\times J,\\ D\partial_3 u &= \tilde{g}_2, && \text{in } \mathbb{R}\times\mathbb{R}\times\{0\}\times J,\\ u(0) &= 0, && \text{in } \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+, \end{aligned}$$

again by [PS16, Theorem 6.2.5] to obtain a unique solution

$$\hat{u} \in W_q^1(J; L^q(\mathbb{R}^2\times\mathbb{R}_+)) \cap L^q(J; W_q^2(\mathbb{R}^2\times\mathbb{R}_+)).$$

Note that $\hat{u}(t, x_1, -x_2, x_3)$ is also a solution of above system. Then uniqueness of solutions implies that

$$\hat{u}(t, x_1, x_2, x_3) = \hat{u}(t, x_1, -x_2, x_3)$$

and hence $\partial_2\hat{u}(t, x_1, 0, x_3) = 0$. Therefore, the restricted $u := \tilde{u} + \hat{u}$ is the desired solution to (4.50). The uniqueness can be derived by the argument in Section 4.3.1 combined with the uniqueness of heat equation in half-spaces ([PS16, Theorem 6.2.5]), which completes the proof. □

4.3.6. The heat equations in bent quarter-spaces. By the same argument in Section 4.3.2, we can find a unique solution to (4.50) with a perturbation of $x_3 = 0$. Hence in turn, one obtains a solution operator $S_{HQS} : \mathbb{F}_{HQS} \to \mathbb{E}_{HQS}$, where $\mathbb{F}_{HQS}$ and $\mathbb{E}_{HQS}$ are the solution space and data space, respectively, for the bent quarter-space and the data in $\mathbb{F}_{QS}$ satisfy the compatibility conditions at the contact line $\{(x_1, 0, \theta(x_1)) : x_1 \in \mathbb{R}\}$.

4.4. General Bounded Cylindrical Domains

In this section, we will show that (4.10), (4.13) and (4.15) respectively admits a unique solution. For system (4.13) and (4.15), a localization procedure does work since they are both standard parabolic equations, while some modifications are needed for (4.10). The point is that for (4.10), the presence of pressure and the divergence equation will bring additional difficulties, for which we proceed as in [PS16] and [Wil20] to obtain extra regularity for the pressure under suitable conditions and reduce the system to a simpler form.

4.4.1. Regularity of the pressure. In general, the pressure π does not have more regularity than that stated in Theorem 4.7. However, in a special situation, one may obtain extra time-regularity of π.

PROPOSITION 4.14. *Let Ω be the bounded domain defined in Theorem 4.7 and $(u, \pi) \in \mathbb{E}(J)$ be a solution of the two-phase problem (4.10) with*

$$(u_0, f_d, g_1 \cdot \nu_\Sigma, g_5 \cdot \nu_S) = 0$$

and $f_u \in {}_0W_q^\alpha(J; L^q(\Omega)^3)$, for some $\alpha \in (0, 1/2 - 1/2q)$. Then $\pi \in {}_0W_q^\alpha(J; L^q(\Omega))$ and there is a constant $C > 0$ independent of J, u, π, f_u, such that

$$\|\pi\|_{W_q^\alpha(J;L^q(\Omega))} \leq C \left(\|u\|_{\mathbb{E}_u} + \|\pi\|_{\mathbb{E}_\pi} + \|f_u\|_{W_q^\alpha(J;L^q(\Omega)^3)} \right).$$

Proof. Fix any $\psi \in L^{q'}(\Omega)$. By Theorem 4.21, the transmission problem

$$\begin{aligned} \Delta\phi &= \psi, && \text{in } \Omega\backslash\Sigma, \\ [\![\rho\phi]\!] &= 0, && \text{on } \Sigma, \\ [\![\partial_{\nu_\Sigma}\phi]\!] &= 0, && \text{on } \Sigma, \\ \rho\phi &= 0, && \text{on } G\backslash\partial\Sigma, \\ \partial_{\nu_S}\phi &= 0, && \text{on } S \end{aligned} \tag{4.51}$$

possesses a unique solution $\phi \in W_{q'}^1(\Omega) \cap W_{q'}^2(\Omega\backslash\Sigma)$, which satisfies the estimate

$$\|\phi\|_{W_{q'}^1(\Omega)\cap W_{q'}^2(\Omega\backslash\Sigma)} \leq C \|\psi\|_{L_{(0)}^{q'}(\Omega)}.$$

By integration by parts, one obtains

$$\begin{aligned} \int_\Omega \pi\psi \mathrm{d}x = \int_\Omega \pi\Delta\phi \mathrm{d}x &= -\int_\Sigma [\![\pi]\!]\, \partial_{\nu_\Sigma}\phi \mathrm{d}\sigma + \int_G \pi\partial_{\nu_G}\phi \mathrm{d}\sigma - \int_\Omega \nabla\pi\cdot\nabla\phi \mathrm{d}x \\ &= -\int_\Sigma [\![\pi]\!]\, \partial_{\nu_\Sigma}\phi \mathrm{d}\Sigma + \int_G \pi\partial_{\nu_G}\phi \mathrm{d}\sigma - \int_\Omega (f_u - \rho\partial_t u + \nu\Delta u)\cdot\nabla\phi \mathrm{d}x \\ &= -\int_\Sigma \left([\![\pi]\!]\, \partial_{\nu_\Sigma}\phi + [\![\mu\partial_{\nu_\Sigma}u\cdot\nabla\phi]\!]\right) \mathrm{d}\sigma + \int_G \pi\partial_{\nu_G}\phi \mathrm{d}\sigma - \int_\Omega f_u\cdot\nabla\phi \mathrm{d}x \\ &\quad - \int_S \mu\partial_{\nu_S}u\cdot\nabla\phi \mathrm{d}\sigma - \int_G \mu\partial_{\nu_G}u\cdot\nabla\phi \mathrm{d}\sigma + \int_\Omega \mu\nabla u : \nabla^2\phi \mathrm{d}x. \end{aligned}$$

Taking the supremum of the left-hand side over all functions $\psi \in L^{q'}(\Omega)$, we have

$$\begin{aligned} \|\pi(t)\|_{L^q(\Omega)} \leq C \Big(&\|\nabla u(t)\|_{L^q(\Omega)} + \|\partial_{\nu_S}u(t)\|_{L^q(S)} + \|\partial_{\nu_G}u(t)\|_{L^q(G)} \\ &+ \|\partial_{\nu_\Sigma}u(t)\|_{L^q(\Sigma)} + \|[\![\pi(t)]\!]\|_{L^q(\Sigma)} + \|\pi(t)\|_{L^q(G)} + \|f_u(t)\|_{L^q(\Omega)} \Big), \end{aligned}$$

for almost all $t \in J$. Furthermore,

$$\begin{aligned}
&\|\pi(t)-\pi(s)\|_{L^q(\Omega)} \\
&\le C\Big(\|\nabla(u(t)-u(s))\|_{L^q(\Omega)} + \|\partial_{\nu_S}(u(t)-u(s))\|_{L^q(S)} + \|\partial_{\nu_G}(u(t)-u(s))\|_{L^q(G)} \\
&\quad + \|\partial_{\nu_\Sigma}(u(t)-u(s))\|_{L^q(\Sigma)} + \|[\![\pi(t)-\pi(s)]\!]\|_{L^q(\Sigma)} + \|(\pi(t)-\pi(s))\|_{L^q(G)} \\
&\quad + \|f_u(t)-f_u(s)\|_{L^q(\Omega)}\Big),
\end{aligned}$$

for almost all $t \in J$. Since ${}_0\mathbb{E}_u \hookrightarrow {}_0H_q^{\frac{1}{2}}(J; W_q^1(\Omega\backslash\Sigma))$, trace theory (see e.g. [PS16, Proposition 6.2.4]) implies that

$$(\partial_k u_l)|_K \in {}_0W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(K)),$$

for $k, l \in \{1,2,3\}$ and $K \in \{\Sigma, G, S\}$. From the embedding $W_q^s \hookrightarrow W_q^{s-\epsilon}$, $\epsilon > 0$, one gets $\pi \in {}_0W_q^{\alpha}(J; L^q(\Omega))$ and the desired estimate. □

Remark 4.15. Note that for unbounded domains, e.g., the (bent) quarter space and the half space with a (bent) interface, one can always reduce the unbounded domain to a bounded domain with the same part of boundary by suitable extension. Therefore, in the following localization procedure as in Section 4.4.3, we will apply Proposition 4.4.1 directly without any further explanations.

4.4.2. Reductions. In this section, we reduce the problem (4.10) to the case

$$(u_0, f_u, f_d, g_1\cdot\nu_\Sigma, g_5\cdot\nu_S) = 0.$$

To this end, consider the parabolic transmission problem

$$\begin{aligned}
\rho\partial_t\bar u - \mu\Delta\bar u &= f_u, && \text{in } \Omega\backslash\Sigma\times J,\\
[\![\bar u]\!] &= g_1, && \text{in } \Sigma\times J,\\
[\![\mu(\nabla\bar u + \nabla\bar u^\top)\nu_\Sigma]\!] &= g_2, && \text{on } \Sigma\times J,\\
(\bar u_1, \bar u_3)^\top &= g_3, && \text{on } G\backslash\partial\Sigma\times J,\\
2\mu\partial_2\bar u_2 &= g_4, && \text{on } G\backslash\partial\Sigma\times J,\\
\bar u &= g_5, && \text{on } S\times J,\\
\bar u(0) &= u_0, && \text{in } \Omega\backslash\Sigma,
\end{aligned} \tag{4.52}$$

with (u_0, f_u, g_j), $j \in \{1, ..., 5\}$, in suitable function spaces. Then by Theorem 4.23, (4.52) admits a unique solution

$$\bar u \in W_q^1(J; L^q(\Omega)^3) \cap L^q(J; W_q^2(\Omega\backslash\Sigma)^3),$$

which implies that $(u-\bar u, \pi)$ solves (4.10) with $(u_0, f_u, g_1, g_3, g_5) = 0$. Next, solving the elliptic transmission problem

$$\begin{aligned}
\Delta\phi &= f_d - \operatorname{div}\bar u, && \text{in } \Omega\backslash\Sigma,\\
[\![\rho\phi]\!] &= 0, && \text{on } \Sigma,\\
[\![\partial_{\nu_\Sigma}\phi]\!] &= 0, && \text{on } \Sigma,\\
\rho\phi &= 0, && \text{on } G\backslash\partial\Sigma,\\
\partial_{\nu_S}\phi &= 0, && \text{on } S
\end{aligned} \tag{4.53}$$

yields a unique solution ϕ with regularity

$$\nabla\phi \in {}_0W^1_q(J; L^q(\Omega)^3) \cap L^q(J; W^2_q(\Omega\backslash\Sigma)^3),$$

thanks to Theorem 4.21. Now for (u, π) the solution of (4.10), define

$$(\tilde{u}, \tilde{\pi}) := (u - \bar{u} - \nabla\phi, \pi + \rho\partial_t\phi - \mu\Delta\phi).$$

It is clear that $(\tilde{u}, \tilde{\pi})$ solves (4.10) with $(u_0, f_u, f_d, g_1 \cdot \nu_\Sigma, g_5 \cdot \nu_S) = 0$ and other modified data in right regularity classes having vanishing time traces at $t = 0$.

4.4.3. Localization procedure. Firstly, we are devoted to prove Theorem 4.7, which is the most crucial part of this work. The proof is based on the localization procedure with the model problems we considered in Section 4.3.

Proof of Theorem 4.7. Following the localization method in e.g. [DHP03], [PS16] and [Wil20], we split the proof into two parts.

(1). Existence of a left inverse. Let (u, π) be a solution of (4.10). By the reduction argument in Section 4.4.2, there exists $(\bar{u}, \bar{\pi})$ such that $(\tilde{u}, \tilde{\pi}) := (u, \pi) - (\bar{u}, \bar{\pi})$ solves the problem

$$\begin{aligned} \rho\partial_t\tilde{u} - \mu\Delta\tilde{u} + \nabla\tilde{\pi} &= 0, && \text{in } \Omega\backslash\Sigma \times J, \\ \operatorname{div}\tilde{u} &= 0, && \text{in } \Omega\backslash\Sigma \times J, \\ [\![\tilde{u}]\!] &= \tilde{g}_1, && \text{on } \Sigma \times J, \\ [\![-\tilde{\pi}\mathbb{I} + \mu(\nabla\tilde{u} + \nabla\tilde{u}^\top)]\!]\,\nu_\Sigma &= \tilde{g}_2, && \text{on } \Sigma \times J, \\ (\tilde{u}_1, \tilde{u}_3)^\top &= \tilde{g}_3, && \text{on } G\backslash\partial\Sigma \times J, \\ -\tilde{\pi} + 2\mu\partial_2\tilde{u}_2 &= \tilde{g}_4, && \text{on } G\backslash\partial\Sigma \times J, \\ \tilde{u} &= \tilde{g}_5, && \text{on } S, \times J \\ \tilde{u}(0) &= 0, && \text{in } \Omega\backslash\Sigma. \end{aligned} \tag{4.54}$$

with $(\tilde{g}_1 \cdot \nu_\Sigma, \tilde{g}_5 \cdot \nu_S) = 0$. Choose open sets $U_k = B_r(x_k)$ with radius $r > 0$ and centers x_k such that $\partial\Sigma \subset \bigcup_{k=5}^{N_1} U_k$, $\partial S \subset \bigcup_{k=N_1+1}^{N} U_k$ and choose $r > 0$ small enough such that corresponding solution operators from Section 4.3.2 and 4.3.4 are well-defined. By Proposition 4.19, there exist open and connected sets such that

- $U_0 \cap \overline{\Omega} \neq \emptyset$, $U_0 \cap \overline{G} = \emptyset$,
- $U_i \cap G_1 \neq \emptyset$, $U_j \cap G_2 \neq \emptyset$, $U_k \cap (\Sigma \cup S) = \emptyset$, $i = 1, 2$, $j = 3, 4$, $k = i, j$,

and a family of functions $\{\varphi_k\}_{k=0}^N \subset C^3_c(\mathbb{R}^3; [0, 1])$ such that $\overline{\Omega} \subset \bigcup_{k=0}^N U_k$, $\operatorname{supp}\varphi_k \subset U_k$, $\sum_{k=0}^N \varphi_k = 1$ and $\partial_{\nu_\Sigma}\varphi_k(x) = 0$, $\partial_{\nu_S}\varphi_k(x) = 0$, for $x \in U_k \cap (\partial\Sigma \cup \partial S)$, $k \geq 5$.

Now define $(\tilde{u}_k, \tilde{\pi}_k) := (\tilde{u}, \tilde{\pi})\varphi_k$, the data $\tilde{g}_{jk} := \tilde{g}_j\varphi_k$, as well as the domain $\Omega^k := \Omega \cap U_k$, where U_k is an open set if $k = 0$, a half space if $k = 1, ..., 4$, a bent quarter space if $k = N_1 + 1, ..., N$, a half space with a bent interface if $k = 5, ..., N_1$. Moreover, let $\Sigma^k := \Sigma \cap U_k$,

$G^k := G \cap U_k$ and $S^k := S \cap U_k$. Then $(\tilde{u}_k, \tilde{\pi}_k)$ satisfies the problem

$$\begin{aligned} \rho\partial_t \tilde{u}_k - \mu\Delta\tilde{u}_k + \nabla\tilde{\pi}_k &= F_k(\tilde{u},\tilde{\pi}), && \text{in } \Omega^k\backslash\Sigma^k \times J,\\ \operatorname{div}\tilde{u}_k &= F_{dk}(\tilde{u}), && \text{in } \Omega^k\backslash\Sigma^k \times J,\\ [\![\tilde{u}_k]\!] &= \tilde{g}_{1k}, && \text{on } \Sigma^k \times J,\\ [\![-\tilde{\pi}_k\mathbb{I} + \mu(\nabla\tilde{u}_k + \nabla\tilde{u}_k^\top)]\!]\,\nu_{\Sigma^k} &= \tilde{g}_{2k} + G_{2k}(\tilde{u}), && \text{on } \Sigma^k \times J,\\ ((\tilde{u}_k)_1, (\tilde{u}_k)_3)^\top &= \tilde{g}_{3k}, && \text{on } G^k\backslash\partial\Sigma^k \times J,\\ -\tilde{\pi}_k + 2\mu\partial_2(\tilde{u}_k)_2 &= \tilde{g}_{4k} + G_{4k}(\tilde{u}), && \text{on } G^k\backslash\partial\Sigma^k \times J,\\ \tilde{u}_k &= \tilde{g}_{5k}, && \text{on } S^k \times J,\\ \tilde{u}_k(0) &= 0, && \text{in } \Omega^k\backslash\Sigma^k, \end{aligned} \tag{4.55}$$

where

$$F_k(\tilde{u},\tilde{\pi}) := [\nabla,\varphi_k]\tilde{\pi} - \mu[\Delta,\varphi_k]\tilde{u}, \quad F_{dk}(\tilde{u}) := \tilde{u}\cdot\nabla\varphi_k,$$
$$G_{2k}(\tilde{u}) := [\![\mu(\nabla\varphi_k \otimes \tilde{u} + \tilde{u}\otimes\nabla\varphi_k)]\!]\,\nu_{\Sigma^k}, \quad G_{4k}(\tilde{u}) := 2\mu\partial_2\varphi_k\tilde{u}_2.$$

For $k = 0$, we extend Σ to $\widetilde{\Sigma}$ and S to $\widetilde{S}$ smoothly, such that

$$\widetilde{\Sigma}\cap U_0 = \Sigma\cap U_0, \quad \widetilde{\Omega}\cap U_0 = \Omega\cap U_0, \quad \widetilde{\Sigma}\subset\widetilde{\Omega}.$$

Then solving a two-phase Stokes problem in a bounded domain with smooth boundary and interface yields the unique solution of this local problem by [PS16, Theorem 8.1.4]. If $k = 1, ..., 4$, we have the half-space Stokes problem with outflow boundary conditions. This problem was investigated in [BP07, Section 6] and [PS16, Section 7.2]. For $k = 5, ..., N_1$ and $k = N_1+1, ..., N$, we obtain a bent quarter-space Stokes problem and respectively, a half-space two-phase Stokes problem with a bent interface, which are solvable according to Section 4.3.2 and 4.3.4. Hence, the solution operators for the charts U_k, $k \geq 5$, are well-defined by the results in Section 4.3.2 and 4.3.4. We denote the corresponding solution operators for each chart by $\mathcal{S}_k$.

Next, we want to reduce $F_{dk}(\tilde{u})$, since we do not have enough time regularity for it, while F_k, G_{2k} and G_{4k} are endowed with extra time regularity due to the regularity of $\tilde{u}$ and Proposition 4.14. More specifically, if $\tilde{u} \in {}_0\mathbb{E}_u(J) \hookrightarrow {}_0W_q^{1/2}(J; W_q^1(\Omega^k)^3)$, then

$$[\Delta,\varphi_k]\tilde{u} \in {}_0W_q^{1/2}(J; L^q(\Omega^k)^3) \cap L^q(J; W_q^1(\Omega^k\backslash\Sigma^k)^3).$$

From Proposition 4.14, we know that

$$\tilde{\pi} \in {}_0W_q^{\alpha}(J; L^q(\Omega^k)) \cap L^q(J; W_q^1(\Omega^k\backslash\Sigma^k)),$$

and hence

$$[\nabla,\varphi_k]\tilde{\pi} \in {}_0W_q^{\alpha}(J; L^q(\Omega^k)^3) \cap L^q(J; W_q^1(\Omega^k\backslash\Sigma^k)^3),$$

for $0 < \alpha < 1/2 - 1/2q$. Then

$$F_k(\tilde{u},\tilde{\pi}) \in {}_0W_q^{\alpha}(J; L^q(\Omega^k)) \cap L^q(J; W_q^1(\Omega^k\backslash\Sigma^k)),$$

as well as

$$G_{2k}(\tilde{u}) \in {}_0W_q^{1-1/2q}(J; L^q(\Sigma^k)^3) \cap {}_0W_q^{1/2}(J; W_q^{1-1/q}(\Sigma^k)^3),$$
$$G_{4k}(\tilde{u}) \in {}_0W_q^{1-1/2q}(J; L^q(G^k)) \cap {}_0W_q^{1/2}(J; W_q^{1-1/q}(G^k\backslash\partial\Sigma^k)),$$

by similar argument and the trace theory (see e.g. [PS16, Proposition 6.2.4]). However, $F_{dk} \in {}_0\mathbb{E}_u(J)$ is an exception without enough regularity. Thus, we reduce it by solving an auxiliary elliptic transmission problem

$$\begin{aligned} \Delta\phi_k &= F_{dk}(\tilde{u}), && \text{in } \Omega^k\backslash\Sigma^k, \\ [\![\rho\phi_k]\!] &= 0, && \text{on } \Sigma^k, \\ [\![\partial_{\nu_\Sigma}\phi_k]\!] &= 0, && \text{on } \Sigma^k, \\ \rho\phi_k &= 0, && \text{on } G^k\backslash\partial\Sigma^k, \\ \partial_{\nu_S}\phi_k &= 0, && \text{on } S^k, \end{aligned} \tag{4.56}$$

with Theorem 4.70 to obtain a unique solution ϕ_k with regularity

$$\begin{aligned} \nabla\phi_k \in {}_0\mathbb{E}_\phi(J) &:= {}_0W^1_q(J; W^1_q(\Omega^k\backslash\Sigma^k)^3) \cap L^q(J; W^3_q(\Omega^k\backslash\Sigma^k)^3) \\ &\hookrightarrow {}_0W^1_q(J; W^1_q(\Omega^k\backslash\Sigma^k)^3) \cap {}_0W^{1/2}_q(J; W^2_q(\Omega^k\backslash\Sigma^k)^3), \end{aligned}$$

and the estimate

$$\|\nabla\phi_k\|_{{}_0\mathbb{E}_\phi(J)} \leq C_N \|\tilde{u}\|_{{}_0\mathbb{E}_u(J)},$$

where C_N depends on N, Ω and Σ but does not depend on J. Moreover, trace theory (see e.g. [PS16, Proposition 6.2.4]) yields that

$$\begin{aligned} [\![\nabla\phi_k]\!] &\in {}_0W^{1-1/2q}_q(J; W^1_q(\Sigma^k)^3) \cap {}_0W^{1/2}_q(J; W^{2-1/q}_q(\Sigma^k)^3), \\ \nabla\phi_k|_{G^k} &\in {}_0W^{1-1/2q}_q(J; W^1_q(G^k\backslash\partial\Sigma^k)^3) \cap {}_0W^{1/2}_q(J; W^{2-1/q}_q(G^k\backslash\partial\Sigma^k)^3), \\ \nabla\phi_k|_{S^k} &\in {}_0W^{1-1/2q}_q(J; W^1_q(S^k)^3) \cap {}_0W^{1/2}_q(J; W^{2-1/q}_q(S^k)^3), \end{aligned}$$

with vanishing normal part on Σ^k and S^k and

$$\begin{aligned} [\![\mu\nabla^2\phi_k]\!] &\in {}_0W^{1-1/2q}_q(J; L^q(\Sigma^k)^{3\times3}) \cap {}_0W^{1/2}_q(J; W^{1-1/q}_q(\Sigma^k)^{3\times3}), \\ \mu\nabla^2\phi_k\big|_{S^k} &\in {}_0W^{1-1/2q}_q(J; L^q(S^k)^{3\times3}) \cap {}_0W^{1/2}_q(J; W^{1-1/q}_q(S^k)^{3\times3}), \end{aligned}$$

whenever they exist. Define

$$(\hat{u}_k, \hat{\pi}_k) := (\tilde{u}_k - \nabla\phi_k, \tilde{\pi}_k + \rho\partial_t\phi_k - \mu\Delta\phi_k).$$

Then we arrive at the system

$$\begin{aligned} \rho\partial_t\hat{u}_k - \mu\Delta\hat{u}_k + \nabla\hat{\pi}_k &= F_k(\tilde{u}, \tilde{\pi}), && \text{in } \Omega^k\backslash\Sigma^k \times J, \\ \operatorname{div}\hat{u}_k &= 0, && \text{in } \Omega^k\backslash\Sigma^k \times J, \\ [\![\hat{u}_k]\!] &= \tilde{g}_{1k} - [\![\nabla\phi_k]\!], && \text{on } \Sigma^k \times J, \\ [\![-\hat{\pi}_k\mathbb{I} + \mu(\nabla\hat{u}_k + \nabla\hat{u}_k^\top)]\!]\nu_{\Sigma^k} &= \tilde{g}_{2k} + \widehat{G}_{2k}(\tilde{u}), && \text{on } \Sigma^k \times J, \\ ((\hat{u}_k)_1, (\hat{u}_k)_3)^\top &= \tilde{g}_{3k} - (\partial_1\phi_k, \partial_3\phi_k)^\top, && \text{on } G^k\backslash\partial\Sigma^k \times J, \\ -\hat{\pi}_k + 2\mu\partial_2(\hat{u}_k)_2 &= \tilde{g}_{4k} + \widehat{G}_{4k}(\tilde{u}), && \text{on } G^k\backslash\partial\Sigma^k \times J, \\ \hat{u}_k &= \tilde{g}_{5k} - \nabla\phi_k, && \text{on } S^k \times J, \\ \hat{u}_k(0) &= 0, && \text{in } \Omega^k\backslash\Sigma^k, \end{aligned} \tag{4.57}$$

where

$$\widehat{G}_{2k}(\tilde{u}) := G_{2k}(\tilde{u}) - [\![2\mu\nabla^2\phi_k]\!]\,\nu_{\Sigma^k} + [\![\mu\Delta\phi_k]\!]\,\nu_{\Sigma^k},$$
$$\widehat{G}_{4k}(\tilde{u}) := G_{4k}(\tilde{u}) - 2\mu\partial_2^2\phi_k + \mu\Delta\phi_k.$$

By the solution operators $\mathcal{S}_k$, one may rewrite (4.57) as

$$(\hat{u}_k, \hat{\pi}_k) = \mathcal{S}_k(\mathcal{D}_k + \mathcal{R}_k(\tilde{u}, \tilde{\pi})), \tag{4.58}$$

where

$$\mathcal{D}_k := (0, 0, \tilde{g}_{1k}, \tilde{g}_{2k}, \tilde{g}_{3k}, \tilde{g}_{4k}, \tilde{g}_{5k})^\top$$

denotes the given data and

$$\mathcal{R}_k(\tilde{u}, \tilde{\pi}) := (F_k(\tilde{u}, \tilde{\pi}), 0, -[\![\nabla\phi_k]\!], \widehat{G}_{2k}(\tilde{u}), -(\partial_1\phi_k, \partial_3\phi_k), \widehat{G}_{4k}(\tilde{u}), -\nabla\phi_k)^\top$$

represents the remaining part on the right-hand side of (4.57).

Choose cutoff functions $\{\eta_k\}_{k=0}^N \subset C_c^\infty(U_k)$ such that $\eta_k|_{\operatorname{supp}\varphi_k} = 0$. Then $\tilde{u} = \sum_{k=0}^N \tilde{u}_k\eta_k$, $\tilde{\pi} = \sum_{k=0}^N \tilde{\pi}_k\eta_k$. Multiplying (4.58) by η_k, replacing $(\hat{u}_k, \hat{\pi}_k)\eta_k$ by $(\tilde{u}_k - \nabla\phi_k, \tilde{\pi}_k + \rho\partial_t\phi_k - \mu\Delta\phi_k)\eta_k$ and rearranging the equation, one obtains

$$(\tilde{u}_k, \tilde{\pi}_k)\eta_k = \mathcal{S}_k(\mathcal{D}_k\eta_k + \mathcal{R}_k(\tilde{u}, \tilde{\pi})\eta_k) + (\nabla\phi_k, -\rho\partial_t\phi_k + \mu\Delta\phi_k)\eta_k.$$

Now recalling the function spaces in Section 4.2.2, we claim that there exist a $\delta > 0$ and a constant C independent of $T > 0$, such that for $\alpha > 1/2q$,

$$\|\mathcal{R}_k(\tilde{u}, \tilde{\pi})\eta_k\|_{{}_0\mathbb{F}(J)} \le CT^\delta \|(\tilde{u}, \tilde{\pi})\|_{{}_0\mathbb{E}(J)}, \tag{4.59}$$

$$\|\nabla\phi_k\eta_k\|_{{}_0\mathbb{E}_u(J)} \le CT^\delta \left(\|(\tilde{u}, \tilde{\pi})\|_{{}_0\mathbb{E}(J)} + \|\mathcal{D}_k\|_{{}_0\mathbb{F}(J)}\right), \tag{4.60}$$

$$\|(-\rho\partial_t\phi_k + \mu\Delta\phi_k)\eta_k\|_{{}_0\mathbb{E}_\pi(J)} \le CT^\delta \left(\|(\tilde{u}, \tilde{\pi})\|_{{}_0\mathbb{E}(J)} + \|\mathcal{D}_k\|_{{}_0\mathbb{F}(J)}\right). \tag{4.61}$$

It is easy to verify (4.59), since we have given associated additional time regularity for the remaining terms. We present the estimate for $F_k(\tilde{u}, \tilde{\pi})$ as an example. For $\alpha > 1/2q$, we have

$$\|\eta_k F_k(\tilde{u}, \tilde{\pi})\|_{\mathbb{F}_1(J)} \le T^{\frac{1}{2q}} \|F_k(\tilde{u}, \tilde{\pi})\|_{L^{2q}(J;L^q(\Omega^k)^3)} \le CT^{\frac{1}{2q}} \|F_k(\tilde{u}, \tilde{\pi})\|_{{}_0W_q^\alpha(J;L^q(\Omega^k)^3)}.$$

By Proposition 4.14, we get

$$\begin{aligned}\|(-\rho\partial_t\phi_k + \mu\Delta\phi_k)\eta_k\|_{{}_0W_q^\alpha(J;L^q(\Omega^k))} &\le \|\tilde{\pi}\eta_k - \hat{\pi}_k\|_{{}_0W_q^\alpha(J;L^q(\Omega^k))} \\ &\le C\left(\|(\tilde{u}, \tilde{\pi})\|_{{}_0\mathbb{E}(J)} + \|\mathcal{D}_k\|_{{}_0\mathbb{F}(J)} + \|\mathcal{R}_k(\tilde{u}, \tilde{\pi})\eta_k\|_{{}_0\mathbb{F}(J)}\right).\end{aligned}$$

Since $\nabla\phi_k \in {}_0W_q^1(J; W_q^1(\Omega^k\backslash\Sigma^k)^3)$, one has

$$\partial_t\phi_k \in {}_0W_q^\alpha(J; L^q(\Omega^k)) \cap L^q(J; W_q^2(\Omega^k\backslash\Sigma^k)) \hookrightarrow {}_0W_q^{\alpha-\epsilon}(J; W_q^1(\Omega^k\backslash\Sigma^k)),$$

for $\epsilon > 0$ small. Then

$$\begin{aligned}&\|\nabla\phi_k\eta_k\|_{{}_0\mathbb{E}_u(J)} \le C\|\nabla\phi_k\|_{L^q(J;W_q^2(\Omega^k\backslash\Sigma^k)^3)} + \|\phi_k\|_{{}_0W_q^1(J;W_q^1(\Omega^k\backslash\Sigma^k)^3)} \\ &\le CT^{\frac{1}{2q}}\|\nabla\phi_k\|_{{}_0W_q^{\frac{1}{2}}(J;W_q^2(\Omega^k\backslash\Sigma^k)^3)} + \|\phi_k\|_{L^q(J;W_q^1(\Omega^k\backslash\Sigma^k))} + \|\partial_t\phi_k\|_{L^q(J;W_q^1(\Omega^k\backslash\Sigma^k))} \\ &\quad\le CT^{\frac{1}{2q}}\|\nabla\phi_k\|_{{}_0W_q^{\frac{1}{2}}(J;W_q^2(\Omega^k\backslash\Sigma^k)^3)} + CT^{\frac{1}{2q}}\|\partial_t\phi_k\|_{{}_0W_q^{\alpha-\epsilon}(J;W_q^1(\Omega^k\backslash\Sigma^k))} \\ &\quad\le CT^{\frac{1}{2q}}\left(\|(\tilde{u}, \tilde{\pi})\|_{{}_0\mathbb{E}(J)} + \|\mathcal{D}_k\|_{{}_0\mathbb{F}(J)}\right),\end{aligned}$$

and

$$\begin{aligned}
&\|(-\rho\partial_t\phi_k+\mu\Delta\phi_k)\eta_k\|_{{}_0\mathbb{E}_\pi(J)}\\
&\quad\le CT^{\frac{1}{2q}}\|\partial_t\phi_k\|_{{}_0W_q^{\alpha-\epsilon}(J;\dot W_q^1(\Omega^k\setminus\Sigma^k))}+CT^{\frac{1}{2q}}\|\nabla\phi_k\|_{{}_0W_q^{\frac12}(J;W_q^2(\Omega^k\setminus\Sigma^k)^3)}\\
&\qquad+CT^{\frac{1}{2q}}\|[\![\Delta\phi_k]\!]\|_{{}_0W_q^{\frac12}(J;W_q^{1-\frac1q}(\Sigma^k))}+CT^{\frac12}\|[\![\Delta\phi_k]\!]\|_{{}_0W_q^{1-\frac{1}{2q}}(J;L^q(\Sigma^k))}\\
&\qquad+CT^{\frac{1}{2q}}\|\Delta\phi_k|_{S^k}\|_{{}_0W_q^{\frac12}(J;W_q^{1-\frac1q}(\Sigma^k))}+CT^{\frac12}\|\Delta\phi_k|_{S^k}\|_{{}_0W_q^{1-\frac{1}{2q}}(J;L^q(\Sigma^k))}\\
&\quad\le CT^{\frac{1}{2q}}\left(\|(\tilde u,\tilde\pi)\|_{{}_0\mathbb{E}(J)}+\|\mathcal{D}_k\|_{{}_0\mathbb{F}(J)}\right),
\end{aligned}$$

where we used

$$\begin{aligned}
\|\partial_t\phi_k\|_{L^q(J;\dot W_q^1(\Omega^k\setminus\Sigma^k))}&\le T^{\frac{1}{2q}}\|\partial_t\phi_k\|_{L^{2q}(J;\dot W_q^1(\Omega^k\setminus\Sigma^k))}\\
&\le CT^{\frac{1}{2q}}\|\partial_t\phi_k\|_{W_q^{\frac{\alpha}{2}-\frac{\epsilon}{2}}(J;\dot W_q^1(\Omega^k\setminus\Sigma^k))}\le CT^{\frac{1}{2q}}\|\partial_t\phi_k\|_{W_q^{\alpha-\epsilon}(J;\dot W_q^1(\Omega^k\setminus\Sigma^k))},
\end{aligned}$$

for some $1/2q<\alpha<1/2-1/2q$ and ϵ small enough. Then we get (4.60) and (4.61). Consequently, there is a constant $\delta>0$, such that

$$\|(\tilde u_k,\tilde\pi_k)\eta_k\|_{{}_0\mathbb{E}(J)}\le C\left(\|\mathcal{D}_k\|_{{}_0\mathbb{F}(J)}+T^\delta\|(\tilde u,\tilde\pi)\|_{{}_0\mathbb{E}(J)}\right),$$

where $C>0$ does not depend on $T>0$. Taking the sum over all charts $k=0,...,N$, one obtains

$$\|(\tilde u,\tilde\pi)\|_{{}_0\mathbb{E}(J)}\le C\left(\|\mathcal{D}\|_{{}_0\mathbb{F}(J)}+T^\delta\|(\tilde u,\tilde\pi)\|_{{}_0\mathbb{E}(J)}\right),$$

where $\mathcal{D}$ denotes the data in (4.54). Then choosing $T>0$ sufficiently small yields the *a priori* estimate

$$\|(\tilde u,\tilde\pi)\|_{{}_0\mathbb{E}(J)}\le C\|\mathcal{D}\|_{{}_0\mathbb{F}(J)}.$$

Hence we may conclude that the operator $\mathcal{L}:{}_0\mathbb{E}\to{}_0\mathbb{F}$ defined by the left-hand side of (4.54) is injective and has closed range and there is a left inverse $\mathcal{S}$ for $\mathcal{L}$ such that $\mathcal{S}\mathcal{L}z=z$ for all $z\in{}_0\mathbb{E}(J)$.

(2). Existence of a right inverse. Now we are in the position to prove the existence of a right inverse. Given data

$$F:=(f_u,f_d,g_1,g_2,g_3,g_4,g_5)^\top\in\mathbb{F}(J),\quad u_0\in X_{\gamma,u},$$

satisfying the compatibility conditions (4.11) and (4.12). As stated in Section 4.4.2, without loss of generality one may assume that $u_0=0$, which means that the time traces of all the data at $t=0$ vanish whenever they exist.

Let $\bar u,\nabla\phi$ be the unique solution of (4.52) and (4.53), respectively. Define

$$(\tilde u,\tilde\pi):=(\bar u-\nabla\phi,-\partial_t\phi+\mu\Delta\phi),\quad \widetilde{\mathcal{S}}F:=(\tilde u,\tilde\pi).$$

Then it follows that

$$\mathcal{L}\widetilde{\mathcal{S}}F=(f_u,f_d,g_1+G_1(\phi),g_2+G_2(\phi),g_3+G_3(\phi),g_4+G_4(\phi),g_5+G_5(\phi),0)^\top,$$

where

$$\begin{gathered}
G_1(\phi):=-[\![\nabla\phi]\!],\quad G_2(\phi):=-[\![2\mu\nabla^2\phi]\!]\nu_\Sigma+[\![\mu\Delta\phi]\!]\nu_\Sigma,\\
G_3(\phi):=(-\partial_1\phi,-\partial_3\phi)^\top,\quad G_4(\phi):=-2\mu\partial_2^2\phi+\mu\Delta\phi,\quad G_5(\phi):=-\nabla\phi.
\end{gathered}$$

In the following, by localization we consider the problem

$$
\begin{aligned}
\rho\partial_t\tilde{u}_k-\mu\Delta\tilde{u}_k+\nabla\tilde{\pi}_k&=0, && \text{in } \Omega^k\backslash\Sigma^k\times J,\\
\operatorname{div}\tilde{u}_k&=0, && \text{in } \Omega^k\backslash\Sigma^k\times J,\\
[\![\tilde{u}_k]\!]&=G_{1k}(\phi), && \text{on } \Sigma^k\times J,\\
[\![-\tilde{\pi}_k\mathbb{I}+\mu(\nabla\tilde{u}_k+\nabla\tilde{u}_k^\top)]\!]\,\nu_{\Sigma^k}&=G_{2k}(\phi), && \text{on } \Sigma^k\times J,\\
((\tilde{u}_k)_1,(\tilde{u}_k)_3)^\top&=G_{3k}(\phi), && \text{on } G^k\backslash\partial\Sigma^k\times J,\\
-\tilde{\pi}_k+2\mu\partial_2(\tilde{u}_k)_2&=G_{4k}(\phi), && \text{on } G^k\backslash\partial\Sigma^k\times J,\\
\tilde{u}_k&=G_{5k}(\phi), && \text{on } S^k\times J,\\
\tilde{u}_k(0)&=0, && \text{in } \Omega^k\backslash\Sigma^k,
\end{aligned}
\tag{4.62}
$$

where $G_{jk}(\phi):=G_j(\phi)\varphi_k$, $j\in\{1,...,5\}$. Now, we are going to check if $G_{jk}(\phi)$ satisfy all the relevant compatibility condition at $\partial\Sigma^k$ and ∂S^k, whenever they exist. For $k=0,...,4$, one does not need compatibility condition for G_{jk}, $j=1,2,5$. Since ϕ has vanishing trace at $t=0$, one obtains $(G_{3k},G_{4k})|_{t=0}=0$. For $k=5,...,N_1$, i.e., the bent quarter-space Stokes problem, it is obviously that $G_{3k}=((G_{5k})_1,(G_{5k})_3)^\top$, which fulfills the compatibility condition on the contact line ∂S^k. For $k=N_1,...,N$, namely, the half-space two-phase Stokes problem with a bent interface, we have $[\![G_{3k}]\!]=((G_{1k})_1,(G_{1k})_3)^\top$ and

$$
\begin{aligned}
(G_{2k})_1=&\,[\![2\mu\partial_1(G_{3k})_1\nu_{\Sigma^k}\cdot e_1+\mu(\partial_1(G_{3k})_2+\partial_3(G_{3k})_1)\nu_{\Sigma^k}\cdot e_3]\!]\\
&+[\![G_{4k}]\!]\,\nu_{\Sigma^k}\cdot e_1+[\![2\mu(\partial_1(G_{3k})_1+\partial_3(G_{3k})_2)]\!]\,\nu_{\Sigma^k}\cdot e_1,\\
(G_{2k})_3=&\,[\![2\mu\partial_3(G_{3k})_2\nu_{\Sigma^k}\cdot e_3+\mu(\partial_1(G_{3k})_2+\partial_3(G_{3k})_1)\nu_{\Sigma^k}\cdot e_1]\!]\\
&+[\![G_{4k}]\!]\,\nu_{\Sigma^k}\cdot e_3+[\![2\mu(\partial_1(G_{3k})_1+\partial_3(G_{3k})_2)]\!]\,\nu_{\Sigma^k}\cdot e_3,
\end{aligned}
$$

at the contact line $\partial\Sigma^k$, which verifies the compatibility condition (4.40).

Therefore, according to the model problems we established in Section 4.3, for each $k\in\{0,1,...,N\}$, there exists a unique solution $(\tilde{u}_k,\tilde{\pi}_k)$ of (4.62) in right regularity class. Choose cutoff functions $\{\eta_k\}_{k=0}^N\subset C_c^\infty(U_k)$ such that $\eta_k|_{\operatorname{supp}\varphi_k}=0$. Solving the elliptic transmission problem

$$
\begin{aligned}
\Delta\phi_k&=(\tilde{u}_k\cdot\nabla\eta_k)|_\Omega\,, && \text{in } \Omega\backslash\Sigma,\\
[\![\rho\phi_k]\!]&=0, && \text{on } \Sigma,\\
[\![\partial_{\nu_\Sigma}\phi_k]\!]&=0, && \text{on } \Sigma,\\
\rho\phi_k&=0, && \text{on } G\backslash\partial\Sigma,\\
\partial_{\nu_S}\phi_k&=0, && \text{on } S,
\end{aligned}
\tag{4.63}
$$

yields a unique solution ϕ_k with regularity

$$
\nabla\phi_k\in{}_0W_q^1(J;W_q^1(\Omega\backslash\Sigma)^3)\cap L^q(J;W_q^3(\Omega\backslash\Sigma)^3).
$$

Finally, define

$$
\overline{\mathcal{S}}F:=\sum_{k=0}^N(\tilde{u}_k\eta_k-\nabla\phi_k,\tilde{\pi}_k\eta_k-\rho\partial_t\phi_k+\mu\Delta\phi_k).
$$

Then it holds that

$$\mathcal{L}\overline{\mathcal{S}}F = \sum_{k=0}^{N} \begin{pmatrix} -\mu[\Delta, \eta_k]\tilde{u}_k + [\nabla, \eta_k]\tilde{u}_k \\ 0 \\ \eta_k G_{1k}(\phi) + G_{1k}(\phi_k) \\ \eta_k G_{2k}(\phi) + \widehat{G}_{2k}(\phi_k) \\ \eta_k G_{3k}(\phi) + G_{3k}(\phi_k) \\ \eta_k G_{4k}(\phi) + \widehat{G}_{4k}(\phi_k) \\ \eta_k G_{5k}(\phi) + G_{5k}(\phi_k) \\ 0 \end{pmatrix},$$

where

$$\begin{aligned} \widehat{G}_{2k}(\phi_k) &:= G_{2k}(\phi_k) - [\![\mu(\nabla\eta_k \otimes \tilde{u}_k + \tilde{u}_k \otimes \nabla\eta_k)]\!]\, \nu_{\Sigma^k}, \\ \widehat{G}_{4k}(\phi_k) &:= G_{4k}(\phi_k) - 2\mu\partial_2\phi_k(\tilde{u}_k)_2. \end{aligned}$$

From the construction of η_k, we know that

$$G_j(\phi) = \sum_{k=0}^{N} \eta_k G_{jk}(\phi).$$

Let $\widehat{\mathcal{S}}F := \widetilde{\mathcal{S}}F - \overline{\mathcal{S}}F$, it follows that

$$\mathcal{L}\widehat{\mathcal{S}}F = \mathcal{L}\widetilde{\mathcal{S}}F - \mathcal{L}\overline{\mathcal{S}}F = F - \mathcal{R}F,$$

where

$$\mathcal{R}F := \sum_{k=0}^{N} \Big(-\mu[\Delta, \eta_k]\tilde{u}_k + [\nabla, \eta_k]\tilde{u}_k, 0, G_{1k}, \widehat{G}_{2k}, G_{3k}, \widehat{G}_{4k}, G_{5k}, 0 \Big)^{\top}.$$

As in the first part of the proof, since we have additional time-regularity for $\tilde{u}_k, \tilde{\pi}_k$ and ϕ, we could conclude that there exists constant $\delta > 0$ such that

$$\|\mathcal{R}F\|_{_0\mathbb{F}(J)} \le CT^{\delta} \|F\|_{_0\mathbb{F}(J)},$$

where $C > 0$ does not depend $T > 0$. Taking $T > 0$ small enough, for example, $CT^{\delta} < 1/2$, the operator $(\mathbb{I} - \mathcal{R})$ is invertible. Substitute F above by $(\mathbb{I} - \mathcal{R})^{-1}F$, one obtains

$$\mathcal{L}\widehat{\mathcal{S}}(\mathbb{I} - \mathcal{R})^{-1}F = F,$$

which defines the right inverse for $\mathcal{L}$ as $\mathcal{S} := \widehat{\mathcal{S}}(\mathbb{I} - \mathcal{R})^{-1}$.

This completes the proof. □

Given $c_s \in L^q(J; W_q^2(\Omega_s)) \cap W_q^1(J; L^q(\Omega_s))$, we obtain the well-posedness of

$$\begin{aligned} \rho \partial_t u - \operatorname{div} S_\mu(u, \pi) &= f_u, && \text{in } \Omega \backslash \Sigma \times J, \\ \operatorname{div} u - \frac{\gamma\beta}{\rho_s} c_s &= f_d, && \text{in } \Omega \backslash \Sigma \times J, \\ [\![u]\!] &= g_1, && \text{on } \Sigma \times J, \\ [\![(-\pi \mathbb{I} + \mu(\nabla u + \nabla u^\top))\nu_\Sigma]\!] &= g_2, && \text{on } \Sigma \times J, \\ (u_1, u_3)^\top &= g_3, && \text{on } G \backslash \partial\Sigma \times J, \\ -\pi + 2\mu \partial_2 u_2 &= g_4, && \text{on } G \backslash \partial\Sigma \times J, \\ u &= g_5, && \text{on } S \times J, \\ u(0) &= u_0, && \text{in } \Omega \backslash \Sigma, \end{aligned} \tag{4.64}$$

with the aid of Theorem 4.7.

Corollary 4.16. *Let $\gamma, \beta > 0$. Given $c_s \in L^q(J; W_q^2(\Omega_s)) \cap W_q^1(J; L^q(\Omega_s))$. Then under the assumptions of Theorem 4.7, (4.64) admits a unique solution*

$$(u, \pi) \in \mathbb{E}(J),$$

if and only if the data are subject to the regularity and compatibility conditions in Theorem 4.7.

Proof. Since $c_s \in \mathbb{E}_{c,s}(J) = L^q(J; W_q^2(\Omega_s)) \cap W_q^1(J; L^q(\Omega_s))$, it follows from the embeddings $W_q^2(\Omega_s) \hookrightarrow W_q^1(\Omega_s), L^q(\Omega_s) \hookrightarrow W_q^{-1}(\Omega_s)$ that $c_s \in \mathbb{F}_2(J)$. For $c_{0,s} \in W_q^{2-2/q}(\Omega_s)$, trace method of real interpolation implies that there exists a function $\tilde{c}_s \in \mathbb{E}_{c,s}(J)$ such that $\tilde{c}_s|_{t=0} = c_{0,s}$. Then one may decompose c_s as $c_s := \tilde{c}_s + \bar{c}_s$, where $\bar{c}_s \in {}_0\mathbb{E}_{c,s}(J)$. Extend $\bar{c}_s$ suitably from ${}_0\mathbb{E}_{c,s}(J)$ to $\bar{c} \in L^q(J; W_q^2(\Omega \backslash \Sigma)) \cap {}_0W_q^1(J; L^q(\Omega))$. By compatibility conditions (4.11) and (4.12), we define a function $\hat{c}$ such that

$$\hat{c} = \begin{cases} \bar{c}|_{\Omega_s} = \bar{c}_s, & \text{in } \Omega_s, \\ \dfrac{\mu_s}{\mu_f}\, \bar{c}|_{\Omega_f}, & \text{in } \Omega_f. \end{cases}$$

Then $\hat{c} \in {}_0\mathbb{F}_2(J)$ and $[\![\mu \hat{c}]\!] = 0$ on Σ. Consequently, the problem

$$\begin{aligned} \rho \partial_t \bar{u} - \operatorname{div} S_\mu(\bar{u}, \bar{\pi}) &= 0, && \text{in } \Omega \backslash \Sigma \times J, \\ \operatorname{div} \bar{u} &= \frac{\gamma\beta}{\rho_s} \hat{c}, && \text{in } \Omega \backslash \Sigma \times J, \\ [\![\bar{u}]\!] &= 0, && \text{on } \Sigma \times J, \\ [\![-\bar{\pi} \mathbb{I} + \mu(\nabla \bar{u} + \nabla \bar{u}^\top)]\!]\, \nu_\Sigma &= 0, && \text{on } \Sigma \times J, \\ (\bar{u}_1, \bar{u}_3)^\top &= 0, && \text{on } G \backslash \partial\Sigma \times J, \\ -\bar{\pi} + 2\mu \partial_2 \bar{u}_2 &= 0, && \text{on } G \backslash \partial\Sigma \times J, \\ \bar{u} &= 0, && \text{on } S \times J, \\ \bar{u}(0) &= 0, && \text{in } \Omega \backslash \Sigma. \end{aligned}$$

admits a unique solution $(\bar{u}, \bar{\pi}) \in {}_0\mathbb{E}(J)$ thanks to Theorem 4.7. Let $(\tilde{u}, \tilde{\pi})$ be the unique solution of (4.10) with f_d substituted by $f_d + \frac{\gamma\beta}{\rho_s} \tilde{c}_s$, then $(\bar{u} + \tilde{u}, \bar{\pi} + \tilde{\pi})$ solves (4.64). □

By standard localization procedure, one can prove the local well-posedness of the heat equation in a cylindrical domain and in a cylindrical ring respectively. We omit the proof of Theorem 4.8 and 4.9 here and refer to Wilke [Wil20, Section 5.3.2] for similar arguments with model problems constructed in Section 4.3.6 and Prüss–Simonett [PS16, Section 6.2].

4.5. Nonlinear Well-posedness

In this section, we aim to prove the local well-posedness of (4.8), namely, to prove Theorem 4.3. To this end, we firstly introduce the function spaces for the data

$$
\begin{gathered}
\mathbb{F}_1(J) := L^q(J; L^q(\Omega)^3), \quad \mathbb{F}_2(J) := W^1_q(J; \dot{W}^{-1}_q(\Omega)) \cap L^q(J; W^1_q(\Omega \backslash \Sigma)), \\
\mathbb{F}_3(J) := W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\Sigma)^3) \cap L^q(J; W_q^{1-\frac{1}{q}}(\Sigma)^3), \\
\mathbb{F}_4(J) := W_q^{1-\frac{1}{2q}}(J; L^q(G)^3) \cap L^q(J; W_q^{2-\frac{1}{q}}(G \backslash \partial\Sigma)^3), \\
\mathbb{F}_5(J) := W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(G)) \cap L^q(J; W_q^{1-\frac{1}{q}}(G \backslash \partial\Sigma)), \\
\mathbb{F}_6(J) := L^q(J; L^q(\Omega_f)), \quad \mathbb{F}_7(J) := W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\Sigma)) \cap L^q(J; W_q^{1-\frac{1}{q}}(\Sigma)), \\
\mathbb{F}_8(J) := W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(G)) \cap L^q(J; W_q^{1-\frac{1}{q}}(G \backslash \partial\Sigma)), \\
\mathbb{F}_9(J) := W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(S)) \cap L^q(J; W_q^{1-\frac{1}{q}}(S)), \\
\mathbb{F}_{10}(J) := L^q(J; W^1_q(\Omega_s)), \quad \mathbb{F}_{11}(J) := L^q(J; W^1_q(\Omega_s)), \quad \mathbb{F}(J) := \Pi_{j=1}^{11} \mathbb{F}_j(J).
\end{gathered}
$$

Let $\mathbf{w} = (\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s^*, \hat{g})$ and $\mathbf{w}_0 = (\hat{\mathbf{v}}^0, \hat{c}^0, 0, 1)$. Recalling the definition of solution and initial spaces in Section 4.2.1, we reformulate (4.8) in the abstract form

$$
\mathscr{L}(\mathbf{w}) = \mathscr{N}(\mathbf{w}, \mathbf{w}_0) \quad \text{for all } \mathbf{w} \in \mathbb{E}(J),\ (\hat{\mathbf{v}}^0, \hat{c}^0) \in X_\gamma, \tag{4.65}
$$

where

$$
\mathscr{L}(\mathbf{w}) := \begin{pmatrix}
\hat{\rho}\partial_t \hat{\mathbf{v}} - \widehat{\operatorname{div}}\, \mathbf{S}(\hat{\mathbf{v}}, \hat{\pi}) \\
\widehat{\operatorname{div}}\, (\hat{\mathbf{v}}) - \dfrac{\gamma\beta}{\hat{\rho}_s} \hat{c}_s \\
[\![\mathbf{S}(\hat{\mathbf{v}}, \hat{\pi})]\!]\, \hat{\mathbf{n}}_\Sigma \\
\mathcal{P}_G(\hat{\mathbf{v}})|_G \\
\mathbf{S}(\hat{\mathbf{v}}, \hat{\pi})\hat{\mathbf{n}}_G \cdot \hat{\mathbf{n}}_G|_G \\
\partial_t \hat{c} - \hat{D}\hat{\Delta}\hat{c} \\
\hat{D}\hat{\nabla}\hat{c} \cdot \hat{\mathbf{n}}_\Sigma \\
\hat{D}\hat{\nabla}\hat{c} \cdot \hat{\mathbf{n}}_G \\
\hat{D}_s \hat{\nabla}\hat{c}_s \cdot \hat{\mathbf{n}}_S \\
\partial_t \hat{c}_s^* - \beta \hat{c}_s \\
\partial_t \hat{g} - \dfrac{\gamma\beta}{n\hat{\rho}_s} \hat{c}_s \\
(\hat{\mathbf{v}}, \hat{c}, \hat{c}_s^*, \hat{g})|_{t=0}
\end{pmatrix}, \quad
\mathscr{N}(\mathbf{w}, \mathbf{w}_0) := \begin{pmatrix}
\mathbf{K}(\mathbf{w}) \\
G(\mathbf{w}) \\
\mathbf{H}^1(\mathbf{w}) \\
\mathbf{H}^2(\mathbf{w}) \\
H^3(\mathbf{w}) \\
F^1(\mathbf{w}) \\
F^2(\mathbf{w}) \\
F^3(\mathbf{w}) \\
F^4(\mathbf{w}) \\
F^5(\mathbf{w}) \\
F^6(\mathbf{w}) \\
\mathbf{w}_0
\end{pmatrix}.
$$

Define

$$
\mathscr{M}(\mathbf{w}) := \left(\mathbf{K}, G, \mathbf{H}^1, \mathbf{H}^2, H^3, F^1, F^2, F^3, F^4, F^5, F^6\right)^\top (\mathbf{w}).
$$

Then it follows from Corollary 4.16 and Theorem 4.8, 4.9 that $\mathscr{L} : \mathbb{E}(J) \to \mathbb{F}(J) \times X_\gamma$ is an *isomorphism* for $J := [0, T]$, $T > 0$. Moreover, the following proposition holds for $\mathscr{M} : \mathbb{E}(J) \to \mathbb{F}(J)$.

Proposition 4.17. *Let $q > 3$, $J = [0, T]$ and $R > 0$. Assume $w = (\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s^*, \hat{g}) \in \mathbb{E}(J)$ with $\hat{g}|_{t=0} = 1$ and $\|\mathbf{w}\|_{\mathbb{E}(J)} \leq R$, then there exist a constant $C = C(R) > 0$, a finite time $T_R > 0$ depending on R and $\delta > 0$ such that for $0 < T < T_R$, $\mathscr{M} : \mathbb{E}(J) \to \mathbb{F}(J)$ is well-defined and bounded along with the estimates:*

$$\|\mathscr{M}(\mathbf{w})\|_{\mathbb{F}(J)} \leq C(R)T^\delta \left(\|\mathbf{w}\|_{\mathbb{E}(J)} + 1\right) \qquad \text{for all } \mathbf{w} \in \mathbb{E}(J). \tag{4.66}$$

Moreover, there exist a constant $C = C(R) > 0$, a finite time $T_R > 0$ depending on R and $\delta > 0$ such that for $0 < T < T_R$,

$$\left\|\mathscr{M}(\mathbf{w}^1) - \mathscr{M}(\mathbf{w}^2)\right\|_{\mathbb{F}(J)} \leq C(R)T^\delta \left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{\mathbb{E}(J)}, \tag{4.67}$$

for $\mathbf{w}^1 = (\hat{\mathbf{v}}^1, \hat{\pi}^1, \hat{c}^1, \hat{c}_s^{*1}, \hat{g}^1), \mathbf{w}^2 = (\hat{\mathbf{v}}^2, \hat{\pi}^2, \hat{c}^2, \hat{c}_s^{*2}, \hat{g}^2) \in Y_T$, $\hat{c}^i\big|_{t=0} = \hat{c}^0$, $\hat{c}_s^*|_{t=0} = 0$, $\hat{g}^i\big|_{t=0} = 1$ and $\left\|\mathbf{w}^i\right\|_{\mathbb{E}(J)} \leq R$ $(i = 1, 2)$.

Proof. This proposition is same as in Proposition 3.20 and the proof is very similar. The different points we need to pay attention are the estimates of $\mathbf{H}^2$ and H^3, where

$$\begin{aligned}
\mathbf{H}^2 &= -\Big(\mathbb{I} - \big((\hat{\mathbf{F}}^{-\top}\hat{\mathbf{n}}_G) \otimes (\hat{\mathbf{F}}^{-\top}\hat{\mathbf{n}}_G) - \hat{\mathbf{n}}_G \otimes \hat{\mathbf{n}}_G\big)\Big)\,\hat{\mathbf{v}},\\
H^3 &= -\hat{\mathbb{T}}\hat{\mathbf{F}}^{-\top}\hat{\mathbf{n}}_G \cdot (\hat{\mathbf{F}}^{-\top}\hat{\mathbf{n}}_G) + \mathbf{S}(\hat{\mathbf{v}}, \hat{\pi})\hat{\mathbf{n}}_G \cdot \hat{\mathbf{n}}_G.
\end{aligned}$$

Note that for matrices $\mathbf{S}, \mathbf{T} \in \mathbb{R}^{3\times 3}$ and vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, we have the tensor algebra property

$$\begin{aligned}
\mathbf{S}(\mathbf{u} \otimes \mathbf{v}) = (\mathbf{S}\mathbf{u}) \otimes \mathbf{v}, \quad (\mathbf{u} \otimes \mathbf{v})\mathbf{S} = \mathbf{u} \otimes (\mathbf{S}^\top \mathbf{v}),\\
\mathbf{T}\mathbf{u} \cdot (\mathbf{S}^\top \mathbf{v}) = (\mathbf{S}\mathbf{T})\mathbf{u} \cdot \mathbf{v} = \mathbf{S}(\mathbf{T}\mathbf{u}) \cdot \mathbf{v}.
\end{aligned}$$

Then one can derive

$$\begin{aligned}
&\mathbf{H}^2(\mathbf{w}^1) - \mathbf{H}^2(\mathbf{w}^2)\\
&= \Big(\big(\hat{\mathbf{F}}^{-\top}(\hat{\nabla}\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-\top}(\hat{\nabla}\hat{\mathbf{v}}^2)\big)(\hat{\mathbf{n}}_G \otimes \hat{\mathbf{n}}_G)\hat{\mathbf{F}}^{-1}(\hat{\nabla}\hat{\mathbf{v}}^1)\\
&\qquad + \hat{\mathbf{F}}^{-\top}(\hat{\nabla}\hat{\mathbf{v}}^2)(\hat{\mathbf{n}}_G \otimes \hat{\mathbf{n}}_G)\big(\hat{\mathbf{F}}^{-1}(\hat{\nabla}\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-1}(\hat{\nabla}\hat{\mathbf{v}}^2)\big)\Big)\hat{\mathbf{v}}^1\\
&\quad + \Big(\big(\hat{\mathbf{F}}^{-\top}(\hat{\nabla}\hat{\mathbf{v}}^2) - \mathbb{I}\big)(\hat{\mathbf{n}}_G \otimes \hat{\mathbf{n}}_G)\hat{\mathbf{F}}^{-1}(\hat{\nabla}\hat{\mathbf{v}}^1)\\
&\qquad + (\hat{\mathbf{n}}_G \otimes \hat{\mathbf{n}}_G)\big(\hat{\mathbf{F}}^{-1}(\hat{\nabla}\hat{\mathbf{v}}^2) - \mathbb{I}\big)\Big)(\hat{\mathbf{v}}^1 - \hat{\mathbf{v}}^2) - (\hat{\mathbf{v}}^1 - \hat{\mathbf{v}}^2)\mathbb{I}
\end{aligned}$$

and

$$\begin{aligned}
&H^3(\mathbf{w}^1) - H^3(\mathbf{w}^2)\\
&= -(\hat{\mathbf{F}}^{-1}(\hat{\nabla}\hat{\mathbf{v}}^1) - \mathbb{I})\big(\hat{\mathbb{T}}(\hat{\mathbf{w}}^1)\hat{\mathbf{F}}^{-\top}(\hat{\nabla}\hat{\mathbf{v}}^1) - \hat{\mathbb{T}}(\hat{\mathbf{w}}^2)\hat{\mathbf{F}}^{-\top}(\hat{\nabla}\hat{\mathbf{v}}^2)\big)\hat{\mathbf{n}}_G \cdot \hat{\mathbf{n}}_G\\
&\quad - \big(\hat{\mathbf{F}}^{-1}(\hat{\nabla}\hat{\mathbf{v}}^1) - \hat{\mathbf{F}}^{-1}(\hat{\nabla}\hat{\mathbf{v}}^2)\big)(\hat{\mathbb{T}}(\hat{\mathbf{w}}^2)\hat{\mathbf{F}}^{-\top}(\hat{\nabla}\hat{\mathbf{v}}^2))\hat{\mathbf{n}}_G \cdot \hat{\mathbf{n}}_G - \mathbf{K}\hat{\mathbf{n}}_G \cdot \hat{\mathbf{n}}_G.
\end{aligned}$$

With the general trace theorem and Proposition 2.18, one can derive the estimates for $\mathbf{H}^2$ and H^3 following the procedure in Proposition 3.20. □

Now we are in the position to show Theorem 4.3.

Proof of Theorem 4.3. For $(\hat{\mathbf{v}}^0, \hat{c}^0) \in X_\gamma$, by the trace method of real interpolation, see e.g. Lunardi [Lun18, Proposition 1.13], there exists a function $\tilde{\mathbf{w}} = (\tilde{\mathbf{v}}, \tilde{c}) \in \mathbb{E}_{\hat{\mathbf{v}}}(\infty) \times \mathbb{E}_{\hat{c}}(\infty)$ such that $\tilde{\mathbf{w}}|_{t=0} = (\hat{\mathbf{v}}^0, \hat{c}^0)$. Then one can reduce (4.8) to the case of trivial initial data by eliminating $\tilde{\mathbf{w}}$. Now we set a well-defined constant for

$$C_{\mathscr{L}} := \sup_{0<T\leq 1} \left\| \mathscr{L}^{-1} \right\|_{\mathcal{L}({}_0\mathbb{F}(J), {}_0\mathbb{E}(J))},$$

which is finite. Choose $R > 0$ large such that $R \geq 2C_{\mathscr{L}} \left\| (\hat{\mathbf{v}}^0, \hat{c}^0) \right\|_{X_\gamma}$. Then

$$\left\| \mathscr{L}^{-1} \mathscr{N}(\bar{\mathbf{0}}, \mathbf{w}_0) \right\|_{\mathbb{E}(J)} \leq C_{\mathscr{L}} \left\| (\hat{\mathbf{v}}^0, \hat{c}^0) \right\|_{X_\gamma} \leq \frac{R}{2}. \tag{4.68}$$

Here $\mathscr{N}(\bar{\mathbf{0}}, \mathbf{w}_0)$ is in the sense of trivial data $\bar{\mathbf{0}} = (0,0,0,0,1)$. For $\left\| \mathbf{w}^i \right\|_{\mathbb{E}(J)} \leq R$, $i = 1, 2$, we take $T_R > 0$ small enough such that $C_{\mathscr{L}} C(R) T_R^\delta \leq 1/2$, where $C(R)$ is the constant in (4.67). Then for $0 < T < T_R$, we infer from Proposition 4.17 that

$$\begin{aligned} &\left\| \mathscr{L}^{-1} \mathscr{N}(\mathbf{w}^1, \mathbf{w}_0) - \mathscr{L}^{-1} \mathscr{N}(\mathbf{w}^2, \mathbf{w}_0) \right\|_{\mathbb{E}(J)} \\ &\qquad \leq C_{\mathscr{L}} C(R) T^\delta \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{\mathbb{E}(J)} \leq \frac{1}{2} \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{\mathbb{E}(J)}, \end{aligned} \tag{4.69}$$

for all $\mathbf{w}^j \in \mathbb{E}(J)$ with $\|\mathbf{w}^j\|_{\mathbb{E}(J)} \leq R$, which implies the contraction property. From (4.68) and (4.69), we have

$$\begin{aligned} &\left\| \mathscr{L}^{-1} \mathscr{N}(\mathbf{w}, \mathbf{w}_0) \right\|_{\mathbb{E}(J)} \\ &\quad \leq \left\| \mathscr{L}^{-1} \mathscr{N}(0, \mathbf{w}_0) \right\|_{\mathbb{E}(J)} + \left\| \mathscr{L}^{-1} \mathscr{N}(\mathbf{w}, \mathbf{w}_0) - \mathscr{L}^{-1} \mathscr{N}(\bar{\mathbf{0}}, \mathbf{w}_0) \right\|_{\mathbb{E}(J)} \leq R. \end{aligned}$$

Define $\mathcal{M}_{R,T}$ by

$$\mathcal{M}_{R,T} := \left\{ \mathbf{w} \in \overline{B_{{}_0\mathbb{E}(J)}(\bar{\mathbf{0}}, R)} : \mathbf{w} = (\hat{\mathbf{v}}, \hat{\pi}, \hat{c}, \hat{c}_s, \hat{g}) \right\},$$

a closed subset of ${}_0\mathbb{E}(J)$. Hence, $\mathscr{L}^{-1}\mathscr{N}(\cdot, \mathbf{w}_0) : \mathcal{M}_{R,T} \to \mathcal{M}_{R,T}$ is well-defined for all $0 < T < T_R$ and a strict contraction. Since ${}_0\mathbb{E}(J)$ is a Banach space, the Banach fixed-point Theorem implies the existence of a unique fixed-point of $\mathscr{L}^{-1}\mathscr{N}(\cdot, \mathbf{w}_0)$ in $\mathcal{M}_{R,T}$, i.e., (4.8) admits a unique strong solution in $\mathcal{M}_{R,T}$ for small time $0 < T < T_R$.

The uniqueness in ${}_0\mathbb{E}(J)$ follows easily by mimicking the continuity argument in Proof of Theorem 3.8, we omit it here and complete the proof. ☐

4.6. Appendix: Partition of Unity with Vanishing Neumann Trace

Proposition 4.18. *Let $G_0 := G_0^+ \cup G_0^- \cup \Gamma \subset \mathbb{R}^2$ with three disjoint components and $\overline{G_0^-} \subset G_0$, be a bounded domain with the boundary $\partial G_0 \in C^{m+1}$ and an interface $\Gamma = \partial G_0^- \in C^{m+1}$, $m \in \mathbb{N}_+$. $\overline{G_0^-} \subset G_0$. Then for each finite open covering $\{U_k\}_{k=1}^N$ of $\Gamma \cup \partial G_0$ with $\{U_k\}_{k=1}^{N_1} \supset \Gamma$ and $\{U_k\}_{k=N_1+1}^N \supset \partial G_0$, there exist open sets $U_0 \subset G_0^-$ and $U_{N+1} \subset G_0^+$ such that $U_0 \cap \Gamma = \emptyset$, $U_{N+1} \cap (\partial G_0 \cup \Gamma) = \emptyset$, $\bigcup_{k=0}^{N+1} U_k \supset \overline{G_0}$. Moreover, there is a subordinated partition of unity $\{\psi_k\}_{k=0}^{N+1} \subset C_c^m(\mathbb{R}^2)$, such that $\operatorname{supp} \psi_k \subset U_k$ and $\partial_{\nu_{G_0}} \psi_k = 0$ on ∂G_0, $\partial_{\nu_\Gamma} \psi_k = 0$, on Γ, where ν_Γ denotes the unit normal vector pointing from G_0^- to G_0^+, ν_{G_0} is the outer unit normal vector on ∂G_0.*

Proof. The proof is based on [Wil20, Proposition 5.3]. For any finite open covering $\{U_k\}_{k=1}^N$ of $\Gamma \cup \partial G_0$, where $\bigcup_{k=1}^{N_1} U_k \supset \Gamma$ and $\bigcup_{k=N_1+1}^{N} U_k \supset \partial G_0$, there exists $U_{N_1+1} \subset G_0^+$, such that $U_{N_1+1} \cap (\Gamma \cup \partial G_0) = \emptyset$ and $\bigcup_{k=1}^{N+1} U_k \supset \overline{G_0^+}$ by [Wil20, Proposition 5.3], while at the same time, there exists $U_0 \subset G_0^-$, such that $U_0 \cap \Gamma = \emptyset$ and $\bigcup_{k=0}^{N_1} U_k \supset \overline{G_0^-}$.

Moreover, there are two subordinated partitions of unity $\{\phi_k\}_{k=0}^{N_1}$, $\{\varphi_k\}_{k=1}^{N+1} \subset C_c^m(\mathbb{R}^2)$ such that $\operatorname{supp} \phi_k \subset U_k$ and $\operatorname{supp} \varphi_k \subset U_k$, $\partial_{\nu_\Gamma} \phi_k = 0$, $-\partial_{\nu_\Gamma} \varphi_k = 0$, on Γ and $\partial_{\nu_{G_0}} \varphi_k = 0$, on ∂G_0. Now we argue by the truncation. Let $V := \bigcup_{k=0}^{N+1} U_k$ and $V^- := \bigcup_{k=0}^{N_1} U_k$. For $x \in V$, define $d(x)$ as the distance from x to V^-. Let $\eta(x) \in C_0^\infty(\mathbb{R}; [0,1])$ be a cutoff function over V such that $\eta(s) = 1$ if $|s| < \epsilon$ and $\eta(s) = 0$ if $|s| > 2\epsilon$ for $\epsilon > 0$ small. Define $\psi_k := \eta(d(x))\phi_k + (1 - \eta(d(x)))\varphi_k$. Then

$$\sum_{k=0}^{N+1} \psi_k = \sum_{k=0}^{N_1} \eta\phi_k + \sum_{k=1}^{N+1} (1-\eta)\varphi_k = 1.$$

Then $\{\psi_k\}_{k=0}^{N+1} \subset C_c^m(\mathbb{R}^2)$ is a partition of unity such that $\operatorname{supp} \psi_k \subset U_k$. Since $\eta(s)$ is constant in both a neighborhood of and far away from $s = 0$, one obtains $\partial_{\nu_\Gamma} \psi_k = 0$, on Γ and $\partial_{\nu_{G_0}} \psi_k = 0$, on ∂G_0. □

Analogous to [Wil20, Section 5.2], the results can be extended to the general cylindrical domain $\Omega := \Omega^+ \cup \Omega^- \cup \Sigma$ with three disjoint components satisfying $\overline{\Omega^-} \subset \Omega$, $\partial\Omega = G \cup \overline{S}$, $\partial\Omega^- = \Sigma \cup G_0^-$, $\partial\Omega^+ = \Sigma \cup S \cup G_0^+$, where $L_1 < L_2 < \infty$ are two constant, $G := \cup_{j=1,2} G_0 \times \{L_j\}$ and $S, \Sigma \subset \mathbb{R}^3$ are general hypersurfaces which will be assigned with certain regularity. In particular, $\partial\Sigma = \cup_{j=1,2} \Gamma \times \{L_j\}$, $\partial S = \cup_{j=1,2} \partial G_0 \times \{L_j\}$ are sub-manifolds of dimension one in $\mathbb{R}^3$. Moreover, we assume that $S \perp G$ at ∂G, $\Sigma \perp G$ at $\partial\Sigma$ and $\Sigma \cap S = \emptyset$.

Proposition 4.19. *Let Ω be a bounded domain defined above with $G, S, \Sigma \in C^m$ and $\partial\Sigma, \partial S \in C^{m+1}$, $m \in \mathbb{N}_+$. Then for each finite open covering $\{U_k\}_{k=1}^N$ of $\partial S \cup \partial\Sigma$ in $\mathbb{R}^3$, there exist open sets $U_j \subset \mathbb{R}^3$, $j \in \{N+1, ..., N+5\}$, such that*

- $U_{N+1} \cap (\overline{\Omega} \backslash \overline{G}) \neq \emptyset$, $U_{N+1} \cap \overline{G} = \emptyset$,
- $U_{N+1+j} \cap U_{N+1} \cap G_j^+ \neq \emptyset$, $U_{N+1+j} \cap (\Sigma \cup S) = \emptyset$, $j = 1, 2$,
- $U_{N+3+j} \cap U_{N+1} \cap G_j^- \neq \emptyset$, $U_{N+3+j} \cap \Sigma = \emptyset$, $j = 1, 2$,
- $\bigcup_{j=1}^{N+5} U_j \supset \overline{\Omega}$.

Moreover, there is a subordinated partition of unity $\{\psi_k\}_{k=1}^{N+5} \subset C_c^m(\mathbb{R}^3)$, such that $\operatorname{supp} \psi_k \subset U_k$ and $\partial_{\nu_S} \psi_k = 0$ on ∂S, $\partial_{\nu_\Sigma} \psi_k = 0$, on $\partial\Sigma$.

Proof. This proposition can be proved easily by mimicking same arguments as in [Wil20, Proposition 5.4]. □

4.7. Appendix: Auxiliary Transmission Problems

In this section, we give the existence and uniqueness of auxiliary elliptic and parabolic transmission problems. Given $G_0 := G_0^+ \cup G_0^- \cup \Gamma \subset \mathbb{R}^2$ with $\overline{G_0^-} \subset G_0$, we define the cylindrical domain for $0 < L_1 < L_2 < \infty$ as $\Omega := \Omega^+ \cup \Omega^- \cup \Sigma$, $\overline{\Omega^-} \subset \Omega$ with three disjoint parts, satisfying $\partial\Omega = G \cup \overline{S}$, $\partial\Omega^- = \overline{\Sigma} \cup G_0^-$, $\partial\Omega^+ = \overline{\Sigma} \cup \overline{S} \cup G_0^+$, where $G := \cup_{j=1,2} G_0 \times \{L_j\}$ and $S, \Sigma \subset \mathbb{R}^3$ are general hypersurfaces which will be assigned with certain regularity. Particularly, curves

$\partial\Sigma = \cup_{j=1,2}\Gamma \times \{L_j\}$, $\partial S = \cup_{j=1,2}\partial G_0 \times \{L_j\}$ are two one dimensional sub-manifolds of $\mathbb{R}^3$. Moreover, we assume that $S \perp G$ at ∂G, $\Sigma \perp G$ at $\partial\Sigma$ and $\Sigma \cap S = \emptyset$, which means ∂S, $\partial\Sigma$ are the contact lines involving with ninety-degree contact angles. In addition, ν denotes the unit outer normal vector on the interface Σ (point from Ω^- to Ω^+) and the boundary $\partial\Omega$.

4.7.1. Elliptic transmission problems. Firstly, for $\lambda > 0$, we consider the elliptic system

$$\begin{aligned} \lambda\phi - \Delta\phi &= f, && \text{in } \Omega\backslash\Sigma, \\ [\![\rho\phi]\!] &= g_1, && \text{on } \Sigma, \\ [\![\partial_{\nu_\Sigma}\phi]\!] &= g_2, && \text{on } \Sigma, \\ \rho\phi &= g_3, && \text{on } G\backslash\partial\Sigma, \\ \partial_{\nu_S}\phi &= g_4, && \text{on } S. \end{aligned} \tag{4.70}$$

Then we have the following theorem.

THEOREM 4.20. *Let $q > 1$, $\rho > 0$, $T > 0$, $J = (0,T)$ and $s \in \{0,1\}$. Assume that Ω, Σ, G, S are defined as above and Σ, G, S are of class C^3, as well as $\partial G \in C^4$. Then there is a constant $\lambda_0 \geq 0$ such that for all $\lambda > \lambda_0$, (4.70) admits a unique solution $\phi \in W_q^{2+s}(\Omega\backslash\Sigma)$ if and only if the data satisfy the following regularity and compatibility conditions:*

(1) $f \in W_q^s(\Omega\backslash\Sigma)$,

(2) $g_1 \in W_q^{2+s-\frac{1}{q}}(\Sigma)$,

(3) $g_2 \in W_q^{1+s-\frac{1}{q}}(\Sigma)$,

(4) $g_3 \in W_q^{2+s-\frac{1}{q}}(G\backslash\partial\Sigma)$,

(5) $g_4 \in W_q^{1+s-\frac{1}{q}}(S)$,

(6) $[\![g_3]\!] = g_1$, $[\![\partial_{\nu_\Sigma}(g_3/\rho)]\!] = g_2$, *at* $\partial\Sigma$,

(7) $\partial_{\nu_S}(g_3/\rho) = g_4$, *on* ∂S.

Proof. We first consider the case $s = 0$. The general tool is the localization procedure, see e.g. [DHP03], [PS16], [Wil20]. To this end, one gives four model problems with respect to (4.70). Namely,

- The half-space elliptic equation with Dirichlet boundary condition
- The quarter-space elliptic equation with Neumann and Dirichlet boundary conditions
- The half-space elliptic transmission problem with a flat interface and Dirichlet boundary condition
- The elliptic transmission problem in smooth domain with a flat interface and Neumann boundary condition.

The first and the last one have been solved well, readers are referred to e.g. Prüss–Simonett [PS16].

Elliptic equation in a quarter space. For a quarter space $\{(x_1, x_2, x_3) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}_+, x_3 \in \mathbb{R}_+\}$, we consider the problem for $\lambda > 0$ large,

$$\begin{aligned} \lambda\phi - \Delta\phi &= f, \quad x_1 \in \mathbb{R},\ x_2 > 0,\ x_3 > 0, \\ \rho\phi &= g_1, \quad x_1 \in \mathbb{R},\ x_2 = 0,\ x_3 > 0, \\ \partial_3\phi &= g_2, \quad x_1 \in \mathbb{R},\ x_2 > 0,\ x_3 = 0. \end{aligned} \tag{4.71}$$

Naturally, the compatibility condition at the contact line

$$\partial_3 \left(\frac{g_1}{\rho}\right) = g_2, \quad x_1 \in \mathbb{R},\ x_2 = 0,\ x_3 = 0,$$

should be satisfied. To solve (4.71), we reduce it to the case $g_1 = 0$. Extend $g_1 \in W_q^{2-\frac{1}{q}}(\mathbb{R} \times \{0\} \times \mathbb{R}_+)$ with respect to x_3 by the extension

$$\tilde{g}_1(x_1, x_2, x_3) = \begin{cases} g_1(x_1, x_2, x_3), & \text{if } x_3 > 0, \\ -g_1(x_1, x_2, -2x_3) + 2g_1(x_1, x_2, -x_3/2), & \text{if } x_3 < 0. \end{cases}$$

Then $\tilde{g}_1 \in W_q^{2-\frac{1}{q}}(\mathbb{R} \times \{0\} \times \mathbb{R})$. Let $\bar{\rho} \equiv \rho$ in $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$. Solving

$$\begin{aligned} \lambda\phi - \Delta\phi &= 0, \quad x_1 \in \mathbb{R},\ x_2 > 0,\ x_3 \in \mathbb{R}, \\ \bar{\rho}\phi &= \tilde{g}_1, \quad x_1 \in \mathbb{R},\ x_2 = 0,\ x_3 \in \mathbb{R}, \end{aligned}$$

yields a unique solution $\bar{\phi} \in W_q^2(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ by [PS16, Section 6.2]. Then $(\phi - \bar{\phi})$ (restricted) solves (4.71) with $g_1 = 0$ and modified data (not to be relabeled) g_2 satisfying the compatibility condition $g_2|_{x_2=0} = 0$. Extend $f \in L^q(\mathbb{R} \times \mathbb{R}_+^2)$ and $g_2 \in W_q^{1-\frac{1}{q}}(\mathbb{R} \times \mathbb{R}_+ \times \{0\})$ to some function $\tilde{f} \in L^q(\mathbb{R}^2 \times \mathbb{R}_+)$ and $\tilde{g}_2 \in W_q^{1-\frac{1}{q}}(\mathbb{R}^2 \times \{0\})$ by odd reflection. Then we solve the half-space elliptic equation

$$\begin{aligned} \lambda\phi - \Delta\phi &= \tilde{f}, \quad x_1 \in \mathbb{R},\ x_2 \in \mathbb{R},\ x_3 > 0, \\ \partial_3\phi &= \tilde{g}_2, \quad x_1 \in \mathbb{R},\ x_2 \in \mathbb{R},\ x_3 = 0, \end{aligned}$$

to obtain a unique solution $\tilde{\phi} \in W_q^2(\mathbb{R}^2 \times \mathbb{R}_+)$ by [PS16, Section 6.2]. By the symmetry, the function $-\tilde{\phi}(x_1, -x_2, x_3)$ is also a solution to the above system and the uniqueness implies $\tilde{\phi}|_{x_2=0} = 0$. Then the restricted function $(\bar{\phi} + \tilde{\phi}) \in W_q^2(\mathbb{R} \times \mathbb{R}_+^2)$ solves (4.71). The uniqueness follows similarly by carrying out argument in Section 4.3.1, with the help of the uniqueness in [PS16, Section 6.2].

Analogously, there is a solution operator with regard to the data and solutions for the elliptic equation in a bent quarter space as in Section 4.3.2.

Elliptic transmission problem in a half space. For a half space

$$\{(x_1, x_2, x_3) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}_+, x_3 \in \mathbb{R}\},$$

we consider the problem for $\lambda > 0$ large,

$$\begin{aligned} \lambda\phi - \Delta\phi &= f, \quad x_1 \in \mathbb{R},\ x_2 > 0,\ x_3 \in \dot{\mathbb{R}}, \\ [\![\rho\phi]\!] &= g_1, \quad x_1 \in \mathbb{R},\ x_2 > 0,\ x_3 = 0, \\ [\![\partial_3\phi]\!] &= g_2, \quad x_1 \in \mathbb{R},\ x_2 > 0,\ x_3 = 0, \\ \rho\phi &= g_3, \quad x_1 \in \mathbb{R},\ x_2 = 0,\ x_3 \in \dot{\mathbb{R}}. \end{aligned} \tag{4.72}$$

The compatibility conditions at the contact line are

$$[\![g_3]\!] = g_1, \quad \left[\!\!\left[\partial_3\left(\frac{g_3}{\rho}\right)\right]\!\!\right] = g_2, \quad x_1 \in \mathbb{R},\ x_2 = 0,\ x_3 = 0.$$

Similarly as above, we reduce (4.72) to the case $(f, g_3) = 0$ firstly. Define $g_3^\pm := g_3|_{x_3 \gtrless 0}$, $f^\pm := f|_{x_3 \gtrless 0}$ and $\rho^\pm := \rho|_{x_3 \gtrless 0}$. Extend $g_3^\pm \in W_q^{2-\frac{1}{q}}(\mathbb{R} \times \{0\} \times \mathbb{R}_\pm)$ to some functions $\tilde{g}_3^\pm \in W_q^{2-\frac{1}{q}}(\mathbb{R} \times \{0\} \times \mathbb{R})$ and respectively, $f^\pm \in L^q(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_\pm)$ to $\tilde{f}^\pm \in L^q(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ by the extension above, $\rho^\pm$ to $\tilde{\rho}^\pm$ by constant. Solving the half-space elliptic equation

$$\begin{aligned} \lambda\phi - \Delta\phi &= \tilde{f}^+, \quad x_1 \in \mathbb{R},\ x_2 > 0,\ x_3 \in \mathbb{R}, \\ \tilde{\rho}^+\phi &= \tilde{g}_3^+, \quad x_1 \in \mathbb{R},\ x_2 = 0,\ x_3 \in \mathbb{R}, \end{aligned}$$

yields a unique solution $\bar{\phi}^+ \in W_q^2(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$ by [PS16, Section 6.2]. Replacing the superscript "+" above by "−", one obtains a unique solution $\bar{\phi}^- \in W_q^2(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$. Define

$$\bar{\phi} := \begin{cases} \bar{\phi}^+, & \text{if } x_3 > 0, \\ \bar{\phi}^-, & \text{if } x_3 < 0. \end{cases}$$

Then $(\phi - \bar{\phi})$ (restricted) solves (4.72) with $(f, g_3) = 0$ and modified data (not to be relabeled) g_1, g_2 satisfying the compatibility condition $g_1|_{x_2=0} = g_2|_{x_2=0} = 0$. Hence one can extend $g_1 \in W_q^{2-\frac{1}{q}}(\mathbb{R} \times \mathbb{R}_+ \times \{0\})$ to some function $\tilde{g}_1 \in W_q^{2-\frac{1}{q}}(\mathbb{R}^2 \times \{0\})$ and $g_2 \in W_q^{1-\frac{1}{q}}(\mathbb{R} \times \mathbb{R}_+ \times \{0\})$ to some function $\tilde{g}_2 \in W_q^{1-\frac{1}{q}}(\mathbb{R}^2 \times \{0\})$ by odd reflection. Then we solve the full-space elliptic transmission problem with a flat interface

$$\begin{aligned} \lambda\phi - \Delta\phi &= 0, \quad x_1 \in \mathbb{R},\ x_2 \in \mathbb{R},\ x_3 \in \dot{\mathbb{R}}, \\ [\![\rho\phi]\!] &= \tilde{g}_1, \quad x_1 \in \mathbb{R},\ x_2 \in \mathbb{R},\ x_3 = 0, \\ [\![\partial_3\phi]\!] &= \tilde{g}_2, \quad x_1 \in \mathbb{R},\ x_2 \in \mathbb{R},\ x_3 = 0, \end{aligned}$$

to obtain a unique solution $\tilde{\phi} \in W_q^2(\mathbb{R}^2 \times \dot{\mathbb{R}})$ by [PS16, Section 6.5]. By the symmetry, the function $-\tilde{\phi}(x_1, -x_2, x_3)$ is also a solution to the above system. Then the uniqueness implies $\tilde{\phi}|_{x_2=0} = 0$. Consequently, the restricted function $(\bar{\phi} + \tilde{\phi}) \in W_q^2(\mathbb{R} \times \mathbb{R}_+ \times \dot{\mathbb{R}})$ solves (4.72). As in Section 4.3.4, one can conclude that there is also a solution operator with regard to the data and solutions for the elliptic transmission problem in a half space with a bent interface.

Now following the procedure in Section 4.4.3, one can complete the proof analogously, with the help of Proposition 4.19.

For the case $s = 1$, the higher-order regularity case, we refer to e.g. Section 6.3.5, Section 6.5.3 in [PS16] for the models problems in a higher-order regularity class. Then proceeding with a similar localization argument gives us the desired higher regularity. □

Based on Theorem 4.70, we have the solvability and regularity for the Laplace transmission problem.

$$\begin{aligned} -\Delta\phi &= f, \quad \text{in } \Omega\backslash\Sigma, \\ [\![\rho\phi]\!] &= 0, \quad \text{on } \Sigma, \\ [\![\partial_{\nu_\Sigma}\phi]\!] &= 0, \quad \text{on } \Sigma, \\ \rho\phi &= 0, \quad \text{on } G\backslash\partial\Sigma, \\ \partial_{\nu_S}\phi &= 0, \quad \text{on } S. \end{aligned} \tag{4.73}$$

THEOREM 4.21. *Under the assumption of Theorem 4.70, (4.73) admits a unique solution $\phi \in W_q^{2+s}(\Omega\backslash\Sigma)$ if and only if $f \in W_q^s(\Omega\backslash\Sigma)$.*

Proof. For $s = 0$, let $H^k := W_2^k$ and $H_K^k := W_{2,K}^k$ be the Sobolev spaces, where

$$W_{q,K}^k(\Omega)^3 = \left\{\mathbf{u} \in W_q^k(\Omega)^3 : \mathbf{u}|_K = 0, \quad K \in \{G, S\}\right\}.$$

Testing (4.73) with a function $\psi \in H_G^1(\Omega)$, one obtains

$$\int_\Omega \nabla\phi \cdot \nabla\psi \mathrm{d}x = \int_\Omega f\psi \mathrm{d}x.$$

Then by Lax–Milgram Theorem, we know that there is a unique weak solution $\phi \in H_G^1(\Omega)$ of (4.73). Equivalently, ϕ solves

$$\begin{aligned} \lambda\phi - \Delta\phi = f + \lambda\phi =: \tilde{f}, &\quad \text{in } \Omega\backslash\Sigma, \\ [\![\rho\phi]\!] = 0, &\quad \text{on } \Sigma, \\ [\![\partial_{\nu_\Sigma}\phi]\!] = 0, &\quad \text{on } \Sigma, \\ \rho\phi = 0, &\quad \text{on } G\backslash\partial\Sigma, \\ \partial_{\nu_S}\phi = 0, &\quad \text{on } S. \end{aligned} \tag{4.74}$$

weakly. From Sobolev embeddings, we have $H^1(\Omega) \hookrightarrow L^6(\Omega)$ in the three dimensional case. Then $\tilde{f} \in L^p(\Omega)$ for $p = \min(q, 6)$. If $q \le 6$, then (4.74) can be solved uniquely in $W_q^2(\Omega\backslash\Sigma) \cap H_G^1(\Omega)$ by Theorem 4.20. Otherwise if $q > 6$, one obtains $\phi \in W_6^2(\Omega\backslash\Sigma) \cap W_{6,G}^1(\Omega) \hookrightarrow L^p(\Omega)$ for any $p \ge 1$. Consequently, $\tilde{f} \in L^q(\Omega)$, which means $\phi \in W_q^2(\Omega\backslash\Sigma) \cap W_{q,G}^1(\Omega)$ by Theorem 4.20.

For $s = 1$, one can proceed the bootstrap argument from $s = 0$, with the help of higher regularity in Theorem 4.20. Namely, we can promote the regularity for $\tilde{f}$ in (4.74) up to $W_q^1(\Omega\backslash\Sigma)$ and get the desired regularity by Theorem 4.20 with $s = 1$. □

Remark 4.22. Note that in Wilke [Wil20, Lemma 5.6], a mean value free condition was imposed for f, while in the present paper, we do not have this one. This is due to the face that if we integrate (4.73) over Ω, we will obtain

$$\int_\Omega f \mathrm{d}x = -\int_G \partial_{\nu_G}\phi \mathrm{d}\sigma,$$

which does not need to vanish since on $G\backslash\partial\Sigma$, $\rho\phi = 0$ instead of $\partial_{\nu_G}\phi = 0$ as in [Wil20].

4.7.2. Parabolic transmission problems. Consider the parabolic transmission problem with mixed boundary conditions

$$\begin{aligned} \rho\partial_t u - \mu\Delta u = f, &\quad \text{in } \Omega\backslash\Sigma \times J, \\ [\![u]\!] = g_1, &\quad \text{on } \Sigma \times J, \\ [\![\mu(\nabla u + \nabla u^\top)\nu_\Sigma]\!] = g_2, &\quad \text{on } \Sigma \times J, \\ (u_1, u_3)^\top = g_3, &\quad \text{on } G\backslash\partial\Sigma \times J, \\ 2\mu\partial_2 u_2 = g_4, &\quad \text{on } G\backslash\partial\Sigma \times J, \\ u = g_5, &\quad \text{on } S \times J, \\ u(0) = u_0, &\quad \text{in } \Omega\backslash\Sigma. \end{aligned} \tag{4.75}$$

Then one obtains the following theorem.

Theorem 4.23. *Let $q > 3$, $\rho, \mu > 0$, $T > 0$ and $J = (0, T)$. Assume that Ω, Σ, G, S are defined as above and Σ, G, S are of class C^3, as well as ∂G of C^4. Then (4.75) admits a unique solution*

$$u \in W_q^1(J; L^q(\Omega)^3) \cap L^q(J; W_q^2(\Omega\backslash\Sigma)^3),$$

if and only if the data satisfy the following regularity and compatibility conditions:

(1) $f \in L^q(J; L^q(\Omega)^3)$,

(2) $g_1 \in W_q^{1-\frac{1}{2q}}(J; L^q(\Sigma)^3) \cap L^q(J; W_q^{2-\frac{1}{q}}(\Sigma)^3)$,

(3) $g_2 \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\Sigma)^3) \cap L^q(J; W_q^{1-\frac{1}{q}}(\Sigma)^3)$,

(4) $g_3 \in W_q^{1-\frac{1}{2q}}(J; L^q(G)^2) \cap L^q(J; W_q^{2-\frac{1}{q}}(G\backslash\partial\Sigma)^2)$,

(5) $g_4 \in W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(G)) \cap L^q(J; W_q^{1-\frac{1}{q}}(G\backslash\partial\Sigma))$,

(6) $g_5 \in W_q^{1-\frac{1}{2q}}(J; L^q(S)^3) \cap L^q(J; W_q^{2-\frac{1}{q}}(S)^3)$,

(7) $u_0 \in W_q^{2-\frac{2}{q}}(\Omega\backslash\Sigma)^3$, $[\![u_0]\!] = g_1|_{t=0}$, $\mathcal{P}_\Sigma([\![\nabla u_0 + \nabla u_0^\top]\!]\,\nu_\Sigma) = \mathcal{P}_\Sigma(g_2)|_{t=0}$,

(8) $((u_0)_1, (u_0)_3)^\top|_G = g_3|_{t=0}$, $2\mu\partial_2(u_0)_2|_G = g_4|_{t=0}$, $u_0|_S = g_5|_{t=0}$,

(9) $[\![g_3]\!] = ((g_1)_1, (g_1)_3)^\top$, *at* $\partial\Sigma$,

(10) $(g_2)_1 = [\![2\mu\partial_1(g_3)_1\nu_\Sigma \cdot e_1 + \mu(\partial_1(g_3)_2 + \partial_3(g_3)_1)\nu_\Sigma \cdot e_3]\!]$ *at* $\partial\Sigma$,

(11) $(g_2)_3 = [\![2\mu\partial_3(g_3)_2\nu_\Sigma \cdot e_3 + \mu(\partial_1(g_3)_2 + \partial_3(g_3)_1)\nu_\Sigma \cdot e_1]\!]$, *at* $\partial\Sigma$,

(12) $g_3 = ((g_5)_1, (g_5)_3)^\top$, $g_4 = 2\mu\partial_2(g_5)_2$, *at* ∂S.

Proof. Again, we are going to prove the theorem by localization procedure. To this end, one considers four model problems with respect to (4.75). Namely,

- The half-space heat equation with outflow boundary condition
- The quarter-space heat equation with outflow and Dirichlet boundary conditions
- The half-space heat transmission problem with a flat interface and Dirichlet boundary condition
- The heat transmission problem in smooth domain with a flat interface and Dirichlet boundary condition.

The first one has been solved, readers are referred to [PS16]. We focus on the remaining ones.

Heat equation in a quarter space. For a quarter space $\{(x_1, x_2, x_3) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}_+, x_3 \in \mathbb{R}_+\}$, we consider the problem

$$\begin{aligned} \rho\partial_t u - \mu\Delta u &= f, && \text{in } \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+\times J,\\ (u_1, u_3)^\top &= g_1, && \text{on } \mathbb{R}\times\{0\}\times\mathbb{R}_+\times J,\\ 2\mu\partial_2 u_2 &= g_2, && \text{on } \mathbb{R}\times\{0\}\times\mathbb{R}_+\times J,\\ u &= g_3, && \text{on } \mathbb{R}\times\mathbb{R}_+\times\{0\}\times J,\\ u(0) &= u_0, && \text{in } \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+, \end{aligned} \tag{4.76}$$

where $\rho, \mu > 0$, $u : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3$ is the unknown quantity. The data g_i, $i = 1, 2, 3$ satisfy the initial compatibility conditions

$$((u_0)_1, (u_0)_3)^\top|_{x_2=0} = g_1|_{t=0}\,, \quad 2\mu\partial_2(u_0)_2|_{x_2=0} = g_2|_{t=0}\,, \quad u_0|_{x_3=0} = g_3|_{t=0}\,,$$

and compatibility condition at the contact line $\mathbb{R} \times \{0\} \times \{0\}$

$$((g_3)_1, (g_3)_3)^\top = g_1, \quad 2\mu\partial_2(g_3)_2 = g_2, \quad x_1 \in \mathbb{R},\ x_2 = 0,\ x_3 = 0.$$

To solve (4.76), we reduce it to the case $(u_0, f, g_3) = 0$. We extend

$$u_0 \in W_q^{2-\frac{2}{q}}(\mathbb{R} \times \mathbb{R}^2_+)^3, \quad f \in L^q(J; L^q(\mathbb{R} \times \mathbb{R}^2_+)^3),$$
$$g_3 \in W_q^{1-\frac{1}{2q}}(J; L^q(\mathbb{R} \times \mathbb{R}_+ \times \{0\})^3) \cap L^q(J; W_q^{2-\frac{1}{q}}(\mathbb{R} \times \mathbb{R}_+ \times \{0\})^3)$$

with respect to x_2 by the similar extension as (4.19) to some functions

$$\tilde{u}_0 \in W_q^{2-\frac{2}{q}}(\mathbb{R}^2 \times \mathbb{R}_+)^3, \quad \tilde{f} \in L^q(J; L^q(\mathbb{R}^2 \times \mathbb{R}_+)^3),$$
$$\tilde{g}_3 \in W_q^{1-\frac{1}{2q}}(J; L^q(\mathbb{R}^2 \times \{0\})^3) \cap L^q(J; W_q^{2-\frac{1}{q}}(\mathbb{R}^2 \times \{0\})^3).$$

Solving

$$\begin{aligned} \rho\partial_t u - \mu\Delta u &= \tilde{f}, && \text{in } \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \times J, \\ u &= \tilde{g}_3, && \text{on } \mathbb{R} \times \mathbb{R} \times \{0\} \times J, \\ u(0) &= \tilde{u}_0, && \text{in } \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+, \end{aligned}$$

yields a unique solution

$$\bar{u} \in W_q^1(J; L^q(\mathbb{R}^2 \times \mathbb{R}_+)^3) \cap L^q(J; W_q^2(\mathbb{R}^2 \times \mathbb{R}_+)^3),$$

by [PS16, Section 6.2]. Then $(u - \bar{u})$ (restricted) solves (4.76) with $(u_0, f, g_3) = 0$ and modified data g_1, g_2 (not to be relabeled) having vanishing trace at $t = 0$ and satisfying $g_1|_{x_3=0} = 0$ and $g_2|_{x_3=0} = 0$. Now by odd reflection with respect to x_3, one can extend

$$g_1 \in {}_0W_q^{1-\frac{1}{2q}}(J; L^q(\mathbb{R} \times \{0\} \times \mathbb{R}_+)^2) \cap L^q(J; W_q^{2-\frac{1}{q}}(\mathbb{R} \times \{0\} \times \mathbb{R}_+)^2),$$
$$g_2 \in {}_0W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\mathbb{R} \times \{0\} \times \mathbb{R}_+)) \cap L^q(J; W_q^{1-\frac{1}{q}}(\mathbb{R} \times \{0\} \times \mathbb{R}_+)),$$

to some functions

$$\tilde{g}_1 \in {}_0W_q^{1-\frac{1}{2q}}(J; L^q(\mathbb{R} \times \{0\} \times \mathbb{R})^2) \cap L^q(J; W_q^{2-\frac{1}{q}}(\mathbb{R} \times \{0\} \times \mathbb{R})^2),$$
$$\tilde{g}_2 \in {}_0W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\mathbb{R} \times \{0\} \times \mathbb{R})) \cap L^q(J; W_q^{1-\frac{1}{q}}(\mathbb{R} \times \{0\} \times \mathbb{R})).$$

Then we solve the half-space heat equation

$$\begin{aligned} \rho\partial_t u - \mu\Delta u &= 0, && \text{in } \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times J, \\ (u_1, u_3)^\top &= \tilde{g}_1, && \text{on } \mathbb{R} \times \{0\} \times \mathbb{R} \times J, \\ 2\mu\partial_2 u_2 &= \tilde{g}_2, && \text{on } \mathbb{R} \times \{0\} \times \mathbb{R} \times J, \\ u(0) &= 0, && \text{in } \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}, \end{aligned} \tag{4.77}$$

to obtain a unique solution

$$\tilde{u} \in {}_0W^1_q(J; L^q(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})^3) \cap L^q(J; W^2_q(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})^3).$$

We remark here that (4.77) is actually assembled by two separated heat equations with Dirichlet boundary condition and Neumann boundary condition respectively. Namely,

$$\begin{aligned} \rho\partial_t(u_1,u_3)^\top - \mu\Delta(u_1,u_3)^\top &= 0, & \rho\partial_t u_2 - \mu\Delta u_2 &= 0, && \text{in } \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times J,\\ (u_1,u_3)^\top &= \tilde{g}_1, & 2\mu\partial_2 u_2 &= \tilde{g}_2, && \text{on } \mathbb{R}\times\{0\}\times\mathbb{R}\times J,\\ (u_1,u_3)^\top(0) &= 0, & u_2(0) &= 0 && \text{in } \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}, \end{aligned}$$

Hence (4.77) is uniquely solved by [PS16, Section 6.2]. By the symmetry, one knows that

$$\bar{u}(t,x_1,x_2,x_3) := (-\tilde{u}_1(t,x_1,x_2,-x_3), -\tilde{u}_2(t,x_1,x_2,-x_3), -\tilde{u}_3(t,x_1,x_2,-x_3))^\top$$

is also a solution to (4.77). Then the uniqueness implies that $\bar{u}(t,x_1,x_2,x_3) = \tilde{u}(t,x_1,x_2,x_3)$, i.e., $\tilde{u}(t,x_1,x_2,0) = 0$. Consequently, the restricted function

$$(\bar{u}+\tilde{u}) \in W^1_q(J; L^q(\mathbb{R}\times\mathbb{R}^2_+)^3) \cap L^q(J; W^2_q(\mathbb{R}\times\mathbb{R}^2_+)^3)$$

solves (4.76) with compatibility condition at the contact line.

Analogously, there is a solution operator with regard to the data and solutions for the heat equation in a bent quarter space as in Section 4.3.2.

Heat transmission problem in a half space. For a half space $\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}$, we consider the problem

$$\begin{aligned} \rho\partial_t u - \mu\Delta u &= f, && \text{in } \mathbb{R}\times\mathbb{R}_+\times\dot{\mathbb{R}}\times J,\\ [\![u]\!] &= g_1, && \text{on } \mathbb{R}\times\mathbb{R}_+\times\{0\}\times J,\\ [\![\mu(\partial_3 u + \nabla u_3)]\!] &= g_2, && \text{on } \mathbb{R}\times\mathbb{R}_+\times\{0\}\times J,\\ (u_1,u_3)^\top &= g_3, && \text{on } \mathbb{R}\times\{0\}\times\dot{\mathbb{R}}\times J,\\ 2\mu\partial_2 u_2 &= g_4, && \text{on } \mathbb{R}\times\{0\}\times\dot{\mathbb{R}}\times J,\\ u(0) &= u_0, && \text{in } \mathbb{R}\times\mathbb{R}_+\times\dot{\mathbb{R}}. \end{aligned} \tag{4.78}$$

Then the data (u_0, g_j), $j = 1, ..., 4$, satisfy the corresponding initial compatibility condition

$$\begin{aligned} [\![u_0]\!] = g_1|_{t=0}, \quad & [\![\partial_3 u_0 + \nabla(u_0)_3]\!] = g_2|_{t=0},\\ ((u_0)_1,(u_0)_3)^\top|_{x_2=0} = g_3|_{t=0}, \quad & 2\mu\partial_2(u_0)_2|_{x_2=0} = g_4|_{t=0}, \end{aligned}$$

and compatibility conditions at the contact line $\{x\in\mathbb{R}^3 : x_2 = 0, x_3 = 0\}$

$$\begin{aligned} ((g_1)_1,(g_1)_3)^\top = [\![g_3]\!], \quad & 2\partial_2(g_1)_2 = \left[\!\!\left[\frac{g_4}{\mu}\right]\!\!\right],\\ (g_2)_1 = [\![\mu(\partial_3(g_3)_1 + \partial_1(g_3)_2)]\!], \quad & (g_2)_3 = [\![2\mu\partial_3(g_3)_2]\!]. \end{aligned}$$

Following the argument in Section 4.3.3, one can easily reduce (4.78) to the case $(u_0, f, g_3, g_4) = 0$ with modified data (not to be relabeled) having vanishing trace at $t = 0$ and satisfying

$$((g_1)_1,(g_1)_3)^\top|_{x_2=0} = 0, \quad \partial_2(g_2)_2|_{x_2=0} = 0, \quad (g_2)_1|_{x_2=0} = 0, \quad (g_2)_3|_{x_2=0} = 0,$$

Subsequently, one can extend $((g_1)_1, (g_1)_3, (g_2)_1, (g_2)_3)$ and $((g_1)_2, (g_2)_2)$ by odd and even reflection respectively to some functions

$$\tilde{g}_1 \in {}_0W_q^{1-\frac{1}{2q}}(J; L^q(\mathbb{R}^2 \times \{0\})^3) \cap L^q(J; W_q^{2-\frac{1}{q}}(\mathbb{R}^2 \times \{0\})^3),$$
$$\tilde{g}_2 \in {}_0W_q^{\frac{1}{2}-\frac{1}{2q}}(J; L^q(\mathbb{R}^2 \times \{0\})^3) \cap L^q(J; W_q^{1-\frac{1}{q}}(\mathbb{R}^2 \times \{0\})^3).$$

Then we solve the full-space heat transmission problem with a flat interface

$$\begin{aligned} \rho\partial_t u - \mu\Delta u &= 0, && \text{on } \mathbb{R} \times \mathbb{R} \times \dot{\mathbb{R}} \times J, \\ [\![u]\!] &= \tilde{g}_1, && \text{on } \mathbb{R} \times \mathbb{R} \times \{0\} \times J, \\ [\![\mu(\partial_3 u + \nabla u_3)]\!] &= \tilde{g}_2, && \text{on } \mathbb{R} \times \mathbb{R} \times \{0\} \times J, \\ u(0) &= 0, && \text{on } \mathbb{R} \times \mathbb{R} \times \dot{\mathbb{R}}, \end{aligned}$$

to obtain a unique solution

$$\tilde{u} \in {}_0W_q^1(J; L^q(\mathbb{R}^2 \times \mathbb{R})^3) \cap L^q(J; W_q^2(\mathbb{R}^2 \times \dot{\mathbb{R}})^3).$$

with the help of [PS16, Section 6.5]. Again by symmetry, we conclude that

$$\hat{u}(t, x_1, x_2, x_3) := (-\tilde{u}_1(t, x_1, x_2, -x_3), \tilde{u}_2(t, x_1, x_2, -x_3), -\tilde{u}_3(t, x_1, x_2, -x_3))^\top$$

is a solution to (4.78) as well. From the uniqueness, that is, $\hat{u}(t, x_1, x_2, x_3) = \tilde{u}(t, x_1, x_2, x_3)$, we know that

$$(\tilde{u}_1, \tilde{u}_3)(t, x_1, 0, x_3) = 0, \quad \partial_2\tilde{u}_2(t, x_1, 0, x_3) = 0.$$

Consequently, the restricted function

$$\tilde{u} \in W_q^1(J; L^q(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})^3) \cap L^q(J; W_q^2(\mathbb{R}^2 \times \dot{\mathbb{R}})^3).$$

solves (4.78) with $(u_0, f, g_3, g_4) = 0$. As in Section 4.3.4, one concludes that there is also a solution operator with regard to the data and solutions for the heat transmission problem in a half space with a bent interface.

Heat transmission problem with a Dirichlet boundary. Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded domain with C^2-boundary, consisting of two parts $\Omega^\pm$ which are also open and such that $\overline{\Omega^+} \subset \Omega$. Let $\Sigma = \partial\Omega^+$ be the interface separating $\Omega^\pm$ such that $\Omega = \Omega^+ \cup \Sigma \cup \Omega^-$. Consider the problem

$$\begin{aligned} \rho\partial_t u - \mu\Delta u &= f, && \text{in } \Omega\backslash\Sigma \times J, \\ [\![u]\!] &= g_1, && \text{on } \Sigma \times J, \\ [\![\mu(\nabla u + \nabla u^\top)\nu]\!] &= g_2, && \text{on } \Sigma \times J, \\ u &= g_3, && \text{on } \partial\Omega \times J, \\ u(0) &= u_0, && \text{in } \Omega\backslash\Sigma, \end{aligned} \tag{4.79}$$

where $\rho, \mu > 0$, ν denotes the outer unit normal vector on Σ pointing from Ω^+ into Ω^-. Note that a similar transmission problem with a Neumann boundary condition was investigated in [PS16, Section 6.5], one could solve (4.79) by a truncation technique. Readers are referred to Proof of Theorem 3.12 for more details, where we handled the two-phase Stokes problem. In this position, we only give a sketch of the procedure. To this end, we choose $\psi(x) \in C_0^\infty(\Omega)$ as a cutoff function over Ω such that

$$\psi(x) = \begin{cases} 1, & \text{in a neighborhood of } \Omega^+, \\ 0, & \text{in a neighborhood of } \partial\Omega. \end{cases} \tag{4.80}$$

We define $u := \psi u_1 + (1-\psi)u_2$, where u_1 is the solution of a parabolic transmission problem with a Neumann boundary condition in Ω, while u_2 solves the heat equation in Ω^- with a Dirichlet boundary condition. Then u solves (4.79) with some remainders. Since systems for u_1 and u_2 are all solvable and hence, following the procedure in Proof of Theorem 3.12 completes the proof of this part.

Finally, by a standard localization procedure as in Section 4.4.3 (see also [PS16]), one can finish the proof analogously, with the help of Proposition 4.19. □

Chapter 5

Quasi-stationary Fluid-Structure Interaction Problem with Growth in Smooth Domain

We address a quasi-stationary fluid-structure interaction problem coupled with cell reactions and growth, which comes from the plaque formation during the stage of the atherosclerotic lesion in human arteries. The blood is modeled by the incompressible Navier–Stokes equation, while the motion of vessels is captured by a quasi-stationary equation of nonlinear elasticity. The growth happens when both cells in fluid and solid react, diffuse and transport across the interface, resulting in the accumulation of foam cells, which are exactly seen as the plaques. Via a fixed-point argument, we derive the local well-posedness of the nonlinear system, which is based on the analysis of the decoupled linear systems.

Overview of This Chapter

Notations. In this chapter, we consider specifically the following notations.

- $\Omega^t = \Omega^t_f \cup \Omega^t_s \cup \Gamma^t$, where $\Omega^t \subset \mathbb{R}^3$ is divided by the interface Γ^t into two disjoint parts, fluid domain Ω^t_f and solid domain Ω^t_s. Γ^t_s denotes the outer boundary of Ω^t. See Figure 3.1.
- $\mathbf{u}$, $\hat{\mathbf{u}}$, the Eulerian and Lagrangian displacement
- $\mathbf{v}$, $\hat{\mathbf{v}}$, the Eulerian and Lagrangian
- c, $\hat{c}$, cell concentrations
- c^*, $\hat{c}^*$ foam cell concentration
- g, $\hat{g}$, growth metrics
- $\hat{\mathbf{F}}$, the deformation gradient in terms of $\hat{\mathbf{v}}$
- $\mathbf{F}$, the inverse deformation gradient

When there is no danger of confusion, we specify the quantities with a subscript "f" and "s" to identify that defined in fluid domain and solid domain respectively. In addition, without a special statement, the quantities or operators with a hat "$\hat{\cdot}$" will indicate those in Lagrangian coordinates.

5.1. Introduction

5.1.1. Model description. In this chapter, we consider a *quasi-stationary* fluid-structure interaction problem for plaque growth, which describes the formation of plaque during the reaction-diffusion and transport of different cells in human blood and vessels. As discussed in Section 1.7, the problem is set up in a smooth domain $\Omega^t \subset \mathbb{R}^3$, with three disjoint parts $\Omega^t = \Omega_f^t \cup \Omega_s^t \cup \Gamma^t$, where $\Gamma^t = \partial\Omega_f^t$, $\overline{\Omega_f^t} \subset \Omega^t$ and Ω_f^t, Ω_s^t denote the domains for the fluid and solid, respectively. $\Gamma_s^t = \partial\Omega^t$ stands for the outer boundary of Ω^t, which is also a free boundary.

Now we recall from (1.38) that the target system reads as

$$
\begin{aligned}
\rho_f\left(\partial_t + \mathbf{v}_f \cdot \nabla\right)\mathbf{v}_f &= \operatorname{div}\mathbb{T}_f, && \text{in } \Omega_f^t,\ t \in (0,T), && (5.1a)\\
\operatorname{div}\mathbf{v}_f &= 0, && \text{in } \Omega_f^t,\ t \in (0,T), && (5.1b)\\
\operatorname{div}\mathbb{T}_s &= 0, && \text{in } \Omega_s^t,\ t \in (0,T), && (5.1c)\\
\rho_s \operatorname{div}\mathbf{v}_s &= \gamma\beta c_s, && \text{in } \Omega_s^t,\ t \in (0,T), && (5.1d)\\
\partial_t c_f + \mathbf{v}_f \cdot \nabla c_f - \operatorname{div}(D_f \nabla c_f) &= 0, && \text{in } \Omega_f^t,\ t \in (0,T), && (5.1e)\\
\partial_t c_s + \mathbf{v}_s \cdot \nabla c_s - \operatorname{div}(D_s \nabla c_s) &= -\beta c_s, && \text{in } \Omega_s^t,\ t \in (0,T), && (5.1f)\\
\partial_t c_s^* + \mathbf{v}_s \cdot \nabla c_s^* &= \beta c_s, && \text{in } \Omega_s^t,\ t \in (0,T), && (5.1g)\\
\partial_t g + \mathbf{v}_s \cdot \nabla g &= \frac{\gamma\beta c_s}{3\rho_s}, && \text{in } \Omega_s^t,\ t \in (0,T), && (5.1h)\\
[\![\mathbf{v}]\!] = 0, \quad [\![\mathbb{T}]\!]\,\mathbf{n}_{\Gamma^t} &= 0, && \text{on } \Gamma^t,\ t \in (0,T), && (5.1i)\\
[\![D\nabla c]\!] \cdot \mathbf{n}_{\Gamma^t} = 0, \quad \zeta\,[\![c]\!] - D_s \nabla c_s \cdot \mathbf{n}_{\Gamma^t} &= 0, && \text{on } \Gamma^t,\ t \in (0,T), && (5.1j)\\
\mathbb{T}_s \mathbf{n}_{\Gamma_s^t} = 0, \quad D_s \nabla c_s \cdot \mathbf{n}_{\Gamma_s^t} &= 0, && \text{on } \Gamma_s^t,\ t \in (0,T), && (5.1k)\\
\mathbf{v}_f|_{t=0} = \mathbf{v}_f^0,\ \mathbf{u}_s|_{t=0} = \mathbf{u}_s^0,\ c|_{t=0} = c^0,\ c_s^*|_{t=0} &= c_*^0,\ g|_{t=0} = g^0, &&&& (5.1l)
\end{aligned}
$$

where $\mathbb{T}_f := -\pi_f \mathbb{I} + \nu_s(\nabla\mathbf{v}_f + \nabla\mathbf{v}_f^\top)$ denotes the Cauchy stress tensor. $\mathbf{v}_f : \bigcup_{t\in(0,T)} \Omega_f^t \times \{t\} \to \mathbb{R}^3$, $\pi_f : \bigcup_{t\in(0,T)} \Omega_f^t \times \{t\} \to \mathbb{R}$ are the unknown velocity and pressure of the fluid. $\rho_f > 0$ stands for the fluid density and ν_s represents the viscosity of the fluid.

Compared to the problems (3.1) and (4.1), where an evolution equation for the incompressible neo-Hookean material was employed, the vessel in this chapter is assumed to be quasi-stationary, in view of the fact that generally it moves far slower than the blood from a macro point of view. Thus, we model the blood vessel by the equilibrium of a nonlinear elastic equation. To mathematically describe the elasticity conveniently, the Lagrangian coordinate was commonly used, see e.g. [Gor17, GFA10]. Thus, we set reference configuration as the initial domain which is defined by $\Omega := \Omega^t|_{t=0}$, as well as $\Omega_f = \Omega_f^0$, $\Omega_s = \Omega_s^0$ and $\Gamma = \Gamma^0$, then $\Omega = \Omega_f \cup \Omega_s \cup \Gamma$. Let X be the spatial variable in the reference configuration. Now we introduce the Lagrangian flow map

$$\varphi : \Omega \times (0,T) \to Q_T,$$

with

$$x(X,t) = \varphi(X,t) = X + \mathbf{u}(x(X,t),t) \tag{5.2}$$

for all $X \in \Omega$ and $x(X,0) = X$, where

$$
\mathbf{u}(x(X,t),t) = \begin{cases} \mathbf{u}_f(x(X,t),t) = \displaystyle\int_0^t \mathbf{v}_f(x(X,t),\tau)\mathrm{d}\tau, & \text{if } X \in \Omega_f,\\ \mathbf{u}_s(x(X,t),t), & \text{if } X \in \Omega_s, \end{cases}
$$

denotes the displacement for the fluid and solid respectively. In the sequel, without special statement, the quantities with a hat will indicate those in Lagrangian reference configuration, e.g., $\hat{\mathbf{u}}(X,t) = \mathbf{u}(x(X,t),t)$, while the operators with a hat means those act on the quantities in Lagrangian coordinate. Then the tensor field

$$\mathbf{F}(x(X,t),t) = \hat{\mathbf{F}}(X,t) := \frac{\partial}{\partial X}\varphi(X,t) = \hat{\nabla}\varphi(X,t) = \mathbb{I} + \hat{\nabla}\hat{\mathbf{u}}(X,t),\ \forall X \in \Omega, \tag{5.3}$$

with $\hat{\mathbf{F}}_f(X,0) = \mathbb{I}$ and $\hat{\mathbf{F}}_s(X,0) = \mathbb{I} + \hat{\nabla}\hat{\mathbf{u}}_s^0$ is referred to be the deformation gradient and $J = \hat{J} := \det(\hat{\mathbf{F}})$ denotes its determinant. For the blood vessels, since the growth is taken into account, we impose the so-called *multiplicative decomposition* for the solid deformation gradient $\hat{\mathbf{F}}_s$ as

$$\hat{\mathbf{F}}_s = \hat{\mathbf{F}}_{s,e}\hat{\mathbf{F}}_{s,g}$$

with $\hat{J}_s = \hat{J}_{s,e}\hat{J}_{s,g}$, where $\hat{\mathbf{F}}_{s,e}$ is the pure elastic deformation tensor and $\hat{\mathbf{F}}_{s,g}$ denotes the growth tensor, which are described as in Section 1.6. Inspired by Goriely [Gor17, Chapter 11–13] and Jones–Chapman [JC12, Section 3.2], a general incompressible hyperelastic material is considered for solid as

$$-\operatorname{div}\mathbb{T}_s = 0, \quad \text{in } \Omega_s^t,\ t \in (0,T), \tag{5.4}$$

where $\mathbb{T}_s := -\pi_s\mathbb{I} + J_{s,e}^{-1}DW(\mathbf{F}_{s,e})\mathbf{F}_{s,e}^\top$ stands for the Cauchy stress tensor. $\pi_s : \bigcup_{t\in(0,T)} \Omega_f^t \times \{t\} \to \mathbb{R}$ is the unknown pressure of the solid. The scalar function $W : \mathbb{R}^{3\times3} \to \mathbb{R}_+$ is called the strain energy density function (also known as the stored energy density), which needs some general assumptions for the sake of analysis.

Assumption 5.1.

- **(A1)** W is frame-indifferent, i.e., $W(\mathbf{RF}) = W(\mathbf{F})$, for all $\mathbf{R} \in SO(3)$ and $\mathbf{F} \in \mathbb{R}^{3\times3}$, where $SO(3) := \{\mathbf{A} \in \mathbb{R}^{3\times3} : \mathbf{A}^\top\mathbf{A} = \mathbb{I}, \det\mathbf{A} = 1\}$ is the set of all proper orthogonal tensors.
- **(A2)** $W \in C^4(\mathbb{R}^{3\times3};\mathbb{R})$.
- **(A3)** $DW(\mathbb{I}) = 0$, $W(\mathbf{R}) = 0$ for all $\mathbf{R} \in SO(3)$.
- **(A4)** There exists a constant $C_0 > 0$, such that $W(\mathbf{F}) \geq C_0 \operatorname{dist}^2(\mathbf{F}, SO(3))$.

Here, $DW(\mathbf{F}) := \frac{\partial W}{\partial F_{ij}}\mathbf{e}_i \otimes \mathbf{e}_j$ for all $\mathbf{F} \in \mathbb{R}^{3\times3}$. $\operatorname{dist}(\mathbf{F}, SO(3)) := \min_{\mathbf{Q}\in SO(d)} |\mathbf{F} - \mathbf{Q}|$.

Remark 5.2. The Assumption **(A4)** implies that

$$D^2W(\mathbb{I})\mathbf{F} : \mathbf{F} \geq C_1 \left|\operatorname{sym}\mathbf{F}\right|^2,$$

for some constant $C_1 > 0$, where $\operatorname{sym}\mathbf{F} := \frac{1}{2}(\mathbf{F} + \mathbf{F}^\top)$. Indeed, by the polar decomposition,

$$\mathbf{F} = \mathbf{RU},$$

for some rotation $\mathbf{R} \in SO(3)$ and positive-definite symmetric tensor $\mathbf{U} \in \mathbb{R}^{3\times3}_{\text{sym}+}$. Then it follows from e.g. [Ame21, Lemma 2.3.3] that

$$\operatorname{dist}^2(\mathbf{F}; SO(3)) = \left|\mathbf{F}^\top\mathbf{F} - \mathbb{I}\right|^2.$$

Substituting $\mathbf{F}$ by $\mathbb{I} + t\mathbf{F}$ for any $t > 0$ and using Assumption **(A4)**, one obtains

$$\begin{aligned}\frac{1}{C_0}W(\mathbb{I} + t\mathbf{F}) &\geq \operatorname{dist}^2(\mathbb{I} + t\mathbf{F}, SO(3)) \\ &= \left|2t\operatorname{sym}\mathbf{F} + t^2\mathbf{F}^\top\mathbf{F}\right|^2 = 4t^2\left|\operatorname{sym}\mathbf{F}\right|^2 + \mathcal{O}(t^3).\end{aligned} \tag{5.5}$$

By a Taylor expansion at $t = 0$, we have

$$\begin{aligned} W(\mathbb{I} + t\mathbf{F}) &= \underbrace{W(\mathbb{I}) + tDW(\mathbb{I})\mathbf{F}}_{=0 \text{ thanks to } \textbf{(A3)}} + \frac{t^2}{2}D^2W(\mathbb{I})\mathbf{F} : \mathbf{F} + \mathcal{O}(t^3) \\ &= \frac{t^2}{2}D^2W(\mathbb{I})\mathbf{F} : \mathbf{F} + \mathcal{O}(t^3). \end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6) together and passing to the limit $t \to 0$, one arrives at

$$D^2W(\mathbb{I})\mathbf{F} : \mathbf{F} \geq C_1 \left|\operatorname{sym} \mathbf{F}\right|^2.$$

Let $\mathbf{F} = \mathbf{a} \otimes \mathbf{b}$, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, then

$$\begin{aligned} D^2W(\mathbb{I})(\mathbf{a} \otimes \mathbf{b}) : (\mathbf{a} \otimes \mathbf{b}) &\geq \frac{C_1}{4} \left|\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a}\right|^2 \\ &= \frac{C_1}{4} \sum_{i,j=1}^{3} (a_ib_ja_ib_j + b_ia_ja_ib_j + a_ib_jb_ia_j + b_ia_jb_ia_j) \\ &= \frac{C_1}{4} \big(2\left|\mathbf{a}\right|^2 \left|\mathbf{b}\right|^2 + \underbrace{2(\mathbf{a} \cdot \mathbf{b})^2}_{\geq 0}\big) \geq \frac{C_1}{2} \left|\mathbf{a}\right|^2 \left|\mathbf{b}\right|^2, \end{aligned}$$

which shows the so-called *Legendre-Hadamard* condition for $-\operatorname{div} D^2W(\mathbb{I})\nabla$. Consequently, it is strongly normally elliptic, see e.g. [PS16, Page 271].

5.1.2. Technical discussions. Under the above setting, the fluid-structure part is of parabolic-elliptic type, while the cells part is similar to the ones in Chapters 3 and 4. To solve the nonlinear problem, our basic strategy is still a fixed-point argument in the framework of maximal regularity theory, while more issues come up when we consider the linearized systems. More precisely, for the linearization of the fluid-structure part, it is hard to solve it directly by the maximal regularity theory due to the parabolic-elliptic type coupling, which results in the unmatched regularity between $\hat{\mathbf{v}}_f$ and $\hat{\mathbf{u}}_s$ on the sharp interface Γ. To overcome the problem, one tries to decouple the system to a nonstationary Stokes equation with respect to fluid velocity $\hat{\mathbf{v}}_s$ and a quasi-stationary Stokes-type equation with regard to the solid displacement $\hat{\mathbf{u}}_s$. Note that one key point is to separate the kinetic and dynamic condition on the interface correctly. Specifically, we impose the Neumann boundary condition for $\hat{\mathbf{v}}_s$ and a Dirichlet boundary condition for $\hat{\mathbf{u}}_s$, see Section 5.3 below. Otherwise if $\hat{\mathbf{v}}_f|_\Gamma = \hat{\mathbf{v}}_s$, one may face the problem that there is no regularity information about the velocity of solid $\hat{\mathbf{v}}_s = \partial_t \hat{\mathbf{u}}_s$, since the solid equation is quasi-stationary without any damping.

Another issue is the choice of function spaces for the elastic equation, i.e., how to assemble suitable function spaces for the data in the linearized solid equation so that the regularity of $\hat{\nabla}\hat{\mathbf{u}}_s$ matches with $\hat{\nabla}\hat{\mathbf{v}}_s$ of fluid on the interface in the nonlinear system. Our choice is

$$\mathbf{f} \in H_q^{1/2}(0, T; W_{q,\Gamma}^{-1}(\Omega_s)^3) \cap L^q(0, T; L^q(\Omega_s)^3).$$

This space is motivated by the observation that the nonstationary Stokes equation (5.12) is uniquely solvable if the Neumann boundary data

$$\mathbf{h} \in W_q^{1/2-1/2q}(0, T; L^q(\Gamma)^3) \cap L^q(0, T; W_q^{1-1/q}(\Gamma)^3)$$

holds, which implies that

$$D^2W(\mathbb{I})\hat{\nabla}\hat{\mathbf{u}}_s \in W_q^{1/2-1/2q}(0, T; L^q(\Gamma)^3) \cap L^q(0, T; W_q^{1-1/q}(\Gamma)^3)$$

as well. In fact, the anisotropic Bessel potential space we assigned is a sharp regularity if one goes back to the anisotropic trace operator, see Lemma 2.25 in Chapter 2, and it is natural to equip $\mathbf{f}$ with the regularity above. Because of this sharp regularity setting, one can not expect the Lipschitz estimates of nonlinear terms only with small time, which gives rise to an additional smallness assumption (5.9) on the initial solid displacement and pressure. Detailed discussion can be found in Remark 5.4 and Proposition 5.22 later.

To solve the quasi-stationary (linearized) elastic equation, one treated it as a Stokes-type problem with respect to the displacement $\hat{\mathbf{u}}_s$ and the pressure $\hat{\pi}_s$, due to the incompressibility. However, as we assigned the certain regularity space for it as above and the Stokes operator is not a standard one (namely, $\operatorname{div}(D^2W(\mathbb{I})\nabla\cdot)$), one needs to consider a generalized stationary Stokes equation with $\mathbf{f}$ in L^q and W^{-1}_{q,Γ_1} respectively, for which the maximal regularity of analytic C_0 semigroups is applied, as well as a complex interpolation method with a *very weak solution* in L^q of a mixed-boundary Stokes-type equation, which can be solved by a duality argument.

For the cells part there is a problem of the positivity for the concentrations, compared to Chapter 3. The idea in Chapter 3 to prove it is to apply the maximum principle to the original equation and deduce a contradiction with the help of Hopf's Lemma. However, due to the lack of regularity of $\mathbf{v}_s = \partial_t \mathbf{u}_s$, one can not expect it to be Hölder continuous in space-time, even continuous. To deal with this trouble, we make use of the idea of mollification, i.e., approximating $\mathbf{v}_s$ by sufficient smooth functions $\mathbf{v}_s^\epsilon$ such that $\int_0^t \mathbf{v}_s^\epsilon \to \mathbf{u}_s$ in certain space. Then arguing by a similar procedure in Chapter 3, we obtain an approximate nonnegative solution c^ϵ. Finally one can show that it converges to a nonnegative function c, which exactly satisfies the original equations of cell concentrations.

5.1.3. Structure of the chapter. The paper is organized as follows. A reformulation of the system is done in Section 5.2.1 and later we give the main result for the reformulated system. Section 5.3 is devoted to three linearized systems in Section 5.3.1 and 5.3.2 respectively. The main results of this section are the L^q-solvability for these linear problems, for which a careful analysis is carried out. In Section 5.4, we first introduce some preliminary Lemmata in Section 5.4.1, which will be frequently used in proving the Lipschitz estimates later in Section 5.4.2. Then by the Banach Fixed-Point Theorem, we derive the short time existence of strong solutions to the nonlinear system in Section 5.4.3. Moreover, the cell concentrations are shown to be nonnegative, provided that the initial concentration is nonnegative. In addition, we establish the solvability of a Stokes-type resolvent problem with mixed boundary condition in Section 5.5.

5.2. Reformulation and Main Result

5.2.1. System in Lagrangian coordinates. In this section, we transform (5.1) in deformed domain Ω^t to the reference domain Ω, whose definition are given in Section 5.1.1.

Similar to Chapters 3 and 4, the reformulated system now reads as:

$$\hat{\rho}_f \partial_t \hat{\mathbf{v}}_f - \widehat{\operatorname{div}}\, \mathbb{P}_f = 0, \quad \hat{\mathbf{F}}_f^{-\top} : \hat{\nabla} \hat{\mathbf{v}}_f = 0 \quad \text{in } \Omega_f \times (0,T), \tag{5.7a}$$

$$\partial_t \hat{c}_f - \hat{D}_f\, \widehat{\operatorname{div}} \left(\hat{\mathbf{F}}_f^{-1} \hat{\mathbf{F}}_f^{-\top} \hat{\nabla} \hat{c}_f \right) = 0 \quad \text{in } \Omega_f \times (0,T), \tag{5.7b}$$

$$-\widehat{\operatorname{div}}\, \mathbb{P}_s = 0, \quad \hat{\mathbf{F}}_s^{-\top} : \hat{\nabla} \hat{\mathbf{u}}_s - \int_0^t \frac{\gamma\beta}{\hat{\rho}_s} \hat{c}_s \,\mathrm{d}\tau = 0 \quad \text{in } \Omega_s \times (0,T), \tag{5.7c}$$

$$\partial_t \hat{c}_s - \hat{D}_s \hat{J}_s^{-1}\, \widehat{\operatorname{div}} \left(\hat{J}_s \hat{\mathbf{F}}_s^{-1} \hat{\mathbf{F}}_s^{-\top} \hat{\nabla} \hat{c}_s \right) + \beta \hat{c}_s \big(1 + \frac{\gamma}{\hat{\rho}_s} \hat{c}_s\big) = 0 \quad \text{in } \Omega_s \times (0,T), \tag{5.7d}$$

$$\partial_t \hat{c}_s^* - \beta \hat{c}_s + \frac{\gamma\beta}{\hat{\rho}_s} \hat{c}_s \hat{c}_s^* = 0, \quad \partial_t \hat{g} - \frac{\gamma\beta}{3\hat{\rho}_s} \hat{c}_s \hat{g} = 0 \quad \text{in } \Omega_s \times (0,T), \tag{5.7e}$$

$$\llbracket \hat{\mathbf{v}} \rrbracket = 0, \quad \llbracket \mathbb{P} \rrbracket \hat{\mathbf{n}}_\Gamma = 0 \quad \text{on } \Gamma \times (0,T), \tag{5.7f}$$

$$\left\llbracket \hat{D}\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}^{-\top}\hat{\nabla}\hat{c} \right\rrbracket \hat{\mathbf{n}}_\Gamma = 0, \; \zeta \llbracket \hat{c} \rrbracket - \hat{D}_s \hat{\mathbf{F}}_s^{-1}\hat{\mathbf{F}}_s^{-\top}\hat{\nabla}\hat{c}_s \cdot \hat{\mathbf{n}}_\Gamma = 0 \quad \text{on } \Gamma \times (0,T), \tag{5.7g}$$

$$\mathbb{P}_s \hat{\mathbf{n}}_{\Gamma_s} = 0, \; \hat{D}_s \hat{\mathbf{F}}_s^{-1}\hat{\mathbf{F}}_s^{-\top}\hat{\nabla}\hat{c}_s \cdot \hat{\mathbf{n}}_{\Gamma_s} = 0 \quad \text{on } \Gamma_s \times (0,T), \tag{5.7h}$$

$$\hat{\mathbf{v}}_f|_{t=0} = \hat{\mathbf{v}}_f^0, \quad \hat{c}_f|_{t=0} = \hat{c}_f^0 \quad \text{in } \Omega_f, \tag{5.7i}$$

$$\hat{\mathbf{u}}_s|_{t=0} = \hat{\mathbf{u}}_s^0, \quad \hat{c}_s|_{t=0} = \hat{c}_s^0, \quad \hat{c}_s^*|_{t=0} = \hat{c}_*^0, \quad \hat{g}|_{t=0} = \hat{g}^0 \quad \text{in } \Omega_s, \tag{5.7j}$$

where $\mathbb{P}_i := \hat{J}_i \hat{\mathbb{T}}_i \hat{\mathbf{F}}_i^{-\top}$, $i \in \{f,s\}$, denotes the first Piola–Kirchhoff stress tensor associated with the Cauchy stress tensor $\mathbb{T}_i$ defined in Section 1.4.1.

5.2.2. Compatibility condition and well-posedness. Before stating our main theorem, one still needs to impose suitable function spaces and compatibility conditions. Following the general setting of maximal regularity, e.g. Chapters 3 and 4 or [AL23a, AL23b, PS16], where the basic space is $L^q(\Omega)$, we assume that

$$\hat{\mathbf{v}}_f^0 \in B_{q,q}^{2-\frac{2}{q}}(\Omega_f)^3 =: \mathcal{D}_q^1, \quad \hat{c}^0 \in B_{q,q}^{2-\frac{2}{q}}(\Omega\backslash\Gamma) =: \mathcal{D}_q^2, \quad \hat{c}_*^0, \hat{g}^0 \in W_q^1(\Omega_s),$$

and $\mathcal{D}_q := \mathcal{D}_q^1 \times \mathcal{D}_q^2$. Moreover, the solution space are defined by $Y_T := \prod_{j=0}^7 Y_T^j$, where

$$Y_T^1 := W_q^1(0,T;L^q(\Omega_f)^3) \cap L^q(0,T;W_q^2(\Omega_f)^3),$$
$$Y_T^2 := H_q^{\frac{1}{2}}(0,T;W_q^1(\Omega_s)^3) \cap L^q(0,T;W_q^2(\Omega_s)^3),$$
$$Y_T^3 := \left\{ \begin{aligned} &\pi \in L^q(0,T;W_q^1(\Omega_f)) : \\ &\qquad \pi|_\Gamma \in W_q^{\frac{1}{2}-\frac{1}{2q}}(0,T;L^q(\Gamma)) \cap L^q(0,T;W_q^{1-\frac{1}{q}}(\Gamma)) \end{aligned} \right\},$$
$$Y_T^4 := L^q(0,T;W_q^1(\Omega_s)) \cap H_q^{\frac{1}{2}}(0,T;L^q(\Omega_s)),$$
$$Y_T^5 := W_q^1(0,T;L^q(\Omega)) \cap L^q(0,T;W_q^2(\Omega\backslash\Gamma)),$$
$$Y_T^6 := W_q^1(0,T;W_q^1(\Omega_s)), \quad Y_T^7 := W_q^1(0,T;W_q^1(\Omega_s)).$$

Analogous to Chapters 3 and 4, the compatibility conditions for $\hat{\mathbf{v}}_f^0$ and $\hat{c}^0$ read as

$$\begin{gathered} \operatorname{div} \hat{\mathbf{v}}_f^0 = 0, \quad \hat{\mathbf{v}}_f^0\big|_\Gamma = 0, \\ \left(\zeta \llbracket \hat{c}^0 \rrbracket - \hat{D}_s \hat{\nabla}\hat{c}_s^0 \cdot \hat{\mathbf{n}}_\Gamma\right)\Big|_\Gamma = 0, \quad \left\llbracket \hat{D}\hat{\nabla}\hat{c}^0 \right\rrbracket \cdot \hat{\mathbf{n}}_\Gamma\Big|_\Gamma = 0, \quad \hat{D}_s \hat{\nabla}\hat{c}_s^0 \cdot \hat{\mathbf{n}}_{\Gamma_s}\Big|_{\Gamma_s} = 0, \end{gathered} \tag{5.8}$$

Generally speaking, one does not need to assign any initial pressure for the Stokes equation. However, in this chapter the coupling on the interface does lead to a condition on the initial fluid pressure since the solid equation is quasi-stationary and holds at $t = 0$. More specifically, we assume that there exists $\hat{\pi}_f^0 \in W_q^{1-3/q}(\Gamma)$ and $(\hat{\mathbf{u}}_s^0, \hat{\pi}_s^0) \in W_q^{2-2/q}(\Omega_s)^3 \times W_q^{1-2/q}(\Omega_s)$ satisfying

$$\left\| \hat{\nabla}\hat{\mathbf{u}}_s^0 \right\|_{W_q^{1-\frac{2}{q}}(\Omega_s)} + \left\| \hat{\pi}_s^0 \right\|_{W_q^{1-\frac{2}{q}}(\Omega_s)} \le \kappa, \tag{5.9}$$

for sufficiently small $\kappa > 0$, such that

$$\begin{aligned} -\widehat{\operatorname{div}}(DW(\mathbb{I} + \hat{\nabla}\hat{\mathbf{u}}_s^0)) + \hat{\nabla}\hat{\pi}_s^0 &= 0, && \text{in } \Omega_s, \\ \widehat{\operatorname{div}}\, \hat{\mathbf{u}}_s^0 &= 0, && \text{in } \Omega_s, \\ \left(-\hat{\pi}_s^0\mathbb{I} + DW(\mathbb{I} + \hat{\nabla}\hat{\mathbf{u}}_s^0)\right)\hat{\mathbf{n}}_\Gamma &= \left(-\hat{\pi}_f^0\mathbb{I} + \nu_f(\hat{\nabla}\hat{\mathbf{v}}_f^0 + (\hat{\nabla}\hat{\mathbf{v}}_f^0)^\top)\right)\hat{\mathbf{n}}_\Gamma, && \text{on } \Gamma, \\ \left(-\hat{\pi}_s^0\mathbb{I} + DW(\mathbb{I} + \hat{\nabla}\hat{\mathbf{u}}_s^0)\right)\hat{\mathbf{n}}_{\Gamma_s} &= 0, && \text{on } \Gamma_s. \end{aligned} \tag{5.10}$$

Remark 5.3. Here, the regularity for $\hat{\pi}_f^0$ on the interface Γ is initiated from the matched regularity of $\hat{\nabla}\hat{\mathbf{v}}_f^0$, $\hat{\pi}_s^0$ and $DW(\mathbb{I}+\hat{\nabla}\hat{\mathbf{u}}_s^0)$ on Γ. Moreover, it coincides with the regularity of $\hat{\pi}_f$ by the trace method of interpolation (see e.g. [PS16, Example 3.4.9]), i.e.,

$$W_q^{\frac{1}{2}-\frac{1}{2q}}(0,T;L^q(\Gamma)) \cap L^q(0,T;W_q^{1-\frac{1}{q}}(\Gamma)) \hookrightarrow C([0,T];W_q^{1-\frac{3}{q}}(\Gamma)).$$

Remark 5.4. In this paper, we need the smallness assumption of initial displacement to guarantee the estimates with respect to the deformation gradient, e.g. (5.24), which is a key element to derive the final contraction property of the certain operator. This is because we consider the general case of $\hat{\mathbf{u}}_s|_{t=0}$ and linearize the elastic equation around the identity $\mathbb{I}$, not the initial deformation gradient $\mathbb{I}+\hat{\nabla}\hat{\mathbf{u}}_s^0$. Specifically, one can not control $(\hat{\mathbf{F}}_s-\mathbb{I})$ by a small constant only with a short time. In particular, for the case $\hat{\mathbf{u}}_s^0=0$, one knows $\hat{\mathbf{F}}_s|_{t=0}=\mathbb{I}$ and hence the estimates later is uniform with respect to time $T>0$. Moreover, for initial pressure it does also need the smallness due to the sharp regularity of pressure, see e.g. (5.32).

Theorem 5.5. *Let $5<q<\infty$ and $\kappa>0$ be a sufficiently small constant. $\Omega\subset\mathbb{R}^3$ is the domain defined above with Γ, Γ_s hypersurfaces of class C^3. Assume that $(\hat{\mathbf{v}}_f^0,\hat{c}^0)\in\mathcal{D}_q$ satisfying the compatibility condition (5.8), $\hat{\pi}_f^0\in W_q^{1-3/q}(\Gamma)$, $\hat{c}_*^0,\hat{g}^0\in W_q^1(\Omega_s)$ and $(\hat{\mathbf{u}}_s^0,\hat{\pi}_s^0)\in W_q^{2-2/q}(\Omega_s)^3\times W_q^{1-2/q}(\Omega_s)$ fulfilling (5.9) and (5.10). Then there is a positive $T_0=T_0(\hat{\mathbf{v}}_f^0,\hat{c}^0,\hat{c}_*^0,\hat{g}^0,\kappa)<\infty$ such that for $0<T<T_0$, the problem (5.7) admits a unique solution $(\hat{\mathbf{v}}_f,\hat{\mathbf{u}}_s,\hat{\pi}_f,\hat{\pi}_s,\hat{c},\hat{c}_s^*,\hat{g})\in Y_T$. Moreoever, $\hat{c},\hat{c}_s^*,\hat{g}\geq 0$ if $\hat{c}^0,\hat{c}_*^0,\hat{g}^0\geq 0$.*

Motivated by [AL23a, AL23b, PS16], we prove Theorem 5.5 via the Banach fixed point theorem. To be more precise, we are going to linearize (5.7) in the first step, show the well-posedness of the linear system, estimate the nonlinear terms in suitable function spaces with small time and then constract a contraction mapping.

Remark 5.6. In fact, Theorem 5.5 still holds true in even more general dimensional case $n\geq 2$ as long as q has a adapted restriction with respect to n. This is also an advantage of making use of maximal regularity theory.

5.2.3. Linearization. Now following the linearization procedure in Chapter 3, we linearize (5.7) first, equate all the lower-order terms to the right-hand side and then arrive at the equivalent system:

$$\hat{\rho}_f\partial_t\hat{\mathbf{v}}_f-\widehat{\operatorname{div}}\,\mathbf{S}(\hat{\mathbf{v}}_f,\hat{\pi}_f)=\mathbf{K}_f \quad \text{in } \Omega_f\times(0,T), \tag{5.11a}$$

$$\widehat{\operatorname{div}}\,\hat{\mathbf{v}}_f=G_f \quad \text{in } \Omega_f\times(0,T), \tag{5.11b}$$

$$\mathbf{S}(\hat{\mathbf{v}}_f,\hat{\pi}_f)\hat{\mathbf{n}}_\Gamma-(D^2W(\mathbb{I})\hat{\nabla}\mathbf{u}_s-\hat{\pi}_s\mathbb{I})\hat{\mathbf{n}}_\Gamma=\mathbf{H}_f^1 \quad \text{on } \Gamma\times(0,T), \tag{5.11c}$$

$$-\widehat{\operatorname{div}}(D^2W(\mathbb{I})\hat{\nabla}\mathbf{u}_s)+\hat{\nabla}\hat{\pi}_s=\mathbf{K}_s \quad \text{in } \Omega_s\times(0,T), \tag{5.11d}$$

$$\widehat{\operatorname{div}}\,\hat{\mathbf{u}}_s-\int_0^t\frac{\gamma\beta}{\hat{\rho}_s}\hat{c}_s\,\mathrm{d}\tau=G_s \quad \text{in } \Omega_s\times(0,T), \tag{5.11e}$$

$$\hat{\mathbf{u}}_s=\mathbf{H}_s^1 \quad \text{on } \Gamma\times(0,T), \tag{5.11f}$$

$$(D^2W(\mathbb{I})\hat{\nabla}\mathbf{u}_s-\hat{\pi}_s\mathbb{I})\hat{\mathbf{n}}_{\Gamma_s}=\mathbf{H}^2 \quad \text{on } \Gamma_s\times(0,T), \tag{5.11g}$$

$$\partial_t\hat{c}_f-\hat{D}_f\hat{\Delta}\hat{c}_f=F_f^1 \quad \text{in } \Omega_f\times(0,T), \tag{5.11h}$$

$$\hat{D}_f\hat{\nabla}\hat{c}_f\cdot\hat{\mathbf{n}}_\Gamma=F_f^2 \quad \text{on } \Gamma\times(0,T), \tag{5.11i}$$

$$\partial_t\hat{c}_s-\hat{D}_s\hat{\Delta}\hat{c}_s=F_s^1 \quad \text{in } \Omega_s\times(0,T), \tag{5.11j}$$

$$\hat{D}_s \hat{\nabla} \hat{c}_s \cdot \hat{\mathbf{n}}_\Gamma = F_s^2 \quad \text{on } \Gamma \times (0,T), \tag{5.11k}$$

$$\hat{D}_s \hat{\nabla} \hat{c}_s \cdot \hat{\mathbf{n}}_{\Gamma_s} = F^3 \quad \text{on } \Gamma_s \times (0,T), \tag{5.11l}$$

$$\partial_t \hat{c}_s^* + \beta(\frac{\gamma \hat{c}_*^0}{\hat{\rho}_s} - 1)\hat{c}_s = F^4, \quad \partial_t \hat{g} - \frac{\gamma\beta\hat{g}^0}{3\hat{\rho}_s}\hat{c}_s = F^5 \quad \text{in } \Omega_s \times (0,T), \tag{5.11m}$$

$$\hat{\mathbf{v}}_f|_{t=0} = \hat{\mathbf{v}}_f^0, \quad \hat{c}_f|_{t=0} = \hat{c}_f^0 \quad \text{in } \Omega_f, \tag{5.11n}$$

$$\hat{\mathbf{u}}_s|_{t=0} = \hat{\mathbf{u}}_s^0, \quad \hat{c}_s|_{t=0} = \hat{c}_s^0, \quad \hat{c}_s^*|_{t=0} = \hat{c}_*^0, \quad \hat{g}|_{t=0} = \hat{g}^0 \quad \text{in } \Omega_s, \tag{5.11o}$$

where $\mathbf{S}(\hat{\mathbf{v}}_f, \hat{\pi}_f) := -\hat{\pi}_f + \nu_f(\hat{\nabla}\hat{\mathbf{v}}_f + \hat{\nabla}\hat{\mathbf{v}}_f^\top)$ and

$$\begin{aligned}
&\mathbf{K}_f = \widehat{\operatorname{div}}\, \tilde{\mathbf{K}}_f, \quad \mathbf{K}_s = \widehat{\operatorname{div}}\, \tilde{\mathbf{K}}_s, \\
&G_f = -\left(\hat{\mathbf{F}}_f^{-\top} - \mathbb{I}\right) : \hat{\nabla}\hat{\mathbf{v}}_f, \quad G_s = -(\hat{\mathbf{F}}_s^{-\top} - \mathbb{I}) : \hat{\nabla}\hat{\mathbf{u}}_s \\
&\mathbf{H}_f^1 = -\tilde{\mathbf{K}}_f \hat{\mathbf{n}}_\Gamma + \tilde{\mathbf{K}}_s \hat{\mathbf{n}}_\Gamma, \quad \mathbf{H}_s^1 = \int_0^t \hat{\mathbf{v}}_f(X,\tau)\mathrm{d}\tau, \quad \mathbf{H}^2 = -\tilde{\mathbf{K}}_s \hat{\mathbf{n}}_{\Gamma_s}, \\
&F_f^1 = \widehat{\operatorname{div}}\, \tilde{F}_f, \quad F_s^1 = \widehat{\operatorname{div}}\, \tilde{F}_s - \beta\hat{c}_s\left(1 + \frac{\gamma}{\hat{\rho}_s}\hat{c}_s\right) - \frac{3\hat{\nabla}\hat{g}}{\hat{g}} \cdot \left(\hat{D}_s \hat{\mathbf{F}}_s^{-1}\hat{\mathbf{F}}_s^{-\top}\hat{\nabla}\hat{c}_s\right), \\
&F_f^2 = \hat{D}_s \nabla \hat{c}_s \cdot \hat{\mathbf{n}}_\Gamma - \left[\!\left[\tilde{F}\right]\!\right] \cdot \hat{\mathbf{n}}_\Gamma, \quad F_s^2 = \zeta\, [\![\hat{c}]\!] - \tilde{F}_s \cdot \hat{\mathbf{n}}_\Gamma, \\
&F^3 = -\tilde{F}_s \cdot \hat{\mathbf{n}}_{\Gamma_s}, \quad F^4 = -\frac{\gamma\beta}{\hat{\rho}_s}\hat{c}_s(\hat{c}_s^* - \hat{c}_*^0), \quad F^5 = -\frac{\gamma\beta}{3\hat{\rho}_s}\hat{c}_s\left(\hat{g} - \hat{g}^0\right),
\end{aligned}$$

with

$$\begin{aligned}
\tilde{\mathbf{K}}_f &= -\hat{\pi}_f(\hat{\mathbf{F}}_f^{-1} - \mathbb{I}) + \nu_f\big(\hat{\mathbf{F}}_f^{-1}\hat{\nabla}\hat{\mathbf{v}}_f + \hat{\nabla}\hat{\mathbf{v}}_f^\top \hat{\mathbf{F}}_f^{-\top}\big)(\hat{\mathbf{F}}_f^{-\top} - \mathbb{I}) \\
&\quad + \nu_f\big((\hat{\mathbf{F}}_f^{-1} - \mathbb{I})\hat{\nabla}\hat{\mathbf{v}}_f + \hat{\nabla}\hat{\mathbf{v}}_f^\top(\hat{\mathbf{F}}_f^{-\top} - \mathbb{I})\big), \\
\tilde{\mathbf{K}}_s &= -\hat{g}^3\hat{\pi}_s(\hat{\mathbf{F}}_s^{-1} - \mathbb{I}) - (\hat{g}^3 - (\hat{g}^0)^3)\hat{\pi}_s\mathbb{I} - ((\hat{g}^0)^3 - 1)\hat{\pi}_s\mathbb{I} \\
&\quad + DW(\hat{\mathbf{F}}_s)\big((\hat{g}^0)^2 - \hat{g}^2\big) + +DW(\hat{\mathbf{F}}_s)\big(1 - (\hat{g}^0)^2\big) + \hat{g}^2\big(DW(\hat{\mathbf{F}}_s) - DW(\hat{\mathbf{F}}_s/\hat{g})\big) \\
&\quad + \int_0^1 D^3W\big((1-s)\mathbb{I} + s\hat{\mathbf{F}}_s\big)(1-s)\mathrm{d}s(\hat{\mathbf{F}}_s - \mathbb{I})(\hat{\mathbf{F}}_s - \mathbb{I}), \\
\tilde{F} &= \hat{D}\big(\hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}^{-\top} - \mathbb{I}\big)\hat{\nabla}\hat{c}.
\end{aligned}$$

Remark 5.7 (Discussions on the linearization).

(1) The linearization can be derived as follows. Let $h(s) := DW((1-s)\mathbb{I} + s\mathbf{F})$. Then $h(0) = DW(\mathbb{I}), h(1) = DW(\mathbf{F})$. Since

$$h(1) = h(0) + h'(0) + \int_0^1 h''(s)(1-s)\,\mathrm{d}s,$$

it follows from **(A3)** that

$$DW(\mathbf{F}) = D^2W(\mathbb{I})(\mathbf{F} - \mathbb{I}) + \mathbf{R}(\mathbf{F}),$$

where

$$\mathbf{R}(\mathbf{F}) := \int_0^1 D^3W((1-s)\mathbb{I} + s\mathbf{F})(1-s)\,\mathrm{d}s(\mathbf{F} - \mathbb{I})(\mathbf{F} - \mathbb{I}).$$

(2) The linearization is similar to the one in Chapter 3, but with several modifications, one of which is deduced above. It is possible to have other kinds of linearizations but we remark that in the present paper a divergence structure of $\widehat{\operatorname{div}}\, \tilde{\mathbf{K}}_s$ plays an essential role when we prove the linear theory and estimate it in a particular function space, see Corollary 5.11 and Proposition 5.22 later. Moreover, for the solid mass balance equation (5.1d), we integrate it over $(0,t)$ as (5.11e) to keep the Stokes-type structure for the elastic equation with respect to the displacement $\hat{\mathbf{u}}_s$.

(3) Noticing that the continuity conditions (5.7f) on the interface are separated to (5.11c) and (5.11f) formally after the linearization, we remark here that this is for the sake of analysis due to the mismatch of the regularity on Γ. For instant, if one replaces (5.11c) with the boundary condition $\hat{\mathbf{v}}_f = \partial_t \hat{\mathbf{u}}_s$, it has no chance to solve the fluid part since we have no first-order temporal derivative information for the solid displacement $\hat{\mathbf{u}}_s$.

5.3. Analysis of the Linear Systems

In this section, we are devoted to solve the linear systems associated with (5.11). Note that we already give the linear heat equation with Neumann boundary condition and ordinary differential equations in Sections 3.3.2 and 3.3.3, here we only consider a nonstationary Stokes equation and a quasi-stationary Stokes equation with mixed boundary conditions.

5.3.1. Nonstationary Stokes equation. Let Ω be a bounded domain with a boundary $\partial\Omega$ of class C^{3-}, $T > 0$. We consider the nonstationary Stokes equation

$$\begin{aligned} \rho\partial_t \mathbf{u} - \operatorname{div} S_\mu(\mathbf{u},\pi) &= \mathbf{f}, && \text{in } \Omega \times (0,T),\\ \operatorname{div} \mathbf{u} &= g, && \text{in } \Omega \times (0,T),\\ S_\mu(\mathbf{u},\pi)\mathbf{n} &= \mathbf{h}, && \text{on } \partial\Omega \times (0,T),\\ \mathbf{u}|_{t=0} &= \mathbf{u}_0, && \text{in } \Omega, \end{aligned} \tag{5.12}$$

where $S_\mu(\mathbf{u},\pi) = -\pi\mathbb{I} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)$. $\rho, \mu > 0$ are the constant density and viscosity. $\mathbf{n}$ denotes the unit outer normal vector on $\partial\Omega$. Then we have the following solvability and regularity result, which can be adapted directly from e.g. Abels [Abe10, Theorem 1.1], Bothe–Prüss [BP07, Theorem 4.1], Prüss–Simonett [PS16, Theorem 7.3.1] by the argument of Proposition 3.21.

Theorem 5.8. *Let $3 < q < \infty$, $T_0 > 0$. Suppose that the initial data is $\mathbf{u}_0 \in W_q^{2-2/q}(\Omega)^3$ satisfying compatibility conditions*

$$\operatorname{div} \mathbf{u}_0 = g|_{t=0}\,, \qquad \mathcal{P}_\mathbf{n}(\mu(\nabla\mathbf{u}_0 + \nabla\mathbf{u}_0^\top)\mathbf{n})\big|_{\partial\Omega} = \mathbf{h}|_{t=0}\,,$$

where $\mathcal{P}_\mathbf{n} := \mathbb{I} - \mathbf{n}\otimes\mathbf{n}$ denotes the tangential projection onto $\partial\Omega$. For given data $(\mathbf{f}, g, \mathbf{h})$ with

$$\begin{aligned} \mathbf{f} &\in \mathbb{F}_\mathbf{f}(T) := L^q(0,T;L^q(\Omega)^3),\\ g &\in \mathbb{F}_g(T) := L^q(0,T;W_q^1(\Omega)) \cap W_q^1(0,T;W_q^{-1}(\Omega)),\\ \mathbf{h} &\in \mathbb{F}_\mathbf{h}(T) := L^q(0,T;W_q^{1-\frac{1}{q}}(\partial\Omega)^3) \cap W_q^{\frac{1}{2}-\frac{1}{2q}}(0,T;L^q(\partial\Omega)^3), \end{aligned}$$

(5.12) admits a unique solution $(\mathbf{u},\pi) \in \mathbb{E}(T) := \mathbb{E}_\mathbf{u}(T) \times \mathbb{E}_\pi(T)$ where

$$\mathbb{E}_\mathbf{u}(T) := L^q(0,T;W_q^2(\Omega)^3) \cap W_q^1(0,T;L^q(\Omega)^3),$$

$$\mathbb{E}_\pi(T) := \left\{ L^q(0,T;W_q^1(\Omega)) : \pi|_{\partial\Omega} \in W_q^{\frac{1}{2}-\frac{1}{2q}}(0,T;L^q(\partial\Omega)) \cap L^q(0,T;W_q^{1-\frac{1}{q}}(\Omega)) \right\}.$$

Moreover, there is a constant $C > 0$ independent of $\mathbf{f}, g, \mathbf{h}, \mathbf{u}_0, T_0$, such that for $0 < T \leq T_0$

$$\|(\mathbf{u}, \pi)\|_{\mathbb{E}(T)} \leq C \left(\|\mathbf{f}\|_{\mathbb{F}_{\mathbf{f}}(T)} + \|g\|_{\mathbb{F}_g(T)} + \|\mathbf{h}\|_{\mathbb{F}_{\mathbf{h}}(T)} + \|\mathbf{u}_0\|_{W_q^{2-2/q}(\Omega)} \right).$$

Remark 5.9. In our case, there will be a term of the form $(D^2W(\mathbb{I})\nabla\mathbf{v} - p\mathbb{I})\mathbf{n}$ in the third equation of (5.12) with certain regularity. It is not a problem since given $(\mathbf{v}, p)$ such that $(D^2W(\mathbb{I})\nabla\mathbf{v} - p\mathbb{I})\mathbf{n}$ is endowed with the same regularity of $\mathbf{h}$, one can solve the original equation with $\mathbf{h} = (D^2W(\mathbb{I})\nabla\mathbf{v} - p\mathbb{I})\mathbf{n}$ and $(\mathbf{f}, g, \mathbf{u}_0) = 0$ by Theorem 5.8 and add the solution above to recover the case.

5.3.2. Quasi-stationary Stokes equation with mixed boundary conditions. Let Ω be a bounded domain with a boundary $\partial\Omega$ of class C^{3-}, $\partial\Omega = \Gamma_1 \cup \Gamma_2$ consisting of two closed, disjoint, nonempty components. Consider the generalized stationary Stokes-type equation

$$\begin{aligned} -\operatorname{div}(D^2W(\mathbb{I})\nabla\mathbf{u}) + \nabla\pi &= \mathbf{f}, && \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= g, && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{h}^1, && \text{on } \Gamma_1, \\ (D^2W(\mathbb{I})\nabla\mathbf{u} - \pi\mathbb{I})\mathbf{n} &= \mathbf{h}^2, && \text{on } \Gamma_2, \end{aligned} \tag{5.13}$$

where $\mathbf{n}$ denotes the unit outer normal vector on $\partial\Omega$. $W : \mathbb{R}^{3\times3} \to \mathbb{R}_+$ is the elastic energy density such that Assumption 5.1 holds. Before going to the quasi-stationary case, we first investigate the weak solution and strong solution in L^q-class of the stationary problem (5.13).

Theorem 5.10. *Let $1 < q < \infty$ and $s \in \{0, -1\}$. Given $\mathbf{f} \in W^s_{q,\Gamma_1}(\Omega)^3$, $g \in W_q^{1+s}(\Omega)$, $\mathbf{h}^1 \in W_q^{2+s-1/q}(\Gamma_1)^3$ and $\mathbf{h}^2 \in W_q^{1+s-1/q}(\Gamma_2)^3$. Then problem (5.13) admits a unique solution $(\mathbf{u}, \pi) \in W_q^{2+s}(\Omega)^3 \times W_q^{1+s}(\Omega)$. Moreover, there is a constant $C > 0$ such that*

$$\|\mathbf{u}\|_{W_q^{2+s}(\Omega)^3} + \|\pi\|_{W_q^{1+s}(\Omega)} \leq C \Big(\|g\|_{W_q^{1+s}(\Omega)} + \left\|\mathbf{h}^1\right\|_{W_q^{2+s-\frac{1}{q}}(\Gamma_1)^3} + \|\mathcal{F}\|_s \Big),$$

where $\|\mathcal{F}\|_s := \|\mathbf{f}\|_{L^q(\Omega)^3} + \left\|\mathbf{h}^2\right\|_{W_q^{1-\frac{1}{q}}(\Omega)^3}$ if $s = 0$ and when $s = -1$,

$$\|\mathcal{F}\|_s := \sup_{\|\mathbf{w}\|_{W^1_{q',\Gamma_1}(\Omega)^3} = 1} \Big(\langle \mathbf{f}, \mathbf{w} \rangle_{W^{-1}_{q,\Gamma_1}(\Omega)^3 \times W^1_{q',\Gamma_1}(\Omega)^3} + \left\langle \mathbf{h}^2, \mathbf{w}|_{\Gamma_2} \right\rangle_{W_q^{-\frac{1}{q}}(\Gamma_2)^3 \times W_{q'}^{1-\frac{1}{q'}}(\Gamma_2)^3} \Big).$$

Proof. First let $s = 0$, we reduce the system (5.13) to the case $(g, \mathbf{h}^1, \mathbf{h}^2) = 0$. To this end, take a cutoff funtion $\psi \in C_0^\infty((0, T))$ such that

$$\int_{T/4}^{3T/4} \psi(t)\, \mathrm{d}t = 1, \quad \text{in } [T/4, 3T/4].$$

Then

$$\begin{gathered} \psi(t)g \in L^p(0, T; W_q^1(\Omega)) \cap W_p^1(0, T; W^{-1}_{q,\Gamma_2}(\Omega)), \\ \psi(t)\mathbf{h}^j \in L^p(0, T; W_q^{3-j-\frac{1}{q}}(\Gamma_j)^3) \cap W_p^{\frac{1}{j}-\frac{1}{2q}}(0, T; L^q(\Gamma_j)^3),\ j = 1, 2. \end{gathered}$$

In view of Remark 5.2 and the maximal L^q-regularity result for the generalized Stokes problems (e.g., [BP07, Theorem 4.1], Prüss–Simonett [PS16, Theorem 7.3.1]), we solve the system

$$\begin{aligned} \partial_t \mathbf{u} - \operatorname{div}(D^2W(\mathbb{I})\nabla\mathbf{u}) + \nabla\pi = 0, &\quad \text{in } \Omega\times(0,T),\\ \operatorname{div}\mathbf{u} = \psi(t)g, &\quad \text{in } \Omega\times(0,T),\\ \mathbf{u} = \psi(t)\mathbf{h}^1, &\quad \text{on } \Gamma_1\times(0,T),\\ (D^2W(\mathbb{I})\nabla\mathbf{u} - \pi\mathbb{I})\mathbf{n} = \psi(t)\mathbf{h}^2, &\quad \text{on } \Gamma_2\times(0,T),\\ \mathbf{u}|_{t=0} = 0, &\quad \text{in } \Omega, \end{aligned}$$

with $3 < p < \infty$, $1 < q < \infty$ to get a pair of solution $(\tilde{\mathbf{u}}, \tilde{\pi})$ fulfilling

$$\tilde{\mathbf{u}} \in W^1_p(0,T;L^q(\Omega)^3) \cap L^p(0,T;W^2_q(\Omega)^3), \quad \tilde{\pi} \in L^p(0,T;W^1_q(\Omega)).$$

Then one infers

$$(\bar{\mathbf{u}}, \bar{\pi}) := \int_{T/4}^{3T/4} (\tilde{\mathbf{u}}, \tilde{\pi})(t)\,\mathrm{d}t \in W^2_q(\Omega)^3 \times W^1_q(\Omega),$$

and

$$\operatorname{div}\bar{\mathbf{u}} = g, \quad \text{in } \Omega, \quad \bar{\mathbf{u}}|_{\Gamma_1} = \mathbf{h}^1, \quad \text{on } \Gamma_1, \quad (D^2W(\mathbb{I})\nabla\bar{\mathbf{u}} - \bar{\pi}\mathbb{I})\mathbf{n}\big|_{\Gamma_2} = \mathbf{h}^2, \quad \text{on } \Gamma_2.$$

Subtracting the solution to (5.13) with $(\bar{\mathbf{u}}, \bar{\pi})$, we are in the position to solve (5.13) with $(g, \mathbf{h}^1, \mathbf{h}^2) = 0$, which can be referred to Theorem 5.24 with $\lambda = 0$. Note that the case $\lambda = 0$ is applicable due to Remark 5.25.

Now we consider $s = -1$, namely the weak solution. In this case we only reduce $(g, \mathbf{h}^1)$ to zero since the Neumann boundary trace need to make sense on Γ_2 correctly. Concerning the Stokes equation with Dirichlet boundary condition

$$\begin{aligned} -\Delta\mathbf{u} + \nabla\pi = 0, &\quad \text{in } \Omega,\\ \operatorname{div}\mathbf{u} = g, &\quad \text{in } \Omega,\\ \mathbf{u} = \mathbf{h}^1, &\quad \text{on } \Gamma_1,\\ \mathbf{u} = \mathbf{c}, &\quad \text{on } \Gamma_2, \end{aligned}$$

where $\mathbf{c} > 0$ is a constant such that

$$\int_\Omega g\,\mathrm{d}x = \int_{\Gamma_1} \mathbf{h}^1\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^2 + \int_{\Gamma_2} \mathbf{c}\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^2,$$

holds, where $\mathcal{H}^d$ with $d \in \mathbb{N}_+$ denotes the d-dimensional Hausdorff measure. It follows from the weak solution theory for stationary Stokes equation, see e.g. Galdi–Simader–Sohr [GSS05, Section 5, (5.12)] in Sobolev spaces, Schumacher [Sch09, Theorem 4.3] in weighted Bessel potential spaces, that one obtains a unique solution denoted by $(\bar{\mathbf{u}}, \bar{\pi})$ such that

$$(\bar{\mathbf{u}}, \bar{\pi}) \in W^1_q(\Omega)^3 \times L^q(\Omega),$$

and

$$\operatorname{div}\bar{\mathbf{u}} = g, \quad \text{in } \Omega, \quad \bar{\mathbf{u}}|_{\Gamma_1} = \mathbf{h}^1, \quad \text{on } \Gamma_1.$$

Then one can subtract the solution of (5.13) with $(\bar{\mathbf{u}}, \bar{\pi})$ and solve (5.13) with reduced data $(g, \mathbf{h}^1) = 0$ and modified $(f, \mathbf{h}^2)$ (not to be relabeled). The idea of the proof is to introduce a L^q-class of *very weak solution* (see e.g. [GSS05, Sch09]), so that one can derive a solution with certain regularity in $W^1_q(\Omega)$ by complex interpolation, see e.g. Schumacher [Sch09] for the stationary Stokes equation in fractional Bessel potential spaces.

Define the solenoidal space

$$L^q_\sigma(\Omega) := \left\{ \mathbf{u} \in L^q(\Omega)^3 : \operatorname{div} \mathbf{u} = 0,\, \mathbf{n} \cdot \mathbf{u}|_{\Gamma_1} = 0 \right\}.$$

For $1 < q, q' < \infty$ satisfying $1/q + 1/q' = 1$, we define a generalized Stokes-type operator with respect to (5.13) as

$$\mathcal{A}_q(\mathbf{u}) := \mathbb{P}_q\big(- \operatorname{div}(D^2W(\mathbb{I})\nabla \mathbf{u})\big) \text{ for all } \mathbf{u} \in \mathcal{D}(\mathcal{A}_q),$$

with

$$\mathcal{D}(\mathcal{A}_q) = \left\{ \mathbf{u} \in W^2_q(\Omega)^3 \cap L^q_\sigma(\Omega) : \mathbf{u}|_{\Gamma_1} = 0,\ \mathcal{P}_{\mathbf{n}}((D^2W(\mathbb{I})\nabla \mathbf{u})\mathbf{n})\big|_{\Gamma_2} = 0 \right\},$$

where $\mathbb{P}_q$ denotes the *Helmholtz–Weyl projection* onto $L^q_\sigma(\Omega)$, see e.g. [Abe10, Appendix A] for the existence of the projection with mixed boundary conditions. $\mathcal{P}_{\mathbf{n}} := \mathbb{I} - \mathbf{n} \otimes \mathbf{n}$ is the tangential projection onto $\partial\Omega$. By the result of $s = 0$ we see that

$$\mathcal{A}_q : \mathcal{D}(\mathcal{A}_q) \to L^q_\sigma(\Omega)$$

is well-defined and bijective. Then one knows that its dual operator

$$\mathcal{A}^*_{q'} : L^{q'}_\sigma(\Omega)' \to \mathcal{D}(\mathcal{A}_{q'})'$$

is bijective as well, which gives rise to the existence of very weak solutions. Note that $\mathcal{A}_q$ and $\mathcal{A}^*_{q'}$ are consistent, namely,

$$\begin{aligned} \left\langle \mathcal{A}^*_{q'} \mathbf{u}, \mathbf{w} \right\rangle_{\mathcal{D}(\mathcal{A}_{q'})' \times \mathcal{D}(\mathcal{A}_{q'})} &= \langle \mathbf{u}, \mathcal{A}_{q'} \mathbf{w} \rangle_{L^q_\sigma(\Omega) \times L^{q'}_\sigma(\Omega)} \\ &= \int_\Omega \nabla \mathbf{u} : D^2W(\mathbb{I}) \nabla \mathbf{w} \,\mathrm{d}x = \int_\Omega D^2W(\mathbb{I}) \nabla \mathbf{u} : \nabla \mathbf{w} \,\mathrm{d}x = \langle \mathcal{A}_q \mathbf{u}, \mathbf{w} \rangle_{L^q_\sigma(\Omega) \times L^{q'}_\sigma(\Omega)}, \end{aligned}$$

for $\mathbf{u} \in \mathcal{D}(\mathcal{A}_q) \subseteq L^q_\sigma(\Omega)$, $\mathbf{w} \in \mathcal{D}(\mathcal{A}_{q'}) \subseteq L^{q'}_\sigma(\Omega)$, where $(D^2W(\mathbb{I}))^{kl}_{ij} = (D^2W(\mathbb{I}))^{ij}_{kl}$, $i, j, k, l = 1, 2, 3$. Then by the complex interpolation of operators, e.g. [Sch09, Theorem 2.6], we record that

$$\mathcal{A}_q : \big(L^q_\sigma(\Omega), \mathcal{D}(\mathcal{A}_q)\big)_{[\frac{1}{2}]} \to \big(L^q_\sigma(\Omega), \mathcal{D}(\mathcal{A}_{q'})'\big)_{[\frac{1}{2}]}$$

is bijective. Since $\mathcal{A}_q$ admits a bounded $\mathcal{H}^\infty$-calculus and has bounded imaginary powers, see e.g. [Prü18, Theorem 1.1], complex interpolation methods can be used to describe domains of fractional power operators. By virtue of [Prü18, Theorem 1.1] and [Sch09, Theorem 2.6], one obtains

$$\begin{aligned} \big(L^q_\sigma(\Omega), \mathcal{D}(\mathcal{A}_q)\big)_{[\frac{1}{2}]} &= \mathcal{D}(\mathcal{A}_q^{1/2}) = W^{1,q}_{\sigma,\Gamma_1}(\Omega), \\ \big(L^q_\sigma(\Omega), \mathcal{D}(\mathcal{A}_{q'})'\big)_{[\frac{1}{2}]} = (L^q_\sigma(\Omega)', \mathcal{D}(\mathcal{A}_{q'}))'_{[\frac{1}{2}]} &= \mathcal{D}(\mathcal{A}_{q'}^{1/2})' = W^{-1,q}_{\sigma,\Gamma_1}(\Omega). \end{aligned}$$

Consequently,

$$\mathcal{A}_q : W^{1,q}_{\sigma,\Gamma_1}(\Omega) \to W^{-1,q}_{\sigma,\Gamma_1}(\Omega)$$

is bijective, which implies there exists a unique solution $\mathbf{u} \in W^{1,q}_{\sigma,\Gamma_1}(\Omega)$ such that

$$\langle \mathcal{A}_q \mathbf{u}, \mathbf{w} \rangle = \langle \mathcal{F}, \mathbf{w} \rangle \text{ for all } \mathbf{w} \in W^{1,q'}_{\sigma,\Gamma_1}(\Omega),$$

with $\mathcal{F} \in W^{-1,q}_{\sigma,\Gamma_1}(\Omega)$ defined by

$$\langle \mathcal{F}, \mathbf{w} \rangle := \langle \mathbf{f}, \mathbf{w} \rangle_{W^{-1}_{q,\Gamma_1}(\Omega)\times W^{1}_{q',\Gamma_1}(\Omega)} + \left\langle \mathbf{h}^2, \mathbf{w}|_{\Gamma_2} \right\rangle_{W_q^{-\frac{1}{q}}(\Gamma_2)\times W_{q'}^{1-\frac{1}{q'}}(\Gamma_2)},$$

for all $\mathbf{w} \in W^{1,q'}_{\sigma,\Gamma_1}(\Omega)$. Moreover, by means of the open mapping theorem, one immediately deduces the estimate

$$\|\mathbf{u}\|_{W^{1,q}_{\sigma,\Gamma_1}(\Omega)^3} \le C\, \|\!|\mathcal{F}|\!\|_{-1},$$

in which

$$\|\!|\mathcal{F}|\!\|_{-1} := \sup_{\|\mathbf{w}\|_{W^{1}_{q',\Gamma_1}(\Omega)^3}=1} \Big(\langle \mathbf{f}, \mathbf{w} \rangle_{W^{-1}_{q,\Gamma_1}(\Omega)^3\times W^{1}_{q',\Gamma_1}(\Omega)^3} + \left\langle \mathbf{h}^2, \mathbf{w}|_{\Gamma_2} \right\rangle_{W_q^{-\frac{1}{q}}(\Gamma_2)^3\times W_{q'}^{1-\frac{1}{q'}}(\Gamma_2)^3} \Big).$$

Up to now, one still needs to recover the pressure in the very weak sense, i.e., solving

$$\int_\Omega \pi \Delta\varphi \,\mathrm{d}x = \langle F, \varphi \rangle, \quad \forall\, \varphi \in \mathcal{D}(\Delta_{q',DN}), \tag{5.14}$$

where

$$\langle F, \varphi \rangle := -\langle \mathbf{f}, \nabla\varphi \rangle + \int_\Omega D^2W(\mathbb{I})\nabla\mathbf{u} : \nabla^2\varphi + \left\langle \mathbf{h}^2\cdot\mathbf{n}, \partial_{\mathbf{n}}\varphi|_{\Gamma_2} \right\rangle,$$
$$\mathcal{D}(\Delta_{q',DN}) := \left\{ \psi \in W^2_{q'}(\Omega) : \partial_{\mathbf{n}}\psi|_{\Gamma_1} = 0,\ \psi|_{\Gamma_2} = 0 \right\}.$$

Since $\mathbf{f} \in W^{-1}_{q,\Gamma_1}(\Omega)^3$, $\mathbf{u} \in W^{1}_{q,\Gamma_1}(\Omega)^3$ and $\mathbf{h}^2 \in W_q^{-1/q}(\Gamma_2)^3$, it is easy to verify that the functional F defined above is well-defined in $\mathcal{D}(\Delta_{q',DN})'$. For every $u \in L^{q'}(\Omega)$, it follows from [PS16, Corollary 7.4.5] that there exists a unique solution $\varphi(u) \in \mathcal{D}(\Delta_{q',DN})$ satisfying $\Delta\varphi = u$. Now we define $\pi \in L^q(\Omega)$ by duality as a linear functional on $L^{q'}(\Omega)$ acting for every u as

$$\langle \pi, u \rangle = \langle F, \varphi \rangle. \tag{5.15}$$

Indeed π is the very weak solution we are looking for, since for all $\varphi \in \mathcal{D}(\Delta_{q',DN})$ we have

$$\langle \pi, \Delta\varphi \rangle = \langle \pi, u \rangle = \langle F, \varphi \rangle.$$

The uniqueness can be showed by letting $F = 0$ in (5.15) so that for all $u \in L^{q'}(\Omega)$, $\langle \pi, u \rangle = 0$, which implies $\pi = 0$ a.e. in Ω. Then we have the estimate

$$\|\pi\|_{L^q(\Omega)} \le C\, \|F\|_{\mathcal{D}(\Delta_{q',DN})'} \le C\Big(\|\mathbf{u}\|_{W^1_q(\Omega)^3} + \|\!|\mathcal{F}|\!\|_{-1} \Big).$$

This completes the proof. □

In Theorem 5.10, we consider the general case of data. In fact, for applications the right-hand side terms sometimes have special structure, which is of much help to derive the estimate in a concise form.

Corollary 5.11. *In the case of $s = -1$, if there is an $\mathbf{F} \in L^q(\Omega)^{3\times3}$ such that*

$$\mathbf{f} = \operatorname{div}\mathbf{F}, \ \text{in } \Omega, \quad \mathbf{h}^2 = -\mathbf{F}\mathbf{n}, \ \text{on } \Gamma_2$$

holds in the sense of distribution, i.e.,

$$\langle \mathbf{f}, \mathbf{w} \rangle_{W^{-1}_{q,\Gamma_1}(\Omega)^3\times W^{1}_{q',\Gamma_1}(\Omega)^3} + \left\langle \mathbf{h}^2, \mathbf{w}|_{\Gamma_2} \right\rangle_{W_q^{-\frac{1}{q}}(\Gamma_2)^3\times W_{q'}^{1-\frac{1}{q'}}(\Gamma_2)^3} = \langle \mathbf{F}, \nabla\mathbf{w} \rangle, \tag{5.16}$$

for all $\mathbf{w} \in W^{1,q'}_{\sigma,\Gamma_1}(\Omega)$*, then the solution* $(\mathbf{u}, \pi)$ *in Theorem 5.10 satisfies*

$$\|\mathbf{u}\|_{W^1_q(\Omega)^3} + \|\pi\|_{L^q(\Omega)} \leq C\Big(\|g\|_{L^q(\Omega)} + \big\|\mathbf{h}^1\big\|_{W_q^{1-\frac{1}{q}}(\Gamma_1)^3} + \|\mathbf{F}\|_{L^q(\Omega)^{3\times 3}} \Big).$$

Proof. On account of (5.16) and the definition of $|||\mathcal{F}|||_{-1}$ above with respect to $(\mathbf{f}, \mathbf{h}^2)$, one has

$$|||\mathcal{F}|||_{-1} = \|\mathbf{F}\|_{L^q(\Omega)^{3\times 3}},$$

which yields the desired estimate. □

Remark 5.12. In fact, Theorem 5.10 can be generalize to $s \in (-2, 0)$ by employing complex interpolation. Namely, since $\mathcal{A}_q$ admits bounded imaginary powers, we have the domains of any fractional powers by complex interpolation

$$\mathcal{D}(\mathcal{A}_q^\theta) = \big(L^q_\sigma(\Omega), \mathcal{D}(\mathcal{A}_q)\big)_{[\theta]}, \quad 0 < \theta < 1.$$

More details can be found in e.g. [Abe10, Prü18, Sch09].

Combining with the temporal regularities, Theorem 5.10 and Corollary 5.11, one arrives at the following theorem.

THEOREM 5.13. *Let* $1 < q < \infty$ *and* $T_0 > 0$*. Given* $(\mathbf{f}, g, \mathbf{h}^1, \mathbf{h}^2)$ *such that*

$$\begin{aligned}
\mathbf{f} \in \mathbb{F}_{\mathbf{f}}(T) &:= L^q(0,T; L^q(\Omega)^3) \cap H_q^{\frac{1}{2}}(0,T; W^{-1}_{q,\Gamma_1}(\Omega)^3),\\
g \in \mathbb{F}_g(T) &:= L^q(0,T; W^1_q(\Omega)) \cap H_q^{\frac{1}{2}}(0,T; L^q(\Omega)),\\
\mathbf{h}^1 \in \mathbb{F}_{\mathbf{h}^1}(T) &:= L^q(0,T; W_q^{2-\frac{1}{q}}(\Gamma_1)^3) \cap H_q^{\frac{1}{2}}(0,T; W_q^{1-\frac{1}{q}}(\Gamma_1)^3),\\
\mathbf{h}^2 \in \mathbb{F}_{\mathbf{h}^2}(T) &:= L^q(0,T; W_q^{1-\frac{1}{q}}(\Gamma_2)^3) \cap H_q^{\frac{1}{2}}(0,T; W_q^{-\frac{1}{q}}(\Gamma_2)^3).
\end{aligned}$$

Then (5.13) *admits a unique solution* $(\mathbf{u}, \pi) \in \mathbb{E}(T) := \mathbb{E}_{\mathbf{u}}(T) \times \mathbb{E}_\pi(T)$ *where*

$$\begin{aligned}
\mathbb{E}_{\mathbf{u}}(T) &:= L^q(0,T; W^2_q(\Omega)^3) \cap H_q^{\frac{1}{2}}(0,T; W^1_q(\Omega)^3),\\
\mathbb{E}_\pi(T) &:= L^q(0,T; W^1_q(\Omega)) \cap H_q^{\frac{1}{2}}(0,T; L^q(\Omega)).
\end{aligned}$$

If additionally, there is an $\mathbf{F} \in L^q(0,T; W^1_q(\Omega)^{3\times 3}) \cap H_q^{1/2}(0,T; L^q(\Omega)^{3\times 3})$ *such that*

$$\mathbf{f} = \operatorname{div}\mathbf{F}, \ \text{in } \Omega, \quad \mathbf{h}^2 = -\mathbf{F}\mathbf{n}, \ \text{on } \Gamma_2$$

holds in the sense of distribution. Then there is a constant $C > 0$ *independent of* $\mathbf{f}, g, \mathbf{h}^1, \mathbf{h}^2, T$*, such that for* $0 < T < \infty$

$$\|(\mathbf{u}, \pi)\|_{\mathbb{E}(T)} \leq C\Big(\|\mathbf{F}\|_{L^q(0,T; W^1_q(\Omega)^{3\times 3}) \cap H_q^{\frac{1}{2}}(0,T; L^q(\Omega)^{3\times 3})} + \|g\|_{\mathbb{F}_g(T)} + \big\|\mathbf{h}^1\big\|_{\mathbb{F}_{\mathbf{h}^1}(T)} \Big).$$

Now given $\gamma > 0$ and $c \in L^q(0,T; W^2_q(\Omega)) \cap W^1_q(0,T; L^q(\Omega))$, one has the solvability of the system

$$\begin{aligned}
-\operatorname{div}(D^2W(\mathbb{I})\nabla\mathbf{u}) + \nabla\pi &= \mathbf{f}, && \text{in } \Omega,\\
\operatorname{div}\mathbf{u} - \gamma\int_0^t c\,\mathrm{d}\tau &= g, && \text{in } \Omega,\\
\mathbf{u} &= \mathbf{h}^1, && \text{on } \Gamma_1,\\
(D^2W(\mathbb{I})\nabla\mathbf{u} - \pi\mathbb{I})\mathbf{n} &= \mathbf{h}^2, && \text{on } \Gamma_2.
\end{aligned} \tag{5.17}$$

COROLLARY 5.14. *Let $\gamma > 0$. Given $c \in L^q(0,T; W_q^2(\Omega)) \cap W_q^1(0,T; L^q(\Omega))$. Then under the assumptions of Theorem 5.13, there is a unique solution (u,π) of (5.17) satisfying*

$$(u,\pi) \in \mathbb{E}(T),$$

Proof. Similar to Corollary 4.16, the only point we need to check is $\gamma \int_0^t c\mathrm{d}\tau \in \mathbb{F}_g(T)$, which is not hard to verify thanks to the regularity of c. Then solving (5.13) with $(\mathbf{f}, \mathbf{h}^1, \mathbf{h}^2) = 0$ and g substituted by $\gamma \int_0^t c\mathrm{d}\tau \in \mathbb{F}_g(T)$, adding the resulted solution and that of (5.13), one completes the proof. □

Remark 5.15. In view of Theorem 5.13 and Lemma 2.25, we know that

$$\big(D^2W(\mathbb{I})\nabla \mathbf{u} - \pi\mathbb{I}\big)\mathbf{n}\big|_{\Gamma_1} \in L^q(0,T; W_q^{1-\frac{1}{q}}(\Gamma_1)^3) \cap W_q^{\frac{1}{2}-\frac{1}{2q}}(0,T; L^q(\Gamma_1)^3)$$

makes sense, which contributes to the nonlinear estimate for the fluid part.

5.4. Nonlinear Well-posdeness

We denote by δ a universal positive function

$$\delta : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that } \delta(t) \to 0_+, \text{ as } t \to 0_+. \tag{5.18}$$

The most common example in the present paper is $\delta(t) = t^\theta$ for different $\theta > 0$ from line to line.

5.4.1. Auxiliary Lemmata. In this section, we give some Lemmata which we shall use later on.

LEMMA 5.16. *Let $f, g \in H_q^{\frac{1}{2}}(\mathbb{R}; L^q(\Omega)) \cap L^\infty(\mathbb{R}; L^\infty(\Omega)) \cap W_{2q}^\alpha(\mathbb{R}; L^{2q}(\Omega))$ with $q > 1$ and $1/4 < \alpha < 1/2$, then $fg \in H_q^{\frac{1}{2}}(\mathbb{R}; L^q(\Omega))$ and*

$$\begin{aligned}\|fg\|_{H_q^{\frac{1}{2}}(\mathbb{R};L^q(\Omega))} \leq C\Big(&\|f\|_{H_q^{\frac{1}{2}}(\mathbb{R};L^q(\Omega))} \|g\|_{L^\infty(\mathbb{R};L^\infty(\Omega))} \\ &+ \|g\|_{H_q^{\frac{1}{2}}(\mathbb{R};L^q(\Omega))} \|f\|_{L^\infty(\mathbb{R};L^\infty(\Omega))} + \|f\|_{W_{2q}^\alpha(\mathbb{R};L^{2q}(\Omega))} \|g\|_{W_{2q}^\alpha(\mathbb{R};L^{2q}(\Omega))}\Big).\end{aligned}$$

Moreover, if additionally $f|_{t=0} = g|_{t=0} = 0$, the assertion is true as well with $\mathbb{R}$ substituted by an interval $(0,T)$ and the constant $C > 0$ in the estimate is independent of $T > 0$.

Proof. First let us recall the equivalent definition of Bessel potential space

$$\|f\|_{H_q^s(\mathbb{R};L^q(\Omega))} = \|f\|_{L^q(\mathbb{R};L^q(\Omega))} + \|(-\Delta)^{\frac{s}{2}} f\|_{L^q(\mathbb{R};L^q(\Omega))},$$

where the fractional Laplace operator is represented by the singular integral

$$(-\Delta)^{\frac{s}{2}} f(t) = C_s \lim_{\epsilon\to 0} \int_{|h|\geq\epsilon} \frac{\Delta_h f(t)}{|h|^{1+s}} \mathrm{d}h,$$

with $C_s > 0$ a constant depending on s, $0 < s < 2$ and $\Delta_h f(t) := f(t) - f(t-h)$, see e.g. [Ste70, Chapter V, Section 6.10]. For $f, g \in H_q^{1/2}(\mathbb{R}; L^q(\Omega)) \cap L^\infty(\mathbb{R}; L^\infty(\Omega)) \cap W_{2q}^\alpha(\mathbb{R}; L^{2q}(\Omega))$ with $q > 1$ and $1/4 < \alpha < 1/2$, we see that the integrand has an algebraic decay rate greater than one with respect to $|h|$, which means the singular integral is actually integrable and one can omit the "lim" in the following. Then we have

$$\begin{aligned}
&\left\|(-\Delta)^{\frac{1}{4}}(fg)(t)\right\|_{L^q(\Omega)} = C\left\|\int_{\mathbb{R}}\frac{\Delta_h(fg)(t)}{|h|^{1+\frac{1}{2}}}\,\mathrm{d}h\right\|_{L^q(\Omega)}\\
&= C\left\|g(t)\int_{\mathbb{R}}\frac{\Delta_h f(t)}{|h|^{1+\frac{1}{2}}}\,\mathrm{d}h + \int_{\mathbb{R}}\frac{f(t)\Delta_h g(t)}{|h|^{1+\frac{1}{2}}}\,\mathrm{d}h + \int_{\mathbb{R}}\frac{\big(\Delta_h f(t)\big)\big(\Delta_h g(t)\big)}{|h|^{1+\frac{1}{2}}}\,\mathrm{d}h\right\|_{L^q(\Omega)}\\
&\le C\underbrace{\left(\left\|g(t)(-\Delta)^{\frac{1}{4}}f(t)\right\|_{L^q(\Omega)} + \left\|f(t)(-\Delta)^{\frac{1}{4}}g(t)\right\|_{L^q(\Omega)}\right)}_{\le\|g(t)\|_{L^\infty(\Omega))}\|(-\Delta)^{\frac{1}{4}}f(t)\|_{L^q(\Omega)}+\|f(t)\|_{L^\infty(\Omega))}\|(-\Delta)^{\frac{1}{4}}g(t)\|_{L^q(\Omega)}}\\
&\qquad + C\underbrace{\left\|\int_{\mathbb{R}}\frac{\big(\Delta_h f(t)\big)\big(\Delta_h g(t)\big)}{|h|^{1+\frac{1}{2}}}\,\mathrm{d}h\right\|_{L^q(\Omega)}}_{=:I(t)}.
\end{aligned}$$

Dividing the region $\mathbb{R}$ into a neighborhood of the origin and its complement, we have

$$\begin{aligned}
I(t) \le& \int_{|h|\le 1}\left\|\big(\Delta_h f(t)\big)\big(\Delta_h g(t)\big)\right\|_{L^q(\Omega)}\frac{1}{|h|^{1+\frac{1}{2}}}\,\mathrm{d}h\\
&+\int_{|h|>1}\left\|\big(\Delta_h f(t)\big)\big(\Delta_h g(t)\big)\right\|_{L^q(\Omega)}\frac{1}{|h|^{1+\frac{1}{2}}}\,\mathrm{d}h =: I_1(t)+I_2(t),
\end{aligned}$$

where

$$\begin{aligned}
I_1(t) &\le C\int_{|h|\le 1}\|\Delta_h f(t)\|_{L^{2q}(\Omega)}\|\Delta_h g(t)\|_{L^{2q}(\Omega)}\frac{1}{|h|^{1+\frac{1}{2}}}\,\mathrm{d}h\\
&= C\int_{|h|\le 1}\|\Delta_h f(t)\|_{L^{2q}(\Omega)}\|\Delta_h g(t)\|_{L^{2q}(\Omega)}\left(\frac{1}{|h|^{1+\frac{q}{2}+\varepsilon\frac{q}{q'}}}\right)^{\frac{1}{q}}\left(\frac{1}{|h|^{1-\varepsilon}}\right)^{\frac{1}{q'}}\mathrm{d}h\\
&\le C\left(\int_{|h|\le 1}\|\Delta_h f(t)\|_{L^{2q}(\Omega)}^q\|\Delta_h g(t)\|_{L^{2q}(\Omega)}^q\frac{\mathrm{d}h}{|h|^{1+\frac{q}{2}+\varepsilon\frac{q}{q'}}}\right)^{\frac{1}{q}}\underbrace{\left(\int_{|h|\le 1}|h|^{-1+\varepsilon}\,\mathrm{d}h\right)^{\frac{1}{q'}}}_{\le C_\varepsilon},
\end{aligned}$$

for every $\varepsilon>0$ and $1/q+1/q'=1$. By the Hölder's inequality,

$$\begin{aligned}
\int_{\mathbb{R}}|I_1(t)|^q\,\mathrm{d}t &\le C_\varepsilon\int_{\mathbb{R}}\int_{|h|\le 1}\|\Delta_h f(t)\|_{L^{2q}(\Omega)}^q\|\Delta_h g(t)\|_{L^{2q}(\Omega)}^q\frac{\mathrm{d}h\mathrm{d}t}{|h|^{1+\frac{q}{2}+\varepsilon\frac{q}{q'}}}\\
&\le C_\varepsilon\left(\int_{\mathbb{R}}\int_{|h|\le 1}\frac{\|\Delta_h f(t)\|_{L^{2q}(\Omega)}^{2q}}{|h|^{1+\frac{q}{2}+\varepsilon\frac{q}{q'}}}\,\mathrm{d}h\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\int_{|h|\le 1}\frac{\|\Delta_h g(t)\|_{L^{2q}(\Omega)}^{2q}}{|h|^{1+\frac{q}{2}+\varepsilon\frac{q}{q'}}}\,\mathrm{d}h\mathrm{d}t\right)^{\frac{1}{2}}\\
&\le C_\varepsilon\|f\|^q_{W_{2q}^{\frac{1}{4}+\frac{\varepsilon}{2q'}}(\mathbb{R};L^{2q}(\Omega))}\|g\|^q_{W_{2q}^{\frac{1}{4}+\frac{\varepsilon}{2q'}}(\mathbb{R};L^{2q}(\Omega))}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{\mathbb{R}}|I_2(t)|^q\,\mathrm{d}t &\le C\int_{\mathbb{R}}\|f(t)\|_{L^{2q}(\Omega)}^q\|g(t)\|_{L^{2q}(\Omega)}^q\left(\int_{|h|>1}\frac{1}{|h|^{1+\frac{1}{2}}}\,\mathrm{d}h\right)^q\mathrm{d}t\\
&\le C\int_{\mathbb{R}}\|f(t)\|_{L^{2q}(\Omega)}^q\|g(t)\|_{L^{2q}(\Omega)}^q\,\mathrm{d}t \le C\|f\|^q_{L^{2q}(\mathbb{R};L^{2q}(\Omega))}\|g\|^q_{L^{2q}(\mathbb{R};L^{2q}(\Omega))}.
\end{aligned}$$

Combining the estimate

$$\|fg\|_{L^q(\mathbb{R};L^q(\Omega))} \le \|f\|_{L^q(\mathbb{R};L^q(\Omega))} \|g\|_{L^\infty(0,T;L^\infty(\Omega))} \le C \|f\|_{H_q^{\frac{1}{2}}(\mathbb{R};L^q(\Omega))} \|g\|_{L^\infty(\mathbb{R};L^\infty(\Omega))},$$

we conclude that for $1/4 < \alpha < 1/2$,

$$\begin{aligned} \|fg\|_{H_q^{1/2}(\mathbb{R};L^q(\Omega))} \le C\Big(& \|f\|_{H_q^{1/2}(\mathbb{R};L^q(\Omega))} \|g\|_{L^\infty(\mathbb{R};L^\infty(\Omega))} \\ & + \|g\|_{H_q^{1/2}(\mathbb{R};L^q(\Omega))} \|f\|_{L^\infty(\mathbb{R};L^\infty(\Omega))} + \|f\|_{W_{2q}^\alpha(\mathbb{R};L^{2q}(\Omega))} \|g\|_{W_{2q}^\alpha(\mathbb{R};L^{2q}(\Omega))} \Big). \end{aligned}$$

Now denoting by $\mathcal{E}$ the extension operator for Bessel potential spaces with vanishing initial value from [MS12, Lemma 2.5], one arrives at

$$\begin{aligned} \|fg\|_{{}_0H_q^{\frac{1}{2}}(0,T;L^q(\Omega))} &= \|\mathcal{E}(f)\mathcal{E}(g)\|_{{}_0H_q^{\frac{1}{2}}(0,T;L^q(\Omega))} \le C \|\mathcal{E}(f)\mathcal{E}(g)\|_{{}_0H_q^{\frac{1}{2}}(\mathbb{R};L^q(\Omega))} \\ &\le C\Big(\|\mathcal{E}(f)\|_{H_q^{\frac{1}{2}}(\mathbb{R};L^q(\Omega))} \|\mathcal{E}(g)\|_{L^\infty(\mathbb{R};L^\infty(\Omega))} + \|\mathcal{E}(g)\|_{H_q^{\frac{1}{2}}(\mathbb{R};L^q(\Omega))} \|\mathcal{E}(f)\|_{L^\infty(\mathbb{R};L^\infty(\Omega))} \\ &\qquad + \|\mathcal{E}(f)\|_{W_{2q}^\alpha(\mathbb{R};L^{2q}(\Omega))} \|\mathcal{E}(g)\|_{W_{2q}^\alpha(\mathbb{R};L^{2q}(\Omega))} \Big) \\ &\le C\Big(\|f\|_{{}_0H_q^{\frac{1}{2}}(0,T;L^q(\Omega))} \|g\|_{L^\infty(0,T;L^\infty(\Omega))} \\ &\qquad + \|g\|_{{}_0H_q^{\frac{1}{2}}(0,T;L^q(\Omega))} \|f\|_{L^\infty(0,T;L^\infty(\Omega))} + \|f\|_{W_{2q}^\alpha(0,T;L^{2q}(\Omega))} \|g\|_{W_{2q}^\alpha(0,T;L^{2q}(\Omega))} \Big), \end{aligned}$$

where the constant $C > 0$ is independent of T. □

Lemma 5.17. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^1 boundary, $R > 0$, $T > 0$ and $5 < q < \infty$. Given*

$$f \in X := H_q^{\frac{1}{2}}(0,T;L^q(\Omega)) \cap L^q(0,T;W_q^1(\Omega))$$

with $\|f\|_X \le R$. Then $f \in L^\infty(0,T;L^\infty(\Omega))$ and there exists some $1/4 < s < 1/2 - 5/4q$ such that $f \in W_{2q}^s(0,T;L^{2q}(\Omega))$. In addition,

$$\|f\|_{L^\infty(0,T;L^\infty(\Omega))} \le C\delta(T), \tag{5.19}$$

$$\|f\|_{W_{2q}^s(0,T;L^{2q}(\Omega))} \le C, \tag{5.20}$$

provided $f|_{t=0} = 0$, where $C > 0$ depends on R. Moreover, for $f^1, f^2, g \in X$ with $(f^1 - f^2)|_{t=0} = 0$, $g|_{t=0} = 0$ and $\|(f^i, g)\|_{X\times X} \le R$, $i \in \{1,2\}$,

$$\|f^1 - f^2\|_{L^\infty(0,T;L^\infty(\Omega))} \le C\delta(T) \|f^1 - f^2\|_X, \tag{5.21}$$

$$\|(f^1 - f^2)g\|_X \le C\delta(T) \|f^1 - f^2\|_X, \tag{5.22}$$

where $C > 0$ depends on R.

Proof. For $f \in X$, by Proposition 2.21 and Proposition 2.24, we have

$$H_q^{\frac{1}{2}}(0,T;L^q(\Omega)) \cap L^q(0,T;W_q^1(\Omega)) \hookrightarrow H_q^{\frac{1}{5}}(0,T;H_q^{\frac{3}{5}}(\Omega)) \hookrightarrow C([0,T];C(\overline{\Omega})),$$

for $q > 5$, which means $f \in L^\infty(0,T;L^\infty(\Omega))$. If $f|_{t=0} = 0$, the first embedding constant above is uniform with regard to T and it follows from Proposition 2.21 that (5.19) holds true, as well as (5.21). By means of the time-space embedding Proposition 2.24 again, one has

$$ {}_0H_q^{\frac{1}{2}}(0,T;L^q(\Omega)) \cap L^q(0,T;W_q^1(\Omega)) \hookrightarrow {}_0H_q^r(0,T;W_q^{1-2r}(\Omega)),$$

for $1/q < r < 1/2$, where the embedding constant does not depend on $T > 0$. By virtue of the embeddings $W^s_q(\Omega) \hookrightarrow L^{2q}(\Omega)$ for $s - 3/q \geq -3/2q$ and $H^s_q(0,T;X) \hookrightarrow W^{s-1/2q}_{2q}(0,T;X)$, one infers the conditions

$$1 - 2r - \frac{3}{q} \geq -\frac{3}{2q}, \quad r - \frac{1}{2q} > \frac{1}{4}.$$

Combining the inequalities above together yields that r should satisfies

$$\frac{1}{4} + \frac{1}{2q} < r \leq \frac{1}{2} - \frac{3}{4q}.$$

Since $q > 5$, it is easy to verify that $\frac{1}{2} - \frac{3}{4q} > \frac{1}{4} + \frac{1}{2q}$, which means that such r does exist and $f \in W^{r-1/2q}_{2q}(0,T;L^{2q}(\Omega))$.

In addition, by means of Lemma 5.16, one obtains for all $1/4 < \alpha < 1/2$,

$$\begin{aligned}
&\left\|(f^1 - f^2)g\right\|_{H^{\frac{1}{2}}_q(0,T;L^q(\Omega))} \\
&\quad \leq C\big(\underbrace{\left\|f^1 - f^2\right\|_{L^\infty(0,T;L^\infty(\Omega))} \|g\|_X + \left\|f^1 - f^2\right\|_X \|g\|_{L^\infty(0,T;L^\infty(\Omega))}}_{\leq C(R)\delta(T)\|f^1 - f^2\|_X \text{ thanks to (5.19) and (5.21)}} \\
&\qquad + \left\|f^1 - f^2\right\|_{W^\alpha_{2q}(\mathbb{R};L^{2q}(\Omega))} \|g\|_{W^\alpha_{2q}(\mathbb{R};L^{2q}(\Omega))} \big),
\end{aligned}$$

Now choosing α such that $1/4 < \alpha < s < 1/2 - 5/4q$, where s is given in (5.20), one deduces that

$$\left\|(f^1 - f^2)g\right\|_{H^{\frac{1}{2}}_q(0,T;L^q(\Omega))} \leq CT^{s-\alpha} \left\|f^1 - f^2\right\|_X.$$

Moreover, we have

$$\begin{aligned}
&\left\|(f^1 - f^2)g\right\|_{L^q(0,T;W^1_q(\Omega))} \\
&\quad = \left\|(f^1 - f^2)g\right\|_{L^q(0,T;L^q(\Omega))} + \left\|\nabla(f^1 - f^2)g\right\|_{L^q(0,T;L^q(\Omega))} + \left\|(f^1 - f^2)\nabla g\right\|_{L^q(0,T;L^q(\Omega))} \\
&\quad \leq \left\|f^1 - f^2\right\|_{L^\infty(0,T;L^\infty(\Omega))} \|g\|_X + \left\|f^1 - f^2\right\|_{L^q(0,T;W^1_q(\Omega))} \|g\|_{L^\infty(0,T;L^\infty(\Omega))} \\
&\qquad + \left\|f^1 - f^2\right\|_{L^\infty(0,T;L^\infty(\Omega))} \|g\|_{L^q(0,T;W^1_q(\Omega))} \leq C(R)\delta(T) \left\|f^1 - f^2\right\|_X.
\end{aligned}$$

which proves (5.22). □

Remark 5.18. If one replaces the $f|_{t=0} = 0$ condition above by $\|f\|_X \leq \kappa$ for $\kappa > 0$, we still have similar estimates above with $\delta(T)$ substituted by $\delta(T) + \kappa$, which can be done by same argument as in (5.28) below.

Lemma 5.19. *Let $q > 5$ and $\hat{\mathbf{F}}$ be the deformation gradient defined by (5.3) with respect to $\hat{\mathbf{v}}_f \in Y^1_T$ in Ω_f and $\hat{\mathbf{u}}_s \in Y^2_T$ in Ω_s respectively. Assume that $\hat{\mathbf{u}}^0_s := \hat{\mathbf{u}}_s|_{t=0} \in W^{2-2/q}_q(\Omega_s)^3$ and $\|\hat{\nabla}\hat{\mathbf{u}}^0_s\|_{W^{1-2/q}_q(\Omega_s)^3} \leq \kappa$ with $\kappa > 0$ small enough. Then for every $R > 0$, there are a constant $C = C(R) > 0$ and a finite time $0 < T_R < 1$ depending on R such that for all $0 < T < T_R$ and $\|(\hat{\mathbf{v}}_f, \hat{\mathbf{u}}_s)\|_{Y^1_T \times Y^2_T} \leq R$, $\hat{\mathbf{F}}^{-1}$ exists a.e. with regularities*

$$\hat{\mathbf{F}}^{-1}_f \in W^1_q(0,T;W^1_q(\Omega_f)^{3\times3}), \quad \hat{\mathbf{F}}^{-1}_s \in H^{\frac{1}{2}}_q(0,T;L^q(\Omega_s)^{3\times3}) \cap L^q(0,T;W^1_q(\Omega_s)^{3\times3}),$$

and satisfies

$$\left\|\hat{\mathbf{F}}_f^{-1}\right\|_{L^\infty(0,T;W^1_q(\Omega_f)^{3\times 3})} \leq C, \quad \left\|\hat{\mathbf{F}}_f^{-1} - \mathbb{I}\right\|_{L^\infty(0,T;W^1_q(\Omega_f)^{3\times 3})} \leq C\delta(T), \tag{5.23}$$

$$\left\|\hat{\mathbf{F}}_s^{-1}\right\|_{L^\infty(0,T;L^\infty(\Omega_s)^{3\times 3})} \leq C, \quad \left\|\hat{\mathbf{F}}_s^{-1} - \mathbb{I}\right\|_{L^\infty(0,T;L^\infty(\Omega_s)^{3\times 3})} \leq C(\delta(T)+\kappa), \tag{5.24}$$

$$\left[\hat{\mathbf{F}}_f^{-1} - \mathbb{I}\right]_{{}_0W^r_q(0,T;W^1_q(\Omega_f)^{3\times 3})} \leq C\delta(T), \quad 0 < r < 1, \tag{5.25}$$

Moreover, for $\hat{\mathbf{w}}_f \in Y^1_T$ *and* $\hat{\mathbf{w}}_s \in Y^2_T$ *with* $\|(\hat{\mathbf{w}}_f, \hat{\mathbf{w}}_s)\|_{Y^1_T\times Y^2_T} \leq R$ *and* $(\hat{\mathbf{w}}_f, \hat{\mathbf{w}}_s)|_{t=0} = (\hat{\mathbf{v}}_f, \hat{\mathbf{v}}_s)|_{t=0}$*, we have*

$$\left[\hat{\mathbf{F}}_f^{-1}(\hat{\nabla}\hat{\mathbf{v}}_f) - \hat{\mathbf{F}}_f^{-1}(\hat{\nabla}\hat{\mathbf{w}}_f)\right]_{{}_0W^r_q(0,T;W^1_q(\Omega_f)^{3\times 3})} \leq C\delta(T)\left\|\hat{\mathbf{v}}_f - \hat{\mathbf{w}}_f\right\|_{Y^1_T}, \ 0 < r < 1, \tag{5.26}$$

$$\left\|\hat{\mathbf{F}}_s^{-1}(\hat{\nabla}\hat{\mathbf{v}}_s) - \hat{\mathbf{F}}_s^{-1}(\hat{\nabla}\hat{\mathbf{w}}_s)\right\|_{L^\infty(0,T;L^\infty(\Omega_s)^{3\times 3})} \leq C(\delta(T)+\kappa)\left\|\hat{\mathbf{v}}_s - \hat{\mathbf{w}}_s\right\|_{Y^2_T}. \tag{5.27}$$

Proof. The proof of this lemma is similar to Lemma 3.16. However, for the solid part the regularity is a bit lower due to the quasi-stationary elastic equation. For $\hat{\mathbf{F}}_f$, one can refer to Lemma 3.16 with Propositions 2.18, 2.19 and 2.21, while (5.25) and (5.26) follows from Lemma 2.20. For $\hat{\mathbf{F}}_s$, it follows from the definition of Y^2_T that

$$\hat{\mathbf{F}}_s - \mathbb{I} = \hat{\nabla}\hat{\mathbf{u}}_s \in H^{\frac{1}{2}}_q(0,T;L^q(\Omega_s)^{3\times 3}) \cap L^q(0,T;W^1_q(\Omega_s)^{3\times 3}).$$

Since $\hat{\nabla}\hat{\mathbf{u}}^0_s \in W^{1-2/q}_q(\Omega_s)^3 \hookrightarrow L^\infty(\Omega_s)^3$ for $q > 5$ and $\|\hat{\nabla}\hat{\mathbf{u}}^0_s\|_{W^{1-2/q}_q(\Omega_s)^3} \leq \kappa$, one obtains from Lemma 5.17 that

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left\|\hat{\mathbf{F}}_s - \mathbb{I}\right\|_{L^\infty(\Omega_s)^{3\times 3}} \\
&\quad = \sup_{0\leq t\leq T}\left\|\hat{\mathbf{F}}_s - (\mathbb{I}+\hat{\nabla}\hat{\mathbf{u}}^0_s)\right\|_{L^\infty(\Omega_s)^{3\times 3}} + \left\|\hat{\nabla}\hat{\mathbf{u}}^0_s\right\|_{L^\infty(\Omega_s)^3} \leq C(\delta(T)+\kappa) \leq \frac{1}{2},
\end{aligned} \tag{5.28}$$

by taking $T_R, \kappa > 0$ small enough such that $\delta(T_R)+\kappa \leq 1/(2C)$. Then by the Neumann series, $\hat{\mathbf{F}}_s^{-1}$ does exist. Note that $\hat{\mathbf{F}}_s^{-1}(\hat{\nabla}\hat{\mathbf{u}}_s) = (\mathbb{I}+\hat{\nabla}\hat{\mathbf{u}}_s)^{-1}$ is in the class of $C^\infty(\mathbb{R}^{3\times 3}\backslash\{-\mathbb{I}\})^{3\times 3}$ with respect to $\hat{\nabla}\hat{\mathbf{u}}_s$, it turns out from Proposition 2.19 and $q > 5$ that

$$(\mathbb{I}+\hat{\nabla}\hat{\mathbf{u}}^0_s)^{-1} \in W^{1-\frac{2}{q}}_q(\Omega_s)^{3\times 3}, \quad \hat{\mathbf{F}}_s^{-1} \in H^{\frac{1}{2}}_q(0,T;L^q(\Omega_s)^{3\times 3}) \cap L^q(0,T;W^1_q(\Omega_s)^{3\times 3}). \tag{5.29}$$

In addition, we have

$$(\mathbb{I}+\hat{\nabla}\hat{\mathbf{u}}^0_s)^{-1} - \mathbb{I} = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}(\mathbb{I}+\tau\hat{\nabla}\hat{\mathbf{u}}^0_s)^{-1}\,\mathrm{d}\tau = \int_0^1 -(\mathbb{I}+\tau\hat{\nabla}\hat{\mathbf{u}}^0_s)^{-1}\hat{\nabla}\hat{\mathbf{u}}^0_s(\mathbb{I}+\tau\hat{\nabla}\hat{\mathbf{u}}^0_s)^{-1}\,\mathrm{d}\tau,$$

then

$$\left\|(\mathbb{I}+\hat{\nabla}\hat{\mathbf{u}}^0_s)^{-1} - \mathbb{I}\right\|_{W^{1-\frac{2}{q}}_q(\Omega_s)^{3\times 3}} \leq C\kappa,$$

provided $\kappa > 0$ sufficiently small, where $C > 0$ is finite. Consequently, similarly to (5.28), one can derive that

$$\sup_{0\leq t\leq T}\left\|\hat{\mathbf{F}}_s^{-1} - \mathbb{I}\right\|_{L^\infty(\Omega_s)^{3\times 3}} \leq C(\delta(T)+\kappa),$$

which proves (5.24) and (5.27). □

Lemma 5.20. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^1 boundary, $0 < s \leq 1$ and $1 < q < \infty$ with $sq > 3$. Let $W : \mathbb{R}^{3\times 3} \to \mathbb{R}_+$ satisfy Assumption (A2). Then for $\mathbf{F} \in K^s_q(\Omega)^{3\times 3}$, $K \in \{W, H\}$, with $\|\mathbf{F}\|_{K^s_q(\Omega)^{3\times 3}} \leq R$, there is a positive constant C depending on R such that*

$$\left\|D^k W(\mathbf{F})\right\|_{K^s_q(\Omega)^{3^{2k}}} \leq C, \quad k \in \{0, 1, 2, 3\}.$$

Moreover, for $\mathbf{F}^1, \mathbf{F}^2 \in K^s_q(\Omega)^{3\times 3}$ with $\left\|\mathbf{F}^1\right\|_{K^s_q(\Omega)^{3\times 3}}, \left\|\mathbf{F}^2\right\|_{K^s_q(\Omega)^{3\times 3}} \leq R$, we have

$$\left\|D^k W(\mathbf{F}^1) - D^k W(\mathbf{F}^2)\right\|_{K^s_q(\Omega)^{3^{2k}}} \leq C \left\|\mathbf{F}^1 - \mathbf{F}^2\right\|_{K^s_q(\Omega)^{3\times 3}}, \quad k \in \{1, 2, 3\}.$$

Proof. One can prove it by Proposition 2.19 for $0 < s \leq 1$ directly. □

Lemma 5.21. *Let $T > 0$, $R > 0$ and $q \in (1, \infty)$. Ω_s is the domain defined in Section 5.1.1. Given $\hat{g} \in W^1_q(0, T; W^1_q(\Omega_s))$ with $\|\hat{g}\|_{W^1_q(0,T;W^1_q(\Omega_s))} \leq R$, $\hat{g}(X, t)|_{t=0} = \hat{g}^0$, there exists a time $T_R > 0$ such that for $T \in (0, T_R)$, one has*

$$\hat{g}(X, t) \geq \frac{1}{2}, \ \forall\, X \in \Omega_s, t \in [0, T].$$

Proof. By calculus,

$$\hat{g}(X, t) = \hat{g}^0(X) + \int_0^t \partial_t \hat{g}(X, \tau) \mathrm{d}\tau, \ \forall\, X \in \Omega_s, t \in [0, T].$$

Then

$$\left\|\hat{g}(t) - \hat{g}^0\right\|_{L^\infty(\Omega_s)} \leq C \left\|\int_0^t \partial_t \hat{g}(\cdot, \tau) \mathrm{d}\tau\right\|_{W^1_q(\Omega_s)} \leq C T^{1-\frac{1}{q}} R \leq \frac{1}{2},$$

where we choose $T_R > 0$ sufficiently small such that $T_R^{1-\frac{1}{q}} \leq \frac{1}{2CR}$. Hence $\hat{g}(X, t) \geq \frac{1}{2}$, for all $X \in \Omega_s, t \in [0, T]$. □

5.4.2. Lipschitz estimates. Now we are in the position to derive the Lipschitz estimates of the nonlinear lower-order terms in (5.11). To this end, let us first define the function spaces for the nonlinear terms $Z_T := \prod_{j=1}^{12} Z^j_T$, where

$$\begin{aligned}
Z^1_T &:= L^q(0, T; L^q(\Omega_f)^3), \\
Z^2_T &:= L^q(0, T; L^q(\Omega_s)^3) \cap H^{\frac{1}{2}}_q(0, T; W^{-1}_{q,\Gamma}(\Omega_s)^3), \\
Z^3_T &:= \left\{ g \in L^q(0, T; W^1_q(\Omega_f)) \cap W^1_q(0, T; W^{-1}_q(\Omega_f)) : \mathrm{tr}_\Gamma(g) \in Z^5_T \right\}, \\
Z^4_T &:= L^q(0, T; W^1_q(\Omega_s)) \cap H^{\frac{1}{2}}_q(0, T; L^q(\Omega_s)), \\
Z^5_T &:= L^q(0, T; W^{1-\frac{1}{q}}_q(\Gamma)^3) \cap W^{\frac{1}{2}-\frac{1}{2q}}_q(0, T; L^q(\Gamma)^3), \\
Z^6_T &:= L^q(0, T; W^{2-\frac{1}{q}}_q(\Gamma)^3) \cap H^{\frac{1}{2}}_q(0, T; W^{1-\frac{1}{q}}_q(\Gamma)^3), \\
Z^7_T &:= L^q(0, T; W^{1-\frac{1}{q}}_q(\Gamma_s)^3) \cap H^{\frac{1}{2}}_q(0, T; W^{-\frac{1}{q}}_q(\Gamma_s)^3), \\
Z^8_T &:= L^q(0, T; L^q(\Omega \backslash \Gamma)), \\
Z^9_T &:= L^q(0, T; W^{1-\frac{1}{q}}_q(\Gamma)) \cap W^{\frac{1}{2}-\frac{1}{2q}}_q(0, T; L^q(\Gamma)), \\
Z^{10}_T &:= L^q(0, T; W^{1-\frac{1}{q}}_q(\Gamma_s)) \cap W^{\frac{1}{2}-\frac{1}{2q}}_q(0, T; L^q(\Gamma_s)), \\
Z^{11}_T &:= L^q(0, T; W^1_q(\Omega_s)), \quad Z^{12}_T := L^q(0, T; W^1_q(\Omega_s)),
\end{aligned}$$

Let

$$\mathbf{w} := (\hat{\mathbf{v}}_f, \hat{\mathbf{u}}_s, \hat{\pi}_f, \hat{\pi}_s, \hat{c}, \hat{c}_s^*, \hat{g}) \tag{5.30}$$

be in the space Y_T given in Section 5.2.1 and define the associated initial data as

$$\mathbf{w}_0 := (\hat{\mathbf{v}}_f, \hat{\mathbf{u}}_s, \hat{\pi}_s, \hat{c}, \hat{c}_s^*, \hat{g})|_{t=0} = (\hat{\mathbf{v}}_f^0, \hat{\mathbf{u}}_s^0, \hat{\pi}_s^0, \hat{c}^0, \hat{c}_*^0, \hat{g}^0). \tag{5.31}$$

Then we have the following Lipschitz estimates for the lower-order terms defined in (5.11).

Proposition 5.22. *Let $q > 5$ and $R > 0$. There exist constants $C, \kappa > 0$ and a finite time $T_R > 0$ both depending on R such that for $0 < T < T_R$ and $\|\hat{\nabla}\hat{\mathbf{u}}_s^0\|_{W_q^{1-2/q}(\Omega_s)^3} + \|\hat{\pi}_s^0\|_{W_q^{1-2/q}(\Omega_s)} \leq \kappa$,*

$$\begin{aligned}
\left\|\mathbf{K}_f(\mathbf{w}^1) - \mathbf{K}_f(\mathbf{w}^2)\right\|_{Z_T^1} &\leq C(\delta(T)+\kappa)\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T},\\
\left\|(\mathbf{K}_s, \mathbf{H}^2)(\mathbf{w}^1) - (\mathbf{K}_s, \mathbf{H}^2)(\mathbf{w}^2)\right\|_{Z_T^2\times Z_T^7} &\leq C(\delta(T)+\kappa)\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T},\\
\left\|G_i(\mathbf{w}^1) - G_i(\mathbf{w}^2)\right\|_{Z_T^3\times Z_T^4} &\leq C(\delta(T)+\kappa)\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T},\ i\in\{f,s\},\\
\left\|\mathbf{H}_i^1(\mathbf{w}^1) - \mathbf{H}_i^1(\mathbf{w}^2)\right\|_{Z_T^5\times Z_T^6} &\leq C(\delta(T)+\kappa)\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T},\ i\in\{f,s\},\\
\left\|F^j(\mathbf{w}^1) - F^j(\mathbf{w}^2)\right\|_{Z_T^{7+j}} &\leq C(\delta(T)+\kappa)\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T},\ j\in\{1,2,3,4,5\},
\end{aligned}$$

for all $\left\|\mathbf{w}^1\right\|_{Y_T}, \left\|\mathbf{w}^2\right\|_{Y_T} \leq R$ with $\mathbf{w}_0^1 = \mathbf{w}_0^2$.

Proof. For the estimates related to the fluid (with a subscript f) except $\mathbf{H}_f^1$ and F^j, $j = 1, ..., 5$, we refer to Proposition 3.20, with the help of Lemma 5.17 and 5.19.

Estimate of $\mathbf{K}_s, \mathbf{H}^2$. Thanks to the divergence form from the linearization, i.e. $\mathbf{K}_s = \operatorname{div}\tilde{\mathbf{K}}_s$ and $\mathbf{H}^2 = -\tilde{\mathbf{K}}_s\hat{\mathbf{n}}_{\Gamma_s}$, one can estimate $\mathbf{K}_s, \mathbf{H}^2$ together with cancellation of boundary data like Corollary 5.11. Namely,

$$\begin{aligned}
&\left\|(\mathbf{K}_s, \mathbf{H}^2)(\mathbf{w}^1) - (\mathbf{K}_s, \mathbf{H}^2)(\mathbf{w}^2)\right\|_{Z_T^2\times Z_T^7}\\
&\qquad \leq \left\|\tilde{\mathbf{K}}_s(\mathbf{w}^1) - \tilde{\mathbf{K}}_s(\mathbf{w}^2)\right\|_{L^q(0,T;W_q^1(\Omega)^{3\times3})\cap H_q^{\frac{1}{2}}(0,T;L^q(\Omega_s)^{3\times3})}.
\end{aligned}$$

Combining the definition of $\tilde{\mathbf{K}}_s$ in Section 5.2.3, Lemma 5.17–5.21 and Assumption 5.1, one obtains for $q > 5$,

$$\left\|(\mathbf{K}_s, \mathbf{H}^2)(\mathbf{w}^1) - (\mathbf{K}_s, \mathbf{H}^2)(\mathbf{w}^2)\right\|_{Z_T^2\times Z_T^7} \leq C(\delta(T)+\kappa)\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T}.$$

In this estimate, the smallness of the initial solid pressure is employed for the term $((\hat{g}^0)^3 - 1)\hat{\pi}_s$ in $\tilde{\mathbf{K}}_s$, i.e.,

$$\|\hat{\pi}_s\|_{L^\infty(0,T;L^\infty(\Omega_s))} \leq \left\|\hat{\pi}_s - \hat{\pi}_s^0\right\|_{L^\infty(0,T;L^\infty(\Omega_s))} + \left\|\hat{\pi}_s^0\right\|_{L^\infty(\Omega_s)} \leq C(\delta(T)+\kappa). \tag{5.32}$$

Estimate of G_s. By means of Lemma 5.17 and 5.19, estimate in $L^q(0,T;W_q^1(\Omega_s))$ is clear. By the definition of G_s,

$$G_s(\mathbf{w}^1) - G_s(\mathbf{w}^2) = (\hat{\mathbf{F}}_s^{-1}(\mathbf{w}^1) - \mathbb{I}) : (\hat{\nabla}\hat{\mathbf{u}}_s^1 - \hat{\nabla}\hat{\mathbf{u}}_s^2) + \left(\hat{\mathbf{F}}_s^{-1}(\mathbf{w}^1) - \hat{\mathbf{F}}_s^{-1}(\mathbf{w}^2)\right) : \hat{\nabla}\hat{\mathbf{u}}_s^2.$$

Then with Lemma 5.17, 5.19 and the regularity (5.29), we have for $q > 5$ that

$$\left\|G_s(\mathbf{w}^1) - G_s(\mathbf{w}^2)\right\|_{H_q^{\frac{1}{2}}(0,T;L^q(\Omega_s))} \leq C(\delta(T)+\kappa)\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T}.$$

Estimate of $\mathbf{H}^1_{f/s}$. With $\hat{\mathbf{v}}_f \in Y^1_T$, one knows

$$\int_0^t \hat{\mathbf{v}}_f(X,\tau)\mathrm{d}\tau \in W^1_q(0,T;W^2_q(\Omega_f)^3) \cap W^2_q(0,T;L^q(\Omega_f)^3) \hookrightarrow H^{\frac{3}{2}}_q(0,T;W^1_q(\Omega_f)^3).$$

It follows from the trace theorem $\mathrm{tr}_\Gamma : W^k_q(\Omega_f) \to W^{k-\frac{1}{q}}_q(\Gamma)$ that

$$\int_0^t \hat{\mathbf{v}}_f(X,\tau)\mathrm{d}\tau\Big|_\Gamma \in H^{\frac{3}{2}}_q(0,T;W^{1-\frac{1}{q}}_q(\Gamma)^3) \cap W^1_q(0,T;W^{2-\frac{1}{q}}_q(\Gamma)^3),$$

whose Lipschitz estimate in Z^6_T can be controlled by $\delta(T)\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T}$ thanks to Lemma 2.20 and 2.25, namely,

$$\left\|\mathbf{H}^1_s(\mathbf{w}^1) - \mathbf{H}^1_s(\mathbf{w}^2)\right\|_{Z^6_T} \leq C\delta(T)\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T}.$$

Now let us recall $\mathbf{H}^1_f = -\tilde{\mathbf{K}}_f\hat{\mathbf{n}}_\Gamma + \tilde{\mathbf{K}}_s\hat{\mathbf{n}}_\Gamma$. The first part can be addressed in the same way as in Proposition 3.20. By Lemma 2.25 the anisotropic trace theorem and C^3 interface that ensures a $\hat{\mathbf{n}}_\Gamma$ of class C^2, the second term can be estimated by

$$\left\|\tilde{\mathbf{K}}_s(\mathbf{w}^1) - \tilde{\mathbf{K}}_s(\mathbf{w}^2)\right\|_{Z^5_T} \leq C\left\|\tilde{\mathbf{K}}_s(\mathbf{w}^1) - \tilde{\mathbf{K}}_s(\mathbf{w}^2)\right\|_{L^q(0,T;W^1_q(\Omega)^{3\times3})\cap H^{\frac{1}{2}}_q(0,T;L^q(\Omega_s)^{3\times3})}$$

Then together with Lemma 5.17–5.21 and Assumption 5.1, one gets

$$\left\|\mathbf{H}^1_f(\mathbf{w}^1) - \mathbf{H}^1_f(\mathbf{w}^2)\right\|_{Z^5_T} \leq C(\delta(T)+\kappa)\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T}.$$

Estimate of F^j. We can estimate F^1_f and $F^j, j = 4,5$ analogously as in Chapter 3. For others, since $Y^5_T \hookrightarrow H^{1/2}_q(0,T;W^1_q(\Omega))$ implies that

$$\hat{\nabla}\hat{c}_s \in H^{\frac{1}{2}}_q(0,T;L^q(\Omega_s)^3) \cap L^q(0,T;W^1_q(\Omega_s)^3),$$

one can apply Lemma 5.17 and 5.19 to derive the corresponding estimates with $q > 5$, combining Lemma 2.25, the regularity of $\hat{\mathbf{F}}^\top_s$, $\hat{c}^*_s$ and $\hat{g}$.

This completes the proof. □

5.4.3. Nonlinear well-posedness. For a $\mathbf{w}$ defined in (5.30), define

$$\mathscr{M}(\mathbf{w}) := \left(\mathbf{K}_f, \mathbf{K}_s, G_f, G_s, \mathbf{H}^1_f, \mathbf{H}^1_s, \mathbf{H}^2, F^1, F^2, F^3, F^4, F^5\right)^\top(\mathbf{w}),$$

where the elements are given by (5.11). Then the following proposition holds for $\mathscr{M}(\mathbf{w}) : Y_T \to Z_T$, where Y_T, Z_T are given in Section 5.2.1, 5.4.2 respectively.

Proposition 5.23. *Let $q > 5$ and $R > 0$. Let $\mathbf{w} \in Y_T$ be the function as in (5.30) with the associated initial data as $\mathbf{w}_0$ as in (5.31). Then there exist constants $C, \kappa > 0$, a finite time $T_R > 0$ both depending on R and $\delta(T)$ as in (5.18) such that for $0 < T < T_R$, $\mathscr{M} : Y_T \to Z_T$ is well-defined and bounded together with the estimates for $\|\mathbf{w}\|_{Y_T} \leq R$ and $\|\hat{\nabla}\hat{\mathbf{u}}^0_s\|_{W^{1-2/q}_q(\Omega_s)^3} + \|\hat{\pi}^0_s\|_{W^{1-2/q}_q(\Omega_s)} \leq \kappa$ that*

$$\|\mathscr{M}(\mathbf{w})\|_{Z_T} \leq C(\delta(T)+\kappa). \tag{5.33}$$

Moreover, there exist a constant $C > 0$, a finite time $T_R > 0$ depending on R and a function $\delta(T)$ as in (5.18) such that for $0 < T < T_R$,

$$\left\|\mathscr{M}(\mathbf{w}^1) - \mathscr{M}(\mathbf{w}^2)\right\|_{Z_T} \leq C(\delta(T)+\kappa)\left\|\mathbf{w}^1 - \mathbf{w}^2\right\|_{Y_T}, \tag{5.34}$$

for all $\mathbf{w}^1, \mathbf{w}^2 \in Y_T$ with $\mathbf{w}^1_0 = \mathbf{w}^2_0$ and $\left\|\mathbf{w}^1\right\|_{Y_T}, \left\|\mathbf{w}^2\right\|_{Y_T} \leq R$.

Proof. The second part follows directly from Proposition 5.22. Then by setting a trivial solution $\mathbf{w}^2 = (0,0,0,0,0,0,1)$ in (5.34), one derives (5.33) immediately in view of the fact that $\mathscr{M}(0,0,0,0,0,0,1) = 0$. □

Now recalling the definition of solution and initial spaces in Section 5.2.1, we rewrite (5.11) in the abstract form

$$\mathscr{L}(\mathbf{w}) = \mathscr{N}(\mathbf{w}, \mathbf{w}_0) \quad \text{for all } \mathbf{w} \in Y_T, \ (\hat{\mathbf{v}}_f^0, \hat{c}^0) \in \mathcal{D}_q, \tag{5.35}$$

where $\mathscr{L}(\mathbf{w})$ denotes the left-hand side of (5.11) and $\mathscr{N}(\mathbf{w}, \mathbf{w}_0)$ is the right-hand side. It follows from the linear theory in Section 5.3 that $\mathscr{L} : Y_T \to Z_T \times \mathcal{D}_q$ is an isomorphism.

Proof of Theorem 5.5. For $(\hat{\mathbf{v}}_f^0, \hat{c}^0) \in \mathcal{D}_q$ satisfying the compatibility conditions, we may solve $\mathscr{L}(\tilde{\mathbf{w}}) = \mathscr{N}(\bar{\mathbf{0}}, \mathbf{w}_0)$ by some $\tilde{\mathbf{w}} \in Y_T$. Here $\mathscr{N}(\bar{\mathbf{0}}, \mathbf{w}_0)$ is in the sense of trivial data $\bar{\mathbf{0}} = (0,0,0,0,0,0,1)$. Then one can reduce the system to the case of trivial initial data by eliminating $\tilde{\mathbf{w}}$ and we are able to set a well-defined constant

$$C_{\mathscr{L}} := \sup_{0<T\le 1} \left\| \mathscr{L}^{-1} \right\|_{\mathcal{L}(Z_T, Y_T)},$$

which is finite by the linear theories in Section 5.3 and the estimate (5.33), as in Chapter 3. Choose $R > 0$ large such that $R \ge 2C_{\mathscr{L}} \left\| (\hat{\mathbf{v}}_f^0, \hat{c}^0) \right\|_{\mathcal{D}_q}$. Then

$$\left\| \mathscr{L}^{-1} \mathscr{N}(\bar{\mathbf{0}}, \mathbf{w}_0) \right\|_{Y_T} \le C_{\mathscr{L}} \left\| (\hat{\mathbf{v}}_f^0, \hat{c}^0) \right\|_{\mathcal{D}_q} \le \frac{R}{2}. \tag{5.36}$$

For $\left\| \mathbf{w}^i \right\|_{Y_T} \le R$, $i = 1,2$, we take $T_R > 0$ and $\kappa > 0$ small enough such that $C_{\mathscr{L}} C(R)(\delta(T_R) + \kappa) \le 1/2$, where $C(R)$ is the constant in (5.34). Then for $0 < T < T_R$, we infer from Theorem 5.23 that

$$\begin{aligned} &\left\| \mathscr{L}^{-1} \mathscr{N}(\mathbf{w}^1, \mathbf{w}_0) - \mathscr{L}^{-1} \mathscr{N}(\mathbf{w}^2, \mathbf{w}_0) \right\|_{Y_T} \\ &\qquad \le C_{\mathscr{L}} C(R) T^{\delta} \left\| \mathbf{w}^1 - w^2 \right\|_{Y_T} \le \frac{1}{2} \left\| \mathbf{w}^1 - \mathbf{w}^2 \right\|_{Y_T}, \end{aligned} \tag{5.37}$$

which implies the contraction property. From (5.36) and (5.37), we have

$$\begin{aligned} &\left\| \mathscr{L}^{-1} \mathscr{N}(\mathbf{w}, \mathbf{w}_0) \right\|_{Y_T} \\ &\qquad \le \left\| \mathscr{L}^{-1} \mathscr{N}(\bar{\mathbf{0}}, \mathbf{w}_0) \right\|_{Y_T} + \left\| \mathscr{L}^{-1} \mathscr{N}(\mathbf{w}, \mathbf{w}_0) - \mathscr{L}^{-1} \mathscr{N}(0, \mathbf{w}_0) \right\|_{Y_T} \le R. \end{aligned}$$

Define a ball in Y_T as

$$\mathcal{M}_{R,T} := \left\{ \mathbf{w} \in \overline{B_{Y_T}(\bar{\mathbf{0}}, R)} : \mathbf{w} \text{ is as in } (5.30) \right\},$$

a closed subset of Y_T. Hence, $\mathscr{L}^{-1} \mathscr{N}(\cdot, \mathbf{w}_0) : \mathcal{M}_{R,T} \to \mathcal{M}_{R,T}$ is well-defined for all $0 < T < T_R$ and a strict contraction. Since Y_T is a Banach space, the Banach fixed-point Theorem implies the existence of a unique fixed-point of $\mathscr{L}^{-1} \mathscr{N}(\cdot, \mathbf{w}_0)$ in $\mathcal{M}_{R,T}$, i.e., (5.11) admits a unique strong solution in $\mathcal{M}_{R,T}$ for small time $0 < T < T_R$.

The uniqueness in Y_T, $0 < T < T_0$, follows easily by repeating the continuity argument in Proof of Theorem 3.8, so we omit it here. In summary, (5.11) admits a unique solution in Y_T, equivalently, (5.7) admits a unique solution in Y_T.

Now we are in the position to prove the positivity of cells concentrations. Since the regularity of $\hat{\mathbf{v}}_s = \partial_t \hat{\mathbf{u}}_s$ is much lower than that in Chapter 3, we can not proceed as in Chapter 3 for $\hat{c}$. To overcome this problem, we take a smooth mollification $\hat{\mathbf{v}}_s^\epsilon$ of $\hat{\mathbf{v}}_s$ for $\epsilon > 0$ such that

$$\int_0^t \hat{\mathbf{v}}_s^\epsilon(\cdot,\tau)\mathrm{d}\tau \to \hat{\mathbf{u}}_s, \text{ in } Y_T^2, \text{ as } \epsilon \to 0.$$

Consider the problem

$$\begin{aligned} \partial_t c_f^\epsilon + \operatorname{div}\left(c_f^\epsilon \mathbf{v}_f\right) - D_f \Delta c_f^\epsilon &= 0, && \text{in } Q_f^T, \\ \partial_t c_s^\epsilon + \operatorname{div}\left(c_s^\epsilon \mathbf{v}_s^\epsilon\right) - D_s \Delta c_s^\epsilon &= -f_s^r, && \text{in } Q_s^T, \end{aligned}$$

with boundary and initial values as in Section 5.1.1. Then with same argument in Proof of Theorem 3.8, one obtains

$$0 \le c^\epsilon(x,t) \in W_q^1(0,T;L^q(\Omega^t)^3) \cap L^q(0,T;W_q^2(\Omega^t \backslash \Gamma^t)^3),$$

which means there is a subsequent still denoted by c^ϵ and a function c such that

$$c^\epsilon \rightharpoonup c \text{ weakly in } W_q^1(0,T;L^q(\Omega^t)^3) \cap L^q(0,T;W_q^2(\Omega^t \backslash \Gamma^t)^3),$$

and

$$c^\epsilon \to c \text{ in } \mathcal{D}'(Q^T \backslash S^T),$$

where $\mathcal{D}'(U)$ denotes the space of distributions on U, Q^T, S^T are defined in Section 5.1.1. It is standard to verify that c solves the same equation with $\mathbf{v}_s^\epsilon$ replaced by $\mathbf{v}_s$. We only give the sketch of proof with respect to $\operatorname{div}(c_s^\epsilon \mathbf{v}_s^\epsilon)$ as an example.

$$\begin{aligned} &\int_0^T \int_{\Omega_s^t} c_s^\epsilon \mathbf{v}_s^\epsilon \cdot \nabla\phi \,\mathrm{d}x\mathrm{d}t - \int_0^T \int_{\Omega_s^t} c_s \mathbf{v}_s \cdot \nabla\phi \,\mathrm{d}x\mathrm{d}t \\ &\quad = \int_0^T \int_{\Omega_s^t} (c_s^\epsilon - c_s)\mathbf{v}_s^\epsilon \cdot \nabla\phi \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Omega_s^t} c_s(\mathbf{v}_s^\epsilon - \mathbf{v}_s) \cdot \nabla\phi \,\mathrm{d}x\mathrm{d}t \to 0, \end{aligned}$$

as $\epsilon \to 0$, for all $\phi \in \mathcal{D}(Q^T \backslash S^T)$, because of the regularity and convergence of c_s^ϵ and $\mathbf{v}_s^\epsilon$. Note that

$$0 \le \int_0^T \int_{\Omega^t \backslash \Gamma^t} c^\epsilon \phi \,\mathrm{d}x\mathrm{d}t \le \limsup_{\epsilon \to 0} \int_0^T \int_{\Omega^t \backslash \Gamma^t} c^\epsilon \phi \,\mathrm{d}x\mathrm{d}t = \int_0^T \int_{\Omega^t \backslash \Gamma^t} c\phi \,\mathrm{d}x\mathrm{d}t,$$

for all $\phi \in \mathcal{D}(Q^T \backslash S^T)$, $\phi \ge 0$, one concludes that $c \ge 0$, a.e. in $Q^T \backslash S^T$. The positivity of $\hat{c}_s^*$ and $\hat{g}$ then follows automatically, as showed in Chapter 3, which completes the proof. □

5.5. Appendix: Stokes Resolvent Problem

In this section, we give a short proof of the solvability of the following Stokes resolvent problem with mixed boundary conditions. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^{3-} with boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ consisting of two closed, disjoint, nonempty components. Consider the resolvent problem

$$\begin{aligned} \lambda \mathbf{u} - \operatorname{div}(D^2 W(\mathbb{I})\nabla \mathbf{u}) + \nabla \pi &= \mathbf{f}, && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0, && \text{in } \Omega, \\ \mathbf{u} &= 0, && \text{on } \Gamma_1, \\ (D^2 W(\mathbb{I})\nabla \mathbf{u} - \pi \mathbb{I})\mathbf{n} &= 0, && \text{on } \Gamma_2, \end{aligned} \tag{5.38}$$

where $\mathbf{n}$ is the outer unit normal on the boundary, $W : \mathbb{R}^{3\times 3} \to \mathbb{R}_+$ is a scalar function with Assumption 5.1 holding.

Theorem 5.24. *Let $1 < q < \infty$. Assume that $\Omega \subset \mathbb{R}^3$ is the domain defined above. Given $\mathbf{f} \in L^q(\Omega)^3$, there exists some $\lambda_0 \in \mathbb{R}$ such that for all $\lambda > \lambda_0$, (5.38) admits a unique solution $(\mathbf{u}, \pi)$ satisfying*

$$\mathbf{u} \in W_q^2(\Omega)^3, \quad \pi \in W_q^1(\Omega).$$

Moreover,

$$\lambda \|\mathbf{u}\|_{L^2(\Omega)^3} + \|\mathbf{u}\|_{W_q^2(\Omega)^3} + \|\pi\|_{W_q^1(\Omega)} \le C \|\mathbf{f}\|_{L^q(\Omega)^3}.$$

Proof. The proof is based on the maximal regularity of a generalized Stokes equation, see e.g. Bothe–Prüss [BP07, Theorem 4.1], Prüss–Simonett [PS16, Theorem 7.3.1]. Let us recall the definition of solenoidal space

$$L^q_\sigma(\Omega) := \{\mathbf{u} \in L^q(\Omega)^3 : \operatorname{div} \mathbf{u} = 0, \, \mathbf{n} \cdot \mathbf{u}|_{\Gamma_1} = 0\}.$$

Then we define a Stokes-type operator as in Section 5.3.2 that

$$\mathcal{A}_q(\mathbf{u}) := \mathbb{P}_q\big(-\operatorname{div}(D^2 W(\mathbb{I})\nabla \mathbf{u})\big) \text{ for all } \mathbf{u} \in \mathcal{D}(\mathcal{A}_q),$$

with

$$\mathcal{D}(\mathcal{A}_q) = \left\{ \mathbf{u} \in W_q^2(\Omega)^3 \cap L^q_\sigma(\Omega) : \mathbf{u}|_{\Gamma_1} = 0, \ \mathcal{P}_{\mathbf{n}}((D^2 W(\mathbb{I})\nabla \mathbf{u})\mathbf{n})\big|_{\Gamma_2} = 0 \right\},$$

where $\mathbb{P}_q$ denotes the *Helmholtz–Weyl projection* on $L^q_\sigma(\Omega)$, see e.g. [Abe10, Appendix A] for the existence of the projection with mixed boundary conditions. $\mathcal{P}_{\mathbf{n}} := \mathbb{I} - \mathbf{n} \otimes \mathbf{n}$ is the tangential projection onto $\partial\Omega$. As in Remark 5.2, the operator $-\operatorname{div}(D^2 W(\mathbb{I})\nabla \cdot)$ is strongly normally elliptic. By e.g. Prüss–Simonett [PS16, Theorem 7.3.2], one knows that $\lambda + \mathcal{A}_q \in \mathcal{MR}_q(L^q_\sigma(\Omega))$ for all $\lambda > \lambda_0 := s(-\mathcal{A}_q)$, where $\mathcal{MR}_q(L^q_\sigma(\Omega))$ means the class of maximal L^q-regularity in $L^q_\sigma(\Omega)$ and $s(-\mathcal{A}_q)$ denotes the spectral bound of $-\mathcal{A}_q$. Consequently, for $\lambda > \lambda_0$ and $\mathbf{f} \in L^q_\sigma(\Omega)$,

$$\lambda \mathbf{u} + \mathcal{A}_q \mathbf{u} = \mathbf{f},$$

is uniquely solvable with $\mathbf{u} \in \mathcal{D}(\mathcal{A}_q)$, which implies that of (5.38).

Now it remains to recover the pressure π. To this end, we solve the Dirichlet–Neumann problem

$$\begin{aligned} \Delta \pi &= \operatorname{div}(\mathbf{f} - \lambda \mathbf{u} + \operatorname{div}(D^2 W(\mathbb{I})\nabla \mathbf{u})), && \text{in } \Omega, \\ \partial_{\mathbf{n}} \pi &= (\mathbf{f} - \lambda \mathbf{u} + \operatorname{div}(D^2 W(\mathbb{I})\nabla \mathbf{u})) \cdot \mathbf{n}, && \text{on } \Gamma_1, \\ \pi &= (D^2 W(\mathbb{I})\nabla \mathbf{u})\mathbf{n} \cdot \mathbf{n}, && \text{on } \Gamma_2, \end{aligned} \tag{5.39}$$

weakly, which is equivalent to the following weak formulation

$$\int_\Omega \nabla \pi \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega \underbrace{(\mathbf{f} - \lambda \mathbf{u} + \operatorname{div}(D^2 W(\mathbb{I})\nabla \mathbf{u}))}_{=:\tilde{\mathbf{f}}} \cdot \nabla \varphi \, \mathrm{d}x, \quad \forall \varphi \in W^1_{q',\Gamma_2}(\Omega). \tag{5.40}$$

Since $\mathbf{f} \in L^q(\Omega)^3$ and $\mathbf{u} \in W_q^2(\Omega)^3 \cap L^q_\sigma(\Omega)$, we have $\tilde{\mathbf{f}} \in L^q(\Omega)^3$. Then (5.40) admits a unique solution $\pi \in W_q^1(\Omega)$, with the aid of [PS16, Theorem 7.4.3]. The boundary regularity is easy due to the third equation of (5.39), and the trace theorem. □

Remark 5.25. As $\mathcal{D}(\mathcal{A}_q)$ embeds compactly into $L^q_\sigma(\Omega)$, the Stokes-type operator $\mathcal{A}_q$ has compact resolvent. Therefore, its spectrum consists only of eigenvalues of finite algebraic multiplicity by spectral theory for compact operators (see e.g. Alt [Alt16]), and is independent of q by Sobolev embeddings. So it is enough to investigate these eigenvalues for the case $q = 2$. Let ω be the eigenvalue of $-\mathcal{A}_2$. Employing the energy method, Lemma 5.2 and the Korn's inequality, we have

$$\begin{aligned}\omega \int_\Omega |\mathbf{u}|^2 \,\mathrm{d}x = -\left\langle \mathcal{A}_2\mathbf{u}, \mathbf{u}\right\rangle_{L^2_\sigma(\Omega)} &= -\int_\Omega D^2W(\mathbb{I})\nabla\mathbf{u} : \nabla\mathbf{u}\,\mathrm{d}x \\ &\le -C\int_\Omega \left|\nabla\mathbf{u} + \nabla\mathbf{u}^\top\right|^2 \,\mathrm{d}x \le -C\int_\Omega |\nabla\mathbf{u}|^2 \,\mathrm{d}x,\end{aligned}$$

which shows that ω is real and nonpositive. Since Ω is bounded and Γ_1 is assumed to be nonempty, the Poincaré's inequality is valid and one obtains

$$\omega \int_\Omega |\mathbf{u}|^2 \,\mathrm{d}x \le -C\int_\Omega |\mathbf{u}|^2 \,\mathrm{d}x,$$

which implies $\omega \le -C$, where $C > 0$ does not depend on λ. Then one conclude that λ_0 defined in Theorem 5.24 is negative and hence if $\lambda = 0$, the theorem still holds true.

Chapter 6

A Diffuse Interface Model for Two-Phase Incompressible Viscoelastic Flows

This chapter concerns a diffuse interface model for the flow of two incompressible viscoelastic fluids in a bounded domain. More specifically, the fluids are assumed to be macroscopically immiscible, but with a small transition region, where the two components are partially mixed. Considering the elasticity of both components, one ends up with a coupled Oldroyd-B/Cahn–Hilliard type system, which describes the behavior of two-phase viscoelastic fluids. In some particular cases, it can be used to describe a class of fluid-structure interaction problems.
We prove the existence of weak solutions to the system in two dimensions for general (unmatched) mass densities, variable viscosities, different shear moduli, and a class of physically relevant and singular free energy densities that guarantee that the order parameter stays in the physically reasonable interval. The proof relies on a combination of a novel regularization of the original system and a new hybrid implicit time discretization for the regularized system together with the analysis of an Oldroyd-B type equation.

Overview of This Chapter

Notations. In this chapter, we consider specifically the following notations.

- $\mathbf{u}$, the Eulerian velocity
- $\mathbb{F}$ or $\mathbf{F}$, the Eulerian deformation gradient
- $\mathbb{B}$, the Eulerian left Cauchy–Green tensor
- ϕ, order parameter

Throughout the chapter, for simplicity we write the induced norms of Lebesgue and Sobolev spaces by $\|\cdot\|_X$ with $X \in \{L^p, W^{k,p}\}$. For the further applications, we give the solenoidal spaces as $L^p_\sigma(\Omega) := \{\mathbf{u} \in L^p(\Omega;\mathbb{R}^d) : \operatorname{div}\mathbf{u} = 0, \mathbf{n}\cdot\mathbf{u}|_{\partial\Omega} = 0\}$ and $W^{k,p}_{0,\sigma}(\Omega) := W^{k,p}_0(\Omega) \cap L^p_\sigma(\Omega)$. Sometimes we use the notation $W^{k,p}_t W^{m,q}_x := W^{k,p}(0,t;W^{m,q}(\Omega))$ with $k, m \in \mathbb{N}_0, 1 \le p, q \le \infty$.

6.1. Introduction

In this chapter we study a so-called *diffuse interface model* (also called *phase field model*) for two incompressible, viscoelastic fluids of different mass densities, viscosities and shear moduli. In the model, a partial mixing of the macroscopically immiscible fluids is considered and elastic effects are taken into account.

This model is quite new and was developed recently in Mokbel–Abels–Aland [MAA18], where they proposed a novel phase-field model for a fluid-structure interaction problem to handle very large deformations as well as topology changes like the contact of a solid to a wall. Under certain assumptions on the system parameters, the model is able to describe a thermodynamically consistent, frame indifferent, incompressible two-phase flow with viscoelasticity of Oldroyd-B type.

Let $T > 0$, $Q_T := \Omega \times (0,T)$ with $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, a sufficiently smooth bounded domain and $S_T := \partial\Omega \times (0,T)$. We consider the following system of Oldroyd-B/Cahn–Hilliard type:

$$\begin{aligned} \partial_t(\rho(\phi)\mathbf{u}) + \operatorname{div}(\rho(\phi)\mathbf{u}\otimes\mathbf{u}) + \operatorname{div}(\mathbf{u}\otimes\mathbf{J}) + \nabla p \qquad & \\ - \operatorname{div}\big(\mathbb{S}(\nabla\mathbf{u},\mathbb{B},\phi)\big) = -\epsilon\tilde{\sigma}\operatorname{div}\big(\nabla\phi\otimes\nabla\phi\big) &\quad \text{in } Q_T, \end{aligned} \tag{6.1a}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } Q_T, \tag{6.1b}$$

$$\partial_t\mathbb{B} + \mathbf{u}\cdot\nabla\mathbb{B} + \frac{\alpha(\phi)}{\lambda(\phi)}(\mathbb{B}-\mathbb{I}) = \mathbb{B}\nabla\mathbf{u}^\top + \nabla\mathbf{u}\mathbb{B} \quad \text{in } Q_T, \tag{6.1c}$$

$$\partial_t\phi + \mathbf{u}\cdot\nabla\phi = \operatorname{div}(m(\phi)\nabla q) \quad \text{in } Q_T, \tag{6.1d}$$

$$q - \tilde{\sigma}\Big(\frac{1}{\epsilon}W'(\phi) - \epsilon\Delta\phi\Big) = \frac{\mu'(\phi)}{2}\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I}) \quad \text{in } Q_T, \tag{6.1e}$$

where $\mathbf{J}$ denotes the relative mass flux associated with the diffusion of the mixture components given by

$$\mathbf{J} := -\rho'(\phi)m(\phi)\nabla q$$

and the stress tensor $\mathbb{S}$ is defined by

$$\mathbb{S}(\nabla\mathbf{u},\mathbb{B},\phi) := \nu(\phi)(\nabla\mathbf{u} + \nabla\mathbf{u}^\top) + \mu(\phi)(\mathbb{B}-\mathbb{I}),$$

as well as the capillary force $\epsilon\tilde{\sigma}\operatorname{div}\big(\nabla\phi\otimes\nabla\phi\big)$ with $\tilde{\sigma} > 0$ related to the surface tension at the interface and $\epsilon > 0$ corresponding to the thickness of the interface. Let ϕ_i be the volume fraction of fluid i, $i \in \{1,2\}$. Define $\phi := \phi_2 - \phi_1$ as the order parameter related to the concentrations of the two fluids. Namely, the value $\phi = -1$ and $\phi = 1$ indicate the unmixed "pure" phases of fluid 1 and fluid 2, respectively. Based on the order parameter $\phi, \mathbf{u}$ and $\rho(\phi)$ are the unknown (volume-averaged) velocity and the density of the mixture of the two fluids given by

$$\mathbf{u} := \frac{1-\phi}{2}\mathbf{u}_1 + \frac{1+\phi}{2}\mathbf{u}_2, \quad \rho(\phi) := \frac{1-\phi}{2}\tilde{\rho}_1 + \frac{1+\phi}{2}\tilde{\rho}_2,$$

where $\mathbf{u}_i$, $\tilde{\rho}_i$, $i \in \{1,2\}$ are the specific velocities and densities of fluid i. Moreover, the matrix $\mathbb{B}$ denotes the left Cauchy–Green tensor related to elasticity, p is the pressure, and q denotes the chemical potential associated to ϕ. In system (6.1), $\nu(\phi) > 0$ denotes the viscosity coefficient, $m(\phi) > 0$ is a (non-degenerate) mobility coefficient, and $\mu(\phi) > 0$ is the shear mudulus. The ratio $\lambda(\phi)/\alpha(\phi) > 0$ refers to the relaxation time of elasticity, which is supposed to be phase-dependent in the case of fluid-structure interaction cf. [MAA18]. Furthermore, $W(\phi)$ is the homogeneous free energy density for the mixture, which is of double-well type. One of the typical examples is the logarithmic potential

$$W(\phi) = \frac{\theta}{2}\big((1+\phi)\ln(1+\phi) + (1-\phi)\ln(1-\phi)\big) - \frac{\theta_c}{2}\phi^2,$$

defined in $[-1,1]$, which leads to a physically relevant value $\phi \in [-1,1]$, with $0 < \theta < \theta_c$ being the absolute temperature and the critical temperature of the mixture. The system is closed with the boundary and initial conditions

$$\mathbf{u} = 0 \qquad \text{on } S_T, \tag{6.1f}$$
$$\partial_{\mathbf{n}}\phi = \partial_{\mathbf{n}} q = 0 \qquad \text{on } S_T, \tag{6.1g}$$
$$(\mathbf{u}, \mathbb{B}, \phi)(0) = (\mathbf{u}_0, \mathbb{B}_0, \phi_0) \quad \text{in } \Omega, \tag{6.1h}$$

where $\partial_{\mathbf{n}} := \mathbf{n} \cdot \nabla$ and $\mathbf{n}$ denotes the exterior normal at $\partial\Omega$. Note that (6.1c) is a transport equation for which no further boundary condition is required, as $\mathbf{n} \cdot \mathbf{u}|_{\partial\Omega} = 0$ ensures that characteristics do not enter the domain.

The total energy of the system (6.1) denoted by $\mathcal{E}(t)$ consists of the kinetic energy, the elastic energy and the Ginzburg–Landau free energy as

$$\mathcal{E}(t) := \underbrace{\int_\Omega \frac{\rho(\phi)}{2} |\mathbf{u}|^2 \, \mathrm{d}x}_{\text{Kinetic energy}} + \underbrace{\int_\Omega \frac{\mu(\phi)}{2} \operatorname{tr}(\mathbb{B} - \ln \mathbb{B} - \mathbb{I}) \, \mathrm{d}x}_{\text{Elastic energy}} + \underbrace{\int_\Omega \tilde{\sigma} \Big(\frac{\epsilon}{2} |\nabla \phi|^2 + \frac{1}{\epsilon} W(\phi) \Big) \, \mathrm{d}x}_{\text{Free energy}} . \tag{6.2}$$

Moreover, every sufficiently smooth solution of (6.1) satisfies the energy dissipation differential inequality

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = &- \int_\Omega \frac{\nu(\phi)}{2} \left| \nabla \mathbf{u} + \nabla \mathbf{u}^\top \right|^2 \, \mathrm{d}x \\ &- \int_{\Omega \setminus \{\lambda = 0\}} \frac{\mu(\phi)\alpha(\phi)}{2\lambda(\phi)} \operatorname{tr}(\mathbb{B} + \mathbb{B}^{-1} - 2\mathbb{I}) \, \mathrm{d}x - \int_\Omega m(\phi) \left| \nabla q \right|^2 \, \mathrm{d}x \le 0. \end{aligned} \tag{6.3}$$

This is referred to [MAA18], where the model was derived via local energy dissipation laws. As they did not provide a detailed comprehension of the elasticity, we give a complete derivation with a more general energy density in Section 1.8. Note that the model (6.1) can be seen as a viscoelastic fluid counterpart of the celebrated Abels–Garcke–Grün (AGG) model [AGG12] where a diffuse interface model was proposed for a two-phase flow of two incompressible fluids with different densities using methods from rational continuum mechanics, which satisfies local and global dissipation inequalities and is frame indifferent.

6.1.1. State of the art. Over the last decades mathematical analysis of fluid dynamics has been developed with an abundant amount of literature. In particular, the problem of two-phase fluids with a free interface fascinated the profound attention of mathematicians. However, the free interface is not easy to track in actual applications and mathematicians still do not have a thorough comprehension of the problems concerning singularities of interfaces and topological changes during the evolution of interfaces. Thus, an effective approximation of the interface has been introduced with the Ginzburg–Landau free energy since, e.g., [GPVn96, HH77], where the authors proposed a so-called "model H" by the phase-field method in terms of the phase variable that indicates the specific phase of the whole system. Later on, this model was investigated and employed explosively in many areas, especially in numerical simulations, since the model has the advantage that topological changes can happen. However, model H was endowed with a basic assumption that the densities of both components are the same, which is not always physically reasonable. To overcome this disadvantage, Abels–Garcke–Grün [AGG12] derived the "AGG model", which considers unmatched densities and is thermodynamically consistent and frame indifferent. For the analysis of weak solutions and strong solutions to the AGG model, we refer to [ADG13a, ADG13b, AGG23, Gio21]. An alternative and thermodynamically consistent model was derived before by Lowengrub–Truskinovski [LT98] by using the mass-averaged (barycentric) velocity. From the mathematical viewpoint, the latter has the disadvantage that the mass-averaged velocity is not divergence free, which is the case in the AGG model based on a volume-averaged velocity. In view of the physical advantages of the AGG model, there came many variants subsequently concerning different aspects. Readers are referred, for example, to [Fri16] for the nonlocal Cahn–Hilliard–Navier–Stokes equations with unmatched densities, to [GGW19, GK23] for diffuse interface models including moving contact lines, to [Sie20] for polymeric fluids, to [KMS23] for magnetohydrodynamic two-phase flow with different densities, and to [MAA18] for a diffuse interface model simulating a fluid-structure interaction problem with viscoelasticity of Oldroyd-B type, which is exactly of our interest.

Now let us recall some facts about viscoelasticity, in particular of Oldroyd-B type. Depending on the deformation gradient $\mathbf{F}$ that is defined in Eulerian coordinates or the corresponding left Cauchy–Green tensor $\mathbb{B} := \mathbf{F}\mathbf{F}^\top$, viscoelasticity can be recognized as two regimes. In the context of $\mathbf{F}$, the incompressible viscoelastic fluid model reads as

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p = \operatorname{div}(\mathbf{F}\mathbf{F}^\top), \\ \partial_t \mathbf{F} + \mathbf{u} \cdot \nabla \mathbf{F} = \nabla \mathbf{u} \mathbf{F}, \\ \operatorname{div} \mathbf{u} = 0, \end{cases} \tag{6.4}$$

whose derivation is referred to, e.g., Giga–Kirshtein–Liu [GKL18] by the energy variational approach. The analysis of (6.4) has been quite popular in the last decade, and the global well-posedness and long time behavior have been established in, e.g., [HW15, LLZ08, LLZ05, LZ08] and their citations. However, due to the lack of compactness for the deformation gradient, problems concerning the existence of weak solutions are still open even in two dimensions. We refer to two recent progressive papers by Hu–Lin [HL16a, HL16b] for the existence of 2D weak solutions with small initial data in L^p and L^2, respectively, where the so-called *effective viscous flux* method was employed. Based on this viscoelastic approach, some very interesting models that describe the motion of two-phase flows have been introduced in the very recent years. Readers are referred to Agosti–Colli–Garcke–Rocca [Ago+23] for a phase-field model coupled with viscoelasticity with large deformations, and Kim–Tawri–Temam [KTT22] for a diffuse interface model describing the interaction between blood flow and a thrombus with Hookean elasticity during the stage of atherosclerotic lesion in human artery.

When it comes to the left Cauchy–Green tensor $\mathbb{B} = \mathbf{F}\mathbf{F}^\top$, the so-called Oldroyd-B equation without dampings or with infinite Weissenberg number can be directly derived from (6.4), with $(6.4)_2$ rewritten in terms of $\mathbb{B}$ as

$$\partial_t \mathbb{B} + \mathbf{u} \cdot \nabla \mathbb{B} - \nabla \mathbf{u} \mathbb{B} - \mathbb{B} \nabla \mathbf{u}^\top = \mathbf{0}.$$

By this approach, one has to face instabilities in applications due to the so-called *high Weissenberg number problem*, which is still not fully understood, see, e.g., [HLL18] and references therein. It is possible to avoid this problem by including additional dissipative terms, for instance, in terms of elastic relaxation. The original formulation of the Oldroyd-B model includes relaxation and is formulated with respect to the elastic stress tensor in linear elasticity $\boldsymbol{\tau} := \mathbb{B} - \mathbb{I}$, i.e.,

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} + \nabla p = \operatorname{div} \boldsymbol{\tau}, \\ \partial_t \boldsymbol{\tau} + \mathbf{u} \cdot \nabla \boldsymbol{\tau} - \nabla \mathbf{u} \boldsymbol{\tau} - \boldsymbol{\tau} \nabla \mathbf{u}^\top + a \boldsymbol{\tau} = \dfrac{b}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top), \\ \operatorname{div} \mathbf{u} = 0, \end{cases} \tag{6.5}$$

where $a, b \geq 0$ and $1/a$ denotes the relaxation time and b/a is the viscosity of the polymers. For a derivation, we refer to [MP18, Old50, RT21]. It can also be recovered from the micro-macroscopic FENE (Finite Extensible Nonlinear Elastic) dumbbell model, see, e.g., [GKL18]. Concerning the analysis of (6.5), the first existence result of weak solutions goes back to Lions–Masmoudi [LM00], where they considered a co-rotational version of the stress tensor equation to get better estimates and compactness. Due to lack of compactness, the existence of weak solutions to (6.5) has been an open problem for a long time. We mention a breakthrough on a related model, for which Masmoudi [Mas13] proved the existence of weak solutions to the FENE dumbbell model with many weak convergence techniques, based on the control of the propagation of strong convergence of some well-chosen quantity by studying a transport equation for its defect measure. Instead of solving (6.5) directly, in recent years mathematicians tried to include some regularization terms for the Oldroyd-B equation, for example the diffusive stress $\Delta\boldsymbol{\tau}$, in order to obtain higher regularity and compactness, which is however still limited to two dimensions. Barrett–Boyaval [BB11] proved the existence of weak solutions to a regularized Oldroyd-B model with stress diffusion in 2D by employing an approximating finite element scheme, together with an entropy regularization, while the global regularity of 2D diffusive Oldroyd-B equations was obtained by Constantin–Kliegl [CK12]. Note that in [BB11], the authors reformulated an equivalent system in terms of the left Cauchy–Green tensor $\mathbb{B}$ instead of $\boldsymbol{\tau}$, and made use of a physically relevant elastic $\operatorname{tr}(\mathbb{B} - \ln \mathbb{B} - \mathbb{I})$ and its estimate, which was derived before by Hu–Lelièvre [HL07]. One of the advantages of the formulation with respect to $\mathbb{B}$ is that one can expect the positive definiteness of it, while in the other formulation this is not clear at all. Later, a compressible counterpart of the (diffusive) Oldroyd-B model was proposed and analyzed in [BLS17] by means of a combination of multilayer approximations from [BB11] and the compressible Navier–Stokes equation [FN17], based on the Galerkin scheme. For other related viscoelastic models, one refers to [BLLM22, LM+17] for Peterlin type models, and to [BBM21] for a mixing of the Oldroyd-B model and the Giesekus model. Along with the development of diffuse interface models, two-phase viscoelasticity came up in recent years. Grün–Metzger [GM16] derived a micro-macro model for two-phase flow of dilute polymeric solutions, where a Fokker – Planck type equation describes orientation and elongation of polymers. Sieber [Sie20] then extended the model in [GM16] to the case with different mass densities for polymeric fluids. Moreover, a viscoelastic phase separation model of Peterlin type was derived in [Bru+21] and investigated in [BLM22a, BLM22b]. By virtue of the Oldroyd-B model, recently Garcke–Kovács–Trautwein [GKT22] established a viscoelastic Cahn–Hilliard model to describe tumor growth and obtained the existence of weak solutions in the case of matched mass densities with the help of finite element approximations.

In contrast to the growing literature on two-phase flows and viscoelastic fluids, the analytical results about two-phase viscoelasticity are quite limited, especially the Oldroyd-B type models with unmatched densities and variable shear moduli. To the best of our knowledge, the only related results of diffuse interface model including viscoelasticity of Oldroyd-B type are [GKT22, Sie20] we mentioned above, where they showed the existence of weak solutions for 2D two-phase Oldroyd-B type fluids with polynomial potential, while none of them concerned the case of unmatched densities and phase-depending shear moduli, which in fact, is rather physical and interesting. The aim of the present paper is to provide a deep understanding on the theory of incompressible two-phase flow with viscoelasticity of Oldroyd-B type with more physical assumptions, in company with the AGG model [ADG13a, AGG12]. More precisely, we are going to prove the existence of weak solutions to the incompressible Oldroyd-B/Cahn–Hilliard model in the presence of variable densities, shear moduli, viscosities, and a singular potential.

6.1.2. Reformulation with a modified pressure and stress diffusion. Before recording our main result, one reformulates the system with a modified pressure which, for the sake of analysis of weak solutions, leads to a more convenient right-hand side of (6.1a). Note that both formulations are equivalent for sufficiently smooth solutions. In particular, we define a new scalar pressure as

$$\pi := p + \frac{\tilde{\sigma}\epsilon}{2}|\nabla\phi|^2 + \frac{\tilde{\sigma}}{\epsilon}W(\phi) + \frac{\mu(\phi)}{2}\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I}).$$

Then we have

$$\nabla\pi = \nabla p + q\nabla\phi + \tilde{\sigma}\epsilon \operatorname{div}\left(\nabla\phi\otimes\nabla\phi\right) + \frac{\mu(\phi)}{2}\nabla\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I}).$$

Additionally, it is necessary for the analysis to introduce a diffusive regularization $\frac{\kappa}{\mu(\phi)}\Delta\mathbb{B}$ for a $\kappa > 0$ in (6.1c) such that the evolution law of the Cauchy–Green tensor $\mathbb{B}$ is of parabolic nature. The system is closed with an additional no-flux boundary condition $\partial_{\mathbf{n}}\mathbb{B} = \mathbf{0}$, see also in [Ago+23, BB11, BBM21, GKT22]. For the purpose of readability, in the following we will let $\alpha(\cdot) = \lambda(\cdot) = \epsilon = \tilde{\sigma} = 1$ since they have no significant contribution to the analysis.

The full system of our interest now reads

$$\begin{aligned}
&\partial_t(\rho(\phi)\mathbf{u}) + \operatorname{div}(\rho(\phi)\mathbf{u}\otimes\mathbf{u}) + \operatorname{div}(\mathbf{u}\otimes\mathbf{J}) + \nabla\pi \\
&\quad - \operatorname{div}\left(\mathbb{S}(\nabla\mathbf{u},\mathbb{B},\phi)\right) = q\nabla\phi + \frac{\mu(\phi)}{2}\nabla\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I}) && \text{in } Q_T, && (6.6a)\\
&\operatorname{div}\mathbf{u} = 0 && \text{in } Q_T, && (6.6b)\\
&\partial_t\mathbb{B} + \mathbf{u}\cdot\nabla\mathbb{B} + (\mathbb{B} - \mathbb{I}) = \mathbb{B}\nabla\mathbf{u}^\top + \nabla\mathbf{u}\mathbb{B} + \frac{\kappa}{\mu(\phi)}\Delta\mathbb{B} && \text{in } Q_T, && (6.6c)\\
&\partial_t\phi + \mathbf{u}\cdot\nabla\phi = \operatorname{div}(m(\phi)\nabla q) && \text{in } Q_T, && (6.6d)\\
&q - W'(\phi) + \Delta\phi = \frac{\mu'(\phi)}{2}\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I}) && \text{in } Q_T, && (6.6e)\\
&\mathbf{u} = 0,\ \partial_{\mathbf{n}}\mathbb{B} = \mathbf{0} && \text{on } S_T, && (6.6f)\\
&\partial_{\mathbf{n}}\phi = \partial_{\mathbf{n}}q = 0 && \text{on } S_T, && (6.6g)\\
&(\mathbf{u},\mathbb{B},\phi)(0) = (\mathbf{u}_0,\mathbb{B}_0,\phi_0) && \text{in } \Omega. && (6.6h)
\end{aligned}$$

Remark 6.1. In comparison to, e.g., [BB11], the special structure of the additional diffusive term $\frac{\kappa}{\mu(\phi)}\Delta\mathbb{B}$ in (6.6c) is of main importance in presence of a phase-depending shear modulus $\mu(\phi)$. The coefficient $\frac{1}{\mu(\phi)}$ is of much help to keep the energy dissipation structure in (6.3), i.e.,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = &-\int_\Omega \frac{\nu(\phi)}{2}\left|\nabla\mathbf{u} + \nabla\mathbf{u}^\top\right|^2 \mathrm{d}x - \frac{\kappa}{2}\int_\Omega |\nabla\operatorname{tr}\ln\mathbb{B}|^2\,\mathrm{d}x \\
&- \int_\Omega \frac{\mu(\phi)}{2}\operatorname{tr}(\mathbb{B} + \mathbb{B}^{-1} - 2\mathbb{I})\,\mathrm{d}x - \int_\Omega m(\phi)\,|\nabla q|^2\,\mathrm{d}x \le 0.
\end{aligned}$$

In general, the additional diffusive term $\frac{\kappa}{\mu(\phi)}\Delta\mathbb{B}$ can also be motivated by means of a nonlocal storage of energy or a nonlocal entropy production mechanisms, see Málek–Průša–Skřivan–Süli [M+18].

Remark 6.2. Here, an additional term $\frac{\mu(\phi)}{2}\nabla\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I})$ appears in (6.6a) after modifying the pressure. In [BLS17], a regularization term $\alpha\nabla\operatorname{tr}\ln\mathbb{B}$ was added in the momentum equation but here we get it for free, since the shear modulus $\mu(\phi)$ is unmatched among two fluids.

6.1.3. Main results. Now we are in the position to state our main result of this chapter. Namely, the global existence of weak solutions to (6.6) in two dimensions, which describes the incompressible viscoelastic two-phase flow with general (unmatched) mass density, variable viscosity, different shear moduli and a singular free energy density in two dimensions. We note that the notation will be explained in the beginning of Section 6.2.

Theorem 6.3 (Proved in Section 6.5.2). *Let $d = 2$ and let Assumption 6.12 hold true (see Section 6.2.2). Assume that the initial data satisfy $(\mathbf{u}_0, \mathbb{B}_0, \phi_0) \in L^2_\sigma(\Omega) \times L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}}) \times W^{1,2}(\Omega)$ with $\mathbb{B}_0$ positive definite a.e. in Ω, $\operatorname{tr}\ln \mathbb{B}_0 \in L^1(\Omega)$, and $|\phi_0| \le 1$ a.e. in Ω and $⨍_\Omega \phi_0 \,\mathrm{d}x \in (-1, 1)$. Then for any $T \in (0, \infty)$, there exists a weak solution $(\mathbf{u}, \mathbb{B}, \phi, q)$ of (6.6) in the sense of Definition 6.14. Moreover, the left Cauchy–Green tensor $\mathbb{B}$ fulfills the following estimate*

$$\|\mathbb{B}(t)\|^2_{L^2} + \int_0^t \|\mathbb{B}(\tau)\|^2_{L^2} \,\mathrm{d}\tau + \int_0^t \|\nabla\mathbb{B}(\tau)\|^2_{L^2} \,\mathrm{d}\tau \le C(\mathcal{E}(0), \|\mathbb{B}_0\|^2_{L^2})$$

for a.e. $t \in (0, T)$.

Remark 6.4. Here we assume that the initial data $\mathbb{B}_0$ additionally satisfies

$$\operatorname{tr}\ln \mathbb{B}_0 \in L^1(\Omega).$$

This is for the sake of the estimate of the elastic energy in (6.2). In general, one can not expect to derive it from $\mathbb{B}_0 \in L^2(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}+})$ since $\int_\Omega \operatorname{tr}\ln \mathbb{B}_0 \,\mathrm{d}x$ is not necessarily bounded from below. This is a rather mild restriction, which was also used in Barrett–Lu–Süli [BLS17]. Note that in Barrett–Boyaval [BB11], a stronger restriction was employed, that is, $\mathbb{B}_0$ is uniformly positive definite, i.e.,

$$\underline{\mathbb{B}}_0 \, |\mathbf{w}|^2 \le \mathbf{w}^\top \mathbb{B}_0(x)\mathbf{w} \le \overline{\mathbb{B}}_0 \, |\mathbf{w}|^2 \quad \text{for all } \mathbf{w} \in \mathbb{R}^2 \text{ and for a.e. } x \in \Omega,$$

where $\underline{\mathbb{B}}_0, \overline{\mathbb{B}}_0 \in \mathbb{R}_+$.

Remark 6.5. In this chapter, we consider an isotropic free energy and a regular mobility. These are simplifications as our main focus lies on the unmatched densities, viscosities and shear moduli. We comment that more general cases can be obtained by suitable modifications in our current framework. See also Remark 6.41 for further discussion.

Remark 6.6. For simplicity of the model, α, λ are assumed to be constant. The existence result also holds true, if $\alpha, \lambda : \mathbb{R} \to \mathbb{R}$ are continuous, positive and bounded functions that depend on the phase-field variable ϕ.

6.1.4. Strategy of the proof (technical discussions). Note that the system (6.6) has a similar structure as the AGG model [AGG12], which was solved with the help of an implicit time discretization scheme in [ADG13a]. In order to combine this with the existence of weak solutions to an Oldroyd-B fluid system proved in [BB11], we introduce a novel regularization to (6.6) (see $\mathscr{R}_\eta$ below, also in (6.40)), such that we can solve the regularized system in a suitable way and have good uniform *a priori* estimates (see Section 6.4.1), inspired by [Abe09b, BS18, LR14]. More specifically, the advantages of the regularization is threefold. Due to the presence of a phase-dependent shear modulus, the regularity of the terms with the coefficient $\mu(\phi)$ will cause problems if one tries the standard testing procedure. For example, multiplying the second term in (6.6c) with $\frac{\mu(\phi)}{2}(\mathbb{I} - \mathbb{B}^{-1})$ and integrating by parts over Ω leads to

$$\begin{aligned}\int_\Omega \mathbf{u}\cdot\nabla\mathbb{B} : \frac{\mu(\phi)}{2}(\mathbb{I} - \mathbb{B}^{-1}) \,\mathrm{d}x &= \int_\Omega \mathbf{u}\cdot\nabla\left(\frac{\mu(\phi)}{2}\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I})\right)\mathrm{d}x \\ &\quad - \int_\Omega \frac{\mu'(\phi)}{2}\mathbf{u}\cdot\nabla\phi \operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I}) \,\mathrm{d}x\end{aligned}$$

where $\mathbf{u} \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;W^{1,2}_0(\Omega;\mathbb{R}^d))$. The first term on the right-hand side vanishes for such $\mathbf{u}$ (as it is solenoidal), while it is not clear if the second integral is well-defined or not under the current regularity setting. With the regularized terms $\mu(\mathscr{R}_\eta \phi)$ and $\mathscr{R}_\eta \mathbf{u}$ in the regularized system, it is not an issue anymore since $\mathscr{R}_\eta \phi$ and $\mathscr{R}_\eta \mathbf{u}$ are sufficiently smooth for any $\eta > 0$. The second motivation of the regularization comes from the solvability of the Oldroyd-B equation, see (6.20) below. Namely, this kind of regularization on the velocities in (6.40c) and $\eta \partial_t \phi$ in (6.40e) allow us to show existence, uniqueness and continuous dependence for it, even in three dimensions, which has been an open problem for a long time for the Oldroyd-B model. Moreover, under such regularization, one does not need to construct multiple layers of approximations for the whole system as in, e.g., [BB11, BLS17]. Instead, we separate the proof into solving two relatively easy systems.

Concerning the solvability of the regularized Oldroyd-B equation in three dimensions, we make use of the standard Galerkin approximation, on accounting to the parabolic nature. In addition, inspired by [BB11], an entropy regularization for $\ln \mathbb{B}$ is introduced, which is of importance since the left Cauchy–Green tensor $\mathbb{B}$ is not necessarily positive definite almost everywhere in Q_T and hence the physical energy $\mathrm{tr}(\mathbb{B} - \ln \mathbb{B} - \mathbb{I})$ is not necessarily well-defined in presence of the logarithmic term. Thanks to the regularization $\mathscr{R}_\eta$, we obtain uniform estimates and then pass to the limit for both the entropy regularization and the Galerkin approximation. Finally, we conclude the positive definiteness of the limit function (i.e. the left Cauchy–Green tensor $\mathbb{B}$) with a contradiction argument.

To solve the regularized system in 3D, one natural idea would be to employ a time discretization scheme for the full system as in [ADG13a]. However, it is not applicable to repeat all the regularization and approximation techniques for practicality and readability. Thus, in this chapter, we propose a novel scheme, we called "*hybrid time discretization*" (see Section 6.4.3), to obtain a suitable approximation of the full system, combined with the solvability of the Oldroyd-B system. Specifically, for the AGG part $(\mathbf{u}, \phi)$ we keep the implicit time discretization as in [ADG13a], namely the piecewise constant discrete solution with time-averaged terms regarding $\mathbb{B}$, while the Oldroyd-B part is solved in each time interval continuously in time with the time-averaged solution from the previous time step as the initial datum. Due to the well-posedness of the Oldroyd-B part, one is able to construct a continuous mapping of $\mathbb{B}$ in terms of $(\mathbf{u}, \phi)$ in the discrete AGG system, which is solvable in a similar fashion as in [ADG13a]. The first essential ingredient of the hybrid time discretization scheme is a uniform estimate, for which we finalize with the energy estimate of $\mathbb{B}$ on a discrete time interval and cancel out all the mixing terms. This is where we use the time-averaged approximation of terms containing $\mathbb{B}$, see (6.55) and (6.56). The second crucial element is the compactness of these time-averaged terms, as the limit passage in these terms requires additional arguments. For this, we provide compactness results for weakly-$*$ convergent and weakly convergent sequences, respectively, with the help of a convolution (in time) with Dirac sequences and the Lebesgue differentiation theorem, see Section 6.4.6. Note that one can not apply the Aubin–Lions lemma for the approximate Cauchy–Green tensor, say $\widetilde{\mathbb{B}}^N$, since it is endowed with jumps across the time intervals and one can not expect the existence of $\partial_t \widetilde{\mathbb{B}}$ over the whole time interval. This is overcome by a compactness argument with time translations, see Section 6.4.7.

The final step is the passage to the limit in the regularized system as the regularization parameter $\eta \to 0$. This can be realized with a compactness argument from the concise uniform *a priori* estimate (6.51) and a stronger estimate (6.78) for $\mathbb{B}$, which are both independent of $\eta > 0$. The stronger estimate is necessary since the energy estimate (6.51) only provides merely L^1 information of $\mathbb{B}$. Note that here the stronger estimate for $\mathbb{B}$ relies on the Gagliardo–Nirenberg inequality which requires the restriction to two dimensions. Then, in light of the compactness of $\mathbf{u}$, which is derived with a Helmholtz projection and the convergence of the kinetic energy, one obtains the existence of a weak solution to the original full system (6.6) by passing to the limit as $\eta \to 0$.

6.1.5. Outline. The rest of the chapter unfolds as follows. In Section 6.2, we explain the notations and provide some auxiliary lemmata. Then, we present the main assumptions and the definition of weak solutions to (6.6). In Section 6.3 we study the matrix-valued equation for the left Cauchy–Green tensor with η-regularization in three dimensions. The well-posedness is of particular significance and is obtained by a Galerkin approximation and an entropy regularization. Section 6.4 is devoted to the analysis of the regularized system, which is solved with a novel hybrid time discretization scheme and the Leray–Schauder principle. In Section 6.5 we finish the proof of Theorem 6.3 (the existence of weak solutions to (6.6)) by passing to the limit in the regularization, i.e., $\eta \to 0$, together with a uniform estimate derived in Section 6.4.1 and a stronger estimate for $\mathbb{B}$ in Section 6.5.1.

6.2. Preliminaries and Weak Solutions

6.2.1. Linear algebra of matrices. For any $\mathbb{B} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, there exists a diagonal decomposition

$$\mathbb{B} = \mathbb{O}^\top \mathbb{D}\mathbb{O}, \text{ satisfying } \operatorname{tr}\mathbb{B} = \operatorname{tr}\mathbb{D}, \tag{6.7}$$

where $\mathbb{O} \in \mathbb{R}^{d\times d}$ is an orthogonal matrix and $\mathbb{D} \in \mathbb{R}^{d\times d}$ is a diagonal matrix. For any symmetric positive definite $\mathbb{B} \in \mathbb{R}^{d\times d}_{\mathrm{sym}+}$, we define its real logarithm $\ln\mathbb{B}$ as a symmetric matrix such that $e^{\ln\mathbb{B}} = \mathbb{B}$. Indeed, in view of (6.7), we have

$$\ln\mathbb{B} = \mathbb{O}^\top \operatorname{diag}\{\ln\lambda_1, \ln\lambda_2, \dots, \ln\lambda_d\}\mathbb{O},$$

where $\mathbb{D} = \operatorname{diag}\{\lambda_1, \lambda_2, \dots, \lambda_d\}$ with $\lambda_k > 0$, $k = 1, \dots, d$, the eigenvalues of $\mathbb{B}$, which implies $\operatorname{tr}\ln\mathbb{B} = \ln\det\mathbb{B}$.

We recall a set of properties for matrix-valued functions.

Lemma 6.7 (Properties of matrix-valued functions). *Let $\mathbb{B} : \mathcal{U} \to \mathbb{R}^{d\times d}_{\mathrm{sym}+}$ with some open $\mathcal{U} \subset \mathbb{R}^n$, $n \le d$, $d, n \in \mathbb{N}$, be differentiable with respect to a first-order differential operator $\partial\cdot$. Then*

$$\mathbb{B} + \mathbb{B}^{-1} - 2\mathbb{I} \textit{ is symmetric and } \operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I}) \ge 0, \tag{6.8}$$

$$\partial\mathbb{B} : \mathbb{B}^{-1} = \operatorname{tr}\left(\mathbb{B}^{-1}\partial\mathbb{B}\right) = \partial\operatorname{tr}\ln\mathbb{B}, \tag{6.9}$$

$$\partial\ln\mathbb{B} : \mathbb{B} = \operatorname{tr}\left(\mathbb{B}\partial\ln\mathbb{B}\right) = \partial\operatorname{tr}\mathbb{B}, \tag{6.10}$$

$$\partial\mathbb{B} : (\mathbb{I} - \mathbb{B}^{-1}) = \partial\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I}). \tag{6.11}$$

Moreover, let $\mathbb{B} = (B_{jk})_{j,k=1}^d, \mathbb{C} = (C_{jk})_{j,k=1}^d \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $\mathbf{v} \in C^1(\mathbb{R}^d; \mathbb{R}^{d\times d})$, then

$$\left(\nabla\mathbf{v}\mathbb{B} + \mathbb{B}\nabla\mathbf{v}^\top\right) : \mathbb{C} = 2\left(\mathbb{C}\mathbb{B}\right) : \nabla\mathbf{v}. \tag{6.12}$$

Proof. We refer to [BB11, BLS17]. □

Auxiliary results.

Lemma 6.8. *Let $E : [0,T) \to [0,\infty)$, $0 < T \le \infty$, be a lower semicontinuous function and let $D : (0,T) \to [0,\infty)$ be an integrable function. Then*

$$E(0)\varsigma(0) + \int_0^T E(t)\varsigma'(t)\,\mathrm{d}t \ge \int_0^T D(t)\varsigma(t)\,\mathrm{d}t$$

holds for all $\varsigma \in W^{1,1}(0,T)$ *with* $\varsigma(T) = 0$ *and* $\varsigma \geq 0$ *if and only if*

$$E(t) + \int_s^t D(\tau)\,\mathrm{d}\tau \leq E(s)$$

holds for all $s \leq t < T$ *and almost all* $0 \leq s < T$ *including* $s = 0$.

Proof. For the proof, one is referred to Abels [Abe09a, Lemma 4.3]. □

Lemma 6.9. *Let* X, Y *be two Banach spaces such that* $Y \hookrightarrow X$ *and* $X' \to Y'$ *densely and let* $0 < T < \infty$. *Then*

$$L^\infty(0,T;Y) \cap C([0,T];X) \hookrightarrow C_w([0,T];Y).$$

Proof. See Abels [Abe09a, Lemma 4.1]. □

Let (X_0, X_1) be a compatible couple of Banach spaces, that is, there is a Hausdorff topological vector space Z such that $X_0, X_1 \hookrightarrow Z$, and let $(\cdot,\cdot)_{[\theta]}$, $(\cdot,\cdot)_{(\theta,r)}$, $\theta \in [0,1]$, $r \in [1,\infty]$, denote the complex and real interpolation functor, respectively, cf. Abels [Abe09b], Bergh–Löfström [BL76]. Then we have the following lemma.

Lemma 6.10 ([BL76, Theorem 5.1.2]). *Let* (X_0, X_1) *be a compatible couple of Banach spaces,* $I \subset \mathbb{R}$ *and* $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$. *For all* $1 \leq p_0 < \infty$, $1 \leq p_1 \leq \infty$, *and* $\theta \in (0,1)$,

$$(L^{p_0}(I;X_0), L^{p_1}(I;X_1))_{[\theta]} = L^p(I;(X_0,X_1)_{[\theta]}),$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. *In particular, for* $p_0 = 2$, $p_1 = \infty$, $p = 4$, $X_0 = W^{1,2}(\Omega)$ *and* $X_1 = L^2(\Omega)$, *it holds*

$$(L^2(I;W^{1,2}(\Omega)), L^\infty(I;L^2(\Omega))_{[\frac{1}{2}]} = L^4(I;H^{\frac{1}{2},2}(\Omega)) \hookrightarrow L^4(I;L^s(\Omega)), \tag{6.13}$$

where $s = 3$ *if* $d = 3$ *and* $s = 4$ *if* $d = 2$. *The space* $H^{s,2}$ *for* $0 < s < 1$ *is the Bessel potential space.*

Mollifiers. For a function $\psi \in C_0^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \psi\,\mathrm{d}x = 1$, $\psi(x) = \psi(-x) \geq 0$ and $\operatorname{supp}\psi \subset \mathbb{R}^d$, define $\psi_\eta(x) := \eta^{-d}\psi(\frac{x}{\eta})$, where $\eta > 0$. Then we introduce a regularization operator $\mathscr{R}_\eta$ by

$$\mathscr{R}_\eta \mathbf{w} := \psi_\eta * \mathbf{w} = \int_{\mathbb{R}^d} \psi_\eta(x-y)\mathbf{w}(y)\,\mathrm{d}y = \int_\Omega \psi_\eta(x-y)\mathbf{w}(y)\,\mathrm{d}y, \tag{6.14}$$

with $\mathbf{w}$ defined on $\Omega \subset \mathbb{R}^d$ extended by 0 outside of Ω. Moreover, the following properties are satisfied.

Lemma 6.11 (Mollification). *Let* X *be a Banach space. If* $\mathbf{v} \in L^1_{loc}(\mathbb{R}^d;X)$, *then we have* $\mathscr{R}_\eta \mathbf{v} \in C^\infty(\mathbb{R}^d;X)$. *Furthermore, the following holds:*

(1) For $\mathbf{u}, \mathbf{v} \in L^1_{loc}(\mathbb{R}^d)$, *it holds that*

$$\int_{\mathbb{R}^d} \mathscr{R}_\eta \mathbf{u} \cdot \mathbf{v}\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathbf{u} \cdot \mathscr{R}_\eta \mathbf{v}\,\mathrm{d}x. \tag{6.15}$$

(2) If $\mathbf{v} \in L^p_{loc}(\mathbb{R}^d;X)$, $1 \leq p < \infty$, *then* $\mathscr{R}_\eta \mathbf{v} \in L^p_{loc}(\mathbb{R}^d;X)$ *and* $\mathscr{R}_\eta \mathbf{v} \to \mathbf{v}$ *in* $L^p_{loc}(\mathbb{R}^d;X)$ *as* $\eta \to 0$.

(3) If $\mathbf{v} \in L^p(\mathbb{R}^d;X)$, $1 \leq p < \infty$, *then* $\mathscr{R}_\eta \mathbf{v} \in L^p(\mathbb{R}^d;X)$, $\|\mathscr{R}_\eta \mathbf{v}\|_{L^p(\mathbb{R}^d;X)} \leq \|\mathbf{v}\|_{L^p(\mathbb{R}^d;X)}$ *and* $\mathscr{R}_\eta \mathbf{v} \to \mathbf{v}$ *in* $L^p(\mathbb{R}^d;X)$ *as* $\eta \to 0$.

(4) If $\mathbf{v} \in L^p(\mathbb{R}^d;X)$*,* $1 \leq p < \infty$*, then* $\|\mathscr{R}_\eta \mathbf{v}\|_{L^\infty(\mathbb{R}^d;X)} \leq C(\eta)\,\|\mathbf{v}\|_{L^p(\mathbb{R}^d;X)}$ *with* $C(\eta) > 0$ *depending on* η*.*

(5) If $\mathbf{v} \in L^\infty(\mathbb{R}^d;X)$*, then* $\mathscr{R}_\eta \mathbf{v} \in L^\infty(\mathbb{R}^d;X)$*, and* $\|\mathscr{R}_\eta \mathbf{v}\|_{L^\infty(\mathbb{R}^d;X)} \leq \|\mathbf{v}\|_{L^\infty(\mathbb{R}^d;X)}$*.*

Proof. The first statement is a direct consequence of $\psi(x) = \psi(-x)$, while the fourth one can be derived by taking the supremum norm of ψ. The remaining statements are referred to [FN17, Theorem 11.3]. □

6.2.2. Assumptions. In the following we summarize the assumptions that are necessary to formulate the notation of a weak solution.

Assumption 6.12. We assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary. Moreover, we impose the following conditions.

(H1) The density of the model is given by $\rho(\phi) = \frac{1}{2}(\tilde{\rho}_1 + \tilde{\rho}_2) + \frac{1}{2}(\tilde{\rho}_2 - \tilde{\rho}_1)\phi$, which is derived in [AGG12, MAA18]. Here $\tilde{\rho}_i > 0$ denote the constant unmatched densities of two fluids and ϕ is the difference of the volume fractions of the fluids.

(H2) We assume $m, \mu \in C^1(\mathbb{R})$, $\nu \in C(\mathbb{R})$ and they have the corresponding constant lower and upper bounds, i.e., $0 < \underline{m} \leq m \leq \overline{m}$, $0 < \underline{\nu} \leq \nu \leq \overline{\nu}$, $0 < \underline{\mu} \leq \mu \leq \overline{\mu}$ and $0 < \underline{\mu}' \leq \mu' \leq \overline{\mu}'$.

(H3) The free energy density is assumed to be a general function $W \in C([-1,1]) \cap C^2((-1,1))$ that satisfies

$$\lim_{s\to -1} W'(s) = -\infty, \quad \lim_{s\to 1} W'(s) = +\infty, \quad W''(s) \geq -\omega \text{ for some } \omega \in \mathbb{R}.$$

Remark 6.13. The Assumption **(H3)** allows for a nonconvex potential W, which has the domain of definition $[-1,1]$. Then, the evaluation of $W(\phi)$ automatically induces the evaluation of a physically relevant $\phi \in (-1,1)$, with $W'(\phi) < \infty$. One typical example is the so-called logarithmic potential as given in Section 6.1.

6.2.3. Weak solutions. Now we give the definition of weak solutions to (6.6).

Definition 6.14. Let $T > 0$ and $(\mathbf{u}_0, \mathbb{B}_0, \phi_0) \in L^2_\sigma(\Omega) \times L^2(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}}) \times W^{1,2}(\Omega)$ with $\mathbb{B}_0$ positive definite, $\operatorname{tr}\ln \mathbb{B}_0 \in L^1(\Omega)$ and $|\phi_0| \leq 1$ almost everywhere in Ω. In addition, let Assumption 6.12 hold true. We call the quadruple $(\mathbf{u}, \phi, q, \mathbb{B})$ a *finite energy* weak solution to (6.6) with initial data $(\mathbf{u}_0, \mathbb{B}_0, \phi_0)$, provided that

(1) the quadruple $(\mathbf{u}, \phi, q, \mathbb{B})$ satisfies

$$\begin{gathered}
\mathbf{u} \in C_w([0,T]; L^2_\sigma(\Omega)) \cap L^2(0,T; W^{1,2}_0(\Omega;\mathbb{R}^2));\\
\phi \in C_w([0,T]; W^{1,2}(\Omega)) \cap L^2(0,T; W^{2,2}(\Omega)) \text{ with } \phi \in (-1,1) \text{ a.e. in } Q_T;\\
W'(\phi) \in L^2(0,T; L^2(\Omega)), \quad q \in L^2(0,T; W^{1,2}(\Omega));\\
\mathbb{B} \text{ is symmetric positive definite a.e. in } Q_T;\\
\mathbb{B} \in C_w([0,T]; L^2(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}})) \cap L^2(0,T; W^{1,2}(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}}));\\
\operatorname{tr}\ln \mathbb{B} \in L^\infty(0,T; L^1(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega));
\end{gathered}$$

(2) for all $t \in (0,T)$ and all $\mathbf{w} \in C^\infty([0,T]; C_0^\infty(\Omega;\mathbb{R}^2))$ with $\operatorname{div}\mathbf{w} = 0$, we have

$$\begin{aligned}\int_0^t \int_\Omega \Big(\rho(\phi)\mathbf{u}\cdot\partial_t\mathbf{w} + (\rho(\phi)\mathbf{u}\otimes\mathbf{u}) : \nabla\mathbf{w} - \rho'(\phi)(\mathbf{u}\otimes m(\phi)\nabla q) : \nabla\mathbf{w}\Big)\,\mathrm{d}x\,\mathrm{d}\tau \\ - \int_0^t \int_\Omega \Big(\nu(\phi)(\nabla\mathbf{u} + \nabla\mathbf{u}^\top) : \nabla\mathbf{w} + \mu(\phi)(\mathbb{B} - \mathbb{I}) : \nabla\mathbf{w}\Big)\,\mathrm{d}x\,\mathrm{d}\tau \\ = -\int_0^t \int_\Omega q\nabla\phi\cdot\mathbf{w}\,\mathrm{d}x\,\mathrm{d}\tau - \int_0^t \int_\Omega \frac{\mu(\phi)}{2}\nabla\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I})\cdot\mathbf{w}\,\mathrm{d}x\,\mathrm{d}\tau \\ + \int_\Omega \rho(\phi(\cdot,t))\mathbf{u}(\cdot,t)\cdot\mathbf{w}(\cdot,t)\,\mathrm{d}x - \int_\Omega \rho(\phi_0)\mathbf{u}_0\cdot\mathbf{w}(\cdot,0)\,\mathrm{d}x;\end{aligned} \tag{6.16}$$

(3) for all $t \in (0,T)$ and all $\xi \in C^\infty([0,T]; C^1(\overline{\Omega};\mathbb{R}))$, we have

$$\begin{aligned}\int_0^t \int_\Omega \phi\big(\partial_t\xi + \mathbf{u}\cdot\nabla\xi\big)\,\mathrm{d}x\,\mathrm{d}\tau - \int_0^t \int_\Omega m(\phi)\nabla q\cdot\nabla\xi\,\mathrm{d}x\,\mathrm{d}\tau \\ = \int_\Omega \phi(\cdot,t)\xi(\cdot,t)\,\mathrm{d}x - \int_\Omega \phi_0\xi(\cdot,0)\,\mathrm{d}x;\end{aligned} \tag{6.17}$$

(4) for a.e. $(x,t) \in Q_T$, we have

$$q = W'(\phi) - \Delta\phi + \frac{\mu'(\phi)}{2}\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I});$$

(5) for all $t \in (0,T)$ and all $\mathbb{C} \in C^\infty(\overline{Q_T}; \mathbb{R}^{2\times 2}_{\mathrm{sym}+})$, we have

$$\begin{aligned}\int_0^t \int_\Omega \Big(\mathbb{B} : \partial_t\mathbb{C} + (\mathbf{u}\otimes\mathbb{B}) : \nabla\mathbb{C}\Big)\,\mathrm{d}x\,\mathrm{d}\tau \\ + \int_0^t \int_\Omega \Big(\big(\nabla\mathbf{u}\mathbb{B} + \mathbb{B}\nabla\mathbf{u}^\top\big) : \mathbb{C} - \kappa\nabla\mathbb{B} : \nabla\frac{\mathbb{C}}{\mu(\phi)}\Big)\,\mathrm{d}x\,\mathrm{d}\tau \\ = \int_0^t \int_\Omega (\mathbb{B} : \mathbb{C} - \operatorname{tr}\mathbb{C})\,\mathrm{d}x\,\mathrm{d}\tau + \int_\Omega \mathbb{B}(\cdot,t) : \mathbb{C}(\cdot,t)\,\mathrm{d}x - \int_\Omega \mathbb{B}_0 : \mathbb{C}(\cdot,0)\,\mathrm{d}x;\end{aligned} \tag{6.18}$$

(6) for a.e. $t \in (0,T)$, the following energy estimate holds

$$\begin{aligned}\mathcal{E}(t) + \frac{1}{2}\int_0^t \left\|\sqrt{\nu(\phi)}(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)(\tau)\right\|_{L^2}^2 \mathrm{d}\tau \\ + \int_0^t \left(\left\|\frac{\mu(\phi)}{2}\operatorname{tr}(\mathbb{B} + \mathbb{B}^{-1} - 2\mathbb{I})(\tau)\right\|_{L^1} + \frac{\kappa}{2}\left\|\nabla\operatorname{tr}\ln\mathbb{B}\right\|_{L^2}^2\right)\mathrm{d}\tau \\ + \int_0^t \left\|\sqrt{m(\phi)}\nabla q(\tau)\right\|_{L^2}^2 \mathrm{d}\tau \le C\mathcal{E}(0),\end{aligned} \tag{6.19}$$

where $\mathcal{E}(t)$ is the energy defined in (6.2).

6.3. Matrix-Valued Equation with Stress Diffusion

In this section, we are going to solve the matrix-valued equation for $\mathbb{B}$ in dimension $d \in \{2,3\}$,

$$\begin{aligned}\partial_t\mathbb{B} + \mathscr{R}_\eta\mathbf{v}\cdot\nabla\mathbb{B} - \mathbb{B}\nabla\mathscr{R}_\eta\mathbf{v}^\top - \nabla\mathscr{R}_\eta\mathbf{v}\mathbb{B} + (\mathbb{B} - \mathbb{I}) &= \frac{\kappa}{\mu_\eta(\xi)}\Delta\mathbb{B}, && \text{in } Q_T \\ \partial_{\mathbf{n}}\mathbb{B} &= \mathbf{0}, && \text{on } S_T \\ \mathbb{B}(0) &= \mathbb{B}_0, && \text{in } \Omega,\end{aligned} \tag{6.20}$$

with $\mathbb{B}_0 \in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}+})$, $d \in \{2,3\}$, and given data $(\mathbf{v},\xi)$ satisfying

$$\mathbf{v} \in C([0,T];W^{1,2}_{0,\sigma}(\Omega;\mathbb{R}^d)), \quad \xi \in C([0,T];W^{2,2}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)). \tag{6.21}$$

Here, $\mathscr{R}_\eta$ is the mollifier defined in (6.14) and $\mu_\eta(\phi) \coloneqq \mu(\mathscr{R}_\eta\phi)$. As usual, we give the precise definition of *finite energy* weak solutions of the system (6.20).

Definition 6.15. Let $d \in \{2,3\}$, $T > 0$, $\mathbb{B}_0 \in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ positive definite a.e. in Ω, $\operatorname{tr}\ln\mathbb{B}_0 \in L^1(\Omega)$, and $(\mathbf{v},\xi)$ be satisfying (6.21). We call $\mathbb{B}$ a finite energy weak solution of (6.20) with data $(\mathbb{B}_0,\mathbf{v},\xi)$, provided that

(1) the left Cauchy–Green tensor $\mathbb{B}$ satisfies

$$\begin{gathered}\mathbb{B} \text{ is symmetric positive definite a.e. in } Q_T;\\ \mathbb{B} \in C_w([0,T];L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})) \cap L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}));\\ \partial_t\mathbb{B} \in L^2(0,T;[W^{1,2}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})]');\\ \operatorname{tr}\ln\mathbb{B} \in L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega));\end{gathered}$$

(2) for all $t \in (0,T)$ and all $\mathbb{C} \in C^\infty(\overline{Q_T};\mathbb{R}^{d\times d}_{\mathrm{sym}})$, we have

$$\begin{aligned}&\int_0^t\int_\Omega\Big(\mathbb{B} : \partial_t\mathbb{C} + (\mathscr{R}_\eta\mathbf{v}\otimes\mathbb{B}) : \nabla\mathbb{C}\Big)\,\mathrm{d}x\,\mathrm{d}\tau\\ &\qquad + \int_0^t\int_\Omega\left(\left(\nabla\mathscr{R}_\eta\mathbf{v}\mathbb{B} + \mathbb{B}\nabla\mathscr{R}_\eta\mathbf{v}^\top\right) : \mathbb{C} - \kappa\nabla\mathbb{B} : \nabla\frac{\mathbb{C}}{\mu_\eta(\xi)}\right)\mathrm{d}x\,\mathrm{d}\tau\\ &\quad = \int_0^t\int_\Omega(\mathbb{B} : \mathbb{C} - \operatorname{tr}\mathbb{C})\,\mathrm{d}x\,\mathrm{d}\tau + \int_\Omega\mathbb{B}(\cdot,t) : \mathbb{C}(\cdot,t)\,\mathrm{d}x - \int_\Omega\mathbb{B}_0 : \mathbb{C}(\cdot,0)\,\mathrm{d}x;\end{aligned} \tag{6.22}$$

(3) for a.e. $t \in (0,T)$ we have the following energy estimate

$$\begin{aligned}&\|\operatorname{tr}(\mathbb{B} - \ln\mathbb{B})(t)\|_{L^1} + \|\mathbb{B}(t)\|^2_{L^2}\\ &\quad + \int_0^t\Big(\|\mathbb{B}(\tau)\|^2_{L^2} + \kappa\|\nabla\mathbb{B}(\tau)\|^2_{L^2}\Big)\,\mathrm{d}\tau\\ &\quad + \int_0^t\Big(\big\|\operatorname{tr}(\mathbb{B} + \mathbb{B}^{-1} - 2\mathbb{I})(\tau)\big\|_{L^1} + \kappa\|\nabla\operatorname{tr}\ln\mathbb{B}\|^2_{L^2}\Big)\,\mathrm{d}\tau\\ &\leq C\Big(1 + \|\operatorname{tr}(\mathbb{B}_0 - \ln\mathbb{B}_0)\|_{L^1} + \|\mathbb{B}_0\|^2_{L^2}\Big).\end{aligned} \tag{6.23}$$

Then main result of this section reads as follows:

Theorem 6.16. *Let $T > 0$ and $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded smooth domain. Assume that $(\mathbf{v},\xi)$ satisfies (6.21), $\mathbb{B}_0 \in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ positive definite a.e. in Ω and $\operatorname{tr}\ln\mathbb{B}_0 \in L^1(\Omega)$. Then, there exists a finite energy weak solution of (6.20) in the sense of Definition 6.15. Moreover, the solution is unique and depends continuously on $(\mathbf{v},\xi,\mathbb{B}_0)$.*

Remark 6.17. This theorem is valid for $(\mathbf{v},\xi)$ satisfying (6.21), which is a quite general and rather strong assumption. Later on, we will only consider the case with piecewise-in-time constant functions satisfying certain spatial regularity, which, for sure, fulfill (6.21) on each subinterval.

We now devote the rest of this section to the proof of Theorem 6.16, which consists of a two-layer approximation, uniform estimates, limit passages and a uniqueness argument.

6.3.1. Galerkin approximation. By the classical theory of eigenvalue problems for symmetric linear elliptic operators, one can carry out the finite space approximation. Let $\{\mathbb{A}_k\}_{k\in\mathbb{N}}$ be the eigenfunctions of the matrix-valued Laplace operator with homogeneous Neumann boundary conditions, namely,

$$-\Delta\mathbb{A}_l = \lambda_l \mathbb{A}_l \quad \text{in } \Omega, \quad \partial_{\mathbf{n}}\mathbb{A}_l = \mathbf{0} \quad \text{on } \partial\Omega,$$

with eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_l \to \infty$, $l \to \infty$. Moreover,

$$\mathbb{A}_l \in W^{2,2}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \cap C^\infty(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}).$$

Then $\{\mathbb{A}_l\}_{l\in\mathbb{N}}$ is an orthonormal basis in $L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and an orthogonal basis in $W^{1,2}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ (see also [BLS17] for the compressible Oldroyd-B case). Define $\mathbb{H}_l = \mathrm{span}\{\mathbb{A}_1, \ldots, \mathbb{A}_l\} \subset L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ with $l \in \mathbb{N}$ and denote by $\mathcal{P}_l : W^{1,2}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \to \mathbb{H}_l$ the orthogonal projection on $\mathbb{H}_l$ with respect to the inner product in $L^2(\Omega; \mathbb{R}^{d\times d})$ such that

$$(\mathbb{A}, \mathbb{B}) = \int_\Omega \mathbb{A} : \mathbb{B} \, \mathrm{d}x, \quad \forall\, \mathbb{A}, \mathbb{B} \in L^2(\Omega; \mathbb{R}^{d\times d}).$$

Now we make the Galerkin ansatz to approximate the solution $\mathbb{B}$ of (6.20) as

$$\mathbb{B}_l = \sum_{k=1}^{l} c_k^l(t)\mathbb{A}_k(x), \quad \mathbb{B}_l(0) = \mathcal{P}_l\mathbb{B}_0,$$

where $\{c_k^l\}$ are scalar functions of time. Then one arrives at the following approximate system

$$\begin{aligned}(\partial_t\mathbb{B}_l, \mathbb{W}) &+ (\mathscr{R}_\eta\mathbf{v}\cdot\nabla\mathbb{B}_l, \mathbb{W}) - \left(\mathbb{B}_l\nabla\mathscr{R}_\eta\mathbf{v}^\top, \mathbb{W}\right) \\ &- (\nabla\mathscr{R}_\eta\mathbf{v}\mathbb{B}_l, \mathbb{W}) + (\mathbb{B}_l - \mathbb{I}, \mathbb{W}) = -\left(\kappa\nabla\mathbb{B}_l, \nabla\frac{\mathbb{W}}{\mu_\eta(\xi)}\right),\end{aligned} \tag{6.24}$$

for all $\mathbb{W} \in \mathbb{H}_l$, with the initial data $\mathbb{B}_l(0) = \mathcal{P}_l\mathbb{B}_0$. Taking $\mathbb{W} = \mathbb{A}_j, j = 1, \ldots, l$ in (6.24) respectively, we have

$$\begin{aligned}(\partial_t\mathbb{B}_l, \mathbb{A}_j) &+ (\mathscr{R}_\eta\mathbf{v}\cdot\nabla\mathbb{B}_l, \mathbb{A}_j) - \left(\mathbb{B}_l\nabla\mathscr{R}_\eta\mathbf{v}^\top, \mathbb{A}_j\right) \\ &- (\nabla\mathscr{R}_\eta\mathbf{v}\mathbb{B}_l, \mathbb{A}_j) + (\mathbb{B}_l - \mathbb{I}, \mathbb{A}_j) = -\left(\kappa\nabla\mathbb{B}_l, \nabla\frac{\mathbb{A}_j}{\mu_\eta(\xi)}\right).\end{aligned} \tag{6.25}$$

Consequently, (6.25) turns into the form of

$$\mathbb{M}^l\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{c}^l = \mathbb{L}^l(t)\mathbf{c}^l + \mathbf{f}^l, \tag{6.26}$$

where $[\mathbf{c}^l(t)]_j = c_j^l(t)$, $[\mathbf{c}^l(0)]_j = (\mathcal{P}_l\mathbb{B}_0, \mathbb{A}_j)$, $j = 1, \ldots, l$. The matrices $\mathbb{M}^l$, $\mathbb{L}^l$ and the vector $\mathbf{f}$ are induced respectively with the entries for $j, k = 1, \ldots, l$,

$$\begin{aligned}[\mathbb{M}^l]_{jk} &= (\mathbb{A}_k, \mathbb{A}_j), \\ [\mathbb{L}^l(t)]_{jk} &= -(\mathscr{R}_\eta\mathbf{v}(t)\cdot\nabla\mathbb{A}_k, \mathbb{A}_j) + \left(\mathbb{A}_k\nabla\mathscr{R}_\eta\mathbf{v}^\top(t), \mathbb{A}_j\right) \\ &\quad + (\nabla\mathscr{R}_\eta\mathbf{v}(t)\mathbb{A}_k, \mathbb{A}_j) - (\mathbb{A}_k, \mathbb{A}_j) + \kappa\lambda_k\left(\frac{1}{\mu_\eta(\xi)}\mathbb{A}_k, \mathbb{A}_j\right), \\ [\mathbf{f}^l]_j &= \int_\Omega \mathrm{tr}(\mathbb{A}_j)\,\mathrm{d}x.\end{aligned}$$

Now we are able to obtain the existence and uniqueness of $\mathbf{c}^l$ by verify the conditions of classical linear ordinary differential equation theory, which thereafter implies that (6.25) admits a unique solution $\mathbb{B}_l$ on the whole interval $[0, T_l]$ for all $l \in \mathbb{N}$.

6.3.2. Entropy regularization. Concerning the energy $\mathrm{tr}(\mathbb{B} - \ln \mathbb{B} - \mathbb{I})$, we notice that it does not necessarily make sence due to the logarithmic term. The *a priori* bounds are usually obtained by assuming that $\mathbb{B}$ is symmetric positive definite, which we do not have *a priori*. Thus, motivated by [BB11], we employ a regularization for the logarithmic function $G(s) = \ln(s)$ to construct a family of symmetric positive definite approximations for $\mathbb{B}$, namely, one defines a C^1 function for $\delta > 0$

$$G_\delta(s) = \begin{cases} \dfrac{s}{\delta} + \ln \delta - 1, & s < \delta, \\ \ln s, & s \geq \delta, \end{cases}$$

and a cut-off function $\beta_\delta(s) = [G'_\delta(s)]^{-1} = \max\{s, \delta\}$, for all $s \in \mathbb{R}$. This kind of regularization was also applied in, e.g., [BLS17] for a compressible Oldroyd-B system, or [GKT22] for a Cahn–Hilliard tumor growth model including viscoelasticity.

Let us recall the following result from [BB11, Lemma 2.1].

Lemma 6.18. *For all $\mathbb{A}, \mathbb{C} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $d \in \{2, 3\}$, and for any $\delta \in (0, 1)$, it holds*

$$\beta_\delta(\mathbb{A})G'_\delta(\mathbb{A}) = G'_\delta(\mathbb{A})\beta_\delta(\mathbb{A}) = \mathbb{I}, \tag{6.27a}$$

$$\mathrm{tr}\left(\beta_\delta(\mathbb{A}) + \beta_\delta^{-1}(\mathbb{A}) - 2\mathbb{I}\right) \geq 0, \tag{6.27b}$$

$$\mathrm{tr}\left(\mathbb{A} - G_\delta(\mathbb{A}) - \mathbb{I}\right) \geq 0, \tag{6.27c}$$

$$(\mathbb{A} - \beta_\delta(\mathbb{A})) : (\mathbb{I} - G'_\delta(\mathbb{A})) \geq 0, \tag{6.27d}$$

$$(\mathbb{A} - \mathbb{C}) : G'_\delta(\mathbb{C}) \geq \mathrm{tr}\left(G_\delta(\mathbb{A}) - G_\delta(\mathbb{C})\right). \tag{6.27e}$$

In addition, if $\delta \in (0, \frac{1}{2}]$, it holds

$$\mathrm{tr}(\mathbb{A} - G_\delta(\mathbb{A})) \geq \begin{cases} \dfrac{1}{2}\,|\mathbb{A}|\,, \\ \dfrac{1}{2\delta}\,|[\mathbb{A}]_-|\,, \end{cases} \tag{6.27f}$$

$$\mathbb{A} : (\mathbb{I} - G'_\delta(\mathbb{A})) \geq \frac{1}{2}\,|\mathbb{A}| - d, \tag{6.27g}$$

where $[\cdot]_-$ denotes the negative part function defined by $[s]_- := \min\{s, 0\}, \forall s \in \mathbb{R}$.

Remark 6.19. The domain of definition of scalar functions is naturally extended to symmetric matrices by the standard functional calculus from spectral theory.

Now employing the regularization from above with respect to $\delta > 0$, we have the regularized Oldroyd-B equation with symmetric positive definite approximations.

$$\begin{aligned} &\left(\partial_t \mathbb{B}^\delta_l, \mathbb{W}\right) + \left(\mathscr{R}_\eta \mathbf{v} \cdot \nabla \beta_\delta(\mathbb{B}^\delta_l), \mathbb{W}\right) - \left(\beta_\delta(\mathbb{B}^\delta_l)\nabla \mathscr{R}_\eta \mathbf{v}^\top, \mathbb{W}\right) \\ &\qquad - \left(\nabla \mathscr{R}_\eta \mathbf{v} \beta_\delta(\mathbb{B}^\delta_l), \mathbb{W}\right) + \left(\mathbb{B}^\delta_l - \mathbb{I}, \mathbb{W}\right) = -\left(\kappa \nabla \mathbb{B}^\delta_l, \nabla \frac{\mathbb{W}}{\mu_\eta(\xi)}\right) \end{aligned} \tag{6.28}$$

for all $\mathbb{W} \in \mathbb{H}_l$, subjected to the initial values $\mathbb{B}^\delta_l(0) = \mathcal{P}_l(\delta\mathbb{I} + \mathscr{R}_\delta \mathbb{B}_0)$, where we use a mollification of the initial data $\mathbb{B}_0$ with respect to the regularization parameter δ and a diagonal shift by $\delta\mathbb{I}$, since $\mathscr{R}_\delta \mathbb{B}_0$ is not necessarily positive definite. Then we have $\delta\mathbb{I} + \mathscr{R}_\delta \mathbb{B}_0 \to \mathbb{B}_0$ in $L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, as $\delta \to 0$, and $\delta\mathbb{I} + \mathscr{R}_\delta \mathbb{B}_0$ is positive definite a.e. in Ω. Note that in the same way as in Section 6.3.1, (6.28) is a system of ordinary differential equations for $\mathbb{B}^\delta_l$ with respect to the time variable, for which the Picard–Lindelöf theorem is applicable. This guaranteees the existence of a unique solution on a local time interval $(0, T_{l,\delta})$, where $T_{l,\delta} \in (0, T)$.

6.3.3. Uniform a priori estimates in Galerkin level. Let $\mathbb{W} = \mathbb{B}_l^\delta$ in (6.28), it follows from $\left|\beta_\delta(\mathbb{B}_l^\delta)\right| \le 1 + \left|\mathbb{B}_l^\delta\right|$, $0 < \delta \le 1$, that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbb{B}_l^\delta\right\|_{L^2}^2 + \int_\Omega \big(\mathscr{R}_\eta \mathbf{v}\cdot\nabla\big)\beta_\delta(\mathbb{B}_l^\delta) : \mathbb{B}_l^\delta \,\mathrm{d}x + \left\|\mathbb{B}_l^\delta\right\|_{L^2}^2 + \frac{\kappa}{\underline{\mu}}\left\|\nabla\mathbb{B}_l^\delta\right\|_{L^2}^2 \\
&\quad\le \int_\Omega \operatorname{tr}\mathbb{B}_l^\delta \,\mathrm{d}x + \kappa\int_\Omega \frac{\mu_\eta'(\xi)}{\mu_\eta^2(\xi)}(\nabla\mathscr{R}_\eta\xi\cdot\nabla)\mathbb{B}_l^\delta : \mathbb{B}_l^\delta\,\mathrm{d}x + 2\int_\Omega \mathbb{B}_l^\delta\beta_\delta(\mathbb{B}_l^\delta) : \nabla\mathscr{R}_\eta\mathbf{v}\,\mathrm{d}x \\
&\quad\le \int_\Omega \left|\operatorname{tr}\mathbb{B}_l^\delta\right|\,\mathrm{d}x + \frac{\kappa\overline{\mu}'}{\underline{\mu}^2}\int_\Omega \left|(\nabla\mathscr{R}_\eta\xi\cdot\nabla)\mathbb{B}_l^\delta : \mathbb{B}_l^\delta\right|\,\mathrm{d}x + 2\int_\Omega (1+\left|\mathbb{B}_l^\delta\right|)\left|\mathbb{B}_l^\delta\right|\left|\nabla\mathscr{R}_\eta\mathbf{v}\right|\,\mathrm{d}x.
\end{aligned}$$

On account of integration by parts, $\operatorname{div}\mathbf{v} = 0$ and $\mathbf{v}|_{\partial\Omega} = 0$, one gets

$$-\int_\Omega \big(\mathscr{R}_\eta\mathbf{v}\cdot\nabla\big)\beta_\delta(\mathbb{B}_l^\delta) : \mathbb{B}_l^\delta\,\mathrm{d}x = \int_\Omega \big(\mathscr{R}_\eta\mathbf{v}\cdot\nabla\big)\mathbb{B}_l^\delta : \beta_\delta(\mathbb{B}_l^\delta)\,\mathrm{d}x.$$

In light of Hölder's and Young's inequalities, we have

$$\begin{aligned}
&\int_\Omega \left|(\nabla\mathscr{R}_\eta\xi\cdot\nabla)\mathbb{B}_l^\delta : \mathbb{B}_l^\delta\right|\,\mathrm{d}x \le \varepsilon\left\|\nabla\mathbb{B}_l^\delta\right\|_{L^2}^2 + C\left\|\nabla\mathscr{R}_\eta\xi\right\|_{L^\infty}^2\left\|\mathbb{B}_l^\delta\right\|_{L^2}^2, \\
&\int_\Omega (1+\left|\mathbb{B}_l^\delta\right|)\left|\mathbb{B}_l^\delta\right|\left|\nabla\mathscr{R}_\eta\mathbf{v}\right|\,\mathrm{d}x \le \big(1+\left\|\nabla\mathscr{R}_\eta\mathbf{v}\right\|_{L^\infty}\big)\left\|\mathbb{B}_l^\delta\right\|_{L^2}^2 + \left\|\nabla\mathbf{v}\right\|_{L^2}^2, \\
&\int_\Omega \left|\big(\mathscr{R}_\eta\mathbf{v}\cdot\nabla\big)\mathbb{B}_l^\delta : \beta_\delta(\mathbb{B}_l^\delta)\right|\,\mathrm{d}x \le \left\|\nabla\mathbb{B}_l^\delta\right\|_{L^2}\left\|\left|\mathscr{R}_\eta\mathbf{v}\right|(1+\left|\mathbb{B}_l^\delta\right|)\right\|_{L^2} \\
&\qquad\qquad \le \varepsilon\left\|\nabla\mathbb{B}_l^\delta\right\|_{L^2}^2 + C\left\|\mathscr{R}_\eta\mathbf{v}\right\|_{L^\infty}^2\left\|\mathbb{B}_l^\delta\right\|_{L^2}^2 + C\left\|\mathbf{v}\right\|_{L^2}^2,
\end{aligned}$$

for some small $\varepsilon > 0$ that will be specified later. In addition,

$$\operatorname{tr}\mathbb{B}_l^\delta \le \sqrt{2}\left|\mathbb{B}_l^\delta\right|.$$

Then integrating it over $(0,t)$, $t \in (0, T_{l,\delta})$ together with choosing $2\varepsilon \le \frac{\kappa}{2\underline{\mu}}$ yields

$$\begin{aligned}
&\frac{1}{2}\left\|\mathbb{B}_l^\delta(t)\right\|_{L^2}^2 + \frac{1}{2}\int_0^t \left\|\mathbb{B}_l^\delta(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau + \frac{\kappa}{2\underline{\mu}}\int_0^t \left\|\nabla\mathbb{B}_l^\delta(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau \\
&\le \frac{1}{2}\left\|\mathbb{B}_l^\delta(0)\right\|_{L^2}^2 + 1 + C\int_0^t (\left\|\mathbf{v}(\tau)\right\|_{L^2}^2 + \left\|\nabla\mathbf{v}(\tau)\right\|_{L^2}^2)\,\mathrm{d}\tau \\
&\quad + C\int_0^t (1 + \left\|\nabla\mathscr{R}_\eta\xi\right\|_{L^\infty}^2 + \left\|\mathscr{R}_\eta\mathbf{v}\right\|_{L^\infty}^2 + \left\|\nabla\mathscr{R}_\eta\mathbf{v}\right\|_{L^\infty} + \left\|\nabla\mathbf{v}\right\|_{L^2}^2)\left\|\mathbb{B}_l^\delta(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau,
\end{aligned}$$

which, by Gronwall's inequality, gives rise to

$$\left\|\mathbb{B}_l^\delta(t)\right\|_{L^2}^2 + \int_0^t \left\|\mathbb{B}_l^\delta(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau + \int_0^t \left\|\nabla\mathbb{B}_l^\delta(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau \le C, \tag{6.29}$$

for all $t \in [0, T_{l,\delta}]$, where $C > 0$ is independent of $l \in \mathbb{N}$, but depends on κ, $\underline{\mu}$, $T_{l,\delta}$, $\mathbb{B}_0$, $\mathbf{v}, \xi$ and the parameters δ, η, which is due to Lemma 6.11.

Moreover, due to the mollification of the initial data and (6.21), one can easily derive the higher regularity estimates by letting $\mathbb{W} = \Delta\mathbb{B}_l^\delta$ and $\mathbb{W} = \partial_t\mathbb{B}_l^\delta$ to obtain

$$\begin{aligned}
\left\|\nabla\mathbb{B}_l^\delta(t)\right\|_{L^2}^2 + \int_0^t \left\|\nabla\mathbb{B}_l^\delta(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau + \int_0^t \left\|\Delta\mathbb{B}_l^\delta(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau \le C, \\
\int_0^t \left\|\partial_t\mathbb{B}_l^\delta(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau + \left\|\mathbb{B}_l^\delta(t)\right\|_{L^2}^2 + \left\|\nabla\mathbb{B}_l^\delta(t)\right\|_{L^2}^2 \le C,
\end{aligned} \tag{6.30}$$

for all $t \in [0, T_{l,\delta}]$, where $C > 0$ is independent of $l \in \mathbb{N}$, but depends on κ, $\underline{\mu}$, $T_{l,\delta}$, $\mathbb{B}_0$, $\mathbf{v}, \xi$ and the parameters δ, η, which is due to Lemma 6.11.

Remark 6.20. The uniform higher regularity estimate is twofold. On one hand, one is able to pass to the limit $l \to \infty$ in the next subsection. On the other hand, this allows us to carry out the energy estimate with a logarithmic bound for the δ-limit, see also Barrett–Lu–Süli [BLS17, Lemma 6.1].

Remark 6.21. Here we have a maximal existence time $T_{l,\delta}$. As shown in [BLS17, Section 7.3], the approximate time $T_{l,\delta}$ actually coincides with the final time T for the compressible Oldroyd-B system. Here in a same manner, one can also prove it for the Oldroyd-B equation (6.28). Thus, in the following we will still denote T for the maximal existence time.

6.3.4. Passing to the limit of the Galerkin approximation. Now, from the uniform bounds (6.29) and (6.30), it follows that the sequence $\mathbb{B}_l$ has the following convergences up to a non-relabeled subsequence, as $l \to \infty$,

$$\begin{aligned}
\mathbb{B}_l^\delta &\to \mathbb{B}^\delta, && \text{weakly-}* && \text{in } L^\infty(0,T; W^{1,2}(\Omega; \mathbb{R}^{d\times d}_{\text{sym}}));\\
\mathbb{B}_l^\delta &\to \mathbb{B}^\delta, && \text{weakly} && \text{in } L^2(0,T; W^{2,2}(\Omega; \mathbb{R}^{d\times d}_{\text{sym}}));\\
\partial_t\mathbb{B}_l^\delta &\to \partial_t\mathbb{B}^\delta, && \text{weakly} && \text{in } L^2(0,T; L^2(\Omega; \mathbb{R}^{d\times d}_{\text{sym}})),
\end{aligned}$$

which thereafter implies the convergence of the weak formulation (6.28). Then, with the higher regularity bounds, we end up with the pointwise formulation

$$\partial_t\mathbb{B}^\delta + \mathscr{R}_\eta\mathbf{v}\cdot\nabla\beta_\delta(\mathbb{B}^\delta) - \beta_\delta(\mathbb{B}^\delta)\nabla\mathscr{R}_\eta\mathbf{v}^\top - \nabla\mathscr{R}_\eta\mathbf{v}\beta_\delta(\mathbb{B}^\delta) + (\mathbb{B}^\delta - \mathbb{I}) = \frac{\kappa}{\mu_\eta(\xi)}\Delta\mathbb{B}^\delta, \tag{6.31}$$

subjected to $\mathbb{B}^\delta(0) = \delta\mathbb{I} + \mathscr{R}_\delta\mathbb{B}_0$.

6.3.5. Uniform energy estimate. In view of the stronger bounds (6.30), the strong formulation (6.31) and [BLS17, Lemma 6.1.], we are allowed to multiply (6.31) with $\frac{\mu_\eta(\xi)}{2}(\mathbb{I} - G'_\delta(\mathbb{B}^\delta))$ and integrate the resulting equation over Ω to get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\mu_\eta(\xi)}{2}\operatorname{tr}(\mathbb{B}^\delta - G_\delta(\mathbb{B}^\delta) - \mathbb{I})\,\mathrm{d}x\\
&\quad + \int_\Omega \frac{\mu_\eta(\xi)}{2}\operatorname{tr}(\beta_\delta(\mathbb{B}^\delta) + \beta_\delta^{-1}(\mathbb{B}^\delta) - 2\mathbb{I})\,\mathrm{d}x + \frac{\kappa}{d}\int_\Omega \left|\nabla\operatorname{tr}\ln\beta_\delta(\mathbb{B}^\delta)\right|^2\,\mathrm{d}x \qquad (6.32)\\
&\le \int_\Omega \frac{\mu'_\eta(\xi)}{2}(\partial_t + \mathscr{R}_\eta\mathbf{v}\cdot\nabla)\mathscr{R}_\eta\xi\operatorname{tr}(\mathbb{B}^\delta - G_\delta(\mathbb{B}^\delta) - \mathbb{I})\,\mathrm{d}x + \int_\Omega \mu_\eta(\xi)(\beta_\delta(\mathbb{B}^\delta) - \mathbb{I}) : \nabla\mathscr{R}_\eta\mathbf{v}\,\mathrm{d}x.
\end{aligned}$$

Here we employed

$$\begin{aligned}
\int_\Omega \mathscr{R}_\eta\mathbf{v}\cdot\nabla\mathbb{B}^\delta : \frac{\mu_\eta(\xi)}{2}(\mathbb{I} - G'_\delta(\mathbb{B}^\delta))\,\mathrm{d}x &= \underbrace{\int_\Omega \mathscr{R}_\eta\mathbf{v}\cdot\nabla\Big(\frac{\mu_\eta(\xi)}{2}\operatorname{tr}(\mathbb{B}^\delta - G_\delta(\mathbb{B}^\delta) - \mathbb{I})\Big)\,\mathrm{d}x}_{=0 \text{ due to } \operatorname{div}\mathbf{v}=0 \text{ and } \mathbf{v}_{|\partial\Omega}=0}\\
&\quad - \int_\Omega \frac{\mu'_\eta(\xi)}{2}\mathscr{R}_\eta\mathbf{v}\cdot\nabla\mathscr{R}_\eta\xi\operatorname{tr}(\mathbb{B}^\delta - G_\delta(\mathbb{B}^\delta) - \mathbb{I})\,\mathrm{d}x,
\end{aligned}$$

$$(\mathbb{B}^\delta - \mathbb{I}) : (\mathbb{I} - G'_\delta(\mathbb{B}^\delta)) \ge \operatorname{tr}\big(\beta_\delta(\mathbb{B}^\delta) + \beta_\delta^{-1}(\mathbb{B}^\delta) - 2\mathbb{I}\big) \ge 0,$$

$$\big(\nabla\mathscr{R}_\eta\mathbf{v}\beta_\delta(\mathbb{B}^\delta) + \beta_\delta(\mathbb{B}^\delta)\nabla\mathscr{R}_\eta\mathbf{v}^\top\big) : \frac{\mu_\eta(\xi)}{2}(\mathbb{I} - G'_\delta(\mathbb{B}^\delta)) = \mu_\eta(\xi)(\beta_\delta(\mathbb{B}^\delta) - \mathbb{I}) : \nabla\mathscr{R}_\eta\mathbf{v},$$

$$-\int_\Omega \nabla\mathbb{B} : \nabla G'_\delta(\mathbb{B})\,\mathrm{d}x \ge \frac{1}{d}\int_\Omega |\nabla\operatorname{tr}\ln\beta_\delta(\mathbb{B})|^2\,\mathrm{d}x,$$

where the second and the third statements can be derived by [BB11, (2.26)], (6.12) respectively, and the last one is referred to [BLS17, Lemma 7.1]. Integrating (6.32) over $(0,t)$, $t\in(0,T)$, one obtains from the inequality $\left|\beta_\delta(\mathbb{B}^\delta)\right|\le 1+\left|\mathbb{B}^\delta\right|$, $0<\delta\le 1$, that

$$\begin{aligned}
&\frac{1}{2}\underline{\mu}\int_\Omega \operatorname{tr}(\mathbb{B}^\delta - G_\delta(\mathbb{B}^\delta))(t)\,\mathrm{d}x\\
&\quad+\int_0^t\int_\Omega \frac{\mu_\eta(\xi)}{2}\operatorname{tr}(\beta_\delta(\mathbb{B}^\delta)+\beta_\delta^{-1}(\mathbb{B}^\delta)-2\mathbb{I})(\tau)\,\mathrm{d}x\,\mathrm{d}\tau+\frac{\kappa}{d}\int_0^t\left\|\nabla\operatorname{tr}\ln\beta_\delta(\mathbb{B}^\delta)\right\|_{L^2}^2\,\mathrm{d}x\\
&\le C\left(1+\int_\Omega \operatorname{tr}(\mathbb{B}^\delta(0)-\ln\mathbb{B}^\delta(0))\,\mathrm{d}x\right)+C\left\|\nabla\mathbf{v}\right\|_{L^2_{tx}}^2+\frac{1}{4}\int_0^t\left\|\mathbb{B}^\delta(\tau)\right\|_{L^2}^2\,\mathrm{d}x\\
&\quad+\int_0^t\frac{\overline{\mu}'}{2}\left\|(\partial_t+\mathscr{R}_\eta\mathbf{v}\cdot\nabla)\mathscr{R}_\eta\xi(\tau)\right\|_{L^\infty}\left(\int_\Omega \operatorname{tr}(\mathbb{B}^\delta-G_\delta(\mathbb{B}^\delta))(\tau)\,\mathrm{d}x\right)\mathrm{d}\tau.
\end{aligned}\tag{6.33}$$

Note that there is a term $\int_0^t\|\mathbb{B}^\delta(\tau)\|_{L^2}^2$ on the right-hand side of (6.33), which can not be controlled for the moment. Thus, we proceed with the similar estimate as (6.29) to reach

$$\begin{aligned}
&\frac{1}{2}\left\|\mathbb{B}^\delta(t)\right\|_{L^2}^2+\frac{1}{2}\int_0^t\left\|\mathbb{B}^\delta(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau+\frac{\kappa}{2\underline{\mu}}\int_0^t\left\|\nabla\mathbb{B}^\delta(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau\\
&\le\frac{1}{2}\left\|\mathbb{B}^\delta(0)\right\|_{L^2}^2+\int_0^t 2\sqrt{2}\int_\Omega \operatorname{tr}(\mathbb{B}^\delta-G_\delta(\mathbb{B}^\delta))(\tau)\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+C\int_0^t(\|\mathbf{v}(\tau)\|_{L^2}^2+\|\nabla\mathbf{v}(\tau)\|_{L^2}^2)\,\mathrm{d}\tau\\
&\quad+C\int_0^t(1+\|\nabla\mathscr{R}_\eta\xi\|_{L^\infty}^2+\|\mathscr{R}_\eta\mathbf{v}\|_{L^\infty}^2+\|\nabla\mathscr{R}_\eta\mathbf{v}\|_{L^\infty}+\|\nabla\mathbf{v}\|_{L^2}^2)\left\|\mathbb{B}^\delta(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau.
\end{aligned}\tag{6.34}$$

Now summing (6.33) and (6.34) gives birth to

$$\begin{aligned}
&\frac{1}{2}\underline{\mu}\left\|\operatorname{tr}(\mathbb{B}^\delta-G_\delta(\mathbb{B}^\delta))(t)\right\|_{L^1}+\frac{1}{2}\left\|\mathbb{B}^\delta(t)\right\|_{L^2}^2+\frac{1}{2}\int_0^t\left\|\mathbb{B}^\delta(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau+\frac{\kappa}{2\underline{\mu}}\int_0^t\left\|\nabla\mathbb{B}^\delta(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau\\
&\quad+\int_0^t\int_\Omega\frac{\mu_\eta(\xi)}{2}\operatorname{tr}(\beta_\delta(\mathbb{B}^\delta)+\beta_\delta^{-1}(\mathbb{B}^\delta)-2\mathbb{I})(\tau)\,\mathrm{d}x\mathrm{d}\tau+\frac{\kappa}{2}\int_0^t\left\|\nabla\operatorname{tr}\ln\beta_\delta(\mathbb{B}^\delta)(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau\\
&\le C\left(1+\int_\Omega\operatorname{tr}(\mathbb{B}^\delta(0)-\ln\mathbb{B}^\delta(0))\,\mathrm{d}x+\frac{1}{2}\left\|\mathbb{B}^\delta(0)\right\|_{L^2}^2\right)\\
&\qquad+C\int_0^t(1+\tilde{g}(\tau))\Big(\left\|\operatorname{tr}(\mathbb{B}^\delta-G_\delta(\mathbb{B}^\delta))(\tau)\right\|_{L^1}+\left\|\mathbb{B}^\delta(\tau)\right\|_{L^2}^2\Big)\,\mathrm{d}\tau,
\end{aligned}\tag{6.35}$$

where $\tilde{g}(t):=\|(\partial_t+\mathscr{R}_\eta\mathbf{v}\cdot\nabla)\mathscr{R}_\eta\xi\|_{L^\infty}+\|\nabla\mathscr{R}_\eta\xi+\mathscr{R}_\eta\mathbf{v}\|_{L^\infty}^2+\|\nabla\mathscr{R}_\eta\mathbf{v}\|_{L^\infty}+\|\nabla\mathbf{v}\|_{L^2}^2$ and C depends on $\|\mathbf{v}+\nabla\mathbf{v}\|_{L^2_{tx}}^2$ and the upper-lower bounds of the coefficients but is uniform with respect to $\delta>0$. Then, by means of Gronwall's lemma, one derives

$$\begin{aligned}
&\left\|\operatorname{tr}(\mathbb{B}^\delta-G_\delta(\mathbb{B}^\delta))(t)\right\|_{L^1}+\left\|\mathbb{B}^\delta(t)\right\|_{L^2}^2+\int_0^t\left\|\mathbb{B}^\delta(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau+\kappa\int_0^t\left\|\nabla\mathbb{B}^\delta(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau\\
&\quad+\int_0^t\int_\Omega\operatorname{tr}(\beta_\delta(\mathbb{B}^\delta)+\beta_\delta^{-1}(\mathbb{B}^\delta)-2\mathbb{I})(\tau)\,\mathrm{d}x\mathrm{d}\tau+\kappa\int_0^t\left\|\nabla\operatorname{tr}\ln\beta_\delta(\mathbb{B}^\delta)(\tau)\right\|_{L^2}^2\,\mathrm{d}\tau\\
&\qquad\le C\left(1+\left\|\operatorname{tr}(\mathbb{B}^\delta(0)-\ln\mathbb{B}^\delta(0))\right\|_{L^1}+\left\|\mathbb{B}^\delta(0)\right\|_{L^2}^2\right)\exp(t+g(t)),
\end{aligned}\tag{6.36}$$

for a.e. $t\in(0,T)$, where $g(t):=\|(\partial_t+\mathscr{R}_\eta\mathbf{v}\cdot\nabla)\mathscr{R}_\eta\xi\|_{L^1_tL^\infty_x}+\|\nabla\mathscr{R}_\eta\xi+\mathscr{R}_\eta\mathbf{v}\|_{L^2_tL^\infty_x}^2+\|\nabla\mathscr{R}_\eta\mathbf{v}\|_{L^1_tL^\infty_x}+\|\nabla\mathbf{v}\|_{L^2_{tx}}^2<\infty$ depends on η, and the constant C is independent of δ.

6.3.6. Passing to the limit of the entropy regularization. From the weak formulation of (6.31) and the estimate (6.36), one directly infers that, up to a non-relabeled subsequence, as $\delta \to 0$,

$$\begin{aligned} \mathbb{B}^\delta &\to \mathbb{B}, && \text{weakly-}* && \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\text{sym}})); \\ \mathbb{B}^\delta &\to \mathbb{B}, && \text{weakly} && \text{in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^{d\times d}_{\text{sym}})); \\ \partial_t\mathbb{B}^\delta &\to \partial_t\mathbb{B}, && \text{weakly} && \text{in } L^2(0,T;[W^{1,2}(\Omega;\mathbb{R}^{d\times d}_{\text{sym}})]'). \end{aligned}$$

In view of the Aubin–Lions lemma and the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, we have

$$\begin{aligned} \mathbb{B}^\delta &\to \mathbb{B}, && \text{strongly} && \text{in } L^2(0,T;L^6(\Omega;\mathbb{R}^{d\times d}_{\text{sym}})); \\ \beta_\delta(\mathbb{B}^\delta) &\to [\mathbb{B}]_+, && \text{strongly} && \text{in } L^2(0,T;L^6(\Omega;\mathbb{R}^{d\times d}_{\text{sym}})). \end{aligned}$$

where the limit matrix-valued function $\mathbb{B}$ is real symmetric as $\mathbb{B}^\delta$ is real symmetric for all $\delta > 0$.

6.3.7. Positive definiteness of the left Cauchy–Green tensor. To make sure that the weak formulation (6.22) is recovered from (6.31) in the limit $\delta \to 0$, and similarly for the energy inequality, we mimic the arguments from [BLS17, Section 8.2] and [BLLM22, Lemma 4.1] to prove the positive definiteness of the limit function $\mathbb{B}$. Here we use the uniform estimate (6.36), also see [BB11, Theorem 6.2].

Lemma 6.22. *Let $\mathbb{B}$ be the limit function of the sequence of positive definite solutions $\mathbb{B}^\delta$. Then $\mathbb{B}$ is positive definite a.e. in Q_T.*

Proof. Assume that $\mathbb{B}$ is not positive definite a.e. in Q_T. Then, there exists a subset $D \subset Q_T$ with nonzero measure and a vector $\mathbf{w} \in L^\infty(Q_T;\mathbb{R}^d)$ satisfying $|\mathbf{w}| = 1$ a.e. in D and $\mathbf{w} = 0$ a.e. in $Q_T \setminus D$ such that

$$[\mathbb{B}]_+\mathbf{w} = 0, \ \text{a.e. in } Q_T. \tag{6.37}$$

With a direct calculation, one has

$$\begin{aligned} |D| = \int_0^T\int_\Omega |\mathbf{w}|^2 \,\mathrm{d}x\mathrm{d}t &= \int_0^T\int_\Omega \left|\mathbf{w}^\top\beta_\delta^{-\frac{1}{2}}(\mathbb{B}^\delta)\beta_\delta^{\frac{1}{2}}(\mathbb{B}^\delta)\mathbf{w}\right| \,\mathrm{d}x\mathrm{d}t \\ &\le \left(\int_0^T\int_\Omega \left|\beta_\delta^{-1}(\mathbb{B}^\delta)\right| \,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^T\int_\Omega \left|\mathbf{w}^\top\beta_\delta(\mathbb{B}^\delta)\mathbf{w}\right| \,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \\ &\le C\left(\int_0^T\int_\Omega \left|\mathbf{w}^\top\beta_\delta(\mathbb{B}^\delta)\mathbf{w}\right| \,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}, \end{aligned}$$

where C does not depend on δ due to the uniform estimate (6.36). In view of the strong convergence in the last subsection, as $\delta \to 0$, and (6.37), we obtain $|D| = 0$, which is a contradiction to the nonzero measure assumption of D. Hence, $\mathbb{B}$ is positive definite a.e. in Q_T. ☐

On account of Lemma 6.22 and the uniform bounds of $\operatorname{tr}\ln\beta_\delta(\mathbb{B}^\delta)$, we can extract a converging subsequence of $\{\nabla\operatorname{tr}\ln\beta_\delta(\mathbb{B}^\delta)\}_{\delta>0}$ in the space $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$ and a limit function $\overline{\nabla\operatorname{tr}\ln\mathbb{B}} \in L^2(0,T;L^2(\Omega;\mathbb{R}^d))$ such that, as $\delta \to 0$,

$$\nabla\operatorname{tr}\ln\beta_\delta(\mathbb{B}^\delta) \to \overline{\nabla\operatorname{tr}\ln\mathbb{B}}, \quad \text{weakly} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)),$$

Indeed, we can identify $\overline{\nabla\operatorname{tr}\ln\mathbb{B}}$ with $\nabla\operatorname{tr}\ln\mathbb{B}$. Note that the energy estimates (6.36) and the definitions of $G_\delta(\cdot)$ and $\beta_\delta(\cdot)$ imply the uniform boundedness of $\operatorname{tr}\ln\beta_\delta(\mathbb{B}^\delta)$ in $L^\infty(0,T;L^1(\Omega))$.

Hence, with the Poincaré–Wirtinger inequality, Lemma 6.22, the continuity of the logarithm and the pointwise convergence $\beta_\delta(\mathbb{B}^\delta) \to \mathbb{B}$ a.e. in Q_T, we obtain for a non-relabeled subsequence, as $\delta \to 0$,

$$\operatorname{tr}\ln\beta_\delta(\mathbb{B}^\delta) \to \operatorname{tr}\ln\mathbb{B}, \quad \text{weakly} \quad \text{in } L^2(0,T;L^2(\Omega)).$$

Based on this, the existence of the weak gradient of $\operatorname{tr}\ln\mathbb{B}$ is established as follows. For any arbitrary test function $\zeta \in C_0^\infty(Q_T;\mathbb{R}^d)$, it holds

$$\begin{aligned}\int_{Q_T} \overline{\nabla\operatorname{tr}\ln\mathbb{B}}\cdot\zeta\,\mathrm{d}x\mathrm{d}t \leftarrow \int_{Q_T} \nabla\operatorname{tr}\ln\beta_\delta(\mathbb{B}^\delta)\cdot\zeta\,\mathrm{d}x\mathrm{d}t \\ = -\int_{Q_T} \operatorname{tr}\ln\beta_\delta(\mathbb{B}^\delta)\operatorname{div}(\zeta)\,\mathrm{d}x\mathrm{d}t \\ \to -\int_{Q_T} \operatorname{tr}\ln\mathbb{B}\operatorname{div}(\zeta)\,\mathrm{d}x\mathrm{d}t,\end{aligned}$$

as $\delta \to 0$, which allows us to identify $\overline{\nabla\operatorname{tr}\ln\mathbb{B}}$ as the weak gradient of $\operatorname{tr}\ln\mathbb{B}$, i.e., $\nabla\operatorname{tr}\ln\mathbb{B} = \overline{\nabla\operatorname{tr}\ln\mathbb{B}} \in L^2(0,T;L^2(\Omega;\mathbb{R}^d))$.

Remark 6.23. For certain smooth solutions, the positive definiteness of the left Cauchy–Green tensor $\mathbb{B}$ can be preserved for all $t > 0$ if $\mathbb{B}$ is positive definite initially. This has been observed for the 2D Oldroyd-B equation with stress diffusion [CK12, Proposition 1] and for other viscoelastic fluid systems in, e.g., [HL07, Lemma 2.1], [LM+17, Remark 3.4]. We remark here that one could obtain a similar result as in [CK12, Proposition 1] for $d \in \{2,3\}$, by means of our argument above, provided with a stronger initial data $\mathbb{B}_0 \in W^{1,2}(\Omega;\mathbb{B}^{d\times d}_{\mathrm{sym}})$ that is positive definite a.e. in Ω and $\operatorname{tr}\ln\mathbb{B}_0 \in L^1(\Omega)$.

6.3.8. Proof of Theorem 6.16. With all the arguments from above, we are able to prove the existence of weak solutions in Theorem 6.16 by verifying Definition 6.15. Noticing that we already have

$$\mathbb{B} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})) \cap W^{1,2}(0,T;[W^{1,2}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})]')$$

and the Sobolev embeddings

$$\begin{aligned} &W^{1,2}(0,T;X) \hookrightarrow C([0,T];X), \quad \text{where } X \text{ is a Banach space,}\\ &W^{1,2}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \hookrightarrow L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \hookrightarrow [W^{1,2}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})]' \text{ densely },\end{aligned}$$

one concludes from Lemma 6.9 that

$$\mathbb{B} \in C_w([0,T);L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})).$$

Combining all the convergences obtained above, one obtains the weak formulation (6.22). The energy inequality (6.23) follows immediately from the uniform estimate (6.36) using the lower semicontinuity of norms and Fatou's lemma. This proves the existence.

In the final step, we show the uniqueness and continuous dependence of weak solutions of (6.20) on the data $(\mathbf{v},\xi,\mathbb{B}_0)$. In fact, one only needs to prove the continuous dependence and, as a consequence, we then have uniqueness. To this end, let $\mathbb{B}_i$, $i = 1,2$ be two weak solutions of (6.20) in the sense of Definition 6.15 with data $(\mathbf{v}_i,\xi_i,\mathbb{B}_0^i)$, respectively, satisfying (6.21) and let

$\overline{\mathbb{B}} = \mathbb{B}_1 - \mathbb{B}_2$ and $(\overline{\mathbf{v}}, \overline{\xi}) = (\mathbf{v}_1 - \mathbf{v}_2, \xi_1 - \xi_2)$. Then, $\overline{\mathbb{B}}$ solves

$$\begin{aligned}
&\partial_t \overline{\mathbb{B}} + \mathscr{R}_\eta \mathbf{v}_1 \cdot \nabla \overline{\mathbb{B}} + \mathscr{R}_\eta \overline{\mathbf{v}} \cdot \nabla \mathbb{B}_2 - \overline{\mathbb{B}} \nabla \mathscr{R}_\eta \mathbf{v}_1^\top - \nabla \mathscr{R}_\eta \mathbf{v}_1 \overline{\mathbb{B}} \\
&-\mathbb{B}_2 \nabla \mathscr{R}_\eta \overline{\mathbf{v}}^\top - \nabla \mathscr{R}_\eta \overline{\mathbf{v}} \mathbb{B}_2 + \overline{\mathbb{B}} = \frac{\kappa}{\mu_\eta(\xi_1)} \Delta \overline{\mathbb{B}} + \kappa \Delta \mathbb{B}_2 \Big(\frac{1}{\mu_\eta(\xi_1)} - \frac{1}{\mu_\eta(\xi_2)} \Big), \text{ in } Q_T, \\
&\partial_{\mathbf{n}} \overline{\mathbb{B}} = \mathbf{0}, \text{ on } S_T, \\
&\overline{\mathbb{B}}(0) = \mathbb{B}_0^1 - \mathbb{B}_0^2, \text{ on } \Omega.
\end{aligned} \tag{6.38}$$

By the standard testing procedure with $\overline{\mathbb{B}}$ in (6.38) and integration by parts over Ω, it follows that

$$\begin{aligned}
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\| \overline{\mathbb{B}} \right\|_{L^2}^2 &+ \int_\Omega \mathscr{R}_\eta \overline{\mathbf{v}} \cdot \nabla \mathbb{B}_2 : \overline{\mathbb{B}} \,\mathrm{d}x - 2 \int_\Omega \overline{\mathbb{B}}^2 : \nabla \mathscr{R}_\eta \mathbf{v}_1 \,\mathrm{d}x \\
&- 2 \int_\Omega \big(\mathbb{B}_2 \overline{\mathbb{B}} \big) : \nabla \mathscr{R}_\eta \overline{\mathbf{v}} \,\mathrm{d}x + \left\| \overline{\mathbb{B}} \right\|_{L^2}^2 \\
&= - \int_\Omega \frac{\kappa}{\mu_\eta(\xi_1)} \left| \nabla \overline{\mathbb{B}} \right|^2 \,\mathrm{d}x + \int_\Omega \frac{\kappa \mu_\eta'(\xi_1)}{\mu_\eta^2(\xi_1)} (\nabla \mathscr{R}_\eta \xi_1 \cdot \nabla) \overline{\mathbb{B}} : \overline{\mathbb{B}} \,\mathrm{d}x \\
&- \int_\Omega \kappa \Big(\frac{1}{\mu_\eta(\xi_1)} - \frac{1}{\mu_\eta(\xi_2)} \Big) \nabla \mathbb{B}_2 : \nabla \overline{\mathbb{B}} \,\mathrm{d}x \\
&+ \int_\Omega \kappa \Big(\Big(\frac{\mu_\eta'(\xi_1) \nabla \xi_1}{\mu_\eta^2(\xi_1)} - \frac{\mu_\eta'(\xi_2) \nabla \xi_2}{\mu_\eta^2(\xi_2)} \Big) \cdot \nabla \Big) \mathbb{B}_2 : \overline{\mathbb{B}} \,\mathrm{d}x.
\end{aligned}$$

Now we estimate the equality above term by term. In view of Hölder's and Young's inequalities and Lemma 6.11, one obtains for some $\epsilon > 0$ to be determined yet,

$$\begin{aligned}
&\int_\Omega \mathscr{R}_\eta \overline{\mathbf{v}} \cdot \nabla \mathbb{B}_2 : \overline{\mathbb{B}} \,\mathrm{d}x \le \| \mathscr{R}_\eta \overline{\mathbf{v}} \|_{L^\infty} \| \nabla \mathbb{B}_2 \|_{L^2} \left\| \overline{\mathbb{B}} \right\|_{L^2} \le C(\eta) \| \overline{\mathbf{v}} \|_{L^2}^2 + \| \nabla \mathbb{B}_2 \|_{L^2}^2 \left\| \overline{\mathbb{B}} \right\|_{L^2}^2, \\
&\int_\Omega \overline{\mathbb{B}}^2 : \nabla \mathscr{R}_\eta \mathbf{v}_1 \,\mathrm{d}x \le \| \nabla \mathscr{R}_\eta \mathbf{v}_1 \|_{L^\infty} \left\| \overline{\mathbb{B}} \right\|_{L^2}^2, \\
&\int_\Omega \big(\mathbb{B}_2 \overline{\mathbb{B}} \big) : \nabla \mathscr{R}_\eta \overline{\mathbf{v}} \,\mathrm{d}x \le \| \nabla \mathscr{R}_\eta \overline{\mathbf{v}} \|_{L^\infty} \| \mathbb{B}_2 \|_{L^2} \left\| \overline{\mathbb{B}} \right\|_{L^2} \le C(\eta) \| \nabla \overline{\mathbf{v}} \|_{L^2}^2 + \| \mathbb{B}_2 \|_{L^2}^2 \left\| \overline{\mathbb{B}} \right\|_{L^2}^2, \\
&\int_\Omega \frac{\kappa \mu_\eta'(\xi_1)}{\mu_\eta^2(\xi_1)} (\nabla \mathscr{R}_\eta \xi_1 \cdot \nabla) \overline{\mathbb{B}} : \overline{\mathbb{B}} \,\mathrm{d}x \le C \| \nabla \mathscr{R}_\eta \xi_1 \|_{L^\infty} \left\| \nabla \overline{\mathbb{B}} \right\|_{L^2} \left\| \overline{\mathbb{B}} \right\|_{L^2} \\
&\qquad\qquad \le \varepsilon \left\| \nabla \overline{\mathbb{B}} \right\|_{L^2}^2 + C \| \nabla \mathscr{R}_\eta \xi_1 \|_{L^\infty}^2 \left\| \overline{\mathbb{B}} \right\|_{L^2}^2, \\
&\int_\Omega \kappa \Big(\frac{1}{\mu_\eta(\xi_1)} - \frac{1}{\mu_\eta(\xi_2)} \Big) \nabla \mathbb{B}_2 : \nabla \overline{\mathbb{B}} \,\mathrm{d}x = \int_\Omega \kappa \frac{\mu_\eta(\xi_2) - \mu_\eta(\xi_1)}{\mu_\eta(\xi_1) \mu_\eta(\xi_2)} \nabla \mathbb{B}_2 : \nabla \overline{\mathbb{B}} \,\mathrm{d}x \\
&\qquad\qquad \le C \left\| \mathscr{R}_\eta \overline{\xi} \right\|_{L^\infty} \| \nabla \mathbb{B}_2 \|_{L^2} \left\| \nabla \overline{\mathbb{B}} \right\|_{L^2} \\
&\qquad\qquad \le \varepsilon \left\| \nabla \overline{\mathbb{B}} \right\|_{L^2}^2 + C(\eta) \| \nabla \mathbb{B}_2 \|_{L^2}^2 \left\| \overline{\xi} \right\|_{L^2}^2, \\
&\int_\Omega \kappa \Big(\Big(\frac{\nabla \mu_\eta(\xi_1)}{\mu_\eta^2(\xi_1)} - \frac{\nabla \mu_\eta(\xi_2)}{\mu_\eta^2(\xi_2)} \Big) \cdot \nabla \Big) \mathbb{B}_2 : \overline{\mathbb{B}} \,\mathrm{d}x \\
&\qquad \le C(\eta) \left\| \mathscr{R}_\eta \overline{\xi} + \nabla \mathscr{R}_\eta \overline{\xi} \right\|_{L^\infty} \| \nabla \mathbb{B}_2 \|_{L^2} \left\| \overline{\mathbb{B}} \right\|_{L^2} \\
&\qquad \le C(\eta) (\left\| \overline{\xi} \right\|_{L^2}^2 + \left\| \nabla \overline{\xi} \right\|_{L^2}^2) + C \| \nabla \mathbb{B}_2 \|_{L^2}^2 \left\| \overline{\mathbb{B}} \right\|_{L^2}^2,
\end{aligned}$$

where $C(\eta)$ depends on η, κ and the upper-lower bounds of μ, μ'. Summarizing all the estimates above, choosing $2\varepsilon \leq \frac{\kappa}{2\overline{\mu}}$ and integrating over $(0,t)$, $t \in (0,T)$, give birth to

$$\begin{aligned}
&\frac{1}{2}\left\|\overline{\mathbb{B}}(t)\right\|_{L^2}^2 + \int_0^t \left\|\overline{\mathbb{B}}(\tau)\right\|_{L^2}^2 \,\mathrm{d}\tau + \frac{\kappa}{2\overline{\mu}}\int_0^t \left\|\nabla\overline{\mathbb{B}}\right\|_{L^2}^2 \,\mathrm{d}\tau \\
&\leq C(\eta)\Big(\|\overline{\mathbf{v}}\|_{L_t^2L_x^2}^2 + \|\nabla\overline{\mathbf{v}}\|_{L_t^2L_x^2}^2 + \left\|\overline{\xi}\right\|_{L_t^\infty L_x^2}^2 + \left\|\nabla\overline{\xi}\right\|_{L_t^2L_x^2}^2 + \left\|\overline{\mathbb{B}}_0\right\|_{L^2}^2 \Big) \\
&\quad + C\int_0^t \Big(\|\nabla\mathscr{R}_\eta\mathbf{v}_1(\tau)\|_{L^\infty} + \|\nabla\mathscr{R}_\eta\xi_1(\tau)\|_{L^\infty}^2 + \|\mathbb{B}_2(\tau)\|_{L^2}^2 + \|\nabla\mathbb{B}_2(\tau)\|_{L^2}^2 \Big) \left\|\overline{\mathbb{B}}(\tau)\right\|_{L^2}^2 \,\mathrm{d}\tau.
\end{aligned}$$

By Gronwall's lemma, one concludes that

$$\begin{aligned}
&\left\|\overline{\mathbb{B}}(t)\right\|_{L^2}^2 + \int_0^t \left\|\overline{\mathbb{B}}(\tau)\right\|_{L^2}^2 \,\mathrm{d}\tau + \int_0^t \left\|\nabla\overline{\mathbb{B}}\right\|_{L^2}^2 \,\mathrm{d}\tau \\
&\quad \leq C(\eta)\Big(\|\overline{\mathbf{v}}\|_{L_t^2L_x^2}^2 + \|\nabla\overline{\mathbf{v}}\|_{L_t^2L_x^2}^2 + \left\|\overline{\xi}\right\|_{L_t^\infty L_x^2}^2 + \left\|\nabla\overline{\xi}\right\|_{L_t^2L_x^2}^2 + \left\|\overline{\mathbb{B}}_0\right\|_{L^2}^2 \Big) \exp(g(t)),
\end{aligned} \tag{6.39}$$

where $C(\eta)$ depends on $\|\nabla\mathbb{B}_2\|_{L_t^2L_x^2}$, η, κ and the upper-lower bounds of the coefficients, the function $g(t) := \int_0^t (\|\nabla\mathscr{R}_\eta\mathbf{v}_1(\tau)\|_{L^\infty} + \|\nabla\mathscr{R}_\eta\xi_1(\tau)\|_{L^\infty}^2 + \|\mathbb{B}_2(\tau)\|_{L^2}^2 + \|\nabla\mathbb{B}_2(\tau)\|_{L^2}^2)\,\mathrm{d}\tau < \infty$ is finite for $t \in (0,T)$ thanks to the existence of weak solutions and the mollification. On account of (6.39), one has the continuous dependence of solutions of (6.20) on the data $(\mathbf{v}, \xi, \mathbb{B}_0)$, which implies the uniqueness.

This completes the proof of Theorem 6.16. □

6.4. Regularized System

As discussed before, we are going to introduce a novel regularization to (6.6), so that one can easily obtain good uniform estimates and pass to the limit in the regularization parameter with enough compactness. In view of the regularization operator $\mathscr{R}_\eta$ defined in (6.14) for $\eta > 0$, we define $\mu_\eta(\phi) := \mu(\mathscr{R}_\eta\phi)$. The aim of this section is therefore to obtain a weak solution to the following regularized system:

$$\begin{aligned}
&\partial_t(\rho(\phi)\mathbf{u}) + \mathbf{u}\cdot\nabla(\rho(\phi)\mathbf{u}) - \rho'(\phi)\operatorname{div}\big(\mathbf{u}\otimes m(\phi)\nabla q\big) + \nabla\pi \\
&\qquad - \operatorname{div}\big(\mathbb{S}_\eta(\nabla\mathbf{u},\mathbb{B},\phi)\big) = q\nabla\phi + \mathscr{R}_\eta\Big[\frac{\mu_\eta(\phi)}{2}\nabla\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I})\Big] && \text{in } Q_T, && (6.40a)\\
&\operatorname{div}\mathbf{u} = 0 && \text{in } Q_T, && (6.40b)\\
&\partial_t\mathbb{B} + \mathscr{R}_\eta\mathbf{u}\cdot\nabla\mathbb{B} + (\mathbb{B} - \mathbb{I}) = \mathbb{B}\nabla\mathscr{R}_\eta\mathbf{u}^\top + \nabla\mathscr{R}_\eta\mathbf{u}\mathbb{B} + \frac{\kappa}{\mu_\eta(\phi)}\Delta\mathbb{B} && \text{in } Q_T, && (6.40c)\\
&\partial_t\phi + \mathbf{u}\cdot\nabla\phi = \operatorname{div}(m(\phi)\nabla q) && \text{in } Q_T, && (6.40d)\\
&q - W'(\phi) + \Delta\phi - \eta\partial_t\phi = \mathscr{R}_\eta\Big[\frac{\mu_\eta'(\phi)}{2}\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I})\Big] && \text{in } Q_T, && (6.40e)\\
&\mathbf{u} = 0,\ \partial_\mathbf{n}\mathbb{B} = \mathbf{0} && \text{on } S_T, && (6.40f)\\
&\partial_\mathbf{n}\phi = \partial_\mathbf{n}q = 0 && \text{on } S_T, && (6.40g)\\
&(\mathbf{u},\mathbb{B},\phi)(0) = (\mathbf{u}_0,\mathbb{B}_0,\phi_0) && \text{in } \Omega, && (6.40h)
\end{aligned}$$

where

$$\mathbb{S}_\eta(\nabla\mathbf{u},\mathbb{B},\phi) = \nu(\phi)(\nabla\mathbf{u} + \nabla\mathbf{u}^\top) + \mathscr{R}_\eta\big[\mu_\eta(\phi)(\mathbb{B} - \mathbb{I})\big].$$

Let us start with the definition of weak solutions to the system (6.40).

Definition 6.24. Let $T > 0$, $d \in \{2, 3\}$, and $(\mathbf{u}_0, \mathbb{B}_0, \phi_0) \in L^2_\sigma(\Omega) \times L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \times W^{1,2}(\Omega)$ with $\mathbb{B}_0$ positive definite a.e. in Ω, $\operatorname{tr}\ln \mathbb{B}_0 \in L^1(\Omega)$ and $|\phi_0| \le 1$ a.e. in Ω. In addition, let Assumption 6.12 hold. We call the quadruple $(\mathbf{u}, \phi, q, \mathbb{B})$ a *finite energy* weak solution to (6.40) with initial data $(\mathbf{u}_0, \mathbb{B}_0, \phi_0)$, provided that

(1) the quadruple $(\mathbf{u}, \phi, q, \mathbb{B})$ satisfies

$$
\begin{gathered}
\mathbf{u} \in C_w([0,T]; L^2_\sigma(\Omega)) \cap L^2(0,T; W^{1,2}_0(\Omega; \mathbb{R}^d));\\
\phi \in C_w([0,T]; W^{1,2}(\Omega)) \cap L^2(0,T; W^{2,2}(\Omega)) \text{ with } \phi \in (-1,1) \text{ a.e. in } Q_T;\\
W'(\phi) \in L^2(0,T; L^2(\Omega)), \quad q \in L^2(0,T; W^{1,2}(\Omega));\\
\mathbb{B} \text{ is symmetric positive definite a.e. in } Q_T;\\
\mathbb{B} \in C_w([0,T]; L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})) \cap L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}));\\
\operatorname{tr}\ln \mathbb{B} \in L^\infty(0,T; L^1(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega));
\end{gathered}
$$

(2) for all $t \in (0,T)$ and all $\mathbf{w} \in C^\infty([0,T]; C^\infty_0(\Omega; \mathbb{R}^d))$ with $\operatorname{div}\mathbf{w} = 0$, we have

$$
\begin{aligned}
&\int_0^t \int_\Omega \Big(\rho(\phi)\mathbf{u} \cdot \partial_t \mathbf{w} + (\rho(\phi)\mathbf{u} \otimes \mathbf{u}) : \nabla \mathbf{w} - \rho'(\phi)(\mathbf{u} \otimes m(\phi)\nabla q) : \nabla \mathbf{w} \Big) \,\mathrm{d}x\mathrm{d}\tau \\
&\qquad - \int_0^t \int_\Omega \Big(\nu(\phi)(\nabla \mathbf{u} + \nabla \mathbf{u}^\top) : \nabla \mathbf{w} + \mathscr{R}_\eta \big[\mu_\eta(\phi)(\mathbb{B} - \mathbb{I}) \big] : \nabla \mathbf{w} \Big) \,\mathrm{d}x\mathrm{d}\tau \\
&\quad = - \int_0^t \int_\Omega q \nabla \phi \cdot \mathbf{w} \,\mathrm{d}x\mathrm{d}\tau - \int_0^t \int_\Omega \mathscr{R}_\eta \Big[\frac{\mu_\eta(\phi)}{2} \nabla \operatorname{tr}(\mathbb{B} - \ln \mathbb{B} - \mathbb{I}) \Big] \cdot \mathbf{w} \,\mathrm{d}x\mathrm{d}\tau \\
&\qquad + \int_\Omega \rho(\phi(\cdot,t))\mathbf{u}(\cdot,t) \cdot \mathbf{w}(\cdot,t) \,\mathrm{d}x - \int_\Omega \rho(\phi_0)\mathbf{u}_0 \cdot \mathbf{w}(\cdot,0) \,\mathrm{d}x;
\end{aligned}
\tag{6.41}
$$

(3) for all $t \in (0,T)$ and all $\xi \in C^\infty([0,T]; C^1(\overline{\Omega}))$, we have

$$
\begin{aligned}
&\int_0^t \int_\Omega \phi\big(\partial_t \xi + \mathbf{u} \cdot \nabla \xi\big) \,\mathrm{d}x\mathrm{d}\tau - \int_0^t \int_\Omega m(\phi) \nabla q \cdot \nabla \xi \,\mathrm{d}x\mathrm{d}\tau \\
&\qquad = \int_\Omega \phi(\cdot,t)\xi(\cdot,t) \,\mathrm{d}x - \int_\Omega \phi_0 \xi(\cdot,0) \,\mathrm{d}x;
\end{aligned}
\tag{6.42}
$$

(4) for a.e. $(x,t) \in Q_T$, we have

$$
q = W'(\phi) - \Delta\phi + \eta \partial_t \phi + \mathscr{R}_\eta \Big[\frac{\mu_\eta'(\phi)}{2} \operatorname{tr}(\mathbb{B} - \ln \mathbb{B} - \mathbb{I}) \Big];
\tag{6.43}
$$

(5) for all $t \in (0,T)$ and all $\mathbb{C} \in C^\infty(\overline{Q_T}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, we have

$$
\begin{aligned}
&\int_0^t \int_\Omega \Big(\mathbb{B} : \partial_t \mathbb{C} + (\mathscr{R}_\eta \mathbf{u} \otimes \mathbb{B}) : \nabla \mathbb{C} \Big) \,\mathrm{d}x\mathrm{d}\tau \\
&\qquad + \int_0^t \int_\Omega \Big(\big(\nabla \mathscr{R}_\eta \mathbf{u} \mathbb{B} + \mathbb{B} \nabla \mathscr{R}_\eta \mathbf{u}^\top \big) : \mathbb{C} - \kappa \nabla \mathbb{B} : \nabla \frac{\mathbb{C}}{\mu_\eta(\phi)} \Big) \,\mathrm{d}x\mathrm{d}\tau \\
&\quad = \int_0^t \int_\Omega (\mathbb{B} : \mathbb{C} - \operatorname{tr}\mathbb{C}) \,\mathrm{d}x\mathrm{d}\tau + \int_\Omega \mathbb{B}(\cdot,t) : \mathbb{C}(\cdot,t) \,\mathrm{d}x - \int_\Omega \mathbb{B}_0 : \mathbb{C}(\cdot,0) \,\mathrm{d}x;
\end{aligned}
\tag{6.44}
$$

(6) for a.e. $t \in (0,T)$, the following energy estimate holds

$$
\begin{aligned}
&\mathcal{E}_\eta(t) + \int_0^t \left\| \sqrt{\nu(\phi)}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)(\tau) \right\|_{L^2}^2 \,\mathrm{d}\tau \\
&\quad + \int_0^t \left\| \sqrt{m(\phi)} \nabla q(\tau) \right\|_{L^2}^2 \,\mathrm{d}\tau + \eta \int_0^t \|\partial_t \phi(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \\
&\quad + \int_0^t \Big(\left\| \operatorname{tr}(\mathbb{B} + \mathbb{B}^{-1} - 2\mathbb{I})(\tau) \right\|_{L^1} + \kappa \|\nabla \operatorname{tr} \ln \mathbb{B}(\tau)\|_{L^2}^2 \Big) \,\mathrm{d}\tau \le C\mathcal{E}(0),
\end{aligned}
\tag{6.45}
$$

where the energy of the regularized system is

$$
\mathcal{E}_\eta(t) = \int_\Omega \frac{\rho(\phi)}{2} |\mathbf{u}|^2 \,\mathrm{d}x + \int_\Omega \frac{\mu_\eta(\phi)}{2} \operatorname{tr}(\mathbb{B} - \ln \mathbb{B} - \mathbb{I}) \,\mathrm{d}x + \int_\Omega \frac{1}{2} |\nabla \phi|^2 + W(\phi) \,\mathrm{d}x, \tag{6.46}
$$

satisfying $\mathcal{E}_\eta(0) \le C(\overline{\mu}, \underline{\mu})\mathcal{E}(0)$ uniformly regarding η with $\mathcal{E}$ defined in (6.2).

Our main result in this section will be as follows.

THEOREM 6.25. *Let Assumption 6.12 hold, $d \in \{2,3\}$, $(\mathbf{u}_0, \mathbb{B}_0, \phi_0) \in L^2_\sigma(\Omega) \times L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \times W^{1,2}(\Omega)$ with $\operatorname{tr} \ln \mathbb{B}_0 \in L^1(\Omega)$, $\mathbb{B}_0$ positive definite and $|\phi_0| \le 1$ a.e. in Ω and $\fint_\Omega \phi_0 \,\mathrm{d}x \in (-1,1)$. Then there exists a finite energy weak solution $(\mathbf{u}, \phi, q, \mathbb{B})$ of (6.40) in the sense of Definition 6.24.*

Remark 6.26. The theorem is proved in Section 6.4.7. Thanks to the regularization $\mathscr{R}_\eta$ we introduced, the existence theorem can be established both in two and three dimensions. This can not be reached when passing to the limit as $\eta \to 0$ in Section 6.5, where we need a stronger estimate of $\mathbb{B}$ (uniform in η) in Section 6.5.1, which is restricted to two dimensions.

Remark 6.27. In the rest of this section, we only proceed with the case $d = 3$, while the other case $d = 2$ is even simpler with better Sobolev embeddings. For this, we refer to the final proof of Theorem 6.3 in Section 6.5.2.

6.4.1. Formal a priori estimates. In this section, we carry out formal energy estimates including the regularization with respect to $\eta > 0$. Note that this energy is not necessarily finite if $\mathbb{B}$ is not positive definite, which is due to the logarithmic term. In Section 6.3.2 we overcome it by means of an entropy regularization to the logarithmic function. Here we do everything formally, which will be justified later. Now we discuss about the *a priori* estimates for (6.40). Testing (6.40a) with $\mathbf{u}$, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{\rho(\phi)}{2} |\mathbf{u}|^2 \,\mathrm{d}x + \int_\Omega \Big(\frac{\nu(\phi)}{2} \left|\nabla \mathbf{u} + \nabla \mathbf{u}^\top\right|^2 + \mathscr{R}_\eta\big[\mu_\eta(\phi)\big(\mathbb{B} - \mathbb{I}\big)\big] : \nabla \mathbf{u} \Big) \,\mathrm{d}x \\
&\quad + \int_\Omega \Big(\underbrace{(\partial_t + \mathbf{u} \cdot \nabla)\rho(\phi) - \operatorname{div}\big(m(\phi)\nabla q\big)\rho'(\phi)}_{=0 \text{ by } (6.40\text{d})} \Big) \frac{|\mathbf{u}|^2}{2} \,\mathrm{d}x \\
&\quad = \int_\Omega q \nabla \phi \cdot \mathbf{u} \,\mathrm{d}x + \underbrace{\int_\Omega \mathscr{R}_\eta \Big[\frac{\mu_\eta(\phi)}{2} \nabla \operatorname{tr}(\mathbb{B} - \ln \mathbb{B} - \mathbb{I}) \Big] \cdot \mathbf{u} \,\mathrm{d}x}_{= - \int_\Omega \frac{\mu'_\eta(\phi)}{2} \mathscr{R}_\eta \mathbf{u} \cdot \nabla \mathscr{R}_\eta \phi \operatorname{tr}(\mathbb{B} - \ln \mathbb{B} - \mathbb{I}) \,\mathrm{d}x},
\end{aligned}
\tag{6.47}
$$

where we used the identity for a first-order differential operator $\partial \cdot$ that

$$
\begin{aligned}
\partial(\rho \mathbf{u}) \cdot \mathbf{u} &= \partial \rho |\mathbf{u}|^2 + \rho \partial \mathbf{u} \cdot \mathbf{u} \\
&= \partial \rho |\mathbf{u}|^2 + \rho \partial \Big(\frac{|\mathbf{u}|^2}{2} \Big) = \partial \rho |\mathbf{u}|^2 + \partial \Big(\rho \frac{|\mathbf{u}|^2}{2} \Big) - \partial \rho \frac{|\mathbf{u}|^2}{2} = \partial \Big(\rho \frac{|\mathbf{u}|^2}{2} \Big) + \partial \rho \frac{|\mathbf{u}|^2}{2}.
\end{aligned}
$$

Multiplying (6.40c) by $\frac{\mu_\eta(\phi)}{2}(\mathbb{I}-\mathbb{B}^{-1})$ and integrating it over Ω, one obtains

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\mu_\eta(\phi)}{2}\operatorname{tr}(\mathbb{B}-\ln\mathbb{B}-\mathbb{I})\,\mathrm{d}x \\
&\qquad+\int_\Omega \frac{\mu_\eta(\phi)}{2}\operatorname{tr}(\mathbb{B}+\mathbb{B}^{-1}-2\mathbb{I})\,\mathrm{d}x+\frac{\kappa}{d}\int_\Omega |\nabla \operatorname{tr}\ln\mathbb{B}|^2\,\mathrm{d}x \\
&\quad\le \int_\Omega \frac{\mu_\eta'(\phi)}{2}(\partial_t+\mathscr{R}_\eta\mathbf{u}\cdot\nabla)\mathscr{R}_\eta\phi\operatorname{tr}(\mathbb{B}-\ln\mathbb{B}-\mathbb{I})\,\mathrm{d}x+\int_\Omega \mu_\eta(\phi)(\mathbb{B}-\mathbb{I}):\nabla\mathscr{R}_\eta\mathbf{u}\,\mathrm{d}x,
\end{aligned}\tag{6.48}$$

where we employed (6.11), (6.12) and

$$-\int_\Omega \nabla\mathbb{B}:\nabla\mathbb{B}^{-1}\,\mathrm{d}x\ge\frac{1}{d}\int_\Omega|\nabla\operatorname{tr}\ln\mathbb{B}|^2\,\mathrm{d}x,$$

which is referred to [BLS17, Lemma 3.1] with $d=2,3$. Testing (6.40d) with q, (6.40e) with $-\partial_t\phi$, integrating and adding both eqations and integrating by parts over Ω yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big(\frac{1}{2}|\nabla\phi|^2+W(\phi)\Big)\,\mathrm{d}x+\int_\Omega\mathbf{u}\cdot\nabla\phi q\,\mathrm{d}x+\eta\int_\Omega|\partial_t\phi|^2\,\mathrm{d}x \\
&\qquad+\int_\Omega\frac{\mu_\eta'(\phi)}{2}\partial_t\mathscr{R}_\eta\phi\operatorname{tr}(\mathbb{B}-\ln\mathbb{B}-\mathbb{I})\,\mathrm{d}x+\int_\Omega m(\phi)|\nabla q|^2\,\mathrm{d}x=0.
\end{aligned}\tag{6.49}$$

Adding (6.47), (6.48) and (6.49) together gives rise to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\eta(t)+\int_\Omega\Big(\frac{\nu(\phi)}{2}\left|\nabla\mathbf{u}+\nabla\mathbf{u}^\top\right|^2+m(\phi)|\nabla q|^2+\eta|\partial_t\phi|^2\Big)\,\mathrm{d}x \\
&\qquad+\int_\Omega\frac{\mu_\eta(\phi)}{2}\operatorname{tr}(\mathbb{B}+\mathbb{B}^{-1}-2\mathbb{I})\,\mathrm{d}x+\frac{\kappa}{d}\int_\Omega|\nabla\operatorname{tr}\ln\mathbb{B}|^2\,\mathrm{d}x\le 0,
\end{aligned}\tag{6.50}$$

with the help of (6.15), where the mixed terms are canceled after the summation. Then, integrating (6.50) over $t\in(0,\tau)$, $\tau\in(0,T)$ and using the upper and lower bounds of ν,μ,m, one finally obtains the *a priori* estimate

$$\begin{aligned}
&\mathcal{E}_\eta(t)+\int_0^\tau\|\nabla\mathbf{u}(t)\|_{L^2}^2\,\mathrm{d}t+\int_0^\tau\|\nabla q(t)\|_{L^2}^2\,\mathrm{d}t+\eta\int_0^\tau\|\partial_t\phi(t)\|_{L^2}^2\,\mathrm{d}t \\
&\qquad+\int_0^\tau\left\|\operatorname{tr}(\mathbb{B}+\mathbb{B}^{-1}-2\mathbb{I})(t)\right\|_{L^1}\,\mathrm{d}t+\int_0^\tau\|\nabla\operatorname{tr}\ln\mathbb{B}(t)\|_{L^2}^2\,\mathrm{d}t\le C\mathcal{E}(0),
\end{aligned}\tag{6.51}$$

for a.e. $\tau\in(0,T)$. Here, the positive constant C comes from $\mathcal{E}_\eta(0)\le\frac{\overline{\mu}}{\underline{\mu}}\mathcal{E}(0)$ and is independent of $\eta>0$.

6.4.2. Reformulation of the chemical potential. In presence of the singular, non-convex potential $W(\phi)$, we first rewrite the chemical potential. The idea is to use the subdifferential of a convex potential, which was also employed in [AW07] for the Cahn–Hilliard equation and in [ADG13a] for the Cahn–Hilliard–Navier–Stokes equation. First we define an extended potential $\widetilde{W}$ through

$$\widetilde{W}:\mathbb{R}\to\mathbb{R},\quad \widetilde{W}(s)=\begin{cases}W(s) & \text{if } s\in[-1,1],\\ +\infty & \text{else,}\end{cases}$$

where W is the potential fulfilling Assumption **(H3)**. Note that $\widetilde{W}$ is not necessarily convex by this assumption. Hence, we define $\widetilde{W}_0(r) := \widetilde{W}(r) + \frac{\omega}{2}r^2$, which satisfies $\widetilde{W}_0 \in C([-1,1]) \cap C^2((-1,1))$ and is convex due to the Assumption **(H3)**. In particular, we have $\widetilde{W}'(r) = \widetilde{W}_0'(r) - \omega r$. Now (6.40e) is equivalent to

$$q + \omega\phi = \widetilde{W}_0'(\phi) - \Delta\phi + \eta\partial_t\phi + \mathscr{R}_\eta\Big[\frac{\mu_\eta'(\phi)}{2}\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I})\Big], \quad \text{a.e. in } Q_T.$$

Following [AW07, ADG13a], we define a modified energy $\widetilde{E} : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$\widetilde{E}(\phi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla\phi|^2 + \int_\Omega \widetilde{W}_0(\phi) & \text{for } \phi \in \mathcal{D}(\widetilde{E}), \\ +\infty & \text{else,} \end{cases}$$

with the domain of definition

$$\mathcal{D}(\widetilde{E}) = \left\{\phi \in W^{1,2}(\Omega) : -1 \le \phi \le 1 \text{ a.e.}\right\}.$$

Then, as in [ADG13a], the domain of the definition of the subgradient $\partial\widetilde{E}$ is

$$\mathcal{D}(\partial\widetilde{E}) = \Big\{\phi \in W^{2,2}(\Omega) : \widetilde{W}_0'(\phi) \in L^2(\Omega),\ \widetilde{W}_0''(\phi)\,|\nabla\phi|^2 \in L^1(\Omega),\ \partial_{\mathbf{n}}\phi|_{\partial\Omega} = 0\Big\}.$$

With these definitions, one has

$$\partial\widetilde{E}(\phi) = -\Delta\phi + \widetilde{W}_0'(\phi) \quad \text{for } \phi \in \mathcal{D}(\partial\widetilde{E}).$$

Note that $\partial\widetilde{E}$ is maximal monotone by the convexity of $\widetilde{W}_0$ and the lower semicontinuity. Moreover, it holds that

$$\|\phi\|_{W^{2,2}}^2 + \|\widetilde{W}_0'(\phi)\|_{L^2}^2 + \int_\Omega \widetilde{W}_0''(\phi(x))\,|\nabla\phi(x)|^2 \,\mathrm{d}x \le C\Big(\|\partial\widetilde{E}(\phi)\|_{L^2}^2 + \|\phi\|_{L^2}^2 + 1\Big). \tag{6.52}$$

Remark 6.28. Note that in [ADG13a] a more complicated subgradient $\partial\widetilde{E}(\phi) = -\Delta A(\phi) + \widetilde{W}_0'(A(\phi))$ was considered to resolve the problem caused by the phase-dependent free energy $\int_\Omega \big(\frac{a(\phi)}{2}|\nabla\phi|^2 + W(\phi)\big)\,\mathrm{d}x$, where $a(\phi)$ is some positive function and $A(\phi)$ is related to $a(\phi)$, see [ADG13a]. However, in this paper, the simpler case of $a(\phi) = 1$ is studied. We still take the advantage of $\partial\widetilde{E}(\phi) = -\Delta\phi + \widetilde{W}_0'(\phi)$, which is a maximal monotone operator, to pass to the limit in the final proof, see Section 6.4.7. Another possible strategy is to approximate the singular potential with a sequence of regular potentials, for which we also refer to, e.g., [GGW19] for the case with dynamic boundary conditions and $a(\phi) = 1$, where the authors take the idea of a convex decomposition of a singular potential and a regular approximation.

6.4.3. Hybrid implicit time discretization. Inspired by the well-posedness result of the Oldroyd-B equation (6.20) and the implicit time discretization argument in [ADG13a] to solve the AGG model with unmatched densities and singular potential, we propose a *hybrid implicit time discretization* for the whole regularized system (6.40). Namely, for the AGG part, we employ a time discretization similarly to [ADG13a], while the Oldroyd-B part is solved continuously in time with the help of Theorem 6.16 on each discrete time interval.

More precisely, let $T > 0$, $h = \frac{T}{N}$ for $N \in \mathbb{N}$, $\mathbb{B}_0 \in L^2(\Omega; \mathbb{R}^{d\times d}_{\text{sym}+})$ with $\operatorname{tr}\ln\mathbb{B}_0 \in L^1(\Omega)$, and, for $k \in \{0, ..., N-1\}$, let $\mathbf{u}_k \in L^2_\sigma(\Omega)$, $\phi_k \in W^{1,2}(\Omega)$ with $W'(\phi_k) \in L^2(\Omega)$, and $\rho_k =$

$\frac{1}{2}(\tilde{\rho}_1 + \tilde{\rho}_2) + \frac{1}{2}(\tilde{\rho}_2 - \tilde{\rho}_1)\phi_k$ be given. We construct $(\mathbf{u}, \phi, q, \widetilde{\mathbb{B}}) = (\mathbf{u}_{k+1}, \phi_{k+1}, q_{k+1}, \widetilde{\mathbb{B}}_{k+1})$ as a solution of the following nonlinear system.

Let $I_{k+1} \coloneqq (t_k, t_{k+1})$ with $t_k = kh$, $k \in \{0, ..., N-1\}$. Find $(\mathbf{u}, \phi, q, \widetilde{\mathbb{B}})$ with $\mathbf{u} \in W^{1,2}_{0,\sigma}(\Omega)$, $\phi \in \mathcal{D}(\partial \widetilde{E})$, $\mu \in W^{2,2}_n = \{u \in W^{2,2}(\Omega) : \partial_{\mathbf{n}} u|_{\partial\Omega} = 0 \text{ on } \partial\Omega\}$, and $\widetilde{\mathbb{B}} \in L^\infty(I_{k+1}; L^2(\Omega; \mathbb{R}^{d\times d}_{\text{sym}})) \cap L^2(I_{k+1}; W^{1,2}(\Omega; \mathbb{R}^{d\times d}_{\text{sym}}))$ with $\widetilde{\mathbb{B}}$ positive definite a.e. in Ω and $\operatorname{tr}\ln\widetilde{\mathbb{B}} \in L^1(I_{k+1}; L^1(\Omega))$, such that $\overline{\mathbb{B}}_{k+1} = \frac{1}{h}\int_{I_{k+1}} \widetilde{\mathbb{B}}(t)\,\mathrm{d}t$, where $\widetilde{\mathbb{B}}$ is the solution of (6.20) on the time interval $\overline{I}_{k+1}$ with initial data $\overline{\mathbb{B}}_k$ and data $(\mathbf{v}, \xi)(t) = (\mathbf{u}, \phi_k)$, $t \in I_{k+1}$, where $\overline{\mathbb{B}}_0 = \mathbb{B}_0$, that is,

$$\begin{aligned} \partial_t\widetilde{\mathbb{B}} + \mathscr{R}_\eta\mathbf{u}\cdot\nabla\widetilde{\mathbb{B}} - \widetilde{\mathbb{B}}\nabla\mathscr{R}_\eta\mathbf{u}^\top - \nabla\mathscr{R}_\eta\mathbf{u}\widetilde{\mathbb{B}} + (\widetilde{\mathbb{B}} - \mathbb{I}) = \frac{\kappa}{\mu_\eta(\phi_k)}\Delta\widetilde{\mathbb{B}}, &\quad \text{in } \Omega\times I_{k+1},\\ \partial_{\mathbf{n}}\widetilde{\mathbb{B}} = \mathbf{0}, &\quad \text{on } \partial\Omega\times I_{k+1},\\ \widetilde{\mathbb{B}}(t_k) = \overline{\mathbb{B}}_k, &\quad \text{in } \Omega. \end{aligned} \tag{6.53a}$$

Note that we have continuous dependence of $\widetilde{\mathbb{B}}$ on $(\mathbf{u}, \phi_k, \overline{\mathbb{B}}_k)$ by Theorem 6.16. Moreover, we consider the following discrete problem with time-averaged terms with respect to $\widetilde{\mathbb{B}}$:

$$\begin{aligned} &\left(\frac{\rho\mathbf{u} - \rho_k\mathbf{u}_k}{h}, \mathbf{w}\right) + (\operatorname{div}(\rho_k\mathbf{u}\otimes\mathbf{u}), \mathbf{w}) + (\operatorname{div}(\mathbf{u}\otimes\mathbf{J}), \mathbf{w})\\ &\quad + \big(\nu(\phi_k)(\nabla\mathbf{u} + \nabla\mathbf{u}^\top), \nabla\mathbf{w}\big) - \frac{1}{h}\int_{I_{k+1}} \Big(\operatorname{div}\mathscr{R}_\eta\big[\mu_\eta(\phi_k)(\widetilde{\mathbb{B}} - \mathbb{I})\big], \mathbf{w}\Big)\,\mathrm{d}t\\ &\quad = (q\nabla\phi_k, \mathbf{w}) + \frac{1}{h}\int_{I_{k+1}} \left(\mathscr{R}_\eta\Big[\frac{\mu_\eta(\phi_k)}{2}\nabla\operatorname{tr}(\widetilde{\mathbb{B}} - \ln\widetilde{\mathbb{B}} - \mathbb{I})\Big], \mathbf{w}\right)\,\mathrm{d}t \end{aligned} \tag{6.53b}$$

for all $\mathbf{w} \in C^\infty_0(\overline{\Omega})$ with $\operatorname{div}\mathbf{w} = 0$, where

$$\mathbf{J} = \mathbf{J}_{k+1} = -\rho'(\phi_k)m(\phi_k)\nabla q_{k+1} = -\rho'(\phi_k)m(\phi_k)\nabla q.$$

In addition, for a.e. $x \in \Omega$,

$$\frac{\phi - \phi_k}{h} + \mathbf{u}\cdot\nabla\phi_k = \operatorname{div}(m(\phi_k)\nabla q), \tag{6.53c}$$

$$\begin{aligned} q + \omega\frac{\phi + \phi_k}{2} &= \widetilde{W}_0'(\phi) - \Delta\phi + \eta\frac{\phi - \phi_k}{h}\\ &\quad + \mathscr{R}_\eta\Big[\frac{1}{2}\frac{\mu_\eta(\phi) - \mu_\eta(\phi_k)}{\mathscr{R}_\eta(\phi - \phi_k)}\frac{1}{h}\int_{I_{k+1}} \operatorname{tr}(\widetilde{\mathbb{B}} - \ln\widetilde{\mathbb{B}} - \mathbb{I})(t)\,\mathrm{d}t\Big]. \end{aligned} \tag{6.53d}$$

Remark 6.29. In the above system, we distinguish between time-discrete functions and functions that are continuous-in-time. That is the reason why we write a tilde on the top of the left Cauchy–Green tensor $\widetilde{\mathbb{B}}$. In order to make sure the time-continuous problem (6.53a) and the time-discrete subsystem (6.53b)–(6.53d) are compatible with each other, we make use of time averages for the terms containing $\widetilde{\mathbb{B}}$ in the time-discrete problem. In fact, it is necessary to take advantage of all the information of $\widetilde{\mathbb{B}}$ in a fixed time interval so that the terms regarding $\widetilde{\mathbb{B}}$ are well-defined concerning the regularity.

Another motivation is to get the uniform energy estimate for the whole hybrid discrete problem with respect to h, for which we approximate $\mu_\eta'(\phi)$ due to the variable shear modulus by the difference quotient $\frac{\mu_\eta(\phi) - \mu_\eta(\phi_k)}{\mathscr{R}_\eta(\phi - \phi_k)}$, and proceed with the energy estimate (6.56) for $\widetilde{\mathbb{B}}$ in the interval I_{k+1}. With all these ingredients, we establish a uniform estimate by canceling the mixed terms corresponding to $\widetilde{\mathbb{B}}$.

Remark 6.30. The approximation of $\mu_\eta'(\phi)$ with the difference quotient $\frac{\mu_\eta(\phi)-\mu_\eta(\phi_k)}{\mathscr{R}_\eta(\phi-\phi_k)}$ is well-defined for $\phi \neq \phi_k$, as $\mu \in C^1(\mathbb{R})$ with bounded derivative in view of Assumption **(H2)**. Indeed, by the mean value theorem, it holds

$$\frac{\mu_\eta(\phi) - \mu_\eta(\phi_k)}{\mathscr{R}_\eta(\phi - \phi_k)} = \mu_\eta'(\xi) \in [\underline{\mu}', \overline{\mu}'],$$

for some $\xi \in (\phi, \phi_k)$, w.l.o.g. $\phi < \phi_k$. Moreover, as $\phi_k \to \phi$ a.e. in Q_T, it holds

$$\frac{\mu_\eta(\phi) - \mu_\eta(\phi_k)}{\mathscr{R}_\eta(\phi - \phi_k)} \to \mu_\eta'(\phi).$$

Remark 6.31. By simply integrating (6.53c) over Ω and applying integration by parts over Ω together with $\operatorname{div} \mathbf{u} = 0$ and the Neumann boundary condition $\partial_{\mathbf{n}} q|_{\partial\Omega} = 0$, one infers that

$$\int_\Omega \phi \,\mathrm{d}x = \int_\Omega \phi_k \,\mathrm{d}x = \int_\Omega \phi_0 \,\mathrm{d}x,$$

which implies that the mass is conserved for the time-discrete problem.

In a similar fashion as in [ADG13a, Lemma 4.2], we have the following lemma, which will be frequently used later. For the reader's convenience, we point out the main differences compared to [ADG13a, Lemma 4.2] arising from the additional terms.

Lemma 6.32. *Let $\phi \in \mathcal{D}(\partial \widetilde{E})$ and $q \in W^{1,2}(\Omega)$ be solving (6.53d) with given $\phi_k \in W^{2,2}(\Omega)$ satisfying $|\phi_k| \leq 1$ and*

$$\sqrt{\eta}\frac{\phi - \phi_k}{h} \in L^2(\Omega), \quad \frac{1}{|\Omega|}\int_\Omega \phi \,\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega \phi_k \,\mathrm{d}x \in (-1, 1).$$

Moreover, let $\widetilde{\mathbb{B}} \in L^2(I_{k+1}; L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}+}))$ with $\operatorname{tr} \ln \widetilde{\mathbb{B}} \in L^2(I_{k+1}; L^2(\Omega))$. Then, there exists a constant $C > 0$ depending on η and $\int_\Omega \phi_k \,\mathrm{d}x$ such that

$$\begin{aligned}\left\|\widetilde{W}_0'(\phi)\right\|_{L^1(\Omega)} + \left|\int_\Omega q \,\mathrm{d}x\right| \leq C\Big(&\|\nabla q\|_{L^2}^2 + \|\nabla \phi\|_{L^2}^2 + \|\nabla \phi_k\|_{L^2}^2 \\ &+ \eta\left\|\frac{\phi - \phi_k}{h}\right\|_{L^2}^2 + \frac{1}{h}\left\|\operatorname{tr}(\widetilde{\mathbb{B}} - \ln \widetilde{\mathbb{B}} - \mathbb{I})\right\|_{L^2(I_{k+1}; L^2(\Omega))}^2 + 1\Big).\end{aligned}$$

Proof. First of all, testing (6.53d) with $\phi - \overline{\phi}$, where $\overline{\phi} \coloneqq \frac{1}{\Omega}\int_\Omega \phi \,\mathrm{d}x$ is the mean value of ϕ over Ω, leads to

$$\begin{aligned}&\int_\Omega q(\phi - \overline{\phi}) \,\mathrm{d}x + \int_\Omega \omega \frac{\phi + \phi_k}{2}(\phi - \overline{\phi}) \,\mathrm{d}x \\ &= \int_\Omega \widetilde{W}_0'(\phi)(\phi - \overline{\phi}) \,\mathrm{d}x + \int_\Omega \nabla\phi \cdot \nabla(\phi - \overline{\phi}) \,\mathrm{d}x + \int_\Omega \eta \frac{\phi - \phi_k}{h}(\phi - \overline{\phi}) \,\mathrm{d}x \\ &\quad + \int_\Omega \frac{\mu_\eta(\phi) - \mu_\eta(\phi_k)}{2\mathscr{R}_\eta(\phi - \phi_k)}\Big(\frac{1}{h}\int_{I_{k+1}} \operatorname{tr}(\widetilde{\mathbb{B}} - \ln \widetilde{\mathbb{B}} - \mathbb{I})(t) \,\mathrm{d}t\Big)\mathscr{R}_\eta(\phi - \overline{\phi}) \,\mathrm{d}x.\end{aligned}$$

By means of Hölder's and Young's inequalities, and the Poincaré–Wirtinger inequality, one has the following estimates:

$$\begin{aligned}
&\int_\Omega q(\phi-\overline{\phi})\,\mathrm{d}x = \int_\Omega (q-\overline{q})\phi\,\mathrm{d}x \le \|q-\overline{q}\|_{L^2}\|\phi\|_{L^2} \le \|\nabla q\|_{L^2}^2 + C(\|\nabla\phi\|_{L^2}+1),\\
&\int_\Omega \omega\frac{\phi+\phi_k}{2}(\phi-\overline{\phi})\,\mathrm{d}x \le C(\|\phi\|_{L^2}+\|\phi_k\|_{L^2})\|\nabla\phi\|_{L^2} \le C(\|\nabla\phi\|_{L^2}^2+\|\nabla\phi_k\|_{L^2}^2+1),\\
&\int_\Omega \nabla\phi\cdot\nabla(\phi-\overline{\phi})\,\mathrm{d}x \le C\|\nabla\phi\|_{L^2}^2,\\
&\int_\Omega \eta\frac{\phi-\phi_k}{h}(\phi-\overline{\phi})\,\mathrm{d}x \le \eta\left\|\frac{\phi-\phi_k}{h}\right\|_{L^2}\|\phi-\overline{\phi}\|_{L^2} \le \eta\left\|\frac{\phi-\phi_k}{h}\right\|_{L^2}^2 + \eta\|\nabla\phi\|_{L^2}^2,\\
&\int_\Omega \frac{\mu_\eta(\phi)-\mu_\eta(\phi_k)}{2\mathscr{R}_\eta(\phi-\phi_k)}\Big(\frac{1}{h}\int_{I_{k+1}}\mathrm{tr}(\widetilde{\mathbb{B}}-\ln\widetilde{\mathbb{B}}-\mathbb{I})(t)\,\mathrm{d}t\Big)\mathscr{R}_\eta(\phi-\overline{\phi})\,\mathrm{d}x\\
&\qquad\qquad \le \frac{\overline{\mu}'}{4}\frac{1}{h}\left\|\mathrm{tr}(\widetilde{\mathbb{B}}-\ln\widetilde{\mathbb{B}}-\mathbb{I})\right\|_{L^2(I_{k+1};L^2(\Omega))}^2 + \frac{\overline{\mu}'}{4}\|\nabla\phi\|_{L^2}^2,
\end{aligned}$$

where C depends on $\int_\Omega \phi_k\,\mathrm{d}x$. Moreover, it follows from the assumption $\lim_{\phi\to\pm1}\widetilde{W}_0'(\phi)\to\pm\infty$ and $\widetilde{W}_0'\in C([-1+\frac{\varepsilon}{2},1-\frac{\varepsilon}{2}])$ due to **(H3)**, together with the fact that $\overline{\phi}\in(-1+\varepsilon,1-\varepsilon)$ for some $\varepsilon>0$, that

$$\widetilde{W}_0'(\phi)(\phi-\overline{\phi}) \ge C\left|\widetilde{W}_0'(\phi)\right| - C_1,$$

by discussing the range of ϕ in three cases: $[-1,-1+\frac{\varepsilon}{2}]$, $[-1+\frac{\varepsilon}{2},1-\frac{\varepsilon}{2}]$, $[1-\frac{\varepsilon}{2},1]$. As a consequence,

$$\int_\Omega \widetilde{W}_0'(\phi)(\phi-\overline{\phi})\,\mathrm{d}x \ge C\int_\Omega\left|\widetilde{W}_0'(\phi)\right|\,\mathrm{d}x - C_2.$$

Collecting all the estimates from above yields

$$\begin{aligned}
\left\|\widetilde{W}_0'(\phi)\right\|_{L^1(\Omega)} \le C\Big(&\|\nabla q\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2+\|\nabla\phi_k\|_{L^2}^2\\
&+\eta\left\|\frac{\phi-\phi_k}{h}\right\|_{L^2}^2+\frac{1}{h}\left\|\mathrm{tr}(\widetilde{\mathbb{B}}-\ln\widetilde{\mathbb{B}}-\mathbb{I})\right\|_{L^2(I_{k+1};L^2(\Omega))}^2+1\Big),
\end{aligned}$$

which thereafter implies that

$$\begin{aligned}
\left|\int_\Omega q\,\mathrm{d}x\right| \le C\Big(&\|\nabla q\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2+\|\nabla\phi_k\|_{L^2}^2\\
&+\eta\left\|\frac{\phi-\phi_k}{h}\right\|_{L^2}^2+\frac{1}{h}\left\|\mathrm{tr}(\widetilde{\mathbb{B}}-\ln\widetilde{\mathbb{B}}-\mathbb{I})\right\|_{L^2(I_{k+1};L^2(\Omega))}^2+1\Big),
\end{aligned}$$

by integrating (6.53d) over Ω, for which we used $\int_\Omega \Delta\phi\,\mathrm{d}x = \int_{\partial\Omega}\partial_{\mathbf{n}}\phi\,\mathrm{d}\mathcal{H}^{d-1}=0$. □

Remark 6.33. The condition $\mathrm{tr}\ln\widetilde{\mathbb{B}}\in L^2(I_{k+1};L^2(\Omega))$ in Lemma 6.32 follows directly from Theorem 6.16 and the Poincaré–Wirtinger inequality.

6.4.4. Existence for the hybrid discrete system. Now we present and prove the existence of solutions to the hybrid discrete system (6.53).

Proposition 6.34. *Let* $\mathbb{B}_0 \in L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}+})$ *with* $\operatorname{tr}\ln \mathbb{B}_0 \in L^1(\Omega)$*, and, for* $k \in \{0, ..., N-1\}$*, let* $\mathbf{u}_k \in L^2_\sigma(\Omega)$*,* $\phi_k \in W^{2,2}(\Omega)$*, and* $\rho_k = \frac{1}{2}(\tilde\rho_1 + \tilde\rho_2) + \frac{1}{2}(\tilde\rho_2 - \tilde\rho_1)\phi_k$ *be given. Then there are some* $(\mathbf{u}, \phi, q, \widetilde{\mathbb{B}}) \in W^{1,2}_{0,\sigma}(\Omega) \times \mathcal{D}(\partial \widetilde{E}) \times W^{2,2}_n \times L^\infty(I_{k+1}; L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})) \cap L^2(I_{k+1}; W^{1,2}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}))$ *with* $\widetilde{\mathbb{B}}$ *positive definite a.e. in* Ω *and* $\operatorname{tr}\ln\widetilde{\mathbb{B}} \in L^1(\Omega \times I_{k+1})$ *solving* (6.53) *and satisfying the discrete energy inequality*

$$\begin{aligned}
&\mathcal{E}_{AGG}(\mathbf{u}, \phi) + \frac{1}{h}\int_{I_{k+1}} \mathcal{E}_B(\widetilde{\mathbb{B}}, \phi)(t)\,\mathrm{d}t + \int_\Omega \rho_k \frac{|\mathbf{u} - \mathbf{u}_k|^2}{2}\,\mathrm{d}x + \int_\Omega \frac{|\nabla\phi - \nabla\phi_k|^2}{2}\,\mathrm{d}x \\
&\quad + h\int_\Omega \frac{\nu(\phi_k)}{2}\left|\nabla\mathbf{u} + \nabla\mathbf{u}^\top\right|^2\mathrm{d}x + h\int_\Omega m(\phi_k)\,|\nabla q|^2\,\mathrm{d}x + \eta h\int_\Omega \left|\frac{\phi - \phi_k}{h}\right|^2\mathrm{d}x \\
&\quad + \frac{\kappa}{d}\int_{I_{k+1}}\int_\Omega \left|\nabla \operatorname{tr}\ln\widetilde{\mathbb{B}}\right|^2\mathrm{d}x\mathrm{d}t + \int_{I_{k+1}}\int_\Omega \frac{\mu_\eta(\phi_k)}{2}\operatorname{tr}(\widetilde{\mathbb{B}} + \widetilde{\mathbb{B}}^{-1} - 2\mathbb{I})\,\mathrm{d}x\,\mathrm{d}t \\
&\le \mathcal{E}_{AGG}(\mathbf{u}_k, \phi_k) + \frac{1}{h}\int_{I_k} \mathcal{E}_B(\widetilde{\mathbb{B}}_k, \phi_k)(t)\,\mathrm{d}t,
\end{aligned} \tag{6.54}$$

where

$$\mathcal{E}_{AGG}(\mathbf{u}, \phi) := \int_\Omega \frac{\rho(\phi)}{2}|\mathbf{u}|^2\,\mathrm{d}x + \int_\Omega \Big(\frac{1}{2}|\nabla\phi|^2 + W(\phi)\Big)\,\mathrm{d}x,$$

and

$$\mathcal{E}_B(\mathbb{B}, \phi) := \int_\Omega \frac{\mu_\eta(\phi)}{2}\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I})\,\mathrm{d}x.$$

Proof. The key idea of the proof is as follows. First, we derive a uniform energy estimate which holds true for any solution $(\mathbf{u}, \phi, q, \widetilde{\mathbb{B}})$ of (6.53). Then, we justify the existence of at least one solution of (6.53) with a fixed point argument. More precisely, we first construct a continuous mapping $(\mathbf{v}, \phi_k, \overline{\mathbb{B}}_k) \mapsto \mathcal{T}_{k+1}(\mathbf{v}, \phi_k, \overline{\mathbb{B}}_k)$, see (6.58), where $\mathcal{T}_{k+1}(\mathbf{v}, \phi_k, \overline{\mathbb{B}}_k)$ denotes the unique solution of the Oldroyd-B equation (6.20) on the time interval I_{k+1}, which depends continuously on a given function triplet $(\mathbf{v}, \phi_k, \overline{\mathbb{B}}_k)$, see Theorem 6.16 and (6.39). Then, we rewrite (6.53) as a fixed point problem for $(\mathbf{u}, \phi, q)$ with $\widetilde{\mathbb{B}}$ implicitly given by $\widetilde{\mathbb{B}} = \mathcal{T}_{k+1}(\mathbf{u}, \phi_k, \overline{\mathbb{B}}_k)$, which is then solved with the Leray–Schauder principle.

Let us start with the energy inequality. With a standard testing procedure as in [ADG13a], one may expect a similar estimate, except for the terms regarding $\widetilde{\mathbb{B}}$. Namely, testing (6.53b) with $\mathbf{u}$, (6.53c) with q and (6.53d) with $\frac{\phi-\phi_k}{h}$, and multiplying everything by h, one has

$$\begin{aligned}
&\mathcal{E}_{AGG}(\mathbf{u}, \phi) + \int_\Omega \rho_k \frac{|\mathbf{u} - \mathbf{u}_k|^2}{2}\,\mathrm{d}x + \int_\Omega \frac{|\nabla\phi - \nabla\phi_k|^2}{2}\,\mathrm{d}x \\
&\quad + h\int_\Omega \frac{\nu(\phi_k)}{2}\left|\nabla\mathbf{u} + \nabla\mathbf{u}^\top\right|^2\mathrm{d}x + h\int_\Omega m(\phi_k)\,|\nabla q|^2\,\mathrm{d}x + \eta h\int_\Omega \left|\frac{\phi - \phi_k}{h}\right|^2\mathrm{d}x \\
&\quad + \int_\Omega \frac{\mu_\eta(\phi) - \mu_\eta(\phi_k)}{2}\Big(\frac{1}{h}\int_{I_{k+1}} \operatorname{tr}(\widetilde{\mathbb{B}} - \ln\widetilde{\mathbb{B}} - \mathbb{I})(t)\,\mathrm{d}t\Big)\,\mathrm{d}x \\
&\le \mathcal{E}_{AGG}(\mathbf{u}_k, \phi_k) + \int_{I_{k+1}}\int_\Omega \frac{\mu_\eta(\phi_k)}{2}\mathscr{R}_\eta\mathbf{u}\cdot\nabla\operatorname{tr}(\widetilde{\mathbb{B}} - \ln\widetilde{\mathbb{B}} - \mathbb{I})\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{I_{k+1}}\int_\Omega \mu_\eta(\phi_k)(\widetilde{\mathbb{B}} - \mathbb{I}) : \nabla\mathscr{R}_\eta\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{6.55}$$

The key point here is to cancel out or control these extra terms associated with $\widetilde{\mathbb{B}}$ uniformly with respect to h, so that the energy will not blow up after summing over $k = 0, ..., N-1$ and sending

$N \to \infty$. Here, we make sufficient use of the entropy structure of $\widetilde{\mathbb{B}}$. Formally multiplying (6.53a) with $\frac{\mu_\eta(\phi_k)}{2}(\mathbb{I} - \widetilde{\mathbb{B}}^{-1})$ and integrating over $\Omega \times (t_k, t)$ for a.e. $t \in I_{k+1}$ yields

$$
\begin{aligned}
&\int_\Omega \frac{\mu_\eta(\phi_k)}{2} \operatorname{tr}(\widetilde{\mathbb{B}} - \ln \widetilde{\mathbb{B}} - \mathbb{I})(t)\,\mathrm{d}x + \int_{t_k}^t \int_\Omega \frac{\mu_\eta(\phi_k)}{2} \operatorname{tr}(\widetilde{\mathbb{B}} + \widetilde{\mathbb{B}}^{-1} - 2\mathbb{I})(s)\,\mathrm{d}x\mathrm{d}s \\
&\quad + \frac{\kappa}{d} \int_{t_k}^t \int_\Omega \left|\nabla \operatorname{tr} \ln \widetilde{\mathbb{B}}\right|^2 (s)\,\mathrm{d}x\mathrm{d}s \\
&\le \int_\Omega \frac{\mu_\eta(\phi_k)}{2} \operatorname{tr}(\overline{\mathbb{B}}_k - \ln \overline{\mathbb{B}}_k - \mathbb{I})\,\mathrm{d}x - \int_{t_k}^t \int_\Omega \frac{\mu_\eta(\phi_k)}{2} \mathscr{R}_\eta \mathbf{u} \cdot \nabla \operatorname{tr}(\widetilde{\mathbb{B}} - \ln \widetilde{\mathbb{B}} - \mathbb{I})\,\mathrm{d}x\mathrm{d}s \\
&\quad + \int_{t_k}^t \int_\Omega \mu_\eta(\phi_k)(\widetilde{\mathbb{B}} - \mathbb{I}) : \nabla \mathscr{R}_\eta \mathbf{u}\,\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

We note that this inequality can be recovered from the limit passage in (6.32) on the δ-regularized level in Section 6.3.5. Now, integrating over $t \in (t_k, t_{k+1}) = I_{k+1}$ and multiplying both sides by $\frac{1}{h}$ lead to

$$
\begin{aligned}
&\frac{1}{h} \int_{I_{k+1}} \int_\Omega \frac{\mu_\eta(\phi_k)}{2} \operatorname{tr}(\widetilde{\mathbb{B}} - \ln \widetilde{\mathbb{B}} - \mathbb{I})\,\mathrm{d}x\mathrm{d}t + \int_{I_{k+1}} \int_\Omega \frac{\mu_\eta(\phi_k)}{2} \operatorname{tr}(\widetilde{\mathbb{B}} + \widetilde{\mathbb{B}}^{-1} - 2\mathbb{I})\,\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{\kappa}{d} \int_{I_{k+1}} \int_\Omega \left|\nabla \operatorname{tr} \ln \widetilde{\mathbb{B}}\right|^2 \mathrm{d}x\mathrm{d}t \\
&\le \int_\Omega \frac{\mu_\eta(\phi_k)}{2} \operatorname{tr}(\overline{\mathbb{B}}_k - \ln \overline{\mathbb{B}}_k - \mathbb{I})\,\mathrm{d}x - \int_{I_{k+1}} \int_\Omega \frac{\mu_\eta(\phi_k)}{2} \mathscr{R}_\eta \mathbf{u} \cdot \nabla \operatorname{tr}(\widetilde{\mathbb{B}} - \ln \widetilde{\mathbb{B}} - \mathbb{I})\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{I_{k+1}} \int_\Omega \mu_\eta(\phi_k)(\widetilde{\mathbb{B}} - \mathbb{I}) : \nabla \mathscr{R}_\eta \mathbf{u}\,\mathrm{d}x\mathrm{d}t,
\end{aligned}
\tag{6.56}
$$

which is with the help of the fundamental theorem of calculus. Using Jensen's inequality, we estimate

$$
\int_\Omega \frac{\mu_\eta(\phi_k)}{2} \operatorname{tr}(\overline{\mathbb{B}}_k - \ln \overline{\mathbb{B}}_k - \mathbb{I})\,\mathrm{d}x \le \frac{1}{h} \int_{I_k} \int_\Omega \frac{\mu_\eta(\phi_k)}{2} \operatorname{tr}(\widetilde{\mathbb{B}}_k - \ln \widetilde{\mathbb{B}}_k - \mathbb{I})(t)\,\mathrm{d}x\mathrm{d}t. \tag{6.57}
$$

Adding (6.55) and (6.56) and noting (6.57) gives birth to the desired uniform estimate, where the mixed terms on the right-hand sides of (6.55)–(6.56) cancel out, i.e.,

$$
\begin{aligned}
&\mathcal{E}_{AGG}(\mathbf{u}, \phi) + \frac{1}{h} \int_{I_{k+1}} \int_\Omega \frac{\mu_\eta(\phi)}{2} \operatorname{tr}(\widetilde{\mathbb{B}} - \ln \widetilde{\mathbb{B}} - \mathbb{I})(t)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{I_{k+1}} \int_\Omega \frac{\mu_\eta(\phi_k)}{2} \operatorname{tr}(\widetilde{\mathbb{B}} + \widetilde{\mathbb{B}}^{-1} - 2\mathbb{I})\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_\Omega \rho_k \frac{|\mathbf{u} - \mathbf{u}_k|^2}{2}\,\mathrm{d}x + \int_\Omega \frac{|\nabla\phi - \nabla\phi_k|^2}{2}\,\mathrm{d}x + \frac{\kappa}{d} \int_{I_{k+1}} \int_\Omega \left|\nabla \operatorname{tr} \ln \widetilde{\mathbb{B}}\right|^2 \mathrm{d}x\mathrm{d}t \\
&\quad + h \int_\Omega \frac{\nu(\phi_k)}{2} \left|\nabla \mathbf{u} + \nabla \mathbf{u}^\top\right|^2 \mathrm{d}x + h \int_\Omega m(\phi_k) \left|\nabla q\right|^2 \mathrm{d}x + \eta h \int_\Omega \left|\frac{\phi - \phi_k}{h}\right|^2 \mathrm{d}x \\
&\le \mathcal{E}_{AGG}(\mathbf{u}_k, \phi_k) + \frac{1}{h} \int_{I_k} \int_\Omega \frac{\mu_\eta(\phi_k)}{2} \operatorname{tr}(\widetilde{\mathbb{B}}_k - \ln \widetilde{\mathbb{B}}_k - \mathbb{I})(t)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

In order to prove the existence of weak solutions, we proceed with the Leray–Schauder principle (see, e.g., [Soh01, Lemma 3.1.1, Chapter II]), which was carried out in [ADG13a] as well. Now we define

$$X = W^{1,2}_{0,\sigma}(\Omega) \times \mathcal{D}(\partial \widetilde{E}) \times W^{2,2}_{n}(\Omega), \quad Y = [W^{1,2}_{0,\sigma}(\Omega)]' \times L^2(\Omega) \times L^2(\Omega),$$
$$Z_{k+1} = \begin{Bmatrix} \mathbb{B} \in L^2(I_{k+1}; W^{1,2}(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})) : \mathbb{B} \text{ is positive definite a.e. in } \Omega, \\ \operatorname{tr}\ln \mathbb{B} \in L^1(\Omega \times I_{k+1}) \end{Bmatrix}.$$

By means of Theorem 6.16, we construct a mapping $\mathcal{T}_{k+1} : X \to Z_{k+1}$ such that

$$\widetilde{\mathbb{B}} = \mathcal{T}_{k+1}(\mathbf{u}, \phi_k, \overline{\mathbb{B}}_k) \text{ solves (6.53a) with } \overline{\mathbb{B}}_k = \frac{1}{h}\int_{I_k} \widetilde{\mathbb{B}}_k(t)\,\mathrm{d}t, \tag{6.58}$$

which are both positive definite a.e. in Ω, and such that the inequality (6.56) is fulfilled.

Next, we are going to deal with (6.53b)–(6.53d) with the help of (6.58). For $\mathbf{w} = (\mathbf{u}, \phi, q) \in X$, we define $\mathcal{L}_k : X \to Y$ as

$$\mathcal{L}_k(\mathbf{w}) = \begin{pmatrix} L_k(\mathbf{u}) \\ -\operatorname{div}(m(\phi_k)\nabla q) + \int_\Omega q\,\mathrm{d}x \\ \partial \widetilde{E}(\phi) + \phi \end{pmatrix},$$

where

$$\langle L_k(\mathbf{u}), \mathbf{v}\rangle_{[W^{1,2}_{0,\sigma}]', W^{1,2}_{0,\sigma}} = \int_\Omega \nu(\phi_k)(\nabla \mathbf{u} + \nabla \mathbf{u}^\top) : \nabla \mathbf{v}\,\mathrm{d}x \quad \text{for } \mathbf{v} \in W^{1,2}_{0,\sigma}(\Omega),$$

and the second and third line are understood in the pointwise sense. Moreover, for $\mathbf{w} = (\mathbf{u}, \phi, q) \in X$ we introduce $\mathcal{F}_k : X \to Y$ as

$$\mathcal{F}_k(\mathbf{w}) = \begin{pmatrix} F_k(\mathbf{w}) \\ -\frac{\phi - \phi_k}{h} - \mathbf{u}\cdot\nabla\phi_k + \int_\Omega q\,\mathrm{d}x \\ -\eta\frac{\phi - \phi_k}{h} + \phi + q + \omega\frac{\phi + \phi_k}{2} - G(\mathbf{w}, \widetilde{\mathbb{B}}) \end{pmatrix},$$

where

$$\begin{aligned} G(\mathbf{w}, \widetilde{\mathbb{B}}) &= \mathscr{R}_\eta\Big[\frac{1}{2}\frac{\mu_\eta(\phi) - \mu_\eta(\phi_k)}{\mathscr{R}_\eta(\phi - \phi_k)}\frac{1}{h}\int_{I_{k+1}} \operatorname{tr}(\widetilde{\mathbb{B}} - \ln\widetilde{\mathbb{B}} - \mathbb{I})(t)\,\mathrm{d}t\Big], \\ F_k(\mathbf{w}) &= -\frac{\rho\mathbf{u} - \rho_k\mathbf{u}_k}{h} - \operatorname{div}(\rho_k \mathbf{u}\otimes\mathbf{u}) + q\nabla\phi_k \\ &\quad - \Big(\operatorname{div}\mathbf{J} - \frac{\rho - \rho_k}{h} - \mathbf{u}\cdot\nabla\rho_k\Big)\frac{\mathbf{u}}{2} - (\mathbf{J}\cdot\nabla)\mathbf{u} \\ &\quad - \operatorname{div}\big(\mathscr{R}_\eta\big[\mu_\eta(\phi_k)(\overline{\mathbb{B}}_{k+1} - \mathbb{I})\big]\big) + \frac{1}{h}\int_{I_{k+1}} \mathscr{R}_\eta\Big[\frac{\mu_\eta(\phi_k)}{2}\nabla\operatorname{tr}(\widetilde{\mathbb{B}} - \ln\widetilde{\mathbb{B}} - \mathbb{I})\Big]\,\mathrm{d}t, \end{aligned}$$

and $\overline{\mathbb{B}}_{k+1}$ and $\widetilde{\mathbb{B}}$ are constructed in (6.58). Consequently, we see that $(\mathbf{u}, \phi, q, \widetilde{\mathbb{B}})$ is a weak solution of the system (6.53) if and only if $\mathbf{w} = (\mathbf{u}, \phi, q) \in X$ satisfies

$$\mathcal{L}_k(\mathbf{w}) - \mathcal{F}_k(\mathbf{w}) = 0. \tag{6.59}$$

Similarly to [ADG13a], one can check that $\mathcal{L}_k : X \to Y$ is invertible with the inverse $\mathcal{L}_k^{-1} : Y \to X$. Note that X is not a Banach space since $\mathcal{D}(\partial\widetilde{E})$ consists of inequality constraints.

To get a continuous and even compact operator, we introduce for $0 < s < \frac{1}{4}$ the following Banach spaces

$$\widetilde{X} \coloneqq W^{1,2}_{0,\sigma}(\Omega) \times W^{2-s,2}(\Omega) \times W^{2,2}_n(\Omega), \quad \widetilde{Y} \coloneqq L^{\frac{3}{2}}(\Omega;\mathbb{R}^2) \times W^{1,\frac{3}{2}}(\Omega) \times W^{1,2}(\Omega).$$

Then we obtain the continuity of $\mathcal{L}_k^{-1} : Y \to \widetilde{X}$ from standard theory and with the above note concerning the continuity of the third line in $\mathcal{L}_k^{-1}$. In view of the Rellich–Kondrachov theorem in two and three dimensions, one knows that $\widetilde{Y} \subset\subset Y$ compactly, which implies that the restriction $\mathcal{L}_k^{-1} : \widetilde{Y} \to \widetilde{X}$ is a compact operator. Moreover, we infer that $\mathcal{T}_{k+1} : \widetilde{X} \to Z$ is well-defined due to Theorem 6.16 and the restriction $\mathcal{F}_k : \widetilde{X} \to \widetilde{Y}$ is continuous and maps bounded sets into bounded sets, which can be verified by the same argument as in [ADG13a]. Here we leave out the details except for the terms regarding $\widetilde{\mathbb{B}}$. Thanks to the regularization operator $\mathscr{R}_\eta$, we have

$$\begin{aligned}
&\left\|\operatorname{div}\left(\mathscr{R}_\eta\left[\mu_\eta(\phi_k)(\overline{\mathbb{B}}_{k+1} - \mathbb{I})\right]\right)\right\|_{L^{\frac{3}{2}}} \\
&\qquad \le C\frac{1}{h}\int_{I_{k+1}} (\|\widetilde{\mathbb{B}}\|^2_{L^2} + \|\nabla\widetilde{\mathbb{B}}\|^2_{L^2})\,\mathrm{d}t + \|\phi_k\|^2_{L^2} + \|\nabla\phi_k\|^2_{L^2} + C \le C_k(1+\frac{1}{h}), \\
&\frac{1}{h}\int_{I_{k+1}} \left\|\mathscr{R}_\eta\left[\frac{\mu_\eta(\phi_k)}{2}\nabla\operatorname{tr}(\widetilde{\mathbb{B}} - \ln\widetilde{\mathbb{B}} - \mathbb{I})\right]\right\|_{L^{\frac{3}{2}}} \mathrm{d}t \\
&\qquad \le C\frac{1}{h}\int_{I_{k+1}} \left\|\nabla\operatorname{tr}(\widetilde{\mathbb{B}} - \ln\widetilde{\mathbb{B}})\right\|^2_{L^2} \mathrm{d}t + C \le C_k(1+\frac{1}{h}), \\
&\left\|G(\mathbf{w},\widetilde{\mathbb{B}})\right\|_{W^{1,2}} \le C\frac{1}{h}\int_{I_{k+1}} \left\|\operatorname{tr}(\widetilde{\mathbb{B}} - \ln\widetilde{\mathbb{B}} - \mathbb{I})\right\|_{W^{1,2}} \mathrm{d}t \le C_k(1+\frac{1}{h}),
\end{aligned}$$

where $C_k = C_k(\|\mathbf{u}\|_{W^{1,2}}) > 0$. Hence, $\mathcal{F}_k$ maps bounded sets into bounded sets.

Next we recall the continuity of $\mathcal{F}_k$, that is, $\mathcal{F}_k(\mathbf{w}^\ell) \to \mathcal{F}_k(\mathbf{w})$ in $\widetilde{Y}$ if $\mathbf{w}^\ell \to \mathbf{w}$ in $\widetilde{X}$ as $\ell \to \infty$. Note that most of the terms of $\mathcal{F}_k$ are similar to [ADG13a], and the main differences are the terms corresponding to $\widetilde{\mathbb{B}}$. In order to prove the continuity, we take an arbitrary sequence $\{\mathbf{w}^\ell\}_{\ell\in\mathbb{N}} \subset \widetilde{X}$ such that $\mathbf{w}^\ell \to \mathbf{w}$ in $\widetilde{X}$, as $\ell \to \infty$, which implies $\mathbf{u}^\ell \to \mathbf{u}$ in $W^{1,2}_{0,\sigma}$, and then we investigate the continuity of $\mathcal{T}_{k+1}$ in terms of $\mathbf{u}^\ell \in W^{1,2}_{0,\sigma}$ for fixed ϕ_k and $\mathbb{B}_k$. It follows from Theorem 6.16 and (6.39) that $\mathbb{B}^\ell \to \mathbb{B}$ in $L^2(\Omega \times I_{k+1})$ and $\nabla\mathbb{B}^\ell \to \nabla\mathbb{B}$ in $L^2(\Omega \times I_{k+1})$. By applying a similar argument as in Section 6.3.7, one is able to show the positive definiteness of $\mathbb{B}$ and $\operatorname{tr}\ln\mathbb{B}^\ell \to \operatorname{tr}\ln\mathbb{B}$ a.e. in $\Omega \times I_{k+1}$ (up to a non-relabeled subsequence). Then, with the uniform boundedness of $\operatorname{tr}\ln\mathbb{B}^\ell$ in $L^2(\Omega\times I_{k+1})$, one concludes that $\operatorname{tr}\ln\mathbb{B}^\ell \to \operatorname{tr}\ln\mathbb{B}$ strongly in $L^1(\Omega\times I_{k+1})$ in view of Vitali's convergence theorem. Hence, $\operatorname{tr}(\mathbb{B}^\ell - \ln\mathbb{B}^\ell - \mathbb{I}) \to \operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I})$ in $L^1(\Omega \times I_{k+1})$, which finally implies the continuity of $\mathcal{F}_k$.

The final step is to apply the Leray–Schauder principle on $\widetilde{Y}$, for which we denote $\mathcal{K}_k \coloneqq \mathcal{F}_k \circ \mathcal{L}_k^{-1} : \widetilde{Y} \to \widetilde{Y}$, rewrite (6.59) as

$$\mathbf{f} - \mathcal{K}_k(\mathbf{f}) = 0 \quad \text{for } \mathbf{f} = \mathcal{L}_k(\mathbf{w}),$$

and find a fixed point of $\mathcal{K}_k$. Note that $\mathcal{K}_k$ is a compact operator because $\mathcal{L}_k^{-1}$ is compact and $\mathcal{F}_k$ is continuous. Now we are in the position to show that

$$\exists R > 0 \text{ such that } \|\mathbf{f}\|_{\widetilde{Y}} \le R, \text{ for } \mathbf{f} \in \widetilde{Y} \text{ fulfilling } \mathbf{f} = \lambda\mathcal{K}_k(\mathbf{f}) \text{ for some } 0 \le \lambda \le 1. \tag{6.60}$$

according to the Leray–Schauder principle (see, e.g., [Soh01, Lemma 3.1.1, Chapter II]). Here λ is a constant used for the proof and has nothing to do with the relaxation time of the model (6.1). By the definition $\mathbf{w} = \mathcal{L}_k^{-1}(\mathbf{f})$, one may see that $\mathbf{f} = \lambda\mathcal{K}_k(\mathbf{f})$ in (6.60) is equivalent to

$$\mathcal{L}_k(\mathbf{w}) - \lambda\mathcal{F}_k(\mathbf{w}) = 0 \quad \text{for some } 0 \le \lambda \le 1. \tag{6.61}$$

The rest of this subsection is devoted to the estimate of $\mathbf{w}$ in $\widetilde{X}$ satisfying (6.61), which thereby gives the estimate of $\mathbf{f}$ in $\widetilde{Y}$ by the fact that $\mathcal{F}_k : \widetilde{X} \to \widetilde{Y}$ is bounded. The subsequent argument is rather lengthy, but is very similar to [ADG13a]. Thus, we point out the main differences associated with $\widetilde{\mathbb{B}}$ in the following. For the reader's convenience, we give the equivalent weak formulation corresponding to (6.61) as follows.

$$\begin{aligned}\lambda\left(\frac{\rho\mathbf{u}-\rho_k\mathbf{u}_k}{h},\mathbf{v}\right) &+ \lambda\left(\operatorname{div}(\rho_k\mathbf{u}\otimes\mathbf{u}),\mathbf{v}\right)+\lambda\left(\operatorname{div}(\mathbf{u}\otimes\mathbf{J}),\mathbf{v}\right)\\ &+\left(\nu(\phi_k)(\nabla\mathbf{u}+\nabla\mathbf{u}^\top),\nabla\mathbf{v}\right)-\lambda\left(\operatorname{div}\mathscr{R}_\eta\left[\mu_\eta(\phi_k)(\mathbb{B}_{k+1}-\mathbb{I})\right],\mathbf{v}\right)\\ &=\lambda\left(q\nabla\phi_k,\mathbf{v}\right)+\lambda\frac{1}{h}\int_{I_{k+1}}\left(\mathscr{R}_\eta\left[\frac{\mu_\eta(\phi_k)}{2}\nabla\operatorname{tr}(\widetilde{\mathbb{B}}-\ln\widetilde{\mathbb{B}}-\mathbb{I})\right],\mathbf{v}\right)\mathrm{d}t,\end{aligned}\tag{6.62a}$$

for all $\mathbf{v}\in W^{1,2}_{0,\sigma}(\Omega)$, and for a.e. $x\in\Omega$,

$$\lambda\frac{\phi-\phi_k}{h}+\lambda\mathbf{u}\cdot\nabla\phi_k=\operatorname{div}(m(\phi_k)\nabla q),\tag{6.62b}$$

$$\lambda q+\lambda\omega\frac{\phi+\phi_k}{2}+\lambda\phi=\partial\widetilde{E}(\phi)+\phi+\lambda\eta\frac{\phi-\phi_k}{h}+\lambda G(\mathbf{w},\widetilde{\mathbb{B}}).\tag{6.62c}$$

By the analogous test procedure as for the energy estimate (also in [ADG13a]), that is, testing (6.62a) with $\mathbf{u}$, (6.62b) with q and (6.62c) with $\frac{\phi-\phi_k}{h}$, we derive similar estimates. The main difference here compared to [ADG13a] is that we have extra terms associated with $\widetilde{\mathbb{B}}$, which in light of (6.56) can be canceled, as

$$\begin{aligned}&\lambda\int_{I_{k+1}}\int_\Omega\mu_\eta(\phi_k)(\widetilde{\mathbb{B}}-\mathbb{I}):\nabla\mathscr{R}_\eta\mathbf{u}\,\mathrm{d}x\mathrm{d}t-\lambda\int_{I_{k+1}}\int_\Omega\frac{\mu_\eta(\phi_k)}{2}\mathscr{R}_\eta\mathbf{u}\cdot\nabla\operatorname{tr}(\widetilde{\mathbb{B}}-\ln\widetilde{\mathbb{B}}-\mathbb{I})\,\mathrm{d}x\mathrm{d}t\\ &+\lambda\int_{I_{k+1}}\int_\Omega\frac{\mathscr{R}_\eta(\phi-\phi_k)}{2h}\frac{\mu_\eta(\phi)-\mu_\eta(\phi_k)}{\mathscr{R}_\eta(\phi-\phi_k)}\operatorname{tr}(\widetilde{\mathbb{B}}-\ln\widetilde{\mathbb{B}}-\mathbb{I})(t)\,\mathrm{d}x\mathrm{d}t\\ \geq{}&\lambda\frac{1}{h}\int_{I_{k+1}}\int_\Omega\frac{\mu_\eta(\phi)}{2}\operatorname{tr}(\widetilde{\mathbb{B}}-\ln\widetilde{\mathbb{B}}-\mathbb{I})(t)\,\mathrm{d}x\mathrm{d}t-\lambda\int_\Omega\frac{\mu_\eta(\phi_k)}{2}\operatorname{tr}(\mathbb{B}_k-\ln\mathbb{B}_k-\mathbb{I})\,\mathrm{d}x\\ &+\lambda\int_{I_{k+1}}\int_\Omega\frac{\mu_\eta(\phi_k)}{2}\operatorname{tr}(\widetilde{\mathbb{B}}+\widetilde{\mathbb{B}}^{-1}-2\mathbb{I})\,\mathrm{d}x\mathrm{d}t+\lambda\frac{\kappa}{2}\int_{I_{k+1}}\int_\Omega\left|\nabla\operatorname{tr}\ln\widetilde{\mathbb{B}}\right|^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Then we reach the estimate

$$\|\mathbf{w}\|_{\widetilde{X}}+\left\|\partial\widetilde{E}(\phi)\right\|_{L^2}\leq C_k,$$

for fixed h, and hence

$$\|\mathbf{f}\|_{\widetilde{Y}}=\|\lambda\mathcal{F}_k(\mathbf{w})\|_{\widetilde{Y}}\leq C_k(\|\mathbf{w}\|_{\widetilde{X}}+1)\leq C_k,$$

which finishes the proof by the Leray–Schauder principle. □

6.4.5. Construction of approximate solutions. Let $T>0$ and $N\in\mathbb{N}$ be given and, for $k\in\{1,...,N\}$, let $(\mathbf{u}_k,\phi_k,q_k,\mathbb{B}_k)$ be chosen successively as a solution to (6.53) with $h=\frac{T}{N}$ and $(\mathbf{u}_0,\phi_0^N,\mathbb{B}_0)$ as the initial data. Here, the regularized initial value $\phi_0^N\in W^{2,2}(\Omega)$ is constructed as in [ADG13a] and satisfies $\phi_0^N\to\phi_0$ in $W^{1,2}(\Omega)$, as $N\to\infty$.

Now we define $f^N(t)$ on $[-h,T)$ through

$$f^N(t)=f_k\quad\text{for } t\in[t_{k-1},t_k),$$

where $k \in \{0, ..., N\}$ and $f \in \{\mathbf{u}, \phi, q, \mathbb{B}\}$. In particular, it holds that

$$f^N((k-1)h) = f_k, \quad f^N(kh) = f_{k+1}, \quad f^N(t) = f_{k+1} \text{ with } t \in [t_k, t_{k+1}), k \in \{0, ..., N-1\}.$$

Moreover, for $t \in [t_{k-1}, t_k)$, $k \in \{1, ..., N\}$, we define $\widetilde{\mathbb{B}}^N(t) := \widetilde{\mathbb{B}}_k(t)$, $\overline{\mathbb{B}}^N(t) := \frac{1}{h}\int_{I_k} \widetilde{\mathbb{B}}_k(t)\,\mathrm{d}t$ and

$$\begin{aligned}
\rho^N := \rho(\phi^N), \quad f_h := f(t-h),\\
(\Delta_h^+ f)(t) := f(t+h) - f(t), \quad \partial_{t,h}^+ f(t) := \frac{1}{h}(\Delta_h^+ f)(t),\\
(\Delta_h^- f)(t) := f(t) - f(t-h), \quad \partial_{t,h}^- f(t) := \frac{1}{h}(\Delta_h^- f)(t).
\end{aligned}$$

By definition, it follows that

$$\int_0^\tau f^N(t)\,\mathrm{d}t = h\sum_{k=0}^{\tau/h} f_{k+1}, \quad \int_0^\tau \widetilde{\mathbb{B}}^N(t)\,\mathrm{d}t = \sum_{k=0}^{\tau/h}\int_{I_{k+1}} \widetilde{\mathbb{B}}_{k+1}(t)\,\mathrm{d}t, \tag{6.63}$$

for $\tau \in h\cdot\{0, ..., N-1\} = \{0, h, 2h, ..., (N-1)h\}$.

Then for arbitrary $\mathbf{w} \in C^\infty([0,T]; C_0^\infty(\Omega;\mathbb{R}^d))$ with $\operatorname{div}\mathbf{w} = 0$ we shall take $\widetilde{\mathbf{w}} := \int_{kh}^{(k+1)h}\mathbf{w}\,\mathrm{d}t$ satisfying $\operatorname{div}\widetilde{\mathbf{w}} = 0$ as the test function in (6.53b) and sum over $k \in \{0, ..., N-1\}$ to get

$$\begin{aligned}
&-\int_0^\tau\int_\Omega \Big(\rho^N\mathbf{u}^N\cdot\partial_{t,h}^+\mathbf{w} - (\rho_h^N\mathbf{u}^N\otimes\mathbf{u}^N) : \nabla\mathbf{w}\Big)\,\mathrm{d}x\mathrm{d}t\\
&\quad + \int_0^\tau\int_\Omega \Big(\nu(\phi_h^N)(\nabla\mathbf{u}^N + (\nabla\mathbf{u}^N)^\top) : \nabla\mathbf{w} - (\mathbf{u}^N\otimes\mathbf{J}^N) : \nabla\mathbf{w}\Big)\,\mathrm{d}x\mathrm{d}t\\
&\quad + \int_0^\tau\int_\Omega \mathscr{R}_\eta\big[\mu_\eta(\phi_h^N)(\overline{\mathbb{B}}^N - \mathbb{I})\big] : \nabla\mathbf{w}\,\mathrm{d}x\mathrm{d}t\\
&= \int_0^\tau\int_\Omega q^N\nabla\phi_h^N\cdot\mathbf{w}\,\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_\Omega \mathscr{R}_\eta\Big[\frac{\mu_\eta(\phi_h^N)}{2}\nabla G^N\Big]\cdot\mathbf{w}\,\mathrm{d}x\mathrm{d}t\\
&\quad - \int_\Omega \rho(\phi^N(\cdot,\tau-h))\mathbf{u}^N(\cdot,\tau-h)\cdot\overline{\mathbf{w}}_\tau\,\mathrm{d}x + \int_\Omega \rho(\phi_0^N)\mathbf{u}_0\cdot\overline{\mathbf{w}}_0\,\mathrm{d}x,
\end{aligned} \tag{6.64a}$$

for all $\tau \in h\cdot\{0, ..., N-1\}$, where

$$\overline{\mathbf{w}}_s := \frac{1}{h}\int_s^{s+h}\mathbf{w}(t)\,\mathrm{d}t \quad \text{for } s \in [0,\tau],$$

and

$$G^N(t) := \frac{1}{h}\int_{I_{k+1}} \operatorname{tr}(\widetilde{\mathbb{B}}_{k+1} - \ln\widetilde{\mathbb{B}}_{k+1} - \mathbb{I})(s)\mathrm{d}s \quad \text{for } t \in [t_k, t_{k+1}).$$

Here the identity

$$\begin{aligned}
&\int_0^\tau\int_\Omega \partial_{t,h}^-(\rho^N\mathbf{u}^N)\cdot\mathbf{w}\,\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_\Omega \rho^N\mathbf{u}^N\cdot\partial_{t,h}^+\mathbf{w}\,\mathrm{d}x\mathrm{d}t\\
&\qquad = \int_\Omega \rho(\phi^N(\cdot,\tau-h))\mathbf{u}^N(\cdot,\tau-h)\cdot\overline{\mathbf{w}}_\tau\,\mathrm{d}x - \int_\Omega \rho(\phi_0^N)\mathbf{u}_0\cdot\overline{\mathbf{w}}_0\,\mathrm{d}x
\end{aligned}$$

is employed for all $\tau \in h \cdot \{0, ..., N-1\}$ and $\mathbf{w} \in C^\infty([0,T]; C_0^\infty(\Omega;\mathbb{R}^d))$. Analogously, it follows from (6.53c) and (6.53d) that

$$-\int_0^\tau \int_\Omega \Big(\phi^N \partial_{t,h}^+ \xi + \phi_h^N \mathbf{u}^N \cdot \nabla \xi\Big) \,\mathrm{d}x\mathrm{d}t \\ = -\int_0^\tau \int_\Omega m(\phi_h^N)\nabla q^N \cdot \nabla \xi \,\mathrm{d}x\mathrm{d}t - \int_\Omega \phi^N(\cdot, \tau - h)\overline{\xi}_\tau \,\mathrm{d}x + \int_\Omega \phi_0^N \overline{\xi}_0 \,\mathrm{d}x \tag{6.64b}$$

for all $\tau \in h \cdot \{0, ..., N-1\}$ and $\xi \in C^\infty([0,T]; C^1(\overline{\Omega}))$, where

$$\overline{\xi}_s := \frac{1}{h}\int_s^{s+h} \xi(t)\,\mathrm{d}t \qquad \text{for } s \in [0,\tau],$$

In addition,

$$\partial \widetilde{E}(\phi^N) = \widetilde{W}_0'(\phi^N) - \Delta\phi^N = q^N + \frac{\omega}{2}(\phi^N + \phi_h^N) - \eta\partial_{t,h}^- \phi^N \\ - \frac{1}{2}\mathscr{R}_\eta\Big[\frac{\mu_\eta(\phi^N) - \mu_\eta(\phi_h^N)}{\mathscr{R}_\eta(\phi^N - \phi_h^N)} G^N\Big] \tag{6.64c}$$

for a.e. $x \in \Omega$, and all $\tau \in h\cdot\{0, ..., N-1\}$. Moreover, testing (6.53a) with any $\mathbb{C} \in C^\infty(\overline{Q_T}; \mathbb{R}^{d\times d}_{\text{sym}})$ and integrating over $\Omega \times I_{k+1}$ with $k \in \{0, ..., N-1\}$, applying integration by parts (for the time derivative) and summing over all $k \in \{0, ..., m\}$ with $m \in \{0, ..., N-1\}$ and $\tau = hm$, one knows

$$\int_0^\tau \int_\Omega \Big(\widetilde{\mathbb{B}}^N : \partial_t \mathbb{C} + (\mathscr{R}_\eta \mathbf{u}^N \otimes \widetilde{\mathbb{B}}^N) : \nabla\mathbb{C}\Big) \,\mathrm{d}x\mathrm{d}t \\ + \int_0^\tau \int_\Omega 2\mathbb{C}\widetilde{\mathbb{B}}^N : \nabla\mathscr{R}_\eta \mathbf{u}^N - \kappa\nabla\widetilde{\mathbb{B}}^N : \nabla\frac{\mathbb{C}}{\mu_\eta(\phi_h^N)}\Big) \,\mathrm{d}x\mathrm{d}t \\ = \int_0^\tau \int_\Omega (\widetilde{\mathbb{B}}^N : \mathbb{C} - \operatorname{tr}\mathbb{C}) \,\mathrm{d}x\mathrm{d}t + \int_\Omega \widetilde{\mathbb{B}}^N(\cdot,\tau) : \mathbb{C}(\cdot,\tau)\,\mathrm{d}x - \int_\Omega \mathbb{B}_0 : \mathbb{C}(\cdot, 0). \tag{6.64d}$$

Now let $\mathcal{E}^N(t)$ be the piecewise linear interpolant of

$$\mathcal{E}_{tot}(\mathbf{u}_k, \phi_k, \widetilde{\mathbb{B}}_k) := \mathcal{E}_{AGG}(\mathbf{u}_k, \phi_k) + \frac{1}{h}\int_{I_k} \mathcal{E}_B(\widetilde{\mathbb{B}}_k, \phi_k)(s)\,\mathrm{d}s$$

at $t_k = kh$ given by

$$\mathcal{E}^N(t) = \frac{(k+1)h - t}{h}\mathcal{E}_{tot}(\mathbf{u}_k, \phi_k, \widetilde{\mathbb{B}}_k) + \frac{t - kh}{h}\mathcal{E}_{tot}(\mathbf{u}_{k+1}, \phi_{k+1}, \widetilde{\mathbb{B}}_{k+1})$$

for $t \in [kh, (k+1)h)$, satisfying $\mathcal{E}^N(0) = \mathcal{E}_{tot}(\mathbf{u}_0, \phi_0^N, \mathbb{B}_0) := \mathcal{E}_{AGG}(\mathbf{u}_0, \phi_0^N) + \mathcal{E}_B(\mathbb{B}_0, \phi_0^N)$. By the discrete energy inequality (6.54), we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^N(t) = \frac{\mathcal{E}_{tot}(\mathbf{u}_k, \phi_k, \widetilde{\mathbb{B}}_k) - \mathcal{E}_{tot}(\mathbf{u}_{k+1}, \phi_{k+1}, \widetilde{\mathbb{B}}_{k+1})}{h} \geq \mathcal{D}^N(t), \tag{6.65}$$

where the piecewise constant dissipation $\mathcal{D}^N(t)$ is given by

$$\mathcal{D}^N(t) := \int_\Omega \frac{\nu(\phi_k)}{2}\left|\nabla\mathbf{u}_{k+1} + \nabla\mathbf{u}_{k+1}^\top\right|^2 \mathrm{d}x + \int_\Omega m(\phi_k)\left|\nabla q_{k+1}\right|^2 \mathrm{d}x \\ + \frac{1}{h}\int_\Omega \rho_k \frac{|\mathbf{u}_{k+1} - \mathbf{u}_k|^2}{2}\,\mathrm{d}x + \frac{1}{h}\int_\Omega \frac{|\nabla\phi_{k+1} - \nabla\phi_k|^2}{2}\,\mathrm{d}x + \eta\int_\Omega \left|\frac{\phi_{k+1} - \phi_k}{h}\right|^2 \mathrm{d}x \\ + \frac{\kappa}{d}\frac{1}{h}\int_{I_{k+1}}\int_\Omega \left|\nabla \operatorname{tr}\ln\widetilde{\mathbb{B}}_{k+1}\right|^2 \mathrm{d}x\mathrm{d}s + \frac{1}{h}\int_{I_{k+1}}\int_\Omega \frac{\mu_\eta(\phi_k)}{2}\operatorname{tr}(\widetilde{\mathbb{B}}_{k+1} + \widetilde{\mathbb{B}}_{k+1}^{-1} - 2\mathbb{I})\,\mathrm{d}x\mathrm{d}s$$

for $t \in I_{k+1}$, $k \in \{0, ..., N-1\}$.

6.4.6. Compactness of time-averaged terms. Before proving the existence of weak solution to the regularized system, we give three technical lemmata, which are of much significance for the limit passage concerning the time-averaged terms, in particular, the compactness of these terms. First, we give a uniform boundedness and compactness property.

Lemma 6.35 (Compactness in L^p). *Let $h = T/N$ with $N \in \mathbb{N}$ and $f \in L^p(0,T;X)$ be a bounded function for $p \in (1,\infty)$, where X is a Banach space. Define $\bar{f}^N(t) = \frac{1}{h}\int_{I_{k+1}} f(s)\,\mathrm{d}s$ for $t \in I_{k+1}$ as a piecewise-in-time constant function in $(0,T)$. Then $\{\bar{f}^N\}_{N\in\mathbb{N}} \subset L^p(0,T;X)$. Moreover, it holds $\bar{f}^N \to f$ in $L^p(0,T;X)$, as $N \to \infty$.*

Proof. By virtue of (6.63) and Jensen's inequality for the time integral (as $p \in (1,\infty)$), one knows that

$$\begin{aligned}
\|\bar{f}^N\|^p_{L^p(0,T;X)} = \int_0^T \|\bar{f}^N(s)\|^p_X \,\mathrm{d}s &= h\sum_{k=0}^{N-1}\left\|\frac{1}{h}\int_{I_{k+1}} f(s)\,\mathrm{d}s\right\|^p_X \\
&\le h\sum_{k=0}^{N-1}\Big(\frac{1}{h}\int_{I_{k+1}}\|f(s)\|_X\,\mathrm{d}s\Big)^p \\
&\le h\sum_{k=0}^{N-1}\frac{1}{h}\int_{I_{k+1}}\|f(s)\|^p_X\,\mathrm{d}s = \int_0^T \|f(s)\|^p_X\,\mathrm{d}s \le C,
\end{aligned}$$

where $C > 0$ is independent of $h > 0$ (and $N \in \mathbb{N}$, respectively). The second part of the lemma follows directly from the uniform boundedness and the Banach–Steinhaus theorem, as the convergence result holds true for any $f \in C^\infty([0,T];X) \subset L^p(0,T;X)$ densely, i.e., for any $f \in C^\infty([0,T];X)$, it holds

$$\begin{aligned}
\|\bar{f}^N - f\|^p_{L^p(0,T;X)} = \int_0^T \|\bar{f}^N(s) - f(s)\|^p_X\,\mathrm{d}s &= \sum_{k=0}^{N-1}\int_{I_{k+1}}\left\|\frac{1}{h}\int_{I_{k+1}} f(s)\,\mathrm{d}s - f(t)\right\|^p_X \mathrm{d}t \\
&= \sum_{k=0}^{N-1}\int_{I_{k+1}}\left\|\frac{1}{h}\int_{I_{k+1}}\big(f(s)-f(t)\big)\,\mathrm{d}s\right\|^p_X \mathrm{d}t \\
&\le Ch\sup_{t\in(0,T)}\|\partial_t f(t)\|^p_X \to 0,
\end{aligned}$$

as $h \to 0$. □

Remark 6.36. The first part of the lemma also holds true for f substituted by a uniformly bounded sequence $\{f^N\}_{N\in\mathbb{N}}$ in $L^p(0,T;X)$, that is, the sequence $\{\bar{f}^N\}_{N\in\mathbb{N}}$ with definition $\bar{f}^N(t) = \frac{1}{h}\int_{I_{k+1}} f^N(s)\,\mathrm{d}s$ for $t \in I_{k+1}$ is uniformly bounded in $L^p(0,T;X)$.

Now concerning the weakly convergent sequences, we introduce the following lemma for the time-averaged functions.

Lemma 6.37 (Weak compactness). *Let $h = T/N$ with $N \in \mathbb{N}$ and $\{f^N\}_{N\in\mathbb{N}} \subset L^2(Q_T)$ be a sequence satisfying $f^N \to f$ weakly in $L^2(Q_T)$, as $N \to \infty$ (resp. $h \to 0$), with $f \in L^2(Q_T)$. Defining a piecewise-in-time constant function $\bar{f}^N(t) = \frac{1}{h}\int_{I_{k+1}} f^N(s)\,\mathrm{d}s$ for $t \in I_{k+1}$, we have for any function $\varphi \in L^2(Q_T)$ independent of h,*

$$\int_{Q_T} \bar{f}^N \varphi \,\mathrm{d}x\mathrm{d}t \to \int_{Q_T} f\varphi\,\mathrm{d}x\mathrm{d}t, \text{ as } N \to \infty.$$

Proof. Define $\bar{\varphi}^N(t) = \frac{1}{h}\int_{I_{k+1}} \varphi(s)\,\mathrm{d}s$ for $t \in I_{k+1}$. In light of Lemma 6.35, $\bar{\varphi}^N$ is uniformly bounded in $L^2(Q_T)$ and it holds

$$\bar{\varphi}^N \to \varphi, \text{ strongly in } L^2(Q_T).$$

With the help of (6.63) and Fubini's theorem, we then have

$$\begin{aligned}
\int_0^T \int_\Omega \bar{f}^N \varphi \,\mathrm{d}x\mathrm{d}t &= \sum_{k=0}^{N-1} \int_{I_{k+1}} \int_\Omega \Big(\Big(\frac{1}{h}\int_{I_{k+1}} f^N(s)\,\mathrm{d}s\Big)\widetilde{\varphi}_{k+1}(t)\Big)\,\mathrm{d}x\mathrm{d}t \\
&= \sum_{k=0}^{N-1} \int_\Omega \Big(\frac{1}{h}\int_{I_{k+1}} f^N(s)\,\mathrm{d}s\Big)\Big(\int_{I_{k+1}} \widetilde{\varphi}_{k+1}(t)\,\mathrm{d}t\Big)\mathrm{d}x \\
&= \sum_{k=0}^{N-1} \int_\Omega \Big(\int_{I_{k+1}} f^N(s)\,\mathrm{d}s\Big)\Big(\frac{1}{h}\int_{I_{k+1}} \widetilde{\varphi}_{k+1}(t)\,\mathrm{d}t\Big)\,\mathrm{d}x \\
&= \int_\Omega \int_0^T f^N(s)\bar{\varphi}^N(s)\,\mathrm{d}s\mathrm{d}x \to \int_0^T \int_\Omega f\varphi\,\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

as $N \to \infty$, where the last convergence holds true for weakly convergent f^N and strongly convergent $\bar{\varphi}^N$. □

We now write the time-averaged terms as a convolution with a Dirac sequence. Let $N \in \mathbb{N}$ and $h = T/N$. We define $\zeta^N(t) = \frac{1}{h}\chi_{(-\frac{h}{2},\frac{h}{2})}(t)$ for $t \in \mathbb{R}$, where $\chi_I(\cdot)$ denotes the characteristic function on a given interval $I \subset \mathbb{R}$. Let X be a Banach space and $p \in [1,\infty)$. Then, for $f \in L^p(0,T;X)$ and $t \in (\frac{h}{2}, T-\frac{h}{2})$, we rewrite the time average over the interval $(t-\frac{h}{2}, t+\frac{h}{2})$ as a convolution with ζ^N, i.e.,

$$(\zeta^N * f)(t) := \int_\mathbb{R} \zeta^N(t-s)f(s)\,\mathrm{d}s = \frac{1}{h}\int_\mathbb{R} \chi_{(t-\frac{h}{2},t+\frac{h}{2})}(s)f(s)\,\mathrm{d}s = \frac{1}{h}\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} f(s)\,\mathrm{d}s.$$

Here, the function f is naturally extended with a zero outside of $(0,T)$. We note some properties from, e.g., [Alt16, Theorems 4.13 and 4.15], that we will use:

- $\{(\zeta^N * f)\}_{N\in\mathbb{N}} \subset L^p(0,T;X)$ with $(\zeta^N * f) \to f$ in $L^p(0,T;X)$, as $N \to \infty$.
- For $f, g \in L^1(0,T;X)$, it holds $\int_\mathbb{R} (\zeta^N * f)(t)g(t)\,\mathrm{d}t = \int_\mathbb{R} f(t)(\zeta^N * g)(t)\,\mathrm{d}t$.

Note that in fact the first one can be derived by the Lebesgue differentiation theorem for $p = 1$ and additionally with Jensen's inequality for $1 < p < \infty$. The second one is a direct consequence, since ζ^N is an even function by construction.

Regarding the weak-$*$ convergence, we have the following lemma.

LEMMA 6.38 (Weak-$*$ compactness). *Let $h = T/N$ with $N \in \mathbb{N}$. Moreover, let $f^N \in L^\infty(-h,T)$, $N \in \mathbb{N}$, with $f^N \to f$ weakly-$*$ in $L^\infty(0,T)$, as $N \to \infty$ (resp. $h \to 0$). Then, for all $\varsigma \in L^1(0,T)$, it holds as $N \to \infty$ (resp. $h \to 0$),*

$$\int_0^T \Big(\frac{1}{h}\int_{t-h}^t f^N(s)\,\mathrm{d}s\Big)\varsigma(t)\,\mathrm{d}t \to \int_0^T f(t)\varsigma(t)\,\mathrm{d}t.$$

Proof. To prove the lemma, we rewrite the inner integral with the help of the Dirac sequence defined above with a shift of $\frac{h}{2}$. Then one applies the associativity of the convolution with the Dirac sequence, the second property above, to move the convolution to the test function $\varsigma \in L^1(0,T)$. Finally, combining with the weak-$*$ convergence of f^N to f in $L^\infty(0,T)$ and the first property above, we finish the proof. Namely,

$$\begin{aligned}\int_0^T \Big(\frac{1}{h}\int_{t-h}^t f^N(s)\,\mathrm{d}s\Big)\varsigma(t)\,\mathrm{d}t &= \int_{\mathbb{R}} (\zeta^N * f^N)(t)\varsigma(t)\,\mathrm{d}t \\ &= \int_{\mathbb{R}} f^N(t)(\zeta^N * \varsigma)(t)\,\mathrm{d}t \to \int_{\mathbb{R}} f(t)\varsigma(t)\,\mathrm{d}t = \int_0^T f(t)\varsigma(t)\,\mathrm{d}t,\end{aligned}$$

as $N \to \infty$ (resp. $h \to 0$). □

Remark 6.39. In principle, it is possible to generalize Lemmata 6.37 and 6.38 for a broader class of Lebesgue spaces. However, we only make use of these special cases in this work.

For future reference, we also note a compactness criterion with time translations, which can be found in, e.g., [Sim87, Section 8, Theorem 5].

Lemma 6.40. *Let $p \in [1,\infty]$ and let X, Y, Z be Banach spaces with $X \subset\subset Y$ compactly and $Y \subset Z$ continuously. Moreover, let $\mathcal{F} \subset L^p(0,T;X)$ be a bounded set and for $f \in \mathcal{F}$ let $\|f(\cdot + h) - f\|_{L^p(0,T-h;Z)} \to 0$ uniformly, as $h \to 0$. Then, $\mathcal{F}$ is relatively compact in $L^p(0,T;Y)$ if $p \in [1,\infty)$, and in $C([0,T];Y)$ if $p = \infty$, respectively.*

6.4.7. Existence of weak solutions for regularized system. Now we are ready to prove Theorem 6.25 by compactness arguments and limit passages.

We obtain the energy inequality for the approximate solution $(\mathbf{u}^N, \phi^N, q^N, \widetilde{\mathbb{B}}^N)$ by integrating (6.65) over I_{k+1} and summing over $k = 0, ..., m$, where $m \in \{0, ..., N-1\}$ and $\tau = hm$, together with (6.63),

$$\begin{aligned}&\mathcal{E}_{AGG}(\phi^N(\tau), \mathbf{u}^N(\tau)) + \frac{1}{h}\int_{\tau-h}^{\tau} \mathcal{E}_B(\widetilde{\mathbb{B}}^N(t), \phi^N(t))\,\mathrm{d}t \\ &\quad + \frac{1}{2h}\int_0^\tau \int_\Omega \Big(\rho_h^N \left|\mathbf{u}^N - \mathbf{u}_h^N\right|^2 + \left|\nabla\phi^N - \nabla\phi_h^N\right|^2\Big)\,\mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^\tau \int_\Omega \Big(\frac{\nu(\phi_h^N)}{2}\left|\nabla\mathbf{u}^N + (\nabla\mathbf{u}^N)^\top\right|^2 + m(\phi_h^N)\left|\nabla q^N\right|^2 + \eta\left|\partial_{t,h}^- \phi^N\right|^2\Big)\,\mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^\tau \int_\Omega \Big(\frac{\kappa}{d}\left|\nabla \operatorname{tr}\ln\widetilde{\mathbb{B}}^N\right|^2 + \frac{\mu_\eta(\phi_h^N)}{2}\operatorname{tr}(\widetilde{\mathbb{B}}^N + (\widetilde{\mathbb{B}}^N)^{-1} - 2\mathbb{I})\Big)\,\mathrm{d}x\mathrm{d}t \\ &\le \mathcal{E}_{AGG}(\phi^N(0), \mathbf{u}^N(0)) + \mathcal{E}_B(\widetilde{\mathbb{B}}^N(0), \phi^N(0))\end{aligned} \tag{6.66}$$

which induces the boundedness of certain norms. However, even with these uniform bounds, the limit passing is still not possible for $N \to \infty$. The reason is that, for $\widetilde{\mathbb{B}}^N$, there is no compactness available at the moment (only $\|\operatorname{tr}\mathbb{B}^N\|_{L^1}$). To overcome this problem, we recall the energy estimate (6.23) derived in Section 6.3, namely,

$$\begin{aligned}&\left\|\operatorname{tr}(\widetilde{\mathbb{B}}^N - \ln\widetilde{\mathbb{B}}^N)(\tau)\right\|_{L^1} + \left\|\widetilde{\mathbb{B}}^N(\tau)\right\|_{L^2}^2 \\ &\quad + \int_0^\tau \Big(\left\|\widetilde{\mathbb{B}}^N(t)\right\|_{L^2}^2 + \kappa\left\|\nabla\widetilde{\mathbb{B}}^N(t)\right\|_{L^2}^2\Big)\,\mathrm{d}t \\ &\quad + \int_0^\tau \Big(\left\|\operatorname{tr}(\widetilde{\mathbb{B}}^N + (\widetilde{\mathbb{B}}^N)^{-1} - 2\mathbb{I})(t)\right\|_{L^1} + \kappa\left\|\nabla \operatorname{tr}\ln\widetilde{\mathbb{B}}^N(t)\right\|_{L^2}^2\Big)\,\mathrm{d}t \\ &\le C\Big(1 + \mathcal{E}_B(\widetilde{\mathbb{B}}^N(0), \phi^N(0)) + \|\mathbb{B}_0\|_{L^2}^2\Big)\end{aligned} \tag{6.67}$$

for all $\tau \in (0,T)$, where $C > 0$ depends on $\eta > 0$ and certain norms of $(\mathbf{u}^N, \phi^N)$, which are bounded uniformly regarding N due to (6.66). The right-hand side of (6.67) is uniformly bounded as well. Combining (6.66) and (6.67), we obtain the following uniform bounds on N (resp. h):

$$\begin{aligned}
\mathbf{u}^N \quad & \text{is bounded in} \quad L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^d)) \text{ and } L^\infty(0,T;L^2_\sigma(\Omega)),\\
\nabla q^N \quad & \text{is bounded in} \quad L^2(0,T;L^2(\Omega;\mathbb{R}^d)),\\
\phi^N \quad & \text{is bounded in} \quad L^\infty(0,T;W^{1,2}(\Omega)),\\
\widetilde{\mathbb{B}}^N \quad & \text{is bounded in} \quad L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})) \text{ and } L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}+})),\\
\operatorname{tr}\ln\widetilde{\mathbb{B}}^N \quad & \text{is bounded in} \quad L^\infty(0,T;L^1(\Omega)),\\
\nabla\operatorname{tr}\ln\widetilde{\mathbb{B}}^N \quad & \text{is bounded in} \quad L^2(0,T;L^2(\Omega;\mathbb{R}^d)),\\
\sqrt{\eta}\partial_{t,h}^-\phi^N \quad & \text{is bounded in} \quad L^2(0,T;L^2(\Omega)),
\end{aligned}$$

and by (6.52) and Lemma 6.32,

$$\begin{aligned}
\phi^N \quad & \text{is bounded in} \quad L^2(0,T;W^{2,2}(\Omega)),\\
\widetilde{W}_0'(\phi^N) \quad & \text{is bounded in} \quad L^2(0,T;L^2(\Omega)),\\
& \int_0^T \left| \int_\Omega q^N \,\mathrm{d}x \right| \mathrm{d}t \le M(T),
\end{aligned}$$

for a certain monotone function $M : \mathbb{R}^+ \to \mathbb{R}^+$. Up to a subsequence (not to be relabeled), one concludes the following convergences:

$$\begin{aligned}
\mathbf{u}^N &\to \mathbf{u}, && \text{weakly} && \text{in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^d)),\\
\mathbf{u}^N &\to \mathbf{u}, && \text{weakly-}* && \text{in } L^\infty(0,T;L^2_\sigma(\Omega)) \cong [L^1(0,T;L^2_\sigma(\Omega))]',\\
\phi^N &\to \phi, && \text{weakly} && \text{in } L^2(0,T;W^{2,2}(\Omega)),\\
\phi^N &\to \phi, && \text{weakly-}* && \text{in } L^\infty(0,T;W^{1,2}(\Omega)) \cong [L^1(0,T;W^{1,2}(\Omega))]',\\
\sqrt{\eta}\partial_{t,h}^-\phi^N &\to \sqrt{\eta}\partial_t\phi, && \text{weakly} && \text{in } L^2(0,T;L^2(\Omega)),\\
q^N &\to q, && \text{weakly} && \text{in } L^2(0,T;W^{1,2}(\Omega)),\\
\widetilde{\mathbb{B}}^N &\to \mathbb{B}, && \text{weakly} && \text{in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})),\\
\widetilde{\mathbb{B}}^N &\to \mathbb{B}, && \text{weakly-}* && \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})) \cong [L^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))]',\\
\nabla\operatorname{tr}\ln\widetilde{\mathbb{B}}^N &\to \overline{\nabla\operatorname{tr}\ln\mathbb{B}}, && \text{weakly} && \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)).
\end{aligned}$$

Here the $L^2(\Omega)$-convergence of q^N is derived by the Poincaré–Wirtinger inequality together with the integrability from above.

Recall the definition of $\widetilde{\mathbb{B}}^N(t) = \widetilde{\mathbb{B}}_{k+1}(t)$ for $t \in [t_k, t_{k+1})$, $k \in \{0,...,N-1\}$, with $\widetilde{\mathbb{B}}_{k+1}(t)$ defined by (6.53a). Thanks to the boundedness of $\mathbf{u}^N, \phi^N$, the mollifier $\mathscr{R}_\eta$ and the weak formulation (6.64d) restricted to each time interval I_{k+1}, $k \in \{0,...,N-1\}$, we get

$$\begin{aligned}
&\sum_{k=0}^{N-1} \int_{I_{k+1}} \left\| \partial_t \widetilde{\mathbb{B}}_{k+1}(t) \right\|^2_{[W^{1,2}]'} \mathrm{d}t \qquad (6.68)\\
&\le C(\eta)\Big(\left\| \mathbf{u}^N \right\|^2_{L^2(0,T;W^{1,2})} \left\| \widetilde{\mathbb{B}}^N \right\|^2_{L^\infty(0,T;L^2)} + \left\| \widetilde{\mathbb{B}}^N \right\|^2_{L^2(0,T;W^{1,2})} \left(1 + \left\| \nabla\phi_h^N \right\|^2_{L^\infty(0,T;L^2)} \right) + 1 \Big),
\end{aligned}$$

where $C(\eta) > 0$ depends on η but not on $N \in \mathbb{N}$. This allows us to employ a time translation compactness argument for $\widetilde{\mathbb{B}}^N$, that is,

$$\begin{aligned}
&\int_0^{T-h} \left\|\widetilde{\mathbb{B}}^N(t+h) - \widetilde{\mathbb{B}}^N(t)\right\|^2_{[W^{1,2}]'} \,\mathrm{d}t \\
&= \sum_{k=0}^{N-2} \int_{I_{k+1}} \left\|\widetilde{\mathbb{B}}_{k+2}(t+h) - \widetilde{\mathbb{B}}_{k+1}(t)\right\|^2_{[W^{1,2}]'} \,\mathrm{d}t \\
&\le \sum_{k=0}^{N-2} \int_{I_{k+1}} 2\Big(\left\|\widetilde{\mathbb{B}}_{k+2}(t+h) - \overline{\mathbb{B}}_{k+1}(t)\right\|^2_{[W^{1,2}]'} + \left\|\overline{\mathbb{B}}_{k+1}(t) - \widetilde{\mathbb{B}}_{k+1}(t)\right\|^2_{[W^{1,2}]'} \Big) \,\mathrm{d}t \\
&= 2\sum_{k=1}^{N-1} \int_{I_{k+1}} \left\|\widetilde{\mathbb{B}}_{k+1}(t) - \widetilde{\mathbb{B}}_{k+1}(t_k)\right\|^2_{[W^{1,2}]'} \,\mathrm{d}t + 2\sum_{k=0}^{N-2} \int_{I_{k+1}} \left\|\overline{\mathbb{B}}_{k+1}(t) - \widetilde{\mathbb{B}}_{k+1}(t)\right\|^2_{[W^{1,2}]'} \,\mathrm{d}t,
\end{aligned}$$

where $\overline{\mathbb{B}}_{k+1}(t) = \frac{1}{h}\int_{I_{k+1}} \widetilde{\mathbb{B}}_{k+1}(s)\,\mathrm{d}s$, as defined in Section 6.4.3. For the first term, using the fundamental theorem of calculus, Jensen's inequality and (6.68), we calculate

$$\sum_{k=1}^{N-1} \int_{I_{k+1}} \left\|\widetilde{\mathbb{B}}_{k+1}(t) - \widetilde{\mathbb{B}}_{k+1}(t_k)\right\|^2_{[W^{1,2}]'} \,\mathrm{d}t \le h^2 \sum_{k=1}^{N-1} \int_{I_{k+1}} \left\|\partial_t \widetilde{\mathbb{B}}_{k+1}(t)\right\|^2_{[W^{1,2}]'} \,\mathrm{d}t \le Ch^2,$$

where C does not depend on $N \in \mathbb{N}$. For the second term, we also use Jensen's inequality, the fundamental theorem of calculus and (6.68) to get

$$\begin{aligned}
&\sum_{k=0}^{N-2} \int_{I_{k+1}} \left\|\overline{\mathbb{B}}_{k+1}(t) - \widetilde{\mathbb{B}}_{k+1}(t)\right\|^2_{[W^{1,2}]'} \,\mathrm{d}t \\
&\le \frac{1}{h} \sum_{k=0}^{N-2} \int_{I_{k+1}} \int_{I_{k+1}} \left\|\widetilde{\mathbb{B}}_{k+1}(s) - \widetilde{\mathbb{B}}_{k+1}(t)\right\|^2_{[W^{1,2}]'} \,\mathrm{d}s\,\mathrm{d}t \\
&\le Ch^2 \sum_{k=0}^{N-2} \int_{I_{k+1}} \left\|\partial_t \widetilde{\mathbb{B}}_{k+1}(t)\right\|^2_{[W^{1,2}]'} \,\mathrm{d}t \\
&\le Ch^2,
\end{aligned}$$

where C is independent of $N \in \mathbb{N}$. Then, one concludes that

$$\int_0^{T-h} \left\|\widetilde{\mathbb{B}}^N(t+h) - \widetilde{\mathbb{B}}^N(t)\right\|^2_{[W^{1,2}]'} \,\mathrm{d}t \to 0,$$

as $N \to \infty$ ($h \to 0$, respectively). By virtue of Lemma 6.40, one obtains the strong convergences

$$\begin{aligned}
&\widetilde{\mathbb{B}}^N \to \mathbb{B}, \quad \text{strongly} \quad \text{in } L^2(0,T;L^p(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})), \quad 2 \le p \le 6, \\
&\widetilde{\mathbb{B}}^N \to \mathbb{B}, \quad \text{a.e.} \quad \text{in } Q_T.
\end{aligned}$$

Proceeding in a similar fashion as in Section 6.3.7 together with the Poincaré–Wirtinger inequality gives birth to

$$\begin{aligned}
&\mathbb{B} \text{ is positive definite a.e. in } Q_T, \\
&\operatorname{tr}\ln \widetilde{\mathbb{B}}^N \to \operatorname{tr}\ln \mathbb{B}, \quad \text{weakly} \quad \text{in } L^2(0,T;W^{1,2}(\Omega)),
\end{aligned}$$

It follows from the continuity of $\operatorname{tr}\ln(\cdot)$ that

$$\operatorname{tr}\ln\widetilde{\mathbb{B}}^N \to \operatorname{tr}\ln\mathbb{B}, \quad \text{a.e.} \quad \text{in } Q_T. \tag{6.69}$$

Arguing as in Section 6.3.8 yields

$$\mathbb{B} \in C_w([0,T]; L^2(\Omega; \mathbb{R}^{d\times d}_{\text{sym}})).$$

Now let $\widetilde{\phi}^N$ be the piecewise linear interpolant of $\phi^N(kh)$, $k \in \{0, ..., N\}$, i.e., $\widetilde{\phi}^N = \frac{1}{h}\chi_{[0,h]} *_t \phi^N$, where the convolution only happens regarding time variable t. Then it follows that $\partial_t\widetilde{\phi}^N = \partial^-_{t,h}\phi^N$ and, for a.e. $t \in (0,T)$,

$$\|\widetilde{\phi}^N - \phi^N\|_{[W^{1,2}(\Omega)]'} \le Ch\|\partial_t\widetilde{\phi}^N\|_{[W^{1,2}(\Omega)]'}. \tag{6.70}$$

In view of the weak formulation (6.64b) and boundedness of $(\mathbf{u}^N, \phi^N, q^N)$, we see that

$$\partial_t\widetilde{\phi}^N \quad \text{is bounded in} \quad L^2(0,T;[W^{1,2}(\Omega)]'),$$

which, together with Aubin–Lions thereby implies the strong convergence

$$\begin{aligned}
&\widetilde{\phi}^N \to \widetilde{\phi}, \quad \text{strongly} \quad \text{in } L^2(0,T;W^{1,p}(\Omega)), \quad 2 \le p \le 6,\\
&\widetilde{\phi}^N \to \widetilde{\phi}, \quad \text{strongly} \quad \text{in } C([0,T];L^p(\Omega)), \quad 2 \le p \le 6,\\
&\widetilde{\phi}^N \to \widetilde{\phi}, \quad \text{a.e.} \quad \text{in } Q_T,
\end{aligned}$$

for some $\widetilde{\phi} \in L^\infty(0,T;W^{1,2}(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega))$, thanks to the boundedness of $\widetilde{\phi}^N$, which can be derived from that of ϕ^N. Note that (6.70) indicates that

$$\widetilde{\phi}^N - \phi^N \to 0 \quad \text{in } L^2(0,T;[W^{1,2}(\Omega)]'), \quad \text{as } N \to \infty,$$

which gives $\widetilde{\phi} = \phi$. Then we have $\partial_t\phi \in L^2(0,T;[W^{1,2}(\Omega)]')$ and hence

$$\phi \in C_w([0,T];W^{1,2}(\Omega))$$

due to Lemma 6.9. Next, we verify the identity $\phi(0) = \phi_0$, which can be recorded from

$$\widetilde{\phi}^N(0) \to \widetilde{\phi}(0) = \phi(0), \quad \text{strongly in } L^p(\Omega),$$

and the fact that $\widetilde{\phi}^N(0) = \phi_0^N$ with $\phi_0^N \to \phi_0$ in $W^{1,2}(\Omega)$. In addition, it holds

$$\int_\Omega \phi^N(\cdot,\tau-h)\overline{\xi}_\tau \,\mathrm{d}x + \int_\Omega \phi_0^N\overline{\xi}_0 \,\mathrm{d}x \to \int_\Omega \phi(\cdot,\tau)\xi(\cdot,\tau)\,\mathrm{d}x + \int_\Omega \phi_0\xi(\cdot,0)\,\mathrm{d}x,$$

as $N \to \infty$ (resp. $h \to 0$) for a.e. $\tau \in (0,T)$, concerning the weak convergence of $\phi^N(\tau)$, and strong convergence of $\overline{\xi}_\tau \to \xi(\tau)$ in $L^2(\Omega)$ for fixed τ. Moreover, the continuity of $m(\cdot)$, $\nu(\cdot)$, $\mu(\cdot)$ and the almost everywhere convergence of ϕ^N yield

$$\begin{aligned}
m(\phi^N) &\to m(\phi), \quad \text{a.e.} \quad \text{in } Q_T,\\
\nu(\phi^N) &\to \nu(\phi), \quad \text{a.e.} \quad \text{in } Q_T,\\
\mu(\phi^N) &\to \mu(\phi), \quad \text{a.e.} \quad \text{in } Q_T,\\
\frac{\mu_\eta(\phi^N) - \mu_\eta(\phi_h^N)}{\mathscr{R}_\eta\phi^N - \mathscr{R}_\eta\phi_h^N} &\to \mu'_\eta(\phi), \quad \text{a.e.} \quad \text{in } Q_T.
\end{aligned}$$

Then one can pass to the limit of (6.64b) to (6.42) and (6.64d) to (6.44) as $N \to \infty$, with the help of the mollifier $\mathscr{R}_\eta$ and the convergence results from above.

By virtue of the weak convergences of q^N, ϕ^N, $\partial_t \phi^N$, $\widetilde{\mathbb{B}}^N$ and $\operatorname{tr} \ln \widetilde{\mathbb{B}}^N$, together with Lemma 6.11, one knows the right-hand side of (6.64c) denoted by Q^N converges weakly in $L^2(0,T;L^2(\Omega))$ to

$$Q \coloneqq q + \omega\phi - \eta\partial_t\phi - \frac{1}{2}\mathscr{R}_\eta\Big[\mu_\eta'(\phi)\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I})\Big],$$

by applying the convergences result above term by term. Here, the weak convergence of the last term on the right-hand side is valid due to Lemmata 6.11 and 6.37. On the other hand,

$$\Big(\partial\widetilde{E}(\phi^N), \phi^N\Big) = \big(Q^N, \phi^N\big) \to (Q, \phi)\,, \text{ as } N \to \infty$$

due to the strong convergence of ϕ^N in $L^2(0,T;L^2(\Omega))$. Therefore, by e.g. [Cia13, Theorem 9.13-2] for monotone operators one knows $\partial\widetilde{E}(\phi) = Q$, which is exactly (6.43).

Next, we are going to get compactness of $\mathbf{u}^N$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$, which implies a pointwise almost everywhere convergence. This is in general not a problem in the case of the matched density (constant ρ), for which one can use the same strategy as of $\mathbb{B}^N$ to achieve the strong convergence by the Aubin–Lions lemma. However, it is not possible to apply the same argument directly for $\mathbf{u}^N$ with unmatched densities, here, instead, we make use of the Helmholtz projection $\mathbb{P}_\sigma$ onto $L^2_\sigma(\Omega)$ as in [ADG13a] (also called Leray projection). With the uniform boundedness of $\mathbf{u}^N$, ϕ^N, q^N and $\mathbb{B}^N$, it follows that

$$\begin{aligned}
\rho_h^N \mathbf{u}^N \otimes \mathbf{u}^N \quad & \text{is bounded in} \quad L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})),\\
\nabla\mathbf{u}^N + (\nabla\mathbf{u}^N)^\top \quad & \text{is bounded in} \quad L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})),\\
\mathbf{u}^N \otimes \nabla q^N \quad & \text{is bounded in} \quad L^{\frac{8}{7}}(0,T;L^{\frac{4}{3}}(\Omega;\mathbb{R}^{d\times d})),\\
q^N \nabla\phi_h^N \quad & \text{is bounded in} \quad L^2(0,T;L^{\frac{3}{2}}(\Omega;\mathbb{R}^d)),\\
\mathscr{R}_\eta[\mu_\eta(\phi_h^N)(\overline{\mathbb{B}}^N - \mathbb{I})] \quad & \text{is bounded in} \quad L^2(0,T;L^6(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})),\\
\mathscr{R}_\eta\Big[\frac{\mu_\eta(\phi_h^N)}{2}\nabla G^N\Big] \quad & \text{is bounded in} \quad L^2(0,T;L^2(\Omega;\mathbb{R}^d)).
\end{aligned}$$

Note that the first four bounds follow directly from [ADG13a, Page 474], while the remaining bounds are valid due to the boundedness of $\widetilde{\mathbb{B}}^N$, the uniform upper bound of μ and Lemma 6.11, Remark 6.36. Then going back to the equation (6.64a), one may infer that

$$\partial_t\big(\mathbb{P}_\sigma(\widetilde{\rho\mathbf{u}}^N)\big) \quad \text{is bounded in} \quad L^{\frac{8}{7}}(0,T;W^{-1,4}(\Omega;\mathbb{R}^d)),$$

where $\widetilde{\rho\mathbf{u}}^N$ is the piecewise linear interpolant of $\rho^N\mathbf{u}^N(kh)$ for $k \in \{0,\dots,N\}$, which fulfulls $\partial_t\big(\widetilde{\rho\mathbf{u}}^N\big) = \partial_{t,h}^-(\rho^N\mathbf{u}^N)$. Moreover,

$$\mathbb{P}_\sigma(\widetilde{\rho\mathbf{u}}^N) \quad \text{is bounded in} \quad L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^d)).$$

In light of the Aubin–Lions lemma, one arrives at the strong convergence

$$\mathbb{P}_\sigma(\widetilde{\rho\mathbf{u}}^N) \to \overline{\mathbb{P}_\sigma(\rho(\phi)\mathbf{u})}, \quad \text{strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)),$$

for some function $\overline{\mathbb{P}_\sigma(\rho(\phi)\mathbf{u})} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))$. Indeed,

$$\widetilde{\rho\mathbf{u}}^N \rightharpoonup \rho(\phi)\mathbf{u}, \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)). \tag{6.71}$$

Then by virtue of the weak continuity of the Leray projection $\mathbb{P}_\sigma : L^2(0,T;L^2(\Omega;\mathbb{R}^d)) \to L^2(0,T;L^2_\sigma(\Omega))$, we obtain

$$\overline{\mathbb{P}_\sigma(\rho(\phi)\mathbf{u})} = \mathbb{P}_\sigma(\rho(\phi)\mathbf{u}).$$

Now in view of the strong convergence of $\mathbb{P}_\sigma(\rho^N\mathbf{u}^N)$ and the weak convergence of $\mathbf{u}^N$ both in $L^2(0,T;L^2(\Omega;\mathbb{R}^d))$, one ends up with

$$\begin{aligned}\int_0^T\int_\Omega \rho^N \left|\mathbf{u}^N\right|^2 \,\mathrm{d}x\mathrm{d}t &= \int_0^T\int_\Omega \mathbb{P}_\sigma(\rho^N\mathbf{u}^N)\cdot\mathbf{u}^N \,\mathrm{d}x\mathrm{d}t \\ &\to \int_0^T\int_\Omega \mathbb{P}_\sigma(\rho(\phi)\mathbf{u})\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega \rho(\phi)\left|\mathbf{u}\right|^2 \,\mathrm{d}x\mathrm{d}t,\end{aligned}$$

which implies

$$\sqrt{\rho^N}\mathbf{u}^N \to \sqrt{\rho(\phi)}\mathbf{u}, \quad \text{strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{6.72}$$

with the weak convergence (6.71). As $\rho(r)$ is affine linear regarding r, one gets

$$\rho(\phi^N)\to\rho(\phi), \quad \text{a.e.} \quad \text{in } Q_T \quad \text{and} \quad \rho(\phi^N)\geq C>0,$$

which together with the strong convergence (6.72) of $\sqrt{\rho^N}\mathbf{u}^N$ gives rise to

$$\mathbf{u}^N = \frac{1}{\sqrt{\rho^N}}\sqrt{\rho^N}\mathbf{u}^N \to \frac{1}{\sqrt{\rho(\phi)}}\sqrt{\rho(\phi)}\mathbf{u} = \mathbf{u}, \quad \text{strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^d)),$$

and hence

$$\mathbf{u}^N \to \mathbf{u}, \quad \text{a.e.} \quad \text{in } Q_T.$$

In addition, it holds

$$\begin{aligned}&\int_\Omega \rho(\phi^N(\cdot,\tau-h))\mathbf{u}^N(\cdot,\tau-h)\cdot\overline{\mathbf{w}}_\tau\,\mathrm{d}x \to \int_\Omega \rho(\phi(\cdot,\tau))\mathbf{u}(\cdot,\tau)\cdot\mathbf{w}(\cdot,\tau)\,\mathrm{d}x,\\ &\int_\Omega \rho(\phi_0^N)\mathbf{u}_0\cdot\overline{\mathbf{w}}_0\,\mathrm{d}x \to \int_\Omega \rho(\phi_0)\mathbf{u}_0\cdot\mathbf{w}(\cdot,0)\,\mathrm{d}x,\end{aligned}$$

as $N\to\infty$ (resp. $h\to 0$) for a.e. $\tau\in(0,T)$, concerning the convergences of $\phi^N(\tau)$ and $\mathbf{u}^N(\tau)$, and the strong convergence of $\overline{\mathbf{w}}_\tau \to \mathbf{w}(\tau)$ in $L^2_\sigma(\Omega)$ for fixed τ. Subsequently, one can pass to the limit in (6.64a) to (6.41) term by term as $N\to\infty$ with the strong convergences of $\mathbf{u}^N, \phi^N$, except for the terms with respect to $\widetilde{\mathbb{B}}^N$. Let us recall the definition of $\overline{\mathbb{B}}^N$ and G^N

$$\begin{aligned}\overline{\mathbb{B}}^N(t) &:= \frac{1}{h}\int_{I_{k+1}} \widetilde{\mathbb{B}}_{k+1}(s)\,\mathrm{d}s = \frac{1}{h}\int_{I_{k+1}} \widetilde{\mathbb{B}}^N(s)\,\mathrm{d}s,\\ G^N(t) &:= \frac{1}{h}\int_{I_{k+1}} \operatorname{tr}(\widetilde{\mathbb{B}}_{k+1} - \ln\widetilde{\mathbb{B}}_{k+1} - \mathbb{I})(s)\,\mathrm{d}s = \frac{1}{h}\int_{I_{k+1}} \operatorname{tr}(\widetilde{\mathbb{B}}^N - \ln\widetilde{\mathbb{B}}^N - \mathbb{I})(s)\,\mathrm{d}s\end{aligned}$$

for $t\in[t_k,t_{k+1})$, where $k\in\{0,...,N-1\}$. In fact, by Lemma 6.35, Remark 6.36 and the uniform bounds of $\widetilde{\mathbb{B}}^N$ and $\operatorname{tr}\ln\widetilde{\mathbb{B}}^N$, we know that $\overline{\mathbb{B}}^N$ and ∇G^N are uniformly bounded in $L^2(Q_T)$. With the help of Lemma 6.37, one concludes

$$\begin{aligned}&\int_0^\tau\int_\Omega \left[\mu_\eta(\phi_h^N)(\overline{\mathbb{B}}^N - \mathbb{I})\right] : \mathscr{R}_\eta\nabla\mathbf{w}\,\mathrm{d}x\mathrm{d}t \to \int_0^\tau\int_\Omega \left[\mu_\eta(\phi)(\mathbb{B}-\mathbb{I})\right] : \mathscr{R}_\eta\nabla\mathbf{w}\,\mathrm{d}x\mathrm{d}t,\\ &\int_0^\tau\int_\Omega \left[\frac{\mu_\eta(\phi_h^N)}{2}\nabla G^N\right]\cdot\mathscr{R}_\eta\mathbf{w}\,\mathrm{d}x\mathrm{d}t \to \int_0^\tau\int_\Omega \left[\frac{\mu_\eta(\phi)}{2}\nabla\operatorname{tr}(\mathbb{B}-\ln\mathbb{B}-\mathbb{I})\right]\cdot\mathscr{R}_\eta\mathbf{w}\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Using integration by parts over Ω and the compactness of ϕ^N, we get the convergence of $\int_0^t \int_\Omega q^N \nabla\phi^N \cdot \mathbf{w}\, dx dt$. Then, it remains to verify that $\mathbf{u} \in C_w([0,T]; L^2_\sigma(\Omega))$ and $\mathbf{u}(t) \to \mathbf{u}_0$ as $t \to 0$, which are clear by proceeding the same argument as in [ADG13a, Section 5.2].

In the final step, one recovers the energy dissipation inequality (6.45). To this end, multiplying the discrete energy inequality (6.65) by a function $\varsigma \in W^{1,1}(0,T)$ with $\varsigma \geq 0$ and $\varsigma(T) = 0$, and integrating the resulting inequality over $(0,T)$ and by parts with respect to the time variable, one obtains

$$\mathcal{E}_{tot}(\mathbf{u}_0, \phi_0^N, \mathbb{B}_0)\varsigma(0) + \int_0^T \mathcal{E}^N(\tau)\varsigma'(\tau)\,\mathrm{d}\tau \geq \int_0^T \mathcal{D}^N(\tau)\varsigma(\tau)\,\mathrm{d}\tau, \tag{6.73}$$

where $\mathcal{E}_{tot}$, $\mathcal{E}^N$ and $\mathcal{D}^N$ are defined in the end of last subsection. Thanks to the compactness of $\mathbf{u}^N$, ϕ^N and $\widetilde{\mathbb{B}}^N$, we deduce that up to a subsequence (not relabeled),

$$\mathbf{u}^N(t) \to \mathbf{u}(t), \quad \text{in } L^2_\sigma(\Omega), \tag{6.74}$$

$$\phi^N(t) \to \phi(t), \quad \text{in } C(\overline{\Omega}), \tag{6.75}$$

$$\widetilde{\mathbb{B}}^N(t) \to \mathbb{B}(t), \quad \text{in } L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}), \tag{6.76}$$

for a.e. $t \in (0,T)$, as $N \to \infty$ (resp. $h \to 0$). On noting the uniform energy estimate (6.66) and $\operatorname{tr}\ln \widetilde{\mathbb{B}}^N \to \operatorname{tr}\ln \mathbb{B}$ a.e. in Q_T, it follows that

$$\int_\Omega \operatorname{tr}\ln \widetilde{\mathbb{B}}^N\,\mathrm{d}x \to \int_\Omega \operatorname{tr}\ln \mathbb{B}\,\mathrm{d}x, \quad \text{weakly-}* \text{ in } L^\infty(0,T).$$

Then by the weak-$*$ compactness Lemma 6.38,

$$\int_0^T \left(\frac{1}{h}\int_{t-h}^t \Big(\int_\Omega \operatorname{tr}\ln \widetilde{\mathbb{B}}^N(\tau)\,\mathrm{d}x\Big)\,\mathrm{d}\tau\right)\varsigma'(t)\,\mathrm{d}t \to \int_0^T \int_\Omega \operatorname{tr}\ln \mathbb{B}(t)\,\mathrm{d}x\,\varsigma'(t)\,\mathrm{d}t. \tag{6.77}$$

for any $\varsigma \in W^{1,1}(0,T)$ with $\varsigma \geq 0$. Therefore with (6.74)–(6.77), we have

$$\int_0^T \mathcal{E}^N(t)\varsigma'(t)\,\mathrm{d}t \to \int_0^T \mathcal{E}_\eta(t)\varsigma'(t)\,\mathrm{d}t,$$

with $\mathcal{E}_\eta(t)$ defined in (6.46). In view of the lower semicontinuity of norms, the positivity of $m(\cdot)$, $\nu(\cdot)$ and $\mu(\cdot)$, and the almost everywhere convergence of ϕ^N to ϕ, one has

$$\liminf_{N\to\infty} \int_0^T \mathcal{D}^N(\tau)\varsigma(\tau)\,\mathrm{d}\tau \geq \int_0^T \mathcal{D}_\eta(\tau)\varsigma(\tau)\,\mathrm{d}\tau,$$

for any $\varsigma \in W^{1,1}(0,T)$ with $\varsigma \geq 0$, where, for a.e. $t \in (0,T)$,

$$\begin{aligned}\mathcal{D}_\eta(t) &:= \int_\Omega \Big(\frac{\nu(\phi)}{2}\left|\nabla\mathbf{u} + \nabla\mathbf{u}^\top\right|^2 + m(\phi)\left|\nabla q\right|^2 + \eta\left|\partial_t\phi\right|^2\Big)\,\mathrm{d}x \\ &\quad + \int_\Omega \frac{\mu_\eta(\phi)}{2}\operatorname{tr}(\mathbb{B} + \mathbb{B}^{-1} - 2\mathbb{I})\,\mathrm{d}x + \frac{\kappa}{d}\int_\Omega \left|\nabla \operatorname{tr}\ln\mathbb{B}\right|^2\,\mathrm{d}x.\end{aligned}$$

Passing to the limit in (6.73), as $N \to \infty$, yields

$$\mathcal{E}(0)\varsigma(0) + \int_0^T \mathcal{E}_\eta(\tau)\varsigma'(\tau)\,\mathrm{d}\tau \geq \int_0^T \mathcal{D}_\eta(\tau)\varsigma(\tau)\,\mathrm{d}\tau,$$

for any $\varsigma \in W^{1,1}(0,T)$ with $\varsigma \geq 0$ and $\varsigma(T) = 0$. On account of Lemma 6.8, we get the desired energy dissipation inequality (6.45).

This completes the proof. □

6.5. Existence of Weak Solutions ($\eta \to 0$)

In this section, we are devoted to proving Theorem 6.3, by virtue of Theorem 6.25 for the regularized system (6.40), compactness discussions and limit passages. Due to technical reasons, the final proof of Theorem 6.3 is restricted to the two dimentional case, as discussed in Sections 6.1.1, 6.1.4 and Remark 6.26.

6.5.1. Stronger uniform estimate. With the *a priori* estimate (6.51) in hand, we are not able to prove the existence of weak solutions by passing to the limit as $\eta \to 0$. This is due to the very weak regularity of $\mathbb{B}$, namely $\|\mathbb{B}(t)\|_{L^1}$, which is of no help to obtain the compactness. Thus, in this section we derive a stronger estimate for the Cauchy–Green tensor, which was also carried out in, e.g., [BB11, BLS17, GKT22]. Note that the restriction of the problem to two dimensions arises precisely from the stronger estimate, even in presence of the stress diffusion term $\frac{\kappa}{\mu(\phi)}\Delta\mathbb{B}$. Multiplying (6.40c) with $\mathbb{B}$, using the chain rule, integrating over Ω and by parts, we have for a.e. $t \in (0,T)$,

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{B}\|_{L^2}^2 + \|\mathbb{B}\|_{L^2}^2 + \int_\Omega \mathscr{R}_\eta\mathbf{u}\cdot\nabla\Big(\frac{1}{2}|\mathbb{B}|^2\Big)\,\mathrm{d}x + \int_\Omega \frac{\kappa}{\mu_\eta(\phi)}|\nabla\mathbb{B}|^2\,\mathrm{d}x \\
&= \int_\Omega \operatorname{tr}\mathbb{B}\,\mathrm{d}x + \kappa\int_\Omega \frac{\mu_\eta'(\phi)}{\mu_\eta^2(\phi)}(\nabla\mathscr{R}_\eta\phi\cdot\nabla)\mathbb{B} : \mathbb{B}\,\mathrm{d}x + 2\int_\Omega(\mathbb{B}\mathbb{B}) : \nabla\mathscr{R}_\eta\mathbf{u}\,\mathrm{d}x.
\end{aligned}
$$

Noting the upper-lower bounds of $\mu(\cdot),\mu'(\cdot)$, integrating by parts over Ω in the convective term, employing Hölder's inequality, the 2D Gagliardo–Nirenberg inequality and Lemma 6.11, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbb{B}\|_{L^2}^2 + \|\mathbb{B}\|_{L^2}^2 + \frac{\kappa}{\overline{\mu}}\|\nabla\mathbb{B}\|_{L^2}^2 \\
&\le \frac{1}{2}\|\mathbb{B}\|_{L^2}^2 + C + \frac{\kappa}{2\underline{\mu}}\|\nabla\mathbb{B}\|_{L^2}^2 + C\Big(\|\nabla\phi\|_{L^4}^4 + \|\nabla\mathbf{u}\|_{L^2}^2\Big)\|\mathbb{B}\|_{L^2}^2 .
\end{aligned}
$$

Here, we used

$$
\begin{aligned}
\int_\Omega (\nabla\mathscr{R}_\eta\phi\cdot\nabla)\mathbb{B} : \mathbb{B}\,\mathrm{d}x &\le \|\nabla\mathbb{B}\|_{L^2}\|\mathbb{B}\|_{L^4}\|\nabla\mathscr{R}_\eta\phi\|_{L^4} \\
&\le \|\nabla\mathbb{B}\|_{L^2}^{\frac{3}{2}}\|\mathbb{B}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathscr{R}_\eta\phi\|_{L^4} \le \frac{\kappa}{4\underline{\mu}}\|\nabla\mathbb{B}\|_{L^2}^2 + C\|\mathbb{B}\|_{L^2}^2\|\nabla\phi\|_{L^4}^4 , \\
\int_\Omega (\mathbb{B}\mathbb{B}) : \nabla\mathscr{R}_\eta\mathbf{u}\,\mathrm{d}x &\le \|\mathbb{B}\|_{L^4}^2\|\nabla\mathscr{R}_\eta\mathbf{u}\|_{L^2} \\
&\le \|\nabla\mathbb{B}\|_{L^2}\|\mathbb{B}\|_{L^2}\|\nabla\mathscr{R}_\eta\mathbf{u}\|_{L^2} \le \frac{\kappa}{4\underline{\mu}}\|\nabla\mathbb{B}\|_{L^2}^2 + C\|\mathbb{B}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2 .
\end{aligned}
$$

Then integrating over $(0,t)$ and applying Gronwall's lemma, one obtains

$$
\|\mathbb{B}(t)\|_{L^2}^2 + \int_0^t \|\mathbb{B}(\tau)\|_{L^2}^2\,\mathrm{d}\tau + \int_0^t \|\nabla\mathbb{B}(\tau)\|_{L^2}^2\,\mathrm{d}\tau \le C(\|\mathbb{B}_0\|_{L^2}^2 + 1)\exp(h(t)), \tag{6.78}
$$

for a.e. $t \in (0,T)$, where $C > 0$ does not depend on $\eta > 0$ and $h(t) := \|\nabla\phi\|_{L^4(0,t;L^4)}^4 + \|\nabla\mathbf{u}\|_{L^2(0,t;L^2)}^2 < \infty$ due to (6.51) and the embedding (6.13).

6.5.2. Proof of Theorem 6.3. Let us denote by $(\mathbf{u}^\eta, \mathbb{B}^\eta, \phi^\eta, q^\eta)$ the corresponding regularized solution of (6.40), where $\eta > 0$. Note that the *a priori* estimate done in Section 6.4.1 and 6.5.1 are uniform in terms of η. We conclude from (6.51) and (6.78) that

$$\begin{aligned}
&\mathcal{E}_\eta(\tau) + \|\mathbb{B}^\eta(\tau)\|_{L^2}^2 + \int_0^\tau \|\nabla \mathbf{u}^\eta(t)\|_{L^2}^2 \,\mathrm{d}t + \int_0^\tau \|\nabla q^\eta(t)\|_{L^2}^2 \,\mathrm{d}t + \int_0^\tau \|\mathbb{B}^\eta(t)\|_{L^2}^2 \,\mathrm{d}t \\
&+ \int_0^\tau \|\nabla \mathbb{B}^\eta(t)\|_{L^2}^2 \,\mathrm{d}t + \int_0^\tau \left\|\mathrm{tr}(\mathbb{B}^\eta + (\mathbb{B}^\eta)^{-1} - 2\mathbb{I})(t)\right\|_{L^1} \mathrm{d}t + \int_0^\tau \|\nabla \operatorname{tr} \ln \mathbb{B}^\eta(t)\|_{L^2}^2 \,\mathrm{d}t \\
&\leq C(\mathcal{E}_\eta(0), \|\mathbb{B}_0\|_{L^2}^2) \leq C(\mathcal{E}(0), \|\mathbb{B}_0\|_{L^2}^2), \qquad (6.79)
\end{aligned}$$

for any $\tau \in (0, T)$, where $C > 0$ is uniform in terms of η, which together with the fact that $\mathcal{E}(0), \|\mathbb{B}_0\|_{L^2}^2$ are bounded, implies the following uniform bounds (in η)

$$\begin{aligned}
\mathbf{u}^\eta &\quad \text{is bounded in} \quad L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^2)) \text{ and } L^\infty(0,T; L^2_\sigma(\Omega)), \\
\nabla q^\eta &\quad \text{is bounded in} \quad L^2(0,T; L^2(\Omega;\mathbb{R}^2)), \\
\phi^\eta &\quad \text{is bounded in} \quad L^\infty(0,T; W^{1,2}(\Omega)), \\
\mathbb{B}^\eta &\quad \text{is bounded in} \quad L^2(0,T; W^{1,2}(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})) \text{ and } L^\infty(0,T; L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}+})),
\end{aligned}$$

and

$$\int_0^T \left| \int_\Omega q^\eta \,\mathrm{d}x \right| \mathrm{d}t \leq M(T),$$

for a certain monotone function $M : \mathbb{R}^+ \to \mathbb{R}^+$. Moreover, testing (6.43) with $\Delta\phi^\eta$ together with $W(r) = W_0(r) - \frac{\omega}{2} r^2$ as in Section 6.4.2, integration by parts and Young's inequality yields

$$\begin{aligned}
&\int_0^T \int_\Omega \left(|\Delta \phi^\eta|^2 + W_0''(\phi^\eta) |\nabla \phi^\eta|^2 \right) \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_\Omega \left(\nabla q^\eta \cdot \nabla \phi^\eta + \omega |\nabla \phi^\eta|^2 - \frac{\mu'}{2} \nabla \operatorname{tr}(\mathbb{B}^\eta - \ln \mathbb{B}^\eta - \mathbb{I}) \cdot \nabla \phi^\eta \right) \mathrm{d}x\mathrm{d}t \\
&\leq \int_0^T \|\nabla q^\eta\|_{L^2}^2 \,\mathrm{d}t + C(\omega, \overline{\mu}') + \int_0^T \left(\|\nabla \mathbb{B}^\eta\|_{L^2}^2 + \|\nabla \operatorname{tr} \ln \mathbb{B}^\eta\|_{L^2}^2 \right) \mathrm{d}t \leq C,
\end{aligned}$$

where $C > 0$ is uniform in η due to $W_0'' > 0$ and (6.79). Therefore, one obtains further

$$\phi^\eta \quad \text{is bounded in} \quad L^2(0,T; W^{2,2}(\Omega)).$$

Next, we gain more bounds for the purpose of compactness. By the Sobolev embedding in two dimensions and the interpolation embedding (6.13) we know

$$\begin{aligned}
\mathbf{u}^\eta &\quad \text{is bounded in} \quad L^4(0,T; L^4(\Omega;\mathbb{R}^2)), \\
\mathbb{B}^\eta &\quad \text{is bounded in} \quad L^4(0,T; L^4(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})),
\end{aligned}$$

which together with bounds above leads to

$$\begin{aligned}
\mathscr{R}_\eta \mathbf{u}^\eta \otimes \mathbb{B}^\eta &\quad \text{is bounded in} \quad L^2(0,T; L^2(\Omega;\mathbb{R}^{2\times 2\times 2})), \\
\nabla \mathscr{R}_\eta \mathbf{u}^\eta \mathbb{B}^\eta &\quad \text{is bounded in} \quad L^{\frac{4}{3}}(0,T; L^{\frac{4}{3}}(\Omega;\mathbb{R}^{2\times 2})).
\end{aligned}$$

By making use of the weak formulation (6.44), one ends up with

$$\partial_t \mathbb{B}^\eta \quad \text{is bounded in} \quad L^{\frac{4}{3}}(0,T; [W^{1,2}(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})]'),$$

Because of (6.42) and the boundedness of $\mathbf{u}^\eta$, ∇q^η, we have

$$\partial_t \phi^\eta \quad \text{is bounded in} \quad L^2(0,T;[W^{1,2}(\Omega)]'),$$

Then up to a subsequence ($\eta_k \to 0$ as $k \to \infty$) still denoted by the superscript η, one obtains

$$\begin{aligned}
\mathbf{u}^\eta &\to \mathbf{u}, && \text{weakly} && \text{in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^2)),\\
\mathbf{u}^\eta &\to \mathbf{u}, && \text{weakly-}* && \text{in } L^\infty(0,T;L^2_\sigma(\Omega)) \cong [L^1(0,T;L^2_\sigma(\Omega))]',\\
\phi^\eta &\to \phi, && \text{weakly} && \text{in } L^2(0,T;W^{2,2}(\Omega)),\\
\phi^\eta &\to \phi, && \text{weakly-}* && \text{in } L^\infty(0,T;W^{1,2}(\Omega)) \cong [L^1(0,T;W^{1,2}(\Omega))]',\\
\partial_t\phi^\eta &\to \partial_t\phi, && \text{weakly} && \text{in } L^2(0,T;[W^{1,2}(\Omega)]'),\\
q^\eta &\to q, && \text{weakly} && \text{in } L^2(0,T;W^{1,2}(\Omega)),\\
\nabla q^\eta &\to \nabla q, && \text{weakly} && \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^2)),\\
\mathbb{B}^\eta &\to \mathbb{B}, && \text{weakly} && \text{in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}})),\\
\mathbb{B}^\eta &\to \mathbb{B}, && \text{weakly-}* && \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}})) \cong [L^1(0,T;L^2(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}}))]',\\
\partial_t\mathbb{B}^\eta &\to \partial_t\mathbb{B}, && \text{weakly} && \text{in } L^{\frac{4}{3}}(0,T;[W^{1,2}(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}})]').
\end{aligned}$$

In view of the Aubin–Lions lemma, one concludes the strong convergence (up to a non-relabeled subsequence)

$$\begin{aligned}
\phi^\eta &\to \phi, && \text{strongly} && \text{in } L^2(0,T;W^{1,p}(\Omega)), \quad 1 \le p < \infty,\\
\phi^\eta &\to \phi, && \text{a.e.} && \text{in } Q_T,\\
\mathbb{B}^\eta &\to \mathbb{B}, && \text{strongly} && \text{in } L^2(0,T;L^p(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}})), \quad 2 \le p < \infty,\\
\mathbb{B}^\eta &\to \mathbb{B}, && \text{a.e.} && \text{in } Q_T.
\end{aligned}$$

Arguing in a similar fashion as in Section 6.3.7 leads us to

$$\begin{aligned}
&\mathbb{B} \text{ is positive definite a.e. in } Q_T,\\
&\operatorname{tr}\ln\mathbb{B}^\eta \to \operatorname{tr}\ln\mathbb{B}, \quad \text{weakly} \quad \text{in } L^2(0,T;W^{1,2}(\Omega)),
\end{aligned}$$

Then it follows from the continuity of $W'(\cdot)$ and $\operatorname{tr}\ln(\cdot)$ that

$$\begin{aligned}
W'(\phi^\eta) &\to W'(\phi), && \text{a.e.} && \text{in } Q_T,\\
\operatorname{tr}\ln\mathbb{B}^\eta &\to \operatorname{tr}\ln\mathbb{B}, && \text{a.e.} && \text{in } Q_T.
\end{aligned}$$

Again by the uniform boundedness of $\operatorname{tr}\ln\mathbb{B}^\eta$ in $L^2(Q_T)$, one concludes the strong convergence of $\operatorname{tr}\ln\mathbb{B}^\eta \to \operatorname{tr}\ln\mathbb{B}$ in $L^{2-\epsilon}(Q_T)$ for $0 < \epsilon < 1$ by Vitali's convergence theorem. Then in view of Lemma 6.11 and strong convergence of ϕ^η, we have

$$\mathscr{R}_\eta\Big[\frac{\mu_\eta'(\phi)}{2}\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I})\Big] \to \frac{\mu'(\phi)}{2}\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I}), \quad \text{in } L^1(Q_T), \tag{6.80}$$

Consequently, up to a non-relabeled subsequence, we end up with

$$q^\eta \to q = W'(\phi) - \Delta\phi + \frac{\mu'(\phi)}{2}\operatorname{tr}(\mathbb{B} - \ln\mathbb{B} - \mathbb{I}), \quad \text{a.e. in } Q_T,$$

as $\eta \to 0$.

Now, we are in the position to get the compactness of $\mathbf{u}^\eta$, by addressing the problem caused by the variable density $\rho(\phi^\eta)$ with the Helmholtz projection $\mathbb{P}_\sigma$. With the boundedness of $\mathbf{u}^\eta$, ϕ^η, q^η and $\mathbb{B}^\eta$ in two dimensions, one infers

$$\begin{aligned}
\rho(\phi^\eta)\mathbf{u}^\eta \otimes \mathbf{u}^\eta \quad &\text{is bounded in} \quad L^2(0,T;L^2(\Omega;\mathbb{R}^{2\times 2})),\\
\nabla\mathbf{u}^\eta + (\nabla\mathbf{u}^\eta)^\top \quad &\text{is bounded in} \quad L^2(0,T;L^2(\Omega;\mathbb{R}^{2\times 2})),\\
\mathbf{u}^\eta \otimes \nabla q^\eta \quad &\text{is bounded in} \quad L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Omega;\mathbb{R}^{2\times 2})),\\
q^\eta \nabla\phi^\eta \quad &\text{is bounded in} \quad L^2(0,T;L^{\frac{2p}{2+p}}(\Omega;\mathbb{R}^2)),\ 2<p<\infty,\\
\mathscr{R}_\eta[\mu_\eta(\phi^\eta)(\mathbb{B}^\eta - \mathbb{I})] \quad &\text{is bounded in} \quad L^2(0,T;L^p(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}})),\ 2<p<\infty,\\
\mathscr{R}_\eta\Big[\frac{\mu_\eta(\phi^\eta)}{2}\nabla \operatorname{tr}(\mathbb{B}^\eta - \ln\mathbb{B}^\eta - \mathbb{I})\Big] \quad &\text{is bounded in} \quad L^2(0,T;L^2(\Omega;\mathbb{R}^2)),
\end{aligned}$$

which means the all terms above are bounded in $L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Omega))$ (without specifying the dimensions). Then in (6.41), the test function can be restricted to

$$\mathbf{w}, \nabla\mathbf{w} \in [L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Omega))]' = L^4(0,T;L^4(\Omega)).$$

Note that $\mathbf{w}$ lies in the solenoidal space. Hence, with the help of the Leray projection $\mathbb{P}_\sigma$, we conclude that

$$\begin{aligned}
\partial_t\big(\mathbb{P}_\sigma(\rho(\phi^\eta)\mathbf{u}^\eta)\big) \quad &\text{is bounded in} \quad [L^4(0,T;W^{1,4}_{0,\sigma}(\Omega;\mathbb{R}^2))]' = L^{\frac{4}{3}}(0,T;W^{-1,4}_\sigma(\Omega;\mathbb{R}^2)),\\
\mathbb{P}_\sigma(\rho(\phi^\eta)\mathbf{u}^\eta) \quad &\text{is bounded in} \quad L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^2)),
\end{aligned}$$

which together with the Aubin–Lions lemma implies the strong convergence

$$\mathbb{P}_\sigma(\rho(\phi^\eta)\mathbf{u}^\eta) \to \overline{\mathbb{P}_\sigma(\rho(\phi)\mathbf{u})}, \quad \text{strongly} \quad \text{in } L^2(0,T;L^2_\sigma(\Omega;\mathbb{R}^2)), \tag{6.81}$$

for some function $\overline{\mathbb{P}_\sigma(\rho(\phi)\mathbf{u})} \in L^\infty(0,T;L^2_\sigma(\Omega))$. Analogously to Section 6.4.7, we identify $\overline{\mathbb{P}_\sigma(\rho(\phi)\mathbf{u})} = \mathbb{P}_\sigma(\rho(\phi)\mathbf{u})$. Indeed, as $\rho(\phi^\eta)\mathbf{u}^\eta \to \rho(\phi)\mathbf{u}$ weakly in $L^2(0,T;L^2(\Omega;\mathbb{R}^2))$, and by virtue of the weak continuity of the Leray projection $\mathbb{P}_\sigma : L^2(0,T;L^2(\Omega;\mathbb{R}^2)) \to L^2(0,T;L^2_\sigma(\Omega))$, we obtain

$$\overline{\mathbb{P}_\sigma(\rho(\phi)\mathbf{u})} = \mathbb{P}_\sigma(\rho(\phi)\mathbf{u}).$$

Once again, we prove the strong convergence of $\mathbf{u}^\eta$ to $\mathbf{u}$ through the convergence of the kinetic energy. Namely,

$$\begin{aligned}
\int_0^T\int_\Omega \rho(\phi^\eta)\,|\mathbf{u}^\eta|^2 \,\mathrm{d}x\mathrm{d}t &= \int_0^T\int_\Omega \mathbb{P}_\sigma(\rho(\phi^\eta)\mathbf{u}^\eta)\cdot\mathbf{u}^\eta \,\mathrm{d}x\mathrm{d}t\\
&\to \int_0^T\int_\Omega \mathbb{P}_\sigma(\rho(\phi)\mathbf{u})\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega \rho(\phi)\,|\mathbf{u}|^2 \,\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

from which we have

$$\sqrt{\rho(\phi^\eta)}\mathbf{u}^\eta \to \sqrt{\rho(\phi)}\mathbf{u}, \quad \text{strongly} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^2)).$$

Because $\rho(r)$ is affine regarding r, one gets

$$\rho(\phi^\eta) \to \rho(\phi), \quad \text{a.e.} \quad \text{in } Q_T \quad \text{and} \quad |\rho(\phi^\eta)| \geq C > 0.$$

Then it holds that

$$\mathbf{u}^\eta = \frac{1}{\sqrt{\rho(\phi^\eta)}}\sqrt{\rho(\phi^\eta)}\mathbf{u}^\eta \to \frac{1}{\sqrt{\rho(\phi)}}\sqrt{\rho(\phi)}\mathbf{u} = \mathbf{u}, \quad \text{strongly} \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^2)),$$

and hence

$$\mathbf{u}^\eta \to \mathbf{u}, \quad \text{a.e.} \quad \text{in } Q_T.$$

With all the compactness above, one can pass to the limit in (6.41) to (6.16) as $\eta \to 0$, combining with

$$\begin{aligned} m(\phi^\eta) &\to m(\phi), \quad \text{a.e.} \quad \text{in } Q_T,\\ \nu(\phi^\eta) &\to \nu(\phi), \quad \text{a.e.} \quad \text{in } Q_T,\\ \mu(\phi^\eta) &\to \mu(\phi), \quad \text{a.e.} \quad \text{in } Q_T, \end{aligned}$$

which can be deduced by means of the continuity of $m(\cdot), \nu(\cdot), \mu(\cdot)$ and the almost everywhere convergence of ϕ^η. Similarly, we use integration by parts and the compactness of ϕ^η to handle the term $\int_0^t\int_\Omega q^\eta\nabla\phi^\eta\cdot\mathbf{w}\,\mathrm{d}x\mathrm{d}t$.

In the final step of the proof, we derive the energy dissipation inequality (6.19). To this end, multiplying the differential inequality (6.50) by a function $\varsigma \in W^{1,1}(0,T)$ with $\varsigma \geq 0$, $\varsigma(T) = 0$, and integrating the resulting inequality over $(0,T)$ and by parts with respect to the time variable, one obtains

$$\mathcal{E}_\eta(0)\varsigma(0) + \int_0^T \mathcal{E}_\eta(\tau)\varsigma(\tau)'\,\mathrm{d}\tau \geq \int_0^T \mathcal{D}_\eta(\tau)\varsigma(\tau)\,\mathrm{d}\tau, \tag{6.82}$$

where $\mathcal{E}_\eta(t)$ is given in (6.46) and, for a.e. $t \in (0,T)$,

$$\begin{aligned} \mathcal{D}_\eta(t) := &\int_\Omega \Big(\frac{\nu(\phi^\eta)}{2}\left|\nabla\mathbf{u}^\eta + (\nabla\mathbf{u}^\eta)^\top\right|^2 + m(\phi^\eta)\left|\nabla q^\eta\right|^2\Big)\,\mathrm{d}x\\ &+ \int_\Omega \frac{\mu(\phi^\eta)}{2}\operatorname{tr}(\mathbb{B}^\eta + (\mathbb{B}^\eta)^{-1} - 2\mathbb{I})\,\mathrm{d}x + \frac{\kappa}{2}\int_\Omega \left|\nabla\operatorname{tr}\ln\mathbb{B}^\eta\right|^2\,\mathrm{d}x. \end{aligned}$$

Thanks to the compactness of $\mathbf{u}^\eta$, ϕ^η, we deduce that up to a subsequence (not relabeled),

$$\begin{aligned} \mathbf{u}^\eta(t) &\to \mathbf{u}(t), \quad \text{in } L^2_\sigma(\Omega),\\ \phi^\eta(t) &\to \phi(t), \quad \text{in } C(\overline{\Omega}),\\ \mathbb{B}^\eta(t) &\to \mathbb{B}(t), \quad \text{in } L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}}),\\ \int_\Omega \operatorname{tr}\ln\mathbb{B}^\eta(t)\,\mathrm{d}x\,\mathrm{d}\tau &\to \int_\Omega \operatorname{tr}\ln\mathbb{B}(t)\,\mathrm{d}x, \end{aligned}$$

for a.e. $t \in (0,T)$, where the last statement holds true by Vitali's convergence theorem in view of the pointwise convergence of $\operatorname{tr}\ln\mathbb{B}^\eta$ and the uniform boundedness in $L^2(Q_T)$. Therefore, it comes up with

$$\mathcal{E}_\eta(t) \to \mathcal{E}(t), \quad \text{a.e. in } (0,T),$$

where $\mathcal{E}(t)$ is defined in (6.2). In view of the lower semicontinuity of norms and the almost everywhere convergence of ϕ^η to ϕ, one has

$$\liminf_{\eta\to 0}\int_0^T \mathcal{D}_\eta(\tau)\varsigma(\tau)\,\mathrm{d}\tau \geq \int_0^T \mathcal{D}(\tau)\varsigma(\tau)\,\mathrm{d}\tau,$$

for any $\varsigma \in W^{1,1}(0,T)$ with $\varsigma \geq 0$, where

$$\mathcal{D}(t) := \int_\Omega \Big(\frac{\nu(\phi)}{2} \left|\nabla \mathbf{u} + \nabla \mathbf{u}^\top\right|^2 + m(\phi) \left|\nabla q\right|^2 \Big) \, \mathrm{d}x \\ + \int_\Omega \frac{\mu(\phi)}{2} \operatorname{tr}(\mathbb{B} + \mathbb{B}^{-1} - 2\mathbb{I}) \, \mathrm{d}x + \frac{\kappa}{2} \int_\Omega \left|\nabla \operatorname{tr} \ln \mathbb{B}\right|^2 \, \mathrm{d}x.$$

Note that here we employed the positivity of $m(\cdot)$, $\nu(\cdot)$ and $\mu(\cdot)$, and

$$\liminf_{\eta \to 0} \int_0^T \eta \left\|\partial_t \phi^\eta(\tau)\right\|_{L^2}^2 \varsigma(\tau) \mathrm{d}\tau \geq 0.$$

Passing to the limit in (6.82) as $\eta \to 0$ yields

$$\mathcal{E}(0)\varsigma(0) + \int_0^T \mathcal{E}(\tau)\varsigma'(\tau) \, \mathrm{d}\tau \geq \int_0^T \mathcal{D}(\tau)\varsigma(\tau) \, \mathrm{d}\tau,$$

for any $\varsigma \in W^{1,1}(0,T)$ with $\varsigma \geq 0$ and $\varsigma(T) = 0$. By virtue of Lemma 6.8, we get the desired energy dissipation inequality (6.19). The additional stronger estimate of $\mathbb{B}$ can be obtained directly from (6.78) together with (6.19).

This finishes the proof. □

Remark 6.41. The case of a general free energy

$$\int_\Omega \Big(\frac{a(\phi)}{2} \left|\nabla \phi\right|^2 + W(\phi) \Big) \, \mathrm{d}x$$

with some positive coefficient $a(\phi)$ can be achieved by our method with slight modifications. Note that the two-phase incompressible flow with different densities and general free energy was already addressed in [ADG13a]. In our framework, one only needs to adopt a more complicated subgradient with respect to a reparametrized potential as in [ADG13a] during the proof of the existence of solutions to the regularized problem in Section 6.4.

Remark 6.42. Our method is capable to deal with other coupled systems, e.g., polymer fluids with a Fokker–Planck type equation or Magnetohydrodynamics fluids. For these, by modifying a little bit of the regularization in the full system and later on by the same argument, one is able to obtain the existence of weak solutions in a nontrival but easier way.

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