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Capillarity-driven Stokes flow: the one-phase problem as small viscosity limit

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Abstract. We consider the quasistationary Stokes flow that describes the motion of a two-dimensional fluid body under the influence of surface tension effects in an unbounded, infinite-bottom geometry. We reformulate the problem as a fully nonlinear parabolic evolution problem for the function that parameterizes the boundary of the fluid with the nonlinearities expressed in terms of singular integrals. We prove well-posedness of the problem in the subcritical Sobolev spaces $H^s(\mathbb{R})$ up to critical regularity, and establish parabolic smoothing properties for the solutions. Moreover, we identify the problem as the singular limit of the two-phase quasistationary Stokes flow when the viscosity of one of the fluids vanishes.

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1. Introduction

In this paper we consider the two-dimensional flow of a fluid layer $\Omega(t)$ of infinite depth in the case when the motion of the incompressible fluid is governed by the quasistationary Stokes equations and the motion is driven by surface tension at the free boundary $\Gamma(t) = \partial\Omega(t)$. We consider the one-phase problem, i.e., no forces are exerted on the liquid by the medium above it. The mathematical model is given by the following system of equations

$$\left.\begin{array}{rl} \mu\Delta v - \nabla p = 0 & \text{in } \Omega(t), \\ \operatorname{div} v = 0 & \text{in } \Omega(t), \\ T_\mu(v,p)\tilde{\nu} = \sigma\tilde{\kappa}\tilde{\nu} & \text{on } \Gamma(t), \\ (v,p)(x) \to 0 & \text{for } |x| \to \infty, \\ V_n = v \cdot \tilde{\nu} & \text{on } \Gamma(t) \end{array}\right\} \tag{1.1a}$$

for $t > 0$, where the interface $\Gamma(t)$ at time t is given as a graph of a function $f(t,\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$, i.e., the fluid domain $\Omega(t)$ and its boundary $\Gamma(t)$ are defined by

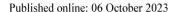$$\Omega(t) := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 < f(t, x_1)\},$$

$$\Gamma(t) := \partial\Omega(t) := \{(\xi, f(t,\xi)) : \xi \in \mathbb{R}\}.$$

Additionally, the interface $\Gamma(t)$ is assumed to be known at time $t = 0$, i.e.

$$f(0, \cdot) = f^{(0)}. \tag{1.1b}$$

In Eq. (1.1a) above, $v = v(t) : \Omega(t) \longrightarrow \mathbb{R}^2$ and $p = p(t) : \Omega(t) \longrightarrow \mathbb{R}$ are the velocity and the pressure of the Newtonian fluid, $\tilde{\nu} = (\tilde{\nu}^1, \tilde{\nu}^2)$ is the unit exterior normal to $\partial\Omega$, $\tilde{\kappa}$ denotes the curvature of the

interface (negative where $\Omega(t)$ is convex), and $T_\mu(v, p) = (T_{\mu,ij}(v, p))_{1 \leq i,\, j \leq 2}$ is the stress tensor which is given by

$$T_\mu(v, p) := -pE_2 + \mu\big[\nabla v + (\nabla v)^\top\big], \quad (\nabla v)_{ij} := \partial_j v_i. \tag{1.2}$$

Moreover, V_n is the normal velocity of the interface $\Gamma(t)$, $a \cdot b$ denotes the Euclidean scalar product of two vectors a, $b \in \mathbb{R}^2$, $E_2 \in \mathbb{R}^{2 \times 2}$ is the identity matrix, and the positive constants μ and σ are the dynamic viscosity of the fluid and the surface tension coefficient at the interface $\Gamma(t)$, respectively.

Previous analysis related to (1.1a) considered mainly the case of a sufficiently regular bounded fluid domain $\Omega(t)$. More precisely, in [14] the authors studied the quasistationary motion of a free capillary liquid drop in $\mathbb{R}^d$ for initial data in $H^{s+1}(\Sigma)$, $s \geq s_1$, s_1 being the smallest integer that satisfies $s_1 > 3 + (d-1)/2$ and $\Sigma \subset \mathbb{R}^d$ the smooth boundary of a strictly star shaped domain in $\mathbb{R}^d$. The authors established in [14] the well-posedness of the problem and they also showed that the equilibria of the problem, which are balls, are exponentially stable. In the context when Σ is the boundary of the unit ball, it is proven in [9] (for $d = 2$) that solutions corresponding to small data in $H^5(\Sigma)$ exist globally and converge to a ball, while in [8] (for $d = 3$) the authors established the same result for small data in $H^6(\Sigma)$. Finally, in three space dimensions and for an initially bounded geometry possessing a $C^{3+\alpha}$-boundary, $\alpha > 0$, it was shown in [27] that the quasistationary Stokes flow is well-posed, and in [28] the same author has rigorously justified this problem as the singular limit of Navier–Stokes flow when the Reynolds number vanishes. The local well-posedness and the stability issue for the two-phase quasistationary Stokes flow (with or without phase transitions) in a bounded geometry in $\mathbb{R}^d$, with $d \geq 2$, has been recently studied in [26] in the phase space $W_p^{2+\mu-2/p}(\Sigma)$, with $1 \geq \mu > (d+2)/p$, by using a maximal L_p-regularity approach. In the context of [26], Σ is a real analytic hypersurface over which the boundary between the two fluid phases is parameterized.

Local well-posedness and stability results for the quasistationary two-phase spatially periodic Stokes flow in $\mathbb{R}^2$ with gravity effects and zero surface tension were established only recently in [11]. We emphasize that in contrast to the present analysis, where we deal with a parabolic problem, see Sect. 4, the problem considered in [11] is of hyperbolic type. A further related and very recent direction of research, see, e.g., [13,17,24], is focused on the study of the transport Stokes system which was derived in [15,23] as a model for the sedimentation of a cloud of rigid particles in a viscous fluid. In this context the inhomogeneous density is transported by the velocity field which solves the gravity-driven Stokes boundary value problem in the entire space $\mathbb{R}^3$. At a formal level the transport Stokes system in $\mathbb{R}^2$ with initial density constant above and below a graph is equivalent to the quasistationary two-phase Stokes flow in $\mathbb{R}^2$ with gravity effects and zero surface tension (in the case of two fluids with equal viscosities).

We emphasize that in the references [8,9,14,26–28] the moving interface is at least of class C^2, whereas the critical L_2-Sobolev space for (1.1) is $H^{3/2}(\mathbb{R})$, see [21,22]. Our goal is to establish the well-posedness of (1.1) in the subcritical spaces $H^s(\mathbb{R})$ with $s \in (3/2, 2)$, see Theorem 1.1. One of the obvious difficulties lies in the fact that, for $f \in H^s(\mathbb{R})$, the curvature term in (1.1a)$_3$ is merely a distribution. To handle this issue we use a strategy inspired by the approach in the papers [4,21,22] where, for the corresponding two-phase problem, potential theory was used to determine the velocity and pressure fields in terms of f. Such a strategy was applied also in the context of the Muskat problem, see the surveys [10,12], and it provides quite optimal results as the mathematical reformulations of the problems obtained by using this strategy require less smallness and regularity assumptions on the data compared to other approaches based on Lagrangian or Hanzawa transformations.

The first goal of this paper is to show that, at each time instant $t > 0$, the free boundary, given via $f = f(t)$, identifies the velocity field $v = v(t)$ and the pressure $p = p(t)$ uniquely. More precisely, as shown in Theorem 2.1, if $f \in H^3(\mathbb{R})$, then (v, p) is given by the hydrodynamic single-layer potential with a density $\beta = (\beta_1, \beta_2)^\top$ which satisfies

$$\left(\frac{1}{2} - \mathbb{D}(f)^*\right)[\beta'] = \sigma\left(-\left(\frac{f'^2}{\omega + \omega^2}\right)', \left(\frac{f'}{\omega}\right)'\right)^\top = \sigma g', \tag{1.3}$$

where $(\cdot)'$ is the derivative with respect to the spatial coordinate $\xi \in \mathbb{R}$ and $g = g(f)$ is defined in (2.4) below. Furthermore, $\mathbb{D}(f)^*$ is the L_2-adjoint of the double layer potential $\mathbb{D}(f)$, see (2.8), and $\omega := (1 + f'^2)^{1/2}$, see (2.2). Concerning (1.3), the following issues need to be clarified:

  (i) The invertibility of the operators $\pm 1/2 - \mathbb{D}(f)$ (and $\pm 1/2 - \mathbb{D}(f)^*$) in $\mathcal{L}(H^1(\mathbb{R})^2)$;
  (ii) the question whether $(\frac{1}{2} - \mathbb{D}(f)^*)^{-1} g'$ is the derivative of some $\beta \in (H^2(\mathbb{R}))^2$.

We remark that these issues are new compared to the treatment of the two-phase problem.

With respect to (i), the main step is performed in Theorem 3.1 where the invertibility in $\mathcal{L}(L_2(\mathbb{R})^2)$ is established for each $f \in \mathrm{BUC}^1(\mathbb{R})$. At this point, we rely on the Rellich identities (3.12)–(3.16) for the Stokes boundary value problem which have been exploited, in a bounded geometry in $\mathbb{R}^n$ with $n \geq 3$, also in [7]. In the unbounded two-dimensional setting considered in the present paper we provide new arguments which use, among others, also a Rellich identity obtained in [19] in the context of the Muskat problem. Based on Theorem 3.1, we then show that these operators are invertible in $\mathcal{L}(H^k(\mathbb{R})^2)$, $k = 1, 2$, provided that $f \in H^{k+1}(\mathbb{R})$, see Lemma 3.3, and in $\mathcal{L}(H^{s-1}(\mathbb{R})^2)$ when $f \in H^s(\mathbb{R})$, see Lemma 3.4.

Concerning (ii), we prove in Lemma 4.1 that, given $f \in H^s(\mathbb{R})$ and $\beta \in H^1(\mathbb{R})^2$, the function $\mathbb{D}(f)[\beta]$ belongs to $H^1(\mathbb{R})^2$ and

$$(\mathbb{D}(f)[\beta])' = -\mathbb{D}(f)^*[\beta']. \tag{1.4}$$

This relation and the observation that the right side of (1.3) is a derivative enables us to essentially replace, for $f \in H^3(\mathbb{R})$, Eq. (1.3) by

$$\left(\frac{1}{2} + \mathbb{D}(f)\right)[\beta] = \sigma g, \tag{1.5}$$

see Corollary 4.3. These properties, in particular Lemma 3.3 and the equivalence of (1.3) and (1.5), are then used to reformulate the one-phase Stokes flow (1.1) as the evolution problem (4.9), which has only f as unknown. Its well-posedness properties are summarized in Theorem 1.1 below.

Our second main result concerns the limit behavior for $\mu^+ \to 0$ of the two-phase quasistationary Stokes problem

$$\left.\begin{aligned}
\mu^\pm \Delta w^\pm - \nabla q^\pm &= 0 && \text{in } \Omega^\pm(t), \\
\operatorname{div} w^\pm &= 0 && \text{in } \Omega^\pm(t), \\
[w] &= 0 && \text{on } \Gamma(t), \\
[T_\mu(w, q)]\tilde{\nu} &= -\sigma \tilde{\kappa} \tilde{\nu} && \text{on } \Gamma(t), \\
(w^\pm, q^\pm)(x) &\to 0 && \text{for } |x| \to \infty, \\
V_n &= w^\pm \cdot \tilde{\nu} && \text{on } \Gamma(t)
\end{aligned}\right\} \tag{1.6a}$$

for $t > 0$ and

$$f(0, \cdot) = f^{(0)}, \tag{1.6b}$$

with $\mu^- = \mu$ fixed. In (1.6a) it is again assumed that $\Gamma(t)$ is the graph of a function $f(t, \cdot)$,

$$\Omega^\pm(t) := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 \gtrless f(t, x_1)\},$$

and $\tilde{\nu}$ is the unit exterior normal to $\partial \Omega^-(t)$. Moreover, $w^\pm(t)$ and $q^\pm(t)$ represent the velocity and pressure fields in $\Omega^\pm(t)$, respectively, and $[w]$ (respectively $[T_\mu(w, q)]$) is the jump of the velocity (respectively stress tensor) across the moving interface, see (2.14) below. We emphasize that the limit $\mu^+ \to 0$ in the formulation (1.6) is singular because ellipticity of the underlying boundary value problem is lost in this limit.

In [22], we reformulated the two-phase Stokes problem (1.6) as a nonlinear evolution equation for f, see (4.10) below. In Sects. 4.3 and 4.4 of the present paper we prove that the right side of (4.10) has a limit for $\mu^+ \to 0$, and the limit is the right side of (4.9). In this sense, we show that the moving boundary

problem (1.1) represents the "regular limit" of (1.6) for $\mu^- = \mu$ and $\mu^+ \to 0$. This property is used in Sect. 4.4 to introduce the common formulation (4.23) that contains both evolution problems. It reads

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \Phi(\mu^+, f), \quad t \geq 0, \qquad f(0) = f^{(0)}, \tag{1.7}$$

where $\mu^+ \geq 0$ is viewed as a parameter. We point out that though this common formulation has been derived from the Stokes flow equations under the assumption that $f(t) \in H^3(\mathbb{R})$, the nonlinear and nonlocal operator Φ is well-defined when assuming only $f \in H^s(\mathbb{R})$, $s \in (3/2, 2)$, and this allows us to consider (1.7) under these lower smoothness assumptions. The regularity of the limit is now seen in the fact that Φ is smooth on $[0, \infty) \times H^s(\mathbb{R})$. For any fixed $\mu^+ > 0$ we investigated the problem (1.7) in [22]. In particular, we showed in [22, Theorem 1.1] that, given $f^{(0)} \in H^s(\mathbb{R})$, there exists a unique maximal solution $(f_{\mu^+}, w^\pm_{\mu^+}, q^\pm_{\mu^+})$ to (1.6) such that

- $f_{\mu^+} = f_{\mu^+}(\cdot, f^{(0)}) \in \mathrm{C}([0, T_{+,\mu^+}), H^s(\mathbb{R})) \cap \mathrm{C}^1([0, T_{+,\mu^+}), H^{s-1}(\mathbb{R}))$,
- $w^\pm_{\mu^+}(t) \in \mathrm{C}^2(\Omega^\pm(t)) \cap \mathrm{C}^1(\overline{\Omega^\pm(t)})$, $q^\pm_{\mu^+}(t) \in \mathrm{C}^1(\Omega^\pm(t)) \cap \mathrm{C}(\overline{\Omega^\pm(t)})$ for all $t \in (0, T_{+,\mu^+})$,
- $w^\pm_{\mu^+}(t)|_{\Gamma(t)} \circ \Xi_{f(t)} \in H^2(\mathbb{R})^2$ for all $t \in (0, T_{+,\mu^+})$,

where $T_{+,\mu^+} = T_{+,\mu^+}(f^{(0)}) \in (0, \infty]$ is the maximal existence time and $\Xi_{f(t)}(\xi) := (\xi, f(t, \xi))$ for $\xi \in \mathbb{R}$.

Our first main result is based on the fact that the properties of $\Phi(\mu^+, \cdot)$ and of its Fréchet derivative $\partial_f \Phi(\mu^+, \cdot)$ which were used to prove [22, Theorem 1.1] remain valid also when $\mu^+ = 0$.

Theorem 1.1. *Let $s \in (3/2, 2)$ be given. Then, the following statements hold true:*

(i) *(Well-posedness) Given $f^{(0)} \in H^s(\mathbb{R})$, there exists a unique maximal solution (f, v, p) to (1.1) such that*
  - $f = f(\cdot; f^{(0)}) \in \mathrm{C}([0, T_+), H^s(\mathbb{R})) \cap \mathrm{C}^1([0, T_+), H^{s-1}(\mathbb{R}))$,
  - $v(t) \in \mathrm{C}^2(\Omega(t)) \cap \mathrm{C}^1(\overline{\Omega(t)})$, $p(t) \in \mathrm{C}^1(\Omega(t)) \cap \mathrm{C}(\overline{\Omega(t)})$ *for all $t \in (0, T_+)$,*
  - $v(t)|_{\Gamma(t)} \circ \Xi_{f(t)} \in H^2(\mathbb{R})^2$ *for all $t \in (0, T_+)$,*

  where $T_+ = T_+(f^{(0)}) \in (0, \infty]$ is the maximal existence time. Moreover, the set

  $$\mathcal{M} := \{(t, f^{(0)}) : f^{(0)} \in H^s(\mathbb{R}), 0 < t < T_+(f^{(0)})\}$$

  is open in $(0, \infty) \times H^s(\mathbb{R})$, and $[(t, f^{(0)}) \longmapsto f(t; f^{(0)})]$ is a semiflow on $H^s(\mathbb{R})$ which is smooth in $\mathcal{M}$.

(ii) *(Parabolic smoothing)*
  (iia) *The map $[(t, \xi) \longmapsto f(t, \xi)] : (0, T_+) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a C^∞-function.*
  (iib) *For any $k \in \mathbb{N}$, we have $f \in \mathrm{C}^\infty((0, T_+), H^k(\mathbb{R}))$.*

(iii) *(Global existence) If*

  $$\sup_{[0,T] \cap [0, T_+(f^{(0)}))} \|f(t)\|_{H^s} < \infty$$

  for each $T > 0$, then $T_+(f^{(0)}) = \infty$.

Observe, in particular, that by Theorem 1.1 (iib) we have $f(t) \in H^3(\mathbb{R})$ for $t > 0$, which justifies the assumptions that were made when deriving the reformulation (4.9). Thus, the solutions we construct correspond to one-phase Stokes flows, starting from initial domains whose boundaries might have a curvature in distribution sense only.

Our second main result gives a precise formulation of the limit result announced above. We point out that Theorem 1.2 is the first result in the literature which addresses the vanishing viscosity limit in the two-phase quasistationary Stokes flow. We recall the notation $(f_{\mu^+}(\cdot; f^{(0)}), w^\pm_{\mu^+}, q^\pm_{\mu^+})$ for solutions to the two-phase problem (1.6).

Theorem 1.2. *Let $s \in (3/2, 2)$ and $f^{(0)} \in H^s(\mathbb{R})$ be given. Let further $(f(\cdot; f^{(0)}), v, p)$ denote the maximal solution to (1.1) identified in Theorem 1.1 and choose $T \in (0, T_+(f^{(0)}))$. Then, there exist constants $\varepsilon > 0$ and $M > 0$ such that for all $\mu^+ \in (0, \varepsilon]$ we have $T < T_{+,\mu^+}(f^{(0)})$ and*

$$\left\| f(\cdot; f^{(0)}) - f_{\mu^+}(\cdot; f^{(0)}) \right\|_{C([0,T], H^s(\mathbb{R}))} + \left\| \frac{\mathrm{d}}{\mathrm{d}t} \left(f(\cdot; f^{(0)}) - f_{\mu^+}(\cdot; f^{(0)}) \right) \right\|_{C([0,T], H^{s-1}(\mathbb{R}))} \leq M \mu^+.$$

The proofs of the main results are presented in Sect. 4.5 which concludes the paper.

2. The Stokes boundary value problem in a fixed domain

In this section we fix $f \in H^3(\mathbb{R})$ and we consider the Stokes boundary value problem

$$\left. \begin{array}{rcll} \mu \Delta v - \nabla p &=& 0 & \text{in } \Omega, \\ \operatorname{div} v &=& 0 & \text{in } \Omega, \\ T_\mu(v, p)\tilde{\nu} &=& \sigma \tilde{\kappa} \tilde{\nu} & \text{on } \Gamma, \\ (v, p)(x) &\to& 0 & \text{for } |x| \to \infty, \end{array} \right\} \tag{2.1}$$

where $\Omega := \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 < f(x_1)\}$ and $\Gamma := \{(\xi, f(\xi)) \,:\, \xi \in \mathbb{R}\}$. The main goal is to show that (2.1) has a unique solution (v, p), see Theorem 2.1 below.

We start by introducing some notation. Since Γ is a graph over $\mathbb{R}$, it is natural to view Γ as is the image of $\mathbb{R}$ under the diffeomorphism $\Xi := \Xi_f := (\mathrm{id}_\mathbb{R}, f)$. Let now ν and τ denote the componentwise pull-back under Ξ of the unit normal $\tilde{\nu}$ on Γ exterior to Ω and of the unit tangent vector $\tilde{\tau}$ to Γ, i.e.

$$\nu := \nu(f) := \frac{1}{\omega}(-f', 1)^\top, \qquad \tau := \tau(f) := \frac{1}{\omega}(1, f')^\top, \qquad \omega := \omega(f) := (1 + f'^2)^{1/2}. \tag{2.2}$$

Observe that the pull-back $\kappa := \omega^{-3} f'' \in H^1(\mathbb{R})$ of the curvature $\tilde{\kappa}$ satisfies

$$\omega \kappa \nu = g', \tag{2.3}$$

where $g := g(f)$ is given by

$$g := (g_1, g_2)^\top := (\omega^{-1} - 1, \omega^{-1} f')^\top = \left(-\frac{f'^2}{\omega + \omega^2}, \frac{f'}{\omega} \right)^\top. \tag{2.4}$$

We further recall that the fundamental solutions $(\mathcal{U}^k, \mathcal{P}^k) : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2 \times \mathbb{R}$, $k = 1, 2$, where $\mathcal{U}^k = (\mathcal{U}_1^k, \mathcal{U}_2^k)^\top$, to the Stokes equations

$$\left. \begin{array}{rcl} \mu \Delta U - \nabla P &=& 0, \\ \operatorname{div} U &=& 0 \end{array} \right\} \qquad \text{in } \mathbb{R}^2 \setminus \{0\} \tag{2.5}$$

are given by

$$\mathcal{U}_j^k(y) = -\frac{1}{4\pi\mu} \left(\delta_{jk} \ln \frac{1}{|y|} + \frac{y_j y_k}{|y|^2} \right), \quad j = 1, 2, \tag{2.6}$$

$$\mathcal{P}^k(y) = -\frac{1}{2\pi} \frac{y_k}{|y|^2}$$

for $y = (y_1, y_2) \in \mathbb{R}^2 \setminus \{0\}$, see [16].

Finally, defining the mapping $r := (r^1, r^2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by the formula

$$r := r(\xi, s) := (\xi - s, f(\xi) - f(s)), \quad (\xi, s) \in \mathbb{R}^2, \tag{2.7}$$

we introduce the double-layer potential $\mathbb{D}(f)$ for the Stokes equations associated to the hypersurface Γ and its L_2-adjoint $\mathbb{D}(f)^*$ by the formulas

$$\mathbb{D}(f)[\beta](\xi) := \frac{1}{\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{r_1 f' - r_2}{|r|^4} \begin{pmatrix} r_1^2 & r_1 r_2 \\ r_1 r_2 & r_2^2 \end{pmatrix} \beta \, \mathrm{d}s,$$

$$\mathbb{D}(f)^*[\beta](\xi) := \frac{1}{\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{-r_1 f'(\xi) + r_2}{|r|^4} \begin{pmatrix} r_1^2 & r_1 r_2 \\ r_1 r_2 & r_2^2 \end{pmatrix} \beta \, \mathrm{d}s \tag{2.8}$$

for $\beta = (\beta_1, \beta_2)^\top \in L_2(\mathbb{R})^2$ and $\xi \in \mathbb{R}$.

In (2.8), the integrals are absolutely convergent whenever f' is Hölder continuous. We prefer the definition as principal value integral because we will consider $f \in \mathrm{BUC}^1(\mathbb{R})$ later. Given $k \in \mathbb{N}$, $\mathrm{BUC}^k(\mathbb{R})$ is the Banach space consisting of functions with bounded and uniformly continuous derivatives up to order k. It is well-known that the intersection of all these spaces, denoted by $\mathrm{BUC}^\infty(\mathbb{R})$, is a dense subspace of $\mathrm{BUC}^k(\mathbb{R})$ for each $k \in \mathbb{N}$.

Both operators $\mathbb{D}(f)$, $\mathbb{D}(f)^*$ can be expressed in terms of the family $\{B_{n,m}^0(f) : n, m \in \mathbb{N}\}$ of singular integral operators introduced in [19,20], see (2.10) and (3.2) below. We now introduce these operators in a more general context. More precisely, given $n, m \in \mathbb{N}$ and Lipschitz continuous functions $a_1, \ldots, a_m, b_1, \ldots, b_n : \mathbb{R} \longrightarrow \mathbb{R}$, we define

$$B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, h](\xi) := \frac{1}{\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{h(\xi - \eta)}{\eta} \frac{\prod_{i=1}^n \left(\delta_{[\xi,\eta]} b_i / \eta \right)}{\prod_{i=1}^m \left[1 + \left(\delta_{[\xi,\eta]} a_i / \eta \right)^2 \right]} \, d\eta, \tag{2.9}$$

where $\delta_{[\xi,\eta]} u := u(\xi) - u(\xi - \eta)$. For brevity we set

$$B_{n,m}^0(f) := B_{n,m}(f, \ldots, f)[f, \ldots, f, \cdot]. \tag{2.10}$$

We note that $B_{0,0}^0 = H$, where H denotes the Hilbert transform.

We now prove that the boundary value problem (2.1) is uniquely solvable and that the solution is given by the hydrodynamic single-layer potential with a suitable density function β.

Theorem 2.1. *Given $f \in H^3(\mathbb{R})$, the boundary value problem* (2.1) *has a unique solution (v, p) such that*

$$v \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^1(\overline{\Omega}), \quad p \in \mathrm{C}^1(\Omega) \cap \mathrm{C}(\overline{\Omega}), \quad v|_\Gamma \circ \Xi \in H^2(\mathbb{R})^2.$$

Moreover, letting $\beta = (\beta_1, \beta_2)^\top \in H^2(\mathbb{R})^2$ denote the unique solution to the equation

$$\left(\frac{1}{2} - \mathbb{D}(f)^* \right)[\beta'] = \sigma g', \tag{2.11}$$

where $g \in H^2(\mathbb{R})^2$ is defined in (2.4), *we have*

$$v(x) := \int_{\mathbb{R}} \partial_s \big(\mathcal{U}^k(x - (s, f(s))) \big) \beta_k(s) \, \mathrm{d}s,$$

$$p(x) := -\int_{\mathbb{R}} \mathcal{P}^k(x - (s, f(s))) \beta_k'(s) \, \mathrm{d}s = \int_{\mathbb{R}} \partial_s \big(\mathcal{P}^k(x - (s, f(s))) \big) \beta_k(s) \, \mathrm{d}s, \quad x \in \Omega. \tag{2.12}$$

Proof. The unique solvability of Eq. (2.11) is established in Sect. 4 and is taken for granted in this proof. We divide the proof in two steps.

Step 1: Uniqueness. It suffices to show that the homogeneous boundary value problem (2.1) (with the right side of (2.1)$_3$ set to be zero) has only the trivial solution. Let thus (v, p) be a solution to the homogeneous system (2.1) with regularity as prescribed above. We then set $\Omega^- := \Omega$, $\Omega^+ := \mathbb{R}^2 \setminus \overline{\Omega}$, and we define $(w^\pm, q^\pm) : \Omega^\pm \longrightarrow \mathbb{R}^2 \times \mathbb{R}$ by

$$(w^-, w^+) := (\mu v, 0) \qquad \text{and} \qquad (q^-, q^+) := (p, 0).$$

Clearly, it holds

$$w^{\pm} \in C^2(\Omega^{\pm}) \cap C^1(\overline{\Omega^{\pm}}), \quad q^{\pm} \in C^1(\Omega^{\pm}) \cap C(\overline{\Omega^{\pm}}), \quad w^{\pm}|_{\Gamma} \circ \Xi \in H^2(\mathbb{R})^2.$$

Moreover, it can be easily checked that $(w^{\pm}, q^{\pm})$ solves the boundary value problem

$$\left.\begin{aligned}
\Delta w^{\pm} - \nabla q^{\pm} &= 0 &&\text{in } \Omega^{\pm}, \\
\operatorname{div} w^{\pm} &= 0 &&\text{in } \Omega^{\pm}, \\
[w] &= \gamma \circ \Xi^{-1} &&\text{on } \Gamma, \\
[T_1(w,q)](\nu \circ \Xi^{-1}) &= 0 &&\text{on } \Gamma, \\
(w^{\pm}, q^{\pm})(x) &\to 0 &&\text{for } |x| \to \infty,
\end{aligned}\right\}
\tag{2.13}$$

with $\gamma := -w^-|_{\Gamma} \circ \Xi \in H^2(\mathbb{R})^2$. Given $z^{\pm} \in C(\overline{\Omega^{\pm}})$, we define $[z]$ as being the jump

$$[z](x) := z^+(x) - z^-(x), \qquad x \in \Gamma. \tag{2.14}$$

According to [22, Proposition 2.1], the system (2.13) has a unique solution. Moreover, we have

$$w^{\pm}|_{\Gamma} \circ \Xi = \left(\pm \frac{1}{2} - \mathbb{D}(f) \right)[\gamma], \tag{2.15}$$

see [22, Lemma A.1]. Since $w^+ = 0$ and $1/2 - \mathbb{D}(f) \in \mathcal{L}(L_2(\mathbb{R})^2)$ is invertible, see Theorem 3.1 below, we conclude that $\gamma = 0$. Consequently $(w^{\pm}, q^{\pm})$ is the trivial solution and this establishes the uniqueness claim.

Step 2: Existence. We are going to verify that (v, p) from (2.12) with β from (2.11) has the announced regularity and satisfies (2.1). Recalling (2.6), we have

$$\partial_1 \mathcal{U}^1(y) = \frac{1}{4\pi\mu|y|^4} \begin{pmatrix} y_1(y_1^2 - y_2^2) \\ y_2(y_1^2 - y_2^2) \end{pmatrix}, \quad \partial_1 \mathcal{P}^1(y) = \frac{y_1^2 - y_2^2}{2\pi|y|^4},$$

$$\partial_2 \mathcal{U}^1(y) = \frac{1}{4\pi\mu|y|^4} \begin{pmatrix} y_2(y_2^2 + 3y_1^2) \\ y_1(y_2^2 - y_1^2) \end{pmatrix}, \quad \partial_2 \mathcal{P}^1(y) = \frac{2y_1 y_2}{2\pi|y|^4},$$

$$\tag{2.16}$$

$$\partial_1 \mathcal{U}^2(y) = \frac{1}{4\pi\mu|y|^4} \begin{pmatrix} y_2(y_1^2 - y_2^2) \\ y_1(y_1^2 + 3y_2^2) \end{pmatrix}, \quad \partial_1 \mathcal{P}^2(y) = \frac{2y_1 y_2}{2\pi|y|^4},$$

$$\partial_2 \mathcal{U}^2(y) = \frac{1}{4\pi\mu|y|^4} \begin{pmatrix} y_1(y_2^2 - y_1^2) \\ y_2(y_2^2 - y_1^2) \end{pmatrix}, \quad \partial_2 \mathcal{P}^2(y) = \frac{y_2^2 - y_1^2}{2\pi|y|^4}$$

for $y \neq 0$. A direct consequence of (2.12) is that (v, p) is defined as an integral of the form

$$(v, p)(x) = \int_{\mathbb{R}} K(x, s) \beta(s) \, \mathrm{d}s, \qquad x \in \Omega,$$

where, for every $\alpha \in \mathbb{N}^2$, we have $\partial_x^{\alpha} K(x, s) = O(s^{-1})$ for $|s| \to \infty$ and locally uniformly in $x \in \Omega$. This shows that v and p are well-defined by (2.12), and that integration and differentiation with respect to x may be interchanged. Hence, $(v, p) \in C^{\infty}(\Omega, \mathbb{R}^2 \times \mathbb{R})$, and, since $\partial_j(\mathcal{U}^k, \mathcal{P}^k)$, $j = 1, 2$, solve (2.5), we deduce that (v, p) is a solution to $(2.1)_1$–$(2.1)_2$.

In view of [5, Lemma A.1] it holds that $p \in C(\overline{\Omega})$ and

$$p|_{\Gamma} \circ \Xi = \frac{B_{0,1}^0(f)[\beta_1'] + B_{1,1}^0(f)[\beta_2']}{2} - \frac{\beta' \cdot \nu}{2\omega}. \tag{2.17}$$

Given $\phi \in H^1(\mathbb{R})$, let $Z_j[\phi] : \Omega \longrightarrow \mathbb{R}$, $j = 0, \dots, 3$, be given by

$$Z_j[\phi](x) := \int_{\mathbb{R}} \frac{(x_1 - s)^{3-j}(x_2 - f(s))^j}{((x_1 - s)^2 + (x_2 - f(s))^2)^2} \phi(s) \, \mathrm{d}s, \quad x \in \Omega.$$

Since

$$\partial_i v(x) = -\int_{\mathbb{R}} \partial_i \mathcal{U}^k(x-(s,f(s)))\beta_k'(s)\,\mathrm{d}s, \qquad i=1,2,\ x\in\Omega,$$

we obtain, due to (2.16), the following formulas

$$\partial_1 v_1 = -\partial_2 v_2 = -\frac{(Z_0-Z_2)[\beta_1'] + (Z_1-Z_3)[\beta_2']}{4\pi\mu},$$

$$\partial_2 v_1 = -\frac{(Z_3+3Z_1)[\beta_1'] + (Z_2-Z_0)[\beta_2']}{4\pi\mu},$$

$$\partial_1 v_2 = -\frac{(Z_1-Z_3)[\beta_1'] + (Z_0+3Z_2)[\beta_2']}{4\pi\mu}.$$

Since $Z_j[\phi] \in \mathrm{C}(\overline{\Omega})$, see the proof of [21, Lemma A.1], we obtain in view of the latter identities that $v \in \mathrm{C}^1(\overline{\Omega})$. Moreover, the formula derived in the proof of [21, Lemma A.1] for the traces $Z_j[\phi]|_\Gamma$, $j=0,\ldots,3$, leads us to

$$\partial_1 v_1|_\Gamma \circ \Xi = -\frac{(B_{0,2}^0(f) - B_{2,2}^0(f))[\beta_1'] + (B_{1,2}^0(f) - B_{3,2}^0(f))[\beta_2']}{4\mu} - \frac{f'\beta'\cdot\tau}{2\mu\omega^3},$$

$$\partial_2 v_1|_\Gamma \circ \Xi = -\frac{(B_{3,2}^0(f) + 3B_{1,2}^0(f))[\beta_1'] + (B_{2,2}^0(f) - B_{0,2}^0(f))[\beta_2']}{4\mu} + \frac{\beta'\cdot\tau}{2\mu\omega^3}, \qquad (2.18)$$

$$\partial_1 v_2|_\Gamma \circ \Xi = -\frac{(B_{1,2}^0(f) - B_{3,2}^0(f))[\beta_1'] + (B_{0,2}^0(f) + 3B_{2,2}^0(f))[\beta_2']}{4\mu} - \frac{f'^2\beta'\cdot\tau}{2\mu\omega^3}.$$

It is now a matter of direct computation to infer from (1.2), (2.3), (2.8), (2.17), and (2.18) that the equation $(2.1)_3$ is equivalent to (2.11), hence also $(2.1)_3$ is satisfied. It remains to check that the far field boundary condition $(2.1)_4$ holds true. To this end we infer directly from [5, Lemma A.4] and $(2.12)_2$ that p vanishes at infinity. Moreover, since by $(2.12)_1$ we have

$$v(x) = \frac{1}{4\pi\mu}\int_{\mathbb{R}} \frac{1}{|R|^2}\begin{pmatrix} -R_2^2 & R_1 R_2 \\ R_1 R_2 & R_2^2 \end{pmatrix}\beta'(s)\,\mathrm{d}s - \frac{1}{4\pi\mu}\int_{\mathbb{R}} \frac{R_1 + f'(s)R_2}{|R|^2}\beta(s)\,\mathrm{d}s$$

for $x\in\Omega$, where $R=(R_1,R_2)$ is given by

$$R := R(s,x) := (x_1-s, x_2-f(s)), \qquad s\in\mathbb{R},\ x\in\Omega,$$

we infer from [5, Lemma A.4] and [21, Lemma B.2] that also v vanishes at infinity.

In order to show that $v|_\Gamma \circ \Xi \in H^2(\mathbb{R})^2$ we conclude from $(2.12)_1$, (2.16), and the formula derived in the proof of [21, Lemma A.1] for the traces $Z_j[\phi]|_\Gamma$, $j=0,\ldots,3$, that

$$v_1|_\Gamma \circ \Xi = \frac{(B_{2,2}^0(f) - B_{0,2}^0(f))[\beta_1 - f'\beta_2] - B_{1,2}^0(f)[3f'\beta_1 + \beta_2] - B_{3,2}^0(f)[f'\beta_1 - \beta_2]}{4\mu},$$

$$v_2|_\Gamma \circ \Xi = \frac{B_{0,2}^0(f)[f'\beta_1 - \beta_2] + (B_{3,2}^0(f) - B_{1,2}^0(f))[\beta_1 - f'\beta_2] - B_{2,2}^0(f)[f'\beta_1 + 3\beta_2]}{4\mu}. \qquad (2.19)$$

Since $B_{n,m}^0(f) \in \mathcal{L}(H^2(\mathbb{R}))$, see Lemma 3.2 (iv) below, we immediately deduce from (2.19) that indeed $v|_\Gamma \circ \Xi \in H^2(\mathbb{R})^2$. $\square$

3. On the invertibility of $\pm 1/2 + \mathbb{D}(f)$ and $\pm 1/2 + \mathbb{D}(f)^*$

In this section we establish the invertibility of the operators $\pm 1/2 - \mathbb{D}(f)$ and $\pm 1/2 - \mathbb{D}(f)^*$ in $\mathcal{L}(H^k(\mathbb{R})^2)$, $k = 0, 1, 2$, and $\mathcal{L}(H^{s-1}(\mathbb{R})^2)$, $s \in (3/2, 2)$, under suitable regularity assumptions on f. These properties are needed on the one hand in the proof of Theorem 2.1, see (2.11), and on the other hand when formulating the Stokes flow as an evolution problem for f, see Sect. 4. The main step is provided by Theorem 3.1 below.

Theorem 3.1. *Given $\delta \in (0, 1)$, there exists a constant $C_0 = C_0(\delta) \geq 1$ such that for all $f \in \mathrm{BUC}^1(\mathbb{R})$ with $\|f'\|_\infty \leq 1/\delta$ and all $\beta \in L_2(\mathbb{R})^2$ we have*

$$C_0 \min \left\{ \left\| \left(\pm \frac{1}{2} - \mathbb{D}(f) \right)[\beta] \right\|_2, \left\| \left(\pm \frac{1}{2} - \mathbb{D}(f)^* \right)[\beta] \right\|_2 \right\} \geq \|\beta\|_2. \tag{3.1}$$

Moreover, $\pm 1/2 - \mathbb{D}(f)^*$ and $\pm 1/2 - \mathbb{D}(f)$ are invertible in $\mathcal{L}(L_2(\mathbb{R}))^2$.

The present section is devoted mainly to the proof of this theorem, which is split up in a number of steps.

3.1. Preliminaries

To start, we reexpress the operators $\mathbb{D}(f)$ and $\mathbb{D}(f)^*$ by using the family of singular integral operators defined in (2.10) as follows

$$\begin{aligned}
\mathbb{D}(f)[\beta] &= \begin{pmatrix} B_{0,2}^0(f) & B_{1,2}^0(f) \\ B_{1,2}^0(f) & B_{2,2}^0(f) \end{pmatrix} \begin{pmatrix} f'\beta_1 \\ f'\beta_2 \end{pmatrix} - \begin{pmatrix} B_{1,2}^0(f) & B_{2,2}^0(f) \\ B_{2,2}^0(f) & B_{3,2}^0(f) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \\
\mathbb{D}(f)^*[\beta] &= -f' \begin{pmatrix} B_{0,2}^0(f) & B_{1,2}^0(f) \\ B_{1,2}^0(f) & B_{2,2}^0(f) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} B_{1,2}^0(f) & B_{2,2}^0(f) \\ B_{2,2}^0(f) & B_{3,2}^0(f) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}
\end{aligned} \tag{3.2}$$

for $\beta = (\beta_1, \beta_2)^\top \in L_2(\mathbb{R})^2$.

Since the operators $B_{n,m}$ are well-studied by now, mapping properties for the operators $\mathbb{D}(f)$ and $\mathbb{D}(f)^*$ can be obtained by using the representation (3.2) and Lemma 3.2 below (which collects some important properties of the operators $B_{n,m}$). In the following, for $n \in \mathbb{N}$ and Banach spaces E and F, we define $\mathcal{L}_{\mathrm{sym}}^n(E, F)$ as the Banach space of n-linear, bounded, and symmetric maps $A : E^n \to F$. Moreover, $\mathrm{C}^{1-}(E, F)$ (resp. $\mathrm{C}^\infty(E, F)$) is the space of locally Lipschitz continuous (resp. smooth) mappings from E to F.

Lemma 3.2. (i) *Given Lipschitz continuous functions $a_1, \ldots, a_m, b_1, \ldots, b_n : \mathbb{R} \longrightarrow \mathbb{R}$, there exists a constant C depending only on n, m and $\max_{i=1,\ldots,m} \|a_i'\|_\infty$, such that*

$$\|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \cdot]\|_{\mathcal{L}(L_2(\mathbb{R}))} \leq C \prod_{i=1}^n \|b_i'\|_\infty.$$

Moreover, $B_{n,m} \in \mathrm{C}^{1-}((W_\infty^1(\mathbb{R}))^m, \mathcal{L}_{\mathrm{sym}}^n(W_\infty^1(\mathbb{R}), \mathcal{L}(L_2(\mathbb{R}))))$.

(ii) *Let $n \geq 1$, $s \in (3/2, 2)$, and $a_1, \ldots, a_m \in H^s(\mathbb{R})$ be given. Then, there exists a constant C, depending only on n, m, s, and $\max_{1 \leq i \leq m} \|a_i\|_{H^s}$, such that*

$$\|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, h]\|_2 \leq C \|b_1\|_{H^1} \|h\|_{H^{s-1}} \prod_{i=2}^n \|b_i\|_{H^s} \tag{3.3}$$

for all $b_1, \ldots, b_n \in H^s(\mathbb{R})$ and $h \in H^{s-1}(\mathbb{R})$.

(iii) *Given $s \in (3/2, 2)$ and $a_1, \ldots, a_m, b_1, \ldots, b_n \in H^s(\mathbb{R})$, there exists a constant C, depending only on n, m, s, and $\max_{1 \le i \le m} \|a_i\|_{H^s}$, such that*

$$\|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, \cdot]\|_{\mathcal{L}(H^{s-1}(\mathbb{R}))} \le C \prod_{i=1}^{n} \|b_i\|_{H^s}.$$

Moreover, $B_{n,m} \in C^{1-}((H^s(\mathbb{R}))^m, \mathcal{L}_{\mathrm{sym}}^n(H^s(\mathbb{R}), \mathcal{L}(H^{s-1}(\mathbb{R}))))$.

(iv) *Let $a_1, \ldots, a_m \in H^2(\mathbb{R})$ be given. Then, there exists a constant C, depending only on n, m, and $\max_{1 \le i \le m} \|a_i\|_{H^2}$, such that*

$$\|B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, h]\|_{H^1} \le C\|h\|_{H^1} \prod_{i=1}^{n} \|b_i\|_{H^2} \tag{3.4}$$

for all $b_1, \ldots, b_n \in H^2(\mathbb{R})$ and $h \in H^1(\mathbb{R})$, with

$$\begin{aligned}
(B_{n,m}&(a_1, \ldots, a_m)[b_1, \ldots, b_n, h])' \\
&= B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_n, h'] \\
&\quad + \sum_{i=1}^{n} B_{n,m}(a_1, \ldots, a_m)[b_1, \ldots, b_{i-1}, b_i', b_{i+1}, \ldots b_n, h] \\
&\quad - 2 \sum_{i=1}^{m} B_{n+2,m+1}(a_1, \ldots, a_i, a_i, \ldots, a_m)[b_1, \ldots, b_n, a_i', a_i, h].
\end{aligned} \tag{3.5}$$

Moreover, $B_{n,m} \in C^{1-}(H^2(\mathbb{R})^m, \mathcal{L}_{\mathrm{sym}}^n(H^2(\mathbb{R}), \mathcal{L}(H^1(\mathbb{R}))))$.

Proof. The claims (i) and (ii) are established in [19, Lemmas 3.1 and 3.2], the property (iii) is proven in [1, Lemmas 5], and (iv) is established in [22, Lemma 4.3]. □

As a direct consequence of (3.2) and Lemma 3.2 (i) we obtain that

$$[f \longmapsto \mathbb{D}(f)], \ [f \longmapsto \mathbb{D}(f)^*] \in C^{1-}(W_\infty^1(\mathbb{R}), \mathcal{L}(L_2(\mathbb{R})^2)). \tag{3.6}$$

Moreover, by Lemma 3.2 (iv), we have

$$[f \longmapsto B_{n,m}^0(f)] \in C^{1-}(H^{k+1}(\mathbb{R}), \mathcal{L}(H^k(\mathbb{R}))) \qquad \text{for } k = 1, 2. \tag{3.7}$$

3.2. Rellich identities on Γ

The proof of Theorem 3.1 relies on several Rellich identities for the Stokes problem, (3.12)–(3.16) below, which hold also in a bounded geometry in $\mathbb{R}^n$, $n \ge 3$, see [7].

Let $f \in \mathrm{BUC}^\infty(\mathbb{R})$ and $\beta = (\beta_1, \beta_2)^\top \in C_c^\infty(\mathbb{R})^2$. Using the notation from Sect. 2, we set $\Omega^- := \Omega$, $\Omega^+ := \mathbb{R}^2 \setminus \overline{\Omega}$, and we define the hydrodynamic single-layer potential (u, Π) by the formula

$$\left. \begin{aligned}
u(x) &:= u(f)[\beta](x) := -\int_{\mathbb{R}} \mathcal{U}^k(x - (s, f(s)))\beta_k(s)\, \mathrm{d}s \\
\Pi(x) &:= \Pi(f)[\beta](x) := -\int_{\mathbb{R}} \mathcal{P}^k(x - (s, f(s)))\beta_k(s)\, \mathrm{d}s
\end{aligned} \right\} \qquad \text{for } x \in \mathbb{R}^2 \setminus \Gamma, \tag{3.8}$$

where $\mathcal{U}^k$ and $\mathcal{P}^k$ are defined in (2.6) (with $\mu = 1$). Since β is compactly supported, it is not difficult to see that the functions (u, Π) are well-defined, smooth in $\Omega^\pm$, and satisfy

$$\left. \begin{aligned}
\Delta u - \nabla \Pi &= 0, \\
\mathrm{div}\, u &= 0
\end{aligned} \right\} \qquad \text{in } \Omega^\pm, \tag{3.9}$$

as well as

$$\Pi, \nabla u = O(|x|^{-1}) \qquad \text{for } |x| \to \infty. \tag{3.10}$$

Moreover, [5, Lemma A.1] and the arguments in the proof of [21, Lemma A.1] enable us to conclude that $\Pi^\pm := \Pi|_{\Omega^\pm}$ and $u^\pm := u|_{\Omega^\pm}$ satisfy $\Pi^\pm \in C(\overline{\Omega^\pm})$ and $u^\pm \in C^1(\overline{\Omega^\pm})$, with

$$\partial_i u_j^\pm|_\Gamma \circ \Xi(\xi) = -\,\mathrm{PV} \int_{\mathbb{R}} \partial_i \mathcal{U}_j^k(r)\beta_k \, ds \pm \frac{-\beta_j \nu^i + \nu^i \nu^j \beta \cdot \nu}{2\omega}(\xi), \quad i, j = 1, 2,$$

$$\Pi^\pm|_\Gamma \circ \Xi(\xi) = -\,\mathrm{PV} \int_{\mathbb{R}} \mathcal{P}^k(r)\beta_k \, ds \pm \frac{\beta \cdot \nu}{2\omega}(\xi) \tag{3.11}$$

for $\xi \in \mathbb{R}$, with $r = r(\xi, s)$ defined in (2.7).

Recalling the definition (1.2) of the stress tensor, we then compute in view of (3.9)

$$\mathrm{div} \begin{pmatrix} 0 \\ \|\nabla u + (\nabla u)^\top\|_F^2 \end{pmatrix} = 4 \,\mathrm{div}\left(T_1(u, \Pi)\partial_2 u\right) \qquad \text{in } \mathbb{R}^2 \setminus \Gamma,$$

$$\mathrm{div} \begin{pmatrix} 0 \\ \|\nabla u\|_F^2 \end{pmatrix} = 2 \,\mathrm{div}\left(((\nabla u)^\top - \Pi E_2)\partial_2 u\right) \qquad \text{in } \mathbb{R}^2 \setminus \Gamma,$$

$$\mathrm{div} \begin{pmatrix} 0 \\ \Pi^2 \end{pmatrix} = 2 \,\mathrm{div}\left((\partial_1 u_2 - \partial_2 u_1)\partial_1 u + \Pi \nabla u_2\right) \qquad \text{in } \mathbb{R}^2 \setminus \Gamma,$$

where $\|\cdot\|_F$ denotes as usual the Frobenius norm of matrices. Using (3.10), we may integrate the latter identities over $\Omega^\pm$ to obtain, in view of Gauss' theorem, the Rellich identities

$$\int_\Gamma \|\nabla u^\pm + (\nabla u^\pm)^\top\|_F^2 \tilde{\nu}^2 \, d\Gamma = 4 \int_\Gamma \partial_2 u^\pm \cdot T_1(u, \Pi)^\pm \tilde{\nu} \, d\Gamma, \tag{3.12}$$

$$\int_\Gamma \|\nabla u^\pm\|_F^2 \tilde{\nu}^2 \, d\Gamma = 2 \int_\Gamma \partial_2 u^\pm \cdot (\nabla u^\pm - \Pi^\pm E_2)\tilde{\nu} \, d\Gamma, \tag{3.13}$$

$$\int_\Gamma |\Pi^\pm|^2 \tilde{\nu}^2 \, d\Gamma = 2 \int_\Gamma (\partial_1 u_2^\pm - \partial_2 u_1^\pm)\partial_{\tilde{\tau}} u_2^\pm + \Pi^\pm \partial_{\tilde{\nu}} u_2^\pm \, d\Gamma. \tag{3.14}$$

We now subtract (3.12) from (3.13) multiplied by 4 to get

$$\int_\Gamma |\partial_1 u_2^\pm - \partial_2 u_1^\pm|^2 \tilde{\nu}^2 \, d\Gamma = 2 \int_\Gamma \Pi^\pm \partial_{\tilde{\tau}} u_1^\pm - (\partial_1 u_2^\pm - \partial_2 u_1^\pm)\partial_{\tilde{\nu}} u_1^\pm \, d\Gamma. \tag{3.15}$$

Furthermore, from

$$(T_1(u^\pm, \Pi^\pm) + \Pi^\pm E_2)\tilde{\nu} = \left(\nabla u^\pm + (\nabla u^\pm)^\top\right)\tilde{\nu} = \begin{pmatrix} \partial_{\tilde{\nu}} u_1^\pm + \partial_{\tilde{\tau}} u_2^\pm \\ \partial_{\tilde{\nu}} u_2^\pm - \partial_{\tilde{\tau}} u_1^\pm \end{pmatrix},$$

we obtain, after taking the difference of (3.14) and (3.15),

$$\int_\Gamma |\partial_1 u_2^\pm - \partial_2 u_1^\pm|^2 \tilde{\nu}^2 \, d\Gamma$$

$$= \int_\Gamma |\Pi^\pm|^2 \tilde{\nu}^2 \, d\Gamma - 2 \int_\Gamma \begin{pmatrix} \partial_1 u_2^\pm - \partial_2 u_1^\pm \\ \Pi^\pm \end{pmatrix} \cdot \left(T_1(u^\pm, \Pi^\pm) + \Pi^\pm E_2\right)\tilde{\nu} \, d\Gamma. \tag{3.16}$$

3.3. Transformation to the real axis

To represent the pull-backs of the one-sided traces of $(\nabla u)_{ij}$ and Π on Γ as singular integral operators, we define, for $f \in W_\infty^1(\mathbb{R})$, $\beta \in L_2(\mathbb{R})^2$, and $\theta \in L_2(\mathbb{R})$, the singular integral operators $\mathbb{T}_i$, $\mathbb{B}_i$, $i = 1, 2$, as follows:

$$\mathbb{T}_1(f)[\beta](\xi) := \frac{1}{4\pi} \operatorname{PV} \int_{\mathbb{R}} \frac{1}{|r|^4} \begin{pmatrix} r_1 r_2^2 - r_1^3 & r_2^3 - r_1^2 r_2 \\ r_2^3 - r_1^2 r_2 & -r_1^3 - 3r_1 r_2^2 \end{pmatrix} \beta \, ds,$$

$$\mathbb{T}_2(f)[\beta](\xi) := \frac{1}{4\pi} \operatorname{PV} \int_{\mathbb{R}} \frac{1}{|r|^4} \begin{pmatrix} -r_2^3 - 3r_1^2 r_2 & r_1^3 - r_1 r_2^2 \\ r_1^3 - r_1 r_2^2 & r_1^2 r_2 - r_2^3 \end{pmatrix} \beta \, ds,$$

$$\mathbb{B}_1(f)[\theta](\xi) := \frac{1}{\pi} \operatorname{PV} \int_{\mathbb{R}} \frac{-r_1 f' + r_2}{|r|^2} \theta \, ds,$$

$$\mathbb{B}_2(f)[\theta](\xi) := \frac{1}{\pi} \operatorname{PV} \int_{\mathbb{R}} \frac{r_1 + r_2 f'}{|r|^2} \theta \, ds,$$

in the notation introduced in (2.7). Since the components of these operators may be expressed by using only the singular operators $B_{n,m}^0(f)$, we infer from Lemma 3.2 (i) that

$$\mathbb{T}_i \in \mathrm{C}^{1-}(W_\infty^1(\mathbb{R}), \mathcal{L}(L_2(\mathbb{R})^2)), \quad \mathbb{B}_i \in \mathrm{C}^{1-}(W_\infty^1(\mathbb{R}), \mathcal{L}(L_2(\mathbb{R}))). \tag{3.17}$$

It follows from (2.6), (2.16) (with $\mu = 1$) and (3.11) that for $f \in \mathrm{BUC}^\infty(\mathbb{R})$ and $\beta \in \mathrm{C}_c^\infty(\mathbb{R})^2$ we have (in matrix notation)

$$\nabla u^\pm(f)[\beta]\big|_\Gamma \circ \Xi = \left(\mathbb{T}_1(f)[\beta] \; \mathbb{T}_2(f)[\beta] \right) \mp \frac{(\beta \cdot \tau)}{2\omega} \tau \nu^\top =: \widetilde{(\nabla u)^\pm}(f)[\beta],$$

$$\Pi^\pm(f)[\beta]\big|_\Gamma \circ \Xi = \frac{(\pm 1 + \mathbb{B}_1(f))[\omega^{-1}\beta \cdot \nu] + \mathbb{B}_2(f)[\omega^{-1}\beta \cdot \tau]}{2} =: \widetilde{\Pi^\pm}(f)[\beta], \tag{3.18}$$

the right sides of (3.18) being meaningful whenever $f \in W_\infty^1(\mathbb{R})$ and $\beta \in L_2(\mathbb{R})^2$. To translate the Rellich identities of the previous subsection to identities for integral operators on $\mathbb{R}$ it is convenient to additionally introduce the operators $\widetilde{T_1^\pm}$ and $\widetilde{\partial_2 u^\pm}$ by

$$\widetilde{T_1^\pm}(f)[\beta] := \widetilde{(\nabla u)^\pm}(f)[\beta] + \widetilde{(\nabla u)^\pm}(f)[\beta]^\top - \widetilde{\Pi^\pm}(f)[\beta] E_2,$$

$$\widetilde{\partial_2 u^\pm}(f)[\beta] := \widetilde{(\nabla u)^\pm}(f)[\beta] e_2,$$

where $e_2 := (0, 1)^\top$.

From (3.17) we immediately get

$$\left. \begin{aligned} \widetilde{(\nabla u)^\pm}, \widetilde{T_1^\pm} &\in \mathrm{C}^{1-}(W_\infty^1(\mathbb{R}), \mathcal{L}(L_2(\mathbb{R})^2, L_2(\mathbb{R})^{2\times 2})), \\ \widetilde{\partial_2 u^\pm} &\in \mathrm{C}^{1-}(W_\infty^1(\mathbb{R}), \mathcal{L}(L_2(\mathbb{R})^2, L_2(\mathbb{R})^2)), \\ \widetilde{\Pi^\pm} &\in \mathrm{C}^{1-}(W_\infty^1(\mathbb{R}), \mathcal{L}(L_2(\mathbb{R})^2, L_2(\mathbb{R}))). \end{aligned} \right\} \tag{3.19}$$

It is not difficult to check that

$$\omega \widetilde{T_1^\pm}(f)[\beta]\nu = \left(\mp \frac{1}{2} - \mathbb{D}(f)^* \right)[\beta]. \tag{3.20}$$

Parameterizing Γ over $\mathbb{R}$ via $[s \longmapsto (s, f(s))]$ and using (3.18) and (3.20), we find from (3.12) that

$$\left\| \widetilde{T_1^\pm}(f)[\beta] + \widetilde{\Pi^\pm}(f)[\beta] E_2 \right\|_2^2 = 4 \left\langle \widetilde{\partial_2 u^\pm}(f)[\beta] \, \middle| \, \left(\mp \frac{1}{2} - \mathbb{D}(f)^* \right)[\beta] \right\rangle_2, \tag{3.21}$$

where $\langle \cdot | \cdot \rangle$ denotes the $L_2(\mathbb{R})^2$ scalar product. Similarly, from (3.13) and (3.16) we get

$$\left\| \widetilde{(\nabla u)}^{\pm}(f)[\beta] \right\|_2^2 = 2 \left\langle \widetilde{\partial_2 u}^{\pm}(f)[\beta] \Big| \omega^{-1} \Big(\widetilde{(\nabla u)}^{\pm}(f)[\beta] - \widetilde{\Pi}^{\pm}(f)[\beta] E_2 \Big) \nu \right\rangle_2 \tag{3.22}$$

and

$$2 \left\langle \left(\begin{array}{c} \Big(\widetilde{(\nabla u)}^{\pm}_{21}(f) - \widetilde{(\nabla u)}^{\pm}_{12}(f) \Big)[\beta] \\ \widetilde{\Pi}^{\pm}(f)[\beta] \end{array} \right) \Bigg| \left(\mp \frac{1}{2} - \mathbb{D}(f)^* \right)[\beta] + \widetilde{\Pi}^{\pm}(f)[\beta] \left(\begin{array}{c} -f' \\ 1 \end{array} \right) \right\rangle_2 \tag{3.23}$$

$$= \left\| \widetilde{\Pi}^{\pm}(f)[\beta] \right\|_2^2 - \left\| \Big(\widetilde{(\nabla u)}^{\pm}_{21}(f) - \widetilde{(\nabla u)}^{\pm}_{12}(f) \Big)[\beta] \right\|_2^2 ,$$

respectively.

By a standard density argument, it follows from (3.6) and (3.19) that (3.21)–(3.23) hold for any functions $f \in \mathrm{BUC}^1(\mathbb{R})$ and $\beta \in L_2(\mathbb{R})^2$.

3.4. Completion of the proof of Theorem 3.1

We divide the remaining arguments in the proof of Theorem 3.1 in three steps.

Step 1. Fix $\delta \in (0,1)$ and $f \in \mathrm{BUC}^1(\mathbb{R})$ such that $\|f'\|_\infty \leq 1/\delta$. In the sequel, we are going to write $C(\delta)$ for different positive constants that depend on δ only. Let $\beta \in L_2(\mathbb{R})^2$. Using Lemma 3.2 (i), we find a constant $C(\delta)$ such that the right side of (3.21) satisfies

$$4 \left\langle \widetilde{\partial_2 u}^{\pm}(f)[\beta] \Big| \left(\mp \frac{1}{2} - \mathbb{D}(f)^* \right)[\beta] \right\rangle_2 \leq C(\delta) \left\| \left(\mp \frac{1}{2} - \mathbb{D}(f)^* \right)[\beta] \right\|_2 \|\beta\|_2 .$$

For the left side of (3.21) we have, in view of (3.20) and Lemma 3.2 (i),

$$\left\| \widetilde{T_1}^{\pm}(f)[\beta] + \widetilde{\Pi}^{\pm}(f)[\beta] E_2 \right\|_2^2 \geq \frac{\delta^2}{2} \int_{\mathbb{R}} \left| \omega \widetilde{T_1}^{\pm}(f)[\beta] \nu + \omega \widetilde{\Pi}^{\pm}(f)[\beta] \nu \right|^2 dx$$

$$\geq \frac{\delta^2}{4} \left\| \widetilde{\Pi}^{\pm}(f)[\beta] \right\|_2^2 - \frac{\delta^2}{2} \left\| \left(\mp \frac{1}{2} - \mathbb{D}(f)^* \right)[\beta] \right\|_2^2$$

$$\geq \frac{\delta^2}{4} \left\| \widetilde{\Pi}^{\pm}(f)[\beta] \right\|_2^2 - C(\delta) \left\| \left(\mp \frac{1}{2} - \mathbb{D}(f)^* \right)[\beta] \right\|_2 \|\beta\|_2 .$$

These estimates show that there exists a constant $C(\delta)$ with the property that

$$C(\delta) \left\| \left(\mp \frac{1}{2} - \mathbb{D}(f)^* \right)[\beta] \right\|_2 \|\beta\|_2 \geq \left\| \widetilde{\Pi}^{\pm}(f)[\beta] \right\|_2^2 \tag{3.24}$$

for all $\beta \in L_2(\mathbb{R})^2$.

Step 2. It follows from (3.23) that

$$\left\| \Big(\widetilde{(\nabla u)}^{\pm}_{21}(f) - \widetilde{(\nabla u)}^{\pm}_{12}(f) \Big)[\beta] \right\|_2^2$$

$$\leq C(\delta) \Bigg[\left\| \left(\mp \frac{1}{2} - \mathbb{D}(f)^* \right)[\beta] \right\|_2^2 + \left\| \widetilde{\Pi}^{\pm}(f)[\beta] \right\|_2^2$$

$$+ \left\| \Big(\widetilde{(\nabla u)}^{\pm}_{21}(f) - \widetilde{(\nabla u)}^{\pm}_{12}(f) \Big)[\beta] \right\|_2 \left(\left\| \left(\mp \frac{1}{2} - \mathbb{D}(f)^* \right)[\beta] \right\|_2 + \left\| \widetilde{\Pi}^{\pm}(f)[\beta] \right\|_2 \right) \Bigg] ,$$

hence

$$\left\| \Big(\widetilde{(\nabla u)}^{\pm}_{21}(f) - \widetilde{(\nabla u)}^{\pm}_{12}(f) \Big)[\beta] \right\|_2^2 \leq C(\delta) \left(\left\| \left(\mp \frac{1}{2} - \mathbb{D}(f)^* \right)[\beta] \right\|_2^2 + \left\| \widetilde{\Pi}^{\pm}(f)[\beta] \right\|_2^2 \right) . \tag{3.25}$$

Furthermore, as

$$2\widetilde{(\nabla u)^{\pm}}(f)[\beta]\nu = \frac{1}{\omega}\Big(\mp\frac{1}{2} - \mathbb{D}(f)^{*}\Big)[\beta] + \widetilde{\Pi^{\pm}}(f)[\beta]\nu - \Big(\widetilde{(\nabla u)^{\pm}_{21}}(f) - \widetilde{(\nabla u)^{\pm}_{12}}(f)\Big)[\beta]\tau,$$

we infer from (3.25) that

$$\Big\|\widetilde{(\nabla u)^{\pm}}(f)[\beta]\nu\Big\|_{2}^{2} \le C(\delta)\Big(\Big\|\Big(\mp\frac{1}{2} - \mathbb{D}(f)^{*}\Big)[\beta]\Big\|_{2}^{2} + \Big\|\widetilde{\Pi^{\pm}}(f)[\beta]\Big\|_{2}^{2}\Big). \tag{3.26}$$

The identity (3.22) implies the estimate

$$\Big\|\widetilde{(\nabla u)^{\pm}}(f)[\beta]\Big\|_{2}^{2} \le C(\delta)\Big\|\widetilde{(\nabla u)^{\pm}}(f)[\beta]\Big\|_{2}\Big(\Big\|\widetilde{(\nabla u)^{\pm}}(f)[\beta]\nu\Big\|_{2} + \Big\|\widetilde{\Pi^{\pm}}(f)[\beta]\Big\|_{2}\Big),$$

and together with (3.26) this yields

$$\Big\|\widetilde{(\nabla u)^{\pm}}(f)[\beta]\Big\|_{2}^{2} \le C(\delta)\Big(\Big\|\Big(\mp\frac{1}{2} - \mathbb{D}(f)^{*}\Big)[\beta]\Big\|_{2}^{2} + \Big\|\widetilde{\Pi^{\pm}}(f)[\beta]\Big\|_{2}^{2}\Big). \tag{3.27}$$

Multiplying the identity $(3.18)_{1}$ by e_{2} and taking subsequently the scalar product with β, we observe that

$$\|\beta \cdot \tau\|_{2}^{2} = \mp 2\omega^{2}\Big(\Big\langle\beta\,\Big|\,\widetilde{(\nabla u)^{\pm}}(f)[\beta]e_{2}\Big\rangle - \langle\beta\,|\,\mathbb{T}_{2}(f)[\beta]\rangle\Big).$$

The second term on the right vanishes as $\mathbb{T}_{2}(f)^{*} = -\mathbb{T}_{2}(f)$, and thus

$$\|\beta \cdot \tau\|_{2}^{2} \le C(\delta)\Big\|\widetilde{(\nabla u)^{\pm}}(f)[\beta]\Big\|_{2}\|\beta\|_{2} \tag{3.28}$$

Next, we rewrite $(3.18)_{2}$ as

$$(\pm 1 + \mathbb{B}_{1}(f)[\omega^{-1}\beta \cdot \nu] = 2\widetilde{\Pi^{\pm}}(f)[\beta] - \mathbb{B}_{2}(f)[\omega^{-1}\beta \cdot \tau].$$

Letting $\mathbb{A}(f) := \mathbb{B}_{1}(f)^{*}$, it follows from the Rellich identity for the Muskat problem established in the proof of [19, Theorem 3.5] that the operator $(\pm 1 - \mathbb{A}(f)) \in \mathcal{L}(L_{2}(\mathbb{R}))$ is an isomorphism with

$$\|(\pm 1 + \mathbb{A}(f))^{-1}\|_{\mathcal{L}(L_{2}(\mathbb{R}))} \le C(\delta).$$

This implies that also its adjoint $(\pm 1 - \mathbb{B}_{1}(f)) \in \mathcal{L}(L_{2}(\mathbb{R}^{2}))$ is an isomorphism and

$$\|(\pm 1 + \mathbb{B}_{1}(f))^{-1}\|_{\mathcal{L}(L_{2}(\mathbb{R}))} \le C(\delta).$$

Using this and Lemma 3.2 (i) we get

$$\|\beta \cdot \nu\|_{2} \le C(\delta)\Big(\|\beta \cdot \tau\|_{2} + \Big\|\widetilde{\Pi^{\pm}}(f)[\beta]\Big\|_{2}\Big),$$

and together with (3.28) and Young's inequality we arrive at

$$\|\beta\|_{2}^{2} = \|\beta \cdot \nu\|_{2}^{2} + \|\beta \cdot \tau\|_{2}^{2} \le C(\delta)\Big(\Big\|\widetilde{(\nabla u)^{\pm}}(f)[\beta]\Big\|_{2}^{2} + \Big\|\widetilde{\Pi^{\pm}}(f)[\beta]\Big\|_{2}^{2}\Big).$$

In view of (3.27) we infer from the latter inequality that

$$\|\beta\|_{2}^{2} \le C(\delta)\Big(\Big\|\Big(\mp\frac{1}{2} - \mathbb{D}(f)^{*}\Big)[\beta]\Big\|_{2}^{2} + \Big\|\widetilde{\Pi^{\pm}}(f)[\beta]\Big\|_{2}^{2}\Big),$$

and together with (3.24) and Young's inequality we finally obtain

$$\|\beta\|_{2} \le C(\delta)\Big\|\Big(\mp\frac{1}{2} - \mathbb{D}(f)^{*}\Big)[\beta]\Big\|_{2}. \tag{3.29}$$

Step 3. In view of the identity

$$(\lambda - \mathbb{D}(f)^{*})[\beta] = \Big(\mp\frac{1}{2} - \mathbb{D}(f)^{*}\Big)[\beta] + \Big(\lambda \pm \frac{1}{2}\Big)\beta, \quad \lambda \in \mathbb{C}, \ \beta \in L_{2}(\mathbb{R})^{2},$$

we deduce from (3.29) that

$$C(\delta)\|(\lambda - \mathbb{D}(f)^{*})[\beta]\|_{2} \ge (1 - C(\delta)|\lambda \pm 1/2|)\|\beta\|_{2}, \quad \lambda \in \mathbb{C}, \ \beta \in L_{2}(\mathbb{R})^{2},$$

and therefore

$$\|\beta\|_2 \le C(\delta)\|(\lambda - \mathbb{D}(f)^*)[\beta]\|_2 \quad \text{for } \lambda \text{ sufficiently close to } \pm 1/2 \text{ and } \beta \in L_2(\mathbb{R})^2. \tag{3.30}$$

Now (3.30) together with the estimate [22, (3.15)] shows there exists a constant $C_0 = C_0(\delta) \ge 1$ such that

$$C_0\|(\lambda - \mathbb{D}(f)^*)[\beta]\|_2 \ge \|\beta\|_2 \qquad \text{for all } \beta \in L_2(\mathbb{R})^2 \text{ and all } \lambda \in \mathbb{R} \setminus (-1/2, 1/2).$$

As $\mathbb{D}(f)^*$ is in $\mathcal{L}(L_2(\mathbb{R})^2)$, the shift $\lambda - \mathbb{D}(f)^* \in \mathcal{L}(L_2(\mathbb{R})^2)$ is an isomorphism if $|\lambda|$ is sufficiently large. The method of continuity, cf. e.g. [2, Proposition I.1.1.1], implies now that $\pm 1/2 - \mathbb{D}(f)^*$, and hence also $\pm 1/2 - \mathbb{D}(f)$, are isomorphisms as well. This completes the proof of Theorem 3.1.

3.5. Spectral properties in Sobolev spaces

In Lemma 3.3 we establish the invertibility of the operators considered in Theorem 3.1 in the Banach algebras $\mathcal{L}(H^k(\mathbb{R})^2)$, $k = 1, 2$.

Lemma 3.3. *For $f \in H^{k+1}(\mathbb{R})$, $k = 1, 2$, the operators $\pm 1/2 - \mathbb{D}(f)$ and $\pm 1/2 - \mathbb{D}(f)^*$ are invertible in $\mathcal{L}(H^k(\mathbb{R})^2)$.*

Proof. Fix $f \in H^{k+1}(\mathbb{R})$. The representation (3.2) and Lemma 3.2 (iv) then immediately imply that $\mathbb{D}(f)$ and $\mathbb{D}(f)^*$ belong to $\mathcal{L}(H^k(\mathbb{R})^2)$.

Let first $k = 1$. Using (3.5), we compute that the components of

$$T[\beta] := (\mathbb{D}(f)[\beta])' - \mathbb{D}(f)[\beta'], \qquad \beta = (\beta_1, \beta_2)^\top \in H^1(\mathbb{R})^2,$$

are (finite) linear combination of terms of the form

$$B_{n,m}(f, \ldots, f)[f', f, \ldots, f, f'^\ell \beta_i] \quad \text{and} \quad B_{n,m}^0(f)[f'' \beta_i]$$

with $n, m \le 5$, $\ell = 0, 1$, and $i = 1, 2$. Choosing $s \in (3/2, 2)$, it follows from Lemma 3.2 (i)-(ii) that there exists a constant $C_1 > 0$ such that

$$\|T[\beta]\|_2 \le C_1\|\beta\|_{H^{s-1}}, \qquad \beta = (\beta_1, \beta_2)^\top \in H^1(\mathbb{R})^2.$$

This property together with (3.1) now leads to

$$\|(\pm 1/2 - \mathbb{D}(f))[\beta]\|_{H^1}^2 = \|(\pm 1/2 - \mathbb{D}(f))[\beta]\|_2^2 + \|((\pm 1/2 - \mathbb{D}(f))[\beta])'\|_2^2$$

$$= \|(\pm 1/2 - \mathbb{D}(f))[\beta]\|_2^2 + \frac{1}{2}\|(\pm 1/2 - \mathbb{D}(f))[\beta']\|_2^2 - \|T[\beta]\|_2^2$$

$$\ge \frac{1}{2C_0^2}\|\beta\|_{H^1}^2 - C_1^2\|\beta\|_{H^{s-1}}^2.$$

The latter estimate, an interpolation argument, and Young's inequality imply there exists a further constant $C_2 = C_2(\delta) \ge 1$ such that

$$C_2\big(\|\beta\|_2^2 + \|(\pm 1/2 - \mathbb{D}(f))[\beta]\|_{H^1}^2\big) \ge \|\beta\|_{H^1}^2$$

for all $\beta \in H^1(\mathbb{R})^2$. This estimate combined with (3.1) now yields

$$C_2(C_0^2 + 1)\|(\pm 1/2 - \mathbb{D}(f))[\beta]\|_{H^1}^2 \ge \|\beta\|_{H^1}^2$$

for all $\beta \in H^1(\mathbb{R})^2$. The invertibility of $\pm 1/2 - \mathbb{D}(f)$ in $\mathcal{L}(H^1(\mathbb{R})^2)$ follows from this estimate and the invertibility property in $\mathcal{L}(L_2(\mathbb{R})^2)$. The invertibility of $\pm 1/2 - \mathbb{D}(f)^*$ in $\mathcal{L}(H^1(\mathbb{R})^2)$ may be established by using the same arguments and therefore we omit the details.

Finally, when $k = 2$, the invertibility of $\pm 1/2 - \mathbb{D}(f)$ and $\pm 1/2 - \mathbb{D}(f)^*$ in $\mathcal{L}(H^2(\mathbb{R})^2)$ may be obtained by arguing along the same lines as above (see the proof of [22, Theorem 4.5] for some details). $\qquad\square$

The next invertibility result is used in Sect. 4.4 when we consider our evolution problems in $H^{s-1}(\mathbb{R})$ with $s \in (3/2, 2)$.

Lemma 3.4. *Given $\delta \in (0,1)$ and $s \in (3/2, 2)$, there exists a positive constant $C = C(\delta, s) \geq 1$ such that for all $f \in H^s(\mathbb{R})$ with $\|f\|_{H^s} \leq 1/\delta$ and all $\beta \in H^{s-1}(\mathbb{R})^2$ we have*

$$C \min \left\{ \left\| \left(\pm \frac{1}{2} - \mathbb{D}(f) \right)[\beta] \right\|_{H^{s-1}}, \left\| \left(\pm \frac{1}{2} - \mathbb{D}(f)^* \right)[\beta] \right\|_{H^{s-1}} \right\} \geq \|\beta\|_{H^{s-1}}. \tag{3.31}$$

Moreover, $\pm 1/2 - \mathbb{D}(f)^*$ and $\pm 1/2 - \mathbb{D}(f)$ are invertible in $\mathcal{L}(H^{s-1}(\mathbb{R})^2)$.

Proof. As a direct consequence of Lemma 3.2 (iii) we get $\mathbb{D}(f), \mathbb{D}(f)^* \in \mathcal{L}(H^{s-1}(\mathbb{R}))^2$. The remaining claims follow from Lemma 3.2, (3.1), and Theorem 3.1, by arguing as in the proof of [22, Theorem 4.2] and Lemma 3.3. □

4. Equivalent formulation and proof of the main results

In this section we formulate the quasistationary Stokes flow (1.1) as an evolution problem for f. The main step is established in Corollary 4.3, which provides the unique solvability of Eq. (2.11), as announced in Theorem 2.1. Using this, in Sect. 4.2 we derive the evolution problem (4.9) for the Stokes flow (1.1). This is in analogy to the problem (4.10) obtained in [22] for the corresponding two-phase Stokes flow (1.6). In Sect. 4.3, Problem (4.9) is then shown to be the limit $\mu_+ \to 0$ of (4.10). This is based on a commutator type identity provided in Proposition 4.4. Finally, in Sect. 4.4, we introduce the general evolution problem (4.23) with parameter $\mu_+ \geq 0$. This formulation enables us to treat both one- and two-phase flows simultaneously for initial data in $H^s(\mathbb{R})$, $s \in (3/2, 2)$, and to establish the main results.

4.1. A relation connecting $(\mathbb{D}(f)[\beta])'$ and $\mathbb{D}(f)^*[\beta']$

The following identity is, besides Lemma 3.3, the main ingredient in the proof of Corollary 4.3.

Lemma 4.1. *Given $f \in H^\tau(\mathbb{R})$, $\tau \in (3/2, 2)$, and $\beta \in H^1(\mathbb{R})^2$, we have $\mathbb{D}(f)[\beta] \in H^1(\mathbb{R})^2$ with*

$$(\mathbb{D}(f)[\beta])' = -\mathbb{D}(f)^*[\beta']. \tag{4.1}$$

In order to prepare the proof of Lemma 4.1, which is presented below, we set

$$D := \{(\xi, \xi) \in \mathbb{R}^2 : \xi \in \mathbb{R}\}$$

and define

$$K(\xi, s) := \frac{r_1 f'(s) - r_2}{|r|^4} \begin{pmatrix} r_1^2 & r_1 r_2 \\ r_1 r_2 & r_2^2 \end{pmatrix}, \qquad (\xi, s) \in \mathbb{R}^2 \setminus D, \tag{4.2}$$

where $r = r(\xi, s)$ is defined in (2.7). The double-layer potential $\mathbb{D}(f)$ and its L_2-adjoint $\mathbb{D}(f)^*$ can now be expressed as follows:

$$\mathbb{D}(f)[\beta](\xi) = \int_{\mathbb{R}} K(\xi, s)\beta(s) \, \mathrm{d}s,$$

$$\mathbb{D}(f)^*[\beta](\xi) = \int_{\mathbb{R}} K(s, \xi)^\top \beta(s) \, \mathrm{d}s = \int_{\mathbb{R}} K(s, \xi)\beta(s) \, \mathrm{d}s$$

for $\beta \in L_2(\mathbb{R})^2$, see (2.8). Both integrals converge when $f \in H^\tau(\mathbb{R})$, with $\tau \in (3/2, 2)$, since there exists a constant $C > 0$ such that

$$\|K(\xi, s)\|_F \leq C|\xi - s|^{\tau - 3/2} \qquad \text{for all } (\xi, s) \in \mathbb{R}^2 \setminus D. \tag{4.3}$$

Here, $\| \cdot \|_F$ is again the Frobenius norm. Motivated by (4.3), we establish the following auxiliary result.

Lemma 4.2. *Let $A \in \mathrm{C}(\mathbb{R}^2) \cap \mathrm{C}^1(\mathbb{R}^2 \setminus D)$, $u \in \mathrm{C}_c(\mathbb{R})$, and assume there exist constants $C > 0$ and $\alpha \in (0,1)$ such that*

$$|\partial_\xi A(\xi,s)| \leq C|\xi - s|^{-\alpha} \quad \text{for all } (\xi,s) \in \mathbb{R}^2 \setminus D.$$

Then, the function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$\psi(\xi) := \int_{\mathbb{R}} A(\xi,s)u(s)\,\mathrm{d}s, \qquad \xi \in \mathbb{R},$$

belongs to $\mathrm{C}^1(\mathbb{R})$ and

$$\psi'(\xi) = \int_{\mathbb{R}} \partial_\xi A(\xi,s)u(s)\,\mathrm{d}s, \qquad \xi \in \mathbb{R}.$$

Proof. Given $\varepsilon \in (0,1)$, let $\psi_\varepsilon \in \mathrm{C}(\mathbb{R})$ be given by

$$\psi_\varepsilon(\xi) := \int_{\{|\xi-s|>\varepsilon\}} A(\xi,s)u(s)\,\mathrm{d}s.$$

Since u has compact support and $A \in \mathrm{C}(\mathbb{R}^2) \cap \mathrm{C}^1(\mathbb{R}^2 \setminus D)$, we have $\psi_\varepsilon \in \mathrm{C}^1(\mathbb{R})$ and $\psi_\varepsilon \to \psi$ for $\varepsilon \to 0$ uniformly on compact subsets of $\mathbb{R}$. By closedness of the differentiation, the lemma is proved once we show

$$\psi_\varepsilon' \to \int_{\mathbb{R}} \partial_\xi A(\cdot,s)u(s)\,\mathrm{d}s =: \varphi \quad \text{for } \varepsilon \to 0,$$

uniformly on compact subsets of $\mathbb{R}$. Indeed, given $\xi \in \mathbb{R}$, it holds that

$$\psi_\varepsilon'(\xi) = \int_{\{|\xi-s|>\varepsilon\}} \partial_\xi A(\xi,s)u(s)\,\mathrm{d}s + A(\xi,\xi-\varepsilon)u(\xi-\varepsilon) - A(\xi,\xi+\varepsilon)u(\xi+\varepsilon),$$

and therefore

$$|(\psi_\varepsilon' - \varphi)(\xi)| \leq C \int_{\{|\xi-s|<\varepsilon\}} |\xi - s|^{-\alpha}\,\mathrm{d}s + |A(\xi,\xi-\varepsilon)u(\xi-\varepsilon) - A(\xi,\xi+\varepsilon)u(\xi+\varepsilon)|,$$

which implies the announced convergence. $\qquad\square$

We are now in a position to prove Lemma 4.1.

Proof of Lemma 4.1. We first establish the result for $\beta \in \mathrm{C}_c^\infty(\mathbb{R})^2$. To this end we define the function $S := S(f) \in \mathrm{C}(\mathbb{R}^2)$ by

$$S(f)(\xi,s) := \begin{cases} \dfrac{f(\xi) - f(s)}{\xi - s}, & \text{if } \xi \neq s, \\ f'(\xi), & \text{if } \xi = s. \end{cases}$$

The function S is continuously differentiable in $\mathbb{R}^2 \setminus D$, where again $D := \{(\xi,\xi) : \xi \in \mathbb{R}\}$, with partial derivatives expressed, in the notation (2.7), as

$$\partial_\xi S(\xi,s) = \frac{f'(\xi)r_1 - r_2}{r_1^2} \quad \text{and} \quad \partial_s S(\xi,s) = \frac{-f'(s)r_1 + r_2}{r_1^2}.$$

Now (with arguments (ξ, s) partly suppressed), the kernel K defined in (4.2) can be expressed as

$$K(\xi, s) = \frac{1}{(1 + S^2)^2} \begin{pmatrix} 1 & S \\ S & S^2 \end{pmatrix} \frac{r_1 f'(s) - r_2}{r_1^2} = -\frac{1}{(1 + S^2)^2} \begin{pmatrix} 1 & S \\ S & S^2 \end{pmatrix} \partial_s S(\xi, s)$$
$$= -\partial_s (G \circ S)(\xi, s),$$

where $G \in \mathrm{C}^\infty(\mathbb{R}, \mathbb{R}^{2 \times 2})$ is a primitive of the smooth matrix valued function

$$\left[r \longmapsto \frac{1}{(1 + r^2)^2} \begin{pmatrix} 1 & r \\ r & r^2 \end{pmatrix} \right] : \mathbb{R} \longrightarrow \mathbb{R}^{2 \times 2}.$$

Thus, after integration by parts,

$$\mathbb{D}(f)[\beta](\xi) = \int_{\mathbb{R}} G(S(\xi, s)) \beta'(s) \, \mathrm{d}s.$$

We next observe that $A := G \circ S$ is continuous on $\mathbb{R}^2$ and $A \in \mathrm{C}^1(\mathbb{R}^2 \setminus D)$ with

$$\partial_\xi A(\xi, s) = \frac{1}{(1 + S^2)^2} \begin{pmatrix} 1 & S \\ S & S^2 \end{pmatrix} \partial_\xi S(\xi, s) = -\frac{1}{(1 + S^2)^2} \begin{pmatrix} 1 & S \\ S & S^2 \end{pmatrix} \frac{-r_1 f'(\xi) + r_2}{r_1^2}$$
$$= -K(s, \xi) = -K(s, \xi)^\top.$$

Moreover, since $f \in \mathrm{BUC}^{\tau - 1/2}(\mathbb{R})$, setting $\alpha := \tau - 3/2 \in (0, 1)$, we obtain from (4.3) that there exists $C > 0$ such that

$$\|\partial_\xi A(\xi, s)\|_F \leq C |\xi - s|^{-\alpha} \qquad \text{for all } (\xi, s) \in \mathbb{R}^2 \setminus D.$$

We may now infer from Lemma 4.2 that $\mathbb{D}(f)[\beta] \in \mathrm{C}^1(\mathbb{R})$ and

$$(\mathbb{D}(f)[\beta])'(\xi) = \int_{\mathbb{R}} \partial_\xi A(\xi, s) \beta'(s) \, \mathrm{d}s = -\int_{\mathbb{R}} K(s, \xi)^\top \beta(s) \, \mathrm{d}s = -\mathbb{D}(f)^*[\beta'](\xi), \quad \xi \in \mathbb{R}.$$

Let now $\beta \in H^1(\mathbb{R})^2$ arbitrary, and let (β_n) be a sequence in $\mathrm{C}_c^\infty(\mathbb{R})^2$ with $\beta_n \to \beta$ in $H^1(\mathbb{R})^2$. Then, by L_2-continuity of $\mathbb{D}(f)$, see (3.6),

$$\mathbb{D}(f)[\beta_n] \to \mathbb{D}(f)[\beta] \quad \text{in } L_2(\mathbb{R})^2$$

and, by L_2-continuity of $\mathbb{D}(f)^*$, see (3.6),

$$(\mathbb{D}(f)[\beta_n])' = -\mathbb{D}(f)^*[\beta_n'] \to -\mathbb{D}(f)^*[\beta'] \quad \text{in } L_2(\mathbb{R})^2.$$

Now the result follows by closedness of the derivative operator. $\qquad \square$

We are now in a position to establish the solvability of (2.11).

Corollary 4.3. *Given $f \in H^3(\mathbb{R})$ and $g \in H^2(\mathbb{R})^2$, let $\beta \in H^2(\mathbb{R})^2$ denote the unique solution to the equation*

$$\left(\frac{1}{2} + \mathbb{D}(f) \right)[\beta] = g. \tag{4.4}$$

Then $\alpha := \beta' \in H^1(\mathbb{R})^2$ is the unique solution to

$$\left(\frac{1}{2} - \mathbb{D}(f)^* \right)[\alpha] = g'. \tag{4.5}$$

Proof. The claim is a direct consequence of Lemmas 3.3 and 4.1. $\qquad \square$

4.2. The evolution problem for f

Let $T_+ > 0$ and (f, v, p) be a solution to (1.1) such that for all $t \in (0, T_+)$ we have $f(t) \in H^3(\mathbb{R})$ and

$$v(t) \in C^2(\Omega(t)) \cap C^1(\overline{\Omega(t)}), \quad p(t) \in C^1(\Omega(t)) \cap C(\overline{\Omega(t)}), \quad v(t)|_{\Gamma(t)} \circ \Xi_{f(t)} \in H^2(\mathbb{R})^2.$$

Define $\beta(t) := \beta(f(t)) \in H^2(\mathbb{R})^2$ by

$$\beta(t) := \left(\frac{1}{2} + \mathbb{D}(f(t)) \right)^{-1} [g(t)], \tag{4.6}$$

with $g(t) = g(f(t))$ as defined in (2.4). Then, by Corollary 4.3, $\beta(t)'$ is the unique solution to

$$\left(\frac{1}{2} - \mathbb{D}(f(t))^* \right) [\beta(t)'] = g(t)',$$

with the prime denoting the spatial derivative along $\mathbb{R}$. Theorem 2.1, in particular (2.19), then implies that

$$v(t)|_{\Gamma(t)} \circ \Xi_{f(t)} = \frac{\sigma}{\mu} \mathbb{V}(f(t))[\beta(t)] \tag{4.7}$$

where, given $f \in H^3(\mathbb{R})$, the operator $\mathbb{V}(f) \in \mathcal{L}(H^2(\mathbb{R})^2)$ (see (3.7)) is defined by

$$\begin{aligned}
\mathbb{V}(f)[\beta] := {} & \frac{1}{4} \begin{pmatrix} B_{2,2}^0(f) - B_{0,2}^0(f) & B_{3,2}^0(f) - B_{1,2}^0(f) \\ B_{3,2}^0(f) - B_{1,2}^0(f) & -3B_{2,2}^0(f) - B_{0,2}^0(f) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \\
& + \frac{1}{4} \begin{pmatrix} -B_{3,2}^0(f) - 3B_{1,2}^0(f) & -B_{2,2}^0(f) + B_{0,2}^0(f) \\ -B_{2,2}^0(f) + B_{0,2}^0(f) & -B_{3,2}^0(f) + B_{1,2}^0(f) \end{pmatrix} \begin{pmatrix} f'\beta_1 \\ f'\beta_2 \end{pmatrix}
\end{aligned} \tag{4.8}$$

for $\beta = (\beta_1, \beta_2)^\top \in H^2(\mathbb{R})^2$. Recalling $(1.1a)_5$ and (1.1b), we may thus recast (1.1) as the following evolution problem

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\sigma}{\mu} \mathbb{V}(f)[\beta] \cdot (-f', 1), \quad t > 0, \qquad f(0) = f^{(0)}, \tag{4.9}$$

where the evolution equation should be satisfied pointwise with values in $H^2(\mathbb{R})$.

4.3. Problem (4.9) as the limit $\mu_+ \to 0$ of the two-phase Stokes problem

In [22] it is shown that if $T_+ > 0$ and $(f, w^\pm, q^\pm)$ is a solution to the two-phase quasistationary Stokes flow (1.6) such that for all $t \in (0, T_+)$ we have $f(t) \in H^3(\mathbb{R})$, $w^\pm(t)|_{\Gamma(t)} \circ \Xi_{f(t)} \in H^2(\mathbb{R})^2$, and

$$w^\pm(t) \in C^2(\Omega^\pm(t)) \cap C^1(\overline{\Omega^\pm(t)}), \quad q^\pm(t) \in C^1(\Omega^\pm(t)) \cap C(\overline{\Omega^\pm(t)}),$$

then $f = f(t)$ solves the evolution problem

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\sigma}{\mu^+ + \mu^-} \gamma \cdot (-f', 1), \quad t > 0, \qquad f(0) = f^{(0)}, \tag{4.10}$$

where $\gamma(t) := \gamma(f(t)) \in H^2(\mathbb{R})^2$ is given by

$$\gamma(t) := \left(\frac{1}{2} + a_\mu \mathbb{D}(f(t)) \right)^{-1} [\mathbb{V}(f(t))[g(t)]] \tag{4.11}$$

and

$$a_\mu := \frac{\mu^+ - \mu^-}{\mu^+ + \mu^-} \in (-1, 1).$$

In (4.11), $g(f(t)) \in H^2(\mathbb{R})^2$ and $\mathbb{V}(f(t)) \in \mathcal{L}(H^2(\mathbb{R})^2)$, $t \in (0, T_+)$, are defined in (2.4) and (4.8), respectively, and the evolution equation should be again satisfied pointwise with values in $H^2(\mathbb{R})$, see [22, Theorem 4.5].

We next prove that the formulation (4.9) coincides with the limit $\mu_+ \to 0$ of (4.10) with $\mu^- = \mu$ fixed. Then $a_\mu \to -1$, and as Lemma 3.3 and [22, Theorem 4.5] show, this limit can be taken by continuity in (4.10).

Proposition 4.4. *Given $f \in H^3(\mathbb{R})$, it holds that*

$$\mathbb{V}(f)\left(\frac{1}{2} + \mathbb{D}(f)\right)^{-1} = \left(\frac{1}{2} - \mathbb{D}(f)\right)^{-1} \mathbb{V}(f). \tag{4.12}$$

Proof. We are going to prove the more compact identity

$$\mathbb{V}(f)\mathbb{D}(f) + \mathbb{D}(f)\mathbb{V}(f) = 0, \tag{4.13}$$

which is equivalent to (4.12) in view of Lemma 3.3.

Let $\beta \in H^2(\mathbb{R})^2$ be arbitrary and let $(v, p) := (v, p)(f)[\beta]$ be defined by (2.12) (with $\mu = 1$). Then, as shown in Theorem 2.1, it holds that

$$v \in C^2(\Omega) \cap C^1(\overline{\Omega}), \quad p \in C^1(\Omega) \cap C(\overline{\Omega}), \quad v|_\Gamma \circ \Xi = \mathbb{V}(f)[\beta] \in H^2(\mathbb{R})^2,$$

and, recalling also Corollary 4.3, (v, p) solves the boundary value problem

$$\left.\begin{array}{rll} \Delta v - \nabla p &= 0 & \text{in } \Omega, \\ \operatorname{div} v &= 0 & \text{in } \Omega, \\ T_1(v, p)\tilde{\nu} &= (\gamma'/\omega) \circ \Xi^{-1} & \text{on } \Gamma, \\ (v, p)(x) &\to 0 & \text{for } |x| \to \infty, \end{array}\right\} \tag{4.14}$$

where

$$\gamma := \left(\frac{1}{2} + \mathbb{D}(f)\right)[\beta] \in H^2(\mathbb{R})^2. \tag{4.15}$$

We next define $(w^\pm, q^\pm) : \Omega^\pm \longrightarrow \mathbb{R}^2 \times \mathbb{R}$ by $(w^-, w^+) = (v, 0)$ and $(q^-, q^+) = (p, 0)$. Then

$$w^\pm \in C^2(\Omega^\pm) \cap C^1(\overline{\Omega^\pm}), \quad q^\pm \in C^1(\Omega^\pm) \cap C(\overline{\Omega^\pm}), \quad w^\pm|_\Gamma \circ \Xi \in H^2(\mathbb{R})^2, \tag{4.16}$$

and $(w^\pm, q^\pm)$ solves the boundary value problem

$$\left.\begin{array}{rll} \Delta w^\pm - \nabla q^\pm &= 0 & \text{in } \Omega^\pm, \\ \operatorname{div} w^\pm &= 0 & \text{in } \Omega^\pm, \\ [w] &= -v|_\Gamma & \text{on } \Gamma, \\ [T_1(w, q)]\tilde{\nu} &= -(\gamma'/\omega) \circ \Xi^{-1} & \text{on } \Gamma, \\ (w^\pm, q^\pm)(x) &\to 0 & \text{for } |x| \to \infty. \end{array}\right\} \tag{4.17}$$

Similarly as in the proof of [22, Proposition 5.1], we decompose $(w^\pm, q^\pm)$ as a sum

$$(w^\pm, q^\pm) = (w_s^\pm, q_s^\pm) + (w_d^\pm, q_d^\pm),$$

where $(w_s^\pm, q_s^\pm)$ solves

$$\left.\begin{array}{rll} \Delta w_s^\pm - \nabla q_s^\pm &= 0 & \text{in } \Omega^\pm, \\ \operatorname{div} w_s^\pm &= 0 & \text{in } \Omega^\pm, \\ [w_s] &= 0 & \text{on } \Gamma, \\ [T_1(w_s, q_s)]\tilde{\nu} &= -(\gamma'/\omega) \circ \Xi^{-1} & \text{on } \Gamma, \\ (w_s^\pm, q_s^\pm)(x) &\to 0 & \text{for } |x| \to \infty. \end{array}\right\} \tag{4.18}$$

The system (4.18) has been studied in [21] and, according to [21, Theorem 2.1 and Lemma A.1], it has a unique solution which satisfies

$$w_s^\pm|_\Gamma \circ \Xi = \mathbb{V}(f)[\gamma] \tag{4.19}$$

and which has the same regularity as (w, q), see (4.16). Consequently, $(w_d^\pm, q_d^\pm)$ enjoys also the regularity (4.16) and moreover it solves

$$\left.\begin{array}{rl}
\Delta w_d^\pm - \nabla q_d^\pm = 0 & \text{in } \Omega^\pm, \\
\operatorname{div} w_d^\pm = 0 & \text{in } \Omega^\pm, \\
[w_d] = -v|_\Gamma & \text{on } \Gamma, \\
[T_1(w_d, q_d)]\tilde{\nu} = 0 & \text{on } \Gamma, \\
(w_d^\pm, q_d^\pm)(x) \to 0 & \text{for } |x| \to \infty.
\end{array}\right\} \tag{4.20}$$

Since $v|_\Gamma \circ \Xi \in H^2(\mathbb{R})^2$, also (4.20) has a unique solution, see [22, Proposition 2.1], and, according to [22, Lemma A.1] we have

$$w_d^-|_\Gamma \circ \Xi = \left(\frac{1}{2} + \mathbb{D}(f)\right)[v|_\Gamma \circ \Xi]. \tag{4.21}$$

Since $\mathbb{V}(f)[\beta] = v|_\Gamma \circ \Xi = (w_s^- + w_d^-)|_\Gamma \circ \Xi$, we infer from (4.15), (4.19), and (4.21) that

$$\mathbb{V}(f)[\beta] = \mathbb{V}(f)\left(\frac{1}{2} + \mathbb{D}(f)\right)[\beta] + \left(\frac{1}{2} + \mathbb{D}(f)\right)[\mathbb{V}(f)[\beta]] \qquad \text{for all } \beta \in H^2(\mathbb{R})^2,$$

and (4.13) is now a direct consequence of this identity. $\qquad\square$

4.4. The common equivalent formulation for (1.1) and (1.6)

We are now able to simultaneously consider the evolution equations (4.9) and (4.10) that, respectively, encode the one-phase problem (1.1) and the two-phase problem (1.6). For this purpose we set

$$\mu^- = \mu$$

and view $\mu^+ \in [0, \infty)$ as a parameter.

For $f \in H^3(\mathbb{R})$, we set

$$\Phi(\mu^+, f) := \frac{\sigma}{\mu^+ + \mu}\left(\frac{1}{2} + \frac{\mu^+ - \mu}{\mu^+ + \mu}\mathbb{D}(f)\right)^{-1}[\mathbb{V}(f)[g(f)]] \cdot (-f', 1), \tag{4.22}$$

with $g(f)$ defined in (2.4) and $\mathbb{V}(f)$ in (4.8).

Now, in view of Proposition 4.4, the parameter-dependent evolution equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \Phi(\mu^+, f), \quad t \geq 0, \qquad f(0) = f^{(0)}, \tag{4.23}$$

is identical to (4.9) for $\mu^+ = 0$ and to (4.10) for $\mu^+ > 0$.

Though (4.23) has been derived under the assumption that the solution lies in $H^3(\mathbb{R})$ for positive times, this equation may be viewed in a more general analytic setting. Indeed, given $f \in H^s(\mathbb{R})$, $s \in (3/2, 2)$, we first deduce from [21, Lemma 3.5] that $g(f) \in H^{s-1}(\mathbb{R})^2$ and

$$g \in C^\infty(H^s(\mathbb{R}), H^{s-1}(\mathbb{R})^2). \tag{4.24}$$

Moreover, since

$$[f \longmapsto B_{n,m}^0(f)] \in C^\infty(H^s(\mathbb{R}), \mathcal{L}(H^{s-1}(\mathbb{R}))), \qquad n, m \in \mathbb{N},$$

see [21, Corollary C.5], $(3.2)_1$ and (4.8) yield

$$[f \longmapsto \mathbb{D}(f)], \; [f \longmapsto \mathbb{V}(f)] \in C^\infty(H^s(\mathbb{R}), \mathcal{L}(H^{s-1}(\mathbb{R})^2)). \tag{4.25}$$

The properties (4.24), (4.25), Lemma 3.4 (for $\mu^+ = 0$) and [22, Theorem 4.2] (for $\mu^+ > 0$) ensure now that the operator $\Phi : [0, \infty) \times H^s(\mathbb{R}) \longrightarrow H^{s-1}(\mathbb{R})$ is well-defined. Moreover, the smoothness of the function which maps an isomorphism to its inverse together with (4.24) and (4.25) implies that

$$\Phi \in C^\infty([0, \infty) \times H^s(\mathbb{R}), H^{s-1}(\mathbb{R})). \tag{4.26}$$

We conclude this section with the observation that $\Phi(\mu^+, \cdot)$ maps bounded sets of $H^s(\mathbb{R})$ to bounded sets of $H^{s-1}(\mathbb{R})$. This property is a consequence of the fact that g maps bounded sets of $H^s(\mathbb{R})$ to bounded sets of $H^{s-1}(\mathbb{R})^2$, of Lemma 3.2 (iii), and of Lemma 3.4 (for $\mu^+ = 0$) or [22, Theorem 4.2] (for $\mu^+ > 0$).

4.5. The proofs of the main results

In the case $\mu^+ > 0$, we shown in [22, Theorem 6.1] that the Fréchet derivative $\partial_f \Phi(\mu^+, f)$ is, for each $f \in H^s(\mathbb{R})$, the generator of an analytic semigroup in $\mathcal{L}(H^{s-1}(\mathbb{R}))$. This property, the observation that $\Phi(\mu^+, \cdot)$ maps bounded sets of $H^s(\mathbb{R})$ to bounded sets of $H^{s-1}(\mathbb{R})$, the smoothness of $\Phi(\mu^+, \cdot)$, the fully nonlinear parabolic theory from [18], and a parameter trick used also for other problems, see, e.g., [3,6,20,25], were then exploited in [22] to establish the well-posedness of the two-phase quasistationary Stokes flow (1.6), see [22, Theorem 1.1]. All these properties are satisfied also when $\mu^+ = 0$. Indeed, using Lemma 3.4 instead of [22, Theorem 4.2], the arguments in the proof of [22, Theorem 6.1] remain valid also if $\mu^+ = 0$, hence also $\partial_f \Phi(0, f)$ is the generator of an analytic semigroup in $\mathcal{L}(H^{s-1}(\mathbb{R}))$ for each $f \in H^s(\mathbb{R})$. In view of these facts we have:

Proof of Theorem 1.1. The proof is identical to that of [22, Theorem 1.1] and therefore we omit the details. □

For the proof of Theorem 1.2 we consider μ^+ as a parameter and we use a result on the continuous dependence of the solutions to abstract parabolic problems on parameters provided in [18, Theorem 8.3.2]

Proof of Theorem 1.2. In view of (4.26) and of the fact that Fréchet derivative $\partial_f \Phi(\mu^+, f)$ is, for each $(\mu^+, f) \in [0, \infty) \times H^s(\mathbb{R})$, the generator of an analytic semigroup in $\mathcal{L}(H^{s-1}(\mathbb{R}))$ we find that all the assumptions of [18, Theorem 8.3.2] are satisfied in the context of (4.23).

Let thus $f(\cdot; f^{(0)}) : [0, T_+(f^{(0)})) \longrightarrow \mathbb{R}$ denote the maximal solution to (4.23) with $\mu^+ = 0$ and fix $T \in (0, T_+(f^{(0)}))$. In view of [18, Theorem 8.3.2] there exist constants $\varepsilon > 0$ and $M > 0$ such that for each $\mu^+ \in (0, \varepsilon]$ the solution $f_{\mu^+}(\cdot; f^{(0)})$ to (4.23) found in [22, Theorem 6.1] satisfies $T_{+,\mu^+}(f^{(0)}) > T$ and

$$\left\| f(\cdot; f^{(0)}) - f_{\mu^+}(\cdot; f^{(0)}) \right\|_{C([0,T], H^s(\mathbb{R}))} + \left\| \frac{\mathrm{d}}{\mathrm{d}t} \left(f(\cdot; f^{(0)}) - f_{\mu^+}(\cdot; f^{(0)}) \right) \right\|_{C([0,T], H^{s-1}(\mathbb{R}))} \leq M\mu^+.$$

This completes the proof. □

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

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