Abstract. Today there is a wealth of fascinating studies of connexive logical systems. But sometimes it looks as if connexive logic is still in search of a convincing interpretation that explains in intuitive terms why the connexive principles should be valid. In this paper I argue that difference-making conditionals as presented in Rott (Review of Symbolic Logic 15, 2022) offer one principled way of interpreting connexive principles. From a philosophical point of view, the idea of difference-making demands full, unrestricted connexivity, because neither logical truths nor contradictions or other absurdities can ever ‘make a difference’ (i.e., be relevantly connected) to anything. However, difference-making conditionals have so far been only partially connexive. I show how the existing analysis of difference-making conditionals can be reshaped to obtain full connexivity. The classical AGM belief revision model is replaced by a conceivability-limited revision model that serves as the semantic base for the analysis. The key point of the latter is that the agent should never accept any absurdities.

Keywords: Conditionals, Conditional logic, Connexive logic, Belief revision, Ramsey test, Difference-making, Relevance, Contingency, Aristotle, Boethius, Abelard.

1. Introduction: Connexive Logic, Conditional Logic and Difference-Making

Connexive logic has a distinguished history. The name ‘connexive logic’ seems to have been introduced in McCall’s Oxford Dissertation [22]. McCall in turn credits the term ‘connexive’ to Bocheński [5] who uses it in his reporting about Sextus Empiricus’ summary of the kinds of implication discussed by the Megarian-Stoic school. However, the main historical sources of connexive logic are Aristotle and Boethius and, to a somewhat lesser extent, Abelard. Connexive logic is most frequently taken to be defined by the idea that certain ‘theses’ or ‘principles’ should be theorems of propositional

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1 ‘Konnexe’ Implikation translated as ‘connexive’ implication, with Bocheński’s scare quotation marks.

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logic—even though they are non-theorems in the propositional logic that we today call classical.

I suggest to use somewhat non-committal and flexible formulations of three central principles of connexivity which are here formulated in a metalinguistic way. They all concern (non-nested) conditionals and negations:

(Arist1) Not $A \rightarrow \neg A$.

(Arist2) Not both $A \rightarrow C$ and $\neg A \rightarrow C$.

(Boet-Abel) Not both $A \rightarrow C$ and $A \rightarrow \neg C$.

Aristotle’s First Thesis (Arist1) also comes in a twin version ‘Not $\neg A \rightarrow A$’ which I will neglect because it is equivalent to the above version in the context of the present paper. Aristotle’s Second Thesis\(^2\) (Arist2) and the thesis (Boet-Abel) of Boethius and Abelard\(^3\) are remarkably similar to each other and exhibit an appealing symmetry: the former concerns a negated antecedent, the latter a negated consequent. The principle of Boethius and Abelard is widely regarded as defining the very concept of connexive logic, while there is only little discussion about (Arist2) in the connexive logic community. This seems unjustified to me, both from a historical and from a systematic point of view. The historical origin of connexive logic is often located in Aristotle’s Analytica Priora 57b3–14 where he argues for (Arist2) (or something very similar) on the grounds of (Arist1) (or something very similar). Boethius and Abelard came nine and fifteen centuries later, respectively, but they had no intention of denying what Aristotle had said.\(^4\) Systematically, the justification of (Boet-Abel) by means of (a version of)

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\(^2\) Since McCall [23, p. 416] these names have become fairly common. Martin [20, pp. 379–381], where the labels were introduced, had the numbering of the Aristotelian theses the other way round, and Martin has kept his numbering in later papers.

\(^3\) According to their most common formalizations in modern connexive logic, (the strong) Boethius Thesis and Abelard’s (First) Principle (a.k.a. Strawson’s Thesis or Angell’s Principle of Subjunctive Contrariety) are similar in spirit, but differ in logical detail. Boethius’ Thesis $(A \rightarrow C) \rightarrow \neg(A \rightarrow \neg C)$ (or its twin version $(A \rightarrow \neg C) \rightarrow \neg(A \rightarrow C)$) features nested conditionals Abelard’s similar principle $\neg((A \rightarrow C) \land \neg(A \rightarrow \neg C))$ does not. I do not think that this divergence can be historically substantiated. If it is granted, my formulation of the principle is closer to Abelard than to Boethius. In this paper, however, I want to avoid any embeddings of conditionals, even Boolean ones.

\(^4\) Here is a telling passage of Abelard’s Dialectica: “Patet itaque ex inductione inconvenientium aristotelicam regulam constare, quae ... negabat ad affirmationem et negationem eiusdem idem sequi; ostendamus quoque et quod nos adieimus affirmationem et negationem eiusdem ad idem consequi non posse”. (Abaelardus [1, p. 291]) Or in English: “And so it is clear from the incidence of the embarrassments that the Aristotelian rule persists which ... denied that from the affirmation and negation of the same thing the
(Arist1) runs exactly parallel to the justification of (Arist2) by means of (the twin version of) (Arist1).\textsuperscript{5} So perhaps the preference that contemporary connexive logicians give to (Boet-Abel) over (Arist2) is only a matter of historical contingency. In any case, one of the fruits of the present paper is to have a perfectly equal treatment of (Arist2) and (Boet-Abel).

Today there is a great wealth of fascinating studies of connexive logical systems. However, connexive logic is only defined by the postulated validity of a few characteristic principles for conditionals and negations. Sometimes it looks as if it is still in search of a convincing interpretation that explains in intuitive terms \textit{why} these connexive principles should be valid. The present paper offers such an interpretation, and in fact one that treats Aristotle’s Second Thesis and the Thesis of Boethius and Abelard in a parallel way. I do not claim, of course, that it is the only reasonable interpretation of connexive logic, or that it is the interpretation that matches the intentions of its classical advocates. But it is one consistent interpretation.

The vantage point of this paper is conditional logic rather than connexive logic, and we’ll mark this by the use of different symbols for the conditional connective. One of the standard readings of a conditional ‘If $A$ then $C$’ is the \textit{suppositional} one that is captured by the \textit{Ramsey Test}:

\begin{enumerate}
\item[(RT)] Accept a (suppositional) conditional ‘If $A$, then $C$’ ($A > C$) just in case you believe $C$ under the supposition of $A$.
\end{enumerate}

The Ramsey Test with ‘Believe a conditional’ substituted for ‘Accept a conditional’ was pioneered by Stalnaker [33, pp. 43–45] and Gärdenfors [8]. We will not interpret conditionals as objects of belief and thus talk of the \textit{acceptance} of conditionals. The main idea of \textit{difference-making} conditionals is expressed by the \textit{Relevant Ramsey Test} (Rott [28, 31]):

\begin{enumerate}
\item[(RRT)] Accept a (difference-making) conditional ‘If $A$, then $C$’ ($A \triangleright C$) just in case
\begin{enumerate}
\item[(i)] you believe $C$ under the supposition of $A$, but
\item[(ii)] you don’t believe $C$ under the supposition of $\neg A$.
\end{enumerate}
\end{enumerate}

\textsuperscript{5}The equal status of (Arist2) and (Boet-Abel) in Abelard [1, pp. 290–292] is nicely highlighted by Martin [21, p. 190], who shows that Abelard’s justification of (Boet-Abel) is entirely parallel to Aristotle’s justification of (Arist2) and presupposes contraposition and transitivity of $\rightarrow$. There is also a direct, immediate link between (Arist2) and (Boet-Abel), if one allows free use of double negation and contraposition (as pointed out by McCall [22, p. 21], and again by Martin [19, p. 568]).
Clause (i) is just the ordinary Ramsey Test for suppositional conditionals, clause (ii) compares the result of the supposition of the negated antecedent $\neg A$ with that of the unnegated $A$. This is a way of encoding the idea that the antecedent of a conditional should make a difference for the consequence, or that the former is (positively) relevant to the latter. ‘If ... then’ is thereby taken as a contrastive connective, much like the factive connective ‘because’.

I want to emphasize a difference in interpretation between conditionals in connexive logic and difference-making conditionals. In historical and systematic studies of connexive logic, conditionals of the form $A \to C$ are frequently glossed as ‘$A$ entails $C$’, ‘$A$ implies $C$’ or ‘$C$ follows from $A$’. It is hard to avoid reading this as ‘$A$ logically entails $C$’, ‘$A$ logically implies $C$’ or ‘$C$ logically follows from $A$’. It is intended that conditionals reflect some metalogical relationship between the antecedent and the consequent (only this, I believe, would justify calling conditionals ‘implications’), and this seems to be a natural view if the task of logic is taken to be the identification of logical truths.

Natural-language conditionals like ‘If Lisa comes, Max will be happy’ (indicative) or ‘If Lisa had come, Max would have been happy’ (counterfactual) are statements expressing the speaker’s view about the world. They do not state logical relationships between antecedent and consequent—as long as one understands logic in the sense of formal, context-free logic. One may say, ‘Given a certain set of facts, it follows from Lisa’s coming that Max will be happy’, or ‘Given my current beliefs, it follows from Lisa’s coming that Max will be happy’, but this involves a non-standard, extended concept of ‘following’ (‘entailment’, ‘implication’), something like following-in-the-context-of-certain-cotenable-propositions.6

Since I won’t interpret conditionals as reflecting logical implication, but at best only some sort of context-dependent implication, it is often difficult to import arguments pertaining to connexive logic into the context of difference-making conditionals. In particular, it is problematic to make free use of inference schemes such as contraposition and transitivity, as it has often been done by Aristotle, Boethius, Abelard, as well as in discussions of connexive logic. Difference-making conditionals are close in kind to the

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6We are then close to the so-called metalinguistic theory of conditionals and counterfactuals associated with philosophers like Roderick Chisholm, Nelson Goodman, John L. Mackie and Nicholas Rescher. Of course this extended concept of ‘implication’ is in much need of explication. Three explications are reviewed by Lewis [18, sections 3.1–3.3] Belief revision theories may be understood as offering an alternative way of explicating ‘cotenable propositions’, viz., as the beliefs (and expectations) that are still held after a doxastic state has been revised.
conditionals studied in conditional logic, and for the founders of this field, contraposition and transitivity are paradigmatically invalid for indicative conditionals and counterfactuals.

Like Rott ([31], ‘DMC’ from now on), the present paper does not aim at presenting a conditional logic in the sense of delineating a set of logical truths or theorems. It rather endorses the view that the task of logic lies in identifying what can be validly inferred from what. And this concept will be explicated semantically, not with the help of possible worlds and truth conditions, but with the help of rational belief states and conditions of belief or acceptance. Something can be validly inferred from something else if and only if the former is believed or accepted whenever the latter is believed or accepted, in every rational belief state. That the inference from \( A \implies B \) and \( A \implies C \) to \( A \implies (B \land C) \) is valid, for instance, means that in every rational belief state in which the former conditionals are accepted, the latter conditional is accepted, too. It will turn out that the logic developed in the present paper has no ‘logical truths’, i.e., no conditionals that are accepted in all rational belief states. None of the typical candidates (like \( A \implies A \), \( (A \land B) \implies A \), \( A \implies (A \lor B) \), \( \top \implies \top \), \( \bot \implies \bot \) or \( \bot \implies \top \)) is universally acceptable.

There is, however, a conditional that is true a priori and depends on the particular belief state the agent is in: if \( X_{bel} \) the agent’s strongest belief, then \( X_{bel} \implies X_{bel} \) is always accepted.

Implementing the idea of relevance into conditionals by means of the Relevant Ramsey Test is a very principled way of obtaining connexivity. As noted in ‘DMC’ (pp. 145, 148), difference-making conditionals have many distinctively connexive features. But they fail to be fully connexive in that they validate (Boet-Abel) only in a restricted form. In the light of the last

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7See Adams [2, pp. 166–168, 177–178, 188–192] Stalnaker [33, pp. 48–49] and Lewis [18, pp. 32–36]. Also cf. the counterexample to contraposition given at the end of section 5.1 of ‘DMC’.—I take it that both connexive logic and the logic of difference-making conditionals are projects of expressing ‘reasonable condition[s] for the behaviour of conditionals’ (Lenzen [16, p. 25]). I do not presuppose that there is just one kind of conditional that is used in ordinary or scientific language, nor do I think that any deviation from classical propositional logic or (normal) modal logic is a mistake.

8This is different from ‘DMC’, which has \( \bot \implies \bot \) as a basic principle.

9Ramsey’s famous footnote indeed contains a formulation which is close to the Abelard-Boethius Thesis. This was first pointed out by McCall [23, p. 420].

10I am using the term ‘full connexivity’ in order to mark the unrestricted validity of the three connexive principles (Arist1), (Arist2) and (Boet-Abel), not in a sense that presupposes embeddings or nestings of conditionals. More specifically, full connexivity concerns the lifting of the restriction of the principle (Boet-Abel) that is present in ‘DMC’.—An important feature of difference-making conditionals as presented in ‘DMC’ is that they
paragraphs, the reasons for this connexivity are likely to be different from
the reasons that drove the historical development of connexive logic. The
questions I want to address in this paper are these: Is it possible to make
difference-making conditionals “more connexive”, i.e., to make them satisfy
all the properties of connexive logic? This is an important question if, as I
have claimed above, the idea of difference-making conditionals offers a very
principled route to connexivity. I will answer this question in the positive.

In terms of general ideas and structure, the remainder of this paper is
quite similar to ‘DMC’. But the reader should not be led into thinking that
this is just a trifling, if somewhat laborious technical exercise. The change
carries considerable philosophical weight. I now think that if one takes the
idea—or rather: my interpretation of the idea—of connexivity seriously, then
one needs to go all the way and endorse unrestricted connexivity. An im-
portant point of ‘DMC’ has been this one:

(Irrelevant $\top$) Tautologies are never relevant to anything, nor is anything
ever relevant to them.

Tautologies have no role to play in difference-making conditionals because
they have no content. They are always true. But the status of contradic-
tions is very different in ‘DMC’. They are (or more precisely, the falsity
constant ‘$\bot$’ is) very frequently used, and indeed they serve the purpose of
defining absurdities and belief in a very convenient way. But intuitively, this
is strange. Because they have ‘much too much content’—so much they are
never true—, I now think that contradictions (and in fact other absurdities
as well) should receive equal treatment. My new thesis thus is:

(Irrelevant $\bot$) Contradictions are never relevant to anything, nor is any-
thing ever relevant to them.

Contradictions should have no role to play in difference-making conditionals
because they have too much content, as it were—so much that they are never
ture. Clearly, this is a reflection of the basically classical background logic

Footnote 10 continued
validate neither ‘(Conjunctive) Simplification’, $(A \land B) \rightarrow A$, nor ‘(Disjunctive) Addition’,
$A \rightarrow (A \lor B)$. Aside from some limiting cases, a difference-making conditional of the form
$(A \land B) \supset A$ is accepted iff $A$ is not believed under the supposition of $\neg(A \land B)$; a difference-
making conditional of the form $A \supset (A \lor B)$ is accepted iff $B$ is not believed under the
supposition $\neg A$. Thus both conditionals make substantive, non-trivial claims and are far
from being universally acceptable. Routley, Meyer, Plumwood and Brady [32, pp. 3–4, 47,
79–85] consider the violation of Simplification (and, to a lesser degree, the violation of
Addition) as a defining feature of connexive logics.
on which the conditional logic to be developed is based. And yet the re-interpreted difference-making conditionals studied in this paper are fully connexive.\textsuperscript{11}

2. \textbf{Two Decisions in the Theory of Belief Revision: What Are Absurdities, and What Is One to Do With Them?}

The analysis of difference-making conditionals in ‘DMC’ is conducted in terms of a belief revision semantics. The classical theory of belief revision is due to Alchourrón, Gärdenfors and Makinson—hence the widely used acronym ‘AGM’ (Alchourrón, Gärdenfors and Makinson \cite{3}, Gärdenfors \cite{9}).

The analysis of difference-making conditionals is sensitive to certain choices made on the level of belief revision theory. We need to distinguish between the genuine revision of the agent’s beliefs that is prompted by the acceptance of new information on the one hand, and a merely hypothetical revision prompted by the making of a supposition on the other. Our starting point is the view that AGM-style belief revision is capable of modelling suppositional belief change, too.\textsuperscript{12}

We will now consider two important decisions for the conceptualization of belief revision. These design decisions concern the way that ‘absurdities’ are defined and dealt with. By an \textit{absurdity} (or \textit{doxastic impossibility}) I informally mean a proposition that is extremely hard—or even impossible—to believe or suppose as true.

We may safely take it that all contradictions (logical falsehoods) are absurdities. The decisions mentioned above can be characterised by the answers they give to the following two questions:

(1) \textit{Are all absurdities contradictions?} Classical AGM theory says ‘yes’. This is a consequence of AGM’s postulate of ‘consistency preservation’: Unless the input is a contradiction, the agent’s posterior belief state is supposed to be consistent. But it is quite easy to say ‘no’ and liberalize AGM here, and

\textsuperscript{11}I apologize to the advocates of \textit{relevance logic} or \textit{relevant logic} for using the same term ‘relevance’ with a very different meaning. There is simply no better word for (positive) relevance in my sense than ‘relevance’. The term ‘difference-making conditionals’ has been chosen in order to avoid confusion with ‘relevance conditionals’ in the sense of Iatridou \cite[p. 50]{13} and other linguists taking up her usage of the term. Finally, let me add that my project is fundamentally different from the projects of \textit{paraconsistent} and \textit{hyperintensional logic}.

\textsuperscript{12}Levi \cite[pp. 5–8]{17} argues that it is even better at modelling suppositional belief change than at modelling genuine change in belief.
this is precisely what was done in ‘DMC’, by using weaker axioms (*5a) and (*5b) rather than AGM’s stronger consistency postulate (*5). I think this latter answer is philosophically desirable for the genuine revision of beliefs: If an agent were forced to accept ‘for real’ that she is the empress of China, her beliefs might ‘explode’ and she might be ready to accept anything. But the liberalization is less compelling for hypothetical reasoning: She will not land in ‘epistemic hell’ (Gärdenfors [9, p. 51]) if she only supposes that she were the empress of China. She can make sense of more propositions as suppositions than she can sensibly digest as genuine beliefs. But even conceivability appears to have its limits. Can an agent suppose that she is a fried egg? Can she, a woman of the 21st century, suppose that she is Napoleon?

Absurdities may alternatively be called doxastic impossibilities. I have not yet said how to formally define an absurdity. This will be done later, but we may introduce already here a bit of further terminology. A sentence $A$ is called conceivable iff it is not an absurdity. It is a doxastic necessity iff its negation $\neg A$ is an absurdity. It is contingent iff it is neither an absurdity nor a doxastic necessity, i.e., iff both $A$ and $\neg A$ are conceivable. More on the notions of absurdity, conceivability, doxastic necessity and contingency later.

(2) Should agents accept absurdities? Classical AGM theory says ‘yes’. This is a consequence of AGM’s unrestricted postulate of ‘success’: Each input for a revision is to be ‘successfully’ accepted, even if the input is absurd. The resulting belief state is epistemic hell, but due to AGM’s design decision regarding (1), only contradictions lead us into epistemic hell. Philosophically, however, it seems a good idea so say ‘no’ here when the genuine revision of beliefs is concerned. Agents should avoid epistemic hell—this seems to be the motivating motivation behind the whole project of belief revision theories. One rather principled way of dealing with the problem of absurdities is to just refuse to change anything and keep one’s doxastic state when presented with an absurd input. So a good option seems to be to restrict the success postulate to conceivable inputs and universally require

13 (*5) says that a belief set revised by $A$ is inconsistent only if $A$ is inconsistent. For (*5a) and (*5b), see the list in Section 4 below.

14 This question is inspired by Priest [25]. Poets have particularly strong powers of imagination: “y si yo fuera sal/ tú serías una lechuga/ una palta o al menos un huevo frito/ y si tú fueras un huevo frito/ yo sería un pedazo de pan” (Bertoni [4], translation: “And if I were salt/ you’d be lettuce/ an avocado or at least a fried egg/ and if you were a fried egg/ I’d be a piece of bread.”)

15 The latter deviates from the rather nonchalant usage of ‘contingent’ in ‘DMC’ which now seems infelicitous to me.
the consistency of an agent’s belief states. Again the situation is somewhat less clear for hypothetical reasoning. It seems quite possible to reason on the basis of a supposition that strikes one as absurd (at least if it is logically consistent). In ‘DMC’, AGM’s design decision is followed, but there is a hint at the alternative as a potential way of connexifying difference-making conditionals. The present paper expands these rather casual side remarks there into a more proper treatment.

So where AGM say ‘yes’ twice, we say ‘no’ in reply to both questions in this paper. While our first decision appears to be desirable from a philosophical point of view, it is not particularly relevant to the problem of connexivity. However, our second decision will turn out to have important consequences for the connexivity of difference-making conditionals, both from a philosophical and a formal point of view.

3. Technical Preliminaries

We denote a belief state by the letter ‘$\mathcal{B}$’. The nature of $\mathcal{B}$ is left completely open, except that we assume that one can always determine, with the help of a uniform method $Bel$, the agent’s set of beliefs in belief state $\mathcal{B}$. The beliefs thus determined are expressed in a certain language. Our object language in this paper features the logical constants $\top$ (verum) and $\bot$ (falsum), the usual truth-functional propositional operators $\neg$, $\land$, $\lor$ and $\supset$, as well as a conditional connective $\supset$. Sentences without any occurrences of $\supset$ are called factual sentences. In this paper, I follow AGM in assuming that the factual language is governed by some reflexive, monotonic and idempotent consequence operation (‘background logic’) $Cn$ which is supraclasical and compact and satisfies the deduction theorem: $\Gamma \subseteq Cn(\Gamma)$; if $\Gamma \subseteq \Delta$ then $Cn(\Gamma) \subseteq Cn(\Delta)$; $Cn(Cn(\Gamma)) = Cn(\Gamma)$; if $Cn_0$ is classical tautological implication then $Cn_0(\Gamma) \subseteq Cn(\Gamma)$; if $A \in Cn(\Gamma)$ then $A \in Cn(\Gamma_0)$ for some finite subset $\Gamma_0$ of $\Gamma$; and finally, $B \in Cn(\Gamma \cup \{A\})$ iff $A \supset B \in Cn(\Gamma)$. I use the notation $A \vdash B$ for $B \in Cn(\{A\})$.

Because I want to remain non-committal as to whether or not conditionals express propositions, conditionals formed with the help of the connective $\supset$ are not embedded in more complex sentences. For the same reason I will say that we accept or reject conditionals, whereas we believe or disbelieve

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17In footnotes 17 and 27 of that paper.
or suspend judgment on factual sentences (or on the propositions expressed by them). Only factual sentences are in the belief set \( Bel \). The ‘logic’ of conditionals as studied in this paper takes place in the meta-language, just as the logic of belief revision does within the AGM approach.

In the following, \( Bel \) is short for \( Bel(\mathfrak{B}) \), and \( Bel*A \) is short for \( Bel(\mathfrak{B}*A) \), which denotes the belief set obtained after revising one’s belief state \( \mathfrak{B} \) by a new piece of information or by a hypothetical assumption \( A \). Throughout this paper, we will make three presuppositions about the initial, unrevised belief set \( Bel \): (i) it is consistent; (ii) it can be characterised by a strongest belief \( X_{bel} \in Bel \) and the condition \( A \in Bel \) just in case \( X_{bel} \) logically implies \( A \); and (iii) there is at least one belief \( A \) such that \( \neg A \) is conceivable for the agent (in other words, her strongest belief \( X_{bel} \) is not a doxastic necessity). Taken together, this means that our beliefs are—or more precisely, that \( X_{bel} \) is—assumed to be contingent.

Since there are no embeddings of conditionals governed by the Ramsey Tests, many of the following considerations will be formulated in the metalanguage. We will use ‘\( A \not\supseteq C \)’ as an abbreviation of the metalinguistic statement ‘not \( A \supseteq C \)’.

4. Classical Belief Revision

For ease of reference, let us repeat the belief revision axioms endorsed in ‘DMC’ which are a slight variation of AGM’s. In the following, \( Bel \) is the agent’s belief set and \( * \) is the revision function that complies with the decisions made in ‘DMC’: no, not all absurdities are contradictions, but yes, agents who are presented with absurdities should accept them, even though this will lead them into a state in which everything is accepted.

\[
\begin{align*}
(*0) & \quad Bel \text{ is consistent.} & \text{(Initial Consistency)} \\
(*1) & \quad Bel*A = \text{Cn}(Bel*A). & \text{(Closure)} \\
(*2) & \quad A \in Bel*A. & \text{(Success)} \\
(*3) & \quad Bel*A \subseteq \text{Cn}(Bel \cup \{A\}). & \text{(Inclusion)} \\
(*4) & \quad \text{If } \neg A \notin Bel, \text{ then } Bel \subseteq Bel*A. & \text{(Preservation)}
\end{align*}
\]

\(^{18}\)In the finite case, we may think of \( X_{bel} \) as a very long conjunction. Note that (ii) implies that \( Bel \) is closed under logical consequences.

\(^{19}\)This applies to the present paper. But it is very well possible to admit right-nested Ramsey Test conditionals if methods for iterated belief revision are available. See Rott [30].
(5a) If $\bot \in Bel^* A$, then $\bot \in Bel^* A \land B$.  
(Absurdity 1)

(5b) If $\bot \in Bel^* A$ and $\bot \in Bel^* B$, then $\bot \in Bel^* A \lor B$.  
(Absurdity 2)

(6) If $A \vdash B$, then $Bel^* A = Bel^* B$.  
(Intensionality)

(7c) If $B \in Bel^* A$, then $Bel^* (A \land B) \subseteq Bel^* A$.  
(Cut)

(7) $Bel^* (A \land B) \subseteq Cn((Bel^* A) \cup \{B\})$.  
(Conditionalisation)

(8c) If $B \in Bel^* A$, then $Bel^* A \subseteq Bel^* (A \land B)$.  
(Cautious Monotonicity)

(8) If $\neg B \notin Bel^* A$, then $Bel^* A \subseteq Bel^* (A \land B)$.  
(Rational Monotonicity)

Some comments on the (minor) deviations from the original AGM axioms are given in ‘DMC’. For the present paper, it is important to emphasize that both (5) and (8) are weakened versions of AGM’s original fourth and eighth axioms which have the consequents $Cn(Bel \cup \{A\}) \subseteq Bel^* A$ and $Cn((Bel^* A) \cup \{B\}) \subseteq Bel^* (A \land B)$, respectively.\footnote{If we endorse the extra postulate that $Bel = Bel^* \top$, (3) follows from (7), and (4) follows from (8).}

Given Closure (1) and Success (2), the weakenings don’t matter, because the stronger versions can easily be recovered. However, in the next section, when we won’t have unrestricted Success any more, we will see that the differences matter. The postulates (5a) and (5b) are weakenings of AGM’s fifth postulate. This change reflects the fact that in ‘DMC’ a sentence $A$ is an \textit{absurdity} of a belief state $\mathfrak{B}$ if and only if $Bel^* A$ is inconsistent, i.e., iff $\bot \in Bel^* A$.\footnote{Thus, in line with the terminology introduced above, a sentence $A$ is \textit{conceivable} iff $\bot \notin Bel^* A$; $A$ is a \textit{doxastic necessity} iff $\bot \in Bel^* \neg A$; and $A$ is \textit{contingent} iff neither $\bot \in Bel^* A$ nor $\bot \in Bel^* \neg A$.}

Postulates (5a) and (5b) allow absurdities that aren’t contradictions. A sentence $A$ is a \textit{doxastic necessity} if $\bot \in Bel^* \neg A$, and it is a \textit{contingency} if neither $\bot \in Bel^* A$ nor $\bot \in Bel^* \neg A$. Postulates (7c) and (8c) are two prominent weakenings of AGM’s ‘supplementary’ postulates (7) and (8) that are well known in the literature of defeasible reasoning as characterising ‘Cumulative Reasoning’.\footnote{More precisely, (8c) is only a weakening of (8) for non-absurd $A$, i.e., only if $\bot \notin Bel^* A$.}
The most striking fact about difference-making conditionals, i.e., conditionals governed by the Relevant Ramsey Test, is that they do not validate Right Weakening which has long seemed entirely innocuous to conditional logicians. That is, for difference-making conditionals, $A \to C$ and $C \vdash B$ together do not imply $A \to B$. In ‘DMC’, the invalidity of Right Weakening is called the hallmark of difference-making conditionals and indeed of the relevance relation. Another important property of difference-making conditionals is that $A \to C$ does not imply that $A \equiv C$ is accepted: neither ‘(Conjunctive) Simplification’ $A \wedge B \to A$ nor ‘(Disjunctive) Addition’ $A \to A \lor B$ is universally acceptable. If $C$ is accepted anyway (like for instance a logical truth $C$ is), then $A$ cannot be relevant to $C$, even if it logically implies $C$.

Supposing that conditionals are meant to express that the antecedent is positively relevant for the consequent, the symmetry between (Arist2) and (Boet-Abel) appears to be plausible: $A$ cannot be positively relevant simultaneously to both $C$ and $\neg C$. And $A$ and $\neg A$ cannot both be positively relevant to $C$.

We now consider the behaviour of difference-making conditionals regarding these principles (the formal proofs are very short and given in ‘DMC’, p. 155).

(i) We have unrestricted (Arist1): Not $A \to \neg A$. The reason for the unacceptability of $A \to \neg A$ here is that we always have $\neg A \in Bel \ast \neg A$ (and $A \in Bel \ast A$), due to the unrestricted success condition.

(ii) We have unrestricted (Arist2): Not both $A \to C$ and $\neg A \to C$. This is a direct consequence of the definition of difference-making conditionals. $A \to C$ says that $C \in Bel \ast A$ and $C \not\in Bel \ast \neg A$, and $\neg A \to C$ essentially says that it is just the other way round.

(iii) We only have a substantially restricted version of (Boet-Abel): Not both $A \to C$ and $A \to \neg C$, unless $A$ is an absurdity (i.e., $\bot \in Bel \ast A$) and neither $C$ nor $\neg C$ is a belief. If, however, $A$ is an absurdity and neither $C$ nor $\neg C$ is a belief, then both $A \to C$ and $A \to \neg C$ are accepted.

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23 This is true as well if relevance is captured by probabilities. From $Pr(C \mid A) > Pr(C)$, both $Pr(\neg C \mid A) < Pr(\neg C)$ and $Pr(C \mid \neg A) < Pr(C)$ follow easily (provided that $0 < Pr(A) < 1$).

24 And similarly, not $\neg A \to A$. In the AGM framework employed, which is based on a supraclassical background logic $\mathcal{Cn}$, there is no real difference between these two versions. Thus we will neglect this and all similar twin theses in the following.

25 It should be mentioned that there is a straightforward, philosophically very different way of making difference-making conditionals fully connexive. As explained in the final section of ‘DMC’, it is not necessary to implement the idea of difference-making in terms of
Summing up: Given the design decisions for belief revision taken in ‘DMC’, the two connexive theses associated with Aristotle are valid in unrestricted form. The connexive thesis associated with Boethius and Abelard, however, is valid only in a restricted, ‘humble’ (Kapsner [14]) form.

5. Conceivability-Limited Belief Revision

Now we embark on the project of making difference-making conditionals “fully connexive”. The central idea is that agents should simply refuse to accept absurdities.26 The crucial step is to adapt the AGM postulates (or the postulates used in ‘DMC’) in order to represent our new design decisions: no, not all absurdities are contradictions, and no, agents who are presented with absurdities should not accept them. In the following axioms, ‘⋆’ is the symbol for the revision function that complies with theses decisions.

\[(⋆0) \text{Bel and Bel}⋆A \text{ are consistent. (Unrestricted Consistency)}\]
\[(⋆0b) \neg X_{\text{bel}} \in \text{Bel}⋆\neg X_{\text{bel}}. \text{ (Contingent Belief)}\]
\[(⋆1) \text{Bel}⋆A = \text{Cn(Bel}⋆A). \text{ (Closure)}\]

a belief revision semantics. One might opt for a possible-worlds semantics for suppositional conditionals \(A > C\) in the style of Stalnaker [33] and Lewis [18], endorse Lewis’s Centering condition, and define the difference-making conditional \(A \gg C\) as the conjunction of \(A > C\) and \(\neg(\neg A > C)\). Many results obtained in the belief revision framework then carry over to the possible worlds framework (see Raidl [26]). But the two kinds of semantics diverge on Boethius/Abelard’s Principle. The unless clause encountered in (iii) includes the condition that neither \(C\) nor \(\neg C\) is a belief. This condition is met whenever the agent suspends judgment on \(C\). But truth is different from belief. Possible worlds are points of evaluation where every (factual) sentence is either true or false. Possible worlds do not offer the option of “suspending truth” on \(C\). If \(A\) is absurd from the perspective of a possible world \(w\), which entails that both \(A > C\) and \(A > \neg C\) are true at \(w\) for arbitrary \(C\), then \(\neg A\) is true at \(w\). Because either \(C\) or \(\neg C\) is true at \(w\), one of the conditionals \(\neg A > C\) or \(\neg A > \neg C\) is true at \(w\). But then either \(A \gg C\) or \(A \gg \neg C\) is false at \(w\). Thus the unless clause (with truth substituted for belief) in the restricted Boethius/Abelard’s Principle mentioned in (iii) cannot be satisfied, and thus the possible-worlds interpretation of difference-making conditionals supports the unrestricted version of the principle of (Boet-Abel). As it also validates (Arist1) and (Arist2), it is fully connexive.

26 This approach has some similarity to the view of negation as cancellation (Strawson 1952, Routley 1978, Routley et al 1982, Routley and Routley 1985, Priest 1999) maintaining that in \(A \land \neg A\), the negation of \(A\) cancels the information provided by \(A\). The contradiction thus provides no information at all. Also compare Wittgenstein’s [34, p. 209] famous dictum: “Well then, don’t draw any conclusions from a contradiction; make that a rule.” In the present approach, a contradiction produces no change in the agent’s belief state at all. The same holds for absurdities of other kinds.
(⋆2) \( A \in \text{Bel} \ast A \) or \( \text{Bel} \ast A = \text{Bel} \). (Restricted Success)

(⋆3) \( \text{Bel} \ast A \subseteq Cn(\text{Bel} \cup \{A\}) \). (Inclusion)

(⋆4) If \( \neg A \notin \text{Bel} \), then \( Cn(\text{Bel} \cup \{A\}) \subseteq \text{Bel} \ast A \). (Preservation+)

(⋆5a) If \( A \land B \in \text{Bel} \ast (A \land B) \), then \( A \in \text{Bel} \ast A \). (Absurdity 1)

(⋆5b) If \( A \lor B \in \text{Bel} \ast (A \lor B) \), then \( A \in \text{Bel} \ast A \) or \( B \in \text{Bel} \ast B \). (Absurdity 2)

(⋆6) If \( A \not\models B \), then \( \text{Bel} \ast A = \text{Bel} \ast B \). (Intensionality)

(⋆7c) If \( B \in \text{Bel} \ast A \), then \( \text{Bel} \ast (A \land B) \subseteq \text{Bel} \ast A \). (Cut)

(⋆7) \( \text{Bel} \ast (A \land B) \subseteq Cn((\text{Bel} \ast A) \cup \{B\}) \). (Conditionalisation)

(⋆8c) If \( B \in \text{Bel} \ast A \), then \( \text{Bel} \ast A \subseteq \text{Bel} \ast (A \land B) \). (Cautious Monotonicity)

(⋆8) If \( \neg B \notin \text{Bel} \ast A \), then \( \text{Bel} \ast A \subseteq \text{Bel} \ast (A \land B) \). (Rational Monotonicity)

Like in classical belief revision, we call (⋆0)–(⋆6) the basic postulates and (⋆7c)–(⋆8) the supplementary postulates. Postulate (⋆0) is new and strengthens AGM’s original fifth postulate which is only a restricted consistency postulate (it is restricted to consistent inputs \( A \)). Obviously, an absurdity cannot be defined with reference to inconsistent belief sets in this adapted context, because (⋆0) rules out such sets. Instead we now say that a sentence \( A \) is an absurdity (or a doxastic impossibility) of a belief state \( \mathfrak{B} \) iff the agent refuses to accept \( A \) when \( A \) is presented to her as an input to the set of beliefs or assumptions, i.e., iff \( A \notin \text{Bel} \ast A \). (⋆0) implies that contradictions are absurdities. Thus, in line with the terminology introduced above, a sentence \( A \) is conceivable iff \( A \in \text{Bel} \ast A \); \( A \) is a doxastic necessity iff \( \neg A \notin \text{Bel} \ast \neg A \); and \( A \) is contingent iff both \( A \in \text{Bel} \ast A \) and \( \neg A \in \text{Bel} \ast \neg A \).

Postulate (⋆0b) is one half of the claim that the agent’s beliefs are, or more precisely: that her strongest belief is contingent. The other half, namely that \( X_{\text{bel}} \in \text{Bel} \ast X_{\text{bel}} \), follows from (⋆2), or alternatively from (⋆0) and (⋆4).

Restricted Success (⋆2) says that the agent should not change her beliefs at all if the input \( A \) is absurd. If a new piece of information or a hypothetical assumption is absurd, then it seems a good strategy not just to refuse to accept it, but simply to ignore it and change nothing. The postulates of Unrestricted Consistency and Restricted Success are the key deviations from AGM.

Postulate (⋆4) is the original fourth AGM postulate, not the weaker postulate of Preservation that is used in ‘DMC’. It is important to have this
stronger version here because it entails a restricted success axiom: no sentence $A$ that is compatible with the agent’s beliefs is absurd.\(^\text{27}\)

Postulates (\(*5a\)) and (\(*5b\)) are counterparts of postulates (\(*5a\)) and (\(*5b\)) of ‘DMC’. They put exactly the same constraints on the set of absurdities, but rely on our new notion of ‘absurdity’.

The supplementary postulates (\(*7c\))–(\(*8\)) are identical to those of the previous section. Rational Monotonicity (\(*8\)) is a sound principle, but the stronger original eighth AGM postulate does not hold for absurd inputs $A$ in conceivability-limited revision, since $B$’s being consistent with $Bel$ does not entail that $B$ is an element of $Bel$.\(^\text{28}\)

Given the background postulates (\(*0\)) and (\(*1\)), it is immediate that (\(*7\)) implies (\(*7c\)) and (\(*8\)) implies (\(*8c\)). Apart from these simple implications, I think this set of postulates is free from redundancies.

**Lemma 1.** [Derived properties of \(*\)] The following conditions are derivable from the principles (\(*0\))–(\(*8\)) of conceivability-limited belief revision.

(r1) $Bel = Bel \star \top = Bel \star X_{bel}$.

(r2) $A \in Bel$ iff $X_{bel} \land \neg A \notin Bel \star (X_{bel} \land \neg A)$.

(r3) Not $Bel \star A \subseteq \neg Bel$.

(r4) If $Bel \subseteq Bel \star A$, then $\neg A \notin Bel$.

(r5) If $A \notin Bel \star A$, then $\neg A \in Bel$.

(r6) If $A \notin Bel \star A$, then $A \notin Bel$.

(r7) If $A \notin Bel \star A$, then $\neg A \in Bel \star A$.

(r8) $Bel \star A = Bel$ iff either $A \in Bel$ or $A \notin Bel \star A$.

(r9) If $A$ is not contingent, then $Bel \star A = Bel = Bel \star \neg A$.

(r10) If $C \in Bel \star A$ and $C \notin Bel \star \neg A$, then $A$ is contingent.

(r11) If $Bel \subseteq Bel \star A$, then $Bel \star A$ is inconsistent with $Bel$.

(Consistent Expansion)

(r12) If $A \in Bel \star B$, then $A \in Bel \star A$.  

(Regularity)

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\(^{27}\) As already mentioned, in the presence of unrestricted Success, the two postulates are equivalent. Here, with Restricted Success only, we need (\(*4\)) in order to derive the conditions (r5)–(r13) listed in Lemma 1.

\(^{28}\) If we endorse the extra postulate that $Bel = Bel \star \top$, (\(*3\)) follows from (\(*7\)). But (\(*4\)) does not quite follow from (\(*8\)). For this we need an additional success postulate saying that if $\neg A \notin Bel$, then $A \in Bel \star A$. This is condition (r5) of Lemma 1, which we will derive from (\(*4\)). On the other hand, (\(*8c\)) follows from (\(*8\)) without restrictions for conceivability-limited belief revision (cf. footnote 22).
(r13) If $A \notin \text{Bel} \star A$, then $\neg A \in \text{Bel} \star B$. \hspace{1cm} \text{(Strong Regularity)}

(r14) $A \in \text{Bel} \star A$ or $\neg A \in \text{Bel} \star \neg A$. \hspace{1cm} \text{(Disjunctive Success)}

(r15) If $C \in \text{Bel} \star A$ and $C \notin \text{Bel} \star \neg A$, then $C$ is contingent.

(r16) If $A \in \text{Bel} \star A$ and $\neg B \notin \text{Bel} \star A$, then
\[
Cn((\text{Bel} \star A) \cup \{B\}) \subseteq \text{Bel} \star (A \land B).
\hspace{1cm} \text{(Guarded Subexpansion)}
\]

(r17) If $A \in \text{Bel} \star B$ and $B \in \text{Bel} \star A$, then $\text{Bel} \star A = \text{Bel} \star B$. \hspace{1cm} \text{(Reciprocity)}

(r18) If $\text{Bel} \star (A \lor B) = \text{Bel} \star A$ or $\text{Bel} \star (A \lor B) = \text{Bel} \star B$ or $\text{Bel} \star (A \lor B) = (\text{Bel} \star B) \cap (\text{Bel} \star B)$. \hspace{1cm} \text{(Disjunctive Factoring)}

The derivations of (r1)–(r11) need only the basic postulates ($\star$0)–($\star$6), those of (r12)–(r18) also need at least one of the supplementary principles ($\star$7c)–($\star$8).

The longer names of the conditions mentioned in Lemma 1 are taken from Hansson, Fermé, Cantwell and Falappa [12].

It is easy to see that the above collection of axioms is consistent. Use a system-of-spheres modelling in the style of Grove [10], but do not require these systems to exhaust the space of all possible worlds, i.e., to be universal in the sense of Lewis [18, p. 16]—first design decision! Then let the set of models of $\text{Bel} \star A$ be intersection of the $A$-models with the smallest sphere that contains any $A$-models if such a sphere exists, and let it be be the smallest sphere otherwise—second design decision!

**Observation 2.** (a) The axioms ($\star$0)–($\star$8) are sound with respect to this simple sphere-based modelling, provided that the system of spheres $\mathcal{S}$ contains at least two spheres (which means that $\bigcap \mathcal{S} \neq \bigcup \mathcal{S}$).

(b) Every revision function satisfying ($\star$0)–($\star$8) can be represented as generated from a system of spheres $\mathcal{S}$ by the recipe mentioned above.

The proviso mentioned in part (a) of the observation is necessary for ($\star$0b). Apart from this addition, the modelling is formally identical to what Hansson, Fermé, Cantwell and Falappa [12] call *non-prioritized* or *credibility limited sphere-based revision*. But there is a clear difference in interpretation here. For Hansson et al. the propositions that initiate successful revisions are *credible* propositions, while in this paper they only need to be conceivable, i.e., non-absurd. It is true that no credible proposition is absurd, but not every proposition that isn’t absurd is credible. So we call the model using $\star$ *conceivability-limited revision*.\(^{29}\)

\(^{29}\)Also compare Garapa [7] on a refined model of credibility-limited belief revision.
We will now summarize what consequences or our deviations from classical belief revision are for the connexivity of difference-making conditionals. A formal development will be given in Section 6.

(i) We again have unrestricted (Arist1): Not $A \not\equiv \neg A$; and also not $\neg A \not\equiv A$ (there is still no real difference between these two versions). The reason for its validity is that the acceptance of $A \not\equiv \neg A$ would require that $\neg A \in Bel^* A$ and $\neg A \notin Bel^* \neg A$, i.e., that both $A$ and $\neg A$ are absurd, which is impossible, by (r14).

(ii) We again have unrestricted (Arist2): Not both $A \not\equiv C$ and $\neg A \not\equiv C$. The reason is exactly the same as in Section 4.

(iii) In contrast to the context of ‘DMC’, now (Boet-Abel) is valid in its unrestricted form: Not both $A \not\equiv C$ and $A \not\equiv \neg C$. This is because we cannot simultaneously have $C \in Bel^* A$ and $\neg C \in Bel^* \neg A$, due to the axiom of Unrestricted Consistency.

Summing up: Given the alternative design decision regarding the axiomatization of belief revision, difference-making conditionals validate all connexive theses in unrestricted (‘unhumble’, ‘ambitious’) form. Difference-making conditionals thus can be fully connexified.

6. The Logic of Difference-Making Conditionals Based on Conceivability-Limited Belief Revision

Remember that we suppose throughout this paper that the belief set $Bel$ is consistent (and that $Bel^* A$ is consistent, too). This is the Relevant Ramsey Test based on $*$:

$$A \not\equiv C @ B \iff C \in Bel^* A \text{ and } C \notin Bel^* \neg A.$$  \hfill (RRT)

We will see later that $A \not\equiv C @ B$ implies that both $A$ and $C$ are contingent. This is in accord with the intuitions captured by (Irrelevant $\top$) and (Irrelevant $\bot$): Only contingent sentences can be relevant to one another.

Notice that RRT has recourse only to revised belief sets. It does not refer to any property of the original belief set $Bel$.

6.1. What Can Be Expressed by Difference-Making Conditionals?

First of all, we specify the ‘meanings’ of some basic difference-making conditionals:
Table 1. The acceptance conditions of some (combinations of) simple
difference-making conditionals

<table>
<thead>
<tr>
<th>Condition</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \gg A$</td>
<td>$A$ is contingent.</td>
</tr>
<tr>
<td>$A \gg A \land C$</td>
<td>$C \in \text{Bel}^* A$ and $A$ is contingent.</td>
</tr>
<tr>
<td>$A \gg A \lor C$</td>
<td>$C \notin \text{Bel}^* \neg A$ and $A$ is contingent.</td>
</tr>
<tr>
<td>$A \land C \gg C$</td>
<td>$C \notin \text{Bel}^* (A \land C)$ and $A \land C$ is contingent.</td>
</tr>
<tr>
<td>$A \lor C \gg C$</td>
<td>$C \in \text{Bel}^* (A \lor C)$ and $A \lor C$ is contingent.</td>
</tr>
<tr>
<td>$A \land \mathit{X}<em>{\mathit{bel}} \gg A \land \mathit{X}</em>{\mathit{bel}}$</td>
<td>$A$ is consistent with $\text{Bel}$.</td>
</tr>
<tr>
<td>$\mathit{X}<em>{\mathit{bel}} \gg \mathit{X}</em>{\mathit{bel}} \land A$</td>
<td>$A \in \text{Bel}$ ($A$ is a belief).</td>
</tr>
</tbody>
</table>
| $\mathit{X}_{\mathit{bel}} \gg \mathit{X}_{\mathit{bel}} \land A$ and $A \not
\gg A$ | $\neg A \notin \text{Bel}^* \neg A$ $(A$ is doxastic necessity). |
| $\mathit{X}_{\mathit{bel}} \gg \mathit{X}_{\mathit{bel}} \land \neg A$ and $A \not
\gg A$ | $A \notin \text{Bel}^* A$ $(A$ is absurd). |

**Lemma 3.** Let the basic conditions $(*0)$–$(*6)$ for conceivability-limited belief revision be given. Then the acceptance conditions of some (combinations of) simple difference-making conditionals are as detailed in Table 1.

Difference-making conditionals lose their contrastive character when the antecedent is logically stronger or weaker than, or equally strong as, the consequent. A conditional of the form $A \gg A \land C$ is a de-relevantised conditional; it expresses a suppositional conditional ‘If $A$ then $C$’ (plus the contingency of $A$ as a side condition) using the difference-making conditional $\gg$. It is almost equivalent to the standard Ramsey conditional $A \gg C$ to which it only adds that $A$ is a contingent sentence. $A \gg A \land C$ is strictly weaker than $A \gg C$. This should come as no surprise, because the acceptance of a difference-making conditional $A \gg C$, in contrast to that of its de-relevantised counterpart, also depends on what happens when its antecedent $A$ is supposed to be false.

There are various things we would like to express in the language of difference-making conditionals. It is more difficult to express some things with difference-making conditionals based on the alternative revision functions $*$ than with those based on $*$ as in ‘DMC’. For instance, it is not directly possible to express that $A$ is an absurdity (with respect to $\mathfrak{B}$), since $A \not
\gg A$ only says that $A$ is an absurdity or $\neg A$ is an absurdity. Similarly,

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$^{30} C \notin \text{Bel}^* (A \land C)$ means that $C$ is not more entrenched than $A$. An explanation of this claim can be found in Rott [29, Ch. 8]
expressing that \( A \) is a doxastic necessity (with respect to \( B \)) is slightly complicated, since \( A \not\propto A \) only says that either \( A \) is a doxastic necessity or \( \neg A \) is a doxastic necessity.

6.2. Basic Principles

If a conditional of the form ‘If \( A \) then \( C \)’ is used in everyday discourse and meant to convey that \( A \) is relevant for \( C \), then \( A \) and \( C \) are hardly ever logically related. What is common usage in a seminar on propositional logic is apt to cause bewilderment in practical contexts. In the sense of ‘relevance’ intended here, it sounds odd to say that a sentence is relevant for some of its subsentences or for some Boolean compounds containing it. Still odder does it sound to say that a sentence is relevant for itself. Yet, if one wants to present a conditional logic, it is just one’s principal task to deal with such statements. We should not expect that the principles valid for difference-making conditionals are generally intuitively appealing. The most important thing to bear in mind is that the Relevant Ramsey Test RRT provides a clear and simple doxastic semantics that is applicable to arbitrary compounds of factual sentences as antecedents and consequents. We are now going to explore the logic of conditionals governed by RRT.

We are now in a position to list the basic principles of difference-making conditionals. All these principles are to be read as quantified over all belief states \( \mathfrak{B} \), but the clause ‘\( \ast \mathfrak{B} \)’ after each conditional is left implicit throughout: ‘\( A \propto C \)’ is short for ‘\( A \propto C \ast \mathfrak{B} \)’ and ‘\( A \not\propto C \)’ is short for ‘not \( A \propto C \ast \mathfrak{B} \)’. I trust that this somewhat sloppy notation will not cause any confusion. A principle of the form ‘If \( \Phi \), then \( \Psi \)’ formulates a validity in the sense that for every belief state \( \mathfrak{B} \), if the conditionals mentioned in \( \Phi \) are all accepted/rejected in \( \mathfrak{B} \), then the conditionals mentioned in \( \Psi \) are accepted/rejected in \( \mathfrak{B} \). The variables \( A, B \) and \( C \) range over propositional sentences without any occurrences of the conditional connective.

\[ (\ast 0) \; X_{\text{bel}} \propto X_{\text{bel}}. \]
\[ (\ast 1) \text{ If } A \propto B \land C, \text{ then } A \propto B \text{ or } A \propto C. \]
\[ (\ast 2a) \; A \propto C \iff (A \propto A \land C \text{ and } A \propto A \lor C). \]
\[ (\ast 2b) \; A \propto A \land C \iff (\neg A \not\propto \neg A \lor C \text{ and } A \propto A). \]
\[ (\ast 3) \text{ If } A \propto A \land C, \text{ then } A \land \neg C \land X_{\text{bel}} \not\propto A \land \neg C \land X_{\text{bel}}. \]
\[ (\ast 4) \text{ If } A \propto A \lor C \text{ and } \neg A \land X_{\text{bel}} \not\propto \neg A \land X_{\text{bel}}, \text{ then } \neg C \land X_{\text{bel}} \propto \neg C \land X_{\text{bel}}. \]
\[ (\ast 5a) \; A \propto A \iff \neg A \not\propto \neg A. \]
(≫5b) If $A \land B \gg A \land B$, then $A \gg A$ or $B \gg B$.

(≫5c) If $A \land B \gg A \land B$ and $A \lor C \gg A \lor C$, then $A \gg A$.

(≫6) If $A \vdash B$ and $C \vdash D$, then: $A \gg C$ iff $B \gg D$.

Let us emphasize again that in the set-up of both ‘DMC’ and the present paper, Reflexivity is not in general valid. In ‘DMC’, the conditional $A \gg A$ is accepted if and only if $\neg A$ is not absurd. In our new set-up, $A \gg A$ is accepted if and only if neither $A$ nor $\neg A$ is absurd, i.e., $A$ is contingent.\(^{30}\)

Condition (≫0) says that the strongest belief is contingent. This implies not only that it is consistent, but also that there are non-absurd non-beliefs.

Condition (≫1) is a weaker replacement of the condition of ‘Right Weakening’ which is valid for suppositional conditionals. It says that when $A$ is a relevant antecedent to $B \land C$, then it must also be a relevant antecedent to at least one of $B$ and $C$.

Condition (≫2a) is valid but close to trivial for reasoning with suppositional conditionals. It says that if one accepts ‘If $A$ then relevantly $C$’, i.e., if $A$ is a relevant antecedent for $C$, then $A$ is also a relevant antecedent for both the conjunction $A \land C$ and and the disjunction $A \lor C$—and vice versa. As we can gather from Table 1, the two conjuncts correspond rather neatly to the two parts of the Relevant Ramsey Test.

Condition (≫2b) says that if $A$ is a relevant antecedent for $A \land C$, then $A$ is also a relevant antecedent for itself, but $\neg A$ is not a relevant antecedent for $\neg A \lor C$—and vice versa. This means that $\neg A \gg \neg A \lor C$ is essentially the negation of $A \gg A \land C$: The one is accepted if and only if the other isn’t (neglecting the contingency condition for $A$). By chaining (≫2a) and (≫2b), we find that only contingent sentences can be relevant antecedents: If $A \gg C$, then $A \gg A$. (We will later derive a similar claim for relevant consequents: If $A \gg C$, then $C \gg C$; see Lemma 11.)

Conditions (≫3) and (≫4) correspond to the Inclusion and Preservation+ postulates for revisions. Like in the numbering of belief revision principles, the third and fourth conditions are the only ones that refer to the agent’s initial beliefs. In view of Lemma 3, we can see that contingencies play a crucial role here.

Conditions (≫5a)–(≫5c) deal with contingencies. The first two say that the set of non-contingencies is closed under negation and conjunction (and thus under disjunction, too, as we will shortly verify). The third one is

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\(^{30}\)See Lemma 3.—It is known that some connexive logics do not satisfy Idempotence in the form of $(A \land A) \rightarrow A$ or $A \rightarrow (A \land A)$ (Estrada-González and Ramírez-Cámara [6]). Given our essentially classical background logic, this is equivalent to a failure of Reflexivity.
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a kind of interpolation property for contingencies. The sentence $A \gg A$ is contingent whenever both a stronger sentence $A \land B$ and a weaker sentence $A \lor C$ are contingent. These three conditions correspond to axiom schemes A1–A3 of the ‘minimal non-contingency logic’ of Kuhn [15].

$(\gg 6)$ is an intensionality principle that corresponds to AGM’s sixth and part of their first postulates. Relevant antecedents and consequents may be replaced by logically equivalent sentences.

Now we need to show that the basic principles for difference-making conditionals are valid.

**Observation 4.** [Basic principles for $\gg$ are valid] Let $*$ be a revision function satisfying the basic postulates mentioned in Section 5, and let $\gg$ be obtained from $*$ by RRT. Then $\gg$ satisfies the basic principles ($\gg 0$), ($\gg 1$), ($\gg 2a$), ($\gg 2b$), ($\gg 3$), ($\gg 4$), ($\gg 5a$), ($\gg 5b$), ($\gg 5c$) and ($\gg 6$).

The following lemma offers a rather long list of derived principles. Some of them are interesting in their own right, some will be useful in later proofs. (All proofs are collected in the Appendix.)

**Lemma 5.** [Derived properties of $\gg$] Let the principles (\(\gg 0\))–(\(\gg 6\)) for difference-making conditionals be given. Then

(c1) If $A \gg C$, then $A \gg A$.
(c2) $A \not\gg \bot$.
(c3) $A \not\gg \top$.
(c4) $\top \not\gg C$.
(c5) $\bot \not\gg C$.
(c6) $A \gg A$ iff ($A \gg A \lor C$ or $A \gg A \lor \neg C$).
(c7) If $A \land B \vdash \bot$, then $A \not\gg B$.
(c8) Not both $A \gg A \land C$ and $A \gg A \land \neg C$.
(c9) If $A \lor B \gg A \lor B$ then $A \gg A$ or $B \gg B$.
(c10) If $\top \vdash A \lor B$, then $A \not\gg B$.
(c11) If $A \gg A \land B \land C$, then $A \gg A \land C$.
(c12) If $A \gg A \lor B \lor C$, then $A \gg A \lor C$.
(c13) If $A \gg B$ and $A \gg C$, then $A \gg B \land C$.
(c14) $A \gg A \lor C$ iff ($\neg A \not\gg \neg A \land C$ and $A \gg A$).
(c15) If $A \not\gg A$ and $A \land B \gg A \land B$, then $\neg A \land B \not\gg \neg A \land B$. 
(c16) If $X_{\text{bel}} \gg X_{\text{bel}} \land A$, then $X_{\text{bel}} \land A \gg X_{\text{bel}} \land A$.

(c17) If $A \not\gg A$ and $A \land B \gg A \land B$, then $X_{\text{bel}} \gg X_{\text{bel}} \land A$.

(c18) If $A \not\gg A$ and $A \lor B \gg A \lor B$, then $X_{\text{bel}} \gg X_{\text{bel}} \land \neg A$.

(c19) If $A \not\gg A$, then $X_{\text{bel}} \gg X_{\text{bel}} \land A$ or $X_{\text{bel}} \gg X_{\text{bel}} \land \neg A$.

(c20) If $X_{\text{bel}} \gg X_{\text{bel}} \land A$ and $A \gg A \land C$, then $X_{\text{bel}} \gg X_{\text{bel}} \land C$.

(c21) If $A \gg A \land C$, then $X_{\text{bel}} \gg X_{\text{bel}} \land (A \supset C)$.

(c22) If $A \gg A \lor C$ and $X_{\text{bel}} \gg X_{\text{bel}} \land C$, then $X_{\text{bel}} \gg X_{\text{bel}} \land A$.

(c23) If $A \gg A$, $X_{\text{bel}} \gg X_{\text{bel}} \land C$ and $X_{\text{bel}} \not\gg X_{\text{bel}} \land \neg A$, then $A \gg A \land C$.

A few brief comments on some of these conditions are in order. It turns out that it does not make sense to place tautologies or contradictions to the left or to the right of difference-making conditionals based on ⋆ (parts (c2)–(c5) of the Lemma). *They can never be relevant antecedents or relevant consequents! And this is indeed what one would expect intuitively. So this is an important advantage over the setup of ‘DMC’ where contradictions are not only taken to make sense, but are very heavily made use of both as antecedents and as consequents. Given our new belief revision semantics, contradictions have no interesting role to play. The reason that contradictions are not acceptable antecedents is that $\text{Bel} \gg \top = \text{Bel} \gg \bot = \text{Bel}$, and the reason that they are not acceptable consequents is that $\top$ is in every and $\bot$ is in no belief set.

Condition (c10) is a qualitative counterpart of the findings of Popper and Miller [24, Theorems 2 and 5] that there can be no positive probabilistic support between $A$ and $A \supset B$ in either direction (notice that $\top \vdash A \lor (A \supset B)$).

As we pointed out above, Right Weakening is in general invalid for difference-making conditionals. Like ($\gg 1$), the principle (c11) is a weakened form of Right Weakening. Condition (c13) says that from ‘If $A$ then relevantly $B’ and ‘If $A$ then relevantly $C$, one can infer ‘If $A$ then relevantly $B$ and $C$. This important principle is called ‘Conjunction in the Consequent’ or simply ‘And’ in the literature, and it is the only inference pattern of suppositional conditionals that remains valid for the difference-making conditional $\gg$!

Conditions (c16)–(c19) say that beliefs are consistent with $\text{Bel}$, that doxastic necessities are beliefs, that absurdities are disbeliefs and that non-contingencies are either beliefs or disbeliefs.
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Figure 1. Some useful implications. Here ‘>’ is the suppositional conditional according to (RT) and ‘≫’ is the difference-making conditional according to (RRT), both based on conceivability-limited belief revision

Conditions (c21) and (c22) look like simpler versions of (≫3) and (≫4) (see Lemma 7(i) below), but as far as I can see, they are not strong enough to replace the latter.

A number of important elementary implications are represented in Figures 1 and 2.

The following observation is our most important result from the point of view of connexivity.

Observation 6. The difference-making conditional ≫ satisfies the following connexivity principles:

(≫Arist1) Not $A \gg \neg A$.

(≫Arist2) Not both $A \gg C$ and $\neg A \gg C$.

(≫Boet-Abel) Not both $A \gg C$ and $A \gg \neg C$.

(≫Arist1) are (≫Arist2) Aristotle’s First and Second Thesis, now in the particular specialization for difference-making conditionals. (≫Boet-Abel) is the principle of Boethius and Abelard. Thus the difference-making conditional ≫ based on * is fully connexive. In particular, it is excluded that ‘If $A$ then relevantly $C$’ and ‘If $\neg A$ then relevantly $C$’ are accepted simultaneously, and also that ‘If $A$ then relevantly $C$’ and ‘If $A$ then relevantly $\neg C$’ are accepted simultaneously.
Figure 2. The meanings of the conditions of Figure 1

The following lemma uses a few facts about the relation between belief, doxastic necessity and absurdity. First, \( A \) is a doxastic necessity if and only if it is a non-contingent belief. Second, \( A \) is an absurdity if and only if \( \neg A \) is a doxastic necessity, i.e., \( \neg A \) is a non-contingent belief, or, again equivalently, \( A \) is a non-contingent non-belief.

**Lemma 7.** [Alternative ways of expressing beliefs, doxastic necessities and absurdities] Let the principles \((\gg 0)\)–\((\gg 6)\) for difference-making conditionals be given.

(i) The following conditions are equivalent:

\begin{itemize}
  \item[(i.0)] \( A \) is a belief;
  \item[(i.1)] \( X_{\text{bel}} \gg X_{\text{bel}} \land A \);
  \item[(i.2)] \( \neg A \land X_{\text{bel}} \not\gg \neg A \land X_{\text{bel}} \);
  \item[(i.3)] for all \( B \), \( \neg A \land B \land X_{\text{bel}} \not\gg \neg A \land B \land X_{\text{bel}} \).
\end{itemize}

(ii) The following conditions are equivalent:

\begin{itemize}
  \item[(ii.0)] \( A \) is a doxastic necessity;
  \item[(ii.1)] \( X_{\text{bel}} \gg X_{\text{bel}} \land A \) and \( A \not\gg A \);
  \item[(ii.2)] \( A \land X_{\text{bel}} \gg A \land X_{\text{bel}} \) and \( A \not\gg A \);
\end{itemize}
(ii.3) there is a \( B \) such that \( A \land B \gg A \land B \) and \( A \nRightarrow A \).

(ii.4) for all \( B \), \( A \lor B \nRightarrow A \lor B \).

(iii) The following conditions are equivalent:

(iii.0) \( A \) is an absurdity;

(iii.1) \( X_{\text{bel}} \gg X_{\text{bel}} \land \neg A \) and \( A \nRightarrow A \);

(iii.2) \( A \land X_{\text{bel}} \nRightarrow A \land X_{\text{bel}} \) and \( A \nRightarrow A \);

(iii.3) \( X_{\text{bel}} \nRightarrow X_{\text{bel}} \land A \) and \( A \nRightarrow A \);

(iii.4) \( \neg A \land X_{\text{bel}} \gg \neg A \land X_{\text{bel}} \) and \( A \nRightarrow A \);

(iii.5) there is a \( B \) such that \( A \lor B \gg A \lor B \) and \( A \nRightarrow A \).

(iii.6) for all \( B \), \( A \land B \nRightarrow A \land B \).

Comments. For the ‘canonical’ ways of expressing beliefs, doxastic necessities and absurdities, see Lemma 3.

On (i). \( A \) is a belief if and only if \( X_{\text{bel}} \land \neg A \) is an absurdity, i.e., if and only if \( X_{\text{bel}} \supset A \) is a doxastic necessity. In this case, since \( X_{\text{bel}} \) is contingent, \( X_{\text{bel}} \land \neg A \) is even a contradiction and \( X_{\text{bel}} \supset A \) is even a logical truth.

It is easy to check that the representation in (i.3) reveals that the set of beliefs is closed under singleton entailment and conjunction.

Concerning (i.1) and (i.2), both ideas are attractive. The former is clearly simpler and thus seems preferable. I will endorse it as the “official” definition of beliefs in terms of conditionals. The latter reduces questions about beliefs to questions about contingencies, and there are particularly perspicuous rules for reasoning about contingencies, even in the absence of the supplementary AGM postulates.

On (ii) and (iii). Weakening a doxastic necessity results in another doxastic necessity, and strengthening an absurdity results in another absurdity, but not the other way round. A comparison of (ii.1) and (i.1) shows immediately that every doxastic necessity is a belief.

(iii.1) and (iii.2) express that absurdities are non-contingent sentences the negations of which are believed. (iii.3) and (iii.4) express that absurdities are non-contingent non-beliefs. In order for the idea of (iii.6) to work, we have to presuppose that there is a contingency. (Otherwise any proper strengthening of a doxastic necessity will result in an absurdity.) Since we have posited in \((\gg 0b)\) that the strongest belief \( X_{\text{bel}} \) is a contingency, this presupposition is met.

The following principles are reminiscent of inference patterns studied intensively in the psychology of reasoning. They concern the interrelations
between belief in the antecedent and belief in the consequent of an accepted difference-making conditional $A \gg C$, interrelations which hold for all belief states:

**Observation 8.** The difference-making conditional $\gg$ satisfies the following principles, in which ‘$A \in Bel$’ is used as a mnemonic for the conditional ‘$X_{\text{bel}} \gg X_{\text{bel}} \land A$’:

- **($\gg$MP)** If $A \gg C$ and $A \in Bel$, then $C \in Bel$. (Modus Ponens)
- **($\gg$MT)** If $A \gg C$ and $\neg C \in Bel$, then $\neg A \in Bel$. (Modus Tollens)
- **($\gg$AC)** If $A \gg C$ and $C \in Bel$, then $A \in Bel$. (Affirming the Consequent)
- **($\gg$DA)\textsuperscript{*}** If $A \gg C$ and $\neg A \in Bel$, then $C \notin Bel$. (weak form of Denying the Antecedent)

Written in extensive form, the four schemes are phrased purely in the language of difference-making conditionals. ($\gg$MP) and ($\gg$MT) are essentially the inference schemes of *Modus Ponens* and *Modus Tollens*. Due to the fact that the difference-making conditional $\gg$ embodies an idea of relevance, it also satisfies a form of *Affirming the Consequent*, ($\gg$AC). This inference scheme is *not* a fallacy here, since the meaning of our conditionals is not the one usually presupposed in formal logic. If $A \gg C$ is accepted and $C$ is a belief, then $A$ is a belief, too.\textsuperscript{31} However, the dual scheme of *Denying the Antecedent* is not satisfied by $\gg$; only a weaker form holds: If $A \gg C$ is accepted and the antecedent $A$ is denied, then the consequent $C$ is not believed to be true by the agent. It does not follow that $C$ is denied by the agent. This weakened scheme is still patently invalid for material or suppositional conditionals. Once more: Alleged logical fallacies may become sound inferences when conditionals are understood in a different, difference-making way.

### 6.3. Compound-Antecedent Principles

We now turn to a systematic discussion of the effects of AGM’s ‘supplementary’ belief revision postulates. It is not very difficult to ‘translate’ the belief revision postulates and their counterparts for the suppositional conditional $>\!\!\!<$ into the language of difference-making conditionals. This is possible by

\textsuperscript{31}This does *not* mean that the difference-making conditional is anything like a biconditional. It actually isn’t, provided we apply the criterion that $\rightarrow$ is a biconditional only if $A \rightarrow B$ is equivalent to $B \rightarrow A$. So difference-making conditionals comply with another basic connexive intuition.
making extensive use of de-relevantised conditionals of the form ‘$A \gg A \land C$’. These are supplementary principles of difference-making conditionals:

($\gg 7c$) If $A \gg A \land B$ and $A \land B \gg A \land B \land C$, then $A \gg A \land C$.

($\gg 7$) If $A \land B \gg A \land B \land C$ and $A \gg A$, then $A \gg A \land (B \supset C)$.

($\gg 7'$) If $A \gg A \land C$ and $B \gg B \land C$ and $A \lor B \gg A \lor B$, then $A \lor B \gg (A \lor B) \land C$.

($\gg 8c$) If $A \gg A \land B$ and $A \gg A \land C$, then $A \land B \gg A \land B \land C$.

($\gg 8$) If not $A \gg A \land \neg B$ and $A \gg A \land C$, then $A \land B \gg A \land B \land C$.

Now we show that the supplementary principles for difference-making conditionals are valid, given the supplementary postulates for conceivability-limited belief revision.

**Observation 9.** [Supplementary principles for $\gg$ are valid] Let $\star$ be a basic revision function in the sense of Section 5, and let $\gg$ be obtained from $\star$ by RRT. If $\star$ satisfies ($\star 7c$), ($\star 7$), ($\star 7'$), ($\star 8c$) or ($\star 8$), then $\gg$ satisfies ($\gg 7c$), ($\gg 7$), ($\gg 7'$), ($\gg 8c$) or ($\gg 8$), respectively.

**Lemma 10.** [Derivabilities within compound-antecedent principles] Given the basic principles, ($\gg 7$) is inter-derivable with ($\gg 7'$), and ($\gg 7c$) is implied by either of them. Given the basic principles, ($\gg 8c$) is implied by ($\gg 8$).

Here are more derived principles that depend on the compound-antecedent principles just introduced.

**Lemma 11.** [Further derived properties of $\gg$] Let the basic principles ($\gg 0$)--($\gg 6$) plus the supplementary principles ($\gg 7c$)–($\gg 8$) for difference-making conditionals be given. Then

(c24) If $A \gg C$, then $A \gg A$.

(c25) If $A \gg C$, then $A \land C$ and $A \lor C$ are contingent.\(^{32}\)

(c26) If $A \gg A$ and $A \land B \nsubseteq A \land B$, then $A \gg A \land \neg B$.

(c27) If $A \nsubseteq A$ and $A \land B \gg A \land B$, then $\neg (A \land B) \gg \neg B$.

(c28) If $B \nsubseteq B$ and $A \land B \gg A \land B \land C$, then $A \gg A \land C$.

(c29) If $B \nsubseteq B$ and $A \lor B \gg (A \lor B) \land C$, then $A \gg A \land C$.

(c30) If $A \nsubseteq A$ and $A \lor B \gg A \lor B$, then $A \lor B \gg B$.

\(^{32}\)One cannot infer from $A \gg C$ that either of $A \land \neg C$ and $\neg A \land C$ is contingent.
(c31) If \( A \gg A \) and \( A \lor B \nloeq A \lor B \), then \( \neg A \gg \neg A \land B \).

I have been unable to find a proof of condition (c24) that does not require the strong principle (\( \gg 8 \)). The simple, but important condition (c24) says that if there is any relevant antecedent to \( C \), then \( C \) is a relevant antecedent to itself, i.e., that \( C \) is contingent. From this and condition (c1) it transpires that only contingent sentences can be relevant antecedents or consequents.

6.4. Constructing Revisions from Difference-Making Conditionals

So far we have taken a revision function \( \star \) in the alternative sense as given, and analysed the conditional connective \( \gg \) as obtained from \( \star \) by RRT. Now we also take the converse perspective. Given the set of difference-making conditionals accepted by the agent in her belief state \( \mathcal{B} \), can we determine the result of the revision of her beliefs by a new sentence \( A \)?

We need to find a way to express, in terms of accepted and rejected conditionals, the inclusion of a sentence \( C \) in a revised belief set \( Bel \star A \). Here is the solution we will use, reminding the reader that both sides of the definition are universally quantified statements about the acceptance or non-acceptance of sentences in belief states:

\[
C \in Bel \star A \iff A \gg A \land C \text{ or } (A \nloeq A \land X_{bel} \gg X_{bel} \land C). \quad (\text{Def}\star)
\]

One can read (Def\( \star \)) as saying that \( C \in Bel \star A \) iff either (i) \( A \) is contingent and an appropriate conditional, viz., \( A \gg A \land C \), is accepted, or (ii) \( A \) is not contingent and \( C \in Bel \). (i) is expressed by the first disjunct and (ii) by the second disjunct of the defining condition in (Def\( \star \)).

According to (Def\( \star \)), a revision by \( A \) is successful, i.e., \( A \in Bel \star A \), if and only if \( (A \gg A \lor X_{bel} \gg X_{bel} \land A) \), i.e., if and only if \( A \) is contingent or a belief.

If we apply the identity (r1), \( Bel = Bel \star \top \), to (Def\( \star \)), we can characterise the agent’s belief set in terms of the difference-making conditionals accepted by her. Since \( \top \gg \top \land A \) and \( \top \gg \top \) are always unacceptable, by (c4), the definition of \( A \in Bel \star \top \) reduces to

\[
A \in Bel \iff X_{bel} \gg X_{bel} \land A. \quad (\text{DefBel})
\]

\(^{33}\) Notice that the non-contingency of \( A \) in (ii) implies that \( A \) is an absurdity or a belief, so either way \( Bel \star A = Bel \), by (r9). (Def\( \star \)) is surely a roundabout way of expressing that \( C \) is in the revision of \( Bel \) by \( A \), but it is not much worse than the condition (Def\( \star \)) that is used in ‘DMC’: \( C \in Bel \star A \) iff \( A \gg A \land C \) or \( \neg A \gg \bot \) and \( \bot \nloeq C \).
This is exactly the characterization of belief we have endorsed in our discussion after Lemma 7.

**Lemma 12.** (i) Using the definition (DefBel), beliefs are closed under conjunction and singleton entailment.
(ii) Using the definitions (DefBel) and (Def⋆), revisions by input that is consistent with the prior beliefs are always successful: If \( \neg A \notin Bel \), then \( A \in Bel \star A \).

The property mentioned in part (ii) of Lemma 12 is exactly what has to be added to Preservation in order to get Preservation+, i.e., (⋆4).

Importantly, the reconstruction (Def⋆) of a revision function is a kind of inverse to the acceptance conditions of conditionals given by the Relevant Ramsey Test.

**Observation 13.** [RRT and (Def⋆) fit together] (i) Let \( \star \) satisfy (⋆0)–(⋆6) and let \( \triangleright \) be the difference-making conditional defined from \( \star \) by RRT. Let \( \star' \) be the revision function defined from \( \triangleright \) by (Def⋆). Then \( \star' \) is identical to \( \star \).
(ii) Let \( \triangleright \) satisfy (\( \triangleright 0 \))–(\( \triangleright 6 \)) and let \( \star \) be the revision function defined from \( \triangleright \) by (Def⋆). Let \( \triangleright' \) be the difference-making conditional defined from \( \star \) by RRT. Then \( \triangleright' \) is identical to \( \triangleright \).

In a converse to Observations 4 and 9, it can be shown that any set of conditionals that satisfies the basic principles characterising difference-making conditionals can be represented as generated by the Relevant Ramsey Test based on a revision function \( \star \) in the sense of Section 5.

**Observation 14.** [Representation theorem] Let \( \triangleright \) be a conditional satisfying the principles (\( \triangleright 0 \)), (\( \triangleright 1 \)), (\( \triangleright 2a \)), (\( \triangleright 2b \)), (\( \triangleright 3 \)), (\( \triangleright 4 \)), (\( \triangleright 5a \))–(\( \triangleright 5c \)) and (\( \triangleright 6 \)), and let \( \star \) be obtained from \( \triangleright \) by (Def⋆). Then \( \star \) is a basic AGM revision function in the sense of Section 5, and RRT is satisfied.

If \( \triangleright \) in addition satisfies (\( \triangleright 8c \)) as well as one of (\( \triangleright 7c \)), (\( \triangleright 7 \)) and (\( \triangleright 7' \)), then \( \star \) satisfies (\( \star 7c \)), (\( \star 7 \)) or (\( \star 7' \)), respectively.

If \( \triangleright \) in addition satisfies (\( \triangleright 8c \)) or (\( \triangleright 8 \)), then \( \star \) satisfies (\( \star 8c \)) or (\( \star 8 \)), respectively.

**7. Conclusion**

I submit that the qualitative positive relevance idea embodied in the Relevant Ramsey Test offers a very clear motivation for connexivity. Some authors (Kapsner [14], Lenzen [16]) have argued that one should restrict
(Arist1) and (Boet-Abel) to consistent antecedents. The idea of difference-making conditionals, however, requires un
restricted, full connexivity, and not only one that is restricted to consistent or conceivable antecedents. We have achieved this aim by basing difference-making conditionals on an alternative, ‘conceivability-limited’ conception of belief revision.

The interpretation of difference-making conditionals in the spirit of connexive logic is already present in ‘DMC’. But the account of conditionals presented there is not fully connexive, because it restricts the Principle of Abelard and Boethius to non-absurd antecedents. Sometimes an absurdity is positively relevant to a proposition and its negation at the same time, viz., if the agent does not have any belief about this proposition. But the account is not entirely ‘humble’ either (in Kapsner’s sense), because it endorses unrestricted versions of Aristotle’s two theses. No proposition is positively relevant for its own negation.

The new set-up based on conceivability-limited revision has a number of advantages over the more conservative treatment of difference-making conditionals in ‘DMC’. Obtaining full connexivity is the first one, the connexive principles are validated without restrictions. A second advantage is that the second Aristotelian thesis (Arist2) is as valid and as important as the thesis associated with Boethius and Abelard (Boet-Abel), just as it had been in Aristotle’s and Abelard’s writings. The third advantage is of a more immediate intuitive nature. Absurdities (and in particular contradictions) should not be understood as relevantly promoting anything, nor should anything be relevant for them. This is exactly the result that the new analysis of difference-making conditionals delivers. We also obtain a perfect symmetry in the roles of logical truths and contradictions and respect both principles (Irrelevant $\top$) and (Irrelevant $\bot$). And fourthly, these symmetries are reflected in the symmetry displayed by Figures 1 and 2.

But there is a price to pay. The new approach offers less expressive power. While in the context of ‘DMC’ it is very easy to express that $A$ is a non-belief ($\bot \rhd A$) or that $A$ is an absurdity ($A \rhd \bot$), this simple expressibility gets lost. In the new set-up, neither $\bot \rhd A$ nor $A \rhd \bot$ is ever acceptable for any $A$, and from a philosophical point of view, this is as it should be. In ‘DMC’, $\bot$ plays an important theoretical role, both as an antecedent and as a consequent. On the one hand, employing $\bot$ in difference-making conditionals is attractive, precisely because we can conveniently avail ourselves of a lot of expressive power. On the other hand, making use of the above-mentioned expressions looks a bit like an abuse of $\bot$, since intuitively—as we have emphasized in the introduction—contradictions are not relevant...
for anything, nor is anything relevant for them. In a way, using contradictions as antecedents and consequents of conditionals appears to be sinning against the very idea of difference-making conditionals. The new set-up has contingency rather than (in-)consistency as a key concept.

We have seen that there are ways of expressing in the new setting of difference-making conditionals that $A$ is a belief (Lemma 7(i)). These ways, however, depend on having a sentential representation $X_{bel}$ of one’s strongest belief, something that is impractical in natural language. In addition, we had to assume that this strongest belief is contingent. This assumption amounts to a kind of enlightened fallibilism: the agent shouldn’t regard it as absurd that at least one of her beliefs is false. The fact that agents cannot easily rephrase their beliefs as conditionals is a limitation of expressive power, but it is not worrying at all. Beliefs are propositional, and agents have a perfect way of expressing them: by uttering them straight away.

There are also several ways of expressing that $A$ is a doxastic necessity or an absurdity (Lemma 7(ii) and (iii)). But most of them are certainly rather unnatural ways of speaking.

There is another question of expressiveness. With the original Ramsey test, it is easy to encode the fact that $C$ is a belief after revising $Bel$ by $A$: this is indeed exactly what is expressed by a simple ‘suppositional’ conditional $A \succ C$. In the set-up of ‘DMC’, there are two ways of expressing this, one less intuitive but rather elegant ($\neg A \notiff \neg A \lor C$), and one more intuitive but less elegant ($A \notiff A \land C$ or ($\neg A \notiff \bot$ and $\bot \notiff A$)). The only way to express the same fact in the new approach, (Def⋆), is even somewhat less elegant than the less elegant way of ‘DMC’.

Absurdities are always somewhat embarrassing, both as pieces of new information and as hypothetical assumptions. In section 2 we saw that there are two ways of making a design decision regarding absurdities. On the first, absurdities lead into the inconsistent ‘epistemic hell’ (AGM and ‘DMC’), on the second, absurdities are simply not processed at all (this paper). Which way of treating absurdities is better, the first or the second?

One way of accounting for the difference between genuine belief revision—‘You are the emperor of China. Believe me!’—and hypothetical reasoning—‘Suppose you were the emperor of China!’—might be the claim that they require different design decisions. For genuine belief revision, the new design

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34 A different solution to a similar expressibility problem is offered by Raidl [27] who introduces an additional primitive belief modality into the formal language and connects it axiomatically with conditionals.
decision seems preferable; absurd testimony, for instance, is simply too incredible to be taken seriously. For hypothetical reasoning, however, allowing suppositions that strike one as absurd appears to make more sense. And it is hypothetical reasoning which is more suitable for the interpretation of conditionals.

But there is another way of dealing with the difference between genuine belief revision and hypothetical reasoning. One might simply hold that for belief revision, more things are absurd than for hypothetical reasoning. In hypothetical reasoning, many things are conceivable. It is not easy to find hypotheses so absurd that rational agents refuse to entertain them at all. Logical contradictions are good candidates, but Priest’s [25] ‘If I were a fried egg...’ is perhaps just as good. It is not implausible to assume that an agent simply refuses to modify her belief state at all in order to accommodate the supposition that she is a fried egg. Considering the philosophical arguments in favour of the new set-up mentioned above, I endorse this view and advocate full, unrestricted connexivity for difference-making conditionals.

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Appendix: Proofs

Proof of Lemma 1. Let $\star$ satisfy the principles $(\star 0)$–$(\star 8)$. 
Difference-Making Conditionals and Connexivity

(r1) Both \( \top \) and \( X_{bel} \) are in \( Bel \), and \( Bel \) is consistent, by \((\ast0)\). So by \((\ast3)\) and \((\ast4)\), \( Bel \ast \top = \text{Cn}(Bel \cup \{ \top \}) = Bel \) and \( Bel \ast X_{bel} = \text{Cn}(Bel \cup \{ X_{bel} \}) = Bel \).

(r2) If \( A \in Bel \), then \( X_{bel} \wedge \neg A \vdash \bot \), so \( X_{bel} \wedge \neg A \notin Bel \ast \ast (X_{bel} \wedge \neg A) \) by \((\ast0)\). For the converse, suppose that \( A \notin Bel \). Then, by \((\ast1)\), \( \neg(X_{bel} \wedge \neg A) \notin Bel \), and so, by \((\ast4)\), \( X_{bel} \wedge \neg A \in Bel \ast \ast (X_{bel} \wedge \neg A) \).

(r3) Suppose for reductio that \( Bel \ast A \subseteq Bel \) and \( Bel \ast A \neq Bel \). From the latter we get, by \((\ast2)\), \( A \in Bel \ast \ast A \). So \( A \in Bel \) and \( \neg A \notin Bel \), by \((\ast0)\). From the latter we get \( Bel \subseteq Bel \ast A \), by \((\ast4)\), and thus \( Bel \ast A = Bel \) after all. We have a contradiction.

(r4) Suppose that \( Bel \subseteq Bel \ast \ast A \) and \( \neg A \in Bel \). Then \( \neg A \in Bel \ast \ast A \) and, by \((\ast0)\), \( A \notin Bel \ast \ast A \). Thus, by \((\ast2)\), \( Bel \ast A = Bel \).

(r5) We show the contraposition. Suppose \( \neg A \notin Bel \). So by \((\ast4)\), \( \text{Cn}(Bel \cup \{ A \}) \subseteq Bel \ast \ast A \) and thus \( A \in Bel \ast \ast A \).

(r6) This follows from \((r5)\) and \((\ast0)\).

(r7) Suppose \( A \notin Bel \ast \ast A \). Then by \((\ast2)\), \( Bel \ast A = Bel \). Since by \((r5)\) \( \neg A \in Bel \), we get \( \neg A \in Bel \ast \ast A \).

(r8) From right to left. If \( A \in Bel \), then \( Bel \ast A = Bel \) by \((\ast0)\), \((\ast3)\) and \((\ast4)\). If \( A \notin Bel \ast \ast A \), then \( Bel \ast A = Bel \) by \((\ast2)\).

From left to right. Suppose that \( A \notin Bel \) and \( A \in Bel \ast \ast A \). This immediately implies that \( Bel \ast A \neq Bel \).

(r9) Let \( A \) be non-contingent, i.e., let either \( A \) or \( \neg A \) be absurd. Suppose without loss of generality that \( A \) is absurd (the other case is similar). Then \( Bel \ast A = Bel \), by \((\ast2)\), and \( Bel \ast \neg A = Bel \), by \((r5)\), \((\ast3)\) and \((\ast4)\).

(r10) This follows immediately from \((r9)\).

(r11) (Consistent Expansion) Let \( Bel \notin Bel \ast \ast A \). Then by \((\ast2)\), \( A \in Bel \ast \ast A \), and by \((\ast4)\), \( \neg A \in Bel \). This proves the claim. (Notice that given Inclusion \((\ast3)\), Consistent Expansion implies Preservation \((\ast4)\): If \( A \) is consistent with \( Bel \), then, by \((\ast3)\), \( Bel \ast A \) is consistent with \( Bel \), and so by \((r11)\) \( Bel \subseteq Bel \ast \ast A \).)

(r12) (Regularity) Let \( A \in Bel \ast \ast B \). Suppose for reductio that \( A \notin Bel \ast \ast A \). From the former we get, by \((\ast8c)\), \( Bel \ast B \subseteq Bel \ast (A \wedge B) \). Since \( A \notin Bel \ast \ast A \), also \( A \wedge B \notin Bel \ast (A \wedge B) \), by \((\ast5a)\). So by \((\ast2)\), \( Bel \ast (A \wedge B) = Bel \). Thus \( Bel \ast B \subseteq Bel \) and further, by \((r3)\), \( Bel \ast B = Bel \). But we know from \((r5)\) that \( \neg A \in Bel \). So \( \neg A \in Bel \ast \ast B \), and we have a contradiction with \((\ast0)\).

(r13) (Strong Regularity) Let \( A \notin Bel \ast \ast A \). Suppose for reductio that \( \neg A \notin Bel \ast \ast B \) for some \( B \). Then by \((\ast8)\), \( Bel \ast B \subseteq Bel \ast (A \wedge B) \). Since \( A \notin Bel \ast \ast A \), also \( A \wedge B \notin Bel \ast (A \wedge B) \), by \((\ast5a)\). So by \((\ast2)\), \( Bel \ast (A \wedge B) = Bel \). Thus \( Bel \ast B \subseteq Bel \) and further, by \((r3)\), \( Bel \ast B = Bel \). But we know from \((r5)\) that \( \neg A \in Bel \). So \( \neg A \in Bel \ast \ast B \), and we have a contradiction.
(r14) (Disjunctive Success) This follows directly from (r13).

(r15) Suppose that $C \in \text{Bel} \ast A$ and $C \notin \text{Bel} \ast \neg A$. From $C \in \text{Bel} \ast A$, we can infer that $C$ is not absurd, by (r12). And from $C \notin \text{Bel} \ast \neg A$, we can infer that $\neg C$ is not absurd, by (r13).

(r16) (Guarded Subexpansion) Let $A \in \text{Bel} \ast A$ and $\neg B \notin \text{Bel} \ast A$. From the latter we get, by ($\ast$8), that $\text{Bel} \ast A \subseteq \text{Bel} \ast (A \land B)$.

We now show that $B \in \text{Bel} \ast (A \land B)$. Suppose this is not the case. Then, by ($\ast$1), $A \land B \notin \text{Bel} \ast (A \land B)$, and by ($\ast$2), $\text{Bel} \ast (A \land B) = \text{Bel}$. So $B \ast A \subseteq \text{Bel}$. Thus, by (r3), $\text{Bel} \ast A = \text{Bel}$. So we have $A \in \text{Bel}$ and $\neg B \notin \text{Bel}$. It follows from ($\ast$1) that $A \land \neg B \notin \text{Bel}$. Thus, by ($\ast$4), $\text{Cn}((\text{Bel} \cup \{A \land B\}) \subseteq \text{Bel} \ast (A \land B)$. So $B \in \text{Bel} \ast (A \land B)$.

From $\text{Bel} \ast A \subseteq \text{Bel} \ast (A \land B)$ and $B \in \text{Bel} \ast (A \land B)$ we get, with the help of ($\ast$1), that $\text{Cn}((\text{Bel} \ast A) \cup \{B\}) \subseteq \text{Bel} \ast (A \land B)$, as desired.

(r17) (Reciprocity) Let $A \in \text{Bel} \ast B$ and $B \in \text{Bel} \ast A$. By ($\ast$0), we get $\neg A \notin \text{Bel} \ast B$ and $\neg B \notin \text{Bel} \ast A$, so by ($\ast$7) and ($\ast$8), we get $\text{Bel} \ast A = \text{Bel} \ast (A \land B) = \text{Bel} \ast B$.

(r18) (Disjunctive Factoring) If $A \lor B \notin \text{Bel} \ast (A \lor B)$, or if $A \notin \text{Bel} \ast A$ and $B \notin \text{Bel} \ast B$, then $\text{Bel} \ast (A \lor B) = \text{Bel} = \text{Bel} \ast A = \text{Bel} \ast B$, by ($\ast$2), ($\ast$5a) and ($\ast$5b).

So suppose without loss of generality that $A \in \text{Bel} \ast A$. Then $A \lor B \in \text{Bel} \ast (A \lor B)$, by ($\ast$5a) and ($\ast$6). By ($\ast$0), we cannot have both $\neg A \in \text{Bel} \ast (A \lor B)$ and $\neg B \in \text{Bel} \ast (A \lor B)$.

Case 1. Suppose $\neg A \in \text{Bel} \ast (A \lor B)$. Then also $B \in \text{Bel} \ast (A \lor B)$, by ($\ast$1), and $\neg B \notin \text{Bel} \ast (A \lor B)$, by ($\ast$0). Thus, by ($\ast$6) and ($\ast$7), $\text{Bel} \ast B = \text{Bel} \ast ((A \lor B) \land (\neg A \lor B)) \subseteq \text{Cn}((\text{Bel} \ast (A \lor B)) \cup \{\neg A \lor B\}) = \text{Bel} \ast (A \lor B)$. On the other hand, we have $\neg (A \lor B) \notin \text{Bel} \ast (A \lor B)$, since $\neg B \notin \text{Bel} \ast (A \lor B)$. So by ($\ast$8), $\text{Bel} \ast (A \lor B) \subseteq \text{Bel} \ast ((A \lor B) \land (\neg A \land B)) = \text{Bel} \ast B$. So $\text{Bel} \ast (A \lor B) = \text{Bel} \ast B$.

Case 2. Suppose $\neg B \in \text{Bel} \ast (A \lor B)$. Then by exactly the same reasoning as in Case 1, $\text{Bel} \ast (A \lor B) = \text{Bel} \ast A$.

Case 3. Suppose $\neg A, \neg B \notin \text{Bel} \ast (A \lor B)$. Then $\text{Bel} \ast (A \lor B) \subseteq \text{Bel} \ast A$ and $\text{Bel} \ast (A \lor B) \subseteq \text{Bel} \ast B$, by (8) and (6). On the other hand, using (6) and (7), we can show $\text{Bel} \ast A \subseteq \text{Cn}((\text{Bel} \ast (A \lor B)) \cup \{A \lor \neg B\})$ and $\text{Bel} \ast B \subseteq \text{Cn}((\text{Bel} \ast (A \lor B)) \cup \{\neg A \lor B\})$. Hence $(\text{Bel} \ast A) \cap (\text{Bel} \ast B) \subseteq \text{Cn}((\text{Bel} \ast (A \lor B)) \cup \{A \lor \neg B\}) \cap \text{Cn}((\text{Bel} \ast (A \lor B)) \cup \{\neg A \lor B\}) = \text{Cn}((\text{Bel} \ast (A \lor B)) \cup \{A \lor \neg B\} \lor (\neg A \lor B)) = \text{Bel} \ast (A \lor B)$, by (1). In sum, $\text{Bel} \ast (A \lor B) = (\text{Bel} \ast A) \cap (\text{Bel} \ast B)$.

Proof of Observation 2. (a) This is left as an exercise to the reader.
(b) Suppose that the revision function \( \star \) satisfies \((\star 0)\)–\((\star 8)\). Then it satisfies the eleven conditions listed in Theorem 13 of Hansson, Fermé, Cantwell and Falappa [12].

Hansson et al. list eight “core postulates”: Closure, Relative Success, Inclusion, Strong Consistency, Extensionality and Disjunctive Distribution are all included (in almost literally the same form) as postulates in the postulate set here. Strict Improvement is equivalent to \((\star 5a)\). Consistent Expansion is \((r11)\).

Hansson et al. have three additional postulates: Vacuity is equivalent to the conjunction of \((\star 3)\) and \((\star 4)\). Strong Regularity and Disjunctive Factoring are \((r13)\) and \((r18)\), respectively.\(^{35}\)

So their representation result (Theorem 13) applies, and the revision function \( \star \) can be represented as a conceivability-limited spheres-based revision.

**Proof of Lemma 3.** Let \( \star \) satisfy the basic conditions \((\star 0)\)–\((\star 6)\) for conceivability-limited belief revision.

\( A \gg A \): We need to show that

\[
A \in \text{Bel}\star A \quad \text{and} \quad A \notin \text{Bel}\star \neg A \quad \text{iff} \quad A \in \text{Bel}\star A \quad \text{and} \quad \neg A \in \text{Bel}\star \neg A.
\]

But \( A \notin \text{Bel}\star \neg A \) is equivalent to \( \neg A \in \text{Bel}\star \neg A \), by \((\star 0)\) and \((r7)\). So \( A \gg A \) is equivalent to the contingency of \( A \).

\( A \gg A \land C \): We need to show that

\[
A \land C \in \text{Bel}\star A \quad \text{and} \quad A \land C \notin \text{Bel}\star \neg A \quad \text{iff} \quad C \in \text{Bel}\star A \quad \text{and} \quad A \in \text{Bel}\star A \land \neg A \in \text{Bel}\star \neg A.
\]

The LHS implies that \( A \) is contingent, by \((r10)\), and that \( C \in \text{Bel}\star A \), by \((\star 1)\). The RHS implies that \( \neg A \) is not absurd, so \( A \land C \notin \text{Bel}\star \neg A \), by \((\star 0)\), and \( A \land C \in \text{Bel}\star A \) follows by \((\star 1)\).

\( A \gg A \lor C \): We need to show that

\[
A \lor C \in \text{Bel}\star A \quad \text{and} \quad A \lor C \notin \text{Bel}\star \neg A \quad \text{iff} \quad C \notin \text{Bel}\star \neg A \quad \text{and} \quad A \in \text{Bel}\star A \land \neg A \in \text{Bel}\star \neg A.
\]

The LHS implies that \( A \) is contingent, by \((r10)\). The RHS implies that \( A \) is not absurd, so \( A \lor C \in \text{Bel}\star A \), by \((\star 1)\). Since \( \neg A \in \text{Bel}\star \neg A \) and \( C \notin \text{Bel}\star \neg A \), it follows that \( A \lor C \notin \text{Bel}\star \neg A \), by \((\star 1)\).

---

\(^{35}\) Disjunctive Factoring is in fact equivalent to the conjunction of \((\star 7)\) and \((\star 8)\), similarly to the situation in AGM belief revision. But proving this here would lead us too far afield.
\[ A \land C \supseteq C; \text{ We need to show that} \]
\[ C \in Bel \ast A \land C \text{ and } C \notin Bel \ast \neg (A \land C) \iff \]
\[ C \notin Bel \ast \neg (A \land C) \text{ and }\]
\[ A \land C \in Bel \ast A \land C \text{ and } \neg (A \land C) \in Bel \ast \neg (A \land C) \].

The LHS implies that \( A \land C \) is contingent, by (r10). The RHS implies that \( A \land C \in Bel \ast A \land C \), so \( C \in Bel \ast A \land C \) follows, by (\ast 1).

\[ A \lor C \supseteq C; \text{ We need to show that} \]
\[ C \in Bel \ast A \lor C \text{ and } C \notin Bel \ast \neg (A \lor C) \iff \]
\[ C \notin Bel \ast (A \lor C) \text{ and }\]
\[ A \lor C \in Bel \ast A \lor C \text{ and } \neg (A \lor C) \in Bel \ast \neg (A \lor C) \].

The LHS implies that \( A \lor C \) is contingent, by (r10). The RHS implies that \( \neg (A \lor C) \) is not absurd, so \( C \notin Bel \ast \neg (A \lor C) \), by (\ast 0).

\[ A \land X_{bel} \supseteq A \land X_{bel}; \text{ We need to show that} \]
\[ A \land X_{bel} \in Bel \ast (A \land X_{bel}) \text{ and } A \land X_{bel} \notin Bel \ast \neg (A \land X_{bel}) \iff \neg A \notin Bel. \]

From left to right. Suppose that \( \neg A \in Bel \). Then \( A \land X_{bel} \) is inconsistent, by the definition of \( X_{bel} \), so \( A \land X_{bel} \notin Bel \ast (A \land X_{bel}) \), by (\ast 0). From right to left. Assume \( \neg A \notin Bel \). Then, since \( X_{bel} \in Bel \), \( \neg (A \land X_{bel}) \notin Bel \), by (\ast 1).

So by (\ast 3) and (\ast 4), \( Bel \ast (A \land X_{bel}) = Con(Bel \cup \{ A \land X_{bel} \}) \). So \( A \land X_{bel} \in Bel \ast (A \land X_{bel}) \). For the second disjunct, we use \( \neg X_{bel} \in Bel \ast \neg X_{bel} \) and get that \( \neg (A \land X_{bel}) \in Bel \ast \neg (A \land X_{bel}) \), by (\ast 5a). So \( A \land X_{bel} \notin Bel \ast \neg (A \land X_{bel}) \), by (\ast 0).

\[ X_{bel} \supseteq X_{bel} \land A; \text{ We need to show that} \]
\[ A \land X_{bel} \in Bel \ast X_{bel} \land A \text{ and } A \land X_{bel} \notin Bel \ast \neg X_{bel} \land A \iff A \in Bel. \]

From left to right. Assume the LHS. We know that \( Bel \ast X_{bel} = Bel \), by (r1), so \( A \land X_{bel} \in Bel \) and \( A \in Bel \) by (\ast 1). From right to left. Assume \( A \in Bel \). Then \( A \land X_{bel} \in Bel \). Since \( X_{bel} \in Bel \), \( A \land X_{bel} \in Bel \ast X_{bel} \), by (\ast 0), (\ast 3) and (\ast 4). Moreover, since \( \neg X_{bel} \in Bel \ast \neg X_{bel} \), by (\ast 0b), we get \( A \land X_{bel} \notin Bel \ast \neg X_{bel} \), by (\ast 0).

\[ X_{bel} \supseteq A \land X_{bel} \text{ and } A \not\supseteq A; \text{ We need to show that} \]
\[ X_{bel} \supseteq A \land X_{bel} \text{ and } A \not\supseteq A \iff \neg A \notin Bel \ast \neg A. \]
From left to right. Suppose that \( X_{\text{bel}} \triangleright A \land X_{\text{bel}} \) and \( A \not\triangleright A \). We already know that \( X_{\text{bel}} \triangleright A \land X_{\text{bel}} \) is equivalent to \( A \in \text{Bel} \). So \( \neg A \notin \text{Bel} \), by \((\star 0)\), and thus \( A \in \text{Bel} \land A \), by \((\star 4)\). But \( A \not\triangleright A \) means that either \( A \notin \text{Bel} \lor A \) or \( A \in \text{Bel} \lor \neg A \). We have just refuted the former. So \( A \in \text{Bel} \lor \neg A \) and thus, by \((r5)\) and \((\star 1)\), \( A \notin \text{Bel} \lor \neg A \). From right to left. Suppose that \( \neg A \notin \text{Bel} \lor \neg A \). Thus, by \((r5)\) and \((\star 1)\), \( A \in \text{Bel} \lor \neg A \) which is sufficient to give us \( A \not\triangleright A \).

\[
X_{\text{bel}} \triangleright \neg A \land X_{\text{bel}} \quad \text{and} \quad A \not\triangleright A : \text{We need to show that}
\]

\[
X_{\text{bel}} \triangleright \neg A \land X_{\text{bel}} \quad \text{iff} \quad A \notin \text{Bel} \lor A.
\]

From left to right. Suppose that \( X_{\text{bel}} \triangleright \neg A \land X_{\text{bel}} \) and \( A \not\triangleright A \). We already know that \( X_{\text{bel}} \triangleright \neg A \land X_{\text{bel}} \) is equivalent to \( \neg A \in \text{Bel} \). So \( A \notin \text{Bel} \), by \((\star 0)\), and thus \( \neg A \in \text{Bel} \lor \neg A \), by \((\star 4)\). So \( A \notin \text{Bel} \lor \neg A \), by \((\star 0)\). But \( A \not\triangleright A \) means that either \( A \notin \text{Bel} \lor A \) or \( A \in \text{Bel} \lor \neg A \). We have just refuted the latter. So \( A \notin \text{Bel} \lor A \). From right to left. Suppose that \( A \notin \text{Bel} \lor A \). Thus, by \((r5)\) and \((\star 1)\), \( \neg A \in \text{Bel} \), which we already know to be equivalent to \( X_{\text{bel}} \triangleright \neg A \land X_{\text{bel}} \). And by \((r7)\) and \((\star 1)\), \( \neg A \in \text{Bel} \lor A \), so \( A \notin \text{Bel} \lor A \), by \((\star 0)\), which is sufficient to give us \( A \not\triangleright A \).

\[\Box\]

**Proof of Observation 4.** Let \( \star \) be a basic revision function in the sense of Section 5, and let \( \triangleright \) be obtained from \( \star \) by RRT.

\((\triangleright 0)\) We know that \( X_{\text{bel}} \triangleright X_{\text{bel}} \) expresses that (our strongest belief) \( X_{\text{bel}} \) is contingent, and this follows from \((\star 0b)\) and \((\star 2)\).

\((\triangleright 1)\) We show that \( A \triangleright B \land C \) implies that \( A \triangleright B \) or \( A \triangleright C \). Suppose that \( B \land C \in \text{Bel} \land A \) and \( B \land C \notin \text{Bel} \land \neg A \). But then, with the help of \((\star 1)\), it follows immediately that either \( B \in \text{Bel} \land A \) and \( B \notin \text{Bel} \land \neg A \) or that \( C \in \text{Bel} \land A \) and \( C \notin \text{Bel} \land \neg A \).

\((\triangleright 2a)\) We show that \( A \triangleright C \) if and only if \( A \triangleright A \land C \) and \( A \triangleright A \lor C \). By RRT, this means that

\[
C \in \text{Bel} \land A \land C \notin \text{Bel} \land \neg A
\]

iff

\[
\left( A \land C \in \text{Bel} \land A \land A \land C \notin \text{Bel} \land \neg A \land A \land C \notin \text{Bel} \land \neg A \right).
\]

For the left-to-right direction, we first infer from the LHS that \( A \) is contingent, by \((r10)\). So we get from \( C \in \text{Bel} \land A \) that \( A \land C \in \text{Bel} \land A \) and \( A \land C \in \text{Bel} \land A \), by \((\star 1)\) and \((\star 2)\). Since \( C \notin \text{Bel} \land \neg A \), \( A \land C \notin \text{Bel} \land \neg A \), by \((\star 1)\). Suppose \( A \land C \in \text{Bel} \land \neg A \). Then since \( \neg A \in \text{Bel} \land \neg A \), \( C \in \text{Bel} \land \neg A \), by \((\star 1)\) and \((\star 2)\), contradicting the supposition.
For the right-to-left direction, we get $C \in \text{Bel} \ast A$ from $A \land C \in \text{Bel} \ast A$, and $C \notin \text{Bel} \ast \lnot A$ from $A \lor C \notin \text{Bel} \ast \lnot A$, both by (⋆1).

(⋆2b) We show that $A \Rightarrow A \land C$ if and only if not $\lnot A \Rightarrow \lnot A \lor C$ and $A \Rightarrow A$. By RRT, this means that

$$A \land C \in \text{Bel} \ast A \quad \text{iff} \quad \left( \begin{array}{c}
A \lor C \notin \text{Bel} \ast \lnot A \\
\lnot A \lor C \in \text{Bel} \ast A \quad \text{and} \quad A \in \text{Bel} \ast A \quad \text{and} \quad A \notin \text{Bel} \ast \lnot A
\end{array} \right).$$

For the left-to-right direction, we first infer from the LHS that $A$ is contingent, by (r10). So $A \in \text{Bel} \ast A$, and also $A \notin \text{Bel} \ast \lnot A$, by (⋆0). Since $A \land C \in \text{Bel} \ast A$, $\lnot A \lor C \in \text{Bel} \ast A$, by (⋆1).

For the right-to-left direction, we first infer from the RHS that $A$ is contingent, by (r10). Thus $\lnot A \lor C \notin \text{Bel} \ast \lnot A$ is impossible, by (⋆1) and (⋆2). So $\lnot A \lor C \in \text{Bel} \ast A$, from which we get $A \land C \in \text{Bel} \ast A$, by (⋆1) and (⋆2). Finally, we get $A \land C \notin \text{Bel} \ast \lnot A$ from $A \notin \text{Bel} \ast \lnot A$, by (⋆1).

(⋆3) We show that $A \land \lnot C \land X_{\text{bel}} \Rightarrow A \land \lnot C \land X_{\text{bel}}$ implies $A \not\Rightarrow A \land C$. By RRT, this means that

$$\text{If} \quad A \land \lnot C \land X_{\text{bel}} \text{ is contingent, then} \quad A \land C \notin \text{Bel} \ast A \quad \text{or} \quad A \land C \in \text{Bel} \ast \lnot A.$$ 

Assume the antecedent. Then, by (⋆0), $A \land \lnot C \land X_{\text{bel}}$ is consistent, so $\lnot (A \land \lnot C) \notin \text{Bel}$. By the logical closure of $\text{Bel}$, then $C \notin \text{Cn} (\text{Bel} \cup \{A\})$, and by (⋆3), this implies that $C \notin \text{Bel} \ast A$ which proves the consequent, using (⋆1).

(⋆4) We show that $\lnot A \land X_{\text{bel}} \Rightarrow \lnot A \land X_{\text{bel}}$ and $A \Rightarrow A \lor C$ together imply $\lnot C \land X_{\text{bel}} \Rightarrow \lnot C \land X_{\text{bel}}$. By RRT, this means that

$$\text{If} \quad \lnot A \land X_{\text{bel}} \text{ is contingent and} \quad A \lor C \in \text{Bel} \ast A \quad \text{and} \quad A \lor C \notin \text{Bel} \ast \lnot A,$$

$$\text{then} \quad \lnot C \land X_{\text{bel}} \text{ is contingent.}$$

Assume the antecedent. Then, by (⋆0), $\lnot A \land X_{\text{bel}}$ is consistent, so $A \notin \text{Bel}$, by the logical closure of $\text{Bel}$. Now we can conclude with (⋆4) from $A \lor C \notin \text{Bel} \ast \lnot A$, that $A \lor C \notin \text{Bel}$. So $C \notin \text{Bel}$, by the logical closure of $\text{Bel}$, and also, since $X_{\text{bel}} \in \text{Bel}$, $\lnot (\lnot C \land X_{\text{bel}}) \notin \text{Bel}$. Now we use (⋆4) once more and conclude $\lnot C \land X_{\text{bel}} \in \text{Bel} \ast (\lnot C \land X_{\text{bel}})$. On the other hand, since $X_{\text{bel}}$ is contingent, $\lnot X_{\text{bel}}$ is not absurd, thus $\lnot (\lnot C \land X_{\text{bel}})$ is not absurd either and so $\lnot (\lnot C \land X_{\text{bel}}) \notin \text{Bel} \ast \lnot (\lnot C \land X_{\text{bel}})$, by (⋆5a). This establishes that $\lnot C \land X_{\text{bel}}$ is contingent.


\( \triangleright 5a \) We show that \( A \triangleright A \) iff \( \neg A \triangleright \neg A \). But as we have already seen, these conditionals just say that \( A \) and \( \neg A \), respectively, are contingent, and this is the same thing.

\( \triangleright 5b \) We show that \( A \land B \triangleright A \land B \) implies \( A \triangleright A \) or \( B \triangleright B \). That is to say, if \( A \land B \) is contingent, so is either \( A \) or \( B \). That is, we have to show that

\[
\text{If } A \land B \in \text{Bel}^* A \land B \text{ and } \neg(A \land B) \in \text{Bel}^* \neg(A \land B) \text{, then }
\begin{cases}
A \in \text{Bel}^* A \text{ and } B \in \text{Bel}^* B & \text{or } \neg B \in \text{Bel}^* \neg B
\end{cases}.
\]

This follows immediately from (⋆5a) and (⋆5b).

\( \triangleright 5c \) We show that \( A \land B \triangleright A \land B \) and \( A \lor C \triangleright A \lor C \) together imply \( A \triangleright A \). That is to say, if both \( A \land B \) and \( A \lor C \) are contingent, so is \( A \). That is, we have to show that

\[
\text{If } A \land B \in \text{Bel}^* A \land B \text{ and } \neg(A \land B) \in \text{Bel}^* \neg(A \land B) \text{ and } A \lor C \in \text{Bel}^* A \lor C \text{ and } \neg(A \lor C) \in \text{Bel}^* \neg(A \lor C) \text{, then }
\begin{cases}
A \in \text{Bel}^* A \text{ and } C \in \text{Bel}^* A \lor C \text{ and } \neg A \in \text{Bel}^* \neg A
\end{cases}.
\]

But by (⋆5a), \( A \land B \in \text{Bel}^* A \land B \) implies \( A \in \text{Bel}^* A \), and \( \neg(A \lor C) \in \text{Bel}^* \neg(A \lor C) \) implies \( \neg A \in \text{Bel}^* \neg A \).

\( \triangleright 6 \) follows straightforwardly from (⋆1) and (⋆6).

\[\text{Proof of Lemma 5.}\] Let the principles \((\triangleright 0)\), \((\triangleright 1)\), \((\triangleright 2a)\), \((\triangleright 2b)\), \((\triangleright 3)\), \((\triangleright 4)\), \((\triangleright 5a)\), \((\triangleright 5b)\), \((\triangleright 5c)\) and \((\triangleright 6)\) for difference-making conditionals be given.

\(\triangleright 1\) Suppose that \( A \triangleright C \). Then, by (\(\triangleright 2a)\), \( A \triangleright A \land C \). Then, by (\(\triangleright 2a)\) again, \( A \triangleright A \lor (A \land C) \). So \( A \triangleright A \), by (\(\triangleright 6)\).

\(\triangleright 2\) Suppose for reductio that \( A \triangleright \bot \). Then, by (\(\triangleright 6)\), \( A \triangleright A \land \neg A \). Using (\(\triangleright 2b)\), we get that \( \neg A \not\triangleright \neg A \lor \neg A \) and \( A \triangleright A \). Using (\(\triangleright 6)\) again, this means that \( A \not\triangleright \neg A \), but \( A \triangleright A \). This contradicts (5a).

\(\triangleright 3\) Suppose for reductio that \( A \triangleright \top \). Then \( A \triangleright A \), by (\(\triangleright 1)\). By (\(\triangleright 5a)\) and (\(\triangleright 6)\), we get \( \neg A \triangleright \neg A \lor \neg A \). By (\(\triangleright 2b)\) and (\(\triangleright 6)\), we can infer that \( A \not\triangleright A \lor \neg A \), and thus, by (\(\triangleright 6)\) again, \( A \not\triangleright \top \), contradicting our supposition.

\(\triangleright 4\) Suppose for reductio that \( \top \triangleright C \). By (\(\triangleright 6)\), \( \top \triangleright \top \land A \). So by (\(\triangleright 2b)\), \( \top \triangleright \top \). But this is impossible, according to (\(\triangleright 3)\).

\(\triangleright 5\) Suppose for reductio that \( \bot \triangleright C \). Then \( \bot \triangleright \bot \land C \), by (\(\triangleright 2a)\). Thus \( \bot \triangleright \bot \), by (\(\triangleright 6)\). But this is impossible, according to (\(\triangleright 2)\).

\(\triangleright 6\) For the left-to-right direction, note that \( A \) is equivalent to \( (A \lor C) \land (A \lor \neg C) \) and apply (\(\triangleright 1)\). For the right-to-left direction, note that \( A \triangleright A \lor C \)
implies \( A \triangleright A \land (A \lor C) \), by \((\triangleright 2a)\), which is equivalent to \( A \triangleright A \), by \((\triangleright 6)\). Similarly for \( A \triangleright A \lor \neg C \).

\((c7)\) Let \( A \land B \vdash \bot \) and suppose for reductio that \( A \triangleright B \). Then, by \((\triangleright 2a)\), \( A \triangleright A \land B \), and thus, by \((\triangleright 6)\), \( A \triangleright \bot \), contradicting \((c2)\).

\((c8)\) follows from \((\triangleright 2b)\) and \((c6)\).

\((c9)\) Let \( A \lor B \triangleright A \lor B \). Then, by \((\triangleright 5a)\), \( \neg(A \lor B) \triangleright \neg(A \lor B) \), and by \((\triangleright 6)\), \( \neg A \land \neg B \triangleright \neg A \land \neg B \). So by \((\triangleright 5b)\), either \( \neg A \triangleright \neg A \) or \( \neg B \triangleright \neg B \). Finally, by \((\triangleright 5a)\) again, either \( A \triangleright A \) or \( B \triangleright B \).

\((c10)\) follows from \((\triangleright 2a)\), \((\triangleright 6)\) and \((c3)\).

\((c11)\) Suppose \( A \triangleright A \land B \land C \). Then, by \((\triangleright 6)\), \( A \triangleright A \land C \land (A \lor B) \). So by \((\triangleright 1)\), either \( A \triangleright A \land C \) or \( A \triangleright A \lor B \). But the latter is impossible, by \((c10)\) and \((\triangleright 6)\).

\((c12)\) Suppose that \( A \triangleright A \lor B \lor C \). Then \( A \triangleright A \), by \((c1)\), and \( \neg A \triangleright \neg A \), by \((\triangleright 5a)\). By \((\triangleright 2b)\), \( \neg A \nsubseteq \neg A \land (B \lor C) \). Hence, by \((c11)\), \( \neg A \nsubseteq \neg A \land (B \lor C) \land (\neg B \lor C) \), and, by \((\triangleright 6)\), \( \neg A \nsubseteq \neg A \land C \). So, by \((\triangleright 2b)\) again, \( A \triangleright A \lor C \).

\((c13)\) Suppose that \( A \triangleright B \) and \( A \triangleright C \). By the left-to-right directions of \((\triangleright 2a)\) and \((\triangleright 2b)\), we get \( A \triangleright A \lor B \) and \( A \triangleright A \lor C \), and \( A \triangleright A \lor \neg A \lor B \) and \( A \triangleright A \lor \neg A \lor C \). From the latter two, we can deduce with \((\triangleright 1)\) that \( \neg A \nsubseteq \neg A \lor B \land (\neg A \lor C) \). Thus by \((\triangleright 6)\), \( \neg A \nsubseteq \neg A \lor (B \lor C) \). Since \( A \triangleright A \), we can apply \((\triangleright 2b)\) in the right-to-left direction and get \( A \triangleright A \lor (B \lor C) \). On the other hand, we know that \( A \triangleright A \lor B \), which is equivalent to \( A \triangleright A \lor (B \land C) \lor (B \land \neg C) \), by \((\triangleright 6)\). So, by \((c12)\), \( A \triangleright A \lor (B \land C) \). Putting this and \( A \triangleright A \lor (B \land C) \) together with the help of the right-to-left direction of \((\triangleright 2a)\), we get \( A \triangleright B \lor C \).

\((c14)\) From left to right. Let \( A \triangleright A \lor C \). Then by \((\triangleright 2b)\) \( \neg A \nsubseteq \neg A \lor C \). Also, by \((c1)\) \( A \triangleright A \). From right to left. Let \( \neg A \nsubseteq \neg A \lor C \) and \( A \triangleright A \). From the latter we get by \((\triangleright 5a)\) \( \neg A \triangleright \neg A \). From this and \( \neg A \nsubseteq \neg A \lor C \) we get \( A \triangleright A \lor C \), by \((\triangleright 2b)\) and \((\triangleright 6)\).

\((c15)\) By \((\triangleright 5a)\) and \((\triangleright 5c)\).

\((c16)\) Suppose that \( X_{bel} \triangleright X_{bel} \land A \). From \((\triangleright 0)\) and \((\triangleright 6)\), we get \( (X_{bel} \land A) \lor (X_{bel} \land \neg A) \equiv (X_{bel} \land A) \lor (X_{bel} \land \neg A) \). So by \((c9)\) either \( X_{bel} \land A \triangleright X_{bel} \land A \) or \( X_{bel} \land \neg A \triangleright X_{bel} \land \neg A \). But from \( X_{bel} \triangleright X_{bel} \land A \) it follows by \((\triangleright 3)\) and \((\triangleright 6)\) that \( X_{bel} \land \neg A \nsubseteq X_{bel} \land \neg A \). So \( X_{bel} \land A \triangleright X_{bel} \land A \), as desired.

\((c17)\) Suppose that \( A \land B \triangleright A \land B \) and \( X_{bel} \nsubseteq X_{bel} \land A \). We show that \( A \triangleright A \). From \( X_{bel} \nsubseteq X_{bel} \land A \) we infer, by \((\triangleright 0)\) and \((\triangleright 2b)\), that \( X_{bel} \triangleright \neg X_{bel} \lor A \). From this we get \( \neg A \land X_{bel} \triangleright \neg A \land X_{bel} \), by \((\triangleright 0)\) and \((\triangleright 4)\). That is, by \((\triangleright 5a)\), \( A \lor \neg X_{bel} \triangleright A \lor \neg X_{bel} \). Finally, from this and \( A \land B \triangleright A \land B \), we get \( A \triangleright A \), by \((\triangleright 5c)\).
(c18) Suppose that \( A \not\succ A \) and \( A \lor B \succ A \lor B \). Thus \( X_{bel} \land A \not\succ X_{bel} \land A \), by \((\succ 5c)\). By \((\succ 0)\) and \((\succ 4)\), we get \( \neg X_{bel} \not\succ \neg X_{bel} \lor \neg A \). By \((\succ 0)\) and \((\succ 2b)\), this gives \( X_{bel} \succ X_{bel} \land \neg A \).

(c19) Suppose that \( A \not\succ A \) and \( X_{bel} \not\succ X_{bel} \land \neg A \). By \((\succ 0)\) and \((\succ 2b)\), \( \neg X_{bel} \succ \neg X_{bel} \lor \neg A \). Since by \((\succ 0)\) \( X_{bel} \land X_{bel} \succ X_{bel} \land X_{bel} \), we can apply \((\succ 4)\) and get \( A \land X_{bel} \succ A \land X_{bel} \). From this and \( A \not\succ A \), we get that \( \neg X_{bel} \lor A \not\succ \neg X_{bel} \lor A \), by \((\succ 5c)\). So by \((\succ 5a)\), \( X_{bel} \land \neg A \not\succ X_{bel} \land \neg A \). Now we use \( X_{bel} \land X_{bel} \not\succ X_{be} \land X_{bel} \) and \((\succ 4)\) again and infer that \( \neg X_{bel} \not\succ \neg X_{bel} \lor A \).

So by \((\succ 0)\) and \((\succ 2b)\) \( X_{bel} \succ X_{bel} \land A \), as desired.

(c20) Suppose that \( X_{bel} \succ X_{bel} \land A \) and \( A \succ A \land C \). From the latter we get, by \((\succ 3)\), that \( X_{bel} \land A \land \neg C \not\succ X_{bel} \land A \land \neg C \). So, by Lemma 7, \((i.2)\) \(\Rightarrow (i.1)\), we get that \( X_{bel} \succ X_{bel} \land (A \supset C) \). From this and \( X_{bel} \succ X_{bel} \land A \), we infer with \((c13)\) and \((c11)\) that \( X_{bel} \succ X_{bel} \land C \), as desired.

(c21) Suppose that \( A \succ A \land C \). Then, by \((\succ 3)\), \( X_{bel} \land A \land \neg C \not\succ X_{bel} \land A \land \neg C \). By \((\succ 0)\) and \((\succ 6)\), we have \( X_{bel} \land X_{bel} \succ X_{bel} \land X_{bel} \). Now we can apply \((\succ 4)\) and get \( \neg X_{bel} \not\succ \neg X_{bel} \lor \neg (A \land \neg C) \). From this and \( X_{bel} \succ X_{bel} \), we get \( X_{bel} \succ X_{bel} \land (A \supset C) \), by \((\succ 2b)\) and \((\succ 6)\).

(c22) Suppose that \( A \succ A \lor C \) and \( X_{bel} \succ X_{bel} \land C \). From the latter we get, by \((\succ 3)\) and \((\succ 6)\), \( X_{bel} \land \neg C \not\succ X_{bel} \land \neg C \). Now we can apply \((\succ 4)\) and get \( X_{bel} \land \neg A \not\succ X_{bel} \land \neg A \). From this and \( X_{bel} \land X_{bel} \succ X_{bel} \land X_{bel} \), we get, by \((\succ 4)\) again, \( \neg X_{bel} \not\succ \neg X_{bel} \lor \neg \neg A \), and thus, by \((\succ 2b)\) and \((\succ 6)\), \( X_{bel} \succ X_{bel} \land A \).

(c23) Suppose that \( A \succ A \), \( X_{bel} \succ X_{bel} \land C \) and \( X_{bel} \not\succ X_{bel} \land \neg A \). From this we get, by \((c22)\), that \( \neg A \not\succ \neg A \lor C \). So, by \((\succ 2b)\), \( A \succ A \lor C \) or \( A \not\succ A \).

Since the latter is excluded by supposition, we get \( A \succ A \land C \). \(\blacksquare\)

**Proof of Observation 6.** For \((\succ \text{Arist1})\), suppose that \( A \succ \neg A \). By \((\succ 2a)\), we get \( A \succ A \land \neg A \). But this is impossible, by \((\ast 6)\) and \((c2)\).

For \((\succ \text{Arist2})\), suppose that \( A \succ C \). By \((\succ 2a)\), we get that \( A \succ A \lor C \).

Thus, by \((\succ 2b)\), \( \neg A \not\succ \neg A \lor C \). So, by \((\succ 2a)\) again, \( \neg A \not\succ C \).

For \((\succ \text{Boet-Abel})\), suppose that both \( A \succ C \) and \( A \succ \neg C \). By \((c13)\), we get that \( A \succ C \land \neg C \). But this is impossible, by \((\ast 6)\) and \((c2)\). \(\blacksquare\)

**Proof of Lemma 7.** Let the principles \((\succ 0)\)–\((\succ 6)\) for difference-making conditionals be given.

(i) **Beliefs.** \((i.0)\) is equivalent to \((i.1)\). See Lemma 3.

\((i.1)\) implies \((i.2)\). This follows immediately from \((\succ 3)\) and \((\succ 6)\).

\((i.2)\) implies \((i.3)\). Suppose for reductio that \( \neg A \land X_{bel} \not\succ \neg A \land X_{bel} \) and \( \neg A \land B \land X_{bel} \succ \neg A \land B \land X_{bel} \). Then on the one hand, by \((\succ 4)\), \( \neg (\neg A \land B) \not\succ \neg (\neg A \land B) \lor \neg A \), i.e., by \((\succ 6)\), \( A \lor B \not\succ A \lor B \). On the
other hand, the supposition gives us, by (\(\geqslant 6\)) and (\(\geqslant 5b\)), \(\neg A \land B \geqslant \neg A \land B\). This contradicts (\(\geqslant 5a\)).

(i.3) implies (i.1). First choose \(B = \neg A\) and get (i.2), \(\neg A \land X_{bel} \not\geq \neg A \land X_{bel}\). By (\(\geqslant 0\)) and (\(\geqslant 6\)), we have \(X_{bel} \land X_{bel} \geqslant X_{bel} \land X_{bel}\). Now we can apply (\(\geqslant 4\)) and get \(\neg X_{bel} \not\geq \neg X_{bel} \lor A\). From this and \(X_{bel} \geqslant X_{bel}\), we get \(X_{bel} \geqslant A \land X_{bel}\), by (\(\geqslant 2b\)).

(ii) \textit{Doxastic necessities.} (ii.0) is equivalent to (ii.1). See Lemma 3.

(ii.1) implies (ii.2). Let \(X_{bel} \geqslant A \land X_{bel}\) and \(A \not\geq\). We have \((A \land X_{bel}) \lor (\neg A \land X_{bel}) \geqslant (A \land X_{bel}) \lor (\neg A \land X_{bel})\), by (\(\geqslant 0\)) and (\(\geqslant 6\)). So by (c9), either \(A \land X_{bel} \geqslant A \land X_{bel}\) or \(\neg A \land X_{bel} \not\geq \neg A \land X_{bel}\). But from \(X_{bel} \geqslant A \land X_{bel}\), we get that \(\neg A \land X_{bel} \not\geq \neg A \land X_{bel}\), by the equivalence of (i.1) and (i.2). So \(A \land X_{bel} \geqslant A \land X_{bel}\), which gives us (ii.2).

(ii.2) implies (ii.3). This is trivial with \(B = X_{bel}\).

(ii.3) implies (ii.4). Let \(A \land B \geqslant A \land B\) and \(A \not\geq\). Suppose for reductio that there is a \(B'\) such that \(A \lor B' \geqslant A \lor B'\). But then, by (\(\geqslant 5c\)), \(A \geqslant A\), and we have a contradiction.

(ii.4) implies (ii.1). Let \(A \lor B \not\geq A \lor B\) for all \(B\). \(A \not\geq\) follows immediately if we choose \(B = A\). We also get \(A \lor \neg X_{bel} \not\geq A \lor \neg X_{bel}\) if we choose \(B = \neg X_{bel}\). Thus \(\neg A \land X_{bel} \not\geq \neg A \land X_{bel}\), by (\(\geqslant 5a\)). This implies \(X_{bel} \not\geq A \land X_{bel}\), by the equivalence of (i.1) and (i.2), and so we have (ii.1).

(iii) \textit{Absurdities.} (iii.0) is equivalent to (iii.1). See Lemma 3.

(iii.1) implies (iii.2), by the equivalence of (i.1) and (i.2) established above.

(iii.2) implies (iii.3), by exactly the same argument that we used for the inference from (ii.1) to (ii.2).

(iii.3) implies (iii.4), by the equivalence of (i.1) and (i.2).

(iii.4) implies (iii.5). Let \(\neg A \land X_{bel} \geqslant \neg A \land X_{bel}\) and \(A \not\geq\). From the former we get, by (\(\geqslant 5a\)) and (\(\geqslant 6\)), that \(A \lor \neg X_{bel} \geqslant A \lor \neg X_{bel}\). So we have (iii.5) with \(B = \neg X_{bel}\).

(iii.5) implies (iii.6). Let \(A \lor B \geqslant A \lor B\) and \(A \not\geq\). Suppose for reductio that there is a \(B'\) such that \(A \land B' \geqslant A \land B'\). But then, by (\(\geqslant 5c\)), \(A \geqslant A\), and we have a contradiction.

(iii.6) implies (iii.1). That (iii.6) implies (iii.2) follows immediately if we first choose \(B = A\) and second choose \(B = X_{bel}\). And (iii.2) implies (iii.1), by the equivalence of (i.1) and (i.2).

\textbf{Proof of Observation 8.} For (\(\geqslant \text{MP}\)), suppose that \(A \geqslant C\) and \(X_{bel} \geqslant X_{bel} \land A\). From the former we get, by (\(\geqslant 2a\)), \(A \geqslant A \land C\). Thus, by (c20), \(X_{bel} \geqslant X_{bel} \land C\).
For \((\gg MT)\), suppose that \(A \gg C\) and \(X_{\text{bel}} \gg X_{\text{bel}} \land \neg C\). From the former we get, by \((\gg 2a)\), \(A \gg A \land C\). Thus, by \((c8)\), \(A \not\gg A \land \neg C\). Since \(A \gg C\) also implies \(A \gg A\), by \((c1)\), we can apply \((\gg 2b)\) and get \(\neg A \gg \neg A \lor \neg C\). By \((c22)\), this taken together with \(X_{\text{bel}} \gg X_{\text{bel}} \land \neg C\) gives us \(X_{\text{bel}} \gg X_{\text{bel}} \land \neg A\), as desired.

For \((\gg AC)\), suppose that \(A \gg C\) and \(X_{\text{bel}} \gg X_{\text{bel}} \land C\). From the former we get, by \((\gg 2a)\), \(A \gg A \lor C\). So by \((c22)\), \(X_{\text{bel}} \gg X_{\text{bel}} \land A\).

For \((\gg DA^w)\), suppose that \(A \gg C\) and \(X_{\text{bel}} \gg X_{\text{bel}} \land \neg A\). Suppose further for reductio that \(X_{\text{bel}} \gg X_{\text{bel}} \land C\). Then we can apply \((\gg AC)\) and get \(X_{\text{bel}} \gg X_{\text{bel}} \land A\). But by \((c8)\), this contradicts \(X_{\text{bel}} \gg X_{\text{bel}} \land \neg A\). So \(X_{\text{bel}} \not\gg X_{\text{bel}} \land C\).

\[\text{Proof of Observation 9.}\] For \((\gg 7c)\), we show that \(A \gg A \land B\) and \(A \land B \gg A \land B \land C\) imply \(A \gg A \land C\). By RRT, this means that

\[
\begin{align*}
\text{If } & A \land B \in \text{Bel}^\ast A \text{ and } A \land B \notin \text{Bel}^\ast \neg A, \\
& A \land B \land C \in \text{Bel}^\ast A \land B \text{ and } A \land B \land C \notin \text{Bel}^\ast \neg A \lor \neg B,
\end{align*}
\]

then

\[
A \land C \in \text{Bel}^\ast A \text{ and } A \land C \notin \text{Bel}^\ast \neg A.
\]

It follows from \(B \in \text{Bel}^\ast A\) and \(C \in \text{Bel}^\ast A \land B\) by \((\ast 7c)\) that \(C \in \text{Bel}^\ast A\), which gives us the upper line. For the lower line, suppose that \(A \land C \in \text{Bel}^\ast \neg A\). Then, by \((\ast 2)\), \(\neg A\) is absurd, and \(A\) is a belief, by \((r5)\). But then \(\text{Bel}^\ast A = \text{Bel} = \text{Bel}^\ast \neg A\), by \((\ast 2)-(\ast 4)\), which contradicts the premises \(A \land B \in \text{Bel}^\ast A\) and \(A \land B \notin \text{Bel}^\ast \neg A\).

For \((\gg 7)\), we show that \(A \land B \gg A \land B \land C\) and \(A \gg A\) imply \(A \gg A \land (B \supset C)\). By RRT, this means that

\[
\begin{align*}
\text{If } & A \land B \land C \in \text{Bel}^\ast A \land B \text{ and } A \land B \land C \notin \text{Bel}^\ast \neg A \lor \neg B, \\
& A \land B \land C \notin \text{Bel}^\ast A \land (B \supset C) \text{ and } A \text{ is contingent},
\end{align*}
\]

then

\[
A \land (B \supset C) \in \text{Bel}^\ast A \text{ and } A \land (B \supset C) \notin \text{Bel}^\ast \neg A.
\]

Since \(A\) is contingent, \(A \in \text{Bel}^\ast A\). And it follows from \(C \in \text{Bel}^\ast A \land B\) by \((\ast 7)\) that \(C \in \text{Cn}(\{\text{Bel}^\ast A\}) \cup \{B\}\). By the deduction theorem and \((\ast 1)\), this entails \(B \supset C \in \text{Bel}^\ast A\). Since \(A\) is contingent, \(\neg A \in \text{Bel}^\ast \neg A\), so \(A \land (B \supset C) \notin \text{Bel}^\ast \neg A\), by \((\ast 0)\).

For \((\gg 7')\), we show that \(A \gg A \land C\), \(B \gg B \land C\) and \(A \lor B \gg A \lor B\) imply \(A \lor B \gg (A \lor B) \land C\). By RRT, this means, after a few routine simplifications
using (⋆1) and (⋆2), that

\[
\begin{align*}
\text{If } & A \land C \in Bel* A \text{ and } A \land C \notin Bel* \neg A \\
& B \land C \in Bel* B \text{ and } B \land C \notin Bel* \neg B \text{,}
\end{align*}
\]

and

\[
\begin{align*}
A \lor B \in Bel* A \lor B \text{ and } A \lor B \notin Bel* \neg A \land \neg B \text{,}
\end{align*}
\]

Then, by (r13), \(\neg A \land \neg B \) is absurd. Then, by (⋆5a), \(\neg A \) is absurd, too, and \(A\) is a belief, by (r5). But then \(Bel* A = Bel = Bel* \neg A\), by (⋆2)-(⋆4), which contradicts the premises \(A \land B \in Bel* A\) and \(A \land B \notin Bel* \neg A\).

For \((\Rightarrow 8)\), we show that \(A \not\supset A \land \neg B\) and \(A \supset A \land C\) imply \(A \land B \supset A \land B \land C\). By RRT, this means that

\[
\begin{align*}
\text{If } & A \land B \in Bel* A \text{ and } A \land B \notin Bel* \neg A \\
& A \land C \in Bel* A \text{ and } A \land C \notin Bel* \neg A \text{,}
\end{align*}
\]

then

\[
\begin{align*}
A \land B \land C \in Bel* A \land B \text{ and } A \land B \land C \notin Bel* \neg A \lor \neg B \text{.}
\end{align*}
\]

It follows from \(B \in Bel* A\) and \(C \in Bel* A\) by (⋆8c) that \(C \in Bel* A \land B\), for arbitrary \(C\). We can also take \(C = A \land B\) and get \(A \land B \in Bel* A \land B\). Thus we have the upper line. For the lower line, suppose that \(A \land B \land C \in Bel* \neg A \lor \neg B\).

Then, by (⋆2), \(\neg A \lor \neg B\) is absurd. But then, by (⋆5a), \(\neg A\) is absurd, too, and \(A\) is a belief, by (r5). But then \(Bel* A = Bel = Bel* \neg A\), by (⋆2)-(⋆4), which contradicts the premises \(A \land B \in Bel* A\) and \(A \land B \notin Bel* \neg A\).

Case 1: \(A \land \neg B \notin Bel* A\). Then, by \(A \land C \in Bel* A\), \(\neg B \notin Bel* A\). Thus, by (⋆8), \(Bel* A \subseteq Bel* A \land B\). Now suppose for reductio that \(A \land B\) is absurd. Then, by (r13), \(\neg (A \land B) \in Bel* A\). Since \(A \land C \in Bel* A\), \(\neg B \in Bel* A\), and we have a contradiction. So \(A \land B \in Bel* A \land B\). But since also \(C \in Bel* A \subseteq Bel* A \land B\), we get \(A \land B \land C \in Bel* A \land B\), as desired. For the second part, suppose for reductio that \(A \land B \land C \in Bel* \neg A \lor \neg B\). Then \(\neg A \lor \neg B\) is absurd, by (⋆0). But then \(\neg A\) is also absurd, by (⋆6) and (⋆5a). So \(Bel* \neg A = Bel* \neg A \lor \neg B = Bel\), by (⋆2), and \(A \land C \notin Bel* \neg A\).
implies $A \land B \land C \notin Bel \ast \neg A \lor \neg B$, by $(\ast1)$. We have a contradiction which proves $A \land B \land C \notin Bel \ast \neg A \lor \neg B$.

Case 2: $A \land \neg B \in Bel \ast \neg A$. Then $\neg A$ is absurd, by $(\ast0)$. So $Bel \ast \neg A = Bel$ and $A \land \neg B \in Bel$. But then $Bel \ast \neg A \lor \neg B = Bel$, by $(\ast3)$ and $(\ast4)$, and we get $A \land B \land C \notin Bel \ast \neg A \lor \neg B$, by $(\ast0)$. ■

Proof of Lemma 10. In the following proofs, we assume the basic conditions $(\geq 0)$–$(\geq 6)$ as given.

$(\geq 7)$ implies $(\geq 7c)$. Let $A \geq A \land B$ and $A \land B \geq A \land B$. We want to show that $A \geq A \land C$. From $A \geq A \land B$ we get $A \geq A$, by (c1), and this taken together with $A \land B \geq A \land B$ gives us $A \geq A \land (B \supseteq C)$, by $(\geq 7)$. Now we take the latter and $A \geq A \land B$, use (c13) to get $A \geq (A \land B) \land (A \land (B \supseteq C))$, which is equivalent to $A \geq A \land B \land C$. By (c11), we get $A \geq A \land C$.

$(\geq 7)$ implies $(\geq 7\prime)$. Let $A \geq A \land C$, $B \geq B \land C$ and $A \lor B \geq A \lor B$. We want to show that $A \lor B \geq (A \lor B) \land C$. From $A \geq A \land C$, which is equivalent to $(A \lor B) \land (A \lor B) \land (A \lor B \land C)$, together with $A \lor B \geq A \lor B$, we derive $A \lor B \geq (A \lor B) \land ((A \lor B) \supseteq C)$, $(\geq 7)$. By an analogous argument, we get $A \lor B \geq (A \lor B) \land ((A \lor B) \supseteq C)$. Using (c13) and $(\geq 6)$, we get $A \lor B \geq (A \lor B) \land C$.

$(\geq 7\prime)$ implies $(\geq 7)$. Let $A \land B \geq A \land B \land C$ and $A \geq A$. We want to show that $A \geq A \land (B \supseteq C)$.

Case 1: Suppose that $A \land \neg B \not\geq A \land \neg B$. This is equivalent to $A \land \neg B \not\geq (A \land \neg B) \land (B \supseteq C)$, by $(\geq 6)$. On the other hand, $A \land B \not\geq A \land B \land C$ is equivalent to $A \land B \not\geq (A \land B) \land (B \supseteq C)$, also by $(\geq 6)$. From the last two facts and $A \geq A$ we get, using $(\geq 7\prime)$, that $(A \land B) \lor (A \land \neg B) \not\geq ((A \land B) \lor (A \land \neg B)) \land (B \supseteq C)$ which reduces to $A \geq A \land (B \supseteq C)$.

Case 2: Suppose that $A \land \neg B \not\geq A \land \neg B$. From this and $A \geq A$, we conclude, using $(\geq 5a)$, that $A \lor B \not\geq A \lor B$. On the other hand, $A \land B \not\geq A \land B \land C$ is equivalent to $A \land (\neg A \lor B) \not\geq A \land (A \land (B \supseteq C))$, by $(\geq 6)$. Now we can apply (c28) and get $A \geq A \land (B \supseteq C)$.

$(\geq 8)$ implies $(\geq 8c)$. Let $A \geq A \land B$ and $A \geq A \land C$. We want to show that $A \land B \geq A \land B$. But $A \geq A \land B$ entails $A \not\geq A \land \neg B$, by (c8), so we can apply $(\geq 8)$ and get the desired claim immediately. ■

Proof of Lemma 11. (c24) Let $A \geq C$. By $(\geq 2a)$, we get $A \geq A \land C$ and $A \geq A \lor C$, and by $(\geq 2b)$, we get $A \geq A$. From $A \geq A \land C$, we get $A \land C \geq A \land C$, by $(\geq 8c)$. From $A \geq A \lor C$ and $A \geq A$, we get $\neg A \not\geq A \land C$, by $(\geq 2b)$. By $(\geq 5a)$ and $(\geq 6)$, we get $\neg A \geq \neg A \land \neg A$. Now we can use $(\geq 8)$ and get that $\neg A \land \neg C \not\geq \neg A \land \neg C \land \neg A$, and thus $\neg A \land \neg C \not\geq \neg A \land \neg C$, by $(\geq 6)$. We can now use $(\geq 5a)$ to get $A \lor C \geq A \lor C$. From this and $A \land C \geq A \land C$, we finally get $A \geq A$, by $(\geq 5c)$. 
(c25) Suppose that $A \supset C$. Then by (≫2a), $A \supset A \land C$ and $A \supset A \lor C$.
So by (c24), $A \land C \supset A \land C$ and $A \lor C \supset A \lor C$.

(c26) Suppose that $A \supset A$ and $A \land B \not\supset A \land B$. From the latter, we get $A \not\supset A \land B$, by (≫8c). So $\neg A \not\supset \neg A \lor \neg B$, by (≫5a). From this and $A \supset A$, we finally get $A \supset A \land \neg B$, by (≫2b).

(c27) Suppose that $A \not\supset A$ and $A \land B \supset A \land B$. From the former, we get $A \land B \not\supset A$, by (c24), and this means $A \land B \not\supset (A \land B) \lor A$, by (≫6).
Since we also have $\neg A \not\supset \neg A$, by (≫5a), we get $\neg (A \land B) \supset \neg (A \land B) \land A$, by (≫2b), or equivalently, $\neg (A \land B) \supset \neg (A \land B) \land A \land \neg B$, by (≫6). So $\neg (A \land B) \supset \neg (A \land B) \land \neg B$, by (c11), and $\neg (A \land B) \supset \neg B$, by (≫6).

(c28) Let $B \not\supset B$ and $A \land B \supset A \land B \land C$. From the latter, we get $A \land B \supset A \land B$, by (c1). From this and $B \not\supset B$, we get two things: First, $A \supset A$, by (≫5b), and second, $A \land \neg B \not\supset A \land \neg B$, by (≫5a) and (≫5c).
From the last two, we get $A \supset A \land B$, by (c26). This, taken together with $A \land B \supset A \land B \land C$, gives us $A \supset A \land C$, by (≫7c).

(c29) Let $B \not\supset B$ and $A \lor B \supset (A \lor B) \land C$. From the latter, we get $A \lor B \supset A \lor B$, by (c1). From this and $B \not\supset B$, we get two things: First, $A \supset A$, by (c9), and second, $A \lor B \supset (A \lor B) \land \neg (\neg A \lor B)$, by (c26).
By (≫6), the latter reduces to $A \lor B \supset (A \lor B) \land A \land \neg B$. Using (c11), we infer $A \lor B \supset (A \lor B) \land A$, which reduces to $A \lor B \supset A$. This, taken together with $A \lor B \supset (A \lor B) \land C$, gives us $A \supset A \land C$, by (≫8c).

(c30) Suppose that $A \not\supset A$ and $A \lor B \supset A \lor B$. From the former, we get $(A \lor B) \land (A \land \neg B) \not\supset (A \lor B) \land (A \land \neg B)$, by (≫6), and thus $A \lor B \not\supset (A \lor B) \land (A \land \neg B)$, by (≫8c). By, (≫5a), $(A \lor B) \not\supset \neg (A \lor B) \lor \neg (A \land \neg B)$.
From this and $A \lor B \supset A \lor B$, (≫2b) gives us $A \lor B \supset (A \lor B) \land \neg (A \land \neg B)$ or, equivalently, $A \lor B \supset (A \lor B) \land \neg A \land B$. So by (c11) and (≫6), $A \lor B \supset B$.

(c31) Suppose that $A \supset A$ and $A \lor B \not\supset A \lor B$. From the latter, we get $A \not\supset A \lor B$, by (c24), and this means $A \not\supset A \lor (A \lor B)$, by (≫6). Since we also have $\neg A \not\supset \neg A$, by (≫5a), we get $\neg A \supset \neg A \land (A \lor B)$, by (≫2b), or equivalently, $\neg A \supset \neg A \land B$, by (≫6).

$\blacksquare$

Proof of Lemma 12. Suppose that the definitions (DefBel) and (Def*) are used.

(i) Closure of Bel under conjunction. Suppose that $A$ and $B$ are beliefs, i.e., both $X_{bel} \supset X_{bel} \land A$ and $X_{bel} \supset X_{bel} \land B$. Then, by (c13), $X_{bel} \supset (X_{bel} \land A) \land (X_{bel} \land B)$, so also $X_{bel} \supset X_{bel} \land (A \land B)$, b (≫6), which means that $A \land B$ is a belief.

Closure of Bel under singleton entailment. We show that if $A \land B$ is a belief, so is $A$. Suppose that $A \land B$ is a belief, i.e., $X_{bel} \supset X_{bel} \land A \land B$.
Hence, by (c11), $X_{bel} \supset X_{bel} \land A$, which means that $A$ is a belief.
(ii) We need to show that $X_{\text{bel}} \not\supset X_{\text{bel}} \land \neg A$ implies that $A \gg A$ or $X_{\text{bel}} \gg A \land X_{\text{bel}}$. Suppose that $X_{\text{bel}} \not\supset X_{\text{bel}} \land A$ and $A \not\gg A$. From the former, we get $A \land X_{\text{bel}} \gg A \land X_{\text{bel}}$, by Lemma 7, part (i.2) $\Rightarrow$ (i.1). Now we apply (c15) and get $\neg A \land X_{\text{bel}} \not\supset \neg A \land X_{\text{bel}}$. Finally, we use Lemma 7, part (i.2) $\Rightarrow$ (i.1), again and get $X_{\text{bel}} \gg A \land X_{\text{bel}}$, as desired.

Proof of Observation 13. (i) Let $\ast$ satisfy (\ast0)--(\ast6). We want to show that $\ast' = \ast$. We have

\begin{itemize}
\item $C \in \text{Bel} \ast' A$ iff (by Def\ast)
\item $A \gg A \land C$ or ($A \not\gg A$ and $X_{\text{bel}} \gg X_{\text{bel}} \land C'$) iff (by RRT)
\item $A \land C \in \text{Bel} \ast A$ and $A \land C \notin \text{Bel} \ast \neg A$ or
\item \left( A \notin \text{Bel} \ast A \text{ or } A \in \text{Bel} \ast \neg A \text{ and } X_{\text{bel}} \land C \in \text{Bel} \ast X_{\text{bel}} \text{ and } X_{\text{bel}} \land C \notin \text{Bel} \ast \neg X_{\text{bel}} \right).$ (i)
\end{itemize}

$C \in \text{Bel} \ast A$ implies $C \in \text{Bel} \ast' A$. Let $C \in \text{Bel} \ast A$.

Case 1. Suppose $A \in \text{Bel} \ast A$. Then $A \land C \in \text{Bel} \ast A$, by (\ast1).

Case 1a: Suppose that $A \notin \text{Bel} \ast \neg A$, then by (\ast1) $A \land C \notin \text{Bel} \ast \neg A$, and we have shown the first disjunct of (i).

Case 1b: Suppose that $A \in \text{Bel} \ast \neg A$, then by (\ast0) $\neg A \notin \text{Bel} \ast \neg A$, so by (\ast2) $\text{Bel} \ast \neg A = \text{Bel}$. So $A \in \text{Bel}$, and $\text{Bel} \ast A = \text{Bel}$, by (\ast3) and (\ast4). So $C \in \text{Bel}$, and thus $X_{\text{bel}} \land C \in \text{Bel} = \text{Bel} \ast X_{\text{bel}}$, by (\ast1) and (r1). On the other hand, $X_{\text{bel}} \land C \notin \text{Bel} \ast \neg X_{\text{bel}}$, by (\ast0) and (\ast0b). So we have shown the second disjunct of (i).

Case 2. Suppose $A \notin \text{Bel} \ast A$. Then $\text{Bel} \ast A = \text{Bel}$, by (\ast2). So $C \in \text{Bel}$, and the rest is as in case 1b.

$C \in \text{Bel} \ast' A$ implies $C \in \text{Bel} \ast A$. Let $C \in \text{Bel} \ast' A$, i.e., suppose that (i) is true.

Case 1. Suppose that $A \land C \in \text{Bel} \ast A$. Then $C \in \text{Bel} \ast A$, by (\ast1), and we are done.

Case 2. Suppose that $A \land C \notin \text{Bel} \ast A$. Thus the second disjunct of (i) must be true, so we have that $X_{\text{bel}} \land C \in \text{Bel} \ast X_{\text{bel}}$. Hence $C \in \text{Bel}$, by (\ast1) and (r1). The second disjunct of (i) says that either $A \notin \text{Bel} \ast A$ or $A \in \text{Bel} \ast \neg A$. If the former, we have $C \in \text{Bel} \ast A$, by (\ast2). If the latter, then $\neg A \notin \text{Bel} \ast \neg A$, by (\gg0), and $A \in \text{Bel}$, by (r5). But this means that $\text{Bel} \ast A = \text{Bel}$, by (r8), and so we have $C \in \text{Bel} \ast A$ again.

(ii) Let $\gg$ satisfy (\gg0)--(\gg6). We want to show that $\gg' = \gg$. We have

$A \gg' C$ iff (by RRT)
\[ C \in Bel* A \text{ and } C \notin Bel* \neg A \text{ iff (by Def*)} \]
\[ A \gg A \land C \text{ or } (A \not\gg A \land X_{bel} \gg X_{bel} \land C) \] and \[ \neg A \not\approx \neg A \land C \quad \text{and} \quad (\neg A \not\approx \neg A \text{ or } X_{bel} \not\approx X_{bel} \land C) \, . \]

\[ A \gg C \text{ implies } A \gg' C \text{. Let } A \gg C \text{. Then } A \gg A \land C \text{, by } (\gg 2a) \text{. Also by } (\gg 2a) \text{, } A \gg A \lor C \text{ and by } (\gg 2b) \text{, } A \not\gg -A \land C \text{. From } (c1) \text{, we get } A \gg A \text{, so } \neg A \not\gg -A \text{, by } (\gg 5a) \text{.} \]

\[ A \gg' C \text{ implies } A \gg C \text{, i.e., suppose that } (\dagger) \text{ is true.} \]

Case 1. Let \( A \gg A \land C \). So by \((c1)\) \( A \gg A \) and by \((\gg 5a)\), \( -A \not\gg -A \). Using \( -A \not\approx -A \land C \) of \((\dagger)\), we get that \( A \gg A \lor C \), by \((\gg 2b)\). Since also \( A \gg A \land C \), we get that \( A \gg C \), by \((\gg 2a)\).

Case 2. Let \( A \not\gg A \) and \( X_{bel} \gg X_{bel} \land C \). But by \((\gg 5a)\), this contradicts \( -A \not\gg -A \) or \( X_{bel} \not\approx X_{bel} \land C \), so the case is impossible.

\[ \square \]

\textbf{Proof of Observation 14.} Let \( \gg \text{ satisfy } (\gg 0), (\gg 1), (\gg 2a), (\gg 2b), (\gg 3), (\gg 4), (\gg 5a)\text{–}(\gg 5c) \text{ and } (\gg 6) \text{, and let } * \text{ be obtained from } \gg \text{ by (Def*)}. \]

\((*1a)\) Closure under conjunction. We show that \( B \in Bel* A \text{ and } C \in Bel* A \text{ imply } B \land C \in Bel* A \). By (Def*), this means
\[ \text{If } A \gg A \land B \text{ or } (A \not\gg A \text{ and } X_{bel} \gg X_{bel} \land (B \lor C)) \text{, then } (A \not\gg A \text{ and } X_{bel} \gg X_{bel} \land C) \text{.} \]

Case 1: \( A \gg A \land B \) and \( A \gg A \land C \) imply \( A \gg A \land B \land C \), by \((c13)\).

Case 2: \( A \gg A \land B \) and \((A \not\gg A \text{ and } X_{bel} \gg C \land X_{bel}) \) is impossible, by \((c1)\).

Case 3: \( A \gg A \land C \) and \((A \not\gg A \text{ and } X_{bel} \gg C \land X_{bel}) \) is similar to case 2.

Case 4: \( A \not\gg A \text{ and } X_{bel} \gg X_{bel} \land B \text{ and } X_{bel} \gg X_{bel} \land C \) imply \( X_{bel} \gg X_{bel} \land (B \lor C) \), by \((c13)\) and \((\gg 6)\).

\((*1b)\) Closure under singleton entailment. We show that \( B \land C \in Bel* A \text{ implies } C \in Bel* A \). By (Def*), this means
\[ \text{If } A \gg A \land B \lor C \text{ or } (A \not\gg A \text{ and } X_{bel} \gg X_{bel} \land (B \lor C)) \text{, then } (A \not\gg A \text{ and } X_{bel} \gg X_{bel} \land C) \text{.} \]

Case 1: \( A \gg A \land B \lor C \) implies \( A \gg A \land C \), by \((c11)\).

Case 2: \( X_{bel} \gg X_{bel} \land (B \lor C) \) implies \( X_{bel} \gg X_{bel} \land C \), by \((c11)\).
\((\ast 2)\) Restricted success. We show that either \(A \in Bel \ast A\) or \(Bel \ast A = Bel\). By (Def\ast) and (DefBel), this means
\[
\left( A \gg A \land A \text{ or } (A \not \gg A \text{ and } X_{bel} \gg X_{bel} \land A) \right) \text{ or } \left( A \gg A \land C \text{ or } (A \not \gg A \text{ and } X_{bel} \gg X_{bel} \land C) \right). 
\]

Suppose the first disjunct is false, i.e., \(A \not \gg A\) and \(X_{bel} \not \gg X_{bel} \land A\). We need to show that the second disjunct is true. From \(A \not \gg A\), we get \(A \not \gg A \land C\), by (c1). But given this and \(A \not \gg A\), the LHS of the second disjunct reduces to the RHS of the second disjunct.

\((\ast 3)\) We showed by proving \((\ast 1a), (\ast 1b)\) and (r1) that \(Bel\) is logically closed. Thus it suffices to show that \(C \in Bel \ast A\) implies \(A \supset C \in Bel\). By (Def\ast) and (DefBel), this means
\[
\text{If } A \gg A \land C \text{ or } (A \not \gg A \text{ and } X_{bel} \gg X_{bel} \land C), \text{ then } X_{bel} \gg X_{bel} \land (A \supset C).
\]

Case 1: \(A \gg A \land C\) implies \(A \land \lnot C \land X_{bel} \not \gg A \land \lnot C \land X_{bel}\), by (\supseteq 3). By Lemma 7, (i.2)\(\Rightarrow\)(i1), we get \(X_{bel} \gg X_{bel} \land (A \supset C)\).

Case 2: \(X_{bel} \gg X_{bel} \land C\) is equivalent to \(X_{bel} \gg X_{bel} \land C \land (A \supset C)\) which implies \(X_{bel} \gg X_{bel} \land (A \supset C)\), by (c11).

\((\ast 4)\) Given that \(Bel\) is logically closed (which we have already shown), it suffices to show that \(\lnot A \not \in Bel\) and \(A \supset C \in Bel\) together imply \(C \in Bel \ast A\). By (Def\ast) and (DefBel), this means
\[
\text{If } X_{bel} \not \gg X_{bel} \land \lnot A \text{ and } X_{bel} \gg X_{bel} \land (A \supset C) \text{ then } A \gg A \land C \text{ or } (A \not \gg A \text{ and } X_{bel} \gg X_{bel} \land C).
\]

From the antecedent we get, by Lemma 7 (i), that \(X_{bel} \land A \gg X_{bel} \land A\) and \(X_{bel} \land (A \land \lnot C) \not \gg X_{bel} \land (A \land \lnot C)\). It follows with (\supseteq 4) that \(\lnot A \not \gg \lnot A \lor \lnot (A \land \lnot C)\), or equivalently, by (\supseteq 6), \(\lnot A \not \gg \lnot A \lor C\).

Case 1: Suppose that \(A \gg A\). Then we get \(A \gg A \land C\), by (\supseteq 2b).

Case 2: Suppose that \(A \not \gg A\). From this and \(X_{bel} \not \gg X_{bel} \land \lnot A\) we get, by Lemma 12 (ii), that \(X_{bel} \gg X_{bel} \land A\). From this and \(X_{bel} \gg X_{bel} \land (A \supset C)\), we get \(X_{bel} \gg X_{bel} \land A \land C\), by (c13) and (\supseteq 6). So \(X_{bel} \gg X_{bel} \land C\), by (c11).

\((\ast 5a)\) We show that \((A \not \gg A \land X_{bel} \not \gg X_{bel} \land A)\) implies \((A \land B \not \gg A \land B\) and \(X_{bel} \not \gg X_{bel} \land (A \land B))\). From \(A \not \gg A\) we get \(\lnot A \not \gg \lnot A\), by (\supseteq 5a), and from \(X_{bel} \not \gg X_{bel} \land A\) we get \(X_{bel} \land \lnot A \gg X_{bel} \land \lnot A\), by Lemma 7, (i.2)\(\Rightarrow\)(i.1). Thus, by (\supseteq 5c), \((\lnot A \lor \lnot B) \not \gg (\lnot A \lor \lnot B)\). So by (\supseteq 5a) again,
A \land B \not\supset A \land B. Moreover, \( X_{\text{bel}} \not\supset X_{\text{bel}} \land A \) implies \( X_{\text{bel}} \not\supset X_{\text{bel}} \land (A \land B) \), by (c11).

(*5b) We show that \( A \lor B \supset A \lor B \) or \( X_{\text{bel}} \supset X_{\text{bel}} \land (A \lor B) \) implies \( A \supset A \lor B \) or \( X_{\text{bel}} \supset X_{\text{bel}} \land B \). First, let that \( A \lor B \supset A \lor B \). Then the claim follows from (c9). Second, let \( X_{\text{bel}} \supset X_{\text{bel}} \land (A \lor B) \). Suppose for reductio that the claim is false. Then it follows from \( A \not\supset A \lor B \) and \( X_{\text{bel}} \not\supset X_{\text{bel}} \land A \) that \( X_{\text{bel}} \supset X_{\text{bel}} \land \neg A \), by Lemma 7, (iii.3)⇒(iii.1). In a similar way, we get \( X_{\text{bel}} \supset X_{\text{bel}} \land \neg B \). By two-fold application of (c13), then, we get \( X_{\text{bel}} \supset X_{\text{bel}} \land (A \lor B) \land \neg A \land \neg B \). So by (≥6), \( X_{\text{bel}} \supset \bot \). But this contradicts (c2).

(*6) follows from (≥6).

(*7c) We show that \( B \in \text{Bel} \ast A \) and \( C \in \text{Bel} \ast (A \land B) \) implies \( C \in \text{Bel} \ast A \). Suppose the hypothesis, i.e., that
\[
A \supset A \land B \quad \text{or} \quad A \land B \supset A \land B \quad \text{and} \quad (A \not\supset A \land B \quad \text{and} \quad X_{\text{bel}} \supset X_{\text{bel}} \land B) \quad \text{or} \quad (A \land B \not\supset A \land B \quad \text{and} \quad X_{\text{bel}} \supset X_{\text{bel}} \land C).
\]
We need to show that \( A \supset A \land C \) or \( A \not\supset A \) and \( X_{\text{bel}} \supset X_{\text{bel}} \land C \).

Case 1. Suppose the two upper rows are true. Then we get \( A \supset A \land C \), by (≥7c) straight away.

Case 2. Suppose the upper left and the lower right rows are true. Since \( A \supset A \land B \) entails \( A \land B \supset A \land B \), by (≥8c), this is impossible.

Case 3. Suppose the lower left and the upper right rows are true. From \( A \land B \supset A \land B \land C \), we get \( A \land B \supset A \land B \), by (c1). From this and \( A \not\supset A \), it follows by (c17) that \( X_{\text{bel}} \supset X_{\text{bel}} \land A \). With \( X_{\text{bel}} \supset X_{\text{bel}} \land B \), we conclude that \( X_{\text{bel}} \supset X_{\text{bel}} \land A \land B \), using (c13). So by (c20), \( X_{\text{bel}} \supset X_{\text{bel}} \land C \), as desired.

Case 4. Suppose the two lower rows are true. Then we have \( A \not\supset A \) and \( X_{\text{bel}} \supset X_{\text{bel}} \land C \).

(*7) We show that \( C \in \text{Bel} \ast (A \land B) \) implies \( B \supset C \in \text{Bel} \ast A \). Suppose the former, i.e., that \( A \land B \supset A \land B \land C \) or \( (A \land B \not\supset A \land B \quad \text{and} \quad X_{\text{bel}} \supset X_{\text{bel}} \land C) \). We need to show that \( A \supset A \land (B \supset C) \) or \( A \not\supset A \) and \( X_{\text{bel}} \supset X_{\text{bel}} \land (B \supset C) \).

Case 1. Suppose that \( A \land B \supset A \land B \land C \) and \( A \supset A \land C \). Then we get \( A \supset A \land (B \supset C) \), by (≥7) straight away.

Case 2. Suppose that \( A \land B \supset A \land B \land C \) and \( A \not\supset A \). We will show that \( X_{\text{bel}} \supset X_{\text{bel}} \land (B \supset C) \). On the one hand, \( A \land B \supset A \land B \land C \) implies \( A \land B \supset A \land B \), by (c1). From this and \( A \not\supset A \), it follows by (c17) that \( X_{\text{bel}} \supset X_{\text{bel}} \land A \). On the other hand, we can infer from \( A \land B \supset A \land B \land C \) that \( A \land B \land \neg C \land X_{\text{bel}} \not\supset A \land B \land \neg C \land X_{\text{bel}} \), by (≥3). So by Lemma 7, (i.2)⇒(i.1), we get \( X_{\text{bel}} \supset X_{\text{bel}} \land ((A \land B) \supset C) \). So by (c13), \( X_{\text{bel}} \supset X_{\text{bel}} \land A \land ((A \land B) \supset C) \). From (≥6) and (c11), we finally get \( X_{\text{bel}} \supset X_{\text{bel}} \land (B \supset C) \), as desired.
Case 3. Suppose that $A \land B \not\supset A \land B$ and $X_{el} \not\supset X_{el} \land C$. Case 3a. Suppose that in addition $A \not\supset A$. Now $X_{el} \supset X_{el} \land C$ gives us $X_{el} \supset X_{el} \land (B \supset C)$, by (≫6) and (c11), so we are done. Case 3b. Suppose that in addition $A \not\supset A$. From this and $A \land B \not\supset A \land B$, we get $A \not\supset A \land \neg B$, by (c26). With the help of (≫6), this can be rewritten as $A \not\supset A \land B \land (B \supset C)$. Thus, by (c11), $A \supset A \land (B \supset C)$, as desired.

(⋆7'). We show that $C \in Bel \star A$ and $C \in Bel \star B$ implies $C \in Bel \star (A \lor B)$. Suppose the hypothesis, i.e., that

$$A \supset A \land C \text{ or } (A \not\supset A \text{ and } X_{el} \supset C \land X_{el}) \quad \text{and} \quad B \supset B \land C \text{ or } (B \not\supset B \text{ and } X_{el} \supset X_{el} \land C)'.$$

We need to show that $A \lor B \supset (A \lor B) \land C$ or $(A \lor B \not\supset (A \lor B) \land C$. Case 1. Suppose the two upper rows are true. Case 1a. Suppose further that $A \lor B \not\supset A \lor B$. Then we can use (≫7') and get $A \lor B \not\supset (A \lor B) \land C$ right away. Case 1b. Suppose further that $A \lor B \not\supset A \lor B$. We will show that $X_{el} \supset X_{el} \land C$. On the one hand, we infer from $A \supset A \land C$ that $A \land \neg C \land X_{el} \not\supset A \land \neg C \land X_{el}$, by (≫3), and use Lemma 7, (i.2)⇒(i.1), to get $X_{el} \supset X_{el} \land (A \supset C)$. Similarly, we get that $X_{el} \supset X_{el} \land (B \supset C)$. On the other hand, we infer from $A \supset A \land C$ that $A \supset A$, by (c1), which together with $A \lor B \not\supset A \lor B$ gives us $X_{el} \supset X_{el} \land (A \lor B)$, by (c17). Finally, we apply (c1) twice and get $X_{el} \supset X_{el} \land (A \lor B) \land (A \lor C) \land (B \lor C)$, which is equivalent to $X_{el} \supset X_{el} \land (A \lor B) \land C$, by (≫6). By (c11), this implies that $X_{el} \supset C$, as desired.

Case 2. Suppose the upper left and the lower right rows are true. Then we have $A \supset A \land C$, $B \not\supset B$ and $X_{el} \supset X_{el} \land C$. If $A \lor B \not\supset A \lor B$, then we are done. So suppose further that $A \lor B \supset A \lor B$. We show that $A \lor B \supset (A \lor B) \land C$. But from $B \not\supset B$ and $A \lor B \supset A \lor B$, we get $A \lor B \supset A$, by (c30), i.e., $A \lor B \supset (A \lor B) \land (A \lor \neg B)$. From this and $A \supset A \land C$, we get $A \lor B \supset (A \lor B) \land C$, by (≫6) and (≫7c).

Case 3. Suppose the lower left and the upper right rows are true. This is similar to Case 2.

Case 4. Suppose both of the lower rows are true. From $A \not\supset A$ and $B \not\supset B$, we get $A \lor B \not\supset A \lor B$, by (c9). But since we also have $X_{el} \supset X_{el} \land C$, we are done.

(⋆8c). We show that $B \in Bel \star A$ and $C \in Bel \star A$ implies $C \in Bel \star (A \land B)$. Suppose the hypothesis, i.e., that

$$A \not\supset A \land B \text{ or } (A \not\supset A \text{ and } X_{el} \supset X_{el} \land B) \quad \text{and} \quad A \supset A \land C \text{ or } (A \not\supset A \text{ and } X_{el} \supset X_{el} \land C)'.$$
We need to show that $A \land B \supseteq A \land B \land C$ or $(A \land B \not\supseteq A \land B$ and $X_{bel} \supseteq X_{bel} \land C$).

Case 1. Suppose the two upper rows are true. Then we get $A \land B \supseteq A \land B \land C$, by $(\supseteq 8c)$, straight away.

Cases 2–3. Suppose one of the upper and one of the lower rows are true. But this is impossible, since each of $A \supseteq A \land B$ and $A \supseteq A \land C$ entails that not $A \not\supseteq A$, by (c1).

Case 4. Suppose the two lower rows are true, i.e., $A \not\supseteq A$, $X_{bel} \supseteq X_{bel} \land B$ and $X_{bel} \supseteq X_{bel} \land C$. Case 1: Suppose that in addition $A \land B \not\supseteq A \land B$. Then we are ready immediately. Case 2: Suppose that in addition $A \land B \not\supseteq A \land B$. From this last supposition and $A \not\supseteq A$ it follows by (c17) that $X_{bel} \supseteq X_{bel} \land A$. So by (c13) $X_{bel} \supseteq X_{bel} \land A \land B$ and by (c8) $X_{bel} \not\supseteq X_{bel} \land \neg(A \land B)$. We can finally apply (c23) and get that $A \land B \supseteq (A \land B) \land C$.

$(\star 8)$. We show that $\neg B \not\in Bel \ast A$ and $C \in Bel \ast A$ together imply $C \in Bel \ast (A \land B)$. Suppose the hypothesis, i.e., that

$$A \not\supseteq A \land \neg B \text{ and } (A \supseteq A \text{ or } X_{bel} \not\supseteq X_{bel} \land \neg B) \text{ and } (A \not\supseteq A \land X_{bel} \supseteq X_{bel} \land C).$$

We need to show that $A \land B \supseteq A \land B \land C$ or $(A \land B \not\supseteq A \land B$ and $X_{bel} \supseteq X_{bel} \land C$).

Case 1. Suppose the upper right row is true. Then we use $A \not\supseteq A \land \neg B$ and get $A \land B \supseteq A \land B \land C$, by $(\supseteq 8)$, straight away.

Case 2. Suppose the lower right row is true. Case 2a. Suppose further that $A \land B \not\supseteq A \land B$. Then the second disjunct of what we need to show is true. Case 2b. Suppose further that $A \land B \supseteq A \land B$. From this last supposition and $A \not\supseteq A$ it follows by (c17) that $X_{bel} \supseteq X_{bel} \land A$. So by (c13) and (c11) $X_{bel} \not\supseteq X_{bel} \land \neg(A \land B)$, because otherwise $X_{bel} \supseteq X_{bel} \land \neg B$ which is excluded by the lower left row. So we can finally apply (c23) and get that $A \land B \supseteq (A \land B) \land C$.

References


Difference-Making Conditionals and Connexivity


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