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On the values taken by slice torus invariants

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Abstract

We study the space of slice torus invariants. In particular we characterise the set of values that slice torus invariants may take on a given knot in terms of the stable smooth slice genus. Our study reveals that the resolution of the local Thom conjecture implies the existence of slice torus invariants without having to appeal to any explicit construction from a knot homology theory.

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1. *Introduction*

A fruitful approach to understanding a group is to construct homomorphisms on it; the group of interest in this paper is the smooth concordance group $\mathcal{C}$ of knots. A classical example of such homomorphisms is given by the Levine–Tristram signatures $\mathcal{C} \to \mathbb{R}$, which were used in Litherland's proof that positive non-trivial torus knots $T_{p,q}$ are linearly independent in $\mathcal{C}$ [**Lit79**]. Signatures also provide lower bounds for the smooth slice genus $g_4(K) \in \mathbb{Z}_{\geqslant 0}$ of a knot K but, in the case of torus knots, these bounds are not sufficient to determine $g_4(T_{p,q})$. In fact, that we have $g_4(T_{p,q}) = (|p| - 1)(|q| - 1)/2$, which is known as the local Thom conjecture, was first shown by Kronheimer and Mrowka, as a consequence of their resolution of the Thom conjecture [**KM93**] using gauge theory. This article is concerned with a class of homomorphisms $\mathcal{C} \to \mathbb{R}$ that is much younger than signatures, namely slice torus invariants, whose definition goes back to Livingston [**Liv04**] (see also [**Lew14**]).

Definition 1. A *slice torus invariant* is a homomorphism $\phi : \mathcal{C} \to \mathbb{R}$ satisfying two conditions:

$$\begin{array}{lll} \text{SLICE:} & \phi(K) \leqslant g_4(K) & \text{for all knots } K \text{ and} \\ \text{TORUS:} & \phi(T_{p,q}) = g_4(T_{p,q}) & \text{for all positive coprime integers } p, q. \end{array}$$

Note that it is quite non-trivial that such invariants do exist. Using suitable normalisations, the first slice torus invariant to be constructed was the τ invariant coming from knot Floer homology [**OSz03**, **Ras03**], followed by the Rasmussen invariant s coming from Khovanov homology [**Ras10**], and the s_n invariants coming from $\mathfrak{sl}_n$ Khovanov–Rozansky homologies [**Wu09**, **Lob09**, **Lob12**]. We study the set $V \subset \mathrm{Hom}(\mathcal{C}, \mathbb{R})$ of all slice torus invariants. Note that V is non-empty and convex. It follows that for each $K \in \mathcal{C}$, the set $V(K) := \{\phi(K) \mid \phi \in V\} \subset \mathbb{R}$ is a nonempty interval. The main result of this note provides a description of these intervals, in terms of the *stable smooth slice genus* (compare [**Liv10**]) $\widehat{g}_4(K) := \lim_{n\to\infty} g_4(K^{\#n})/n$.

PROPOSITION 2. *For every knot K, the sequence $t_1(K), t_2(K), \ldots$ defined as*

$$t_p(K) := \widehat{g}_4(T_{p,p+1} \# K) - \widehat{g}_4(T_{p,p+1})$$

is decreasing and convergent. Its limit $\ell(K)$ satisfies $-\ell(-K) \leqslant \ell(K)$.

THEOREM 3. *For every knot K, the set $V(K) = \{\phi(K) \mid \phi \in V\} \subset \mathbb{R}$ of values taken by all slice torus invariants on K equals $[-\ell(-K), \ell(K)]$.*

Remark 4. Our proof of Theorem 3 uses the fact that $g_4(T_{p,p+1}) = p(p-1)/2$ for all integers $p \geqslant 1$ [**KM93**], but we do not use the *a priori* existence of any slice torus invariant. Thus, it follows from our proof that the local Thom conjecture implies the existence of slice torus invariants without the need of any explicit construction of a slice torus invariant. However, we note that from our proof it is not clear that there exist integer valued slice torus invariants such as s or τ (suitably normalised), or even that $[-\ell(K), \ell(-K)]$ contains an integer for all knots K.

Example 5. Let us explicitly calculate $V(K)$ for K the $(2, -3, 5)$ pretzel knot, which is 10_{125} in the knot table. As above, let s_n be the concordance invariant coming from $\mathfrak{sl}_n$ Khovanov–Rozansky homology. Then $\tilde{s}_n := s_n/2(n-1)$ is a slice torus invariant, and s_2 is equal to the original Rasmussen invariant $s_2 = s$. One may calculate that $\tilde{s}_2(K) = 1$ and furthermore, for all $n \geqslant 3$, that $\tilde{s}_n(K) \in \{0, 1/(n-1)\}$; see [**Lew14**]. Since $\lim_{n\to\infty} \tilde{s}_n(K) = 0$, it follows from Theorem 3 that $[0, 1] \subset V(K)$.

To show the converse inclusion $V(K) \subset [0, 1]$, let us use the sharpened slice-Bennequin inequalities [**Lob11**, **Lew14**]. We will only need the braid version of the inequalities as stated in $(*)$ below. Denote by $\sigma_1, \ldots, \sigma_{k-1}$ the standard generators of the braid group B_k on k strands. For β a word in these generators, let

$$O_\pm(\beta) = \#\{i \in \{1, \ldots, k-1\} \mid \sigma_i^{\pm 1} \text{ does not appear in } \beta\}.$$

Then for all slice torus invariants $\phi \in V$, we have that the value taken on the closure $\mathrm{cl}(\beta)$ satisfies

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$$2\phi(\mathrm{cl}(\beta)) \quad \in \quad [1+w(\beta)-k+2O_+(\beta),\ -1+w(\beta)+k-2O_-(\beta)], \qquad (*)$$

where $w(\beta)$ denotes the writhe of β. The knot $K=10_{125}$ is the closure of $\beta=\sigma_1^5\sigma_2^{-1}\sigma_1^{-3}\sigma_2^{-1}\in B_3$. Since $w(\beta)=0$, $O_+(\beta)=1$ and $O_-(\beta)=0$, ($*$) implies that $\phi(K)\in[0,1]$ for all $\phi\in V$, and thus $V(K)\subset[0,1]$. All in all, we have shown that $V(K)=[0,1]$.

This computation of $V(K)$ yields further examples. Namely, for all $a,b\in\mathbb{Z}$ with $a\geqslant 0$ it immediately follows that

$$V(K^{\#a}\#T_{2,3}^{\#b})=[b,a+b].$$

Hence we have the following result.

PROPOSITION 6. *Every nonempty compact interval with integral endpoints is realised as V(J) for some knot J.*

Beyond Proposition 6, we do not know if any further intervals can be realised. The following geography question thus remains open.

Question 7. Which nonempty compact intervals arise as V(J) for some knot J?

2. *Squeezed knots*

Slice torus invariants all agree on the following class of knots.

Definition 8 ([**FLL22**]). A knot K is called *squeezed* if and only if there exists a smooth oriented connected cobordism C_+ between a positive torus knot T^+ and K, and a smooth oriented connected cobordism C_- between K and a negative torus knot T^- such that $C_+\cup C_-$ is a smooth oriented connected cobordism between T^+ and T^- that is genus-minimising.

The reader may wish to try to prove the following proposition directly from the definitions. It says, roughly speaking, that squeezed knots are boring from the point of view of slice torus invariants.

PROPOSITION 9 ([**FLL22**]). *If ϕ_1 and ϕ_2 are slice torus invariants and K is squeezed then we have that $\phi_1(K)=\phi_2(K)$.*

By Theorem 3, Proposition 9 also follows from the following.

PROPOSITION 10. *If a knot K is squeezed, then $-\ell(-K)=\ell(K)$.*

Proof. Let $C_\pm$ and $T^\pm$ be chosen as in the definition of squeezedness applied to K. We may assume that for some large $p>0$, the $T^\pm$ satisfy $T^+=T:=T(p,p+1)$ and $T^-=-T$. This is because for any positive torus knot L there exists a $p>0$ such there is a genus-minimising slice surface for $T(p,p+1)$ that factors through L (see Lemma 14 (i)). Then, we have

$$\ell(-K)+\ell(K)\leqslant t_p(-K)+t_p(K)$$

by the monotonicity of $t_p(K)$ shown in Proposition 2. By definition of t_p this equals

$$=\widehat{g_4}(T\#-K)+\widehat{g_4}(T\#K)-2\widehat{g_4}(T).$$

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It is well known (and we provide a proof in Lemma 14 (iv)) that slice genus and stable slice genus of torus knots agree; therefore we find the equality

$$= \widehat{g_4}(T\# -K) + \widehat{g_4}(T\#K) - g_4(T\#T).$$

Since $\widehat{g_4} \leqslant g_4$ for all knots,

$$\leqslant g_4(T\# -K) + g_4(T\#K) - g_4(T\#T).$$

For all J, J', $g_4(J\# -J')$ equals the cobordism distance between J and J', and so

$$\leqslant g(C_+) + g(C_-) - g_4(T\#T),$$

which equals 0 because of the assumption that $C_+ \cup C_-$ is a genus-minimising cobordism between T and $-T$.

The proof of Proposition 10 shows that if K is squeezed, then the sequence $t_p(K)$ is constant for sufficiently large p. We do not know whether this is the case for all knots:

Question 11. We ask the following:

(i) is $\ell(K)$ an integer for all knots K?

(ii) (stronger) Is $\widehat{g_4}(T_{p,p+1}\#K)$ an integer for all but finitely many p for every fixed knot K?

(iii) (strongest) Does $\widehat{g_4}(T_{p,p+1}\#K) = g_4(T_{p,p+1}\#K)$ hold for all but finitely many p for every fixed knot?

Remark 12. If (i) can be answered positively, then Question 7 is resolved: the intervals that occur as $V(J)$ for some knot are $[a,b]$ with $a \leqslant b$ integers.

If (iii) can be answered positively, then K satisfying $-\ell(-K) = \ell(K)$ implies that K is squeezed. This is seen as follows. If a knot K satisfies $-\ell(-K) = \ell(K)$, then we have for some $p > 0$

$$g_4(T_{p,p+1}\#K) - g_4(T_{p,p+1}) = \ell(K) = -\ell(-K) = g_4(T_{p,p+1}) - g_4(T_{p,p+1}\# -K)$$

and hence

$$g_4(T_{p,p+1}\#K) + g_4(T_{p,p+1}\# -K) = 2g_4(T_{p,p+1}).$$

The left-hand side of the equation is the genus of a cobordism from $T_{p,p+1}$ to $-T_{p,p+1}$ that factors through K. On the other hand, the right-hand side is the minimal genus of a cobordism from $T_{p,p+1}$ to $-T_{p,p+1}$. Thus we see that K must be squeezed.

In light of this, we conjecture the converse of Proposition 10.

CONJECTURE 13. *For all knots K, K is squeezed if and only if $-\ell(K) = \ell(-K)$.*

3. *Proof of the main theorem*

The stable 4-genus $\widehat{g_4}$ induces a seminorm on the vector space $\mathcal{C} \otimes \mathbb{R}$, as discussed by Livingston [**Liv10**] (Livingston states the result for the vector space $\mathcal{C} \otimes \mathbb{Q}$, but it

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easily extends to $\mathcal{C}\otimes\mathbb{R}$). Moreover, every slice torus invariant y gives rise to a homomorphism $y:\mathcal{C}\otimes\mathbb{R}\to\mathbb{R}$, with $y\leqslant\widehat{g_4}$. Here, our slightly abusive notation does not differentiate between $\widehat{g_4}$ and the induced seminorm, y and the induced homomorphism, nor between knots and the vectors they represent in $\mathcal{C}\otimes\mathbb{R}$.

In what follows, let $\mathcal{T}$ be the real subspace of $\mathcal{C}\otimes\mathbb{R}$ generated by torus knots, and let $\mathcal{T}_+\subset\mathcal{T}$ be the closed convex cone consisting of linear combinations of positive torus knots with non-negative coefficients.

Let us emphasize, as mentioned in Lemma 4, that we only use the fact that $g_4(T_{p,p+1})=p(p-1)/2$. The realization that this fact is enough to determine g_4 for much larger classes of knots is due to Rudolph [**Rud93**]. For the sake of self-containedness, we include a short proof.

LEMMA 14. *We have the following.*

(i) For all knots $K\in\mathcal{T}_+$, there is an integer $p\geqslant 1$ and a smooth cobordism C between K and $T_{p,p+1}$ such that $g(C)=g_4(T_{p,p+1})-g_4(K)$.

(ii) For all integers $p\geqslant 2$, there is a smooth cobordism C between $T_{p-1,p}$ and $T_{p,p+1}$ of genus $p-1$.

(iii) For all knots $K,K'\in\mathcal{T}_+$, we have $g_4(K\#K')=g_4(K)+g_4(K')$.

(iv) For all knots $K\in\mathcal{T}_+$, we have $\widehat{g_4}(K)=g_4(K)$.

Proof.

(i) Since K is a connected sum of positive torus knots, it may in particular be written as closure of a positive braid word $\beta\in B_k$ for some k. Assume that β is the product of l generators. Replace each σ_i in β with $\sigma_1\cdots\sigma_{k-1}$ to find a cobordism (consisting of $l(k-2)$ 1-handles) from K to the torus link $T(k,l)$. Set $p=\max\{k,l-1\}$. Compose this first cobordism with a cobordism from $T(k,l)$ to $T(k,p+1)$ given by $(p+1-l)(k-1)$ 1-handles, and then with a further cobordism from $T(k,p+1)$ to $T(p,p+1)$ given by $(p-k)p$ 1-handles. In total, this yields a cobordism C of genus $g(C)=p(p-1)/2-(1+l-k)/2$. The triangle inequality implies $g_4(K)\geqslant g_4(T_{p,p+1})-g(C)=(1+l-k)/2$. On the other hand, Seifert's algorithm applied to β results in a Seifert surface of genus $(1+l-k)/2$ for K. Thus, $g_4(K)=(1+l-k)/2$, and $g(C)=g_4(T_{p,p+1})-g_4(K)$ as desired.

(ii) Note that $T_{p-1,p}$ is the closure of the braid $\beta=(\sigma_1\cdots\sigma_{p-1})^{p-1}\in B_p$. The desired cobordism C consists of $2(p-1)$ 1-handles and may be constructed by appending $(\sigma_1\cdots\sigma_{p-1})^2$ to β, thus obtaining the braid $(\sigma_1\cdots\sigma_{p-1})^{p+1}$, whose closure is $T_{p,p+1}$. The existence of C is also implicit in [**Baa12**, proof of theorem 2], [**Fel16**, example 20], or follows from [**Fel14**, theorem 2].

(iii) As in (i), K and K' may be written as closures of positive braid words $\beta\in B_k$, $\beta'\in B_{k'}$ that are the product of l and l' generators, respectively. Then, $K\#K'$ is the closure of a positive braid word $\beta''\in B_{k+k'-1}$ that is the product of $l+l'$ generators. As shown in (i), this implies that

$$g_4(K\#K')=\frac{1+l+l'-(k+k'-1)-1}{2}$$

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$$= \frac{1+l-k}{2} + \frac{1+l'-k'}{2} = g_4(K) + g_4(K').$$

(iv) This directly follows from (iii) and the definition of $\widehat{g_4}$.

We are now ready to proceed to prove Proposition 2 and Theorem 3.

Proof of Proposition 2. Let us first show that $t_p(K)$ is monotonically decreasing. By Lemma 14 (ii), for $p \geqslant 2$ there exists a smooth cobordism C of genus $g_4(T_{p,p+1}) - g_4(T_{p-1,p}) = p-1$ between $T_{p-1,p}$ and $T_{p,p+1}$. Let F be a genus-minimising slice surface of $(T_{p-1,p}\#K)^{\#n}$. Gluing F to $C^{\#n}$ gives a slice surface F' of $(T_{p,p+1}\#K)^{\#n}$ of genus $g(F') = g(F) + n(p-1)$. Thus

$$g_4((T_{p,p+1}\#K)^{\#n}) \leqslant g_4((T_{p-1,p}\#K)^{\#n}) + n(p-1)$$

$$\implies \frac{g_4((T_{p,p+1}\#K)^{\#n})}{n} - \frac{p(p-1)}{2} \leqslant \frac{g_4((T_{p-1,p}\#K)^{\#n})}{n} - \frac{(p-1)(p-2)}{2}$$

$$\implies t_p(K) \leqslant t_{p-1}(K).$$

Next, we observe that $t_p(K)$ is bounded below, and thus converges. Indeed,

$$\begin{aligned} t_p(K) + t_p(-K) &= \widehat{g_4}(T_{p,p+1}\#K) + \widehat{g_4}(T_{p,p+1}\# -K) - 2g_4(T_{p,p+1}) \\ &\geqslant \widehat{g_4}(T_{p,p+1}\#K\#T_{p,p+1}\# -K) - 2g_4(T_{p,p+1}) \\ &= 2\widehat{g_4}(T_{p,p+1}) - 2g_4(T_{p,p+1}), \end{aligned}$$

which is zero by Lemma 14 (iv). Hence we have $t_p(K) \geqslant -t_p(-K) \geqslant -t_1(-K)$. Finally, taking the limit $p \to \infty$ of $t_p(K) + t_p(-K) \geqslant 0$ also yields $\ell(K) + \ell(-K) \geqslant 0$, as desired.

Proof of Theorem 3. We first check that $\phi(K) \in [-\ell(-K), \ell(K)]$ for every slice torus invariant ϕ. For every p, we have

$$\begin{aligned} \phi(-K) &= \phi(T_{p,p+1}\# -K) + \phi(-T_{p,p+1}) \\ &= \phi(T_{p,p+1}\# -K) - \widehat{g_4}(T_{p,p+1}) \\ &\leqslant \widehat{g_4}(T_{p,p+1}\# -K) - \widehat{g_4}(T_{p,p+1}) \\ &= t_p(-K), \end{aligned}$$

where we used that $\phi(J) \leqslant \widehat{g_4}(J)$ for all knots J. Taking the limit gives $-\phi(K) = \phi(-K) \leqslant \ell(-K)$, and, by replacing K by $-K$, we find $\phi(K) \leqslant \ell(K)$. Hence, we have $\phi(K) \in [-\ell(-K), \ell(K)]$ as desired.

As last step of the proof, for a given knot K and a given real number $\lambda \in [-\ell(K), \ell(-K)]$, we need to construct a slice torus invariant ϕ with $\phi(K) = \lambda$. Positive non-trivial torus knots have linearly independent Levine–Tristram signatures [**Lit79**]. Therefore they are linearly independent in $\mathcal{C} \otimes \mathbb{R}$ and form a basis of $\mathcal{T}$. Thus there is a unique homomorphism $\phi'' : \mathcal{T} \to \mathbb{R}$ with $\phi''(T_{p,q}) = g_4(T_{p,q})$ for all coprime positive p, q. We claim that

$$\phi''(P) = \widehat{g_4}(P) \qquad (\dagger)$$

holds for all vectors $P \in \mathcal{T}_+$. If P is a knot, then (†) is true by Lemma 14 (iii). Since $\phi''(\xi P) = \xi\phi''(P) = \xi\widehat{g_4}(P) = \widehat{g_4}(\xi P)$ for all positive rationals ξ, (†) also holds for P equal to a rational

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multiple of a knot. Thus we have that

$$\phi''|_{\mathcal{T}_+\cap\mathcal{C}\otimes\mathbb{Q}} = \widehat{g_4}|_{\mathcal{T}_+\cap\mathcal{C}\otimes\mathbb{Q}},$$

but $\mathcal{T}_+ \cap \mathcal{C} \otimes \mathbb{Q}$ is a dense subset of $\mathcal{T}_+$ endowed with the subspace topology arising from the colimit topology of the Euclidean topologies on all finite-dimensional subspaces of $\mathcal{T}$. The colimit topology is the finest topology such that for all finite-dimensional subspaces of $\mathcal{T}$, equipped with the Euclidean topology, the inclusion homomorphism into $\mathcal{T}$ is continuous. Moreover, ϕ'' and the restriction of $\widehat{g_4}$ to $\mathcal{T}$ are continuous functions with respect to the colimit topology since their restrictions to all finite dimensional subspaces are continuous. Thus (†) holds for all $P \in \mathcal{T}_+$.

Now, all $T \in \mathcal{T}$ can be written as $P - P'$ with $P, P' \in \mathcal{T}_+$. We have

$$\phi''(T) = \widehat{g_4}(P) - \widehat{g_4}(P') \leqslant \widehat{g_4}(P - P') = \widehat{g_4}(T),$$

so the homomorphism ϕ'' is dominated by $\widehat{g_4}$, i.e. $\phi''(T) \leqslant \widehat{g_4}(T)$ for all $T \in \mathcal{T}$.

We now proceed to construct the desired slice torus invariant ϕ. Let us first consider the case that the given knot K lies in $\mathcal{T}$. Then it follows from Lemma 14 (i) that K is squeezed, and so $-\ell(-K) = \ell(K)$ by Proposition 10. Therefore $\lambda = \ell(K) = \phi(K)$ for all slice torus invariants ϕ. So it is enough to show the existence of any slice torus invariant. The Hahn–Banach theorem implies that ϕ'' extends to a homomorphism $\phi : \mathcal{C} \otimes \mathbb{R} \to \mathbb{R}$ that satisfies $\phi \leqslant \widehat{g_4}$ on all of $\mathcal{C} \otimes \mathbb{R}$. Precomposing ϕ with the canonical map $\mathcal{C} \to \mathcal{C} \otimes \mathbb{R}, K \mapsto K \otimes 1$, gives a slice torus invariant.

Now, let us take care of the case that $K \notin \mathcal{T}$. Consider the space $\mathcal{T}_K = \mathcal{T} + \langle K\rangle$. Set $\phi'(T + \mu K) = \phi''(T) + \mu \cdot \lambda$ for all vectors $T \in \mathcal{T}$ and reals $\mu \in \mathbb{R}$. This is clearly a homomorphism $\mathcal{T}_K \to \mathbb{R}$. Let us check that it is dominated by $\widehat{g_4}$, i.e. $\phi'(T + \mu K) \leqslant \widehat{g_4}(T + \mu K)$ for all $T \in \mathcal{T}$ and $\mu \in \mathbb{R}$. We claim that the case $\mu = 1$ quickly implies the general case. Indeed, for $\mu > 0$, assuming the case $\mu = 1$, we have

$$\phi'(T + \mu K) = \mu\phi'(T/\mu + K) \leqslant \mu\widehat{g_4}(T/\mu + K) = \widehat{g_4}(T + \mu K).$$

The case $\mu < 0$ follows from the case $\mu > 0$ since $T + \mu K = T + (-\mu)(-K)$. So let us now show the case $\mu = 1$, i.e. that for all $T \in \mathcal{T}$ we have

$$\phi'(T + K) \leqslant \widehat{g_4}(T + K). \tag{‡}$$

Let us first consider the case that T is a knot in $\mathcal{T}_+$. Then by Lemma 14 (i), there exists a cobordism C from T to some $T_{p,p+1}$ with genus $g(C) = \widehat{g_4}(T_{p,p+1}) - \widehat{g_4}(T)$. We then have

$$\begin{aligned}
\phi'(T + K) &= \widehat{g_4}(T) + \lambda \\
&\leqslant \widehat{g_4}(T) + \ell(K) \\
&\leqslant \widehat{g_4}(T) + t_p(K) \\
&= \widehat{g_4}(T) + \widehat{g_4}(T_{p,p+1}\#K) - \widehat{g_4}(T_{p,p+1}) \\
&\leqslant \widehat{g_4}(T) + \widehat{g_4}(T_{p,p+1}\# -T) + \widehat{g_4}(T\#K) - \widehat{g_4}(T_{p,p+1}) \\
&\leqslant \widehat{g_4}(T) + g(C) + \widehat{g_4}(T\#K) - \widehat{g_4}(T_{p,p+1}) \\
&= \widehat{g_4}(T\#K).
\end{aligned}$$

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So, we have shown (‡) in case that T is a knot in $\mathcal{T}_+$. If $T \in \mathcal{T}_+$ such that $n\,T$ is a knot for some positive integer n, then

$$\phi'(T+K) = \tfrac{1}{n}\phi'(nT + K^{\#n}) \leqslant \tfrac{1}{n}\widehat{g_4}(nT + K^{\#n}) = \widehat{g_4}(T+K).$$

Thus (‡) holds for all T in $\mathcal{T}_+ \cap \mathcal{C} \otimes \mathbb{Q}$. Similarly as in the proof of (†), the denseness of $\mathcal{T}_+ \cap \mathcal{C} \otimes \mathbb{Q}$ in $\mathcal{T}_+$ and the continuity of ϕ' and $\widehat{g_4}$ now imply that (‡) holds for all $T \in \mathcal{T}_+$. In the general case that $T \in \mathcal{T}$, we may again write T as $P - P'$ with $P, P' \in \mathcal{T}_+$. Applying linearity of ϕ' and the triangle inequality for $\widehat{g_4}$, we find

$$\begin{aligned}\phi'(T+K) &= \phi'(-P') + \phi'(P+K)\\ &\leqslant -\widehat{g_4}(P') + \widehat{g_4}(P+K)\\ &\leqslant \widehat{g_4}(T+K).\end{aligned}$$

This concludes the proof that ϕ' is dominated by $\widehat{g_4}$ on $\mathcal{T}_K$. By the Hahn–Banach theorem, ϕ' extends to a homomorphism $\phi : \mathcal{C} \otimes \mathbb{R} \to \mathbb{R}$ that is dominated on all of its domain by $\widehat{g_4}$. Precomposing ϕ with $\mathcal{C} \to \mathcal{C} \otimes \mathbb{R}$ gives the desired slice torus invariant.

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