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A Note on Clarke's Generalized Jacobian for the Inverse of Bi-Lipschitz Maps

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Abstract

Clarke's inverse function theorem for Lipschitz mappings states that a bi-Lipschitz mapping f is locally invertible about a point x_0 if the generalized Jacobian $\partial f(x_0)$ does not contain singular matrices. It is shown that under these assumptions the generalized Jacobian of the inverse mapping at $f(x_0)$ is the convex hull of the set of matrices that can be obtained as limits of sequences $J_f(x_k)^{-1}$ with f differentiable in x_k and x_k converging to x_0. This identity holds as well if f is assumed to be locally bi-Lipschitz at x_0.

Keywords Inverse mapping · Lipschitz continuous mapping · Clarke Jacobian

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1 Introduction

F. H. Clarke introduced in [1, 2] the concepts of generalized gradients and generalized Jacobians as a tool to investigate Lipschitz continuous real-valued functions and proved a version of the inverse function theorem. To state this result precisely, we denote the Jacobian matrix of a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$, which is differentiable in x_0, by $J_f(x_0)$. By Rademacher's theorem, every Lipschitz mapping is λ^n-a.e. differentiable and the norm of the Jacobian matrix is bounded by the Lipschitz constant. The convex hull of a set $A \subseteq \mathbb{R}^n$ is denoted by $\mathrm{conv}(A)$. Throughout this note, the Euclidean norm of a

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vector $x \in \mathbb{R}^n$ is written as $\|x\|$ and the induced matrix norm is denoted by the same symbol.

Definition 1.1 For $m, n \in \mathbb{N}$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ locally Lipschitz continuous, the set of points in which f is not differentiable is denoted by N_f. For any $x_0 \in \mathbb{R}^n$ define

$$\mathcal{J}_f(x_0) := \Big\{ \lim_{\ell\to\infty} J_f(x_\ell) \in \mathbb{R}^{m\times n} \mid x : \mathbb{N} \to \mathbb{R}^n \setminus N_f,\ x_\ell \xrightarrow{\ell\to\infty} x_0, \lim_{\ell\to\infty} J_f(x_\ell) \text{ exists} \Big\}$$

and the Clarke generalized Jacobian of f at x_0 by $\partial f(x_0) := \operatorname{conv}(\mathcal{J}_f(x_0))$.

Since bounded sets in $\mathbb{R}^{m\times n}$ are precompact, this set is not empty. Clarke's inverse function theorem gives a sufficient condition for local invertibility of a Lipschitz continuous mapping $f : \mathbb{R}^d \to \mathbb{R}^d$ at some point $x_0 \in \mathbb{R}^d$ by requiring that every matrix in the Clarke generalized Jacobian $\partial f(x_0)$ be invertible.

Theorem 1.1 ([2, Theorem 1]) *Assume that $x_0 \in \mathbb{R}^d$ and that $f : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous in a neighborhood of x_0 with any matrix in $\partial f(x_0)$ being invertible. Then there exist neighborhoods $U \subseteq \mathbb{R}^d$ of x_0, $V \subseteq \mathbb{R}^d$ of $f(x_0)$ and a Lipschitz continuous mapping $g : V \to \mathbb{R}^d$, such that*

(a) $g(f(u)) = u$ for every $u \in U$,
(b) $f(g(v)) = v$ for every $v \in V$.

The fundamental assertion of the theorem is the existence of an inverse mapping which is Lipschitz continuous. Clarke did not include a statement about the generalized Jacobian of the inverse mapping, and it was recently shown in [3] that the assumptions of Theorem 1.1 imply the relations

$$\mathcal{J}_f(x_0)^{-1} \subseteq \mathcal{J}_g(f(y_0)), \tag{1}$$

$$\operatorname{conv}(\mathcal{J}_f(x_0)^{-1}) = \partial g(f(y_0)). \tag{2}$$

It was left open whether equality holds in (1), an assertion, which gives (2) as an immediate consequence by taking the convex hull on both sides. In the next section it is shown that equality holds in (1) even if only the existence of a Lipschitz continuous local inverse mapping g is assumed instead of the invertibility of any matrix in $\partial f(x_0)$. A crucial observation is that in fact $f(N_f) = N_g$ and $g(N_g) = N_f$. Under these weaker assumptions, a formula for the Clarke generalized Jacobian is, to the best of our knowledge, not available in the literature.

2 Main Result

Theorem 2.1 *Assume that $U, V \subseteq \mathbb{R}^d$ are open and that $f : U \to \mathbb{R}^d$ and $g : V \to \mathbb{R}^d$ are Lipschitz continuous with $g \circ f = \mathrm{Id}_U$ and $f \circ g = \mathrm{Id}_V$. Then $f(N_f) = N_g$,*

$g(N_g) = N_f$ and for any $x_0 \in U$ the set $\mathcal{J}_f(x_0)$ is invertible with

$$\mathcal{J}_f(x_0)^{-1} = \mathcal{J}_g(f(x_0)) .$$

In particular

$$\partial g(f(x_0)) = \mathrm{conv}(\mathcal{J}_g(f(x_0))) = \mathrm{conv}(\mathcal{J}_f(x_0)^{-1}) . \tag{3}$$

Proof Let $L > 0$ be a Lipschitz constant for g and fix any $x \in U \backslash N_f$. Then $J_f(x)$ is injective, since for every $h \in \mathbb{R}^d$ and $t \in \mathbb{R}\backslash\{0\}$ with $x + th \in U$

$$\frac{\|f(x+th) - f(x)\|}{|t|} \geq \frac{\|g(f(x+th)) - g(f(x))\|}{L|t|} = \frac{\|th\|}{L|t|} = \frac{\|h\|}{L} .$$

This lower bound implies injectivity of $J_f(x)$ since

$$\begin{aligned} \|J_f(x)h\| &= \left\| \lim_{t\to 0} \frac{f(x+th) - f(x)}{t} \right\| \\ &= \lim_{t\to 0} \frac{\|f(x+th) - f(x)\|}{|t|} \geq \frac{\|h\|}{L} . \end{aligned} \tag{4}$$

Once $J_f(x)$ is shown to be injective, according to [7, III, Theorem 3.3], this matrix is invertible. By [3, Lemma 1], or more generally by [4, Corollary 3.1], g is differentiable at $f(x)$ with $J_g(f(x)) = J_f(x)^{-1}$. In particular, $f(U\backslash N_f) \subseteq V\backslash N_g$ and consequently

$$N_g = V \setminus (V \setminus N_g) \subseteq f(U) \setminus f(U \setminus N_f) = f(N_f) .$$

With the same argument for g, we have $N_f \subseteq g(N_g)$ and

$$N_g \subseteq f(N_f) \subseteq f(g(N_g)) = N_g ,$$

which implies $f(N_f) = N_g$. Analogously we obtain $g(N_g) = N_f$. Fix now $x_0 \in U$, $M \in \mathcal{J}_f(x_0)$ and a sequence $x : \mathbb{N} \to U\backslash N_f$ converging to x_0 with $J_f(x_\ell)$ converging to M for $\ell \to \infty$. Then M is injective, since by (4) for any $h \in \mathbb{R}^d$ the uniform lower bound

$$\|Mh\| = \| \lim_{\ell\to\infty} J_f(x_\ell)h\| = \lim_{\ell\to\infty} \|J_f(x_\ell)h\| \geq \frac{\|h\|}{L}$$

is satisfied. Hence by [7, III, Theorem 3.3], M is invertible. By the continuity of the matrix inversion on the set of invertible matrices, $(J_g(f(x_\ell)))_{\ell\in\mathbb{N}} = (J_f(x_\ell))^{-1}$ converges to $M^{-1} \in \mathcal{J}_g(f(x_0))$ for $\ell \to \infty$. Consequently $\mathcal{J}_f(x_0)^{-1} \subseteq \mathcal{J}_g(f(x_0))$. By the same argument for g we have that the set $\mathcal{J}_g(f(x_0))$ is invertible with $\mathcal{J}_g(f(x_0))^{-1} \subseteq \mathcal{J}_f(g(f(x_0)))$. □

For a convenience of a reader we present a special case of the example constructed in [6, Example 2.2] and [5, Example 3.9], which provides a piecewise linear bi-Lipschitz mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$, to which Theorem 1.1 cannot be applied at the origin due to singularity of the Clarke generalized Jacobian. Nevertheless the inverse function and its Clarke generalized Jacobian can be given explicitly, which verifies formula (3) from Theorem 2.1.

Example 2.1 Setting $i = \sqrt{-1}$, we use polar coordinates in the plane with $x = r\exp(i\phi)$, $\phi \in [0, 2\pi)$, $r > 0$. Define

$$\alpha := \left(0, \frac{\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6}, \frac{4\pi}{6}, \frac{7\pi}{6}, 0\right), \quad \beta := \left(0, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6}, \frac{10\pi}{6}, \frac{11\pi}{6}, 0\right)$$

and $a_j = \exp(i\alpha_j)$, $b_j = \exp(i\beta_j)$, $j = 1, \dots, 7$. For $j = 1, \dots, 6$, define the matrix M_j by $M_j a_j = b_j$ and $M_j a_{j+1} = b_{j+1}$, which leads to

$$M_1 = \mathrm{Id}\,,\ M_2 = \frac{1}{2}\begin{pmatrix} 3+\sqrt{3} & -3-\sqrt{3} \\ -1+\sqrt{3} & -1+\sqrt{3} \end{pmatrix}\,,\ M_3 = \begin{pmatrix} -\sqrt{3} & 0 \\ 1+\sqrt{3} & -1 \end{pmatrix}$$

and

$$M_4 = -\mathrm{Id}\,,\ M_5 = \frac{1}{2}\begin{pmatrix} -1-\sqrt{3} & -1+\frac{1}{\sqrt{3}} \\ 1+\sqrt{3} & -1+\frac{1}{\sqrt{3}} \end{pmatrix}\,,\ M_6 = \begin{pmatrix} 1 & -1-\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Since $\alpha_{j+1} - \alpha_j, \beta_{j+1} - \beta_j \in (0, \pi)$, all the matrices M_j are invertible with positive determinant. The mapping

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad x \longmapsto \begin{cases} M_j x & \text{if } x \neq 0, \alpha_j \leq \phi(x) < \alpha_{j+1}, \\ 0 & \text{if } x = 0 \end{cases}$$

is bi-Lipschitz with the inverse mapping

$$g : \mathbb{R}^2 \to \mathbb{R}^2, \quad x \longmapsto \begin{cases} M_j^{-1} x & \text{if } x \neq 0, \beta_j \leq \phi(x) < \beta_{j+1}, \\ 0 & \text{if } x = 0. \end{cases}$$

The mappings f and g are λ_2-a.e. differentiable with

$$N_f = \{t \cdot a_j \mid j \in \{1, \dots, 6\}, t \geq 0\},\ N_g = \{t \cdot b_j \mid j \in \{1, \dots, 6\}, t \geq 0\},$$

the range of the derivative consists for f and g of the six matrices M_j and M_j^{-1}, respectively. Moreover, we have $N_g = f(N_f)$, $N_f = g(N_g)$ and

$$\mathcal{J}_f(0) = \{M_j \mid j \in \{1, \dots, 6\}\}$$
$$\mathcal{J}_g(0) = \{M_j^{-1} \mid j \in \{1, \dots, 6\}\} = \mathcal{J}_f(0)^{-1},$$

as asserted in Theorem 2.1. Since $0 = \frac{1}{2}(\mathrm{Id} + (-\mathrm{Id})) \in \partial f(0) \cap \partial g(0)$, the assumptions in Clarke's inverse function theorem are not satisfied for f or g. However, since the inverse mapping is known to exist, Theorem 2.1 is applicable and this example nicely illustrates formula (3).

3 Conclusions

We presented a formula for the Clarke generalized Jacobian of the inverse f^{-1} of a bi-Lipschitz map $f : \mathbb{R}^d \to \mathbb{R}^d$ which relates the Clarke generalized Jacobian of f^{-1} to the Clarke generalized Jacobian of f. In particular, if the assumptions of Clarke's inverse function theorem hold, then our result provides a formula for the subgradient of the inverse map. Of course, the most important part in Clarke's theorem is the existence of the inverse map and it is a challenging open problem to identify sufficient conditions for the existence of an inverse map which is Lipschitz continuous if the Clarke generalized Jacobian of a Lipschitz function contains singular matrices.

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