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Fermion Systems in Discrete Space-Time – Outer Symmetries and Spontaneous Symmetry Breaking

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Abstract

A systematic procedure is developed for constructing fermion systems in discrete space-time which have a given outer symmetry. The construction is illustrated by simple examples. For the symmetric group, we prove that fermion systems exist only if the number of particles satisfies certain constraints. When applied to physical systems, this result shows that the permutation symmetry of discrete space-time is always spontaneously broken by the fermionic projector.

1 Discrete Fermion Systems with Outer Symmetry

We briefly recall the mathematical setting of the fermionic projector in discrete space-time as introduced in [1] (see also [2] or [3]). Let H be a finite-dimensional complex vector space endowed with a non-degenerate sesquilinear form $<.|.>$. We call $(H, <.|.>)$ an indefinite inner product space. To every element x of a finite set $M = \{1, \ldots, m\}$ we associate a projector E_x. We assume that these projectors are orthogonal and complete,

$$E_x \, E_y \;=\; \delta_{xy} \, E_x \,, \qquad \sum_{x \in M} E_x \;=\; \mathbb{1} \,, \tag{1}$$

and that the images of the E_x are non-degenerate subspaces of H. We denote the signature of the subspace $E_x(H) \subset H$ by (p_x, q_x) and refer to it as the *spin dimension* at x. We call the structure $(H, <.|.>, (E_x)_{x \in M})$ *discrete space-time*. M are the *discrete space-time points* and E_x the *space-time projectors*. The *fermionic projector* P is defined as a projector on a subspace of H which is negative definite and of dimension f. The vectors in the image of P have the interpretation as the quantum states of the particles of the system, and f is the number of particles. In what follows, we refer to $(H, <.|.>, (E_x)_{x \in M}, P)$ as a *fermion system in discrete space-time* or, for brevity, a **discrete fermion system**.

We point out that in [1, 2] we assumed furthermore that the spin dimension is equal to (n, n) at every space-time point. Here we consider a more general spin dimension (p_x, q_x) for two reasons. First, a constant spin dimension (n, n) would not be a major simplification for what follows. Second, even if we started with constant spin dimension (n, n), the corresponding simple systems (see Section 4) will in general have a spin dimension which varies in space-time, and therefore it is more elegant to begin right away with a non-constant spin dimension (p_x, q_x).

In this paper we consider discrete fermion systems which have a space-time symmetry, as described by the next definition. We denote the symmetric group of M (= the group of all permutations of M) by $\mathcal{S}_m$.

Def. 1.1 *A subgroup $\mathcal{O}$ of the symmetric group $\mathcal{S}_m$ is called* **outer symmetry group** *of the discrete fermion system if for every $\sigma \in \mathcal{O}$ there is a unitary transformation U such that*

$$UPU^{-1} \;=\; P \qquad \textit{and} \qquad UE_xU^{-1} \;=\; E_{\sigma(x)} \quad \forall\, x \in M\,. \tag{2}$$

Our aim is to characterize the discrete fermion systems for a given outer symmetry group $\mathcal{O}$.

2 Reduction of the Proper Free Gauge Group

The transformation U in Def. 1.1 is determined only up to transformations which leave both the fermionic projector and the space-time projectors invariant, i.e.

$$UPU^{-1} \;=\; P \qquad \text{and} \qquad UE_xU^{-1} \;=\; E_x \quad \forall\, x \in M\,. \tag{3}$$

In simple terms, our goal is to "fix" such transformations, thereby making the transformation U in (2) unique. Then the resulting mapping $\sigma \mapsto U(\sigma)$ will be a representation of the outer symmetry group on H, making it possible to apply the representation theory for finite groups (see Section 3). In preparation, we need to study the transformations of the form (3). As in [2], we introduce the *gauge group* $\mathcal{G}$ as the group of all unitary transformations U which leave discrete space-time invariant, i.e.

$$UE_xU^{-1} \;=\; E_x \qquad \forall\, x \in M\,.$$

A transformation of the fermionic projector

$$P \;\rightarrow\; UPU^{-1} \qquad \text{with } U \in \mathcal{G}$$

is called a *gauge transformation*. Clearly, the transformations (3) are gauge transformations, and they form the following subgroup of $\mathcal{G}$.

Def. 2.1 *We define the* **free gauge group** $\mathcal{F}$ *by*

$$\mathcal{F} \;=\; \left\{U \in \mathcal{G} \textit{ with } UPU^{-1} = P\right\}.$$

The free gauge group describes symmetries of the fermionic projector which do not involve a transformation of the space-time points, and which are therefore sometimes referred to as *inner symmetries*. Unfortunately, the free gauge group is in general not completely reducible, as the following example shows.

Example 2.2 Consider the case $m = 2$, spin dimension $(1,1)$ and $f = 1$. As in [2] we represent the scalar product $<.|.>$ with a signature matrix S. More specifically,

$$<u|v> \;=\; (u \,|\, Sv) \qquad \forall u, v \in H\,,$$

where $(.|.)$ denotes the canonical scalar product on $\mathbb{C}^4$ and $S = S^\dagger$, $S^2 = \mathbb{1}$. By choosing a suitable basis, we can arrange that

$$S \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad E_1 \;=\; \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 \;=\; \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix},$$

where for $E_{1/2}$ we used a block matrix notation (thus every matrix entry stands for a 2×2-matrix). We represent the fermionic projector in bra/ket notation as

$$P \;=\; -|\,u\rangle\langle u\,| \qquad \text{with} \qquad \langle u\,|\,u\rangle \;=\; -1\,. \tag{4}$$

We choose $u = (1, 0, 0, 1)$. The free gauge group consists of all gauge transformations U which change u at most by a phase. A short calculation yields that such U are precisely of the form

$$U \;=\; e^{i\alpha} \begin{pmatrix} 1 & i\gamma & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\beta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{with } \alpha, \beta, \gamma \in \mathbb{R}. \tag{5}$$

Hence $\mathcal{F}$ is group isomorphic to $S^1 \times S^1 \times \mathbb{R}$ (where $\mathbb{R}$ denotes the additive group $(\mathbb{R}, +)$). The subspace spanned by the vector $(1, 0, 0, 0)$ is invariant, but it has no invariant complement (this is indeed quite similar to the standard example of the triangular matrices as mentioned for example in [5, Section 2.2]). Hence the group representation (5) is not completely reducible. ♦

Our method for avoiding this problem is to mod out the subgroup of the free gauge group which leaves every vector of $P(H)$ invariant.

Def. 2.3 *The* **trivial gauge group** $\mathcal{F}_0$ *is defined by*

$$\mathcal{F}_0 \;=\; \{U \in \mathcal{G} \text{ with } UP = P\}\,.$$

Taking the adjoint of the relation $UP = P$ we find that $P = PU^{-1}$ and thus $UPU^{-1} = UP = P$, showing that $\mathcal{F}_0$ really is a subgroup of $\mathcal{F}$. Furthermore, for every $g \in \mathcal{F}$,

$$g\mathcal{F}_0 g^{-1} P \;=\; g\mathcal{F}_0 g^{-1} P^2 \;=\; g(\mathcal{F}_0 P) g^{-1} P \;=\; gPg^{-1}P \;=\; P\,,$$

proving that $g\mathcal{F}_0 g^{-1} \subset \mathcal{F}_0$. Hence $\mathcal{F}_0$ is a normal subgroup, and we can form the quotient group.

Def. 2.4 *The* **proper free gauge group** $\hat{\mathcal{F}}$ *is defined by*

$$\hat{\mathcal{F}} \;=\; \mathcal{F}/\mathcal{F}_0\,.$$

In order to make $\hat{\mathcal{F}}$ to a metric space, we introduce the distance function

$$d(\hat{g}, \hat{h}) \;=\; \inf_{g,h \in \mathcal{F}} \|g - h\|_H\,, \tag{6}$$

where g and h run over all representatives of $\hat{g}, \hat{h} \in \hat{\mathcal{F}}$, and $\|.\|_H$ is the sup-norm corresponding to a given norm on H. Clearly, the topology generated by this metric coincides with the quotient topology.

Example 2.5 In the setting of Example 2.2, $\mathcal{F}_0$ consists of all unitary transformations U of the form (5) with $\alpha = 0$. Hence the equivalence class $\hat{U}$ corresponding to a unitary transformation of the form (5) is the set

$$\hat{U} \;=\; \left\{ e^{i\alpha} \begin{pmatrix} 1 & i\gamma & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\beta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ with } \beta, \gamma \in \mathbb{R} \right\}.$$

These equivalence classes are described completely by the parameter α, and thus $\hat{\mathcal{F}}$ is group isomorphic to $U(1)$. Moreover, it is easy to verify that the topology induced by the norm (6) coincides with the standard topology of $U(1)$. Hence we can identify $\hat{\mathcal{F}}$ with the compact Lie group $U(1)$. This group can be obtained even without forming equivalence classes simply by restricting U to the image of P, because

$$U_{|P(H)} \;=\; e^{i\alpha}\, \mathbb{1}_{P(H)}\,. \qquad \blacklozenge$$

The last example illustrates and motivates the following general constructions. It will be crucial that $I := P(H)$ is a *definite* subspace of H. Thus the inner product $<.|.>$ makes I to a Hilbert space. We denote the corresponding norm by

$$\|u\|_I \;:=\; \sqrt{-<u\,|\,u>}\,.$$

Furthermore, we denote the unitary endomorphisms of I by $U(I)$. Choosing an orthonormal basis of I, one sees that $U(I)$ can be identified with the *compact* Lie group $U(f)$. The condition $P = UPU^{-1}$ in Def. 2.1 means that every $U \in \mathcal{F}$ maps I to itself, and thus the restriction to I gives a mapping

$$\varphi \;:\; \mathcal{F} \rightarrow U(I) \;:\; U \mapsto U_{|I}\,.$$

Since every $U_0 \in \mathcal{F}_0$ is trivial on I, the mapping φ is well-defined on the equivalence classes $\mathcal{F}/\mathcal{F}_0$. Furthermore, $\varphi(U') = \varphi(U)$ if and only if $U'U^{-1} \in \mathcal{F}_0$. Thus φ gives rise to the injection

$$\varphi \;:\; \hat{\mathcal{F}} \hookrightarrow U(I)\,. \tag{7}$$

Since every free gauge transformation $U \in \mathcal{F}$ maps the subspaces $E_x(H)$ into themselves, the corresponding $\varphi(U) \in U(I)$ is locally unitary in the following sense.

Def. 2.6 *A linear map $U \in U(I)$ is called* **locally unitary** *if for all $u, v \in I$ and all $x \in M$ the following conditions are satisfied:*

(i) $E_x\, v = 0 \iff E_x\, U\, v = 0.$

(ii) $<E_x\, U\, u\,|\,U\, v> \;=\; <E_x\, u\,|\,v>.$

The group of all locally unitary transformations is denoted by $U_{loc}(I)$.

Lemma 2.7 *$U_{loc}(I)$ is a compact Lie-subgroup of $U(I)$.*

Proof. Let $\mathcal{A}$ be the set of all symmetric operators A on I which satisfy for all $u, v \in I$ and $x \in M$ the conditions

$$E_x\, v = 0 \iff E_x\, A\, v = 0 \qquad \text{and} \qquad <E_x\, A\, u\,|\,v> \;=\; <E_x\, u\,|\,A\, v>\,.$$

Obviously, $\mathcal{A}$ is a linear subspace of $L(I)$ (where $L(I)$ denotes the linear endomorphisms of I). Furthermore, the above conditions are compatible with the Lie bracket $\{A, B\} = i[A, B]$, and thus $\mathcal{A}$ is a Lie algebra. The exponential map $A \mapsto \exp(iA)$ maps $\mathcal{A}$ into $U_{\rm loc}(I)$. In a neighborhood of $\mathbb{1} \in U(I)$, we can define the logarithm by the power series

$$\log(V) \;=\; \log(\mathbb{1} - (\mathbb{1} - V)) \;=\; -\sum_{n=1}^{\infty} \frac{(\mathbb{1} - V)^n}{n}\,, \tag{8}$$

showing that the exponential map is locally invertible near $0 \in \mathcal{A}$. Hence the exponential map gives a chart near $\mathbb{1} \in U_{\text{loc}}(I)$. Using the group structure, we can "translate" this chart to the neighborhood of any $\hat{V} \in U_{\text{loc}}(I)$ to get a smooth atlas. We conclude that $U_{\text{loc}}(I)$ is a Lie-subgroup of $U(I)$. Finally, the conditions (i) and (ii) in Def. 2.6 are preserved if one takes limits, proving that $U_{\text{loc}}(I)$ is closed in $U(I)$ and thus compact. ■

The construction of the next lemma allows us to extend every locally unitary map to a free gauge transformation on H.

Lemma 2.8 (extension lemma) *There is a constant $C > 0$ (depending only on I and the norm $\|.\|_H$) such that for every locally unitary $U \in U(I)$ there is a $V \in \mathcal{F}$ with $\varphi(V) = U$ and*

$$\|\mathbb{1} - V\|_H \;\leq\; C\, \|\mathbb{1} - U\|_I \,.$$

This V can be chosen to depend smoothly on U, giving rise to a smooth injection

$$\lambda \;:\; U_{loc}(I) \;\hookrightarrow\; \mathcal{F} \subset L(H)\,, \tag{9}$$

which is a group homomorphism.

Proof. The first step is to "localize" U at a given $x \in M$ to obtain an operator

$$U_x \;:\; E_x(I) \to E_x(I)\,.$$

Introducing the abbreviations $I_x := E_x(I)$ and $H_x := E_x(H)$, we choose an injection $\iota_x : I_x \hookrightarrow I$ such that

$$E_x\, \iota_x \;=\; \mathbb{1}_{I_x}\,. \tag{10}$$

We define U_x by

$$U_x = E_x U \iota_x \;:\; I_x \to I_x\,.$$

Let us verify that this definition is independent of the choice of ι_x. For two different injections ι_x and ι'_x, we know from (10) that for all $u_x \in I_x$,

$$E_x\, (\iota_x - \iota'_x)\, u_x \;=\; 0\,.$$

Using that U is locally unitary, we conclude from Def. 2.6 (i) that

$$0 \;=\; E_x\, U\, (\iota_x - \iota'_x)\, u_x \;=\; (U_x - U'_x)\, u_x\,.$$

Let us collect some properties of U_x. First of all, choosing for a given $u \in I$ the injection ι'_x such that $\iota'_x E_x u = u$, the above independence of U_x of the choice of the injection implies that for all $u \in I$,

$$E_x\, U\, u \;=\; U_x\, E_x\, u \tag{11}$$

and thus for all $u_x \in I_x$,

$$E_x\, U\, \iota_x\, u_x \;=\; U_x\, u_x\,. \tag{12}$$

As a consequence,

$$(U^{-1})_x\, U_x\, u_x \;\stackrel{(12)}{=}\; (U^{-1})_x\, E_x\, U\, \iota_x\, u_x \;\stackrel{(11)}{=}\; E_x\, U^{-1}\, U\, \iota_x\, u_x \;=\; u_x\,.$$

In a more compact notation,

$$(U_x)^{-1} \;=\; (U^{-1})_x \;,$$

and thus it is unambiguous to simply write U_x^{-1}. By restriction, we can also consider the norm $\|.\|_H$ on the subspace H_x. Since every unitary map in the Hilbert space I has norm one, we can estimate the corresponding norm of U_x by

$$\|U_x\|_H \;\leq\; \|E_x\| \, \|U\|_I \, \|\iota_x\| \;=\; \|E_x\| \, \|\iota_x\| \;;$$

note that the resulting upper bound is independent of U_x. Applying the same argument to U_x^{-1}, we conclude that there is a constant c independent of U_x such that

$$\|U_x\|_H + \|U_x^{-1}\|_H \;\leq\; c \,. \tag{13}$$

Furthermore, we have the following estimates,

$$\begin{aligned} \|\mathbb{1} - U_x\| \;&=\; \|E_x \, (\mathbb{1} - U) \, \iota_x\| \;\leq\; c \, \|\mathbb{1} - U\| && (14) \\ \|\mathbb{1} - U_x^{-1}\| \;&\leq\; \|U_x^{-1}\| \, \|U_x - \mathbb{1}\| \;\leq\; c^2 \, \|\mathbb{1} - U\| \,. && (15) \end{aligned}$$

Finally, U_x is isometric on I_x. Namely, using the properties of the space-time projectors together with Def. 2.6 (ii), we obtain that for all $u_x, v_x \in I_x$,

$$<\!U_x u_x \,|\, U_x v_x\!> \;=\; <\!E_x U \iota_x u_x \,|\, U \iota_x v_x\!> \;=\; <\!E_x \iota_x u_x \,|\, \iota_x v_x\!> \;=\; <\!u_x \,|\, v_x\!> .$$

Our goal is to construct a unitary operator $V_x : H_x \to H_x$ which coincides on I_x with U_x and satisfies the inequality

$$\|\mathbb{1} - V_x\|_H \;\leq\; C \, \|\mathbb{1} - U\|_I \,. \tag{16}$$

Namely, provided that the operator V_x can be constructed for every $x \in M$, we can introduce V by

$$V \;=\; \sum_{x \in M} V_x \, E_x \;:\; H \to H \,.$$

This operator is obviously unitary and invariant on the subspaces H_x, thus $V \in \mathcal{F}$. Furthermore, for all $x \in M$ and $u \in I$,

$$E_x \, \varphi(V) \, u \;=\; E_x \, V \, u \;=\; E_x \, V_x \, E_x \, u \;=\; E_x \, U_x \, E_x \, u \;\stackrel{(11)}{=}\; E_x \, U \, u \,,$$

proving that $\varphi(V) = U$. Hence V really has all the required properties.

In order to construct V_x, we choose in I_x a non-degenerate subspace of maximal dimension and in this subspace a pseudo-orthonormal basis (e_i). We extend this basis by vectors (f_j) to a basis of I_x (thus the vectors f_j are all null and orthogonal to I_x). Next we choose vectors $h_j \in H_x$ which are orthogonal to the (e_i) and conjugate to the (f_j) in the sense that $<\!f_i \,|\, h_j\!> = \delta_{ij}$. Then the span of the vectors e_i, f_j and h_j is non-degenerate, and we can choose on its orthogonal complement a pseudo-orthonormal basis (g_k). We thus obtain a basis (e_i, f_j, g_k, h_j) of H_x. Using a block matrix notation in this basis, the signature matrix takes the form

$$S \;=\; \begin{pmatrix} S_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & S_2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} ,$$

where S_1 and S_2 are diagonal matrices with entries equal to ± 1. Without loss of generality, we choose the norm on H_x such that it coincides in this basis with the standard Euclidean norm on $\mathbb{C}^{p_x+q_x}$.

We represent operators on I_x as 2×2 block matrices in the basis (e_i, f_j). Since the operators U_x and U_x^{-1} are isometric on I_x and inverses of each other, one easily verifies that they must be of the form

$$U_x \;=\; \begin{pmatrix} W & 0 \\ C & A \end{pmatrix}, \qquad U_x^{-1} \;=\; \begin{pmatrix} W^{-1} & 0 \\ D & A^{-1} \end{pmatrix}, \tag{17}$$

where $D = -A^{-1} C W^{-1}$. The matrix W is unitary, meaning that $W^{-1} = S_1 W^\dagger S_1$. We choose V_x as

$$V_x \;=\; \begin{pmatrix} W & 0 & 0 & S_1 D^\dagger \\ C & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (A^{-1})^\dagger \end{pmatrix} \qquad \text{with} \qquad B = -\frac{1}{2}\, A D S_1 D^\dagger\,. \tag{18}$$

Obviously, V_x coincides on I_x with U_x, and a direct calculation shows that V_x is unitary on H_x, i.e.

$$V_x \, S \, V_x^\dagger \, S \;=\; \mathbb{1}\,.$$

Using that, according to (13), the norms of all the matrix entries appearing in (17) can be estimated in terms of c, we find that

$$\begin{aligned} \|\mathbb{1} - V_x\| \;&\leq\; (1+c^2) \left(\|\mathbb{1} - W\| + \|\mathbb{1} - A\| + \|\mathbb{1} - A^{-1}\| + \|C\| + \|D\| \right) \\ &\leq\; (1+c^2) \left(\|\mathbb{1} - U_x\| + \|\mathbb{1} - U_x^{-1}\| \right). \end{aligned}$$

Applying (14, 15) give the desired inequality (16).

Finally, it is obvious from the explicit formulas (17, 18) that our choice of V depends smoothly on U and that the mapping λ is a group homomorphism. ∎

The last lemma shows in particular that (7) gives a one-to-one correspondence between proper free gauge transformations and locally unitary transformations. Since $U_{\text{loc}}(I)$ is a compact Lie group, one might expect that $\hat{\mathcal{F}}$ is itself compact. This is really the case, as we now prove.

Lemma 2.9 *$\hat{\mathcal{F}}$ is a compact Lie group. The mapping*

$$\varphi \;:\; \hat{\mathcal{F}} \;\to\; U_{loc}(I) \tag{19}$$

is a Lie group homomorphism.

Proof. We first consider the infinitesimal generators of the groups. We thus introduce the following families of linear operators on H,

$$\begin{aligned} \mathcal{A} \;&=\; \{A \text{ with } A^* = A,\; [A, E_x] = 0 \;\forall x \in M \text{ and } [A, P] = 0\} \\ \mathcal{A}_0 \;&=\; \{A \text{ with } A^* = A,\; [A, E_x] = 0 \;\forall x \in M \text{ and } AP = 0\}\,. \end{aligned}$$

Obviously, these families are linear subspaces of $L(H)$ which, together with the Lie bracket $\{A, B\} = i[A, B]$, form real Lie algebras. Furthermore, $\mathcal{A}_0$ is a subalgebra of $\mathcal{A}$, and the calculation

$$[A_0, A]\, P \;=\; A_0 A P - A A_0 P \;=\; (A_0 P)\, A - A\, (A_0 P) \;=\; 0$$

shows that $\mathcal{A}_0$ is an ideal of $\mathcal{A}$. Hence $\hat{\mathcal{A}} := \mathcal{A}/\mathcal{A}_0$ is again a Lie algebra. (Since $\hat{\mathcal{A}}$ is a finite-dimensional vector space, we need not worry about introducing a norm or topology on it.)

The exponential map $a \mapsto \exp(ia)$ gives a mapping from $\hat{\mathcal{A}}$ to $\hat{\mathcal{F}}$ which is obviously continuous. Assume conversely that $\hat{V} \in B_\varepsilon(\mathbb{1}) \subset \hat{\mathcal{F}}$ (corresponding to the distance function (6)). Since restricting an operator on H the subspace I decreases its norm, we know that for any representative $V \in \mathcal{F}$ of $\hat{V}$,

$$\|\mathbb{1} - \varphi(\hat{V})\|_I \;\leq\; c\, \|\mathbb{1} - V\|_H$$

(with c independent of $\hat{V}$ and V), and taking the infimum over all representatives, we find that

$$\|\mathbb{1} - \varphi(\hat{V})\|_I \;\leq\; c\varepsilon\,.$$

Since the map $\varphi(\hat{V})$ is locally unitary, Lemma 2.8 allows us to choose a representative V of $\hat{V}$ satisfying the inequality

$$\|\mathbb{1} - V\|_H \;\leq\; Cc\,\varepsilon\,.$$

Hence, after choosing ε sufficiently small, the logarithm of V may again be defined by the power series (8). We conclude that the exponential map is invertible locally near $0 \in \hat{\mathcal{A}}$, and that its inverse is continuous. Hence the exponential map gives a chart near $\mathbb{1} \in \hat{\mathcal{F}}$. Using the group structure, get a smooth atlas. We conclude that $\hat{\mathcal{F}}$ is a Lie group.

According to Lemma 2.8, the image of φ consists precisely of all locally unitary maps, which by Lemma 2.7 form a closed subset of $U(I)$. Furthermore, restricting the above exponential map to I,

$$\varphi \exp(ia) \;=\; \exp\left(ia_{|I}\right), \tag{20}$$

we obtain precisely the chart near $\mathbb{1} \in U_{\mathrm{loc}}(I)$ constructed in Lemma 2.7. Hence φ is a smooth map from $\hat{\mathcal{F}}$ to $U_{\mathrm{loc}}(I)$. Its inverse can be written as $\varphi^{-1} = \pi\lambda$ with λ as given by (9) and $\pi : \mathcal{F} \to \hat{\mathcal{F}}$ the natural projection. Hence the smoothness of φ^{-1} follows from the smoothness of λ. ∎

The last lemma allows us to identify $\hat{\mathcal{F}}$ with the compact subgroup $U_{\mathrm{loc}}(I)$ of $U(I)$. As the next lemma shows, compactness implies complete reducibility into definite subspaces.

Lemma 2.10 *Let $\mathcal{E}$ be a finite group or a compact Lie group, and U a unitary representation of $\mathcal{E}$ on an indefinite inner product space H of signature (p,q). Then H can be decomposed into a direct sum of irreducible subspaces, which are all definite and mutually orthogonal.*

Proof. We introduce on $(H, <.|.>)$ in addition a positive definite scalar product $(.|.)$. By averaging over the group,

$$(u \,|\, v)_{\mathcal{E}} \;:=\; \left\{ \begin{array}{ll} \displaystyle \frac{1}{\#\mathcal{E}} \sum_{g \in \mathcal{E}} (U(g)\, u \,|\, U(g)\, v) & \text{if } \mathcal{E} \text{ is a finite group} \\ \displaystyle \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} (U(g)\, u \,|\, U(g)\, v)\, dg & \text{if } \mathcal{E} \text{ is a compact Lie group}\,, \end{array} \right.$$

we obtain an invariant scalar product $(.|.)_{\mathcal{E}}$. Hence the representation U is unitary with respect to both $<.|.>$ and $(.|.)_{\mathcal{E}}$.

In a suitable basis, $(.|.)_\mathcal{E}$ coincides with the Euclidean scalar product on $\mathbb{C}^{p+q}$, whereas $<.|.>$ takes the form

$$<u|v> \;=\; (u \,|\, S\, v)_\mathcal{E} \qquad \text{with } \; S \;=\; \mathrm{diag}(\underbrace{1,\ldots,1}_{p \text{ times}}, \underbrace{-1,\ldots,-1}_{q \text{ times}}) \,.$$

Let $H^+ \subset H$ be the positive definite subspace of all vectors whose last q components vanish. Then for every $v \in H^+$ and every representation matrix $U = U(g)$,

$$\begin{aligned} \sum_{i=1}^p |v^i|^2 &\;=\; (v \,|\, v)_\mathcal{E} \;=\; (Uv \,|\, Uv)_\mathcal{E} \;=\; \sum_{i=1}^{p+q} \left|(Uv)^i\right|^2 \\ \sum_{i=1}^p |v^i|^2 &\;=\; <v \,|\, v> \;=\; <Uv \,|\, Uv> \;=\; \sum_{i=1}^{p} \left|(Uv)^i\right|^2 \;-\; \sum_{i=p+1}^{q} \left|(Uv)^i\right|^2 \end{aligned}$$

Subtracting the two lines, we find that

$$\sum_{i=p+1}^{q} \left|(Uv)^i\right|^2 \;=\; 0$$

and thus $Uv \in H^+$. We conclude that H^+ is an invariant subspace.

Similarly, the subspace H^- of all vectors whose first p components vanish is also invariant. In this way, we have decomposed H into an orthogonal direct sum of two invariant definite subspaces. We finally decompose these invariant definite subspaces in the standard way into mutually orthogonal, irreducible subspaces. ∎

We are now ready to prove the main result of this section. We always endow the tensor product $\mathbb{C}^l \otimes H$ of $\mathbb{C}^l$ with an indefinite inner product space H with the natural inner product

$$<(u_i) \,|\, (v_j)> \;=\; \sum_{i=1}^l <u_i \,|\, v_i>_H \,. \tag{21}$$

Theorem 2.11 *There are integers $(l_r)_{l=1,\ldots,R}$,*

$$1 \;\leq\; l_1 \;\leq\; \cdots \;\leq\; l_R \,,$$

such that $\hat{\mathcal{F}}$ is Lie group isomorphic to the product of the corresponding unitary groups,

$$\hat{\mathcal{F}} \;\simeq\; U(l_1) \times \cdots \times U(l_R) \,. \tag{22}$$

The inner product space $(H, <.|.>)$ is isomorphic to the orthogonal direct sum

$$H \;\simeq\; H^{(0)} \oplus \left(\bigoplus_{r=1}^R \mathbb{C}^{l_r} \otimes H^{(r)} \right) \,, \tag{23}$$

where $H^{(r)}$ are inner product spaces of signature $(p^{(r)}, q^{(r)})$. Under the isomorphism (23), the projectors P and $(E_x)_{x\in M}$ take the form

$$P \;\simeq\; \quad 0 \quad \oplus \left(\bigoplus_{r=1}^R \mathbb{1}_{\mathbb{C}^{l_r}} \otimes P^{(r)} \right) \tag{24}$$

$$E_x \;\simeq\; E_x^{(0)} \oplus \left(\bigoplus_{r=1}^R \mathbb{1}_{\mathbb{C}^{l_r}} \otimes E_x^{(r)} \right) \,, \tag{25}$$

where $P^{(r)}$ and $E_x^{(r)}$ are projectors on $H^{(r)}$. None of the operators $P^{(r)}$ vanishes. Furthermore, $\hat{\mathcal{F}}$ acts only on the factors $\mathbb{C}^{l_r}$ in the sense that for every representative $V \in \mathcal{F}$ of a $\hat{V} = (V_1, \ldots, V_r) \in \hat{\mathcal{F}}$,

$$V_{|I} \;=\; \bigoplus_{r=1}^{R} V_r \otimes \mathbb{1}_{I^{(r)}} \,, \tag{26}$$

where we set $I = P(H)$ and $I^{(r)} = P^{(r)}(H^{(r)})$.

Choosing $H^{(0)}$ maximal in the sense that every subspace $J \subset H$ satisfies the condition

$$J \text{ definite, } P(J) = 0 \quad \text{and} \quad E_x(J) \subset J \;\; \forall x \in M \quad \Longrightarrow \quad J \subset H^{(0)} \,, \tag{27}$$

the above representation is unique.

Note that we do not exclude the case $p^{(0)} = 0 = q^{(0)}$, and thus $H^{(0)}$ might be zero dimensional. The situation is different if $r \geq 1$, because in this case we know that $P^{(r)}$ does not vanish, and therefore the dimension of $H^{(r)}$ must be at least one, $p^{(r)} + q^{(r)} \geq 1$. We also point out that P vanishes on $H^{(0)}$ and thus $I^{(0)} = \{0\}$; this is why in (26) we could leave out the direct summand corresponding to $H^{(0)}$.

Proof of Theorem 2.11. The mapping $\lambda \circ \varphi$ (with φ and λ according to (19, 9)) is a unitary representation of $\hat{\mathcal{F}}$ on H. According to Lemma 2.10, this representation splits into irreducible representations on definite, mutually orthogonal subspaces. We denote the appearing non-trivial, non-equivalent irreducible representations by $V_1, \ldots V_R$ and let V_0 be the trivial representation on $\mathbb{C}$. We let these irreducible representations act unitarily on the respective vector spaces $\mathbb{C}^{l_r}$, $l_r \geq 1$, endowed with the standard Euclidean scalar product. Collecting the direct summands of H corresponding to equivalent irreducible representations, we obtain an orthogonal decomposition of the form

$$H \;\simeq\; \bigoplus_{r=0}^{R} \mathbb{C}^{l_r} \otimes H^{(r)} \tag{28}$$

with inner product spaces $H^{(r)}$ of signature $(p^{(r)}, q^{(r)})$ together with the representation

$$(\lambda \circ \varphi)(g) \;=\; \bigoplus_{r=0}^{R} V_r(g) \otimes \mathbb{1}_{H^{(r)}} \qquad \forall g \in \hat{\mathcal{F}} \,. \tag{29}$$

Schur's lemma yields that the operators P and E_x take the form

$$P \;\simeq\; \bigoplus_{r=0}^{R} \mathbb{1}_{\mathbb{C}^r} \otimes P^{(r)} \,, \qquad E_x \;\simeq\; \bigoplus_{r=0}^{R} \mathbb{1}_{\mathbb{C}^r} \otimes E_x^{(r)} \,. \tag{30}$$

By restricting to I, (29) gives

$$\varphi(g) \;\simeq\; \bigoplus_{r=0}^{R} V_r(g) \otimes \mathbb{1}_{I^{(r)}} \,,$$

and according to Lemma 2.9 this is simply the fundamental representation of $U_{\text{loc}}(I)$.

Suppose that $P^{(r)} = 0$. Then replacing V_r by the trivial representation, we get a new group homomorphism $\tilde{\lambda} \;:\; \hat{\mathcal{F}} \hookrightarrow \mathcal{F}$ with $\varphi \circ \tilde{\lambda} = \tilde{\lambda}_{|I} = \lambda_{|I} = \mathbb{1}$, for which the above construction applies just as well. Then $H^{(r)}$ will be combined with $H^{(0)}$. In this way, we can arrange that $P^{(r)} \neq 0$ unless $r = 0$.

Using the representation (28, 30), it is obvious that every transformation of the form

$$\bigoplus_{r=0}^{R} U_r \otimes \mathbb{1}_{I^{(r)}} \quad \text{with} \quad U_r \in U(l_r) \tag{31}$$

is locally unitary. Comparing with (26), one sees that the $V_r(g)$ can be chosen independently and arbitrarily in $U(l_r)$. However, one must keep in mind that if $I^{(r)} = \{0\}$, the corresponding summand drops out of both (26) and (31). We conclude that U_{loc} coincides with the product of all those groups $U(l_r)$ for which $I^{(r)} \neq \{0\}$. This implies that $P^{(0)}$ must vanish, because otherwise $V_0 = U(l_0)$ would be a non-trivial representation. After reordering the l_r, we obtain (22) as well the desired representations (23–26).

It is obvious that every subspace J which satisfies the conditions on the left of (27) can be combined with $H^{(0)}$. The only arbitrariness in the construction is the choice of the embedding λ. Choosing $H^{(0)}$ maximal corresponds to choosing λ equal to the identity on a non-degenerate subspace of maximal dimension. Then the signature of each subspace $E_x \lambda(\hat{\mathcal{F}})$ coincides with the signature of the smallest non-degenerate subspace containing I_x and is therefore fixed. As a consequence, two different choices of λ can be related to each other by a free gauge transformation. This proves uniqueness of our representation. ∎

We denote the signature of $E_x^{(r)}(P^{(r)})$ by $(p_x^{(r)}, q_x^{(r)})$ and set $f^{(r)} = \dim P^{(r)}(H^{(r)})$. Computing dimensions and signatures, we immediately obtain the following result.

Corollary 2.12 *The parameters in Theorem 2.11 are related to the spin dimensions (p_x, q_x), the number of space-time points m and the number of particles f by*

$$\begin{aligned}
\sum_{r=1}^{R} l_r\, f^{(r)} &= f \\
p^{(0)} + \sum_{r=1}^{R} l_r\, p^{(r)} &= \sum_{x\in M} p_x\,, \qquad & q^{(0)} + \sum_{r=1}^{R} l_r\, q^{(r)} &= \sum_{x\in M} q_x \\
p_x^{(0)} + \sum_{r=1}^{R} l_r\, p_x^{(r)} &= p_x \qquad , & q_x^{(0)} + \sum_{r=1}^{R} l_r\, q_x^{(r)} &= q_x\,.
\end{aligned}$$

3 A Representation of the Outer Symmetry Group

The goal of this section is to construct for a given discrete fermion system with outer symmetry a representation of the outer symmetry group defined as follows.

Def. 3.1 *A unitary representation $\sigma \mapsto U(\sigma)$ of $\mathcal{O}$ on $(H, <.|.>)$ is called a* **representation of the outer symmetry group** *if*

$$U(\sigma)\, P\, U(\sigma)^{-1} \;=\; P \qquad \text{and} \qquad U(\sigma)\, E_x\, U(\sigma)^{-1} \;=\; E_{\sigma(x)} \quad \forall\, x \in M\,. \tag{32}$$

A representation of the outer symmetry group is very useful because it allows us to split up the discrete fermion system by standard methods into irreducible components:

Proposition 3.2 *Assume that U is a representation of the outer symmetry group $\mathcal{O}$. Then there are inequivalent irreducible representations $(R_l, \mathbb{C}^{d_l})_{l=1,\ldots,L}$ of $\mathcal{O}$ such that H has an orthogonal decomposition of the form*

$$H \simeq \bigoplus_{l=1}^{L} \mathbb{C}^{d_l} \otimes H^{[l]} ,$$

where $H^{[l]}$ are inner product spaces of signature $(p^{[l]}, q^{[l]})$. The representation of the outer symmetry group and the fermionic projector take the form

$$U(\sigma) \simeq \bigoplus_{l=1}^{L} R_l(\sigma) \otimes \mathbb{1}_{H^{[l]}} , \qquad P \simeq \bigoplus_{l=1}^{L} \mathbb{1}_{\mathbb{C}^{d_l}} \otimes P^{[l]} ,$$

where the $P^{[l]}$ are projectors in $H^{[l]}$ with negative definite image.

Proof. The proposition follows immediately from Lemma 2.10 and Schur's lemma. ∎

Before entering the construction of the representation, we give two simple examples.

Example 3.3 As in Example 2.2 we consider two space-time points and spin dimension $(1,1)$, but now for convenience in the matrix representation

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{33}$$

We choose the one-particle fermionic projector (4) with $u = 2^{-\frac{1}{2}}(0,1,0,1)$ and thus

$$P = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \tag{34}$$

The free gauge transformations are of the form

$$U = \mathrm{diag}(e^{i\alpha}, e^{i\varphi}, e^{i\gamma}, e^{i\varphi}) \quad \text{with } \alpha, \beta, \varphi \in \mathbb{R}$$

and thus $\mathcal{F} = U(1) \times U(1) \times U(1)$. When restricting to $P(H)$, this transformation simplifies to $U = e^{i\varphi}\, \mathbb{1}$, and thus $\hat{\mathcal{F}} \simeq U(1)$. Theorem 2.11 gives the decomposition

$$H \simeq H^{(0)} \oplus \mathbb{C} \otimes H^{(1)} ,$$

where $\hat{\mathcal{F}}$ acts on the factor $\mathbb{C}$ and

$$H^{(0)} = \{(a,0,c,0) \,:\, a,c \in \mathbb{C}\} , \quad H^{(1)} = \{(0,b,0,d) \,:\, b,d \in \mathbb{C}\}$$

are both two-dimensional definite subspaces.

The system (33, 34) is symmetric under permutations of the two space-time points. Thus we choose $\mathcal{O} = \{\mathbb{1}, \sigma\}$ with $\sigma(1) = 2$ and $\sigma(2) = 1$. The corresponding unitary transformations U as in Def. 1.1 are of the general form

$$U(\mathbb{1}) \in \mathcal{F} , \qquad U(\sigma) \in \mathcal{F} \cdot \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{35}$$

The subspace $H^{(0)}$ is trivial in the sense that it is invariant under E_1 and E_2, and that P vanishes on it. The fact that $\mathcal{O}$ has a representation on $H^{(0)}$ boils down to the statement that the subspaces $E_x(H^{(0)})$ have constant signature on the orbits of $\mathcal{O}$. Since this situation is very simple, we do not need to consider $H^{(0)}$ further. Thus, restricting attention to $H^{(1)}$, the transformation U becomes unique up to a phase,

$$U(\mathbb{1})_{|H^{(1)}} \;=\; e^{i\varphi}\,\mathbb{1}_{|H^{(1)}}\,, \qquad U(\sigma)_{|H^{(1)}} \;=\; e^{i\varphi}\cdot \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}_{|H^{(1)}}\,.$$

This phase can be fixed by imposing that U should have determinant one, $U \in SU(I^{(1)})$. This yields the desired representation of the outer symmetry group on $H^{(1)}$,

$$U^{(1)}(\mathbb{1}) \;=\; \mathbb{1}_{|H^{(1)}}\,, \qquad U^{(1)}(\sigma) \;=\; \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}_{|H^{(1)}}\,.$$

In our next example we consider in the discrete space-time (33) the two-particle fermionic projector

$$P \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{36}$$

Now the free gauge transformations are of the form

$$U \;=\; \mathrm{diag}(e^{i\alpha}, e^{i\beta}, e^{i\gamma}, e^{i\delta}) \quad \text{with } \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

and thus $\mathcal{F} = U(1)^4$. When restricting to $P(H)$, the factors $e^{i\alpha}$ and $e^{i\gamma}$ drop out, and thus $\hat{\mathcal{F}} \simeq U(1) \times U(1)$. Theorem 2.11 gives the decomposition

$$H \;\simeq\; H^{(0)} \oplus \left(\mathbb{C} \otimes H^{(1)}\right) \oplus \left(\mathbb{C} \otimes H^{(2)}\right),$$

where $\hat{\mathcal{F}}$ acts on the factors $\mathbb{C}$ and

$$H^{(0)} \;=\; \{(a,0,c,0) \,:\, a,c \in \mathbb{C}\}\,, \quad H^{(1)} \;=\; \langle\, (0,1,0,0)\,\rangle, \quad H^{(2)} \;=\; \langle\, (0,0,0,1)\,\rangle\,.$$

This system is again symmetric under permutations of the two space-time points, $\mathcal{O} = \{\mathbb{1}, \sigma\}$ with $\sigma(1) = 2$ and $\sigma(2) = 1$. The corresponding unitary transformations U as in Def. 1.1 are again of the form (35). The subspace $H^{(0)}$ is again trivial. Restricting attention to its complement $H^{(0)\perp} = H^{(1)} \oplus H^{(2)}$, there remains a $U(1) \times U(1)$-freedom,

$$U(\mathbb{1})_{|H^{(0)\perp}} \;=\; \begin{pmatrix} e^{i\beta}\,\mathbb{1} & 0 \\ 0 & e^{i\delta}\,\mathbb{1} \end{pmatrix}_{|H^{(0)\perp}}, \qquad U(\sigma)_{|H^{(0)\perp}} \;=\; \begin{pmatrix} 0 & e^{i\beta}\,\mathbb{1} \\ e^{i\delta}\,\mathbb{1} & 0 \end{pmatrix}_{|H^{(0)\perp}}\,.$$

In order to fix the phases, we impose that U should be of the form

$$U_{|H^{(0)\perp}} \;=\; V\,(U_1 \oplus U_2)$$

with $U_k \in SU(I^{(k)})$ and V a permutation matrix, i.e. a 2×2-matrix with the entries $V_{k'k} = \delta_{k',\pi(k)}$, where $\pi \in \mathcal{S}_2$ is a permutation. Then the U become a representation of the outer symmetry group on $H^{(0)\perp}$,

$$U(\mathbb{1})_{|H^{(0)\perp}} \;=\; \mathbb{1}_{|H^{(0)\perp}}, \qquad U(\sigma)_{|H^{(0)\perp}} \;=\; \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}_{|H^{(0)\perp}}\,.$$

This example explains why it is in general impossible to arrange that the mappings U are invariant on the subspaces $H^{(k)}$. ♦

These examples illustrate the statement of the following general theorem.

Theorem 3.4 *In the representation of Theorem 2.11, every unitary transformation U as in Def. 1.1 restricted to I can be represented as*

$$U_{|I} \;=\; f \cdot V \cdot \left(\bigoplus_{r=1}^{R} \mathbb{1}_{\mathbb{C}^{l_r}} \otimes U_r \right) \tag{37}$$

with $f \in \hat{\mathcal{F}}$ and unitary operators $U_r \in SU(I^{(r)})$. Here the operator V is a permutation operator in the sense that there is a permutation $\pi \in \mathcal{S}_R$ such that for all $u_r \in \mathbb{C}^{l_r} \otimes I^{(r)}$,

$$V\left(\oplus_{r=1}^{R} u_r\right) \;=\; \oplus_{r=1}^{R} u_{\pi(r)} \;.$$

The permutation π satisfies the constraints

$$l_r \;=\; l_{\pi(r)}\;, \qquad \dim I^{(r)} \;=\; \dim I^{(\pi(r))}\;, \tag{38}$$

and we identify $I^{(r)}$ with $I^{(\pi(r))}$ via an (arbitrarily chosen) isomorphism. For a given choice of these isomorphisms, the operators V and U_r are unique.

Proof. For given $\sigma \in \mathcal{O}$ we let U be a unitary transformation satisfying (2). Then for every $f \in \mathcal{F}$, the conjugated matrix $f^U := U f U^{-1}$ satisfies the conditions

$$\begin{aligned} f^U P (f^U)^{-1} &= U f U^{-1} P U f^{-1} U^{-1} \;=\; P \\ f^U E_x (f^U)^{-1} &= U f U^{-1} E_x U f^{-1} U^{-1} \;=\; U f E_{\sigma^{-1}(x)} f^{-1} U^{-1} \\ &= U E_{\sigma^{-1}(x)} U^{-1} \;=\; E_x \;, \end{aligned}$$

showing that $f^U \in \mathcal{F}$. We write the relation between f and f^U in the form

$$U f \;=\; f^U\, U \;. \tag{39}$$

According to (26), f and f^U can be represented as

$$f_{|I} \;=\; \bigoplus_{r=1}^{R} f_r \otimes \mathbb{1}_{I^{(r)}}\;, \quad f^U_{|I} \;=\; \bigoplus_{r=1}^{R} f^U_r \otimes \mathbb{1}_{I^{(r)}} \qquad \text{with } f_r, f^U_r \in U(l_r)\;. \tag{40}$$

We choose $r, s \in \{1, \ldots R\}$. Restricting U to $\mathbb{C}^{l_r} \otimes I^{(r)}$ and orthogonally projecting its image to $\mathbb{C}^{l_s} \otimes I^{(s)}$, we get a mapping

$$U_{sr} \;:\; \mathbb{C}^{l_r} \otimes I^{(r)} \to \mathbb{C}^{l_s} \otimes I^{(s)} \;.$$

If this mapping vanishes identically, it can clearly be written in the form

$$U_{sr} \;=\; M_{sr} \otimes A_{sr} \tag{41}$$

with linear maps

$$M_{sr} \;:\; \mathbb{C}^{l_r} \to \mathbb{C}^{l_s}\;, \qquad A_{sr} \;:\; I^{(r)} \to I^{(s)}\;. \tag{42}$$

Our goal is to show that U_{sr} can also be represented in the form (41, 42) if it does *not* vanish identically. In this case, we define for any non-zero vectors $u^{(r)} \in I^{(r)}$ and $u^{(s)} \in I^{(s)}$ the following injection and projection operators,

$$\begin{aligned} \iota_r(u^{(r)}) &\;:\; \mathbb{C}^{l_r} \hookrightarrow I \;:\; v \mapsto v \otimes u^{(r)} \\ \pi_s(u^{(s)}) &\;:\; I \to \mathbb{C}^{l_s} \;:\; w \mapsto \Big(<e_i \otimes u^{(s)} \mid w> \Big)_{i=1,\ldots,l_s} \;, \end{aligned}$$

where e_i denotes the canonical basis of $\mathbb{C}^{l_s}$. Since U_{sr} is non-trivial, we can choose $u^{(r)}$ such that the product $U\iota_r$ is not identically equal to zero. Thus we can choose $u^{(l)}$ such that the operator

$$M_{sr} \;:=\; \pi_s \, U \, \iota_r \;:\; \mathbb{C}^{l_r} \to \mathbb{C}^{l_s}$$

does not vanish identically. Using the representation (40) together with (39) and the definitions of ι_r and π_s, we obtain for every $f \in \mathcal{F}$,

$$\pi_s \, U \, \iota_r \, f_r \;=\; \pi_s \, U f \, \iota_r \;=\; \pi_s \, f^U U \, \iota_r \;=\; f_s^U \, \pi_s \, U \, \iota_r$$

and thus

$$M_{sr} \, f_r \;=\; f_s^U \, M_{sr} \qquad \forall f \in \mathcal{F} \,. \tag{43}$$

Let us show that (43) and the fact that $M_{sr} \not\equiv 0$ implies that M_{sr} is bijective: We choose a vector $u \in \mathbb{C}^{l_r}$ which is not in the kernel of M_{sr} and set $v = M_{sr} u$. Then for all $f \in \mathcal{F}$,

$$M_{sr} \, f_r \, u \;=\; f_s^U \; v \;\neq\; 0,$$

and since $f_r \in U(l_r)$ is arbitrary, it follows that M_{sr} is injective. Moreover, for all $f \in \mathcal{F}$,

$$f_s^U \; v \;=\; M_{sr} \, (f_r \, u) \,,$$

and since $f_s^U \in U(l_s)$ is arbitrary, we see that M_{sr} is surjective.

The bijectivity of M_{sr} clearly implies that $l_s = l_r$. Furthermore, we can relate f_r and f_s^U by

$$f_s^U \;=\; M_{sr} \, f_r \, M_{sr}^{-1} \,. \tag{44}$$

Restricting both sides of (39) to $\mathbb{C}^{l_r} \otimes I^{(r)}$ and orthogonally projecting their image to $\mathbb{C}^{l_s} \otimes I^{(s)}$, we get the relation

$$U_{sr} \, (f_r \otimes \mathbb{1}_{I^{(r)}}) \;=\; \left(f_s^U \otimes \mathbb{1}_{I^{(s)}}\right) U_{sr} \,.$$

Using (44), we obtain

$$B \, (f_r \otimes \mathbb{1}_{I^{(r)}}) \;=\; (f_r \otimes \mathbb{1}_{I^{(s)}}) \, B$$

with $B := (M_{sr}^{-1} \otimes \mathbb{1}_{I^{(s)}}) \, U_{sr}$. Now we can apply Schur's lemma to conclude that B is trivial in its first factor,

$$\left(M_{sr}^{-1} \otimes \mathbb{1}_{I^{(s)}}\right) U_{sr} \;=\; \mathbb{1}_{\mathbb{C}^{l_r}} \otimes A_{sr}$$

for some linear operator $A_{sr} : I^{(r)} \to I^{(s)}$. Multiplying both sides by $(M_{sr} \otimes \mathbb{1}_{I^{(s)}})$ proves the representations (41, 42).

Suppose that for a given r there are two $s, s' \in \{1, \ldots, R\}$ with $U_{sr} \neq 0 \neq U_{s'r}$. Then we obtain from (44) that

$$M_{sr}^{-1} \, f_s^U \, M_{sr} \;=\; M_{s'r}^{-1} \, f_{s'}^U \, M_{s'r} \qquad \forall f \in \mathcal{F} \,.$$

Since the $f_s^U \in U(l_s)$ can be chosen independently, this relation can hold only if $s = s'$. Hence U_{sr} vanishes except for at most one s. On the other hand, the surjectivity of U implies that for every r there is at least one r such that $U_{rs} \not\equiv 0$. We conclude that the mapping $r \mapsto s$ is a permutation. We introduce $\pi \in \mathcal{S}_R$ such that $s = \pi(r)$. We conclude that

$$U_{|I^{(r)}} \;=\; M_{\pi(r)\,r} \otimes A_{\pi(r)\,r} \;:\; \mathbb{C}^{l_r} \otimes I^{(r)} \;\to\; \mathbb{C}^{l_{\pi(r)}} \otimes I^{(\pi(r))} \,,$$

and due to the unitarity of U, this mapping must be bijective and isometric. In particular, $I^{(r)}$ and $I^{(\pi(r))}$ are isomorphic. Choosing an arbitrary isomorphism $\kappa : I^{(r)} \to I^{(\pi(r))}$, we can write the above mapping as

$$U_{|I^{(r)}} \;=\; \left(M_{\pi(r)\,r} \otimes \mathbb{1}_{I^{(\pi(r))}}\right) (\mathbb{1}_{\mathbb{C}^{l_r}} \otimes \kappa)\, (\mathbb{1}_{\mathbb{C}^{l_r}} \otimes U_r) \tag{45}$$

with $U_r \in U(I^{(r)})$. For fixed κ, this representation is obviously unique up to the phase transformation

$$M_{\pi(r)\,r} \;\mapsto\; e^{i\vartheta}\, M_{\pi(r)\,r}\,, \quad U_r \;\mapsto\; e^{-i\vartheta}\, U_r \qquad \text{with } \vartheta \in \mathbb{R}\,.$$

These phase transformations can be fixed by imposing that $U_r \in SU(I^{(r)})$. This makes the representation (45) unique, and by restricting (37) to $I^{(r)}$, one sees that it coincides precisely with the desired representation of $U_{|I}$. ■

Corollary 3.5 *Let $(H, <.|.>, (E_x)_{x\in M}, P)$ be a discrete fermion system with outer symmetry group $\mathcal{O}$. Provided that $H^{(0)}$ is chosen maximally (27), there are group homomorphisms $\lambda^{(r)} : \mathcal{O} \to U(H^{(r)})$ such that the operators*

$$U(\sigma) \;=\; V(\sigma) \cdot \left(\bigoplus_{r=1}^{R} \mathbb{1}_{\mathbb{C}^{l_r}} \otimes \lambda^{(r)}(\sigma) \right)$$

form a representation of the outer symmetry group on $H^{(0)\perp}$.

Proof. Let us choose a convenient basis in every subspace $H_x^{(r)} := E_x^{(r)}(H^{(r)})$, $x \in M$, $r \in \{1, \ldots, R\}$. We closely follow the construction of the special basis of H_x in the proof of Lemma 2.8. First, in every subspace $I_x^{(r)} := E_x^{(r)} P^{(r)}(H^{(r)})$ we choose a non-degenerate subspace of maximal dimension and in this subspace a pseudo-orthonormal basis $(e_i^{(x,r)})$. We extend this basis by vectors $f_j^{(x,r)}$ to a basis of $I_x^{(r)}$. Next we choose vectors $h_j^{(x,r)} \in H_x^{(r)}$ which are conjugate to the $f_j^{(x,r)}$ in the sense that $<f_i^{(x,r)} \,|\, h_j^{(x,r)}> = \delta_{ij}$. Then $(e_i^{(x,r)}, f_j^{(x,r)}, h_j^{(x,r)})$ is a basis of $H_x^{(r)}$, as the following argument shows. Suppose that $u \in H_x^{(r)}$ is a vector in the orthogonal complement of the span of $(e_i^{(x,r)}, f_j^{(x,r)}, h_j^{(x,r)})$. Then the vector space $\mathbb{C}^{l_r} \otimes \{v\} \subset H$ is orthogonal to I and thus in the kernel of P. Furthermore, it is invariant under the projectors E_x. Using that $H^{(0)}$ is maximal (27), we conclude that $\mathbb{C}^{l_r} \otimes \{v\} \subset H^{(0)}$ and and thus $v = 0$.

The calculation

$$U(I_x^{(r)}) \;=\; (U E_x P)(H^{(r)}) \;=\; (E_{\sigma(x)}\, PU)(H^{(r)}) \;=\; (E_{\sigma(x)}\, P)(H^{(\pi(r))}) \;=\; I_{\sigma(x)}^{(\pi(r))}$$

shows that U maps $I_x^{(r)}$ to $I_{\sigma(x)}^{(\pi(r))}$, and this mapping is clearly isometric and bijective. Introducing the isomorphism κ in (45) by mapping the basis vectors $(e_i^{(x,r)}, f_j^{(x,r)}, h_j^{(x,r)})$ to the corresponding basis vectors $(e_i^{(\sigma(x),\pi(r))}, f_j^{(\sigma(x,\pi(r))}, h_j^{(\sigma(x),\pi(r))})$, the mapping $U_r \in U(I^{(r)})$ in (45) is locally unitary according to Def. 2.6. Thus we can apply Lemma 2.8 to unitarily extend U_r to $H^{(r)}$. More precisely, we choose the extension on the subspaces $H_x^{(r)}$

according to (18). The resulting mapping $\lambda^{(r)} : U(I^{(r)}) \to U(H^{(r)})$ allows us to define $U(\sigma)$ by

$$U(\sigma)_{|H^{(r)}} \;=\; (\mathbb{1}_{\mathbb{C}^{l_r}} \otimes \kappa)\,(\mathbb{1}_{\mathbb{C}^{l_r}} \otimes \lambda(U_r))\;.$$

Unfortunately, in this formula the mappings κ and λ depends on σ. But we can remove this dependence by redefining λ. Namely, for any given isomorphism $\kappa_{sr} : I^{(r)} \to I^{(s)}$ we rewrite $U(\sigma)$ as

$$U(\sigma)_{|H^{(r)}} \;=\; (\mathbb{1}_{\mathbb{C}^{l_r}} \otimes \kappa_{sr})\left(\mathbb{1}_{\mathbb{C}^{l_r}} \otimes \lambda^{(r)}(\sigma)\right) \quad \text{with} \quad \lambda^{(r)}(\sigma) \;:=\; (\kappa_{sr})^{-1} \circ \kappa(\sigma) \circ \lambda$$

and $s = (\pi(\sigma))(r)$. Since we fixed the bases $(e_i^{(x,r)}, f_j^{(x,r)}, h_j^{(x,r)})$ and constructed our extensions simply by mapping corresponding basis vectors onto each other, it is obvious that the mapping $\sigma \mapsto U(\sigma)$ is compatible with the group operations. ■

The just-constructed isomorphisms $I^{(r)} \simeq I^{(\pi(r))}$ and $H_x^{(r)} \simeq H_{\sigma(x)}^{(\pi(r))}$ immediately imply the following relations between dimensions and signatures.

Corollary 3.6 *Assume that in the representation of Theorem 2.11 the vector space $H^{(0)}$ is chosen maximally (27). Then the parameters in Theorem 2.11 and Theorem 3.4 are related to each other for all $x \in M$ and $r \in \{1, \ldots, R\}$ by*

$$\begin{aligned} f^{(r)} &= f^{(\pi(r))} \\ (p^{(r)}, q^{(r)}) &= (p^{(\pi(r))}, q^{(\pi(r))}) \\ (p_x^{(r)}, q_x^{(r)}) &= (p_{\sigma(x)}^{(\pi(r))}, q_{\sigma(x)}^{(\pi(r))})\;. \end{aligned}$$

4 Simple Subsystems and Simple Systems

In the previous sections we constructed representations of the inner and outer symmetries, and by completely reducing these representations we decomposed our discrete fermion system (see Theorem 2.11, Proposition 3.2 and Theorem 3.4). Using these results, we now turn attention to the opposite problem of building up a fermion system from smaller components. Our goal is to give a systematic method for building up a general discrete fermion system from smaller "building blocks," which should be as simple as possible.

We always keep the discrete space-time points $M = \{1, \ldots, m\}$ as well as the outer symmetry group $\mathcal{O} \subset \mathcal{S}_m$ fixed. We begin with the definitions of our smallest building blocks.

Def. 4.1 *Let $(H, <.|.>, (E_x)_{x\in M}, P)$ be a discrete space-time. Assume that the spin dimension is constant on the orbits of $\mathcal{O}$,*

$$p_x \;=\; p_{\sigma(x)}\;, \quad q_x \;=\; q_{\sigma(x)} \qquad \forall x \in M\;.$$

We set $P = 0$. Then the system $(H, <.|.>, (E_x)_{x\in M}, P)$ is called a **trivial system**.

Since the spin dimension is constant on the orbits of $\mathcal{O}$, we can choose pseudo-orthonormal bases of the subspaces $E_x(H)$ and identify the corresponding basis vectors to obtain isomorphisms between $E_x(H)$ and $E_{\sigma(x)}(H)$. This gives rise to a unitary representation U of the outer symmetry group (see Def. 3.1).

Def. 4.2 *Let $(H, <.|.>, (E_x)_{x \in M}, P)$ be a discrete space-time. Assume that we are given a subgroup $\mathcal{N}$ of $\mathcal{O}$ together with a unitary representation U of $\mathcal{N}$ on H. Assume furthermore that the following conditions are satisfied:*

(i) *U represents $\mathcal{N}$ as an outer symmetry in the sense that for every $\sigma \in \mathcal{N}$, the corresponding $U(\sigma)$ satisfies the relations (32).*

(ii) *The system contains no trivial subsystems, i.e.*

$$J \subset H \text{ definite, } P(J) = 0 \text{ and } E_x(J) \subset J \;\; \forall x \in M \quad \Longrightarrow \quad J \;=\; \varnothing\,.$$

(iii) *The proper free gauge group is simply the $U(1)$ of global phase transformations,*

$$\hat{\mathcal{F}} \;=\; \{e^{i\vartheta}\, \mathbb{1} \textit{ with } \vartheta \in \mathbb{1}\}\,.$$

Then the structure $(H, <.|.>, (E_x)_{x \in M}, P, \mathcal{N} \subset \mathcal{O}, U)$ is called a **simple subsystem**.

We point out that only the subgroup $\mathcal{N}$ of $\mathcal{O}$ is a symmetry of the simple subsystem.

Our next step is to construct out of a simple subsystem a discrete fermion system which has $\mathcal{O}$ as outer symmetry. We denote the cosets $\{\sigma\mathcal{N} \text{ with } \sigma \in \mathcal{O}\}$ by $C_1, \ldots, C_K$; they form a partition of the set $\mathcal{O}$. Of each coset we choose one representative $\sigma_k \in C_k$. For convenience, we set $C_1 = \mathcal{N}$ and choose $\sigma_1 = \mathbb{1}$. Every $\sigma \in \mathcal{N}$ defines via $\tau\mathcal{N} \mapsto (\sigma\tau)\mathcal{N}$ a permutation of the cosets $C_1, \ldots, C_K$. This yields a homomorphism of $\mathcal{O}$ to the symmetric group $\mathcal{S}_K$, which we denote by π,

$$\pi \;:\; \mathcal{O} \rightarrow \mathcal{S}_K\,. \tag{46}$$

Clearly, $(\pi(\sigma_k))(1) = k$, and thus the subgroup $\pi(\mathcal{O}) \subset \mathcal{S}_K$ acts transitively on the set $\{1, \ldots, K\}$.

We introduce the inner product space $\tilde{H} = \mathbb{C}^K \otimes H$ (with the natural inner product (21)). On $\tilde{H}$ we introduce the projectors $\tilde{P}$ and $\tilde{E}_x$ by

$$\begin{aligned} \tilde{P}|_{\langle e_k\rangle \otimes H} &= (\mathbb{1} \otimes P)\,|_{\langle e_k\rangle \otimes H} && (47)\\ \tilde{E}_{\sigma_k(x)}|_{\langle e_k\rangle \otimes H} &= (\mathbb{1} \otimes E_x)\,|_{\langle e_k\rangle \otimes H}\,, && (48) \end{aligned}$$

where (e_k) denotes the canonical basis of $\mathbb{C}^K$. Furthermore, we introduce for all $k, l \in \{1, \ldots K\}$ the canonical identification maps

$$\kappa_{l,k} \;:\; \langle e_k\rangle \otimes H \subset \tilde{H} \;\rightarrow\; \langle e_l\rangle \otimes H \;:\; e_k \otimes u \mapsto e_l \otimes u\,.$$

In order to define a unitary representation $\tilde{U}$ of $\mathcal{O}$ on $\tilde{H}$, we introduce for any $\sigma \in \mathcal{O}$ and $k \in \{1, \ldots, K\}$ the parameter $l = (\pi(\sigma))(k)$. Then the group element $\tau := \sigma_l^{-1}\sigma\sigma_k$ satisfies the condition

$$(\pi(\tau))(1) \;=\; (\pi(\sigma_l^{-1}\sigma\sigma_k))(1) \;=\; (\pi(\sigma_l^{-1}\sigma)(k) \;=\; (\pi(\sigma_l^{-1}))(l) \;=\; 1$$

and thus $\tau \in \mathcal{N}$. Hence we may define $\tilde{U}(\sigma)$ by

$$\tilde{U}(\sigma)\Big|_{\langle e_k\rangle \otimes H} \;=\; \kappa_{l,k} \circ (\mathbb{1} \otimes U(\tau))\Big|_{\langle e_k\rangle \otimes H}\,. \tag{49}$$

Lemma 4.3 *The unitary operators $\tilde{U}$, (49), are a representation of the outer symmetry group $\mathcal{O}$ on $(\tilde{H}, <.|.>, (\tilde{E}_x)_{x \in M}, \tilde{P})$. The definitions (47–49) are, up to isomorphisms, independent of the choice of the group elements $\sigma_k \in \mathcal{O}$.*

Proof. We first verify that $\tilde{U}$ is a representation of $\mathcal{O}$. For given $\sigma, \bar{\sigma} \in \mathcal{O}$ we set $l = (\pi(\sigma))(k)$ and $\bar{l} = (\pi(\sigma))(l)$. Then

$$\begin{aligned}
\tilde{U}(\bar{\sigma}) \circ \tilde{U}(\sigma)\Big|_{\langle e_k \rangle \otimes H} &= \kappa_{\bar{l},l} \circ (\mathbb{1} \otimes U(\bar{\tau}))\Big|_{\langle e_l \rangle \otimes H} \circ \kappa_{l,k} \circ (\mathbb{1} \otimes U(\tau))\Big|_{\langle e_k \rangle \otimes H} \\
&= \kappa_{\bar{l},k} \circ \Big(\mathbb{1} \otimes U(\sigma_{\bar{l}}^{-1} \bar{\sigma} \sigma_l)\Big) \left(\mathbb{1} \otimes U(\sigma_l^{-1} \sigma \sigma_k)\right)\Big|_{\langle e_k \rangle \otimes H} \\
&= \kappa_{\bar{l},k} \circ \Big(\mathbb{1} \otimes U(\sigma_{\bar{l}}^{-1} \bar{\sigma} \sigma \sigma_k)\Big)\Big|_{\langle e_k \rangle \otimes H} = \tilde{U}(\bar{\sigma}\sigma)\Big|_{\langle e_k \rangle \otimes H} .
\end{aligned}$$

For the proof of (32) we can restrict attention to the space-time projectors $\tilde{E}_x$, because the fermionic projector transforms in exactly the same way, except for the simplification that it does not carry a space-time index. We again choose any $\sigma \in \mathcal{O}$ and $k \in \{1, \ldots, K\}$. We set $l = (\pi(\sigma))(k)$ and $\tau = \sigma_l^{-1} \sigma \sigma_k$. Then, setting $x = \sigma_k^{-1} y$, we have for all $y \in M$,

$$\begin{aligned}
\tilde{U}(\sigma)\, \tilde{E}_y\, \tilde{U}(\sigma)^{-1}\Big|_{\langle e_l \rangle \otimes H} &= (\mathbb{1} \otimes U(\tau))\, (\mathbb{1} \otimes E_x)\, (\mathbb{1} \otimes U(\tau))^{-1}\Big|_{\langle e_l \rangle \otimes H} \\
&= \big(\mathbb{1} \otimes E_{\tau(x)}\big)\,|_{\langle e_l \rangle \otimes H} = \tilde{E}_{(\sigma_l \circ \tau)(x)} = \tilde{E}_{\sigma(y)} ,
\end{aligned}$$

where in the last line we used that $\tau \in \mathcal{N}$ and that $\mathcal{N}$ is a symmetry of the simple subsystem represented by U.

Suppose that $\bar{\sigma}_k \in \mathcal{O}$ is another choice of group elements with $(\pi(\bar{\sigma}_k))(1) = k$. Then

$$(\pi(\bar{\sigma}_k^{-1}\, \sigma_k))(1) = (\pi(\bar{\sigma}_k)^{-1} \circ \pi(\sigma_k))(1) = (\pi(\bar{\sigma}_k)^{-1})(k) = 1$$

and thus $\tau_k := \bar{\sigma}_k^{-1} \sigma_k \in \mathcal{N}$. Using that $\mathcal{N}$ is a symmetry of the simple subsystem, we find that

$$\begin{aligned}
\tilde{\tilde{E}}_x\Big|_{\langle e_k \rangle \otimes H} &= \Big(\mathbb{1} \otimes E_{\bar{\sigma}_k^{-1}(x)}\Big)\Big|_{\langle e_k \rangle \otimes H} = \Big(\mathbb{1} \otimes E_{(\tau_k \circ \sigma_k^{-1})(x)}\Big)\Big|_{\langle e_k \rangle \otimes H} \\
&= \Big(\mathbb{1} \otimes U(\tau_k) E_{\sigma_k^{-1}(x)} U(\tau_k)^{-1}\Big)\Big|_{\langle e_k \rangle \otimes H} ,
\end{aligned}$$

showing that the objects defined using σ_k and those defined using $\bar{\sigma}_k$ are related to each other by the unitary transformation V given by

$$V|_{\langle e_k \rangle \otimes H} = (\mathbb{1} \otimes U(\tau_k))\,|_{\langle e_k \rangle \otimes H} .$$

Hence our definitions are unique up to isomorphisms. ∎

Def. 4.4 *We call the system $(\tilde{H}, <.|.>, (\tilde{E}_x)_{x \in M}, \tilde{P}, \mathcal{O}, \tilde{U})$ the* **simple system** *corresponding to the simple subsystem of Def. 4.2.*

Theorem 4.5 *Let $(H, <.|.>, (E_x)_{x \in M}, P)$ be a discrete fermion system with outer symmetry $\mathcal{O}$. Then there is a trivial system $(H^{(0)}, (E_x^{(0)})_{x \in M}, P^{(0)}, U^{(0)})$ as well as a collection* *of simple systems* $(\tilde{H}^{(a)}, (\tilde{E}_x^{(a)})_{x\in M}, \tilde{P}^{(a)}, \tilde{U}^{(a)}, K_a)_{a=1,\ldots,A}$*,* $A \geq 1$*, together with parameters* $n_a \in \mathbb{N}$ *such that we have the following isomorphisms,*

$$\begin{aligned}
H &\simeq H^{(0)} \oplus \left(\bigoplus_{a=1}^{A} \mathbb{C}^{n_a} \otimes \tilde{H}^{(a)} \right) && (50)\\
E_x &\simeq E_x^{(0)} \oplus \left(\bigoplus_{a=1}^{A} \mathbb{1}_{\mathbb{C}^{n_a}} \otimes \tilde{E}_x^{(a)} \right) && (51)\\
P &\simeq \bigoplus_{a=1}^{A} \mathbb{1}_{\mathbb{C}^{n_a}} \otimes \tilde{P}^{(a)}\,. && (52)
\end{aligned}$$

The unitary operators

$$U(\sigma) = U^{(0)} \oplus \left(\bigoplus_{a=1}^{A} \mathbb{1}_{\mathbb{C}^{n_a}} \otimes \tilde{U}^{(a)}(\sigma) \right)$$

are a representation of the outer symmetry group.

Proof. We present the discrete fermion system as in Theorem 2.11 with $H^{(0)}$ maximal (27) and represent the outer symmetry as in Theorem 3.4 and Corollary 3.5. The orbits of the permutation matrices $V(\sigma)$ form a partition of the set $\{1, \ldots, R\}$ into disjoint subsets. Let $Q \subset \{1, \ldots, R\}$ be one of these orbits. By reordering the space-time points we can arrange that $Q = \{1, \ldots, K\}$ with a parameter K in the range $1 \leq K \leq R$. From (38) and Corollary 3.6 we know that

$$\bigoplus_{r\in Q} \mathbb{C}^{l_r} \otimes H^{(r)} = \mathbb{C}^{l_1} \otimes \mathbb{C}^K \otimes H^{(1)}\,. \qquad (53)$$

The action of $V(\sigma)$ on Q defines a transitive group homomorphism $\pi : \mathcal{O} \to P_K$. We introduce the subsets $C_1, \ldots C_K$ by

$$C_k = \{\sigma \in \mathcal{O} \,|\, (\pi(\sigma))(1) = k\}\,.$$

Clearly, $\mathcal{N} := C_1$ is a subgroup of $\mathcal{O}$. Let us verify that the C_k coincide with the cosets of $\mathcal{N}$ in $\mathcal{O}$: For $\sigma, \tau \in C_k$, the calculation

$$\big(\pi(\sigma^{-1}\,\tau)\big)\,(1) = \big(\pi(\sigma)^{-1} \circ \pi(\sigma)\big)\,(1) = (\pi(\sigma)^{-1})(k) = 1$$

shows that $\sigma^{-1}\tau \in \mathcal{N}$, and thus σ and τ belong to the same coset. If conversely σ and τ are in the same coset, we know that $\sigma^{-1}\tau \in \mathcal{N}$ and thus $(\pi(\sigma)^{-1}\pi(\sigma))(1) = 1$. In other words,

$$(\pi(\sigma))(1) = (\pi(\tau))(1) =: k\,,$$

meaning that $\sigma, \tau \in C_k$.

Identifying the C_k with the cosets of $\mathcal{S}$, the above homomorphism coincides precisely with the action of $\mathcal{O}$ on the cosets as described by (46). According to (24, 25), the projectors P and E_x act only on the last factor in the decomposition (53) and can thus be regarded as operators on $H^{(1)}$. For the resulting subsystem $(H^{(1)}, P, E_x)$, the group $\mathcal{N}$ is a symmetry, which is represented by U_1. The maximality condition (27) implies Def. 4.2 (ii), whereas Def. 4.2 (iii) follows from the representation of the proper free gauge group (26). We conclude that $(H^{(1)}, E_x, P, \mathcal{N}, U_1)$ is a simple subsystem.

Let $(\tilde{H}^{(1)}, \tilde{E}_x, \tilde{P}, \tilde{U}_1)$ be the corresponding simple system. Then the tensor product of $\mathbb{C}^{l_1}$ with this system is obviously isomorphic to the restriction of our original system to the subspace (53). Taking the direct sum of $H^{(0)}$ with the so-obtained systems corresponding to the different orbits of V yields our original discrete fermion system. ■

Example 4.6 Let us build up the discrete fermion systems considered in Example 3.3. In both examples (34) and (36), we obtain the trivial system by restricting the fermion system to the subspace

$$H^{(0)} = \{(a, 0, c, 0) \,:\, a, c \in \mathbb{C}\} .$$

In the example (34), the simple subsystem is constructed as follows. We set $H = \mathbb{C}^2$ with $-<.|.>$ equal to the canonical scalar product on $\mathbb{C}^2$ and introduce the projectors

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We again let $\mathcal{O} = \{\mathbb{1}, \sigma\}$ with $\sigma(1) = 2$ and $\sigma(2) = 1$. We choose $\mathcal{N} = \mathcal{O}$ with the representation

$$U(\mathbb{1}) = \mathbb{1}\,, \qquad U(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{54}$$

Then there is only one coset, $K = 1$. Furthermore, π is the trivial mapping $\pi(\sigma) = \mathbb{1}$. This system is a simple subsystem according to Def. 4.2. Since $K = 1$, this subsystem coincides with the corresponding simple system. Obviously, the direct sum of this system with $H^{(0)}$ is isomorphic to the system (33, 34).

In the example (36), we construct the simple subsystem by choosing $H = \mathbb{C}$ with $<u|v> = -\overline{u}v$. Furthermore, we choose

$$E_1 = 1\,, \quad E_2 = 0\,, \qquad P = 1\,.$$

We again let $\mathcal{O} = \{\mathbb{1}, \sigma\}$ with $\sigma(1) = 2$ and $\sigma(2) = 1$. But now we choose $\mathcal{N} = \{\mathbb{1}\}$ equal to the trivial subgroup. Then its representation is also trivial, $U(\mathbb{1}) = \mathbb{1} = U(\sigma)$. There are two cosets of $\mathcal{N}$ in $\mathcal{O}$, $K = 2$. The mapping $\pi : \mathcal{O} = \mathcal{S}_2 \to \mathcal{S}_2$ is the identity map. This system satisfies all the conditions in Def. 4.2. Since $K = 2$, the corresponding simple system lives in the inner product space $\tilde{H} = \mathbb{C}^2 \times H \simeq \mathbb{C}^2$, where $-<.|.>$ coincides with the canonical scalar product on $\mathbb{C}^2$. The resulting representation $\tilde{U}$ given by (49) coincides with (54). A short calculation using (48, 49) yields

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is a simple fermion system with outer symmetry $\mathcal{O}$ consisting of two simple subsystems. Taking the direct sum with the trivial system $H^{(0)}$ gives precisely the system (33, 36). ♦

We finally give a useful characterization of simple systems which does not refer to simple subsystems.

Proposition 4.7 *Let $(\tilde{H}, <.|.>, (\tilde{E}_x)_{x\in M}, \tilde{P}, \mathcal{O}, \tilde{U})$ be a discrete fermion system with outer symmetry group $\mathcal{O}$ represented by $\tilde{U}$. This system can be realized as a simple fermion system according to Def. 4.4 if and only if the following two conditions are satisfied:*

(a) *The system contains no trivial subsystems according to Def. 4.2 (ii).*

(b) *The system cannot be decomposed into the direct sum of two non-trivial fermion systems $(H^1, E^1_x P^1$ and $(H^2, E^2_x P^2)$,*

$$\tilde{H} \;=\; H^1 \oplus H^2\,, \quad \tilde{E}_x \;=\; E^1_x \oplus E^2_x\,, \quad \tilde{P} \;=\; P^1 \oplus P^2\,,$$

which both have $\mathcal{O}$ as an outer symmetry group.

Proof. It is obvious from Def. 4.2 (ii) and our above construction that a simple system contains no trivial subsystems. Furthermore, a simple subsystem cannot be decomposed into non-trivial subsystems because otherwise the proper free gauge group would contain independent phase transformations of both subsystems and thus $\hat{\mathcal{F}} \supset U(1) \times U(1)$, in contradiction to Def. 4.2 (iii). The corresponding simple system is by construction the smallest system with outer symmetry group $\mathcal{O}$ which contains the simple subsystem, and therefore it cannot be decomposed into smaller systems with these properties.

Assume conversely that a discrete fermion system $(\tilde{H}, <.|.>, (\tilde{E}_x)_{x\in M}, \tilde{P}, \mathcal{O}, \tilde{U})$ satisfies the assumptions stated in the proposition. We again present the discrete fermion system as in Theorem 2.11 with $H^{(0)}$ maximal (27) and represent the outer symmetry as in Theorem 3.4 and Corollary 3.5. Then the assumption **(a)** implies that $H^{(0)}$ is trivial. Furthermore, the group $\pi(\mathcal{O}) \subset \mathcal{S}_R$ must act transitively on the set $\{1, \ldots, R\}$ because otherwise the orbits of $\pi(\mathcal{O})$ would give a splitting of the fermion system into non-trivial smaller systems with outer symmetry $\mathcal{O}$, in contradiction to assumption **(b)**. Hence there is only one orbit $Q = \{1, \ldots, R\}$, and the construction in the proof of Theorem 4.5 shows how the system $(\tilde{H}, <.|.>, (\tilde{E}_x)_{x\in M}, \tilde{P}, \mathcal{O}, \tilde{U})$ is realized as the simple system corresponding to a suitable simple subsystem. ■

5 The Pinned Symmetry Group

In Def. 4.2 we assumed a unitary representation U of a finite group $\mathcal{N} \subset \mathcal{O}$ on an inner product space $(H, <.|.>)$ which satisfies for all $x \in M$ the conditions

$$U(\sigma)\, E_x\, U(\sigma)^{-1} \;=\; E_{\sigma(x)} \tag{55}$$

plus the symmetry condition for the fermionic projector $UPU^{-1} = P$. In this section we disregard the symmetry condition for the fermionic projector and consider unitary representations of $\mathcal{N}$ which only satisfy (55). Our goal is to use the gauge freedom to bring such representations into a simple form.

Because of the completeness of the space-time projectors, we can consider instead of $U(\sigma)$ the operator products $E_x U(\sigma) E_y$ for $x, y \in M$. We denote the orbits of the action of $\mathcal{N}$ on M by $M_1, \ldots, M_J$, $J \geq 1$. The orbits form a partition of M, and we can introduce an equivalence relation $x \simeq y$ by identifying the points on the same orbit. Rewriting (55) as $U(\sigma) E_y = E_{\sigma(y)} U(\sigma)$ and multiplying from the left by E_x, we find that

$$E_x\, U(\sigma)\, E_y \;=\; 0 \quad \text{unless } x \simeq y. \tag{56}$$

Therefore, it suffices to consider the case that x and y are on the same orbit. Without loss of generality we can assume that $x, y \in M_1$. In other words, it remains to consider the following restriction of U,

$$U_{|H_1} \quad \text{with} \quad H_1 = \bigoplus_{x \in M_1} E_x(H) \,. \tag{57}$$

Furthermore, it is no loss of generality to distinguish one point of M_1, because this point can be mapped to any other point of M_1 by applying $\mathcal{N}$. For simplicity, we assume that $1 \in M_1$. We now form the subgroup of the outer symmetry group which leaves this distinguished point invariant.

Def. 5.1 *The* **pinned symmetry group** $\mathcal{R} \subset \mathcal{N}$ *is the group of all* $\sigma \in \mathcal{N}$ *with* $\sigma(1) = 1$.

In the case when $\mathcal{O}$ is the group of all affine orientation-preserving transformations of $\mathbb{R}^3$, pinching the origin gives the group of all rotations around the origin. This is the reason for denoting the pinned symmetry group by the letter "$\mathcal{R}$."

For every $\sigma \in \mathcal{R}$, we find that $U(\sigma) E_1 = E_{\sigma(1)} U(\sigma) = E_1 U(\sigma)$. In other words, $U(\sigma)$ maps the subspace $\hat{H} := E_1(H)$ into itself. Hence

$$V(\sigma) \;:=\; U(\sigma)_{|\hat{H}} \qquad \text{is a unitary representation of } \mathcal{R} \text{ on } \hat{H} \,. \tag{58}$$

The next proposition gives a procedure to reconstruct $U_{|H_1}$ from a given representation V.

Proposition 5.2 *Let* $(H, <.|.>, (E_x)_{x \in M})$ *be a discrete space-time. Assume that we are given a group* $\mathcal{N} \subset \mathcal{O}$ *such that the spin dimension is constant on the orbits of* $\mathcal{N}$. *Let* $M_1 \subset M$ *be the orbit containing the point* $1 \in M$. *Suppose that* V *is a unitary representation of the corresponding pinned symmetry group* $\mathcal{R}$ *(see Def. 5.1) on* $\hat{H} := E_1(H)$. *Then there is, up to gauge transformations, a unique unitary representation* U *of* $\mathcal{N}$ *on* H_1 *(see (57)) which satisfies for all* $x \in M_1$ *the conditions (55) and which, when restricted to* $\mathcal{R}$ *and* $\hat{H}$, *coincides with* V.

Proof. Since $\mathcal{N}$ acts transitively on M_1, we can for every $x \in M_1$ choose a group element $\sigma_x \in \mathcal{N}$ with the property that $\sigma_x(1) = x$. For convenience, we choose $\sigma_1 = \mathbb{1}$. Since the spin dimension is by assumption constant on the orbits of $\mathcal{N}$, the spaces $E_x(H)$, $x \in M_1$, are all isomorphic. Thus for every $x \in M_1$ we can choose an isomorphism $\kappa_x : \hat{H} \to E_x(H)$. For convenience we choose $\kappa_1 = \mathbb{1}$. We define $U(\sigma_x)$ restricted to $\hat{H}$ by

$$U(\sigma_x)_{|\hat{H}} \;:\; \hat{H} \to E_x(H) \;:\; u \mapsto \kappa_x(u) \,. \tag{59}$$

Together with the given representation of $\mathcal{R}$ on $\hat{H}$, (59) uniquely determines a representation of $\mathcal{N}$ on H. Namely, suppose that for a given $\sigma \in \mathcal{N}$ and $x \in M_1$, we want to construct $U(\sigma)_{|E_x(H)}$. Setting $y = \sigma(x)$, we rewrite σ in the form $\sigma = \sigma_y \rho \sigma_x^{-1}$. Then ρ is an element of $\mathcal{R}$ and, using that U should be a group representation,

$$U(\sigma)_{|E_x(H)} \;=\; U(\sigma_y)_{|\hat{H}}\, V(\rho)_{|\hat{H}}\, U(\sigma_x)^{-1}{}_{|E_x(H)} \,. \tag{60}$$

All the operators on the right side are given. It is straightforward to verify that the operators (60) form a representation of $\mathcal{N}$ on H satisfying (55).

For the uniqueness question we let U be any unitary representation of $\mathcal{N}$ on H satisfying (55). Then for all $x \in M_1$, the operator $U(\sigma_x)_{|\hat{H}}$ is a unitary operator from $\hat{H}$

to $E_x(H)$. By a local gauge transformation at x we can arrange that this operator coincides with κ_x. Thus we can achieve by a suitable gauge transformation that U satisfies the conditions (59). But then U is uniquely determined according to (60). ■

6 Building up General Systems: A Constructive Procedure

The constructions of the previous sections yield a systematic procedure for constructing all discrete fermion systems for a given outer symmetry group $\mathcal{O}$ and for given values of the parameters (p_x, q_x), m and f. We denote the maximal spin dimension by $n = \max_{x \in M}\{p_x, q_x\}$.

1. Choose a subgroup $\mathcal{N}$ of $\mathcal{O}$.

2. Determine the orbits $M_1, \ldots M_J$, $J > 0$, of the action of $\mathcal{N}$ on M.

3. Choose in every orbit one representative $x_j \in M_j$ and determine the corresponding pinned symmetry groups $\mathcal{R}_j$ (see Def. 5.1).

4. Choose a unitary representation of each pinned symmetry group on a corresponding indefinite inner product space $\hat{H}_j$ of signature (p_j, q_j) and $p_j, q_j \leq n$. The irreducible subspaces of this representation can be chosen to be definite (see Lemma 2.10). Since the dimension of the irreducible subspaces is bounded a-priori by n, there is only a finite (typically small) number of possible choices of these representations.

5. The construction of Proposition 5.2 gives a unitary representation U of $\mathcal{N}$ on a discrete space-time $(H, <.|.>, (E_x)_{x \in M})$ satisfying (55).

6. After completely reducing the obtained representation U on each of the invariant subspaces
$$H_j \;=\; \bigoplus_{x \in M_j} E_x(H)\,,$$
one can characterize all projectors P which satisfy the condition $UPU^{-1} = P$ (see Proposition 3.2).

7. Restricting attention to projectors P which satisfy the conditions Def. 4.2 (ii) and (iii), we obtain simple subsystems according to Def. 4.2. Carrying out the construction (47–49) yields corresponding simple systems (see Def. 4.4).

8. According to Theorem 4.5, a general discrete fermion system is obtained from simple systems by taking tensor products with $\mathbb{C}^k$ and by taking direct sums. We must satisfy the conditions that the spin dimension of the resulting system must nowhere exceed (n, n) and that the number of particles should be equal to f. These conditions reduce the allowed constructions in this step to a finite (typically small) number of possibilities.

For systems with few particles it may be easier to avoid simple subsystems by constructing right away the simple systems. This gives rise to an alternative construction procedure, for which we only need to replace the above construction steps 1 and 7 by

1'. Choose $\mathcal{N} = \mathcal{O}$.

7'. Restricting attention to projectors P for which the corresponding discrete fermion system satisfies the conditions (a) and (b) of Proposition 4.7, we obtain simple systems.

7 Examples

We now illustrate the construction steps of the previous section in a few examples.

Def. 7.1 *A discrete fermion system* $(H, <.|.>, (E_x)_{x \in M}, P)$ *with outer symmetry group* $\mathcal{O}$ *is said to be* **homogeneous** *if* $\mathcal{O}$ *acts transitively on* M*.*

Example 7.2 (homogeneous systems with Abelian outer symmetry)
Let us consider the case of a homogeneous discrete fermion system with Abelian outer symmetry group. Then $\mathcal{O}$ acts transitively on M, and thus for every $x \in M$ we can choose a group element $\sigma_x \in \mathcal{O}$ with $\sigma_x(1) = x$. The corresponding pinned symmetry group $\mathcal{R}$ is trivial, because for every $\sigma \in \mathcal{R}$,

$$\sigma(x) \;=\; (\sigma_x \sigma \sigma_x^{-1})(x) \;=\; (\sigma_x \sigma)(1) \;=\; \sigma_x(1) \;=\; x \qquad \forall x \in M \;,$$

and thus $\sigma = \mathbb{1}$. As a consequence, for every $x \in M$ the choice of σ_x is unique. In particular, the order $\#\mathcal{O}$ of the symmetry group equals the number m of space-time points, and we can use the mapping $x \mapsto \sigma_x$ to identify M with $\mathcal{O}$.

According to the basis theorem (see [4, Chap. II, §10]), every finite Abelian group is the direct sum of cyclic groups of prime power order. Thus there are parameters $(l_n)_{n=1,\ldots,N}$, each being a power of a prime p_n, and corresponding group elements g_n with the properties that the $(g_n)_{n=1,\ldots,N}$ generate $\mathcal{O}$ and that each of the groups $\{g_n^k \;:\; k \in \mathbb{Z}\}$ is cyclic of order l_n. Introducing the group $\mathcal{T} = l_1\mathbb{Z} \oplus \cdots \oplus l_N\mathbb{Z}$, we can write $\mathcal{O}$ as the quotient group

$$\mathcal{O} \;=\; \mathbb{Z}^N / \mathcal{T} \;.$$

Identifying the points $x \in M$ with the corresponding group elements $\sigma_x \in \mathcal{O}$, we can regard M as an N-dimensional lattice with side lengths l_n.

Let $(\hat{H}, <.|.>)$ be an indefinite inner product space of signature (p, q). Since $\mathcal{R}$ is trivial, its only representation on $\hat{H}$ is $V \equiv \mathbb{1}$. The construction of Proposition 5.2 yields that the corresponding discrete space-time $(H, (E_x)_{x \in M})$ and the representation U of the outer symmetry group $\mathcal{O}$ can be given as follows,

$$\begin{aligned} H \;&=\; \mathbb{C}^M \otimes \hat{H} \;, \qquad M \;=\; \mathbb{Z}^N / \mathcal{T} \\ E_x \;&:\; e_y \otimes u \;\mapsto\; \delta_{xy}\, e_y \otimes u \\ U(\sigma) \;&:\; e_y \otimes u \;\mapsto\; e_{\sigma(y)} \otimes u \;. \end{aligned}$$

In other words, H consists of $\hat{H}$-valued functions on M, and U acts on these functions by translating the points of M by the group $\mathcal{O}$. It is convenient to use the short notation

$$u(x) \;=\; E_x u \;\in\; \langle e_x \rangle \otimes \hat{H} \;\simeq\; \hat{H} \;,$$

where in the last step we identify the vector spaces in the natural way.

In order to completely reduce U, we first note that, since $\mathcal{O}$ is Abelian, its irreducible representations are all one-dimensional. Thus our task is to decompose H into one-dimensional subspaces which are invariant under the action of U. An easy calculation shows that the subspaces spanned by the vectors

$$u(x) \;=\; \hat{u}\, \exp\left(i \sum_{n=1}^{N} k_n x_n\right) \quad \text{with} \quad \hat{u} \in \hat{H},\; k_n \in \left\{0, 1\,\frac{2\pi}{l_n}, 2\,\frac{2\pi}{l_n}, \ldots, (l_n - 1)\,\frac{2\pi}{l_n}\right\} \tag{61}$$

are invariant under the action of U. Also, counting dimensions one sees that these vectors form a basis of H, and therefore the subspaces spanned by the vectors (61) completely reduce U. The fermionic projectors which satisfy the conditions $UPU^{-1} = P$ must be invariant on the irreducible subspaces, i.e.

$$P \;=\; \sum_{x,y \in M} \sum_{k \in \mathcal{K}} \kappa_{x,y} P^{(k)} E_y \exp\left(i \sum_{n=1}^{N} k_n (x_n - y_n)\right), \tag{62}$$

where $\mathcal{K}$ is a set of vectors $k = (k_n)_{n=1,\ldots,N}$ with components in the range as in (61). Here the $P^{(k)}$ are projectors on negative definite subspaces in $\hat{H}$, and $\kappa_{x,y}$ is the natural isomorphism from $E_y(H)$ to $E_x(H)$.

Clearly, the vectors (61) are plane waves on the lattice M with periodic boundary conditions, and (62) is the general form of a projector which is "diagonal in momentum space." We conclude that the construction procedure of Section 6 reduces to the usual discrete Fourier transform on a finite lattice, with the only difference that the side lengths l_n are always prime powers. ♦

Example 7.3 (general systems with Abelian outer symmetry)
As in the previous example we consider an Abelian group $\mathcal{O}$, but which now does not necessarily act transitively on M. We denote the orbits of the action of $\mathcal{O}$ on M by $M_1, \ldots, M_L$. We let $\mathcal{K}_l$ be the subgroups of $\mathcal{O}$ which keep the sets M_l fixed. Since every subgroup of an Abelian group is normal, we can form the quotient groups $\mathcal{O}_l = \mathcal{O}/\mathcal{K}_l$. Then the groups $\mathcal{O}_l$ can be regarded as a group of permutations on the sets M_l, which act transitively. Therefore, on each of the orbits M_l we can use the construction of Example 7.2 to construct a discrete "sub-space-time" $(H_l, (E_x)_{x \in M_l})$ together with a unitary representation U_l of the outer symmetry group $\mathcal{O}_l$. Since a representation of an outer symmetry is trivial between different orbits (56), the discrete space-time is obtained simply by taking the direct sums of the sub-space-times.

In order to construct the fermionic projector, we first note that the irreducible subspaces of H are precisely the span of the plane waves (61) of all the sub-space-times. Let κ be an irreducible representation of $\mathcal{O}$. We form the subspace $H_\kappa \subset H$ spanned by all those invariant subspaces on which U is equivalent to κ. According to Lemma 2.10, H_κ is a non-degenerate subspace of H. The most general fermionic projector satisfying the symmetry condition $UPU^{-1} = P$ is the operator which is invariant on the subspaces H_κ corresponding to the different irreducible representations of $\mathcal{O}$ and is on each of these subspaces a projector on a negative-definite subspace (see Proposition 3.2). ♦

Example 7.4 (two-dimensional lattice with pinned symmetry)
To give an example with a non-trivial pinned symmetry group, we next consider a discrete

space-time which, similar to Example 7.2, is a finite lattice, but now with a larger, non-Abelian symmetry group. For a prime power $l > 2$ we introduce the group $\mathcal{T} = l\,\mathbb{Z} \oplus l\,\mathbb{Z}$ as well as the square lattice

$$M \;=\; \mathbb{Z}^2/\mathcal{T}\,.$$

We let $\mathcal{S}$ be the group of all isometries of $\mathbb{R}^2$ which map the lattice points $\mathbb{Z}^2 \subset \mathbb{R}^2$ onto themselves (thus $\mathcal{S}$ is the group of all translations, reflections and rotations about multiples of the angle 90°). A short consideration shows that $\mathcal{T}$ is a normal subgroup of $\mathcal{S}$. We let $\mathcal{O}$ be the corresponding quotient group,

$$\mathcal{O} \;=\; \mathcal{S}/\mathcal{T}.$$

This group has a natural action on M which corresponds to translations, reflections and rotations on a square lattice whose opposite sides are identified.

Since $\mathcal{O}$ contains the translations, which act transitively on M, our system is clearly homogeneous. Thus we can arbitrarily distinguish one point of M; for convenience we denote the origin in $\mathbb{Z}^2/\mathcal{T}$ by 1. To construct the corresponding pinned symmetry group, we introduce the two unitary matrices

$$\alpha \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \rho \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{63}$$

These matrices describe a rotation by 90° and the reflection at the x_2-axis of $\mathbb{R}^2$, respectively. Since they are compatible with the lattice structure of $\mathbb{Z}^2$ and the action of $\mathcal{T}$, they can be regarded as elements of $\mathcal{O}$. Furthermore, they leave the origin of $\mathbb{Z}^2$ fixed and thus $\alpha, \rho \in \mathcal{R}$. Since by composing 90°-rotations with reflections we obtain all elements of the pinned symmetry group, It is obvious that $\mathcal{R}$ is generated by α and ρ. Note that α and ρ do not commute and thus $\mathcal{R}$ is non-Abelian.

The next step is to construct a representation V of $\mathcal{R}$ on an indefinite inner product space $(\hat{H}, <.|.>)$. The possibilities depend on the signature (p,q) of $\hat{H}$. One possible choice clearly is the trivial representation

$$V(\alpha) \;=\; \mathbb{1} \;=\; V(\beta)\,. \tag{64}$$

Another possibility is to choose the sign representation

$$V(\alpha) \;=\; \mathbb{1}\,, \qquad V(\beta) \;=\; -\mathbb{1}\,. \tag{65}$$

If $p > 1$ or $q > 1$, more complicated representations are possible. For example, one can take direct sums of the one-dimensional representations (64, 65). In this case, the corresponding representation U of $\mathcal{O}$ will also split into a direct sum of representations corresponding to the irreducible summands of V, and therefore this case is straightforward. Moreover, one can choose higher-dimensional irreducible representations of $\mathcal{R}$. To give a simple example, we consider the two-dimensional irreducible representation by the matrices in (63),

$$V(\alpha) \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad V(\rho) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{66}$$

Let us construct the corresponding representations U on H. For every $x \in M$, we choose the unique translation $\sigma_x \in \mathcal{O}$ with $\sigma_x(1) = x$. Carrying out the construction of Proposition 5.2 for the trivial representation (64), we obtain $H = \mathbb{C}^M \otimes \hat{H}$ and

$$E_x \;:\; e_y \otimes u \;\mapsto\; \delta_{xy}\, e_y \otimes u\,, \qquad U(\sigma) \;:\; e_y \otimes u \;\mapsto\; e_{\sigma(y)} \otimes u\,.$$

In the case of the sign representation (65), we obtain the same discrete space-time as for the trivial representation, with the only difference that the resulting representation U also involves signs,

$$U(\sigma) \;:\; e_y \otimes u \;\mapsto\; \mathrm{sgn}(\sigma)\, e_{\sigma(y)} \otimes u\;,$$

where $\mathrm{sgn}(\sigma)$ equals -1 if σ changes the orientation and equals $+1$ otherwise. In the case of the two-dimensional irreducible representation (66), we obtain the same discrete space-time as for the trivial representation, but now with $\hat{H} = \mathbb{C}^2$ and the resulting representation U given by

$$U(\sigma) \;:\; e_y \otimes u \;\mapsto\; e_{\sigma(y)} \otimes V(\sigma)(u)\;,$$

where in order to define $V(\sigma)$ we compose σ by a translation in order to arrange that the origin is fixed. The resulting group element is in the pinned symmetry group, and taking its representation matrix V defines us $V(\sigma)$.

It remains to completely reduce U. To this end, we first note that for the subgroup of translations, U coincides precisely with the representation U in Example 7.2. Thus the invariant subspaces of this subgroup are again the plane waves $\phi_{k,\hat{u}}$ of the form

$$\phi_{k,\hat{u}}(x) \;=\; \hat{u}\, \exp\left(i{<}k,x{>}\right)$$

with $\hat{u} \in \hat{H}$ and ${<}.,.{>}$ the canonical scalar product on $\mathbb{R}^2$. Here the momentum vector $k = (k_1, k_2)$ must be in the "dual lattice" $\mathcal{K}$,

$$k \;\in\; \mathcal{K} \;:=\; \left\{0, 1\,\frac{2\pi}{l}, 2\,\frac{2\pi}{l}, \ldots, (l-1)\,\frac{2\pi}{l}\right\}^2 .$$

In order to get the invariant subspaces of the whole group $\mathcal{O}$, we form the subspaces of plane wave solutions which are mapped into each other by the action of $\mathcal{R}$,

$$H_k \;:=\; \left\{\phi_{\rho(k),\hat{u}} \,|\, \rho \in \mathcal{R}, \hat{u} \in \hat{H}\right\} \subset H\;,$$

where $\rho(k)$ is the action of $\mathcal{R}$ induced on the dual lattice via the relation ${<}k,x{>} = {<}\rho(k),\rho(x){>}$. If $k = 0$, the dimension of H_k coincides with the dimension d of $\hat{H}$ (i.e., it is equal to one if V is the trivial or sign representation, and it equals two for the representation (66)). In the cases $k_1 = 0$, $k_2 = 0$ or $k_1 = k_2$ (and $k = (k_1, k_2) \neq 0$), H_k is of dimension $4d$. In the remaining case $0 \neq k_1 \neq k_2 \neq 0$, the orbit of $\mathcal{R}$ on k consists of eight points and thus $\dim H_k = 8d$. On these low-dimensional subspaces, U can be completely reduced in a straightforward way; we leave the details to the reader. ♦

8 Spontaneous Breaking of the Permutation Symmetry

In this section we consider the case when $\mathcal{O} = \mathcal{S}_m$ is the symmetric group describing permutations of the space-time points. The discrete fermion system is clearly homogeneous (see Def. 7.1). This implies that spaces $E_x(H)$ must all be isomorphic, and thus the spin dimension is constant in space-time,

$$(p_x, q_x) \;=\; (n, n) \qquad \forall x \in M\;.$$

We first give a physical motivation of our result. If a physical system is modeled by a discrete fermion system, the parameter n is known (for example, $n = 2$ for the simplest system involving Dirac spinors [3]), whereas the number m of space-time points will be very large. The number f of particles will also be very large, but much smaller than the number of space-time points (note that we also count the states of the Dirac sea as being occupied by particles, see [1, 3], and as these states lie on a 3-dimensional surface in 4-dimensional momentum space, their number scales typically like $f \sim m^{\frac{3}{4}}$). Hence the case of physical interest is

$$n \ll f \ll m .$$

Theorem 8.2 will show that in this case no discrete fermion systems with outer symmetry group $\mathcal{S}_m$ exist. In other words, the permutation symmetry of discrete space-time is necessarily destroyed by the fermionic projector, and thus a *spontaneous symmetry breaking* occurs. Our result can be understood in non-technical terms as follows: One possibility to build up fermion systems with the required symmetry is to take fermions which are "spread out" over all of space-time. The orthogonality of the fermionic states implies that the number of such states can be at most as large as the spin dimension. Hence not all the particles can be "completely delocalized" in this way. Another method is to "localize" the particles at individual space-time points. But then the permutation symmetry implies that there must be a particle at *every* space-time point, and the number of particles will be as large as m, which is impossible. Roughly speaking, the arguments below show that there is no other way of building in fermions and make the orthogonality conditions precise.

The symmetric group has two obvious one-dimensional representations: the *trivial representation* $U(\sigma) = 1$ and the *sign representation* $U(\sigma) = \mathrm{sgn}(\sigma)$. The next lemma gives a lower bound for the dimensions of all other irreducible representations.

Lemma 8.1 *Suppose that U is an irreducible representation of $\mathcal{S}_k$ on $\mathbb{C}^N$, which is neither the trivial nor the sign representation. Then*

$$N \geq \frac{k}{2} .$$

Proof. The representation theory for the symmetric group is formulated conveniently using Young diagrams (for a good introduction see for example [5, Section 2.8]). Every irreducible representation of $\mathcal{S}_k$ corresponds to a Young diagram with k positions. The Young diagram λ corresponding to U has more than one row (otherwise U would be the trivial representation) and more than one column (otherwise U would be the sign representation). The hook formula (see [5, Section 2.8 and Appendix C.5]) states that the dimension N of the representation is given by

$$N = \frac{k!}{\prod (\text{all hook lengths in } \lambda)} , \tag{67}$$

where the hook length of any position in a Young diagram is defined as the sum of positions to its right plus the number of positions below it plus one.

We consider the subdiagram μ of all the positions which are to the right of the last column having more than one position. In the following example, the subdiagram μ is marked by stars:

$$\lambda = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & * & * & * \\ \hline 3 & 4 & * & \multicolumn{2}{c}{} \\ \cline{1-3} 5 & 6 & * & \multicolumn{2}{c}{} \\ \cline{1-3} 7 & 8 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}$$

We denote the number of positions of μ by l and the number of its rows by r. Obviously, $l \geq r \geq 2$. We compute the hook lengths of all positions of μ and subsitute them into (67),

$$N = \frac{k!}{l\,(r-1)!\,(l-r)!} \, \frac{1}{\prod(\text{all hook lengths } \textit{not} \text{ in } \mu)} \,.$$

When computing the hook lengths of any position which is not in μ, at most $(l-r+1)$ of the "stared squares" of μ contribute (because at most the stared squares in one row are counted). Furthermore, ordering the positions of $\lambda \setminus \rho$ beginning from the upper left corner as indicated in the figure, one can arrange that the hook length of any position does not involve all the previous positions. Hence the hook length of the first position is at most $(k-l)+(l-r+1) = k-r+1$, the hook length of the second position is at most $k-r$, and so on. We conclude that

$$\begin{aligned} N &\geq \frac{k!}{l\,(r-1)!\,(l-r)!} \, \frac{1}{(k-r+1)(k-r)\cdots(l-r+2)} \\ &= \frac{k!\,(l-r+1)}{l\,(r-1)!\,(k-r+1)!} = \frac{l-r+1}{l} \begin{pmatrix} k \\ r-1 \end{pmatrix} . \end{aligned}$$

We consider two cases. If $k=l$, the diagrams λ and μ coincide, and since our Young diagram has more than one column, we know that $k>r$. This allows us to simplify and estimate the above inequality as follows,

$$N \geq \begin{pmatrix} k-1 \\ r-1 \end{pmatrix} \geq k-1 \geq \frac{k}{2} \,.$$

In the remaining case $k>l$, we can exploit that the number of positions in each column decreases from the left to the right to conclude that $k-l \geq r$. In the subcase $r=2$, we obtain

$$N \geq \frac{l-1}{l}\, k \geq \frac{k}{2} \,.$$

If conversely $r>2$, we have the inequalities $1 < r-1 < k$ as well as $l \geq r$ and $k-1 \geq l$. Hence

$$N \geq \frac{l-r+1}{l} \, \frac{k\,(k-1)}{2} \geq (l-r+1)\, \frac{k-1}{l} \, \frac{k}{2} \geq \frac{k}{2} \,. \qquad \blacksquare$$

Theorem 8.2 *Suppose that* $(H, <.|.>, (E_x)_{x \in M}, P)$ *is a discrete fermion system of spin dimension* (n,n) *with outer symmetry group* $\mathcal{O} = \mathcal{S}_m$. *Then the number of particles* f *satisfies one of the inequalities*

$$f \leq n \qquad \textit{or} \qquad f > \frac{m-1}{2} \,.$$

Proof. We can clearly assume that

$$n < \frac{m-1}{2} \,, \tag{68}$$

because otherwise there is nothing to prove. In particular, we can assume that $m \geq 3$. Our aim is to get a contradiction to the assumption

$$n < f \leq \frac{m-1}{2} \,. \tag{69}$$

Distinguishing the point $1 \in M$, the corresponding pinned symmetry group $\mathcal{R}$ is the group $\mathcal{S}_{m-1}$ of permutations of the other points $\{2, \ldots, M\}$. Due to our assumption $m \geq 3$, we can for every $x \in M$ choose an *even* permutation $\sigma_x \in \mathcal{O}$ with $\sigma_x(1) = x$.

According to Corollary 3.5, we can construct a representation U of the outer symmetry on H. Let V be the corresponding representation of $\mathcal{R}$ on $\hat{H} := E_1(H)$ as given by (58). According to Lemma 2.10, the irreducible subspaces of V can be chosen to be definite. Using Lemma 8.1 together with (68), one sees that V must be the direct sum of trivial and sign representations. Since $\hat{H}$ has signature (n, n), we can decompose it into a direct sum of the one-dimensional invariant subspaces

$$\hat{H} \;=\; \bigoplus_{j=1}^{n} \hat{H}_j^+ \;\oplus\; \bigoplus_{j=1}^{n} \hat{H}_j^- \tag{70}$$

where the spaces H_j^+ and H_j^- are positive and negative definite, respectively.

Proposition 5.2 allows us to reconstruct U from V. Let us consider what we get in the two cases when V is the trivial or sign representation. For the trivial representation, we can assume that $\hat{H} = \mathbb{C}$. The construction of Proposition 5.2 yields $H = \mathbb{C}^M$ and

$$E_x \;:\; (u_x)_{x \in M} \;\mapsto\; (\delta_{xy}\, u_x)_{x \in M}\;, \qquad U(\sigma) \;:\; (u_x)_{x \in M} \;\mapsto\; (u_{\sigma(x)})_{x \in M}\;. \tag{71}$$

In other words, U is the standard representation of $\mathcal{O}$ on the complex-valued functions on M. Suppose that $\langle u \rangle \subset H$ is an irreducible subspace corresponding to the trivial or the sign representation. Then, since the group elements $\sigma_x \in \mathcal{O}$ are even,

$$E_x u \;=\; E_x\, U(\sigma_x)\, u \;=\; U(\sigma_x)\, E_1 u\;,$$

and using (71) one sees that u is a multiple of the vector $(1, \ldots, 1) \in \mathbb{C}^m$. Clearly, U acts trivially on this vector. We conclude that H has a one-dimensional invariant subspace where U acts trivially, and it has no invariant subspace corresponding to the sign representation. According to Lemma 8.1, every other irreducible subspace has dimension at least $m/2$. In view of (69), the fermionic projector must vanish identically on each of these irreducible subspaces. We conclude that the subsystem corresponding to our one-dimensional representation of V must contain either zero or one particles.

In the case when V is the sign representation, we can again assume that $\hat{H} = \mathbb{C}$. The construction of Proposition 5.2 yields the same discrete space-time as for the trivial representation, but now, using that the permutations σ_x are all even,

$$U(\sigma) \;:\; (u_x)_{x \in M} \;\mapsto\; \big(\mathrm{sgn}(\sigma)\, u_{\sigma(x)}\big)_{x \in M}\;.$$

Since changing the $U(\sigma)$ by the sign $\mathrm{sgn}(\sigma)$ has no effect on whether a subspace in invariant, this representation has the same irreducible subspaces as the representation corresponding to a trivial V. Again, our subsystem must contain either zero or one particles.

The uniqueness statement in Proposition 5.2 yields that H is, in a suitable gauge, the direct sum of the scalar product spaces $\mathbb{C}^m$ obtained from each direct summand in (70). Since the spaces corresponding to the H_j^+ are positive definite, they must not contain any particles. As we saw above, each of the spaces corresponding to the H_j^- may contain at most one particle. Hence the total number particles is at most n, contradicting (69). ■

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