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Admissible Semi-Linear Representations

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ADMISSIBLE SEMI-LINEAR REPRESENTATIONS

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ABSTRACT. The category of admissible (in the appropriately modified sense of representation theory of totally disconnected groups) semi-linear representations of the automorphism group of an algebraically closed extension of infinite transcendence degree of the field of algebraic complex numbers is described.

Let k be a field of characteristic zero containing all ℓ -primary roots of unity for a prime ℓ , F be a universal domain over k , i.e., an algebraically closed extension of k of countable transcendence degree, and $G_{F/k}$ be the field automorphism group of F over k . We consider $G_{F/k}$ as a topological group with the base of open subgroups generated by $\{G_{F/k(x)} \mid x \in F\}$.

Denote by \mathcal{C} the category of smooth (with open stabilizers) F -semi-linear representations of $G_{F/k}$, i.e., F -vector spaces V endowed with an additive semi-linear ($g(fv) = gf \cdot gv$ for any $f \in F$, $g \in G_{F/k}$ and $v \in V$) action $G_{F/k} \times V \rightarrow V$ of $G_{F/k}$.

Denote by \mathcal{A} the full sub-category of \mathcal{C} whose objects V are *admissible*: $\dim_{F^U} V^U < \infty$ for any open subgroup $U \subseteq G_{F/k}$. Clearly, \mathcal{A} is an additive category and it is shown in [R] that it is a tensor (but not rigid) category. In the present paper one proves that the category \mathcal{A} is abelian (Theorem 3.6), and F is its projective object (Proposition 3.4).

Let the ideal $\mathfrak{m} \subset F \otimes_{k_0} F$ be the kernel of the multiplication map $F \otimes_{k_0} F \xrightarrow{\times} F$, where $k_0 = k \cap \overline{\mathbb{Q}}$ is the number subfield of k . Consider the powers $\mathfrak{m}^s \subseteq F \otimes_{k_0} F$ of the ideal \mathfrak{m} for all $s \geq 0$ as objects of \mathcal{C} with the F -multiplication via $F \otimes_{k_0} k_0$.

In this paper we study the category \mathcal{A} and describe it if k is a number field. Namely, in the case $k = \overline{\mathbb{Q}}$ we prove the following:

- The sum of the images of the F -tensor powers $\bigotimes_{F}^{\geq \bullet} \mathfrak{m}$ under all morphisms in \mathcal{C} defines a decreasing filtration W^\bullet on the objects of \mathcal{A} such that its graded quotients gr_W^q are finite direct sums of direct summands of $\bigotimes_F^q \Omega_F^1$ (cf. §4.1, p.17 and Theorem 4.10). This filtration is evidently functorial and multiplicative: $(W^p V_1) \otimes_F (W^q V_2) \subseteq W^{p+q}(V_1 \otimes_F V_2)$ for any $p, q \geq 0$ and any $V_1, V_2 \in \mathcal{A}$.
- \mathcal{A} is equivalent to the direct sum of the category of finite-dimensional k -vector spaces and its abelian full subcategory \mathcal{A}° with objects V such that $V^{G_{F/k}} = 0$ (Lemma 4.13).
- Any object V of \mathcal{A}° is a quotient of a direct sum of objects (of finite length) of type $\bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^s)$ for some $q, s \geq 1$ (Theorem 4.10).
- If $V \in \mathcal{A}$ is of finite type then it is of finite length and $\dim_k \text{Ext}_{\mathcal{A}}^j(V, V') < \infty$ for any $j \geq 0$ and any $V' \in \mathcal{A}$; if $V \in \mathcal{A}$ is irreducible and $\text{Ext}_{\mathcal{A}}^1(\mathfrak{m}/\mathfrak{m}^q, V) \neq 0$ for some $q \geq 2$ then $V \cong \text{Sym}_F^q \Omega_F^1$ and $\text{Ext}_{\mathcal{A}}^1(\mathfrak{m}/\mathfrak{m}^q, V) \cong k$ (Corollary 4.17).
- \mathcal{A}° has no projective objects (Corollary 4.14), but $\bigotimes_F^q \mathfrak{m}$ are its “projective pro-generators”: the functor $\text{Hom}_{\mathcal{C}}(\bigotimes_F^q \mathfrak{m}, -)$ is exact on \mathcal{A} for any q (Corollary 4.16).

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To describe the objects of \mathcal{A} , one studies first their “restrictions” to projective groups ($\cong \mathrm{PGL}_m k$), considered as subquotients of $G_{F/k}$. It is known ([R]) that such semi-linear representations are related to homogeneous vector bundles on projective spaces.

Let $n \geq 1$ be an integer, $K_n = k(\mathbb{P}_k^n)$ be the function field of an n -dimensional projective k -space \mathbb{P}_k^n and $G_n = \mathrm{Aut}(\mathbb{P}_k^n/k)$ be its automorphism group.

Fix a k -field embedding $K_n \hookrightarrow F$. We show, in particular, that if V is an admissible F -semi-linear representation of $G_{F/k}$ with no sub-objects isomorphic to F then any irreducible subquotient of the K_n -semi-linear representation $V^{G_{F/K_n}}$ of G_n is a direct summand of $\bigotimes_{K_n}^{\geq 1} \Omega_{K_n/k}^1$ (this is Theorem 2.4 and Proposition 3.4 below).

0.1. Some motivation. The study of semi-linear representations comes from the study of \mathbb{Q} -linear representations of $G_{F/k}$, that are related to geometry, cf. [R].

Let $\mathcal{S}m_G$ be the category of smooth representations of $G_{F/k}$ over k . Extending of coefficients to F gives a faithful functor $F \otimes_k : \mathcal{S}m_G \rightarrow \mathcal{C}$. It is not full: if $U \subset G_{F/k}$ is an open subgroup and $\bar{f} \in (F^\times/k^\times)^U - \{1\}$ then $[\sigma] \mapsto \sigma \bar{f} \cdot [\sigma]$ defines an element of $\mathrm{End}_{\mathcal{C}}(F[G_{F/k}/U])$ which is not in $\mathrm{End}_{\mathcal{S}m_G}(k[G_{F/k}/U])$. However, its restriction to the subcategory $\mathcal{I}_G \otimes k$ of “homotopy invariant” representations¹ is.

Lemma 0.1. *If $k = \bar{k}$ then the functor $\mathcal{I}_G \otimes k \xrightarrow{F \otimes_k} \mathcal{C}$ is fully faithful.*

Proof. More generally, let us show that $\mathrm{Hom}_{\mathcal{S}m_G}(W, W') = \mathrm{Hom}_{\mathcal{C}}(F \otimes_k W, F \otimes_k W')$ for any $W \in \mathcal{I}_G \otimes k$ and any $W' \in \mathcal{S}m_G$. Let $\varphi \in \mathrm{Hom}_G(W, F \otimes_k W')$ and $\varphi(w) = \sum_{j=1}^N f_j \otimes w_j$ for some $w \in W$, $w_j \in W'$, $f_j \in F$ and minimal possible $N \geq 1$. We have to show that $f_j \in k$.

Choose a smooth proper model X of $k(f_1, \dots, f_N)$ over k . If it is not a point, choose a generically finite rational dominant map π to a projective space Y over k which is

- well-defined at the generic points of the irreducible components D_α of the divisors of poles of f_1, \dots, f_N ,
- induces on each D_α a birational map and
- separates D_α .

Then the trace $\pi_* \varphi(w)$ has poles. On the other hand, $\pi_* \varphi(w)$ is in the image of $W^{G_{F/k}(Y)} = W^{G_{F/k}}$, so $\pi_* \varphi(w) \in (F \otimes_k W')^{G_{F/k}} = k \otimes_k (W')^{G_{F/k}}$ by Lemma 7.5 of [R]. This contradiction implies that $f_j \in k$, and therefore, $\varphi(W) \subseteq k \otimes_k W' \subseteq F \otimes_k W'$. \square

The \mathbb{Q} -linear representations of $G_{F/k}$ of particular interest are admissible representations, forming a full subcategory in $\mathcal{I}_G \otimes k$. Though tensoring with F does not transform them to admissible semi-linear representations,² there exists, at least if $k = \overline{\mathbb{Q}}$, a similar faithful functor in the opposite direction.

Namely, it is explained in Corollary 5.2 that, for any object V of \mathcal{A} and any smooth k -variety Y , embedding of the generic points of Y into F determines a locally free coherent sheaf \mathcal{V}_Y on Y . Any dominant morphism $X \xrightarrow{\pi} Y$ of smooth k -varieties induces an injection of coherent sheaves $\pi^* \mathcal{V}_Y \hookrightarrow \mathcal{V}_X$, which is an isomorphism if π is étale.

This gives an equivalence $\mathcal{S} : \mathcal{A} \xrightarrow{\sim} \{\text{“coherent” sheaves in the smooth topology}\}$, $V \mapsto (Y \mapsto \mathcal{V}_Y(Y))$. More generally, the “coherent” sheaves are contained in the category $\mathcal{F}l$ of the flat “quasi-coherent” sheaves in the smooth topology, cf. §5, p.18. For any flat

¹i.e. such that $W^{G_{F/L}} = W^{G_{F/L'}}$ for any purely transcendental extension L'/L of subfields in F

²and moreover, there are irreducible objects of \mathcal{C} outside of \mathcal{A} (Corollary 3.5).

“quasi-coherent” sheaf \mathcal{V} in the smooth topology the space $\Gamma(Y, \mathcal{V}_Y)$ is a birational invariant of a proper Y (Lemma 5.3). Then we get a left exact functor $\mathcal{F}l \xrightarrow{\Gamma} \mathcal{S}m_G$ given by $V \mapsto \varinjlim \Gamma(Y, \mathcal{V}_Y)$, where Y runs over the smooth proper models of subfields in F of finite type over k .

The functor $\Gamma \circ \mathcal{S}$ is faithful, since $\Gamma(Y', \mathcal{V}_{Y'})$ generates the (generic fibre of the) sheaf $\mathcal{V}_{Y'}$ for appropriate finite covers Y' of Y (Lemma 5.3), if \mathcal{V} is “coherent”. But it is not full, and the objects in its image are highly reducible. If $\Gamma(Y, \mathcal{V}_Y)$ has the Galois descent property then $\Gamma(V)$ is admissible. However, there is no Galois descent property in general.

0.2. Notation. Let k, F and $G_{F/k}$ be as above. For a subfield L of F we denote by \bar{L} its algebraic closure in F . We fix a transcendence basis x_1, x_2, x_3, \dots of F over k .

For each $n \geq 1$ set $Y_n = \mathbf{Spec}k[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \subset \mathbb{A}_k^n = \mathbf{Spec}k[x_1, \dots, x_n] \subset \mathbb{P}_k^n = \mathbf{Proj}k[X_0, \dots, X_n]$ with $x_j = X_j/X_0$, $K_n = k(x_1, \dots, x_n)$, $G_n = \text{Aut}(\mathbb{P}_k^n/k)$.

Let $\text{Aff}_n = G_n \cap \text{Aut}(\mathbb{A}_k^n/k)$ be the affine subgroup, and $(\text{Aff}_n)_u$ be its unipotent radical, i.e., the translation subgroup. Let $H_n = \text{Aff}_n \cap \text{Aut}(\mathbb{A}_k^n/\mathbb{A}_k^{n-1})$ be the subgroup fixing the coordinates x_1, \dots, x_{n-1} on \mathbb{A}_k^n . Let $T_n \subset G_n$ be the maximal torus acting freely on Y_n .

Denote by T_n^{tors} the torsion subgroup in T_n .

For a field extension L/K we denote by $\text{Der}(L/K)$ the Lie K -algebra of derivations of L over K . For an integer $\ell \geq 2$, the group of ℓ -th roots of unity in \bar{k} is denoted by μ_ℓ , and the corresponding cyclotomic number subfield in \bar{k} is denoted by $\mathbb{Q}(\mu_\ell)$.

0.3. Structure of the paper. As it is mentioned above, we consider the projective groups G_n as subquotients of $G_{F/k}$. In §1 we identify irreducible subquotients of “restrictions” of objects of \mathcal{A} to G_n with the generic fibres of the G_n -equivariant coherent sheaves on \mathbb{P}_k^n . Main ingredients there come from [BT] and [R]. In §2 we exclude some cases, thus showing that these irreducible subquotients are direct summands of $\bigotimes_{K_n}^\bullet \Omega_{K_n/k}^1$. In §3 we show that \mathcal{A} is abelian and calculate Ext^1 -groups between the irreducible objects of a tannakian category $\mathfrak{S}\mathcal{L}_u^n$ (defined at the beginning of §1, p.3) of semi-linear representations of G_n , containing “restrictions” of objects of \mathcal{A} to G_n . The latter part uses [LR]. After showing principal structural results on \mathcal{A} (in §4) we identify (in §5) \mathcal{A} with the category of “coherent” sheaves in smooth topology. Finally (in §6), we define a descending filtration $\mathcal{A}_{>\bullet}$ of \mathcal{A} by Serre “ideal” subcategories. Then we localize the quotients $\mathcal{A}/\mathcal{A}_{>m}$ for each $m \geq 0$ to get a tannakian subcategory of finite-dimensional semi-linear representations of $G_{F'/k}$ over F' for an algebraically closed extension F' of k in F of transcendence degree m .

1. EQUIVARIANTNESS OF IRREDUCIBLE PGL -SHEAVES

Let $\mathfrak{S}\mathcal{L}_n^u$ be the category of finite-dimensional semi-linear representations of G_n over K_n whose restrictions to the maximal torus T_n in G_n are of type $K_n \otimes_k W$ for unipotent representations W of T_n (where T_n is considered as a discrete group).

Note that $V = V^{T_n^{\text{tors}}} \otimes_k K_n$ for any $V \in \mathfrak{S}\mathcal{L}_n^u$.

In [R], for $n > 1$, a fully faithful functor $\mathfrak{S}\mathcal{L}_n^u \xrightarrow{\mathcal{S}} \{\text{coherent } G_n\text{-sheaves on } \mathbb{P}_k^n\}$ is constructed. (A G_n -sheaf is G_n -equivariant sheaf if G_n is considered as a discrete group. In other words, \mathcal{V} is a G_n -sheaf if it is endowed with a collection of isomorphisms $\alpha_g : \mathcal{V} \xrightarrow{\sim} g^*\mathcal{V}$ for each $g \in G_n$ satisfying the chain rule: $\alpha_{hg} = g^*\alpha_h \circ \alpha_g$ for any $g, h \in G_n$. The term “ G_n -equivariant” is reserved for G_n -vector bundles with algebraic G_n -action on their total

spaces.) The composition of \mathcal{S} with the generic fibre functor is the identical full embedding of \mathfrak{SL}_n^u into the category of finite-dimensional K_n -semi-linear G_n -representations.

In this section we show that the category \mathfrak{SL}_n^u is abelian and its irreducible objects are generic fibres of irreducible coherent G_n -equivariant sheaves on \mathbb{P}_k^n , i.e., direct summands of $\mathrm{Hom}_{K_n}((\Omega_{K_n/k}^n)^{\otimes M}, \bigotimes_{K_n}^{\bullet} \Omega_{K_n/k}^1)$ for appropriate integer $M \geq 0$.

Lemma 1.1. *The category \mathfrak{SL}_n^u is closed under taking K_n -semi-linear subquotients.*

Proof. Let $V \in \mathfrak{SL}_n^u$ and $0 \rightarrow V_1 \rightarrow V \xrightarrow{\pi} V_2 \rightarrow 0$ be a short exact sequence of semi-linear representations of G_n over K_n . As the k -vector space $V^{T_n^{\mathrm{tors}}}$ (of the elements in V fixed by the torsion subgroup T_n^{tors} in T_n) spans the K_n -vector space V , the k -vector space $\pi(V^{T_n^{\mathrm{tors}}}) \subseteq V_2^{T_n^{\mathrm{tors}}}$ spans the K_n -vector space V_2 .

This means that $V_2 = V_2^{T_n^{\mathrm{tors}}} \otimes_k K_n$ and $\pi(V^{T_n^{\mathrm{tors}}}) = V_2^{T_n^{\mathrm{tors}}}$.

In other words, the sequence of T_n^{tors} -invariants $0 \rightarrow V_1^{T_n^{\mathrm{tors}}} \rightarrow V^{T_n^{\mathrm{tors}}} \rightarrow V_2^{T_n^{\mathrm{tors}}} \rightarrow 0$ is exact, and extending its coefficients to K_n gives the exact sequence $0 \rightarrow V_1^{T_n^{\mathrm{tors}}} \otimes_k K_n \rightarrow V = V^{T_n^{\mathrm{tors}}} \otimes_k K_n \xrightarrow{\pi'} V_2 = V_2^{T_n^{\mathrm{tors}}} \otimes_k K_n \rightarrow 0$. As π coincides with π' , we get $V_1 = V_1^{T_n^{\mathrm{tors}}} \otimes_k K_n$.

Clearly, any subquotient of a unipotent representation of T_n is again unipotent, and thus, $V_1, V_2 \in \mathfrak{SL}_n^u$. \square

Lemma 1.2. *Let E be the total space of a vector bundle on \mathbb{P}_k^n , $\mathrm{Aut}_{\mathrm{lin}}(E)$ be the group of automorphisms of E over k inducing linear transforms between the fibres, and $\tau : G_n \rightarrow \mathrm{Aut}_{\mathrm{lin}}(E)$ be an irreducible G_n -structure on E , i.e., a discrete group homomorphism splitting the projection $\mathrm{Aut}_{\mathrm{lin}}(E) \rightarrow G_n$. Then the Zariski closure $\overline{\tau(G_n)}$ is reductive.*

Proof. Let Aut_{τ} be the kernel of the projection $\overline{\tau(G_n)} \xrightarrow{\pi} G_n$.

For each point $p \in \mathbb{P}_k^n$ let $\rho_p : R_p := \pi^{-1}(\mathrm{Stab}_p) \rightarrow \mathrm{GL}(E_p)$ be the natural representation.

As we suppose that E is an irreducible G_n -bundle, ρ_p is irreducible, since otherwise $B := \overline{\tau(G_n)}B_p \subset E$ is a G_n -subbundle for any proper R_p -invariant k -subspace $B_p \subset E_p$.

In particular, ρ_p is trivial on the unipotent radical of R_p . The unipotent radical of any algebraic group contains the unipotent radical of its arbitrary normal subgroup, so ρ_p is trivial on the unipotent radical of Aut_{τ} . As the action of Aut_{τ} on E is faithful, $\bigcap_p \ker \rho_p|_{\mathrm{Aut}_{\tau}} = \{1\}$, i.e., Aut_{τ} is reductive. As G_n is also reductive, so is $\overline{\tau(G_n)}$. \square

For a commutative finite k -algebra A denote by $R_{A/k}$ the Weil functor of restriction of scalars on A -schemes, cf. [DG], I, §1, 6.6.

We need the following particular case of Théorème 8.16 of [BT].

Theorem 1.3. *Let G be a simply connected absolutely almost simple k -group, and G' be a reductive k -group. Let $\tau : G(k) \rightarrow G'(k)$ be a homomorphism with Zariski dense image. Let G'_1, \dots, G'_m be the almost simple normal subgroups of G' .*

Then there exist finite field extensions k_i/k , field embeddings $\varphi_i : k \rightarrow k_i$, a special isogeny $\beta : \prod_{i=1}^m R_{k_i/k}^{\varphi_i} G \rightarrow G'$ (here $\varphi_i G := G \times_{k, \varphi_i} k_i$) and a homomorphism $\mu : G(k) \rightarrow Z_{G'}(k)$ such that $\beta(R_{k_i/k}^{\varphi_i} G) = G'_i$ and $\tau(h) = \mu(h) \cdot \beta(\prod_{i=1}^m \varphi_i^{\circ}(h))$ for any $h \in G(k)$ (here $\varphi_i^{\circ} : G(k) \rightarrow (R_{k_i/k}^{\varphi_i} G)(k)$ is the canonical homomorphism). \square

Corollary 1.4. *Under assumptions of Theorem 1.3, for any torus $T \subset G$ the Zariski closure of $\tau(T(k))$ is a torus in G' . \square*

Proposition 1.5. *If $n \geq 2$ then any irreducible object of \mathfrak{SL}_n^u is a direct summand of $\mathrm{Hom}_{K_n}((\Omega_{K_n/k}^n)^{\otimes M}, \bigotimes_{K_n}^{\bullet} \Omega_{K_n/k}^1)$ for an appropriate M .*

Proof. The functor \mathcal{S} , mentioned in the beginning of this §, associates to an irreducible object V of $\mathfrak{S}\mathcal{L}_n^u$ a coherent G_n -sheaf \mathcal{V} on \mathbb{P}_k^n with generic fibre V .

Let, as before, T_n be a maximal torus in G_n and $Y_n \subset \mathbb{P}_k^n$ be the n -dimensional T_n -orbit. As $V^{T_n^{\text{tors}}} = \Gamma(Y_n, \mathcal{V})^{T_n^{\text{tors}}}$, cf. [R], is a unipotent representation of T_n , Lemma 1.2 and Corollary 1.4 imply that $\Gamma(Y_n, \mathcal{V})^{T_n^{\text{tors}}}$ is a trivial representation of T_n .

In a k -basis of $V^{T_n^{\text{tors}}}$ the G_n -action on V determines a 1-cocycle $(g_\sigma) \in Z^1(G_n, \text{GL}_M K_n)$, where $M = \dim_{K_n} V$. There is an integer $N > n + 2$ and elements $\alpha_1, \dots, \alpha_N \in G_n$ such that the morphism $(T_n)^N \xrightarrow{\pi} G_n$, given by $(h_1, \dots, h_N) \mapsto \alpha_1 h_1 \alpha_1^{-1} \cdots \alpha_N h_N \alpha_N^{-1}$, is surjective. Namely, using the Gauß elimination algorithm, one shows that any element of G_n is a product of $\leq (n + 1)^2$ elementary matrices and an element of T_n . On the other hand, it follows from the identity $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b - a & b \end{pmatrix}$ that for any elementary matrix α the product $T_n \cdot \alpha T_n \alpha^{-1} \cdot T_n$ contains all elementary matrices of the same type as α . This gives a surjection $(T_n \times \prod_{i \neq j} \alpha_{ij} T_n \alpha_{ij}^{-1})^{(n+1)^2} \times T_n \xrightarrow{\times} G_n$, where α_{ij} is the elementary matrix with 1 in the i -th row and j -th column.

Then

$$g_\pi(h_1, \dots, h_N) = g_1(x) g'_1(\alpha_1 h_1(x)) g_2(\alpha_1 h_1 \alpha_1^{-1}(x)) g'_2(\alpha_1 h_1 \alpha_1^{-1} h_2(x)) \cdots \\ g_N(\alpha_1 h_1 \alpha_1^{-1} \cdots \alpha_{N-1} h_{N-1} \alpha_{N-1}^{-1}(x)) g'_N(\alpha_1 h_1 \alpha_1^{-1} \cdots \alpha_{N-1} h_{N-1} \alpha_{N-1}^{-1} \alpha_N h_N(x)),$$

where $g_j(x) := g_{\alpha_j}$ and $g'_j(x) := g_{\alpha_j^{-1}}$ for all $1 \leq j \leq N$. In other words, the lifting of the G_n -action on $\text{tot}(\mathcal{V})$ to $(T_n)^N$ -“coupling” via π determines a rational map $(T_n)^N \times \mathbb{P}_k^n \dashrightarrow \text{GL}_M k$. Clearly, it corresponds to a regular morphism $(T_n)^N \times \text{tot}(\mathcal{V}) \rightarrow \text{tot}(\mathcal{V})$ and factors through a regular morphism $G_n \times \text{tot}(\mathcal{V}) \rightarrow \text{tot}(\mathcal{V})$ of k -varieties, i.e., we see that \mathcal{V} is equivariant.

The generic fibres of irreducible G_n -equivariant sheaves on \mathbb{P}_k^n are exactly of the desired type. \square

REMARK. There can exist, a priori, non-equivariant irreducible coherent G_n -sheaves on \mathbb{P}_k^n , e.g. the extension of coefficients to $\mathcal{O}_{\mathbb{P}_k^n}$ of a non-rational representation of G_n is seemingly of this type.

2. “POSITIVITY”

In this section we show that for any admissible F -semi-linear representation V of $G_{F/k}$ any irreducible subquotient of the K_n -semi-linear representation $V^{G_{F/K_n}}$ of G is a direct summand of $\bigotimes_{K_n}^\bullet \Omega_{K_n/k}^1$.

It is shown in [R] that any finite-dimensional K_n -semi-linear G_n -representation extendable to $\text{End}(K_n/k)$, e.g. $V^{G_{F/K_n}}$, is an object of $\mathfrak{S}\mathcal{L}_n^u$. By Proposition 1.5, we only need to eliminate the negative twists by $\Omega_{K_n/k}^n$ in irreducible subquotients of $V^{G_{F/K_n}}$.

To do that we show first that the generic fibres of irreducible coherent G_n -equivariant sheaves are determined by their restrictions to the subgroup $\text{Aff}_n = G_n \cap \text{Aut}(\mathbb{A}_k^n/k)$.

Lemma 2.1. *Let Aff_n be the group of affine transformations of an affine space \mathbb{A}_k^n with the function field K_n . Then the natural morphism*

$$(1) \quad \{\text{rational } k\text{-linear } \text{Aff}_n\text{-representations}\} \xrightarrow{\otimes_k K_n} \{K_n\text{-semi-linear } \text{Aff}_n\text{-representations}\}$$

transforms isomorphism classes of irreducible k -representations of Aff_n to isomorphism classes of irreducible K_n -semi-linear representations of $\text{Aff}_n^{(1)}\mathbb{Q}$, the subgroup of Aff_n consisting of \mathbb{Q} -affine substitutions of x_1, \dots, x_n with Jacobian equal to 1.

Proof. Let W be an irreducible k -representations of Aff_n , and $U \subset W \otimes_k K_n$ a non-zero K_n -semi-linear subrepresentation of $\text{Aff}_n^{(1)}\mathbb{Q}$. Let $\alpha = \sum_{j=1}^N w_j \alpha_j \in U$ be a non-zero element with minimal possible N , where $w_j \in W$ and $\alpha_j \in K_n$. Multiplying α by an element of K_n we may assume that all α_j are polynomials: $\alpha = \sum_I w'_I x^I$. Since W is irreducible, the elements of the unipotent radical $(\text{Aff}_n)_u$ of Aff_n , i.e., σ such that $\sigma z - z \in k$ for any linear function z on \mathbb{A}_k^n , act trivially on W .

Applying an appropriate composition of difference operators $\sigma - \tau$ for some σ, τ in the unipotent radical of $\text{Aff}_n^{(1)}\mathbb{Q}$ to α , we can lower the degrees of the polynomials α_j and eventually get a non-zero element of W . As $W = W_0 \otimes_{\mathbb{Q}} k$ for an irreducible representation W_0 of $\text{Aff}_n^{(1)}\mathbb{Q}$, any non-zero element of W generates $W \otimes_k K_n$, which means that $U = W \otimes_k K_n$. \square

Corollary 2.2. *Let Aff_n , $(\text{Aff}_n)_u$, \mathbb{A}_k^n and K_n be as in Lemma 2.1. Then $V \mapsto V^{(\text{Aff}_n)_u}$ gives a natural bijection*

$$(2) \quad \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{irreducible } G_n\text{-subrepresentations in} \\ \bigoplus_M \text{Hom}_K((\Omega_{K_n/k}^n)^{\otimes M}, \bigotimes_{K_n} \Omega_{K_n/k}^1) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{irreducible rational } k\text{-linear} \\ \text{Aff}_n\text{-representations} \end{array} \right\}$$

such that its composition with the morphism (1) is the inclusion map. \square

Let W be an $(n+1)$ -dimensional k -vector space, $L \subset W$ be a one-dimensional subspace, and $H_{\text{lin}} = \ker[\text{GL}(W, L) \rightarrow \text{GL}(W/L)] \cong k^\times \times \text{Hom}(W/L, L)$ be the group preserving L and inducing the identity automorphism of W/L .

Lemma 2.3. *For any Young diagram λ with no columns of height $\geq n+1$ one has*

$$(S^\lambda W^\vee \otimes_k (\det W)^{\otimes s})^{H_{\text{lin}}} = \begin{cases} S^\lambda(W/L)^\vee & \text{if } s = 0, \\ 0 & \text{otherwise} \end{cases}$$

Proof. Denote by $X = AF(W) \cong \text{GL}(W)/R_u(B)$, the variety of complete affine flags in W . An affine flag is a filtration $W_\bullet = (0 = W_0 \subset W_1 \subset W_2 \subset \dots \subset W_{n+1} = W)$ with $\dim_k W_j = j$ and a collection of $l_j \in W_j/W_{j-1} - \{0\}$.

Let $Y \cong \text{GL}(W)/B$ be the variety of complete linear flags in W . Then the natural projection $X \xrightarrow{\pi} Y$ is a principal $(\mathbb{G}_m)^{n+1}$ -bundle, and there is a decomposition $\pi_* \mathcal{O}_X = \bigoplus_\mu \mathcal{M}(\mu)$ into a direct sum of invertible sheaves on Y , where μ runs over the group \mathbb{Z}^{n+1} of characters of $(\mathbb{G}_m)^{n+1}$, so $\mathcal{O}(X) = \bigoplus_\mu \Gamma(Y, \mathcal{M}(\mu))$.

Set $X^\circ = \{(V_\bullet, l_\bullet) \in X \mid V_n \cap L = 0 \Leftrightarrow "l_{n+1} \in L"\}$. Then reduction modulo L defines a principal $L^{\oplus n} \times \mathbb{G}_m$ -bundle $X^\circ \rightarrow AF(W/L)$, $(l_1, \dots, l_{n+1}) \mapsto (l_1, \dots, l_n)$.

Let $X^\circ \xrightarrow{"l_{n+1}"} L - \{0\}$ be the natural H - (or \mathbb{G}_m -) equivariant map, and \bar{l}_{n+1} be the composition of $"l_{n+1}"$ with a fixed isomorphism $L - \{0\} \cong \mathbb{G}_m$.

Set $SH := H_{\text{lin}} \cap \text{SL}(W) = \text{Hom}(W/L, L)$. Then $\mathcal{O}(X)^{SH} = \mathcal{O}(X) \cap \mathcal{O}(X^\circ)^{SH} = \mathcal{O}(X) \cap \mathcal{O}(AF(W/L))[\bar{l}_{n+1}, \bar{l}_{n+1}^{-1}]$, so $\mathcal{O}(X)(\mu)^{SH} = \mathcal{O}(X) \cap \mathcal{O}(AF(W/L))(\mu') \bar{l}_{n+1}^{\mu_{n+1}}$, where $\mu' \in \mathbb{Z}^n$ is the restriction of μ to the first n multiples of $(\mathbb{G}_m)^{n+1}$.

For any μ this is an irreducible representation of $\text{GL}(W/L)$, and thus, $\mathcal{O}(X)(\mu)^{SH} = \mathcal{O}(AF(W/L))(\mu') \bar{l}_{n+1}^{\mu_{n+1}}$ if $\mathcal{O}(X)(\mu) \neq 0$.

As any irreducible representation of $\mathrm{SL}(W)$ coincides with $\mathcal{O}(X)(\mu)$ for some μ , this implies that $(S^\lambda W^\vee)^{S^H} = S^\lambda(W/L)^\vee$. \square

Theorem 2.4. *For any F -semi-linear $G_{F/k}$ -representation $V \in \mathcal{A}$ any irreducible subquotient of the K_n -semi-linear G_n -representation $V^{G_{F/K_n}}$ is a direct summand of $\bigotimes_{K_n}^\bullet \Omega_{K_n/k}^1$.*

Proof. Let $W = \mathbb{A}_k^{n+1}$ be the vector space with coordinates x_1, \dots, x_{n+1} , so $k(W) = K_{n+1}$. By Proposition 1.5 and Corollary 2.2, the restrictions to $\mathrm{Aff}_{n+1} = \mathrm{Aff}(W)$ of irreducible subquotients of the K_{n+1} -semi-linear G_{n+1} -representation $V^{G_{F/K_{n+1}}}$ are of type $(S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K_{n+1}$ for a Young diagram λ with no columns of height $n+1$ and some integer s , where Aff_{n+1} acts on W via its reductive quotient $\mathrm{GL}(W)$.

Let $H \subset \mathrm{Aff}_{n+1}$ be the subgroup fixing the functionals x_1, \dots, x_n in W^\vee vanishing on L . Let Aff_u be the unipotent radical of Aff_{n+1} , i.e. the group of translations of W .

Then the restrictions to Aff_n of the irreducible subquotients of the K_n -semi-linear G_n -representation $V^{G_{F/K_n}}$ are contained in $((S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K_{n+1})^H$. As $H \cap \mathrm{Aff}_u = \langle 1 \rangle_k \cong k$, we get $((S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K_{n+1})^{H \cap \mathrm{Aff}_u} = (S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K_n$, so $((S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K_{n+1})^H = (S^\lambda W^\vee \otimes (\det W)^{\otimes s})^H \otimes_k K_n$.

By Lemma 2.3, $(S^\lambda W^\vee \otimes (\det W)^{\otimes s})^H$ coincides with $S^\lambda(W/L)^\vee$ if $s = 0$, and vanishes otherwise. This means that any representation of Aff_n obtained this way is a direct summand of the tensor algebra of the representation $(W/L)^\vee = (\Omega_{K_n/k}^1)^{\{\text{translations}\}}$ of $\mathrm{GL}_n k$. As any irreducible subquotient U of the K_n -semi-linear G_n -representation $V^{G_{F/K_n}}$ is determined by its restriction $U|_{\mathrm{Aff}_n}$ to Aff_n and $U|_{\mathrm{Aff}_n}$ is a direct summand of $\bigotimes_{K_n}^\bullet \Omega_{K_n/k}^1$, the same holds for U . \square

3. EXTENSIONS IN $\mathfrak{S}\mathcal{L}_n^u$ AND IN \mathcal{A}

For an integer $\ell \geq 2$ such that $\mu_\ell \subset k$ (see §0.2), denote by $\mathrm{Aff}_n^{(\ell)}\mathbb{Q}$ the subgroup of Aff_n consisting of the $\mathbb{Q}(\mu_\ell)$ -affine substitutions of x_1, \dots, x_n with Jacobian in μ_ℓ : $x_i \mapsto \sum_{j=1}^n a_{ij}x_j + b_i$, where $a_{ij}, b_i \in \mathbb{Q}(\mu_\ell) \subset k$ and $\det(a_{ij}) \in \mu_\ell$; and by $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$ the subgroup of index ℓ consisting of elements with Jacobian equal to 1: $\det(a_{ij}) = 1$.

Lemma 3.1. *Let $n, \ell \geq 2$ be integers. Assume that $\mu_\ell \subset k$. Let U_0 be the unipotent radical of $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$. Then for any object $V \in \mathfrak{S}\mathcal{L}_n^u$ there is a rational representation W of the reductive quotient $\mathrm{SL}_n\mathbb{Q}(\mu_\ell) = \mathrm{SAff}_n^{(\ell)}\mathbb{Q}/U_0$ of $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$, and an isomorphism of semi-linear $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$ -modules $W \otimes_{\mathbb{Q}(\mu_\ell)} K_n \xrightarrow{\sim} V$.*

Irreducible rational representations of $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$ with coefficients extended to K_n are irreducible semi-linear representations of $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$ over K_n . In particular, any extension in $\mathfrak{S}\mathcal{L}_n^u$ splits as an extension of K_n -semi-linear representations of $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$.

Proof. It is shown in Lemma 6.3 (1) of [R] that $H^0(U_0, -)$ is a fibre functor on $\mathfrak{S}\mathcal{L}_n^u$ independent of ℓ , so $V = V^{U_0} \otimes_k K_n$, i.e. the restriction of V to $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$ is a k -linear representation V^{U_0} of $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$ with coefficients extended to K_n , for any $V \in \mathfrak{S}\mathcal{L}_n^u$.

As it follows from Proposition 1.5, the irreducible subquotients V_α of V restricted to $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$ are of the form $W_\alpha \otimes_{\mathbb{Q}(\mu_\ell)} K_n$, where W_α are rational irreducible representations of $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$. Then the irreducible subquotients of V^{U_0} are $V_\alpha^{U_0} = W_\alpha \otimes_{\mathbb{Q}(\mu_\ell)} k$, and V_α are irreducible semi-linear representations of $\mathrm{SAff}_n^{(\ell)}\mathbb{Q}$ by Lemma 2.1.

If V^{U_0} is not semi-simple then it admits a non-semi-simple subquotient W of length 2. Let in Theorem 3.9 $\kappa = \mathbb{Q}(\mu_\ell)$, $K = k$, $G = \mathrm{SL}_n$, \mathcal{G} be the Zariski closure of the

image of $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$ in $\mathrm{GL}_k(W)$ and let τ be given by the $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$ -action on W . Then the unipotent radical of \mathcal{G} is commutative. As the derivations of $\kappa = \mathbb{Q}(\mu_\ell)$ are zero, we see that the k -linear representation W of $\mathrm{SL}_n\mathbb{Q}(\mu_\ell)$ is semi-simple. \square

REMARKS. 1. Using Theorems 1.3 and 3.9 it is not hard to show that any representation of SL_nK over any field of characteristic zero is semi-simple for any number field K .

2. Let $V = \Omega_{K_n}^1/\Lambda \otimes_k K_n$, where $\Lambda \subset \Omega_k^1$ is a proper k -subspace. Let the extension $0 \rightarrow V \rightarrow U \rightarrow K_n \rightarrow 0$ be given by the cocycle $(\omega_\sigma = d \log \frac{\sigma\omega}{\omega}) \in Z^1(G_n, V)$, where $\omega = dx_1 \wedge \cdots \wedge dx_n \in \Omega_{K_n/k}^n$. Then the restriction of (ω_σ) to $\mathrm{GL}_n k$ is non-trivial.

3. The convolution with the Euler vector field $\sum_{j=1}^n x_j \partial/\partial x_j$ defines a $\mathrm{GL}_n k$ -equivariant morphism $\Omega_{K_n/k}^1 \rightarrow K_n$ given by $dx_j \mapsto x_j$. It is non-split for $n \geq 3$, since $(\Omega_{K_n/k}^1)^{\mathrm{SL}_n\mathbb{Q}} = 0$.

Lemma 3.2. *Let $n, \ell \geq 2$ and s be some integers, and λ be a Young diagram with columns of height $< n$ such that ℓ does not divide $s + \frac{|\lambda|}{n-1}$, if λ is rectangular of height $n-1$ and non-empty. Let $V = S_{K_n}^\lambda \Omega_{K_n/k}^1 \otimes_{K_n} (\Omega_{K_n/k}^n)^{\otimes s}$.*

Then $(V^{\mathcal{H}_n^{(\ell)}})^{\mathrm{Aff}_n^{(\ell)}\mathbb{Q}} = V^{\mathrm{Aff}_n^{(\ell)}\mathbb{Q}}$, where $\mathcal{H}_n^{(\ell)} := G_{K_n/K_{n-1}} \cap \mathrm{Aff}_n^{(\ell)}\mathbb{Q}$.

Proof. Let W be the k -span of dx_1, \dots, dx_n in $\Omega_{K_n/k}^1$. Then $S_{K_n}^\lambda \Omega_{K_n/k}^1 = S_k^\lambda W \otimes_k K_n$ and $\Omega_{K_n/k}^n = \det_k W \otimes_k K_n$. Set $S\mathcal{H}_n^{(\ell)} = \mathcal{H}_n^{(\ell)} \cap \mathrm{SAff}_n^{(\ell)}\mathbb{Q}$. Then $\mathcal{H}_n^{(\ell)} \cong \mu_\ell \times S\mathcal{H}_n^{(\ell)}$, and therefore, $V^{\mathcal{H}_n^{(\ell)}} = (V^{S\mathcal{H}_n^{(\ell)}})^{\mu_\ell}$.

One has $V^{S\mathcal{H}_n^{(\ell)}} = (S_k^\lambda W \otimes_k K_n)^{S\mathcal{H}_n^{(\ell)}} \otimes_k (\det_k W)^{\otimes s}$. As the intersection of the unipotent radical of $\mathrm{Aff}_n^{(\ell)}\mathbb{Q}$ with $S\mathcal{H}_n^{(\ell)}$ (i.e. the $\mathbb{Q}(\mu_\ell)$ -translations of x_n) acts trivially on $S_k^\lambda W$ and fixes exactly K_{n-1} in K_n , if $n \geq 1$, we get

$$V^{S\mathcal{H}_n^{(\ell)}} = (S_k^\lambda W)^{S\mathcal{H}_n^{(\ell)}} \otimes_k (\det_k W)^{\otimes s} \otimes_k K_{n-1} = S_k^\lambda (W^{S\mathcal{H}_n^{(\ell)}}) \otimes_k (\det_k W)^{\otimes s} \otimes_k K_{n-1}.$$

Then

$$\begin{aligned} V^{\mathcal{H}_n^{(\ell)}} &= S_k^\lambda (W^{S\mathcal{H}_n^{(\ell)}}) \otimes_k ((\det_k W)^{\otimes s})^{\mu_\ell} \otimes_k K_{n-1} \\ &= \begin{cases} S_{K_{n-1}}^\lambda \Omega_{K_{n-1}/k}^1 \otimes_{K_{n-1}} (\Omega_{K_{n-1}/k}^{n-1})^{\otimes s} & \text{if } \ell|s \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand, $V^{\mathrm{Aff}_n^{(\ell)}\mathbb{Q}} = (S_k^\lambda W \otimes_k (\det_k W)^{\otimes s} \otimes_k K_n)^{\mathrm{Aff}_n^{(\ell)}\mathbb{Q}}$ coincides with $(S_k^\lambda W \otimes_k (\det_k W)^{\otimes s})^{\mathrm{Aff}_n^{(\ell)}\mathbb{Q}}$, since the unipotent radical of $\mathrm{Aff}_n^{(\ell)}\mathbb{Q}$ acts trivially on $S_k^\lambda W \otimes_k (\det_k W)^{\otimes s}$ and fixes exactly k in K_n . Thus, for $n \geq 1$, we get $V^{\mathrm{Aff}_n^{(\ell)}\mathbb{Q}} = \begin{cases} k & \text{if } \lambda = 0 \text{ and } \ell|s, \\ 0 & \text{otherwise.} \end{cases}$ This

implies that $(V^{\mathcal{H}_n^{(\ell)}})^{\mathrm{Aff}_n^{(\ell)}\mathbb{Q}} = \begin{cases} k & \text{if } \lambda \text{ is rectangular of height } n-1 \text{ and } \ell|(s + \frac{|\lambda|}{n-1}), \\ 0 & \text{otherwise} \end{cases}$ for $n \geq 2$ (assuming that empty λ is $(0 \times (n-1))$ -rectangular). \square

Lemma 3.3 ([R], Lemma 7.1). *Let $n > m \geq 0$ be integers and H be a subgroup of $G_{F/k}$ preserving K_n and projecting onto a subgroup of $G_{K_n/k}$ containing the permutation group of the set $\{x_1, \dots, x_n\}$. Then the subgroup in $G_{F/k}$ generated by G_{F/K_m} and H is dense. \square*

We note that $\mathrm{Aff}_n^{(\ell)}\mathbb{Q} \subset G_n \subset G_{K_n/k}$ does indeed contain the permutation group of the set $\{x_1, \dots, x_n\}$ for any even $\ell \geq 2$.

For any $U \in \mathcal{A}$ and $m \geq 0$ set $U_m = U^{G_{F/K_m}}$. Using smooth cochains, one defines the smooth cohomology $H_{\text{smooth}}^j(G_{F/k}, -) := \text{Ext}_{\mathcal{S}m_{G_{F/k}}}^j(\mathbb{Q}, -)$.

Proposition 3.4. *If $U \in \mathcal{A}$ and there is a subquotient of $U_n \in \mathfrak{S}\mathcal{L}_n^u$ isomorphic to K_n then there is an embedding $F \hookrightarrow U$ in \mathcal{A} . One has $H_{\text{smooth}}^1(G_{F/k}, V) = 0$ for any $V \in \mathcal{A}$.*

Proof. By Lemma 3.3, $U^{G_{F/k}} = U_{n+1}^{\text{Aff}^{(\ell)}\mathbb{Q}} \cap U_n$ for any even $\ell \geq 2$. By Theorem 2.4, Lemma 3.1 and Lemma 3.2, $(U_{n+1}^{\mathcal{H}_{n+1}^{(\ell)}})^{\text{Aff}^{(\ell)}\mathbb{Q}} = U_{n+1}^{\text{Aff}^{(\ell)}\mathbb{Q}}$ for any $n \geq 1$ and any sufficiently big ℓ (where $\mathcal{H}_n^{(\ell)}$ is defined in Lemma 3.2). Then, as $U_n \subseteq U_{n+1}^{\mathcal{H}_{n+1}^{(\ell)}}$, one has $U^{G_{F/k}} = U_n^{\text{Aff}^{(\ell)}\mathbb{Q}}$ for any sufficiently big even ℓ , and thus, $U^{G_{F/k}} \neq 0$ if there is a subquotient of $U_n \in \mathfrak{S}\mathcal{L}_n^u$ isomorphic to K_n .

Clearly, $\text{Ext}_{\mathcal{S}m_{G_{F/k}}}^j(\mathbb{Q}, -) = \text{Ext}_{\mathcal{C}}^j(F, -)$ on \mathcal{C} for any $j \geq 0$,³ so we have to show that any smooth F -semi-linear extension $0 \rightarrow V \rightarrow U \rightarrow F \rightarrow 0$ splits.

Fix $u \in U$ in the preimage of $1 \in F$. The stabilizer of u contains a subgroup of type $G_{F/L}$ such that the elements of L are algebraic over K_m for some $m > 1$. Then the normalized trace $\text{tr}_{/K_m} u \in U_m$ belongs again to the preimage of $1 \in K_m$, so U_m surjects onto K_m .

By Theorem 2.4 and Lemma 3.1 the semi-linear representation U_m of $\text{Aff}_m^{(\ell)}\mathbb{Q}$ over K_m splits as $K_m \oplus V_m$, and thus, $U^{G_{F/k}}$ projects onto k . Then sending $1 \in k \subset F$ to one of its preimages in $U^{G_{F/k}}$ extends to a splitting of $U \rightarrow F$. \square

Corollary 3.5. *For an integer $n \geq 1$ let $H \subseteq G_{F/k}$ be a subgroup containing G_{F/K_n} such that $G_{F/\overline{K_n}}$ is a normal subgroup in H . Consider $H/G_{F/K_n}$ as a subset in the set $\{K_n \xrightarrow{/k} \overline{K_n}\}$ of field embeddings of K_n into its algebraic closure in F over k . Suppose that $H/G_{F/K_n}$ contains Aff_n . Let $V = F[G_{F/k}/H]^\circ \in \mathcal{C}$ consist of formal degree-zero F -linear combinations of elements in $G_{F/k}/H$.⁴ Then any quotient of V which lies in \mathcal{A} is zero.*

Proof. V is generated by $\alpha = [1] - [\sigma] \in V_{2n}^{(\langle \text{Aff}_{2n} \rangle_u, T_{2n})}$, where σ sends x_j to x_{2n+1-j} for each $1 \leq j \leq 2n$. Any admissible semi-linear quotient of V is generated by the image of α , which is, by Propositions 1.5 and 3.4, fixed by the whole $G_{F/k}$. On the other hand, $\sigma\alpha = -\alpha$, so any admissible semi-linear quotient of V is zero. \square

Theorem 3.6. *The category \mathcal{A} is abelian. The functor $H^0(G_{F/L}, -)$ is exact on \mathcal{A} for any subfield L in F containing k .*

Proof. We have to check that \mathcal{A} is stable under taking quotients. Let $V \in \mathcal{A}$ and $V \xrightarrow{\pi} V'$ be a surjection of F -semi-linear representations of $G_{F/k}$. By Proposition 3.4, for any $K \subset F$ of finite type over k and any $v \in (V')^{G_{F/\overline{K}}} - \{0\}$, the extension $0 \rightarrow \ker \pi \rightarrow \pi^{-1}(F \cdot v) \rightarrow F \rightarrow 0$ of F -semi-linear representations of $G_{F/\overline{K}}$ splits. This implies that

³Any class in $\text{Ext}_{\mathcal{C}}^j(F, V)$ represented by $0 \rightarrow V \rightarrow V_j \rightarrow \cdots \rightarrow V_1 \rightarrow F \rightarrow 0$ is sent to the class of $0 \rightarrow V \rightarrow V_j \rightarrow \cdots \rightarrow V_2 \rightarrow V_1 \times_F \mathbb{Q} \rightarrow \mathbb{Q} \rightarrow 0$ in $\text{Ext}_{\mathcal{S}m_{G_{F/k}}}^j(\mathbb{Q}, V)$. Conversely, the class of $0 \rightarrow V \rightarrow U_j \rightarrow \cdots \rightarrow U_1 \rightarrow \mathbb{Q} \rightarrow 0$ in $\text{Ext}_{\mathcal{S}m_{G_{F/k}}}^j(\mathbb{Q}, V)$ is sent to the class of $0 \rightarrow V \rightarrow (U_j \otimes F)/K \rightarrow U_{j-1} \otimes F \rightarrow \cdots \rightarrow U_1 \otimes F \rightarrow F \rightarrow 0$, where K is the kernel of the surjection $\text{forget}(V) \otimes F \rightarrow V$ and $\text{forget} : \mathcal{C} \rightarrow \mathcal{S}m_{G_{F/k}}$ is the forgetful functor.

⁴In particular, if $H = G_{\{F, \overline{K_n}\}/k}$ then $V \cong F[\{L \subset F \mid L \cong \overline{K_n}\}]^\circ$ consists of formal degree-zero F -linear combinations of algebraically closed subfields in F of transcendence degree n over k .

the natural projection $V^{G_{F/K}} \xrightarrow{\pi_K} (V')^{G_{F/K}}$ is surjective, and thus, V' is also an admissible semi-linear representation.

The functor $H^0(G_{F/L}, -)$ on \mathcal{A} is the composition of the forgetful functor $\Phi : \mathcal{A}_k \rightarrow \mathcal{C}_L$, the functor $H^0(G_{F/\bar{L}}, -)$ on \mathcal{C}_L and the exact functor $H^0(G_{\bar{L}/L}, -)$ on $\mathcal{S}m_{G_{\bar{L}/L}}$. If L is of finite transcendence degree over k then the forgetful functor Φ factors through $\mathcal{A}_{\bar{L}}$, so the composition $H^0(G_{F/\bar{L}}, -) \circ \Phi$ is exact. If L is of infinite transcendence degree over k then $H^0(G_{F/\bar{L}}, -)$ induces an equivalence of categories $\mathcal{S}m_{G_{F/k}} \xrightarrow{\sim} \mathcal{S}m_{G_{\bar{L}/k}}$, so $H^0(G_{F/\bar{L}}, -)$ is also exact. \square

Corollary 3.7. $H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k, \text{closed}}^\bullet) = H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k, \text{exact}}^\bullet) = 0$.

Proof. By Proposition 3.4, $H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k}^\bullet) = 0$. Then a piece of the long cohomological sequence of the short exact sequence $0 \rightarrow \Omega_{F/k, \text{closed}}^q \rightarrow \Omega_{F/k}^q \xrightarrow{d} \Omega_{F/k, \text{exact}}^{q+1} \rightarrow 0$ looks as $H^0(G_{F/k}, \Omega_{F/k, \text{exact}}^{q+1}) \rightarrow H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k, \text{closed}}^q) \rightarrow H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k}^q) = 0$. Evidently, $H^0(G_{F/k}, \Omega_{F/k, \text{exact}}^{q+1}) = 0$, so $H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k, \text{closed}}^q) = 0$.

Clearly, $H^0(G_{F/k}, H^q_{\text{dR}/k}(F)) = 0$.⁵ A piece of the long cohomological sequence of short exact sequence $0 \rightarrow \Omega_{F/k, \text{exact}}^q \rightarrow \Omega_{F/k, \text{closed}}^q \rightarrow H^q_{\text{dR}/k}(F) \rightarrow 0$ looks as

$$H^0(G_{F/k}, H^q_{\text{dR}/k}(F)) \rightarrow H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k, \text{exact}}^q) \rightarrow H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k, \text{closed}}^q) = 0,$$

so $H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k, \text{exact}}^\bullet) = 0$. \square

3.1. Extensions in $\mathfrak{S}\Omega_n^u$. Now we need the following particular case of Bott's theorem.

Theorem 3.8 ([B], cf. also [D]). *If \mathcal{V} is an irreducible G_n -equivariant coherent sheaf on \mathbb{P}_k^n then there exists at most one $j \geq 0$ such that $H^j(\mathbb{P}_k^n, \mathcal{V}) \neq 0$. If $H^j(\mathbb{P}_k^n, \mathcal{V})^{G_n} \neq 0$ then $\mathcal{V} \cong \Omega_{\mathbb{P}_k^n/k}^j$.* \square

We also need the following explicit description of the homomorphisms in the case of commutative unipotent radicals of the target groups. It confirms general expectations, sketched in Remark 8.19 of [BT] and in [T], §5.1.

Theorem 3.9 ([LR], Theorem 3). *Let G be a simple simply connected Chevalley group over a field κ of characteristic zero. Let \mathcal{G} be a connected algebraic group over a field extension K of κ . Let $\tau : G(\kappa) \rightarrow \mathcal{G}(K)$ be a homomorphism with Zariski dense image. Assume that the unipotent radical \mathcal{G}_u of \mathcal{G} is commutative and the composition $G(\kappa) \xrightarrow{\tau} \mathcal{G}(K) \rightarrow G'(K)$, where $G' = \mathcal{G}/\mathcal{G}_u$, is induced by a rational K -homomorphism $\lambda : G \times_\kappa K \rightarrow G'$.*

Then \mathcal{G}_u splits over a finite field extension L/K into a direct sum of r copies of the adjoint representation of G' , so $r = \dim \mathcal{G}_u / \dim G'$.

Let $A = \kappa[\varepsilon_1, \dots, \varepsilon_r]/(\varepsilon_1^2, \dots, \varepsilon_r^2)$ and $\mathcal{H} = R_{A/\kappa}(G \times_\kappa A) \cong G \ltimes \mathfrak{g}^{\oplus r}$, where $\mathfrak{g} = \text{Lie}(G)$ is the adjoint representation of G .

Then there exist derivations $\delta_1, \dots, \delta_r : \kappa \rightarrow L$ and an L -isogeny $\mu : \mathcal{H} \times_\kappa L \rightarrow \mathcal{G} \times_\kappa L$ such that $\tau = \mu \circ \eta_\delta$, where $\eta_\delta : G(\kappa) \rightarrow \mathcal{H}(L)$ is induced by the ring embedding $\text{id} + \sum_{j=1}^r \varepsilon_j \delta_j : \kappa \rightarrow A \otimes_\kappa L$. \square

⁵Let $\omega \in \Omega_{A/k}^q \subset \Omega_{F/k}^q$ represent a $G_{F/k}$ -fixed element for a smooth finitely generated k -subalgebra $A \subset F$. Fix $\sigma \in G_{F/k}$ such that A and $\sigma(A)$ are algebraically independent over k . Then $\omega - \sigma\omega = d\eta$ for some $\eta \in \Omega_{B/k}^{q-1}$, where $B \subset F$ is a smooth finitely generated $(A \otimes_k \sigma(A))$ -subalgebra. Fix a k -algebra homomorphism $\varphi : \sigma(A) \rightarrow \bar{k} \subset F$ and extend $\text{id} \cdot \varphi : A \otimes_k \sigma(A) \rightarrow A \otimes_k \bar{k} \subset F$ to $\psi : B \rightarrow F$. Then ψ induces a morphism of differential graded k -algebras $\psi_* : \Omega_{B/k}^\bullet \rightarrow \Omega_{F/k}^\bullet$ identical on $\Omega_{A/k}^\bullet$, so $\omega = d\psi_*(\eta)$.

Lemma 3.10. *Let $n \geq 2$. Suppose that $\text{Ext}_{\mathfrak{S}\mathcal{L}_n^u}^1(K_n, V_\circ) \neq 0$ for some irreducible object V_\circ of $\mathfrak{S}\mathcal{L}_n^u$. Then either $V_\circ \cong \Omega_{K_n/k}^1$, or $V_\circ \cong \text{Der}(K_n/k)$. One has $\text{Ext}_{\mathfrak{S}\mathcal{L}_n^u}^1(K_n, \Omega_{K_n/k}^1) = k$ and $\text{Ext}_{\mathfrak{S}\mathcal{L}_n^u}^1(K_n, \text{Der}(K_n/k)) = \text{Der}(k)$.*

Proof. Let $\mathcal{V} = \mathcal{S}(V_\circ)$ be the irreducible coherent G_n -equivariant sheaf on $\mathbb{P}(Q) = \mathbb{P}_k^n$ with the generic fibre V_\circ , and let $0 \rightarrow V_\circ \rightarrow V \rightarrow K_n \rightarrow 0$ be an extension in $\mathfrak{S}\mathcal{L}_n^u$.

Suppose that the short exact sequence $0 \rightarrow \mathcal{V} \rightarrow \mathcal{S}(V) \rightarrow \mathcal{O} \rightarrow 0$ of coherent sheaves on $\mathbb{P}(Q)$ splits. Let E be the total space of $\mathcal{S}(V) \cong \mathcal{O} \oplus \mathcal{V}$. Then, as $\text{Aut}(\mathcal{S}(V), \mathcal{V}) \cong (\mathbb{G}_m \times \mathbb{G}_m) \ltimes \Gamma(\mathbb{P}_k^n, \mathcal{V})$, the G_n -structure on V corresponds to a splitting of the sequence

$$(3) \quad 1 \rightarrow (\mathbb{G}_m \times \mathbb{G}_m) \ltimes \Gamma(\mathbb{P}(Q), \mathcal{V}) \rightarrow \text{Aut}_{\text{lin}}(E, \text{tot}(\mathcal{V})) \rightarrow G_n \rightarrow 1.$$

As $H^1(G_n, \mathbb{G}_m \times \mathbb{G}_m) = 1$, Theorem 3.9 (with $G = \text{SL}_{n+1}k$, $G' = G_n$ and $\mathcal{G}_u \subseteq \Gamma(\mathbb{P}_k^n, \mathcal{V})$) implies that a non-standard splitting of (3) can exist only if $\Gamma(\mathbb{P}_k^n, \mathcal{V})$ is isomorphic to the adjoint representation of G_n , i.e., if $\mathcal{V} \cong \mathcal{T}_{\mathbb{P}_k^n/k}$. The identity $\text{Ext}_{\mathfrak{S}\mathcal{L}_n^u}^1(K_n, \text{Der}(K_n/k)) = \text{Der}(k)$ follows also from Theorem 3.9.

If $\mathcal{V} \cong \Omega_{\mathbb{P}_k^n/k}^1$ then the target of the homomorphism $\text{Ext}_{\mathfrak{S}\mathcal{L}_n^u}^1(K_n, \mathcal{V}_\circ) \xrightarrow{\alpha} \text{Ext}_{\mathcal{O}}^1(\mathcal{O}_{\mathbb{P}_k^n}, \mathcal{V}) = k$ induced by the functor \mathcal{S} is generated by the class of the Euler extension $0 \rightarrow \Omega_{\mathbb{P}(Q)/k}^1 \rightarrow Q^\vee \otimes_k \mathcal{O}_{\mathbb{P}(Q)}(-1) \rightarrow \mathcal{O}_{\mathbb{P}(Q)} \rightarrow 0$. Let E be the total space of the vector bundle with the sheaf of sections $Q^\vee \otimes_k \mathcal{O}(-1)$. Any G_n -structure on the middle term of this extension corresponding to an element of $\text{Ext}_{\mathfrak{S}\mathcal{L}_n^u}^1(K_n, \Omega_{K_n/k}^1)$ is a splitting of the short exact sequence

$$(4) \quad 1 \rightarrow \text{Aut}(Q^\vee \otimes_k \mathcal{O}(-1), \Omega_{\mathbb{P}(Q)/k}^1) \rightarrow \text{Aut}_{\text{lin}}(E, \text{tot}(\Omega_{\mathbb{P}(Q)/k}^1)) \rightarrow G_n \rightarrow 1.$$

As any point of $\mathbb{P}(Q)$ determines a hyperplane in Q^\vee , the group $\text{Aut}(Q^\vee \otimes_k \mathcal{O}(-1), \Omega_{\mathbb{P}(Q)}^1)$ coincides with the subgroup of $\text{GL}(Q)$ stabilizing all hyperplanes in Q^\vee , i.e., with the centre \mathbb{G}_m of $\text{GL}(Q)$. Then $\text{Aut}_{\text{lin}}(E, \text{tot}(\Omega_{\mathbb{P}(Q)/k}^1))$ is a central \mathbb{G}_m -extension of G_n , so the splitting of (4) is unique and corresponds to the usual G_n -equivariant structure. \square

Corollary 3.11. *If $k = \overline{\mathbb{Q}}$ then $\mathfrak{S}\mathcal{L}_n^u$ is equivalent to the category of G_n -equivariant vector bundles on \mathbb{P}_k^n .*

Proof. The category of G_n -equivariant vector bundles on \mathbb{P}_k^n is a full sub-category of $\mathfrak{S}\mathcal{L}_n^u$ with the same irreducible objects. As it is mentioned at the beginning of §1, p.3, the objects of $\mathfrak{S}\mathcal{L}_n^u$ are generic fibres of coherent G_n -sheaves on \mathbb{P}_k^n . Suppose that $V \in \mathfrak{S}\mathcal{L}_n^u$ is the generic fibre of a non-equivariant vector bundle on \mathbb{P}_k^n of minimal possible rank. Then it fits into an exact sequence $0 \rightarrow B \rightarrow V \rightarrow A \rightarrow 0$, where A, B are the generic fibres of G_n -equivariant vector bundles on \mathbb{P}_k^n and A is irreducible. Let $0 \neq C \subseteq B$ be an irreducible sub-object and $D = B/C$. Then the rows in the following commutative diagram are exact:

$$\begin{array}{ccccccccc} \text{Hom}_u(A, D) & \rightarrow & \text{Ext}_u^1(A, C) & \rightarrow & \text{Ext}_u^1(A, B) & \rightarrow & \text{Ext}_u^1(A, D) & \rightarrow & \text{Ext}_u^2(A, C) \\ & & \parallel & & \cup & & \parallel & & \uparrow \xi \\ \text{Hom}_{\text{eq}}(A, D) & \rightarrow & \text{Ext}_{\text{eq}}^1(A, C) & \rightarrow & \text{Ext}_{\text{eq}}^1(A, B) & \rightarrow & \text{Ext}_{\text{eq}}^1(A, D) & \rightarrow & \text{Ext}_{\text{eq}}^2(A, C) \end{array}$$

where subscript u refers to the category $\mathfrak{S}\mathcal{L}_n^u$ and eq refers to the category of equivariant vector bundles.

Let us show that ξ is injective for any irreducible A and C . As $C \otimes_{K_n} A^\vee$ is semi-simple (as it follows from Proposition 1.5) and $\text{Ext}_?^2(A, C) = \text{Ext}_?^2(K_n, C \otimes_{K_n} A^\vee)$, where $? = u$ or eq, we may assume that $A = K_n$ and C is still irreducible. By Bott's Theorem 3.8, $\dim_k \text{Ext}_{\text{eq}}^2(K_n, C) \leq 1$ with equality only if $C \cong \Omega_{K_n/k}^2$, so we assume further that

$C = \Omega_{K_n/k}^2$. The forgetful functor from $\mathfrak{S}\mathcal{L}_n^u$ to the category of coherent sheaves on \mathbb{P}_k^n induces a homomorphism $\text{Ext}_u^2(A, C) \rightarrow H^2(\mathbb{P}_k^n, \Omega_{\mathbb{P}_k^n/k}^2)$. Clearly, its composition with ξ is an isomorphism.

Then the 5-lemma implies that $\text{Ext}_u^1(A, B) = \text{Ext}_{\text{eq}}^1(A, B)$, and thus, V is equivariant. \square

4. THE CATEGORY \mathcal{A} IN THE CASE $k = \overline{\mathbb{Q}}$

In this section we determine (in Theorem 4.10) the structure of the objects of \mathcal{A} in the case $k = \overline{\mathbb{Q}}$, the field of algebraic numbers. The objects V of \mathcal{A} are quotients of sums of representations of G over k induced by rational representations of $\text{GL}_m k$'s (considered as subquotients of G) with coefficients extended to F (cf. Lemma 4.1). Then we find (in Lemma 4.2) a supply of elements in the induced representations vanishing in V , and use them in Lemmas 4.3–4.7 to show that the objects of \mathcal{A} are sums of quotients of $\bigotimes_F^\bullet \mathfrak{m}$. In §4.1 we study extensions in \mathcal{A} .

Let $V \in \mathcal{A}$ and $m \geq 0$ be such that $V_m \neq 0$. Then there is a non-zero morphism $F[G_{F/k}/G_{F/K_m}] \otimes_{K_m[\text{PGL}_{m+1}k]} V_m \rightarrow V$ in \mathcal{C} . The object V_m of $\mathfrak{S}\mathcal{L}_m^u$ admits an irreducible sub-object $A \neq 0$. By Theorem 2.4, $A \cong S_{K_m}^\lambda \Omega_{K_m/k}^1$ for a Young diagram λ . Then $F[G_{F/k}/G_{F/K_m}] \otimes_{K_m[\text{PGL}_{m+1}k]} A \rightarrow V$ is also non-zero. Clearly, $A = B \otimes_k K_m$, where $B := A^{(\text{Aff}_m)_u} \cong (S_{K_m}^\lambda \Omega_{K_m/k}^1)^{(\text{Aff}_m)_u} \cong S_k^\lambda(k^m)$ is a rational irreducible representation of $\text{GL}_m k := \text{Aff}_m/(\text{Aff}_m)_u$.

This implies that there is a non-zero morphism $U := F[W^\circ] \otimes_{k[\text{GL}_m k]} B \xrightarrow{\varphi} V$ and a surjection $U \rightarrow S_F^\lambda \Omega_{F/k}^1$, where $W^\circ := \{K_m \xrightarrow{/k} F\}/(\text{Aff}_m)_u$ is considered as a $G_{F/k}$ -set.

As any embedding $K_m \xrightarrow{/k} F$ is determined by the images of x_1, \dots, x_m , one can consider W° as a subset of $(F/k)^m$ consisting of m -tuples with entries algebraically independent over k . More invariantly, let $W := \text{Hom}_k((K_m/k)^{(\text{Aff}_m)_u}, F/k) \cong (F/k)^m$ be the group (a k -vector space) generated by W° . The isomorphism is given by restriction of the homomorphisms to the basis $\{\overline{x_1}, \dots, \overline{x_m}\}$ of $(K_m/k)^{(\text{Aff}_m)_u}$. Define a homogeneous map $\varkappa : W \rightarrow \Omega_{F/k}^m \otimes_k \det_k \text{Hom}_G(F/k, W)$ of degree m by inverting the first isomorphism in the sequence $W \xleftarrow{\sim} (F/k) \otimes_k \text{Hom}_G(F/k, W) \xrightarrow{d \otimes \text{id}} \Omega_{F/k}^1 \otimes_k \text{Hom}_G(F/k, W) \rightarrow \text{Sym}_F^m(\Omega_{F/k}^1 \otimes_k \text{Hom}_G(F/k, W)) \rightarrow \Omega_{F/k}^m \otimes_k \det_k \text{Hom}_G(F/k, W)$. Then $W^\circ = \{w \in W \mid \varkappa(w) \neq 0\}$.

Let $(y_1, \dots, y_m) \mapsto [y_1, \dots, y_m]$ be the map $(F/k)^m \rightarrow \{0\} \cup W^\circ$ sending (y_1, \dots, y_m) to $[x_j \mapsto y_j]$ if y_1, \dots, y_m are algebraically independent over k , and to 0 otherwise. Then $[\mu y_1, \dots, \mu y_m] \otimes b = \mu^{|\lambda|} [y_1, \dots, y_m] \otimes b$ in U for any $\mu \in k$. If y_1, \dots, y_m belong to the k -linear envelope of x_1, \dots, x_M for some integer $M \geq 1$ then $[y_1, \dots, y_m] \otimes b \in U_M^{(\text{Aff}_M)_u}$ is a weight $|\lambda|$ eigenvector of the centre of $\text{GL}_M k$.

Let $U_M^! \subseteq U_M^{(\text{Aff}_M)_u}$ be the $k[\text{GL}_M k]$ -envelope of $[x_1, \dots, x_m] \otimes b$ for some $b \neq 0$ (which is the same as k -envelope of all $[y_1, \dots, y_m] \otimes c$ for algebraically independent y_1, \dots, y_m in the k -linear envelope of x_1, \dots, x_M and all c). Clearly, $U_m^! \cong B$ as $k[\text{GL}_m k]$ -modules, and any non-zero morphism $U \rightarrow V$ induces an embedding $U_m^! \hookrightarrow V$.

Lemma 4.1. *If $k = \overline{\mathbb{Q}}$ then for any $V \in \mathfrak{S}\mathcal{L}_M^u$ the representation $V^{(\text{Aff}_M)_u}$ of $\text{GL}_M k$ is rational semi-simple, and $V^{(\text{Aff}_M)_u} \otimes_k K_M = V$.*

Proof. By Corollary 3.11, V is the generic fibre of a $\text{PGL}_{M+1} k$ -equivariant vector bundle.

Then, by Lemma 6.3 (1) of [R], $V = V^{U_0} \otimes_k K_M$, where U_0 is a \mathbb{Q} -lattice in $(\text{Aff}_M)_u$. The group $(\text{Aff}_M)_u$ acts rationally on V^{U_0} . As the action of the \mathbb{Q} -lattice U_0 is trivial, the action of the entire $(\text{Aff}_M)_u$ is trivial, i.e., $V = V^{(\text{Aff}_M)_u} \otimes_k K_M$. Then the action of $\text{GL}_M k$ on $V^{(\text{Aff}_M)_u}$ is rational, and thus, semi-simple. \square

REMARK. There is no semisimplicity if k contains a transcendental element.

Indeed, let the underlying K_n -vector space of $V \in \mathfrak{S}\mathfrak{L}_n^u$ be $K_n \oplus \Omega_{K_n}^1 / (\Lambda \otimes_k K_n)$ for a proper k -vector subspace $\Lambda \subset \Omega_k^1$ of finite codimension, and the G_n -action be given by $\sigma(f, \omega) = (\sigma f, \sigma \omega + \sigma f \cdot d \log(\sigma \eta / \eta))$ for any $\sigma \in G_n$, where $\eta = dx_1 \wedge \cdots \wedge dx_n \in \Omega_{K_n/k}^n$. (So V fits into a non-split exact sequence $0 \rightarrow \Omega_{K_n}^1 / (\Lambda \otimes_k K_n) \rightarrow V \rightarrow K_n \rightarrow 0$.) Then $\sigma(f, \omega) = (\sigma f, \sigma \omega)$ if $\sigma \in (\text{Aff}_n)_u$, and therefore, $V^{(\text{Aff}_n)_u} = k \oplus \Omega_k^1 / \Lambda$ is a non-trivial extension of trivial representations of $\text{GL}_n k$.

Lemma 4.2. *The kernel of $U_M^! \rightarrow S_F^\lambda \Omega_{F/k}^1$ is contained in the kernel of $U \xrightarrow{\varphi} V$.*

Proof. By Lemma 4.1, the image $\overline{U_M^!}$ of $U_M^!$ in V is isomorphic to $\bigoplus_{|\nu|=|\lambda|} (S_k^\nu(k^M))^{m_\nu}$. As $\overline{U_m^!} \subseteq \overline{U_M^!}^{\text{GL}_M k \cap G_{K_M/K_m}} \cong \bigoplus_{|\nu|=|\lambda|} (S_k^\nu(k^m))^{m_\nu}$ and $U_M^!$ is generated by $U_m^! \xrightarrow{\sim} S_k^\lambda(k^m) \subseteq S_k^\lambda(k^M)$, we see that $\overline{U_M^!}$ is isomorphic to $S_k^\lambda(k^M)$. Then the kernel of $U_M^! \rightarrow S_F^\lambda \Omega_{F/k}^1$ is contained in the kernel of the morphism $U \xrightarrow{\varphi} V$. \square

Let $W^{M^\circ} \subset W^M$ be the subset consisting of M -tuples (y_1, \dots, y_M) such that $\sum_{i \in I} y_i \in W^\circ$ for any non-empty subset $I \subseteq \{1, \dots, M\}$.

Let $k[W^{M^\circ}] \rightarrow k[W^\circ] \otimes_{k[k^\times]} k(M)$ be the k -linear map sending (y_1, \dots, y_M) to

$$\langle y_1, \dots, y_M \rangle := \sum_{I \subseteq \{1, \dots, M\}} (-1)^{\#I} \left[\sum_{i \in I} y_i \right] \in k[W^\circ] \otimes_{k[k^\times]} k(M).$$

Here $k(M)$ denotes a one-dimensional k -vector space with k^\times -action by M -th powers. As (y, \dots, y) is sent to

$$\sum_{j \geq 0} (-1)^j \binom{M}{j} j^M [y] = (t \frac{d}{dt})^M (1-t)^M |_{t=1} \cdot [y] = (-1)^M M! \cdot [y],$$

it is surjective. Clearly, $\langle y_1, \dots, y_M \rangle = \langle y_{\theta(1)}, \dots, y_{\theta(M)} \rangle$ for any permutation $\theta \in \mathfrak{S}_M$. Let $\tilde{U} := F[W^{|\lambda|^\circ}] \rightarrow U$ be the F -linear surjection sending $(y_1, \dots, y_{|\lambda|})$ to $\langle y_1, \dots, y_{|\lambda|} \rangle \otimes b$.

Lemma 4.3. *Let the k -linear map $\alpha : k[W^\circ] \rightarrow \bigotimes_k^M W$ be given by $[w] \mapsto w^{\otimes M}$. Then α factors through $k[W^\circ] \otimes_{k[k^\times]} k(M)$ and $\langle y_1, \dots, y_M \rangle \mapsto (-1)^M \sum_{\theta \in \mathfrak{S}_M} y_{\theta(1)} \otimes \cdots \otimes y_{\theta(M)}$ if $(y_1, \dots, y_M) \in W^{M^\circ}$.*

Proof. The element $\langle y_1, \dots, y_M \rangle$ is sent to

$$\sum_{I \subseteq \{1, \dots, M\}} (-1)^{\#I} \left(\sum_{i \in I} y_i \right)^{\otimes M} = \sum_{1 \leq i_1, \dots, i_M \leq M} A_{i_1, \dots, i_M} y_{i_1} \otimes \cdots \otimes y_{i_M}.$$

If $S = \{1, \dots, M\} \setminus \{i_1, \dots, i_M\}$ then $A_{i_1, \dots, i_M} = \sum_{J \subseteq S} (-1)^{M-\#J}$, so $A_{i_1, \dots, i_M} = 0$ if S is non-empty, and $A_{i_1, \dots, i_M} = (-1)^M$ if $\{1, \dots, M\} = \{i_1, \dots, i_M\}$. \square

Lemma 4.4. *If $M = |\lambda|$, $\mu \in k$, $y_0, y_1, y_0 + y_1 \in W^\circ$ and all coordinates of $t_2, \dots, t_M \in W$ are algebraically independent over $k(y_0, y_1)$ then*

$$\langle y_0 + y_1, t_2, \dots, t_M \rangle \otimes b \equiv \langle y_0, t_2, \dots, t_M \rangle \otimes b + \langle y_1, t_2, \dots, t_M \rangle \otimes b \pmod{\ker \varphi},$$

and $\langle \mu y_1, t_2, \dots, t_M \rangle \otimes b \equiv \mu \langle y_1, t_2, \dots, t_M \rangle \otimes b \pmod{\ker \varphi}$.

Proof. It follows from Lemmas 4.2 and 4.3, that $\langle z_0 + z_1, z_2, \dots, z_M \rangle \otimes b - \langle z_0, z_2, \dots, z_M \rangle \otimes b - \langle z_1, \dots, z_M \rangle \otimes b$ and $\langle \mu z_1, z_2, \dots, z_M \rangle \otimes b - \mu \cdot \langle z_1, \dots, z_M \rangle \otimes b$ are sent to zero by φ , where the coordinates of z_j are $x_{jm+1}, \dots, x_{jm+m}$. As the G -orbits of these elements are also sent to zero by φ , for some $u, v \in W^\circ$ with coordinates algebraically independent over the subfield in F generated over k by $y_0, y_1, t_2, \dots, t_M$, one has the following congruences modulo the kernel of φ :

$$(5) \quad \langle y_0 + y_1, t_2, \dots, t_M \rangle \otimes b \equiv \langle y_0 + y_1 + u, t_2, \dots, t_M \rangle \otimes b - \langle u, t_2, \dots, t_M \rangle \otimes b,$$

$$(6) \quad \langle y_0, t_2, \dots, t_M \rangle \otimes b \equiv \langle y_0 + u - v, t_2, \dots, t_M \rangle \otimes b - \langle u - v, t_2, \dots, t_M \rangle \otimes b,$$

$$(7) \quad \langle y_1, t_2, \dots, t_M \rangle \otimes b \equiv \langle y_1 + v, t_2, \dots, t_M \rangle \otimes b - \langle v, t_2, \dots, t_M \rangle \otimes b.$$

As $\langle y_0 + y_1 + u, t_2, \dots, t_M \rangle \otimes b \equiv \langle y_0 + u - v, t_2, \dots, t_M \rangle \otimes b + \langle y_1 + v, t_2, \dots, t_M \rangle \otimes b$, and $\langle u, t_2, \dots, t_M \rangle \otimes b \equiv \langle u - v, t_2, \dots, t_M \rangle \otimes b + \langle v, t_2, \dots, t_M \rangle \otimes b$, the left hand side of the congruence (5) is congruent to the sum of the left hand sides of the congruences (6) and (7) modulo $\ker \varphi$. \square

Lemma 4.5. *Let $(y_1, \dots, y_M) \in \{0\} \times W^{(M-1)^\circ} \cup W^{M^\circ}$ and let the coordinates of $t_{ij} \in W^\circ$ be algebraically independent over $k(y_1, \dots, y_M)$, where $1 \leq i \leq M$ and $2 \leq j \leq M$. Set $[0] := 0$ and $\langle 0, y_2, \dots, y_M \rangle := 0$. Then*

$$(8) \quad \langle y_1, \dots, y_M \rangle \otimes b \equiv \sum_{J \subseteq \{2, \dots, M\}} (-1)^{\#J} \langle y_1, \sum_{s \in \{1\} \cup J} t_{s2}, \dots, \sum_{s \in \{1\} \cup J} t_{sM} \rangle \otimes b \\ - \sum_{\emptyset \neq I \subseteq \{2, \dots, M\}} (-1)^{\#I} \langle y_1, y_2 + \sum_{i \in I} t_{2i}, \dots, y_M + \sum_{i \in I} t_{Mi} \rangle \otimes b \pmod{\ker \varphi}.$$

Proof. It follows from the identities

$$[\sum_{s \in J} y_s] = \sum_{\emptyset \neq I \subseteq \{2, \dots, M\}} (-1)^{\#I} \left([\sum_{s \in J} \sum_{i \in I} t_{si}] - [\sum_{s \in J} (y_s + \sum_{i \in I} t_{si})] \right) \\ - \langle \sum_{s \in J} y_s, \sum_{s \in J} t_{s2}, \dots, \sum_{s \in J} t_{sM} \rangle$$

that

$$(9) \quad \langle y_1, \dots, y_M \rangle = \sum_{\emptyset \neq I \subseteq \{2, \dots, M\}} (-1)^{\#I} \left(\langle \sum_{i \in I} t_{1i}, \dots, \sum_{i \in I} t_{Mi} \rangle \right. \\ \left. - \langle y_1 + \sum_{i \in I} t_{1i}, \dots, y_M + \sum_{i \in I} t_{Mi} \rangle \right) - \sum_{J \subseteq \{1, \dots, M\}} (-1)^{\#J} \langle \sum_{s \in J} y_s, \sum_{s \in J} t_{s2}, \dots, \sum_{s \in J} t_{sM} \rangle.$$

Then Lemma 4.4, applied to the summands containing y_1 , implies that

$$\langle y_1, \dots, y_M \rangle \otimes b \equiv \sum_{J \subseteq \{2, \dots, M\}} (-1)^{\#J} \langle y_1, \sum_{s \in \{1\} \cup J} t_{s2}, \dots, \sum_{s \in \{1\} \cup J} t_{sM} \rangle \otimes b \\ - \sum_{\emptyset \neq I \subseteq \{2, \dots, M\}} (-1)^{\#I} \langle y_1, y_2 + \sum_{i \in I} t_{2i}, \dots, y_M + \sum_{i \in I} t_{Mi} \rangle \otimes b + \langle 0, y_2, \dots, y_M \rangle \otimes b,$$

so we get (8). \square

Lemma 4.6. *If $M = |\lambda|$, $\mu \in k$ and $(z_j, y_2, \dots, y_M), (\sum_{i=1}^N z_i, y_2, \dots, y_M), (\mu z_1, y_2, \dots, y_M) \in W^{M^\circ}$ for all $1 \leq j \leq N$ then*

$$(10) \quad \left\langle \sum_{j=1}^N z_j, y_2, \dots, y_M \right\rangle \otimes b \equiv \sum_{j=1}^N \langle z_j, y_2, \dots, y_M \rangle \otimes b \pmod{\ker \varphi},$$

and $\langle \mu z_1, y_2, \dots, y_M \rangle \otimes b \equiv \mu \langle z_1, y_2, \dots, y_M \rangle \otimes b \pmod{\ker \varphi}$.

Proof. If $N = 2$ then (10) follows from Lemmas 4.4 and 4.5. If $N \geq 3$ then

$$\left\langle \sum_{j=1}^N z_j, y_2, \dots, y_M \right\rangle \otimes b \equiv \left\langle \sum_{j=3}^N z_j - u, y_2, \dots, y_M \right\rangle \otimes b + \langle z_1 + z_2 + u, y_2, \dots, y_M \rangle \otimes b$$

for any sufficiently general $u \in W^\circ$. By the induction assumption, this is congruent to $\sum_{j=3}^N \langle z_j, y_2, \dots, y_M \rangle \otimes b - \langle u, y_2, \dots, y_M \rangle \otimes b + \langle z_1, y_2, \dots, y_M \rangle \otimes b + \langle z_2 + u, y_2, \dots, y_M \rangle \otimes b \equiv \sum_{j=1}^N \langle z_j, y_2, \dots, y_M \rangle \otimes b$. \square

Lemma 4.7. *The k -linear map $k[W^{M^\circ}] \rightarrow \bigotimes_k^M W$, given by $[(y_1, \dots, y_M)] \mapsto y_1 \otimes \dots \otimes y_M$, is surjective and its kernel is spanned over k by $[(y_0, \dots, y_{j-1} + y_j, \dots, y_M)] - [(y_0, \dots, \widehat{y}_{j-1}, \dots, y_M)] - [(y_0, \dots, \widehat{y}_j, \dots, y_M)]$ and $\mu[(y_1, \dots, y_M)] - [(y_1, \dots, \mu y_j, \dots, y_M)]$ for all $y_0, \dots, y_M \in W^\circ$ and all $\mu \in k^\times$.*

Proof. By Zorn's lemma, there exists a maximal subset S in W° consisting of k -linear independent elements. If S does not generate W then the k -linear envelope of S does not contain W° , i.e., an element $y \in W^\circ$ k -linear independent over S , so $S \cup \{y\}$ is a bigger subset in W° consisting of k -linear independent elements. This contradiction shows that S is a k -basis of W .

For any $y \in W^\circ$ and any $z \in W$ there exist at most m values of $\mu \in k$ such that $y + \mu z \notin W^\circ$, since this condition is equivalent to vanishing of the $\Omega_{F/k}^m$ -valued polynomial $(dy_1 + \mu dz_1) \wedge \dots \wedge (dy_m + \mu dz_m)$ of degree $\leq m$ in μ with non-zero constant term. Let us show that the map $(k^\times S)^M \cap W^{M^\circ} \rightarrow S^M$ given by the projectivization is surjective. Indeed, let $(s_1, \dots, s_M) \in S^M$. For all but $\leq m$ values of $\mu \in k^\times$ one has $s_1 + \mu s_2 \in W^\circ$. Fix one of such μ and set $s'_2 := \mu s_2$. For all but $\leq 3m$ values of $\mu \in k^\times$ one has $s_1 + \mu s_3, s'_2 + \mu s_3, s_1 + s'_2 + \mu s_3 \in W^\circ$. Fix one of such μ and set $s'_3 := \mu s_3$. Proceeding further this way, we get an element $(s_1, s'_2, \dots, s'_M) \in ((k^\times S)^M) \cap W^{M^\circ}$ projecting onto (s_1, \dots, s_M) .

Fix a section of the projection $(k^\times S)^M \cap W^{M^\circ} \rightarrow S^M$. Denote by $\widetilde{S^M}$ the image of S^M under this section. Then $\widetilde{S^M}$ considered as a subset in $k[W^{M^\circ}]$ maps to a basis of $\bigotimes_k^M W$, which shows the surjectivity.

For the injectivity it suffices to check that $k[\widetilde{S^M}]$ maps onto $k[W^{M^\circ}]$ modulo the relations. For an element $w = (w_1, \dots, w_M) \in W^{M^\circ}$ set $l(w) := \sum_{j=1}^M l_j(w) \geq M$, where $l_j(w)$ is the number of non-zero coordinates of w_j in the basis S .

By induction on $l(w)$ we are going to show that $[w]$ is in the image of $k[\widetilde{S^M}]$.

If $l(w) = M$ then $w_j = \mu_j s_j$ for all $1 \leq j \leq M$, where $(s_1, \dots, s_M) \in \widetilde{S^M}$. For any sufficiently general $\nu_2, \dots, \nu_M \in k^\times$ one has

$$\begin{aligned} [w] &\equiv \nu_2^{-1} \cdots \nu_M^{-1} [(w_1, \nu_2 w_2, \dots, \nu_M w_M)] \equiv \mu_1 \nu_2^{-1} \cdots \nu_M^{-1} [(s_1, \nu_2 w_2, \dots, \nu_M w_M)] \\ &\equiv \mu_1 \mu_2 \nu_3^{-1} \cdots \nu_M^{-1} [(s_1, s_2, \nu_3 w_3, \dots, \nu_M w_M)] \equiv \cdots \equiv \mu_1 \cdots \mu_M [(s_1, \dots, s_M)]. \end{aligned}$$

The induction step: if, for instance, $l_1(w) \geq 2$ then for all but $\leq l_1(w)m + m$ values of $\mu \in k^\times$ one has $[w] \equiv \mu^{-1} [(\mu w_1, w_2, \dots, w_M)] \equiv \mu^{-1} \sum_{s \in S} [(\mu \mu_s s, w_2, \dots, w_M)]$, where $w_1 = \sum_{s \in S} \mu_s s$ is a finite sum. By the induction assumption, the summands $[(\mu \mu_s s, w_2, \dots, w_M)]$ are in the image of $k[\widetilde{S^M}]$, and thus, $[w]$ is also there. \square

Let \mathfrak{m} be the kernel of the multiplication map $F \otimes_k F \xrightarrow{\times} F$. The map $F \otimes_k (F/k) \rightarrow \mathfrak{m}$, given by $\sum_j z_j \otimes \bar{y}_j \mapsto \sum_j z_j \otimes y_j - (\sum_j z_j y_j) \otimes 1$ is clearly an isomorphism, so we can use the notation \mathfrak{m} instead of $F \otimes_k (F/k)$, and the multiplicative structure of the ideal \mathfrak{m} .

Lemma 4.8. *The element $\alpha_q := (x_1 \otimes 1 - 1 \otimes x_1)^{s_1} \otimes \cdots \otimes (x_q \otimes 1 - 1 \otimes x_q)^{s_q} \in \bigotimes_F^q \mathfrak{m}$ generates the sub-object $\mathfrak{m}^{s_1} \otimes_F \cdots \otimes_F \mathfrak{m}^{s_q}$.*

Proof. We need to show that for any collection of $\beta_i \in \mathfrak{m}^{s_i}$ the element $\beta_1 \otimes \cdots \otimes \beta_q$ belongs to the $F[G_{F/k}]$ -submodule generated by α_q . Set $\alpha := x_1 \otimes 1 - 1 \otimes x_1$. Then the $G_{F/k}$ -orbit of α^s contains $(\sum_{j=1}^s a_j (y_j \otimes 1 - 1 \otimes y_j))^s$ for any $a_j \in k$ and $y_j \in F$ such that $\sum_{j=1}^s a_j y_j \notin k$. The k -span of such elements with fixed y_1, \dots, y_s contains $\prod_{j=1}^s (y_j \otimes 1 - 1 \otimes y_j)$. Such products generate \mathfrak{m}^s as an ideal. Moreover, they generate \mathfrak{m}^s as a $F \otimes_k k$ -module: $(1 \otimes b) \prod_{j=1}^s (y_j \otimes 1 - 1 \otimes y_j) = ((by_1 \otimes 1 - 1 \otimes by_1) - (y_1 \otimes 1)(b \otimes 1 - 1 \otimes b)) \prod_{j=2}^s (y_j \otimes 1 - 1 \otimes y_j) = (by_1 \otimes 1 - 1 \otimes by_1) \prod_{j=2}^s (y_j \otimes 1 - 1 \otimes y_j) - (y_1 \otimes 1)(b \otimes 1 - 1 \otimes b) \prod_{j=2}^s (y_j \otimes 1 - 1 \otimes y_j)$.

This implies that $\beta_i = \sum_{j=1}^{s_i} f_{ij} \cdot \sigma_{ij} \alpha^{s_i}$ for some $\sigma_{ij} \in G_{F/k}$ and $f_{ij} \in F$. The $G_{F/k}$ -orbit of α_q contains $\alpha' := (z_1 \otimes 1 - 1 \otimes z_1)^{s_1} \otimes \cdots \otimes (z_q \otimes 1 - 1 \otimes z_q)^{s_q}$, where $z_1, \dots, z_q \in F$ are algebraically independent over the subfield in F generated over k by all $f_{ij}, \sigma_{ij} x_1$.

For each pair (i, j) such that $1 \leq i \leq q$ and $1 \leq j \leq s_i$ there exists an element $\xi_{ij} \in G_{F/k}$ fixing all $f_{\lambda\mu}, \sigma_{\lambda\mu} x_1$ and the elements z_{i+1}, \dots, z_q , such that $\xi_{ij} z_\mu = z_\mu + \sigma_{ij} x_1$. Then $(\sum_{j=1}^{s_q} f_{qj} (\xi_{qj} - 1)^{s_q}) \cdots (\sum_{j=1}^{s_1} f_{1j} (\xi_{1j} - 1)^{s_1}) (\alpha') = \beta_1 \otimes \cdots \otimes \beta_q$. \square

Corollary 4.9. *Any homomorphism $F \otimes_k \bigotimes_k^M (F/k) \rightarrow V$ factors through $\bigotimes_F^M (\mathfrak{m}/\mathfrak{m}^s)$ for some $s \geq 1$.*

Proof. For any integer $s \geq 1$ the element $\alpha_s := (x_1 \otimes 1 - 1 \otimes x_1)^s \otimes x_2 \otimes \cdots \otimes x_M = \sum_{j=0}^s (-1)^j \binom{s}{j} x_1^{s-j} \otimes x_1^j \otimes x_2 \otimes \cdots \otimes x_M \in (\mathfrak{m}^s \otimes_k \bigotimes_k^{M-1} (F/k))_M^{(\text{Aff}_M)_u}$ is homogeneous of degree $s+M-1$. As V_M is finite-dimensional, the image of α_s in V_M is zero for all sufficiently big s . Note that α_s generates $\mathfrak{m}^s \otimes_k \bigotimes_k^{M-1} (F/k)$ as an F -semi-linear representation of G .

This implies that the image of U is a quotient of $\mathfrak{m}/\mathfrak{m}^s \otimes_k \bigotimes_k^{M-1} (F/k)$ for some $s \geq 1$, and therefore, any homomorphism $F \otimes_k \bigotimes_k^M (F/k) = \bigotimes_F^M \mathfrak{m} \rightarrow V$ factors through $\bigotimes_F^M (\mathfrak{m}/\mathfrak{m}^s)$ for some $s \geq 1$. \square

Theorem 4.10. *Any (finitely generated) object V of \mathcal{A} is a quotient of a (finite) direct sum of objects of type $\bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^s)$ for some $q, s \geq 1$ and F , if $k = \overline{\mathbb{Q}}$. In particular, any irreducible object of \mathcal{A} is a direct summand of the tensor algebra $\bigotimes_F^\bullet \Omega_{F/k}^1$.*

Proof. V is generated by V_m for some $m \geq 0$. By Lemma 4.1, $V_m^{(\text{Aff}_m)_u}$ is a semi-simple $\text{GL}_m k$ -module generating V . As it is explained at the beginning of this section, V is a

quotient of a direct sum of U 's corresponding to irreducible direct summands of $V_m^{(\text{Aff}_m)_u}$. By Lemmas 4.6 and 4.7, V is a quotient of a direct sum of $F \otimes_k \bigotimes_k^M (F/k)$ for some $M \geq 0$. Then the conclusion follows from Corollary 4.9 and the identities $\mathfrak{m}^j/\mathfrak{m}^{j+1} = \text{Sym}_F^j(\mathfrak{m}/\mathfrak{m}^2)$ and $\mathfrak{m}/\mathfrak{m}^2 = \Omega_{F/k}^1$. \square

Corollary 4.11. *Any finitely generated object of \mathcal{A} is of finite length, if $k = \overline{\mathbb{Q}}$.* \square

4.1. Ext's in \mathcal{A} .

Lemma 4.12. $\text{Hom}_C(\bigotimes_F^r \mathfrak{m}, \bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^{N+1})) = \text{Hom}_{\mathcal{A}}(\bigotimes_F^r (\mathfrak{m}/\mathfrak{m}^{N+1}), \bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^{N+1}))$ admits a natural k -basis identified with the set $P = P(q, r, N)$ of the surjections $\{1, \dots, r\} \rightarrow \{1, \dots, q\}$ with fibres of cardinality $\leq N$, if $N \geq 1$, $q, r \geq 0$ and $q + r \geq 1$. In particular (take $r \geq q = 1$), any subobject of $\mathfrak{m}/\mathfrak{m}^{N+1}$ is of type $\mathfrak{m}^r/\mathfrak{m}^{N+1}$.

EXAMPLE. $P = \emptyset$ if $q > r$, or if $r > qN$; $\#P = q!$ if $q = r$; $\#P = 1$ if $N \geq r \geq q = 1$.

Proof. By Lemma 4.8, $\mathfrak{m}^{s_1} \otimes_F \dots \otimes_F \mathfrak{m}^{s_r}$ is generated by the element $\otimes_{j=1}^r (x_j \otimes 1 - 1 \otimes x_j)^{s_j} \in (\bigotimes_{j=1}^r \mathfrak{m}^{s_j})_r^{(\text{Aff}_r)_u}$ of weight (s_1, \dots, s_r) with respect to $(k^\times)^r \subseteq \text{GL}_r k := \text{Aff}_r/(\text{Aff}_r)_u$. The central weights of $(\bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^{N+1}))_r^{(\text{Aff}_r)_u}$ are contained in the interval $[q, qN]$, so $\bigotimes_{j=1}^r \mathfrak{m}^{s_j}$ is mapped to 0 if $\sum_{j=1}^r s_j \notin [q, qN]$. In particular, the morphisms factor through $\bigotimes_F^r (\mathfrak{m}/\mathfrak{m}^{qN-r+2})$, and are zero if $r > qN$.

The elements $\pi_\varphi := \otimes_{i=1}^q \prod_{\varphi(u)=i} (x_u \otimes 1 - 1 \otimes x_u)$ for all surjections $\varphi \in P$ span the $(\underbrace{1, \dots, 1}_r)$ -eigenspace of $(\bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^{N+1}))_r^{(\text{Aff}_r)_u}$. Any morphism of $\bigotimes_F^r \mathfrak{m}$ is determined by the image $\sum_{\varphi \in P} \lambda_\varphi \pi_\varphi$ of the generator $\otimes_{j=1}^r (x_j \otimes 1 - 1 \otimes x_j)$ for some collection of $\lambda_\varphi \in k$. \square

Lemma 4.13. \mathcal{A} splits as $\text{Vec}_k \oplus \mathcal{A}^\circ$, where Vec_k is the category of finite-dimensional k -vector spaces and \mathcal{A}° is the full subcategory of \mathcal{A} with objects V such that $V^{G_{F/k}} = 0$.

Proof. For any $V \in \mathcal{A}$ set $V^\circ := \bigcap_{\varphi \in \text{Hom}_C(V, F)} \ker \varphi$. It follows from Theorem 4.10 and Lemma 4.12 that $V = (V^{G_{F/k}} \otimes_k F) \oplus V^\circ$, and $\text{Ext}_{\mathcal{A}}^*(F, \mathcal{A}^\circ) = \text{Ext}_{\mathcal{A}}^*(\mathcal{A}^\circ, F) = 0$. The equivalence is given by $V \mapsto (V^{G_{F/k}}, V^\circ)$. \square

Define the following decreasing ‘‘weight’’ filtration on the objects V of \mathcal{A} : $W^q V$ is the sum of the images of all morphisms to V from $\bigotimes_F^{\geq q} \mathfrak{m}$. Clearly, W^\bullet is functorial and multiplicative. By Theorem 4.10, $gr_W^q V$ is a finite direct sum of direct summands of $\bigotimes_F^q \Omega_F^1$.

Corollary 4.14. \mathcal{A}° has no non-zero projective objects.

Proof. Let $P \in \mathcal{A}^\circ$ be a projective object and $\xi_2 : P \rightarrow S_F^\lambda \Omega_F^1$ be its irreducible quotient for a Young diagram λ , where $|\lambda|$ is minimal such that $W^{|\lambda|+1} P \neq P$. Then, for any $s \geq 2$, there is a lifting $\xi_s : P \rightarrow S_F^\lambda (\mathfrak{m}/\mathfrak{m}^s)$ of ξ_2 . By Theorem 4.10, there exist $q, a \geq 1$ and a morphism $\bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^a) \rightarrow P$ such that its composition with ξ_2 is non-zero. Then its composition with any ξ_s is also non-zero. By Lemma 4.12, $\text{Hom}_{\mathcal{A}}(\bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^a), S_F^\lambda (\mathfrak{m}/\mathfrak{m}^N)) = 0$ for any $N \geq a + q$, leading to contradiction. \square

Lemma 4.15. *One has $\text{Ext}_{\mathcal{A}}^1(\bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^N), V) = 0$ for any $V \in \mathcal{A}$ of finite length, $q \geq 1$ and $N >$ the maximal weight of V .*

Proof. Induction on the length of V reduces the problem to the case of irreducible V . Let $0 \rightarrow V \rightarrow E \xrightarrow{\pi} \bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^N) \rightarrow 0$ be an extension. By Theorem 4.10, there

is a surjection of a direct sum of objects of type $\bigotimes_F^p(\mathfrak{m}/\mathfrak{m}^a)$ onto E . By Lemma 4.12, $\mathrm{Hom}_{\mathcal{A}}(\bigotimes_F^{\neq q}(\mathfrak{m}/\mathfrak{m}^a), \bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^2)) = 0$, so there is a morphism of a direct sum of objects of type $\bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^a)$ to E surjective over $\bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^2)$. As the latter is semi-simple, there is a morphism of $\bigoplus_{|\lambda|=q} S_F^\lambda(\mathfrak{m}/\mathfrak{m}^a)$ to E surjective over $\bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^2)$. By Lemma 4.12, its composition with π is surjective, and therefore, the weights of its kernel are $\geq N$, so it does not intersect V . In other words, the extension splits. \square

Corollary 4.16. *The following pro-representable functor on \mathcal{A}*

$$\mathrm{Hom}_{\mathcal{C}}(\mathfrak{m}^{s_1} \otimes_F \cdots \otimes_F \mathfrak{m}^{s_q}, -) = \varinjlim \mathrm{Hom}_{\mathcal{A}}((\mathfrak{m}^{s_1}/\mathfrak{m}^N) \otimes_F \cdots \otimes_F (\mathfrak{m}^{s_q}/\mathfrak{m}^N), -)$$

is exact if and only if $s_1 = \cdots = s_q = 1$.

Proof. Let $V \rightarrow V'$ be a surjection in \mathcal{A} and $\xi : \bigotimes_F^q \mathfrak{m} \rightarrow V'$ be a morphism in \mathcal{C} . We have to show that ξ factors through V . By Lemma 4.8, the image of ξ is cyclic. Let V'' be the cyclic sub-object of V generated by a pre-image of a generator of the image of ξ . Then the kernel K of $V'' \rightarrow \mathrm{Im}(\xi)$ is of finite length. As ξ factors through $\bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^N)$ for some $N \gg 0$, and Lemma 4.15 implies that $\mathrm{Ext}_{\mathcal{A}}^1(\bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^N), K) = 0$, ξ factors through V .

The rest follows from the fact that the projection $\mathfrak{m}^s \rightarrow \mathfrak{m}^s/\mathfrak{m}^{N+s}$ does not lift to $\mathfrak{m}^s \rightarrow \bigotimes_F^s(\mathfrak{m}/\mathfrak{m}^{N+1})$, if $s \geq 2$: neither non-zero morphism $\bigotimes_F^s \mathfrak{m} \rightarrow \bigotimes_F^s(\mathfrak{m}/\mathfrak{m}^{N+1})$ factors through \mathfrak{m}^s , if $N \geq 2$. \square

Corollary 4.17. *If $V \in \mathcal{A}$ is of finite type then $\dim_k \mathrm{Ext}_{\mathcal{A}}^j(V, V') < \infty$ for any $j \geq 0$ and any $V' \in \mathcal{A}$. If $V \in \mathcal{A}$ is irreducible and $\mathrm{Ext}_{\mathcal{A}}^1(\mathfrak{m}/\mathfrak{m}^q, V) \neq 0$ for some $q \geq 2$ then $V \cong \mathfrak{m}^q/\mathfrak{m}^{q+1}$ and $\mathrm{Ext}_{\mathcal{A}}^1(\mathfrak{m}/\mathfrak{m}^q, V) \cong k$.*

Proof. If $V \in \mathcal{A}$ is of finite type then, by Theorem 4.10, it admits a resolution $\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0$ whose terms are finite direct sums of objects of type $\bigotimes_F^s \mathfrak{m}$. By Lemma 4.8, the terms of the complex $\mathrm{Hom}_{\mathcal{C}}(P_\bullet, V')$ are finite-dimensional over k and, by Corollary 4.16, it calculates $\mathrm{Ext}_{\mathcal{A}}^\bullet(V, V')$. \square

Corollary 4.18. *The filtration W^\bullet is strictly compatible with the surjections.*

Proof. Let $V \rightarrow V'$ be a surjection in \mathcal{A} . Then, by Corollary 4.16, any morphism $\bigotimes_F^q \mathfrak{m} \rightarrow V'$ factors through V . \square

5. "COHERENT" SHEAVES IN SMOOTH TOPOLOGY

Let $\mathfrak{S}m_k$ be the category of locally dominant morphisms of smooth k -schemes. Consider on $\mathfrak{S}m_k$ the (pre-)topology, where the covers are surjective smooth morphisms. Clearly, the covers are stable under the base changes.

By definition, the structure presheaf \mathcal{O} of $\mathfrak{S}m_k$ associates to any $Y \in \mathfrak{S}m_k$ its k -algebra of regular functions $\mathcal{O}(Y)$. Clearly, \mathcal{O} is a sheaf in this topology.

A sheaf \mathcal{F} on $\mathfrak{S}m_k$ is "(quasi-)coherent" if its values $\mathcal{F}(Y)$ are endowed with $\mathcal{O}(Y)$ -module structures and its restriction to the small étale site of Y is a (quasi-)coherent sheaf for any $Y \in \mathfrak{S}m_k$.

Lemma 5.1. *Let $X \rightarrow Y$ be an étale morphism of smooth varieties over k sending a point $q \in X$ to a point $p \in Y$. Then $\mathfrak{m}_q^s/\mathfrak{m}_q^N = \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p^s/\mathfrak{m}_p^N)$ for any $s \leq N$, where $\mathfrak{m}_q := \ker(\mathcal{O}_q \otimes_k \mathcal{O}_q \xrightarrow{\times} \mathcal{O}_q)$.*

Proof. One has $\mathfrak{m}_q/\mathfrak{m}_q^2 = \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p/\mathfrak{m}_p^2)$, so applying $\mathrm{Sym}_{\mathcal{O}_q}^s$ we get $\mathfrak{m}_q^s/\mathfrak{m}_q^{s+1} = \mathrm{Sym}_{\mathcal{O}_q}^s(\mathfrak{m}_q/\mathfrak{m}_q^2) = \mathcal{O}_q \otimes_{\mathcal{O}_p} \mathrm{Sym}_{\mathcal{O}_p}^s(\mathfrak{m}_p/\mathfrak{m}_p^2) = \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p^s/\mathfrak{m}_p^{s+1})$. The induction on $N - s$ gives the conclusion:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathfrak{m}_q^{s+1}/\mathfrak{m}_q^N & \rightarrow & \mathfrak{m}_q^s/\mathfrak{m}_q^N & \rightarrow & \mathfrak{m}_q^s/\mathfrak{m}_q^{s+1} & \rightarrow & 0 \\ & & \parallel & & \cup & & \parallel & & \\ 0 & \rightarrow & \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p^{s+1}/\mathfrak{m}_p^N) & \rightarrow & \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p^s/\mathfrak{m}_p^N) & \rightarrow & \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p^s/\mathfrak{m}_p^{s+1}) & \rightarrow & 0 \end{array}$$

□

Corollary 5.2. *The category \mathcal{A} is equivalent to the category of “coherent” sheaves on $\mathfrak{S}m_k$, if $k = \overline{\mathbb{Q}}$.*

Proof. Fix an embedding over k of the function field of each connected component of each smooth k -variety into F . Then, for any $V \in \mathcal{A}$, $Y \in \mathfrak{S}m_k$ and a point $q \in Y$ define an \mathcal{O}_q -lattice $\mathcal{V}_q \subset V^{G_{F/k(Y)}}$ as follows. Let $\mathcal{O}_p \subseteq \mathcal{O}_q$ be an étale extension of a local subring in F of a closed point p of a projective space.

Any object V of \mathcal{A} is a quotient of a direct sum of objects of type $\bigotimes_F^s(\mathfrak{m}/\mathfrak{m}^N)$. Then, as it is true for $\bigotimes_F^s(\mathfrak{m}/\mathfrak{m}^N)$ (Lemma 5.1), it follows that the module $\mathcal{V}_p \subset V$ provided by the exact functor \mathcal{S} , cf. §1, is independent of the choice of the projective space, and $\mathcal{V}_q := \mathcal{O}_q \otimes_{\mathcal{O}_p} \mathcal{V}_p \subset V$ is independent of \mathcal{O}_p .

This determines a locally free coherent sheaf \mathcal{V}_Y on Y with the generic fibre $V^{G_{F/k(Y)}}$.

It follows also that, for any dominant morphism $X \xrightarrow{\pi} Y$ of smooth k -varieties, the inclusion of the generic fibres $k(X) \otimes_{k(Y)} V^{G_{F/k(Y)}} \subseteq V^{G_{F/k(X)}}$ induces an injection of the coherent sheaves $\pi^* \mathcal{V}_Y \hookrightarrow \mathcal{V}_X$ on X , which is an isomorphism if π is étale.

To check that \mathcal{V} is a sheaf on $\mathfrak{S}m_k$, we need to show that for any surjective smooth morphism $X \rightarrow Y$ the sequence $0 \rightarrow \mathcal{V}(Y) \xrightarrow{\beta} \mathcal{V}(X) \xrightarrow{p_1^* - p_2^*} \mathcal{V}(X \times_Y X)$ is exact. As \mathcal{V}_X is a sheaf in Zariski topology on X , it suffices to treat the case of affine X and Y . In the case $V = \bigotimes_F^s(\mathfrak{m}/\mathfrak{m}^N)$, which is sufficient by Theorem 4.10 and Lemma 4.12, this amounts to the exactness of the sequence $0 \rightarrow \bigotimes_B^s(\mathfrak{m}_B/\mathfrak{m}_B^N) \rightarrow \bigotimes_A^s(\mathfrak{m}_A/\mathfrak{m}_A^N) \rightarrow \bigotimes_{A \otimes_B A}^s(\mathfrak{m}_{A,B}/\mathfrak{m}_{A,B}^N)$, where B is a smooth k -algebra of finite type, A is a smooth B -algebra of finite type, $\mathfrak{m}_C := \ker(C \otimes_k C \xrightarrow{\times} C)$ for any k -algebra C , and $\mathfrak{m}_{A,B} := \mathfrak{m}_{A \otimes_B A}$. But this is clear.

Conversely, a “coherent” sheaf \mathcal{V} on $\mathfrak{S}m_k$ is sent to the object $\varinjlim \mathcal{V}(U)$, where U runs over the spectra of regular subalgebras in F of finite type over k . (As F is the union of its regular subalgebras of finite type over k , $\varinjlim \mathcal{V}(U)$ is an $(F = \varinjlim \mathcal{O}(U))$ -module. The action of an element $\sigma \in G$ comes as the limit of isomorphisms $\sigma^* : \mathcal{V}(U) \xrightarrow{\sim} \mathcal{V}(U')$, where $U = \mathbf{Spec}(A)$ and $U' = \mathbf{Spec}(\sigma(A))$ induced by the isomorphism $U' \xrightarrow{\sim} U$.) □

Lemma 5.3. *For any “quasi-coherent” flat (as \mathcal{O} -module) sheaf \mathcal{V} on $\mathfrak{S}m_k$ the k -space $\mathcal{V}(Y)$ is a birational invariant of proper Y . If \mathcal{V} is “coherent” then $\mathcal{V}(Y')$ generates the (generic fibre of the) sheaf $\mathcal{V}_{Y'}$ for appropriate finite covers Y' of Y .*

Proof. According to Hironaka, for any pair of smooth proper birational k -varieties Y, Y'' there is a smooth proper k -variety Y' and birational k -morphisms $Y' \xrightarrow{\pi} Y$ and $Y' \rightarrow Y''$. Let $Z \subset Y$ be the subset consisting of points z such that $\pi : \pi^{-1}(z) \rightarrow z$ is not an isomorphism. It is a subvariety of codimension ≥ 2 . As \mathcal{V} is torsion-free, one has $\mathcal{V}(Y) \rightarrow \mathcal{V}(Y') \xrightarrow{i^*} \mathcal{V}(U)$,⁶ where $U := Y - Z \xrightarrow{i} Y'$ is the section of π . It suffices

⁶To show that i^* is also injective, choose an affine covering $\{U_j\}$ of Y' , and a dense affine subset $U' \subseteq U$. As sum of ample divisors is ample, any intersection of open affine subsets is again affine, so $\{U_j \cap U'\}$ is an

to check that for any affine Y one has $\mathcal{V}(Y) \xrightarrow{\sim} \mathcal{V}(U)$. Choose an affine covering $\{U_j\}$ of U . Then $0 \rightarrow \mathcal{V}(U) \rightarrow \bigoplus_j \mathcal{O}(U_j) \otimes_{\mathcal{O}(Y)} \mathcal{V}(Y) \rightarrow \bigoplus_{i,j} \mathcal{O}(U_i \cap U_j) \otimes_{\mathcal{O}(Y)} \mathcal{V}(Y)$ is exact, so, as $0 \rightarrow \mathcal{O}(U) = \mathcal{O}(Y) \rightarrow \bigoplus_j \mathcal{O}(U_j) \rightarrow \bigoplus_{i,j} \mathcal{O}(U_i \cap U_j)$ is also exact, we get $\mathcal{V}(Y) = \mathcal{V}(U)$. \square

REMARK. If $\mathcal{V} : Y \mapsto \Omega_{k(Y)}^j / \Omega^j(Y)$ then the sequence $0 \rightarrow \Omega^j(Y) \rightarrow \Omega_{k(Y)}^j \rightarrow \mathcal{V}(Y) \rightarrow H^1(Y, \Omega_Y^j) \rightarrow 0$ is exact, so $\mathcal{V}(Y)$ is birationally invariant if and only if $j = 0$: for any closed smooth $Z \subset \mathbb{P}^{j+1} = Y$ of codimension 2 such that $\Omega^{j-1}(Z) \neq 0$ one has $H^1(Y', \Omega_{Y'}^j) \cong H^1(Y, \Omega_Y^j) \oplus \Omega^{j-1}(Z)$, where Y' is the blow-up of Y along Z .

Then, using Lemma 5.3, we get a left exact (non faithful) functor (with faithful restriction to the subcategory of “coherent” sheaves)

$$\{\text{flat “quasi-coherent” sheaves on } \mathfrak{S}m_k\} \xrightarrow{\Gamma} \{\text{smooth representations of } G_{F/k} \text{ over } k\}$$

given by $\mathcal{V} \mapsto \varinjlim \Gamma(Y, \mathcal{V}_Y)$, where Y runs over the smooth proper models of subfields in F of finite type over k . This functor is not full, and the objects in its image are highly reducible, e.g., $\Gamma(\Omega_{F/k}^1) \cong \bigoplus_A (A(F)/A(k)) \otimes_{\text{End}(A)} \Gamma(A, \Omega_{A/k}^1)$, where A runs over the set of isogeny classes of simple abelian varieties over k . If \mathcal{V} is “coherent” and $\Gamma(Y, \mathcal{V}_Y)$ has the Galois descent property then $\Gamma(\mathcal{V})$ is admissible. However, there is no Galois descent property in general.

EXAMPLE. Let Y' be a smooth projective hyperelliptic curve $y^2 = P(x)$, considered as a 2-fold cover of the projective line Y . Then, for $\mathcal{V}_Y = (\Omega_{Y/k}^1)^{\otimes 2}$, the section $y^{-2}(dx)^2 = P(x)^{-1}(dx)^2$ is a Galois invariant element of $\Gamma(Y', \mathcal{V}_{Y'})$, which is not in $\Gamma(Y, \mathcal{V}_Y) = 0$.

6. $\mathcal{A}/\mathcal{A}_{>m}$

The only finite-dimensional objects of \mathcal{A} are direct sums of copies of F , so the category \mathcal{A} is far from being tannakian. However, \mathcal{A} admits a decreasing filtration by Serre subcategories $\mathcal{A}_{>m}$ such that all $\mathcal{A}/\mathcal{A}_{>m}$ are again abelian tensor categories and their objects are finite-dimensional. The category $\mathcal{A}/\mathcal{A}_{>m}$ is not rigid.

Let $\mathcal{A}_{>m}$ be the full subcategory of \mathcal{A} with objects V such that $V_m = 0$. Clearly, $\mathcal{A}_{>m}$ is a Serre subcategory of \mathcal{A} . Moreover, it is an “ideal” in \mathcal{A} in the sense that the tensor product functor $\mathcal{A}_{>m} \times \mathcal{A} \rightarrow \mathcal{A}$ factors through $\mathcal{A}_{>m}$, so the quotient abelian category $\mathcal{A}/\mathcal{A}_{>m}$ carries a tensor structure.

By definition, the objects of $\mathcal{A}/\mathcal{A}_{>m}$ are the objects of \mathcal{A} , but the morphisms are defined by $\text{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V, V') = \text{Hom}_{\mathcal{A}}(\langle V_m \rangle, V' / \langle V' \rangle_{>m}) = \text{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V, \langle V'_m \rangle)$, where $\langle V_m \rangle$ denotes the semi-linear subrepresentation of V generated by V_m and $\langle V' \rangle_{>m}$ is the maximal subobject of V' in $\mathcal{A}_{>m}$.⁷ In particular, $V \cong \langle V_m \rangle$ in $\mathcal{A}/\mathcal{A}_{>m}$.

EXAMPLE. $\mathcal{A}/\mathcal{A}_{>0}$ is equivalent to the category of finite-dimensional k -vector spaces.

affine covering of U' . Then the diagram

$$\begin{array}{ccc} \mathcal{V}(Y') & \hookrightarrow & \bigoplus_i \mathcal{V}(U_i) \\ i^* \swarrow & & \downarrow \varphi \\ \mathcal{V}(U) & \rightarrow & \mathcal{V}(U') \hookrightarrow \bigoplus_i \mathcal{V}(U_i \cap U') \end{array}$$

is commutative, and φ is injective since $\mathcal{V}(U_i \cap U') = \mathcal{O}(U_i \cap U') \otimes_{\mathcal{O}(U_i)} \mathcal{V}(U_i)$ and \mathcal{V} is torsion-free.

⁷The functor $\mathcal{A} \rightarrow \mathcal{A}_{>m}$, $V \mapsto \langle V \rangle_{>m}$ is right adjoint to inclusion functor $\mathcal{A}_{>m} \rightarrow \mathcal{A}$. In particular, it is left exact.

The functor $\mathcal{A}/\mathcal{A}_{>m} \rightarrow \mathfrak{S}\mathcal{L}_m^u$, $V \mapsto V_m$ is exact, faithful and tensor. Note also that the objects of $\mathcal{A}/\mathcal{A}_{>m}$ are *finite-dimensional*. Namely, $\bigwedge^{\dim_{K_m} V_m+1} V = 0$.

Let Φ be a monoid of one-dimensional objects of $\mathcal{A}/\mathcal{A}_{>m}$, such as $(\Omega_{F/k}^m)^{\otimes N}$ for any $N \geq 0$. The set Φ is partially ordered: $\omega \leq \eta$ if there is $\xi \in \mathcal{A}/\mathcal{A}_{>m}$ such that $\eta \cong \omega \otimes \xi$. In particular, $\omega \leq \omega \otimes \eta$ and $\eta \leq \omega \otimes \eta$. If $k = \overline{\mathbb{Q}}$ then Φ consists of some (symmetric) F -tensor powers of $\Omega_{F/k}^m$.

Lemma 6.1. *The k -vector space $\text{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V \otimes \omega, V' \otimes \omega)$ is finite-dimensional and independent of $\omega \in \Phi$ for ω sufficiently big.*

Proof. For any $\omega, \eta \in \Phi$ such that $\omega \leq \eta$ (i.e., $\eta \cong \omega \otimes \xi$ for some $\xi \in \mathcal{A}/\mathcal{A}_{>m}$) the twist by ξ defines a canonical inclusion $\text{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V \otimes \omega, V' \otimes \omega) \subseteq \text{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V \otimes \eta, V' \otimes \eta)$. The k -vector spaces

$$\begin{aligned} \text{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V, V') &= \text{Hom}_{\mathcal{A}}(\langle V_m \rangle, V'/(V')_{>m}) \rightarrow \text{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V \otimes U, V' \otimes U) \\ &= \text{Hom}_{\mathcal{A}}(\langle V_m \otimes_{K_m} U_m \rangle, (V' \otimes U)/(V' \otimes U)_{>m}) \subseteq \text{Hom}_{K_m \langle G_{K_m/k} \rangle}(V_m \otimes_{K_m} U_m, V'_m \otimes_{K_m} U_m) \end{aligned}$$

are finite-dimensional. On the other hand,

$$\begin{aligned} \text{Hom}_{\mathcal{A}}(\langle V_m \rangle, V'/(V')_{>m}) &\subseteq \text{Hom}_{K_m \langle G_{K_m/k} \rangle}(V_m, V'_m) \\ &\subseteq \text{Hom}_{K_m \langle G_{K_m/k} \rangle}(V_m \otimes_{K_m} U_m, V'_m \otimes_{K_m} U_m) \end{aligned}$$

for any $U \in \mathcal{C}$ with $U_m \neq 0$, where the second equality takes place if and only if $\dim_{K_m} U_m = 1$, e.g., for $U \in \Phi$. \square

Let $Ob(\mathcal{A}_{\Phi,m}^+) := Ob(\mathcal{A})$ and $\text{Hom}_{\mathcal{A}_{\Phi,m}^+}(V, V') := \text{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V \otimes \omega, V' \otimes \omega)$ for sufficiently big $\omega \in \Phi$. Then $\otimes \omega : \mathcal{A}_{\Phi,m}^+ \rightarrow \mathcal{A}_{\Phi,m}^+$ is a fully faithful functor, so we can invert objects in Φ to get a category $\mathcal{A}_{\Phi,m} := \mathcal{A}_{\Phi,m}^+[\Phi^{-1}]$. If Φ is the set of all one-dimensional objects of $\mathcal{A}/\mathcal{A}_{>m}$ then $\mathcal{A}_m := \mathcal{A}_{\Phi,m}$ is tannakian.

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