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On Maximal Proper Subgroups of Field Automorphism Groups

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ON MAXIMAL PROPER SUBGROUPS OF FIELD AUTOMORPHISM GROUPS

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Abstract. Let G be the automorphism group of an extension F/k of algebraically closed fields of characteristic zero of transcendence degree n, $1 \le n \le \infty$. In this paper we

- construct some maximal closed non-open, and some (all, in the case of countable transcendence degree) maximal open subgroups of G;
- describe, in the case of countable transcendence degree, the automotphism subgroups over the intermediate subfields (a question of Krull, [K2]);
- construct a subfunctor Γ of the identity functor on the category of smooth representations of G, coincident (via the forgetful functor) with the functor Γ on the category of smooth admissible semi-linear representations of G constructed in [R2] in the case $n = \infty$ and $k = \overline{\mathbb{Q}}$.

1. Structure of G

Let F/k be an extension of countable or finite transcendence degree n, $1 \le n \le \infty$, of algebraically closed fields of characteristic zero, and $G = G_{F/k}$ be its automorphism group.

Following [Jac, ПШ-Ш, Sh, I] (and generalizing the case [K1] of algebraic extension), consider G as a topological group with the base of open subgroups given by the stabilizers of finite subsets of F. Then G is a totally disconnected Hausdorff group, and for any intermediate subfield $k \subseteq K \subseteq F$ the topology on $G_{F/K}$ coincides with the restriction of the topology on G. G is locally compact if and only if $n < \infty$.

The classical morphism β : {subfields F over k} $\hookrightarrow$ {closed subgroups of G}, given by $K \mapsto G_{F/K}$, is injective, inverts the inclusions, transforms the compositum of subfields to the intersection of subgroups, and respects the units: $k \mapsto G$. The image of β is stable under the passages to sup-/sub- groups with compact quotients; β identifies the subfields over which F is algebraic with the compact subgroups of G ([Jac, ПШ-Ш, Sh, I]).

In particular, the proper subgroups in the image of β are the compact subgroups in the case $n = 1$.

The map $H \mapsto F^H$, left inverse of β, preserves the order, but it is not a morphism.

Let $A\Pi$ be the set of algebraically closed subfields of F of finite transcendence degree over k. There is a morphism of partially ordered commutative associative unitary monoids (transforming the intersection of subgroups to the algebraic closure of the compositum of subfields) α : {open subgroups of G} $\longrightarrow\!\!\!\!\rightarrow A\Pi$, uniquely determined by the condition $G_{F/\alpha(U)} \subseteq U$ and the transcendence degree of $\alpha(U)$ over k is minimal. Details are in Proposition 2.6.

Theorem 1.1 ([R1]). 1. *The subgroup G° of G, generated by the compact subgroups, is open and topologically simple, if $n < \infty$. If $n = \infty$ then G° is dense in G.*

2. *Any closed normal proper subgroup of G is trivial, if $n = \infty$.*

If $n < \infty$, the left G-action on the one-dimensional oriented $\mathbb{Q}$-vector space of right-invariant measures on G gives rise to a surjective homomorphism, the modulus, $\chi : G \longrightarrow \mathbb{Q}_+^\times$, which is trivial on G°. However, I know nothing about the discrete group $\ker \chi / G^\circ$.

Then one can characterize the image of β in the case $n = \infty$ in the following 4 steps.

1. The normalizers $G_{\{F,L\}/k}$ of $G_{F/L}$ in G for all $L \in A\Pi\backslash\{k\}$ are the maximal open proper subgroups of G. (This follows from Proposition 2.6.)
2. The subgroups $G_{F/L}$ of G for all $L \in A\Pi\backslash\{k\}$ are minimal closed non-trivial normal subgroups in $G_{\{F,L\}/k}$ from (1). (This follows from Theorem 1.1 (2).)
3. The subgroups $G_{F/L}$ of G for all non-trivial extensions $L \subset F$ of k of finite type are the open subgroups containing normal co-compact subgroups of type $G_{F/\overline{L}}$ from (2). (This follows from the above Galois theory for $\overline{L}/k$.)
4. The proper subgroups in the image of β are intersections of subgroups from (3).

Remark. The subgroups $G_{F/L}$ of G for all extensions $L \subset F$ of k of finite type and transcendence degree one are the subgroups $G_{F/L}$ from (3) with the only maximal proper subgroup of G containing them.

2. Maximal open subgroups

In this section we assume that $\mathrm{tr.deg}(F/k) = n \leq \infty$.

Lemma 2.1. *If $\overline{L_1}, \overline{L_2}$ are proper subfields of F and $\mathrm{tr.deg}(\overline{L_1}/\overline{L_1} \cap \overline{L_2}) > 1$ then the group H, generated by $G_{F/\overline{L_1}}$ and $G_{F/\overline{L_2}}$, contains $G_{F/\overline{L}}$, where $\overline{L}$ is a subfield of F, not containing L_1, and such that $\overline{L_1} \cap \overline{L}$ contains $\overline{L_1} \cap \overline{L_2}$ as a proper subfield.*

Proof. Set $k := \overline{L_1} \cap \overline{L_2}$. Clearly, H contains $G_{F/\sigma(\overline{L_1})}$ for any $\sigma \in G_{F/\overline{L_2}}$. As $\sigma(\overline{L_1})$ is algebraically closed and $\overline{L_1} \cap \sigma(\overline{L_1}) \supseteq k$, it remains to choose $\sigma \in G_{F/\overline{L_2}}$ such that $\sigma^{-1}(\overline{L_1}) \not\subseteq \overline{L_1}$ and $\overline{L_1} \cap \sigma(\overline{L_1}) \neq k$, and to set $L := \sigma(\overline{L_1})$.

Let $x \in \overline{L_1} - k$ and $y \in \overline{L_1} - \overline{k(x)}$. We need $\sigma \in G_{F/\overline{L_2}}$ such that $\sigma x \in \overline{L_1}$ and $\sigma^{-1} y \notin \overline{L_1}$. If y is not in $\overline{L_2(x)}$ then such $\sigma \in G_{F/\overline{L_2}}$, clearly, exists.

Let now y be algebraic over $L_2(x)$. Then x and y are related by a polynomial $P(x, y) = 0$ with coefficients in $\overline{L_2}$. Let us enumerate the non-zero coefficients of P by a set $\{0, \dots, N\}$. We shall assume that the polynomial $P(X, Y) = \sum_{s=0}^{N} p_s X^{i_s} Y^{j_s}$ is irreducible over $\overline{L_2}$, and $p_0 = 1$. Suppose that $\sigma^{-1}(\overline{L_1}) \subseteq \overline{L_1}$ for any $\sigma \in G_{F/\overline{L_2}}$ such that $\sigma x \in \overline{L_1}$. Then $-(dP)(\sigma x, \sigma y) = P_I(\sigma x, \sigma y) d(\sigma x) + P_{II}(\sigma x, \sigma y) d(\sigma y) \in F \otimes_{L_1} \Omega^1_{L_1/k}$, which can be rewritten as $\sum_{s=1}^{N} \sigma x^{i_s} \sigma y^{j_s} dp_s \in F \otimes_{L_1} \Omega^1_{L_1/k}$. It remains to show that the whole space F^N is the linear envelope of the vectors $(x^{i_1} \sigma y^{j_1}, \dots, x^{i_N} \sigma y^{j_N})$ for $\sigma \in G_{F/\overline{L_2}}$ and $\sigma x \in \overline{L_1}$ (since then $dp_s \in F \otimes_{L_1} \Omega^1_{L_1/k}$ for all s, i.e. $p_s \in k$).

If $\sum_{s=1}^{N} \lambda_s \sigma x^{i_s} \sigma y^{j_s} = 0$ for $\sigma \in G_{F/\overline{L_2}}$ as above, then we may assume that $\lambda_1 = 1$ and the number M of non-zero λ_s is minimal: $\sum_{s=1}^{M} \lambda_s \sigma x^{i_s} \sigma y^{j_s} = 0$. Subtracting from this equality its image under the action of an appropriate σ', we get that $\lambda_s \in \bigcap_{z \in \overline{L_1} - k} \overline{L_2(z)} \subseteq \bigcap_{b=1}^{\infty} \overline{L_2(x^b)} = \overline{L_2}$, contradicting the minimality of the polynomial P. $\square$

Corollary 2.2. *If $\mathrm{tr.deg}(L_1/L_1 \cap L_2) < \infty$, L_1, L_2 are proper algebraically closed subfields in F and L_1 is not contained in L_2 then $H \supseteq G_{F/L}$ for some algebraically closed subfield L such that $\mathrm{tr.deg}(L_1/L_1 \cap L) = 1$.*

Proof. The set of algebraically closed subfields of L, not containing L_1 and such that $H \supseteq G_{F/L}$, is non-empty, since contains L_2. Let $\mathrm{tr.deg}(L_1/L_1 \cap L)$ be minimal. According to Lemma 2.1, $\mathrm{tr.deg}(L_1/L_1 \cap L) = 1$. $\square$

Lemma 2.3. *If L_1, L_2 are algebraically closed subfields in F and $\mathrm{tr.deg}(L_1/L_1 \cap L_2) = 1$ then the subgroup H generated by G_{F/L_1} and G_{F/L_2} is dense in $G_{F/L_1 \cap L_2}$.*

Proof. This is a version of Lemma 2.16 from [R1]. We may assume that L_2 is a proper subfield of F not contained in L_1. Thus, L_1 is an algebraic closure of $k(x)$ for some $x \in F - L_2$, where $k := L_1 \cap L_2$. Then for any $z \in F - L_2$, $y \in F - L_1$ there exist $\sigma \in G_{F/L_2}$, $\tau \in G_{F/L_1}$ such that $\sigma x = z$, $\tau x = y$. Then for any $y \in F - k$ there exists $\sigma \in G_{F/L_1}G_{F/L_2}$ such that $\sigma x = y$. It follows that H is a normal subgroup of G. If $n = \infty$ then the topological group G is simple (Theorem 2.9, [R1]), i.e., H is dense in G. If $n < \infty$ then, according to the same Theorem 2.9, [R1], the topological group G° is also simple, and thus, the closure of H contains G°. It is known (Lemma 2.15, [R1]) that $G = G_{F/L_1}G^\circ$, i.e., the closure of H again coincides with G. $\square$

Corollary 2.4. *If L_1, L_2 are proper algebraically closed subfields of F, L_1 is not contained in L_2 and $\mathrm{tr.deg}(L_1/L_1 \cap L_2) < \infty$ then the closure of H contains $G_{F/L}$ for some proper algebraically closed subfield L in L_1.*

Proof. According to Corollary 2.2, there exists an algebraically closed subfield L' in F such that $\mathrm{tr.deg}(L_1/L_1 \cap L') = 1$ and $H \supseteq G_{F/L'}$. It follows from Lemma 2.3 that the subgroup generated by G_{F/L_1} and $G_{F/L'}$ is dense in $G_{F/L}$, where $L := L_1 \cap L'$. We deduce from this that the closure of H contains some $G_{F/L}$ of the desired type. $\square$

Proposition 2.5. *The subgroup $H = \langle G_{F/\overline{L_1}}, G_{F/\overline{L_2}}\rangle$ is dense in $G_{F/\overline{L_1}\cap\overline{L_2}}$ for any subfields L_1 and L_2 in F such that $\mathrm{tr.deg}(\overline{L_1}/\overline{L_1} \cap \overline{L_2}) < \infty$.*

Proof. We may assume that $\overline{L_2}$ does not contain L_1, and $\overline{L_1} \neq F$. Set $k := \overline{L_1} \cap \overline{L_2}$. Let $L = \overline{L} \supseteq k$ be such that $H \supseteq G_{F/L}$ and let $\mathrm{tr.deg}(L/k) (\leq \mathrm{tr.deg}(\overline{L_1}/k))$ be minimal. If $L \not\subseteq \overline{L_j}$ ($j = 1$ or $j = 2$) then, according to Corollary 2.4, ($\mathrm{tr.deg}(L/L \cap \overline{L_j}) \leq \mathrm{tr.deg}(L/k) < \infty$), the closure of the subgroup generated by $G_{F/L}$ and $G_{F/\overline{L_j}}$, (and thus, the closure of H as well) contains $G_{F/L'}$, where L' is a proper algebraically closed subfield in L, contradicting the minimality of $\mathrm{tr.deg}(L/k)$. Thus, $L = k$. $\square$

Proposition 2.6. *There is a morphism of commutative associative unital monoids inverting inclusions (transforming the intersection of subgroups to the algebraic closure of the compositum of subfields, and the identity G to the identity k)*

$$\left\{ \begin{array}{c} \text{open} \\ \text{subgroups of } G \end{array} \right\} \stackrel{\alpha}{\twoheadrightarrow} \left\{ \begin{array}{c} \text{algebraically closed subfields of } F \\ \text{of finite transcendence degree over } k \end{array} \right\} =: A\Pi,$$

uniquely determined by the following equivalent conditions:

- $G_{F/\alpha(H)}$ *is a normal subgroup of H and, if possible, $\alpha(H) \neq F$;*
- $G_{F/\alpha(H)} \subseteq H$ *and $\mathrm{tr.deg}(\alpha(H)/k)$ is minimal.*

In particular, for any $L \in A\Pi$ distinct from F and k, the normalizer $G_{\{F,L\}/k}$ of $G_{F/L}$ is maximal among proper subgroups of G.

If $n = \infty$ then any proper open subgroup of G is contained in a maximal proper subgroup of G, and any maximal proper open subgroup of G is of type $G_{\{F,L\}/k}$ for some $L \in A\Pi$, $L \neq k$.

Proof. If a subgroup H of G is open then there exists $L \in A\Pi$ such that $G_{F/L} \subseteq H$. Let L be of minimal possible transcendence degree. For any $\sigma \in H$ the group H contains the closure of the subgroup generated by $G_{F/L}$ and $G_{F/\sigma(L)}$. Then according to Proposition 2.5, H contains $G_{F/L\cap\sigma(L)}$, and therefore, $\sigma(L) \supseteq L$. Thus, $H \subseteq G_{\{F,L\}/k}$. In other words, $G_{F/L} \triangleleft H$.

Let us check the maximality of $G_{\{F,L\}/k}$ for $L \neq k, F$. If $G_{\{F,L\}/k} \subseteq U$ then $G_{F/K} \subseteq U \subseteq G_{\{F,K\}/k}$ for some $K = \overline{K}$ such that $\mathrm{tr.deg}(K/k) \leq \mathrm{tr.deg}(L/k)$, since $G_{F/L} \subseteq U$. The inclusion $G_{\{F,L\}/k} \subseteq G_{\{F,K\}/k}$ takes place only if either $K = L$, or $K = k$. In the second case $U = G$.

If, moreover, $G_{F/L'} \subseteq H$ for another $L' \in A\Pi$ then either $L \subseteq L'$, or $L = F$. If, in addition, $H \subseteq G_{\{F,L'\}/k}$ then either $L = L'$, or $L = F$, or $L' = F$. This shows that α is well-defined.

Let us check that α is a morphism. If $G_{F/\overline{K}} \subseteq U \subseteq G_{\{F,\overline{K}\}/k}$ and $G_{F/\overline{L}} \subseteq V \subseteq G_{\{F,\overline{L}\}/k}$ then $G_{F/\overline{K}\ \overline{L}} \subseteq U \cap V \subseteq G_{\{F,\overline{K},\overline{L}\}/k}$. As $G_{F/\overline{K}\ \overline{L}} \supseteq G_{F/\overline{KL}}$ and $G_{\{F,\overline{K},\overline{L}\}/k} \subseteq G_{\{F,\overline{KL}\}/k}$, it remains to show that $\alpha(U \cap V) = F$ in the case $\overline{KL} = F$. If $\alpha(U) = F$ then evidently $\alpha(U \cap V) = F$. If $\overline{K}$ and $\overline{L}$ are distinct from F then $\alpha(G_{\{F,\overline{K},\overline{L}\}/k}) = \overline{KL}$, i.e. again $\alpha(U \cap V) = F$. □

Remark. If $H \subset G$ is contained in a group of type $G_{\{F,L\}/k}$ for neither $L \in A\Pi \backslash \{k, F\}$ then, for any $x \in F - k$, F is algebraic over the subfield generated by the H-orbit of x.

Corollary 2.7. *G is not the union of its proper open subgroups if $n = \infty$. In particular, there exist $\sigma \in G$ such that $W^{\langle\sigma\rangle} = W^G$ for any smooth (i.e. with open stabilizers) representation W of G.*

Proof. For instance, if $\{x_i \mid i \in \mathbb{Z}\}$ is a transcendence basis of F over k and $\sigma x_i = x_{i+1}$ for any $i \in \mathbb{Z}$ then G is the only open subgroup containing σ. □

Questions. 1. The preimage of any subgroup of a prime index in $\mathbb{Q}_+^\times$ under the modulus character gives, if $n < \infty$, an example of a maximal open proper subgroup, not encounted by Proposition 2.6. Are there any other?

2. Do there exist closed subgroups not contained in maximal proper ones?

3. Valuation subgroups

Let $\mathcal{O}_v$ be a valuation ring in F, $\mathfrak{m}_v = \mathcal{O}_v - \mathcal{O}_v^\times$ be the maximal ideal, and $\kappa(v)$ be the residue field. If $k \subseteq \mathcal{O}_v$, fix a subfield $k \subseteq F' \subseteq \mathcal{O}_v$ identified with $\kappa(v)$ by the reduction modulo $\mathfrak{m}_v$. In this case $\kappa(v)$ is of characteristic zero (and algebraically closed).

Set $G_v := \{\sigma \in G \mid \sigma(\mathcal{O}_v) = \mathcal{O}_v\}$. This is a closed subgroup in G.

The group $\Gamma := F^\times/\mathcal{O}_v^\times \cong \mathbb{Q}^r$ is totally ordered: $v(x) \geq v(y)$ if and only if $xy^{-1} \in \mathcal{O}_v$, where $v : F^\times \longrightarrow \Gamma$ is the natural projection. The rank of v is $r = \dim_{\mathbb{Q}} \Gamma$.

Let us call the transcendence degree of F over F' the codimension of v.

The basic example of $\mathcal{O}_v$ is, after a choice of $x_1, \dots, x_n$ algebraically independent over k and of an embedding $F_j \hookrightarrow \lim\limits_{N \longrightarrow} F_{j-1}((x_j^{1/N}))$ over $F_{j-1}(x_j)$, where $F_j = \overline{F_{j-1}(x_j)}$, preimage in F of the ring $\widehat{\mathcal{O}}_n = k \oplus \widehat{\wp}_1 = \lim\limits_{N \longrightarrow} k[[x_1^{1/N}]] \oplus \widehat{\wp}_2 = \widehat{\mathcal{O}}_{j-1} \oplus \widehat{\wp}_j$, where $\widehat{\wp}_{n+1} = 0$ and $\widehat{\wp}_j = \lim\limits_{N \longrightarrow} x_j^{1/N} k((x_1^{1/N})) \dots ((x_{j-1}^{1/N}))[[x_j^{1/N}]] \oplus \widehat{\wp}_{j+1}$ are prime ideals for all $1 \leq j \leq n+1$. In this case $v(x_1^{m_1}) < \dots < v(x_n^{m_n})$ for all $m_1, \dots, m_n > 0$.

If $r < \infty$ and $\sigma(\mathcal{O}_v) \subseteq \mathcal{O}_v$ for some $\sigma \in G$ then $\sigma \in G_v$, since σ induces surjective endomorphism Γ, i.e. an automorphism.

It is well-known, Exercise 32, Chapter 5, [AM], that $v(\mathcal{O}_p - \wp)$ is the set of all non-negative elements in some isolated subgroup, and

$$\mathbf{Spec}\mathcal{O}_p \xrightarrow{\sim} \{\text{isolated subgroups in } L^\times/\mathcal{O}_p^\times\},$$

$\wp \mapsto \langle v(\mathcal{O}_p - \wp)\rangle$, so there are exactly $r+1$ prime ideals in $\mathcal{O}_v$.

Remarks. 1. If $\wp \neq 0$ is a non-maximal prime ideal of finite codimension in $\mathcal{O}_v$ and $\mathcal{O}_{v'} := (\mathcal{O}_v)_\wp$ then $G_v \subseteq G_{v'}$ (since any element $\sigma \in G_v$ preserves $\wp$, thus also $\mathcal{O}_v - \wp$, i.e. induces an automorphism $\mathcal{O}_{v'}$).

2. There are proper inclusions $G_{\{F,\mathcal{O}_v[x_1^{-1}]\}/\overline{k(x_1)}} \subset G_v \subset G_{\{F,\mathcal{O}_v[x_1^{-1}]\}/k}$ (since $x_j x_1^{\mathbb{Z}} \subset \mathcal{O}_v$ for any $j \geq 2$, and ...) for $r > 1$, i.e. G_v is not maximal.

Let $\mathcal{P}_L^r$ is the set of valuation rings or rank and codimension r in L, containing k, admitting also the following description. Let $\mathcal{C}_X^r$ be the set of chains of irreducible normal subvarieties up to codimension r on an irreducible proper normal variety X over k. Any birational morphism $X' \xrightarrow{\pi} X$ induces an embedding $\mathcal{C}_X^r \hookrightarrow \mathcal{C}_{X'}^r$, $(Z^1 \supset \cdots \supset Z^r) \mapsto (W^1 \supset \cdots \supset W^r)$, where $W^0 := X'$ and $W^j := (\pi|_{W^{j-1}})^{-1}_{\text{prop}}(Z^j)$ for $1 \leq j \leq r$ (and $\pi|_{W_1} : W_1 \xrightarrow{\sim} Z_1$). If L is of finite type over k then $\mathcal{P}_L^r \cong \lim\limits_{X \longrightarrow} \mathcal{C}_X^r$, where X runs over the models of L/k, and $\mathcal{P}_F^r = \lim\limits_{\longleftarrow L} \mathcal{P}_L^r$. In particular, $\mathcal{P}_L^1 = C(k)$, if $n = 1$, where C is a smooth proper model of the field L over k.

Lemma 3.1. *If $0 \leq r < n+1 \leq \infty$ then the G-action on $\mathcal{P}_F^r$ is transitive.*

Proof. Let $v : F^\times \twoheadrightarrow \Gamma \cong \mathbb{Q}^r$ be an element of $\mathcal{P}_F^r$.

Let $0 = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_{r-1} \subset \Gamma_r = \Gamma$ be the isolated subgroups in Γ. Choose $x_1, x_2, \dots \in \mathcal{O}_v$ such that $v(x_j) \in \Gamma_j - \Gamma_{j-1}$ for $1 \leq j \leq r$ and $x_{r+1}, x_{r+2}, \dots$ modulo $\mathfrak{m}_v$ form a transcendence basis of $\mathcal{O}_v/\mathfrak{m}$ over k. Clearly, $x_1, x_2, \dots$ are algebraically independent over k. Set $k' = \overline{k(X_{r+1}, X_{r+2}, \dots)}$ and $\widehat{F}_r := \lim\limits_{N \longrightarrow} k'((X_1^{1/N}))\dots((X_r^{1/N}))$.

Consider the set S of embeddings into $\widehat{F}_r$ of subfields in F containing $k(x_1, \dots, x_n)$, extending φ and respecting the valuation. The set S is partially ordered. Clearly, S contains maximal elements. Let $L' \overset{\xi}{\hookrightarrow} \widehat{F}_r$ be a maximal element of S. If $\sum_s a_s y^s = 0$ is a minimal polynomial of $y \in F$ over L' then there exist $i < j$ such that $v(y) = \frac{v(a_i/a_j)}{j-i}$. There exists $z \in \widehat{F}_r$ such that $\sum_s \xi(a_s) z^s = 0$ and $v(z) = \frac{v(\xi(a_i/a_j))}{j-i}$. Therefore, ξ extends to $L'(z)$, and thus, $L' = F$.

This shows that the embedding $k(x_1, x_2, \dots) \overset{\varphi}{\hookrightarrow} k'((X_1))\dots((X_r))$, $x_j \mapsto X_j$, respects the valuation, and extends to an embedding $F \hookrightarrow \widehat{F}_r$, respecting the valuation.

Thus, any element of $\mathcal{P}_F^r$ is determined by a choice of $x_1, x_2, \dots \in F$ algebraically independent over k and by an embedding of F into $\widehat{F}_r$ over $k(x_1, x_2, \dots)$. Clearly, the G-action is transitive on these data. □

The G_v-action on $\kappa(v)$ induces a homomorphism $G_v \twoheadrightarrow G_{\kappa(v)/k}$.

Let $G_v^\dagger := \{\sigma \in G_v \mid \sigma x \equiv x \bmod \mathfrak{m}_v \text{ for any } x \in \mathcal{O}_v\}$ be its kernel.

A continuous[1] section of $G_{\kappa(v)/k} \hookrightarrow G_v$ is determined by a subfield $F' \subseteq \mathcal{O}_v$, identified with $\kappa(v)$ by the reduction modulo $\mathfrak{m}_v$, and by an embedding of F into the field of iterated Puiseux series $\lim_{N \longrightarrow} F'((t_1^{1/N}))\dots((t_r^{1/N}))$ over F' compatible with valuations (i.e. by a choice of a section $\Gamma \hookrightarrow F^\times$ of the valuation v).

As the centralizer of $G_v^\dagger$ is trivial, such sections form a $G_v^\dagger$-torsor. One has a decomposition $G_v = G_v^\dagger(G_v \cap G_{\{F,F'\}/k})$, where $G_v^\dagger \cap G_{\{F,F'\}/k} = G_v \cap G_{F/F'}$.

Let B_+ be the group of linear transformations of Γ (isomorphic to the group of upper-triangular rational $r \times r$ matrices with positive diagonal entries), respecting the order, i.e. the flag of isolated subgroups $0 = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_r = \Gamma$ and the orientation of each $\Gamma_j/\Gamma_{j-1} \cong \mathbb{Q}$, $1 \le j \le r$. Then $\mathcal{P}_L^r$ consists of (some) valuations $(L^\times/k^\times) \otimes \mathbb{Q} \longrightarrow\!\!\!\!\to \mathbb{Q}^r$ modulo the B_+-action. The G_v-action on $F^\times$ induces a linear action on Γ, preserving the order, i.e. a homomorphism $G_v \longrightarrow B_+$, surjective and splitting, if the rank of v coincides with its codimension.

Let $G_v^\circ := \{\sigma \in G_v \mid \sigma x/x \in \mathcal{O}_v^\times \text{ for any } x \in F^\times\}$ be its kernel.

There is a homomorphism $G_v^\dagger \cap G_v^\circ \longrightarrow \mathrm{Hom}(\Gamma, \kappa(v)^\times)$,[2] $\sigma \longmapsto (v(x) \mapsto \sigma x/x \bmod \mathfrak{m}_v)$, surjective and splitting, if the rank of v coincides with its codimension.

Lemma 3.2. *Its kernel $G_v^1 := \{\sigma \in G_v \mid \sigma x/x \in 1 + \mathfrak{m}_v \text{ for any } x \in F^\times\}$ is a discrete subgroup if $n < \infty$. The functors $H^0(G, -)$ and $H^0(G_v^1, -)$ coincide on the category $\mathcal{S}m_G$ of smooth representations of G, if $n = \infty$ and the rank of v is finite, non-zero and coincides with its codimension.*

Proof. The G_v-action on F extends to a continuous G_v-action on the completion F_v of F, and the continuous G_v^1-action on F_v is determined by the action on some r-tuple of elements, representing each of isolated subgroups of Γ and by the action on some maximal subfield with the trivial valuation. Clearly, the latter action is determined by its restriction to any transcendence basis over k. The second assertion follows from the fact that any open subgroup containing G_v^1 coincides with G, which follows from Proposition 2.6. □

Proposition 3.3. *The set $\{L \overset{/k}{\hookrightarrow} F\}$ of embeddings into F over k of the function field L of a d-dimensional variety over k is a disjoint union of (non-empty) G_v-orbits $O_{p,v} := \{L \overset{\sigma}{\hookrightarrow} F \mid \sigma^{-1}(\mathcal{O}_v) = \mathcal{O}_p\}$, where $v \in \mathcal{P}_F^r$, $p \in \mathcal{P}_L^j$ and $m := \max(0, r + d - n) \le j \le M := \min(d, r)$. In particular, $\mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F\}]_{G_v^\dagger} = \bigoplus_{j=m}^M \bigoplus_{p \in \mathcal{P}_L^j} \mathbb{Q}[\{\kappa(p) \overset{/k}{\hookrightarrow} \kappa(v)\}]$ and $G_v \backslash \{L \overset{/k}{\hookrightarrow} F\} = \coprod_{j=m}^M \mathcal{P}_L^j$. If $d = n$ then the stabilizers are compact and isomorphic to $\widehat{\mathbb{Z}}^r$.*

Proof. Any element $x \in \sigma(L^\times)$ is either contained in $\mathcal{O}_v$, or its inverse is contained in $\mathcal{O}_v$, i.e., $\mathcal{O}_p := \sigma^{-1}(\mathcal{O}_v)$ is a valuation ring.[3]

Clearly, $\mathrm{rk}(L^\times/\mathcal{O}_p^\times) \le M$. Let us check that $\mathrm{rk}(L^\times/\mathcal{O}_p^\times) \ge m$, and in particular, the inclusion $(L^\times/\mathcal{O}_p^\times) \otimes \mathbb{Q} \overset{\sigma}{\hookrightarrow} F^\times/\mathcal{O}_v^\times$ is bijective for $d = n$. Let $x \in F^\times$ and $\sum_{j=0}^m a_j x^j = 0$

[1] One can apply the well-known fact that the coefficients of any algebraic (over $k(t)$) element of $k((t))$ generate in k a subfield of finite type over $\mathbb{Q}$.

[2] Clearly, $\mathrm{Hom}(\Gamma, \kappa(v)^\times) \cong (\lim_{\longleftarrow N} M_N)^r$, where $M_{NN'} := \kappa(v)^\times \longrightarrow\!\!\!\!\to M_{N'} := \kappa(v)^\times$ is the raising to the N-th power.

[3] Note, that the henselian property does not suffice. For instance, if $k(X) = k(x_1, \dots, x_d)$ is embedded into the field of fractions of $\mathcal{O} := k[[T_1, \dots, T_N]]$, $x_j \mapsto T^{\alpha_j}$, $|\alpha_j| = 0$ then $k(X) \cap \mathcal{O} = k$, since it consists of homogeneous functions in $T_1, \dots, T_N$ of degree 0.

be a minimal polynomial over $\sigma(\mathcal{O}_p)$. Then there exist $0 \le i < j \le m$ such that $a_i a_j \neq 0$ and $v(a_i x^i) = v(a_j x^j)$, so $v(x) = \frac{v(a_i/a_j)}{j-i}$, i.e., the element $v(x) \in F^\times/\mathcal{O}_v^\times$ is in the image of $(L^\times/\mathcal{O}_p^\times) \otimes \frac{1}{m!}\mathbb{Z}$.

We may assume that $a_j = 1$ for some j. Thus, if $x \in \mathcal{O}_v$ then its image in $\mathcal{O}_v/\mathfrak{m}_v$ is algebraic over $\sigma(\mathcal{O}_p)/\sigma(\mathfrak{m}_p)$, i.e. $\mathcal{O}_v/\mathfrak{m}_v \cong \overline{\mathcal{O}_p/\mathfrak{m}_p}$, and therefore, $p \in \mathcal{P}_L^r$.

To check that $O_{p,v}$ is a G_v-orbit, we have to show that for any pair of embeddings $\sigma, \tau : L \overset{/k}{\hookrightarrow} F$ such that $\sigma(\mathcal{O}_p), \tau(\mathcal{O}_p) \subset \mathcal{O}_v$ there exists $\xi \in G_v$, extending $\tau\sigma^{-1} : \sigma(\mathcal{O}_p) \xrightarrow{\sim} \tau(\mathcal{O}_p)$. According to Zorn's lemma, there are maximal elements in the set

$$S := \left\{ A \overset{\varphi}{\hookrightarrow} \mathcal{O}_v \;\middle|\; \begin{array}{l} v(\varphi(x)) \ge v(\varphi(y)), \text{ if } v(x) \ge v(y) \text{ for } x, y \in A, \\ \sigma(\mathcal{O}_p) \subseteq A \subseteq \mathcal{O}_v, \quad \varphi|_{\sigma(\mathcal{O}_p)} = \tau\sigma^{-1} \end{array} \right\} \ni (\sigma(\mathcal{O}_p) \overset{\tau\sigma^{-1}}{\hookrightarrow} \mathcal{O}_v),$$

for instance, $B \overset{\psi}{\hookrightarrow} \mathcal{O}_v$. Then B is integrally closed in $\mathcal{O}_v$, since for any $x \in \mathcal{O}_v$ integral over B an embedding ψ extends to $B[x] \subseteq \mathcal{O}_v$, even respecting the order,[4] since $\mathcal{O}_v$ is integrally closed in F. Thus, any element of $\mathcal{O}_v$ can be presented as a/b, where $a, b \in B$ and $v(a) \ge v(b)$. As ψ respects the order, it maps $\mathcal{O}_v$ into $\mathcal{O}_v$, i.e., $\mathcal{O}_v = B$.

The set $O_{p,v}$ is non-empty, since the valuation p extends to $\overline{L} \cong F$, and the group G permutes the elements of $\mathcal{P}_F^r$ (Lemma 3.1). Then $O_{p,v} \cong G_v/St_{p,v}$, where $St_{p,v} = G_v \cap G_{F/\sigma(L)}$ is an open compact subgroup. The compact subgroups of G_v are contained in G_v° and they project to $\mathrm{Hom}(\Gamma, k^\times)$ injectively (Lemma 3.2), so $St_{p,v} \cong \widehat{\mathbb{Z}}(1)^r$. □

3.1. **Valuations and maximal subgroups.** Example. If $n = 1$ then to any valuation v the decomposition $\{k(C) \overset{/k}{\hookrightarrow} F\} = C(F) - C(k) = \coprod_{C(k)}(\mathfrak{m}_v - \{0\})$ is associated.

Proposition 3.4. *If $n = 1$ then $O_{p,v} \cap O_{q,v'}$ is non-empty for any pair of distinct $v, v' \in \mathcal{P}_F^1$ and any pair of distinct points $p, q \in C(k)$ on a smooth proper curve C over k, i.e. there is an embedding $\sigma : k(C) \overset{/k}{\hookrightarrow} F$ such that $\sigma(\mathcal{O}_p) \subset \mathcal{O}_v$ and $\sigma(\mathcal{O}_q) \subset \mathcal{O}_{v'}$.*

Proof. We consider v and v' as a compatible system of points on smooth proper curves over k with the function fields embedded into F: if $C_\beta \twoheadrightarrow C_\alpha$ then $v_\beta \mapsto v_\alpha$ and $v'_\beta \mapsto v'_\alpha$. One needs to check that there exist β and a map $C_\beta \twoheadrightarrow C$ such that $v_\beta \mapsto p$ and $v'_\beta \mapsto q$.

Choose a non-constant function $x \in \mathcal{O}(C \backslash \{p\})$, i.e. a surjective morphism $C \longrightarrow \mathbb{P}^1$ sending to ∞ only p. Let $C' \longrightarrow C$ be a cover such that the composition $C' \longrightarrow C \longrightarrow \mathbb{P}^1$ is a Galois cover with the group A. For some α such that $v_\alpha \neq v'_\alpha$ choose a surjection $C_\alpha \longrightarrow \mathbb{P}^1$ such that $v_\alpha \mapsto \infty$ and $v'_\alpha \mapsto x(q)$. Consider the normalization D of an irreducible component of $C' \times_{\mathbb{P}^1} C_\alpha$. The surjection $D \longrightarrow C_\alpha$ is isomorphic to the surjection $C_\beta \longrightarrow C_\alpha$ for some β. Let $\pi : C_\beta \longrightarrow C'$ be the projection. As A acts transitively on the fibres of the composition $C' \longrightarrow C \longrightarrow \mathbb{P}^1$, there is an element $\gamma \in A$ such that $\gamma\pi(v'_\beta)$ belongs to the preimage of q. Then the composition $C_\beta \overset{\gamma\pi}{\longrightarrow} C' \longrightarrow C$ maps v_β to p, and v'_β to q. □

Proposition 3.5. *If $r = 1$ then the subgroup H, generated by G_v and $G_{v'}$, acts transitively on $\{L \overset{/k}{\hookrightarrow} F\}$, i.e. H is dense in G.*

Proof. Let L be an extension of k and $v, v' : L^\times/k^\times \longrightarrow \mathbb{Q}$ be a pair of discrete valuations. If $\mathcal{O}_v^\times \subseteq \mathcal{O}_{v'}^\times$ in L then $\mathcal{O}_{v'}^\times/\mathcal{O}_v^\times$ is the kernel of $\mathbb{Q} \supseteq L^\times/\mathcal{O}_v^\times \twoheadrightarrow L^\times/\mathcal{O}_{v'}^\times \subseteq \mathbb{Q}$, which is

[4] i.e., if $\sum_{s=0}^m a_s x^s = 0$, $a_s \in B$ and $v(x) = \frac{v(a_i/a_j)}{j-i}$ for some $i < j$ then one can choose $y \in \mathcal{O}_v$ such that $\sum_{s=0}^m \psi(a_s) y^s = 0$ and $v(y) = \frac{v(\psi(a_i/a_j))}{j-i}$.

evidently injective, so $v = v'$. Let $v \neq v'$. Then for any $x \in \mathcal{O}_v^\times$ one has either $x \in \mathcal{O}_{v'}$, or $x^{-1} \in \mathcal{O}_{v'}$, that is there is some $t \in \mathcal{O}_v^\times \cap \mathcal{O}_{v'}$ such that $t \notin \mathcal{O}_{v'}^\times$. Fix such t and $t^{1/N}$ for all integers $N \geq 1$. Let $x_1, x_2, \dots \in \mathcal{O}_v^\times$ be a lifting of a transcendence basis of $\kappa(v)$ over $k(t)$. Set $k_0 := k$ and define inductively a strictly increasing sequence of algebraically closed subfields in $\{0\} \cup (\mathcal{O}_v^\times \cap \mathcal{O}_{v'}^\times) \subset L$ as follows. For any $i \geq 1$ there exist $P \in k_{i-1}[T]$, an integer $N \geq 1$ and $M \in \mathbb{Q}_+^\times$ such that $y_i := t^{-M}(x_i^{v'(t)} - t^{v'(x_i)} P(t^{1/N})) \in \mathcal{O}_{v'}^\times$ and $y_i \notin k_{i-1} + \mathfrak{m}_{v'}$. Then $y_1, y_2, \dots$ is another lifting of a transcendence basis of $\kappa(v)$ over $k(t)$ in $\mathcal{O}_v^\times$. Set $k_i := \overline{k_{i-1}(y_i)}$ and $k' = \bigcup_{i \geq 1} k_i$. Then $\kappa(v)$ is algebraic over the reduction of k' modulo $\mathfrak{m}_v$ and $\kappa(v')$ is algebraic over the reduction of k' modulo $\mathfrak{m}_{v'}$. This shows that we are reduced to the case of $n = 1$.

By Proposition 3.4, for any $\xi : L \overset{/k}{\hookrightarrow} F$ the map $G_v \times G_{v'} \longrightarrow \{L \overset{/k}{\hookrightarrow} F\}$, given by $(\sigma, \tau) \mapsto \sigma\tau\xi$, is surjective. □

If $n = 1$ define $\varphi : G_v^1 \longrightarrow \Gamma$ by $\sigma \mapsto v(\sigma x/x - 1)$ for any $x \in \mathfrak{m} - \{0\}$, or $x \in F - \mathcal{O}_v$. Independence of x: if $y = \sum_{j \geq 1} a_j x^{j/N}$ and $\sigma x/x = 1 + a x^\alpha + o(x^\alpha)$, where $a \in k^\times$, then $\sigma y - y = \sum_{j \geq 1} a_j x^{j/N}((1 + a x^\alpha + o(x^\alpha))^{j/N} - 1)$, i.e.

$$\sigma y/y - 1 = \frac{v(y)}{v(x)} a x^\alpha + o(x^\alpha). \tag{1}$$

Thus, $v(\sigma y - y) = v(y) + [x^\alpha]$, and therefore, $v(\sigma y/y - 1) = v(\sigma x/x - 1)$.

From the equality $\sigma\tau x/x - 1 = (\sigma(\tau x)/\tau x - 1)\tau x/x + \tau x/x - 1$ we get that $\varphi(\sigma\tau) \geq \min(\varphi(\sigma), \varphi(\tau))$, and $\varphi(\sigma) = \varphi(\sigma^{-1})$: $v(\sigma^{-1}x/x - 1) = v(x/\sigma x - 1) = \varphi(\sigma)$.

Let $G_v^1(\beta) := \{\sigma \in G_v^1 \mid \varphi(\sigma) \geq \beta\}$, where $\beta \in \Gamma \otimes \mathbb{R}$. This is a normal subgroup in G_v°. Then $G_v^1 = G_v^1(0) = G_v^1(0)^+$, where $G_v^1(\beta)^+ := \bigcup_{\gamma > \beta} G_v^1(\gamma) = \{\sigma \in G_v^1 \mid \varphi(\sigma) > \beta\}$. Clearly, $G_v^1(\beta) \neq G_v^1(\gamma)$, if $\beta \neq \gamma$.

Lemma 3.6. *There is a canonical isomorphism* $G_v^1(\beta)/G_v^1(\beta)^+ \xrightarrow{\sim} \mathrm{Hom}(\Gamma, \mathfrak{m}^{[\beta]})$, *where*

$$\mathfrak{m}^{[\beta]} = \{x \in \mathfrak{m} \mid v(x) \geq \beta\}/\{x \in \mathfrak{m} \mid v(x) > \beta\} \cong \begin{cases} k, & \text{if } \beta \in \Gamma \text{ and } \beta > 0 \\ 0, & \text{otherwise} \end{cases}$$

Proof. It is clear from the formula (1) that any element $\sigma \in G_v^1(\beta)$ induces a homomorphism $\Gamma \longrightarrow \mathfrak{m}^{[\beta]}$, $[y] \mapsto \sigma y/y - 1$. This gives a surjective homomorphism $G_v^1(\beta)/G_v^1(\beta)^+ \longrightarrow \mathrm{Hom}(\Gamma, \mathfrak{m}^{[\beta]})$: $\sigma\xi \longmapsto ([x] \mapsto \sigma\xi x/x - 1 = \sigma(\xi x/x - 1)\sigma x/x + (\sigma x/x - 1) \equiv (\xi x/x - 1) + (\sigma x/x - 1))$.

Let us check its injectivity: $\sigma x = x(1 + a x^\beta + x^\beta h)$, $h \in \mathfrak{m}$. Let $\tau[x(1 + a x^\beta + x^\beta h)] := x(1 + a x^\beta)$. Then $\tau\sigma x/x = 1 + a x^\beta$ and, if we set $y := x(1 + a x^\beta)$ then $x(1 + a x^\beta + x^\beta h) = y + o(y^{\beta+1})$, and thus, $\tau \in G_v^1(\beta)^+$. Finally, $G_v^1(\beta)/G_v^1(\beta)^+ = \mathrm{Hom}(\Gamma, \mathfrak{m}^{[\beta]})$. □

Lemma 3.7. $G_v^1(\beta)$ *is surjective over* G/G° *for any* $\beta \in \Gamma \otimes \mathbb{R}$.

Proof. As G_v is maximal and does not contain G°, it suffices to show that the subgroup generated by $G_v^1(\beta)$ and G° contains G_v. A choice of a section $s : \Gamma \longrightarrow F^\times$ of the projection $v : F^\times \longrightarrow \Gamma$ determines an additive section $\mathrm{Hom}(\Gamma, k^\times) \longrightarrow G_v^\circ \cap G^\circ$ of the projection $G_v^\circ \longrightarrow \mathrm{Hom}(\Gamma, k^\times)$. A choice of $x \in \mathfrak{m} \cap s(\Gamma)$ gives a bijection of sets G_v^1 and $\mathfrak{m}$: $\mathfrak{m} \ni m : x \mapsto x(1 + m)$. The elements $(x \mapsto (1 + x^\alpha)^{1/\alpha} - 1) \in G_v \cap G^\circ$, $\alpha \in \mathbb{Q}_+^\times$, determine a set-theoretic section of the projection $G_v \longrightarrow \mathbb{Q}_+^\times$. It remains to check that the subgroup generated by $G_v^1(\beta)$ and $G_v^1 \cap G^\circ$ contains G_v^1. Clearly, $x(1 + a x^\alpha)^{-1/\alpha} \in G_v^1 \cap G^\circ$. If an element of G_v^1 is presented by $h = x(1 + a x^\alpha + o(x^\alpha)) \in x(1 + \mathfrak{m}) \cap k[[x^{1/N}]]$ then

$h(1+a\alpha h^{\alpha})^{-1/\alpha}=x(1+o(x^{\alpha}))\in x(1+\mathfrak{m})\cap k[[x^{1/N}]]$. So the composition of h with an appropriate element of $G_v^1\cap G^{\circ}$ will be in $G_v^1(\beta)$. $\square$

4. Valuations and associated functors

4.1. **The "globalization" functor.** For a discrete valuation $v\in\mathcal{P}_F^r$ and for any $W\in\mathcal{S}m_G$ set $W_v:=\sum_{\sigma\in G_v}W^{G_{F/\sigma(F')}}=\sum_{\sigma\in G_v^{\dagger}}W^{G_{F/\sigma(F')}}\subseteq W$. Clearly, the additive functor $\mathcal{S}m_G\longrightarrow\mathcal{S}m_{G_v}$, $W\mapsto W_v$, is faithful and preserves surjections if $n=\infty$ (and injections in general). Set $\Gamma_r(W):=\bigcap_{v\in\mathcal{P}_F^r}W_v$ and $\Gamma:=\Gamma_1$, so $\Gamma_r:\mathcal{S}m_G\longrightarrow\mathcal{S}m_G$ are additive functors.

Denote by $\mathcal{I}_G$ is the full subcategory in $\mathcal{S}m_G$ consisting of representations W such that $W^{G_{F/M}}=W^{G_{F/M'}}$ for any extension M of k in F and any purely transcendental extension M' of M in F, Denote by $\mathcal{I}=\mathcal{I}_{/k}=\mathcal{I}_{F/k}:\mathcal{S}m_G\longrightarrow\mathcal{I}_G$ the left adjoint to the inclusion functor $\mathcal{I}_G\hookrightarrow\mathcal{S}m_G$, and set $C_L:=\mathcal{I}_{F/k}\mathbb{Q}[\{L\overset{/k}{\hookrightarrow}F\}]$ for any extension L of k of finite type, cf. [R1], §6.

Lemma 4.1. *If $n=\infty$ and $r=1$ then the projection $\mathbb{Q}[\{k(X)\overset{/k}{\hookrightarrow}\mathcal{O}_v\}]\longrightarrow C_{k(X)}$ is surjective for any irreducible variety X over k.*

Proof. Let $P:k(X)\overset{/k}{\hookrightarrow}F$ be a generic point. Choose a subfield $k'\subset\mathcal{O}_v\cap\overline{P(k(X))}$ over k projecting isomorphically onto $(\mathcal{O}_v\cap\overline{P(k(X))})/(\mathfrak{m}_v\cap\overline{P(k(X))})\subset\kappa(v)$.

Let $k'(C):=k'P(k(X))\subset F$. If P does not factor through $\mathcal{O}_v$ then $k'(C)$ is the function field of a smooth proper curve over k'. As the class of P in $C_{k(X)}$ belongs to the image of $\mathbb{Q}[\{k'(C)\overset{/k'}{\hookrightarrow}F\}]\longrightarrow\mathcal{I}_{/k'}\mathbb{Q}[\{k'(C)\overset{/k'}{\hookrightarrow}F\}]=\mathrm{Pic}(C_F)_{\mathbb{Q}}\longrightarrow C_{k(X)}$, it remains only to check the surjectivity of $\mathbb{Q}[\{k'(C)\overset{/k'}{\hookrightarrow}\mathcal{O}_v\}]\longrightarrow\mathrm{Pic}(C_F)_{\mathbb{Q}}$.

By Corollary 3.5 from [R1], the generic points $k'(C)\overset{/k'}{\hookrightarrow}\kappa(v)$ generate $\mathrm{Pic}(C_{\kappa(v)})$.

Let us check the transitivity of the $(G_v^{\dagger}\cap G_{F/k'})$-action on the generic fibres of the specialization $\mathrm{Pic}(C_F)\overset{s}{\twoheadrightarrow}\mathrm{Pic}(C_{\kappa(v)})$ (i.e. over all $k'(\mathrm{Pic}^jC)\overset{\sigma}{\hookrightarrow}\kappa(v)$). Let $x_1,\dots,x_g$ be a transcendence basis of $k'(\mathrm{Pic}^jC)$ over k', and $\xi_1,\xi_2:k'(\mathrm{Pic}^jC)\overset{/k}{\hookrightarrow}\mathcal{O}_v$ be a pair of liftings of σ. Then there exists an element $\tau\in G_v^{\dagger}\cap G_{F/k'}$ such that $\tau\xi_1x_j=\xi_2x_j$ for all $1\le j\le g$. The field $k'(\mathrm{Pic}^jC)$ is generated over $k'(x_1,\dots,x_g)$ by an algebraic element f. According to Hensel's lemma, $\tau\xi_1f=\xi_2f$. (Note, that one can replace Pic$C$ by an arbitrary smooth variety over k'.)

Thus, the kernel of s coincides with $\{\tau\widetilde{\sigma}-\widetilde{\sigma}\mid\tau\in G_v^{\dagger}\cap G_{F/k'}\}$, and therefore, the generic points $k'(C)\overset{/k'}{\hookrightarrow}\mathcal{O}_v$ (i.e. whose specializations are also generic) generate $\mathrm{Pic}(C_F)$. $\square$

Corollary 4.2. *$\Gamma(W)=W_v=W$ for any $W\in\mathcal{I}_G$.*

Proof. This follows from Lemma 4.1 in the case of $W=C_{k(X)}$. As $C_{k(X)}$ for all irreducible varieties X over k form a system of generators of $\mathcal{I}_G$, and the functor $W\mapsto W_v$ preserves the surjections, we get that $W_v=W$ for arbitrary $W\in\mathcal{I}_G$. $\square$

Lemma 4.3. *Suppose that $n=\infty$. Then $(W_1\otimes W_2)_v\subseteq(W_1)_v\otimes(W_2)_v$ and $\Gamma(W_1\otimes W_2)\subseteq\Gamma(W_1)\otimes\Gamma(W_2)$ for any $W_1,W_2\in\mathcal{S}m_G$. However, $(W\otimes W)_v\neq W_v\otimes W_v$ if $W=\mathbb{Q}[F-k]$.*

If either W_1 is a quotient of $A(F)$ for a commutative algebraic k-group A, or $W_1 \in \mathcal{I}_G$ then $(W_1 \otimes W_2)_v = (W_1)_v \otimes (W_2)_v$ for any $W_2 \in \mathcal{S}m_G$.

Proof. The first follows from $(W_1 \otimes W_2)^{G_{F/F'}} = W_1^{G_{F/F'}} \otimes W_2^{G_{F/F'}}$, the second follows from the first. One has $(W \otimes W)_v \not\ni [x] \otimes [x'] \in W_v \otimes W_v$ for any pair of distinct $x, x' \in \mathcal{O}_v^\times$ such that $x \equiv x' \bmod \mathfrak{m}_v$.

Let $w \in W_2^{G_{F/F'}}$. Then $G_{F/F'} \subset G_{F/L} \subseteq \mathrm{Stab}_w$, where $L \subset F'$ is of finite type over k.

If F'' is an algebraically closed subfield in $\mathcal{O}_v$ and either F'' and L are algebraically independent over k in $\kappa(v)$, or $L \subseteq F''$ then $\overline{LF''} \subset \mathcal{O}_v$, so $W_1^{G_{F/F''}} \otimes w \subseteq (W_1 \otimes W_2)^{G_{F/\overline{LF''}}} \subseteq (W_1 \otimes W_2)_v$. If W_1 is of type as above then $W_1^{G_{F/F''}}$, with F'' and L algebraically independent over k in $\kappa(v)$, generate $(W_1)_v$, i.e., $(W_1)_v \otimes w \subseteq (W_1 \otimes W_2)_v$, thus finally, $(W_1)_v \otimes (W_2)_v \subseteq (W_1 \otimes W_2)_v$. $\square$

REMARKS. 1. By Corollary 4.2 and Lemma 4.3, $\Gamma(W \otimes F) = W \otimes k$ for any $W \in \mathcal{I}_G$, so $\Gamma(V) \otimes F \twoheadrightarrow V$ for any semi-linear quotient V of $W \otimes F$.

2. If W is an F-vector space then the F-vector space structure $F \otimes W \longrightarrow W$ on W induces an $\mathcal{O}_v$-module structure $(F \otimes W)_v = \mathcal{O}_v \otimes W_v \longrightarrow W_v$ on W_v. Clearly, $F \otimes_{\mathcal{O}_v} W_v \longrightarrow W$ is injective, but not surjective, as shows the example of $W = F[\{L \overset{/k}{\hookrightarrow} F\}]$.

3. Clearly, Γ_r preserves the injections, but not the surjections. Namely, let $W := \bigotimes_k^N F \longrightarrow \Omega^{N-1}_{F/k}$ be given by $a_1 \otimes \cdots \otimes a_N \mapsto a_1 da_2 \wedge \cdots \wedge da_N$. Then $W_v = \bigotimes_k^N \mathcal{O}_v$ if $n \geq 2N$, so $(\bigotimes_F^{N-1} \Omega^1_{F/k_0})_v = \bigotimes_F^{N-1} \Omega^1_{\mathcal{O}_v/k_0}$ for any $k_0 \subseteq k$; and $\Gamma(\bigotimes_k^N F) = k$, but $\Gamma_r(\Omega^\bullet_{F/k}) = \Omega^\bullet_{F/k,\mathrm{reg}}$ for any $r \geq 1$.

Proof. Clearly, $W^{G_{F/F'}} = \bigotimes_k^N F'$, so $W_v \subseteq \bigotimes_k^N \mathcal{O}_v$. Let $\omega = x_1 \otimes \cdots \otimes x_N$ for some $x_1, \dots, x_N \in F'$ whose images in $\kappa(v)$ algebraically independent over $k(\overline{y_1}, \dots, \overline{y_N})$ for some $y_1, \dots, y_N \in \mathcal{O}_v$. Then for each $1 \leq i \leq N$ there exists an element $\sigma_i \in G_v \cap G_{F/k(x_1,\dots,\widehat{x_i},\dots,x_N,y_1,\dots,y_N)}$ such that $\sigma_i x_i = x_i + y_i$. Then $\prod_{i=1}^N (\sigma_i - 1)\omega = y_1 \otimes \cdots \otimes y_N$. $\square$

In the case $n = \infty$ one can also apply Lemma 4.3.

4. $\mathbb{Q}[\{L \overset{/k}{\hookrightarrow} F\}]_v = \mathbb{Q}[\{L \overset{/k}{\hookrightarrow} \mathcal{O}_v\}]$.

For an integral normal k-variety X with $k(X) \subset F$ let $\mathfrak{V}(X)$ be the set of all discrete valuations of F of rank one trivial on k such that their restrictions to $k(X)$ are either trivial, or correspond to divisors on X. Set $\mathcal{W}(X) := W^{G_{F/k(X)}} \cap \bigcap_{v \in \mathfrak{V}(X)} W_v \subseteq W$.

REMARK. $W^{G_{F/k(X)}} \cap W_v$ depends only on the restriction of v to $k(X)$, since the set of $G_{F/k(X)}$-orbits $G_{F/k(X)} \backslash G/G_v$ of the valuations of F coincides, by Proposition 3.3, with the set of valuations of $k(X)$ of rank $\leq r$. E.g., if the restriction of v to $k(X)$ is trivial then $W^{G_{F/k(X)}} \subseteq W_v$.

Consider the following site $\mathfrak{H}$. Objects of $\mathfrak{H}$ are the smooth varieties over k. Morphisms in $\mathfrak{H}$ are the locally dominant morphisms, transforming non-dominant divisors to divisors. Coverings are smooth morphisms surjective over the generic point of any divisor on the target.

Lemma 4.4. *A choice of k-embeddings into F of all generic points of all smooth k-varieties defines a sheaf $\mathcal{W}$ on $\mathfrak{H}$ for any $W \in \mathcal{S}m_G$.*

Proof. Clearly, if a dominant morphism $f : U \longrightarrow X$ transforms divisors on U, non-dominant over X, to divisors on X then $\mathfrak{V}(U) \subseteq \mathfrak{V}(X)$, so $\mathcal{W}(X) \subseteq \mathcal{W}(U)$.

If, moreover, the pull-back of any divisor on X is a divisor on U then $\mathfrak{V}(X)=\mathfrak{V}(U)$.

By Lemma 1.1 of [JR], the sequence $0\longrightarrow\prod_{x\in X^0}W^{G_{F/k(x)}}\longrightarrow\prod_{x\in U^0}W^{G_{F/k(x)}}\longrightarrow\prod_{x\in(U\times_XU)^0}W^{G_{F/k(x)}}$ is exact (in fact, $X\mapsto\prod_{x\in X^0}W^{G_{F/k(x)}}$ is a sheaf on a topology $\mathfrak{D}m_k$). As $\mathfrak{V}(X)=\mathfrak{V}(U)=\mathfrak{V}(U\times_XU)$, the sheaf property for the covering f amounts to the exactness of the above sequence restricted to $\prod_{x\in U^0}\bigcap_{v\in\mathfrak{V}(X)}W_v$. $\square$

4.2. The "specialization" functor.

Lemma 4.5. *If $r=1$ and $n=\infty$ then $H_0(G_v^{\dagger},-)=-_{G_v^{\dagger}}$ gives functors $\mathcal{S}m_G\longrightarrow\mathcal{S}m_{G_{\kappa(v)/k}}$ and $\mathcal{I}_G\longrightarrow\mathcal{I}_{G_{\kappa(v)/k}}$. More precisely, there are natural surjections $W^{G_{F/F'}}\twoheadrightarrow H_0(G_v^{\dagger},W_v)$ and $\mathcal{I}_{\kappa(v)/k}H_0(G_v^{\dagger},W)\twoheadrightarrow H_0(G_v^{\dagger},\mathcal{I}W)$ for any $W\in\mathcal{S}m_G$.*

Proof. For any smooth representation W of G the stabilizer of any vector $\overline{w}\in H_0(G_v^{\dagger},W)$ contains the stabilizer in G_v of its preimage $w\in W$, which implies the smoothness of $H_0(G_v^{\dagger},W)$, since the projection $G_v\twoheadrightarrow G_{\kappa(v)/k}$ is open.

The first surjection is evident. Clearly, it is $G_{F'/k}$-equivariant.

By Lemma 6.7 of [R1], the functor $H^0(G_{F/F'},-)$ induces equivalences of categories $\mathcal{S}m_G\xrightarrow{\sim}\mathcal{S}m_{G_{F'/k}}$ and $\mathcal{I}_G\xrightarrow{\sim}\mathcal{I}_{G_{F'/k}}$, so $H_0(G_v^{\dagger},W_v)\in\mathcal{S}m_{G_{\kappa(v)/k}}$ and, by Corollary 4.2, $H_0(G_v^{\dagger},\mathcal{I}W)=H_0(G_v^{\dagger},(\mathcal{I}W)_v)\in\mathcal{I}_{G_{\kappa(v)/k}}$. $\square$

Lemma 4.6. *Let X be a smooth proper variety over k and $q\geq 0$.*

Then $H_0(G_v^{\dagger},CH^q(X_F))=CH^q(X_{\kappa(v)})$ and $H_0(G_v^{\dagger},\Omega^{\bullet}_{F/k,\mathrm{reg}})=\Omega^{\bullet}_{\kappa(v)/k,\mathrm{reg}}$.

Proof. The specialization homomorphism $CH^q(X_F)\twoheadrightarrow CH^q(X_{\kappa(v)})$ (cf. [S]) is $G_v^{\dagger}$-invariant, so it factors through the coinvariants $H_0(G_v^{\dagger},CH^q(X_F))$. Clearly, the composition of the surjection $CH^q(X_{F'})=CH^q(X_F)^{G_{F/F'}}\overset{\xi}{\twoheadrightarrow}H_0(G_v^{\dagger},CH^q(X_F))$ from Lemma 4.5 with the specialization coincides with the natural isomorphism $CH^q(X_{F'})\xrightarrow{\sim}CH^q(X_{\kappa(v)})$. The case of $\Omega^{\bullet}_{F/k,\mathrm{reg}}$ is similar. $\square$

REMARK. Let us show that $CH^q(X\times_k\mathcal{O}_v)=CH^q(X_F)$. One has a short exact sequence of Gersten complexes of flasque sheaves on $X\times_k\mathcal{O}_v$:

$$0\longrightarrow\coprod_{x\in X^{\bullet-1}}K_{q-\bullet}(\kappa(x))\longrightarrow\coprod_{x\in(X\times_k\mathcal{O}_v)^{\bullet}}K_{q-\bullet}(\kappa(x))\longrightarrow\coprod_{x\in X_F^{\bullet}}K_{q-\bullet}(\kappa(x))\longrightarrow 0,$$

the end of whose long exact cohomological sequence looks as

$$H^{q-1}(X_F,\mathcal{K}_q)\xrightarrow{\partial}CH^{q-1}(X)\longrightarrow CH^q(X\times_k\mathcal{O}_v)\longrightarrow CH^q(X_F)\longrightarrow 0.$$

The composition of the surjections $CH^{q-1}(X_F)\otimes F^{\times}\overset{id\cdot v}{\twoheadrightarrow}CH^{q-1}(X_F)\overset{\mathrm{sp}}{\twoheadrightarrow}CH^{q-1}(X)$ factors through $CH^{q-1}(X_F)\otimes F^{\times}\longrightarrow H^{q-1}(X_F,\mathcal{K}_q)$, which implies the surjectivity of ∂, and therefore, that $CH^q(X\times_k\mathcal{O}_v)\longrightarrow CH^q(X_F)$ is an isomorphism. $\square$

Let $\mathcal{I}_G^r$ be the maximal full subcategory of $\mathcal{I}_G$, for whose objects W the natural map $W^{G_{F/F'}}\longrightarrow H_0(G_v^{\dagger},W)$ is an isomorphism.

Corollary 4.7. *If $n=\infty$ then $\mathcal{I}_G^1$ is an abelian category closed under taking subquotients in $\mathcal{I}_G$.*

Proof. For any $W \in \mathcal{I}^1_G$ and any short exact sequence $0 \longrightarrow W_1 \longrightarrow W \longrightarrow W_2 \longrightarrow 0$ the rows in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_1^{G_{F/F'}} & \longrightarrow & W^{G_{F/F'}} & \longrightarrow & W_2^{G_{F/F'}} & \longrightarrow & 0 \\ & & \alpha_1 \downarrow & & \downarrow\cong & & \downarrow \alpha_2 & & \\ & & H_0(G_v^\dagger, W_1) & \longrightarrow & H_0(G_v^\dagger, W) & \longrightarrow & H_0(G_v^\dagger, W_2) & \longrightarrow & 0 \end{array}$$

are exact (the upper – since, $H^0(G_{F/F'}, -) : \mathcal{S}m_G \longrightarrow \mathcal{S}m_{G_{F'/k}}$ is an equivalence of categories, cf. [R1], Lemma 6.7). By Corollary 4.2 and Lemma 4.5, α_1 and α_2 are surjective, so they are isomorphisms. □

REMARKS. 1. So far I do not know any smooth representation W with non-injective $W^{G_{F/F'}} \longrightarrow H_0(G_v^\dagger, W_v)$. But first, I do not know, whether $W \mapsto W_v$ is exact.

2. The functor $H_0(G_v^\dagger, -) : \mathcal{S}m_G \longrightarrow \mathcal{S}m_{G_{\kappa(v)/k}}$ is neither full, since $\mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}]^G = 0$ for $r \geq d > 0$, but $H_0(G_v^\dagger, \mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}])^{G_{\kappa(v)/k}} = \mathbb{Q}[\mathcal{P}^d_{k(X)}]$, nor faithful, since $(F^\times)^G = k^\times$ and $F^G = k$, but $H_0(G_v^\dagger \cap G_{F/F'}, F^\times) = H_0(G_v^\dagger \cap G_{F/F'}, F) = 0$.[5] Also one has $H_0(G_v^\dagger, V) = 0$ for any semi-linear representation V of G over F, if $\mathrm{tr.deg}(\kappa(v)/k) = \infty$.

Proof. Let $w \in V$ and $G_{F/L} \subseteq \mathrm{Stab}_w$ for some L of finite type over k. Choose $t \in \mathfrak{m}_v \backslash \{0\}$ and $f \in F'$ transcendental over $L(t)$. Then there is $\sigma \in G_{F/L(t)} \cap G_v^\dagger$ such that $\sigma f - f = t$, i.e. $w = \sigma(t^{-1} f w) - t^{-1} f w$. □

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[5] since for any $x \in F \backslash \mathcal{O}_v$ there are $\sigma, \tau \in G_v^\dagger \cap G_{F/F'}$ such that $\sigma t = t x^{-1}$ and $\tau x = 2x$, and $F \backslash \mathcal{O}_v$ generates both $F^\times$ and F.

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