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Nuclear Physics B 992 (2023) 116210

NUCLEAR PHYSICS B

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Renormalization of three-quark operators at two loops in the RI′/SMOM scheme

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Received 18 July 2022; received in revised form 3 April 2023; accepted 20 April 2023
Available online 5 May 2023
Editor: Hong-Jian He

Abstract

We consider the renormalization of the three-quark operators without derivatives at next-to-next-to-leading order in QCD perturbation theory at the symmetric subtraction point. This allows us to obtain conversion factors between the $\overline{\text{MS}}$ scheme and the regularization invariant symmetric MOM (RI/SMOM, RI′/SMOM) schemes. The results are presented both analytically in R_ξ gauge in terms of a set of master integrals and numerically in Landau gauge. They can be used to reduce the errors in determinations of baryonic distribution amplitudes in lattice QCD simulations.

1. Introduction

Light cone distribution amplitudes (DAs) play an important rôle in the analysis of hard exclusive reactions involving large momentum transfer from the initial to the final state. The cases of baryon asymptotic states have been considered already long ago [1–3].

The theoretical description of DAs is based on the relation of their moments to matrix elements of local operators. Such matrix elements involve long-distance dynamics and, thus, cannot be accessed via perturbation theory alone.

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https://doi.org/10.1016/j.nuclphysb.2023.116210

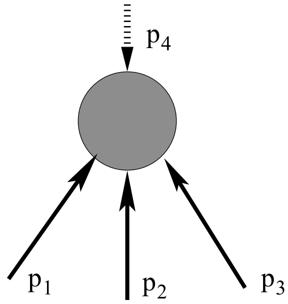

Fig. 1. Matrix element $\langle \mathcal{O}(p_4)\bar{u}(p_1)\bar{d}(p_2)\bar{s}(p_3)\rangle$ of a three-quark baryonic operator in momentum space, where we omit all spinor and color indices. The momentum $p_4 = -(p_1 + p_2 + p_3)$ is the one coming into the operator.

First estimates of the lower moments of the baryon DAs have been obtained more than 30 years ago using QCD sum rules [4–8]. An alternative way to access the moments is to calculate them from first principles using lattice QCD. Such studies for the nucleon DAs have a long history [9–12]. More recently, this analysis has been extended to include the full SU(3) octet of baryons [13].

To renormalize the matrix elements on the lattice, the RI′/SMOM scheme [14] was used in Ref. [13]. However, in order to embed lattice estimations of hadronic matrix elements into the complex of other studies and to assure comparability, it is necessary to present the result in the widely used $\overline{\text{MS}}$ scheme. Since the RI′/SMOM prescription can be used in both perturbative and nonperturbative calculations, the conversion from the RI′/SMOM to the $\overline{\text{MS}}$ scheme can be evaluated perturbatively as a power series in the strong-coupling constant $\alpha_s(\mu)$ at some typical mass scale μ of the order of a few GeV.

The letter “S” in RI′/SMOM stands for symmetric configuration of kinematics. For the bilinear quark operator, this implies that the virtualities of the momenta of the quarks and the operator itself are all taken at the same euclidean point μ^2. Such analyses including operators with zero, one, and two derivatives, were done at one loop in Refs. [14,15] and at two loops in Refs. [16–20]. These calculations were performed fully analytically. In our previous works [21,22], we have evaluated the matching constants for the bilinear quark operators with up to two derivatives and up to three loops with RI′/SMOM subtraction numerically. Our results for the mass operator [21] have been confirmed later in Ref. [23]. Similar results in the RI′/MOM scheme may be found in Refs. [16,24].

By contrast, baryonic operators consist of three quark fields, which we may call u, d, s, the actual flavor structure being irrelevant for the following discussion, and a number of covariant derivatives D_μ:

$$\epsilon^{ijk}\left(D_{\mu_1}\ldots D_{\mu_l}u\right)^i_{\xi_1}\left(D_{\mu_{l+1}}\ldots D_{\mu_{l+m}}d\right)^j_{\xi_2}\left(D_{\mu_{l+m+1}}\ldots D_{\mu_{l+m+n}}s\right)^k_{\xi_3},$$

where i, j, k are color indices, μ_i are Lorentz indices, and ξ_i are spinor indices. The matrix element of such an operator is shown schematically in Fig. 1.

The simplest baryonic operators, without derivatives, were studied in perturbative QCD a long time ago. In Refs. [25,26], the anomalous dimensions of the octet baryonic currents were evaluated at two loops in the $\overline{\text{MS}}$ scheme working in Feynman gauge. In Ref. [27], the renormalization of the three-quark operators with open indices, which are considered also here, was performed at two loops in Feynman gauge as well. In Ref. [28], this was extended to arbitrary gauge, and the anomalous dimensions of the baryonic operators were provided at three loops, thus lifting the results of Refs. [25–27] by one loop.

In addition, in Ref. [28], two-loop matrix elements were evaluated at the RI′/MOM subtraction point with zero-momentum operator insertion. Specifically, the kinematics of the MOM scheme adopted in Ref. [28] implies that $p_1^2 = p_2^2 = p_3^2 = \mu^2$ for some euclidean point μ^2, while $p_4 = 0$. However, it has been argued that the presence of zero momentum can generate additional sensitivity to infrared dynamics, which aggravates lattice QCD analyses [14]. This problem may be avoided by selecting a more symmetric kinematic configuration of the baryonic current matrix elements, where all four virtualities p_i^2 are taken to coincide at some euclidean point μ^2. This leaves residual freedom of how to fix the scalar products $p_i \cdot p_j$ with $i \neq j$. The most symmetric setting would be $p_i \cdot p_j = -\mu^2/3$ for $i \neq j$. However, this choice of kinematics turns out to be technically inconvenient in lattice QCD analyses. A recent such analysis [13] used the following kinematics:

$$\begin{aligned} p_1^2 = p_2^2 = p_3^2 = p_4^2 = \mu^2\,, \\ p_1 \cdot p_2 = p_3 \cdot p_1 = -\frac{\mu^2}{2}\,, \\ p_2 \cdot p_3 = 0\,, \end{aligned} \tag{1}$$

where μ^2 is the euclidean subtraction point of the SMOM scheme. To allow for direct comparisons of our results with those of Ref. [13], we adopt Eq. (1) in this paper.

An auxiliary analysis, based on differential equations, has revealed that even in the most symmetric kinematics, let alone the kinematics of Eq. (1), the analytic results beyond one loop cannot be expressed in terms of polylogarithms, but include a more complicated class of special functions, involving elliptic structures. In this work, we present our two-loop results in numerical form. We perform the renormalization of the three-quark operators for RI′/SMOM kinematics, which allows for the lowest moments of the baryonic DAs to be converted between the RI′/SMOM and $\overline{\text{MS}}$ schemes.

It is well known that the $\overline{\text{MS}}$ renormalization prescription is ambiguous for operators with more than one open spinor chain. This happens because there are more tensor structures in d dimensions than in four. In particular, there are operators in d dimensions that have no counterparts in four dimensions. These operators vanish as $d \to 4$ and are traditionally called *evanescent operators*. One cannot simply ignore these structures, since they mix with the physical operators under renormalization [29,30], thus leading to finite contributions.

To take these contributions properly into account, we use the renormalization scheme proposed in Ref. [27]. The idea is to express all d-dimensional operators in terms of d-dimensional tensor structures built from antisymmetrized products of n gamma matrices, $\Gamma_n = (1/n!)\gamma_{[\mu_1} \ldots \gamma_{\mu_n]}$. All structures involving Γ_n with $n > 4$ are evanescent and vanish for $d = 4$. We first renormalize the coefficients in front of these tensor structures and then take the limit $d \to 4$. Since the γ matrix structures are now convoluted with renormalized, finite quantities, we can safely put $\Gamma_n = 0$ for $n > 4$.

This scheme has another very useful property: we completely avoid any problems with γ_5 in dimensional regularization. However, the disadvantage of this scheme is that we are confronted with a large number of different spin tensor structures.

This paper is organized as follows. In Section 2, we introduce our notations and definitions. In Section 3, we discuss the tensor decomposition and the renormalization procedure. In Section 4, we present sample results, while our complete results are provided in ancillary files submitted to the ArXiv along with this paper. In Section 5, we present our conclusions. In Appendix A, we expose the relevant spin tensor structures. In Appendix B, we list useful four-dimensional identities for the spin tensor structures.

2. Basic setup

The basic object for the three-quark operators without derivatives located at the origin is the amputated four-point function,

$$\begin{aligned} H_{\beta_1\beta_2\beta_3,\alpha_1\alpha_2\alpha_3}(p_1,p_2,p_3) = & -\int d^4x_1\, d^4x_2\, d^4x_3 e^{i(p_1\cdot x_1+p_2\cdot x_2+p_3\cdot x_3)}\epsilon^{b_1b_2b_3}\epsilon^{a_1a_2a_3} \\ & \times \langle u^{b_1}_{\beta_1}(0) d^{b_2}_{\beta_2}(0) s^{b_3}_{\beta_3}(0) \bar{u}^{a_1}_{\alpha_1'}(x_1) \bar{d}^{a_2}_{\alpha_2'}(x_2) \bar{s}^{a_3}_{\alpha_3'}(x_3)\rangle \\ & \times G_2^{-1}(p_1)_{\alpha_1'\alpha_1} G_2^{-1}(p_2)_{\alpha_2'\alpha_2} G_2^{-1}(p_3)_{\alpha_3'\alpha_3}\,, \end{aligned} \tag{2}$$

where all quantities are to be understood as Euclidean. The quark flavors are called u, d, and s, but the only essential feature is that they are all different. All masses are supposed to vanish. α_i and β_j are spinor indices, a_k and b_l are color indices in the fundamental representation, and p_m are external momenta. The matrix element of the three-quark operator is shown schematically in Fig. 1.

The two-point function $G_2(p)$ required for the amputation of the external legs is defined by

$$\delta^{a'a}\, G_2(p)_{\alpha'\alpha} = \int d^4x\, e^{ip\cdot x} \langle u^{a'}_{\alpha'}(0) \bar{u}^{a}_{\alpha}(x)\rangle\,. \tag{3}$$

Our goal is to evaluate the matrix element (2) in the kinematics defined by Eq. (1) at the two-loop order.

3. Tensor decomposition and projection

As was already mentioned in the Introduction, we renormalize Eq. (2) without contracting the spinor indices and projecting on some particular baryonic currents. For this purpose, let us decompose the tensor in Eq. (2) as

$$H_{\beta_1\beta_2\beta_3,\alpha_1\alpha_2\alpha_3}(p_1,p_2,p_3) = \sum_{n=1}^{N} T_{n,\beta_1\beta_2\beta_3,\alpha_1\alpha_2\alpha_3}(p_1,p_2,p_3)\, f_n\left(\{p_i\cdot p_j\}\right)\,, \tag{4}$$

where T_n are spin tensor structures and f_n are scalar form factors. The explicit construction of these structures is discussed in Appendix A. The form factors f_n generally depend on six kinematic invariants, p_1^2, p_2^2, p_3^2, $p_1\cdot p_2$, $p_2\cdot p_3$, and $p_3\cdot p_1$. In the following discussion, we omit spinor indices and arguments, and simply write

$$H = \sum_{n=1}^{N} T_n\, f_n\,. \tag{5}$$

The upper limit N of summation in Eqs. (4) and (5) is the number of the linearly independent spin tensor structures. It depends on the number of loops. We also have to distinguish between the decompositions in d and four dimensions. In d dimensions, the number of independent structures is larger, owing to the presence of evanescent operators. The values N of independent form factors through two loops are given in Table 1.

Table 1
Number of form factors for different numbers of loops in d and four dimensions.

# of loops	0	1	2
N (in d dimensions)	1	67	581
N (in 4 dimensions)	1	64	247

Let us introduce the following notation. If $X_{\beta_1\beta_2\beta_3,\alpha_1\alpha_2\alpha_3}$ is an object with six spinor indices, we denote by $\mathrm{tr}_3(X)$ the trace over three pairs of indices, i.e.,

$$\mathrm{tr}_3(X) = \sum_{\alpha_1,\alpha_2,\alpha_3=1}^{4} X_{\alpha_1\alpha_2\alpha_3,\alpha_1\alpha_2\alpha_3} \,. \tag{6}$$

Using this definition, we can introduce the symmetric $N \times N$ matrix

$$M_{kn} = \mathrm{tr}_3(T_k T_n)\,, \tag{7}$$

where T_j are the spin tensor structures from Eqs. (4) and (5). Then, the projectors on the form factors f_j take the form

$$P_l = \sum_{k=1}^{N} M^{-1}_{lk}\, T_k\,, \tag{8}$$

where M^{-1} is the inverse matrix, and we obviously have

$$f_l = \mathrm{tr}_3\,(P_l H)\;. \tag{9}$$

The use of Eqs. (8) and (9) for unrenormalized amplitudes is delicate within dimensional regularization, since the projectors P_l depend nontrivially on the dimension d. A better way is to first renormalize the amplitude H in Eq. (2) and then to use the projectors in four dimensions. In order to achieve this, we construct N scalar amplitudes a_k as

$$a_k = \mathrm{tr}_3(T_k H)\,, \qquad k = 1, \ldots, N\,. \tag{10}$$

After renormalization of all amplitudes a_k in the $\overline{\mathrm{MS}}$ scheme, the form factors can be obtained as

$$f_l = \sum_{k=1}^{N} M^{-1}_{lk}\, a_k\,, \tag{11}$$

where M^{-1} is now taken in four dimensions. In this limit, all elements of M^{-1} are just rational numbers.

However, in four dimensions, we cannot apply the formula in Eq. (11) directly, since the determinant of the matrix M_{lk} is then zero. This may be understood from Table 1 by observing that the number of independent structures in four dimensions is less than in d dimensions. In this case, we need to solve for the unknown $\vec{f}$ the system (in matrix notation)

$$M\vec{f} = \vec{a}\,, \tag{12}$$

where $\vec{f} = (f_1, \ldots, f_N)^T$, etc.

The system (12) is over-determined, but consistent by construction. We find the solution for $\vec{f}$ in the form

$$\vec{f} = \vec{f}_0 + \sum_{j=0}^{N_d - N_4} C_j \vec{y}_j \,, \tag{13}$$

where $\vec{f}_0$ is some particular solution of system (12), the vectors $\vec{y}_j$ form a basis of the $N_d - N_4 = 334$ dimensional null space of the matrix M, and C_j are arbitrary constants.

After renormalization, we have 581 two-loop form factors f_n in four dimensions, 247 of which are linearly independent. We have calculated all of them analytically in R_ξ gauge in terms of a set of complicated master integrals, which we have evaluated numerically.

Our calculational procedure is similar to Refs. [21,22]. As usual, the evaluation of the Feynman diagrams is organized in two steps: the reduction to master integrals and the evaluation of the latter. After the projection and the evaluation of the color and Dirac traces, we first reduce the large number of Feynman integrals using integration-by-parts (IBP) relations [31] to a small set of master integrals. This is done with the help of the computer package FIRE [32]. Besides the IBP relations, we have additional relations arising from the symmetric kinematics. With these new relations, we can further reduce the number of master integrals. Finally, we can express all the Feynman diagrams in terms of 4 one-loop master integrals, to be evaluated analytically, and 44 two-loop master integrals, to be evaluated numerically. As a by-product of our analytic two-loop calculation, we reproduce the well-known renormalization constant of the baryonic operator in the $\overline{\text{MS}}$ scheme at this order [25–28].

For the numerical evaluation of the two-loop master integrals, we adopt the method of sector decomposition [33,34], which is based on the analytic resolution of singularities and the successive numerical integration of the parametric integrals by Monte Carlo methods. This is done with the help of the program package FIESTA [35]. At the two-loop level, we have up to seven-fold parametric integrals resulting usually in several hundreds of so-called sector integrals, which are then evaluated numerically using the program library CUBA [36]. With a typical sample of 10^8 function calls, we achieve a relative accuracy of order 10^{-6} for individual master integrals. However, due to large cancellations between different terms, the resulting relative accuracy is expected to be worse.

4. Results

Because of their large number, we refrain from listing the renormalized two-loop form factors f_n here, but supply them in ancillary files submitted to the ArXiv along with this manuscript. Specifically, we present our analytic results in R_ξ gauge in the form of Eq. (13) including explicit expressions with the constants C_i, and our numerical results for $C_i = 0$. To obtain the results in Landau gauge, one sets $\xi = 0$.

To illustrate the structure and typical size of the corrections, we present here, in numerical form, the two-loop form factor f_1, corresponding to the structure $\Gamma_0 \otimes \Gamma_0 \otimes \Gamma_0$, in R_ξ gauge. We have

$$\begin{aligned}\frac{f_1}{f_{1,\text{Born}}} &= 1 + \frac{a}{3}[3 - c_2 + \ln 2 + \xi(9 - 2c_1 - c_2 - \ln 2)] \\ &\quad + a^2\left[10.4515(4) + 3.5881(4)\xi + 1.4232(2)\xi^2 - 0.68933(2)n_f\right] \\ &= 1 + a(0.6204053307351691 + 0.5957023845688996\,\xi)\end{aligned}$$

$$+a^2\left(10.4515(4)+3.5881(4)\xi+1.4232(2)\xi^2-0.68933(2)n_f\right), \tag{14}$$

where $f_{1,\mathrm{Born}}=\epsilon^{ijk}\epsilon^{ijk}=6$ is the Born result, $a=\alpha_s/\pi$, n_f is the number of light quark flavors, ξ is the gauge parameter, and

$$c_1=\frac{4}{\sqrt{3}}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)=2.3439072386894588906\ldots, \tag{15}$$

$$c_2=2\,\mathrm{Cl}_2\left(\frac{\pi}{2}\right)=1.8319311883544380301\ldots, \tag{16}$$

with

$$\mathrm{Cl}_2(\theta)=-\int_0^\theta d\phi\,\ln\left(2\sin\frac{\phi}{2}\right) \tag{17}$$

being Clausen's integral. The errors quoted in Eq. (14) reflect the uncertainties from the numerical integration of the two-loop master integrals.

Having all spin tensor components of the matrix element H in Eq. (2) at our disposal, we are now in a position to build any baryonic current without derivatives. As examples, let us evaluate the matrix elements of the baryonic currents previously considered in Refs. [25,26,28],

$$(O_1)_\alpha=\epsilon^{ijk}u^i_\alpha\left[(u^j)^T C d^k\right], \tag{18}$$

$$(O_2)_\alpha=\epsilon^{ijk}\gamma_5 u^i_\alpha\left[(u^j)^T C\gamma_5 d^k\right], \tag{19}$$

where C is the unitary, antisymmetric matrix, with $C^{-1}=C^\dagger=-C^*$, of charge conjugation,

$$C\gamma_\mu C^{-1}=-\gamma_\mu^T, \tag{20}$$

$$C\gamma_5 C^{-1}=\gamma_5^T. \tag{21}$$

The matrix γ_5 in Eqs. (19) and (21) is taken to be four-dimensional, since we are working here with finite renormalized quantities.

The matrix elements of the baryonic currents O_1 and O_2 in Eqs. (18) and (19) may be decomposed as

$$\left\langle (O_{1,2})_\delta(p_4)\bar{u}_\alpha(p_1)\bar{u}_\beta(p_2)\bar{d}_\gamma(p_3)\right\rangle=\sum_n\left(\Gamma_{\alpha\delta}\otimes\Gamma_{\beta\gamma}\right)_n f_n^{O_{1,2}}, \tag{22}$$

with some scalar form factors $f_n^{O_{1,2}}$ being linear combinations of f_l as defined in Eq. (9). It is actually sufficient to consider O_1, since the expression for O_2 may be obtained by multiplying that for O_1 with $\gamma_5\otimes\gamma_5$. Exploiting the four-dimensional identities listed in Appendix B, we may reduce the number n of form factors in Eq. (22) down to 32, some of which are zero.

Omitting spinor indices, we thus obtain

$$\begin{aligned}
O_1=I\otimes I[&1+a(2.00280429+0.549441049\,\xi)\\
&+a^2(26.486(1)+3.03333(7)\xi+1.3064(3)\xi^2-2.79941(6)n_f)]\\
&+\left(I\otimes\sigma_{p_1p_2}+I\otimes\sigma_{p_3p_1}\right)[-a\times 0.52086828\\
&+a^2(-10.85904(3)-0.03004(3)\xi-0.12631(9)\xi^2+0.75237(3)n_f)]\\
&+I\otimes\sigma_{p_2p_3}[a(1.38629436+0.462098120\,\xi)
\end{aligned}$$

$$
\begin{aligned}
&+ a^2(34.345(1) + 5.98739(6)\xi + 1.95648(2)\xi^2 - 2.4645(1)n_f)] \\
&+ \gamma_5 \otimes \gamma_5\, a^2(-1.59871(1) - 0.02545(2)\xi) \\
&+ \left(\gamma_5 \otimes \gamma_5\sigma_{p_1p_2} + \gamma_5 \otimes \gamma_5\sigma_{p_3p_1}\right) a^2(0.14070(2) + 0.22190(3)\xi) \\
&+ \gamma_5 \otimes \gamma_5\sigma_{p_2p_3}\, a^2(0.19179(7) + 0.03528(2)\xi) \\
&+ \left(\sigma_{p_1p_2} \otimes I - \sigma_{p_3p_1} \otimes I\right) [-a \times 0.52086828 \\
&+ a^2(12.9088(4) + 0.6163(4)\xi + 0.1125(1)\xi^2 - 0.752365(4)n_f)] \\
&+ \left(\gamma_5\sigma_{p_1p_2} \otimes \gamma_5 - \gamma_5\sigma_{p_3p_1} \otimes \gamma_5\right) a^2(-0.3571(1) - 0.02859(4)\xi) \\
&+ \left(\sigma_{p_1p_2} \otimes \sigma_{p_1p_2} - \sigma_{p_3p_1} \otimes \sigma_{p_3p_1}\right) [a(0.076423831 - 0.076423831\,\xi) \\
&+ a^2(0.6196(9) - 0.652(2)\xi - 0.196(7)\xi^2 - 0.084915(3)n_f)] \\
&+ \left(\sigma_{p_2p_3} \otimes \sigma_{p_1p_2} - \sigma_{p_2p_3} \otimes \sigma_{p_3p_1}\right) a^2(-0.0041(4) - 0.0301(1)\xi) \\
&+ \left(\sigma_{p_1p_2} \otimes \sigma_{p_3p_1} - \sigma_{p_3p_1} \otimes \sigma_{p_1p_2}\right) a^2(-0.5875(8) - 0.1445(5)\xi + 0.0346(1)\xi^2) \\
&+ \left(\sigma_{p_1p_2} \otimes \sigma_{p_2p_3} - \sigma_{p_3p_1} \otimes \sigma_{p_2p_3}\right) a^2(1.1520(6) + 0.7480(8)\xi + 0.1593(1)\xi^2) \\
&+ \sigma_{p_2p_3} \otimes \sigma_{p_2p_3}\, a^2(2.95198(6) - 6.2222(3)\xi) \\
&+ \left(\sigma_{\mu p_2} \otimes \sigma^{\mu p_1} - \sigma_{\mu p_3} \otimes \sigma^{\mu p_1}\right) [a \times 0.111111111 \\
&+ a^2(2.6434(7) - 0.06095(1) + 0.0387(1)\xi)] \\
&+ \left(\sigma_{\mu p_1} \otimes \sigma^{\mu p_2} - \sigma_{\mu p_1} \otimes \sigma^{\mu p_3}\right) [a(0.37154525 + 0.26043414\,\xi) \\
&+ a^2(9.6843(8) + 3.0743(2)\xi + 0.9510(1)\xi^2 - 0.697562(2)n_f)] \\
&+ \left(\sigma_{\mu p_2} \otimes \sigma^{\mu p_3} - \sigma_{\mu p_3} \otimes \sigma^{\mu p_2}\right) a^2(1.3642(2) + 0.5083(1)\xi) \\
&+ \left(\sigma_{\mu p_2} \otimes \sigma^{\mu p_2} - \sigma_{\mu p_3} \otimes \sigma^{\mu p_3}\right) [a \times 0.22222222 \\
&+ a^2(2.9803(8) + 0.1734(1)\xi - 0.208701(1)n_f)] \\
&+ \sigma_{\mu p_1} \otimes \sigma^{\mu p_1}\, a^2 \times 4.2853(1)\,.
\end{aligned} \tag{23}
$$

5. Conclusion

In this work, we have established a framework for the evaluation of the corrections to the baryonic current without derivatives through the two-loop order. The main difficulty in the study of the baryonic operators is the presence of evanescent operators that mix under renormalization with the physical operators. This leads to a large mixing matrix and the necessity for finite renormalizations. On the other hand, if we use the open-indices approach, there is no ambiguity in the interpretation within the $\overline{\mathrm{MS}}$ scheme. Exploiting this observation, we have evaluated all the form factors appearing through two loops and presented them in a numerical form that is ready for use in lattice QCD simulations.

CRediT authorship contribution statement

Bernd Kniehl: research, manuscript preparation. **Oleg Veretin:** research, manuscript preparation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

We are grateful to Vladimir M. Braun, Meinulf Göckeler, and Alexander N. Manashov for fruitful discussions. O.L.V. is grateful to the University of Hamburg for the warm hospitality. This work was supported in part by the German Research Foundation DFG through Research Unit FOR 2926 "Next Generation Perturbative QCD for Hadron Structure: Preparing for the Electron-Ion Collider" with Grant No. KN 365/13-1.

Appendix A. Spin tensor structures

In this Section, we explicitly enumerate all linearly independent spin tensor structures T_n through two loops in d dimensions. All tensors T_n are represented as tensor products of three Dirac structures, as

$$T_{\alpha_1\alpha_2\alpha_3,\beta_1\beta_2\beta_3} = \Gamma_{\alpha_1\beta_1} \otimes \Gamma_{\alpha_2\beta_2} \otimes \Gamma_{\alpha_3\beta_3} \,. \tag{24}$$

The building blocks Γ are antisymmetric products of Dirac γ matrices,

$$\Gamma_0 = \mathbb{1} \,, \tag{25}$$

$$\Gamma_{\mu_1\mu_2} = \frac{1}{2!}\gamma_{[\mu_1}\gamma_{\mu_2]} \,, \tag{26}$$

$$\Gamma_{\mu_1\mu_2\mu_3\mu_4} = \frac{1}{4!}\gamma_{[\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4]} \,, \tag{27}$$

where $\mathbb{1}$ is the unit Dirac matrix and square brackets $[\dots]$ denote antisymmetrization. Notice that Dirac structures with odd numbers of Dirac matrices do not appear in our calculation.

We also introduce the following notation for the contraction of a vector and a tensor (Schoonship notation):

$$p^\mu \Gamma_{\dots\mu\dots} = \Gamma_{\dots p\dots} \,. \tag{28}$$

Furthermore, we introduce the following wildcards: **p** can take one of p_1, p_2, p_3, **pp** can take one of $p_1 p_2$, $p_2 p_3$, $p_3 p_1$, and **ppp** stands for $p_1 p_2 p_3$.

For the sake of systematics, we assign to each tensor structure a signature, which is an ordered triplet of the numbers 0, 2, and 4 of γ matrices appearing in each Γ factor, and a number $[p]$ counting the overall appearances of momenta. Furthermore, we distinguish between symmetric and non-symmetric structures. The symmetric structures do not have co-partners arising under the change of order of the Γ factors in the tensor products, while the non-symmetric ones do. So, the numbers of non-symmetric structures should be multiplied by 3. In Tables 2 and 3, we systematically list the symmetric and non-symmetric tensor structures, respectively, and specify the number (#) of entities for each signature and each value of $[p]$. We also give the total number (##) of entities for each signature.

Table 2
Symmetric structures ordered according to their signatures and values of $[p]$, numbers # of entities for given signature and value of $[p]$, and total numbers ## of entities for given signature.

signature	$[p]$	tensor structure	#	##
000	0	$\Gamma_0 \otimes \Gamma_0 \otimes \Gamma_0$	1	1
222	0	$\Gamma_{\mu_1\mu_2} \otimes \Gamma_{\mu_2\mu_3} \otimes \Gamma_{\mu_3\mu_1}$	1	
222	6	$1/(-\mu^2)^3\, \Gamma_{\mathbf{pp}} \otimes \Gamma_{\mathbf{pp}} \otimes \Gamma_{\mathbf{pp}}$	27	28

Table 3
Non-symmetric structures. The meaning of the columns is the same as in Table 2.

signature	$[p]$	tensor structure	#	##
200	2	$1/(-\mu^2)^1\, \Gamma_{\mathbf{pp}} \otimes \Gamma_0 \otimes \Gamma_0$	3	3
220	0	$\Gamma_{\mu_1\mu_2} \otimes \Gamma_{\mu_1\mu_2} \otimes \Gamma_0$	1	
220	2	$1/(-\mu^2)^1\, \Gamma_{\mathbf{p}\mu_1} \otimes \Gamma_{\mathbf{p}\mu_1} \otimes \Gamma_0$	9	
220	4	$1/(-\mu^2)^2\, \Gamma_{\mathbf{pp}} \otimes \Gamma_{\mathbf{pp}} \otimes \Gamma_0$	9	19
222	2	$1/(-\mu^2)^1\, \Gamma_{\mathbf{p}\mu_1} \otimes \Gamma_{\mathbf{p}\mu_2} \otimes \Gamma_{\mu_1\mu_2}$	9	
222	2	$1/(-\mu^2)^1\, \Gamma_{\mathbf{pp}} \otimes \Gamma_{\mu_1\mu_2} \otimes \Gamma_{\mu_1\mu_2}$	3	
222	4	$1/(-\mu^2)^2\, \Gamma_{\mathbf{pp}} \otimes \Gamma_{\mathbf{p}\mu_1} \otimes \Gamma_{\mathbf{p}\mu_1}$	27	39
402	2	$1/(-\mu^2)^1\, \Gamma_{\mathbf{pp}\mu_1\mu_2} \otimes \Gamma_0 \otimes \Gamma_{\mu_1\mu_2}$	3	
402	4	$1/(-\mu^2)^2\, \Gamma_{\mathbf{ppp}\mu_1} \otimes \Gamma_0 \otimes \Gamma_{\mathbf{p}\mu_1}$	3	6
420	2	$1/(-\mu^2)^1\, \Gamma_{\mathbf{pp}\mu_1\mu_2} \otimes \Gamma_{\mu_1\mu_2} \otimes \Gamma_0$	3	
420	4	$1/(-\mu^2)^2\, \Gamma_{\mathbf{ppp}\mu_1} \otimes \Gamma_{\mathbf{p}\mu_1} \otimes \Gamma_0$	3	6
440	0	$\Gamma_{\mu_1\mu_2\mu_3\mu_4} \otimes \Gamma_{\mu_1\mu_2\mu_3\mu_4} \otimes \Gamma_0$	1	
440	2	$1/(-\mu^2)^1\, \Gamma_{\mathbf{p}\mu_1\mu_2\mu_3} \otimes \Gamma_{\mathbf{p}\mu_1\mu_2\mu_3} \otimes \Gamma_0$	9	
440	4	$1/(-\mu^2)^2\, \Gamma_{\mathbf{pp}\mu_1\mu_2} \otimes \Gamma_{\mathbf{pp}\mu_1\mu_2} \otimes \Gamma_0$	9	
440	6	$1/(-\mu^2)^3\, \Gamma_{\mathbf{ppp}\mu_1} \otimes \Gamma_{\mathbf{ppp}\mu_1} \otimes \Gamma_0$	1	20
422	0	$\Gamma_{\mu_1\mu_2\mu_3\mu_4} \otimes \Gamma_{\mu_1\mu_2} \otimes \Gamma_{\mu_3\mu_4}$	1	
422	2	$1/(-\mu^2)^1\, \Gamma_{\mathbf{p}\mu_1\mu_2\mu_3} \otimes \Gamma_{\mathbf{p}\mu_1} \otimes \Gamma_{\mu_2\mu_3}$	9	
422	2	$1/(-\mu^2)^1\, \Gamma_{\mathbf{p}\mu_1\mu_2\mu_3} \otimes \Gamma_{\mu_2\mu_3} \otimes \Gamma_{\mathbf{p}\mu_1}$	9	
422	2	$1/(-\mu^2)^1\, \Gamma_{\mathbf{pp}\mu_1\mu_2} \otimes \Gamma_{\mu_2\mu_3} \otimes \Gamma_{\mu_3\mu_1}$	3	
422	4	$1/(-\mu^2)^2\, \Gamma_{\mathbf{pp}\mu_1\mu_2} \otimes \Gamma_{\mathbf{p}\mu_1} \otimes \Gamma_{\mathbf{p}\mu_2}$	27	
422	4	$1/(-\mu^2)^2\, \Gamma_{\mathbf{pp}\mu_1\mu_2} \otimes \Gamma_{\mathbf{pp}} \otimes \Gamma_{\mu_1\mu_2}$	9	
422	4	$1/(-\mu^2)^2\, \Gamma_{\mathbf{pp}\mu_1\mu_2} \otimes \Gamma_{\mu_1\mu_2} \otimes \Gamma_{\mathbf{pp}}$	9	
422	4	$1/(-\mu^2)^2\, \Gamma_{\mathbf{ppp}\mu_1} \otimes \Gamma_{\mathbf{p}\mu_2} \otimes \Gamma_{\mu_1\mu_2}$	3	
422	4	$1/(-\mu^2)^2\, \Gamma_{\mathbf{ppp}\mu_1} \otimes \Gamma_{\mu_1\mu_2} \otimes \Gamma_{\mathbf{p}\mu_2}$	3	
422	6	$1/(-\mu^2)^3\, \Gamma_{\mathbf{ppp}\mu_1} \otimes \Gamma_{\mathbf{pp}} \otimes \Gamma_{\mathbf{p}\mu_1}$	9	
422	6	$1/(-\mu^2)^3\, \Gamma_{\mathbf{ppp}\mu_1} \otimes \Gamma_{\mathbf{p}\mu_1} \otimes \Gamma_{\mathbf{pp}}$	9	91

Appendix B. Four-dimensional identities

In $d = 4$ dimensions, all Γ matrices with more than four indices vanish. For Eqs. (26) and (27), we may write

$$\begin{aligned} \Gamma_{\mu_1\mu_2} &= \sigma_{\mu_1\mu_2}\,, \\ \Gamma_{\mu_1\mu_2\mu_3\mu_4} &= \varepsilon_{\mu_1\mu_2\mu_3\mu_4}\gamma_5\,, \end{aligned} \tag{29}$$

where $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}$ is the totally antisymmetric tensor.

For any four-momenta $q_1, \ldots, q_4$, the following identities hold:

$$\begin{aligned}
\Gamma_{\mu_1\mu_2\mu_3\mu_4} \otimes \Gamma^{\mu_1\mu_2\mu_3\mu_4} &= 24\gamma_5 \otimes \gamma_5\,, \\
\Gamma_{\mu_1\mu_2\mu_3 q_1} \otimes \Gamma^{\mu_1\mu_2\mu_3 q_2} &= 6(q_1\cdot q_2)\gamma_5 \otimes \gamma_5\,, \\
\Gamma_{\mu_1\mu_2 q_1 q_2} \otimes \Gamma^{\mu_1\mu_2 q_3 q_4} &= 2\left[(q_1\cdot q_3)(q_2\cdot q_4) - (q_1\cdot q_4)(q_2\cdot q_3)\right]\gamma_5 \otimes \gamma_5\,, \\
\Gamma_{\mu_1 q_1 q_2 q_3} \otimes \Gamma^{\mu_1 q_1 q_2 q_3} &= \Big[q_1^2 q_2^2 q_3^2 + 2(q_1\cdot q_2)(q_2\cdot q_3)(q_3\cdot q_1) \\
&\quad - q_1^2(q_2\cdot q_3)^2 - q_2^2(q_3\cdot q_1)^2 - q_3^2(q_1\cdot q_2)^2\Big]\gamma_5 \otimes \gamma_5\,, \\
\Gamma_{\mu_1\mu_2 q_1 q_2} \otimes \Gamma^{\mu_1\mu_2} &= -2\gamma_5 \otimes \gamma_5\sigma_{q_1 q_2}\,, \\
\Gamma_{\mu_1 q_1 q_2 q_3} \otimes \Gamma^{\mu_1 q_4} &= -(q_1\cdot q_4)\gamma_5 \otimes \gamma_5\sigma_{q_2 q_3} - (q_2\cdot q_4)\gamma_5 \otimes \gamma_5\sigma_{q_3 q_1} \\
&\quad - (q_3\cdot q_4)\gamma_5 \otimes \gamma_5\sigma_{q_1 q_2}\,.
\end{aligned} \tag{30}$$

There is one more identity that relates the 19 structures of the form $\sigma_{\mu\nu} \otimes \sigma^{\mu\nu}$, $\sigma_{\mu\mathbf{p}} \otimes \sigma^{\mu\mathbf{p}}$, and $\sigma_{\mathbf{pp}} \otimes \sigma^{\mathbf{pp}}$ in four dimensions. This relation is quite cumbersome for arbitrary kinematics. We present it here for the particular kinematics of Eq. (1):

$$\begin{aligned}
0 = \sigma_{\mu\nu} \otimes \sigma^{\mu\nu} &+ \frac{4}{(-\mu^2)^2}\left(\sigma_{p_1p_2} \otimes \sigma^{p_1p_2} + \sigma_{p_2p_3} \otimes \sigma^{p_2p_3} + \sigma_{p_3p_1} \otimes \sigma^{p_3p_1}\right) \\
&- \frac{2}{(-\mu^2)^2}\left(\sigma_{p_1p_2} \otimes \sigma^{p_2p_3} + \sigma_{p_3p_1} \otimes \sigma^{p_2p_3} + \sigma_{p_2p_3} \otimes \sigma^{p_1p_2} + \sigma_{p_2p_3} \otimes \sigma^{p_3p_1}\right) \\
&- \frac{2}{(-\mu^2)}\left(\sigma_{\mu p_1} \otimes \sigma^{\mu p_2} + \sigma_{\mu p_2} \otimes \sigma^{\mu p_1} + \sigma_{\mu p_1} \otimes \sigma^{\mu p_3} + \sigma_{\mu p_3} \otimes \sigma^{\mu p_1}\right) \\
&- \frac{1}{(-\mu^2)}\left(\sigma_{\mu p_2} \otimes \sigma^{\mu p_3} + \sigma_{\mu p_3} \otimes \sigma^{\mu p_2}\right) \\
&- \frac{3}{(-\mu^2)}\left(\sigma_{\mu p_2} \otimes \sigma^{\mu p_2} + \sigma_{\mu p_3} \otimes \sigma^{\mu p_3}\right) - \frac{4}{(-\mu^2)}\sigma_{\mu p_1} \otimes \sigma^{\mu p_1}\,.
\end{aligned} \tag{31}$$

Appendix C. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.nuclphysb.2023.116210.

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